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Floer Homology for Oriented 3-Manifolds

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Dedicated to Professor Akio Hattori on his sixtieth birthday

Contents

§1. Introduction

In [F], A. Floer introduced a new invariant for homology 3-spheres. In this paper we generalize his invariant to arbitrary closed and oriented 3-manifolds. In the case when the first homology group of the manifold is torsion free and nonzero, we also define invariants $I_k^s(M)$ for $s < 3$, which, in the case $s = 0$, is a generalization of Floer's one. The construction of this invariant is closely related also to the Donaldson's polynomial for closed 4-manifolds [D4]. The construction is based on the study of the moduli space of selfdual connections over $M \times \mathbf{R}$ and its compactification.

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In this section, we describe briefly the construction of our invariant. Throughout this paper, we let M be an oriented 3-manifold, σ a Riemannian metric on it. It induces the Hodge $*$-operator, $*_\sigma : \Lambda^k(M) \to \Lambda^{3-k}(M)$. We consider the trivial $SU(2)$ bundle over M. Let

$$\mathcal{A}(M) = \{d + a \mid a \in \Gamma(M, \Lambda^1 \otimes su(2))\}$$

be the set of all smooth connections of it. (In later sections, we work with Sobolev spaces but in this section we omit those details.) Put

$$\begin{aligned}
\widehat{\mathcal{G}}(M) &= \{g : M \to SU(2) \mid C^\infty\text{-maps}\},\\
\mathcal{G}(M) &= \{g \in \widehat{\mathcal{G}}(M) \mid \deg g = 0\},\\
\mathcal{B}(M) &= \mathcal{A}(M)/\widehat{\mathcal{G}}(M),\\
\widetilde{\mathcal{B}}(M) &= \mathcal{A}(M)/\mathcal{G}(M),
\end{aligned}$$

where $\mathcal{G}(M)$ acts on $\mathcal{A}(M)$ by

$$g^*(d + a) = d + g^{-1}dg + g^{-1}ag.$$

Following Taubes [T4] and Floer [F], we define a functional $\mathfrak{cs} : \widetilde{\mathcal{B}}(M) \to \mathbf{R}$ by

$$\mathfrak{cs}(a) = \int_M \mathrm{Tr}(\frac{1}{2} a \wedge da + \frac{1}{3} a \wedge a \wedge a) \tag{1.1}$$

(Here and hereafter, we shall write a in place of $d + a$.) It is well known that the right hand side is $\mathcal{G}(M)$-invariant. The gradient flow of this functional is described by

$$\frac{\partial a_t}{\partial t} = *_\sigma F^{a_t}. \tag{1.2}$$

The idea of Floer and Taubes is to use this gradient flow in order to define the $\infty/2$-dimensional homology group of $\mathcal{B}(M)$. It is not in general true that $\mathrm{grad}\,\mathfrak{cs}$ is a Morse-Smale flow, then in [T4], [F], they used a perturbation of it. In their case, where M is a homology sphere, the singular locus $\mathcal{SB}(M)$ and the set of critical points of the flow $\mathrm{grad}\,\mathfrak{cs}$ intersect at one point, the trivial connection. (Recall that the singular locus of $\mathcal{B}(M)$ is the set of reducible connections, and a critical point of the flow $\mathrm{grad}\,\mathfrak{cs}$ is a flat connection.) In our case the intersection is

$$\mathrm{Hom}(\pi_1(M), U(1))/\mathbf{Z}_2. \tag{1.3}$$

which is $b_1(M)$-dimensional. In §2, using the sum of the traces of the holonomy along the generators of $H_1(M;\mathbf{Z})$, we shall find a functional $f : \mathcal{B}(M) \to \mathbf{R}$, such that the equation

$$*_\sigma F^a - \operatorname{grad}_a f = 0 \tag{1.4}$$

has only a finite number of solutions, each of which is nondegenerate (see §2 for definition.) A connected component of elements of the set of elements $\mathcal{SB}(M)$, the reducible connections, satisfying (1.4) is identified to an element of

$$\operatorname{Hom}(\operatorname{Tor} H_1(M;\mathbf{Z}), U(1))/\mathbf{Z}_2. \tag{1.5.1}$$

And each connected component is identified to

$$\operatorname{Hom}\left(\frac{H_1(M;\mathbf{Z})}{\operatorname{Tor} H_1(M;\mathbf{Z})}, \mathbf{Z}_2\right) \tag{1.5.2}$$

or its quotient by $\mathbf{Z}_2$. Put

$$Fl = \{a \in \widetilde{\mathcal{B}}(M) \mid a \text{ satisfy } (1.4)\}, \tag{1.6.1}$$

$$Fl_0 = \{a \in Fl \mid a \text{ is irreducible}\}. \tag{1.6.2}$$

For $a, b \in Fl_0$, we set

$$\mathcal{M}(a,b) = \left\{ a_t \,\middle|\, \begin{array}{l} a_t : (-\infty,\infty) \to \widetilde{\mathcal{B}}(M), a_t \text{ satisfies } (1.7), \\ \lim_{t\to\infty} a_t = b, \lim_{t\to-\infty} a_t = a \end{array} \right\}.$$

(The precise definition is in §3.) Here

$$\frac{\partial a_t}{\partial t} = *_\sigma F^a - \operatorname{grad}_{a_t} f. \tag{1.7}$$

In a way similar to [F], we can find a map $\mu : Fl_0 \to \mathbf{Z}$ such that

$$\dim \mathcal{M}(a,b) = \mu(a) - \mu(b),$$

for $a, b \in Fl_0$ (§5.) We can also prove that $\mathcal{M}(a,b)$ is orientable (§6). Then, following Witten [W1] and Floer [F], we put

$$C_k^0 = \bigoplus_{\substack{a \in Fl_0 \\ \mu(a)=k}} \mathbf{Z}[a] \tag{1.8}$$

We define a boundary operator $\partial : C_k^0 \to C_{k-1}^0$ as follows. (Again our construction is the same as Floer's.) The action of $\mathbf{R}$ on $M \times \mathbf{R}$ induces a free action of $\mathbf{R}$ on $\mathcal{M}(a,b)$. We put, for $a \in Fl_0$, $\mu(a) = k$,

$$\partial([a]) = \sum_{\mu(b)=k-1} \langle \partial a, b \rangle [b],$$

where $\langle \partial a, b\rangle$ is the difference of the number of connected components of $\mathcal{M}(a,b)$ for which the direction of its orientation and the $\mathbf{R}$ action coincide and the number of connected components for which the orientation is the opposite direction to the $\mathbf{R}$-action. In a way similar to [F], we can prove $\partial\partial = 0$. Then we define

$$I_k^0(M) = \frac{\operatorname{Ker}\partial : C_k^0 \to C_{k-1}^0}{\operatorname{Im}\partial : C_{k+1}^0 \to C_k^0},$$

which, we shall prove, is an invariant of M. (In fact, we need to fix a basis of $H_1(M;\mathbf{Z})$.)

As is pointed out by Donaldson, Atiyah [A] and Witten [W2], Floer homology is closely related to the Donaldson polynomial [D4]. In fact, in the case when M is a homology sphere and is a boundary of a 4-manifold satisfying some additional assumptions, it is possible to define a relative Donaldson polynomial, which has a value in $I_k^0(M)$. But in the case when the first Betti number of M is positive, it seems that the above boundary operator is not enough for such a purpose. Then we construct other boundary operators. To motivate our construction we recall the definition of relative Donaldson polynomial very briefly. (Our description is not precise since it is anounced that the precise description will appear in [DFK].) Let X be a 4 manifold such that its boundary $\partial X = M$ is a homology sphere. Let $[\Sigma_1], \cdots, [\Sigma_\ell] \in H_2(X)$, $a \in Fl_0$. By $\mathcal{M}_k(X;a)$, we denote the set of all gauge classes of self dual connections ∇ with $c^2(\nabla) = k$, $\nabla|_{\partial X} = a$. Define a line bundle $\mathcal{L}_{\Sigma_i}$ on it by

$$\mathcal{L}_{\Sigma_i}(\nabla) = \bigwedge^{\text{top}} \left(\operatorname{Ker} \eth_{\nabla|_{\Sigma_i}}\right)^* \otimes \bigwedge^{\text{top}} \operatorname{Coker} \eth_{\nabla|_{\Sigma_i}},$$

where $\eth_{\nabla|_{\Sigma_i}}$ is a Dirac operator on Σ_i twisted by the restriction of ∇ to Σ_i. We put

$$Q_\ell([\Sigma_1], \cdots, [\Sigma_\ell])(a) = \int_{\mathcal{M}_k(X;a)} c^1(\mathcal{L}_{\Sigma_1}) \cup \cdots \cup c^1(\Sigma_\ell).$$

Here we choose k, ℓ so that $\dim \mathcal{M}_k(X,a) = 2\ell$. We regard $Q_\ell([\Sigma_1], \cdots, [\Sigma_\ell])$ as a cochain, an element of $\operatorname{Hom}(C_m, 0)$ with $m =$

$\mu(a)$. Under an appropriate assumtion this cochain is a cocycle and its cohomology class is an invariant of X.

In case $\partial X_1 = \partial X_2 = M$, $X = X_1 \amalg_M X_2$, $\Sigma_1 \cdots \Sigma_{\ell_1} \subset X_2$, $\Sigma'_1 \cdots \Sigma'_{\ell_2} \subset X_2$, one can prove, under appropriate assumption, that

$$\begin{aligned} Q_{\ell_1+\ell_2}(\Sigma_1, \cdots, \Sigma_{\ell_1}, \Sigma'_1, \cdots, \Sigma'_{\ell_2}) \\ = \langle Q_{\ell_1}(\Sigma_1, \cdots, \Sigma_{\ell_1}), Q_{\ell_2}(\Sigma'_1, \cdots, \Sigma'_{\ell_2})\rangle, \end{aligned} \tag{1.9}$$

where $\langle \ , \ \rangle$ is a coupling between Floer cohomologies of M and M^-, (M with opposite orientation). Note that in case $H_1M = 0$, we have $H_2X = H_2X_1 \oplus H_2X_2$.

Now we remove the assumption $H_1M = 0$. Assume, for example $H_1X_1 = H_1X_2 = 0$. Then we have Mayer-Vietoris exact sequence:

$$H_2X_1 \oplus H_2X_2 \longrightarrow H_2X \longrightarrow H_1M \longrightarrow 0.$$

Fix a section $s : H_1M \to H_2X$. This is equivalent to choose, for each $[\gamma] \in H_1M$, surfaces $\Sigma_{(i)}(\gamma) \subset X_i$ with $\partial\Sigma_i(\gamma) = \gamma$ such that $s([\gamma]) = [\Sigma_{(1)}(\gamma) \cup \Sigma_{(2)}(\gamma)] = [\Sigma(\gamma)]$. To generalize (1.9) one needs to calculate

$$Q_{\ell_1+\ell_2+\ell_3}(\Sigma_1, \cdots, \Sigma_{\ell_1}, \Sigma(\gamma_1), \cdots, \Sigma(\gamma_{\ell_3}), \Sigma'_1, \cdots, \Sigma'_{\ell_2}),$$

in terms of invariants of X_1,X_2. So it is natural to consider cochains such as

$$\begin{aligned} &Q_{\ell+\ell'}(\Sigma_{(1)}, \cdots, \Sigma_\ell, \Sigma_{(1)}(\gamma_1), \cdots, \Sigma_1(\gamma_{\ell'}))(a) \\ &\qquad = \int_{\mathcal{M}_k(X_1,a)} \mathcal{L}_{\Sigma_1} \cup \cdots \cup \mathcal{L}_{\Sigma_\ell} \cup \mathcal{L}_{\Sigma_{(1)}(\gamma_1)} \cup \cdots \cup \mathcal{L}_{\Sigma_{(1)}(\gamma_{\ell'})}. \end{aligned}$$

But one finds that this cochain is *not* a cocycle in general. Hence in our situation, the relative Donaldson polynomial should not take a value on usual Floer cohomology but a generalization of it. Our purpose is to find such a generalization.

We assume that $H_1(M;\mathbf{Z})$ is torsion free. Choose a set of closed loops $\{\gamma_1, \cdots, \gamma_d\}$ representing a basis of $H_1(M;\mathbf{Z})$. Put $\Sigma_i = \gamma_i \times \mathbf{R} \subset M \times \mathbf{R}$. Let $a_t \in \mathcal{M}(a,b)$, $a, b \in Fl_0$. It induces a connection of a trivial $SU(2)$ bundle over Σ_i. Let $\eth_{a_t}$ be the Dirac operator on Σ_i twisted by the connection. We may assume that $a(\gamma_i) \neq 1$ for each $a \in Fl_0$. It implies that $\eth_{a_t}$ is Fredholm. Put

$$\operatorname{Det} \eth_{a_t} = \bigwedge^{\text{top}} (\operatorname{Ker} \eth_{a_t})^* \otimes \bigwedge^{\text{top}} \operatorname{Coker} \eth_{a_t}.$$

By taking $(\operatorname{Det}\bar{\partial}_{a_t})^{\otimes 2}$ and moving a_t on $\mathcal{M}(a,b)$, we obtain a complex line bundle on $\mathcal{M}(a,b)$, which is denoted by $\mathcal{L}_i^{(2)}$. (The reason why we have to take the square will be explained in §7.) Now, let $a, b \in Fl_0$ with $\mu(a) - \mu(b) = 2\ell + 1$. Put $\overline{\mathcal{M}}(a,b) = \mathcal{M}(a,b)/\mathbf{R}$. Then we can "define" the Chern number

$$\int_{\overline{\mathcal{M}}(a,b)} c^1(\mathcal{L}_{i_1}^{(2)}) \cup \cdots \cup c^1(\mathcal{L}_{i_\ell}^{(2)}) \in \mathbf{Z}.$$

This number is denoted by $\langle \partial_{i_1,\cdots,i_\ell} a, b\rangle$. (Since $\overline{\mathcal{M}}(a,b)$ has a boundary, the above number is, in fact, *not* well defined. This problem is discussed in §12.) We define $\partial_{i_1,\cdots,i_\ell} : C_k^0 \to C_{k-2\ell-1}^0$ by

$$\partial_{i_1,\cdots,i_\ell}([a]) = \sum_b \langle \partial_{i_1,\cdots,i_\ell} a, b\rangle [b].$$

Now we can state the main result of this paper. Let $\alpha \in \{1,\cdots,d\}^\ell / S_\ell$. (Here S_ℓ stands for the symmetric group.) We put $\partial_\alpha = \partial_{\alpha_1,\cdots,\alpha_\ell}$.

Theorem 1.10. *If $\sharp\alpha < 3$, and if $H_1(M;\mathbf{Z})$ is torsion free, then*

$$\sum_{\alpha^1 \cup \alpha^2 = \alpha} \partial_{\alpha^1}\partial_{\alpha^2} = 0.$$

Remark 1.11. In case when $\alpha = (1,1)$ the formula is:

$$\partial\partial_{1,1} + 2\partial_1\partial_1 + \partial_{1,1}\partial = 0.$$

Remark 1.12. For $\sharp\alpha > 2$ the formula is *not* correct. We discuss the reason in §12. There we also discuss why the formula may not be correct for $s > 0$ if $H_1(M;\mathbf{Z})$ has a torsion.

Now let $S^\ell H_1(M;\mathbf{Z})$ be the symmetric power. We put

$$C_k^s = \bigoplus_{\ell \le s} S^\ell H_1(M;\mathbf{Z}) \otimes C_{k+2\ell}^0.$$

Define $\partial_k^s : C_k^s \to C_{k-1}^s$ by

$$\partial_k^s(\gamma_\alpha \otimes [a]) = \sum_{\alpha^1 \cup \alpha^2 = \alpha} \gamma_{\alpha^1} \otimes \partial_{\alpha^2}[a],$$

where $\gamma_\alpha = \gamma_{\alpha_1} \otimes \cdots \otimes \gamma_{\alpha_\ell}$. Theorem 1.10 immediately implies

Corollary 1.13. *Suppose that $H_1(M;\mathbf{Z})$ is torsion free. For $s<3$ we have*

$$\partial_{k-1}^s \partial_k^s = 0.$$

We put

$$I_k^s(M) = \frac{\operatorname{Ker} \partial_k^s}{\operatorname{Im} \partial_{k-1}^s}.$$

Theorem 1.14. *Suppose that $H_1(M;\mathbf{Z})$ is torsion free. $I_k^s(M)$ does not depend on the choices of the metrics, γ_i's, etc, and is an invariant of M, equipped with a basis of $H_1(M;\mathbf{Z})$.*

By construction we have an exact sequence of complexes

$$0 \longrightarrow C_k^s \longrightarrow C_k^{s+1} \longrightarrow S^{s+1}(H_1(M;\mathbf{Z})) \otimes C_{k+2s+2}^0 \longrightarrow 0$$

It follows that:

Theorem 1.15. *Suppose that $H_1(M;\mathbf{Z})$ is torsion free. There exists a long exact sequence*

$$\longrightarrow I_k^s(M) \longrightarrow I_k^{s+1}(M) \longrightarrow S^{s+1}(H_1(M,\mathbf{Z})) \otimes I_{k+2s+2}^0(M) \longrightarrow$$

for $s=0$ or 1. The exact sequence is also an invariant of M.

The proof of these theorems is based on the detailed analysis of the end of the moduli space $\mathcal{M}(a,b)$. The results on it is in §7. In fact, we shall prove more general results than we need to construct our invariants. In the course, we develop various techniques, which might be useful in other situations.

Using our invariant $I_k^s(M)$, we can partially generalize the definition of relative Donaldson polynomial to the case when the boundary is not necessary a homology sphere. Those applications will appear elsewhere.

The organization of this paper is as follows.

In §2,3, we perturb the equation.

In §4, we review the sum formula for the index of the elliptic operators. We also discuss the sum formula of the family of indices.

This result is used in §5 to define the degree μ. In §5 we study also neighborhoods of various reducible connections.

In §6 we define the orientation of the moduli space. The fact that every oriented 3-manifolds bounds an oriented 4-manifold, is essentially used in the proof.

§§7–11 are devoted to the study of the end of moduli space $\mathcal{M}(a,b)$. The results of these sections are stated in §7.

In §8, we prove that the patching procedure of selfdual connections as in [T1] is possible in our situation, where various reducible connections must be dealt with.

In §9, we shall prove that the selfdual connections constructed in §8, contains all the connections in the end of the moduli space, except the concentrated ones. For this purpose, we establish a decay estimate such as in [FU].

Combining the results of §§8,9 we obtain a chart for a neighborhood of each point at infinity. In order to patch those charts, we introduce, in §10, the local action of the groups. This notion is a generalization of one introduced in [CG] to study the end of Riemannian manifolds. We use it to study the end of the moduli space.

The line bundle $\mathcal{L}_i^{(2)}$ is constructed and is extended to the boundary in §11. For this purpose we use the sum theorem for index bundles in §4 and the existence of the lift of the local action to the bundle.

Using the results of §§7–11, we define the boundary operator in §12 and prove Theorem 1.10. As is remarked before, the Chern number of the bundle $\mathcal{L}_i^{(2)}$ is not well defined. We shall prove in §12 that the boundary operator is well defined modulo isomorphism. In §12, we also discuss the case when $s = 3$ and describe why Theorem 1.10 does not hold in that case.

Finally we shall prove Theorems 1.14 and 1.15 in §13.

As the reader can find easily, this paper heavily depends on the brilliant ideas due to Donaldson, Floer, Taubes e.t.c. in their papers. Before this work is completed the author is informed (without the precise statement) that A. Floer generalized his invariant to homology $S^1 \times S^2$.

§2. Perturbation

Let L^p_ℓ be the Sobolev space of the sections, namely the set of sections L^p-norms of whose ℓ-th derivatives are finite. Put

$$\mathcal{A}^p_\ell(M) = \{d + a \mid a \in L^p_\ell(M, \wedge^1 \otimes su(2))\}$$
$$\mathcal{G}_\ell(M) = \text{the set of maps} : M \to SU(2) \text{ of } L^2_\ell\text{-class}.$$

$\mathcal{A}^2_\ell$ is denoted by $\mathcal{A}_\ell$. We choose sufficiently large ℓ and fix it throughout this paper. $\mathcal{G}_{\ell+1}$ acts on $\mathcal{A}_\ell$. (See [FU].) Put

$$\mathcal{B}_\ell(M) = \mathcal{A}_\ell(M)/\mathcal{G}_{\ell+1}(M).$$

Let $a \in \mathcal{A}_\ell(M)$. Then the set

$$\{u \in L^2_\ell(M, \wedge^1 \otimes su(2)) \mid d_a^* u = 0\} \tag{2.1}$$

is the orthonormal complement of $T_a\mathcal{G}_{\ell+1}a$ in $T_a\mathcal{A}_\ell(M)$. In the case when a is irreducible, the set (2.1) can be identified to $T_{[a]}\mathcal{B}_\ell(M)$. (See [FU].) We let the set (2.1) be denoted by $T_{[a]}\mathcal{B}_\ell(M)$ also in the case when a is reducible. In that case, $[a]$ is a singular point of $\mathcal{B}_\ell(M)$.

The purpose of this section is to perturb the functional $\mathfrak{cs}$ and the equation (1.2), so that (1.4) has only a finite number of solutions each of which is nondegenerate. We put

$$H_1'(M;\mathbf{Z}) = \frac{H_1(M;\mathbf{Z})}{\text{Torsion}}.$$

First we deal with singular points on

$$\operatorname{Hom}(H_1'(M;\mathbf{Z}), SU(2))/\text{conjugate} \subset \mathcal{B}_\ell(M).$$

Choose a set of loops $\{\ell_1^0, \cdots, \ell_d^0\}$ representing a basis of $H_1'(M;\mathbf{Z})$. Extend ℓ_i^0 to an embedding $\ell_i^0 : S^1 \times D^2 \to M$. Choose a nonnegative function u on D^2 with compact support such that

$$\int_{D^2} u(x)dx = 1.$$

For a loop $\ell : S^1 \to M$ and $a \in \mathcal{A}(M)$, let $h_\ell(a) \in SU(2)$ be the holonomy along ℓ. Define a functional f_0 on $\mathcal{B}_\ell(M)$ by

$$f_0(a) = \epsilon \sum_{i=1}^{d} \int \operatorname{Tr}\left(h_{\ell_i^0(\cdot,x)}(a)\right) u(x)dx, \tag{2.2}$$

where ϵ is a small positive number. Then by [F] 1b.1, $\operatorname{grad}_a f_o \in T_a\mathcal{B}_\ell(M)$ is well defined. Similarly we can define the hessian, $\operatorname{Hess}_a f_0 : T_{[a]}\mathcal{B}_\ell(M) \to T_{[a]}\mathcal{B}_\ell(M)$.

Here we examine the set, FR, of the flat reducible connections in $\mathcal{B}_\ell(M)$. The set of the conjugacy classes of the elements of $\operatorname{Hom}(\operatorname{Tor} H_1(M,\mathbf{Z}), U(1))$ has a one to one correspondence to $\pi_0(FR)$. For $\varphi \in \operatorname{Hom}(\operatorname{Tor} H_1(M,\mathbf{Z}), U(1))$, let FR_φ be the corresponding component. FR_φ is diffeomorphic to T^d if $\operatorname{Im}(\varphi) \not\subset \{\pm 1\}$, and is diffeomorphic to $T^d/\mathbf{Z}_2$ if $\operatorname{Im}(\varphi) \subset \{\pm 1\}$. Let $1 \in \operatorname{Hom}(\operatorname{Tor} H_1(M,\mathbf{Z}), \mathbf{Z})$ be the trivial representation.

Lemma 2.3. *There exists a neighborhood U of FR_1 such that, for sufficiently small ϵ, the set of elements of U satisfying*

$$*_\sigma F^a - \operatorname{grad}_a f_0 = 0 \tag{2.4}$$

is identified to $\mathrm{Hom}(H_1'(M,\mathbf{Z}),\mathbf{Z}_2) \simeq \{\pm 1\}^d$.

Proof. By identifying $FR_1 = \{(e^{i\theta_1},\cdots,e^{i\theta_d})\}/\mathbf{Z}_2$, we have

$$f_0(e^{i\theta_1},\cdots,e^{\theta_d}) = 2\epsilon\sum\cos\theta_i. \tag{2.5}$$

The lemma follows immediately.

Lemma 2.6. *Let* $a \in \mathrm{Hom}(H_1'(M,\mathbf{Z}),\mathbf{Z}_2)$. *Then* $\mathfrak{cs} - f_0$ *is non-degenerate at* a. *In other words*

$$*_\sigma d_a - \mathrm{Hess}_a\, f_0 : T_{[a]}\mathcal{B}_\ell(M) \to T_{[a]}\mathcal{B}_{\ell-1}(M)$$

is invertible.

Remark 2.7. $\mathrm{Hess}_a\, \mathfrak{cs} = *_\sigma d_a$. See [F],[T4].

Proof. We have

$$\mathrm{Ker}\, *_\sigma d_a \simeq H^1(M;\mathbf{R}) \otimes su(2) \simeq su(2)^d.$$

On this space $\mathrm{Hess}_a\, f_0$ is given by $-\epsilon\sum x_i^2$. Hence the lemma follows from the invertibility of the matrix

$$\begin{pmatrix} A+\epsilon E & \epsilon B \\ \epsilon C & \epsilon D \end{pmatrix}$$

for small ϵ and invertible A and D.

We take ϵ in (2.2) such that Lemma 2.6 holds and fix it.

Next we use a method similar to [D3] and [F]. Let $p_0 \in M$ and $v_0 \in T_{p_0}M$. Choose an embedding $I : D^2 \to M$, such that $I(0) = p_0$, and that $I_*(T_0D^2)$ is transversal to v_0. Let $\Gamma_1(p_0, I, v)$ be the set of smooth embeddings such that $\ell(1,0) = p_0$, $\frac{D\ell}{dt}(1,0) = v_0$, $\ell(0,x) = I(x)$. We put

$$\Gamma_m = \bigcup_{(p_0,v_0,I)} (\Gamma_1(p_0,v_0,I))^m.$$

Let $L_m = SU(2)^m/SU(2)$, where $SU(2)$ acts by conjugation. Define a map

$$\widetilde{\Phi}' : \mathcal{A}_\ell(M) \times \Gamma_m \to \mathrm{Map}(D^2, SU(2)^m)$$

by

$$\widetilde{\Phi}'(a,(\ell_1,\cdots,\ell_m))(x) = (h_{\ell_1(\cdot,x)}(a),\cdots,h_{\ell_m(\cdot,x)}(a)).$$

$\widetilde{\Phi}'$ induces a map

$$\Phi' : \mathcal{B}_\ell(M) \times \Gamma_m \to \mathrm{Map}(D^2, L_m).$$

Following [F], we choose $(\beta_i)_{i\in\mathbf{Z}_+}$ $(\beta_i > 0)$. and put

$$C^\beta(L_m, \mathbf{R}) = \{\psi \in C^\infty(L_m, \mathbf{R}) \mid \|\psi\|_\beta < \infty\},$$

where

$$\|\psi\|_\beta = \sum_{i=1}^{\infty} \beta_i \max_{x\in L_m} |D^i\psi(x)|.$$

Fix a function $u : D^2 \to [0, \infty)$ as before and define

$$\Phi : \mathcal{B}_\ell(M) \times \Gamma_m \times C^\beta(L_m, \mathbf{R}) \to \mathbf{R}$$

by

$$\Phi([a], (\ell_1, \cdots, \ell_m), \psi) = \int_{D^2} \psi(\Phi'([a], (\ell_1, \cdots, \ell_m)(x))u(x)dx.$$

For $v \in \Gamma_m \times C^\beta(L_m, \mathbf{R})$, we put $f_v([a]) = \Phi([a], v)$. For $\lambda = (\ell_1, \cdots, \ell_m) \in L_m$ and $\lambda' = (\ell'_1, \cdots, \ell'_{m'}) \in L_{m'}$, we say $\lambda \prec \lambda'$ if $\{\ell_1, \cdots, \ell_m\} \subset \{\ell'_1, \cdots, \ell'_{m'}\}$

Lemma 2.8. *There exists $\lambda_0 \in \Gamma_{m_0}$ and $\delta > 0$ such that for each $\lambda_0 \prec \lambda$, the set of $\psi \in C^\beta((L_m), \mathbf{R})$ satisfying the following conditions is of first category in $\{\psi \mid \|\psi\|_\beta < \delta\}$.*

(2.8.1) *The set $Fl(\psi)$ of the solution of*

$$*_\sigma F^a = \mathrm{grad}_a(f_0 + f_{(\lambda,\psi)}).$$

is finite.
(2.8.2) *For each $a \in Fl(\psi)$ the map*

$$*_\sigma d - \mathrm{Hess}_{[a]}(f_0 + f_{(\lambda,\psi)}) : T_a\mathcal{B}_\ell(M) \to T_a\mathcal{B}_{\ell-1}(M)$$

is invertible.

Proof. As is well known, (2.8.2) implies (2.8.1). Hence the problem is local on $\mathcal{B}_\ell(M)$. The argument in a neighborhood of irreducible connections is the same as [F] 2c.1. Then we study the neighborhood of the set of reducible connections. Precisely, we first take a perturbation so that (2.8.2) holds in a neighborhood of the set of the reducible connections, next we perturb again so that (2.8.1) and (2.8.2) holds, in the set of irreducible connections, as well.

Let $\varphi \in \mathrm{Hom}(\mathrm{Tor}\, H_1(M,\mathbf{Z}), SU(2))$. In the case when $\mathrm{Im}\,\varphi \subset \{\pm 1\}$, the proof of Lemma 2.6 works in a neighborhood of FR_φ. Then we assume that $\mathrm{Im}(\varphi) \not\subset \{\pm 1\}$. By the proof of Lemma 2.3, f_0 is a Morse function on Fl_φ and has exactly 2^d singular points on it. The same holds for $f_0 + f_{\lambda,\psi}$ if $\|\psi\|_\beta$ is small. Hence it suffices to work at a neighborhood of each singular point a_0. Choose a neighborhood U of a_0 with is of bounded L^2_ℓ norm.

Sublemma 2.9. *The set of ψ such that $*_\sigma d_a - \mathrm{Hess}_a(f_0 + f_{\lambda,\psi})$ is invertible for each $a \in U \cap Fl(\psi)$, is open.*

Proof. First we remark that the set

$$Fl(\psi) = \{[a] \in \mathcal{B}_\ell(M) \mid *_\sigma F^a = \mathrm{grad}_a(f_0 + f_{\lambda+\psi})\}$$

is independent of ℓ because the equation is elliptic modulo gauge transformation. Hence we can find a bounded subset L in $L^2_{\ell+2}(M, \wedge^1 \otimes su(2))$ such that if

$$\|\psi' - \psi\|_\beta < \delta \tag{2.10.1}$$

$$[a] \in Fl(\psi) \tag{2.10.2}$$

$$[a] \in U \tag{2.10.3}$$

then $[a] = [a_0 + u]$ for some $u \in L$. Now, if the sublemma is false, then, there exists ψ,ψ_i and a_i such that

(2.11.1) $\lim_{i\to\infty} \|\psi_i - \psi\|_\beta = 0$,
(2.11.2) $[a_i] \in Fl(\psi_i)$,
(2.11.3) $[a_i] \in U$,
(2.11.4) $*_\sigma d_{a_i} - \mathrm{Hess}_{a_i}(f_0 + f_{\lambda,\psi_i})$ is not invertible,
(2.11.5) $*_\sigma d_a - \mathrm{Hess}_a(f_0 + f_{\lambda,\psi})$ is invertible for each $a \in Fl(\psi) \cap U$.

We can choose $u_i \in L$ such that $[a_0 + u_i] = [a_i]$. By Rellich's Theorem, we can find a subsequence such that u_i converges to u_∞ in $L^2_{\ell+1}$. Hence by (2.11.1),(2.11.2) and (2.11.3), we have $[a_0 + u_\infty] = [a_\infty] \in U \cap Fl(\psi)$. Therefore $*_\sigma d_{a_\infty} - \mathrm{Hess}_{a_\infty}(f_0 + f_{\lambda,\psi})$ is invertible. On the other hand, we remark that the map

$$\begin{aligned} \mathcal{A}_{\ell+1}(M) \times L^2_\ell(M, \wedge^1 \otimes su(2)) &\to L^2_{\ell-1}(M, \wedge^1 \otimes su2)) \\ : (a,u) &\mapsto *_\sigma d_a u - \mathrm{Hess}_a(f_0 + f_{\lambda,\psi})u \end{aligned}$$

is continuous. (See [FU]). It follows that $*_\sigma d_{a_i} - \mathrm{Hess}_{a_i}(f_0 + f_{\lambda,\psi_i})$ is invertible for sufficiently large i. This contradicts (2.11.4). The proof of Sublemma 2.9 is now complete.

Hence it suffices to show that the set of ψ for which

$$*_\sigma d_{a_0} - \mathrm{Hess}_{a_0}(f_0 + f_{(\lambda,\psi)})$$

is surjective, is dense. We can choose a loop ℓ_0 so that $\varphi(\ell_0) \notin \{\pm 1\}$ and assume $\{\ell_0\} \prec \lambda = (\ell_1, \cdots, \ell_m)$. Put

$$\widetilde{\Phi}'(a_0, \lambda)(0) = (g_1, \cdots, g_m).$$

We have

$$\{g \in SU(2) \mid g^{-1}(g_1, \cdots, g_m)g = (g_1. \cdots, g_m)\} \simeq U(1) \tag{2.12}$$

Hence $[g_1, \cdots, g_m]$ is contained in $U(1)^m/\mathbf{Z}_2 \subset SU(2)^m/SU(2)$ and is a regular point of $U(1)^m/\mathbf{Z}_2$. Put

$$\mathcal{B}_\ell^{\mathrm{red}}(M) = \{[a] \in \mathcal{B}_\ell^{\mathrm{red}}(M) \mid a \text{ is reducible.}\}$$

It follows from (2.12) that $[a_0]$ is a regular point of $\mathcal{B}_\ell^{\mathrm{red}}(M)$. Therefore, by a $U(1)$ analogue of [F] 2c.1, we may assume that

$$*_\sigma d_{a_0} - \mathrm{Hess}_{a_0}(f_0 + f_{\lambda,\psi}) : T_{[a_0]}(\mathcal{B}_\ell^{\mathrm{red}}(M)) \to T_{[a_0]}(\mathcal{B}_\ell^{\mathrm{red}}(M)) \tag{2.13}$$

is invertible. Put

$$K_\psi = \{u \in T_{[a_0]}\mathcal{B}_\ell(M) \mid *_\sigma d_{a_0} u - \mathrm{Hess}_{a_0}(f_0 + f_{\lambda,\psi})u = 0\}$$

By the invertibility of (2.13) we have

$$K_\psi \cap T_{[a_0]}\mathcal{B}_\ell^{\mathrm{red}}(M) = \{0\}. \tag{2.14}$$

The group

$$U(1) = \{g \in \mathcal{G}_\ell(M) \mid g^* a_0 = a_0\} \tag{2.15}$$

acts on K_ψ. By (2.14) and the finite dimensionality of K_ψ, we can identify $K_\psi \simeq \mathbf{C}^k$. Therefore by taking sufficiently large λ and m we may assume that

$$P : K_\psi \to T_{(g_1,\cdots,g_m)} SU(2)^m$$

is injective, where P is the differential at $[a_0]$ of the map : $[a] \mapsto \widetilde{\Psi}'(a, \lambda)(0) : \mathcal{A}_\ell(M) \to SU(2)^m$. By (2.8), $U(1)$ acts on $T_{(g_1,\cdots,g_m)}SU(2)^m$, which we can identify to $\mathbf{C}^m \oplus \mathbf{R}^m$. The map P

is $U(1)$ invariant. Hence we may assume that $P(K_\psi) \subset \mathbf{C}^m$. We define a function ψ' in a neighborhood of $(g_1, \cdots, g_m)$ by

$$\psi'(\exp_{(g_1,\cdots,g_m)}(z_1,\cdots,z_m,t_1,\cdots,t_m)) = -\sum |z_i|^2, \tag{2.16}$$

and extend it to a $SU(2)$ invariant function on $SU(2)^m$. We obtain a function on L_m, for which we use the same symbol. Now it is easy to see that

$$*_\sigma d_{a_0} - \mathrm{Hess}_{a_0}(f_0 + f_{\lambda,\psi+\epsilon\psi'})$$

is invertible for each sufficiently small ϵ. The proof of Lemma 2.7 is now completed.

Note that a linear function is used in [F] for the perturbation in a neighborhood of an irreducible connection. Here we use quadratic function to perturb the equation in a neighborhood of a reducible connection.

Remark 2.17. We choose the perturbation so that the zero eigenvalues of $*_\sigma d - \mathrm{Hess}_a(f_0 + f_{(\lambda,\mu)})$ is perturbed to positive one, if a is a reducible connection and if the corresponding eigenspace is identified to $\mathbf{C}^k$ with respect to the $U(1)$ action. The set of such connections is a subset of first category in an open set. This choice is used in the proof of Theorem 5.6. (See Remark 5.7.)

Now we put $f = f_0 + f_{\lambda,\psi}$ for generic ψ, and define Fl and Fl_0 by (1.6.1) and (1.6.2).

§3. Local structure of moduli space

Let $p : M \times \mathbf{R} \to M$ be the projection, $p^*(\wedge^i M)$ be the pull back of the vector bundles on $M \times \mathbf{R}$. Let δ be a number sufficiently close to 0. Choose a C^∞-map $\|\| : \mathbf{R} \to [0,\infty)$, such that $\|t\| = |t|$ outside a compact subset, put $e_\delta(t) = e^{\delta\|t\|}$. For a smooth section u of $p^*(\wedge^i M) \otimes su(2)$ with compact support, we put

$$\left(\|u\|^p_{\ell,\delta}\right)^p = \sum_{k\le\ell} \int_{M\times\mathbf{R}} e_\delta(t)|\nabla^k u|^p dxdt.$$

Let $L^p_{\ell,\delta}(M \times \mathbf{R}, su(2) \otimes p^*(\wedge^i M))$ be the completion with respect to this norm. We put

$$\mathcal{L}^i_{\ell,\delta} = L^2_{\ell,\delta}(M \times \mathbf{R}, su(2) \otimes p^*(\wedge^i M)).$$

Define $L^p_{\ell,\delta}(M \times \mathbf{R}, su(2) \otimes \wedge^i(M \times \mathbf{R}))$ in a similar way. Let $L^p_{\ell,\delta}(M \times \mathbf{R}, su(2) \otimes \wedge^2_\pm(M \times \mathbf{R}))$ be the subspace of $L^p_{\ell,\delta}(M \times \mathbf{R}, su(2) \otimes \wedge^2(M \times \mathbf{R}))$ consisting of the elements u satisfying $\widetilde{*}_\sigma u = \pm u$, respectively. Here and hereafter $\widetilde{*}_\sigma$ denotes the Hodge $*$ operator on $M \times \mathbf{R}$ with respect to the product metric $\sigma \oplus dt^2$. The Hodge operator on M induces $*_\sigma$: $p^*(\wedge^k M) \to p^*(\wedge^{3-k} M)$. We define isomorphisms

$$\begin{aligned}
I^2_\pm &: L^p_{\ell,\delta}(M \times \mathbf{R}, su(2) \otimes p^*(\wedge^1 M)) \to \\
&\qquad\qquad L^p_{\ell,\delta}(M \times \mathbf{R}, su(2) \otimes \wedge^2_\pm(M \times \mathbf{R})) \\
I^1 &: L^p_{\ell,\delta}(M \times \mathbf{R}, su(2) \otimes p^*(\wedge^0 M \oplus \wedge^1 M)) \to \\
&\qquad\qquad L^p_{\ell,\delta}(M \times \mathbf{R}, su(2) \otimes \wedge^1(M \times \mathbf{R})) \\
I^0 &: L^p_{\ell,\delta}(M \times \mathbf{R}, su(2)) \to L^p_{\ell,\delta}(M \times \mathbf{R}, su(2))
\end{aligned}$$

by

$$\begin{aligned}
I^2_\pm(\alpha) &= \alpha \pm (*_\sigma \alpha) \wedge dt \\
I^1(\varphi, \alpha) &= \varphi\, dt + \alpha \\
I^0 &= \text{identify}.
\end{aligned}$$

We put

$$\begin{aligned}
\Omega^0_{\ell,\delta} &= L^2_{\ell,\delta}(M \times \mathbf{R}, su(2)) \\
\Omega^1_{\ell,\delta} &= L^p_{\ell,\delta}(M \times \mathbf{R}, su(2) \otimes \wedge^1(M \times \mathbf{R})) \\
\Omega^2_{\ell,\delta} &= L^2_{\ell,\delta}(M \times \mathbf{R}, su(2) \otimes \wedge^2_-(M \times \mathbf{R}))
\end{aligned}$$

and identify $\mathcal{L}^0_{\ell,\delta} \simeq \Omega^0_{\ell,\delta}$, $\mathcal{L}^0_{\ell,\delta} \oplus \mathcal{L}^1_{\ell,\delta} \simeq \Omega^1_{\ell,\delta}$, $\mathcal{L}^1_{\ell,\delta} \simeq \Omega^2_{\ell,\delta}$, by I^i.

For $a, b \in Fl$, choose a connection $d + A^{a,b}$ of the trivial $SU(2)$ bundle on $M \times \mathbf{R}$ such that $A^{a,b} = b$ if $t > 1$ and that $A^{a,b} = a$ if $t < -1$. We put

$$\mathcal{A}_{\ell,\delta}(a, b) = \{d + A^{a,b} + \alpha \mid \alpha \in \Omega^1_{\ell,\delta}\}.$$

Clearly this space is independent of the choice of $A^{a,b}$. Hereafter we write A in place of $d + A$. Let $\mathcal{G}^0_{\ell,\delta}(M \times \mathbf{R})$ be the set of all locally L^2_ℓ map $g : M \times \mathbf{R} \to SU(2)$ such that there exists $\psi \in \mathcal{L}_{\ell,\delta}$ satisfying $\exp \psi = g$ outside a compact subset.

Lemma 3.1. *$\mathcal{G}^0_{\ell+1,\delta}(M \times \mathbf{R})$ acts on $\mathcal{A}_{\ell,\delta}(a, b)$ by*

$$g^* A = g^{-1} dg + g^{-1} A g.$$

The action is free if δ is positive or $a, b \in Fl_0$.

We omit the proof. (See [FU],[T3],[F].)
For $a \in \mathcal{A}_\ell(M)$, $A \in \mathcal{A}_{\ell,\delta}(a,b)$, we put

$$G_a = \{g \in \mathcal{G}_{\ell+1}(M) \mid g^* a = a\}$$
$$G_A = \{g : M \times \mathbf{R} \to G \mid g \text{ is a locally } L^2_{\ell+1} \text{ map satisfying } g^* A = A.\}$$

Remark 3.2. $G_A \subset G_a \cap G_b$.

Put

$$\mathcal{B}^{reg}_{\ell,\delta}(a,b) = \{[A] \mid A \in \mathcal{A}_{\ell,\delta}(a,b), G_A \neq \{\pm 1\}\}$$
$$T_{[A]}\mathcal{B}_{\ell,\delta}(a,b) = \{\alpha \in \Omega^1_{\ell,\delta} \mid e_\delta d^*_A e^{-1}_\delta \alpha = 0\}.$$

G_A acts on $\mathcal{B}_{\ell,\delta}(a,b)$ and $T_{[A]}\mathcal{B}_\ell(a,b)$.

Lemma 3.3. *The map $T_{[A]}\mathcal{B}_{\ell,\delta}(a,b) \to \mathcal{B}_{\ell,\delta}(a,b) : \alpha \mapsto [A+\alpha]$, induces a G_A-invariant diffeomorphism from a neighborhood of* 0 *onto a neighborhood of A, if $a, b \in Fl_0$, or if $\delta > 0$.*

The proof is in [FU], [T3], [F].

Lemma 3.4. *$G_a \times G_b$ acts on $\mathcal{B}_{\ell,\delta}(a,b)$. The action is compatible with the diagonal inclusion : $G_A \to G_a \times G_b$.*

Proof. For each $g_1 \in G_a$ and $g_2 \in G_b$ choose a map $g : M \times \mathbf{R} \to SU(2)$ such that $g_t = g_1$ if $t < -1$ and that $g_t = g_2$ if $t > 1$. For $[A] \in \mathcal{B}_{\ell,\delta}(a,b)$ the element g^*A is contained in $\mathcal{A}_{\ell,\delta}(a,b)$, and $[g^*A]$ depends only on $[A]$ and g_1, g_2. Clearly this induces a desired action.

Hereafter we put

$$g_1[A]g_2^{-1} = (g_1, g_2)[A]$$

for $A \in \mathcal{B}_{\ell,\delta}(a,b)$, $g_1 \in G_a$, $g_2 \in G_b$. Then G_a and G_b act from left and right on $\mathcal{B}_{\ell,\delta}(a,b)$, respectively.

Remark 3.5. The action is trivial if $\delta < 0$.

Now we consider a differential equation

$$F^A - \widetilde{*}_\sigma F^A - \operatorname{grad}_{a_t} f \wedge dt + *_\sigma \operatorname{grad}_{a_t} f = 0, \tag{3.6}$$

for $A \in \mathcal{A}_{\ell,\delta}(a,b)$. Here we put $A = I^1(a_t,\varphi)$. Let $\widehat{\mathcal{M}}_{\ell,\delta}(a,b)$ be the set of all solutions of (3.6) in $\mathcal{A}_{\ell,\delta}(a,b)$. Since $\operatorname{grad}_{g_t^* a_t} f = g_t^{-1}(\operatorname{grad}_{a_t} f)g_t$, it follows that

$$F^{g^*A} - \widetilde{*}_\sigma F^{g^*A} - \operatorname{grad}_{g_t^* a_t} f \wedge dt + *_\sigma \operatorname{grad}_{g_t^* a_t} f = \\ g^{-1}\left(F^A - \widetilde{*}_\sigma F^A - \operatorname{grad}_{a_t} f \wedge dt + *_\sigma \operatorname{grad}_{a_t} f\right) g.$$

Therefore $\widehat{M}_{\ell,\delta}(a,b)$ is $\mathcal{G}^0_{\ell+1,\delta}$ invariant. We put

$$\mathcal{M}_{\ell,\delta}(a,b) = \widehat{M}_{\ell,\delta}(a,b)/\mathcal{G}^0_{\ell+1,\delta}.$$

By a standard elliptic regularity estimate, $\mathcal{M}_{\ell,\delta}(a,b)$ is independent of ℓ. Then we omit ℓ and write $\mathcal{M}_\delta(a,b)$.

Here we remark that the set $G_a \backslash \mathcal{M}_\delta(a,b)/G_b$ is identified to the set $\mathcal{M}(a,b)$ in §1. In fact, the elements of the set $\mathcal{M}(a,b)$ have a one to one correspondence to the set of a_t's satisfying (1.7) and $\lim_{t\to-\infty} a_t = a$, $\lim_{t\to\infty}[a_t] = [b]$. Put $\lim_{t\to\infty} a_t = b'$. There exists g_∞ such that $g_\infty^* b' = b$. Choose g_t such that $\lim_{t\to-\infty} g_t = 1$, $\lim_{t\to\infty} g_t = g_\infty$. It is easy to see that $g^*(d+a_t) \in \mathcal{M}_\delta(a,b)$. This element depends only on $[a_t]$ and is independent of a_t. Conversely, if $A \in \widehat{\mathcal{M}}_\delta(a,b)$, we can find g such that g^*A has no dt factor. Let $(g^*A)(\cdot,t) = a_t$. Then $[a_t] \in \mathcal{M}(a,b)$.

Remark 3.7. It is *not* in general true that the set of loops joining $[a]$ and $[b]$ in $\mathcal{B}_\ell(M)$ has one to one correspondence to $\mathcal{B}_{\ell,\delta}(a,b)$. This is valid if the loop is contained in $\mathcal{B}_\ell(M) - \mathcal{SB}_\ell(M)$

For $A \in \mathcal{A}_\ell(a,b)$, we define $\mathcal{D}_A : \Omega^1_\ell \to \Omega^2_{\ell-1}$ by

$$\mathcal{D}_A\alpha = (d_A - \widetilde{*}_\sigma d_A)\alpha - \operatorname{Hess}_{a_t} f(u_t),$$

where $\alpha = I_1(u_t,\varphi)$, $d+A = d+a_t+\psi dt$. If we identify $\Omega^1_{\ell,\delta} \simeq \mathcal{L}^1_{\ell,\delta} \oplus \mathcal{L}^0_{\ell,\delta}$, $\Omega^2_{\ell-1,\delta} \simeq \mathcal{L}^1_{\ell-1,\delta}$, we have

$$\mathcal{D}_A(u,\varphi) = -\frac{\partial u}{\partial t} + (*_\sigma d_{a_t} - \operatorname{Hess}_{a_t} f - \psi_t\wedge)u + d_{a_t}\varphi. \tag{3.8}$$

Recall that $\mathcal{M}(a,b)$ is a C^∞-manifold in a neighborhood of $[A]$ if $\mathcal{D}_A$ is surjective.

Lemma 3.9. *There exists λ_0 and m_0 such that, for each $\lambda_0 \prec \lambda$, the set of $\psi \in C^\beta(L_m,\mathbf{R})$ satisfying the following is of first category in an open set. Let $a,b \in Fl$, $f = f_{\lambda,\psi}$.*

(3.9.1) *$\mathcal{M}_\delta(a,b)$ is a finite dimensional smooth manifold.*

(3.9.2) *For each* $[A] \in \mathcal{M}_\delta(a,b)$, $\mathcal{D}_A$ *is surjective.*

Proof. We write $\mathcal{M}_\delta^\psi(a,b)$, $\mathcal{D}_A^\psi$ while proving Lemma 3.9. In the set of irreducible connections, the proof of [F] 2c.2 works. Hence we study $\mathcal{M}_\delta^\psi(a,b)$ in the neighborhood of reducible connections. Put

$$\mathcal{B}_{\ell,\delta}^{\mathrm{red}}(a,b) = \{[A] \in \mathcal{B}_{\ell,\delta}(a,b) \mid G_A = U(1)\}$$
$$\mathcal{M}_\delta^{\mathrm{red},\psi}(a,b) = \mathcal{B}_{\ell,\delta}^{\mathrm{red}}(a,b) \cap \mathcal{M}_\delta^\psi(a,b)$$

Then by a $U(1)$ analogue of the argument by Floer [F] 2c.2, we may assume that $\mathcal{M}_\delta^{\mathrm{red},\psi}(a,b)$ is a C^∞-manifold, and, for each $[A] \in \mathcal{M}_{\ell,\delta}^{\mathrm{red},\psi}(a,b)$, the map

$$\mathcal{D}_A^{\mathrm{red}} : L_{\ell,\delta}^2(M \times \mathbf{R}, u(1) \otimes \wedge^1(M \times \mathbf{R})) \to L_{\ell-1,\delta}^2(M \times \mathbf{R}, u(1) \otimes \wedge_-^2(M \times \mathbf{R}))$$

is surjective. Let $[A] \in \mathcal{M}_\delta^{\mathrm{red},\psi}$. Choose a neighborhood U of $[A]$ in $\mathcal{B}_{\delta,\ell}^\psi(a,b)$, which is bounded in L_ℓ^2 norm.

Sublemma 3.10. *The set of all* ψ' *such that* $\mathcal{D}_A^{\psi'}$ *is surjective for all* $A \in U \cap \mathcal{M}_\delta^{\psi'}(a,b)$, *is open.*

The proof is similar to one for Sublemma 2.9 and is omited.

Sublemma 3.11. *For each* $\epsilon > 0$ *and* ψ, *there exists* ψ' *and a neighborhood* U' *of* A, *such that* $\|\psi'\|_\beta < \epsilon$ *and that* $\mathcal{D}_{A'}^{\psi+\psi'}$ *is surjective for each* $[A'] \in U' \cap \mathcal{M}_\delta^{\psi+\psi'}(a,b)$.

Proof. By an argument similar to the proof of Sublemma 2.9, it suffices to find ψ' such that $\|\psi'\|_\beta < \epsilon$, and that $\mathcal{D}_A^{\psi+\psi'}$ is surjective. We put

$$Cok = \mathrm{Ker}\,(D_A^\psi)^* \subset \mathcal{L}_{\ell,\delta}^1,$$
$$Ker = \{u \in \mathcal{L}_{\ell,\delta}^1 \mid \mathcal{D}_A u = 0, d_{a_t}^* u_t = 0\}$$

The group $U(1) \simeq G_A$ acts on Ker and Cok. By the surjectivity of $\mathcal{D}_A^{\psi,\mathrm{red}}$, we have $Cok \simeq \mathbf{C}^k$ as $U(1)$ module. By the index calculation in §5, we can find a $U(1)$ invariant subspace K of Ker which is isomorphic to $\mathbf{C}^k$ as $U(1)$ module. (See Remark 5.7.) Choose an isomorphism $Q : Cok \to K$. For each t, let $K_t, Cok_t \subset T_{[a_t]}\mathcal{B}_\ell(M)$ be the projection of

K and Cok. By the unique continuation theorem ([Ar]), the projections $K \to K_t, Cok \to Cok_t$ are isomorphisms. Let $Q_t : Cok_t \to K_t$ be the projection of Q. We can choose sufficiently large m and λ such that the curve $t \mapsto \widetilde{\Psi}'(a_t, \lambda)(0) = a'_t$ is injective, and $P_t : T_{[a_t]}(\mathcal{B}_\ell(M)) \to T_{a'_t} SU(2)^m$ is injective on $K_t + Cok_t$ for each t. Since the action of $U(1)$ has no trivial component on Cok_t, it follows that $P_t(K_t + Cok_t)$ is transversal to the tangent vector of the curve a'_t. Hence we can find a function $\psi_0 \in C^\beta(L_m, \mathbf{R})$ such that

$$\left(\mathrm{Hess}_{a'_t} \psi_0\right)(P_t V, P_t W) = \langle Q_t V, W \rangle,$$

for each $V \in Cok_t$ and $W \in K_t$. It is easy to see that $\psi' = \psi + \delta\psi_0$ has the required property.

Lemma 3.9 follows easily from Sublemmas 3.10 and 3.11.

§4. Sum formula for index bundles

It seems that many parts of this section are well known to experts. But we include it here because of the lack of appropriate reference and because we need a part of the proof in §11. However we omit the detail of the proof since the results are essentially known. First we shall work in the following situation.

Situation 4.1. Let X^{n+1} be an oriented complete Riemannian manifold, E, F be vector bundles on it, K a compact subset. Suppose that $X - K$ is isometric to the direct product $M \times (0, \infty)$. Let V be a vector bundle on M and $\Psi_E : E \to p^*V$, and $\Psi_F : F \to p^*V$ be isomorphisms of vector bundles. (Here $p : M \times (0, \infty) \to M$ is the projection.) Let $\mathcal{D}^0 : \Gamma(V) \to \Gamma(V)$ and $\mathcal{D} : \Gamma(E) \to \Gamma(F)$ be elliptic operators of first order. Suppose that $\mathcal{D}^0$ is selfadjoint. Assume that M is decomposed to $M_+ \amalg M_-$ such that

$$\mathcal{D} = \Psi_F^{-1}(\pm\frac{\partial}{\partial t} + \mathcal{D}^0)\Psi_E$$

respectively on $M_\pm \times (0, \infty)$. Let $\{\lambda_i | i \in \mathbf{Z}\}$ be the set of all eigenvalues of $\mathcal{D}^0$. Put $\lambda_0 = \min_{i \in \mathbf{Z}} \lambda_i^2$.

Theorem 4.2. *Suppose $\lambda_0 > 0$. Then $\mathcal{D}$ is Fredholm. Moreover, for $\lambda < \lambda_0$, there exists a finite dimensional subspace L_λ of $L^2(E)$, such that*

(4.2.1) *If $u \in L_\lambda^\perp$ then $|\mathcal{D}u| > \sqrt{\lambda}|u|.$. Here $L_\lambda^\perp$ is a orthonormal complement of L_λ*

(4.2.2) *L_λ is generated by the vectors v satisfying $\mathcal{D}^*\mathcal{D}v = \lambda' v$ with $\lambda' \le \lambda$.*

We omit the proof. See [LM],[T3]. Theorem 4.2 implies that

$$\operatorname{Index}\mathcal{D} = \dim\operatorname{Ker}\mathcal{D} - \dim\operatorname{Ker}\mathcal{D}^*$$

is well defined.

Situation 4.3. Let $X_i, M_i, E_i, F_i, V_i, \mathcal{D}_i, \mathcal{D}_i^0$ be as in Situation 4.1. We assume that there are unions of connected components, say $M^0_{1,+}$ and M^0_{2-}, of $M_{1,+}$ and $M_{2,-}$ respectively, and an orientation reversing diffeomorphism from $M^0_{1,+}$ to $M^0_{2,-}$, by which we can identity $V_1, \mathcal{D}^0_1$ and $V_2, \mathcal{D}^0_1$. We patch $X_1 - M^0_{1,+} \times (T,\infty)$ and $X_2 - M^0_{2,-} \times (T,\infty)$ by the diffeomorphism $M^0_{1,+} \times \{T\} \to M^0_{2,-} \times \{T\}$ to obtain $X(T)$. (Figure 1)

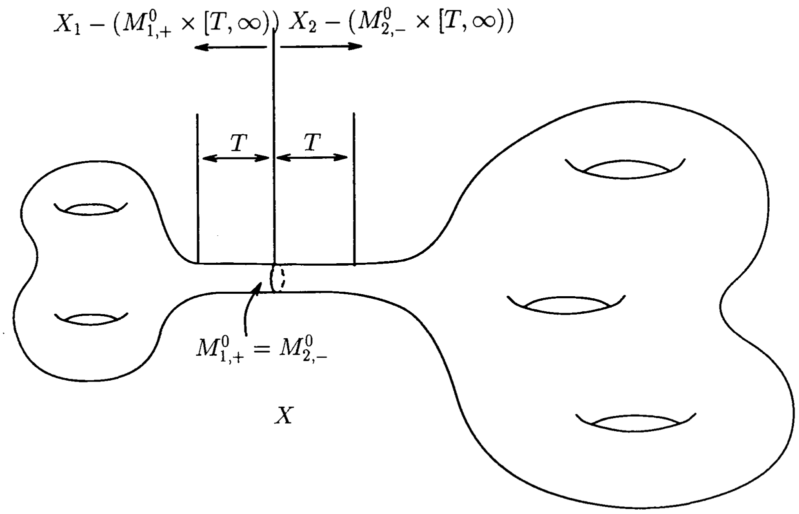

Figure 1.

Let $E(T)$ (resp. $F(T)$) be a vector bundle on $X(T)$ obtained by patching E_1 and E_2 (resp. F_i) by $\Psi_{E_2}^{-1}\Psi_{E_1}$ (resp. $\Psi_{F_2}^{-1}\Psi_{F_1}$). Define an operator $\mathcal{D} : \Gamma(E(T)) \to \Gamma(F(T))$ by

$$\mathcal{D} = \begin{cases} \mathcal{D}_1 & \text{on} \quad X_1 \\ \mathcal{D}_2 & \text{on} \quad X_2 \end{cases}$$

Theorem 4.4. *If $\lambda_0 > 0$ then we have*

$$\operatorname{Index}\mathcal{D} = \operatorname{Index}\mathcal{D}_1 + \operatorname{Index}\mathcal{D}_2.$$

Proof. Let $0 < \lambda < \lambda_0$. We may assume that λ is not an eigenvalue of $\mathcal{D}^*\mathcal{D}$ or $\mathcal{D}_i^*\mathcal{D}_i$. Let $L_\lambda \subset L^2(E)$ be the vector space generated by the vectors v such that $\mathcal{D}^*\mathcal{D}v = \lambda' v$ with $\lambda' < \lambda$. Define $L_\lambda^* \subset L^2(F)$, L_λ^i, L_λ^{i*} in the same way. Note that an embedding $X_1 - M_{1,+}^0 \times [T,\infty) \to X$ can be extended to an embedding $X_1 - M_{1,+}^0 \times [2T,\infty)$. Let $M_{1,+}^0 \times [0,2T] \to X$ be its restriction. Put $d(t) = \min(|t|, |2T-t|)$.

Lemma 4.5. *If $u \in L_\lambda$ then*

$$|\nabla^k \varphi|(I(x,t)) < C_k e^{-\sqrt{\lambda_0-\lambda}d(t)}\|u\|_{L^2}.$$

Proof. We may assume $\mathcal{D}^*\mathcal{D}u = \lambda' u$, $\lambda' < \lambda$. Let $\varphi_1, \cdots$ be the eigenvectors of $\mathcal{D}_0^*\mathcal{D}_0$. We put

$$u(I(x,t)) = \sum_{i=1}^{\infty} u_i(t)\varphi_i(x).$$

Since

$$\mathcal{D}^*\mathcal{D} = -\frac{\partial^2}{\partial t^2} + (\mathcal{D}^0)^2,$$

we have

$$-\frac{d^2 u_i}{dt^2} + \lambda_i^2 u_i = \lambda' u_i.$$

It follows that

$$|u_i(t)| \le C e^{-\sqrt{\lambda_0-\lambda'}d(t)} \max\{|u_i(0)|, |u_i(T)|\},$$

from which the lemma follows by the standard estimates for elliptic operators.

Let $\chi : [-1,1] \to [0,1]$ be a nondecreasing C^∞ function such that

$$\chi(t) = \begin{cases} 0 & \text{if} \quad t < -1 \\ 1 & \text{if} \quad t > 1. \end{cases}$$

We define $P_i' : L_\lambda \to \Gamma_c(X_i, E_i)$ as follows. (Here Γ_c stands for the set of smooth sections with compact support.)

$$\begin{cases} (P_1'u)(x,t) = (1-\chi(\frac{t-T}{T}))u(x,t) & \text{if} \quad (x,t) \in M_{1,+}^0 \times [0,2T] \\ (P_1'u)(x,t) = 0 & \text{if} \quad (x,t) \in M_{1,+}^0 \times [2T,\infty) \\ (P_1'u)(z) = u(z) & \text{if} \quad z \notin M_{1,+}^0 \times [0,\infty) \end{cases}$$

$$\begin{cases} (P_2'u)(x,t) = \chi(\frac{t-T}{T})u(x,t) & \text{if} \quad (x,t) \in M_{2,-}^0 \times [0,2T] \\ (P_1'u)(x,t) = 0 & \text{if} \quad (x,t) \in M_{2,-}^0 \times [2T,\infty) \\ (P_1'u)(z) = u(z) & \text{if} \quad z \notin M_{2,-}^0 \times [0,\infty) \end{cases}$$

Let $P_i(u)$ be the orthonormal projection of $P_i'(u)$ to L_λ^i. Put $P_\lambda = (P_1, P_2) : L_\lambda \to L_\lambda^1 \oplus L_\lambda^2$. Then using Lemma 4.5 we can prove that P_λ is an isomorphism for large T. Similarly we can construct an isomorphism $P_\lambda^* : L_\lambda^* \to L_\lambda^{1*} \oplus L_\lambda^{2*}$. On the other hand, $\mathcal{D}$ defines an isomorphism: $L_\lambda \cap (\operatorname{Ker} \mathcal{D})^\perp \to L_\lambda^* \cap (\operatorname{Ker} \mathcal{D}^*)^\perp$. Therefore

$$\operatorname{Index} \mathcal{D} = \dim L_\lambda - \dim L_\lambda^*.$$

Similarly, we have

$$\operatorname{Index} \mathcal{D}_i = \dim L_\lambda^i - \dim L_\lambda^{i*}.$$

The theorem follows immediately. (Recall that $\operatorname{Index} \mathcal{D}^T$ does not depend on T.)

Remark 4.6. By the same method, we can prove that, if $\mathcal{D}_0$ is invertible, then the $Ce^{-\sqrt{\lambda_0 - \lambda}T/C}$-neighborhood of the set

$$\{\text{eigenvalues of } \mathcal{D}^{T*}\mathcal{D}^T \text{ smaller than } \lambda_0\}$$

contains the set

$$\{\text{eigenvalues of } \mathcal{D}_1\mathcal{D}_1^* \text{ smaller than } \lambda_0\}$$
$$\cup \{\text{eigenvalues of } \mathcal{D}_2\mathcal{D}_2^* \text{ smaller than } \lambda_0\}.$$

Also the $Ce^{-\sqrt{\lambda_0 - \lambda}T/C}$-neighborhood of the later set contains the former set.

Moreover we can prove the following:

Corollary 4.7. *In Situation* 4.1, *let M_+^0, M_-^0 be unions of components of M_+, M_-, respectively. Suppose that M_+^0, together with $\mathcal{D}_0$,V on it, is diffeomorphic to M_-^0. Construct $X(T), E(T), F(T), \mathcal{D}^T$, e.t.c. as before.* (*Figure* 2) *Then we have*

$$\operatorname{Index} \mathcal{D}^T = \operatorname{Index} \mathcal{D}.$$

In §6 and §11, we need also a family version of Theorem 4.4.

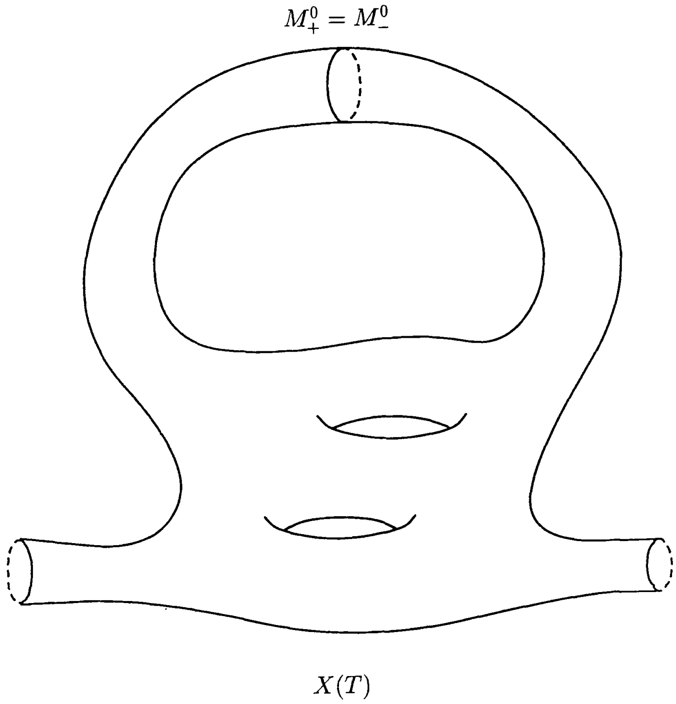

Figure 2.

Situation 4.8. Let Y be a manifold, $p_i : W_i \to Y$, $q : Z \to Y$ be fibre bundles. Let $\widetilde{E}_i, \widetilde{F}_i :\to W_i$, $\widetilde{V} \to Z$ be vector bundles and $\widetilde{\mathcal{D}}_i : \Gamma(\widetilde{E}_i) \to \Gamma(\widetilde{F}_i)$, $\widetilde{D}^0 : \Gamma(\widetilde{V}) \to \Gamma(\widetilde{V})$ be families of elliptic operators. Suppose that $p_i^{-1}(y) = X_i(y)$, $q^{-1}(y) = M(y)$, $\widetilde{E}_i\Big|_{X_i(y)} = E_i(y)$, $F_i(y)$, $V(y)$, $\mathcal{D}_i(y)$, $\mathcal{D}^0(y)$ are as in Situation 4.3, for each $y \in Y$. As before, we can construct, $W(T) \to Y$, $\widetilde{E}(T), \widetilde{F}(T) \to W(T)$, $\mathcal{D}(T) : \Gamma(\widetilde{E}(T)) \to \Gamma(\widetilde{F}(T))$. As in [AS], the index bundles

$$\operatorname{Index} \mathcal{D}_i, \operatorname{Index} \mathcal{D}^T \in K(Y),$$

are well defined if $\mathcal{D}^0(y)$ is invertible.

Theorem 4.9. *Suppose $\mathcal{D}^0(y)$ is invertible for each y, then we have*

$$\operatorname{Index} \mathcal{D}_1 + \operatorname{Index} \mathcal{D}_2 = \operatorname{Index} \mathcal{D}^T,$$

in $K(Y)$.

Theorem 4.9 follows from the proof of Theorem 4.4, since P_λ and L_λ, e.t.c. there depend smoothly on operators.

Remark 4.10. The results of this section hold in the case when, for example, in Situation 4.1 the operator $\mathcal{D}$ is not exactly equal to $\Psi_F^{-1}(\pm\frac{\partial}{\partial t}+\mathcal{D}^0)\Psi_E$, but the difference is estimated by $Ce^{-|t|/C}$. (See [T3].)

§5. Dimension of moduli space

We put $\overline{\mathcal{M}}_\delta(a,b) = G_a\backslash\mathcal{M}_\delta(a,b)/G_b$. Recall that the action of $G_a\times G_b$ is trivial if $\delta<0$. We can prove that $\overline{M}_\delta(a,b)$ is independent of δ. Hence we write $\overline{\mathcal{M}}(a,b)$.

Theorem 5.1. *There exists a map $\mu : Fl \to \mathbf{Z}$ such that $\mu(1)=0$ and that*

$$\dim\overline{\mathcal{M}}(a,b) = \mu(a)-\mu(b)-\dim G_a, \tag{5.1}$$

except the component containing no irreducible connection.

Proof. First we assume that $a,b\in Fl_0$. In this case $\dim\overline{\mathcal{M}}(a,b) = \dim\mathcal{M}_\delta(a,b)$. We can use the perturbed Atiyah-Hitchin-Singer complex

$$\Omega^0_{\ell+1,0}\xrightarrow{d_A}\Omega^1_{\ell,0}\xrightarrow{\mathcal{D}_A}\Omega^2_{\ell-1,0}. \tag{5.2}$$

(definitions of operators and spaces are in §3), to calculate the dimension as

$$\dim\mathcal{M}_\delta(a,b) = \dim\frac{\operatorname{Ker}\mathcal{D}_A}{\operatorname{Im}d_A}.$$

Since $a\in Fl_0$, it follows that d_A is injective. By Lemma 3.9, $\mathcal{D}_A$ is surjective. Hence $\dim\mathcal{M}_\delta(a,b)$ is equal to the index of the complex (5.2). We put

$$(\mathcal{D}_A, d_A^*) : \Omega^1_{\ell,0}\to\Omega^2_{\ell,0}\oplus\Omega^0_{\ell-1,0}.$$

Then we have:

$$\dim\mathcal{M}_\delta(a,b) = \operatorname{Index}(\mathcal{D}_A, d_A^*).$$

We identify Ω^1_ℓ and $\Omega^2_\ell\oplus\Omega^0_\ell$ to $\mathcal{L}^1_{\ell,\delta}\oplus\mathcal{L}^0_{\ell,\delta}$ as in §3. For $a\in\mathcal{A}_\ell(M)$, define

$$D_a : L^2_\ell(M,(\wedge^1\oplus\wedge^2)\otimes su(2))\to L^2_\ell(M,(\wedge^1\oplus\wedge^2)\otimes su(2))$$

by

$$D_a(u,\varphi) = (*_\sigma d_a u - \operatorname{Hess}_a u + d_a\varphi, d_a^*u).$$

Then when $t \to \infty$ the operator $(\mathcal{D}_A, d_A^*)$ is asymptotic to $-\frac{\partial}{\partial t} + D_b$ and when $t \to -\infty$ it is asymptotic to $-\frac{\partial}{\partial t} + D_a$. Since $a, b \in Fl_0$ it follows that

$$d_a : L^2(M, su(2)) \to L^2(M, \wedge^1 \otimes su(2))$$

is injective. Hence by (2.8.2), D_a and D_b are invertible. Therefore by Theorem 4.3, $(\mathcal{D}_A, d_A^*)$ is Fredholm for each $A \in \mathcal{B}_{\ell,\delta}(a,b)$. Since $\mathcal{B}_{\ell,\delta}(a,b)$ is connected, it follows that its index is independent of A. Therefore, we can use Theorem 4.4 to show

$$\mathrm{Index}(\mathcal{D}_C, d_C^*) = \mathrm{Index}(\mathcal{D}_A, d_A^*) + \mathrm{Index}(\mathcal{D}_B, d_B^*),$$

for $A \in \mathcal{M}_\delta(a,b)$, $B \in \mathcal{M}_\delta(b,c)$, $C \in \mathcal{M}_\delta(a,c)$, $a, b, c \in Fl_0$. In the case when b is reduced, way we can prove

$$\begin{aligned}\mathrm{Index}(\mathcal{D}_C, e_\delta d_C^* e_\delta^{-1}) = &\mathrm{Index}(\mathcal{D}_A, e_\delta d_A^* e_\delta^{-1}) + \mathrm{Index}(\mathcal{D}_B, e_\delta d_B^* e_\delta^{-1}) \\ &- \dim G_b,\end{aligned}$$

in a similar way, for $\delta > 0$. Therefore the theorem follows by putting

$$\mu(a) = \mathrm{Index}(\mathcal{D}_A, e_\delta d_A^* e_\delta^{-1}) - 3,$$

for an element $[A] \in \mathcal{B}_{\ell,\delta}(1,a)$.

Next we study the neighborhood of a reducible connection $A \in \mathcal{M}_\delta(a,b)$. There are two cases:

Case I. $\dim G_a = \dim G_b = 3$, $G_A = U(1)$.
Case II. $\dim G_a = \dim G_b = 1$, $G_A = U(1)$.

In case I, there exists $\varphi : \mathrm{Tor}\, H_1(M, \mathbf{Z}) \to \{\pm 1\} \subset U(1)$ such that $a, b \in RF_\varphi$. (See §2.) Then we can renumber the loops $\ell_1^0, \cdots, \ell_d^0$, which we choose at the beginning of §2, such that

$$\begin{aligned} a(\ell_i^0) = 1 &\iff i \le p \\ b(\ell_i^0) = 1 &\iff i \le p + k. \end{aligned}$$

(At this point, it is not yet clear that $k > 0$.)

Replacing the element b by a gauge equivalent one, we may assume that there exists $a_t \in \mathcal{A}_\ell(M)$ such that $d + A = d + a_t$. (Namely A has no dt component.) The group $U(1) = G_A$ acts on the complex $(\mathcal{D}_A, e_\delta d_A^* e_\delta^{-1})$. It follows that its index is a $U(1)$ module.

Lemma 5.3.

$$\mathrm{Index}(\mathcal{D}_A, e_\delta d_A^* e_\delta^{-1}) \simeq \begin{cases} \mathbf{C}^{k+1} \oplus \mathbf{R}^{k+1} & \text{if} \quad \delta > 0 \\ \mathbf{C}^{k-1} \oplus \mathbf{R}^{k-1} & \text{if} \quad \delta < 0. \end{cases}$$

Proof. We replace the complex $(\mathcal{D}_A, d_A^*)$ by $(\mathcal{D}_{A,1}+\epsilon, d_A^*+\epsilon)$, where

$$\mathcal{D}_{A,1}(u,\varphi) = -\frac{\partial u}{\partial t} + *_\sigma d_{a_t} u + d_{a_t}\varphi.$$

Put

$$\mathrm{Index}(\mathcal{D}_{A,1}+\epsilon, d_A^*+\epsilon) = \mathbf{C}^{k_1} \oplus \mathbf{R}^{k_2}.$$

The trivial $su(2)$ bundle together with (nontrivial) connection $d + a_t$ on $M \times \mathbf{R}$ splits into a real line bundle $\mathcal{L}^{\mathbf{R}}$ and a complex line bundle $\mathcal{L}^{\mathbf{C}}$, since $d+a_t$ is reducible. Note that the image of holonomy representation of a and b is contained in $\{\pm 1\}$, the center of $SU(2)$. Therefore the line bundles together with their connections, have canonical trivializations on their ends. Hence we can apply Corollary 4.7 to obtain bundles $\overline{\mathcal{L}}^{\mathbf{R}}$ and $\overline{\mathcal{L}}^{\mathbf{C}}$ on $M \times S^1$ such that

$$\begin{aligned} k_1 &= \dim_{\mathbf{C}} \mathrm{Index}\left((P_- d_A, d_A^*) \otimes \overline{\mathcal{L}}^{\mathbf{C}}\right) \\ k_2 &= \dim_{\mathbf{R}} \mathrm{Index}\left((P_- d_A, d_A^*) \otimes \overline{\mathcal{L}}^{\mathbf{R}}\right). \end{aligned}$$

Here

$$\overline{\mathcal{L}}^{\mathbf{C}} \xrightarrow{d_A} \wedge^1(M \times S^1) \otimes \overline{\mathcal{L}}^{\mathbf{C}} \xrightarrow{P_- d_A} \wedge^2_-(M \times S^1) \otimes \overline{\mathcal{L}}^{\mathbf{C}},$$

and similarly for $\overline{\mathcal{L}}^{\mathbf{R}}$. Therefore, as in Atiyah-Hichin-Singer [AHS], we have

$$\begin{aligned} k_1 &= \int_{M \times S^1} \left(2 + \frac{p^1(M \times S^1)}{3}\right)\left(1 + c^1(\overline{\mathcal{L}}^{\mathbf{C}}) + \frac{c^1(\overline{\mathcal{L}}^{\mathbf{C}}) \wedge c^1(\overline{\mathcal{L}}^{\mathbf{C}})}{2}\right) \\ &= 0, \end{aligned}$$

since

$$c^1(\overline{\mathcal{L}}^{\mathbf{C}}) = \sum_{i=p+1}^{p+k} [\ell_i^0] \cup [S^1].$$

Similarly $k_2 = 0$.

Next we compare the index of $(\mathcal{D}_{A,1} + \epsilon, d_A^* + \epsilon)$ to one of $(\mathcal{D}_A, e_\delta d_A^* e_\delta^{-1})$. For this purpose, we use the notion of spectral flow due to Atiyah-Patodi-Singer [APS]. Put

$$D_{a_t,1}(u,\varphi) = (*_\sigma d_{a_t} u + d_{a_t}\varphi, d_{a_t}^*\varphi).$$

The spectral flow of the operator $D_{a_t,1} + \epsilon$ gives the index of $(\mathcal{D}_A + \epsilon, d_A^* + \epsilon)$. The operator $D_{a,1}$ has zero as eigenvalue. The eigenspace is identified to $(\mathbf{C} \oplus \mathbf{R})^{d+1} \simeq (H_0(M;\mathbf{R}) \oplus H_1(M;\mathbf{R})) \otimes su(2)$. Replacing $\mathcal{D}_A$ by $\mathcal{D}_A + \epsilon$ is equivalent to push these eigenvalues a bit to positive direction. Next we examine the effect of the perturbation. We put

$$D_{a_t,2}(u,\varphi) = (*_\delta d_{a_t} u - \mathrm{Hess}_{a_t} f(u) + d_{a_t}\varphi, d_{a_t}^*\varphi).$$

We take the basis $(z_1, \cdots, z_d, t_1, \cdots, t_d)$ of $H_1(M;\mathbf{R}) \otimes su(2)$ such that z_i and t_i correspond to ℓ_i^0. Then, by (2.5) and our choice of a and b, replacement of $D_{a_t,1}$ by $D_{a_t,2}$ is equivalent to push the zero eigenvalues corresponding $z_1, \cdots, z_p$ and $t_1, \cdots, t_p$ a bit to positive direction and the others to negative direction while $t \to -\infty$, and to push the zero eigenvalue corresponding to $z_1, \cdots, z_{p+k}$ and $t_1, \cdots, t_{p+k}$ a bit to positive direction and the others to negative direction while $t \to \infty$. It follows from $k_1 = k_2 = 0$ that the index of the spectral flow $D_{a_t,2}$ is $\mathbf{C}^k \oplus \mathbf{R}^k$.

Finally we examine the effect replacing D_{a_t} by $(\mathcal{D}_{a_t}, e_\delta d_A^* e_\delta^{-1})$. If $\delta > 0$, this is equivalent to push the zero eigenvalues in $H_0(M;\mathbf{R}) \otimes su(2)$ to positive direction while $t \to \infty$ and push them to negative direction while $t \to -\infty$. If $\delta < 0$ this is equivalent to the perturbation to the opposite direction. Lemma 5.3 follows.

Lemma 5.3 implies $k > 0$. Using Lemma 5.3, we have a description of the moduli space in a neighborhood of reducible connections. First let $k = 1$, $\delta > 0$. The group $SU(2) \times SU(2) \times \mathbf{R}$ acts on $\mathcal{M}_\delta(a,b)$. Here $SU(2) \times SU(2) \simeq G_a \times G_b$ acts on $\mathcal{M}_\delta(a,b)$ by Lemma 3.4, and the action of $\mathbf{R}$ is induced by its action on $M \times \mathbf{R}$. Since $G_A = U(1)$ there exists an embedding

$$\frac{SU(2) \times SU(2)}{U(1)} \times \mathbf{R} \to \mathcal{M}_\delta(a,b).$$

By Lemma 5.3, this map is a diffeomorphism onto a connected component containing $[A]$. It follows that all the connections on this component is reducible. In the case $k \geq 2$ we can use a similar argument. Summing up we obtain

Theorem 5.4. *Suppose* $\dim G_a = \dim G_b = 3$, $\dim G_A = 1$, $[A] \in \mathcal{M}_\delta(a,b)$, $\delta > 0$. *Then* $\mu(a) = 3k + \mu(b)$ *for some* $k \le d$ *and that there exists a diffeomorphism from*

$$\frac{SU(2) \times \mathbf{C}^{k-1} \times SU(2)}{U(1)} \times \mathbf{R}^k$$

onto a neighborhood of the $G_a \times G_b \times \mathbf{R}$ *orbit of* $[A]$. *The diffeomorphism is compatible with* $G_a \times G_b \times \mathbf{R} \simeq SU(2) \times SU(2) \times \mathbf{R}$ *action.*

Remark 5.5. In case $k = 1$ the formula (5.1) does not hold for this component. This is similar to the fact that the virtual dimension of the trivial connection on S^3 is -3. In case $k > 1$ the neighborhood of $[A]$ in $\overline{\mathcal{M}}(a,b)$ is diffeomorphic to the product of $C\mathbf{C}P^{k-1} \times \mathbf{R}^k$. Here C means the cone. (Compare [D1].)

By a similar but simpler argument we can examine the case when $G_0 = U(1)$ and obtain:

Theorem 5.6. *Let* $G_a = G_b = G_A = U(1)$, $A \in \mathcal{M}_\delta(a,b)$ *and* $\delta > 0$. *Then* $\mu(a) = \mu(b) + k$ *for some* $k \le d$. *All the connections contained in the connected component of* $\mathcal{M}_\delta(a,b)$ *containing* $[A]$ *are reducible.*

Remark 5.7. We used the above index calculation in the proof of Sublemma 3.10. The fact we used there is that the $\mathbf{C}$-part of the index is always of nonnegative dimension.

If we use different perturbation from one we gave in §§2,3, (for example if we change the sign in Formula (2.16) from point to point) then the above fact is no longer true. As the consequence, Lemma 3.9 does not necessary hold in that case, and we have an obstruction in second homology of Atiyah-Hitchin-Singer complex.

Finally we remark:

Lemma 5.8. *Let* $[a], [b] \in Fl$, $b = g^*a$, *where* $g : M \to SU(2)$ *and* $\deg g = k$. *Then,*

$$\mu(b) = 8k + \mu(a).$$

For the proof see [F].

§6. Orientation of moduli space

Lemma 6.1. $\mathcal{M}_\delta(a,b)$ *is orientable.*

Proof. Let $\mathcal{DET}(a,b) = \mathcal{DET}(\mathcal{D}_A, e_\delta d_A^* e_\delta^{-1})$ be the determinant bundle of the Atiyah-Hitchin-Singer complex (5.2). We can extend $\mathcal{DET}(a,b)$ to a real line bundle on $\mathcal{B}_{\ell,\delta}(a,b)$. On $\mathcal{M}_\delta(a,b)$, the bundle $\mathcal{DET}(a,b)$ is isomorphic to the bundle of $\dim \mathcal{M}_\delta(a,b)$-forms. Hence it suffices to show :

Lemma 6.2. *The bundle $\mathcal{DET}(a,b)$ on $\mathcal{B}_{\ell,\delta}(a,b)$ is trivial.*

Proof. Since $\mathcal{M}_{\ell,\delta}(a,b)$ is not simply connected, the argument in [D1],[F], can not be applied directly to our situation. Instead we shall proceed as follows. Since 3-dimensional oriented cobordism group is trivial, we can find oriented manifolds $\overline{X}_\pm$ such that $\partial\overline{X}_+ = M$, $\partial\overline{X}_- = M^-$, where M^- is the manifold M with opposite orientation. Let W be a closed oriented 4-manifold obtained by patching X_+ and X_- along M. Take trivial $SU(2)$ bundles on them. Let $\mathcal{A}_\ell(W)$ be the set of all L^2_ℓ connection on W, and $\mathcal{G}_\ell(W)$ be the group of transformations. We put $\mathcal{B}_\ell(W) = \mathcal{A}_\ell(W)/\mathcal{G}_{\ell+1}(W)$. Put a metric on $X_\pm = \overline{X}_\pm - \partial\overline{X}_\pm$, such that $X_\pm - K_\pm$ is isometric to $M \times (0,\infty)$ for some compact subset $K_\pm$. Let e_δ be a function on $X_\pm$ such that $e_\delta(x,t) = e^{-\delta\|t\|}$ outside $K_\pm$. For $a \in Fl$ choose a connection $d + A^a$ on $X_\pm$ such that $A^a = a$ outside $K_\pm$. Put

$$L^2_{\ell,\delta}(X_\pm, \wedge^1 \otimes su(2)) = \left\{ u \;\middle|\; \begin{matrix} u \text{ is a locally } L^2_\ell \text{ section} \\ \text{of } \wedge^1 \otimes su(2) \\ \displaystyle\sum_{k=0}^{\ell} \int_{X_\pm} e_\delta |\nabla^k u| < \infty \end{matrix} \right\}$$

$$\mathcal{A}_{\ell,\delta}(X_\pm, a) = \{d + A^a + u \mid u \in L^2_{\ell,\delta}(X_\pm, \wedge^1 \otimes su(2))\}.$$

Define $\mathcal{G}^0_{\ell,\delta}$ as in §2. Put

$$\mathcal{B}_{\ell,\delta}(X_\pm, a) = \mathcal{A}_{\ell,\delta}(X_\pm, a)/\mathcal{G}^0_{\ell+1,\delta}(X_\pm).$$

Let $\mathcal{DET}_\pm(a)$ be the determinant bundle of Atiyah-Hitchin-Singer complex on $\mathcal{B}_{\ell,\delta}(X_\pm, a)$. First we shall prove that $\mathcal{DET}_\pm(a)$ is trivial. For simplicity, we assume that $a \in Fl_0$. It suffices to show that $\mathcal{DET}_\pm(a)$ is trivial on each compact subset $L_\pm$ of $\mathcal{B}_{\ell,\delta}(X_\pm, a)$. We define a map Pat : $L_+ \times L_- \to \mathcal{B}_{\ell,\delta}(W)$ as follows. Define a Riemannian manifold $X(T)$ by patching X_+ and X_- along M as in Situation 4.3. Then $M \times [0, 2T]$ is embedded in $X(T)$. Choose a C^∞ function $\chi : [-1,1] \to [0,1]$ by

$$\chi(t) = \begin{cases} 0 & \text{if} \quad t < -1 \\ 1 & \text{if} \quad t > 1. \end{cases}$$

For $[d+A] \in L_+, [d+B] \in L_-$ define $\mathrm{Pat}([A],[B])$ by

$$\begin{cases} \mathrm{Pat}([A],[B])(z) = A(z) \quad \text{if} \quad z \in X_+ - M \times (0,\infty) \\ \mathrm{Pat}([A],[B])(x,t) = \left(1 - \chi\left(\frac{t-T}{T}\right)\right) A(x,t) + \chi\left(\frac{t-T}{T}\right) B(x,t) \\ \mathrm{Pat}([A],[B])(z) = B(z) \quad \text{if} \quad z \in X_- - M \times (0,\infty) \end{cases}$$

Let $\mathcal{DET}_{X(T)} \to \mathcal{B}_\ell(X(T))$ be the determinant bundle of the Atiyah-Hitchin-Singer complex on $X(T)$. By Theorem 4.9, we have

$$\mathrm{Pat}^*(\mathcal{DET}_X(T)) \simeq \mathcal{DET}_+(a) \otimes \mathcal{DET}_-(a).$$

For sufficiently large T. By [D3], $\mathcal{DET}_{X(T)}$ is trivial. It follows that $\mathcal{DET}_\pm(a)$ is trivial.

Next, Let $L \subset \mathcal{B}_{\ell,\delta}(a,b)$, $L' \subset \mathcal{B}_{\ell,\delta}(X^+,a)$ be compact subsets. In a similar way, we define a map $\mathrm{Pat} : L \times L' \to \mathcal{B}_{\ell,\delta}(X^+,b)$. By Theorem 4.9, we have

$$\mathrm{Pat}^*(\mathcal{DET}_+(b)) \simeq \mathcal{DET}(a,b) \otimes \mathcal{DET}_+(a).$$

Therefore the trivializations of $\mathcal{DET}_+(a)$ and $\mathcal{DET}_+(b)$ induces a trivialization of $\mathcal{DET}(a,b)$, if $a,b \in Fl_0$. The case when a and/or b are reducible can be proved in a similar way, by using a perturbation of the complex around the boundaries. The proof of Lemma 6.2 is now complete.

§7. Partial compactification of moduli space

Let $\mathcal{M}'_\delta(a,b)$, $\overline{\mathcal{M}}'(a,b)$ be the quotients of $\mathcal{M}_\delta(a,b)$ and $\overline{\mathcal{M}}(a,b)$ by the $\mathbf{R}$-action. The proof of the theorems in §1 is based on the following Theorems 7.1 and 7.3 on the structure of the ends of $\overline{\mathcal{M}}'(a,b)$. Hereafter we fix sufficiently small positive number δ and write $\mathcal{M}(a,b)$ e.t.c. in place of $\mathcal{M}_\delta(a,b)$.

Theorem 7.1. *For $a,b \in Fl$, let $C\overline{\mathcal{M}}'(a,b)$ be the disjoint union of*

$$\overline{\mathcal{M}}'(a,c_0) \times \prod_{i=0}^{k-1} \overline{\mathcal{M}}'(c_i,c_{i+1}) \times \overline{\mathcal{M}}'(c_k,b),$$

for $c_0,\cdots,c_k \in Fl$, with $\mu(a) > \mu(c_0) > \cdots > \mu(c_k) > \mu(b)$. Put $m = \dim \overline{\mathcal{M}}'(a,b)$.

Then we can define a smooth structure on $\mathcal{C}\overline{\mathcal{M}}'(a,b)$ such that the following holds.

(7.1.1) *If*

$$x \in \overline{\mathcal{M}}'(a,c_0) \times \prod_{i=0}^{k-1} \overline{\mathcal{M}}'(c_i,c_{i+1}) \times \overline{\mathcal{M}}'(c_k,b),$$

with $G_{c_i} = \{\pm 1\}$. Then a neighborhood of x in $\mathcal{C}\overline{\mathcal{M}}'(a,b)$ is diffeomorphic to $[0,\infty)^{k+1} \times \mathbf{R}^{m-k-1}$.

(7.1.2) *If $x = ([A],[B]) \in \overline{\mathcal{M}}'(a,c) \times \overline{\mathcal{M}}'(c,b)$, with $G_c = U(1), G_A = G_B = \{\pm 1\}$. Then a neighborhood of x is diffeomorphic to $\mathbf{R}^m$.*

(7.1.3) *If $x = ([A],[B]) \in \overline{\mathcal{M}}'(a,c) \times \overline{\mathcal{M}}'(c,b)$, with $G_c = SU(2)$, $G_A = G_B = \{\pm 1\}$. Then a neighborhood of x is diffeomorphic to*

$$\frac{\mathbf{C}^2}{\mathbf{Z}_2} \times \mathbf{R}^{m-4}.$$

(7.1.4) *If $x = (A,B,C) \in \overline{\mathcal{M}}'(a,c_1) \times \overline{\mathcal{M}}'(c_1,c_2) \times \overline{\mathcal{M}}'(c_2,b)$, with $G_{c_1} = G_{c_2} = SU(2)$, $G_B = U(1)$, $G_A = G_C = \{\pm 1\}$, $3k = \mu(c_1) - \mu(c_2)$. Then a neighborhood of x is diffeomorphic to*

$$\left(\left(\frac{SO(3) \times \mathbf{C}^{k-1} \times SO(3)}{U(1)} \times (0,\infty]^2 \right) / \sim \right) \times \mathbf{R}^{m-2k-5},$$

where $\sim$ is defined by

$$([g_1,z,g_2],(\infty,t)) \sim [g_1 g,z,g_2],(\infty,t))$$
$$([g_1,z,g_2],(t,\infty)) \sim [g_1,z,g g_2],(t,\infty)).$$

(7.1.5) *If $x = ([A],[B],[C]) \in \overline{\mathcal{M}}'(a,c_1) \times \overline{\mathcal{M}}'(c_1,c_2) \times \overline{\mathcal{M}}'(c_2,b)$, with $G_{c_1} = G_{c_2} = G_B = U(1)$, $G_A = G_C = \{\pm 1\}$. Then a neighborhood of x is diffeomorphic to $\mathbf{R}^m$.*

(7.1.6) *Let $\Lambda \in \mathbf{R}_+$. Then the set*

$$\overline{\mathcal{M}}'(a,b;\Lambda) = \{[A] \in \overline{\mathcal{M}}'(a,b) \mid \sup |F^A| < \Lambda\}$$

is relatively compact in $\mathcal{C}\overline{\mathcal{M}}'(a,b)$.

(7.1.7) *The orientations of $\overline{\mathcal{M}}'(c_i,c_{i+1})$ are compatible in $\mathcal{C}\overline{\mathcal{M}}'(a,b)$.*

Remark 7.2. (7.1.1) ... (7.1.5) above do not cover all the possible cases. The general case is the combination of them and the reader can easily supply it.

Next we construct the bundles in §1. Choose a set of loops $\{\gamma_1, \cdots, \gamma_d\}$ representing a basis of $H_1'(M;\mathbf{Z})$. Put $\Sigma_i = \gamma_i \times \mathbf{R} \subset M \times \mathbf{R}$. The surface Σ_i has a canonical spin structure. For $A \in \mathcal{A}_{\ell,\delta}(a,b)$, we let

$$\eth_A^i : \Gamma_c(\Sigma_i, su(2)\otimes \mathbf{C}) \to \Gamma_c(\Sigma_i, su(2)\otimes \mathbf{C})$$

be the Dirac operator twisted by the connection A. For each $a, b \in Fl$, $\eth_A^i + \epsilon$ is a Fredholm operator. (We add ϵ since $\eth_A^i$ is not Fredholm when a or b is reducible.) Then we obtain a complex line bundle

$$\mathcal{L}_i(a,b) \to \mathcal{B}_{\ell,\delta}(a,b)$$

by

$$\mathcal{L}_i(a,b)|_{[A]} = \bigwedge^{\mathrm{top}} \left(\mathrm{Ker}(\eth_A^i + \epsilon)\right)^* \otimes \bigwedge^{\mathrm{top}} \mathrm{Coker}(\eth_A^i + \epsilon).$$

(Note the action of $\mathcal{G}_{\ell,\delta}$ is free on $\mathcal{A}_{\ell,\delta}(a,b)$). The action of $G_a \times G_b$ on $\mathcal{B}_{\ell,\delta}(a,b)$ is lifted to this line bundle. The group $\{\pm 1\}$ acts trivially on $\mathcal{B}_{\ell,\delta}(a,b)$. The lift of the action of $\{\pm 1\}$ to $\mathcal{L}_i(a,b)$ is not necessary trivial. (Compare [D2], where the similar action is trivial because the numerical index of the Dirac operator on a *closed* surface is zero.) Then we consider the tensor product $\mathcal{L}_i(a,b)\otimes\mathcal{L}_i(a,b)$. It induces a complex line bundle $\overline{\mathcal{L}}_i^{(2)}(a,b)$ on $\overline{\mathcal{M}}'_*(a,b)$, the set of irreducible connections in $\overline{\mathcal{M}}'(a,b)$. (If we want to "define" the first Chern class $c^1(\mathcal{L}_i(a,b))$ itself, we have to invert 2.)

Theorem 7.3. *Collection of line bundles*

$$\mathcal{L}_i^{(2)}(a,c_0)\otimes\cdots\otimes\mathcal{L}_i^{(2)}(c_k,b) \to \overline{\mathcal{M}}'_*(a,c_0)\times\prod_{i=0}^{k-1}\overline{\mathcal{M}}'_*(c_i,c_{i+1})\times\overline{\mathcal{M}}'_*(c_k,b),$$

can be patched together to give a complex line bundle on $C\overline{\mathcal{M}}'_*(a,b)$.

Here and hereafter $\mathcal{M}_*$ stands for the set of irreducible connections. We can not extend the line bundle to the neighborhood of the connections described in Theorems 5.4 and 5.6. This is the reason why Theorem 1.10 does not hold for $s>2$ when $H_1(M;\mathbf{Z})$ is torsion free and $s>0$ when $H^1(M;\mathbf{Z})$ has a torsion. (We shall explain this point a bit more detail in §12.)

The proofs of Theorems 7.1 and 7.3 occupy §§7–11. We include the analysis of the structure of moduli space and the line bundle on it in the neighborhood of the connection described in Theorems 5.4 and 5.6, though the author does not know how to use it to deduce a topological information. In order to explain the outline of the proofs of Theorems 7.1 and 7.3, we introduce the following notion. (Compare Donaldson [D2].)

Definition 7.4. Let $K_0 \subset \overline{\mathcal{M}}'(a, c_0), \cdots, K_k \subset \overline{\mathcal{M}}'(c_k, b)$ be compact subsets and $\epsilon, T, C > 0$. We say that $[A] \in \overline{\mathcal{M}}'(a, b)$ is a *standard model of type* $(K_0, \cdots, K_k, T, \epsilon, C)$, if there exist $[A_i] \in K_i$, $S_{i+1} > T + S_i$, and $[A'] = [A]$, with the following property.

Let $I_i : M \times [-T, T] \to M \times \mathbf{R}$ be the embedding defined by $I_i(x, t) = (x, t + S_i)$. Then we have

(7.4.1) $\qquad \|I_i^*(A') - A_i\|_{C^\ell}(x, t) < \epsilon,$

(7.4.2) $\qquad |A' - c_i|_{C^\ell}(x, t) <$
$$C \exp\{-\min\{|S_i + T/2 - t|, |S_{i+1} - T/2 - t|\}/C\},$$

if $t \in [S_i + T/2, S_{i+1} - T/2]$.

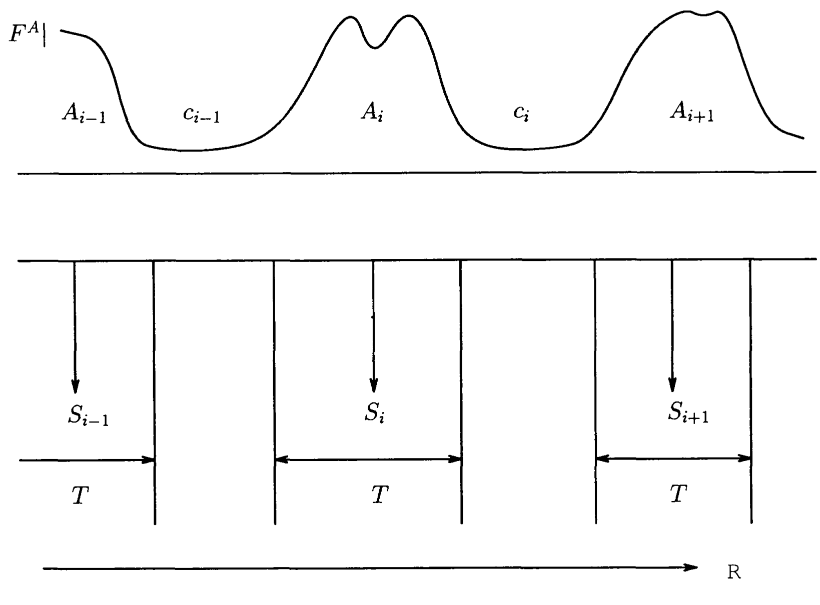

Figure 3.

The proof of Theorem 7.1 is based on the following two Theorems 7.5 and 7.6.

Theorem 7.5. *There exists C such that, for each $T, \Lambda, \epsilon > 0$, we can find a compact subset $K_{a,b}$ of $\overline{\mathcal{M}}(a, b)$ for each $a, b \in Fl$, with the* *following property. If* $[A] \in \overline{\mathcal{M}}(a,b)$, $\sup|F^A| < \Lambda$, *and if* $[A] \notin K_{a,b}$, *then there exist* $c_0, \cdots, c_k \in Fl$ *such that* $[A]$ *is a standard model of type* $(K_{a,c_0}, \cdots, K_{c_k,b}, T, \epsilon, C)$.

Theorem 7.6. *For each compact set* $K_0 \subset \overline{\mathcal{M}}'(a, c_0), \cdots, K_k \subset \overline{\mathcal{M}}'(c_k, b)$ *and* C, *there exist* $\epsilon = \epsilon(K_0, \cdots, K_k, C)$ *and* $T = T(K_0, \cdots, K_k, C)$, *such that the set of elements of* $\mathcal{M}'(a,b)$ *which is a standard form of type* $(K_0, \cdots, K_k, \epsilon, T, C)$ *is parametrized by*

$$\widetilde{K}_0 \times_{G_{c_0}} \widetilde{K}_1 \times_{G_{c_1}} \cdots \times_{G_{c_k}} \widetilde{K}_k \times (T, \infty)^{k+1}.$$

Here $\widetilde{K}_i \subset \mathcal{M}'(c_{i-1}, c_i)$ *is the lift of* K_i.

Here $\widetilde{K}_0 \times_{G_0} \widetilde{K}_1$ is the quotient of $\widetilde{K}_0 \times \widetilde{K}_1$ by the action $g([A],[B]) = ([A]g^{-1}, g[B])$ of G_0. The proof of Theorem 7.6 is in §8. For the proof of Theorem 7.1, we need a bit more complicated version of Theorem 7.5.

Theorem 7.5′. *For each* $\Lambda > 0$ *we can find* $K_{a,b} \subset \overline{\mathcal{M}}(a,b)$ *and* C_k *such that the conclusion of Theorem 7.5 holds for*

$$\epsilon_k = \epsilon(K_{a,c_0}, \cdots, K_{c_k,b}, C_k),$$
$$T_k = T(K_{a,c_0}, \cdots, K_{c_k,b}, C_k)$$

where $\epsilon(\cdots)$, $T(\cdots)$, *and* $C(\cdots)$ *are as in Theorem 7.6.*

The proof of Theorem 7.5′ is in §9. Now we are ready to explain the outline of the proof of Theorem 7.1. Let $a, b \in Fl_0$. Choose $K_{c,c'}$ for $\mu(a) \geq \mu(c) \geq \mu(c') \geq \mu(b)$, as in Theorem 7.5′. For $\mathfrak{c} = (c_0, \cdots, c_k)$, Let $\epsilon(\mathfrak{c})$ and $T(\mathfrak{c})$ be the number in Theorem 7.6. Define an equivalence relation $\sim$ on

$$\widetilde{K}_{a,c_0} \times \cdots \times \widetilde{K}_{c_k,b} \times (T(\mathfrak{c}), \infty]^{k+1}$$

by

$$\begin{cases} (x_0, \cdots, x_{k+1}, t_0, \cdots, t_{k+1}) \sim (x_0, \cdots, x_i g, g^{-1}x_{i+1}, \cdots, t_{k+1}) \\ \qquad\qquad \text{for each } t_0, \cdots, t_{k+1} \\ (x_0, \cdots, x_{k+1}, t_0, \cdots, t_{k+1}) \sim (x_0, \cdots, x_i g, x_{i+1}, \cdots, t_{k+1}) \\ \qquad\qquad \text{if} \quad t_i = \infty. \end{cases}$$

Put

$$\begin{aligned}
\widetilde{X}(\mathfrak{c}) &= \frac{\widetilde{K}_{a,c_0} \times \cdots \times \widetilde{K}_{c_k,b} \times (T(\mathfrak{c}), \infty]^{k+1}}{\sim}, \\
X(\mathfrak{c}) &= G_a \backslash \widetilde{X}(\mathfrak{c}) / G_b, \\
\widetilde{X}^\circ(\mathfrak{c}) &= \frac{\widetilde{K}_{a,c_0} \times \cdots \times \widetilde{K}_{c_k,b} \times (T(\mathfrak{c}), \infty)^{k+1}}{\sim}, \\
\overset{\circ}{X}(\mathfrak{c}) &= G_a \backslash \widetilde{X}^\circ(\mathfrak{c}) / G_b.
\end{aligned}$$

By Theorem 7.6, we have a diffeomorphism

$$\Phi_{\mathfrak{c}} : \overset{\circ}{X}(\mathfrak{c}) \to \overline{\mathcal{M}}'(a, b).$$

to its image. If $\mathfrak{c}' \subset \mathfrak{c}$, we have, by Theorem 7.6,

$$\Phi_{\mathfrak{c},\mathfrak{c}'} : X(\mathfrak{c}) \to G_a \backslash \mathcal{M}'(a, c'_0) \times_{G_{c'_0}} \cdots \times_{G_{c'_{k'}}} \mathcal{M}'(c'_{k'}, b)/G_b \times [T, \infty]^{k'+1}.$$

We put

$$U(\mathfrak{c}, \mathfrak{c}') = \{z \in X(\mathfrak{c}) \mid \Phi_{\mathfrak{c},\mathfrak{c}'}(z) \in \overset{\circ}{X}(\mathfrak{c}')\}.$$

If $\Phi_{\mathfrak{c}'}\Phi_{\mathfrak{c},\mathfrak{c}'} = \Phi_{\mathfrak{c}}$ is true, then we are able to use these maps to define the smooth structure on $\mathcal{CM}'(a, b)$. But the above equality does not exactly hold but holds modulo some small difference. Hence we have to perturb them. The argument needed for it is in §10, where we define the notion of local action and construct it on the end of $\mathcal{M}'(a, b)$. To extend line bundle we use an argument similar to the proof of the theorems in §4 and a lift of the local action to the line bundle.

§8. Taubes construction

We prove Theorem 7.6 in this section. Theorem 7.6 corresponds Donaldson [D2] §4. There Donaldson used the "alternating method". His method might work in our situation, where we have to deal with various types of reducible connections. But, since the organization needed for alternating method is a bit complicated, we use here more direct argument. (Maybe this is one Donaldson suggested in [D2] p 302.)

For simplicity of notation, we shall prove a special (but the most difficult) case. Let $a, c_1, c_2, b \in Fl$ such that $G_a = G_b = \{\pm 1\}$, $G_{c_1} = G_{c_2} = SU(2)$, $\mu(c_1) = \mu(c_2) + 3$, and $\widetilde{K} \subset \mathcal{M}'(c_1, c_2)$ be a component consisting of reducible connections. (We have, by Theorem 5.4,

$$\widetilde{K} \simeq \frac{SU(2) \times SU(2)}{U(1)}.)$$

Let $K_1 \subset \overline{\mathcal{M}}'(a,c_1)$, $K_2 \subset \overline{\mathcal{M}}'(c_2,b)$ be compact subsets and $\widetilde{K}_1 \subset \mathcal{M}'(a,c_1)$, $\widetilde{K}_2 \subset \mathcal{M}'(c_2,b)$ be their lifts. We shall construct a diffeomorphism $\Phi_{K,K_1,K_2} : \widetilde{K}_1 \times_{G_{c_1}} \widetilde{K} \times_{G_{c_2}} \widetilde{K}_2 \times [T,\infty)^2 \times \mathbf{R} \to \mathcal{M}(a,b)$, whose image contains all standard model of type $(K_1, K, K_2, T, \epsilon, C)$.

Choose a finite open covering

$$\begin{aligned} U_1^1 \cup \cdots \cup U_N^1 &\supseteq K_1 \\ U_1^2 \cup \cdots \cup U_N^2 &\supseteq K_2, \end{aligned}$$

and sections $\overline{s}_j^i : U_j^i \to \widetilde{K}_i$. Let $s_j^1 : U_j^1 \to \mathcal{A}_{\ell,\delta}(a,c_1)$, $s_j^2 : U_j^2 \to \mathcal{A}_{\ell,\delta}(c_2,b)$ be their lifts. Choose also an open covering

$$V_1 \cup \cdots \cup V_N = SU(2),$$

such that V_k is contractible. We have maps

$$\begin{aligned} J_k^1 &: V_k \times \mathbf{R} \to SU(2) \\ J_k^2 &: V_k \times \mathbf{R} \to SU(2) \end{aligned}$$

such that

$$\begin{cases} J_k^1(g,t) = 1 & \text{if} \quad t < -1 \\ J_k^1(g,t) = g & \text{if} \quad t > 0 \end{cases}$$
$$\begin{cases} J_k^2(g,t) = 1 & \text{if} \quad t > 1 \\ J_k^2(g,t) = g & \text{if} \quad t < 0. \end{cases}$$

Let $d + a_t^0 \in \mathcal{A}_{\ell,\delta}(c_1,c_2)$ be a representative of $G_{c_1}\backslash\widetilde{K}/G_{c_2} =$ one point. Choose a nonincreasing smooth function $\chi : \mathbf{R} \to [0,1]$ such that

$$\chi(t) = \begin{cases} 1 & \text{if} \quad t < 0 \\ 0 & \text{if} \quad t > 1. \end{cases}$$

Now, we define a map

$$\widetilde{\Phi}'_{j_1,j_2,k_1,k_2} : U_{j_1}^1 \times V_{k_1} \times V_{k_2} \times U_{j_2}^2 \times [T,\infty)^2 \times \mathbf{R} \to \mathcal{A}_{\ell,\delta}(a,b),$$

as follows. Let $A_i = s^i_{j_i}([A_i])$, $S_i \in [T, \infty)$, $S \in \mathbf{R}$, $g_i \in V_{k_i}$. Then

$$
\begin{aligned}
&\widetilde{\Phi}'_{j_1,j_2,k_1,k_2}([A_1], g_1, g_2, [A_2], S_1, S_2, S) \\
&\quad = (J_{k_1}(g_1,\cdot)^* A_1)(x, t-S) \qquad \text{for} \quad t < S + S_1/3 \\
&\quad = \chi\left(\frac{t - S - S_1/3}{S_1/3}\right) g_1^* A_1(x, t-S) \\
&\qquad\quad + \left(1 - \chi\left(\frac{t - S - S_1/3}{S_1/3}\right)\right) a^0_{t-S-S_1} \\
&\qquad\quad \text{for} \quad t \in [S + S_1/3, S + 2S_1/3] \\
&\quad = a^0_{t-S-S_1} \qquad \text{for} \quad t \in [S + 2S_1/3, S + S_1 + S_2/3] \\
&\quad = \chi\left(\frac{t - S - S_1 - S_2/3}{S_2/3}\right) a^0_{t-S_1-S} \\
&\qquad\quad + \left(1 - \chi\left(\frac{t - S - S_1 - S_2/3}{S_2/3}\right)\right) g_2^* A_2(x, t - S - S_1 - S_2) \\
&\qquad\quad \text{for} \quad t \in [S + S_1 + S_2/3, S + S_1 + 2S_2/3] \\
&\quad = (J^2_{k_2}(g_2,\cdot)^* A_2)(s, t - S - S_1 - S_2) \qquad \text{for} \quad t > S + S_1 + 2S_2/3.
\end{aligned}
$$

Here $J^i_k(g,\cdot)$ is regarded as a map $M \times \mathbf{R} \to SU(2)$ and a gauge transformation.

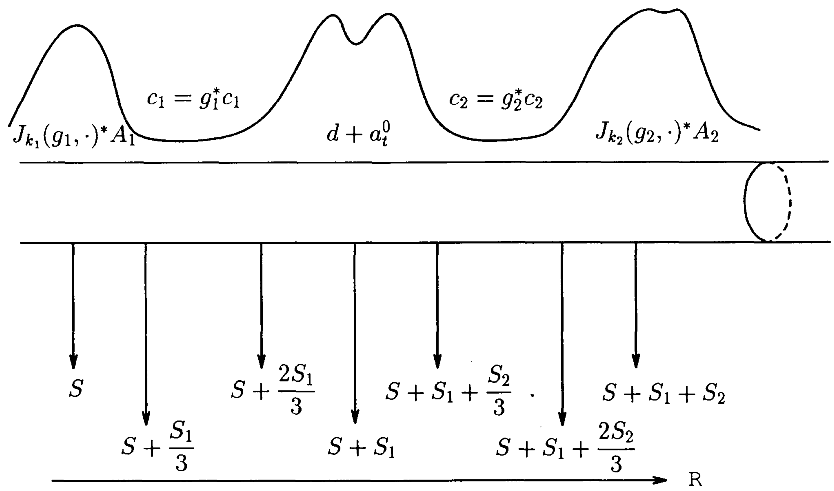

Figure 4

We remark that, by the compactness of K_1, we have a constant C such that

$$
\begin{cases}
|(d + A_1) - (d + a)| < Ce^{t/C}, \\
|(d + A_1) - (d + c_1)| < Ce^{-t/C},
\end{cases}
\tag{8.1}
$$

for $A_1 \in K_1$. (Compare the decay estimate in next section.) A similar estimate holds for K_2 and K. Using (8.1) we can prove the following:

Lemma 8.2. *If*

$$[A_1] \in U^1_{j_1} \cap U^1_{j_1'},$$
$$[A_2] \in U^1_{j_2} \cap U^2_{j_2'},$$
$$g_1 \in V_{k_1} \cap V'_{k_2},$$
$$g_2 \in V_{k_2} \cap V'_{k_2},$$

then there exists a gauge transformation $\widehat{g}$, such that

$$\widehat{g}^* \widetilde{\Phi}'_{j_1,j_2,k_1,k_2}([A_1], g_1, g_2, [A_2], S_1, S_2, S)(t,x) = \widetilde{\Phi}'_{j_1',j_2',k_1',k_2'}([A_1], g_1, g_2, [A_2], S_1, S_2, S)(t,x),$$

if $t \notin [S + S_1/3, S + 2S_1/3] \cup [S + S_1 + S_2/3, S + S_1 + 2S_2/3]$, and

$$|\widehat{g}^* \widetilde{\Phi}'_{j_1,j_2,k_1,k_2} - \widetilde{\Phi}'_{j_1',j_2',k_1',k_2'}| < e(S_1, S_2).$$

Here and hereafter, we put

$$e(S_1, S_2) = C \exp(-\min\{S_1, S_2\}/C).$$

Choose an embedding $U(1) \subset SU(2)$ such that a^0_t is invariant by the image. By Lemma 8.2 and the construction, we can apply the partition of unity associated to the coverings $\{U^1_j\}$ and $\{U^2_j\}$ to prove the following:

Lemma 8.3. *There exists*

$$\widetilde{\Phi}''_{j_1,j_2,k_1,k_2} : U^1_{j_1} \times V_{k_1} \times V_{k_2} \times U^2_{j_2} \times [T,\infty)^2 \times \mathbf{R} \to \mathcal{A}_{\ell,\delta}(a,b),$$

such that

(8.3.1) $$|\widetilde{\Phi}''_{j_1,j_2,k_1,k_2} - \widetilde{\Phi}'_{j_1,j_2,k_1,k_2}| < e(S_1, S_2),$$

(8.3.2) *the maps $\widetilde{\Phi}''_{j_1,j_2,k_1,k_2}$ can be patched together to give a map*

$$\Phi'_{K_1,K,K_2} : \widetilde{K}_1 \times_{SU(2)} \frac{SU(2) \times SU(2)}{U(1)} \times_{SU(2)} \widetilde{K}_2 \times [T,\infty)^2 \times \mathbf{R} \to \mathcal{B}_{\ell,\delta}(a,b).$$

By (8.1) we have:

Lemma 8.4. *Let* $[A] \in \operatorname{Im} \Phi'_{K_1,K,K_2}$ *then*

$$|F^A + \widetilde{*}_\sigma F^A - \operatorname{grad}_{a_t} f \wedge dt - *_\sigma \operatorname{grad}_{a_t} f|_{L^2_\ell} < e(S_1, S_2).$$

We put

$$|u|_{\ell,S_1,S_2,S} = |u|_{L^2_\ell(M\times\mathbf{R})} + |u|_{L^1_\ell(M\times S,S+S_1+S_2)}.$$

Then we have also

Lemma 8.4′. *Let* $[A] \in \operatorname{Im} \Phi'_{K_1,K,K_2}$ *then*

$$|F^A + \widetilde{*}_\sigma F^A - \operatorname{grad}_{a_t} f \wedge dt - *_\sigma \operatorname{grad}_{a_t} f|_{\ell,S_1,S_2,S} < e(S_1, S_2).$$

We shall apply Taubes' method as in [FU], to deform Φ'_{K_1,K,K_2} to a map to $\mathcal{M}(a,b)$. For this purpose, the following estimate is essential.

Lemma 8.5. *There exists* $\lambda > 0$ *independent of* S_i *such that if* $A \in \operatorname{Im} \Phi'_{K_1,K,K_2}$, $u \in \Omega^2_\ell$ *we have*

$$|\mathcal{D}_A \mathcal{D}_A^* u|_{L^2_{\ell-2}} > \lambda |u|_{L^2_\ell}.$$

This lemma is an immediate consequence of Lemma 3.9 and Remark 4.6. Furthermore since $a \to \operatorname{grad}_a f$ is a C^2 map with respect to the L^2_ℓ norm for large ℓ, it follows that

$$\operatorname{grad}_{a_t+u_t} f = \operatorname{grad}_{a_t} f + (\operatorname{Hess}_{a_t} f)(u_t) + E(a,u)$$

with

$$\begin{aligned}|E(a,u)|_{L^2_\ell} &\le C|u|^2_\ell \\ |E(a,u)|_{\ell,S_1,S_2,S} &\le C|u|^2_{\ell,S_1,S_2,S}.\end{aligned}$$

Hence we can apply the argument of [FU] pp.132–139, and obtain

Lemma 8.6. *There exists* T_0, *and* $\widetilde{\Phi}_{j_1,j_2,k_1,k_2} : U^1_{j_1} \times V_{k_1} \times V_{k_2} \times U^2_{j_2} \times [T_0,\infty) \times \mathbf{R} \to \widehat{M}(a,b)$ *such that*

(8.6.1) $\widetilde{\Phi}_{j_1,j_2,k_1,k_2}$ *can be patched together to give a map*

$$\Phi_{K_1,K,K_2} : \widetilde{K}_1 \times_{SU(2)} \frac{SU(2)\times SU(2)}{U(1)} \times_{SU(2)} \widetilde{K}_2 \times [T,\infty)^2 \times \mathbf{R} \\ \to \mathcal{M}(a,b).$$

$$(8.6.2) \qquad \left|\widetilde{\Phi}''_{j_1,j_2,k_1,k_2} - \widetilde{\Phi}_{j_1,j_2,k_1,k_2}\right|_{C^1,\ell,S_1,S_2,S} < e(S_1,S_2).$$

The definition of the norm in (8.6.2) is as follows. $U^1_{j_1} \times V_{k_1} \times V_{k_2} \times U^2_{j_2} \times [T_0,\infty) \times \mathbf{R}$ has a natural Riemannian metric. We define a norm on $\mathcal{A}_{\ell,\delta}(a,b)$ by using (ℓ,S_1,S_2,S)-norm. Then the norm in (8.6.2) is the C^1-norm with respect to this metric and norm.

Note that the linear equation solved in [FU] pp.132–139 is gauge invariant. (8.6.1) follows from this fact.

We shall prove that the map Φ_{K_1,K,K_2} is an immersion, surjective to the set of standard model, and that injective.

Let $g_1, g_2 \in V_{k_1}, V_{k_2}$, and $\Pi \subset T_{(g_1,g_2)}(V_{k_1},V_{k_2})$ be an orthonormal complement of $T_{(g_1,g_2)}(U(1)\cdot(g_1,g_2))$.

Lemma 8.7. *There exists C independent of S_1, S_2 such that, for each $v \in \Pi$ we have:*

$$\left|\Phi'_{j_1,j_2,k_1,k_2*}(v)\right|_{\ell,S_1S_2,S} \geq C|v|,$$

for sufficiently large S_i. Here we choose $[A_i] \in U^i_{j_i}$, S_i,S and regard

$$\Pi \subset T_{([A_1],g_1,g_2,[A_2],S_1,S_2,S)}(U^1_{j_1} \times V_{k_1} \times V_{k_2} \times U^2_{j_2} \times [T,\infty)^2 \times \mathbf{R}).$$

Remark 8.8. The lemma does not hold if we replace the $||_{\ell,S_1,S_2,S}$-norm by L^2_ℓ-norm, since c_1 and c_2 are reducible.

Proof. For simplicity, we put $g_1 = g_2 = 1$. Set

$$A = \widetilde{\Phi}'_{j_i,j_2,k_1,k_2}([A_1],1,1,[A_2],S_1,S_2,S)$$
$$v = (\overline{v}_1,\overline{v}_2) \in su(2) \oplus su(2).$$

Define $v_i : \mathbf{R} \to su(2)$, by

$$v_i(t) = \frac{d}{ds} J^i_{k_i}(1 + s\overline{v}_i, t)\Big|_{s=0}.$$

Then by definition

$$(8.9) \qquad \widetilde{\Phi}'_{j_1,j_2,k_1,k_2*}(v_1,v_2) = \begin{cases} (d^{A_1}v_1)(x,t-S) \\ \qquad \text{for} \quad t < S \\ (d^{A_2}v_2)(x,t-S_1-S_2-S) \\ \qquad \text{for} \quad t > S+S_1+S_2 \\ 0 \qquad \text{otherwise.} \end{cases}$$

Let the differential form in the above formula be denoted by w. Lemma 8.7 is a consequence of the following:

Lemma 8.10. *There exists C such that*

$$\left|w - d^A u\right|_{\ell,S_1,S_2,S} > C(|v_1| + |v_2|)$$

for each $u \in \Omega^0_{\ell+1}$ and sufficiently large S_i.

(In the statement we omit δ, since a and b are irreducible.)

Proof. We prove by construction. Then we assume that we have $\overline{v}_i^n \in su(2)$ with $|\overline{v}_i^n| = 1$, and $S_i^n \to \infty$, $[A_i^n]$,u^n such that

$$\lim_{n\to\infty} \left|w^n - d^{A_i^n} u^n\right|_{\ell,S_1^n,S_2^n,S} = 0.$$

Since $[A_i^n]$ and $\overline{v}_i^n$ move on compact sets, we may assume that they are independent of n. Hence we have

$$\begin{aligned} &S_i^n \to \infty \\ &\left|w^n - d^{A^n} u^n\right|_{\ell,S_1^n,S_2^n,S} \to 0. \end{aligned}$$

Here w^n is as in (8.9) with $S_i = S_i^n$, and

$$A^n = \widetilde{\Phi}'_{j_1,j_2,k_1,k_2}([A_1], 1, 1, [A_2], S_1^n, S_2^n, S).$$

(Since everything is invariant by the $\mathbf{R}$ action, we may assume that S is independent of n.) By construction, there exists α independent of n such that

(8.11)
$$\begin{aligned} &|d + A^n - d|_{C^{\ell'}} < Ce^{-\beta_1(t)/C} \quad \text{if} \quad t \in S + \alpha, [S + S_1^n - \alpha] \\ &|d + A^n - d|_{C^{\ell'}} < Ce^{-\beta_2(t)/C} \quad \text{if} \quad t \in [S + S_1^n + \alpha, S + S_1^n + S_2^n - \alpha], \end{aligned}$$

where

$$\begin{aligned} \beta_1(t) &= d(t, \partial[S + \alpha, S + S_1^n - \alpha]) \\ \beta_2(t) &= d(t, \partial[S + S_1^n + \alpha, S + S_1^n + S_2^n - \alpha]). \end{aligned}$$

Hence, by (8.9), we have, for each $\alpha' > \alpha$, that

$$|du^n|_{L^1_\ell(S+\alpha'.S+S_1^n+S_2^n-\alpha')} < \epsilon_n + Ce^{-\alpha'/C},$$

where $\epsilon_n \to 0$. Therefore there exists $s_1^n, s_2^n \in su(2)$ such that

$$\begin{aligned}&|u^n - s_1^n|_{C^{\ell'}}(x,t) < C\epsilon_n + Ce^{-\beta_1(t)/C}\\ &\qquad \text{if} \quad t \in [S+\alpha', S+S_1^n - \alpha']\\ &|u^n - s_2^n|_{C^{\ell'}}(x,t) < C\epsilon_n + Ce^{-\beta_2(t)/C}\\ &\qquad \text{if} \quad t \in [S+S_1^n+\alpha', S+S_1^n+S_2^n-\alpha'].\end{aligned}$$

(This is the step we can not work with L^2 norm.)

Then patching u with s_1^n and s_2^n, we have $u_1^n, u_2^n, u_3^n \in L^2_{\ell+1}(M \times \mathbf{R}, su(2))$ such that

$$|d^{A_1}(v_1 - u_1^n)|_{C^{\ell'}} < C\epsilon_n \tag{8.12.1}$$

$$|d^{A_2}(v_2 - u_2^n)|_{C^{\ell'}} < C\epsilon_n \tag{8.12.2}$$

$$|d^{a_t^0} u_3|_{C^{\ell'}} < C\epsilon_n \tag{8.12.3}$$

$$|u_1^n(t,x) - s_1^n|_{C^{\ell'}} < Ce^{-t/C} \tag{8.12.4}$$

$$|u_2^n(t,x) - s_2^n|_{C^{\ell'}} < Ce^{t/C} \tag{8.12.5}$$

$$|u_3^n(t,x) - s_2^n|_{C^{\ell'}} < Ce^{-t/C} \tag{8.12.6}$$

$$|u_3^n(t,x) - s_1^n|_{C^{\ell'}} < Ce^{t/C} \tag{8.12.7}$$

(u_1^n,u_2^n, and u_3^n are constructed from the restrictions of u^n to $(-\infty, S+S_1^n/3]$, $[S+S_1^n+2S_2^n/3, \infty)$, $[S+2S_1^n/3, S+S_1^n+S_2^n/3]$, respectively.)

We may assume that $\lim s_1^n = s_1$ and $\lim s_2^n = s_2$. Therefore, by (8.12.3), (8.12.6),(8.12.7) and the fact $G_{a_t^0} = U(1)$ imply that $s_1 = s_2 \in u(1) \subset su(2)$. ($u(1)$ is a Lie algebra of $G_{a_y^0} = U(1)$.) Hence, using the fact that $(\overline{v}_1, \overline{v}_2)$ is perpendicular to $u(1) \subset su(2) \oplus su(2)$, we can find t_0 such that

$$|v_1 - u_1^n|(x,t_0) > C \tag{8.13}$$

or

$$|v_2 - u_2^n|(x,-t_0) > C,$$

for some C independent of n. Suppose, for example (8.13) holds. By scaling, we can find $(u^n)'$ such that

$$\begin{aligned}&\infty > C_2 > |(u^n)'|(x,t_0) > C_1 > 0\\ &\left|d^{A_1}(u^n)'\right|_{C^\ell} < \epsilon_n \to 0.\end{aligned}$$

Therefore, by taking a subsequence, $(u^n)'$ converges to u' such that $d^{A_1}u' = 0$, with respect to the compact uniform topology. This contradicts the irreducibility of A_1. The proof of Lemma 8.10 is now complete.

An estimate similar to Lemma 8.7 for TK_i direction and $[T,\infty)^2 \times \mathbf{R}$ direction is easier. Then, combined with (8.6.2), they imply:

Lemma 8.14. *If V is a tangent vector of*

$$\widetilde{K}_1 \times_{SU(2)} \frac{SU(2)\times SU(2)}{U(1)} \times_{SU(2)} \widetilde{K}_2 \times [T,\infty)^2 \times \mathbf{R},$$

at $([A_1], g_1, g_2, [A_2], S_1, S_2, S)$, then we have

$$|\Phi_{K_1,K,K_2,*}(V)|_{\ell,S_1,S_2,S} > C|V|.$$

Lemma 8.14 implies that Φ_{K_1,K,K_2} is of maximal rank.

Remark 8.15. By Hölder's inequality, we have

$$||_{\ell,S_1,S_2,S} < C(S_1+S_2)||_{L^2}.$$

Hence, Lemma 8.14 implies

$$|\Phi_{K_1,K,K_2*}(v)|_{L^2} > \frac{C|v|}{S_1+S_2}.$$

It seems that this reflects the fact that the sectional curvature K of $\mathcal{M}(a,b)$ at $\Phi(A_1,g_1,g_2,A_2,S_1,S_2,S)$ is estimated as $|K| < C(S_1+S_2)^2$.

Lemma 8.16. *For each C, there exist T,S,ϵ, such that if $[A]$ is a standard model of type (K_1,K,K_2,T,ϵ,C), then*

$$[A] \in \Phi_{K_1,K,K_2}(\widetilde{K}_1 \times_{G_{c_1}} \widetilde{K} \times_{G_{c_2}} \widetilde{K}_2 \times [S,\infty)^2 \times \mathbf{R}).$$

Proof. The definition of the standard model implies that there exist $[A_1]$,$[A_2]$,g_1,g_2,S_1,S_2,S such that

$$|\widetilde{\Phi}'_{i_1,i_2,k_1,k_2}([A_1],g_1,g_2,[A_2],S_1S_2,S) - A|_{L^2_\ell} < e(S_1,S_2).$$

Here A is a representative of A, and $A_j \in U_{i_j}$, $g_j \in V_{i_j}$. Let $\ell : [0,1] \to \mathcal{A}_{\ell,\delta}(a,b)$ be the straight line connecting them. The length of ℓ is smaller than $e(S_1,S_2)$. By [FU] pp.132–139, we can deform this path to a path ℓ' in $\widehat{\mathcal{M}}(a,b)$ connecting $\widetilde{\Phi}_{i_1,i_2,k_1,k_2}([A_1],g_1,g_2,[A_2],S_1,S_2,S)$ and A. The length of ℓ' is also estimated by $e(S_1,S_2)$. By using Lemma 8.14, we can lift this path to $\widetilde{\ell}: [0,1] \to \widetilde{K}_1 \times_{G_{c_1}} \widetilde{K} \times_{G_{c_2}} \widetilde{K}_2 \times [T,\infty)^2 \times \mathbf{R}$ such that $\widetilde{\ell}(0) = ([A_1], g_1, g_2, [A_2], S_1, S_2, S)$. Therefore

$$\Phi_{K_1,K,K_2}(\widetilde{\ell}(1)) = [A],$$

as required.

Finally we shall prove that Φ_{K_1,K,K_2} is injective.

Lemma 8.17. *If*

$$\Phi_{K_1,K,K_2}([A_1], g_1, g_2, [A_2], S_1, S_2, S) = \\ \Phi_{K_1,K,K_2}([A_1'], g_1', g_2', [A_2'], S_1', S_2', S')$$

then

$$\begin{aligned} &|A_i - A_i'|_{\ell,S_1,S_2,S} < e(S_1,S_2) \\ &|S_i - S_i'| < e(S_1,S_2) \\ &|S - S'| < e(S_1,S_2), \end{aligned}$$

and there exists $h \in SU(2)$ such that

$$|hg_i - g_i'| < e(S_1,S_2).$$

Proof. The proof is similar to the proof of Lemma 8.7. Suppose $A_j \in U_{i_j}$, $A_j' \in U_{i_j'}$, $g_j \in V_{k_j}$, $g \in V_{k_j'}$. The proof of the statement on S_i and S is easy, then we assume that $S_i = S_i'$, $S = S'$, for simplicity. By assumption, there exists a gauge transformation $\widehat{g} : M \times \mathbf{R} \to SU(2)$ such that

$$\widehat{g}^* \widetilde{\Phi}_{i_1,i_2,k_1,k_2}([A_1], g_1, g_2, [A_2], S_1, S_2, S) = \\ \widetilde{\Phi}_{i_1',i_2',k_1',k_2'}([A_1'], g_1', g_2', [A_2'], S_1, S_2, S).$$

Then

$$|\widehat{g}^* \widetilde{\Phi}'_{i_1,i_2,k_1,k_2}([A_1], g_1, g_2, [A_2], S_1, S_2, S) - \\ \widetilde{\Phi}'_{i_1',i_2',k_1',k_2'}([A_1'], g_1', g_2', [A_2'], S_1, S_2, S)|_{\ell,S_1,S_2,S} < e(S_1,S_2).$$

Therefore, we have

$$|d\widehat{g}|_{C^\ell} < \begin{cases} Ce^{-\beta_1(t)/C} & \text{if} \quad t \in [S+\alpha, S+S_1-\alpha] \\ Ce^{-\beta_2(t)/C} & \text{if} \quad t \in [S+S_1+\alpha, S+S_1+S_2-\alpha]. \end{cases}$$

Here β_i is as in (8.11). Hence we have $g_i^0 \in SU(2)$ such that

$$|\widehat{g} - g_1^0| < Ce^{-\beta_1(t)/C} \quad \text{if} \quad t \in [S+\alpha, S+S_1-\alpha]$$
$$|\widehat{g} - g_2^0| < Ce^{-\beta_2(t)/C} \quad \text{if} \quad t \in [S+S_1+\alpha, S+S_1+S_2-\alpha].$$

Hence as in the proof of Lemma 8.10, we obtain $\widehat{g}_i : M \times \mathbf{R} \to SU(2)$, $i = 1, 2, 3$, such that

$$(8.18.1) \qquad |(\widehat{g}_1 J^1_{k_1}(g_1, \cdot))^* A_1 - J^1_{k'_1}(g'_1, \cdot)^* A'_1|_{L^2_\ell} < e(S_1, S_2)$$

$$(8.18.2) \qquad |(\widehat{g}_2 J^2_{k_2}(g_2, \cdot))^* A_2 - J^2_{k'_2}(g'_2, \cdot)^* A'_2|_{L^2_\ell} < e(S_1, S_2)$$

$$(8.18.3) \qquad |\widehat{g}_3^* a_t^0 - a_t^0|_{L^2_\ell} < e(S_1, S_2)$$

and

$$(8.18.4) \qquad |\widehat{g}_1(x,t) - g_1^0|_{C^\ell} < Ce^{-t/C}$$

$$(8.18.5) \qquad |\widehat{g}_2(x,t) - g_2^0|_{C^\ell} < Ce^{t/C}$$

$$(8.18.6) \qquad |\widehat{g}_3(x,t) - g_2^0|_{C^\ell} < Ce^{-t/C}$$

$$(8.18.7) \qquad |\widehat{g}_3(x,t) - g_1^0|_{C^\ell} < Ce^{t/C}$$

(8.18.3),(8.18.6),(8.18.7) and $G_{a_t^0} = U(1)$ implies that we have $h \in U(1)$ such that

$$|g_i^0 - h| < e(S_1, S_2).$$

Hence (8.18.1),(8.18.2),(8.18.4),(8.18.5) and the irreducibility of A_i, A'_i imply

$$|g'_i - hg_i| < e(S_1, S_2)$$
$$|A_i - A'_i|_{L^2_\ell} < e(S_1, S_2).$$

The proof of Lemma 8.17 is now complete.

Lemma 8.19. *For sufficiently large T, the map Φ_{K_1,K,K_2} is injective.*

Proof. Let A_i, A'_i, g_i, g'_i, S_i, S'_i, S, S' be as in the proof of Lemma 8.17. Replacing g_i by hg_i, we may assume that $|g_i - g'_i| < e(S_1, S_2)$. Hence we can find a path $\ell : [0,1] \to \widetilde{K}_1 \times_{G_{c_1}} \widetilde{K} \times_{G_{c_2}} \widetilde{K}_2 \times [T, \infty)^2 \times \mathbf{R}$ connecting $([A_1], g_1, g_2, [A_2], S_1, S_2, S)$ and $(A'_1, g'_1, g'_2, A'_2, S'_1, S'_2, S')$. The length of ℓ is smaller than $e(S_1, S_2)$. We may assume that A_j and A'_j are in the same $U^i_{j_i}$, and that g_j and g'_j are in the same V_{k_j}. Therefore the map

$$\overline{\ell} = \widetilde{\Phi}_{U^1_{j_1}, U^2_{j_2}, V_{j_1}, V_{j_2}} \circ \ell : [0,1] \to \widehat{\mathcal{M}}(a, b)$$

is well defined. Note $\overline{\ell}(0) = \overline{\ell}(1)$ and the length of $\overline{\ell}$ with respect to the $||_{\ell,S_1,S_2}$-norm is smaller than $e(S_1,S_2)$. Hence we can find $H : D^2 \to \mathcal{A}_{\ell,\delta}(a,b)$ such that $H|_{\partial D^2} = \overline{\ell}$. By [FU] pp.132–139, we can deform H to $H' : D^2 \to \widehat{\mathcal{M}}_{\ell,\delta}(a,b)$ such that $H = H'$ on ∂D^2. Since the diameter of $H'(D^2)$ is smaller than $e(S_1,S_2)$, we can lift H' to $\widetilde{K}_1 \times_{G_{c_1}} \widetilde{K} \times_{G_{c_2}} \widetilde{K}_2 \times [T,\infty)^2 \times \mathbf{R}$, by Lemma 8.14. We conclude $\ell(0) = \ell(1)$. The proof of Lemma 8.19 is complete.

Thus, we have proved that the set of the standard model of type $(K_1, K, K_0, T, \epsilon, C)$ in $\mathcal{M}'(a,b)$ is parametrized by

$$\widetilde{K}_1 \times_{SU(2)} \frac{SU(2) \times SU(2)}{U(1)} \times_{SU(2)} \widetilde{K}_2.$$

We divide it by $G_a \times G_b = \{\pm 1\} \times \{\pm 1\}$ and obtain

$$\widetilde{K}_1 \times_{SU(2)} \frac{SO(3) \times SO(3)}{U(1)} \times_{SU(2)} \widetilde{K}_2.$$

This proves Theorem 7.6, in our case. The proof of the general case is the same, but the notations will be more complicated.

Remark 8.20. It seems that the proofs of Lemmas 8.17 and 8.19 reflect the fact that the injectivity radius of $\overline{\mathcal{M}}'(a,b)$ at $\Phi_{K_1,K,K_2}([A_1], g_1, g_2, [A_2], S_1, S_2, S)$ is larger than $C(\frac{1}{|S_1|+|S_2|})$.

§9. Decay estimate

In this section we shall prove Theorem 7.5$'$. This theorem corresponds to [FU] §9. There Weitzenbeck formula was used for the proof. We can not use it here because, in our case, M is not S^3 and because we perturbed the equation.

Lemma 9.1. *There exist ϵ,λ and C independent of T such that if $d + a_t$ is a $su(2)$ connection on $M \times [-T,T]$ without dt component, $c \in Fl$ and if*

(9.2.1) $$|a_t - c|_{L^2_\ell} < \epsilon$$

(9.2.2) $$\frac{\partial a_t}{dt} = *_\sigma F^{a_t} - \mathrm{grad}_{a_t} f$$

(9.2.3) $$d_c^* a_0 = 0,$$

then we have

$$|a_t - c|_{L^2_\ell} \leq Ce^{-\lambda\beta_T(t)}. \tag{9.3}$$

Here $\beta_T(t) = \inf\{T-t, T+t\}$.

Proof. We put $u(t) = a_t - c$. We have

$$\begin{aligned}*_\sigma F^{c+u(t)} &- \mathrm{grad}_c f \\ &= *_\sigma d_c u(t) - \mathrm{Hess}_c f\,(u(t)) + E(u(t)),\end{aligned}$$

with

$$|E(u(t))|_{L^2_\ell} \leq C|u(t)|^2_{L^2_\ell}, \tag{9.4}$$

for sufficiently large ℓ. Decompose $u(t) = \alpha(t) + \beta(t)$ with

$$\begin{cases} d_c^*\alpha(t) = 0 \\ \beta(t) \in \mathrm{Im}\, d_c \end{cases}$$

Then we have

$$|\alpha(t)|_{L^2_\ell} < C\epsilon, \quad |\beta|_{L^2_\ell} < C\epsilon, \tag{9.5.1}$$

$$\frac{\partial\alpha(t)}{\partial t} = *_\sigma d_c\alpha(t) - \mathrm{Hess}_c f(\alpha(t)) + E_1(\alpha(t), \beta(t)) \tag{9.5.2}$$

$$\frac{\partial\beta(t)}{\partial t} = E_2(\alpha(t), \beta(t)), \tag{9.5.3}$$

with

$$|E_i(\alpha(t), \beta(t))|_{L^2_\ell} < C\left(|\alpha(t)|_{L^2_\ell} + |\beta(t)|_{L^2_\ell}\right)^2. \tag{9.6}$$

We decompose

$$\alpha(t) = \alpha_+(t) + \alpha_-(t),$$

where α_+, α_- belong to the spaces spanned by positive and negative eigenspaces of $*_\sigma d_c - \mathrm{Hess}_c f$, respectively. (Note that by Lemma 2.8, zero is not an eigenvalue of $*_\sigma d_c - \mathrm{Hess}_c f$.) We put $g_\pm(t) = |\alpha_\pm(t)|_{L^2}$, $h(t) = |\beta(t)|_{L^2}$. By (9.2.2) and (9.4), we have

$$|E_1(\alpha(t), \beta(t))|_{L^\infty} < C(g_+(t) + g_-(t) + h(t))^2.$$

Therefore, we have

$$\frac{dg_+}{dt} \geq \lambda g_+ - C_0(g_- + h)^2, \tag{9.7.1}$$

$$\frac{dg_-}{dt} \leq -\lambda g_- + C_0(g_+ + h)^2, \tag{9.7.2}$$

$$\left|\frac{dh}{dt}\right| \leq C_0(g_+ + g_- + h)^2. \tag{9.7.3}$$

Hence, by elliptic regularity, it suffices to show the following:

Sublemma 9.8. *There exists a constant C and ϵ depending only on C_0 and λ and is independent of T such that if g_+,g_- and h be non-negative functions satisfying* (9.7.1)–(9.7.3) *and*

$$|g_\pm(t)| < \epsilon, |h(t)| < \epsilon, \tag{9.7.4}$$

$$h(0) = 0, \tag{9.7.5}$$

then

$$|g_\pm(t)|, |h(t)| < Ce^{-\lambda\beta_T(t)}. \tag{9.9}$$

Proof. First we replace the assumption (9.7.5) by $|h(0)| < \delta$, and prove

$$|g_\pm(t)|, |h(t)| < C(e^{-\lambda\beta_T(t)} + \delta).$$

when $\delta^2 T < \mu_0$, $\epsilon T < \mu_0$ for some μ_0 depending only on C_0 and λ. For this purpose we prove

$$|h| < C_0(\epsilon^n + \epsilon e^{-\lambda\beta_T(t)} + \delta) \tag{9.10.2n}$$

$$|g_\pm| < C_0(\epsilon^n + \epsilon e^{-\lambda\beta_T(t)} + \delta) \tag{9.11.2n.$\pm$}$$

by an induction on n. (Here n is a half integer.) Assume (9.10.2n). Let $t_0 \in [-T, T]$. We put

$$\widehat{g}_+(t) = e^{-\lambda(t-t_0)}g_+(t).$$

Then, by (9.7.1),(9.7.4),(9.10.2n), and (9.11.2$n-1$,$\pm$), we have:

$$\begin{aligned}\epsilon e^{-\lambda(T-t_0)} &\geq \widehat{g}_+(T)\\ &\geq g_+(t_0) - \int_{t_0}^{T} C_0^3 e^{-\lambda(t-t_0)}(\epsilon^{n-1/2} + \epsilon e^{-\lambda\beta_T(t)} + \delta)^2 dt.\end{aligned}$$

(9.11.2n,+) follows. For the proof of (9.11.2n,−), we use $\widehat{g}_- = e^{\lambda(t-t_0)}g_-(t)$ in a similar way.

It is easy to see that (9.10.2n) and (9.11.2n) imply (9.10.2n+1).

For general T, we proceed as follows. Apply the first step to $T_0 = \mu_0/\epsilon$, and $\delta = 0$. We have $h(3T_0/4) < C_0 e^{-T_0\lambda/4}$. Then we apply the first step to $g_\pm(t - 3T_0/4)$, $h(t - 3T_0/4)$ and $T = T_0$. We obtain

$$\sup_{0<t<4T_0/3} |g_\pm(t)| < C_0 e^{-5T_0\lambda/12}$$

$$\sup_{0<t<4T_0/3} |h(t)| < C_0 e^{-5T_0\lambda/12},$$

if $3T_0/2 < T$. And similarly for $-4T_0/3 < t < 0$. Hence we can apply the first step to $T = 4T_0/3$. Iterating this, we obtain the desired result. The proof of Lemma 9.1 is now complete.

Lemma 9.12. *For each δ, C, there exists ϵ such that if $a \in \mathcal{A}_\ell(M)$,*

$$| *_\sigma F^a - \mathrm{grad}_a f|_{L^2_\ell} < \epsilon$$

$$|a|_{L^2_\ell} < C,$$

then there exists $c \in Fl$ and $g \in \mathcal{G}_{\ell+1}$ such that

$$|g^* a - c|_{L^2_\ell} < \delta.$$

Proof. If not, there exists $a_i \in \mathcal{A}_\ell(M)$ and $\delta > 0$, such that

$$\lim_{i\to\infty} | *_\sigma F^{a_i} - \mathrm{grad}_{a_i} f|_{L^2_\ell} = 0, \tag{9.13.1}$$

$$|a_i|_{L^2_\ell} < C, \tag{9.13.2}$$

$$|g_i^* a_i - c|_{L^2_\ell} > \delta \tag{9.13.3}$$

for each i, $g_i \in G_{\ell+1}$, and $c \in Fl$. (9.13.2) implies that, by taking a subsequence, a_i converges to an element a_∞ of $\mathcal{A}_{\ell-1,\delta}(a,b)$. Then, (9.13.1) implies that

$$| *_\sigma F^{a_\infty} - \mathrm{grad}_{a_\infty} f|_{L^2_\ell} = 0.$$

Hence there exists $g_i \in \mathcal{G}_{\ell+1}(M)$ and $c \in Fl$ such that $g_i^* a_i$ converges to c in $\mathcal{A}_{\ell-1}(M)$. By replacing g_i if necessary, we may assume that

$$d_c^*(g_i^* a_i - c) = 0. \tag{9.14}$$

(See FU.) By (9.13.1) we have

(9.15) $$\lim_{i\to\infty} | *_\sigma F^{g_i^* a_i} - \mathrm{grad}_{g_i^* a_i} f|_{L^2_\ell} = 0.$$

By (9.14),(9.15), $\lim |g_i^* a_i - c|_{L^2_{\ell-1}} = 0$, and an elliptic estimate, we have

(9.16) $$\lim_{i\to\infty} |g_i^* a_i - c|_{L^2_\ell} = 0.$$

(9.16) contradicts (9.13.3).

Using this lemma, we can improve Lemma 9.1 as follows.

Lemma 9.17. *There exists T_0,ϵ,λ, and C, such that if $d + a_t$ be a $su(2)$-connection on $M \times [-T,T]$ without dt component, and if*

(9.18.1) $$T > T_0$$

(9.18.2) $$\frac{\partial a_t}{\partial t} = *_\sigma F^{a_t} - \mathrm{grad}_{a_t} f$$

(9.18.3) $$\left| \frac{\partial a_t}{\partial t} \right|_{L^2_\ell} < \epsilon,$$

then there exists $c \in Fl$ and $g \in \mathcal{G}_{\ell+1}(M)$ such that

(9.19) $$|g^* a_t - c|_{L^2_\ell} < Ce^{-\lambda \beta_T(t)}.$$

Here g is regarded as a gauge transformation on $M \times \mathbf{R}$ independent of the $\mathbf{R}$ factor. The constants C,ϵ,λ are independent of T.

Proof. Let ϵ_0 be the number determined in Lemma 9.1, and S be a sufficiently large positive number determined later. Put $\delta = \epsilon_0/2S$. Then we obtain ϵ by Lemma 9.12. We may assume that $\epsilon < \delta$. By Lemma 9.12, we obtain $c \in Fl$. Replacing a_t by gauge transformation independent of t, we may assume that

(9.20.1) $$|a_0 - c|_{L^2_\ell} < \delta$$

(9.20.2) $$d_c^*(a_0 - c) = 0.$$

By (9.20.1),(9.18.3), and $2S\epsilon < \epsilon_0$, we can apply Lemma 9.1 to $M \times [-S,S]$, and obtain

$$|a_t - c|_{L^2_\ell} < Ce^{-\lambda \beta_S(t)}.$$

Hence by taking S sufficiently large, we have

(9.21.1) $$|a_{3S/4} - c|_{L^2_\ell} < \epsilon_0/K$$

(9.21.2) $$|a_{-3S/4} - c|_{L^2_\ell} < \epsilon_0/K.$$

Here K is a sufficiently large positive number determined later. Therefore there exists $g \in \mathcal{G}_{\ell+1}(M)$ such that

$$\begin{aligned}
&|g-1|_{L^2_\ell} < C\epsilon_0/K \\
&d_c^*(g^*a_{3S/4} - c) = 0 \\
&|g^*a_{3S/4} - c|_{L^2_\ell} < C\epsilon_0/K.
\end{aligned}$$

Here C depends only on M. Hence we can apply Lemma 9.1 to $g^*a_{t+3S/4}$, on $M \times [-S, S]$. By choosing S sufficiently large, we obtain

$$|g^*a_t - c|_{L^2_\ell} < C\epsilon_0/K,$$

for $t \in [0, 4S/3]$, provided $3S/2 < T$. By taking K sufficiently large, we have

$$|a_t - c|_{L^2_\ell} < \delta,$$

for $t \in [0, 4S/3]$. By using (9.21.2) we have the same estimate for $t \in [-3S/4, 0]$. Hence we can apply Lemma 9.1 to $M \times [-4S/3, 4S/3]$ if $3S/2 < T$. Repeating this we obtain the lemma.

Lemma 9.22. *There exists $\theta > 0$ such that, if $[A] \in \mathcal{M}_\delta(a, b)$ with $\mu(a) \neq \mu(b)$, and if $g^*A = d + a_t$, where $d + a_t$ is a connection without dt factor, then we have*

$$\int_{M\times\mathbf{R}} \left|\frac{\partial a_t}{dt}\right|^2 dxdt > \theta.$$

Proof. By [F] p122, the integral in the lemma is independent of A but depends only on a and b. Hence the lemma follows from (2.8.1).

Proof of Theorem 7.5$'$. Fix $a, b \in Fl$. Put $k_0 = \mu(a) - \mu(b)$. We shall prove that, for each $\mu(a) \geq \mu(c) \geq \mu(c') \geq \mu(b)$ there exists $K_{c,c'}$, such that the conclusion of Theorem 7.5 holds for

$$\begin{aligned}
\epsilon &= \frac{\epsilon(K_{a,c_0}, \cdots, K_{c_k,b})}{2^k} \\
T &= \frac{T(K_{a,c_0}, \cdots, K_{c_k,b})}{2^k}.
\end{aligned}$$

The proof is by induction on k. The first step is obvious, since $\overline{\mathcal{M}}'(c, c')$ is a finite set if $\mu(c) = \mu(c') + 1$. Hence it is enough to show the last step of the induction. We assume that the last step is false. Then we have $A_i \in \overline{\mathcal{M}}'(a,b)$, such that

(9.23.1) $\sup |F^{A_i}| < \Lambda$,

(9.23.2) $[A_i]$ is unbounded in $\overline{\mathcal{M}}'(a,b)$,

(9.23.3) non of A_i is a standard model.

Let g_i be a gauge transform such that $g_i^* A_i = d + a_t^i$ has no dt component. We have

$$\frac{da_t^i}{dt} = *_\sigma F^{a_t^i} - \operatorname{grad}_{a_t^i} f.$$

If

$$\left| \frac{\partial a_t^i}{dt} \right|_{L^2_\ell} < \epsilon,$$

were true for each t, then Lemma 9.17 would imply that $a_t^i = c$ for some $c \in Fl$. It would follow that $a = b$. This is a contradiction. Hence there exists t_i^1 such that

$$\left| \frac{\partial a_{t_i^1}^i}{dt} \right|_{L^2_\ell} > \epsilon.$$

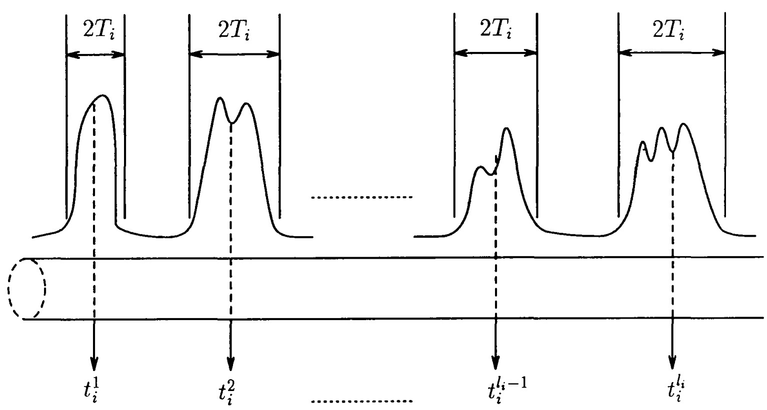

Figure 5.

Lemma 9.24. *There exists L independent of i, and there exist T_i,*

$t_i^1, \dots, t_i^{\ell_i}$, *such that*

(9.24.1) $\ell_i < L$,

(9.24.2) $\lim T_i = \infty$,

(9.24.3) $\left| \frac{\partial a_t^i}{\partial t} \right|_{L^2_\ell} < \epsilon$

if $|t - t_i^j| > T_i$ *for each* i,

(9.24.4) $|t_i^j - t_i^{j'}| > T_i$ *if* $j \neq j'$.

Proof. The existence of the upperbound L of ℓ_i independent of i is the essential part of the statement. Hence, if Lemma 9.24 is not true, then, by taking a subsequence, we may assume that there exist $t_i^1, \cdots, t_i^{\ell_i} \in \mathbf{R}$, T_i such that (9.24.2),(9.24.4) and

(9.24.5) $\lim \ell_i = \infty$

(9.24.6) $\left| \frac{\partial a^i_{t_i^j}}{dt} \right|_{L^2_\ell} > \epsilon$

hold. By $|a_t| < \Lambda$, and by Uhlenbeck's theorem [FU] p117, we can find $g_i^j \in \mathcal{G}_{\ell+1}(M)$ such that a subsequence of the connection

$$t \mapsto g_i^{j*} a^i_{t - t_i^j},$$

converges to an element $d + a^\infty_{j,t}$ of $\mathcal{M}(c_j, c_j')$, for fixed j, (in C^2 topology on any compact set.) Here $c_j, c_{j'} \in Fl$. By (7.24.6), we have $c_j \neq c_{j'}$. Hence by Lemma 9.22

$$\int_{M \times \mathbf{R}} \left| \frac{\partial a^\infty_{j,t}}{dt} \right|^2 dt > \theta,$$

for each j. Therefore, Fatou's lemma implies

$$\begin{aligned} &\lim_{i \to \infty} \int \left| \frac{\partial a_t^i}{\partial t} \right|^2 dt \\ &\geq \sum_{j=1}^{\infty} \int_{M \times \mathbf{R}} \left| \frac{\partial a^\infty_{j,t}}{dt} dt \right|^2 dt \\ &= \infty. \end{aligned}$$

This contradicts the fact that

$$\int \left| \frac{\partial a_t^i}{dt} \right|^2 dt$$

is independent of i but depends only on a and b. The proof of the lemma is complete.

By Lemma 9.24 and $|F^{a_i}| < \Lambda$, we can take a subsequence such that the following holds : $\ell_i = \ell$ is independent of i : let $\widehat{a}_t^{i,j} = a^i_{t-t_i^j}$: there exists $g_{i,j}$ such that $\lim_{i\to\infty} g_{i,j}^* \widehat{a}_t^{i,j}$ converges to an element $a_i^{\infty,j}$ of $\mathcal{M}(c'_j, c''_j)$ uniformly on every compact set, for some c'_j, c''_j. If $\ell = 1$, we can easily prove that A_i is bounded in $\mathcal{M}'(a,b)$. This contradicts (9.23.2). On the other hand, by induction hypothesis, $\widehat{a}_t^{\infty,j}$ is either an element of $K_{c'_j,c''_j}$, or a standard model. Therefore, using Lemma 9.17 and (9.24.3), we can prove that A_i is a standard model for large i. This contradicts (9.23.3). The proof of Theorem 7.5′ is now complete.

§10. Local action on the end of moduli space

Using the results in §§8,9, we obtain charts $\Phi_{\mathfrak{c}} : X(\mathfrak{c}) \to \overline{\mathcal{M}}'(a,b)$ for each $\mathfrak{c}$. As we pointed out in §7 these charts are not compatible. Then we have to perturb them. Also, in order to extend bundles $\mathcal{L}_i^{(2)}$ to the boundary, we have to examine its behaviour on the image of each chart. For these purposes, it is useful to use the notion, local action of groups, which is a generalization of one introduced by Cheeger-Gromov [CG]. They used the local action to study the end of Riemannian manifolds with bounded curvature. In their case, a special kind of local action, F-structure, (that is the local action of Torus,) arises, and the direction of the orbits is the collapsed one. In our case, the curvature is not bounded from above. (It might be bounded from below.) Hence the group acting on the end is not necessary Abelian. (The group $SU(2)$ arises as well.) However the end is also collapsed and the collapsed direction is homogeneous. (For example, in the case we studied in §8, the collapsed direction is parametrized by $SO(3) \times SO(3)/S^1$.

Before stating our result we shall discuss examples. First consider the case, when $G_a = G_b = \{\pm 1\}$, $G_c = G_{c'} = U(1)$, $\mu(a) > \mu(c) > \mu(c') > \mu(b)$. Choose a compact subset $K_{c,c'}$ of $\overline{\mathcal{M}}'(c,c')$, consisting of irreducible connections. Then, by Theorem 7.6, the intersection of $\overline{\mathcal{M}}'(a,b)$ and a neighborhood of $K_{a,c} \times K_{c,c'} \times K_{c',b}$ in $\mathcal{C}\overline{\mathcal{M}}'(a,b)$ is diffeomorphic to

$$G_a \backslash \widetilde{K}_{a,c} \times_{G_c} \widetilde{K}_{c,c'} \times_{G_{c'}} \widetilde{K}_{c',b}/G_b \times (T,\infty)^2.$$

On this set we can define an action of $U(1) \times U(1) = G_c \times G_{c'}$ by

$$(h,h')([x,y,z],t,s) = ([xh,y,h'z],t,s).$$

Note that $\widetilde{K}_{a,c} \to K_{a,c}$ is a principal $U(1)$ bundle, hence $U(1)$ acts on $\widetilde{K}_{a,c}$. As in §7, we have a map

$$\begin{aligned}
\Phi_{(c,c'),(c')} &: G_a \backslash \widetilde{K}_{a,c} \times_{G_c} \widetilde{K}_{c,c'} \times_{G_{c'}} \widetilde{K}_{c',b}/G_b \times (T,\infty)^2 \\
&\to G_a \backslash \mathcal{M}'(a,c') \times_{G_{c'}} \widetilde{K}_{c',b}/G_b \times (T,\infty) \\
\Phi_{(c,c'),(c)} &: G_a \backslash \widetilde{K}_{a,c} \times_{G_c} \widetilde{K}_{c,c'} \times_{G_{c'}} \widetilde{K}_{c',b}/G_b \times (T,\infty)^2 \\
&\to G_a \backslash \mathcal{M}'(a,c) \times_{G_c} \widetilde{K}_{c,b}/G_b \times (T,\infty)
\end{aligned}$$

Let Z_2, Z_1 be inverse images of $G_a \backslash \widetilde{K}_{a,c'} \times_{G_{c'}} \widetilde{K}_{c',b}/G_b \times (T,\infty)$ and $G_a \backslash \widetilde{K}_{a,c} \times_{G_c} \widetilde{K}_{c,b}/G_b \times (T,\infty)$ respectively. (See Figure 6.) $G_a \backslash \widetilde{K}_{a,c'} \times_{G_{c'}} \widetilde{K}_{c',b}/G_b \times (T,\infty)$ has a $U(1)$ action. This action is identified to the action on the second factor of $U(1) \times U(1)$ on Z_2. Similarly the $U(1)$ action of $G_a \backslash \widetilde{K}_{a,c} \times_{G_c} \widetilde{K}_{c,b}/G_b \times (T,\infty)$ is identified to the action of the first factor of $U(1) \times U(1)$ on Z_1. This is exactly the situation of T-structure defined in [CG].

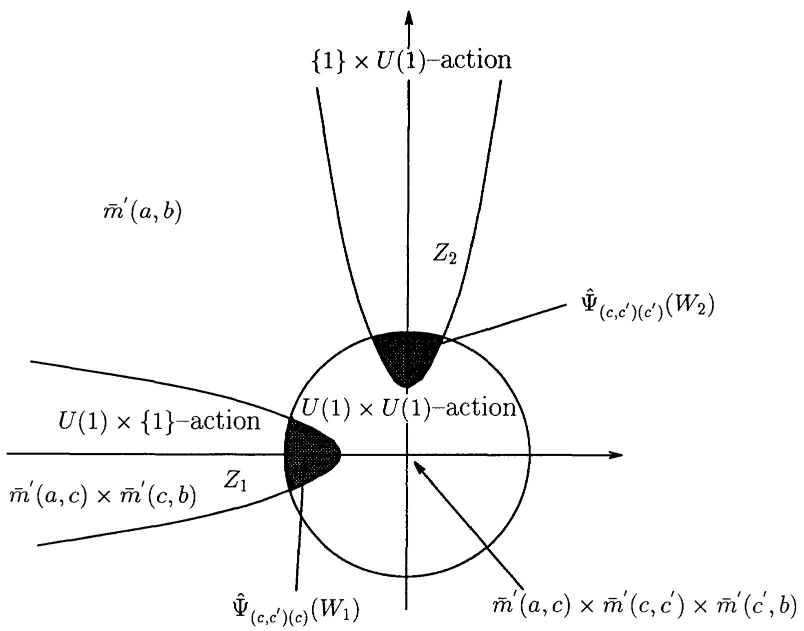

Figure 6

Next, consider the case, $G_a = G_b = \{\pm 1\}$, $G_c = SU(2)$. A neighborhood of $K_{a,c} \times K_{c,b}$ in $\mathcal{C}\overline{\mathcal{M}}'(a,b)$ is diffeomorphic to

$$G_a \backslash \widetilde{K}_{a,c} \times_{SU(2)} \widetilde{K}_{c,b}/G_b \times (T,\infty).$$

On this set $SU(2)$ does not has a global action, but has a local action in the following sense. Consider the principal $SU(2)$ bundle : $\widetilde{K}_{a,c} \to \widetilde{K}_{a,c}/SU(2)$. Let $SU(2)$ act on itself by conjugation, and $P' \to \widetilde{K}_{a,c}/SU(2)$ be the associated bundle. P' has a structure of Lie group bundle. P' induces a bundle $P \to \widetilde{K}_{a,c}/G_c \times G_c\backslash\widetilde{K}_{c,b}$. P has a fibrewise action to

$$\widetilde{K}_{a,c} \times_{G_c} \widetilde{K}_{c,b} \to \widetilde{K}_{a,c}/G_c \times G_c\backslash\widetilde{K}_{c,b},$$

induced from the fibrewise action of P' to $\widetilde{K}_{a,b}$ from left. (Note $SU(2)$ act globally on $\widetilde{K}_{a.b}$ from right.) This fibrewise action defines a local action. If $\mu(c) > \mu(c') > \mu(b)$, the local action of $G_c = SU(2)$ can be made to be compatible with the local action of $G_c \times G_{c'}$.

Note that this action is *not* an action of a sheaf of groups in the sense of [CG], because the fibre bundle $P \to \widetilde{K}_{a,c}/G_c \times G_c\backslash\widetilde{K}_{c,b}$ is not flat, in general.

Take a principal bundle $\widetilde{K}_{c,b} \to SU(2)\backslash\widetilde{K}_{c,b}$ and construct a Lie group bundle $Q \to \widetilde{K}_{a,c}/G_c \times G_c\backslash\widetilde{K}_{c,b}$ in a similar way. Q has also a fibrewise action on

$$G_a \backslash \widetilde{K}_{a,c} \times_{SU(2)} \widetilde{K}_{c,b}/G_b \times (T,\infty).$$

This action does not coincide to the action of P. But they have the same orbits. By convention, we use only the action of P.

Definition 10.1. Let X be a C^∞ manifold. A local action on X is a collection $(U_i, G_i, \varphi_{i,j})$ such that

(10.1.1) U_i is an open covering of X.

(10.1.2) $\cdot : G_i \times U_i \to U_i$ is a smooth action of a Lie group G_i on U_i.

(10.1.3) $U_i \cap U_j$ is G_i and G_j invariant.

(10.1.4) Let $Em(G_i, G_j)$ be the set of all injective homomorphisms. For $i < j$, there exists a smooth map $\varphi_{i,j} : \frac{U_i \cap U_j}{G_i} \to Em(G_i, G_j)$ such that

$$g(x) = \varphi_{i,j}([x])(g)(x)$$

holds for each $x \in U_i \cap U_j$, $g \in G_i$.

Example 10.2. Let $X \to N$ be a principal G bundle. (G acts on X from right.) Let $P = X \times_{ad} G$. P is a Lie group bundle and has a fibrewise left action on X. This gives a local action on X.

Example 10.3. Let $\widetilde{X}^o(c)$ be as in §7. There exists a fibration

$$\widetilde{X}^o(c) \to G_a \backslash \widetilde{K}_{a,c_0}/G_{c_0} \times \cdots \times G_{c_k} \backslash \widetilde{K}_{c_k,b}/G_b \times (T(c),\infty)^{k+1}$$

the fibre of which is $G_a \times G_{c_0} \times \cdots \times G_{c_k} \times G_b$. We have a Lie group bundle

$$P \to G_a \backslash \widetilde{K}_{a,c_0}/G_{c_0} \times \cdots \times G_{c_k} \backslash \widetilde{K}_{c_k,b}/G_b \times (T(c),\infty)^{k+1}$$

whose fibre is $G_a \times G_{c_0} \times \cdots \times G_{c_k} \times G_b$. The bundle P has a fibrewise action to $\widetilde{X}^o(c)$. This gives a local action on $\widetilde{X}^o(c)$.

Theorem 10.4. *There exist a local action on $\overline{\mathcal{M}}'(a,b)$ and maps*

$$\begin{aligned} \Psi_{\mathfrak{c}} &: \overset{\circ}{X}(\mathfrak{c}) \to \overline{\mathcal{M}}'(a,b), \\ \Psi_{\mathfrak{c},\mathfrak{c}'} &: U(\mathfrak{c},\mathfrak{c}') \to X(\mathfrak{c}), \end{aligned}$$

such that

(10.4.1) *The restriction by Ψ_c of the local action on $\overset{\circ}{X}(c)$ of the local action coincides to one in Example* 10.3.

(10.4.2) $\Psi_{\mathfrak{c}'}\Psi_{\mathfrak{c},\mathfrak{c}'} = \Psi_{\mathfrak{c}}$. *(The subset $U(\mathfrak{c},\mathfrak{c}') \subset X(\mathfrak{c})$ is as in* §7.)

Theorem 7.1 follows immediately from Theorem 10.4. We have also

$$|\Phi_{\mathfrak{c}} - \Psi_{\mathfrak{c}}|(z) < e(S_1,\cdots,S_k). \tag{10.5}$$

Here $\Phi_{\mathfrak{c}}$ is the map constructed in §8, $z = ([A_1,\cdots,A_k], S_1,\cdots,S_k)$ and

$$e(S_1,\cdots,S_k) = \sum Ce^{-S_i/C}.$$

To prove Theorem 10.4, we modify the maps $\Psi_{\mathfrak{c}}$ inductively on $\mathfrak{c}$. First we take $\mathfrak{c}$ which is maximal with respect to the inclusion and put $\Psi_{\mathfrak{c}} = \Phi_{\mathfrak{c}}$. We do not change $\Psi_{\mathfrak{c}'}$ while modifying $\Phi_{\mathfrak{c}}$ with $\mathfrak{c}' \supset \mathfrak{c}$. For simplicity of the notation, we discuss one step of modifications. We consider the following case. Let $\mu(a) < \mu(c) < \mu(c') < \mu(b)$, with $G_a = \{\pm 1\}$, $G_c = G_{c'} = G_b = U(1)$, and consider the component of $K_{c,c'}$ consisting of irreducible connections. Suppose, by induction hypothesis, we have

$$\begin{aligned}\Psi_{(c,c')} &: \widetilde{K}_{a,c} \times_{G_c} \widetilde{K}_{c,c'} \times_{G_{c'}} \widetilde{K}_{c',b} \times (T,\infty)^2 \\ &\to \mathcal{M}'(a,b) \\ \Psi_{(c,c'),(c)} &: \widetilde{K}_{a,c} \times_{G_c} \widetilde{K}_{c,c'} \times_{G_{c'}} \widetilde{K}_{c',b} \times (T,\infty)^2 \\ &\to \widetilde{\mathcal{M}}'(a,c) \times_{G_c} \mathcal{M}'(c,b) \times (T,\infty) \\ \widehat{\Psi}_{(c,c'),(c')} &: \widetilde{K}_{a,c} \times_{G_c} \widetilde{K}_{c,c'} \times_{G_{c'}} \widetilde{K}_{c',b} \times (T,\infty)^2 \\ &\to \widetilde{M}'(a,c') \times_{G_{c'}} \mathcal{M}'(c',b) \times (T,\infty),\end{aligned}$$

and a local action on the image of $\Psi_{(c,c')}$. We shall define $\Psi_{(c)}$ and $\Psi_{(c')}$ such that

$$\Psi_{(c)}\Psi_{(c,c'),(c)} = \Psi_{(c,c')}$$

on

$$W_1 = \Psi^{-1}_{(c,c'),(c)}(\widetilde{K}_{a,c} \times_{G_c} \widetilde{K}_{c,b} \times [T,\infty)),$$

and

$$\Psi_{(c')}\Psi_{(c,c'),(c')} = \Psi_{(c,c')}$$

on

$$W_2 = \Psi^{-1}_{(c,c'),(c')}(\widetilde{K}_{a,c'} \times_{G_{c'}} \widetilde{K}_{c',b} \times [T,\infty)).$$

(See Figure 6.) By induction hypothesis, $\Psi_{(c,c'),(c)}$ and $\Psi_{(c,c'),(c')}$ preserves $G_c \times G_b$ and $G_{c'} \times G_b$ actions respectively. (In this case, those actions are defined globally since the groups are abelian.) The maps $\Psi_{(c)}$ and $\Psi_{(c')}$ we shall construct must be G_b invariant. Once we obtain such maps $\Psi_{(c)}$ and $\Psi_{(c')}$ we can define a local action on their images by pushing out one by those maps. These local actions can be patched together with one on the image of $\Psi_{(c,c')}$ by the $G_c \times G_b$ and $G_{c'} \times G_b$ invariance of the maps $\Psi_{(c,c'),(c)}$ and $\Psi_{(c,c'),(c')}$.

We begin the construction of Ψ_c. We choose an open coverings U_j^1, U_j^2, U_j^3, U_j^4, of $\widetilde{K}_{a,c}/G_c$, $K_{c,c'}$, $G_{c'}\backslash\widetilde{K}_{c',b}/G_b$, $\widetilde{K}_{a,c'}/G_{c'}$, respectively. Let V_k be an open covering of $U(1)$. Take maps J_k^1 and J_k^2 as in §8. Choose sections $s_j^1 : U_j^1 \to \mathcal{A}_\ell(a,c)$ and s_j^2,s_j^3.s_j^4. As in §8, define a map

$$\widetilde{\Phi}'_{j_1,j_2,j_3,k_1,k_2} : U_{j_1}^1 \times U_{j_2}^2 \times U_{j_3}^3 \times V_{k_1} \times V_{k_2} \times (T,\infty) \times \mathbf{R} \to \mathcal{A}_{\ell,\delta}(a,b)$$

by

$$\widetilde{\Phi}'_{j_1,j_2,j_3,k_1,k_2}([A_1],[A_2],[A_3],g_1,g_2,S_1,S_2,S)$$

$$\begin{cases} = (J_{k_1}(g_1,\cdot)^*A_1)(x,t-S) \quad \text{for} \quad t < S + S_1/3 \\ = \chi\left(\dfrac{t-S-S_1/3}{S_1/3}\right) g_1^* A_1(x,t-S) \\ \qquad + \left(1 - \chi\left(\dfrac{t-S-S_1/3}{S_1/3}\right)\right) A_2(t-S-S_1) \\ \qquad\qquad \text{for} \quad t \in [S+S_1/3, S+2S_1/3] \\ = A_2(t-S-S_1) \quad \text{for} \quad t \in [S+2S_1/3, S+S_1+S_2/3] \\ = \chi\left(\dfrac{t-S-S_1-S_2/3}{S_2/3}\right) A_2(t-S_1-S) \\ \qquad + \left(1 - \chi\left(\dfrac{t-S-S_1-S_2/3}{S_2/3}\right)\right) g_2^* A_3(x,t-S-S_1-S_2) \\ \qquad\qquad \text{for} \quad t \in [S+S_1+S_2/3, S+S_1+2S_2/3] \\ = (J_{k_2}^2(g_2,\cdot)^*A_3)(s,t-S-S_1-S_2) \quad \text{for} \quad t > S+S_1+2S_2/3. \end{cases}$$

By perturbing this map as in §8, we obtain a map

$$\widetilde{\Phi}_{j_1,j_2,j_3,k_1,k_2} : U^1_{j_1} \times U^2_{j_2} \times U^3_{j_3} \times V_{k_1} \times V_{k_2} \times (T,\infty) \times \mathbf{R} \to \widehat{\mathcal{M}}_{\ell,\delta}(a,b)$$

which is a lift of the map $\Phi_{(c,c')}$ of Theorem 7.6. By construction in §8, we have

$$|\widetilde{\Phi}'_{j_1,j_2,j_3,k_1,k_2} - \widetilde{\Phi}_{j_1,j_2,j_3,k_1,k_2}| < e(S_1,S_2).$$

Similarly we have

$$\begin{aligned} \widetilde{\Phi}'^{(1)}_{j_1,j_2,k_1} &: U^1_{j_1} \times U^2_{j_2} \times V_{k_1} \times (T,\infty) \times \mathbf{R} \to \mathcal{A}_\ell(a,c') \\ \widetilde{\Phi}^{(1)}_{j_1,j_2,k_1} &: U^1_{j_1} \times U^2_{j_2} \times V_{k_1} \times (T,\infty) \times \mathbf{R} \to \widehat{\mathcal{M}}_\ell(a,c'), \end{aligned}$$

such that $\widetilde{\Phi}^{(1)}_{j_1,j_2,k_1}$ is a lift of

$$\Phi_{(c)} : G_a \backslash \widetilde{K}_{a,c} \times_{G_c} \widetilde{K}_{c,c'}/G_{c'} \times (T,\infty) \times \mathbf{R} \to \overline{\mathcal{M}}(a,c').$$

Here $\widetilde{\Phi}^{(1)\prime}_{j_1,j_2,k_1}$ is obtained by a similar patching procedure as $\widetilde{\Phi}'_{j_1,j_2,j_3,k_1,k_2}$, and that

$$|\widetilde{\Phi}'^{(1)}_{j_1,j_2,k_1} - \widetilde{\Phi}^{(1)}_{j_1,j_2,k_1}| < e(S_1).$$

We may assume that for each j_1,j_2 with

$$G_a \backslash \widetilde{U}^1_{j_1} \times_{G_c} \widetilde{U}^2_{j_2} \times_{G_{c'}} \widetilde{K}_{c',b} \times (T,\infty)^2 \subset W_1,$$

there exists $j = j(j_1, j_2, k_1)$ such that

$$\operatorname{Im} \widetilde{\Phi}_{j_1,j_2,k_1} \subset U_j^4.$$

We have maps

$$\widetilde{\Phi}'^{(2)}_{j,j_3,k_2} : U_j^4 \times U_{j_3}^3 \times V_{k_2} \times (T,\infty) \times \mathbf{R} \to \mathcal{A}_\ell(a,b)$$
$$\widetilde{\Phi}^{(2)}_{j,j_3,k_1} : U_j^4 \times U_{j_3}^3 \times V_{k_2} \times (T,\infty) \times \mathbf{R} \to \widehat{\mathcal{M}}_\ell(a,b)$$

such that $\widetilde{\Phi}^{(2)}_{j,j_3,k_2}$ is a lift of

$$\Phi_{(c')} : \widetilde{K}_{a,c'} \times_{G_{c'}} \widetilde{K}_{c',b}/G_b \times (T,\infty) \times \mathbf{R} \to \overline{\mathcal{M}}(a,b),$$

Here $\widetilde{\Phi}^{(2)\prime}_{j,j_3,k_2}$ is obtained by a similar patching procedure as $\widetilde{\Phi}'_{j_1,j_2,j_3,k_1,k_2}$, and that

$$|\widetilde{\Phi}'^{(2)}_{j,j_3,k_2} - \widetilde{\Phi}^{(2)}_{j,j_3,k_2}| < e(S_2).$$

By construction, we can choose lifts s_j^1 e.t.c. so that

$$\begin{aligned}&\widetilde{\Phi}'^{(2)}_{j(j_1,j_2,k_1),j_3,k_2}\left(\widetilde{\Phi}'^{(1)}_{j_1,j_2,k_1}([A_1],[A_2],g_1,S_1,S),[A_3],g_2,S_2,S'\right)\\&\quad= \widetilde{\Phi}'_{j_1,j_2,j_3,k_1,k_2}([A_1],[A_2],[A_3],g_1,g_2,S_1,S_2,S'').\end{aligned}$$

(Here S'' is determined by S,S',S_1 and S_2.) It follows that

$$|\Phi_{(c')}\Phi_{(c,c'),(c')} - \Phi_{(c,c')}| < e(S_1,S_2).$$

Using induction hypothesis (10.5), we obtain

$$|\Phi_{(c')}\Psi_{(c,c'),(c')} - \Psi_{(c,c')}| < e(S_1,S_2).$$

Let $\widetilde{\Psi}_{j_1,j_2,j_3,k_1,k_2}$ and $\widetilde{\Psi}^{(1)}_{j_1,j_2,k_1}$ be the lifts of $\Psi_{c,c'}$ and $\Psi_{(c,c')(c')}$, respectively. Then we have

$$\begin{aligned}&|\widetilde{\Phi}^{(2)}_{j(j_1,j_2,k_1),j_3,k_2}\left(\widetilde{\Psi}^{(1)}_{j_1,j_2,k_1}([A_1],[A_2],g_1,S_1,S),[A_3],g_2,S_2,S'\right)\\&\qquad- \widetilde{\Psi}_{j_1,j_2,j_3,k_1,k_2}([A_1],[A_2],[A_3],g_1,g_2,S_1,S_2,S'')| < e(S_1,S_2).\end{aligned}$$

Therefore we can define

$$\widetilde{\Xi}'_{j_1,j_2,j_3,k_1,k_2} : U_{j_1} \times U_{j_2} \times U_{j_3} \times V_{k_1} \times V_{k_2} \times (T,\infty)^2 \times [0,1] \to \mathcal{A}_\ell(a,b)/\mathbf{R}$$

by

$$\widetilde{\Xi}'_{j_1,j_2,j_3,k_1,k_2}([A_1],[A_2],[A_3],g_1,g_2,S_1,S_2,s) =$$
$$(1-s)\cdot\widetilde{\Phi}^{(2)}_{j(j_1,j_2,k_1),j_3,k_2}\left(\widetilde{\Psi}^{(1)}_{j_1,j_2,k_1}([A_1],[A_2],g_1,S_1,S),[A_3],g_2,S_2,S'\right)$$
$$+\,s\cdot\widetilde{\Psi}_{j_1,j_2,j_3,k_1,k_2}([A_1],[A_2],[A_3],g_1,g_2,S_1,S_2,S'').$$

Since gauge transformation is an affine map (namely $g^*(sA+(1-s)B) = sg^*A+(1-s)g^*B$ holds for each connections A, B and gauge transformation g), it follows from an argument similar to the proof of Lemma 8.3 that we can perturb $\widetilde{\Xi}'_{j_1,j_2,j_3,k_1,k_2}$ so that it defines a map Ξ' : $W_1\times[0,1]\to\mathcal{B}_\ell(a,b)$, which is G_b invariant. Using Taubes' method as in §8, we can perturb this map and obtain $\Xi : W_1\times[0,1]\to\mathcal{M}'_\ell(a,b)$. This map Ξ is an isotopy between $\Psi_{(c,c')}$ and $\Phi_{(c')}\Psi_{(c,c'),(c')}$. Take a small open neighborhood W_1' of W_1 in

$$\mathcal{M}(a,c)\times_{G_c}\mathcal{M}(c,c')\times_{G_{c'}}\mathcal{M}(c',b)\times(T,\infty)^2.$$

Ξ can be extend to W_1'. Let $\varphi : W_1'\to[0,1]$ be a G_b-invariant function such that

$$\begin{cases}\varphi(x)=0 & \text{if}\quad x\in\partial W_1', \qquad\qquad \text{and if}\quad \Psi_{(c,c'),(c')}(x)\in X_{(c)}\\ \varphi(x)=1 & \text{if}\quad x\in W_1.\end{cases}$$

(See Figure 7.) Define $\Psi_{(c')}$ on $\Psi_{(c,c'),(c')}(W_1')$ by

$$\Psi_{(c')}(\Psi_{(c,c'),(c')}(x)) = \Xi(x,\varphi(x)).$$

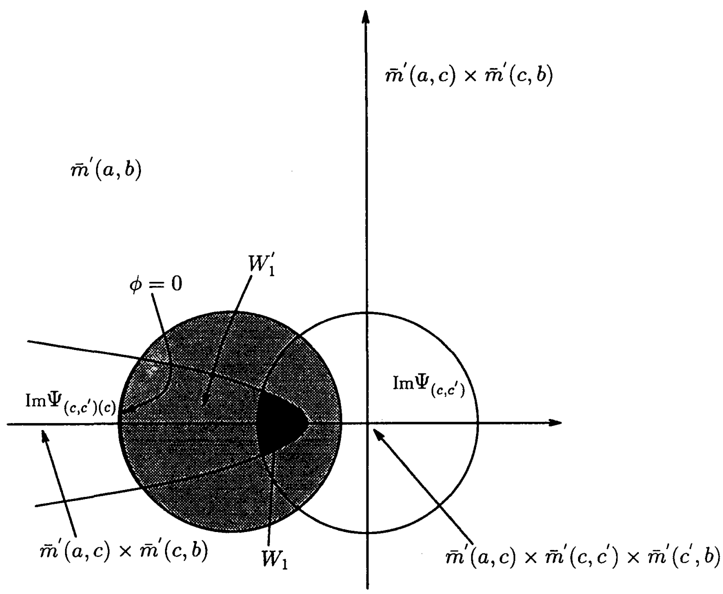

Figure 7.

Since

$$\Xi(x,0) = \Phi_{(c')}\Psi_{(c,c'),(c')}(x),$$

we can extend $\Psi_{(c')}$, by putting $\Psi_{(c')} = \Phi_{(c')}$ outside $\Psi_{(c,c'),(c')}(W_1')$. Since

$$\Xi(x,1) = \Psi_{(c,c')}(x),$$

we have $\Psi_{(c')}\Psi_{(c,c'),(c')} = \Psi_{(c,c')}$, on W_1. The inequality (10.5) holds by construction. Using Lemma 8.14, we can prove that $\Psi_{(c')}$ is a diffeomorphism to its image. Thus the patching argument for the proof of Theorem 10.2 is completed in our case. The proof of general case is the same, but the notation will be more complicated.

Remark 10.6. If we can establish rigorously what we suggested in Remarks 8.15 and 8.20 we might be able to prove Theorem 10.2 using the center of mass technique in Riemannian geometry. (See [GK].) But the direct argument we gave above might be simpler.

§11. Extension of the line bundle to the boundary

In this section, we shall prove Theorem 7.3. First we consider the case when none of c_i are reducible. We put

$$\mathcal{C}_1\overline{\mathcal{M}}'(a,b) = \bigcup_{c_0,\cdots,c_k,G_{c_i}=\{\pm 1\}} \overline{\mathcal{M}}'(a,c_0) \times \prod_{i=0}^{k-1} \overline{\mathcal{M}}'(c_i,c_{i+1}) \times \overline{\mathcal{M}}'(c_k,b).$$

Lemma 11.1. *Let* $\mathfrak{c} = (c_0,\cdots,c_k)$, $\mu(a) > \mu(c_0) > \cdots > \mu(c_k) > \mu(b)$, $G_{c_i} = \{\pm 1\}$, *and*

$$\Psi_{\mathfrak{c}} : K_{a,c_0} \times \prod K_{c_i,c_{i+1}} \times K_{c_k,b} \times (T,\infty)^k \to \overline{\mathcal{M}}'(a,b)$$

be the map given in §10. Then there exists an isomorphism of line bundles

$$\varphi^i_{\mathfrak{c}} : \Psi^*_{\mathfrak{c}}\mathcal{L}^{(2)}_i(a,b) \to \mathcal{L}^{(2)}_i(a,c_0) \otimes \cdots \otimes \mathcal{L}^{(2)}_i(c_k,b).$$

This lemma follows from Theorem 4.9 and the construction of $\Psi_{\mathfrak{c}}$. Hereafter we write

$$\mathcal{L}^{(2)}_i(\mathfrak{c}) = \mathcal{L}^{(2)}_i(a,c_0) \otimes \cdots \otimes \mathcal{L}^{(2)}_i(c_k,b).$$

Similarly, for $\mathfrak{c}' \subset \mathfrak{c}$, we have an isomorphism

$$\varphi^i_{\mathfrak{c},\mathfrak{c}'} : \Psi^*_{\mathfrak{c},\mathfrak{c}'}\mathcal{L}^{(2)}_i(\mathfrak{c}') \to \mathcal{L}^{(2)}_i(\mathfrak{c}).$$

Lemma 11.2. *On*

$$K_{a,c_0} \times \prod K_{c_i,c_{i+1}} \times K_{c_k,b} \times \{(S_1, \cdots, S_k)\}$$

we have

$$\|\varphi^i_{\mathfrak{c},\mathfrak{c}'} \circ \varphi^i_{\mathfrak{c}'} - \varphi^i_{\mathfrak{c}}\| < e(S_1, \cdots, S_k).$$

This lemma follows from the construction of $\varphi^i_{\mathfrak{c}}$. By Lemma 11.2, we can perturb $\varphi^i_{\mathfrak{c}}$, $\varphi^i_{\mathfrak{c},\mathfrak{c}'}$ such that

$$\varphi^i_{\mathfrak{c},\mathfrak{c}'} \circ \varphi^i_{\mathfrak{c}'} = \varphi^i_{\mathfrak{c}}.$$

Using these isomorphisms, we can patch the bundles $\mathcal{L}^{(2)}_i(\mathfrak{c})$ and obtain a line bundle over $\mathcal{C}_1\overline{\mathcal{M}}'(a,b)$.

Next we consider the case when some c_i are reducible. The following three results are used for this purpose.

Theorem 11.3. *The local action on $\overline{\mathcal{M}}'(a,b)$ constructed in* §10, *can be lifted to $\mathcal{L}^{(2)}_i(a,b)$.*

Hence, for each $\mathfrak{c}$, the line bundle $\Psi^*_{\mathfrak{c}}\mathcal{L}^{(2)}_i(a,b)$ on $\widetilde{K}_{a,c_0} \times_{G_{c_0}} \cdots \times_{G_{c_k}} \widetilde{K}_{c_k,b} \times (T,\infty)^k$ has a local $G_a \times G_{c_0} \times \cdots \times G_{c_k} \times G_b$ action. Therefore we obtain a bundle $\overline{\Psi^*_{\mathfrak{c}}\mathcal{L}^{(2)}_i(a,b)}$ on

$$K^*_{a,c_0} \times \prod K^*_{c_i,c_{i+1}} \times K^*_{c_k,b} \times (T,\infty)^k.$$

Here $K^*_{c_i,c_{i+1}}$ denotes the set of reducible connections. As before we put

$$\mathcal{L}^{(2)}_i(\mathfrak{c}) = \mathcal{L}^{(2)}_i(a,c_0) \otimes \cdots \otimes \mathcal{L}^{(2)}_i(c_k,b),$$

which is a line bundle on

$$K^*_{a,c_0} \times \prod K^*_{c_i,c_{i+1}} \times K^*_{c_k,b} \times (T,\infty)^k.$$

Lemma 11.4. *There exist isomorphisms*

$$\varphi^i_{\mathfrak{c}} : \overline{\Psi^*_{\mathfrak{c}}\mathcal{L}^{(2)}_i(a,b)} \to \mathcal{L}^{(2)}_i(\mathfrak{c})$$
$$\varphi^i_{\mathfrak{c},\mathfrak{c}'} : \overline{\Psi^*_{\mathfrak{c},\mathfrak{c}'}\mathcal{L}^{(2)}_i(\mathfrak{c}')} \to \mathcal{L}^{(2)}_i(\mathfrak{c}).$$

Lemma 11.5. *On*

$$K^*_{a,c_0} \times \prod K^*_{c_i,c_{i+1}} \times K^*_{c_k,b} \times \{(S_1, \cdots, S_k)\}$$

we have

$$\|\varphi^i_{\mathfrak{c},\mathfrak{c}'} \circ \varphi^i_{\mathfrak{c}'} - \varphi^i_{\mathfrak{c}}\| < e(S_1, \cdots, S_k).$$

Using these results, we can prove Theorem 7.3 in a way similar to the case when none of c_i are reducible. The proof of Lemmas 11.4 and 11.5 are similar to one of Lemma 11.1 and 11.2 respectively. In the rest of this section, we prove Theorem 11.3.

First we lift the action on the image $\Psi_{\mathfrak{c}}(\widetilde{X}^\circ(\mathfrak{c})) \subset \mathcal{M}'(a,b)$. We are studying the determinant bundle of the operator $\mathfrak{d}^i_A + \epsilon$ defined on $\Sigma_i \simeq S^1 \times \mathbf{R} \subset M \times \mathbf{R}$. On their ends, these operators are asymptotic to $\frac{\partial}{\partial t} + \mathfrak{d}^i_a + \epsilon$, for some $a \in Fl$. Here the operator $\mathfrak{d}^i_a$ is defined on S^1. We choose λ_0 such that the first eigenvalue of $(\mathfrak{d}^i_a + \epsilon)^*(\mathfrak{d}^i_a + \epsilon)$ is larger than λ_0 for each a.

For simplicity, we shall consider the case where $\mathfrak{c} = (c)$, $G_c \neq \{\pm 1\}$. In this case, $\Psi_{\mathfrak{c}}$ is a perturbation of the map Φ defined below. (See §8.)

Choose an open covering

$$\begin{aligned} &U^1_1 \cup \cdots \cup U^1_N \supseteq K_{a,c}, \\ &U^2_1 \cup \cdots \cup U^2_N \supseteq K_{c,b}, \\ &V_1 \cup \cdots \cup V_N = G_c, \end{aligned}$$

and sections $s^1_j : U^1_j \to \mathcal{A}_{\ell,.\delta}(a,c)$, $s^2_j : U^2_j \to \mathcal{A}_{\ell,\delta}(c,b)$. Let $J_k : V_k \times \mathbf{R} \to G_c$ be a map such that

$$J_k(g,t) = \begin{cases} 1 & \text{if } t < -1 \\ g & \text{if } t > 0 \end{cases}.$$

Then the map

$$\widetilde{\Phi}'_{j_1,j_2,k} : U^1_{j_1} \times V_k \times U^2_{j_2} \times [T,\infty) \times \mathbf{R} \to \mathcal{A}_{\ell,\delta}(a,b)$$

is defined by

$$\begin{aligned} &\widetilde{\Phi}'_{j_1,j_2,k}([A_1], g, [A_2], S', S) \\ &= \begin{cases} (J_k(g,\cdot)^* A_1)(x, t-S) & \text{if } t < S + S'/3. \\ \chi\left(\dfrac{t-S-S'/3}{S'/3}\right) g^* A_1(x, t-S) & \\ \qquad + \left(1 - \chi\left(\dfrac{t-S-S'/3}{S'}\right)\right) A_2(t-S-S') & \\ \qquad\qquad \text{if } S + S'/3 < t < S + 2S'/3 & \\ A_2(t-S-S') & \text{if } t > S + 2S'/3. \end{cases} \end{aligned}$$

Here χ is the cut function in §8. The maps $\widetilde{\Phi}'_{j_1,j_2,k}$ induce a map Φ : $\widetilde{X}^\circ((c)) \to \mathcal{B}_{\ell,\delta}(a,c)$. They satisfy

$$\|(\Psi_{(c)} - \Phi)([A_1], g, [A_2], S', S)\|_{L^2_\ell} < Ce^{-S'/C}.$$

Therefore, there exists an isomorphism $\Psi^*_{(c)}\mathcal{L}^{(2)}_i(a,b) \to \Phi^*\overline{\mathcal{L}}^{(2)}_i(a,b)$. We shall lift the local action of G_c on $\widetilde{K}_{a,c} \times_{G_c} \widetilde{K}_{c,b}$, to a local action on $\Phi^*\mathcal{L}^{(2)}_i(a,c)$.

Replacing $U^1_{j_1}$ and $U^2_{j_2}$ by a smaller one if necessary, we can find positive numbers $\lambda_{j_1,j_2} < \lambda_0$, such that the following holds.

(11.6.1) If $[a_t] \in U^1_{j_1}$ then λ_{j_1,j_2} is not an eigenvalue of $(\eth_{a_t}+\epsilon)^*(\eth_{a_t}+\epsilon)$ on Σ_i.
(11.6.2) If $[a_t] \in U^2_{j_2}$ then λ_{j_1,j_2} is not an eigenvalue of $(\eth_{a_t}+\epsilon)^*(\eth_{1,t}+\epsilon)$ on Σ_i.

Then, by Remark 4.6, λ_{j_1,j_2} is not an eigenvalue of $(\eth_A+\epsilon)^*(\eth_A+\epsilon)$ on Σ_i, if

$$[A] \in \Phi(U^1_{j_1} \times G_c \times U^2_{j_2} \times (T,\infty) \times \mathbf{R})$$

for sufficiently large T. Let $[A_1] \in U^1_{j_1}$, $[A_2] \in U^2_{j_2}$, $g \in V_k \subset G_c$, and $A = \widetilde{\Phi}'_{j_1,k,j_2}([A_1], g, [A_2], S', S)$, we put

$$L(A_1,g,A_2,S',S) = \bigoplus_{\lambda<\lambda_{j_1,j_2}} \{u \mid (\eth_A+\epsilon)^*(\eth_A+\epsilon)u = \lambda u\},$$

$$L'(A_1,g,A_2,S',S) = \bigoplus_{\lambda<\lambda_{j_1,j_2}} \{u \mid (\eth_A+\epsilon)(\eth_A+\epsilon)^*u = \lambda u\},$$

$$L = \bigcup_{A_1,g,A_2,S',S} L(A_1,g,A_2,S',S),$$

$$L' = \bigcup_{A_1,g,A_2,S',S} L'(A_1,g,A_2,S',S).$$

By (11.6.1) and (11.6.2), the dimensions of L and L' are constant. By definition,

$$\Phi^*(\mathcal{L}^{(2)}_i(a,b))|_{([A_1],g,[A_2],S',S)}$$
$$\simeq \left(\bigwedge^{\mathrm{top}} (L(A_1,g,A_2,S',S))^* \otimes \bigwedge^{\mathrm{top}} L'(A_1,g,A_2,S',S)\right)^{\otimes 2}.$$

Lemma 11.7. *Let* $t \in [S+S'/3, S+2S'/3]$, $u \in L(A_1, g, A_2, S', S)$. *Then*

$$|u(x,t)|_{C^\ell} < Ce^{-\sqrt{\lambda_0 - \lambda_{j_1,j_2}}\beta(t)}\|u\|_{L^2}$$

Here $\beta(t) = d(t, \partial[S + S'/3, S + 2S'/3])$

The proof of the lemma is similar to one of Lemma 4.5.

For $u \in L(A_1, g, A_2, S', S)$, $g, h \in G_c$ with $g, hg \in V_k$, we put

$$I_1(h)(u)(t,x) =$$
$$\begin{cases} J_k(hg, t-S)J_k(g, t-S)^{-1}u(x,t) \qquad \text{if } t < S + S'/3. \\ \chi\left(\dfrac{t-S-S'/3}{S'/3}\right) hu(x,t) + \left(1 - \chi\left(\dfrac{t-S-S'/3}{S'/3}\right)\right) u(x,t) \\ \qquad \text{if } S + S'/3 < t < S + 2S'/3, \\ u(x,t) \qquad \text{if } t > S + 2S'/3. \end{cases}$$

Let $I_2(h)(u)$ is the orthonormal projection of $I_1(h)(u)$ to $L(A_1, hg.A_2, S', S)$. Lemma 11.7 implies:

Lemma 11.8.

$$\|I_2(h)(u) - I_1(h)(u)\|_{L^2} < Ce^{-S'/C}\|u\|_{L^2}.$$

Lemma 11.9. *If* $g \in V_k$, $hg \in V_k$ *and* $h'hg \in V_k$, *then*

$$\|I_2(h'h)(u) - I_2(h')I_2(h)(u)\|_{L^2} < Ce^{-S'/C}\|u\|_{L^2}.$$

Next we extend I_2 to I_5 which is defined also for h such that $g \in V_k$ and $hg \notin V_k$. Note that $G_c = U(1)$ or $= SU(2)$. Hence, in fact, we need only two charts V_1 and V_2 to cover G_c. (This fact is not essential for the proof but we use it to simplify the notation.) Choose $g_0 \in V_1 \cap V_2$. For $g \in V_1$, $hg \in V_2$, we take h_1 and h_2 such that $h_1 g = g_0$ and $h_2 h_1 = h$. Then, for $h \in L(A_1, g, A_2, S', S)$, the element $I_2(h_1)(u) \in L(A_1, g_0, A_2, S', S)$ is well defined. We put

$$I_3(h)(u) = I_2(h_2)I_2(h_1)(u).$$

Since $h_2(h_1 g), h_1 g \in V_2$, it follows that $I_2(h_2)$ in the above formula is well defined. Choose $\chi : G_c \to [0,1]$ such that

$$\chi(g) = \begin{cases} 1 & \text{if } g \in V_1 - (V_1 \cap V_2). \\ 0 & \text{if } g \in V_2 - (V_1 \cap V_2). \end{cases}$$

Put

$$I_4^1(u) = \begin{cases} I_2(h)(u) & \text{if } hg \in V_1 - (V_1 \cap V_2), \\ \chi(hg)I_2(h)(u) + (1 - \chi(hg))I_3(h)(u) & \\ \qquad \text{if } hg \in V_1 \cap V_2, & \\ I_3(h)(u) & \text{if } hg \in V_2 - (V_1 \cap V_2). \end{cases}$$

In the case when $g \in V_2$, we define $I_4^2(h)$ in a similar way. Finally we put, for $u \in L(A_1, g, A_2, S', S)$

$$I_5(h)(u) = \begin{cases} I_4^1(h)(u) & \text{if } g \in V_1 - (V_1 \cap V_2), \\ \chi(g)I_4^1(h)(u) + (1 - \chi(g))I_4^2(h)(u) & \\ \qquad \text{if } g \in V_1 \cap V_2, & \\ I_4^2(h)(u) & \text{if } g \in V_2 - (V_1 \cap V_2). \end{cases}$$

Then I_5 is defined for every h and g and depends smoothly on them. By perturbing I_5 a bit we obtain $I_6(h)$ which is a linear isometry

$$L(A_1, g, A_2, S', S) \to L(A_1, hg, A_2, S', S).$$

By construction, we have

$$\|I_6(h'h)(u) - I_6(h')I_6(h)(u)\|_{L^2} < Ce^{-S'/C}\|u\|_{L^2}. \tag{11.10}$$

Next we use the center of mass technique, to perturb I_6 and obtain I satisfying $I(h)I(h') = I(hh')$. Namely we use the following:

Lemma 11.11. *For each compact Lie group G and $n, \epsilon > 0$, there exists $\delta_n(G, \epsilon) > 0$, such that the following holds.*

Let $\pi : L \to X$ be a hermitian vector bundle of rank n, G act on X, and $\varphi : G \times L \to L$ be a map. Suppose

(11.12.1) $\quad \pi(\varphi(g, v)) = g(\pi(v))$,

(11.12.2) $\quad \varphi$ *is a linear isometry on each fibre,*

(11.12.3) $\quad |\varphi(g_1, g_2, v) - \varphi(g_1(\varphi(g_2, v))| < \delta_n(G, \epsilon)$.

Then, there exists a lift of the action of G to L, such that

$$|\varphi(g, v) - g \cdot v| < \epsilon.$$

In the case when X is a point, Lemma 11.11 means that an almost homomorphism $G \to U(n)$ is approximated by a homomorphism. This case is proved in [GKR]. The proof of Lemma 11.11 is identical to that case and hence is omitted. (See also [BK] p138.) Note that $\delta_n(G,\epsilon)$ in the lemma is independent of X.

Now, using Lemma 11.11, we can perturb I_6 to obtain a lift I of the local action on $U^1_{j_1} \times G_c \times U^2_{j_2} \times (T,\infty) \times \mathbf{R}$ to the vector bundle $L(A_1,g,A_2,S',S)$ on it. In a similar way, we can lift the action to $L'(A_1,g,A_2,S',S)$. Hence we obtain a lift of the action to the restriction of $\Phi^*\overline{\mathcal{L}}^{(2)}_{a,b}$ to $\widetilde{U}^1_{j_1} \times_{G_c} \widetilde{U}^2_{j_2} \times (T,\infty) \times \mathbf{R} = U^1_{j_1} \times G_c \times U^2_{j_2} \times (T,\infty) \times \mathbf{R}$. (Here $\widetilde{U}^1_{j_1}$ and $\widetilde{U}^2_{j_2}$ are the inverse images of $U^1_{j_1}$ and $U^2_{j_2}$ in $\widetilde{K}_{a,c}$ and $\widetilde{K}_{c,b}$, respectively.) We denote the lift by I_{j_1,j_2}. By construction, we have, on $(\widetilde{U}^1_{j_1} \times_{G_c} \widetilde{U}^2_{j_2}) \cap (\widetilde{U}^1_{j_1'} \times_{G_c} \widetilde{U}^2_{j_2'}) \times (T,\infty) \times \mathbf{R}$,

$$d(I_{j_1,j_2}(h), I_{j_1',j_2'}(h)) < Ce^{-T/C}.$$

Hence using a partition of unity, we can patch them as an almost action. Therefore, using Lemma 11.11, we obtain a lift of the local action to $\Phi^*\mathcal{L}^{(2)}_i(a,b)$.

In order to lift the local action on $\mathcal{M}(a,b)$, we have to patch those lifts we constructed above. By construction, they are compatible modulo a difference estimated by $e(S_1,\cdots,S_k)$ on $\cdots \times \{(S_1,\cdots,S_k)\} \times \mathbf{R}$. Hence we can apply a similar patching procedure as above. The proof of Theorem 11.3 is now complete.

§12. Boundary operators

In this section, we define the boundary operators

$$\begin{aligned} \partial &: C^0_k \to C^0_{k-1} \\ \partial_\gamma &: C^0_k \to C^0_{k-3} \\ \partial_{\gamma_1,\gamma_2} &: C^0_k \to C^0_{k-5}. \end{aligned}$$

The definition of ∂ is the same as Floer's. Let $a, b \in Fl$, with $\mu(a) = \mu(b) + 1$. Then, $\overline{\mathcal{M}}'(a,b)$ consists of finitely many points each of which is given an orientation $+$ or $-$. We let $\langle \partial a, b\rangle$ be the number of the points with $+$ orientation minus the number of points with $-$ orientation. Put

$$\partial[a] = \sum \langle \partial a, b\rangle [b].$$

Next we define ∂_γ. For a closed loop γ on M we obtain a line bundles $\mathcal{L}^{(2)}_\gamma(c,c')$, over $\overline{\mathcal{M}}'(c,c')$. We choose sections $s_\gamma(c,c')$ to $\mathcal{L}^{(2)}_\gamma(c,c')$, such that the following holds.

(12.1.1) For each $a, b \in Fl$, the collection of the sections

$$s_\gamma(a, c_0) \otimes \cdots \otimes s_\gamma(c_k, b)$$

to

$$\mathcal{L}_\gamma^{(2)}(a, c_0) \otimes \cdots \otimes \mathcal{L}_\gamma^{(2)}(c_k, b)$$

can be patched together to give a smooth section on $C\overline{\mathcal{M}}'(a, b)$. (We use the symbol $s_\gamma(a, b)$ also for this extension.)
(12.1.2) The zeros of $s_\gamma(c, c')$ are transversal and transversal to each other.

Since we restrict ourselves to the case when $s < 3$ if $H_1(M; \mathbf{Z})$ is torsion free, and when $s = 0$ otherwise, then we need only to study the case when $\mu(a) < \mu(b) + 8$, $H_1(M; \mathbf{Z})$ is torsion free and a and b are irreducible. In this case, if $\mu(a) \geq \mu(c) \geq \mu(c') \geq \mu(b)$, and if $\mathcal{M}(a, c) \neq \emptyset$, $\mathcal{M}(c', b) \neq \emptyset$, then $\mathcal{M}(c, c')$ does not contain a reducible connection. Also in our case, Lemma 5.8 implies that bubbling off of instanton does not happen. Hence (7.1.6) implies that the set $C\overline{\mathcal{M}}'(a, b)$ is compact. The later fact is not really necessary for the argument. (We can discuss as in Donaldson [D4], in case when a and b are irreducible.) However the former point is essential. We discuss it at the end of this section.

Now, let $\mu(a) = \mu(b) + 3$. Set

$$\Sigma_\gamma(a, b) = \left\{ x \in C\overline{\mathcal{M}}'(a, b) | s_\gamma(a, b)(x) = 0 \right\}.$$

Dimension counting, the compactness of $C\overline{\mathcal{M}}'(a, b)$ and the transversality (12.1.2) imply

$$\Sigma_\gamma(a, b) \cap \partial C\overline{\mathcal{M}}'(a, b) = \emptyset$$
$$\sharp \Sigma_\gamma(a, b) < \infty.$$

The orientation of $\overline{\mathcal{M}}'(a, b)$ induces an orientation of each point of Σ_i. We define $\langle \partial_\gamma a, b \rangle$ by

$$\langle \partial_\gamma a, b \rangle = \sharp \Sigma_\gamma.$$

Here and hereafter $\sharp$ stands for the number of points with $+$ orientation minus the number of points with $-$ as orientation. We set

$$\partial_\gamma [a] = \sum_b \langle \partial_\gamma a, b \rangle [b].$$

For $\mu(b) = \mu(a) + 5$, and loops γ_1 and γ_2, we put

$$\Sigma_{\gamma_1,\gamma_2}(a,b) = \{x \in \mathcal{C}\overline{\mathcal{M}}'(a,b) | s_{\gamma_1}(a,b)(x) = s_{\gamma_2}(a,b)(x) = 0.\},$$

and define

$$\begin{aligned}\langle \partial_{\gamma_1,\gamma_2} a, b\rangle &= \sharp \Sigma_{\gamma_1,\gamma_2}(a,b) \\ \partial_{\gamma_1,\gamma_2}[a] &= \sum_b \langle \partial_{\gamma_1,\gamma_2} a, b\rangle [b].\end{aligned}$$

Now we prove Theorem 1.10. For simplicity, we discuss the case $\alpha = \{\gamma\}$, and prove $\partial_\gamma \partial + \partial\partial_\gamma = 0$. Let $a, b \in Fl$ with $\mu(a) = \mu(b) + 4$. The line bundle $\mathcal{L}^{(2)}_\gamma(a,b) \to \overline{\mathcal{M}}'(a,b)$ can be extended to $\mathcal{C}\overline{\mathcal{M}}'(a,b)$ by Theorem 7.3. Since $\dim \overline{\mathcal{M}}'(a,b) = 3$, the set

$$\Sigma_\gamma(a,b) = \{x \in \mathcal{C}\overline{\mathcal{M}}'(a,b) | s_\gamma(a,b)(x) = 0\}$$

is one dimensional oriented manifold. And

$$\partial \Sigma_\gamma(a,b) = \Sigma_\gamma(a,b) \cap \partial \overline{\mathcal{M}}'(a,b).$$

By transversality and dimension counting we have

$$\begin{aligned}\partial \Sigma_\gamma(a,b) = &\{(x,y) \in \overline{\mathcal{M}}'(a,b) \times \overline{\mathcal{M}}'(c,b) | \\ &\quad s_\gamma(a,c)(x) \cdot s_\gamma(c,b)(y) = 0, c \text{ is irreducible.}\}. \\ = &\coprod_{\mu(c)=\mu(b)+1} \Sigma_\gamma(a,b) \times \overline{\mathcal{M}}'(c,b) \cup \\ &\coprod_{\mu(c')=\mu(b)+2} \overline{\mathcal{M}}'(a,c') \times \Sigma_\gamma(c',b).\end{aligned}$$

The orientations are also compatible. Therefore we have

$$\sum_c \langle \partial_\gamma a, c\rangle \langle \partial c, b\rangle + \sum_{c'} \langle \partial a, c'\rangle \langle \partial_\gamma c', b\rangle = 0.$$

Hence $\partial_\gamma \partial + \partial \partial_\gamma = 0$, as required.

The proof of $\partial_{\gamma_1,\gamma_2}\partial + \partial_{\gamma_1}\partial_{\gamma_2} + \partial_{\gamma_2}\partial_{\gamma_1} + \partial\partial_{\gamma_1,\gamma_2} = 0$ is similar.

Now put

$$C^s_k = \bigoplus_{\ell \le s} S^\ell H_1(M, \mathbf{Z}) \otimes C^0_{k-2\ell},$$

and define $\widehat{\partial} : C^s_k \to C^s_{k-1}$, by

$$\widehat{\partial}(\gamma_\alpha \otimes [a]) = \sum_{\alpha^1 \cup \alpha^2 = \alpha} \gamma_{\alpha^1} \otimes \partial_{\alpha^2}[a].$$

(Here we fix a basis $\gamma_1, \cdots, \gamma_d$ of the first homology group and put

$$\partial_\alpha = \sum_{j_1, \cdots, j_\ell} \prod_i C_{i,j_i} \partial_{\gamma_{j_1} \cdots \gamma_{j_\ell}}$$

if $\alpha = (\sum_{j_1} C_{1,j_1}[\gamma_{j_1}], \cdots, \sum_{j_\ell} C_{\ell,j_\ell}[\gamma_{j_\ell}])$. Later, in Lemma 12.10, we shall prove that ∂_γ are additive with respect to γ.) Theorem 1.10 implies $\widehat{\partial}\widehat{\partial} = 0$.

As we pointed out in §1, the boundary operator $\widehat{\partial}$ itself *does* depend on the choice of the sections $s_\gamma(c, c')$, because the spaces $\mathcal{C}\overline{\mathcal{M}}'(c, c')$ have boundaries. Next we prove that the chain complex $(C^s_\cdot, \widehat{\partial})$ is independent of the choice of the section.

Theorem 12.2. *Suppose $H_1(M; \mathbf{Z})$ is torsion free and $s < 3$. Let $s_\gamma(a, b)$ and $s'_\gamma(a, b)$ are the sections satisfying* (12.1.1) *and* (12.1.2). *Let $(C^s, \widehat{\partial})$ and $(C^s, \widehat{\partial}')$ be the corresponding chain complexes. Then there exist maps $\psi, \varphi : C^s \to C^s$ such that*

(12.2.1) $$\widehat{\partial}'\varphi = \varphi\widehat{\partial}$$

(12.2.2) $$\widehat{\partial}\psi = \psi\widehat{\partial}'$$

(12.2.3) $$\varphi\psi = \psi\varphi = \textit{identity}.$$

Proof. For each loop γ and $c, c' \in Fl$, we choose a section $\widetilde{s}_\gamma(c, c')$ to $\mathcal{L}^{(2)}_\gamma(c, c') \times [0, 1] \to \overline{\mathcal{M}}'(c, c') \times [0, 1]$ such that

(12.3.1) $$\begin{aligned} \widetilde{s}_\gamma(c, c')(x, 0) &= s_\gamma(c, c')(x) \\ \widetilde{s}_\gamma(c, c')(x, 1) &= s'_\gamma(c, c')(x) \end{aligned}$$

(12.3.2) For each $a, b \in Fl$, the collections of sections

$$\widetilde{s}_\gamma(a, c_0) \otimes \cdots \otimes \widetilde{s}_\gamma(c_k, b)$$

can be patched together to give a smooth section on $\mathcal{C}\overline{\mathcal{M}}'(a, b) \times [0, 1]$.
(12.3.3) The zeros of $\widetilde{s}_{\gamma_i}$ are transversal and are transversal to each other.

Now, let $\mu(a) = \mu(b) + 3$, and put

$$\widetilde{\Sigma}_\gamma(a,b) = \{(x,t) \in \mathcal{C}\overline{\mathcal{M}}'(a,b) \times [0,1] | \widetilde{s}_\gamma(a,b)(x,t) = 0\}.$$

Then $\dim \widetilde{\Sigma}_\gamma(a,b) = 1$. Note that (12.3.2) implies that

$$\widetilde{\Sigma}_\gamma(a,b) \cap (\overline{\mathcal{M}}'(a,c) \times \overline{\mathcal{M}}'(c,b) \times [0,1]) \neq \emptyset$$

only if c is irreducible and $\mu(c) = \mu(b) + 1$ or 2. Therefore

(12.4)

$$\begin{aligned}
&\partial\widetilde{\Sigma}_\gamma(a,b) = \\
&\quad \{(x,0)|\widetilde{s}_\gamma(a,b)(x,0) = 0\} \cup \{(x,1)|\widetilde{s}_\gamma(a,b)(x,1) = 0\} \cup \\
&\quad \coprod_c \{(x_1,x_2,t)|\widetilde{s}_\gamma(c,b)(x_1,t) \cdot \widetilde{s}_\gamma(a,c)(x_2,t) = 0\}.
\end{aligned}$$

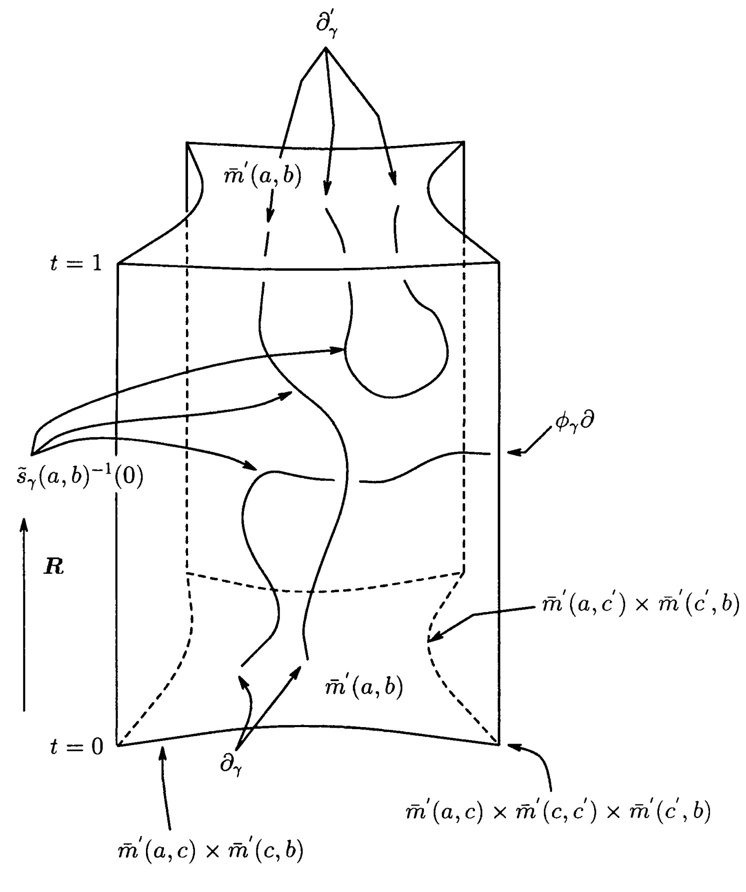

Figure 8.

For each $a, c \in Fl$, with $\mu(a) = \mu(c) + 2$, we put

$$\langle \varphi_\gamma a, c \rangle = \sharp\{(x,t) \in \overline{\mathcal{M}}'(a,c) \times [0,1] | \widetilde{s}_\gamma(x,t) = 0\}.$$

Note the set in the right hand side is a finite set, by (12.3.3) and dimension counting. Define $\varphi_\gamma : C_k^0 \to C_{k-2}^0$ by

$$\varphi_\gamma[a] = \sum \langle \varphi_\gamma a, c \rangle [c].$$

Then (12.4) implies

$$\partial_\gamma - \partial'_\gamma + \partial \varphi_\gamma - \varphi_\gamma \partial = 0. \tag{12.5}$$

Now define $\varphi, \psi : C^1 \to C^1$ by

$$\begin{aligned}
&\varphi(1 \otimes [a]) = 1 \otimes [a] \\
&\varphi(\gamma \otimes [a]) = \gamma \otimes [a] + 1 \otimes \varphi_\gamma[a], \\
&\psi(1 \otimes [a]) = 1 \otimes [a] \\
&\psi(\gamma \otimes [a]) = \gamma \otimes [a] - 1 \otimes \varphi_\gamma[a].
\end{aligned}$$

Then using (12.5), it is easy to verify (12.2.1),(12.2.2), and (12.2.3).

Next we consider the case $s = 2$. Let $\mu(a) = \mu(b) + 5$. Put

$$\widetilde{\Sigma}_{\gamma_1,\gamma_2}(a,b) = \{(x,t) \in \mathcal{C}\overline{\mathcal{M}}'(a,b) \times [0,1] | \widetilde{s}_{\gamma_1}(x,t) = \widetilde{s}_{\gamma_2}(x,t) = 0\}.$$

We have

(12.6)

$$
\begin{aligned}
&\partial\widetilde{\Sigma}_{\gamma_1,\gamma_2}(a,b) = \\
&\quad \{(x,0) | s_{\gamma_1}(a,b)(x) = s_{\gamma_2}(a,b)(x) = 0\} \\
&\quad \cup \{(x,1) | s'_{\gamma_1}(a,b)(x) = s'_{\gamma_2}(a,b)(x) = 0\} \\
&\quad \cup \coprod_{\mu(c_1)=\mu(b)+1} \{(x,y,t) \in \overline{\mathcal{M}}'(a,c_1) \times \overline{\mathcal{M}}'(c_1,b) \times [0,1] | \\
&\qquad\qquad \widetilde{s}_{\gamma_1}(a,c_1)(x,t) = \widetilde{s}_{\gamma_2}(a,c_1)(x,t) = 0\}, \\
&\quad \cup \coprod_{\mu(c_4)=\mu(b)+4} \{(x,y,t) \in \overline{\mathcal{M}}'(a,c_4) \times \overline{\mathcal{M}}'(c_4,b) \times [0,1] | \\
&\qquad\qquad \widetilde{s}_{\gamma_1}(c_4,b)(x,t) = \widetilde{s}_{\gamma_2}(c_4,b)(y,t) = 0\}, \\
&\quad \cup \coprod_{\mu(c_2)=\mu(b)+2} \left\{ (x,y,t) \;\middle|\; \begin{matrix} (x,y,t) \in \overline{\mathcal{M}}'(a,c_2) \times \overline{\mathcal{M}}'(c_2,b) \times [0,1] \\ \widetilde{s}_{\gamma_1}(a,c_2)(x,t) = 0 = \widetilde{s}_{\gamma_2}(c_2,b)(y,t) \\ \text{or} \\ \widetilde{s}_{\gamma_1}(c_2,b)(x,t) = 0 = \widetilde{s}_{\gamma_2}(a,c_2)(y,t) \end{matrix} \right\} \\
&\quad \cup \coprod_{\mu(c_3)=\mu(b)+3} \left\{ (x,y,t) \;\middle|\; \begin{matrix} (x,y,t) \in \overline{\mathcal{M}}'(a,c_3) \times \overline{\mathcal{M}}'(c_3,b) \times [0,1] \\ \widetilde{s}_{\gamma_1}(a,c_3)(x,t) = 0 = \widetilde{s}_{\gamma_2}(c_3,b)(y,t) \\ \text{or} \\ \widetilde{s}_{\gamma_1}(c_3,b)(x,t) = 0 = \widetilde{s}_{\gamma_2}(a,c_3)(y,t) \end{matrix} \right\}.
\end{aligned}
$$

Let $\Lambda_0, \Lambda_5, \Lambda_1, \Lambda_4, \Lambda_2, \Lambda_3$ be the sets in the above formula, respectively. We have

$$\sharp\Lambda_0 = \langle \partial_{\gamma_1,\gamma_2} a, b\rangle, \tag{12.7.1}$$

$$\sharp\Lambda_5 = -\langle \partial'_{\gamma_1,\gamma_2} a, b\rangle. \tag{12.7.2}$$

For $a, c \in Fl$ with $\mu(a) = \mu(c) + 4$, we put

$$\langle \varphi_{\gamma_1,\gamma_2} a, c\rangle = \sharp\{(x,t) \in \overline{\mathcal{M}}'(a,c) \times [0,1] | \widetilde{s}_{\gamma_1}(x,t) = \widetilde{s}_{\gamma_2}(x,t) = 0\}.$$

Then we have

$$\sharp\Lambda_1 = \sum_{c_1} \langle \varphi_{\gamma_1,\gamma_2} a, c_1\rangle \langle \partial c_1, b\rangle, \tag{12.7.3}$$

$$\sharp\Lambda_4 = -\sum_{c_4} \langle \partial a, c_4\rangle \langle \varphi_{\gamma_1,\gamma_2} c_4, b\rangle. \tag{12.7.4}$$

To examine $\sharp\Lambda_2$ and $\sharp\Lambda_3$, we remark that the sections $\widetilde{s}_\gamma(c,c')$ can be defined by an induction on $\mu(c)-\mu(c')$. Then, we can assume the following conditions (12.8). For $c,c'\in Fl$ with $\mu(c)=\mu(c')+2$, we put

$$T(c,c')=\sup\{t|\exists x\ \ (x,t)\in\widetilde{\Sigma}_\gamma(c,c')\},$$
$$S(c,c')=\inf\{t|\exists x\ \ (x,t)\in\widetilde{\Sigma}_\gamma(c,c')\}.$$

(12.8.1) If $\mu(c)=\mu(c')+3=\mu(c'')+5$, and if $t>T(c',c'')$ then

$$\widetilde{s}'_\gamma(c,c')(x,t)=\widetilde{s}'_\gamma(c,c')(x,1)$$

(12.8.2) If $\mu(c)=\mu(c')+2=\mu(c'')+5$, and if $t<S(c,c')$, then

$$\widetilde{s}'_\gamma(c',c'')(x,t)=\widetilde{s}'_\gamma(c',c'')(x,0)$$

Using (12.8.1), we can prove:

$$\begin{aligned}\Lambda_2=&\coprod_{c_2}\{x\in\overline{\mathcal{M}}'(a,c_2)|s'_{\gamma_1}(x)=0\}\times\\&\qquad\{(y,t)\in\overline{\mathcal{M}}'(c_2,b)\times[0,1]|\widetilde{s}_{\gamma_2}(y,t)=0\}\\&\cup\coprod_{c_2}\{x\in\overline{\mathcal{M}}'(a,c_2)|s'_{\gamma_2}(x)=0\}\times\\&\qquad\{(y,t)\in\overline{\mathcal{M}}'(c_2,b)\times[0,1]|\widetilde{s}_{\gamma_1}(y,t)=0\}.\end{aligned}$$

Therefore

(12.9.1) $$\sharp\Lambda_2=-\sum_{c_2}\langle\partial'_{\gamma_1}a,c_2\rangle\langle\varphi_{\gamma_2}c_2,b\rangle-\sum_{c_2}\langle\partial'_{\gamma_2}a,c_2\rangle\langle\varphi_{\gamma_1}c_2,b\rangle.$$

Similarly, using (12.8.2), we can prove:

(12.9.2) $$\sharp\Lambda_3=\sum_{c_3}\langle\varphi_{\gamma_1}a,c_3\rangle\langle\partial_{\gamma_2}c_3,b\rangle+\sum_{c_3}\langle\varphi_{\gamma_2}a,c_3\rangle\langle\partial_{\gamma_1}c_3,b\rangle.$$

By (12.6.1),(12.7),(12.9), we have

(12.10)
$$\partial_{\gamma_1,\gamma_2}+\varphi_{\gamma_1}\partial_{\gamma_2}+\varphi_{\gamma_1}\partial_{\gamma_2}+\varphi_{\gamma_1,\gamma_2}\partial=\partial'_{\gamma_1,\gamma_2}+\partial'_{\gamma_1}\varphi_{\gamma_2}+\partial'_{\gamma_2}\varphi_{\gamma_1}+\partial'\varphi_{\gamma_1,\gamma_2}.$$

Now we put

$$\begin{aligned}\varphi(\gamma_1\gamma_2\otimes[a])&=\gamma_1\gamma_2\otimes[a]+\gamma_1\otimes\varphi_{\gamma_2}[a]+\gamma_2\otimes\varphi_{\gamma_1}[a]+1\otimes\varphi_{\gamma_1,\gamma_2}[a]\\\psi(\gamma_1\gamma_2\otimes[a])&=\gamma_1\gamma_2\otimes[a]-\gamma_1\otimes\varphi_{\gamma_2}[a]-\gamma_2\otimes\varphi_{\gamma_1}[a]\\&\quad-1\otimes(\varphi_{\gamma_1,\gamma_2}+\varphi_{\gamma_1}\varphi_{\gamma_2}+\varphi_{\gamma_2}\varphi_{\gamma_1})[a].\end{aligned}$$

Formulas (12.2.1),(12.2.2),(12.2.3) follow immediately from (12.5) and (12.10). The proof of Theorem 12.2 is now complete.

Next we shall prove the following:

Lemma 12.11. *Let $\gamma_1, \gamma_2, \gamma, \gamma'$ be closed loops on M with $[\gamma_1] + [\gamma_2] = [\gamma]$ in $H_1(M; \mathbf{Z})$. Then we can find collections of sections $s_{\gamma_1}(c,c')$, $s_{\gamma_2}(c,c')$, $s_\gamma(c,c')$, $s_{\gamma'}(c,c')$ with* (12.1.1), (12.1.2) *such that the corresponding boundary operators satisfy*

$$\partial_{\gamma_1} + \partial_{\gamma_2} = \partial_\gamma \tag{12.11.1}$$

$$\partial_{\gamma_1,\gamma'} + \partial_{\gamma_2,\gamma'} = \partial_{\gamma,\gamma'}. \tag{12.11.2}$$

Proof. Let $\mu(a) = \mu(b) + 3$. Consider $\mathcal{C}\overline{\mathcal{M}}(a,b)$. (We do not divide it by the $\mathbf{R}$ action.) Let Σ be a surface on $M \times \mathbf{R}$ which is asymptotic to $(\gamma_1 \cup \gamma_2) \times \mathbf{R}$ as $t \to -\infty$, and to $\gamma \times \mathbf{R}$ as $t \to \infty$. Using the Dirac operator on Σ, we can define a line bundle $\mathcal{L}^{(2)}_\Sigma(a,b)$ on $\mathcal{C}\overline{\mathcal{M}}(a,b) = \mathcal{C}\overline{\mathcal{M}}'(a,b) \times \mathbf{R}$. We put

$$\mathcal{C}\mathcal{C}\overline{\mathcal{M}}(a,b) = \mathcal{C}\overline{\mathcal{M}}(a,b) \times [-\infty, \infty].$$

By construction and Theorem 4.9, the bundles $\mathcal{L}^{(2)}_\Sigma(a,b)$ on $\mathcal{C}\overline{\mathcal{M}}(a,b)$, and $\mathcal{L}^{(2)}_{\gamma_1}(a,b) \otimes \mathcal{L}^{(2)}_{\gamma_2}(a,b)$ on $\mathcal{C}\overline{\mathcal{M}}'(a,b) \times \{-\infty\}$, and $\mathcal{L}^{(2)}_\gamma(a,b)$ on $\mathcal{C}\overline{\mathcal{M}}'(a,b) \times \{\infty\}$ can be patched together to give a line bundle over $\mathcal{C}\mathcal{C}\overline{\mathcal{M}}(a,b)$. We extend the sections $s_{\gamma_1}(a,b) \otimes s_{\gamma_2}(a,b)$ and $s_\gamma(a,b)$ to a section on $\mathcal{C}\mathcal{C}\overline{\mathcal{M}}(a,b)$. Then, by an argument similar to the proof of Theorem 12.2, we can find φ_γ such that

$$\partial_\gamma - (\partial_{\gamma_1} + \partial_{\gamma_2}) = \partial\varphi_\gamma - \varphi_\gamma\partial.$$

Using this map φ_γ, we can modify the section s_γ such that (12.11.1) is satisfied. The proof of (12.11.2) is similar.

Finally, we discuss what happens when $s \geq 1$ in case $H_1(M; \mathbf{Z})$ has a torsion, and when $s \geq 3$ in case $H_1(M; \mathbf{Z})$ is torsion free.

Suppose first that $H_1(M; \mathbf{Z})$ has a torsion, and $\mu(a) = \mu(b) + 5$. In this case, there may be reducible connections c and c' such that $G_c = G_{c'} = U(1)$ and that $\mu(c) = \mu(c') + 1 = \mu(b) + 2$. Then

$$\dim \overline{\mathcal{M}}'(a,c) = \dim \overline{\mathcal{M}}'(c,c') = \dim \overline{\mathcal{M}}'(c',b) = 0.$$

The set $\overline{\mathcal{M}}'(c,c')$ may have a 0 dimensional orbit $\overline{\mathcal{M}}'_{red}(c,c')$ which consists only of reducible connections. (See Theorem 5.6.) A neighborhood of each point of $\overline{\mathcal{M}}'(a,c) \times \overline{\mathcal{M}}'_{red}(c,c') \times \overline{\mathcal{M}}'(c',b)$, in $\mathcal{C}\overline{\mathcal{M}}'(a,b)$ is identified to $(0,\infty] \times (0,\infty] \times U(1)/\sim$, where $(t,s,g_1) \sim (t,s,g_2)$ if and only if $t = \infty$ or $s = \infty$. Here $\{\infty\} \times (0,\infty) \times U(1)/\sim$ and $(0,\infty) \times \{\infty\} \times U(1)/\sim$ are identified to $\overline{\mathcal{M}}'(a,c) \times \overline{\mathcal{M}}'(c,b)$ and $\overline{\mathcal{M}}'(a,c') \times \overline{\mathcal{M}}'(c',b)$ respectively. The bundle $\mathcal{L}^{(2)}_\gamma(a,b)$ is extended outside $\infty \times \infty \times U(1)/\sim = point$. The neighborhood of this point is a cone of S^2. (It may be more natural to regard that this S^2 has two singular points.)

Using the basis $[\ell_i]$ of $H_1'(M;\mathbf{Z})$, chosen at the beginning of §2, we can find ℓ_{i_0} such that

$$c(\ell_i) = c'(\ell_i) \qquad \text{if } i \neq i_0. \tag{12.12.1}$$

$$c(\ell_{i_0}) = 1, \quad c'(\ell_{i_0}) = -1. \tag{12.12.2}$$

In this case we can prove that the restriction of the line bundle $\mathcal{L}^{(2)}_{\ell_{i_0}}(a,b)$ to this S^2 is nontrivial. (Its chern number is ± 1.) (See the proof of Lemma 12.13 below.) Then the formula

$$\partial_\gamma \partial + \partial_\gamma \partial = 0$$

does not hold in general.

Next suppose that $H_1(M;\mathbf{Z})$ is torsion free. Let c and c' be reducible connections such that $G_c = G_{c'} = SU(2)$, $A \in \overline{\mathcal{M}}'(c,c')$, $G_A = U(1)$, $\mu(c) = \mu(c') + 3$. Then, if $a,b \in Fl$ and if $\mathcal{M}(a,c) \neq \emptyset, \mathcal{M}(c',b) \neq \emptyset$, then $\mu(a) \geq \mu(c) + 4$,$\mu(b) \leq \mu(c') - 1$. Hence, the first case we are to examine is the case when $\mu(a) = \mu(b) + 8 = \mu(c') + 7 = \mu(c) + 4$. In this case,

$$\dim \overline{\mathcal{M}}'(a,c) = \dim \overline{\mathcal{M}}'_{red}(c,c') = \dim \overline{\mathcal{M}}'(c',b) = 0.$$

Here $\overline{\mathcal{M}}'_{red}(c,c')$ is the component of $[A]$, which consists of one point. By Theorem 7.1 a neighborhood of each point of

$$\overline{\mathcal{M}}'(a,c) \times \overline{\mathcal{M}}'_{red}(c,c') \times \overline{\mathcal{M}}'(c',b)$$

in $\mathcal{C}\overline{\mathcal{M}}'(a,b)$ is

$$\left(\frac{SO(3) \times SO(3)}{U(1)} \times (0,\infty]^2\right) / \sim$$

where $\sim$ is as in (7.1.4). In other words, it is a cone of $\mathbf{C}P^3/\mathbf{Z}_2 = X$. (See the proof of Lemma 12.13.) Here $\mathbf{Z}_2$ acts by

$$\tau[z_0, z_1, z_2, z_3] = [z_0, z_1, -z_2, -z_3].$$

The fixed points set of this action has two components. The fixed points correspond to the singular points of X. Those singular locus are identified to

$$\begin{aligned}&\left(\frac{SO(3)\times SO(3)}{U(1)}\times\{\infty\}\times(0,\infty)\right)/\sim\\ &\quad\subset\overline{\mathcal{M}}'(a,c)\times\overline{\mathcal{M}}'(c,b),\end{aligned}$$

and

$$\begin{aligned}&\left(\frac{SO(3)\times SO(3)}{U(1)}\times(0,\infty)\times\{\infty\}\right)/\sim\\ &\quad\subset\overline{\mathcal{M}}'(a,c')\times\overline{\mathcal{M}}(c',b),\end{aligned}$$

respectively. We can find ℓ_{i_0} such that (12.12.1) and (12.12.2) are satisfied.

Lemma 12.13.

$$\int_X c^1(\mathcal{L}^{(2)}_{\ell_{i_0}}(a,b))^3 = \pm 4.$$

Proof. Let a_t^0 be a representative of $\overline{\mathcal{M}}'(c,c') = point$, (used in §8.) On $\ell_{i_0}\times\mathbf{R}$, a_t^0 converges to the trivial connection as t goes to $-\infty$, and, as t goes to ∞, it converges to a flat connection -1 whose holonomy, $\rho_{-1}:\mathbf{Z}=\pi_1(S^1)\to SU(2)$ is given by $\rho_{-1}(1)=-1$.

Sublemma 12.14.

$$\mathrm{Index}(\eth_{a_t^0}+\epsilon) = -1.$$

Proof. We put $S^1 = \mathbf{R}/2\pi\mathbf{Z}$. Let x be the coordinate of S^1. We have

$$\eth_{trivial} = \frac{\partial}{\partial t} + i\frac{\partial}{\partial x}.$$

We can perturb a_t^0 so that it is a connection with holonomy

$$\begin{pmatrix} e^{\pi i t} & 0\\ 0 & e^{-\pi i t}\end{pmatrix}.$$

(a_0^0 is a trivial connection and $a_1^0 = -1$.) Then the spectral flow corresponding to the operator $\eth_{a_t^0} + \epsilon$ is as in Figure 9. (Here we take $\epsilon > 0$.)

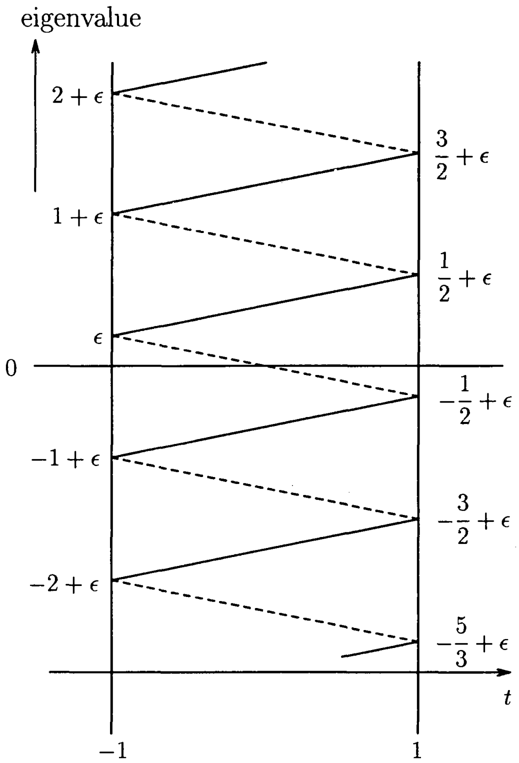

Figure 9.

The sublemma follows.

Remark 12.15. In our case, the half spin bundle $\otimes \mathbf{C}^2$ together with connection a_t^0 splits to the direct sum of two complex line bundles. The dotted lines in Figure 9 correspond to the second factor and the others to the first factor.

The group $U(1) = I_{a_t^0}$ acts on the eigenspaces, and the index in Sublemma 12.14 can be regarded as an element of the representation ring $R(U(1)) \sim \mathbf{Z}[t, t^{-1}]$. Here t be the representation corresponding to $z \mapsto z$ and t^{-1} to $z \to z^{-1}$, where we identify $U(1) = \{z \| z| = 1\}$. By Figure 9, The index is equal to $-t^{-1}$.

If we choose $\epsilon < 0$ then the index is t.

Now we consider the map $\pi : SU(2) \times SU(2) \to \mathcal{M}'(c, c')$ constructed in Theorem 5.4. Let $\mathcal{L}_i(c, c')$ be the line bundle defined in §7. (We have not yet divided it by $G_c \times G_{c'}$.) $\pi^* \mathcal{L}_i(c, c')$ is trivial.

On $SU(2) \times SU(2)$, the group $U(1) = I_{a_t^0}$ acts by

$$h(g_1, g_2) = (g_1 h, h^{-1} g_2).$$

This action lifts to $\pi^*(\mathcal{L}_i(c, c'))$. The quotient is identified to the restriction of $\mathcal{L}_i(c, c')$ to the image of π, which is diffeomorphic to $SU(2) \times SU(2)/U(1)$. By Sublemma 12.14 and Remark 12.15, the action of $U(1)$ on $\pi^*(\mathcal{L}_i(c, c'))$ is given by

$$h((g_1, g_2), v) = ((g_1 h^{-1}, h g_2), hv), \tag{12.16}$$

(in both cases $\epsilon > 0$ and $\epsilon < 0$.)

We put

$$\widehat{X} = \frac{SU(2) \times SU(2) \times [0,1]}{\sim},$$

where

$$\begin{aligned}(g_1, g_2, 0) &= (g_1', g_2, 0),\\ (g_1, g_2, 1) &= (g_1, g_2', 1).\end{aligned}$$

$\widehat{X}$ is diffeomorphic to S^7. By Theorem 7.1,

$$X = \frac{\widehat{X}}{U(1) \times \mathbf{Z}_2}.$$

Here $h \in U(1)$ and $\tau = -1 \in \mathbf{Z}_2$ acts on $\widehat{X}$ by

$$\begin{aligned}h([g_1, g_2, t]) &= [g_1 h, h^{-1} g_2, t],\\ \tau([g_1, g_2, t]) &= [-g_1, g_2, t].\end{aligned}$$

Hence $\widehat{X}/U(1) \simeq \mathbf{C}P^3$. By (12.16), the bundle $\mathcal{L}_i(a, b)$ on $\widehat{X}/U(1) \subset \mathcal{CM}'(a, b)$ is isomorphic to the canonical bundle on $\mathbf{C}P^3$. Hence, its Chern class is equal to the generator, u. Therefore,

$$\int_X c^1\left(\mathcal{L}_i^{(2)}(a, b)\right)^3 = \int_{\mathbf{C}P^3} (2u)^3/2 = 4.$$

The proof of Lemma 12.13 is now complete.

Using Lemma 12.13, we can discuss as in the proof of Theorem 1.10, to show

$$\sum_{\alpha_1 \cup \alpha_2 = \alpha} \partial_{\gamma_{\alpha_1}} \partial_{\gamma_{\alpha_2}} = 4 \sum_{c, c'} \sharp\overline{\mathcal{M}}'(a, c) \cdot \sharp\overline{\mathcal{M}}'(c', b),$$

in the case when $\alpha = (\ell_{i_0}, \ell_{i_0}, \ell_{i_0})$.

It might be possible to define an invariant mod 4 using the above formula. But the author does not try to do it here, because he suspects if it is a correct way.

From the above observation, it seems that we need to examine the reducible connections more seriously when we generalize the invariant for larger s.

§13. Independence of the metrics and the perturbations

The proof of Theorem 1.14 is based on an argument similar to one in §§7–12 and [F]. Let σ_1, σ_2 be two metrics on M and f_1,f_2 be two perturbations as in §§2,3. Let Fl_1 and Fl_2 be the set of solutions of

$$*_{\sigma_1} F^a - \operatorname{grad}_a f_1 = 0,$$

and

$$*_{\sigma_2} F^a - \operatorname{grad}_a f_2 = 0,$$

respectively. Let $(C^s_{(1)}, \partial^1)$ and $(C^s_{(2)}, \partial^2)$ be corresponding complexes constructed in §12. Choose a family of metrics g_t such that

$$\sigma_t = \sigma_1 \quad \text{for } t < -1. \tag{13.1.1}$$

$$\sigma_t = \sigma_2 \quad \text{for } t > 1. \tag{13.1.2}$$

Choose χ such that

$$\begin{aligned} \chi(t) &= 1 \quad \text{for } t > 1, \\ \chi(t) &= 0 \quad \text{for } t < 0. \end{aligned}$$

Let σ_t be the metric $\sigma_t \oplus dt^2$ on $M \times \mathbf{R}$. We consider the equation

$$\begin{aligned} F^A - \widetilde{*}_{\sigma_t} F^A - &\chi(-t)\left(\operatorname{grad}_{a_t} f_1 \wedge dt - *_{\sigma_t} \operatorname{grad}_{a_t} f_1\right) \\ &- \chi(t)\left(\operatorname{grad}_{a_t} f_2 \wedge dt - *_{\sigma_t} \operatorname{grad} f_2\right) = 0, \end{aligned} \tag{13.2}$$

for $A \in \mathcal{A}_{\ell,\delta}(a,b)$. (Compare (3.6).) Here $a \in Fl_1$ and $b \in Fl_2$. The linearization of (13.2) is given by

$$\begin{aligned} &0 = \mathcal{D}_A(u,\varphi) = \\ &-\frac{\partial u}{\partial t} + (*_{\sigma_t} d_{a_t} - \psi_t - \chi(-t)\operatorname{Hess}_{a_t} f_1 - \chi(t)\operatorname{Hess}_{a_t} f_2) \wedge u + d_{a_t}\varphi \end{aligned}$$

Here u, φ e.t.c are the same as in (3.8). Let $\mathcal{D}^1_A$ and $\mathcal{D}^2_A$ be the operators in (3.8) for $\sigma = \sigma_1 \oplus dt^2, \sigma_2 \oplus dt^2$ and $f = f_1, f_2$, respectively.

Lemma 13.3. *If $A \in \mathcal{A}_{\ell,\gamma}(a,b)$ with $a \in Fl_1$ $b \in Fl_2$, then*

$$\dim \operatorname{Coker} \mathcal{D}_A < \infty.$$

Proof. If not we have (u_i, φ_i) such that

$$\mathcal{D}_A^*(u_i, \varphi_i) = 0,$$
$$< (u_i, \varphi_i), (u_j, \varphi_j) >= \delta_{i,j}.$$

Then, by elliptic regularity, we have $|t_i| \to \infty$ such that

$$|(u_i(x_0, t_i), \varphi_i(x_0, t_i))| > C_0 > 0.$$

We may assume that $t_i \to \infty$. Put $u_i'(t,x) = u_i(t - t_i, x)$, $\varphi_i'(t,x) = \varphi_i(t - t_i, x)$. By taking a subsequence we may assume that (u_i', φ_i') converges to $(\widehat{u}, \widehat{\varphi})$ with respect to the C^∞ topology on each compact set. Then we have

$$\mathcal{D}_b^{(2)*}(\widehat{u}, \widehat{\varphi}) = 0$$
$$(\widehat{u}, \widehat{\varphi}) \neq 0.$$

This contradicts (2.6).

Using Lemma 13.3, we can apply the argument of [D3] to obtain a perturbation $Q(\cdot)$, such that the linearized operator $\mathcal{D}_A'$ of

$$\begin{aligned}(13.4)\quad F^A - \widetilde{*}_{\sigma_t} F^A &- \chi(-t)(\operatorname{grad}_{a_t} f_1 \wedge dt - *_{\sigma_t} \operatorname{grad}_{a_t} f_1) \\ &- \chi(t)(\operatorname{grad}_{a_t} f_2 \wedge dt - *_{\sigma_t} \operatorname{grad}_{a_t} f_2) + Q(A) = 0.\end{aligned}$$

is surjective. Here $Q(A)$ depends only on a restriction of A to $M \times [-1,1]$ and its support is also contained in it. Let $\overline{\mathcal{M}}(a,b)$ be the set of solutions of (13.4) divided by gauge transformations. Let $\overline{\mathcal{M}}'_{(1)}(a,b)$ and $\overline{\mathcal{M}}'_{(2)}(a,b)$ be the set of solutions of (3.6) for $\sigma = \sigma_1$, $f = f_1$ and $\sigma = \sigma_2$, $f = f_2$, divided by the gauge transformations and $\mathbf{R}$ action, respectively.

Theorem 13.5. *For $a \in Fl_1$ and $b \in Fl_2$, let $C\overline{\mathcal{M}}(a,b)$ be the* *disjoint union of*

$$\overline{\mathcal{M}}(a,b),$$

$$\overline{\mathcal{M}}(a,c_0) \times \prod_{i=0}^{k-1} \overline{\mathcal{M}}'_{(2)}(c_i,c_{i+1}) \times \overline{\mathcal{M}}'_{(2)}(c_k,b),$$

$$\overline{\mathcal{M}}'_{(1)}(a,c_0) \times \prod_{i=0}^{k-1} \overline{\mathcal{M}}'_{(1)}(c_i,c_{i+1}) \times \overline{\mathcal{M}}(c_k,b),$$

$$\overline{\mathcal{M}}'_{(1)}(a,c_0) \times \prod_{i=1}^{k_0-1} \overline{\mathcal{M}}'_{(1)}(c_i,c_{i+1}) \times \overline{\mathcal{M}}(c_{k_0},c_{k_0+1})$$

$$\times \prod_{i=k_0+1}^{k-1} \overline{\mathcal{M}}'_{(2)}(c_i,c_{i+1}) \times \overline{M}'_{(2)}(c_k,b).$$

Then $C\overline{\mathcal{M}}(a,b)$ has a smooth structure with properties similar to (7.1.1) -(7.1.7).

The proof is similar to the proof of Theorem 7.1 and is omitted.

We remark here the reason why we need to fix a basis of $H_1'(M;\mathbf{Z})$. Let μ_1,μ_2 be the maps defined in Theorem 5.1 for metrics σ_1,σ_2 and let f_1 and f_2 be functions we used in sections 2 and 3. If we use the same basis of $H_1'(M;\mathbf{Z})$ (or more precisely $H_1'(M;\mathbf{Z})\otimes\mathbf{Z}_2$), then we have $\mu_1(c)=\mu_2(c)$ for each reducible connection c. This fact is essential for the argument of the rest of this section. In fact, suppose, for example, there exists reducible c such that

$$\mu_1(c)=\mu_2(c)-10.$$

Then for some $a\in Fl_1$,$b\in Fl_2$ with $\mu_1(a)=\mu_2(b)+1$, the space $\overline{\mathcal{M}}(a,b)$ may have an end described by

$$\overline{\mathcal{M}}'_{(1)}(a,c)\times\overline{\mathcal{M}}(c,c)\times\overline{\mathcal{M}}'_{(2)}(c,b).$$

And $\mu_1(a)-\mu_1(c)$ can be greater than 7. Therefore, in the compactification of $\overline{\mathcal{M}}'_{(1)}$ the end we discussed at the end of §12 can appear. These ends can cause serious problem for the argument of the well definedness. The point is that the virtual dimension of $\overline{\mathcal{M}}(a,b)$ is -10 but we can not find perturbation to make it empty

The author has no explicit example which shows that our invariant does depend on the choice of the basis of $H_1(M;\mathbf{Z})$. But it seems quite unlikely that it is independent.

We return to the proof of invariance. For $\gamma \simeq S^1 \subset M$, we define bundles

$$\begin{aligned} &\mathcal{L}^{(2)}_{\gamma,1}(a,a') \quad \text{on } \overline{\mathcal{M}}_{(1)*}(a,a'),\\ &\mathcal{L}^{(2)}_{\gamma,2}(b,b') \quad \text{on } \overline{\mathcal{M}}_{(2)*}(b,b'),\\ &\mathcal{L}^{(2)}_{\gamma}(a,b) \quad \text{on } \overline{\mathcal{M}}(a,b). \end{aligned}$$

Theorem 13.6. *The tensor products of $\mathcal{L}^{(2)}_{\gamma,1}$,$\mathcal{L}^{(2)}_{\gamma,2}$, and $\mathcal{L}^{(2)}_{\gamma}$ can be patched together to give a line bundle on $\mathcal{C}\overline{\mathcal{M}}_*(a,b)$.*

The proof is the same as the proof of Theorem 7.3.

Now we define $\varphi : (C^s_{(1)}, \partial^1) \to (C^s_{(2)}, \partial^2)$. We put

$$< \varphi_\emptyset(a), b >= \sharp\overline{\mathcal{M}}(a,b)$$

if $\mu(a) = \mu(b)$. (Here $\sharp$ is the same as in §12.) Set

$$\varphi[a] = \sum_b < \varphi_\emptyset a, b > [b].$$

This defines the map $\varphi : C^0_{(1)} \to C^0_{(2)}$.

Next we fix sections $s_\gamma(a,b)$, $s_{\gamma,1}(a,a')$, $s_{\gamma,2}(b,b')$ to $\mathcal{L}^{(2)}_\gamma(a,b)$, $\mathcal{L}^{(2)}_{\gamma,1}(a,a')$, $\mathcal{L}^{(2)}_{\gamma,2}(b,b')$ such that (12.1.2) holds and that they can be patched together to give a section of the line bundle obtained in Theorem 13.6. Now, for $\mu(a) = \mu(b) + 2$, we put

$$< \varphi_\gamma a, b >= \sharp\{x \in \overline{\mathcal{M}}(a,b) | s_\gamma(x) = 0.\}.$$

For $\mu(a) = \mu(b) + 4$, we put

$$< \varphi_{\gamma_1,\gamma_2} a, b >= \sharp\{x \in \overline{\mathcal{M}}(a,b) | s_{\gamma_1}(x) = s_{\gamma_2}(x) = 0\}.$$

Set

$$\varphi_\gamma[a] = \sum_b < \varphi_\gamma a, b > [b],$$

$$\varphi_{\gamma_1,\gamma_2}[a] = \sum_b < \varphi_{\gamma_1,\gamma_2} a, b > [b].$$

Lemma 13.7. *If $|\alpha| < 3$, then*

$$\sum_{\alpha_1 \cup \alpha_2 = \alpha} \partial^2_{\alpha_1} \varphi_{\alpha_2} = \sum_{\alpha_1 \cup \alpha_2 = \alpha} \varphi_{\alpha_1} \partial^1_{\alpha_2}.$$

(If $|\alpha| > 0$ we assume that $H_1(M; \mathbf{Z})$ is torsion free.)

The proof is the same as the proof of Theorem 1.10 in §12. Put

$$\varphi(\gamma_\alpha \otimes a) = \sum_{\alpha_1 \cup \alpha_2 = \alpha} \gamma_{\alpha_1} \otimes \gamma_{\alpha_2} a.$$

Lemma 13.7 implies that $\varphi : (C^s_{(1)}, \partial^1) \to (C^s_{(2)}, \partial^2)$ is a chain map.

Lemma 13.8. *The chain map φ modulo chain homotopy is independent to the choice of the homotopy σ_t of the metrics and the perturbation Q in* (13.4).

Proof. Let σ^1_t,σ^2_t,Q_1,Q_2 be the homotopies and perturbations and φ_1, φ_2 be corresponding chain maps. Choose homotopies σ^u_t and Q_u among them. Let $\overline{\mathcal{M}}'_u(a,b)$ be the set of solutions of (13.4) for $\sigma_t = \sigma^u_t$ and $Q = Q_u$. Let $\mathcal{C}\overline{\mathcal{M}}'_u(a,b)$ be the disjoint union of

$$\overline{\mathcal{M}}_u(a,b)$$

$$\overline{\mathcal{M}}_u(a,c_0) \times \prod_{i=0}^{k-1} \overline{\mathcal{M}}'_{(2)}(c_i, c_{i+1}) \times \overline{\mathcal{M}}'_{(2)}(c_k, b),$$

$$\overline{\mathcal{M}}'_{(1)}(a,c_0) \times \prod_{i=0}^{k-1} \overline{\mathcal{M}}'_{(1)}(c_i, c_{i+1}) \times \times \overline{\mathcal{M}}_u(c_k, b),$$

$$\overline{\mathcal{M}}'_{(1)}(a,c_0) \times \prod_{i=0}^{k_0-1} \overline{\mathcal{M}}'_{(1)}(c_i, c_{i+1}) \times \overline{\mathcal{M}}_u(c_{k_0}, c_{k_0+1})$$

$$\times \prod_{i=k_0+1}^{k-1} \overline{\mathcal{M}}'_{(2)}(c_i, c_{i+1}) \times \overline{\mathcal{M}}'_{(2)}(c_k, b).$$

(Here we do *not* assume that $\mu(a) > \mu(c_0) > \cdots > \mu(c_k) > \mu(b)$.) (Note that $\mathcal{M}_{(1)}(a,b) \neq \mathcal{M}_1(a,b)$.)

Put

$$\mathcal{H}\overline{\mathcal{M}}(a,b) = \bigcup_u \overline{\mathcal{M}}_u(a,b) \times \{u\},$$

$$\mathcal{C}\mathcal{H}\overline{\mathcal{M}}(a,b) = \bigcup_u \mathcal{C}\overline{\mathcal{M}}_u(a,b) \times \{u\}.$$

Theorem 13.9. *We can take σ^u_t and Q_u such that $\mathcal{C}\mathcal{H}\overline{\mathcal{M}}(a,b)$ has a smooth structure which has properties similar to* (7.1.1)–(7.1.7).

The proof of Theorem 13.9 is a bit more difficult than that of Theorem 7.1. The reason is that we can *not* assume that the operator $\mathcal{D}_A^{(u)}$ obtained by linearizing (13.4) is surjective for every u, (even if we choose σ_t^u and Q_u to be generic.) Then we have to use the Kuranishi map as in [T2], [D2]. For simplicity we prove the case $\mu(a) = \mu(b)$. Here $a \in Fl_1, b \in Fl_2$. Then $\dim \mathcal{HM}'(a,b) = 1$. In this case, Theorem 13.9 follows immediately from the following two lemmas.

Lemma 13.10. *Suppose that the sequence $(A_i, u_i) \in \mathcal{H}\overline{\mathcal{M}}(a,b)$ is unbounded. Then, by taking a subsequence if necessary, there exist either $c \in Fl_1$, t_i, $B \in \overline{\mathcal{M}}_u(a,c)$, $C \in \overline{\mathcal{M}}'_{(2)}(c,b)$ with $\mu(c) = \mu(a)+1$ or $c' \in Fl_2$, t'_i, $B' \in \overline{\mathcal{M}}_{(1)}(a,c')$, $C' \in \overline{\mathcal{M}}_u(c',b)$ with $\mu(c') = \mu(a)-1$ such that the Conditions* (13.10.1)–(13.10.3) *or* (13.10.1) –(13.10.3)$'$ *below hold.*

(13.10.1) $\qquad u_i \to u$

(13.10.2) $\qquad |A_i(x,t) - B(x,t)| \to 0$

(13.10.3) $\qquad |A_i(x,t-t_i) - C(x,t)| \to 0$

(13.10.2)$'$ $\qquad |A_i(x,t+t_i) - B'(x,t)| \to 0$

(13.10.3)$'$ $\qquad |A_i(x,t) - C'(x,t)| \to 0.$

(See Figure 10.) Note that $\overline{\mathcal{M}}_u(a,c) = \emptyset = \overline{\mathcal{M}}_u(c',b)$ for generic u. (The virtual dimension of them is -1.) But "1-parameter family of -1-dimensional spaces is a finite set". Hence by a generic choice of σ_t^u and Q_u there exist a finite number of u's, for which $\overline{\mathcal{M}}_u(a,c)$ or $\overline{\mathcal{M}}_u(c',b)$ is nonempty.

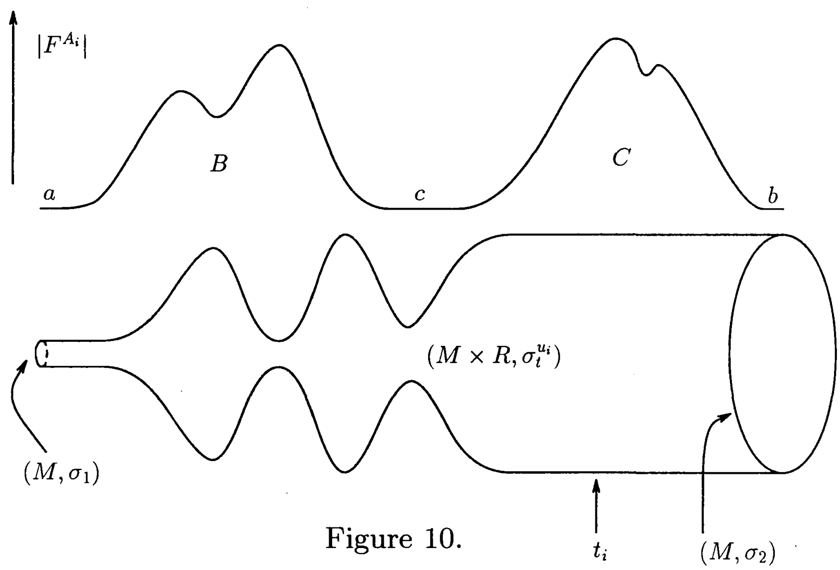

Figure 10.

Lemma 13.11. *Let $B \in \overline{\mathcal{M}}_u(a,c)$, $C \in \overline{\mathcal{M}}'_{(2)}(c,b)$. Then there exist $u(v) : (0,\infty) \to 0,1$, $A(v) \in \overline{\mathcal{M}}_{u(v)}(a,b)$ and $t(v), t'(v) \in \mathbf{R}$, such that*

(13.11.1) $\lim_{v\to\infty} u(v) = u$

(13.11.2) $\lim_{v\to\infty} |A(v)(x, t - t(v)) - B(x,t)| = 0$

(13.11.3) $\lim_{v\to\infty} |A(v)(x, t + t'(v)) - C(x,t)| = 0.$

Moreover, if A_i satisfies (13.10.1) - (13.10.3) *then* $[A_i] = [A(v_i)]$ *for large i. A similar statement holds for c'.*

The proof of Lemma 13.10 is similar to the proof in §9 and is omitted. Before proving Lemma 13.11 we complete the proof of Lemma 13.8 in the case when $s = 0$.

In this case, Theorem 13.9 implies

$$\begin{aligned}&\partial \mathcal{H}\overline{\mathcal{M}}(a,b) - \overline{\mathcal{M}}_1(a,b) - \overline{\mathcal{M}}_2(a,b)\\ &\qquad = \bigcup_{u,c} \overline{\mathcal{M}}_u(a,c) \times \overline{\mathcal{M}}'_{(2)}(c,b) \cup \bigcup_{u,c'} \overline{\mathcal{M}}'_{(1)}(a,c') \times \overline{\mathcal{M}}_u(c',b).\end{aligned}$$

We put

$$< \Phi a, c > = \sum_u \sharp \overline{\mathcal{M}}_u(a,c)$$

$$< \Phi c', b > = \sum_u \sharp \overline{\mathcal{M}}_u(c',b),$$

and

$$\Phi[a] = \sum_c < \Phi a, c > [c]$$

$$\Phi[c'] = \sum_b < \Phi c', b > [b].$$

Then we have

$$\varphi_1 - \varphi_2 = \partial\Phi - \Phi\partial.$$

Here φ_1 and φ_2 are the chain maps constructed using σ_t^1, Q_1 and σ_t^2, Q_2, respectively. This proves Lemma 13.8 when $s = 0$. The case when $s > 0$ can be proved by combining the methods of §§7 - 12 and Theorem 13.9. (In fact, the case $s > 0$ is simpler, because we do not have to use Kuranishi map in that case.)

Proof of Lemma 13.11. Let $\mathcal{D}_A^u$ be the operator obtained by linearizing the equation (13.4) for $\sigma_t = \sigma_t^u$ and $Q = Q_u$. By the generic choice of σ_t^u and Q_u we have $\dim \operatorname{Coker} \mathcal{D}_B^u = 1$. We consider the set X of the connections which is a standard form of type $(\{B\}, \{C\}, \epsilon, T)$. By Remark 4.6, there exists a positive number λ_0, such that, if $A \in X$ and if $|u-u'| < \epsilon$, then, there is exactly one eigenvalue of $\mathcal{D}_A^{u'} \mathcal{D}_A^{u'*}$ smaller than λ_0. Let Π_I be the orthonormal projection to this eigenspace, (which is isomorphic to $\mathbf{R}$). Put $\Pi_{II} = identify - \Pi_I$. For $A \in \mathcal{A}(a,b)$, $u' \in [0,1]$ we consider the equation

(13.12)
$$\begin{aligned}\Pi_{II}(F^A - \widetilde{*}_{\sigma_t^{u'}} F^A - \chi_{u'}(-t)(\operatorname{grad}_{a_t} f_1 \wedge dt - *_{\sigma_t^{u'}} \operatorname{grad}_{a_t} f_1)\\ - \chi_{u'}(t)(\operatorname{grad}_{a_t} f_2 \wedge dt - *_{\sigma_t^{u'}} \operatorname{grad}_{a_t} f_2) + Q_{u'}(A)) = 0.\end{aligned}$$

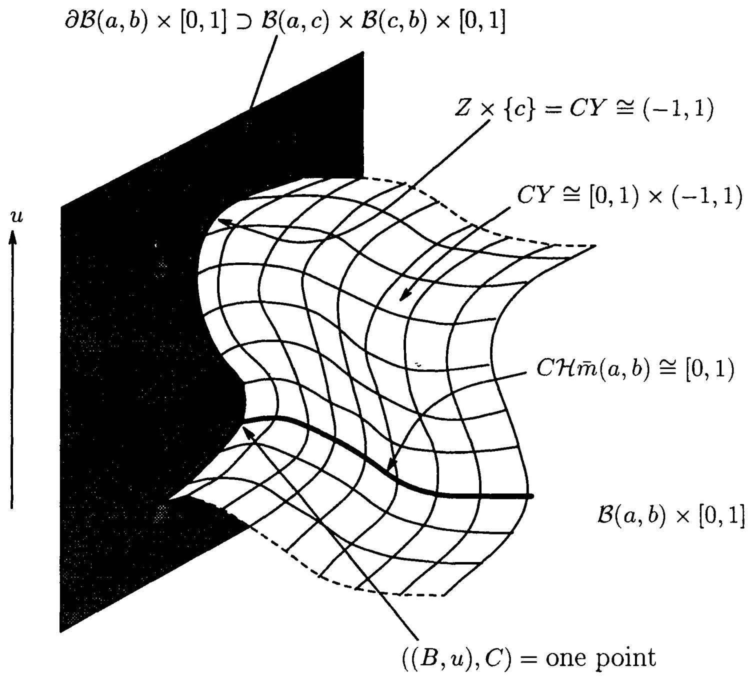

Figure 11.

The set of solutions of (13.12) divided by gauge transformations consists a 2-dimensional family Y. Let Z be the set of solutions of (13.12) for $A \in \mathcal{A}(a,c)$ and $u' \in [0,1]$. ($\dim Z = 1$.) Then, using the method of the proof of Theorem 7.1, we can compactify Y by adding $Z \times \{C\}$. Put $\mathcal{C}Y = Y \cup (Z \times \{C\})$. A neighborhood of $((B,u),C)$ in $\mathcal{C}Y$ is identified to $[0,1) \times (0,1)$, where $\{0\} \times (0,1) \subset Z \times \{C\}$. (See Figure 11.) For

(A, u'), we put

$$\begin{aligned} f(A, u') =& \Pi_I(F^A - \widetilde{*}_{\sigma_t^{u'}} F^A - \chi_{u'}(-t)(\mathrm{grad}_{a_t} f_1 \wedge dt - *_{\sigma_t^{u'}} \mathrm{grad}_{a_t} f_1) \\ &- \chi_{u'}(t)(\mathrm{grad}_{a_t} f_2 \wedge dt - *_{\sigma_t^{u'}} \mathrm{grad}_{a_t} f_2) + Q_{u'}(A)). \end{aligned}$$

We identify the image of Π_I to $\mathbf{R}$ and regard f as a function. Using the decay estimate in §9 we can extend the function f to a *smooth* function on $\mathcal{C}Y$. The set of zero's of f is identified to a neighborhood of $((B, u), C)$ in $\mathcal{CH}\overline{\mathcal{M}}(a, b)$. We consider the restriction of f to $\{0\} \times (0, 1) \subset Z$. If we choose g_t^u and Q_u generic, we may assume that the derivative of this restriction is nonzero at $((B, u), C) \in \{0\} \times (0, 1)$. It follows from implicit function theorem that the zero of f in $\mathcal{C}Y$ is diffeomorphic to $[0, 1)$ where $0 \in [0, 1)$ corresponds to $((B, u), C)$. Lemma 13.11 follows immediately.

The proof of Lemma 13.8 is now complete.

Next we take another metric σ_3 and another perturbation f_3. Choose homotopies $\sigma_t^{1,2}$ and $\sigma_t^{2,3}$ from σ_1 to σ_2 and from σ_2 to σ_3. Choose also perturbations $Q_{1,2}$ and $Q_{2,3}$. Let $\varphi_{1,2}$ and $\varphi_{2,3}$ be the chain maps obtained by them, respectively.

Lemma 13.12. *We can find homotopy of metric $\sigma_t^{1,3}$ from σ_1 to σ_3 and a perturbation $Q_{1,3}$ such that the chain map $\varphi_{1,3} : C^s_{(1)} \to C^s_{(3)}$ satisfies*

$$\varphi_{3,2}\varphi_{1,2} = \varphi_{1,3}.$$

Proof. We put

$$\sigma_t^s = \chi(-t - s)\sigma^{1,2}_{t+2s} + \chi(t - s)\sigma^{2,3}_{t-2s}.$$

We shift the perturbation $Q_{1,2}$ by $2s$ to the negative direction and shift $Q_{2,3}$ by $2s$ to the positive direction. Let $Q^s_{1,3}$ be the sum of them. We consider the equation

(13.13)

$$\begin{aligned} F^A - \widetilde{*}_{\sigma_t^s} F^A &- \chi(-t - s)(\mathrm{grad}_{a_t} f_1 \wedge dt - *_{\sigma_t^s} \mathrm{grad}_{a_t} f_1) \\ &- \chi(t + s)\chi(s - t)(\mathrm{grad}_{a_t} f_2 \wedge dt - *_{\sigma_t^s} \mathrm{grad}_{a_t} f_2) \\ &- \chi(t - s)(\mathrm{grad}_{a_t} f_3 \wedge dt - *_{\sigma_t^s} \mathrm{grad}_{a_t} f_3) + Q_{1,3}(A) = 0 \end{aligned}$$

Let $\overline{\mathcal{M}}(s; a, e)$ be the set of solutions of (13.13) divided by gauge transformations. Let $\overline{\mathcal{M}}_{1,2}(a, b)$ and $\overline{\mathcal{M}}_{2,3}(b, e)$ be the moduli spaces used in the definitions of $\varphi_{1,2}$ and $\varphi_{2,3}$ respectively. (Here $a \in Fl_1$, $b \in Fl_2$, $e \in Fl_3$.)

By using Remark 4.6, we can prove that the linearized equation for (13.13) is surjective for sufficiently large s. Consider the disjoint union of

$$\mathcal{C}\overline{\mathcal{M}}(s; a, e) \times \{s\} \qquad s \in [s_0, \infty)$$

and

$$\begin{aligned}&\prod_{i=-1}^{k_0-1} \overline{\mathcal{M}}'_{(1)}(c_i, c_{i+1}) \times \overline{\mathcal{M}}_{1,2}(c_{k_0}, c_{k_0+1})\\ &\quad\times \prod_{i=k_0+1}^{k_1-1} \overline{\mathcal{M}}'_{(2)}(c_i, c_{i+1}) \times \overline{\mathcal{M}}_{2,3}(c_{k_1}, c_{k_1+1})\\ &\quad\times \prod_{i=k_1+1}^{k_2} \overline{\mathcal{M}}'_{(3)}(c_i, c_{i+1}) \times \{\infty\}.\end{aligned}$$

(Here we put $a = c_{-1}$, $e = c_{k_2+1}$.) The later one is a compactification of $\cup_b \overline{\mathcal{M}}_{1,2}(a, b) \times \overline{\mathcal{M}}_{2,3}(b, e)$. Let $\mathcal{C}\mathcal{C}\overline{\mathcal{M}}(a, e)$ be the union. Using this moduli space, the proof of the lemma goes in a way similar to the argument of §§7 - 13.

Now we are in the position to complete the proof of Theorem 1.14. Suppose $\sigma_1 = \sigma_3$, in Lemma 13.12. Then we can take a trivial homotopy $\sigma^{1,3} = \sigma_1$ and $Q_{1,3} = 0$. In this case, it is easy to see that the corresponding chain map is the identity map. Therefore by Lemma 13.12 and Lemma 13.8, $\varphi_{2,3}\varphi_{1,2}$ is chain homotopic to identity. (In this case $\varphi_{2,3} = \varphi_{2,1}$.) Thus the chain map $\varphi_{1,2}$ we constructed gives an isomorphisms on the homology groups. Also the isomorphism is canonical because of Lemma 13.8. The proof of Theorem 1.14 is now complete. The proof of the independence of the exact sequence 1.15 is similar.

References

[Ar] N. Arondazajn, A unique continuation theorem for solutions of elliptic partial differential equations or inequalities of the second order, J. Math. Pures Appl., **36(9)** (1957), 235–249.

[A] M. Atiyah, New invariants of three and four dimensional manifolds, in "The Mathematical Heritage of Herman Weyl", Proc. Sym. Pure. Math. 48, American mathematical Society, 1988.

[AHS] M. Atiyah, N. Hitchin, I. Singer, Self-duality in four dimensional topology, Proc. R. Soc. London A, **362** (1978), 425 –461.

[APS] M. Atiyah, N. Patodi, I. Singer, Spectral asymmetry and Riemannian geometry III, Proc. Camb. Philos. Soc., **79** (1976), 71–99.

[AS] M. Atiyah, I. Singer, Index of Elliptic operators IV, Ann. of Math., **93** (1971), 97–118.

[BK] P. Buser, H. Karcher, Gromov's almost flat manifolds, Asterisque, **81** (1981), 1–148.

[CG] J. Cheeger, M. Gromov, Collapsing Riemannian manifolds while keeping their curvatures bounded I, J. Differential Geom., **23** (1986), 309–346.

[D1] S. Donaldson, An application of gauge theory to the topology of 4-manifolds, J. Differential Geom., **18** (1983), 269 –316.

[D2] ——, Connection, cohomology, and intersection forms of 4-manifolds, J. Differential Geom., **24** (1986), 275 – 341.

[D3] ——, The orientation of Yang-Mills moduli spaces and 4-manifold topology, J. Differential Geom., **26** (1986), 397 –428.

[D4] ——, Polynomial invariants for smooth 4-manifolds, Topology, **29** (1990), 257–315.

[DFK] S. Donaldson, M. Furuta, D. Kotshick, Floer homology groups in Yang-Mills theory, in preparation.

[DK] S. Donaldson, P. Kronheimer, The geometry of four manifolds, Oxford Science Publications, Oxford, (1990).

[F] A. Floer, An instanton invariant for three manifolds, Commun. Math. Phys., **118** (1988), 215–240.

[FU] D. Freed, K. Uhlenbeck, "Instantons and Four manifolds", Springer, New York, 1984.

[GK] K. Grove, H. Karcher, How to conjugate C^1-close actions, Math. Z., **132** (1973), 11–20.

[GKR] K. Grove, H. Karcher, E. Ruh, Group actions and curvature, Invent. Math., **23** (1974), 31–48.

[LM] R. Lockhart, R. McOwen, Elliptic operators on noncompact manifolds, Ann. Sci. Ec. Norm. Sup. Pisa, **IV-12** (1985), 409–446.

[T1] C. Taubes, Self-dual connections on non-selfdual manifolds, J. Differential Geom., **17** (1982), 139–170.

[T2] ——, Self dual connections on manifolds with indefinite intersection matrix, J. Differential Geom., **19** (1984), 517–560.

[T3] ——, Gauge theory on asymptotically periodic 4-manifolds, J. Differential Geom., **25** (1987), 364–430.

[T4] ——, Casson's invariant and gauge theory, J. Differential Geom., **31** (1989), 547–599.

[W1] E. Witten, Supersymmetry and Morse theory, J. Differential Geom., **17** (1982), 661–692.

[W2] ——, Topological field theory, Comm. Math. Phys., **117** (1988), 353–386.

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Polyhedral Decomposition of Hyperbolic 3-Manifolds with Totally Geodesic Boundary

Sadayoshi Kojima

Dedicated to Professor Kunio Murasugi on his sixtieth birthday

§1. Introduction

A hyperbolic manifold will be a riemannian manifold with constant sectional curvature -1. It is shown by Epstein and Penner [1] that every noncompact complete hyperbolic manifold of finite volume, hence having cusps, is decomposed by ideal polyhedra. The decomposition supplies a quite convenient block to study several geometries of the cusped manifold especially in dimension three. See [4] for instance.

A variant of the construction by Epstein and Penner would establish a decomposition of a compact hyperbolic manifold with nonempty geodesic boundary by truncated polyhedra as well, which we plan to discuss in a forthcoming paper [3]. However the process will be rather unseen in the manifold.

In this paper, taking advantage of working only in dimension three, we give a more visible construction of this decomposition. In fact we directly show

Theorem. *Let N be a compact hyperbolic 3-manifold with nonempty totally geodesic boundary. Then the topological decomposition of N dual to the cut locus of ∂N modulo boundary is homotopic by straightening to a polyhedral decomposition.*

The visible process is expected to lead us to the deep understanding of geometry of those manifolds. We apply it for example to find the minimum of their volumes in [2].

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We describe the rule of the decompostion in the next section with some detailed accounts of truncated polyhedra. We study the cut locus of the boundary and its topological dual decomposition in §3. Then we show in the subsequent sections that the straightening of the dual along its internal edges yields the final polyhedral decomposition. The proof of Proposition 5.2 thus finishes the proof of the theorem.

I am grateful to Tomoyoshi Yoshida for showing his idea to decompose cusped manifolds by ideal polyhedra.

§2. Truncated polyhedra

We start with describing a basic piece of truncated polyhedra, called truncated tetrahedra. An ideal tetrahedron is a hyperbolic polyhedron identified with a finite volume region in the hyperbolic 3-space $\mathbf{H}^3$ bounded by four geodesic planes, every two of which intersect each other, and every three of which intersect at infinity. An ultra ideal tetrahedron is one identified with a similar region bounded by four planes, every two of which intersect each other again but no three of which intersect even at infinity. If we are in the projective model, an ultra ideal tetrahedron is one whose vertices are located outside of the model disk.

An ultra ideal tetrahedron is of infinite volume. The truncation is the device to cut off its thick end by a geodesic plane which intersects three planes towards the end perpendicularly. Such truncation is always uniquely possible since

Lemma 2.1. *For any three metric disks on the euclidean plane which have no points in common but each two of which have a common region, there is a unique circle intersecting their boundaries perpendicularly.*

Proof. Let us name three disks by A, B and C. By conformal change, we may assume that one of the intersection points of ∂A and ∂B is located at infinity. Then ∂A and ∂B are the lines intersecting say at the origin. By the assumption on the position of disks, C does not contain the origin. Hence we have a unique circle centered at the origin intersecting ∂C perpendicularly. This circle automatically intersects both ∂A and ∂B perpendicularly. Q.E.D.

Regard the boundaries of these disks as the ends of the geodesic planes which make up a thick end of an ultra ideal tetrahedron. The circle obtained in Lemma 2.1 will be the boundary of the plane for truncation. This plane intersects three planes perpendicularly. Cutting off each thick end by truncation, we get a compact polyhedron. This is a truncated tetrahedron. The surface of a truncated tetrahedron consists of four right angle hexagons on the planes to bound the region, and four triangles produced by the truncation.

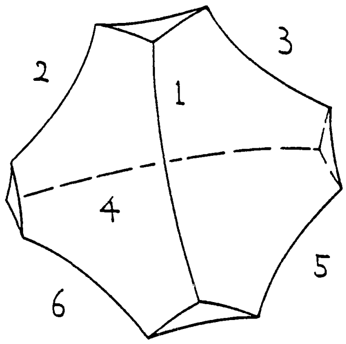

Fig. 1.

A convex truncated polyhedron can be described in a similar manner. Start with a finite set of geodesic planes in $\mathbf{H}^3$, no three of which intersect even at infinity. Assume that it bounds a noncompact convex region each thick end of which admits truncation. Then cutting off each end of the region by truncation, we get a compact polyhedron. This is a convex truncated polyhedron. The surface of a convex truncated polyhedron consists of right angle polygons on the planes, which we call internal faces, and the other polygons produced by the truncation, which we call external faces. The union of internal faces is connected, while external faces are mutually disjoint.

A tetrahedron is a basic piece of a polyhedron even in this situation.

Lemma 2.2. *A convex truncated polyhedron is decomposed by truncated tetrahedra without producing vertices in the interior.*

Proof. Choose an external face τ and introduce the shortest geodesic paths from the face to the other external faces. Such a path uniquely exists for each face. It lies on the boundary if τ and the terminal face are joined by just one face. Obviously it lies on this joining face then. Otherwise the paths go through interior of the polyhedron.

The internal faces touching τ are now subdivided into right angle hexagons. Subdivide then the other internal faces by a geodesic path into right angle hexagons arbitrarily. Each geodesic path introduced here joins two external faces. It together with the shortest paths assigned to the terminal faces span a right angle hexagon in the interior of the polyhedron. Because two paths determine a geodesic plane intersecting three external faces involved perpendicularly and hence this plane must contain the last path. The collection of these hexagons divides the original polyhedron into truncated tetrahedra. Q.E.D.

A *polyhedral decomposition* of a hyperbolic 3-manifold with totally geodesic boundary is a geometric cellular decomposition by (convex) truncated polyhedra so that their external faces form the boundary. This justifies our naming for faces. We also call an edge internal if it is an intersection of two internal faces, and external otherwise. Notice in this decomposition that every internal edge is a geodesic path from the boundary to the boundary.

Let us describe a parametrization of isometry classes of labelled truncated tetrahedra to show its variation, though the result is not needed for the proof of the theorem. The isometry class of a truncated tetrahedron is determined by the mutual position of the internal faces, since the truncation is unique. Label the internal edges as in Figure 1.1 and denote the dihedral angle along the edge j by θ_j. θ_j's are quantities to describe mutual position. The sum of three dihedral angles having a common external face must be less than π because otherwise three planes towards the end meet in the real world. Thus we have a necessary condition,

$$\begin{cases} \theta_1 + \theta_2 + \theta_3 < \pi \\ \theta_1 + \theta_5 + \theta_6 < \pi \\ \theta_2 + \theta_4 + \theta_6 < \pi \\ \theta_3 + \theta_4 + \theta_5 < \pi. \end{cases}$$

Conversely,

Lemma 2.3. *For $\theta_1, ..., \theta_6$ satisfying the above inequalitites, there is a unique labelled truncated tetrahedron with these dihedral angles.*

Proof. Make four geodesic triangles using $\theta_1, ..., \theta_6$ which would form external faces. They have twelve edge lengths as data we can use. Choose a triple from these twelve lengths that would be assigned to the external edges of an internal face we expect to make. Then there is a unique right angle hexagon having these as non adjacent edge lengths, which is a candidate of the internal face. Applying the same for the other triples, we get four right angle hexagons.

The expected truncated tetrahedra should be obtained by gluing these faces in $\mathbf{H}^3$, and what we need to show now is that the length of a common internal edge for each pair of hexagons made are the same. We do this for the internal edge 1. By the hyperbolic cosine rule, we have

$$(*) \qquad \cosh \ell_{ij} = \frac{\cos\theta_i \cos\theta_j + \cos\theta_k}{\sin\theta_i \sin\theta_j},$$

where $\{i, j, k\}$ corresponds to 3 angles of a triangle and ℓ_{ij} is the length of the external edge connecting edges i and j. We made two hexagons having the edge 1. By the hexagon rule [4], the length ℓ_1 of the edge 1 computed in the hexagon having the edges 2 and 6 is given by

$$\cosh \ell_1 = \frac{\cosh \ell_{12} \cosh \ell_{16} + \cosh \ell_{26}}{\sinh \ell_{12} \sinh \ell_{16}},$$

and the same having the edges 3 and 5 is given by

$$\cosh \ell_1 = \frac{\cosh \ell_{13} \cosh \ell_{15} + \cosh \ell_{35}}{\sinh \ell_{13} \sinh \ell_{15}}.$$

It is then easy to check by substitution of (*) that right hand sides of both identities are the same. Q.E.D.

§3. Cut locus

Studying several properties of the cut locus of the boundary in this section, we will find a topological cellular decomposition of a hyperbolic manifold with totally geodesic boundary. It is dual to the cut locus modulo boundary and turns out to be equivalent to the final one. The decomposition will be denoted by K.

Here we start with making a few conventions used throughout the sequel. Let N be a compact hyperbolic 3-manifold with totally geodesic boundary ∂N. Let $\pi : \widetilde{N} \to N$ be the universal covering of N. We use the symbol $\widetilde{X}$ to denote the preimage of a subspace X of N in $\widetilde{N}$. We always identify the universal cover $\widetilde{N}$ with a subspace in $\mathbf{H}^3$. Then the boundary $\partial\widetilde{N}$ of the universal cover $\widetilde{N}$ or the preimage $\widetilde{\partial N}$ of the boundary ∂N is formed by geodesic planes in $\mathbf{H}^3$. We often identify a cell complex with its underlying polyhedron. The symbol $Y^{(k)}$ will be used to denote the k-skeleton of a cell complex Y as usual.

We define three terminologies for our convenience. To each pair of components of $\partial\widetilde{N}$, associated is a unique shortest path connecting them. We call this path a *short cut*. Also there is an associated bisectorial geodesic plane to the short cut in $\mathbf{H}^3$. We call this plane a *middle fence.* A short cut descends to the geodesic path in N from the boundary to the boundary. We call such a path a *return path.* Though it may come back to a different component, we wish to emphasize by this name that it comes back to the boundary anyway. These are the terminologies we shall use frequently.

The cut locus $\mathbf{C}$ of ∂N in N is a subset in int N which consists of the points that admit at least two distinct shortest paths to ∂N. Obviously a point on $\mathbf{C}$ lifts to a point on the middle fence of some short cut. $\mathbf{C}$ is canonically stratified by grouping the points which have the same number of shortest paths to the boundary. This stratification is quite nice in our case since

Proposition 3.1. *The stratification defines a convex cellular decomposition of the cut locus* $\mathbf{C}$.

A point on $\mathbf{C}$ is in a 2-cell if it admits precisely two shortest paths to the boundary, however the number of shortest paths the point admits is rather unrelated with the dimension of the cell in the other case. To see this proposition, we need a few preliminaries.

Lemma 3.2. *Suppose that* A *and* B *are ultra parallel planes of distance* d *in* $\mathbf{H}^3$. *Then the orthogonally projected image of* A *to* B *is an open metric disk of radius* $\operatorname{arccosh}(\coth d)$.

Proof. This is an easy consequence of length calculus for a hyperbolic rectangle with one ideal vertex and three vertices of right angle as in Figure 2.

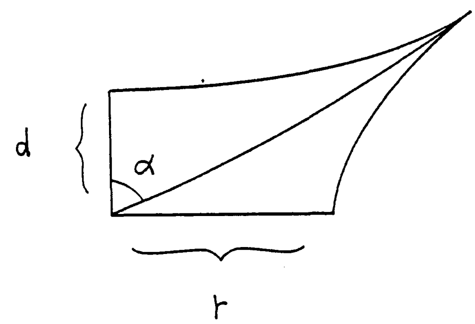

Fig. 2.

The hyperbolic cosine rule shows the identities;

$$\cosh d = \frac{1}{\sin \alpha},$$
$$\cosh r = \frac{1}{\sin(\pi/2 - \alpha)} = \frac{1}{\cos \alpha}.$$

Then we are done by solving the relation between d and r in terms of r. Q.E.D.

Corollary 3.3. *There exist only finitely many return paths with bounded length.*

Proof. Choose a component S of $\partial\widetilde{N}$, and project the other boundary components orthogonally to S. Then we get an open disk packing on S invariant under the action of the covering transformations preserving S. Hence $\pi(S) \subset N$ is packed by open balls. It is obvious by definition that the packing on $\pi(S)$ does not depend on the choice of a component S of $\pi^{-1}(\pi(S))$. Applying the same process to all the other components, we get a ball packing on ∂N. The radius of each ball is related to the length of the associated return path by Lemma 3.2. Since ∂N is compact, the number of balls packing ∂N with bounded radius away from zero is obviously finite. Hence there are only finitely many return paths of bounded length. Q.E.D.

Proof of Proposition 3.1. Choose a component U of the complement of $\widetilde{\mathbf{C}}$ in $\widetilde{N}$ and let S be its boundary in $\partial\widetilde{N}$. U is invariant under the action of covering transformations preserving S. We are interested in the internal boundary of the closure $\overline{U}$ of U not meeting $\partial\widetilde{N}$. It is a part of $\widetilde{\mathbf{C}}$ and formed by a part of middle fences. Since N is compact, its diameter is bounded, and the points on $\mathbf{C}$ have bounded distance to ∂N. The shortest arc from a point on $\mathbf{C}$ to ∂N is lifted to an arc in $\overline{Y}$. In particular, the distance between S and any point on the internal boundary of $\overline{U}$ is bounded. Hence the middle fences involved in this boundary are associated with the short cuts of bounded length.

By Corollary 2.3, there are only finitely many return paths with bounded length. Hence the middle fences involved in the internal boundary of $\overline{U}$ belong to only finitely many orbits of middle fences by the action of covering transformations preserving S. The internal boundary of $\overline{U}$ thus gets a locally finite invariant cellular decomposition induced by the intersection of middle fences involved. It descends to a cellular decomposition of the internal boundary of $\pi(\overline{U})$.

We can apply the same argument to the other component. It is an exercise to check that the cell structures for the common part of different internal boundaries are identical. Hence we get an invariant cellular decomposition of $\widetilde{\mathbf{C}}$ and hence a cell complex structure of $\mathbf{C}$. Each 2-cell of $\mathbf{C}$ is convex since it lifts to a convex polygon on some middle fence bounded by the intersections with a finite number of the other middle fences. Q.E.D.

From now on, let us mean by $\mathbf{C}$ not only the cut locus itself but endowed with this cellular decomposition by virtue of the proposition. In the universal cover, we say a 2-cell of $\widetilde{\mathbf{C}}$ faces a component of $\partial\widetilde{N}$ if the cell can be projected orthogonally to the component by the shortest paths to $\partial\widetilde{N}$. Each 2-cell faces two boundary components associated to the middle fence containing it. The set of orthogonal projections for each 2-cell to these components gives rise to an equivariant one-to-finite orthogonal projection : $\widetilde{\mathbf{C}} \to \partial\widetilde{N}$. The number of the image of $p \in \widetilde{\mathbf{C}}$ is equal to the number of the shortest paths from p to $\partial\widetilde{N}$. The cellular decomposition of $\widetilde{\mathbf{C}}$ is conveyed to an invariant convex polygonal decomposition of $\partial\widetilde{N}$. In particular, the cellular decomposition of $\mathbf{C}$ induces a convex polygonal decomposition of ∂N.

Now, we would like to build up a topological cellular decomposition K of N dual to $\mathbf{C}$ modulo boundary. Start with defining a compact 3-cell, which we call a block, in the universal cover. Its interior will be a 3-cell in the precise definition of the cell complex K. Take an invariant graph G on $\widetilde{\mathbf{C}}$ under the action of $\pi_1(N)$ which is dual to the 1-skeleton $\widetilde{\mathbf{C}}^{(1)}$. Here we mean by dual, the 1-dimensional subcomplex of the barycentric like subdivision of $\widetilde{\mathbf{C}}$ spanned by vertices not in $\widetilde{\mathbf{C}}^{(0)}$. Then project it by the one-to-finite orthogonal projection to $\partial\widetilde{N}$. The trace of the projection determines a fence which divides $\widetilde{N}$ into equivariant pieces homeomorphic to a ball. This is a block to built up $\widetilde{K}$.

Let us next define a compact cell which we call a face, an edge or a vertex according to its dimension. The intersection of two blocks is the trace of the star subgraph of a vertex of G on a 1-cell of $\widetilde{\mathbf{C}}$ by the orthogonal projection. Hence take it as a dual face to the 1-cell of $\widetilde{\mathbf{C}}$ on which the center is located, and call it an internal face. We also take a component of the intersection of a block and $\partial\widetilde{N}$ as an external face. A face will be either an internal or external face. The intersection of two internal faces is the trace of a vertex of G on a 2-cell of $\widetilde{\mathbf{C}}$. Hence take it as a dual edge to the 2-cell containing the vertex, and call it an internal edge. We also take a component of the intersection of an internal face and $\partial\widetilde{N}$ as an external edge. An edge will be either an internal or external edge. Finally a vertex will be a terminal point of an edge.

Then let $\widetilde{K}$ be a cellular decomposition of $\widetilde{N}$ by the interior of blocks, faces, edges and vertices. Since it is invariant under the action of $\pi_1(N)$, it determines a cellular decomposition $K = \pi(\widetilde{K})$ of N. This is what we call a dual to $\mathbf{C}$ modulo boundary. Notice that ∂K is dual to the convex polygonal decomposition of ∂N induced by the cut locus.

We describe the compact cells of $\widetilde{K}$ more locally to visualize the situation. Each block contains a unique 0-cell of $\widetilde{\mathbf{C}}$. We call this a center. Choose a block σ with the center p and let us describe its combinatorial structure of the boundary by identifying p with the origin of the 3-dimensional Poincaré disk. p has the shortest rays to finitely many components of $\partial\widetilde{N}$, say $S_1, S_2, ..., S_m$. σ can be identified with a regular neighborhood of the union of these rays. The ray extends and terminates in the sphere at infinity S^2_∞. The terminal point q_j is the center of the metric circle ∂S_j on S^2_∞ with respect to the canonical spherical metric, where $j = 1, 2, ..., m$. Notice that the radii of circles are the same because the distances from the origin are the same.

Take the cut locus $\mathbf{D}$ of the point set $\{q_1, ..., q_m\}$ on S^2_∞. $\mathbf{D}$ consists of the points on S^2_∞ which admit at least two shortest paths to the set $\{q_1, ..., q_m\}$. $\mathbf{D}$ is unit tangentially equivalent to $\mathbf{C}$ at p and hence determines a convex polygonal decomposition on S^2_∞.

A topological dual decomposition $\mathbf{D}^*$ of $\mathbf{D}$ on S^2_∞ with vertices $q_1, ..., q_m$ is identified with one obtained from the cellular decomposition of $\partial\sigma$ by collapsing each external face to q_j. Notice by the definition of the cut locus that the vertices of a face of $\mathbf{D}^*$ have the same distance to the vertex of $\mathbf{D}$ in this face. This fact will be used later.

We may assume that each edge of $\mathbf{D}^*$ is straight at least in the disks bounded by ∂S_j's. Replacing the part of $\mathbf{D}^*$ in each disk by ∂S_j, we get a cellular decomposition $\mathbf{D}^{**}$ on S^2_∞. $\mathbf{D}^{**}$ is equivalent to $\partial\sigma$.

There are several immediate correspondence by the identification of $\partial\sigma$ and $\mathbf{D}^{**}$. The external faces correspond to the faces bounded by ∂S_j's, and the internal faces do to the others. The external edges correspond to the edges on ∂S_j's, while the internal edges do to the others. The vertices on the circle ∂S_j correspond to 2-cells of $\mathbf{C}$ which touches p and faces S_j. Both are arranged in the same order.

The final decomposition is obtained by straightening each edges of $\widetilde{K}$. The straightening here is the device first to replace each internal edges by homotopic short cuts, and then to replace external edges by geodesic paths using their end points. The straight map we get is sup-

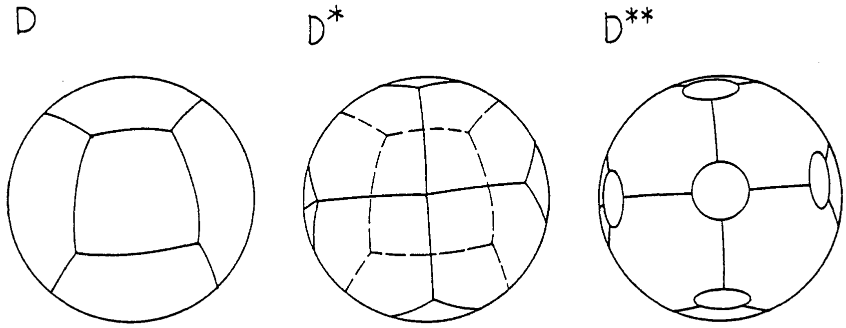

Fig. 3.

ported on the 1-skeleton $K^{(1)}$ at the beginning and there is no obvious reasons why it creates something good. The rest of this paper is to check the reason why it does.

The first step is to observe that the image of $\partial K^{(1)}$ by the straight map, which will be denoted by $\partial \Delta^{(1)}$, turns out to be a 1-skeleton of a convex polygonal decomposition of ∂N, denoted by $\partial \Delta$. This will be done in the next section. The second step starts by showing that the map can be straightened over the 2-skeleton $K^{(2)}$. The main step is then to observe that the straightened image of $K^{(2)}$, denoted by $\Delta^{(2)}$, turns out to be a 2-skeleton of a convex polyhedral decomosition of N, denoted by Δ. Since we define the final decomposition Δ from the lower dimensional skeletons, the accessories for Δ we use is not appropriate in fact, but will be justified by the end of the paper.

§4. Polygonal decomposition

We study the effect of straightening on the boundary in this section, and prove that the straightening defines a convex polygonal decomposition of ∂N equivalent to ∂K. The argument will be given mainly in the universal cover.

An internal edge of $\widetilde{K}$ bridges two components of $\partial \widetilde{N}$. Hence to each internal edge, assigned is a unique middle fence and a unique short cut. Recall that an internal edge is a dual to a 2-cell of $\widetilde{\mathbf{C}}$ which lies on this middle fence. The number of orbits of short cuts associated to 2-cells of $\widetilde{\mathbf{C}}$ by the action of $\pi_1(N)$ was finite. Let $\widetilde{\mathbf{R}}$ be the set of these short cuts, and $\mathbf{R} = \pi(\widetilde{\mathbf{R}})$ be the set of descending return paths in N. $\mathbf{R}$ is a finite set.

The straightening device at this stage is precisely to connect vertices of $\widetilde{\mathbf{R}}$ by geodesic paths if the corresponding two vertices in $\partial\widetilde{K}^{(0)}$ are joined by an external edge. We denote the resultant geodesic 1-complex by $\partial\widetilde{\Delta}^{(1)}$ in the abstract sense. The accessories in the notation should be ignored for the moment. The definition does not immediately tell us that $\partial\widetilde{\Delta}^{(1)}$ is an embedded 1-complex. What we obviously know by definition is that $\partial\widetilde{\Delta}^{(1)}$ is invariant under the action of $\pi_1(N)$, and that there is an equivariant graph isomorphism $h : \partial\widetilde{K}^{(1)} \to \partial\widetilde{\Delta}^{(1)}$.

Since the connection rule to build up $\partial\widetilde{\Delta}^{(1)}$ was followed by the rule for $\partial\widetilde{K}^{(1)}$, $\partial\widetilde{\Delta}^{(1)}$ should be very similar to $\partial\widetilde{K}^{(1)}$. The claim to be proved is that $\partial\widetilde{\Delta}^{(1)}$ is in fact a 1-skeleton of an invariant convex polygonal decomposition $\partial\widetilde{\Delta}$ of $\partial\widetilde{N}$, and h extends to an equivariant cellular isomorphism of $\partial\widetilde{K}$. The statement in ∂N is hence

Proposition 4.1. $\partial\Delta^{(1)} = \pi(\partial\widetilde{\Delta}^{(1)})$ *turns out to be a* 1*-skeleton of a convex polygonal decomposition* $\partial\Delta = \pi(\partial\widetilde{\Delta})$ *of* ∂N *equivalent to* ∂K.

To see this, we need a few observations about local structure of edges in $\partial\widetilde{\Delta}^{(1)}$. The first one is about the image of the boundary of a face of $\partial\widetilde{K}$.

Lemma 4.2. *The image of the boundary of a face of* $\partial\widetilde{K}$ *by* h *bounds a convex polygon on* S. *The canonical extension of* h *to the face preserves the orientation.*

Proof. Choose a face τ of $\partial\widetilde{K}$ and assume that it lies on a block σ with the center p. The cellular decomposition of $\partial\sigma$ was described by $\mathbf{D}^{**}$. The external face τ is identified with a face bounded by a metric circle ∂S on S^2_∞. The center q of ∂S is the terminal point of an extension of the shortest path from p to S.

Label the vertices of τ by v_j with $j = 0, 1, ..., n-1$ in counter-clockwise order. Each vertex is a projected image of a dual vertex to a 2-cell in $\widetilde{\mathbf{C}}$ touching p and facing S. Hence we also label the 2-cell of $\widetilde{\mathbf{C}}$ corresponding to v_j by F_j.

Each F_j is on the middle fence of a short cut from a point on S since F_j faces S. Hence we let its starting point on S by w_j. Because of the definition of labeling, any adjacent w_j's are joined by an edge in $\partial\widetilde{\Delta}^{(1)}$. $h(\partial\tau)$ is then a 1-complex formed by geodesic paths $w_j w_{j+1}$ with $j = 0, 1, ..., n-1$, where j counts modulo n as usual.

We show that the vertices $w_{j_0}, w_{j_1}, w_{j_2}$ span a triangle $\triangle w_{j_0} w_{j_1} w_{j_2}$, and its orientation assigned by how the vertices round induces the counterclockwise orientation on S as long as $j_0 < j_1 < j_2$ up to cyclic permutation. Then using this property, we will get the conclusion by contradiction.

Identify S with the 2-dimensional Poincaré disk and q with the origin. The middle fences containing $F_{j_0}, F_{j_1}, F_{j_2}$ respectively are orthogonally projected to three open metric disks $B_{j_0}, B_{j_1}, B_{j_2}$ on S including the origin. The vertices $w_{j_0}, w_{j_1}, w_{j_2}$ are the centers of these disks. The outside of ∂S is reflected into the inside by the orthgonal projection to S. The picture of the projection is shown in Figure 4.

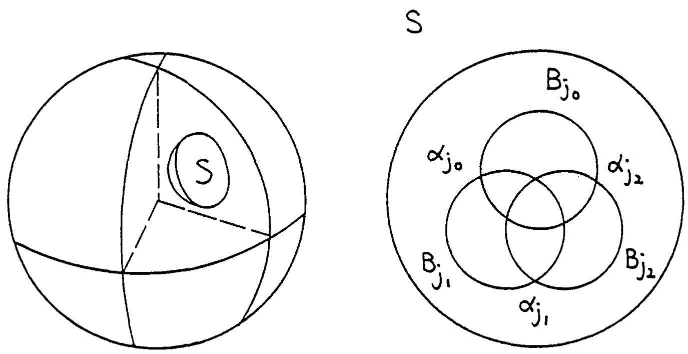

Fig. 4.

By the convexity of $\mathbf{D}$, B_{j_0}, B_{j_1} and B_{j_2} are arranged in counterclockwise order as in the second picture in Figure 4. We named the intersections of the boundary of balls as in the figure. Then α_{j_0}, α_{j_1}, and α_{j_2} determine the oriented triangle $\triangle \alpha_{j_0} \alpha_{j_1} \alpha_{j_2}$ inducing the counterclockwise orientation on S.

Here is an elementary geometry. Let $\gamma_{j_0}, \gamma_{j_1}, \gamma_{j_2}$ be the bisectors to the segments $\alpha_{j_2}\alpha_{j_0}$, $\alpha_{j_0}\alpha_{j_1}$ and $\alpha_{j_1}\alpha_{j_2}$ on S respectively. These three lines meet at the center β of the circumscribed circle of the triangle $\triangle \alpha_{j_0}\alpha_{j_1}\alpha_{j_2}$. w_j is on γ_j, where $j = j_0, j_1, j_2$. Since $B_{j_0}, B_{j_1}, B_{j_2}$ do not contain α_{j_1}, α_{j_2}, α_{j_0} respectively, the direction of the vector βw_j is the same as that of the outward vector from the triangle $\triangle \alpha_{j_0}\alpha_{j_1}\alpha_{j_2}$ along γ_j. Hence the centers $w_{j_0}, w_{j_1}, w_{j_2}$ are arranged in counterclockwise order from the viewpoint β, and determines an oriented triangle $\triangle w_{j_0} w_{j_1} w_{j_2}$ inducing the counterclockwise orientation on S.

Suppose now that the union of geodesic paths $w_j w_{j+1}$ with $j =$

$0, ..., n-1$ does not bound a convex polygon on S. Then since any three vertices determines a nondegenerate triangle, there are suffices $0 \leq j_3, j_4 \leq n-1$ so that $w_{j_3}w_{j_3+1}$ meets the biinfinite extension of $w_{j_4}w_{j_4+1}$ at an interior point. Then the induced orientations on S by the triangles $\triangle w_{j_3}w_{j_4}w_{j_4+1}$ and $\triangle w_{j_3+1}w_{j_4}w_{j_4+1}$ are different each other. This contradicts what we have proved.

Since the vertices v_j's of τ and the corresponding vertices w_j's of the convex polygon mapped by h are both arranged in counterclockwise order, a canonical extension of h preserves the orientation. Q.E.D.

We need one more observation about the structure around the vertex of $\partial\widetilde{K}$. Label all the edges coming to the vertex of $\partial\widetilde{K}$. Then by the connection rule of $\partial\widetilde{\Delta}^{(1)}$, this labeling is canonically conveyed to the labeling of the edges of $\partial\widetilde{\Delta}^{(1)}$ which terminate at the corresponding vertex.

Lemma 4.3. *The counterclockwise orders of the labeling at a vertex of $\partial\widetilde{K}^{(1)}$ and the corresponding vertex of $\partial\widetilde{\Delta}^{(1)}$ are the same up to cyclic permutation.*

Proof. Choose a vertex v of $\partial\widetilde{K}$ and assume that it lies on a block σ. The cellular decomposition of $\partial\sigma$ is described by $\mathbf{D}^{**}$. v is identified with a vertex on the metric circle ∂S.

Choose an adjacent vertex v' to v on the same circle ∂S. v and v' correspond to 2-cells F and F' in $\widetilde{\mathbf{C}}$ facing S and touching the center of σ. Recall that the adjacency is reflected by the property that these 2-cells F and F' have a common 1-cell. Denote by w and w' the vertices of $\partial\widetilde{\mathbf{R}}$ corresponding to v and v' respectively. Here is a geometric relation between the adjacency of v, v' and w, w'. The middle fence L containing F has an intersection line l with the middle fence L' containing F'. F is orthogonally projected to a convex polygon P on S and l is to a geodesic l_S which is an biinfinite extension of an edge of P. The plane determined by short cuts from w and w' is orthogonal to both L and L', and in particular to l. Hence the geodesic path connecting w and w' extends to a biinfinite path ω orthogonal to l_S.

What we have seen is that to each pair of v and v', and hence to each edge coming to v, associated is an biinfinite extension l_S of an edge of P, and that w' lies on the geodesic ω through w and orthogonal to l_S. Furthermore, though the vertices w and w' may not be separated by l_S, the vector from w to w' is directed towards the component of $S - l_S$ not containing P, as the vector from v to v' obviously is.

Now identify S with the 2-dimensional Poincaré disk. The biinfinite extensions of the edges of P determine a line configulation on S. Each line inherits a label from the associated edge coming to v. To each labelled line, we assign the orthogonal ray from v endowed with the same label. The counterclockwise order of the labeling for orthogonal rays is the same as that for edges of $\partial\widetilde{K}$ coming to v.

Then for each point on S, draw orthogonal rays to the geodesic lines again keeping the outward direction from P. Then the assignment of the counterclockwise order of the labeling of rays is a continuous function on S to the set of cyclic orders possibly with singularities. The singularity occurs only if two rays coincide. This may happen when two geodesic lines are ultra parallel. However in this case, the direction of associated two rays must be opposite since the region bounded by such lines contains a convex polygon P. Hence this continuous function has no singularities with discrete image. In particular, the order at w is the same as one at v. Q.E.D.

Proof of Proposition 4.1. By Lemma 4.2, extending a graph isomorphism $h : \partial\widetilde{K}^{(1)} \to \partial\widetilde{\Delta}^{(1)}$, we get a map h (still using the same notation) of $\partial\widetilde{K}$ by assigning to each face of $\partial\widetilde{K}$ a polygon bounded by corresponding edges of $\partial\widetilde{\Delta}^{(1)}$. Here h is a local homeomorphism on the interior of faces. Since h preserves the orientation for each face, it must be a homeomorphism also around edges. Lemma 4.3 shows that the corners of convex polygons fill up a neighborhood of the vertices. Hence h is a local homeomorphism also around the vertices. It is easy to see that h is surjective. Since the image is simply connected, h is a global homeomorphism.

$\partial\widetilde{\Delta}^{(1)}$ now determines a convex polygonal decomposition $\partial\widetilde{\Delta}$ of $\partial\widetilde{N}$. The decomposition is invariant under the action of $\pi_1(N)$, and the map h can be chosen to be equivariant. Hence it determines a convex polygonal decomponsition $\partial\Delta = \pi(\partial\widetilde{\Delta})$ of ∂N with a descending equivalence from ∂K to $\partial\Delta$. Q.E.D.

§5. Polyhedral decomposition

In this section, we study the effect of straightening in the interior and finish to prove that the straightening determines a convex polyhedral decomposition of N, which we promised to denote by Δ. The argument will be given again mainly in the universal cover.

The map $h : \partial\widetilde{K}^{(1)} \to \partial\widetilde{\Delta}^{(1)} \subset \partial\widetilde{N}$ we had at the beginning was a graph isomorphism. The main claim in §4 was that h extends to a cellular map h on $\partial\widetilde{K}$ to $\partial\widetilde{\Delta}$. It obviously further extends as a cellular isomorphism to $h : \partial\widetilde{K} \cup \widetilde{K}^{(1)} \to \partial\widetilde{\Delta} \cup \widetilde{\mathbf{R}} \subset \widetilde{N}$. We then will see first that the map h extends as a straight map over the 2-skeleton $\widetilde{K}^{(2)}$, showing that the image of the boundary of each internal face of $\widetilde{K}$ spans a geodesic polygon. Namely, $\partial\widetilde{\Delta} \cup \widetilde{\mathbf{R}}$ extends to a geodesic 2-complex $\widetilde{\Delta}^{(2)}$ in $\widetilde{N}$ in the abstract sense.

Lemma 5.1. *The image of the boundary of an internal face of $\widetilde{K}$ by h bounds a right angle polygon on a geodesic plane in $\widetilde{N}$.*

Proof. Choose an internal face τ and assume that it lies on a block σ. The cell decomposition of $\partial\sigma$ was described in $\mathbf{D}^{**}$ by identifying the center p of σ with the origin of the 3-dimensional Poincaré disk. There are metric circles $\partial S_1, ..., \partial S_m$ on S^2_∞ which are the boundaries of the nearest components of $\partial\widetilde{N}$ from p. The centers $q_1, ..., q_m$ of these metric circles are also the endpoints of the rays extending the shortest path from the origin to the component S_j. The circles $\partial S_1, ..., \partial S_m$, having the same radius, lie in the complement of the cut locus $\mathbf{D}$ of $\{q_1, ..., q_m\}$ on S^2_∞.

The face τ is identified with a face not bounded by ∂S_j's. We rearrange ∂S_j's so that $\partial\tau$ passes through $\partial S_1, \partial S_2, ..., \partial S_k$ in counterclockwise order. τ contains a vertex u of a cut locus $\mathbf{D}$. Recall as we noted in the description of $\mathbf{D}$ and $\mathbf{D}^*$ that every ∂S_j has the same distance from u. In particular, there is a circle ∂H on S^2_∞, bounding a geodesic plane H in the 3-dimensional Poincaré disk, that intersects orthogonally to each $\partial S_1, ..., \partial S_k$ simultaneously. Moreover, ∂H passes through $\partial S_1, ..., \partial S_k$ in counterclockwise order also.

$h(\partial\tau)$ is a piecewise geodesic whose bent occurs only at the end of external and hence internal edges. Each internal edge is mapped to the short cut between S_j and S_{j+1}. It must lie on the plane H since it intersects both S_j and S_{j+1} orthogonally. In particular, the image of internal edges is on a geodesic plane H. The image of external edges is on S_j's and on H since the intersection of S_j and H is a geodesic passing two end points of the short cuts. It is then obvious by the order of intersections to ∂S_j's that $h(\partial\tau)$ bounds a convex polygon on H.

Q.E.D.

Denote by $\widetilde{\Delta}^{(2)}$ the collection of the straight image of each internal faces by Lemma 5.1 and $\partial\widetilde{\Delta} \cup \widetilde{\mathbf{R}}$. The accessories in this notation should be ignored for the moment. The definition does not immediately tell us that $\widetilde{\Delta}^{(2)}$ is an embedded 2-complex. What we obviously know by definition is that $\widetilde{\Delta}^{(2)}$ is invariant under the action of $\pi_1(N)$, and that there is an equivariant cellular isomorphism $h : \widetilde{K}^{(2)} \to \widetilde{\Delta}^{(2)}$ which extends the original h. The claim to be proved is that $\widetilde{\Delta}^{(2)}$ is in fact a 2-skeleton of an invariant convex polyhedral decomosition $\widetilde{\Delta}$ of $\widetilde{N}$, and h extends to an equivariant cellular isomorphism of $\widetilde{K}$. The statement in N is our final goal.

Proposition 5.2. *$\Delta^{(2)} = \pi(\widetilde{\Delta}^{(2)})$ turns out to be a 2-skeleton of a convex polyhedral decomposition $\Delta = \pi(\widetilde{\Delta})$ of N equivalent to K.*

We have shown so far that if we restrict the map h to the set of external faces or to each internal face, then h is an embedding. What we still do not know is if the image of some internal faces intersect. To see our final proposition, we proceed further to a local study.

Lemma 5.3. *The image of the boundary of a block of $\widetilde{K}$ by h bounds a convex polyhedron in $\widetilde{N}$.*

Proof. Choose a block σ and recall that the cell decomposition of $\partial\sigma$ is described by $\mathbf{D}^{**}$ on S^2_∞. Assigned to each external face was a geodesic boundary S_j, and assigned to each internal face τ_i now by Lemma 5.1 is a geodesic plane H_i in $\mathbf{H}^3$. Using this description, we will define a continuous deformation $\{h_t\}$ of a restriction of h to $\partial\sigma$, $h|_{\partial\sigma} = h_0$, so that it eventually pushes the image of internal faces out to S^2_∞. Then by referring to the fact that $h_{\pi/2}$ is a homeomorphism, we will establish the stable cellularity of h_t to conclude the claim.

For each internal face τ_i, a neighborhood of $h_0(\tau_i)$ in $h_0(\partial\sigma)$ is contained in one side of $\mathbf{H}^3$ separated by H_i. We call the other side of H_i outwards. The outside of S_j's is similarly defined using the image of external faces. Let ${H_i}^t$ be the equidistant surface outside of H_i with the distance $\int_0^t \sec\theta\, d\theta$. This is not a geodesic plane but is a surface which intersects $H_i = {H_i}^0$ at S^2_∞ with dihedral angle t. It can be seen also as an intersection of an euclidean metric sphere with the Poincaré disk meeting the unit sphere S^2_∞ with dihedral angle t. The angle t varies from 0 to $\pi/2$. As t increases, ${H_i}^t$ is gradually pushed out towards S^2_∞.

To define the image of an internal face $\tau_0 = \tau$, let us rearrange τ_i's in such a way that $\partial\tau$ passes through $h_0(\tau_1)$, S_1, $h_0(\tau_2)$, S_2, ..., $h_0(\tau_k)$ and S_k in cyclic order. $h_0(\tau)$ and $h_0(\tau_i)$ meet on the intersection of ${H_0}^0 = H^0$ and ${H_i}^0$. Take two internal faces $h_0(\tau)$ and $h_0(\tau_{i_0})$ having a common internal edge, and identify the edge with a segment on the z-axis in the upper half space model so that it meets S_{i_0} at the bottom end. See Figure 5 which shows the situation locally. ${H_{i_0}}^0$ is a geodesic plane

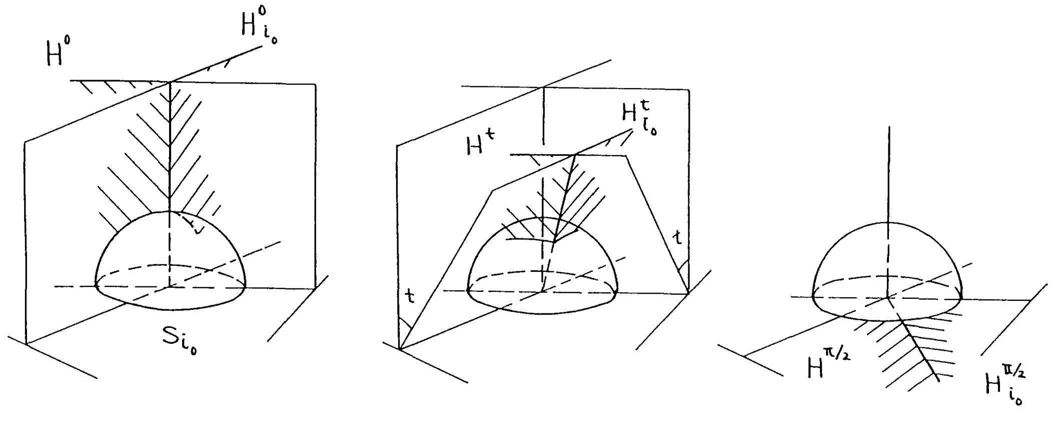

Fig. 5.

containing $h_0(\tau_{i_0})$. $H_0{}^t$ and $H_{i_0}{}^t$ are described in the second picture. In this coordinate, they are euclidean hyperplanes through the origin.

The original $h_0(\tau)$ is a convex polygon on H^0 bounded by the intersections with $H_i{}^0$'s and S_j's where $i, j = 1, ..., k$. As t increases, this region is gradually slided to again a convex region on H^t bounded by the intersections with $H_i{}^t$'s and S_j's, and eventually reaches to a circular polygon on S^2_∞. This bounded region on each H^t is the image of τ by h_t. We have not ruled out the possibility that H^t intersects S_j for some $j > k$, but it will turn out that this never happen.

We next describe how to map the external faces. The trace of the deformation of $H_i{}^t$'s on the external boundary S_{i_0} viewed from the above is described in Figure 6. The image of an external face on S_{i_0} by h_t is a convex region on S_{i_0} bounded by the intersections with $H_i{}^t$'s. As t increases, the region is getting enlarged keeping convexity and finally fills up S_{i_0}.

h_t is obviously a continuous deformation for $0 \leq t < \pi/2$, and is still continuous at $t = \pi/2$ if we topologize $\mathbf{H}^3 \cup S^2_\infty$ as a 3-ball. What is saved in this deformation is the property that h_t is an embedding on the set of external faces or on each internal face.

Modify $h_{\pi/2}$ a bit to $\widehat{h}_{\pi/2} : \partial\sigma \to S^2_\infty$ by pushing each S_j outward to the disk on S^2_∞ bounded by ∂S_j. We claim that $\widehat{h}_{\pi/2}$ and hence $h_{\pi/2}$, and moreover h_t with t near $\pi/2$ is a homeomorphism. $\widehat{h}_{\pi/2}$ is a local homeomorphism on the interior of each faces of $\partial\sigma$ by the definition. It is also a local homeomorphism around edges and around vertices by the definition of h_t (see Figures 5, 6). Hence it is a local homeomorphism to S^2_∞. Since $\partial\sigma$ is compact and the image is simply connected, it must be

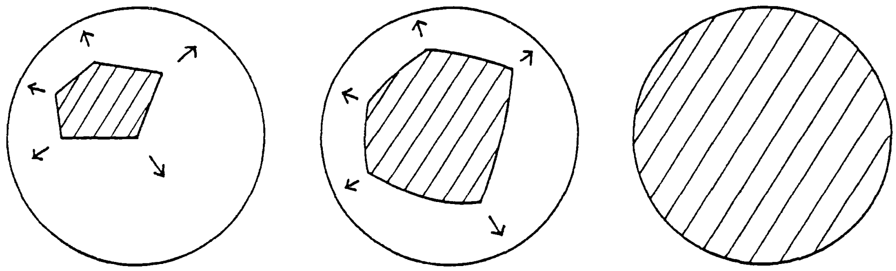

Fig. 6.

a homeomorphism. $h_{\pi/2}$ is not quite different from $\widehat{h}_{\pi/2}$ and is clearly a homeomorphism since so is $\widehat{h}_{\pi/2}$. The bent of the image by h_t is mild for t near $\pi/2$, and therefore, h_t is also necessarily to be a homeomorphism up to some moment.

For any $0 \leq t < \pi/2$, each ${H_i}^t$ separates $\mathbf{H}^3$ into a convex inward region and its complementary outward region. The intersections of any two of ${H_i}^t$'s look quite simple and are classified by the intersection of their boundaries on S^2_∞. If the intersection on S^2_∞ is nonempty and transversal, then surfaces intersect transversely for all t. If the intersection on S^2_∞ is empty, then as t decreases, the intersection of surfaces is gradually changed from a circle, a point of contact to an empty set. It may be empty from the beginning. If the boundaries of surfaces on S^2_∞ are the same, then the intersection is empty for $0 < t < \pi/2$ unless they are the same surface. The transversality of the intersections of ${H_i}^t$'s is missed only when either different surfaces without intersections for $t > 0$ coinside at $t = 0$, or surfaces with circular intersection at the beginning contact at some moment.

We thus have a family of very visible stratifications of $\mathbf{H}^3$ defined by the intersections of ${H_i}^t$'s and S_j's. The intersection of their convex inward regions in $\mathbf{H}^3$ is a compact convex stratum. The convex stratum bounded by ${H_i}^t$'s and S_j's is certainly nonempty for t near $\pi/2$. On the other hand, by the continuity of the deformation, $h_t(\partial\sigma)$ bounds a locally convex and hence a convex region in $\mathbf{H}^3$ also for t near $\pi/2$. It is the same as the stratum bounded by ${H_i}^t$'s and S_j's because of its convexity. Hence h_t is a cellular map : $\partial\sigma \to h_t(\partial\sigma)$ with respect to the stratification of $\mathbf{H}^3$ for t close enough to $\pi/2$.

In this stratification, every surface, that is any one of ${H_i}^{t}$'s and S_j's, plays a role to determine a face of the convex stratum for t near $\pi/2$. A consequence to this stable property is that the map h_t is cellular with respect to the stratification of $\mathbf{H}^3$ on an open interval in $[0, \pi/2]$ including $\pi/2$. It also concludes that ${H_i}^{t}$'s are different each other for all $0 < t < \pi/2$.

As t decreases form $\pi/2$, this compact convex stratum is continuously compressed. If the stratum does not degenerate and the structure of the stratification on the boundary is kept in the deformation up to $t = 0$, then we are done since h_0 turns out to be an embedding and the image bounds a convex polyhedron.

Otherwise, there is the first moment $t_0 \geq 0$ at which h_t fails to be cellular since the cellularity is open. Then by continuity of h_t, $h_{t_0}(\partial\sigma)$ either still bounds a convex region, which is the convex stratum bounded by ${H_i}^{t_0}$'s and S_j's, or degenerates to a convex set on some geodesic plane in $\mathbf{H}^3$. In the first case, the surfaces still in fact intersect transversely at t_0, but some edge of the stratification on the boundary of the convex stratum degenerates. Then two vertices must be close each other if t is near t_0. However the vertices of the stratification on $h_t(\partial\sigma)$ for $t > t_0$ is the image of the vertices of $\partial\sigma$ by the definition of t_0, and hence their mutual distance is bounded away from zero by the definition of h_t. This is contradiction. In the second case, the faces of $\partial\sigma$ are mapped on the same geodesic plane by h_{t_0}. Hence three vectors from a vertex of $\partial\sigma$ to adjacent vertices in the image of h_{t_0} must be linearly dependent. However they are always independent by the definition of h_t. This is also a contradiction. Q.E.D.

Proof of Proposition 5.2 *and Theorem.* Assigning to each block of $\widetilde{K}$ a polyhedron bounded by the image of its boundary, we get a map from $\widetilde{K}$ extending $h : \widetilde{K}^{(2)} \to \widetilde{\Delta}^{(2)}$. It is a local homeomorphism on the interior of blocks. We have already seen that it is a homeomorphism on the boundary. Hence it is a local homeomorphism everywhere since there is no vertices in the interior and every cell meets the boundary. The surjectivity is obvious. Since the image is simply connected, it must be a homeomorphism.

$\widetilde{\Delta}^{(2)}$ now determines a convex polyhedral decomposition $\widetilde{\Delta}$ of N. The decomposition is invariant under the action of $\pi_1(N)$ and the map can be chosen to be equivariant. Hence it determines a convex polyhedral decomposition Δ on N with a descending equivalence from K to Δ.

Q.E.D.

References

[1] Epstein, D. B. A. and Penner, R., Euclidean decomposition of noncompact hyperbolic manifolds, J. Differential Geom., **27** (1988), 67–80.
[2] Kojima, S. and Miyamoto, Y., The smallest hyperbolic 3-manifolds with totally geodesic boundary, J. Differential Geom., **34** (1991), 175–192.
[3] Kojima, S., Polyhedral decomposition of hyperbolic manifolds with boundary, preprint, Tokyo Inst. Tech. (1991).
[4] Thurston, W. P., "The Geometry and Topology of 3-Manifolds", Lecture Note, Princeton University, 1977.

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Aspects of Low Dimensional Manifolds
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Behavior of Knots under Twisting

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§1. Introduction

This paper is a continuation of [6] in the study of the twist move of knots. First we recall some notations. Let K be an unoriented smooth knot in the oriented 3-sphere S^3, and V a solid torus endowed with a preferred framing which contains K in its interior and satisfies $w_V(K) \geq 2$. ($w_V(K)$ denotes the geometric intersection number of K and a meridian disk of V.) Let f_n be an orientation preserving homeomorphism of V satisfying $f_n(\text{meridian}) = (\text{meridian})$ and $f_n(\text{longitude}) = (\text{longitude}) + n(\text{meridian})$ in $H_1(\partial V)$. (We shall not distinguish notationally between a homeomorphism and an isomorphism on a homology group induced by it.) We denote the knot $f_n(K)$ in S^3 by $K_{V,n}$. If there exsists an orientation preserving homeomorphism of S^3 carrying K_1 to K_2, then we write $K_1 \cong K_2$. Note that $K_1 \cong K_2$ is the same as saying that K_1 and K_2 are ambient isotopic in S^3. We note that for a given knot K, a solid torus V and an integer n determine a unique knot type. For a given knot K, we have an abundant solid tori which contain K to carry out a twist move. Sect.2 is directed towards the following question : for a given knot K, is it possible to obtain the same knot by twistings along distinct solid tori from K? Concerning the case when an original knot is trivial, we give Example 2.1 and Theorem 2.2. In the case when both solid tori are knotted, we shall give Theorem 2.6 and Examples (see Figures 4, 5). In Sect.3, the behavior of Gromov invariants under twistings will be studied. In Sect.4, we study the effects of twistings on primeness of knots. Throughout this paper $N(X)$, ∂X and $\operatorname{int} X$ denote the tubular neighborhood of X, the boundary of X and the interior of X respectively.

§2. On twistings along distinct solid tori

Let V_1 and V_2 be solid tori containing a knot K. We write $V_1 \cong V_2$ provided that there exists an orientation preserving homeomorphism f

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of S^3 such that $f(V_1) = V_2$, $f(K) = K$. Note that $K_{V_1,n} \cong K_{V_2,n}$ holds for any integer n when $V_1 \cong V_2$. To begin with, we give an example as follows.

Example 2.1. In Figure 1, $V_1 \not\cong V_2$ because the winding number of O in V_1 equals 2 and that of O in V_2 equals 3. But $O_{V_1,-1} \cong O_{V_2,-1}$.

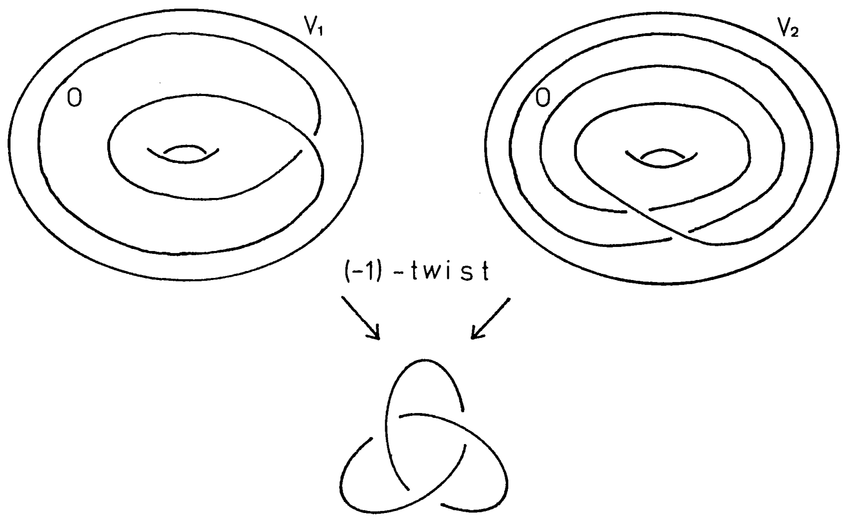

Fig. 1.

For twistings of the unknot, we prove the following theorem.

Theorem 2.2. *Let O be the unknot and V_i $(i = 1, 2)$ a solid torus containing O with $w_{V_i}(O) \geq 1$. If $O_{V_1,n_j} \cong O_{V_2,n_j}$ holds for infinitely many integers n_j, then $V_1 \cong V_2$.*

To prove this, we prepare some lemmas. Let V be a solid torus containing a knot K in its interior with $w_V(K) \geq 1$. Then $V - \operatorname{int} N(K)$ is a boundary irreducible Haken manifold. Consider the torus decomposition of $V - \operatorname{int} N(K)$ in the sense of Jaco-Shalen [3] and Johannson [4]. Combining Thurston's uniformization theorem [7], they assert that $V - \operatorname{int} N(K)$ is uniquely decomposed by a family of tori into pieces each of which is Seifert fibred or admits a complete hyperbolic structure of finite volume in its interior. Moreover each Seifert piece is one of torus knot spaces, cable spaces and composing spaces (see [3]). We denote the piece which contains ∂V by P_0, and the piece containing $\partial N(K)$ by P.

If V is an unknotted solid torus in S^3 which contains K, then $S^3 - \operatorname{int} V$ is also a solid torus, and we denote it by V_J. When we perform $(-1/n)$-Dehn surgery on the unknot J (the core of V_J), then the result is also S^3 and the image of K becomes a new knot K_n^*. The next lemma is an interpretation of a twisting.

Lemma 2.3. $K_{V,n} \cong K_n^*$.

It follows that $S^3 - \operatorname{int} N(K_{V,n})$ is homeomorphic to $V_J \bigcup_{m_J = \ell m^{-n}} (V - \operatorname{int} N(K))$.

Lemma 2.4 ([6]). *If P_0 is a cable space in which a regular fibre is presented by $\ell^p m^q$ $(p \geq 2)$, then $V_J \bigcup_{m_J = \ell m^{-n}} P_0$ is a Seifert fibred manifold with two exceptional fibres of indices p, $|pn+q|$. The dual knot of J, J_n^* in $V_J \bigcup_{m_J = \ell m^{-n}} P_0$ is a fibre of index $|pn+q|$.*

Lemma 2.5 ([6]). *If P_0 is hyperbolic, then there exists $N_{V,K}$ such that $V_J \bigcup_{m_J = \ell m^{-n}} P_0$ is also hyperbolic for $|n| \geq N_{V,K}$. Moreover for any $\varepsilon > 0$, there exsits $N_{V,K}(\varepsilon)$ such that J_n^* is a closed geodesic of length $< \varepsilon$ in $V_J \bigcup_{m_J = \ell m^{-n}} P_0$ for $|n| \geq N_{V,K}(\varepsilon)$.*

Proof of Theorem 2.2. If $w_{V_1}(O) = 1$ (resp. $w_{V_2}(O) = 1$), then by the assumption and Theorem 4.2 in [6], $w_{V_2}(O) = 1$ (resp. $w_{V_1}(O) = 1$) must hold. In this case O is a core of both V_1 and V_2, so we have $V_1 \cong V_2$. Assume $w_{V_i}(O) \geq 2$ and consider the torus decomposition of $V_i - \operatorname{int} N(O)$. Let P_i be the piece containing ∂V_i. Since O is trivial, P_i can not be a composing space. We remark that V_i is necessarily unknotted by the assumption (see [9]), and $S^3 - \operatorname{int} V_i$ is also a solid torus V_{J_i}. Then we can characterize the core of V_{J_i} in $E(O_{V_i,n}) = V_{J_i} \bigcup_{m_{J_i} = \ell_i m_i^{-n}} (V_i - \operatorname{int} N(O))$, which is denoted by $J_{i,n}^*$, as follows. There exists a constant $N_{V_i,O}$ such that $J_{i,n}^*$ is an exceptional fibre of unique maximal index or a unique shortest closed geodesic in $E(O_{V_i,n})$ by Lemmas 2.4 and 2.5 for $|n| \geq N_{V_i,O}$. Now we take n as above. Let f be an orientation preserving homeomorphism of S^3 sending $O_{V_1,n}$ to $O_{V_2,n}$. Then by an ambient isotopy, we may assume f maps $N(O_{V_1,n})$ to $N(O_{V_2,n})$ and maps $J_{1,n}^*$ to $J_{2,n}^*$ (see also [8]). From this, we see that $f|_{V_1}$ is an orientation preserving homeomorphism from V_1 to V_2 with $f|_{V_1}(O) = O$. Moreover $f|_{V_1}$ maps $\ell_1 m_1^{-n}$ to $\ell_2^{\varepsilon} m_2^{-\varepsilon n}$ $(\varepsilon = \pm 1)$. This implies that $f|_{V_1}$ maps ℓ_1 to ℓ_2^{ε}. By extending $f|_{V_1}$ to S^3, we get a required homeomorphism. This completes the proof of Theorem 2.2.

Q.E.D.

If we require both V_1 and V_2 are knotted, the following result holds.

Theorem 2.6. *Let K be a knot in S^3 and V_i a knotted solid torus containing K. Suppose that $V_1 \subset V_2$ and the core C_1 of V_1 satisfies $w_{V_2}(C_1) \geq 2$ and $w_{V_1}(K) \geq 2$. Then $K_{V_1,m} \not\cong K_{V_2,n}$ for any pair $(m,n) \neq (0,0)$ (Figure 2).*

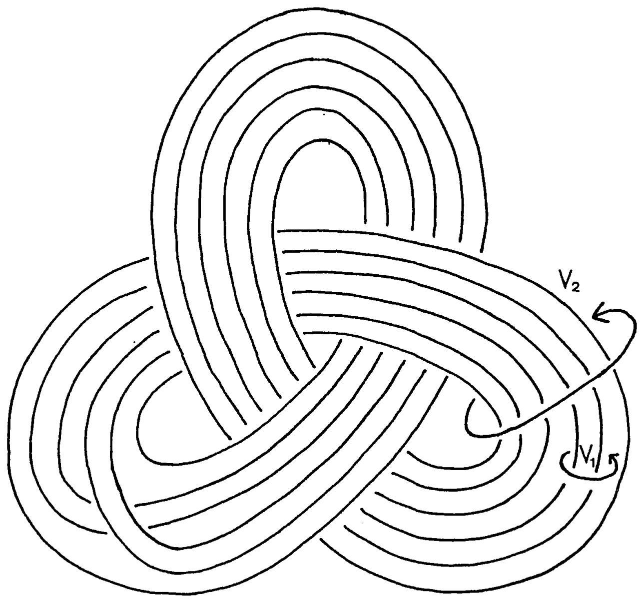

Fig. 2.

Proof. Let $f_m : V_1 \longrightarrow V_1$ and $g_n : V_2 \longrightarrow V_2$ be twist homeomorphisms with m-twist and n-twist respectively. By Theorem 2.1 in [6], $g_n(C_1) \not\cong C_1$ for any integer $n \neq 0$. Meanwhile $f_m(C_1) \cong C_1$ for any integer m. So the composition $g_n \circ f_m^{-1} : V_1 \longrightarrow g_n(V_1)$ sends C_1 to $g_n(C_1) \not\cong C_1$. We remark that C_1 and $g_n(C_1)$ are knotted in S^3, because they are geometrically essential in the knotted solid torus V_2. Also $g_n \circ f_m^{-1}$ satisfies $g_n \circ f_m^{-1}(K_{V_1,m}) = K_{V_2,n}$. Using Theorem [5], we can conclude $K_{V_1,m} \not\cong K_{V_2,n}$, if $n \neq 0$. In the case of $n = 0$, $K_{V_2,n} \cong K$ but $K_{V_1,m} \cong K$ holds only when $m = 0$ by Theorem 2.1 [6]. It follows that $K_{V_1,m} \not\cong K_{V_2,n}$ for any pair $(m,n) \neq (0,0)$. Q.E.D.

Remark. In the above theorem, the condition $w_{V_2}(C_1) \geq 2$ excludes the following trivial example.

Also in general, if both solid tori V_1 and V_2 are knotted then by Schubert's Satz 1 ([12]), we may assume one of the following occurs by

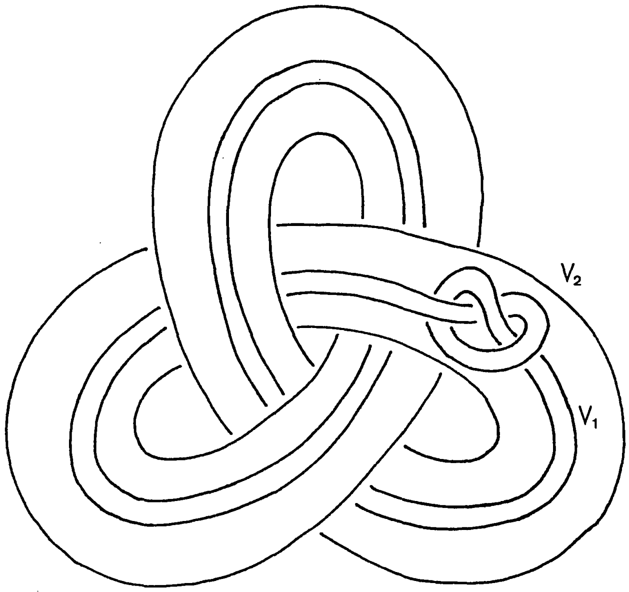

Fig. 3.

an ambient isotopy of S^3 which leaves K fixed. (1) $V_1 \subset V_2$ or $V_2 \subset V_1$, (2) $V_1 \cup V_2 = S^3$, and (3) there exists a solid torus W in $\operatorname{int} V_1 \cap \operatorname{int} V_2$ such that $w_{V_1}(C_W) = w_{V_2}(C_W) = 1$ for the core of C_W of W.

Theorem 2.6 corresponds to the case (1). As for cases (2) and (3), there exist inessential examples as in Figure 4 and Figure 5 respectively.

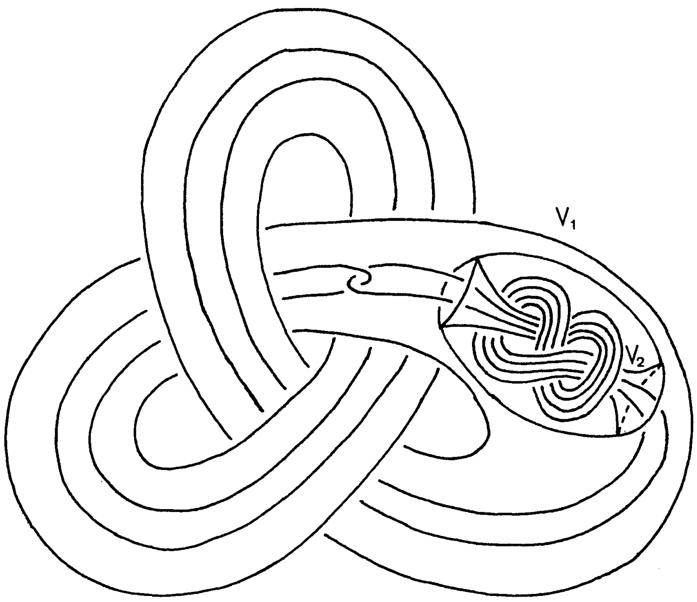

Fig. 4.

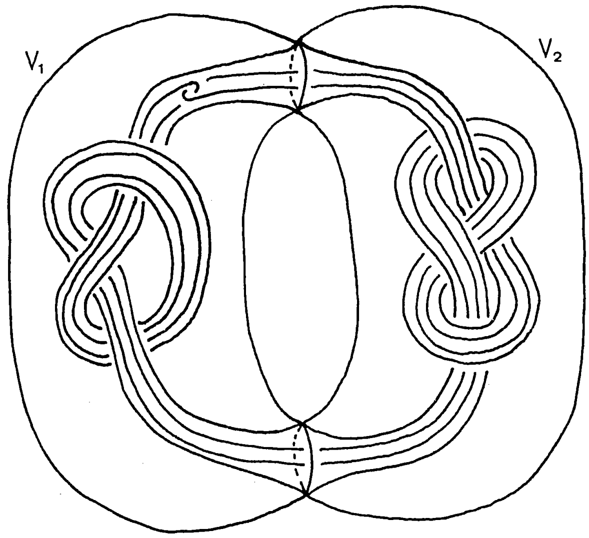

Fig. 5.

§3. Gromov invariants

The notion of the Gromov invariant of closed manifolds was introduced by Gromov [1]. In the 3-dimensional case, Thurston defined the Gromov invariant of compact 3-manifolds whose boundaries consists of tori [14]. In this section we shall study the Gromov invariant of the exterior of a knot K in S^3 which we simply call the Gromov invariant of K and we denote it by $\|K\|$. For the definition of the Gromov invariant, the reader is referred to [1], [14] and [13].

First we prove the following.

Theorem 3.1. *Let K be a knot in S^3 and V a knotted solid torus containing K. Then $\|K_{V,n}\| = \|K\|$ holds for any integer n.*

Proof. If $w_V(K) \leq 1$, then $K_{V,n} = K$ for any integer n. So we assume $w_V(K) \geq 2$. The exterior of $K_{V,n}$ ($K_{V,0} \cong K$) is described as $(S^3 - \operatorname{int} V) \bigcup_{h_n} (V - \operatorname{int} N(K))$ for some gluing homeomorphism h_n. Since V is knotted, $\partial(S^3 - \operatorname{int} V)$ is an incompressible torus. Also ∂V is an incompressible torus in $V - \operatorname{int} N(K)$ because $w_V(K) \geq 2$. Hence we have the following equality independent of n by Soma's theorem [13].

$$\|K_{V,n}\| = \|E((K_{V,n})\| = \|(S^3 - \operatorname{int} V) \amalg (V - \operatorname{int} N(K))\|.$$

It follows that $\|K_{V,n}\| = \|K\|$. Q.E.D.

Hence, in Theorem 2.6, $K_{V_1,m}$ and $K_{V_2,n}$ have the same Gromov invariants for any pair (m, n).

The following is straightforward from Theorem 3.1.

Corollary 3.2. *Suppose that K_1 and K_2 are knots with $\|K_1\| \neq \|K_2\|$. Then K_2 can not be obtained by a sequence of twistings along knotted solid tori from K_1.*

On the other hand, if V is unknotted we have:

Proposition 3.3. *Let O be the unknot in S^3. For any real number r, there exists an unknotted solid torus V containing O such that $\|O_{V,1}\| > r$.*

Proof. Consider a solid torus V as in Figure 6. Then in the exterior of $O_{V,1}$, there exist incompressible tori which decompose it into k figure eight knot spaces, 1 Whitehead link space and 1 composing space. Hence $\|O_{V,1}\| = 1/v_3(k\,\text{Vol(figure eight knot complement)} + \text{Vol(Whitehead link complement)})$, where v_3 is the volume of the regular ideal simplex (see [14] [13]). Thus the result holds for some integer $k > 0$. Q.E.D.

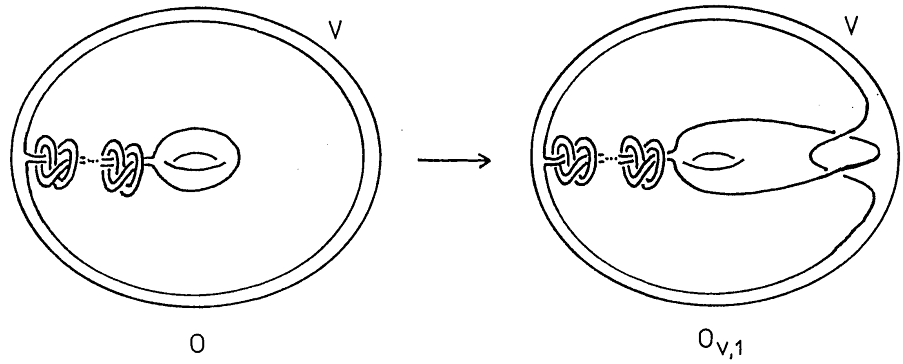

Fig. 6.

This also shows that for any knot K and any real number r, there exists an unknotted solid torus V such that $K_{V,1} > r$.

But the Gromov invariants behave as follows once V is fixed.

Proposition 3.4. *Let K be a knot in S^3 and V an unknotted solid torus containing K. Then $\|K_{V,n}\|$ is less than a constant $C_{V,K}$ for any integer n.*

Proof. We may assume $w_V(K) \geq 2$. If P_0 is a cable space, $\|K_{V,n}\|$ is constant for all but at most two integers n such that a regular fibre is presented by $\ell^p m$ for some p. If P_0 is a composing space, then twisting along V is reduced to that along a knotted solid torus W bounded by the torus ($\subset \partial P_0$) which separates K and ∂V (see Sublemma 3.7 [6]). Hence Theorem 2.1 in [6] implies the result. Suppose that P_0 is hyperbolic, by Lemma 2.3 $V_J \bigcup_{m_J=\ell m^{-n}} P_0$ is also hyperbolic for $|n| \geq N_{V,K}$. Then we have $\mathrm{Vol}(\mathrm{int}(V_J \bigcup_{m_J=\ell m^{-n}} P_0)) < \mathrm{Vol}(\mathrm{int}\, P_0)$ by Thurston's theorem (6.5.6 Theorem [14]), and from this we have the following inequality for $|n| \geq N_{V,K}$,

$$\begin{aligned}\|K_{V,n}\| &= 1/v^3\Big(\sum_{\substack{P_i:\text{hyperbolic}\\ i\neq 0}} \mathrm{Vol}(\mathrm{int}\, P_i) + \mathrm{Vol}(\mathrm{int}(V_J \bigcup_{m_J=\ell m^{-n}} P_0))\Big)\\ &< 1/v^3\Big(\sum_{\substack{P_i:\text{hyperbolic}\\ i\neq 0}} \mathrm{Vol}(\mathrm{int}\, P_i) + \mathrm{Vol}(\mathrm{int}\, P_0)\Big)\\ &= \|K \amalg J\|.\end{aligned}$$

Now we set $C_1 = \max\{\|K_{V,n}\| : |n| < N_{V,K}\}$ and we take $C_{V,K} = \max\{C_1, \|K \amalg J\|\}$, then $C_{V,K}$ is the required constant. Q.E.D.

Example 3.5 (Thurston [14]).

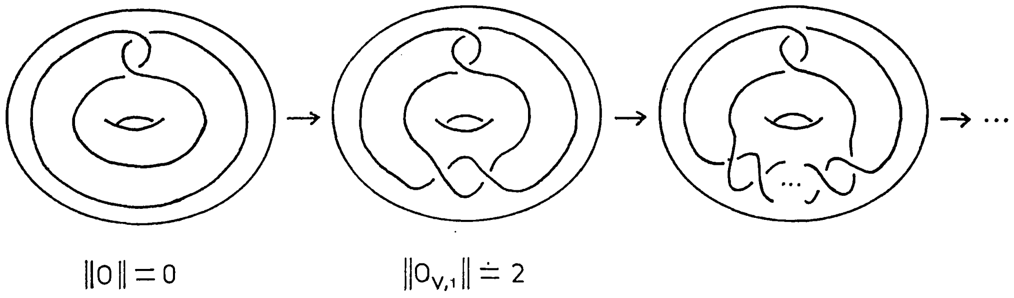

Fig. 7.

The Gromov invariants of these knots tend from below to a finite limit ($\doteq 3.6$).

§4. Primeness of knots under twistings

In this section, we investigate the effects of twistings on primeness of knots. To begin with, we consider the case when a twisting solid torus is knotted.

Theorem 4.1. *Let K be a knot in S^3 and V a knotted solid torus containing K. Then K is prime if and only if $K_{V,n}$ is prime for any integer n.*

Proof. We may assume $w_V(K) \geq 2$. Consider the torus decomposition of $V - \operatorname{int} N(K)$ and denote the piece containing $\partial N(K)$ by P. Suppose that K is a prime knot, then it turns out P is not a composing space. Now we consider the torus decomposition of $E(K_{V,n}) = (S^3 - \operatorname{int} V) \bigcup_{h_n} (V - \operatorname{int} N(K))$. In $E(K_{V,n})$, P is also a decomposing piece. It follows that $K_{V,n}$ is also prime for any integer n. Q.E.D.

If V is unknotted, then the following example exists.

Example 4.2. In Figure 8, K is a prime knot, but $K_{V,n}$ is a composite knot for any nonzero integer n.

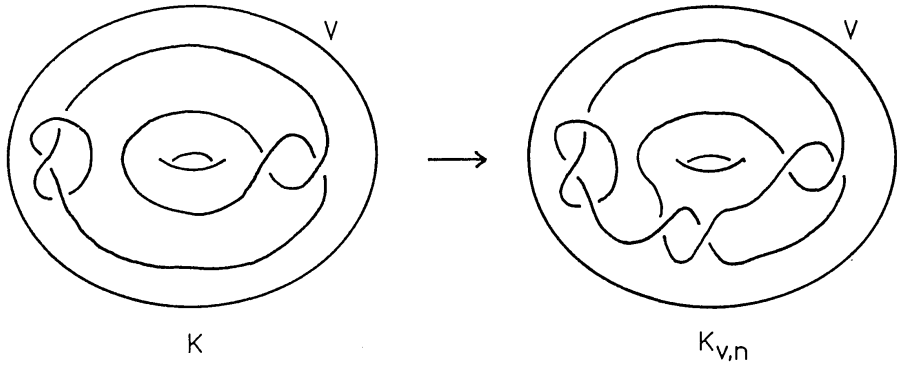

Fig. 8.

In this example K has a locally knotted arc in V (*i.e.* there is a 3-ball $B \subset V$ such that $(B,\ B \cap K)$ is a knotted ball pair). If K does not have a locally knotted arc in V, then we get the following.

Theorem 4.3. *Let V be an unknotted solid torus containing K without a locally knotted arc. Then $K_{V,n}$ is prime for all but at most finitely many integers n.*

Proof. Consider the torus decomposition of $V - \operatorname{int} N(K)$, and let P be a piece containing $\partial N(K)$ and P_0 a piece containing ∂V.

Sublemma. *Suppose that $K \subset V$ does not have a local knot. Then P can not be a composing space.*

Proof of Sublemma. Suppose that P is a composing space. Let T be a component of ∂P which does not separate ∂V and $\partial N(K)$. Note that T

bounds a nontrivial knot exterior E, and a regular fibre of P coincides a boundary of a meridian disk of $N(K)$. Hence we have a saturated annulus A' which joins T and $\partial N(K)$. Then $D' = A' \cup D$ becomes a meridian disk of $W = S^3 - \operatorname{int} E$. Since $K \cap D'$ and $K \cap D$ consist of one point, K has a locally knotted arc in V. This is a contradiction.

Q.E.D.

If P_0 is a cable space, $V_J \bigcup_{m_J=\ell m^{-n}} P_0$ is a (nontrivial) torus knot exterior except for at most only two integers n by Lemma 2.4. If P_0 is a k-fold composing space, then $V_J \bigcup_{m_J=\ell m^{-n}} P_0$ is a $(k-1)$-fold composing space for any integer n. Finally we consider the case when P_0 is hyperbolic. By Lemma 2.5, we see that $V_J \bigcup_{m_J=\ell m^{-n}} P_0$ is also hyperbolic except for at most finitely many integers n. It follows that in any case, $V_J \bigcup_{m_J=\ell m^{-n}} P_0$ is boundary irreducible Haken manifold. Now we divide into two cases depending upon whether $P = P_0$ or not. If $P = P_0$, then $V_J \bigcup_{m_J=\ell m^{-n}} P = V_J \bigcup_{m_J=\ell m^{-n}} P_0$ can not be a composing space by Sublemma and the above, and it becomes a decomposing piece in $E(K_{V,n})$. Thus $K_{V,n}$ is prime except for at most finitely many integers n. If $P \neq P_0$, then it turns out that P is still a decomposing piece in $E(K_{V,n})$. Since P is not a composing space, $K_{V,n}$ is prime except for at most finitely many integers n. Q.E.D.

Remark 4.4. Even if K does not have a locally knotted arc in V, there is an example such that $K_{V,n}$ is a composite knot for some integer n (see Figure 9).

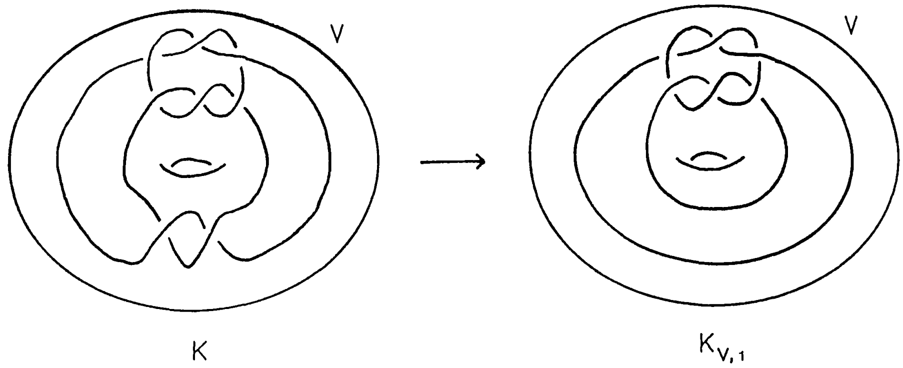

Fig. 9.

When an original knot is trivial, Scharlemann-Thompson [11], Eudave-Munoz and Gordon have shown the following result, which is a generalization of the theorem — "Unknotting number one knots are prime [10]".

Theorem 4.5 ([11]). *Let V be a solid torus containing the unknot O with $w_V(O) \leq 2$. Then $O_{V,n}$ is prime for any integer n.*

Since the unknot can not have a locally knotted arc, as an application of Theorem 4.3, we have the following.

Corollary 4.6. *Let V be a solid torus containing the unknot O. Then $O_{V,n}$ is prime for all but at most finitely many integers n.*

We conclude this paper with the following question.

Question. *Is the result of twisting of the unknot always prime?*

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References

[1] M. Gromov, Volume and bounded cohomology, Inst. Hautes Etudes Sci. Publ. Math., **56** (1983), 213–307.

[2] W. Jaco, "Lectures on three manifold topology", Conference board of Math. Science, Regional Conference Series in Math. 43. Amer. Math. Soc., 1980.

[3] W. Jaco and P. Shalen, "Seifert fibered spaces in 3-manifolds", Mem. Amer. Math. Soc. 220, 1979.

[4] K. Johannson, "Homotopy equivalences of 3-manifolds with boundaries", Lecture Notes in Math., Vol.761. Springer-Verlag, 1979.

[5] M. Kouno, On knots with companions, Kobe J. Math., **2** (1985), 143–148.

[6] M. Kouno, K. Motegi and T. Shibuya, Twisting and knot types, preprint.

[7] J. Morgan and H. Bass, "The Smith conjecture", Pure and Applied Math. Academic Press, 1984.

[8] K. Motegi, Homology 3-spheres which are obtained by Dehn surgeries on knots, Math. Ann., **281** (1988), 483–493.

[9] D. Rolfsen, "Knots and links", Mathematics Lecture Series, No.7. Publish or Perish, Berkeley, Calif., 1976.

[10] M. Scharlemann, Unknotting number one knots are prime, Invent. Math., **82** (1985), 37–55.

[11] M. Scharlemann and A. Thompson, Unknotting number, genus, and companion tori, Math. Ann., **280** (1988), 191–205.

[12] H. Schubert, Knoten und Vollringe, Acta Math., **90** (1953), 131–286.

[13] T. Soma, The Gromov invariant of links, Invent. Math., **64** (1981), 445–454.

[14] W. Thurston, "The geometry and topology of 3-manifolds", Lecture Note Princeton Univ., 1978.

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Polynomial Invariants of 2-Bridge Links through 20 Crossings

Taizo Kanenobu* and Toshio Sumi

In this paper, we calculate the homfly polynomial $P_L(v,z)$, Kauffman polynomial $F_L(a,z)$, Jones polynomial $V_L(t)$, Q polynomial $Q_L(z)$, 2-variable Conway polynomial $\nabla_L(t_1,t_2)$, and reduced Conway polynomial $\widetilde{\nabla}_L(z)$ of a 2-bridge link L with crossing number ≤ 20 and list all the pairs sharing the same polynomial invariants (Table 2). This paper is a continuation of [9], where these polynomial invariants except for the 2-variable Conway polynomial for 2-bridge knots through 22 crossings are calculated and all the pairs having the same polynomial invariants are listed. The total number of the links is 44,118, where we ignore the orientations of both a link and its ambient space. If we consider them, this amounts 175,788. The program is written in Turbo Pascal for the NEC PC-9801 Series as before.

We observe the following for 2-bridge links through 20 crossings:

Fact 1. *$P_L(v,z) = P_{L'}(v,z)$ iff $V_L(t) = V_{L'}(t)$ and $\widetilde{\nabla}_L(z) = \widetilde{\nabla}_{L'}(z)$.*

Fact 2. *If $P_L(v,z) = P_{L'}(v,z)$ and $P_{L^\wedge}(v,z) = P_{L'^\wedge}(v,z)$, then $\nabla_L(t_1,t_2) = \nabla_{L'}(t_1,t_2)$.*

Fact 3. *The number of links having the same homfly or Kauffman polynomial is at most two.*

Fact 4. *$P_L(v,z) = P_{L^\wedge}(v,z)$ iff $\nabla_L(t_1,t_2) = \nabla_{L^\wedge}(t_1,t_2)$ $\big(= -\nabla_L(t_1^{-1},t_2)\big)$.*

Here $L^\wedge$ is a 2-bridge link obtained from L by reversing the orientation of one of the 2 components. Facts 1 and 3 are the same as those in [9]. For Fact 3, we do not consider the pair of 2-bridge links L and

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L' which share the same Kauffman polynomial and have linking number zero (such as "3800 1669, 2429" of 19 crossing links in Table). For these links, it holds $F_L = F_{L'} = F_{L^\wedge} = F_{L'^\wedge}$. The "only if" part of Fact 4 cannot be deduced from only Table 2. We must check the 2-variable Conway polynomials. The example as in Fact 4 is constructed in [5, Theorem 9].

For the pair L and L', where $L \neq L', L'^\wedge$, sharing the same Q polynomial, the cases given in Table 1 occur, where the last column gives an example for each case from Table 2. For example, Case 5 indicates the pair such that $V_L = V_{L'}$ $(V_{L^\wedge} = V_{L'^\wedge})$, $P_L = P_{L'}$, $P_{L^\wedge} \neq P_{L'^\wedge}$, $F_L = F_{L'}$ $(F_{L^\wedge} = F_{L'^\wedge})$, $\nabla_/ = \nabla_{L'}$ $(\nabla_{L^\wedge} \neq \nabla_{L'^\wedge})$, $\widetilde{\nabla}_L = \widetilde{\nabla}_{L'}$, $\widetilde{\nabla}_{L^\wedge} \neq \widetilde{\nabla}_{L'^\wedge}$. Cases 3–5 explain Fact 2. Relating to Cases 2 and 3, we can construct the following examples:

(i) Arbitrarily many skein equivalent fibered 2-bridge links with the same 2-variable Conway polynomial ([5, Theorem 7]).

(ii) Arbitrarily many skein equivalent 2-bridge links which have mutually distinct 2-variable Conway polynomials ([6, Theorem 2]).

Relating to Cases 4–6, we can construct the following examples:

(iii) A pair of skein equivalent 2-bridge links with the same Kauffman polynomial but distinct 2-variable Conway polynomials ([6, Theorem 7]).

(iv) Arbitrarily many skein equivalent fibered 2-bridge links which have the same Kauffman and 2-variable Conway polynomial ([8, Theorem 2]).

Relating to Case 7, we can construct the following example:

(v) Arbitrarily many 2-bridge links which have the same Q and 2-variable Conway polynomial, but distinct Jones polynomials ([7, Theorem]).

Table 1

Case	V_L	P_L	$P_{L^\wedge}$	F_L	∇_L	$\widetilde{\nabla}_L$	$\widetilde{\nabla}_{L^\wedge}$	cr	p	q, r
1	$=$	$\neq$	$\neq$	$\neq$	$\neq$	$\neq$	$\neq$	14	196	69,−155
2	$=$	$=$	$\neq$	$\neq$	$\neq$	$=$	$\neq$	11	98	29,−55
3	$=$	$=$	$=$	$\neq$	$=$	$=$	$=$	15	392	139,−309
4	$=$	$=$	$=$	$=$	$=$	$=$	$=$	18	1010	313,293
5	$=$	$=$	$\neq$	$=$	$\neq$	$=$	$=$	17	1250	−799,−699
6	$=$	$\neq$	$\neq$	$=$	$\neq$	$\neq$	$\neq$	12	130	57,−47
7	$\neq$	$\neq$	$\neq$	$\neq$	$=$	$=$	$=$	15	504	181,197
8	$\neq$	$\neq$	$\neq$	$\neq$	$\neq$	$=$	$\neq$	17	930	421,601
9	$\neq$	$\neq$	$\neq$	$\neq$	$\neq$	$\neq$	$\neq$	9	24	5,11

§1. 2-bridge link

The 2-bridge links are classified in Schubert's normal form $S(p,q)$ [10], where $p>0, -p<q<p$, and p and q are coprime integers.

Proposition 1. *$S(p,q)$ and $S(p',q')$ are isotopic as oriented (resp. unoriented) links if and only if:*

$$p=p',\ q^{\pm1}\equiv q' \mod 2p \quad (\textit{resp.} \mod p).$$

The following properties are easily seen from Schubert's normal form (cf. [1, Proposition 12.5]):

Proposition 2. (1) *A 2-bridge link $L=K_1\cup K_2$ is interchangeable, that is, there is an isotopy φ of S^3 such that $\varphi(K_i)=K_j$, $i\neq j$.*

(2) *A 2-bridge link $L=K_1\cup K_2$ is invertible, that is, there is an isotopy ψ of S^3 such that $\psi(K_i)=-K_i$, $i=1,2$.*

Let L be an oriented 2-bridge link. Then we denote by $L^\wedge$ a 2-bridge link obtained by reversing the orientation of one of the two components of L, and by $\overline{L}$ a mirror image of L. So if $L=S(p,\pm q)$, $q>0$, then $L^\wedge=S(p,\pm(q-p))$ and $\overline{L}=S(p,\mp q)$. Note that $\overline{(L^\wedge)}=\left(\overline{L}\right)^\wedge=S(p,\pm(p-q))$, which we denote by $\overline{L}^\wedge$. Thus according as the isotopy types of the four oriented 2-bridge links $L, L^\wedge, \overline{L}$, and $\overline{L}^\wedge$, $L=S(p,q)$, there are three types for the 2-bridge link:

Type A: $L=\overline{L}^\wedge\neq\overline{L}=L^\wedge$, that is, $q(p-q)\equiv1 \mod 2p$.

Type B: $L=L^\wedge\neq\overline{L}=\overline{L}^\wedge$, that is, $q(p-q)\equiv-1 \mod 2p$.

Type C: No two of $L, \overline{L}, L^\wedge$, and $\overline{L}^\wedge$ are isotopic, that is, $q(p-q) \not\equiv \pm 1 \mod 2p$.

Given a 2-bridge link in Schubert's normal form $S(p,q)$, it can be put in Conway's normal form $C(a_1, a_2, \ldots, a_k)$, (cf. [9, Fig. 3]), where

$$\frac{p}{q} = a_1 + \frac{1}{a_2} + \cdots + \frac{1}{a_k}. \tag{1}$$

Note that this is a normal form for an unoriented 2-bridge link.

Let $p, q > 0$ and $a_1, a_2, \ldots, a_k > 0$. Since $a_k = (a_k - 1) + 1/1$, if we suppose $a_k > 1$ or fix the parity of k, this expression is unique and the crossing number is $a_1 + a_2 + \cdots + a_k$.

Proposition 3. *Every 2-bridge link $S(p,q)$, $q > 0$, of Type A can be expressed as $C(a_1, a_2, \ldots, a_n, a_n, \ldots, a_2, a_1)$, $a_i > 0$, and vice versa.*

Proof. Suppose that

$$\frac{p}{q} = b_1 + \frac{1}{b_2} + \cdots + \frac{1}{b_\ell},$$

where $b_i > 0$ and ℓ is even. Then we have

$$\begin{pmatrix} s & q \\ r & p \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & b_1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & b_2 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ 1 & b_\ell \end{pmatrix},$$

where $p > q > s > 0$, $p > r > s > 0$, and $ps - rq = 1$ (cf. [11]). Since $(q^2+1)/p \in \mathbb{Z}$,

$$\frac{r^2+1}{p} = \frac{r^2 + (ps-qr)^2}{p} = \frac{q^2+1}{p} r^2 - 2qrs + ps^2 \in \mathbb{Z}.$$

Let $x = (q^2+1)/p$ and $y = (r^2+1)/p$. Since $xp - qq = 1$ and $sp - rq = 1$, there exists an integer a such that $s - x = aq$ and $r - q = ap$. Since $yp - rr = 1$ and $sp - qr = 1$, there exists an integer b such that $s - y = br$ and $q - r = bp$. Then we have $a = b = 0$ and $q = r$. From the uniqueness of the continued fraction, we have $b_1 = b_\ell$, $b_2 = b_{\ell-1}$, $\ldots$, $b_{\ell/2} = b_{\ell/2+1}$. The converse is easy, and the proof is complete.

Proposition 4. *Every 2-bridge link $S(p,q)$, $q > 0$, of Type B can be expressed as $C(a_1, a_2, \ldots, a_n, 2a-1, a_n, \ldots, a_2, a_1)$, $a_i > 0$, $a > 0$, and vice versa.*

Proof. Since $q(p-q) \equiv -1 \mod 2p$, there is an integer b such that

$$q^2 - 1 = p(q + 2b). \tag{2}$$

First we show that there exist positive integers x, y, z, w satisfying:

$$\begin{pmatrix} w & y \\ z & x \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} w & z \\ y & x \end{pmatrix} = \begin{pmatrix} q+2b & q \\ q & p \end{pmatrix} \tag{3}$$

and

$$xw - yz = \varepsilon = \pm 1, \ x > y. \tag{4}$$

From (2), we have

$$\frac{q+1}{2}\frac{q-1}{2} = \frac{p}{4}(q+2b),$$

which is even, and thus $p/4$ is also even, that is, $p \equiv 0 \mod 8$. Let $p_{\pm} = \text{g.c.d.}(p/4, (q \pm 1)/2) > 0$. Since $\text{g.c.d.}((q+1)/2, (q-1)/2) = 1$, we have $p_+ p_- = p/4$. Let $z = p_+ + p_-$, which is an odd integer. Let

$$(x, y) = \begin{cases} (2p_+, (q-1)/2p_-) & \text{if } p_+ < p_-, \\ (2p_-, (q+1)/2p_+) & \text{if } p_- < p_+. \end{cases}$$

Since $\frac{q+1}{2p_+}\frac{q-1}{2p_-} = q + 2b$ is odd, both $(q+1)/2p_+$ and $(q-1)/2p_-$ are odd, so let $w = \frac{1}{2}(\frac{q+1}{2p_+} + \frac{q-1}{2p_-})$. Then x, y, z, w satisfy (3) and (4). Since $z > x > 0$, there are integers a and u such that $z = ax + u$, $a > 0$ and $x > u > 0$. Let $v = w - ay$. Then $xv - yu = \varepsilon$, and there exist positive integers $a_1, a_2, \ldots, a_n$ such that

$$\begin{pmatrix} v & y \\ u & x \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & a_1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & a_2 \end{pmatrix}\cdots\begin{pmatrix} 0 & 1 \\ 1 & a_n \end{pmatrix}.$$

Then $\varepsilon = (-1)^n$ and

$$\begin{pmatrix} v & y \\ u & x \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 2a-1 \end{pmatrix}\begin{pmatrix} v & u \\ y & x \end{pmatrix} = \begin{pmatrix} q+2b & q \\ q & p \end{pmatrix},$$

and so $S(p,q)$ can be expressed as $C(a_1, a_2, \ldots, a_n, 2a-1, a_n, \ldots, a_2, a_1)$. Conversely, if n is odd, a rotation through π about the axis E as shown in Fig. 1, where $\alpha = S_2^{a_1} S_1^{-a_2} \ldots S_2^{a_n} S_1^{1-a}$ and $\alpha' = S_1^{1-a} S_2^{a_n} \ldots S_1^{-a_2} S_2^{a_1}$, gives an isotopy of S^3 which reverses the orientation of one of the two components. If n is even, we have a similar isotopy of S^3. This completes the proof.

Let $\mathcal{L}_n$ be the set of the unoriented 2-bridge links $C(a_1, a_2, \ldots, a_k)$'s satisfying the following:

$$a_1, a_k \geq 2, \ a_2, \ldots, a_{k-1} \geq 1. \tag{5}$$

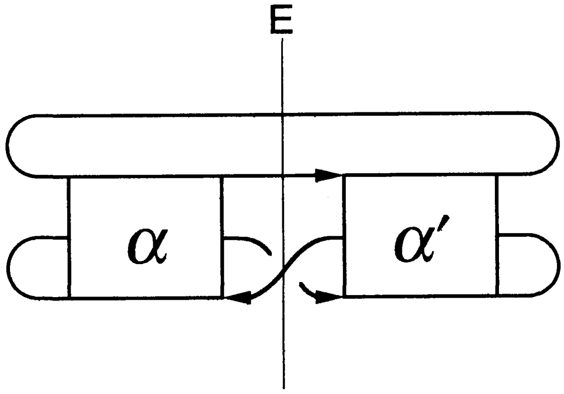

Fig. 1.

(6) Either $a_i = a_{k-i+1}$ for all $i \geq 1$ or $a_1 = a_k, a_2 = a_{k-1}, \ldots, a_{i-1} = a_{k+2-i}$, $a_i > a_{k+1-i}$ for some $i \geq 1$.

(7) $$a_1 + a_2 + \cdots + a_k = n.$$

In other words, this is the set of representatives of unoriented 2-bridge links with n crossings up to chirality. Let $\mathcal{AL}_n$ and $\mathcal{BL}_n$ be the subsets of $\mathcal{L}_n$ consisting of the unoriented 2-bridge links of the form $C(b_1, \ldots, b_\ell, b_\ell, \ldots, b_1)$ and $C(c_1, \ldots, c_\ell, 2c-1, c_\ell, \ldots, c_1)$, respectively. There is a bijective mapping

$$\psi\colon \mathcal{AL}_{2m} \longrightarrow \mathcal{BL}_{2m-1}$$

defined by

$$\psi(C(b_1, \ldots, b_\ell, b_\ell, \ldots, b_1)) = \begin{cases} C(b_1, \ldots, b_\ell - 1, 1, b_\ell - 1, \ldots, b_1) & \text{if } b_\ell > 1 \\ C(b_1, \ldots, b_{\ell-2}, 2b_{\ell-1} + 1, b_{\ell-2}, \ldots, b_1) & \text{if } b_\ell = 1. \end{cases}$$

The explicit numbers of $\mathcal{L}_n$ and $\mathcal{AL}_n$ are given by Ernst and Sumners [3], in which they are denoted by TL_n and ATL_n. Thus we can know the number of $\mathcal{BL}_n$, which equals ATL_n. Let TL_n^{**} denote the number of oriented 2-bridge links of n crossings up to isotopy. Since

$$TL_n^{**} = 4TL_n - 2ATL_{n+1} - 2ATL_n,$$

we have:

Proposition 5.

$$TL_n^{**} = \begin{cases} (2^{n-2} + 2^{\frac{n+2}{2}} - 2^{\frac{n-2}{2}} + 2)/3 & \textit{if } n \equiv 0 \mod 2, \\ (2^{n-2} - 2)/3 & \textit{if } n \equiv 1 \mod 2. \end{cases}$$

Remark. $TL_0^{**} = 2$ and $TL_1^{**} = 0$. $\sum_{n=0}^{20} TL_n = 44,118$ and $\sum_{n=0}^{20} TL_n^{**} = 175,788$.

§2. Conway polynomial

Let L be a 2-component link and $\nabla_L(t_1, t_2) \in \mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1}]$ its Conway polynomial, where the components correspond to the labels t_1 and t_2. This is a uniquely determined invariant of the isotopy type of an oriented link and is related to the 2-variable Alexander polynomial $\Delta(x_1, x_2)$ by

$$\Delta(t_1^2, t_2^2) = \pm t_1^{n_1} t_2^{n_2} \nabla(t_1, t_2),$$

where $\pm t_1^{n_1} t_2^{n_2}$ is a unit (cf. [2,4]). Let L_n, $n \in \mathbb{Z}$, be the 2-component links with labels t_1 and t_2, which contain a 2-braid σ_1^n and are identical except near the 2-braid. Let $\nabla_n(t_1, t_2)$ be the Conway polynomial of L_n.

(8) Suppose that the 2-braid consists of the different components with orientation not parallel. Then

$$\nabla_2(t_1, t_2) + \nabla_{-2}(t_1, t_2) = (t_1 t_2^{-1} + t_1^{-1} t_2) \nabla_0(t_1, t_2).$$

(9) Suppose that the 2-braid consists of the same component having label t_i and parallel orientation. Then

$$\nabla_1(t_1, t_2) = \nabla_{-1}(t_1, t_2) + (t_i - t_i^{-1}) \nabla_0(t_1, t_2).$$

(10) Let $L \sharp L'$ be the connected sum of two 2-component links L and L' such that the connection takes place between the components with the same label t_i. Then

$$\nabla_{L \sharp L'} = (t_i - t_i^{-1}) \nabla_L \nabla_{L'}.$$

(11) For the split 2-component link L, $\nabla_L = 0$.
(12) For the Hopf link L with linking number ± 1, $\nabla_L = \pm 1$.

Let $\nabla(b_1, b_2, \ldots, b_m)$ be the Conway polynomial of the 2-bridge link $D(b_1, b_2, \ldots, b_m)$, m odd (cf. [9, Fig. 2]). Hartley [4, (6.4)] shows that $\nabla(b_1, b_2, \ldots, b_m)$ is an integral polynomial in $f = t_1 t_2 + t_1^{-1} t_2^{-1}$ and $g = t_1 t_2^{-1} + t_1^{-1} t_2$. More precisely we have:

Proposition 6.

$$\nabla(b_1, b_2, \cdots, b_m) = (1,0)A^{b_m}B^{b_{m-1}}\cdots A^{b_3}B^{b_2}A^{b_1}\begin{pmatrix}0\\1\end{pmatrix},$$

where

$$A = \begin{pmatrix} g & -1 \\ 1 & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 0 \\ f-g & 1 \end{pmatrix}.$$

Note that $z^{-1}\widetilde{\nabla}(z)$, where $\widetilde{\nabla}(z)$ is the reduced Conway polynomial [2, p.340], is obtained from $\nabla(b_1, b_2, \ldots, b_m)$ by substituting $f = z^2+1$ and $g = 2$.

Proof. Apply (9) and (10) to one of the crossings in the 2-braid with $2b_{m-1}$ crossings. Then

$$\begin{aligned}\nabla(b_1, \ldots, b_{m-2}, b_{m-1}, b_m) &= \nabla(b_1, \ldots, b_{m-2}, b_{m-1}-1, b_m) \\ &\quad + (t_1 - t_1^{-1})(t_2 - t_2^{-1})\nabla(b_1, \ldots, b_{m-2})\nabla(b_m).\end{aligned}$$

So by induction on b_{m-1}, we have:

$$\begin{aligned}\nabla(b_1, \ldots, b_{m-2}, b_{m-1}, b_m) &= \nabla(b_1, \ldots, b_{m-2}+b_m) \\ &\quad + b_{m-1}(t_1 - t_1^{-1})(t_2 - t_2^{-1})\nabla(b_1, \ldots, b_{m-2})\nabla(b_m).\end{aligned}$$

Apply (8) to the 2-braid with $2b_m$ crossings. Then

$$\begin{aligned}&\nabla(b_1, \ldots, b_{m-1}, b_m) + \nabla(b_1, \ldots, b_{m-1}, b_m - 2) \\ &= g\nabla(b_1, \ldots, b_{m-1}, b_m - 1),\end{aligned}$$

and so we have

$$\begin{pmatrix} \nabla(b_1, \ldots, b_m) \\ \nabla(b_1, \ldots, b_m - 1) \end{pmatrix} = A \begin{pmatrix} \nabla(b_1, \ldots, b_m - 1) \\ \nabla(b_1, \ldots, b_m - 2) \end{pmatrix}.$$

Then we have

$$\begin{aligned}&\begin{pmatrix} \nabla(b_1, \ldots, b_{m-3}, b_{m-2}+b_m) \\ \nabla(b_1, \ldots, b_{m-3}, b_{m-2}+b_m - 1) \end{pmatrix} \\ &\qquad = A^{b_m}\begin{pmatrix} \nabla(b_1, \ldots, b_{m-3}, b_{m-2}) \\ \nabla(b_1, \ldots, b_{m-3}, b_{m-2} - 1) \end{pmatrix},\end{aligned}$$

and

$$\begin{pmatrix} \nabla(b_m) \\ \nabla(b_m - 1) \end{pmatrix} = A^{b_m}\begin{pmatrix}0\\1\end{pmatrix},$$

since $\nabla(0)=0$ by (11) and $\nabla(-1)=1$ by (12). Therefore

$$\begin{aligned}
&\begin{pmatrix} \nabla(b_1,\dots,b_{m-1},b_m) \\ \nabla(b_1,\dots,b_{m-1},b_m-1) \end{pmatrix} \\
&= A^{b_m}\begin{pmatrix} \nabla(b_1,\dots,b_{m-2}) \\ \nabla(b_1,\dots,b_{m-2}-1) \end{pmatrix} \\
&\qquad\qquad + b_{m-1}(t_1-t_1^{-1})(t_2-t_2^{-1})\nabla(b_1,\dots,b_{m-2})A^{b_m}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
&= A^{b_m}\begin{pmatrix} 1 & 0 \\ b_{m-1}(t_1-t_1^{-1})(t_2-t_2^{-1}) & 1 \end{pmatrix}\begin{pmatrix} \nabla(b_1,\dots,b_{m-2}) \\ \nabla(b_1,\dots,b_{m-2}-1) \end{pmatrix} \\
&= A^{b_m}B^{b_{m-1}}\begin{pmatrix} \nabla(b_1,\dots,b_{m-2}) \\ \nabla(b_1,\dots,b_{m-2}-1) \end{pmatrix} \\
&= A^{b_m}B^{b_{m-1}}\cdots B^{b_2}A^{b_1}\begin{pmatrix} 0 \\ 1 \end{pmatrix},
\end{aligned}$$

and we have the desired formula.

§3. Computational process

From [9, Sect.2, Step 1], we have the set $\mathcal{L}_n$. Let $C(a_1,a_2,\dots,a_k) \in \mathcal{L}_n$ and p,q be the integers obtained from the continued fraction (1). Let

$$\frac{p}{q} = 2b_1 + \frac{1}{2b_2} + \cdots + \frac{1}{2b_m}$$

and

$$\frac{p}{q-p} = 2c_1 + \frac{1}{2c_2} + \cdots + \frac{1}{2c_\ell},$$

where m and ℓ are odd. If let $L = D(b_1,b_2,\dots,b_m)$, then $L^\wedge = D(c_1,c_2,\dots,c_\ell)$. We denote these 2-bridge links by $T(p,q)$ and $T(p,q-p)$ $\left(= T(p,q)^\wedge\right)$. Then $T(p,q)$ is isotopic to either $S(p,q)$ or $S(p,q-p)$.* We first compute the homfly polynomials $P_L = P(b_1,b_2,\dots,b_m)$, $P_{L^\wedge} = P(c_1,c_2,\dots,c_\ell)$, the Kauffman polynomials $F_L = F(b_1,b_2,\dots,b_m)$, and the Conway polynomials $\nabla_L = \nabla(b_1,b_2,\dots,b_m)$ using [9, Propositions 1 and 4] and Proposition 6. Then we compute: $P_{\overline{L}}$, $P_{L^\wedge}$, $F_{L^\wedge}$, $F_{\overline{L}}$, $F_{\overline{L}^\wedge}$, $\nabla_{L^\wedge}$, $\nabla_{\overline{L}}$, and $\nabla_{\overline{L}^\wedge}$, using the following:

$$P_{\overline{L}}(v,z) = P_L(v^{-1},z),$$

*Note added in proof. T. Kanenobu and Y. Miyazawa proved that $T(p,q) = S(p,q-p)$.

$$F_{L^\wedge}(a,z) = a^{4\lambda} F_L(a,z),$$
$$F_{\overline{L}}(a,z) = F_L(a^{-1},z),$$
$$\nabla_{L^\wedge}(t_1,t_2) = -\nabla_L(t_1,t_2^{-1}),$$
$$\nabla_{\overline{L}}(t_1,t_2) = -\nabla_L(t_1,t_2),$$

where $\lambda = -b_1 - b_2 - \cdots - b_m = c_1 + c_2 + \cdots + c_\ell$ is the linking number of L.

Next we compute the Jones, Q, and reduced Conway polynomials by suitable substitutions. Finally we search all the pairs of 2-bridge links through 20 crossings having the same homfly, Kauffman, Jones, and Q polynomials as in [9, Sect.3, Step 3]. For the Conway and reduced Conway polynomials, we examine for the pairs having the same Q polynomials.

§4. Computational results

In Table 2, the three numbers "p q,r" represent the pair of the 2-bridge links $\{T(p,q), T(p,r)\}$ sharing the same Q polynomial. If there is an entry "V" (resp. "P", "F", "A", "C"), they also share the same Jones (resp. homfly, Kauffman, 2-variable Conway, reduced Conway) polynomial. We do not list the pairs L and $L^\wedge$ having the linking number zero if they are not contained in Cases 1–5. These links have the same Kauffman polynomial.

The two numbers "p q" represent the pair of the 2-bridge links $\{T(p,q), T(p,q)^\wedge\}$ sharing the same homfly and 2-variable Conway polynomials (cf. Fact 4). Note that we do not list the pair sharing only the same 2-variable Conway or reduced Conway polynomial. The entries "a" and "b" indicate that the links are of types A and B, respectively.

Table 2

9 crossing		
24	5,11	b
11 crossing		
78	17,35	
84	19,25	
98	29,-55	P
98	-69,43	V
128	47	
12 crossing		
60	11,19	
130	57,-47	Fa
130	-73,47	Fa
13 crossing		
110	19,51	
124	39,23	
132	25,29	
132	25,59	
132	29,59	
138	31,43	
162	37,73	P
162	-125,-89	V
196	57,-111	P
196	-139,85	V
200	61	
232	101	
240	71,89	b
242	65,-155	V
242	-177,87	P
248	109	
256	95	
264	115	
280	123	
14 crossing		
188	35,59	
196	69,-155	V
196	-127,41	V
196	45,37	
220	61,39	
252	71,55	A
264	71,49	
324	127,-233	V
324	-197,91	V
370	153,-207	F
370	-217,163	F
380	137,-167	F
380	-243,213	F
15 crossing		
120	29,19	b
186	41,83	
192	43,61	
228	59,47	
234	101,43	
238	109,75	
242	111,-197	V
242	-131,45	P
252	115,47	
252	115,79	

Table 2 (continued)

252	47,79	
260	61,49	
294	127,-209	V
294	-167,85	P
304	79,63	
308	65,87	
308	83,97	
324	73,145	P
324	-251,-179	V
336	89,103	
338	79,-233	V
338	-259,105	P
350	93,-243	V
350	-257,107	V
352	161	
352	161,63	
368	169	
374	69,169	
380	103,87	
384	143	
388	85,89	
392	139,83	V
392	139,-309	PA
392	-253,83	PA
392	-253,-309	V
400	121	
400	183	
402	125,143	
406	93,121	P
406	-313,-285	V
408	121,127	
416	191	
418	111,89	
434	177,115	
448	137,201	
450	97,133	P
450	-353,-317	V
468	101,-211	P
468	-367,257	V
476	109,277	P
476	-367,-199	V
484	131,-309	V
484	-353,175	P
488	213	
494	105,131	
504	221	
504	221,-115	P
504	221,389	V
504	-283,-115	P
504	-283,389	V
504	181,197	Ab
512	223	
520	227	
522	119,155	P
522	-403,-367	V
536	235	
574	131,159	P
574	-443,-415	V
578	169,237	V
578	-409,-341	P
648	253,181	PA
648	253,-467	V

648	-395,181	V
648	-395,-467	PA
722	305,-455	P
722	-417,267	V

16 crossing

256	81,49	V
256	-175,-207	V
296	47,137	
316	59,99	
322	71,57	
324	77,61	
352	163,291	V
352	-189,-61	V
354	73,163	
374	167;303	V
374	-207,-71	V
378	137,67	
396	73,91	
402	83,113	
406	73,143	
456	107,125	
462	127,83	
462	127,97	
462	83,97	
484	109,197	V
484	-375,-287	V
506	137,93	
508	135,119	A
510	107,233	
516	121,223	
516	121,113	A
516	223,113	
564	245,131	
572	125,333	V
572	-447,-239	V
572	155,131	
576	107,125	
588	209,-463	V
588	-379,125	V
594	163,-413	V
594	-431,181	V
620	253,-347	F
620	-367,273	F
624	145,175	
630	193,-227	V
630	-437,403	V
630	193,277	V
630	-437,-353	V
630	227,-277	F
630	-403,353	F
638	139,371	V
638	-499,-267	V
644	289,473	V
644	-355,-171	V
666	241,-203	V
666	-425,463	V
676	287,183	V
676	-389,-493	V
702	197,-487	V
702	-505,215	V
704	149,299	

728	215,327	V
728	-513,-401	V
742	303,515	V
742	-439,-227	V
748	159,317	
972	271,-593	V
972	-701,379	V
1016	397,-651	A
1032	379,-661	A
1130	437,-467	Fa
1130	-693,467	Fa

17 crossing

240	107,53	
246	55,79	
338	53,157	P
338	-285,-181	V
342	53,109	
370	89,59	
380	119,71	
380	119,61	
380	71,61	
388	73,93	
390	101,61	
392	113,-223	P
392	-279,169	V
406	187,-277	V
406	-219,129	V
462	79,101	
464	101,-379	V
464	-363,85	V
472	221	
476	151,-257	V
476	-325,219	V
484	221,-395	V
484	-263,89	P
486	217,109	P
486	-269,-377	V
488	229	
496	157,405	V
496	-339,-91	V
512	191	
512	161,97	V
512	161,-415	PA
512	-351,97	PA
512	-351,-415	V
528	163,427	V
528	-365,-101	V
536	251	
536	93,85	
552	259	
560	107,-437	V
560	-453,123	V
564	179,197	
570	181,169	
578	203,-477	P
578	-375,101	V
594	271,107	
598	113,425	V
598	-485,-173	V
600	181	
602	163,191	V

602	-439,-411	V
616	279,113	
636	167,151	
638	135,-525	V
638	-503,113	V
644	141,153	
672	209,239	b
672	211,197	
676	209,-519	P
676	-467,157	V
686	181,209	V
686	-505,-477	V
714	155,127	
720	317,133	
722	151,-533	P
722	-571,189	V
726	263,-529	P
726	-463,197	V
728	333	
728	333,229	
736	337	
738	137,331	
742	233,339	
744	325	
748	141,163	
752	345	
754	199,225	
756	235,163	P
756	-521,-593	V
760	333	
760	349	
764	203,179	
770	137,277	
772	181,177	
774	349,-167	P
774	-425,607	V
776	355	
776	339	
784	359	
784	279,167	V
784	279,-617	PA
784	-505,167	PA
784	-505,-617	V
786	163,361	
792	347	
798	251,223	
798	143,283	
800	367	
808	371	
814	173,-663	V
814	-641,151	V
834	233,173	
840	379,181	b
846	193,-371	P
846	-653,475	V
854	153,181	
864	269,197	P
864	-595,-667	V
868	353,-639	V
868	-515,229	V
874	245,-675	V
874	-629,199	V

Table 2 (continued)

894	247,187	
896	205,261	P
896	-691,-635	V
924	415,283	
930	421,601	C
936	205,277	P
936	205,-659	V
936	-731,277	V
936	-731,-659	V
938	409,275	
942	431,197	
942	203,215	
950	199,249	
954	427,-209	P
954	-527,745	V
966	409,745	V
966	-557,-221	P
968	395,219	PA
968	395,-749	V
968	-573,219	V
968	-573,-749	PA
990	223,-437	P
990	-767,553	V
992	447,639	C
994	303,-549	V
994	-691,445	V
994	431,767	V
994	-563,-227	P
996	227,449	
996	275,233	
1008	227,299	P
1008	-781,-709	V
1022	285,313	
1022	285,-751	V
1022	-737,271	V
1022	285,299	
1022	313,271	
1022	313,-723	V
1022	-709,299	V
1022	271,299	
1024	447	
1026	215,269	
1034	285,219	
1036	317,303	
1036	317,275	
1036	317,-747	V
1036	-719,289	V
1036	303,-761	V
1036	-733,275	V
1036	303,289	
1036	275,289	
1056	241,463	b
1056	247,457	
1058	459,-737	P
1058	-599,321	V
1058	231,415	V
1058	-827,-643	P
1064	299,243	P
1064	299,-821	V
1064	-765,243	V
1064	-765,-821	V
1102	251,-793	V
1102	-851,309	V
1104	257,479	
1120	297,457	
1122	245,-811	V
1122	-877,311	V
1134	509,347	
1140	241,301	
1144	309,243	
1156	339,475	V
1156	-817,-681	P
1164	271,325	
1190	321,349	
1216	257,321	
1242	379,343	PA
1242	-863,-899	PA
1250	451,551	F
1250	-799,-699	PF
1278	391,-461	PA
1278	-887,817	PA
1292	295,-929	V
1292	-997,363	V
1296	505,361	PA
1296	505,-935	V
1296	-791,361	V
1296	-791,-935	PA
1298	349,-1015	V
1298	-949,283	V
1314	401,-475	PA
1314	-913,839	PA
1316	543,355	
1330	389,579	
1350	377,413	PA
1350	-973,-937	PA
1352	365,573	V
1352	365,-779	PA
1352	-987,573	PA
1352	-987,-779	V
1372	405,-995	V
1372	-967,377	V
1444	533,-835	V
1444	-911,609	P
1456	393,407	
1458	541,433	P
1458	-917,-1025	V
1528	549,-931	A
1544	555,571	A
1682	637,-1219	V
1682	-1045,463	P
1784	653	
1800	659	
1922	805,-1179	V
1922	-1117,743	P
2024	741	
2040	781,749	Ab
2056	755	
2296	843	
2312	885	
	18 crossing	
210	29,41	
400	139,-341	V
400	-261,59	V
444	83,139	
452	109,85	
532	109,137	
544	93,189	
558	131,-301	V
558	-427,257	V
576	119,263	V
576	119,-313	V
576	-457,263	V
576	-457,-313	V
684	145,107	
686	141,-531	V
686	-545,155	V
688	123,307	
702	163,-557	V
702	-539,145	V
704	161,129	V
704	-543,-575	V
704	127,193	
720	169,151	
732	337,151	
736	135,503	V
736	-601,-233	V
738	173,-583	V
738	-565,155	V
748	203,137	
760	159,121	
764	183,199	A
768	241,145	V
768	-527,-623	V
770	159,-541	V
770	-611,229	V
772	185,169	A
780	161,239	
782	135,169	
784	141,-475	V
784	-643,309	V
800	153,553	V
800	-647,-247	V
812	151,-633	V
812	-661,179	V
832	191,159	V
832	-641,-673	V
836	217,-543	V
836	-619,293	V
858	301,-635	V
858	-557,223	V
858	181,233	
868	179,-381	V
868	-689,487	V
870	353,-487	F
870	-517,383	F
880	317,-387	F
880	-563,493	F
882	199,163	
882	205,-479	V
882	-677,403	V
896	375,-185	V
896	375,711	V
896	-521,-185	V
896	-521,711	V
900	209,-511	V
900	-691,389	V
900	247,-617	V
900	-653,283	V
936	295,-329	V
936	-641,607	V
942	287,221	
948	289,199	
952	345,-775	V
952	-607,177	V
952	205,171	
956	227,251	
964	229,221	
976	181,213	V
976	-795,-763	V
980	209,-631	V
980	-771,349	V
994	275,-789	V
994	-719,205	V
996	209,455	
1002	235,433	
1008	187,-653	V
1008	-821,355	V
1008	187,691	V
1008	-821,-317	V
1008	355,-317	V
1008	-653,691	V
1010	313,293	PFAa
1010	-697,-293	PFAa
1020	239,271	A
1024	225,289	V
1024	225,-735	V
1024	-799,289	V
1024	-799,-735	V
1032	185,271	
1040	197,717	V
1040	-843,-323	V
1044	329,-751	V
1044	-715,293	V
1062	233,197	
1064	277,221	
1072	235,203	V
1072	-837,-869	V
1078	493,885	V
1078	-585,-193	V
1078	475,-225	V
1078	-603,853	V
1100	203,603	V
1100	-897,-497	V
1102	235,293	
1106	197,239	
1118	245,-787	V
1118	-873,331	V
1148	241,-935	V
1148	-907,213	V
1156	307,-645	V
1156	-849,511	V
1158	269,503	
1162	263,-409	V
1162	-899,753	V
1162	417,207	
1164	515,527	
1164	433,343	

Table 2 (continued)

1166	303,-841	V	1484	409,-1159	V	1998	557,-1387	V	754	353,111	
1166	-863,325	V	1484	-1075,325	V	1998	-1441,611	V	800	119,279	V
1188	211,545		1488	439,409		2014	845,-1131	V	800	119,-521	PA
1190	431,-269	V	1536	689,335		2014	-1169,883	V	800	-681,279	PA
1190	-759,921	V	1536	359,425		2028	859,547	V	800	-681,-521	V
1190	423,213		1542	349,679		2028	-1169,-1481	V	800	241	
1218	559,253		1582	345,-1111	V	2040	797,-1307	A	808	167,127	
1218	373,283		1582	-1237,471	V	2056	739,-1293	A	816	139,173	
1232	229,555		1590	473,587		2116	873,-1335	V	836	151,381	
1246	223,265		1596	691,373		2116	-1243,781	V	840	193,263	
1260	263,-913	V	1620	631,-1169	V	2142	593,-1675	V	850	133,303	
1260	-997,347	V	1620	-989,451	V	2142	-1549,467	V	858	389,157	
1266	347,377		1634	347,433		2198	957,649	V	882	211,-713	V
1266	263,353		1634	617,579		2198	-1241,-1549	V	882	-671,169	P
1272	571,277		1644	611,485		2210	863,-837	Fa	884	139,309	
1274	537,269		1644	713,383		2210	-1347,837	Fa	892	215,231	
1274	279,-561	V	1650	463,373		2212	933,1565	V	900	197,217	
1274	-995,713	V	1666	377,-1303	V	2212	-1279,-647	V	928	163,-733	V
1278	343,361		1666	-1289,363	V	2318	1017,-1339	V	928	-765,195	V
1288	269,381	V	1666	699,-365	V	2318	-1301,979	V	936	149,427	
1288	269,-907	V	1666	-967,1301	V	2500	1051,-1549	F	942	329,299	
1288	-1019,381	V	1690	759,359		2500	-1449,951	F	944	221,-771	V
1288	-1019,-907	V	1704	397,475		2546	935,-1117	V	944	-723,173	V
1296	397,-827	V	1716	727,703		2546	-1611,1429	V	948	295,301	
1296	-899,469	V	1758	523,493		2610	719,701		968	351,-705	P
1316	571,403	V	1782	389,1037	V	3064	1133,-1899	A	968	-617,263	V
1316	-745,-913	V	1782	-1393,-745	V	3080	1131,-1941	A	976	457	
1320	371,349		1786	661,-783	V		19 crossing		976	179,667	V
1320	371,389		1786	-1125,1003	V				976	-797,-309	V
1320	349,389		1804	477,-1163	V	294	131,65		984	461	
1326	277,367		1804	-1327,641	V	300	97,67		988	173,211	
1330	353,733	V	1804	391,479	V	426	83,59		992	173,-787	V
1330	-977,-597	V	1804	-1413,-1325	V	438	67,61		992	-819,205	V
1342	377,-987	V	1826	679,-1313	V	450	61,-239	P	1000	469	
1342	-965,355	V	1826	-1147,513	V	450	-389,211	V	1000	437	
1342	379,291	V	1846	391,495		472	215,73		1008	473	
1342	-963,-1051	V	1848	773,-403	V	490	211,-349	V	1014	235,-701	V
1374	311,377		1848	773,1445	V	490	-279,141	P	1014	-779,313	P
1374	629,287		1848	-1075,-403	V	508	159,95		1014	161,265	
1378	513,293		1848	-1075,1445	V	516	125,97		1016	445	
1380	379,301		1860	1081,841	C	592	281		1024	193,321	V
1414	635,411	V	1870	763,-1097	F	608	277,-107	V	1024	193,-703	PA
1414	-779,-1003	V	1870	-1107,773	F	608	-331,501	V	1024	-831,321	PA
1428	401,311		1876	823,-409	V	608	289		1024	-831,-703	V
1428	583,991	V	1876	-1053,1467	V	620	149,99		1032	451	
1428	-845,-437	V	1880	737,-767	F	630	101,121		1040	487	
1430	607,303		1880	-1143,1113	F	640	239		1048	459	
1434	529,427		1890	523,-1313	V	648	289,145	P	1048	491	
1452	329,593	V	1890	-1367,577	V	648	-359,-503	V	1050	487,163	
1452	-1123,-859	V	1904	557,837	V	666	101,137		1058	183,275	P
1456	431,319	V	1904	-1347,-1067	V	672	319		1058	-875,-783	V
1456	431,-1137	V	1918	835,1383	V	676	99,177		1062	229,337	P
1456	-1025,319	V	1918	-1083,-535	V	676	105,313	P	1062	-833,-725	V
1456	-1025,-1137	V	1926	517,743		676	-571,-363	V	1064	337,167	
1462	575,-309	V	1926	695,-589	V	688	327		1064	337,489	
1462	-887,1153	V	1926	-1231,1337	V	722	115,-493	V	1064	167,489	
1462	607,1015	V	1936	747,-1365	V	722	-607,229	P	1064	499	
1462	-855,-447	V	1936	-1189,571	V	726	133,-395	P	1072	503	
1470	617,-307	V	1962	599,-709	V	726	-593,331	V	1072	205,741	V
1470	-853,1163	V	1962	-1363,1253	V	726	251,233		1072	-867,-331	V
1474	313,625		1962	527,769		728	113,153		1078	501,893	V
1482	335,653		1984	895,1151	C	732	235,217		1078	-577,-185	V

Table 2 (continued)

1080	233,-487	P	1364	245,-1075	V	1554	355,439	P	1708	481,523	
1080	-847,593	V	1364	-1119,289	V	1554	-1199,-1115	V	1708	481,495	
1092	491,251		1386	317,401	P	1554	277,557		1708	481,-1255	V
1100	189,589	V	1386	-1069,-985	V	1562	337,359		1708	-1227,453	V
1100	-911,-511	V	1394	487,241		1562	295,863	V	1708	523,-1213	V
1102	175,347		1414	381,325		1562	-1267,-699	V	1708	-1185,495	V
1104	211,-845	V	1422	295,331	V	1564	703,473		1708	523,453	
1104	-893,259	V	1422	-1127,-1091	V	1566	487,343	P	1708	495,453	
1120	209,239	b	1422	421,313	P	1566	-1079,-1223	V	1708	521,451	
1120	257,513		1422	-1001,-1109	V	1568	281,617	PA	1710	401,781	
1120	513		1426	255,301		1568	281,-951	V	1722	457,527	
1136	521		1440	643,-317	P	1568	-1287,617	V	1722	457,-1223	V
1148	275,299		1440	-797,1123	V	1568	-1287,-951	PA	1722	-1265,499	V
1152	239		1444	379,-1141	V	1568	487,-297	V	1722	457,485	
1152	239,527	V	1444	-1065,303	P	1568	487,1271	V	1722	527,499	
1152	239,-625	V	1456	317,-1123	V	1568	-1081,-297	V	1722	527,-1237	V
1152	-913,527	V	1456	-1139,333	V	1568	-1081,1271	V	1722	-1195,485	V
1152	-913,-625	V	1456	317,-619	V	1576	723		1722	499,485	
1152	527		1456	-1139,837	V	1582	563,283		1734	319,455	
1156	203,-749	V	1456	333,837	P	1582	419,363		1734	713,509	V
1156	-953,407	P	1456	-1123,-619	V	1584	709,1285	V	1734	-1021,-1225	P
1156	265,277		1458	593,269	V	1584	-875,-299	V	1748	367,459	
1156	531,251		1458	-865,-1189	P	1584	709,-347	P	1758	487,367	
1162	475,267		1458	305,341	V	1584	-875,1237	V	1758	385,787	
1162	341,369	V	1458	-1153,-1117	V	1584	299,-1237	V	1760	373,483	
1162	-821,-793	V	1472	337,273	V	1584	-1285,347	V	1768	315,485	
1162	337,365		1472	337,-1199	V	1586	329,277		1778	545,405	
1168	535		1472	-1135,273	V	1596	331,733		1786	407,467	
1176	251,419	V	1472	-1135,-1199	V	1600	303,367	V	1792	389,333	V
1176	251,-757	PA	1476	653,-331	P	1600	303,-1233	V	1792	-1403,-1459	V
1176	-925,419	PA	1476	-823,1145	V	1600	-1297,367	V	1800	419,779	V
1176	-925,-757	V	1484	641,1145	V	1600	-1297,-1233	V	1800	419,-1021	PA
1178	245,207		1484	-843,-339	P	1602	733,373	V	1800	-1381,779	PA
1184	543		1484	471,415	V	1602	-869,-1229	V	1800	-1381,-1021	V
1188	271,-521	P	1484	-1013,-1069	V	1610	507,367		1804	767,381	
1188	-917,667	V	1488	277,-1163	V	1610	493,723		1806	377,827	
1190	377,547		1488	-1211,325	V	1616	371,-1261	V	1806	479,737	
1206	383,275	P	1494	335,443	P	1616	-1245,355	V	1812	397,391	
1206	-823,-931	V	1494	-1159,-1051	V	1624	471,703		1812	553,379	
1206	551,283		1496	533,269		1628	309,727		1820	817,557	
1206	223,533		1496	685		1638	349,293		1820	543,-1433	V
1218	557,383		1498	267,323		1644	341,503		1820	-1277,387	V
1224	553,281		1504	279,-473	V	1650	343,757		1826	773,389	
1224	547,227		1504	279,1031	V	1652	379,505		1848	491,421	
1232	563,387		1504	-1225,-473	V	1656	373,-731	P	1860	401,419	
1276	335,303		1504	-1225,1031	V	1656	373,925	V	1860	389,851	
1278	587,299	V	1512	479,409		1656	-1283,-731	V	1862	519,491	V
1278	-691,-979	V	1518	703,263		1656	-1283,925	V	1862	-1343,-1371	V
1284	301,305		1520	411,1171	V	1672	299,365		1862	519,393	
1288	363,405		1520	-1109,-349	V	1672	439,351		1862	519,421	
1288	409,-695	V	1520	477,1237	V	1674	521,377	P	1862	491,393	
1288	-879,593	V	1520	-1043,-283	V	1674	-1153,-1297	V	1862	491,421	
1298	345,-931	V	1528	701		1680	733,-1187	V	1862	393,421	PA
1298	-953,367	V	1536	671		1680	-947,493	V	1862	-1469,-1441	PA
1300	383,583	V	1540	283,-1213	V	1682	753,521	V	1870	507,397	
1300	-917,-717	V	1540	-1257,327	V	1682	-929,-1161	P	1872	589,517	PA
1312	229,-603	V	1544	707		1682	737,365		1872	-1283,-1355	PA
1312	-1083,709	V	1550	461,411		1692	355,731		1908	671,-601	PA
1330	607,303		1552	293,1069	V	1698	353,473		1908	-1237,1307	PA
1358	625,-927	V	1552	-1259,-483	V	1700	467,297		1914	431,787	
1358	-733,431	V	1552	421,1197	V	1702	363,-1385	V	1914	431,863	
1360	613,237		1552	-1131,-355	V	1702	-1339,317	V	1914	787,863	

Table 2 (continued)

1922	557,433	V	2108	951,375		2358	697,733	P	2632	-1907,557	V
1922	-1365,-1489	P	2114	873,485		2358	-1661,-1625	V	2632	-1907,-2075	V
1922	683,-1053	V	2116	919,-1473	P	2366	851,1831	PA	2646	769,-1919	V
1922	-1239,869	P	2116	-1197,643	V	2366	-1515,-535	PA	2646	-1877,727	V
1924	515,-1357	V	2116	829,461	V	2398	519,1007		2646	737,809	PA
1924	-1409,567	V	2116	-1287,-1655	P	2412	751,-857	P	2646	-1909,-1837	PA
1932	359,695	V	2128	395,451	V	2412	-1661,1555	V	2660	737,793	P
1932	-1573,-1237	V	2128	-1733,-1677	V	2436	743,1091		2660	-1923,-1867	V
1936	439,791	PA	2132	577,-1503	V	2448	761,-871	P	2660	787,563	P
1936	439,-1145	V	2132	-1555,629	V	2448	761,1577	V	2660	-1873,-2097	V
1936	-1497,791	V	2136	481,487		2448	-1687,-871	V	2676	1201,583	
1936	-1497,-1145	PA	2136	925,499		2448	-1687,1577	V	2678	1107,-1961	V
1938	409,511		2142	871,-1577	V	2450	531,1371	P	2678	-1571,717	V
1944	757,541	PA	2142	-1271,565	V	2450	-1919,-1079	V	2680	821	
1944	757,-1403	V	2156	601,657	V	2450	687,-1777	V	2684	581,1127	
1944	-1187,541	V	2156	-1555,-1499	V	2450	-1763,673	V	2686	727,795	V
1944	-1187,-1403	PA	2166	913,-1367	P	2450	911,-1889	V	2686	-1959,-1891	P
1946	579,411	P	2166	-1253,799	V	2450	-1539,561	P	2704	729,1145	V
1946	-1367,-1535	V	2170	647,-1213	P	2464	667,723	V	2704	729,-1559	PA
1946	593,-1075	V	2170	-1523,957	V	2464	-1797,-1741	V	2704	-1975,1145	PA
1946	-1353,871	V	2170	883,573		2464	1103,751	V	2704	-1975,-1559	V
1958	449,797		2178	925,-1715	P	2464	1103,-1713	V	2724	809,587	
1958	427,361		2178	-1253,463	V	2464	-1361,751	V	2724	761,803	
1960	573,517	V	2184	509,947		2464	-1361,-1713	V	2728	1587,-1525	C
1960	573,-1443	V	2198	907,593		2482	725,-1723	V	2730	739,-1601	V
1960	-1387,517	V	2210	467,597		2482	-1757,759	V	2730	-1991,1129	V
1960	-1387,-1443	V	2212	957,641		2494	659,975		2738	591,1035	P
1968	449,431		2238	971,521		2496	673,737		2738	-2147,-1703	V
1976	451,413		2240	473,-983	V	2500	1101,901	F	2738	815,-1997	V
1976	535,-1545	V	2240	473,1257	P	2500	-1399,-1599	PF	2738	-1923,741	P
1976	-1441,431	V	2240	-1767,-983	V	2502	779,743	P	2744	811,755	P
1978	613,-1107	V	2240	-1767,1257	V	2502	-1723,-1759	V	2744	811,-1989	V
1978	-1365,871	V	2244	625,523		2508	679,-1961	V	2744	-1933,755	V
1980	623,-697	PA	2254	687,659	V	2508	-1829,547	V	2744	-1933,-1989	V
1980	-1357,1283	PA	2254	-1567,-1595	V	2530	899,711		2756	1237,813	V
1984	415,353		2254	993,1777	V	2538	775,703	PA	2756	-1519,-1943	V
1988	705,369	V	2254	-1261,-477	P	2538	-1763,-1835	PA	2758	597,821	P
1988	-1283,-1619	V	2268	925,517		2546	673,581		2758	-2161,-1937	V
1990	617,-1383	PFA	2268	949,1789	V	2546	689,-1991	V	2758	625,1605	P
1990	-1373,607	PFA	2268	-1319,-479	P	2546	-1857,555	V	2758	-2133,-1153	V
1992	587,421		2296	947,523		2552	917,-1555	A	2772	1031,733	
2010	583,1387	PFA	2298	703,535		2552	917,675		2778	767,827	
2010	-1427,-623	PFA	2310	521,499		2552	-1555,675		2782	1179,589	
2014	435,-1473	V	2312	613,1021	PA	2562	709,541	P	2784	769,607	
2014	-1579,541	V	2312	613,-1291	V	2562	-1853,-2021	V	2784	601,823	
2016	563,635	PA	2312	-1699,1021	V	2568	757,923		2786	1215,817	
2016	-1453,-1381	PA	2312	-1699,-1291	PA	2568	757,955		2794	993,751	
2022	905,461		2314	645,-1695	V	2568	923,955	A	2800	821,1221	
2022	617,473		2314	-1669,619	V	2574	787,-929	PA	2828	613,837	P
2028	599,-1585	V	2320	913,607		2574	-1787,1645	PA	2828	-2215,-1991	V
2028	-1429,443	V	2332	641,-1823	V	2592	793,937	PA	2832	791,617	
2048	577,449	PA	2332	-1691,509	V	2592	793,-1655	V	2842	797,-2003	V
2048	577,-1599	V	2336	989,1053		2592	-1799,937	V	2842	-2045,839	V
2048	-1471,449	V	2338	533,1049		2592	-1799,-1655	PA	2842	643,615	PA
2048	-1471,-1599	PA	2338	697,-1307	P	2600	567,-1097	V	2842	-2199,-2227	PA
2072	601,559		2338	-1641,1031	V	2600	-2033,1503	V	2852	1841,1289	C
2074	439,371		2338	659,631		2610	943,-797	PA	2856	835,1243	V
2076	485,575		2338	829,1809	PA	2610	-1667,1813	PA	2856	835,-1613	V
2080	453,-1523	V	2338	-1509,-529	PA	2622	1183,565		2856	-2021,1243	V
2080	-1627,557	V	2352	533,701	P	2626	555,1109		2856	-2021,-1613	P
2082	901,487		2352	-1819,-1651	V	2632	725,557	P	2856	619,787	P
2082	919,469		2356	537,499		2632	725,-2075	V	2856	619,-2069	V

Table 2 (continued)

2856	-2237,787	V
2856	-2237,-2069	V
2886	1301,623	
2886	653,797	
2892	853,799	
2914	1315,1879	C
2916	865,1081	P
2916	-2051,-1835	V
2924	1273,865	V
2924	-1651,-2059	P
2924	1285,619	
2926	1283,2263	V
2926	-1643,-663	P
2940	869,671	
2940	641,1319	
2946	877,1087	
2994	677,1319	
2996	809,683	
3014	895,807	V
3014	-2119,-2207	P
3038	1275,2395	V
3038	-1763,-643	P
3038	883,-2141	V
3038	-2155,897	V
3038	1761,-1699	C
3040	853,-2347	V
3040	-2187,693	V
3042	1327,-1793	V
3042	-1715,1249	P
3048	899,1133	
3052	853,895	
3054	691,1345	
3054	697,1339	
3058	663,1775	V
3058	-2395,-1283	V
3074	1105,909	
3078	833,-2407	V
3078	-2245,671	V
3080	901,859	
3094	863,-2257	V
3094	-2231,837	V
3094	863,-2189	V
3094	-2231,905	V
3094	837,905	V
3094	-2257,-2189	P
3102	707,1361	
3102	1145,923	
3122	845,873	V
3122	-2277,-2249	V
3122	859,-1817	P
3122	-2263,1305	V
3162	715,883	
3172	841,-2487	V
3172	-2331,685	V
3198	677,859	
3200	879	
3216	733,955	
3220	1347,2523	V
3220	-1873,-697	P
3222	865,901	
3230	737,-2323	V
3230	-2493,907	V
3232	1407	
3234	1357,2533	V
3234	-1877,-701	P
3248	1425,-703	V
3248	1425,2545	V
3248	-1823,-703	P
3248	-1823,2545	V
3268	917,-2523	V
3268	-2351,745	V
3270	973,913	
3298	1009,-2323	V
3298	-2289,975	V
3332	755,923	P
3332	-2577,-2409	V
3346	1467,-2357	V
3346	-1879,989	P
3352	1243	
3360	991	
3362	901,1229	V
3362	-2461,-2133	P
3362	1311,1475	P
3362	-2051,-1887	V
3362	985,-1967	P
3362	-2377,1395	V
3364	927,-2089	P
3364	-2437,1275	V
3374	1423,941	
3388	1475,991	
3440	1049	
3444	1411,1459	
3456	1519	
3458	971,-2669	V
3458	-2487,789	V
3468	1421,1433	
3576	1309	
3586	1065,2297	V
3586	-2521,-1289	P
3592	1333	
3600	1319	
3650	1609,1509	F
3650	-2041,-2141	PF
3674	1351,2583	P
3674	-2323,-1091	V
3696	1609	
3698	1547,1031	P
3698	-2151,-2667	V
3700	1329,-1631	F
3700	-2371,2069	PF
3710	1027,803	P
3710	-2683,-2907	V
3712	1617	
3728	1623	
3800	1669,-1371	F
3800	1669,2429	F
3800	-2131,-1371	F
3800	-2131,2429	PF
3816	1397	
3824	1401	
3832	1421	
3844	1487,-2233	P
3844	-2357,1611	V
3848	1427	
3850	1591,1691	F
3850	-2259,-2159	PF
3870	1039,1499	
3872	1495,1143	V
3872	1495,-2729	PA
3872	-2377,1143	PA
3872	-2377,-2729	V
3906	1049,1525	
3936	1729	
3952	1737	
3984	1751	
4064	1489	
4088	1517	
4088	1517,1549	A
4088	1549	
4096	1503	
4104	1507	
4104	1507,1523	A
4104	1523	
4120	1219	
4192	1825	
4208	1833	
4232	1563,1747	V
4232	1563,-2485	PA
4232	-2669,1747	PA
4232	-2669,-2485	V
4240	1847	
4246	1261,1173	V
4246	-2985,-3073	P
4328	1605	
4336	1591	
4344	1595	
4360	1651	
4418	1693,-3195	P
4418	-2725,1223	V
4464	1961	
4480	1969	
4496	1975	
4554	1223,1259	
4600	1749,1701	Ab
4600	1707	
4616	1749	
4624	1769	
4736	2063	
4802	1863,-3037	V
4802	-2939,1765	P
4888	1851	
5000	2101,1901	PFA
5000	2101,-3099	F
5000	-2899,1901	F
5000	-2899,-3099	PFA
5024	2207	

20 crossing

484	67,155	V
484	-417,-329	V
572	179,107	
580	141,109	
610	159,89	
642	125,89	
654	103,91	
744	115,131	
836	153,265	
858	155,131	A
884	415,129	
890	131,409	
918	157,-659	V
918	-761,259	V
946	259,171	
956	183,175	
990	181,269	
1020	247,263	A
1024	159,415	V
1024	-865,-609	V
1028	249,225	A
1064	339,899	V
1064	-725,-165	V
1078	481,205	
1084	343,207	
1092	337,209	
1100	509,191	
1102	345,925	V
1102	-757,-177	V
1110	169,229	
1110	169,511	
1110	229,511	
1118	165,295	
1120	453,-627	F
1120	-667,493	F
1130	407,-497	F
1130	-723,633	F
1158	365,239	
1184	217,281	V
1184	-967,-903	V
1200	419,-1021	V
1200	-781,179	V
1206	331,-821	V
1206	-875,385	V
1216	249,313	
1216	385,993	V
1216	-831,-223	V
1240	567,193	
1242	431,-397	V
1242	-811,845	V
1248	199,329	
1256	509,195	
1258	327,191	Va
1258	-931,-191	Va
1276	219,241	
1280	401,241	V
1280	-879,-1039	V
1292	593,-223	V
1292	-699,1069	V
1304	205,531	
1308	607,269	
1340	323,347	
1344	415,1087	V
1344	-929,-257	V
1348	309,325	
1350	431,-469	V
1350	-919,881	V
1368	253,307	
1372	293,-883	V
1372	-1079,489	V

Table 2 (continued)

1376	263,327	V
1376	-1113,-1049	V
1386	443,-997	V
1386	-943,389	V
1390	211,361	
1406	221,297	
1422	293,-1003	V
1422	-1129,419	V
1428	335,449	
1434	451,301	
1444	341,645	V
1444	-1103,-799	V
1444	265,417	V
1444	-1179,-1027	V
1482	677,235	
1482	677,311	
1482	235,311	
1488	349,643	
1496	313,233	
1508	267,-661	V
1508	-1241,847	V
1532	367,399	A
1540	369,361	A
1548	713,-319	V
1548	-835,1229	V
1548	341,287	
1554	289,275	
1562	271,723	
1564	245,279	
1566	341,-1243	V
1566	-1225,323	V
1568	275,851	V
1568	-1293,-717	V
1576	247,327	
1584	301,707	
1586	713,-899	V
1586	-873,687	V
1596	419,379	
1598	733,-287	V
1598	-865,1311	V
1600	281,681	V
1600	-1319,-919	V
1652	341,-1171	V
1652	-1311,481	V
1692	383,-913	V
1692	-1309,779	V
1710	353,-787	V
1710	-1357,923	V
1712	299,771	
1728	791,359	V
1728	791,-1369	V
1728	-937,359	V
1728	-937,-1369	V
1734	373,-1259	V
1734	-1361,475	V
1748	515,1435	V
1748	-1233,-313	V
1750	361,-759	V
1750	-1389,991	V
1752	407,761	
1764	799,463	V
1764	-965,-1301	V
1778	753,-367	V
1778	-1025,1411	V
1802	335,505	
1804	823,1479	V
1804	-981,-325	V
1812	571,421	
1816	317,285	
1826	827,1491	V
1826	-999,-335	V
1832	841,287	
1836	793,-431	V
1836	-1043,1405	V
1836	379,-1421	V
1836	-1457,415	V
1854	565,-1271	V
1854	-1289,583	V
1854	853,-383	V
1854	-1001,1471	V
1862	333,1313	V
1862	-1529,-549	V
1876	527,-1489	V
1876	-1349,387	V
1886	335,499	V
1886	-1551,-1387	V
1888	331,395	V
1888	-1557,-1493	V
1890	337,407	
1890	863,-397	V
1890	-1027,1493	V
1904	333,299	
1924	865,-695	V
1924	-1059,1229	V
1926	677,-607	V
1926	-1249,1319	V
1936	331,419	V
1936	-1605,-1517	V
1946	429,359	
1946	361,767	
1946	347,697	
1950	581,529	
1952	341,405	V
1952	-1611,-1547	V
1952	425,457	V
1952	425,-1495	V
1952	-1527,457	V
1952	-1527,-1495	V
1962	691,-617	V
1962	-1271,1345	V
1968	365,461	V
1968	-1603,-1507	V
1972	557,-1619	V
1972	-1415,353	V
1974	367,773	
1974	613,703	
1974	613,353	
1974	703,353	
1978	537,365	
1980	409,-1391	V
1980	-1571,589	V
1998	467,-1081	V
1998	-1531,917	V
1998	595,-1529	V
1998	-1403,469	V
2000	373,437	V
2000	-1627,-1563	V
2000	371,1371	V
2000	-1629,-629	V
2002	523,607	P
2002	-1479,-1395	V
2002	613,865	V
2002	-1389,-1137	V
2016	535,-473	V
2016	535,1543	V
2016	-1481,-473	V
2016	-1481,1543	V
2016	535,-905	V
2016	535,1111	V
2016	-1481,-905	V
2016	-1481,1111	V
2016	473,905	V
2016	473,-1111	V
2016	-1543,905	V
2016	-1543,-1111	V
2030	433,363	
2032	637,1653	V
2032	-1395,-379	V
2034	929,-427	V
2034	-1105,1607	V
2034	623,-1393	V
2034	-1411,641	V
2050	443,607	V
2050	-1607,-1443	V
2052	431,-1441	V
2052	-1621,611	V
2054	929,449	
2058	631,547	V
2058	-1427,-1511	V
2064	643,1675	V
2064	-1421,-389	V
2068	555,731	V
2068	-1513,-1337	V
2076	569,647	
2080	553,-487	V
2080	553,1593	V
2080	-1527,-487	V
2080	-1527,1593	V
2082	955,433	
2090	359,579	V
2090	-1731,-1511	V
2096	459,395	V
2096	-1637,-1701	V
2096	651,1699	V
2096	-1445,-397	V
2100	607,943	V
2100	-1493,-1157	V
2112	595,485	
2114	647,1251	V
2114	-1467,-863	V
2128	635,373	
2128	499,403	V
2128	-1629,-1725	V
2130	443,593	
2142	445,-1643	V
2142	-1697,499	V
2142	461,-1555	V
2142	-1681,587	V
2144	503,471	V
2144	503,-1673	V
2144	-1641,471	V
2144	-1641,-1673	V
2156	457,915	
2166	493,-1559	V
2166	-1673,607	V
2178	455,-1669	V
2178	-1723,509	V
2178	511,457	
2178	511,-1685	V
2178	-1667,493	V
2178	511,475	
2178	457,493	
2178	457,-1703	V
2178	-1721,475	V
2178	493,475	
2190	481,979	
2190	457,607	
2196	497,479	
2196	497,-1735	V
2196	-1699,461	V
2196	497,515	
2196	479,461	
2196	479,-1681	V
2196	-1717,515	V
2196	461,515	
2200	779,581	
2200	933,467	
2204	657,387	
2204	933,1009	
2222	599,-1645	V
2222	-1623,577	V
2238	1025,467	
2238	629,695	
2240	513,417	V
2240	-1727,-1823	V
2254	1021,685	V
2254	-1233,-1569	V
2266	609,389	V
2266	-1657,-1877	V
2268	883,-1637	V
2268	-1385,631	V
2272	397,1229	V
2272	-1875,-1043	V
2272	423,1559	V
2272	-1849,-713	V
2278	399,801	
2282	827,421	
2288	525,931	
2288	809,-647	V
2288	809,1641	V
2288	-1479,-647	V
2288	-1479,1641	V
2292	515,629	
2298	505,631	
2304	529,625	V
2304	-1775,-1679	V
2304	535,679	V
2304	535,-1625	V

Table 2 (continued)

2304	-1769,679	V
2304	-1769,-1625	V
2322	685,-1835	V
2322	-1637,487	V
2322	541,-1259	V
2322	-1781,1063	V
2336	441,1609	V
2336	-1895,-727	V
2340	529,-1271	V
2340	-1811,1069	V
2352	491,-1021	V
2352	-1861,1331	V
2352	733,835	
2352	421,-1427	V
2352	-1931,925	V
2356	1039,-1393	V
2356	-1317,963	V
2358	553,-1019	V
2358	-1805,1339	V
2366	439,425	
2368	447,543	V
2368	-1921,-1825	V
2376	701,557	V
2376	701,-1819	V
2376	-1675,557	V
2376	-1675,-1819	V
2380	673,503	
2398	677,853	
2398	1095,431	
2400	1061,539	
2406	871,733	
2412	1057,-551	V
2412	-1355,1861	V
2420	549,989	V
2420	-1871,-1431	V
2422	447,853	
2430	523,-1097	V
2430	-1907,1333	V
2442	1105,449	
2448	1103,-529	V
2448	1103,1919	V
2448	-1345,-529	V
2448	-1345,1919	V
2448	965,-1339	V
2448	-1483,1109	V
2450	1107,743	V
2450	-1343,-1707	V
2454	689,539	
2460	511,1129	
2484	541,-1907	V
2484	-1943,577	V
2484	1135,-521	V
2484	-1349,1963	V
2486	571,659	
2502	1087,-581	V
2502	-1415,1921	V
2508	767,521	
2508	767,653	
2508	521,653	
2514	691,781	
2516	441,543	
2530	887,-1873	V
2530	-1643,657	V
2530	669,449	V
2530	-1861,-2081	V
2534	711,-2005	V
2534	-1823,529	V
2538	1097,-595	V
2538	-1441,1943	V
2546	677,543	
2550	703,533	
2552	895,-1889	V
2552	-1657,663	V
2552	453,717	V
2552	453,-1835	V
2552	-2099,717	V
2552	-2099,-1835	V
2552	575,1049	
2562	785,755	
2562	745,-2027	V
2562	-1817,535	V
2562	563,1145	
2568	577,1135	
2616	725,611	
2616	815,929	
2616	779,797	
2620	1083,-1557	F
2620	-1537,1063	F
2622	1177,571	
2626	783,1187	
2628	1159,1087	
2630	1037,-1067	F
2630	-1593,1563	F
2640	553,1097	
2652	575,745	
2664	1195,619	V
2664	1195,-2045	V
2664	-1469,619	V
2664	-1469,-2045	V
2676	601,751	
2676	1159,625	
2676	817,559	
2678	799,1203	
2682	623,-1165	V
2682	-2059,1517	V
2686	579,477	
2698	565,707	
2700	629,-1531	V
2700	-2071,1169	V
2702	821,709	P
2702	-1881,-1993	V
2704	571,987	V
2704	-2133,-1717	V
2718	745,-2099	V
2718	-1973,619	V
2724	1181,635	
2724	1247,569	
2728	615,769	
2736	625,769	V
2736	625,-1967	V
2736	-2111,769	V
2736	-2111,-1967	V
2744	573,-1163	V
2744	573,1581	V
2744	-2171,-1163	V
2744	-2171,1581	V
2744	601,-1191	V
2744	-2143,1553	V
2744	839,727	V
2744	-1905,-2017	V
2754	1243,-593	V
2754	-1511,2161	V
2758	579,509	
2760	859,509	
2772	599,-2257	V
2772	-2173,515	V
2772	599,-1993	V
2772	-2173,779	V
2772	491,601	
2772	515,779	V
2772	-2257,-1993	V
2778	863,989	
2778	605,851	
2782	1257,829	
2784	853,1003	
2784	775,649	
2794	519,739	V
2794	-2275,-2055	V
2794	641,1137	
2826	613,649	
2832	641,863	
2834	837,-2215	V
2834	-1997,619	V
2838	865,619	
2844	895,-2057	V
2844	-1949,787	V
2844	661,-1235	V
2844	-2183,1609	V
2852	1009,1561	V
2852	-1843,-1291	P
2860	787,-2117	V
2860	-2073,743	V
2862	665,-2215	V
2862	-2197,647	V
2880	1013,-907	V
2880	-1867,1973	V
2882	1073,765	
2892	901,811	
2912	1045,-2371	V
2912	-1867,541	V
2914	1037,1601	V
2914	-1877,-1313	P
2924	1035,519	
2926	851,-2285	V
2926	-2075,641	V
2944	899,853	
2952	929,-1039	V
2952	-2023,1913	V
2954	647,-2125	V
2954	-2307,829	V
2968	551,-1913	V
2968	-2417,1055	V
2988	941,-2155	V
2988	-2047,833	V
2990	907,927	PFA
2990	-2083,-2063	PFA
2992	565,829	V
2992	-2427,-2163	V
2992	653,685	V
2992	-2339,-2307	V
2992	653,1741	V
2992	-2339,-1251	V
2992	685,1741	V
2992	-2307,-1251	V
2994	653,1343	
3000	679,1321	
3010	873,2077	PFA
3010	-2137,-933	PFA
3014	1117,2213	V
3014	-1897,-801	V
3014	1239,679	
3038	687,-2393	V
3038	-2351,645	V
3042	655,-2189	V
3042	-2387,853	V
3042	707,-2137	V
3042	-2335,905	V
3048	689,1343	
3054	931,901	
3056	701,2229	V
3056	-2355,-827	V
3058	667,1779	V
3058	-2391,-1279	V
3060	661,-2219	V
3060	-2399,841	V
3064	1197,-1963	A
3072	673,865	V
3072	673,-2207	V
3072	-2399,865	V
3072	-2399,-2207	V
3078	1351,-701	V
3078	-1727,2377	V
3080	1147,-1973	A
3088	707,2251	V
3088	-2381,-837	V
3096	851,707	V
3096	851,-2389	V
3096	-2245,707	V
3096	-2245,-2389	V
3102	725,947	
3104	1297,1393	
3104	919,951	
3108	577,-2447	V
3108	-2531,661	V
3122	663,-2193	V
3122	-2459,929	V
3130	1277,-1227	F
3130	-1853,1227	Fa
3136	953,1289	V
3136	-2183,-1847	V
3136	1329,-687	V
3136	1329,2449	V
3136	-1807,-687	V
3136	-1807,2449	V
3146	593,681	V
3146	-2553,-2465	V
3152	691,723	V
3152	-2461,-2429	V

Table 2 (continued)

3162	1369,739	
3168	1127,841	
3178	1397,921	V
3178	-1781,-2257	V
3182	845,-2251	V
3182	-2337,931	V
3186	973,-2303	V
3186	-2213,883	V
3192	1381,-899	V
3192	1381,2293	V
3192	-1811,-899	V
3192	-1811,2293	V
3210	1157,983	
3212	597,685	
3216	985,727	
3222	985,-1163	V
3222	-2237,2059	V
3230	847,677	
3256	689,711	
3258	1177,-995	V
3258	-2081,2263	V
3264	995,1181	
3268	691,863	
3278	871,1179	V
3278	-2407,-2099	V
3294	1007,-2377	V
3294	-2287,917	V
3300	623,887	V
3300	-2677,-2413	V
3306	1295,-2533	V
3306	-2011,773	V
3318	1439,773	
3322	881,749	
3328	981,1493	V
3328	-2347,-1835	V
3330	1193,1373	
3358	623,715	
3378	1033,787	
3380	911,1431	V
3380	-2469,-1949	V
3384	731,-1525	V
3384	731,1859	V
3384	-2653,-1525	V
3384	-2653,1859	V
3402	775,-1493	V
3402	-2627,1909	V
3404	987,1033	
3406	735,1423	
3420	781,-1499	V
3420	-2639,1921	V
3430	1511,-729	V
3430	-1919,2701	V
3432	1051,1021	
3444	1003,-2693	V
3444	-2441,751	V
3458	927,1031	
3468	749,-2515	V
3468	-2719,953	V
3468	1531,-2549	V
3468	-1937,919	V
3484	1029,-2723	V
3484	-2455,761	V

3484	759,2007	V
3484	-2725,-1477	V
3498	1261,-2555	V
3498	-2237,943	V
3498	799,755	A
3514	795,-1249	V
3514	-2719,2265	V
3520	763,653	
3520	931,-2269	V
3520	-2589,1251	V
3542	801,-1271	V
3542	-2741,2271	V
3562	961,753	
3600	781,1381	V
3600	-2819,-2219	V
3612	767,-1585	V
3612	-2845,2027	V
3626	1597,-783	V
3626	-2029,2843	V
3640	773,-1523	V
3640	773,2117	V
3640	-2867,-1523	V
3640	-2867,2117	V
3668	1587,1083	V
3668	-2081,-2585	V
3682	1563,2615	V
3682	-2119,-1067	V
3710	809,-1571	V
3710	-2901,2139	V
3738	1621,-815	V
3738	-2117,2923	V
3762	1055,-2905	V
3762	-2707,857	V
3782	2197,1709	C
3800	1003,803	
3838	815,1017	
3844	1673,1425	PA
3844	-2171,-2419	PA
3850	871,1579	A
3888	1189,-2483	V
3888	-2699,1405	V
3894	889,1589	A
3906	2267,1763	C
3918	1453,1159	
3934	849,-2763	V
3934	-3085,1171	V
3952	1061,1165	V
3952	-2891,-2787	V
3952	1733,-1459	V
3952	-2219,2493	V
3962	1719,1159	V
3962	-2243,-2803	V
3972	1477,1171	
3976	1115,-1725	V
3976	1115,2251	V
3976	-2861,-1725	V
3976	-2861,2251	V
3978	1681,841	
3982	1119,1053	
3982	1119,1075	
3982	1119,-2885	V
3982	-2863,1097	V

3982	1053,-2907	V
3982	-2929,1075	V
3982	1053,1097	
3982	1075,1097	
4002	1651,1697	
4004	1103,-2945	V
4004	-2901,1059	V
4004	1103,1125	
4004	1103,873	
4004	1103,1081	
4004	1103,-3153	V
4004	-2901,851	V
4004	1059,1125	
4004	1059,873	
4004	1059,1081	
4004	1059,851	V
4004	-2945,-3153	V
4004	1125,-3131	V
4004	-2879,873	V
4004	1125,-2923	V
4004	-2879,1081	V
4004	1125,851	
4004	873,1081	V
4004	-3131,-2923	V
4004	873,851	
4004	1081,851	
4010	1183,1223	PFAa
4010	1223,2827	PFAa
4018	867,-3109	V
4018	-3151,909	V
4020	1193,1103	
4032	1193,857	V
4032	1193,-3175	V
4032	-2839,857	V
4032	-2839,-3175	V
4080	889,1831	
4096	1215,1727	V
4096	-2881,-2369	V
4096	1601,1473	PA
4096	1601,-2623	V
4096	-2495,1473	V
4096	-2495,-2623	PA
4108	1103,895	V
4108	-3005,-3213	V
4128	1217,1151	
4130	877,-2987	V
4130	-3253,1143	V
4134	1751,875	
4142	1751,-2429	V
4142	-2391,1713	V
4160	1227,1123	V
4160	-2933,-3037	V
4180	1757,-2347	V
4180	-2423,1833	V
4186	1731,1501	
4188	1159,913	
4200	1243,907	V
4200	1243,-3293	V
4200	-2957,907	V
4200	-2957,-3293	V
4228	913,-3063	V
4228	-3315,1165	V

4236	923,1181	
4242	1165,1255	
4246	917,961	A
4256	1853,1237	V
4256	-2403,-3019	V
4270	1241,2461	V
4270	-3029,-1809	V
4278	1765,1535	
4290	1277,1187	
4296	1267,1189	
4318	1773,2997	V
4318	-2545,-1321	V
4332	989,-3115	V
4332	-3343,1217	V
4344	1949,947	
4344	1289,1607	
4352	1331,2555	V
4352	-3021,-1797	V
4356	1651,1915	PA
4356	-2705,-2441	PA
4394	1981,1305	
4398	1217,995	
4398	1301,1235	
4452	1655,1313	
4458	1645,1327	
4462	1891,1845	
4466	1587,-3285	V
4466	-2879,1181	V
4484	1891,-2517	V
4484	-2593,1967	V
4488	1261,1669	V
4488	1261,-2819	V
4488	-3227,1669	V
4488	-3227,-2819	V
4488	1237,973	V
4488	1237,-3515	V
4488	-3251,973	V
4488	-3251,-3515	V
4506	1985,1019	
4512	1339,1669	
4550	963,1223	
4566	2011,1033	
4592	991,-1921	V
4592	991,2671	V
4592	-3601,-1921	V
4592	-3601,2671	V
4606	979,-1933	V
4606	-3627,2673	V
4614	1703,1373	
4620	997,-2027	V
4620	-3623,2593	V
4624	1701,1293	V
4624	-2923,-3331	V
4628	1955,3619	V
4628	-2673,-1009	V
4674	1303,1057	
4674	1687,-2741	V
4674	-2987,1933	V
4708	1305,1019	
4712	1733,-2067	V
4712	1733,2645	V
4712	-2979,-2067	P

Table 2 (continued)

4712	-2979,2645	V	4978	-3641,1075	V	5434	-3187,-3967	V	5828	2131,3259	C
4722	1751,1397		5018	1399,1347	V	5434	2247,-3473	V	5850	1571,2281	
4756	1315,1257		5018	-3619,-3671	V	5434	-3187,1961	V	5966	2467,-3461	V
4782	1333,1423		5050	1969,-2071	F	5434	1467,-3473	V	5966	-3499,2505	V
4784	1411,2147	V	5050	-3081,2979	F	5434	-3967,1961	V	6094	1677,-4395	V
4784	-3373,-2637	V	5074	1981,1803		5456	1521,2017	V	6094	-4417,1699	V
4796	1337,-3503	V	5112	1933,-3211	A	5456	1521,-3439	V	6136	2325,-3859	A
4796	-3459,1293	V	5128	1907,-3245	A	5456	-3935,2017	V	6152	2275,-3869	A
4802	2015,-1037	V	5150	2161,-3039	F	5456	-3935,-3439	V	6348	2345,-3727	V
4802	-2787,3765	V	5150	-2989,2111	F	5456	1181,1467		6348	-4003,2621	V
4836	1435,1357		5166	1441,1387		5546	2339,-3113	V	6498	1799,1745	
4850	2039,-2861	F	5172	2119,2191		5546	-3207,2433	V	6578	2569,-4035	F
4850	-2811,1989	F	5196	2129,2153		5704	3189,2085	C	6578	-4009,2543	F
4898	2149,3097	P	5278	1139,-3817	V	5776	1595,2203	V	6604	2501,-2579	F
4898	-2749,-1801	V	5278	-4139,1461	V	5776	-4181,-3573	V	6604	-4103,4025	F
4950	2029,-1931	F	5302	1425,1469		5798	1565,1617	V	6900	2899,2851	A
4950	-2921,3019	F	5336	1469,1411		5798	-4233,-4181	V	7500	3151,-4649	F
4950	1073,1337	V	5382	1445,2225	V	5808	2243,-4093	V	7500	-4349,2851	F
4950	-3877,-3613	V	5382	-3937,-3157	V	5808	-3565,1715	V			
4978	1337,-3903	V	5434	2247,1467	V	5814	1561,2255				

References

[1] G.Burde and H. Zieschang, "Knots", de Gruyter, Berlin and New York, 1986.

[2] J. H. Conway, An enumeration of knots and links, in "Computational Problems in Abstract Algebra", (ed. J. Leech) Pergamon Press, New York, 1969, pp. 329–358.

[3] C. Ernst and D. W. Sumners, The growth of the number of prime knots, Math. Proc. Cambridge Philos. Soc., **102** (1987), 303–315.

[4] R. Hartley, The Conway potential function for links, Comment. Math. Helv., **58** (1983), 365–378.

[5] T. Kanenobu, Examples on polynomial invariants of knots and links, Math. Ann., **275** (1986), 555–572.

[6] T. Kanenobu, Examples on polynomial invariants of knots and links II, Osaka J. Math., **26** (1989), 465–482.

[7] T. Kanenobu, Jones and Q polynomials for 2-bridge knots and links, Proc. Amer. Math. Soc., **110** (1990), 835–841.

[8] T. Kanenobu, Kauffman polynomials for 2-bridge knots and links, Yokohama Math. J, **38** (1991), 145–154.

[9] T. Kanenobu and T. Sumi, Polynomial invariants of 2-bridge knots through 22 crossings, preprint.

[10] H. Schubert, Knoten mit zwei Brücken, Math. Z., **65** (1956), 133–170.

[11] L. Siebenmann, Exercices sur les nœuds rationels, preprint.

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Invariants of Spatial Graphs

Jun Murakami

§1. Introduction

The purpose of this paper is to construct invariants of spatial graphs from regular isotopy invariants of non-oriented link diagrams of *knit trace type*. Kauffman's bracket polynomial [4], which is a version of the Jones polynomial, is of knit trace type. The Dubrovnik polynomial [5], which is used in the definition of the Kauffman polynomial, is also of knit trace type [6]. Hence these two invariants are generalized to invariants of spatial graphs by our method. The Yamada polynomial introduced in [10] is the non-trivial simplest one of our invariants. A similar invariants are introduced in [9] for ribbon graphs. They use quasi-triangular Hopf algebras. But we use representations of knit semigroups or braid groups instead of Hopf algebras.

To introduce regular isotopy invariants of link diagrams of *knit trace type*, we need notion of a *Markov knit sequence*. Let $\mathbb{C}$ be the field of complex numbers. Knit semigroups K_n, $(n = 1, 2, \cdots)$ are introduced in [6] defined by the following generators and relations.

$$\begin{aligned} K_n = \langle \tau_1, \cdots, \tau_{n-1},\ \tau_1^{-1}, \cdots, \tau_{n-1}^{-1},\ \varepsilon_1, \cdots, \varepsilon_{n-1} \quad | \\ \tau_i \tau_i^{-1} = \tau_i^{-1} \tau_i = 1, \quad \tau_i \tau_j = \tau_j \tau_i \ (|i-j| \geq 2), \\ \tau_i \tau_{i+1} \tau_i = \tau_{i+1} \tau_i \tau_{i+1}, \quad \tau_i \varepsilon_j = \varepsilon_j \tau_i \ (|i-j| \geq 2), \\ \varepsilon_i \varepsilon_{i\pm 1} \varepsilon_i == \varepsilon_i, \quad \varepsilon_i \varepsilon_j = \varepsilon_j \varepsilon_i \ (|i-j| \geq 2), \\ \varepsilon_i \tau_{i\pm 1} = \varepsilon_i \varepsilon_{i\pm 1} \tau_i^{-1}, \quad \varepsilon_i \tau_{i\pm 1}^{-1} = \varepsilon_i \varepsilon_{i\pm 1} \tau_i, \\ \tau_{i\pm 1} \varepsilon_i = \tau_i^{-1} \varepsilon_{i\pm 1} \varepsilon_i, \quad \tau_{i\pm 1}^{-1} \varepsilon_i = \tau_i \varepsilon_{i\pm 1} \varepsilon_i \rangle \end{aligned}$$

The generators of K_n are presented graphically as in Figure 1. In the graphical presentation, the product of two elements of K_n corresponds to the composite of two diagrams as in the case of braid groups. Let

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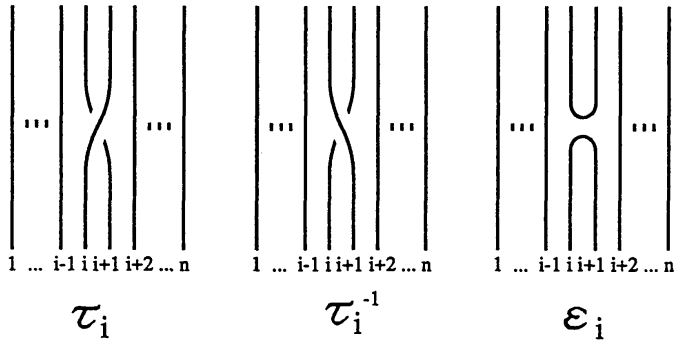

Fig. 1. Generators of K_n.

$\mathbb{C}K_n$ be the semigroup algebra of K_n over $\mathbb{C}$. We regard the braid group B_n as a subsemigroup of K_n generated by $\tau_1, \tau_2, \cdots, \tau_{n-1}$.

Let γ be a non-zero complex number. Knit semigroup algebra with writhe factor γ, denoted by $K_n(\gamma)$, is a quotient algebra of $\mathbb{C}K_n$ defined by the following.

$$K_n(\gamma) = \mathbb{C}K_n/(\tau_i^{\pm 1}\varepsilon_i - \gamma^{\pm 1}\varepsilon_i, \quad \varepsilon_i\tau_i^{\pm 1} - \gamma^{\pm 1}\varepsilon_i \quad (1 \leq i \leq n-1)\,).$$

Let A be a semisimple $\mathbb{C}$-algebra. Let $\hat{A}$ be the set of equivalence classes of irreducible representations of A. A $\mathbb{C}$-linear map T from A to $\mathbb{C}$ is called a trace if T is a linear combination of irreducible characters of A, i.e.

$$T(x) = \sum_{\rho \in \hat{A}} a_\rho \ \mathrm{Trace}(\rho(x)) \quad (a_\rho \in \mathbb{C}) \tag{1.1}$$

The trace T is called *faithful* if all the coefficients a_ρ are not equal to 0. A sequence A_1, A_2, $\cdots$, A_n, $\cdots$ of semisimple $\mathbb{C}$-algebras are called *a knit type sequence* if they satisfy the following.

(1) There is an algebra epimorphism p_n from $K_n(\gamma)$ to A_n and monomorphism j_n from A_n to A_{n+1} such that $j_n \circ p_n = p_{n+1} \circ i_n$ for $n = 1, 2, \cdots$, where i_n is an inclusion from $K_n(\gamma)$ to $K_{n+1}(\gamma)$ which sends $\tau_i^{\pm 1} \in K_n(\gamma)$ to $\tau_i^{\pm 1} \in K_{n+1}(\gamma)$ and $\varepsilon_i \in K_n(\gamma)$ to $\varepsilon_i \in K_{n+1}(\gamma)$ for $1 \leq i \leq n-1$.

(2) There are a complex number μ and a faithful trace T_n from A_n to $\mathbb{C}$ which satisfy the following. For any $x \in A_n$, $T_{n+1}(j_n(x)) = \mu\, T_n(x)$, $T_n(x) = \gamma^{\pm 1}\, T_{n+1}(j_n(x)\, p_{n+1}(\tau_n^{\pm 1}))$ and $T_n(x) = T_{n+1}(x\, p_{n+1}(\varepsilon_n))$.

For $x \in K_n$, let $\hat{x}$ denote the link diagram obtained from the closure of x (Figure 2). A regular isotopy invariant X of link diagrams is called of *knit trace type* if there is a Markov knit sequence and X is obtained by the traces of it, i.e. $X(\hat{x}) = T_n(p_n(x))$ for $x \in K_n$. Kauffman's bracket polynomial [4] is of knit trace type (see Section 3 of [7]). The Dubrovnik polynomial is also of knit trace type [6].

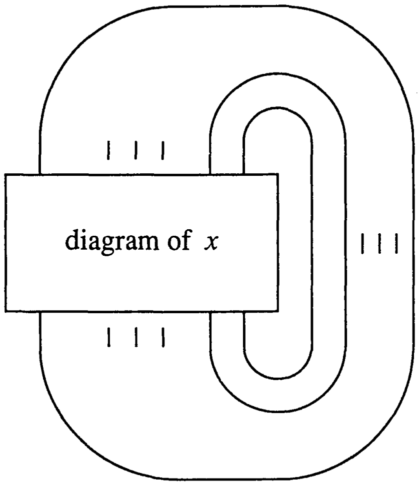

Fig. 2. Closure of $x \in K_n$.

Remark. Let X be a regular isotopy invariant of knit trace type with writhe factor γ. For an oriented link diagram x, there are a positive integer n and $y \in K_n$ such that $\hat{y}$ is equal to x without orientation. Let $w(x)$ be the sum of signatures of the crossings of x. Let $X'(x) = \gamma^{w(x)} X(\hat{y})$. Then X' is an invariant of links.

Now we define spatial graphs in S^3. Let $\mathcal{V}$ is a set of 2-disks and $\mathcal{E}$ be a set of edges homeomorphic to $[0,1]$ in S^3. Each edge has an orientation induced by the orientation of $[0,1]$. The terminal points of an edge corresponding to 0 and 1 are called the initial point and the final point of the edge respectively. The pair $\Gamma = (\mathcal{V}, \mathcal{E})$ is called an oriented spatial graph if it satisfies the following. The disks in $\mathcal{V}$ are mutually disjoint and the edges in $\mathcal{E}$ are mutually disjoint. Also assume that the interiors of the disks in $\mathcal{V}$ and edges in $\mathcal{E}$ are mutually disjoint. Terminal points of edges in $\mathcal{E}$ are contained in the boundaries of disks in $\mathcal{V}$. Two spatial graphs Γ and Γ' are called equivalent if there is an isotopy of S^3 which sends Γ to Γ'. A spatial graph Γ is called an embedding of a

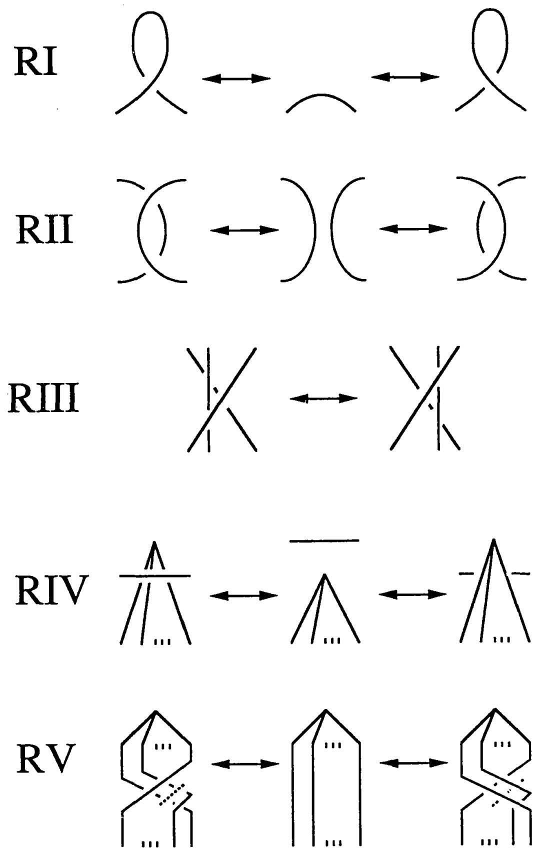

Fig. 3. Reidemeister moves.

tri-valent graph if the degree of all the vertices of Γ are equal to 3. A diagram of a spatial graph is defined as in the case of a link.

Proposition 1. *Two spatial graphs* Γ *and* Γ' *are equivalent if and only if there is a sequence of Reidemeister moves of types* (SRI)–(SRV) *sending a diagram of* Γ *to a diagram of* Γ'.

For a spatial graph Γ, we define a diagram of Γ as in the case of links. Let A_1, A_2, $\cdots$ be a Markov knit sequence. For each edge E of Γ, we associate a non-negative integer $N(E)$, an irreducible representation $R(E) \in \hat{A}_{n(E)}$ and a signature $S(E)$. The triple (N, R, S) is called a coloring of Γ if it satisfies the following. For a vertex v of Γ, let $\mathcal{E}_v$ be a set of edges with terminal point v Then

$$(1.2) \qquad \sum_{E \in \mathcal{E}_v} N(E) = \textit{even} \text{ and } 2N(E) \leq \sum_{E' \in \mathcal{E}_v} N(E') \text{ for all } E \in \mathcal{E}_v.$$

We construct an invariant of spatial graphs colored as above. First, we generalize link invariants of *braid trace type* to invariants of colored oriented tri-valent graph embeddings in S^3 in §2. And then we generalize invariants of *knit trace type* to invariants of colored spatial graphs in §3. By attaching the same color to all the edges of graphs, we get invariants of spatial graphs. In §4, we give some examples.

§2. Invariants of colored oriented tri-valent graphs

In this section, we generalize link invariants of braid trace type to invariants of embeddings of colored oriented tri-valent graphs in S^3. To introduce link invariants of braid trace type, we need notion of a *Markov braid sequence.*

Definition. A sequence (A_1, T_1), (A_2, T_2), $\cdots$, (A_n, T_n), $\cdots$ of pairs of a semisimple $\mathbb{C}$-algebra and its trace are called a *Markov braid sequence* if they satisfy the following.

(1) There is an algebra homomorphism p_n from $\mathbb{C}B_n$ to A_n and j_n from A_n to A_{n+1} such that $j_n \circ p_n = p_{n+1} \circ i_n$ for $n = 1, 2, \cdots$, where i_n is an inclusion from $\mathbb{C}B_n$ to $\mathbb{C}B_{n+1}$ which sends $\sigma_i \in \mathbb{C}B_n$ to $\sigma_i \in \mathbb{C}B_{n+1}$ for $1 \leq i \leq n-1$.

(2) There is a faithful trace T_n from A_n to $\mathbb{C}$ and $\mu, c \in k \backslash \{0\}$ which satisfy $\mu\, T_n(x) = T_{n+1}(j_n(x))$, $T_n(x) = c\, T_{n+1}(x\, p_{n+1}(\sigma_n))$ and $T_n(x) = c^{-1}\, T_{n+1}(x\, p_{n+1}({\sigma_n}^{-1}))$ for any $x \in A_n$.

From a Markov braid sequence, we get a $\mathbb{C}$-valued link invariant. For a braid $b = {\sigma_{i(1)}}^{\varepsilon(1)} {\sigma_{i(2)}}^{\varepsilon(2)} \cdots {\sigma_{i(r)}}^{\varepsilon(r)} \in B_n$, let $w(b) = \sum_{i=1}^{r} \varepsilon(i)$. Then $w(b)$ is a sum of signatures of all the crossings of b. For a braid b, let $\hat{b}$ denote the link obtained from the closure of b. Let

$$X(\hat{b}) = c^{-w(b)}\, T_n(p_n(b)).$$

Then Alexander's theorem and Markov's theorem ([1], Theorem 2.1 and 2.2) implies that X is an invariant of links. Link invariant obtained from a Markov braid sequence as above is called of *braid trace type.* Jones polynomial, HOMFLY polynomial and Kauffman polynomial are all of braid trace type and the associated braid type sequences are Jones algebras, Iwahori's Hecke algebras and a q-analogue of Brauer's algebras respectively ([2], [3], [6], [8]).

From now on, fix an invariant X of braid trace type and let (A_1, T_1), (A_2, T_2), $\cdots$ be the Markov braid sequence of X. Since A_n is a semisimple algebra, we have

$$A_n = \bigoplus_{\rho \in \hat{A}_n} M_{d(\rho)}(\mathbb{C})$$

where $d(\rho)$ is the degree of ρ. Let q_ρ be an element of A_n such that

$$\nu(q_\rho) = \delta_{\nu\rho}\ \mathrm{id} \in M_{d(\nu)}(\mathbb{C}) \qquad \text{for} \quad \nu \in \hat{A}_n.$$

Let $\tilde{q}_\rho$ be an element of $\mathbb{C}B_n$ such that $p_n(\tilde{q}_\rho) = q_\rho$. Note that $\tilde{q}_\rho$ is not unique. Let $h_n = \sigma_1\sigma_2\cdots\sigma_{n-1}\sigma_1\cdots\sigma_{n-2}\cdots\sigma_1\sigma_2\,\sigma_1$. We call h_n the *half twist* of B_n. Let $f_n = {h_n}^2$ and we call f_n the *full twist* of B_n. It is known that f_n commute with every element of B_n and so $\rho(p_n(f_n))$ is a scalar matrix, i.e. $\rho(p_n(f_n)) = \alpha_\rho$ id.

A formal $\mathbb{C}$-linear combination of link diagrams are called *a virtual link diagram*. We generalize the link invariant X to a function from virtual link diagrams to $\mathbb{C}$ formally as follows. For a virtual link diagram $L = \sum_{i=1}^r a_i\, L_i$ ($a_i \in k$, L_i is a link diagram), let $X(L) = \sum_{i=1}^r a_i\, X(L_i)$.

As in the case of links, we define a diagram of an oriented tri-valent graph embedded in S^3. Let G be an oriented tri-valent graph. We define a *coloring* of G. For each edge E of G, associate a non-negative integer $N(E)$, an irreducible representation $R(E) \in \hat{A}_{n(E)}$ and a signature $S(E) = \pm 1$. The triple (N, R, S) is called a coloring of G if it satisfies the following. For a vertex v of G, let E_v^- be a set of edges with end point v and E_v^+ a set of edges with start point v. Then

$$\sum_{E \in E_v^-} N(E) = \sum_{E \in E_v^+} N(E).$$

Let Γ be a diagram of an embedding of an oriented tri-valent graph G colored by (N, R, S). We identify the edge sets of Γ and G. For an edge E of Γ, let $\beta(E) = \frac{1}{2}\,\tilde{q}_{R(E)}\,(1 + S(E)\,\alpha_{R(E)}^{-1/2}\,h_n) \in \mathbb{C}B_{N(E)}$. Replace every vertices and edges as in Figure 4, we get a virtual link diagram $\Gamma^{(N,R,S)}$. For a edge E of Γ, let $c(E) = S(E)\,\alpha_{R(E)}^{1/2}$.

Theorem 2. *Let Γ and Γ' be equivalent embeddings of an oriented tri-valent graph G colored by (N, R, S). Then, for every edge E of G, there is an integer $d(E)$ such that*

$$X(\Gamma^{(N,R,S)}) = \prod_{E \in \mathcal{E}} c(E)^{d(E)} X({\Gamma'}^{(N,R,S)}). \tag{2.1}$$

Proof. We check (2.1) for Reidemeister moves (SRI)–(SRV). Let Γ and Γ' be diagrams of embeddings of G. We identify the sets of edges of Γ and Γ' with that of G.

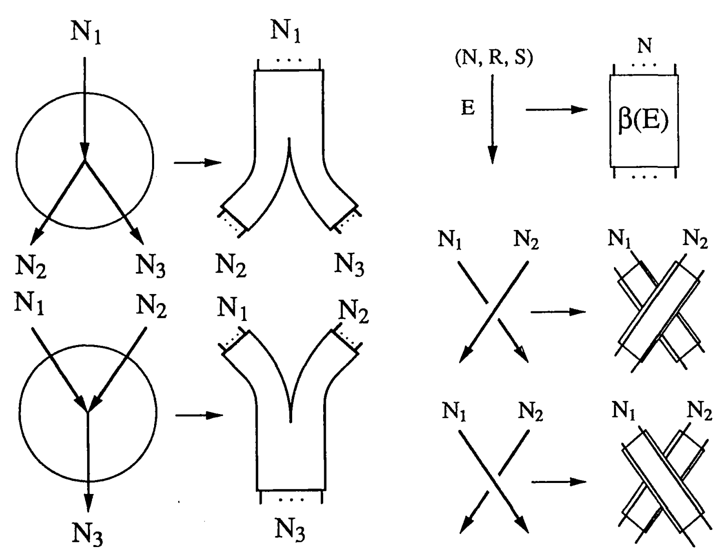

Fig. 4. Replace vertices and edges.

Case 1. Assume that Γ and Γ' are regular isotopic, i. e. there is a sequence of Reidemeister moves of types (SRII), (SRIII), (SRIV) sending Γ to Γ'. Then the associated virtual link diagrams $\Gamma^{(N,R,S)}$ and $\Gamma'^{(N,R,S)}$ are equivalent. Hence we have

$$X(\Gamma^{(N,R,S)}) = X(\Gamma'^{(N,R,S)}). \tag{2.2}$$

Case 2. In this and the next cases, we check (2.1) for (SRI) moves. Assume that Γ and Γ' are identical except within a ball where they are as shown in Figure 5. Let E be the edge of G embedded differently by Γ and Γ'. Let $n = N(E)$, $\rho = R(E)$, $s = S(E)$ and $\beta = \beta(E)$. Then there are positive integer N and a braid $b \in \mathbb{C}B_N$ such that the associated link diagrams $\Gamma^{(N,R,S)}$ and $\Gamma'^{(N,R,S)}$ are equivalent to the closures of $b_1 = b\,\eta(\beta)$ and $b_2 = b\,\eta(\beta)\,f_n$ where η is an algebra homomorphism from $\mathbb{C}B_n$ to $\mathbb{C}B_N$ defined by $\eta(\sigma_i) = \sigma_i$ for $1 \leq i \leq n-1$. Since X is an invariant of trace type, there is an algebra homomorphism J from A_n to A_N such that $p_N \circ \eta = J \circ p_n$. From the definition of trace type invariants, we have

$$X(\hat{b}_2) = T_N(p_N(b_2)) = T_N(p_N(b\,\eta(\beta)\,f_n)).$$

The definitions of q_ρ and β imply that $p_n(\beta\, h_n^{\pm 1}) = (s\,\alpha_\rho^{1/2})^{\pm 1} p_n(\beta)$. Hence we have

$$T_N(p_N(b\,\eta(\beta)\,f_n)) = T_N(p_N(b)\,J(p_n(\beta\,{h_n}^2)))$$

$$=T_N(p_N(b)J(\alpha_\rho\, p_n(\beta))) = \alpha_\rho\, T_N(p_N(b)J(p_n(\beta))),$$

and so we get

$$X(\hat{b}_2) = \alpha_\rho\, X(\hat{b}_1).$$

In other words,

$$X(\Gamma^{(N,R,S)}) = \alpha_\rho\, X(\Gamma'^{(N,R,S)}). \tag{2.3}$$

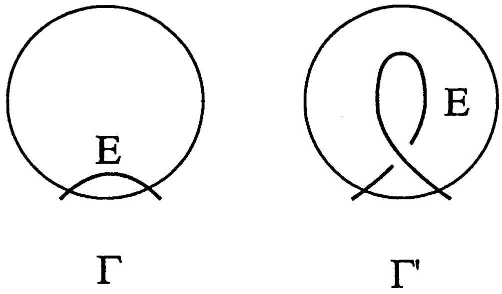

Fig. 5.

Case 3. Let Γ and Γ' be diagrams of colored tri-valent graphs identical except within a ball where they are as shown in Figure 6. Then, as in Case 2, we have

$$X(\Gamma^{(N,R,S)}) = \alpha_{R(E)}^{-1}\, X(\Gamma'^{(N,R,S)}). \tag{2.4}$$

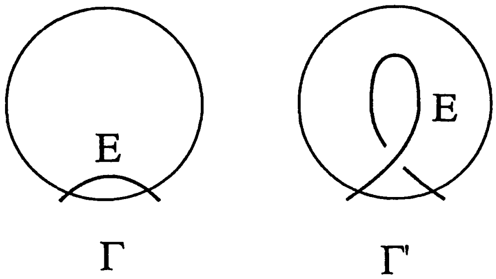

Fig. 6.

Case 4. To check (SRV), it is suffice to verify the theorem for moves illustrated in Figures 7–10. Assume that Γ and Γ' are identical except within a ball where they are as shown in Figure 7. Let $n(i) = N(E_i)$, $\rho(i) = R(E_i)$, $s(i) = S(E_i)$, $\tilde{q}_i = \tilde{q}_{\rho(i)}$, $q_i = q_{\rho(i)}$, $p_i = p_{n(i)}$, $h_i = h_{n(i)}$ and $\beta_i = \beta(E_i)$ for $i = 1, 2, 3$. Then there are positive integer N and $b \in \mathbb{C}B_N$ such that the associated link diagrams $\Gamma^{(N,R,S)}$ and $\Gamma'^{(N,R,S)}$ are equivalent to the closures of

$$\begin{aligned} b_1 =& b\eta_1(\beta_1)\,\eta_2(\beta_2)\,\eta_3(\beta_3), \\ b_2 =& b\eta_1(\beta_1)\,\eta_2(f_{n(2)}\,\beta_2)\,\sigma_{n(1),n(2)}\,\eta_3(\beta_3), \end{aligned}$$

where $\sigma_{n(1),n(2)} = \sigma_{n(1)}\sigma_{n(1)+1}\cdots\sigma_{n(1)+n(2)-1}\ \sigma_{n(1)-1}\cdots\sigma_{n(1)+n(2)-2}\cdots\sigma_1\sigma_2\cdots\sigma_{n(2)}$ and η_1, η_2 η_3 are algebra homomorphisms from $\mathbb{C}B_{n(1)}$, $\mathbb{C}B_{n(2)}$ $\mathbb{C}B_{n(3)}$ to $\mathbb{C}B_N$ defined by the following. $\eta_1(\sigma_i) = \sigma_i$ for $1 \leq i \leq n(1) - 1$, $\eta_2(\sigma_i) = \sigma_{n(1)+i}$ for $1 \leq i \leq n(2) - 1$ and $\eta_3(\sigma_i) = \sigma_i$ for $1 \leq i \leq n(3) - 1$. We know that $\eta_1(h_{n(1)})\eta_2(h_{n(2)}\sigma_{n(1),n(2)}) = \eta_3(h_{n(3)})$. Hence we have

$$b_2 \doteq b\,\eta_1(\beta_1\,h_1^{-1})\eta_2(\beta_2\,h_2)\eta_3(h_3\beta_3).$$

Since X is an invariant of trace type, there are algebra homomorphisms J_1, J_2 and J_3 from $A_{n(1)}$, $A_{n(2)}$ and $A_{n(3)}$ to A_N such that $p_N \circ \eta_s = J_s \circ p_{n(s)}$ for $s = 1, 2, 3$. From the definition of the trace type, we have

$$\begin{aligned} X(\hat{b}_2) =& T_N(p_N(b_2)) \\ =& T_N(p_N(b\eta_1(\beta_1\,h_1^{-1})\,\eta_2(\beta_2\,h_2)\,\eta_3(h_3\,\beta_3)) \\ =& T_N(p_N(b)\,J_1(p_1(\beta_1\,h_1^{-1}))\,J_2(p_2(\beta_2\,h_2))\,J_3(p_3(h_3\,\beta_3))). \end{aligned}$$

The definition of q_R and $\beta(E)$ implies that

$$p_t(\beta(t)\,h_t^{\pm 1}) = S(t)\,\alpha_{\rho(t)}^{\pm 1/2}\,p_t(\beta_t) \qquad (t = 1, 2, 3).$$

Hence we have

$$\begin{aligned} & T_N(p_N(b)\,J_1(p_1(\beta_1\,h_1^{-1}))\,J_2(p_2(\beta_2\,h_2))\,J_3(p_3(h_3\,\beta_3))\,) \\ =& (\prod_{t=1}^{3} s(t))\,\alpha_{\rho(1)}^{-1/2}\,\alpha_{\rho(2)}^{1/2}\,\alpha_{\rho(3)}^{1/2}\,T_N(p_N(b)\,J_1(p_1(\beta_1))\,J_2(p_2(\beta_2))\,J_3(p_3(\beta_3))), \end{aligned}$$

and so we get

$$X(\hat{b}_2) = s(1)\,\alpha_{\rho(1)}^{-1/2}\,s(2)\,\alpha_{\rho(2)}^{1/2}\,s(3)\,\alpha_{\rho(3)}^{1/2}\,X(\hat{b}_1).$$

In other words,

$$X(\Gamma^{(N,R,S)}) = s(1)\,\alpha_{\rho(1)}^{-1/2}\,s(2)\,\alpha_{\rho(2)}^{1/2}\,s(3)\,\alpha_{\rho(3)}^{1/2} X(\Gamma'^{(N,R,S)}). \tag{2.5}$$

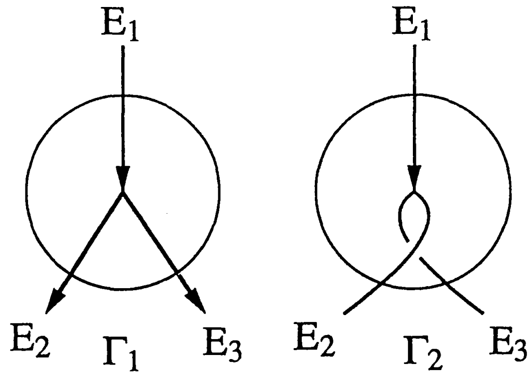

Fig. 7.

Case 5. Assume that Γ and Γ' are identical except within a ball where they are as shown in Figure 8. Then, as in Case 4, we have

$$X(\Gamma^{(N,R,S)}) = s(1)\,\alpha_{\rho(1)}^{1/2}\,s(2)\,\alpha_{\rho(2)}^{-1/2}\,s(3)\,\alpha_{\rho(3)}^{-1/2}\,X(\Gamma'^{(N,R,S)}). \tag{2.6}$$

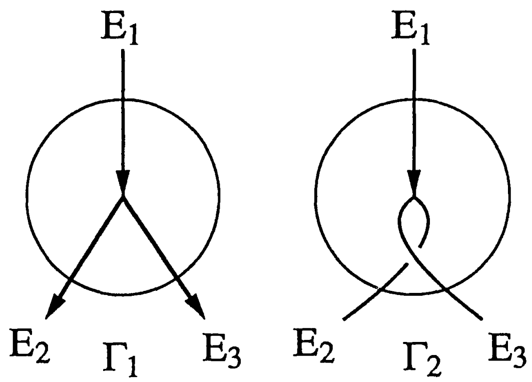

Fig. 8.

Case 6. Assume that Γ and Γ' are identical except within a ball where they are as shown in Figure 9. Then, as in Case 4, we have

$$X(\Gamma^{(N,R,S)}) = s(1)\,\alpha_{\rho(1)}^{1/2}\,s(2)\,\alpha_{\rho(2)}^{1/2}\,s(3)\,\alpha_{\rho(3)}^{-1/2}\,X(\Gamma'^{(N,R,S)}). \tag{2.7}$$

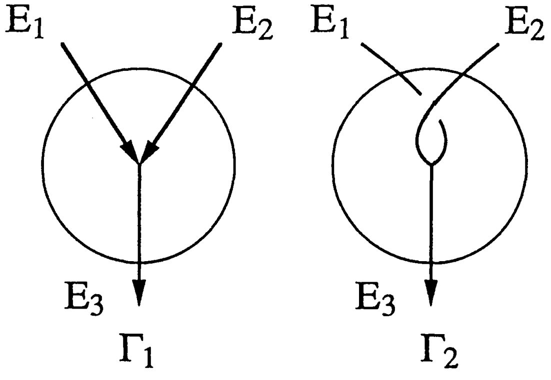

Fig. 9.

Case 7. Let Γ and Γ' be diagrams of colored tri-valent graphs identical except within a ball where they are as shown in Figure 10. Then, as in Case 4, we have

$$X(\Gamma^{(N,R,S)}) = s(1)\,\alpha_{\rho(1)}^{-1/2}\,s(2)\,\alpha_{\rho(2)}^{-1/2}\,s(3)\,\alpha_{\rho(3)}^{1/2}\,X(\Gamma'^{(N,R,S)}). \tag{2.8}$$

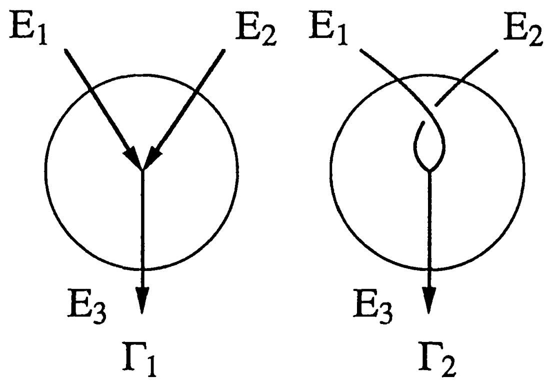

Fig. 10.

The above formulas (2.2)–(2.8) implies Theorem 2. Q.E.D.

§3. Invariants of non-oriented spatial graphs

Let X be a regular isotopy invariant of link diagrams of knit trace type with writhe factor γ. Let G be an abstract graph. For each edge E of G, we attach a non-negative integer $N(E)$, an irreducible representation $R(E) \in \check{A}_{N(E)}$ and a signature $S(E) = \pm 1$. If these data satisfy

(1.2) in §1, they are called a *coloring* of G and denoted by (N, R, S). Let $\mathcal{E}_v$ be the subset of edges of G with a terminal point v.

From now on, fix an invariant X of knit trace type and let (A_1, T_1), (A_2, T_2), $\cdots$ be the Markov knit sequence of X. Since A_n is a semisimple algebra, we have

$$A_n = \bigoplus_{\rho \in \check{A}_n} M_{d(\rho)}(\mathbb{C})$$

where $d(\rho)$ is the degree of ρ. Let q_ρ be an element of A_n such that

$$\nu(q_\rho) = \delta_{\nu\rho} \text{ id} \in M_{d(\nu)}(\mathbb{C}) \qquad \text{for} \quad \nu \in \check{A}_n.$$

Let $\tilde{q}_\rho$ be an element of $\mathbb{C}K_n$ such that $p_n(\tilde{q}_\rho) = q_\rho$. Note that $\tilde{q}_\rho$ is not unique. Let $h_n = \tau_1 \tau_2 \cdots \tau_{n-1} \tau_1 \cdots \tau_{n-2} \cdots \tau_1 \tau_2 \tau_1$. We call h_n the *half twist* of K_n. Let $f_n = {h_n}^2$ and we call f_n the *full twist* of K_n. It is known that f_n commute with every element of K_n and so $\rho(p_n(f_n))$ is a scalar matrix, i.e. $\rho(p_n(f_n)) = \alpha_\rho$ id.

Let G be an abstract graph colored by (N, R, S). Let Γ be a colored non-oriented spatial graph equal to G as an abstract graph. We identify the sets of edges of Γ and G. Let v be a vertex of Γ. Let E_1, E_2, $\cdots$, E_r be the edges with a terminal point v. Let ξ_1, ξ_2, $\cdots$, ξ_r be the terminal points of E_1, E_2, $\cdots$, E_r on the boundary of v and $N(i) = N(E_i)$ for $i = 1, 2, \cdots, r$. Replace these points by $\zeta_1^{(1)}$, $\zeta_1^{(2)}$, $\cdots$, $\zeta_1^{(N(1))}$, $\zeta_2^{(1)}$, $\cdots$, $\zeta_2^{(N(2))}$, $\cdots$, $\zeta_r^{(1)}$, $\cdots$, $\zeta_r^{(N(r))}$ as in Figure 11. Let $n_v = (\sum_{i=1}^r N(i))/2$. A diagram D on v is a set of mutually disjoint n_v curves connecting $\gamma_{i(1)}^{j(1)}$ to $\gamma_{i(2)}^{j(2)}$. Two diagrams D and D' on v are called equivalent if there is an isotopy of v sending D to D' which fixes the boundary of v. A diagram D on v is called *essential* if D satisfies the following.

(*) Let $\gamma_{i(1)}^{j(1)}$ and $\gamma_{i(2)}^{j(2)}$ be distinct boundary points of a curve of D. Then $i(1) \neq i(2)$.

We denote by $\mathcal{D}_v$ the set of equivalence classes of essential diagrams on v. If the valency of v is equal to 3, then $\mathcal{D}_v$ has only one element. If the valency of v is equal to 4 and $N(E_i) = 2$ for $i = 1, \cdots, 4$, then $\mathcal{D}_v$ consists of 3 elements as in Figure 12.

Let $\beta(E) = \frac{1}{2}\, \tilde{q}_{R(E)}\, (1 + S(E)\, \alpha_{R(E)}^{-1/2}\, h_n) \in \mathbb{C}B_{N(E)}$. Let $\Gamma^{(N,R,S)}$ be the virtual link diagram obtained by replacing each vertex v by a sum of the all elements of $\mathcal{D}_v$ and each edge E by $\beta(E)$ as in the case of embeddings of oriented tri-valent graphs. For a edge E of Γ, let $c(E) = S(E)\, \alpha_{R(E)}^{1/2}$.

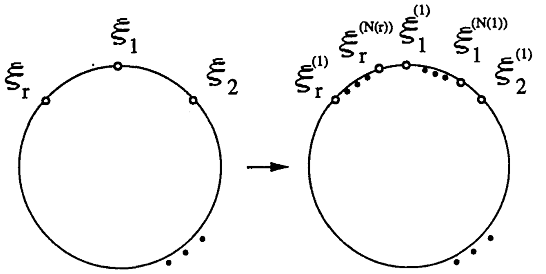

Fig. 11. Replace $\xi_1, \cdots, \xi_r$ by $\zeta_1^{(1)}, \cdots, \zeta_1^{(N(1))}, \zeta_2^{(1)}, \cdots, \zeta_2^{(N(2))}, \cdots, \zeta_r^{(1)}, \cdots, \zeta_r^{(N(r))}$.

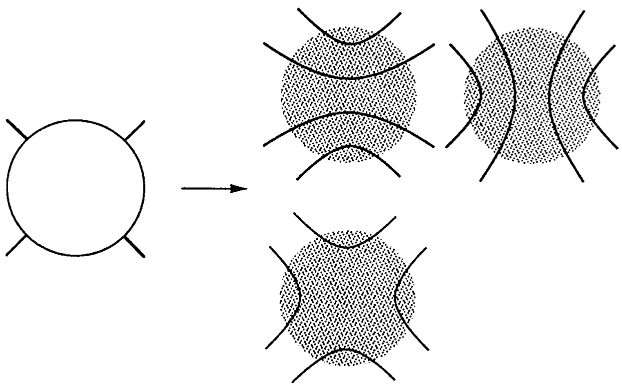

Fig. 12. Elements of $\mathcal{D}_v$.

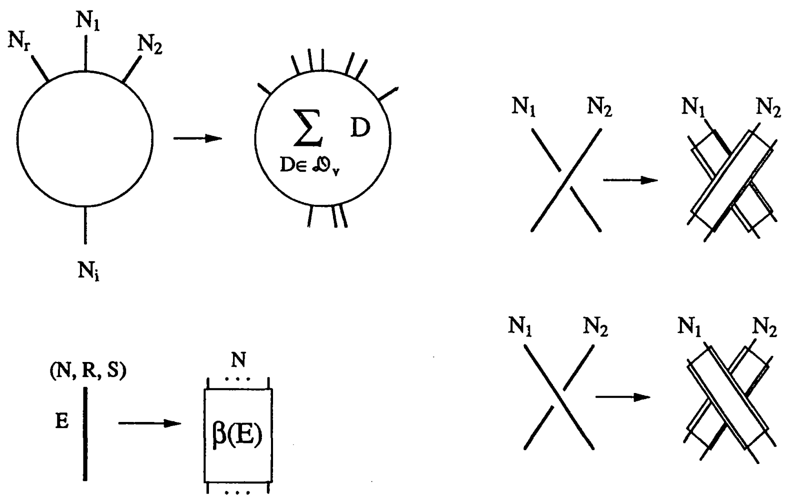

Fig. 13. Replace edges and vertices.

Theorem 3. *Let* Γ *and* Γ' *be colored spatial graphs isomorphic to* *a graph G colored by (N,R,S) as abstruct graphs. Identify the sets of edges of Γ and Γ' with that of G. If Γ and Γ' are equivalent as spatial graphs, then there are integers d and $d(E)$ for every edge E of G such that*

$$X(\Gamma^{(N,R,S)}) = \gamma^d \prod_{E\in\mathcal{E}} c(E)^{d(E)} X({\Gamma'}^{(N,R,S)}). \tag{3.1}$$

Proof. We check (3.1) for Reidemeister moves (SRI)–(SRV). Let Γ and Γ' be diagrams of colored spatial graphs isomorphic to G. We identify the sets of edges of Γ and Γ' with that of G.

Case 1. Assume that Γ and Γ' are regular isotopic, i. e. there is a sequence of reidemeister moves of types (SRII), (SRIII), (SRIV) sending Γ to Γ'. Then the associated virtual link diagrams $\Gamma^{(N,R,S)}$ and ${\Gamma'}^{(N,R,S)}$ are equivalent and we have

$$X(\Gamma^{(N,R,S)}) = X({\Gamma'}^{(N,R,S)}). \tag{3.2}$$

Case 2. In this and the next cases, we check (2.1) for (SRI) moves. Assume that Γ and Γ' are identical except within a ball where they are as shown in Figure 5. Let $n = N(E)$, $\rho = R(E)$, $s = S(E)$ and $\beta = \beta(E)$. Then there are positive integer N and $b \in \mathbb{C}K_N$ such that the associated link diagrams $\Gamma^{(N,R,S)}$ and ${\Gamma'}^{(N,R,S)}$ are equivalent to the closures of $b_1 = b\,\eta(\beta)$ and $b_2 = b\,\eta(\beta)\,h_n^2$ where η is an algebra homomorphism from $\mathbb{C}K_n$ to $\mathbb{C}K_N$ defined by $\eta(\sigma_i) = \sigma_i$ for $1 \le i \le n-1$. Since X is a regular isotopy invariant of knit trace type, there is an algebra homomorphism J from A_n to A_N such that $p_N \circ \eta = J \circ p_n$. From the definition of trace type invariants, we have

$$X(\hat{b}_2) = T_N(p_N(b_2)) = T_N(p_N(b\,\eta(\beta)\,{h_n}^2)).$$

The definition of β implies that

$$p_n(\beta\, h_n^{\pm 1}) = s\,\alpha_\rho^{\pm 1/2} p_n(\beta).$$

Hence we have

$$\begin{aligned} T_N(p_N(b\,\eta(\beta)\,{h_n}^2)) =& T_N(p_N(b)\,J(p_n(\beta\,{h_n}^2))) \\ =& T_N(p_N(b) J(\alpha_\rho\, p_n(\beta))) \\ =& \alpha_\rho\, T_N(p_N(b) J(p_n(\beta))), \end{aligned}$$

and so we get

$$X(\hat{b}_2) = \alpha_\rho\, X(\hat{b}_1).$$

In other words,

$$X(\Gamma^{(N,R,S)}) = \alpha_\rho\, X(\Gamma'^{(N,R,S)}).$$

Case 3. Assume that Γ and Γ' are identical except within a ball where they are as shown in Figure 6. Then, as in Case 2, we have

$$X(\Gamma^{(N,R,S)}) = \alpha_\rho^{-1}\, X(\Gamma'^{(N,R,S)}). \tag{3.3}$$

Case 4. Assume that Γ and Γ' are identical except within a ball where they are as shown in Figure 14. Let E_1, E_2, $\cdots$, E_r be edges around the vertex v. Let $n(i) = N(E_i)$ for $i = 1, 2, \cdots, r$ and $n = \sum_{i=1}^{r} n(i)$. Let $\varepsilon_{1,n} = \varepsilon_1\varepsilon_3\cdots\varepsilon_{2n-1} \in K_n$.

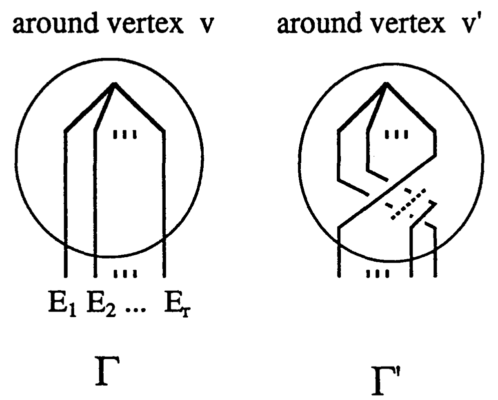

Fig. 14.

Let h_v, e_v and e'_v be the element of K_n corresponding to the diagram in Figure 15. Let $\eta_{i,j,k}$ $(i, j > 0,\ k \geq 0,\ i + k \leq j)$ be a semigroup homomorphism from K_i to K_j which sends $\tau_i^{\pm 1}$, $\varepsilon_i \in K_i$ to $\tau_{i+k}^{\pm 1}$, $\varepsilon_{i+k} \in K_j$ and $\phi_{i,j} = \eta_{n(i),j,n(1)+n(2)+\cdots+n(i-1)}$. Note that

$$h_n e_v = \gamma^n e_v,$$

$$h_n e_v = e'_v\, \phi_{1,n}(h_{n(1)})\, \phi_{2,n}(h_{n(2)}) \cdots \phi_{r,n}(h_{n(r)}),$$

and so we have

$$e'_v = \gamma^n\, e_v\, \phi_{1,n}(h_{n(1)}^{-1})\, \phi_{2,n}(h_{n(2)}^{-1}) \cdots \phi_{r,n}(h_{n(r)}^{-1}). \tag{3.4}$$

Let $\rho(i) = R(E_i)$, $s(i) = S(E_i)$ and $\beta(i) = \beta(E_i)$ for $i = 1, 2, \cdots, r$. Then there are an integer N and an element $b \in \mathbb{C}K_n$ such that the

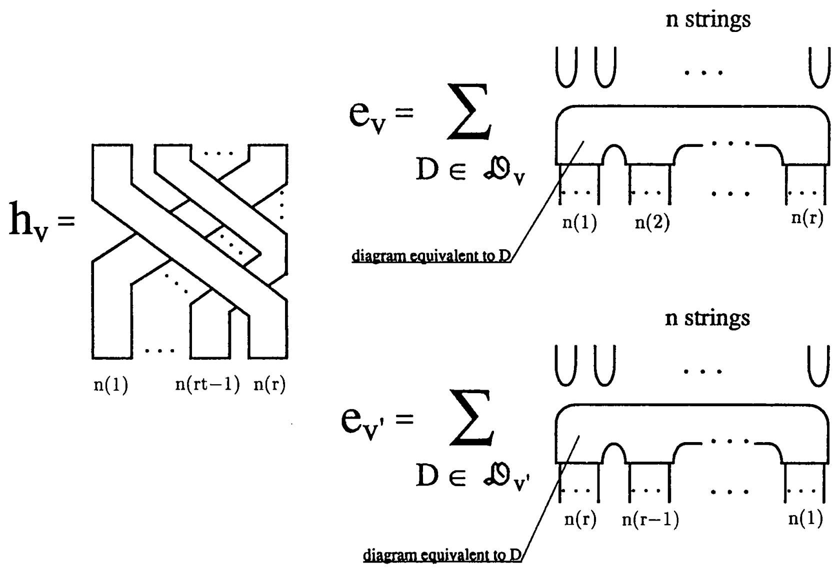

Fig. 15. Diagrams of h_v, e_v and e'_v.

associated link diagrams $\Gamma^{(N,R,S)}$ and $\Gamma'^{(N,R,S)}$ are equivalent to the closures of

$$\begin{aligned} b_1 &= b\,\eta_{n,N,0}(e_v)\,\phi_{1,N}(\beta(1)\tilde{q}_{\rho(1)})\,\phi_{2,N}(\beta(2)\tilde{q}_{\rho(2)})\,\cdots\,\phi_{r,N}(\beta(r)\tilde{q}_{\rho(r)}). \\ b_2 &= b\,\eta_{n,N,0}(e'_v h_v)\,\phi_{1,N}(\beta(1)\tilde{q}_{\rho(1)})\,\phi_{2,N}(\beta(2)\tilde{q}_{\rho(2)})\,\cdots\,\phi_{r,N}(\beta(r)\tilde{q}_{\rho(r)}). \end{aligned}$$

From (3.4), we have

$$(3.5)\quad b_2 = \gamma^n\, b\,\eta_{n,N,0}(e_v)\,\phi_{1,N}(h_{n(1)}^{-1}\beta(1)\tilde{q}_{\rho(1)})\,\cdots\,\phi_{r,N}(h_{n(r)}^{-1}\beta(r)\tilde{q}_{\rho(r)}).$$

Recall that the definition of $q_{R(E)}$ and $\beta(E)$ implies that

$$q_{\rho(t)}p_{n(t)}(\beta(t)\,{h_{n(t)}}^{\pm 1}) = s(t)\,\alpha_{\rho(t)}^{\pm 1/2}\,q_{\rho(t)}\,p_{n(t)}(\beta(t))$$

for $t = 1, 2, \cdots, r$. Hence formula (3.5) implies

$$(3.6)\qquad X(\hat{b}_2) = \prod_{i=1}^{r} S(t)\,\alpha_{\rho(t)}^{-1/2}\,X(\hat{b}_1),$$

because X is of knit trace type.

Case 5. Assume that Γ and Γ' are identical except within a ball where they are as shown in Figure 16. Then, as in Case 4, we have

$$X(\hat{b}_2) = \prod_{i=1}^{r} s(t)\, \alpha_{\rho(t)}^{1/2}\, X(\hat{b}_1). \tag{3.7}$$

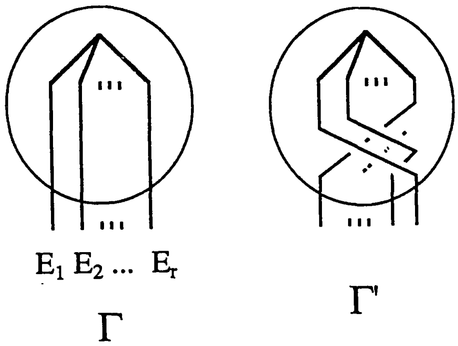

Fig. 16.

Formulas (3.2), (3.3), (3.6), (3.7) show Theorem 3. Q.E.D.

Let N be a positive even number. Let R be an irreducible representation of the algebra A_N associated with the link invariant X. Let S be 1 or -1. For a spatial graph Γ, let (N', R', S') be the coloring of Γ defined by $N'(E) = E$, $R'(E) = R$ and $S'(E) = S$ for every edge E of Γ. Let $X^{(N,R,S)}(\Gamma) = X(\Gamma^{(N',R',S')})$. Then $X^{(N,R,S)}$ is a regular isotopy invariant of diagrams of spatial graphs.

Corollary 4. *Let Γ and Γ' be diagrams of the same spatial graph G. Then, there are integers d and d' such that*

$$X^{(N,R,S)}(\Gamma) = \gamma^d\, {\alpha_R}^{d'}\, X^{(N,R,S)}(\Gamma').$$

The proof is similar to that of Theorem 2.

§4. Examples

KauffmanUs bracket polynomial $\langle\,.\,\rangle$ is a regular isotopy invariant of knit trace type and the Jones polynomial is obtained from $\langle\,.\,\rangle$ as in Remark in §1. To fix the notation, we give the definition of the bracket polynomial $\langle\,.\,\rangle$ [4]. Let $A \in \mathbb{C} \setminus \{0\}$ which is not equal to any roots of unity. The bracket polynomial with parameter A is a regular isotopy invariant of non-oriented link diagrams defined by the following relations.

$$\langle L_O \rangle = 1,$$
$$\langle L_x \rangle = A\langle L_{||} \rangle + A^{-1}\langle L_\infty \rangle,$$

where L_O is a trivial knot and L_x, $L_{||}$, L_∞ are link diagrams identical except within a ball where they are as shown in Figure 17.

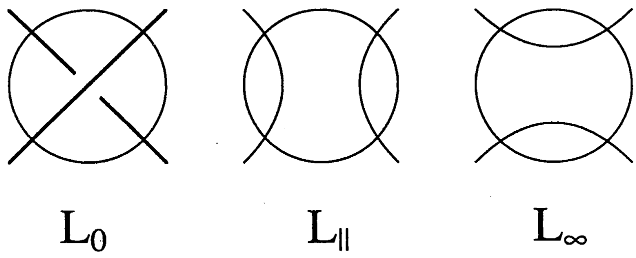

Fig. 17. Diagrams of L_x, $L_{||}$, L_∞.

Let A be a non-zero complex number which is not equal to any roots of unity. Let $J_n(A)$ be the Jones algebra defined over $\mathbb{C}$ by the following.

$$J_n(A) = \langle e_1, e_2, \cdots, e_{n-1} \,|\, e_i\, e_{i\pm 1}\, e_i = e_i,\ e_i\, e_j = e_j\, e_i\ (|i-j| \geq 2),$$
$$e_i^2 = -(A^2 + A^{-2})\, e_i \rangle.$$

The Markov knit sequence of KauffmanUs bracket polynomial $\langle\,.\,\rangle$ is $J_1(A)$, $J_2(A)$, $\cdots$. The algebra homomorphism p_n from $\mathbb{C}K_n$ to $J_n(A)$ is defined by $p_n(\varepsilon_i) = e_i$, $p_n(\tau_i) = A + A^{-1}\, e_i$ and $p_n(\tau_i^{-1}) = A^{-1} + A\, e_i$. Let ρ_n be the linear representation of $J_n(A)$ sending e_1, e_2, $\cdots$, e_{n-1} to 0. Since $\rho_n(p_n(\tau_i)) = A$, we have

$$(4.1) \qquad \rho_n(h_n) = A^{n(n-1)/2}.$$

Let $\alpha_n = A^{n(n-1)}$ and $\sqrt{\alpha_n} = A^{n(n-1)/2}$. The Yamada polynomial in [10] is coming from $\langle\,.\,\rangle$ as in Corollary 4 with $N = 2$, $R = \rho_2$ and $S = 1$.

Let Γ_1 and Γ_2 be two diagrams of spatial graphs as in Figure 18.

The diagrams Γ_1 and Γ_2 are colored as in the figure. Let C_1, C_2 denote the above coloring for Γ_1 and Γ_2 respectively. Since $p_2(1 + (A^2 +$

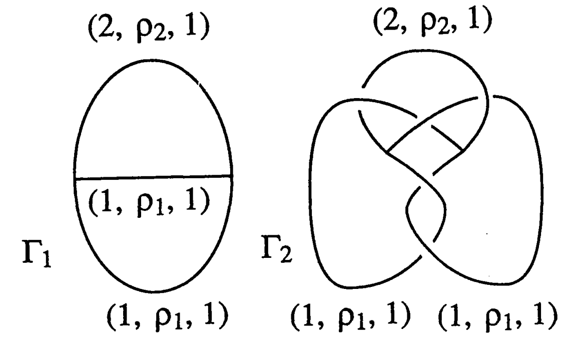

Fig. 18. Diagrams of spatial graphs Γ_1 and Γ_2.

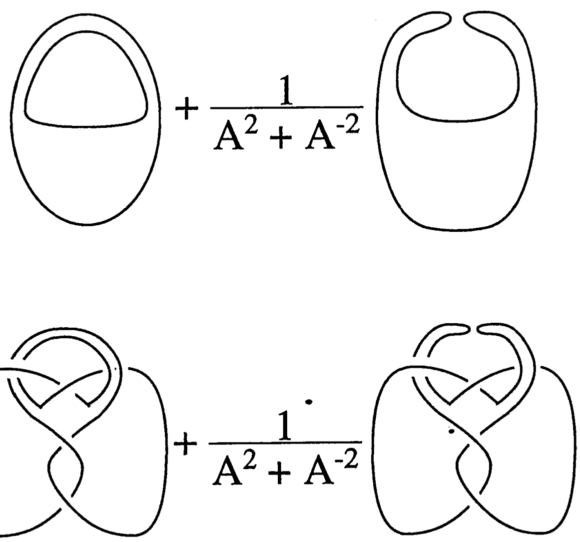

Fig. 19. Virtual link diagrams $\Gamma_1^{C_1}$ and $\Gamma_2^{C_2}$.

$A^{-2})^{-1}\,\varepsilon_1) = q_{R_2}$, the virtual diagrams $\Gamma_1^{C_1}$ and $\Gamma_2^{C_2}$ associated to the colorings are given in Figure 19.

Hence we have

$$\langle \Gamma_1 \rangle^{C_1} = -\frac{A^8 + A^4 + 1}{A^2\,(A^4+1)}$$

and

$$\langle \Gamma_2 \rangle^{C_2} = -\frac{-A^{32} + A^{28} + A^{20} + A^8 + 1}{A^{13}\,(A^4+1)}.$$

By (4.1) and Theorem 3, we know that Γ_1 and Γ_2 are not equivalent as spatial graphs.

To investigate the invariants associated with the Jones polynomial more closely, Section 4 of [7] may be helpful.

The HOMFLY polynomial P is an oriented link invariant of trace type. Hence we get invariants of colored oriented tri-valent graph embeddings from the HOMFLY polynomial.

The Kauffman polynomial F is an oriented link invariant obtained from the Dubrovnik polynomial [5], which is a regular isotopy invariant of unoriented link diagrams. It is shown in [2], [7], [8] that the Dubrovnik polynomial is of knit trace type. Hence we get invariants of spatial graphs from the Dubrovnik polynomial. To investigate properties of these invariants, Section 5 of [7] may be helpful.

References

[1] Birman, J.S., "Braids, Links and Mapping Class Groups", Princeton University Press, Princeton, 1975.

[2] Birman, J.S. and Wenzl, H., Braids, link polynomials and a new algebra, Trans. Amer. Math. Soc., **313** (1989), 249–273.

[3] Jones, V.F.R., A polynomial invariant for knots via von Neumann algebras, Bull. Amer. Math. Soc., **12** (1985), 103–111.

[4] Kauffman, L.H., State models and the Jones polynomial, Topology, **26** (1987), 395–407.

[5] ———, State models and the Jones polynomial — An introduction, preprint, University of Illinois at Chicago, 1987.

[6] Murakami, J., The Kauffman polynomial of links and representation theory, Osaka J. Math, **24** (1987), 745–758.

[7] ———, The parallel version of polynomial invariants of links, Osaka J. Math., **26** (1989), 1–55.

[8] ———, The representations of the q-analogue of Brauer's centralizer algebras and the Kauffman polynomial of links, Publ. Res. Inst. Math. Sci., Kyoto Univ., **26** (1990), 935–945.

[9] Reshetikhin, N.Yu. and Turaev, V.G., Ribbon graphs and their invariants derived from quantum groups, Commun. Math. Phys., **127** (1990), 1–26.

[10] Yamada, S., An invariant of spatial graphs, J. Graph Theory, **13** (1989), 537–551.

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Foundations of Flat Conformal Structure

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Dedicated to Professor Masahisa Adachi

Introduction

A flat conformal structure on an n-dimensional manifold N is a maximal system of local charts taking values on S^n, with transition functions Moebius transformations. In short it is a geometric structure modelled on $(\mathcal{M}(S^n), S^n)$, where $\mathcal{M}(S^n)$ denotes the group of Moebius transformations on S^n. Equivalently, it is a conformal equivalence class of conformally flat Riemannian metrics on N if $n \geq 3$. See §1 for Liouville's theorem. By certain abuse we denote a flat conformal structure by the same letter as the underlying manifold.

In dimension 2, flat conformal structures are usually called projective structures and have been extensively studied by various authors in the field of function theory. Analytic methods such as the theory of quasiconformal maps often play crucial roles there. In dimension ≥ 3, however, the situation is quite different. Topology, instead of analysis, provides major tools of study.

The concept of flat conformal structures was first introduced by Kuiper ([35],[36],[37]) around 1950. Thereafter it had been forgotten for some time, until it was revived by Kulkarni ([40],[41],[42],[43]), related with his study of discrete group actions in general. Then came an important turning point when Fried ([13]) established a remarkable theorem concerning closed similarity manifolds. It solved a fundamental and annoying problem which one encounters in the primary stage of the theory, thereby making it possible to have a good grip on elementary flat conformal structures, with Goldman ([15]) and Kamishima ([25]) contributing significantly to this direction.

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At the same time various interesting examples have been piled up by many authors including Thurston [56], Bestvina-Cooper [4], Freedman-Skora [10], Gromov-Lawson-Thurston [19], Kuiper [38] and, quite recently,
Kapovich-Potyagailo [32], making the field even more active.

This article has two objectives. One is to provide the basic knowledge of flat conformal structures and to serve as an introductory guide of the field. The other is to show some new pieces of knowledge. §1 $\sim$ §3 are devoted to the former purpose, where the reader can find exposition of fundamental properties of Moebius transformations and flat conformal structures. No original results are included in these early sections. However for the full understanding of later sections, they are helpful, or even indispensable.

§4 and §5 are also mainly expository, though they include some slightly improved (new) results. Hereafter let N be a connected closed flat conformal manifold of dimension ≥ 3. In §4, we prove the following version of Fried's theorem.

Theorem (4.4). *If the holonomy group of* N *has a fixed point in* S^n, *then* N *is either* S^n, *an Euclidean space form or a Hopf manifold.*

Unlike the original theorem ([13]), we no longer postulate that the developing map misses the fixed point. This yields clearer understanding of the limit set (§5) and a wider range of applications. Using Theorem (4.4), various results (mostly known) can be proved by elementary and straightforward arguments. Although the proof of Theorem (4.4) is nothing but a small modification of the argument in [13], it might be worth while to record it. The same result was obtained independently by R. Miner [58], who mainly worked in the context of spherical CR structures.

In §5, we define the limit set $L(N)$ of a flat conformal manifold N. Five different ways are possible and in Theorem (5.18), they are shown to coincide eventually. Especially we get that the limit set defined by means of the holonomy group is identical to the one obtaind by looking at the behaviour of the developing map. (Most of these facts are already known to Kulkarni-Pinkall [43].) As immediate corollaries we have the followings.

Corollary (5.23). *If the developing map of* N *is not onto* S^n, *then it is a covering map onto its image.*

Corollary (5.24). *Suppose the following* (1) *and* (2).

(1) *$S^n \setminus L(N)$ is connected and the fundamental group $\pi_1(S^n \setminus L(N))$ is finitely generated.*

(2) *For any point $x \in L(N)$, there exists an arbitrarily small neighbourhood U of x such that $U \setminus L(N)$ is connected.*

Then the developing map is a covering map onto $S^n \setminus L(N)$.

In dimension 2, Corollary (5.23) is well known and easy to show using hyperbolic metric. For higher dimension, it was first proved by Kamishima. Again our method is short and straightforward. Corollary (5.24) can be found in Kulkarni-Pinkall [45], where condition (2) is mistakingly dropped. In §5, we also characterize those flat conformal manifolds whose developing maps are covering maps (onto the images) and whose holonomy groups are indiscrete. (Theorem (5.26).) In dimension 3, this was first obtained by Kamishima ([24]) and independently by Gusevskiĭ-Kapovich ([20]) in dimension 3.

N is called elementary if the limit set is finite. N is called a C-structure if it is a connected sum of elementary structures and is not itself elementary. In dimension 3, we have the following result.

Theorem (6.12). *Suppose* $\dim(N) = 3$. *Then N is a C-structure if and only if the limit set $L(N)$ is a tame Cantor set.*

Recall that a Cantor set Υ in S^n is called tame if there exists a self homeomorphism of S^n which carries Υ into S^1. Otherwise it is called wild.

The above theorem is proved along the argument of Kulkarni ([43]), in which Stalling's theorem ([54],[55]) concerning ends of groups plays a central part. The theory of ends are summarized in the appendix for the convenience of the reader.

After preparing Poincaré's polyhedral theorem in §7 (in the framework of flat conformal manifolds), we shall show the following theorem in §8.

Theorem (8.1). *There exists a flat conformal manifold N of dimension* 3 *whose limit set $L(N)$ is a wild Cantor set.*

This theorem is an improvement of the work of Bestvina-Cooper ([4]) who constructed such examples for open 3-manifolds. Our example in Theorem (8.1) is compact.

Literature concerning flat conformal structures is extensively collected in the reference, though not complete, of course.

The author wishes to express his thanks to Y. Kamishima, S. Kojima, W.M. Goldman, N.A. Gusevskiĭ, L.D. Potyagailo and M.E. Kapovich for helpful conversations.

CONTENTS

§1. Conformal map and Liouville's theorem

In this section, we give definitions of a conformal map and a Moebius transformation of the n-sphere. After providing fundamental properties, we show that a locally defined conformal map is the restriction of a Moebius transformation if $n \geq 3$. (Liouville's theorem.)

Definition (1.1). A real $n \times n$ matrix A is called a *conformal matrix* if $A = \lambda P$ for $\lambda > 0$ and an orthogonal matrix P.

Thus A is conformal precisely when A preserves the angle of given two vectors. Notice that the products and the inverses of conformal matrices are again conformal.

Let $\widehat{\mathbf{R}}^n = \mathbf{R}^n \cup \{\infty\}$ be the one point compactification of $\mathbf{R}^n$. Points in $\widehat{\mathbf{R}}^n$ is indicated by letters a, x and so forth. For $x = (x_1, \ldots, x_n) \in \mathbf{R}^n$,

$$|x| = \left(\sum_{i=1}^{n} x_i^2\right)^{1/2}$$

denotes the Euclidean norm of x. To endow $\widehat{\mathbf{R}}^n$ the structure of an oriented manifold, the following local charts (U_i, q_i) are commonly used $(i = 1, 2)$.

$$\begin{cases} U_1 = \mathbf{R}^n, & q_1 = id : U_1 \longrightarrow \mathbf{R}^n, \\ U_2 = \widehat{\mathbf{R}}^n \setminus \{0\}, & q_2 : U_2 \longrightarrow \mathbf{R}^n, \end{cases}$$

where q_2 is defined by

$$q_2(x_1, \ldots, x_n) = \frac{1}{|x|^2}(x_1, \ldots, x_{n-1}, -x_n).$$

In the above definition and in all that follows, if the image of ∞ by a map is clear by the continuity, we do not explicitly state it. An important property of q_2 is that the differential matrix $D_a q_2$ at any point $a \in \mathbf{R}^n \setminus \{0\}$ is a conformal matrix. Verification is left to the reader.

Let U be a domain (i.e. a connected open subet) of $\widehat{\mathbf{R}}^n$.

Definition (1.2). A C^1 map $f : U \to \widehat{\mathbf{R}}^n$ is called a *conformal map* if the following condition is satisfied. For any $a \in U$, if $a \in U_i$ and $f(a) \in U_j$, then the differential $D_{q_i(a)}(q_j \circ f \circ q_i^{-1})$ is a conformal matrix.

Since for any $b \in \mathbf{R}^n \setminus \{0\}$, $D_b(q_2 \circ q_1^{-1})$ is a conformal matrix, Definition (1.2) is invariant under possible changes of local charts around a and $f(a)$. A conformal map is a submersion and thus has a local inverse, which is again a conformal map. Also the composite of two conformal maps is conformal.

Lemma (1.3). *Suppose $f : U \to \widehat{\mathbf{R}}^n$ is a C^1 submersion, where U is a domain of $\widehat{\mathbf{R}}^n$. If $D_a f$ is a conformal matrix for any $a \in U \cap \mathbf{R}^n \cap f^{-1}(\mathbf{R}^n)$, then f is a conformal map.*

Proof. This follows at once from the fact that the conformal matrices form a closed subset in the general linear group. Q.E.D.

Let us give examples of conformal maps. Let $0 < p < n$. By a *dimension p sphere* in $\widehat{\mathbf{R}}^n$, we mean either a dimension p metric sphere in $\mathbf{R}^n$ or a dimension p plane in $\mathbf{R}^n$ plus $\{\infty\}$. A dimension p sphere is sometimes called a codimension $n - p$ sphere.

Definition (1.4). Let σ be a codimension one sphere in $\widehat{\mathbf{R}}^n$. The *inversion at σ*

$$J_\sigma : \widehat{\mathbf{R}}^n \longrightarrow \widehat{\mathbf{R}}^n$$

is defined as follows.

(1) If σ is the sphere of radius r centered at a, then for any $x \in \mathbf{R}^n \setminus \{a\}$,

$$J_\sigma(x) = \frac{r^2}{|x-a|}(x-a) + a.$$

(2) If σ contains a codimension one plane, J_σ is the reflexion at that plane.

See Figure (1.1). The inversion is an orientation reversing involution with the fixed point set σ.

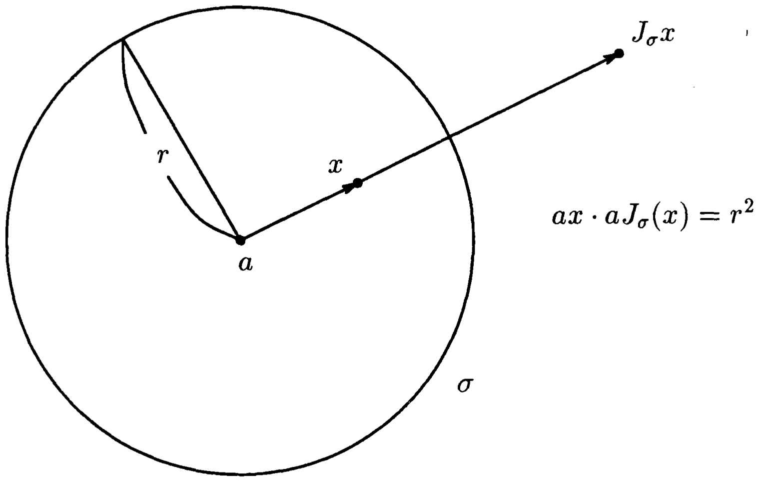

Figure (1.1)

Definition (1.5). Composite of inversions is called a *Moebius transformation.* The group of all the Moebius transformations of $\widehat{\mathbf{R}}^n$ is denoted by $\mathcal{M}(\widehat{\mathbf{R}}^n)$.

Proposition (1.6). *Moebius transformation is a conformal map and carries a sphere in* $\widehat{\mathbf{R}}^n$ *to a sphere of the same dimension.*

Proof. Computaion shows that an inversion is a conformal map. Also it is well known, very easy to show by Euclidean geometry, that an inversion maps a codimension one sphere to a codimension one sphere. Therefore a sphere of arbitrary dimension, the intersection of several codimension one spheres, is mapped to a sphere of the same dimension. The proposition follows from this. Q.E.D.

Proposition (1.7). *The following maps are Moebius transformations.*

(a) *Translation by* a, $x \mapsto x + a$.

(b) *Magnification by* $\lambda > 0$, $x \mapsto \lambda x$.

(c) *Orthogonal transformation by* $P \in O(n)$, $x \mapsto Px$.

Proof. Translation is the composite of two inversions at parallel planes. This shows (a). Likewise positive magnification is the composite of two inversions at concentric spheres and orthogonal transformation is the composite of several inversions at planes through 0, showing (b) and (c). Q.E.D.

Lemma (1.8). *Let* $f : \widehat{\mathbf{R}}^n \to \widehat{\mathbf{R}}^n$ *be a Moebius transformation. If* $f(0) = 0$, $f(\infty) = \infty$, $D_0 f = E$, *then* $f = \mathrm{id}$.

Proof. Moebius transformations carry circles to circles. Since f keeps 0 and ∞ fixed, f preserves the (singular) dimension one foliation $\mathcal{L}$ formed by the straight lines through 0. Since f is a conformal map, f also preserves the codimension one foliation $\mathcal{L}^\perp$ of spheres centered at 0. See Figure (1.2). Notice also that f keeps the leaf of $\mathcal{L}$ invariant, since $D_0 f = E$. Thus we obtain

$$f(x) = \frac{R}{r} x.$$

on the sphere $|x| = r$. The conformality of f implies

$$\frac{dR}{dr} = \frac{R}{r}.$$

Therefore we have $R = ar$. But $a = 1$ since $D_0 f = E$. This shows $f = \mathrm{id}$. Q.E.D.

Proposition (1.9).

(1) f *is a Moebius transformation such that* $f(\infty) = \infty$ *if and only if*

$$f(x) = Ax + b.$$

(2) f *is a Moebius transformtion such that* $f(\infty) \neq \infty$ *if and only if*

$$f(x) = AJ(x - b) + c.$$

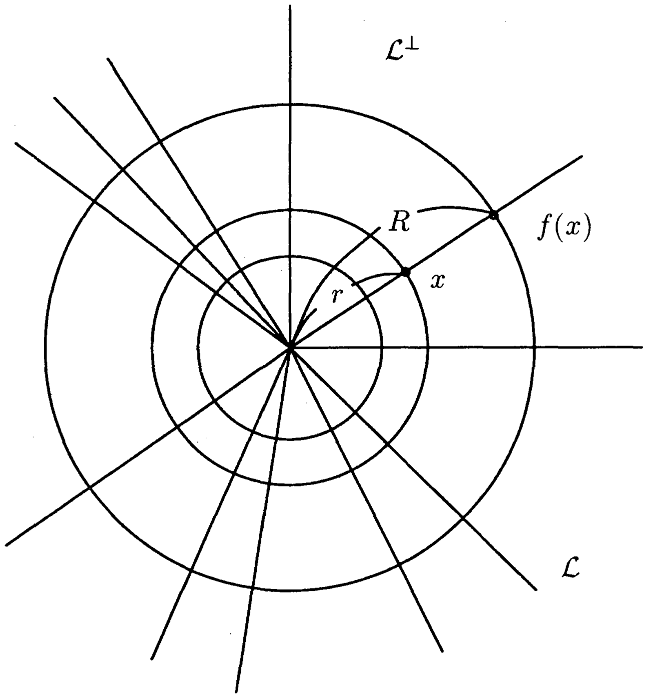

Figure (1.2)

Here A is a conformal matrix, b and c are points of $\mathbf{R}^n$ and J is the inversion at the unit sphere $\{|x| = 1\}$.

Proof. It is a direct consequence of Lemma (1.7) that the transformations of the above expressions are Moebius transformations. Conversely suppose that f is a Moebius transformation with $f(\infty) = \infty$. Let $f(0) = b$ and $D_0 f = A$. Define $g(x) = Ax + b$. Then $g^{-1} \circ f$ satisfies the hypothesis of Lemma (1.8). Thus $g = f$. This completes the proof of (1). On the other hand, suppose that f is a Moebius transformation with $f(\infty) \neq \infty$. Let $f(b) = \infty$. Define h by $h(x) = J(x - b)$. Then $f \circ h^{-1}$ is a Moebius transformation which keeps ∞ fixed. By (1), we have

$$f \circ h^{-1}(x) = Ax + c.$$

This completes the proof of (2). Q.E.D.

We shall finish this section with the following celebrated theorem of Liouville.

Theorem (1.10). *Let $n \geq 3$. Suppose $f : U \to \widehat{\mathbf{R}}^n$ is a conformal map, where U is a domain of $\widehat{\mathbf{R}}^n$. Then f is the restriction of a Moebius transformation.*

As a matter of fact, this theorem does not hold for $n = 2$. In fact the Riemann mapping theorem asserts the abundance of conformal maps which are not restrictions of Moebius transformations.

Theorem (1.10) was first proved by J. Liouville in his 1850 paper ([46]), under the additional assumption that f be of class C^3. Since then, it had been an open problem, astonishingly difficult, to weaken the differentiability assumption, until at last in 1969, P. Hartman gave a complete proof for C^1 maps ([21]).

Independently, F.W. Gehring, among others, developed the theory of quasiconformal maps in dimension ≥ 3. Specifically he defined 1-quasiconformal maps, which is a genaralization of conformal maps, where no differentiability assumption is made. In [14], Gehring showed that a locally defined 1-quasiconformal map is the restriction of a Moebius transformation.

However these results need involvement in deep general treatment and cannot be collected here. Instead, we give a simple elementary proof essentially due to R. Nevanlinna ([49]) assuming that the given conformal map f is C^3. (Nevanlinna postulated that f is C^4.)

Proof of Theorem (1.10). We use the following convention. x_i denotes the i-th coordinate of $\mathbf{R}^n$ and for $f : U \to \widehat{\mathbf{R}}^n$, f_{x_i}, $f_{x_i x_j}$ and so forth denote the first and the second partial derivatives and so forth. They are vectors of $\mathbf{R}^n$. In the first place, since f is conformal, we have

$$(f_{x_i}, f_{x_j}) = r^2 \delta_{ij},$$

where $r(x) = \|D_x f\|$ is the mapping norm of the Jacobi matrix. Differentiating by x_k, we get for $i = j$,

$$(f_{x_i x_k}, f_{x_i}) = r r_{x_k}$$

and for $i \neq j$,

$$(f_{x_i x_k}, f_{x_j}) + (f_{x_i}, f_{x_j x_k}) = 0.$$

For mutually distinct indices i, j and k, by permuting the indices, we have

$$(f_{x_i x_k}, f_{x_j}) = 0.$$

Since j can be any index except i and k and $f_{x_1}, \ldots, f_{x_n}$ are mutually orthogonal, we have

$$f_{x_i x_k} = \mu f_{x_i} + \nu f_{x_k},$$

where

$$\mu = (f_{x_i x_k}, f_{x_i})/r^2 = r_{x_k}/r$$

$$\nu = r_{x_i}/r.$$

Letting $\rho = 1/r$, we have

$$\rho f_{x_i x_k} + \rho_{x_i} f_{x_k} + \rho_{x_k} f_{x_i} = 0.$$

Differentiating by x_j, we obtain

$$\rho f_{x_i x_j x_k} + \rho_{x_j} f_{x_i x_k} + \rho_{x_i} f_{x_j x_k} + \rho_{x_k} f_{x_i x_j}$$

$$+\rho_{x_i x_j} f_{x_k} + \rho_{x_j x_k} f_{x_i} = 0.$$

By permutation of the indices, we obtain for $j \neq k$,

$$\rho_{x_j x_k} = 0.$$

By rotating the coordinates by 45 degrees in the (x_j, x_k)-plane, we have

$$\rho_{x_j x_j} = \rho_{x_k x_k}.$$

Now since $\rho_{x_j x_k} = 0$ for any $k \neq j$ ρ_{x_j} is constant on the hyperplane $\{x_j = c\}$. Thus it follows that $\rho_{x_j x_j}$ is constant on $\{x_j = c\}$. That is, $\rho_{x_1 x_1} = \cdots\cdots = \rho_{x_n x_n}$ is constant in U.

By composing f with a suitable Moebius transformation if necessary, we may assume that $0 \in U$ and $f(0) = \infty$. Then the image by f of an arbitrarily small ball $|x| < \varepsilon$ contains $|x| > K$ for some large $K > 0$. By the volume formula, this implies that $\rho(a_m) \to 0$ for some sequence $a_m \to 0$. On the other hand, since $\rho_{x_i x_j} = 2\alpha\delta_{ij}$ for some $\alpha > 0$, ρ is a quadratic function on $U \setminus \{0\}$, with the leading term $\alpha|x|^2$. Since ρ is positive valued on $U \setminus \{0\}$ and $\rho(a_m) \to 0$, we have

$$\rho(x) = \alpha|x|^2.$$

Notice that the same value of ρ is also attained by the inversion g which is defined by

$$g(x) = \frac{x}{\alpha|x|^2}.$$

Thus by the chain rule, the composite $h = g \circ f^{-1} : f(U) \to \widehat{\mathbf{R}}^n$ satisfies $\|D_p h\| = 1$ for any $p \in f(U) \setminus \{\infty\}$. That is, h is an isometry with respect to the Euclidean metric on $\mathbf{R}^n$. This implies that $h(x) = Px + b$ for some orthogonal matrix P and $b \in \mathbf{R}^n$. In fact, all that needs proof is that h is an affine transformation. But since

$$(h_{x_i}, h_{x_k}) = \delta_{ij},$$

by differentiating we get

$$(h_{x_i x_j}, h_{x_k}) = 0,$$

showing that $h_{x_i x_j} = 0$. This implies that h is an affine transformation. Thus h and hence f are the restrictions of Moebius transformations, as is required. Q.E.D.

§2. More on Moebius transformation

Denote by $\mathcal{M}(\widehat{\mathbf{R}}^n)$ the group of Moebius transformations of $\widehat{\mathbf{R}}^n$.

Lemma (2.1). *Let* $f \in \mathcal{M}(\widehat{\mathbf{R}}^n)$ *and let* $\sigma \subset \widehat{\mathbf{R}}^n$ *be a codimension one sphere. Then,*

$$f \circ J_\sigma \circ f^{-1} = J_{f(\sigma)}.$$

Proof. Clearly $g = f \circ J_\sigma \circ f^{-1} \circ J_{f(\sigma)}$ is an orientation preserving Moebius transformation which keeps points in $f(\sigma)$ fixed. Thus for an arbitrary Moebius transformation h such that $h(f(\sigma)) = \{x_n = 0\}$, we have that $k = h \circ g \circ h^{-1}$ keeps $\{x_n = 0\}$ pointwise fixed. Especially we obtain that $k(0) = 0$, $k(\infty) = \infty$ and $D_0 k = E$ since k is orientation preserving. Therefore by (1.8), we obtain $k = \mathrm{id}$. This shows (2.1). Q.E.D.

Let $\iota : \widehat{\mathbf{R}}^n \to \widehat{\mathbf{R}}^{n+1}$ be the standard embedding, i.e.,

$$\iota(x_1, \ldots, x_n) = (x_1, \ldots, x_n, 0).$$

As usual $\widehat{\mathbf{R}}^n$ is considered to be a subset of $\widehat{\mathbf{R}}^{n+1}$ by ι. Let σ be an $(n-1)$-dimensional sphere in $\widehat{\mathbf{R}}^n$. Then the inversion $J_\sigma : \widehat{\mathbf{R}}^n \to \widehat{\mathbf{R}}^n$ can be extended to the inversion $J_\tau : \widehat{\mathbf{R}}^{n+1} \to \widehat{\mathbf{R}}^{n+1}$ at the n-dimensional sphere τ orthogonal to $\widehat{\mathbf{R}}^n$ such that $\widehat{\mathbf{R}}^n \cap \tau = \sigma$. This yields an injection.

$$i : \mathcal{M}(\widehat{\mathbf{R}}^n) \to \mathcal{M}(\widehat{\mathbf{R}}^{n+1}).$$

Again $\mathcal{M}(\widehat{\mathbf{R}}^n)$ is considered to be a subgroup of $\mathcal{M}(\widehat{\mathbf{R}}^{n+1})$ by i.

On the other hand let

$$S^n = \{x \in \mathbf{R}^{n+1} \mid |x| = 1\}.$$

Let τ be an n-dimensional sphere in $\widehat{\mathbf{R}}^{n+1}$ which is perpendicular to S^n. Since inversions are conformal maps which send spheres to spheres, J_τ is a transformation which keeps S^n invariant. Composites of such inversions constitute a Lie group $\mathcal{M}(S^n)$ of *Moebius transformations of* S^n. Denote the inclusion by

$$j : \mathcal{M}(S^n) \longrightarrow \mathcal{M}(\widehat{\mathbf{R}}^{n+1}).$$

Define $v \in \mathcal{M}(\widehat{\mathbf{R}}^{n+1})$ by $v = T \circ J_2 \circ J_1$. where J_1 is the reflexion at the plane $x_{n+1} = -1/2$, J_2 is the inversion at the sphere $|x| = 2$ and T is the translation by $(0, \dots, 0, 1)$. See Figure (2.1).

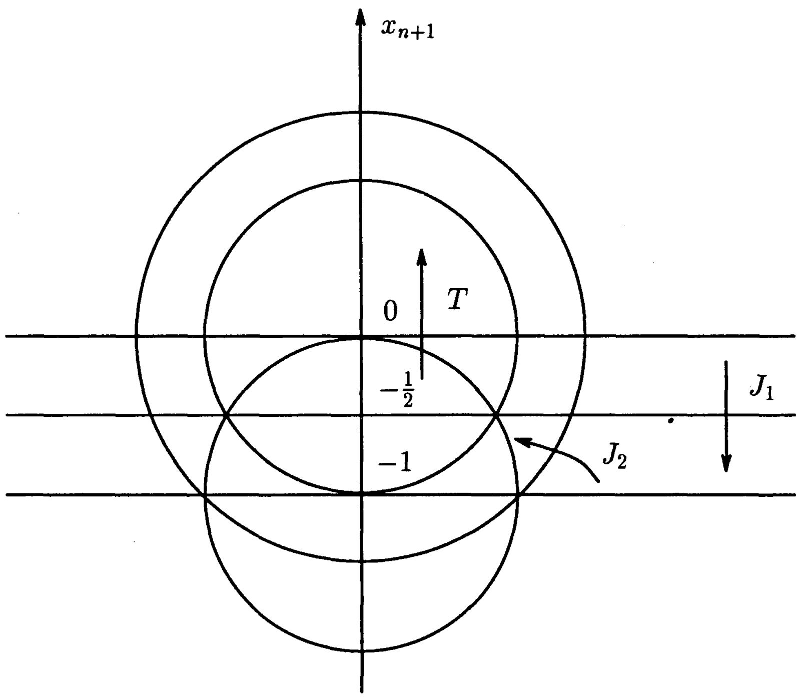

Figure (2.1)

Notice that $v(\widehat{\mathbf{R}}^n) = S^n$. Define

$$c_v : \mathcal{M}(\widehat{\mathbf{R}}^{n+1}) \to \mathcal{M}(\widehat{\mathbf{R}}^{n+1})$$

by

$$c_v(f) = v \circ f \circ v^{-1}.$$

Proposition (2.2). *c_v maps the subgroup $\mathcal{M}(\widehat{\mathbf{R}}^n)$ isomorphically onto the subgroup $\mathcal{M}(S^n)$.*

Proof. v maps an n-sphere τ perpendicular to $\widehat{\mathbf{R}}^n$ to the n-sphere $v(\tau)$ which is perpendicular to S^n. On the other hand it follows from (2.1) that $c_v(J_\tau) = J_{v(\tau)}$. This shows (2.2). Q.E.D.

Let

$$D^{n+1} = \{x \in \mathbf{R}^{n+1} \mid |x| < 1\},$$
$$H^{n+1} = \{x \in \mathbf{R}^{n+1} \mid x_{n+1} > 0\}.$$

Proposition (2.3). *We have*

$$\mathcal{M}(\widehat{\mathbf{R}}^n) = \{f \in \mathcal{M}(\widehat{\mathbf{R}}^{n+1}) \mid f(H^{n+1}) = H^{n+1}\},$$
$$\mathcal{M}(S^n) = \{f \in \mathcal{M}(\widehat{\mathbf{R}}^{n+1}) \mid f(D^{n+1}) = D^{n+1}\}.$$

Proof. By virtue of (2.2), it suffices to show the statement only for $\widehat{\mathbf{R}}^n$. (Notice that $v(H^{n+1}) = D^{n+1}$.) The inclusion $\subset$ is clear. Conversely, suppose that $f \in \mathcal{M}(\widehat{\mathbf{R}}^{n+1})$ satisfies that $f(H^{n+1}) = H^{n+1}$. First of all, consider the case where $f(\infty) = \infty$. Then by (1.9), $f(x) = \lambda Px + b$, where $\lambda > 0$, $P \in O(n+1)$ and $b \in \mathbf{R}^{n+1}$. Since $f(\mathbf{R}^n) = \mathbf{R}^n$, we have that $b \in \mathbf{R}^n$. Further since f preserves H^{n+1}, we also obtain that

$$P = \begin{pmatrix} Q & 0 \\ 0 & 1 \end{pmatrix},$$

where $Q \in O(n)$. Thus it follows from (1.7) that $f \in \mathcal{M}(\widehat{\mathbf{R}}^n)$. The remaining case can easily be reduced to this case. Details are left to the reader. Q.E.D.

We need some standard terminologies in geometry.

Definition (2.4). Two Riemannian metrics g_1 and g_2 on a manifold M are said to be *conformally equivalent*, if there exists a positive valued function μ on M such that $g_2 = \mu g_1$.

Definition (2.5). A C^1 map $f : (M_1, g_1) \to (M_2, g_2)$ of Riemannian manifolds is called a *Riemannian conformal map* if the induced metric $f^* g_2$ is conformally equivalent to g_1.

Riemannian conformal maps are usually called conformal maps in the literature. However in order to avoid confusion with Definition (1.2), we call them Riemannian conformal maps in this article. The following three Riemannian metrics are important in what follows.

Definition (2.6). Denote by g_E the *Euclidean metric* on $\mathbf{R}^n$, i.e. $g_E = \sum_{i=1}^n dx_i^2$, by g_S the *spherical metric* on S^n, that is, the restriction of the Euclidean metric on $\mathbf{R}^{n+1}$ to the submanifold S^n and by g_H the *hyperbolic metric* on D^n, i.e.,

$$g_H = \frac{4\sum_{i=1}^n dx_i^2}{(1-|x|^2)^2}.$$

It is well known that g_S has constant sectional curvature 1 and that (D^n, g_H) is a complete Riemannian manifold with constant sectional curvature -1.

Proposition (2.7). Let U be a domain in $\widehat{\mathbf{R}}^n$. A C^1 map $f : U \to \widehat{\mathbf{R}}^n$ is a conformal map in the sense of Definition (1.2) if and only if $v \circ f \circ v^{-1} : v(U) \to S^n$ is a Riemannian conformal map w.r.t. the spherical metric.

Proof. First notice that for a domain $U \subset \mathbf{R}^n$,

$$f : (U, g_E) \longrightarrow (\mathbf{R}^n, g_E)$$

is a Riemannian conformal map if and only if $D_a f$ is a conformal matrix for any $a \in U$. On the other hand, the following two maps

$$v : \mathbf{R}^n \to S^n,$$

$$v \circ q_2 : \mathbf{R}^n \to S^n$$

are Riemannian conformal maps from $(\mathbf{R}^n, g_E)$ to (S^n, g_S), where q_2 is the coordinate chart of $\widehat{\mathbf{R}}^n$ defined in §1. (2.7) follows from this. Q.E.D.

Thus Liouville's theorem can be rephrased as follows.

Let $U \subset S^n$ $(n \geq 3)$ be a domain. Then a Riemannian conformal map $f : U \to S^n$ w.r.t. the spherical metric is the restriction of a transformation in $\mathcal{M}(S^n)$.

Hereafter we focus our attention to the action of $\mathcal{M}(S^n)$ on S^n and D^{n+1}. Thus Moebius transformations are considered primarily as acting on S^n. However there are some occasions where the coordinates of $\widehat{\mathbf{R}}^n$ is more convenient. In what follows, frequent use will be made of the following lemma, which is a special case of (2.1). As before $J \in \mathcal{M}(\widehat{\mathbf{R}}^{n+1})$ denotes the inversion at S^n.

Lemma (2.8). *For $f \in \mathcal{M}(S^n)$, we have $J \circ f = f \circ J$.*

To study the action of $\mathcal{M}(S^n)$, the transformations are classified according to whether they preserve ∞ or not. In the first place, we have the following proposition.

Proposition (2.9). *For $f \in \mathcal{M}(S^n)$, the following statements are equivalent.*

(1) $f(\infty) = \infty$.
(2) $f(0) = 0$.
(3) f *induces an isometry of* (S^n, g_S).
(4) $f(x) = Px$ *for some* $P \in O(n+1)$.

Proof. By virtue of (2.8), We have (1) $\Leftrightarrow$ (2). (1) $\Rightarrow$ (4) follows from the expression of (1.9), (4) $\Rightarrow$ (1) and (4) $\Rightarrow$ (3) is clear and (3) $\Rightarrow$ (4) follows from the next lemma. Q.E.D.

Lemma (2.10). *Suppose that a Lie group G acts on a connected n-dimensional Riemannian manifold N transitively and isometrically. Suppose also that the first derivative gives an isomorphism $G_x \cong O(n)$, where G_x is the isotropy subgoup at some $x \in N$. Then G is precisely the group of all the isometries of N.*

Proof. For any isometry f, there exists a unique element $g \in G$ such that $g^{-1} \circ f(x) = x$ and $D_x(g^{-1} \circ f) = E$. Then $g^{-1} \circ f$ keeps any point on any geodesic ray at x fixed. That is, $g^{-1} \circ f = \mathrm{id}$.
Q.E.D.

Next for f with $f(\infty) \neq \infty$, we define the isometric sphere and use it to describe a geometric decomposition of f. For an $n \times n$ matrix A, $\|A\|$ denotes the mapping norm. In particular if A is a conformal matrix, then we have $\|A\| = (\det A)^{1/n}$.

Definition (2.11). For a transformation $f \in \mathcal{M}(\widehat{\mathbf{R}}^{n+1})$ with $f(\infty) \neq \infty$, the *isometric sphere* $I(f)$ of f is defined by

$$I(f) = \{x \in \mathbf{R}^{n+1} \mid \|D_x f\| = 1\}.$$

The isometric sphere cannot be defined for transformations which keep ∞ fixed. Recall that by (1.9), f can be expressed as

$$f(x) = \lambda P J(x-b) + c,$$

where $\lambda > 0$, $P \in O(n+1)$ and $b, c \in \mathbf{R}^{n+1}$. Note that $f(b) = \infty$ and $f(\infty) = c$. For $x \in \mathbf{R}^{n+1}$, we have

$$\|D_x f\| = \frac{\lambda}{|x-b|^2}.$$

Thus the isometric sphere $I(f)$ is the codimension one sphere of radius $\lambda^{1/2}$, centered at $f^{-1}(\infty)$. We summarize fundamental properties of isometric sphere in the following proposition. The proof is left to the reader.

Proposition (2.12). *For $f \in \mathcal{M}(\widehat{\mathbf{R}}^{n+1})$ such that $f(\infty) \neq \infty$, we have the following.*

(1) *The center of the isometric sphere $I(f)$ is the point $f^{-1}(\infty)$.*
(2) *f carries $I(f)$ to $I(f^{-1})$ and induces an isometry there. In particular, $I(f)$ and $I(f^{-1})$ have the same radius.*
(3) *f carries the interior of $I(f)$ to the exterior of $I(f^{-1})$.*
(4) *The interior of the isometric sphere $I(f)$ consists precisely of those points x for which $\|D_x f\| > 1$ holds.*

Proposition (2.13). *For $f \in \mathcal{M}(S^n)$ such that $f(\infty) \neq \infty$, the isometric sphere $I(f)$ is perpendicular to S^n.*

Proof. Since the action of f on S^n is not an isometry, there are points in S^n where the norms of the derivatives of f are less than or greater than 1. This implies that $I(f)$ intersects S^n in an $(n-1)$ sphere. f induces an isometry from $I(f)$ to $I(f^{-1})$ which sends the sphere $I(f) \cap S^n$ to the sphere $I(f^{-1}) \cap S^n$. Thus for $x \in I(f)$, the spherical distance in $I(f)$ between x and $I(f) \cap S^n$ coincides with the spherical distance in $I(f^{-1})$ between $f(x)$ and $I(f^{-1}) \cap S^n$. That is, for $x \in I(f)$, we have $|x| = |f(x)|$ and consequently $\|D_x J\| = \|D_{f(x)} J\|$. See Figure (2.2). Differentiating the equation $J \circ f = f \circ J$, we obtain that $\|D_x f\| = 1$ implies $\|D_{J(x)} f\| = 1$. That is, $J(I(f)) = I(f)$. This shows (2.13). Q.E.D.

Proposition (2.14). *A transformation $f \in \mathcal{M}(S^n)$ such that $f(\infty) \neq \infty$ can be decomposed as*

$$f = J_{\pi(f)} \circ J_{I(f)} \circ P(f),$$

where $P(f)$ is a transformation in $O(n+1)$ which preserves $I(f)$ and $\pi(f)$ is the bisector of the centers of $I(f)$ and $I(f^{-1})$ if $I(f) \neq I(f^{-1})$

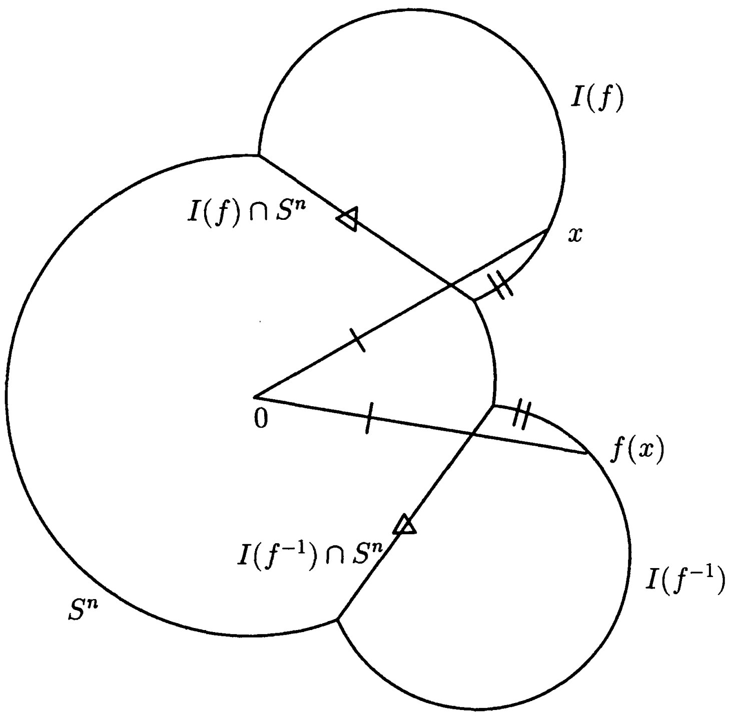

Figure (2.2)

and an arbitrary hyperplane which passes through the center of $I(f)$ *and* 0 *if* $I(f) = I(f^{-1})$. *See Figure* (2.3).

Proof. The transformation $g = J_{\pi(f)} \circ J_{I(f)}$ clearly carries $I(f)$ to $I(f^{-1})$ and there the norm of the differential is 1. That is, $I(g) = I(f)$ and $I(g^{-1}) = I(f^{-1})$. It follows that $g^{-1} \circ f$ preserves the sphere $I(f)$ and is an isometry there. Notice also that $g^{-1} \circ f$ preserves the interior of $I(f)$. Applying (2.9) to a transformation of $I(f)$, it follows that $g^{-1} \circ f = P(f)$ keeps ∞ fixed. Since $P(f)$ preserves S^n, $P(f)$ is a transformation in $O(n+1)$. Q.E.D.

It is a well known fact that $\mathcal{M}(S^n)$ is a Lie group of dimension $\frac{1}{2}(n+1)(n+2)$ with two connected components.

Definition (2.15). Let $\{f_k\}_{k=1,2,\ldots}$ be a sequence of elements of $\mathcal{M}(S^n)$. We say $f_k \to \infty$ if and only if for any compact subset C of $\mathcal{M}(S^n)$, there exists $k_0 > 0$ such that $f_k \notin C$ for $k \geq k_0$.

Thus $f_k \to \infty$ if and only if f_k has no subsequence which converges to an element of $\mathcal{M}(S^n)$.

For $f \in \mathcal{M}(S^n)$, we define

$$\|Df\|_{S^n} = \sup\{\|D_x f\| \mid x \in S^n\}.$$

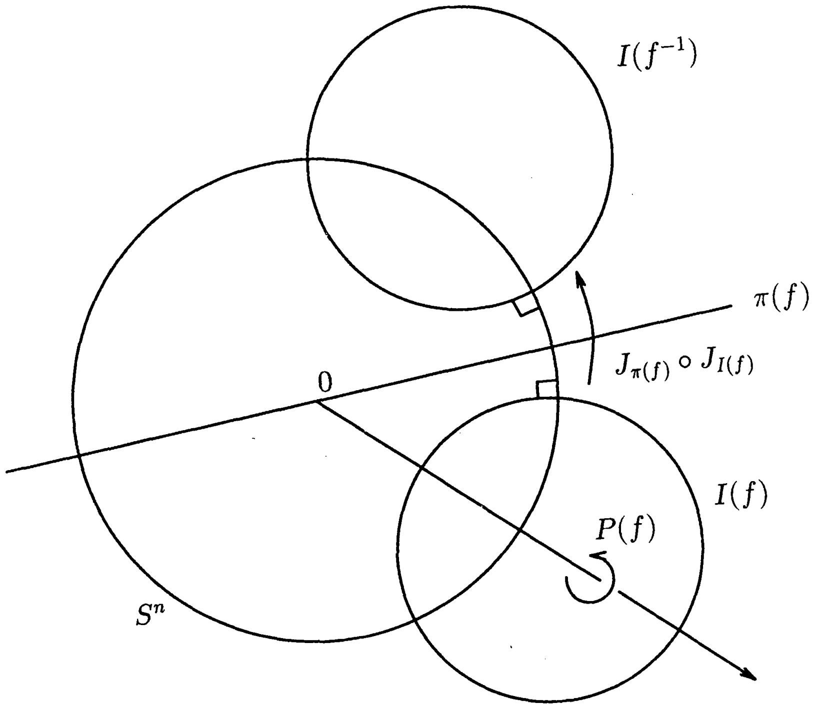

Figure (2.3)

Proposition (2.16). *For a sequence* $\{f_k\}$ *in* $\mathcal{M}(S^n)$, *the following conditions are equivalent.*

(1) $f_k \to \infty$.
(2) $\|Df_k\|_{S^n} \to \infty$.
(3) *Except for finite* k, $f_k(\infty) \neq \infty$ *and* radius $I(f_k) \to 0$.

Proof. First we shall show the equivalence of (2) and (3). Assume for simplicity that $f_k(\infty) \neq \infty$ for any k. Let

$$f_k(x) = r_k^2 P_k J(x - b_k) + c_k.$$

We have

$$\|D_x f_k\| = \frac{r_k^2}{|x - b_k|^2},$$

where $r_k =$ radius $I(f_k)$. Since $I(f_k)$ is perpendicular to S^n, we obtain

$$\|Df_k\|_{S^n} = \frac{r_k^2}{(\sqrt{1 + r_k^2} - 1)^2} = \frac{(\sqrt{1 + r_k^2} + 1)^2}{r_k^2}.$$

See Figure (2.4). From this follows the equivalence of (2) and (3).

Next, (2) $\Rightarrow$ (1) is obvious. To show the converse, we assume that (2) , hence (3), does not hold and will show that (1) fails, that is, f_k has

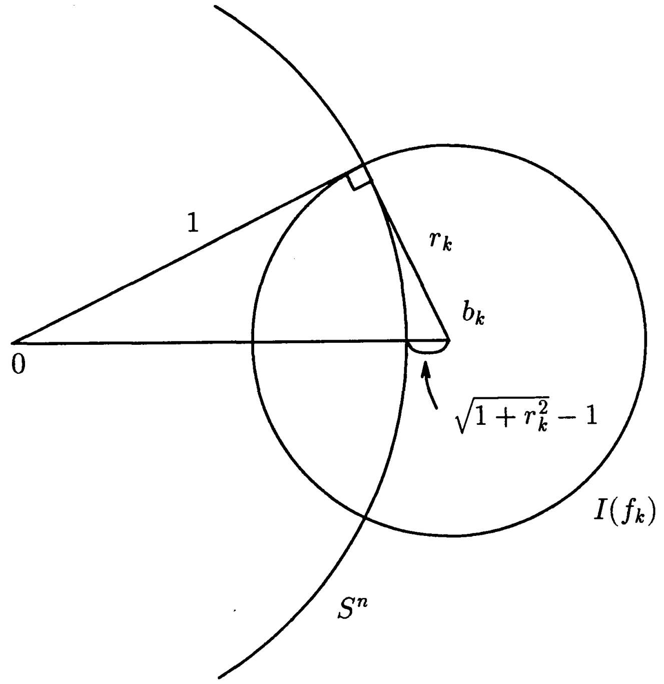

Figure (2.4)

a subsequence which converges in $\mathcal{M}(S^n)$. Thus in the course of the proof, we are free to pass to a subsequence, if necessary. If $f_k(\infty) = \infty$ for infinitely many k, then such f_k belongs to a compact subgroup $O(n+1)$ of $\mathcal{M}(S^n)$, showing that (1) does not hold. Therefore we may assume (passing to a subsequence) that $f_k(\infty) \neq \infty$ for any $k \geq 1$ and $r_k \to \rho$ for some $0 < \rho \leq \infty$.

Assume for a while that $0 < \rho < \infty$. Then in the decomposition of (2.14), the sphere $I(f_k)$ may be assumed to converge. That is, the inversion $J_{I(f_k)}$ converges in $\mathcal{M}(S^n)$. Likewise we may assume that $J_{\pi(f_k)}$ and $P(f_k)$ also converge in $\mathcal{M}(S^n)$. This shows that (1) does not hold.

Next consider the case where $\rho = \infty$. Notice that $\rho = \infty$ if and only if $f_k^{-1}(\infty) \to \infty$, since the sphere $I(f_k)$ centered at $f_k^{-1}(\infty)$ is always perpendicular to the fixed sphere S^n. Take an arbitrary transformation g of $\mathcal{M}(S^n)$ such that $g(b) = \infty$ for some $b \neq \infty$ and consider the sequence $f_k \circ g$. Then $g^{-1} \circ f_k^{-1}(\infty) \to b$. That is, radius $I(f_k \circ g) \to r$ $(0 < r < \infty)$. Therefore this case can be reduced to the former case. Q.E.D.

Next we shall show that a Moebius transformation in $\mathcal{M}(S^n)$ induces an isometry of (D^{n+1}, g_H). The key step is the following lemma.

Lemma (2.17). *Let* $f \in \mathcal{M}(S^n)$ *and let* $x \in \mathbf{R}^{n+1} \setminus S^n$. *Then*

$$\|D_x f\| = \frac{1 - |f(x)|^2}{1 - |x|^2}.$$

Proof. Both hand sides decompose as products when f decomposes as a composite. Thus it is sufficient to show (2.17) only for the inversion J_τ at an n-dimensional sphere $\tau = \{|x - a| = r\}$ which is perpendicular to S^n. We have

$$J_\tau(x) = r^2 \frac{(x-a)}{|x-a|^2} + a$$

and

$$\|D_x J_\tau\| = \frac{r^2}{|x-a|^2}.$$

Since the sphere τ is perpendicular to S^n, we have

$$|a|^2 = 1 + r^2.$$

Then it is easy to show by calculation that

$$|J_\tau(x)|^2 - 1 = \frac{r^2}{|x-a|^2}(|x|^2 - 1).$$

This shows (2.17). Q.E.D.

Corollary (2.18). *An element* $f \in \mathcal{M}(S^n)$ *induces an isometry of* (D^{n+1}, g_H).

The converse can also be shown using (2.10), once we establish the following lemma.

Lemma (2.19). *For any point* $a \in D^{n+1}$, *there exists a transformation* $f \in \mathcal{M}(S^n)$ *such that* $f(0) = a$.

Proof. Let l be the radius through a. For any $x \in l$, let σ_x be the codimension one sphere perpendicular to l at x and orthogonal to S^n. Then $J_{\sigma_x} \in \mathcal{M}(S^n)$ sends 0 to some point in l. Clearly we have

$$\lim_{x \to 0} J_{\sigma_x}(0) = 0, \quad \lim_{x \to b} J_{\sigma_x}(0) = b,$$

where b is the end point of l. By the continuity of $J_{\sigma_x}(0)$, we obtain a point x in l such that $J_{\sigma_x}(0) = a$. Q.E.D.

Theorem (2.20). *$\mathcal{M}(S^n)$ is precisely the group of isometries of (D^{n+1}, g_H).*

Theorem (2.21). *In (D^{n+1}, g_H), the geodesics are the circles that are orthogonal to S^n. Denoting the distance in (D^{n+1}, g_H) by d_H, we also have for $a \in D^{n+1}$*

$$d_H(0, a) = \log \frac{1 + |a|}{1 - |a|}.$$

Proof. First let us find the shortest path combining 0 and a $(a \neq 0)$. Let $\gamma(t)$ be an arbitrary smooth arc such that $\gamma(0) = 0$ and $\gamma(1) = a$. Schwartz's inequality yields

$$||\gamma(t)|'| \leq |\gamma'(t)|.$$

Thus we have

$$\begin{aligned} \text{length}(\gamma) &= \int_0^1 \frac{2|\gamma'(t)|dt}{1 - |\gamma(t)|^2} \geq \int_0^1 \frac{2||\gamma(t)|'|dt}{1 - |\gamma(t)|^2} \\ &\geq \int_0^{|a|} \frac{2ds}{1 - s^2} = \log \frac{1 + |a|}{1 - |a|}. \end{aligned}$$

This shows the last part of (2.21) and that the geodesic through 0 and a are the radius.

Now consider the general case. Let $a, b \in D^{n+1}$. By (2.19), there exists $f \in \mathcal{M}(S^n)$ such that $f(0) = a$. Since f^{-1} is an isometry, f^{-1} maps the geodesics to the geodesics. Further since f^{-1} is a Moebius transformation, f^{-1} maps the diameter through $f(b)$ to the circle through a and b which is orthogonal to S^n. Q.E.D.

Finally we shall classify transformations in $\mathcal{M}(S^n)$ according to its dynamics on $\mathrm{Cl}(D^{n+1})$. By (2.3), they keep $\mathrm{Cl}(D^{n+1})$ invariant, where Cl denotes the closure.

Proposition (2.22). *Let $f \in \mathcal{M}(S^n)$. For the induced transformation*

$$f : \mathrm{Cl}(D^{n+1}) \to \mathrm{Cl}(D^{n+1}),$$

we have the followings.

(1) *f has at least one fixed point in $\mathrm{Cl}(D^{n+1})$.*
(2) *If f has three or more fixed points in S^n, then f has a fixed point in D^{n+1}.*

Proof. (1) follows from Brouwer's fixed point theorem. To show (2), coordinates of $\widehat{\mathbf{R}}^n$ and H^{n+1} are more convenient. By conjugating

$$g = c_v^{-1}(f) \in \mathcal{M}(\widehat{\mathbf{R}}^n)$$

by a suitable element of $\mathcal{M}(\widehat{\mathbf{R}}^n)$, we may assume that g keeps fixed 0, ∞ and another point a. By (1.9), we have for $x \in \mathbf{R}^n$, $g(x) = \lambda P x$, where $\lambda > 0$ and

$$P = \begin{pmatrix} Q & 0 \\ 0 & 1 \end{pmatrix},$$

where $Q \in O(n)$. Since g also keeps a fixed, it follows that $\lambda = 1$. Thus for example, $(0, y) \in H^{n+1}$ $(y > 0)$ is fixed by g. This completes the proof of (2). Q.E.D.

Definition (2.23). $f \in \mathcal{M}(S^n)$ (resp. $\mathcal{M}(\widehat{\mathbf{R}}^n)$) is called *elliptic* if f has fixed points in D^{n+1} (resp. H^{n+1}), *loxodromic* if f is not elliptic and has exactly two fixed points in S^n (resp. $\widehat{\mathbf{R}}^n$) and *parabolic* otherwise.

Notice that by (2.22), a parabolic transformation has precisely one fixed point in S^n (resp. $\widehat{\mathbf{R}}^n$).

Next we shall describe the standard forms of conjugacy classes of these three types of transformations. For elliptic transformations, it is convenient to work with the coordinates of S^n and to conjugate so that 0 is the fixed point. However for the other types, the coordinates of $\widehat{\mathbf{R}}^n$ is preferable. Notice that parabolic (resp. loxodromic) transformations can be conjugated so that they keep ∞ (resp. ∞ and 0) fixed.

Proposition (2.24).

(1) *Let $f \in \mathcal{M}(S^n)$ be an elliptic transformation such that $f(0) = 0$. Then we have $f(x) = Px$ for some $P \in O(n+1)$.*

(2) *Let $f \in \mathcal{M}(\widehat{\mathbf{R}}^n)$ be a loxodromic transformation such that $f(\infty) = \infty$ and $f(0) = 0$. Then we have $f(x) = \lambda P x$ for some $\lambda \neq 1, > 0$ and $P \in O(n)$.*

(3) *Let $f \in \mathcal{M}(\widehat{\mathbf{R}}^n)$ be a parabolic transformation such that $f(\infty) = \infty$. Then by conjugating with a translation of $\mathbf{R}^n$, we have $f(x) = Px + b$ for some $P \in O(n)$ and $b \in \mathbf{R}^n \setminus \{0\}$ such that $Pb = b$.*

Proof. To show (2), notice that $\lambda \neq 1$ since otherwise f would fix points of the straight line perpendicular to $\mathbf{R}^n$ which passes through 0, contrary to the hypothesis that f is loxodromic.

To prove (3), let $f(x) = \lambda Px + b'$. Since f cannot have a fixed point in $\mathbf{R}^n$, we have $\lambda = 1$ and $b' \notin \mathrm{Image}(P - I)$. But $b' = (P-I)a + b$, for some $a \in \mathbf{R}^n$ and $b \perp \mathrm{Image}(P - I)$. It is a standard exercise in linear algebra to show $Pb = b$. Conjugating f by the translation by a, we obtain the transformation $x \longmapsto Px + b$, as is required. Q.E.D.

Definition (2.25). For a loxodromic transformation $f \in \mathcal{M}(S^n)$, the geodesic which combines the two fixed points of f is called an *axis* of f.

Definition (2.26). A codimension one sphere in $\mathrm{Cl}(D^{n+1})$ which is tangent to S^n at $a \in S^n$ is called a *horosphere* at a.

Proposition (2.27). *A loxodromic transformation of $\mathcal{M}(S^n)$ preserves its axis. A parabolic transformation preserves the horospheres at the fixed point.*

Proof. To prove the first part, notice that the standard form (2) of (2.24) preserves the x_{n+1}-axis in H^{n+1}. The transformation $v \in \mathcal{M}(\widehat{\mathbf{R}}^{n+1})$ (defined just before (2.2)) maps x_{n+1}-axis to a diameter in D^{n+1}. Any transformation of $\mathcal{M}(S^n)$ maps a diameter to a geodesic of D^{n+1}. Therefore by conjugating the standard form, we get the desired result. The latter part can be shown likewise. Notice that the standard form (3) of (2.24) preserves the plane $\{x_{n+1} = c\}$ $(c > 0)$, which is mapped by v to a horosphere. Q.E.D.

§3. Flat conformal structure

In this section we define a flat conformal structure, its developing map and holonomy homomorphism. We study their fundamental properties.

In the first place, we define a (G, X)-structure in general circumstances. Let X be a real analytic manifold and let G be a Lie group acting real analytically, transitively and effectively on X. In this study, all the group actions are to be on the left, unless otherwise specified. Let N be a connected topological manifold of the same dimension as X.

Definition (3.1). A collection $\mathcal{U} = \{(U_\alpha, q_\alpha)\}_{\alpha \in \Lambda}$ is called a (G, X)-*atlas* if

(1) $\{U_\alpha\}$ is an open covering of N.
(2) $q_\alpha : U_\alpha \to X$ is an embedding.
(3) For each component V of $U_\alpha \cap U_\beta$, there exists $g \in G$ such that $q_\beta(x) = g q_\alpha(x)$, $x \in V$.

An element (U_α, q_α) is called a *$\mathcal{U}$-chart.*

Definition (3.2). A maximal (G,X)-atlas is called a (G,X)-*structure* on N or a *geometric structure* vaguely. A manifold equipped with a (G,X)-structure is called a (G,X)-*manifold.*

Let $p : M \to N$ be a covering map.

Definition (3.3). Let $\{(U_\alpha, q_\alpha)\}_{\alpha\in\Lambda}$ be a (G,X)-atlas on N for a (G,X)-structure $\mathcal{U}$ such that U_α is homeomorphic to an n-ball. Let V_α^i be a connected component of $p^{-1}(U_\alpha)$. Then $\{(V_\alpha^i, q_\alpha \circ p)\}$ is a (G,X)-atlas on M. The (G,X)-structure which contains $\{(V_\alpha^i, q_\alpha \circ p)\}$ is called the *lift* of $\mathcal{U}$ by p and is denoted by $p^*\mathcal{U}$. Especially when p is a homeomorphism, $p^*\mathcal{U}$ and $\mathcal{U}$ are called *isomorphic.*

Given a (G,X)-structure $\mathcal{U}$ on N, the associated developing map and holonomy homomorphism are defined as follows.

Let $p : \widetilde{N} \to N$ be the universal covering space with the base point $x_0 \in \widetilde{N}$. Let $\pi_1(N)$ be the fundamental group at the base point $p(x_0)$. As usual, $\pi_1(N)$ is identified via x_0, with the group of deck transformations of $\widetilde{N}$. Denote by $\widetilde{\mathcal{U}}$ the lift of $\mathcal{U}$ by p. Fix once and for all a $\widetilde{\mathcal{U}}$-chart (U_0, q_0) around x_0.

Definition (3.4). A sequence $((U_i, q_i), g_i)$, $(1 \le i \le r)$ is called a *chart chain from* (U_0, q_0) if for $1 \le i \le r$, we have

(a) $(U_i, q_i) \in \widetilde{\mathcal{U}}$, $g_i \in G$,
(b) $U_{i-1} \cap U_i$ is nonempty and connected,
(c) $q_{i-1}(x) = g_i q_i(x)$, $x \in U_{i-1} \cap U_i$.

Given a chart chain as above, it is possible to extend the base map q_0 to a continuous map $D : U_0 \cup U_1 \to X$ by

$$D(x) = g_1 q_1(x), \quad x \in U_1.$$

Successively D can be extended to $U_0 \cup U_1 \cup U_2$ by

$$D(x) = g_1 g_2 q_2(x), \quad x \in U_2.$$

See Figure (3.1). This motivates the following definition.

Definition (3.5).

(1) The *developing map* $D : \widetilde{N} \to X$ *w.r.t. the base chart* (U_0, q_0) is defined by

$$D(x) = g_1 g_2 \cdots g_r \cdot q_r(x), \quad x \in \widetilde{N},$$

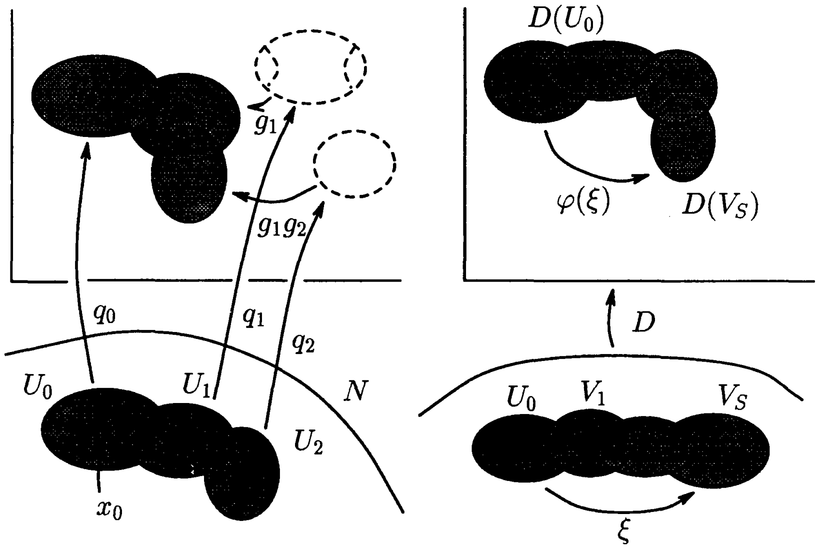

Figure (3.1)

where $((U_i, q_i), g_i)$, $(1 \le i \le r)$ is a chart chain from (U_0, q_0) such that $x \in U_r$.

(2) The *holonomy homomorphism* $\varphi : \pi_1(N) \to G$ *w.r.t. the base chart* (U_0, q_0) is defined by

$$\varphi(\xi) = h_1 h_2 \cdots h_s, \quad \xi \in \pi_1(N)$$

where $((V_j, p_j), h_j)$, $1 \le j \le s$ is a chart chain from (U_0, q_0) such that

$$(V_s, p_s) = (\xi U_0, q_0 \circ \xi^{-1}).$$

D and ϕ are well defined since $\widetilde{N}$ is simply connected. The proof is routine and is omitted. Also it is clear that D is a submersion (or immersion).

Definition (3.6). A pair (D, φ) is called a *DH pair* if the following is satisfied.

(1) $D : \widetilde{N} \to X$ is a submersion.
(2) $\varphi : \pi_1(N) \to G$ is a homomorphism.
(3) $D(\xi x) = \varphi(\xi) D(x)$, $\xi \in \pi_1(N)$, $x \in \widetilde{N}$.

Proposition (3.7). *Let D and φ be the developing map and the holonomy homomorphism for a base chart (U_0, q_0). Then (D, φ) is a DH pair.*

Proof. To show

$$D(\xi x) = \varphi(\xi) D(x), \quad \xi \in \pi_1(N),\ x \in \widetilde{N},$$

let

$$\sigma = \{((U_i, q_i), g_i)\}_{1 \leq i \leq r}$$

be a chart chain from (U_0, q_0) such that $x \in U_r$ and let

$$\tau = \{((V_j, p_j), h_j)\}_{1 \leq j \leq s}$$

be a chart chain from (U_0, q_0) such that

$$(V_s, p_s) = (\xi U_0, q_0 \circ \xi^{-1}).$$

Let

$$\xi_\sharp \sigma = \{((\xi U_i, q_i \circ \xi^{-1}), g_i)\}.$$

$\xi_\sharp \sigma$ is a chart chain from $(\xi U_0, q_0 \circ \xi^{-1}) = (V_s, p_s)$. Thus τ followed by $\xi_\sharp \sigma$ is a chart chain from (U_0, q_0) to the point ξx. That is, we have

$$D(\xi x) = h_1 h_2 \cdots h_s \cdot g_1 g_2 \cdots g_r \cdot q_r \circ \xi^{-1}(\xi x) = \varphi(\xi) D(x).$$

Finally let us show that φ is a homomorphism. We have

$$\varphi(\xi_1 \xi_2) D(x) = D(\xi_1 \xi_2 x) = \varphi(\xi_1) D(\xi_2 x) = \varphi(\xi_1) \varphi(\xi_2) D(x).$$

It follows that $\varphi(\xi_1 \xi_2) = \varphi(\xi_1)\varphi(\xi_2)$, since the action of G on X is effective and real analytic. (Note that Image(D) is a domain since D is a submersion.) Likewise we have $\varphi(1) = 1$. Q.E.D.

Definition (3.8). Two DH pair (D, φ) and (D', φ') are said to be *equivalent* if there exists $g \in G$ such that $D'(x) = gD(x)$ and $\varphi'(\xi) = g\varphi(\xi)g^{-1}$ for $x \in \widetilde{N}$ and $\xi \in \pi_1(N)$.

Proposition (3.9). *The correspondence of (3.7) gives a bijection between the set of (G, X)-structures on N and the set of the equivalence classes of DH pairs.*

Proof. Let (D, φ) (resp. (D', φ')) be the DH pair associated to the base chart (U_0, q_0) (resp. (U_0', q_0'))of a given (G, X)-structure.

Consider a chart chain

$$((U_i, q_i), g_i), \quad 1 \leq i \leq r$$

from (U_0', q_0') such that $(U_r, q_r) = (U_0, q_o)$. Let $g = g_1 g_2 \cdots g_r$. Then it is easy to show that

$$D'(x) = gD(x), \quad \varphi'(\xi) = g\varphi(\xi)g^{-1}.$$

Conversely given an equivalence class of DH pairs, one can get a (G, X)-structure on N by restricting the developing map to small domains of $\widetilde{N}$ and projecting down by $p : \widetilde{N} \to N$. Q.E.D.

By certain abuse, (G, X)-structures are sometimes denoted by their DH pairs as $[D, \varphi]$.

Definition (3.10). For a (G, X)-structure $[D, \varphi]$ on N,

$$H = \text{Image}(\varphi) \subset G$$

is called the *holonomy group* of $[D, \varphi]$.

By (3.9), the holonomy group of a (G, X)-structure is unique up to conjugations in G.

Let Γ be a discrete group which acts on N.

Definition (3.11). Γ is said to act *discontinuously* on N, if for any $x \in N$, there exists a neighbourhood U of x such that

$$\text{Card}\{\gamma \in \Gamma \mid \gamma U \cap U \neq \phi\} < \infty.$$

The proof of the following proposition is left to the reader.

Proposition (3.12). *Γ acts freely and discontinuously on N if and only if for any $x \in N$, there exists a neighbourhood U such that if $\gamma \neq 1$, then $\gamma U \cap U = \phi$.*

Suppose $N \to P$ be a regular covering with the group of deck trasformations Γ. Then the action of Γ on N is free and discontinuous. Conversely, if Γ acts freely and discontinuously on a manifold N, then the canonical projection $\pi : N \to N/\Gamma$ is a regular covering with the group of deck transformations Γ.

Proposition (3.13). *Suppose that* Γ *acts on* N *freely and discontinuously. Then*

$$\widetilde{\Gamma} = \{\widetilde{\gamma} : \widetilde{N} \to \widetilde{N} \mid \widetilde{\gamma} \text{ is a lift of } \gamma, \quad \gamma \in \Gamma\}$$

acts on $\widetilde{N}$ *freely and discontinuously.* $\widetilde{\Gamma}$ *is the group of deck transformations of the following universal covering.*

$$\pi \circ p : \widetilde{N} \to N \to N/\Gamma.$$

We have the following exact sequence;

$$1 \to \pi_1(N) \to \widetilde{\Gamma} \to \Gamma \to 1.$$

Proof. We only show that the action of $\widetilde{\Gamma}$ is free and discontinuous. The rest is left to the reader. Let $x \in \widetilde{N}$. Take a small neighbourhood $\widetilde{U}$ of x such that

(1) $U = p(\widetilde{U})$ is evenly covered by p and
(2) $\gamma U \cap U = \phi$ if $\gamma \neq 1$, $\gamma \in \Gamma$.

Suppose $\widetilde{\gamma}(\widetilde{U}) \cap \widetilde{U} \neq \phi$ for $\widetilde{\gamma} \in \widetilde{\Gamma}$. Then we have $\gamma(U) \cap U \neq \phi$, where $\widetilde{\gamma}$ is a lift of γ. This shows that $\gamma = 1$ by (2). Thus $\widetilde{\gamma}$ is a deck transformation of p. But by (1), we have $\widetilde{\gamma} = 1$. Q.E.D.

Let $\mathcal{U}$ be a (G, X)-structure on N.

Definition (3.14). An action of Γ on N is called a *$\mathcal{U}$-action* if and only if for any $\gamma \in \Gamma$, we have $\gamma^* \mathcal{U} = \mathcal{U}$.

Suppose that an action of Γ on N is a free and discontinuous $\mathcal{U}$-action. As before, $\pi : N \to N/\Gamma$ is the canonical projection.

Definition (3.15). A (G, X)-strucure $\pi_* \mathcal{U}$, called the *projection* of $\mathcal{U}$, is defined as follows. Let (D, φ) be the DH pair associated to a base chart (U_0, q_0). Since the action of the lift $\widetilde{\Gamma}$ is a $\widetilde{\mathcal{U}}$-action, we have that $(\widetilde{\gamma} U_0, q_0 \circ \widetilde{\gamma}^{-1})$ is a $\widetilde{\mathcal{U}}$-chart for any $\widetilde{\gamma} \in \widetilde{\Gamma}$. Thus as in Definition (3.5) (2), we can define a homomorphism

$$\psi : \widetilde{\Gamma} \to G$$

by using a chart chain to $(\widetilde{\gamma} U_0, q_0 \circ \widetilde{\gamma}^{-1})$. Then (D, ψ) is a DH pair for N/Γ. $\pi_* \mathcal{U}$ is defined to be the (G, X)-structure corresponding to this DH pair.

Clearly $\psi : \widetilde{\Gamma} \to G$ is an extension of the holonomy homomorphism $\varphi : \pi_1(N) \to G$.

As is shown later, there are many examples of pair (G, X) such that the isotropy subgroup

$$G_x = \{g \in G \mid gx = x\}$$

is compact for any $x \in X$. Then the corresponding (G, X)-structures have the following striking feature.

Proposition (3.16). *Let N be a closed (G, X)-manifold. Suppose the isotropy subgroup G_x is compact for $x \in X$. Then the developing map $D : \widetilde{N} \to X$ is a covering map onto X. In particular, if X is simply connected, then D is a homeomorphism.*

Proof. Since G_x is compact, there exists a G_x-invariant, positive definite, symmetric, bilinear form on the tangent space T_xX. Distributing it by the action of G, we obtain a G-invariant Riemannian metric g of X. Since $\bar{g} = D^*g$ is $\pi_1(N)$-invariant, it projects down to a Riemannian metric on N. Therefore $\bar{g}$ is complete.

For small $\varepsilon > 0$, we have that D maps any 2ε-ball in $\widetilde{N}$ isometrically onto a 2ε-ball in X. Then clearly any ε-ball in X is evenly covered by D. Q.E.D.

We shall raise some examples of (G, X)-structures.

Example (3.17). Denote by $\mathrm{Isom}(S^n)$, $\mathrm{Isom}(\mathbf{R}^n)$ or $\mathrm{Isom}(D^n)$ the group of isometries of the Riemannian manifold (S^n, g_S), $(\mathbf{R}^n, g_E)$ or (D^n, g_H). The corresponding (G, X)- structure (resp. manifold) is called *spherical, Euclidean* or *hyperbolic* structure (resp. manifold). Specifically, closed spherical or Euclidean manifold is called *spherical* or *Euclidean space form.*

Notice that $\mathrm{Isom}(S^n) = O(n+1)$ and $\mathrm{Isom}(D^n) = \mathcal{M}(S^{n-1})$. $\mathrm{Isom}(\mathbf{R}^n)$ consists of transformations, called *Euclidean motions*,

$$x \mapsto Px + b, \quad (P \in O(n),\ b \in \mathbf{R}^n).$$

All the three satisfy the hypothesis of (3.16). Therefore if the manifolds are compact, their universal covering spaces can be identified with S^n (if $n > 1$), $\mathbf{R}^n$ or D^{n+1}. A spherical space form is isomorphic to S^n/Γ if $n > 1$, where Γ is a finite group of $SO(n+1)$. The following theorem is due to Bieberbach ([5]). A neat proof, quite short, is found in P. Buser ([6]).

Theorem (3.18). *An Euclidean space form has n-torus as a finite covering.*

The main object of our study is the following (G, X)-structure.

Definition (3.19). A $(\mathcal{M}(S^n), S^n)$-structure (resp. manifold) is called a *flat comformal structure* (resp. *manifold*). A group action preserving a flat conformal structure is called a *conformal action.*

There is another way to get to the same concept.

Definition (3.20). A Riemannian manifold (N, g) of dimension n is called *conformally flat* if for any point $x \in N$, there exist a neighbourhood U and an embedding $f : U \to \mathbf{R}^n$ such that f^*g_E is conformally equivalent to $g|_U$.

Notice that the above definition does not change if we use as a model space (S^n, g_S) instead of $(\mathbf{R}^n, g_E)$. In fact, they are conformally equivalent as we saw in §2.

Now let $\mathcal{U}$ be a flat conformal structure on N. For each $\mathcal{U}$-chart (U_α, q_α), there is the induced Riemannian metric $q_\alpha^* g_S$ on U_α. In a component V of $U_\alpha \cap U_\beta$, we have $q_\beta = g \circ q_\alpha$ for some $g \in \mathcal{M}(S^n)$. Since g is a conformal map w.r.t. g_S, $q_\alpha^* g_S$ and $q_\beta^* g_S$ are conformally equivalent on V. Take a locally finite partition of unity $\{t_\alpha\}$ associated with the covering $\{U_\alpha\}$ of $\mathcal{U}$-charts. The Riemannian metric

$$g = \sum_\alpha t_\alpha q_\alpha^* g_S$$

is a conformally flat metric.

Conversely suppose $n \geq 3$. Let g be a conformally flat metric on an n-dimensional manifold N. Then we have a family $\{(U_\alpha, f_\alpha)\}$ such that $\{U_\alpha\}$ is an open covering of N, that f_α is an embedding of U_α into S^n and that $f_\alpha^* g_S$ is conformally equivalent to g. Thus for any component V of $U_\alpha \cap U_\beta$,

$$f_\beta \circ f_\alpha^{-1}|_{f_\alpha(V)} : f_\alpha(V) \to f_\beta(V)$$

is a Riemannian conformal map in (S^n, g_S). Thus by Liouville's theorem, we have that

$$f_\beta \circ f_\alpha^{-1}|_{f_\alpha(V)} \in \mathcal{M}(S^n).$$

We obtain a flat conformal structure. In summary, we have;

Proposition (3.21). *Flat conformal structure on a manifold N yields a conformally equivalence class of conformally flat metrics. Further if $n \geq 3$, this correspondence is bijective.*

For $n = 2$, the above two concepts are in fact different. In this dimension, flat conformal structure is often called *(complex) projective* *structure* since

$$(\mathcal{M}(S^2), S^2) = (PGL(2:\mathbf{C}), \mathbf{C}P^1),$$

while conformally flat Riemannian metric corresponds to complex structures.

If $(G', X') \subset (G, X)$, that is, $G' \subset G$, $X' \subset X$ and the G'-action on X' is the restriction of the G-action on X, then, as a matter of fact, a (G', X')-structure is naturally considered as a (G, X)-structure. Thus spherical manifolds, Euclidean manifolds and hyperbolic manifolds are considered to be flat conformal manifolds. In fact we have the following inclusions of (G, X)-pairs.

$(\mathrm{Isom}(S^n), S^n) \subset (\mathcal{M}(S^n), S^n)$.

$(\mathrm{Isom}(\mathbf{R}^n), \mathbf{R}^n) \subset (\mathcal{M}(\widehat{\mathbf{R}}^n), \widehat{\mathbf{R}}^n) \xrightarrow[\approx]{c_v} (\mathcal{M}(S^n), S^n)$.

$(\mathrm{Isom}(D^n), D^n) \subset (\mathcal{M}(S^{n-1}), D^n) \subset (\mathcal{M}(\widehat{\mathbf{R}}^n), \widehat{\mathbf{R}}^n) \xrightarrow[\approx]{c_v} (\mathcal{M}(S^n), S^n)$.

A significant feature of these examples is that the developing maps are homeomorphisms onto their images (except the case of $(\mathrm{Isom}(S^1), S^1)$). However for a point $a \in S^n$, the isotropy group $\mathcal{M}(S^n)_a$ is not compact. (Compare that $\mathcal{M}(S^n)_a$ is compact for $a \in D^{n+1}$.) Therefore flat conformal manifolds in general do not enjoy this kind of good properties. In fact there are many such examples as we shall show in what follows. We make the following definition.

Definition (3.22). Let $\mathcal{U} = [D, \varphi]$ be a flat conformal structure and let $H =$ Image(φ) be the holonomy group. $\mathcal{U}$ is said to be of *type* 1 if D is a covering map onto its image and H is discrete, of *type* 2 if D is a covering map but H is indiscrete, of *type* 3 if H is discrete but D is not a covering map and *type* 4 otherwise.

Before starting the study of type 1 flat conformal structures, we need some preparations. Let Γ be a subgroup of $\mathcal{M}(S^n)$.

Definition (3.23). A subset $A \subset S^n$ is called Γ-invariant if $\gamma(A) = A$ for any $\gamma \in \Gamma$.

Definition (3.24). Let Ω_Γ be the set of points $x \in S^n$ such that there exists a neighbourhood U of x such that $\gamma U \cap U = \phi$ but for finitely many $\gamma \in \Gamma$. Ω_Γ is called the *domain of discontinuity* of Γ.

Ω_Γ is the maximal Γ-invariant open subset of S^n on which Γ acts discontinuously.

Definition (3.25). Γ is called a *Kleinian group* if $\Omega_\Gamma \neq \phi$.

Clearly we have:

Proposition (3.26). *A Kleinian group is discrete in* $\mathcal{M}(S^n)$.

It is known that the converse does not hold. However we have:

Proposition (3.27). *If* Γ *is discrete, then* Γ *acts on* D^{n+1} *discontinuously.*

Proof. Assume Γ is infinite and let $\Gamma = \{\gamma_n\}$. Since Γ is discrete, $\gamma_n \to \infty$, that is, $\gamma_n \notin O(n+1)$ for but finitely many n and radius $I(\gamma_n) \to 0$. It follows that any compact subset of D^{n+1} is outside $I(\gamma_n)$ except finite n. (Recall $I(\gamma_n) \cap S^n \neq \phi$.) (3.27) follows from (2.12)(3). Q.E.D.

The following fact is helpful in our study of flat conformal structures of type 1. The proof is more or less the same as (3.27). The reader will find it in §5, after the definition of limit set is made.

Corollary (5.16). *Suppose a discrete group* Γ *admits an invariant open set* Ω *such that* $S^n \setminus \Omega$ *is neither empty nor a singleton. Then* Γ *acts on* Ω *discontinuously.*

Flat conformal structure of type 1 is constructed as follows. Let Γ be a Kleinian group in $\mathcal{M}(S^n)$ which acts freely and discontinuously on a Γ-invariant domain Ω. The action is of course a conformal action on a flat conformal manifold Ω. Hence the quotient manifold Ω/Γ admits a flat conformal structure $\mathcal{U}$. The developing map D is the universal covering followed by the inclusion;

$$D : \widetilde{\Omega} \xrightarrow{\pi} \Omega \subset S^n$$

and the holonomy group is Γ. Concrete examples of this construction will be given in later sections.

Definition (3.28). The flat conformal structure (manifold) constructed as above is called a *Kleinian structure* (*manifold*).

Definition (3.29). Two flat conformal manifolds are called *commensurable* if they have isomorphic finite coverings.

Proposition (3.30). *Any type 1 flat conformal compact manifold* N *is commensurable to a Kleinian manifold.*

The proof needs the following theorem due to Selberg. See e.g. ([53]).

Theorem (3.31). *Any finitely generated subgroup of* $GL(n, \mathbf{R})$ *has a torsion free subgroup of finite index.*

As is well known, $\mathcal{M}(S^n)$ is isomorphic to the projectivised Lorentz group $PO(n+1,1)$. Thus (3.31) is applicable to a subgroup of $\mathcal{M}(S^n)$.

Proof of (3.30). If the developing map D is onto S^n, then D is a homeomorphism and N is isomorphic to a spherical space form. Likewise if D misses only one point, then N is isomorphic to an Euclidean space form. Otherwise, by (5.16), the holonomy group H acts on $\Omega =$ Image(D) discontinuously. Let Γ be a torsion free finite index subgroup of H. Γ acts on Ω freely. We have the following two covering maps.

$$p : \widetilde{N}/\varphi^{-1}(\Gamma) \to N,$$
$$\bar{D} : \widetilde{N}/\varphi^{-1}(\Gamma) \to \Omega/\Gamma.$$

p is a finite covering since $\varphi^{-1}(\Gamma)$ is a finite index subgroup of $\pi_1(N)$. Therefore $\widetilde{N}/\varphi^{-1}(\Gamma)$ is compact and $\bar{D}$ is also a finite covering.

Q.E.D.

One can show by examples that Proposition (3.30) cannot be sharpened in general.

Next an example of type 2 flat conformal structure is in order.

Example (3.32). Let $P(x) = \lambda R_\theta x$ be a conformal linear transformation on $\mathbf{R}^2$ ($\lambda > 0$, R_θ; the rotation by θ). For $t \in \mathbf{R}$, let

$$P^t(x) = \lambda^t R_{t\theta}(x).$$

Let Q be another conformal transformation which keeps 0 fixed such that $Q \neq P^t$ for any $t \in \mathbf{R}$.

Let $\mathbf{R}^2/\mathbf{Z}^2 = T^2$. Define $\varphi : \mathbf{Z}^2 \to \mathcal{M}(\widehat{\mathbf{R}}^2)$ by $\varphi(l,m) = P^l Q^m$ and $D : \mathbf{R}^2 \to \widehat{\mathbf{R}}^2$ by $D(x,y) = P^x Q^y a$ for some $a \in \mathbf{R} \setminus \{0\}$. Since $PQ = QP$, we have (D, φ) is a DH pair. D is clearly a covering map onto $\mathbf{R}^2 \setminus \{0\}$. But often $H =$ Image(φ) is not discrete, for example when $\lambda = 1$ and $\theta \notin \mathbf{Q}$.

See Figure (3.2). This example cannot be generalized to higher dimensions, since $\mathbf{R}^n \setminus \{0\}$ is simply connected if $n \geq 3$. However, in §5, we give examples of type 2 flat conformal compact manifolds of dimension ≥ 3 and give a characterization of such manifolds.

The following is an example of type 3 flat conformal structure.

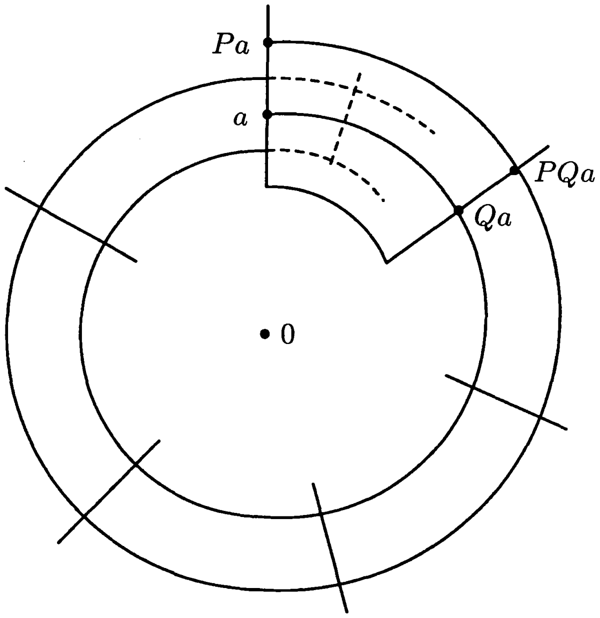

Figure (3.2)

Example (3.33). Let Σ be a closed Riemann surface of genus ≥ 2, that is, a hyperbolic manifold of dimension 2. The developing map D is a homeomorphism onto a disk in S^2. We shall alter D without changing the holonomy homomorphism φ. Let α be a simple closed geodesic in Σ and let V be the ε-neighbouhood of α for small $\varepsilon > 0$. Then a lift $\widetilde{V}$ of V in the universal covering $\widetilde{\Sigma} \cong D^2$ is the mutually disjoint ε-neighbourhood of a lift of α. See Figure (3.3). D is altered inside $\widetilde{V}$ to a new map D' in such a way that it coincides with D near the boundary of $\widetilde{V}$ and it goes extra once around S^2. Clearly D' can be constructed so that (D', φ) is a DH pair. See Figure (3.4). It is easy to show that D' is onto S^n. Thus it is not a covering map. For more detail, see Goldman ([16]). The same construction is possible for higher dimension if we start with a compact hyperbolic manifold which admits a totally geodesic closed submanifold of codimension 1. See Kourouniotis ([33]).

Finally an example of type 4.

Example (3.34). Prepare two copies of type 2 flat conformal manifolds N_1 and N_2 constructed in Example (3.34). Inside an atlas (U_i, q_i) of N_i, take a small disk V_i which is mapped by q_i to a metric disk in S^2. There exists an element $g \in \mathcal{M}(S^2)$ such that g maps V_1 to the exterior of V_2 . Consider the connected sum

$$N_1 \sharp N_2 = (N_1 \setminus \mathrm{Int} V_1) \cup (N_2 \setminus \mathrm{Int} V_2)/ \sim .$$

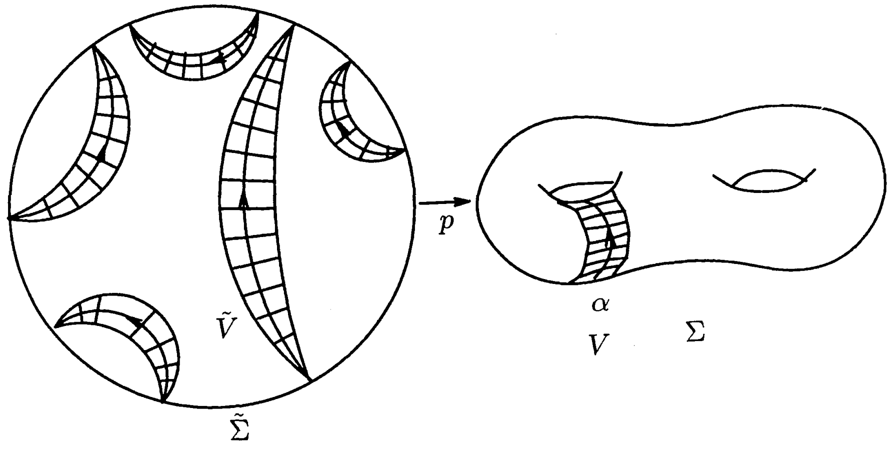

Figure (3.3)

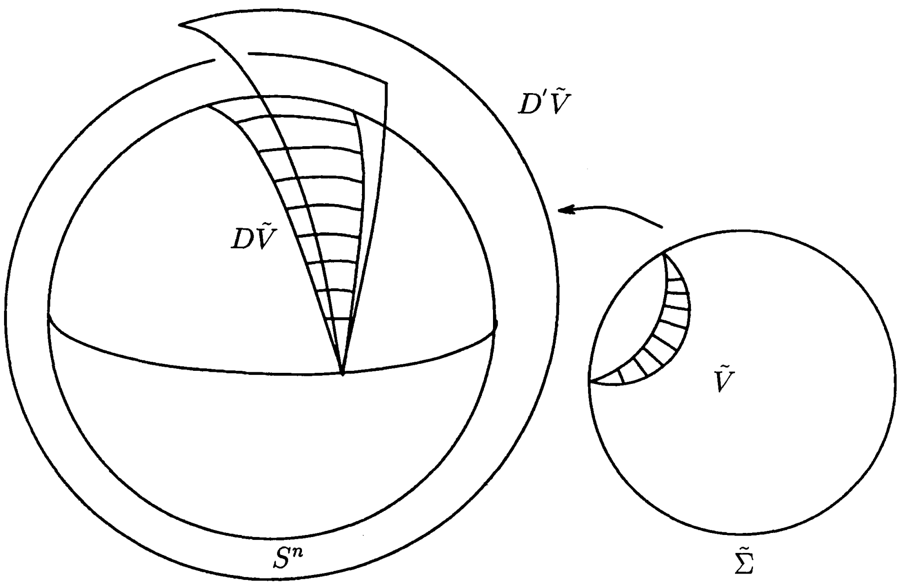

Figure (3.4)

If we chose the above identification appropriately, we obtain a continuous map

$$(g \circ q_1) \cup q_2 : (U_1 - \mathrm{Int}V_1) \cup (U_2 - \mathrm{Int}V_2)/ \sim \longrightarrow S^2.$$

Using this we get in an obvious way a flat conformal structure on $N_1 \sharp N_2$. It is not difficult to show that the developing map of this structure is onto S^2 and therefore is not a covering map. The holonomy group is indiscrete since we started with type 2 examples.

The above operation, called *connected sum* of the structure, will be described in more detail in §6.

§4. Closed similarity manifolds

In this section we assume $n \geq 3$ and mainly work with $\widehat{\mathbf{R}}^n$, instead of S^n. As before $\mathcal{M}(\widehat{\mathbf{R}}^n)$ denotes the group of Moebius transformations of $\widehat{\mathbf{R}}^n$. As shown in §1, the isotropy subgroup $\mathcal{M}(\widehat{\mathbf{R}}^n)_\infty$ at ∞ consists of transformations

$$f(x) = \lambda P x + b, \quad \lambda > 0, \ P \in O(n), \ b \in \mathbf{R}^n.$$

λ (resp. P) is called the *norm* (resp. *orthogonal part*) of f and is denoted by $\|f\|$ (resp. $P(f)$). Clearly a transformation $f \in \mathcal{M}(\widehat{\mathbf{R}}^n)_\infty$ induces a transformation of $\mathbf{R}^n$. When viewed as a transformation of $\mathbf{R}^n$, f is called an *Euclidean similarity.* The group of Euclidean similarities is denoted by $ES(\mathbf{R}^n)$. We have an isomorphism

$$ES(\mathbf{R}^n) \approx \mathcal{M}(\widehat{\mathbf{R}}^n).$$

Definition (4.1). An $(ES(\mathbf{R}^n), \mathbf{R}^n)$-structure (manifold) is called a *similarity structure (manifold).*

Euclidean space forms are examples of similarity manifolds. Other examples are Hopf manifolds to be defined below.

Definition (4.2). A closed similarity manifold N is called a *Hopf manifold* if the developing map D is a homeomorphism onto $\mathbf{R}^n \setminus \{0\}$.

Then the holonomy group H is discrete and is contained in the isotropy subgroup $ES(\mathbf{R}^n)_0$. By taking norm and orthogonal part, we obtain the isomorphism

$$ES(\mathbf{R}^n)_0 \cong \mathbf{R}_+ \times O(n).$$

$\|H\| = \{\|f\| \mid f \in H\}$ is nontrivial since N is closed, and is discrete since $O(n)$ is compact. Therefore it is infinite cyclic. Let $\|h\|$ $(h \in H)$ be a generator. Since the kernel $\{\|h\| = 1\}$ is finite, $\langle h^2 \rangle$ is a finite index subgroup of H. Clearly $(\mathbf{R}^n \setminus \{0\})/\langle h^2 \rangle$ is homeomorphic to $S^{n-1} \times S^1$. Thus we have;

Proposition (4.3). *Hopf manifold has a finite covering which is homeomorphic to* $S^{n-1} \times S^1$.

In [13], Fried has shown that these two examples of similarity manifolds are the only examples. That is, an arbitrary similarity manifold is isomorphic to either an Euclidean space form or a Hopf manifold. See also Kuiper ([36]). The purpose of this section is to give an improved version of Fried's theorem. Instead of confining ourselves to similarity manifolds, we consider flat conformal manifolds in general.

Theorem (4.4). *Let N be a closed flat conformal manifold of dimension ≥ 3 such that the holonomy group H is contained in the isotropy subgroup $\mathcal{M}(\widehat{\mathbf{R}}^n)_\infty$. Then N is isomorphic to either $\widehat{\mathbf{R}}^n$, a Hopf manifold or an Euclidean space form.*

One can state the original Fried's theorem as a corollary.

Corollary (4.5). *Closed similarity manifold of dimension ≥ 3 is isomorphic to a Hopf manifold or an Euclidean space form.*

What is new in Theorem (4.4) is that the developing map is allowed to cover the point ∞, while in the original Fried's theorem (Corollary (4.5)) it is postulated to miss ∞. Although the difference is apparently not significant and the proof is in fact almost the same, Theorem (4.4) brings forth a far wider range of applications in practice (as far as flat conformal structures are concerned). To the best knowledge of the author, (4.4) cannot be found in the literature. Therefore it is obviously worth while to give a complete proof of (4.4).

The rest of this section is devoted to the proof of (4.4). In way of contradiction, we assume that N is isomorphic to neither of the three structures in (4.4). Denote by D the developing map , by φ the holonomy homomorphism and by H the holonomy group. The proof consists of three steps.

Step 1. Clearly $D^{-1}(\infty)$ is discrete and invariant by the deck transformation. Thus $N(\infty) = \pi(D^{-1}(\infty))$ is a finite set. Then $N^* = N \setminus N(\infty)$ is a similarity manifold.

Definition (4.6). A domain $U^* \subset \widetilde{N}^* = \pi^{-1}(N^*)$ is called a *copy* of $U \subset \mathbf{R}^n$ if $D|_{U^*} : U^* \to U$ is a homeomorphism.

Points in $\widetilde{N}^*$ are denoted by a^*, x^* and so forth and their images by D by a, x and so forth. Thus , $B^*(a^*, r)$ denotes a copy containing $a^* \in \widetilde{N}^*$ of $B(a, r)$, the open ball of radius r centered at a. We call a^* and r the *center* and *radius* of $B^*(a^*, r)$.

Definition (4.7). A closed subset $l^* \subset \widetilde{N}^*$ is called a *complete half line* if for any copy of ball $B^* \subset \widetilde{N}^*$, $B^* \cap l^*$ is mapped by D to $B \cap k$, where k is a complete half line in $\mathbf{R}^n$.

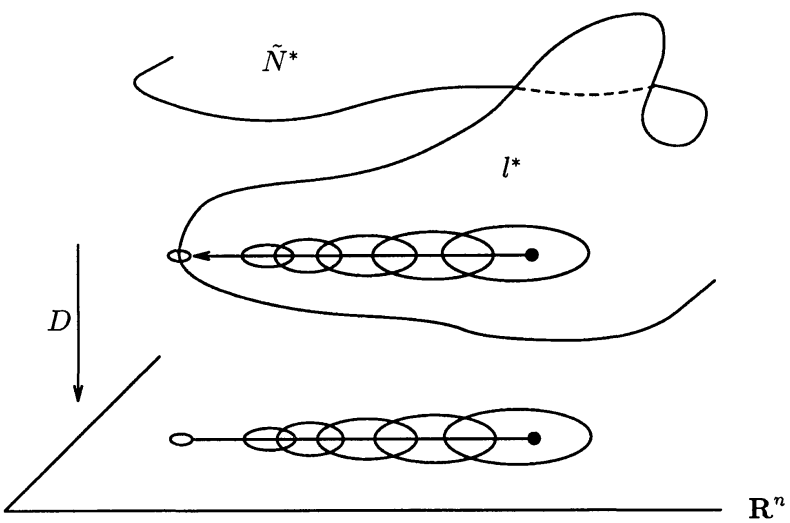

Figure (4.1)

See Figure (4.1).

By certain abuse, the parametrization of a complete half line l^* is denoted by the same letter, as

$$l^* : [0, \infty) \to l^*.$$

Notice that given any point $x^* \in \widetilde{N}^*$ and a tangent vector v at x^*, there exists a unique complete half line l^* such that $l^*(0) = x^*$ and tangent to v. Clearly deck transformation carries a complete half line to a complete half line.

Definition (4.8). A complete half line l^* is called *short* if $D(l^*)$ is not a complete half line in $\mathbf{R}^n$.

Claim (4.9). *Given a short complete half line l^*, there exists a neighbourhood U of $N(\infty)$ such that $\pi(l^*) \cap U = \phi$.*

Proof. For any point $c_i \in N(\infty)$, choose a compact neighbourhood U_i such that

(a) $\pi(l^*(0)) \notin U_i$,
(b) U_i is evenly covered by π,
(c) For any component E^* of $\pi^{-1}(U_i)$, there exist $a \in \mathbf{R}^n$ and $R > 0$ such that the following map is a homeomorphism.

$$D|_{E^*} : E^* \to E = E(a, R) = \{|x - a| \geq R\} \cup \{\infty\}$$

Notice that in (c), if one component of $\pi^{-1}U_i$ is mapped to $E(a, R)$, then all the other is also mapped to some $E(a', R')$. Thus (c) is attained if one chooses U_i small and appropriate.

Let us show that $l^* \cap E^*$ is empty. If not, the image $l \cap E$ is a half line starting at ∂E. ($l = D(l^*)$.) Since l^* is short, l is not a complete half line of $\mathbf{R}^n$. Then one can choose a ball $B \subset E \setminus \{\infty\}$ centered at the point $\lim_{t\to\infty} l(t)$. Then B has a copy B^* in E^*. But $B \cap l$ is not the intersection of B with a complete half line in $\mathbf{R}^n$. See Figure (4.2). This contradicts the hypothesis that l^* is complete. Let $U = \cup_i U_i$. We have $\pi(l^*) \cap U = \phi$. Q.E.D.

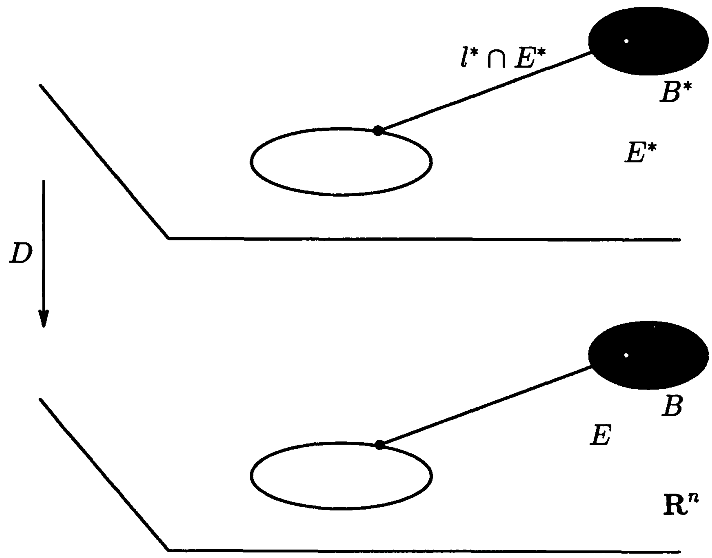

Figure (4.2)

For any $x^* \in \widetilde{N}^*$, let $r(x^*)$ be the maximal radius of a copy of ball centered at x^*. See Figure (4.3).

Claim (4.10). $r(x^*) < \infty$.

Proof. If not, x^* is contained in a copy of $\mathbf{R}^n$, say P. If $P = \widetilde{N}$, then N would be an Euclidean space form, contradicting the hypothesis. Suppose $P \neq \widetilde{N}$. Take a point $y^* \in \mathrm{Fr}(P)$ and a sequence $\{y_n^*\} \subset P$ such that $y_n^* \to y^*$. Clearly we have $D(y_n^*) \to \infty$. It follows from the continuity of D that $D(y^*) = \infty$. Therefore there is a neighbourhood Q of y^* which is mapped by D homeomorphically onto $E(0, R)$ for some large $R > 0$. Then $D : P \cup Q \to \widehat{\mathbf{R}}^n$ is a

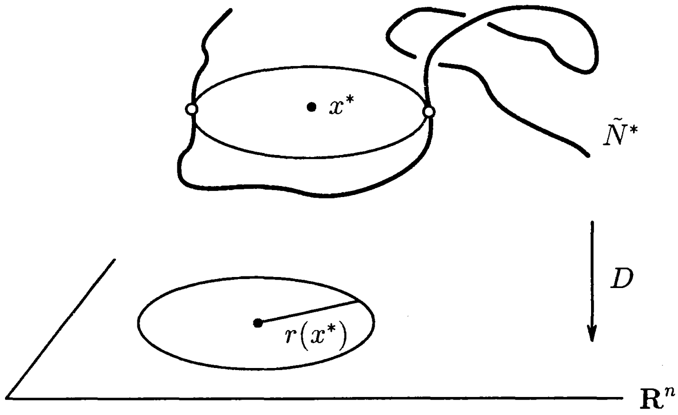

Figure (4.3)

homeomorphism. We have

$$N \cong \widetilde{N} = P \cup Q \cong \widehat{\mathbf{R}}^n.$$

Again a contradiction. Q.E.D.

Definition (4.11). The *Fried metric* is a continuous Riemannian metric on $\widetilde{N}^*$ defined by

$$g_F = \frac{D^* g_E}{r(x^*)^2} \quad \text{on} \quad T_{x^*}\widetilde{N}^*.$$

Let ξ be a deck transformation of $\widetilde{N}$ and $x^* \in \widetilde{N}^*$. We have

$$\xi(B^*(x^*, r)) = B^*(\xi x^*, \|\varphi(\xi)\| r).$$

This shows $r(\xi x^*) = \|\varphi(\xi)\| r(x^*)$. That is, the deck transformation ξ is an isometry for the Fried metric g_F. Thus g_F induces a Riemannian metric of N^*, which is also called the Fried metric. The distance functions of Fried metrics both on $\widetilde{N}^*$ and on N^* are denoted by d_F.

The following is the aim of Step 1.

Claim (4.12). *Let $B^* = B^*(a^*, r(a^*))$ be the maximal copy of ball centered at $a^* \in \widetilde{N}^*$. Then there exists a copy of half space H^* such that $B^*(a^*, r(a^*)) \subset H^*$.*

Proof. For simplicity let us assume $r(a^*) = 1$ and $D(a^*) = e_n = (0, \cdots, 0, 1)$. By D, we identify B^* with $B = \{|x - e_n| < 1\}$. By this identifidation, we consider the function r or the Fried metric g_F to be defined on B. By the maximality of B^*, there exists a radius l^* which is a complete half line. Assume

$$l = D(l^*) = \{x_1 = 0, \cdots, x_{n-1} = 0, 0 < x_n \leq 1\}.$$

See Figure (4.4).

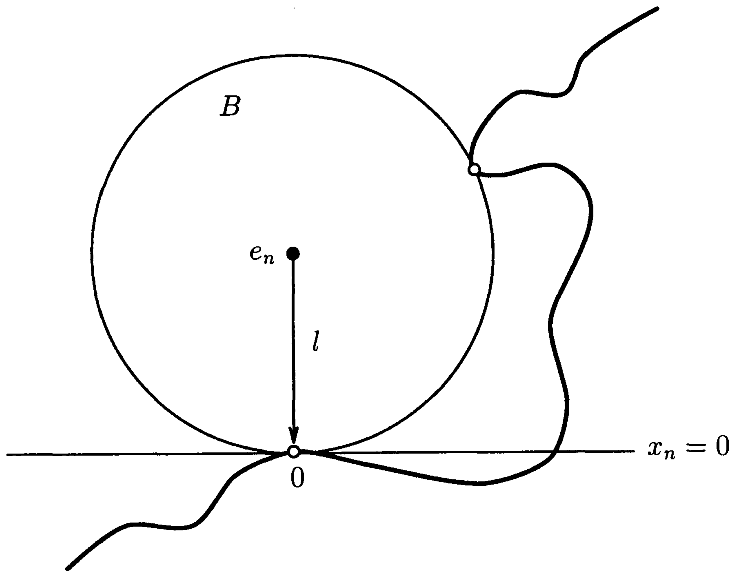

Figure (4.4)

Let us study for a while the Fried metric on B. First of all for any $x_0 \in B$, we have $r(x_0) \leq |x_0|$. In fact if not, the origin 0 is contained in

$$A = B \cup \{|x - x_0| < r(x_0)\}.$$

A has a copy containing a^*. This contradicts the completeness of l^*. Thus we have $g_F \geq g_G$ on B, where

$$g_G = \frac{g_E}{|x|^2}.$$

For any $x \in B$, let $\theta = \theta(x)$ be the angle of the vector $\overrightarrow{0x}$ and l. We have

Subclaim (4.12.1). $\qquad d_F(x, l) \geq d_G(x, l) = \theta.$

Proof. Let $\gamma(t)$ be a smooth path in B combining x and a point in l. Denote by $\text{length}_G(\gamma)$ the length of γ w.r.t. d_G. Let

$$\gamma(t) = |\gamma(t)| p(t).$$

We have

$$|\gamma'(t)| \geq |\gamma(t)||p'(t)|.$$

In fact, since $|p(t)| = 1$, we have $(p(t), p'(t)) = 0$ and

$$\gamma'(t) = |\gamma(t)|' p(t) + |\gamma(t)| p'(t).$$

Therefore we obtain the following equality.

$$\text{length}_G(\gamma) = \int_0^1 \frac{|\gamma'(t)|}{|\gamma(t)|} dt \geq \int_0^1 |p'(t)| dt \geq \theta.$$

On the other hand it is easy to show that for a suitable choice of γ, one has $\text{length}_G \gamma = \theta$. Q.E.D.

Now by Claim (4.9), There exist a compact submanifold $N_C = N - \text{Int} U$ which contains $\pi(l^*)$ and a sequence $t_i \uparrow \infty$ such that for some $c \in N_C$,

$$d_F(\pi(l^*(t_i)), c) \downarrow 0.$$

Also assume that $d_F(\pi(l^*(t_1)), c)$ is sufficiently small. Then by (4.12.1), there exists a point $b^* \in B^*$ such that $c = \pi(b^*)$ and

$$d_F(l^*(t_1), b^*) = d_F(\pi(l^*(t_1)), c).$$

Now there exists a sequence $\{\xi_i\}$ of deck transformations such that

$$d_F(l^*(t_i), \xi_i b^*) \downarrow 0.$$

See Figure (4.5).

Thus passing to the model $B \subset \mathbf{R}^n$, we may assume the following. Let $f_i = \varphi(\xi_i) \in ES(\mathbf{R}^n)$ and $b = D(b^*) \in B$.

(1) $f_i(b) \in B$.
(2) $\theta(f_i(b)) \to 0$.
(3) $f_i(b) \to 0$.
(4) $P(f_i) \to P_0 \in O(n)$.
(5) $\|f_i\| \to 0$.

Notice that (5) follows from (3) since

$$\|f_i\| = \frac{r(f_i(b))}{r(b)} \leq \frac{|f_i(b)|}{r(b)} \to 0.$$

See Figure (4.6).

Now for $i \gg 1$, taking $j \gg i$, we may assume

(6) $P(f_i f_j^{-1})$ is very near E ,
(7) $\|f_i f_j^{-1}\|$ is very large.

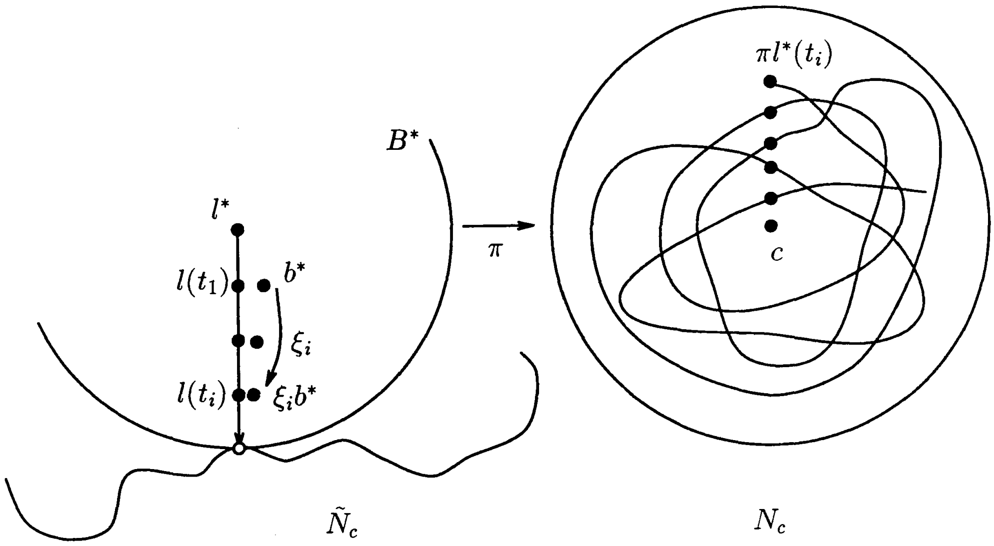

Figure (4.5)

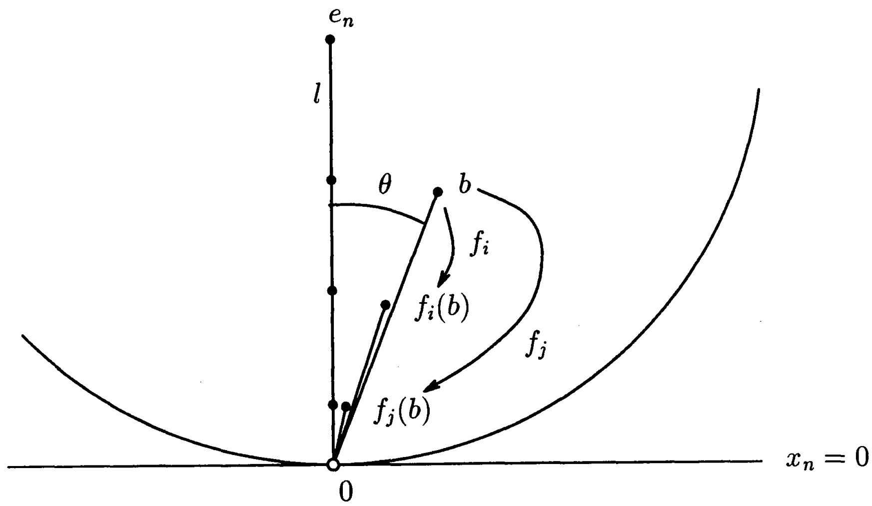

Figure (4.6)

Next by (2), $\overrightarrow{0f_j(b)}$ is almost parallel to $\overrightarrow{e_n}$ and is almost perpendicular to ∂B. Applying $f_i f_j^{-1}$, we still have that

(8) $\overrightarrow{f_i f_j^{-1}(0) f_i(b)}$ is almost parallel to $\overrightarrow{e_n}$,

(9) $\overrightarrow{f_i f_j^{-1}(0) f_i(b)}$ is almost perpendicular to $f_i f_j^{-1}(\partial B)$.

In fact (8) follows from (6) and (9) from the fact that $f_i f_i^{-1}$ is an

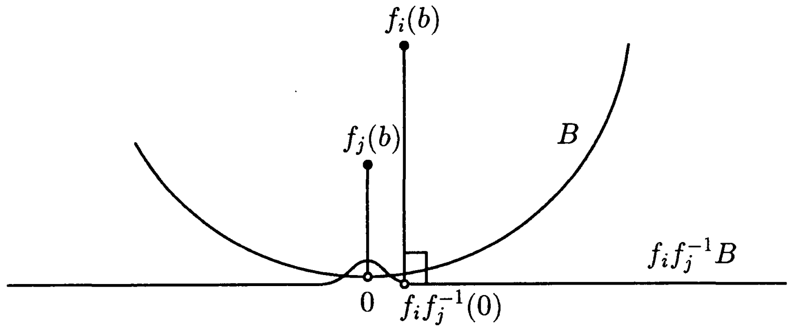

Figure (4.7)

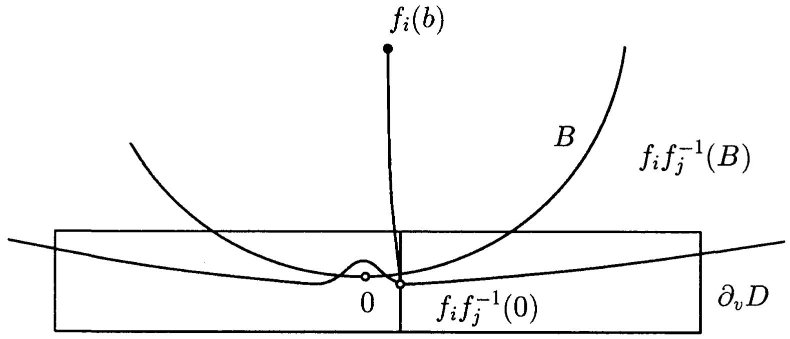

Figure (4.8)

Euclidean similarity. See Figure (4.7).

On the other hand, notice that $\xi_i\xi_j^{-1}B^* \cap B^*$ is nonempty and $\xi_i\xi_j^{-1}B^* \cup B^*$ is a copy of $f_if_j^{-1}B \cup B$. Therefore by the completeness of l^*, we have that

(10) $f_if_j^{-1}(0) \notin B$,

(11) $0 \notin f_if_j^{-1}(B)$.

Let

$$f_if_j^{-1}(0) = (\alpha_1, \cdots, \alpha_n)$$

and for $M \gg 1$ and $0 < \varepsilon \ll 1$, let

$$D = \{|x_i - \alpha_i| \le M \ (1 \le i \le n-1), \ |x_n - \alpha_n| \le \varepsilon\}.$$

Then by (5), (8) and (9), (taking $j \gg i \gg 1$ even greater) we have

$$\partial(f_if_j^{-1}B) \cap \partial D = \partial(f_if_j^{-1}B) \cap \partial_v D,$$

where ∂_v denotes the vertical boundary. See Figure (4.8).

We have by (11) that $\alpha_n > -\varepsilon$ and by (10) that $\alpha_n < \varepsilon$. It also follows that $f_i f_j^{-1}(0)$ is very near 0.

This shows that any x in the half space $\{x_n > 0\}$ is in $f_i f_j^{-1} B$ for some $j \gg i \gg 1$. Since $f_i f_j^{-1} B \cup B$ has a copy containing a^*, the proof of (4.12) is now complete. Q.E.D.

Step 2. In Step 1, for any point $a^* \in \widetilde{N}^*$, we have found a copy of half space containing $B^*(a^*, r(a^*))$. We have;

Claim (4.13). *A copy of half space H^* containing $B^*(a^*, r(a^*))$ is unique.*

Proof. Clearly H is tangent to $B(a, r(a^*))$ and the radius to the point of tangency is the developing image of a short complete line. In other words, there exists a unique short complete line in $\widetilde{N}^*$ which is contained in $B^*(a^*, r(a^*))$. This shows the uniqueness of H^*. Q.E.D.

Definition (4.14). H^* of (4.13) is denoted by $H^*(a^*)$ and its image by D by $H(a^*)$. The point of tangency of $H(a^*)$ and $B(a, r(a^*))$ is denoted by $p(a^*)$.

Notice that maximal copy of half space containing a^* may not be unique. Since D is a submersion,

$$D|_{\,\mathrm{Cl}H^*(a^*)} : \mathrm{Cl}H^*(a^*) \to \mathbf{R}^n$$

is injective and $D(\mathrm{Fr}H^*(a^*))$ is an open subset of $\partial H(a^*)$.

Definition (4.15). For $a^* \in \widetilde{N}^*$, denote

$$L(a^*) = \partial H(a^*) \setminus D(\mathrm{Fr}H^*(a^*)) \subset \mathbf{R}^n.$$

In other words, $x \in L(a^*)$ if and only if $x = \lim_{t\to\infty} l(t)$ for some short complete line l^* such that $l^*(0) = a^*$. See Figure (4.9).

For $b \in \mathrm{Cl}H(a^*) \setminus L(a^*) \subset \mathbf{R}^n$, we denote by b^* the unique point in $\mathrm{Cl}H^*(a^*) \subset \widetilde{N}^*$ such that $D(b^*) = b$.

Claim (4.16). *For $b \in \mathrm{Cl}H(a^*) \setminus L(a^*)$, $\partial H(b^*)$ passes through $p(a^*)$.*

Proof. Suppose not. We have $a \notin H(b^*)$ since $H(a^*) \cup H(b^*)$ has a copy in $\widetilde{N}^*$. Consider the transformation

$$f_j f_i^{-1} = (f_i f_j^{-1})^{-1} \in ES(\mathbf{R}^n)$$

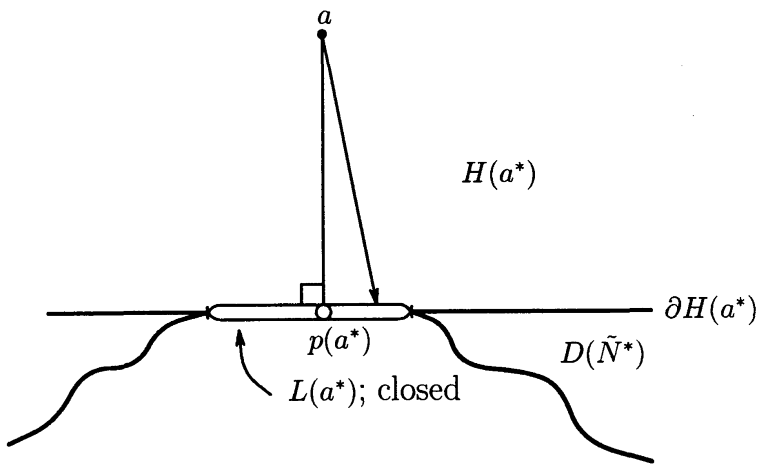

Figure (4.9)

of Step 1. Recall that $\|f_j f_i^{-1}\|$ is very small, $P(f_j f_i^{-1})$ is very near E and $f_j f_i^{-1}$ has a fixed point near $p(a^*)$. Thus $f_j f_i^{-1}(\partial H(b^*))$ is almost parallel to $\partial H(b^*)$ and much near $p(a^*)$. Clearly

$$H(a^*) \cup H(b^*) \cup f_j f_i^{-1}(H(b^*))$$

has a copy in $\widetilde{N}^*$. This contradicts that $p(b^*) \in L(b^*)$. See Figure (4.10). Q.E.D.

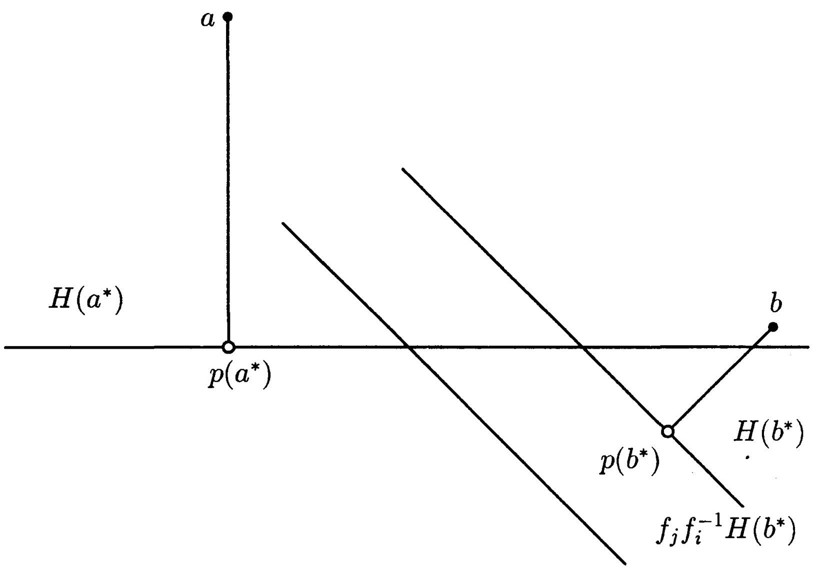

Figure (4.10)

Claim (4.17). *$L(a^*)$ is an affine subspace of $\mathbf{R}^n$.*

Proof. Let $x, y \in L(a^*)$. Clearly $x = p(c^*)$ for some $c^* \in H(a^*)$. Likewise for y. If a point b on the line passing x and y does not belong to $L(a^*)$, apply (4.16) to b. Then $\partial H(b^*)$ passes through x and y, that is, through b. A contradiction. Q.E.D.

Claim (4.18). *The correspondence $a^* \mapsto L(a^*)$ is locally constant.*

Proof. Take $b^* \in H^*(a^*)$. Then $\partial H(b^*)$ passes through $p(a^*)$ by (4.16) and $p(a^*) \in L(b^*)$. Since $L(b^*) \cap H(a^*) = \phi$, we have $L(b^*) \subset \partial H(a^*) \cap \partial H(b^*)$. Likewise, $L(a^*) \subset \partial H(a^*) \cap \partial H(b^*)$. It follows easily that $L(a^*) = L(b^*)$. Q.E.D.

Since $\widetilde{N}^*$ is connected, $L(a^*)$ is independent of the choice of $a^* \in \widetilde{N}^*$. Denote $L = L(a^*)$.

Claim (4.19). *The developing map D is a covering map onto a component of $\mathbf{R}^n \setminus L$.*

Proof. Clearly no points of $\widetilde{N}^*$ are mapped by D into L. Also we have that points in $\mathbf{R}^n \setminus L$ are evenly covered by D. Let us consider the point ∞. For $\dim L \geq 1$, $\infty \in \mathrm{Cl} L$ cannot be in $\mathrm{Image}(D)$. For $\dim(L) = 0$ (say $L = \{0\}$), if $\infty \in \mathrm{Image}(D)$, then one can show that

$$D : \widetilde{N} \to \widehat{\mathbf{R}}^n \setminus \{0\}$$

is a homeomorphism. But $H \subset ES(\mathbf{R}^n)$ has ∞ as a fixed point. A contradiction. Q.E.D.

Step 3.

Lemma (4.20). *Let $\Gamma = \langle f, g \rangle \subset ES(\mathbf{R}^n)$, where*

(1) $\|f\| \neq 1$, $f(a) = a$ $(a \in \mathbf{R}^n)$,
(2) $g(a) \neq a$.

Then Γ is indiscrete.

Proof. Assume $\|f\| < 1$. Let $h = g \circ f \circ g^{-1}$. Then $\|h\| = \|f\|$ and $h(g(a)) = g(a)$. Let $h_n = f^n \circ h \circ f^{-n}$. We have $\|h_n\| = \|h\| = \|f\|$, the fixed point of h_n is $f^n(g(a))$ and $f^n(g(a)) \to a$ $(n \to \infty)$. That is, $h_n \to f$. This shows (4.20). Q.E.D.

Now we shall complete the proof of Theorem (4.4). First of all if $L = \{0\}$, then by Step 2, $D : \widetilde{N} \to \mathbf{R}^n \setminus \{0\}$ is a homeomorphism. That is, N is a Hopf manifold. This contradicts our hypothesis.

Consider the case $\dim L \neq n-2$. Suppose for simplicity that $L = \mathbf{R}^q$. By (4.19), D is a homeomorphism onto a component V of $\mathbf{R}^n \setminus \mathbf{R}^q$. Thus the holonomy group H must be discrete. By Step 1, there exists $f \in H$ such that $\|f\| \neq 1$. Clearly $f(L) = L$. Assume $f(0) = 0$. By (4.20), we have $g(0) = 0$ for any $g \in H$. Therefore $g(\mathbf{R}^{n-q}) = \mathbf{R}^{n-q}$, where $\mathbf{R}^{n-q}$ is the orthogonal complement of $\mathbf{R}^q$. When identified by D , Fried metric is given by

$$g_F = \frac{g_E}{|x_2|^2},$$

where $x = (x_1, x_2)$ $(x_1 \in \mathbf{R}^q, x_2 \in \mathbf{R}^{n-q})$. Since $N = N^*$ is compact, d_F is totally bounded. That is, there exists $K > 0$ such that for any $x, y \in V$, $d_F(x, gy) < K$ for some $g \in H$. But this is impossible if we choose $y \in \mathbf{R}^{n-q} \cap V$ and x_1 as large as desired.

Finally suppose $L = \mathbf{R}^{n-2}$. This case needs extra care. Since $\mathbf{R}^n \setminus \mathbf{R}^{n-2}$ is not simply connected, D is not a homeomorphism and H may not be discrete. Denote by $R_\theta \in ES(\mathbf{R}^n)$ the rotation by angle θ around $\mathbf{R}^{n-2}$. Let

$$\mathrm{Stab}(\mathbf{R}^{n-2}) = \{f \in ES(\mathbf{R}^n) \mid f(\mathbf{R}^{n-2}) = \mathbf{R}^{n-2}\}.$$

Notice that R_θ commutes with an element of $\mathrm{Stab}(\mathbf{R}^{n-2})$. Let

$$H^{n-1} = \{x_{n-1} > 0, x_n = 0\}.$$

Define a homeomorphism

$$h : H^{n-1} \times S^1 \to \mathbf{R}^n \setminus \mathbf{R}^{n-2}$$

by $h(x,t) = R_{2\pi t}x$. The universal covering of $\mathbf{R}^n \setminus \mathbf{R}^{n-2}$ is identified with $H^{n-1} \times \mathbf{R}$. Then as is easily shown, the lift of $\mathrm{Stab}(\mathbf{R}^{n-2})$ is identified with $ES(\mathbf{R}^{n-2}) \times \mathbf{R}$. That is, we have the following equivariant mapping of (G, X)-pairs

$$(ES(\mathbf{R}^{n-2}) \times \mathbf{R},\ H^{n-1} \times \mathbf{R}) \longrightarrow (\mathrm{Stab}(\mathbf{R}^{n-2}),\ \mathbf{R}^n \setminus \mathbf{R}^{n-2}).$$

The DH-pair

$$(D, \varphi) : (\widetilde{N}, \pi_1(N)) \longrightarrow (\mathbf{R}^n \setminus \mathbf{R}^{n-2},\ \mathrm{Stab}(\mathbf{R}^{n-2}))$$

clearly lifts to a DH-pair

$$(D', \varphi') : (\widetilde{N}, \pi_1(N)) \longrightarrow (H^{n-1} \times \mathbf{R},\ ES(\mathbf{R}^{n-2}) \times \mathbf{R}).$$

Since D' is a homeomorphism, the image $\widetilde{H} = \varphi'(\pi_1(N))$ is discrete. As before, $\widetilde{H}$ contains

$$(f, t) \in ES(\mathbf{R}^{n-2}) \times \mathbf{R}$$

such that $\|f\| \neq 1$. Let $f(0) = 0$. Since $ES(\mathbf{R}^{n-2})$ and $\mathbf{R}$ commute, the argument of (4.20) is also valid and we have $g(0) = 0$ for any $(g, s) \in \widetilde{H}$. The rest of the proof is similar.

§5. Limit set

The purpose of this section is to define limit set for flat conformal manifolds of an arbitrary type. In this section flat conformal manifolds are to be connected and compact, unless otherwise specified.

First of all consider an arbitrary subgroup Γ of $\mathcal{M}(S^n)$. (Γ may not be discrete. It may not be even finitely generated.) Let us begin by defining the limit set for the group Γ by looking at its action on S^n. There are four different ways and all of them are natural and useful.

Definition (5.1). Let $L_F = L_F(\Gamma)$ be the closure of the set of the fixed points of loxodromic or parabolic elements of Γ.

Definition (5.2). Let $L_J = L_J(\Gamma)$ be the set of points $x \in S^n$ such that for any neighbourhood U of x, the family $\{f|_U\}_{f \in \Gamma}$ is not equicontinuous.

Definition (5.3). Let $L_P = L_P(\Gamma)$ be the set of points $x \in S^n$ such that for any neighbourhood U of x, the set $\{f \in \Gamma \mid fU \cap U \neq \phi\}$ is not precompact in $\mathcal{M}(S^n)$.

By definition L_F, L_J and L_P are closed Γ-invariant subsets of S^n. Of course L_J is an analogy of Julia set in one dimensional complex dynamical system. Notice that if Γ is discrete, then $S^n \setminus L_P$ coincides with the domain of discontinuity Ω_Γ defined in (3.24).

Definition (5.4). Let $L_\omega = L_\omega(\Gamma)$ be the set of accumulation points in S^n of the orbit Γa of a certain point $a \in D^{n+1}$.

This definition is independent of the choice of $a \in D^{n+1}$. In fact, for another point $b \in D^{n+1}$ and for $\gamma_k \in \Gamma$, we have $d_H(\gamma_k(a), \gamma_k(b)) =$

$d_H(a,b)$, where d_H denotes the hyperbolic distance. By the difference between the hyperbolic distance and the Euclidean distance, we have

$$\lim_{k\to\infty} \gamma_k(a) = x \iff \lim_{k\to\infty} \gamma_k(b) = x.$$

Also note that L_ω is closed and Γ-invariant. In fact if $\lim_{k\to\infty} \gamma_k(a) = x$, then we have $\lim_{k\to\infty} \gamma\gamma_k(a) = \gamma(x)$ for $\gamma \in \Gamma$. Below we shall prove the *minimality* of L_ω.

Definition (5.5). Let A be a Γ-invariant closed subset of S^n such that $\mathrm{Card}(A) \geq 2$. The *convex hull* of A, denoted by $C(A)$, is defined to be the convex hull in (D^{n+1}, g_H) of all the geodesics combining two points of A.

Clearly $C(A)$ is a closed Γ-invariant subset of D^{n+1}. See Figure (5.1).

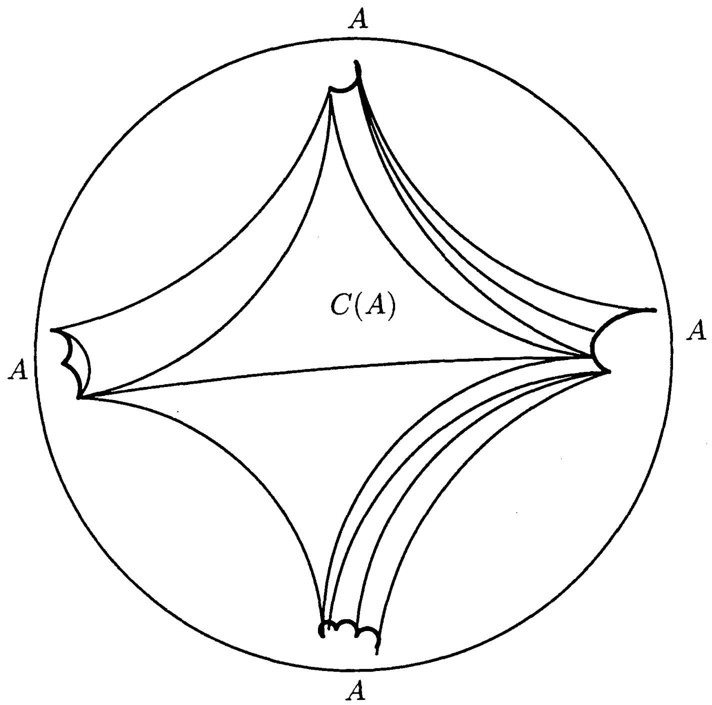

Figure (5.1)

Lemma (5.6). *Let A be an arbitrary Γ-invariant closed set such that* $\mathrm{Card}(A) \geq 2$. *Then we have* $L_\omega(\Gamma) \subset A$.

Proof. Take the point $a \in D^{n+1}$ of (5.4) inside $C(A)$. Then the orbit of a cannot evade $C(A)$. This shows (5.6). Q.E.D.

Corollary (5.7). *If* Γ *has no fixed point in* S^n, *then* $L_\omega(\Gamma)$ *is the unique minimal set, i.e., it is contained in any nonempty closed* Γ*-invariant subset of* S^n.

For $\gamma \in \mathcal{M}(S^n)$, denote by $CI(\gamma)$ the convex hull in $\mathbf{R}^{n+1}$ of the isometric sphere $I(\gamma)$.

Lemma (5.8). *For* $\{\gamma_k\} \subset \Gamma \subset \mathcal{M}(S^n)$ *such that* $\gamma_k \to \infty$, *we have* $d(CI(\gamma_k), L_\omega(\Gamma)) \to 0$.

Proof. For the properties of isometric spheres, see (2.11)~(2.16). We shall prove (5.8) by establishing $d(CI(\gamma_k^{-1}), L_\omega) \to 0$. Recall that $\gamma_k \to \infty$ if and only if $\text{radius}\, I(\gamma_k^{-1}) = \text{radius}\, I(\gamma_k) \to 0$ and that $I(\gamma_k)$ is always orthogonal to S^n. Therefore given a point $a \in D^{n+1}$, we have $a \notin CI(\gamma_k)$ for large k. That is, $\gamma_k a \in CI(\gamma_k^{-1})$. See Figure (5.2). Since $d(\gamma_k a, L_\omega) \to 0$, it follows that $d(CI(\gamma_k^{-1}), L_\omega) \to 0$.

Q.E.D.

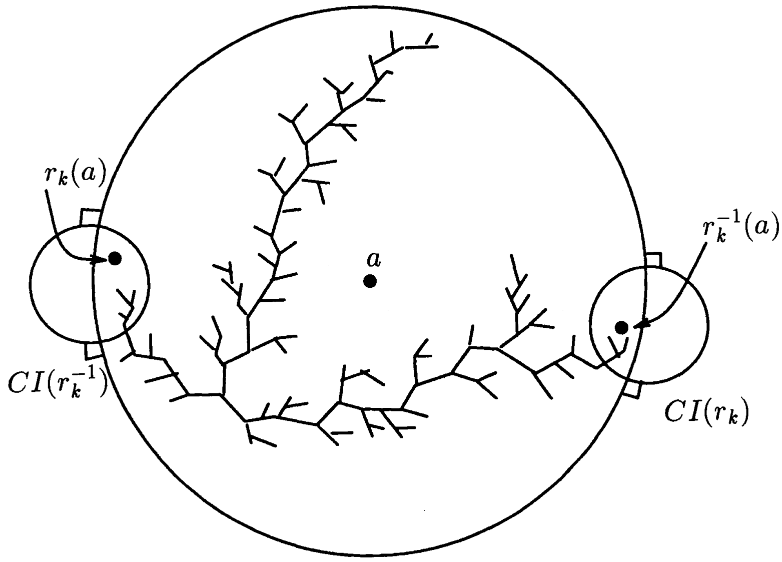

Figure (5.2)

Definition (5.9). Two points $x, y \in L_\omega$ are called *dual* in case there exists $\gamma_k \in \Gamma$ such that $\gamma_k(a) \to x$ and $\gamma_k^{-1}(a) \to y$ $(a \in D^{n+1})$.

This is also independent of the choice of a. For $x \in L_\omega$, let D_x be the set of points in L_ω which are dual to x. Diagonal argument shows that D_x is a closed subset. Also if $\gamma_k a \to x$ and $\gamma_k^{-1} a \to y$, then for $\gamma \in \Gamma$, $\gamma_k\gamma^{-1}a \to x$ and $\gamma\gamma_k^{-1}a \to \gamma y$. That is, D_x is Γ-invariant.

Lemma (5.10). *If* Γ *has no fixed point in* S^n, *then any two points of* L_ω *are dual.*

Proof. For $L_\omega \neq \phi$, the condition assures that $\mathrm{Card}(D_x) \geq 2$. Since D_x is closed and Γ-invariant, we have by (5.6) that $D_x \supset L_\omega$.
Q.E.D.

Proposition (5.11). *If* Γ *has no fixed point in* S^n, *then for any pair of distinct points* $x, y \in L_\omega(\Gamma)$, *there exists a loxodromic transformation whose two fixed points are arbitrarily near* x *and* y.

Proof. By (5.10), we have that x and y are dual. Let $\gamma_k(a) \to x$ and $\gamma_k^{-1}(a) \to y$ $(\gamma_k \in \Gamma,\ a \in D^{n+1})$. We have clearly $\gamma_k \to \infty$. By applying the argument of (5.8), we obtain that $CI(\gamma_k^{-1})$ is sufficiently near x and $CI(\gamma_k)$ is sufficiently near y. Since $x \neq y$, we may assume that $CI(\gamma_k^{-1}) \cap CI(\gamma_k) = \phi$. It is easy to show that γ_k is a loxodromic transformation with one fixed point in $CI(\gamma_k^{-1})$ and the other in $CI(\gamma_k)$. This shows (5.11). Q.E.D.

Lemma (5.12). $L_\omega(\Gamma) = \phi$ *if and only if* Γ *is precompact.*

Proof. This follows at once from the fact that for any $a \in D^{n+1}$, the isotropy subgroup of $\mathcal{M}(S^n)$ at a is isomorphic to a compact group $O(n+1)$. Q.E.D.

Proposition (5.13). *A subgroup* Γ *of* $\mathcal{M}(S^n)$ *is precompact if and only if it has a common fixed point in* D^{n+1}. *In particular, maximal compact subgroups of* $\mathcal{M}(S^n)$ *are conjugate to* $O(n+1)$.

Proof. The if part is trivial. Let us show that a compact subgroup Γ has a fixed point in D^{n+1}. (Pass to $\mathrm{Cl}\Gamma$ if Γ is noncompact.) Choose an arbitrary point $a \in D^{n+1}$. Let $d = \mathrm{diam}_H(\Gamma a)$ and let $d_H(a, ga) = d$ $(g \in \Gamma)$. Let a_1 be the middle point of a and ga. For any $h \in \Gamma$, consider the hyperbolic tetrahedron with vertices a, ga, ha and hga. All the edges have length $\leq d$. Easy hyperbolic trigonometry shows $d(a_1, ha_1) \leq cd$ for some (computable) $c \in (0,1)$. That is, $\mathrm{diam}_H(\Gamma a_1) \leq cd$. Likewise construct a_2, a_3 etc. Let $a_\infty = \lim_{k\to\infty} a_k$. We have $\mathrm{diam}_H(\Gamma a_\infty) = 0$. That is, a_∞ is a fixed point of Γ. Q.E.D.

Corollary (5.14). *Unless* Γ *has a fixed point in* $D^{n+1} \cup S^n$, Γ *contains a loxodromic transformation.*

Proof. The condition implies that $\mathrm{Card} L_\omega \geq 2$. Therefore (5.14) follows from (5.11). Q.E.D.

Theorem (5.15). *For an arbitrary subgroup* $\Gamma \subset \mathcal{M}(S^n)$, *we have*

$$L_F(\Gamma) \subset L_J(\Gamma) = L_P(\Gamma) = L_\omega(\Gamma).$$

Moreover unless $L_F(\Gamma) = \phi$ *and* $L_\omega(\Gamma)$ *is a singleton, we have* $L_F(\Gamma) = L_\omega(\Gamma)$.

Proof. $L_F \subset L_J \cap L_P$: This follows at once from the local models of loxodromic and parabolic transformations.

$L_J \cup L_P \subset L_\omega$: Suppose $x \notin L_\omega$. Then by (5.8), for small $\varepsilon > 0$ and for a small neighbourhood U of x, we have that $CI(\gamma) \cap U = \phi$ if $\mathrm{radius}\, I(\gamma) < \varepsilon$ and $\gamma \in \Gamma$. But the set of γ such that $\mathrm{radius}\, I(\gamma) \geq \varepsilon$ is precompact by (2.16). It follows from (2.12) that $x \notin L_J \cup L_P$.

We shall divide the proof of the remaining part into four cases.

Case 1. Γ has no fixed point in $D^{n+1} \cup S^n$.

By (5.14) we have $L_F \neq \phi$. Therefore it follows from (5.7) that $L_\omega \subset L_F$. Together with the inclusion we have already established, we obtain that $L_F = L_J = L_P = L_\omega$.

Case 2. Γ has a fixed point in D^{n+1}.

By (5.12) and (5.13), this is equivalent to $L_\omega = \phi$. We have $L_F = L_J = L_P = L_\omega = \phi$.

Case 3. Γ has a fixed point $y \in S^n$ and that $L_\omega \setminus \{y\} \neq \phi$.

Let $x \in L_\omega \setminus \{y\}$. Notice that parabolic and elliptic transformations of the isotropy group Γ_y keep horospheres at y invariant. Therefore there must exist loxodromic transformations $\gamma_n \in \Gamma$ such that $\gamma_n a \to x$ $(a \in D^{n+1})$. Then $\gamma_n^{-1} a \to y$. That is, we have $y \in L_\omega$ and $L_\omega \subset L_F$, showing that $L_F = L_J = L_P = L_\omega$.

Case 4. $L_\omega = \{y\}$.

This is the only case where we cannot prove $L_\omega \subset L_F$. In order to complete the proof of (5.15), it suffices to show that $y \in L_F \cap L_P$. Since $L_\omega \neq \phi$, there exists a sequence $\{\gamma_k\} \subset \Gamma$ such that $\gamma_k \to \infty$. Since $\gamma_k y = y$ and γ_k are not loxodromic, we have $y \in CI(\gamma_k) \cup CI(\gamma_k^{-1})$

and $CI(\gamma_k) \cap CI(\gamma_k^{-1}) \neq \phi$. Hence for any neighbourhood U of y, $CI(\gamma_k) \subset U$ for sufficiently large $k > 0$. But by (2.12) and (2.16), we have that $\{\gamma_k|_U\}$ is not equicontinuous. That is, $y \in L_J$. Clearly we have $\gamma_k(U) \cap U \neq \phi$. Therefore $y \in L_P$. Q.E.D.

The following corollary was already used in §3.

Corollary (5.16). *Suppose a discrete group* Γ *admits an invariant open set* Ω *such that* $S^n \setminus \Omega$ *is neither empty nor a singleton. Then* Γ *acts on* Ω *discontinuously.*

Proof. Since Γ is discrete, $S^n \setminus L_P$ coincides with the domain of discontinuity. By (5.6), we have $S^n \setminus \Omega \supset L_\omega = L_P$. Therefore Ω is contained in the domain of discontinuity. Q.E.D.

We will give an example of Γ for which $L_F(\Gamma) = \phi$ and $L_\omega(\Gamma)$ is a singleton. The same example can be found in Kulkarni ([44]).

Example (5.17). Let us work with $\mathcal{M}(\widehat{\mathbf{R}}^4)$. We shall construct a subgroup Γ such that $L_F(\Gamma) = \phi$ and that $L_\omega(\Gamma) = \{\infty\}$. Equivalently, the group Γ consists purely of elliptic elements, keeps ∞ fixed and does not have a fixed point in H^5. By (1.9) and (2.24), any element $f \in \Gamma$ has the form

$$(*) \qquad f(x) = Px + b \quad (P \in O(4),\ \ b \in \mathbf{R}^4).$$

Notice that f is elliptic if and only if f has a fixed point $a \in \mathbf{R}^4$. In fact, then, the point $(a, x) \in H^5$ $(x > 0)$ is kept fixed by the extended action of f. Likewise the group Γ has a fixed point in H^5 if and only if it has a fixed point in $\mathbf{R}^4$. Therefore our purpose is to construct a group Γ consisting of transformations f of $(*)$ such that

$f \in \Gamma$ has a fixed point in $\mathbf{R}^4$.

Γ does not have a fixed point in $\mathbf{R}^4$.

First of all let us show that there exist $P, Q \in SO(4)$ such that for any nontrivial reduced word $w(P, Q)$, we have $|w(P, Q) - E| \neq 0$. Notice that for a (possibly real) algebraic group G, if G contains a free group of two generators, then for any nontrivial reduced word $w(x, y)$, the equation $w(x, y) = \mathrm{id}$ defines a proper subvariety (that is, a subvariety of positive codimension) of $G \times G$. The converse also holds since the complements of subvarieties of positive codimension are open dense subsets and their countable intersection is nonempty. Therefore a real algebraic group contains a free subgroup of two generators if and only if its complexification does. Now it is well known that $SO(2, 1)$

has a free subgroup of two generators. Clearly $SO(2,1)_{\mathbf{C}} = SO(3)_{\mathbf{C}}$. Therefore by the above consideration, $SO(3)$, hence its universal covering $SU(2)$, has a free group of two generators also. Considering the inclusion of $SU(2)$ into $SO(4)$, we obtain the desired P and Q.

Let
$$f : x \mapsto Px \quad \text{and} \quad g : x \mapsto Qx + b \ \ (b \neq 0).$$
Now $\Gamma = \langle f, g\rangle$ consists purely of elliptic transformations, since any element of Γ has the linear part without eigenvalue 1 and hence has a fixed point in $\mathbf{R}^4$. However f and g have no common fixed points in $\mathbf{R}^4$.

As a matter of fact, (5.17) implies that $L_F = L_\omega$ does not hold in higher dimension. However in low dimension, we have;

Theorem (5.18). *For* $\Gamma \subset \mathcal{M}(\widehat{\mathbf{R}}^n)$ *($n \leq 3$), we have*
$$L_F(\Gamma) = L_J(\Gamma) = L_P(\Gamma) = L_\omega(\Gamma).$$

Proof. All that need proof is that if $L_\omega = \{\infty\}$, then $L_F = \{\infty\}$. Equivalently, if Γ keeps ∞ fixed and if Γ does not have a fixed point in $\mathbf{R}^n$, then Γ contains nonelliptic transformations.

First of all for $n = 1$, there exist no elliptic transformations that keep ∞ fixed and there is nothing to prove.

For $n = 2$, assume that $f, g \in \mathcal{M}(\widehat{\mathbf{R}}^2)_\infty$ have no common fixed points in $\mathbf{R}^3$. Computation shows that $[f, g] = fgf^{-1}g^{-1}$ is parabolic, since the linear parts commute.

Finally let $n = 3$. It clearly suffices to verify for a group Γ consisting of orientation preserving transformations. Orientation preserving elliptic transformations in $\mathcal{M}(\widehat{\mathbf{R}}^3)_\infty$ are rotations around their axes. Let us show first that if two rotations f, g have disjoint axes, then the group $\langle f, g\rangle$ they generate contains a parabolic transformations. In fact, if the axes are parallel, then $[f, g]$ is parabolic. Suppose they are not parallel and assume for contradiction that fg^{-1} has a fixed point $x \in \mathbf{R}^n$. Then we have $f(x) = g(x) = y$. By Euclidean geometry, we have that the bisector of x and y contains the axes of f and g. (See Figure (5.3).) A contradiction.

Therefore if $\Gamma \subset \mathcal{M}(\widehat{\mathbf{R}}^3)_\infty$ is purely elliptic and have no common fixed points, then all the axes of transformarions of Γ must lie in a plane and all their rotation angles must be π. Therefore there exists an index two subgroup of Γ consisting of parabolic transformations. This contradiction shows (5.18). Q.E.D.

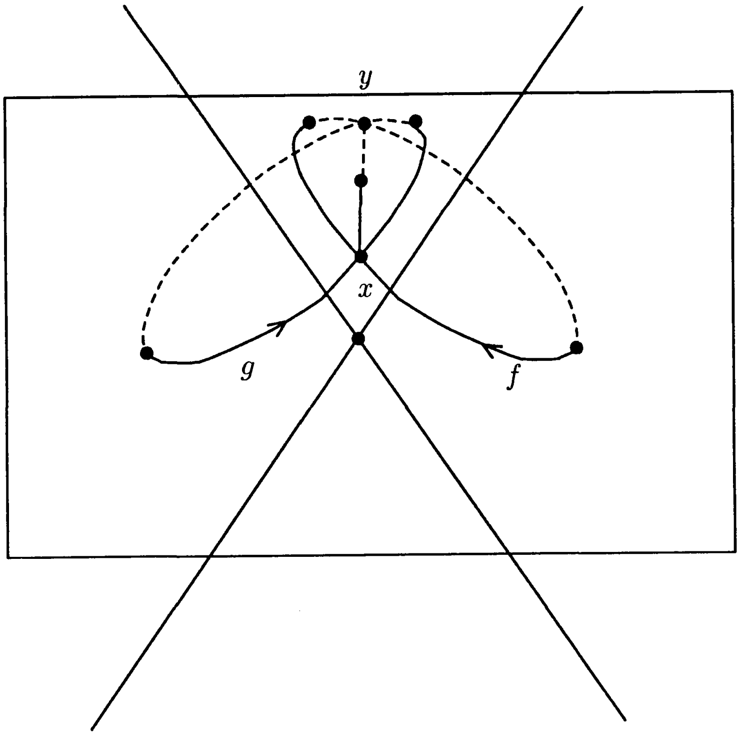

Figure (5.3)

Now let N be a connected closed flat conformal manifold modeled on $(\mathcal{M}(S^n), S^n)$. As before denote by D the developing map, by φ the holonomy homomorphism and by H the holonomy group. Hereafter by certain abuse, we consider a flat conformal manifold N to be equipped with a particular choice of developing map, holonomy homomorphism and holonomy groups. Our purpose is to define the limit set of N. So far, we already had four kinds of limit set in terms of the holonomy group H. For a flat conformal manifold, they are denoted by $L_F(N) = L_F(H)$ and so forth. We need one more definition, which is obtained by looking at the developing map.

Definition (5.19). Let $L_O = L_O(N)$ be the set of points x such that for any compact neighbourhood $\overline{U}$ of x, the inverse image $D^{-1}(\overline{U})$ has a nonempty and noncompact component.

As is shown easily, L_O is precisely the set of points which are not evenly covered by D.

For general closed (G, X)-manifolds, Kulkarni-Pinkall([45]) defined L_J and L_P and showed $L_J \supset L_P$ and $L_J \supset L_O$. They also showed that $L_J = L_P$ for closed flat conformal manifolds. The following is an elaboration of their rerult.

Theorem (5.20). *For a connected compact flat conformal manifold N, we have*

$$L_F(N) = L_\omega(N) = L_J(N) = L_P(N) = L_O(N).$$

Proof. $L_F = L_\omega$: If not, we have $L_F = \phi$, $L_\omega = \{a\}$ and H has a fixed point $a \in S^n$. But then by (4.4), N is isomorphic to either S^n, a Hopf manifold or a Euclidean space form. In any case we have $L_F = L_\omega$.

$L_\omega \subset L_O$: If $\mathrm{Card} L_O \geq 2$, then this follows at once from (5.6). If $L_O = \{a\}$, then a is a fixed point of H and again by (4.4), we have $L_\omega = L_O$. If $L_O = \phi$, then D is a covering map onto S^n. That is, N is a spherical space form and we have $L_\omega = \phi$.

$L_O \subset L_J$: Denote by $\overline{B}(x, r)$ the closed disk centered at $x \in S^n$ of radius $r > 0$ w.r.t. the spherical metric. The proof is by contradiction. Suppose $b \in L_O \setminus L_J$. That is, we assume

(1) For some $r_k \downarrow 0$, $D^{-1}\overline{B}(b, r_k)$ has a noncompact component E_k.

(2) $\{f|_{\overline{B}(b,r_1)}\}_{f \in H}$ is equicontinuous.

Choose $a_k \in E_k$. (Note that $D(a_k) \to b$.) Then since N is compact, there exists $\xi_k \in \pi_1(N)$ such that $\xi_k a_k$ is in some compact region of $\widetilde{N}$. Assume $\xi_k a_k \to c$. Choose a compact neighbourhood $\overline{V}$ of c such that

$$D|_{\overline{V}} : \overline{V} \to \overline{B}(D(c), 2\varepsilon)$$

is a homeomorphism for some $\varepsilon > 0$. Assume also $D(\xi_k a_k) \in \overline{B}(D(c), \varepsilon)$ for any $k > 0$. Choose $\delta > 0$ so

$$x, y \in \overline{B}(b, r_1), \quad d(x, y) < 2\delta \implies d(f(x), f(y)) < \varepsilon \quad \text{for any} \quad f \in H.$$

For $r_k < \delta$, we have

$$\begin{aligned} &\overline{B}(b, r_k) \subset \overline{B}(D(a_k), 2\delta), \\ &\varphi(\xi_k)(\overline{B}(D(a_k), 2\delta)) \subset \overline{B}(D(\xi_k a_k), \varepsilon), \\ &\overline{B}(D(\xi_k a_k), \varepsilon) \subset \overline{B}(D(c), 2\varepsilon). \end{aligned}$$

Therefore

$$\varphi(\xi_k)(\overline{B}(b, r_k)) \subset \overline{B}(D(c), 2\varepsilon),$$

Now $\xi_k E_k$ is the component containing $\xi_k a_k$ of $D^{-1}(\varphi(\xi_k)(\overline{B}(b, r_k))$ and is contained in $\overline{V}$. Therefore $\xi_k E_k$, hence E_k, is compact. A contradiction. Q.E.D.

Definition (5.21). The set in (5.20) is called the *limit set* of N and is denoted by $L = L(N)$.

We summarize fundamental properties of L in the following proposition.

Proposition (5.22). *$L(N)$ is a closed H-invariant subset of S^n. Further if N is not isomorphic to a Hopf manifold, then $L(N)$ is the unique minimal set. In particular, if $L(N) \neq S^n$, then we have* $\mathrm{Int}L(N) = \phi$.

Below we shall give applications of (5.20). The first one (5.23) is originally due to Kamishima([25]). See also Gusevskiĭ-Kapovich([20]).

Corollary (5.23). *If the developing map D of a connected compact flat conformal manifold is not onto S^n, then D is a covering map onto its image.*

Proof. We need only consider the case where N is not a Hopf manifold. Then by (5.22), we have $L = L_O$ is contained in the complement of Image(D). That is, Image(D) is evenly covered by D.
Q.E.D.

The next application is found in Kulkarni-Pinkall ([45]), in which condition(2) below is mistakingly dropped.

Corollary (5.24). *Let N be a connected compact flat conformal manifold and let $\Omega = S^n \setminus L(N)$. Suppose*

(1) *Ω is connected and its fundamental group $\pi_1(\Omega)$ is finitely generated.*
(2) *For any point $x \in S^n$, there exists an arbitrarily small neighbourhood U such that $U \setminus L$ is connected.*

Then the developing map D is a covering map onto its image.

Proof. First of all let us prove that $D^{-1}(\Omega)$ is connected. In fact given any two points $a, b \in D^{-1}(\Omega)$, choose a path p in $\widetilde{N}$ joining a and b. The path p is covered by a finite union of small open set V_i. We may assume by (2) that $V_i \setminus D^{-1}(L) \approx D(V_i) \setminus L$ is connected. Then we can make a small change of p within $\bigcup_i V_i$ fixing the boundary points so that p is contained in $D^{-1}(\Omega)$. Therefore $D^{-1}(\Omega)$ is connected.

Now by (4.4), we need only consider the case where H has no fixed points in S^n. We need only show that $D(\widetilde{N}) \cap L = \phi$. Suppose the contrary. Choose a small compact ball $\overline{V}$ such that D is a homeomorphism on $\overline{V}$, that $\mathrm{Int}D(\overline{V})\cap L \neq \phi$ and that $\mathrm{Ext}D(\overline{V})\cap L \neq \phi$. Since $\pi_1(\Omega)$ is finitely generated, it is supported on some compact subset K of Ω. By (5.11), there exists a loxodromic transformation $f \in H$ with an attracting fixed point in $\mathrm{Int}D(\overline{V})$ and with a repelling fixed point outside $D(\overline{V})$. We have $f^n(K) \subset D(\overline{V})$ for some $n > 0$. Therefore $\pi_1(\Omega)$ is supported on $D(\overline{V}) \cap \Omega$. Now D gives a homeomorphism from $\overline{V} \cap D^{-1}(\Omega)$ onto $D(\overline{V}) \cap \Omega$. This shows $D_* : \pi_1(D^{-1}(\Omega)) \to \pi_1(\Omega)$ is an epimorphism. Since points in Ω are evenly covered by D, D gives a homeomorphism from $D^{-1}(\Omega)$ onto Ω. However $D^{-1}(D(\overline{V}))$ has a noncompact component, which is of course disjoint from $\overline{V}$. A contradiction. Q.E.D.

The condition (2) of (5.24) is in fact necessary. For, let Σ be a closed flat conformal 2-manifold corresponding to a B-group Γ ([3]). That is, $\Omega = S^n \setminus L$ is connected and simply connected and Σ is isomorphic to Ω/Γ. Apply the construction of (3.37) to Σ. We obtain a flat conformal structure with the same holonomy group and surjective developing map. All this is of course well known. For more general treatment, see e.g. Goldman ([16]).

We shall finish this section by studying type 2 flat conformal structures, i.e., with the developing maps covering maps and with indiscrete holonomy groups. First we give examples in dimension ≥ 3. (2-dimensional examples were already given in (3.32).) For our purpose the coordinates of $\widehat{\mathbf{R}}^n$ is convenient.

Consider $\widehat{\mathbf{R}}^{n-2} \subset \widehat{\mathbf{R}}^n$. As before, denote by $R_\theta \in \mathcal{M}(\widehat{\mathbf{R}}^n)$ the rotation by angle θ around $\widehat{\mathbf{R}}^{n-2}$. Let

$$H^{n-1} = \{x_{n-1} > 0, x_n = 0\}.$$

Define

$$h : H^{n-1} \times \mathbf{R} \to \widehat{\mathbf{R}}^n \setminus \widehat{\mathbf{R}}^{n-2}$$

by $h(x,t) = R_{2\pi t}x$. h is a universal covering. By (2.3), we have

$$\mathcal{M}(\widehat{\mathbf{R}}^{n-2}) = \{g \in \mathcal{M}(\widehat{\mathbf{R}}^n) \mid g(H^{n-1}) = H^{n-1}\}.$$

Let us define

$$\mathcal{S}(\widehat{\mathbf{R}}^{n-2}) = \{f \in \mathcal{M}(\widehat{\mathbf{R}}^n) \mid f(\widehat{\mathbf{R}}^{n-2}) = \widehat{\mathbf{R}}^{n-2}\}.$$

An element $f \in \mathcal{S}(\widehat{\mathbf{R}}^{n-2})$ carries H^{n-1} to a half plane bounded by $\widehat{\mathbf{R}}^{n-2}$. Clearly f is determined by $f(H^{n-1})$ and $f|_{\widehat{\mathbf{R}}^{n-2}}$. This shows that f commutes with R_θ. Therefore we have the epimorphism

$$\psi : \mathcal{M}(\widehat{\mathbf{R}}^{n-2}) \times \mathbf{R} \longrightarrow \mathcal{S}(\widehat{\mathbf{R}}^{n-2})$$

defined by $\psi(g,t) = R_{2\pi t}g$.

Consider the diagonal action of $\mathcal{M}(\widehat{\mathbf{R}}^{n-2}) \times \mathbf{R}$ on $H^{n-1} \times \mathbf{R}$. Then

$$(\psi, h) : (\mathcal{M}(\widehat{\mathbf{R}}^{n-2}) \times \mathbf{R},\ H^{n-1} \times \mathbf{R}) \longrightarrow (\mathcal{S}(\widehat{\mathbf{R}}^{n-2}),\ \widehat{\mathbf{R}}^{n} \setminus \widehat{\mathbf{R}}^{n-2})$$

is an equivariant mapping of (G, X)-pairs.

Example (5.25). Let $\Gamma \subset \mathcal{M}(\widehat{\mathbf{R}}^{n-2})$ be a discrete subgroup which acts freely on H^{n-1}. Suppose $M = H^{n-1}/\Gamma$ is compact. For any $\theta \in \mathbf{R} \setminus \{0\}$ and any homomorphism $\mu : \Gamma \to \mathbf{R}$, define

$$\overline{\varphi} : \Gamma \times \mathbf{Z} \to \mathcal{M}(\widehat{\mathbf{R}}^{n-2}) \times \mathbf{R}$$

by $\overline{\varphi}(\gamma, m) = (\gamma, \mu(\gamma) + m\theta)$. Using a triangulation of H^{n-1}/Γ, one can construct a continuous map $u : H^{n-1} \to \mathbf{R}$ such that $u(\gamma x) = \mu(\gamma) + u(x)$ for $\gamma \in \Gamma$ and $x \in H^{n-1}$. Define a homeomorphism

$$\overline{D} : \widetilde{M} \times \mathbf{R} \to H^{n-1} \times \mathbf{R}$$

by $\overline{D}(x,t) = (x, u(x) + \theta t)$, where we identify the universal covering $\widetilde{M}$ with H^{n-1}. Let $\varphi = \psi \circ \overline{\varphi}$ and $D = h \circ \overline{D}$. Then (D, φ) is clearly a DH pair for $M \times S^1$. Therefore it defines a flat conformal structure on $M \times S^1$. Since $\overline{D}$ is a homeomorphism, D is a covering map onto $\widehat{\mathbf{R}}^{n} \setminus \widehat{\mathbf{R}}^{n-2}$ and the holonomy group H is indiscrete e.g., if we choose $\theta \in \mathbf{R} \setminus \mathbf{Q}$. (Moreover for a suitable choice of μ, the "rotation part" of H is not even infinite cyclic.) Thus this is a type 2 flat conformal structure.

Conversely we have the following theorem which was first obtained by Gusevskiĭ-Kapovich ([20]) in dimension 3.

Theorem (5.26). *Suppose N is a type 2 connected closed flat conformal manifold modeled on $(\mathcal{M}(\widehat{\mathbf{R}}^{n}), \widehat{\mathbf{R}}^{n})$, where $n \geq 3$. Then by changing the DH pair within the equivalence class, we have* $L(N) =$

$\widehat{\mathbf{R}}^{n-2}$. *Moreover* N *is a hyperbolic manifold bundle over the circle whose holonomy map is an isometry.*

Proof. *Step* 1. $L(N) = \widehat{\mathbf{R}}^{n-2}$.

Let $\mathrm{Cl}H_0$ be the identity component of the closure $\mathrm{Cl}H$ of H. Since H is indiscrete, we have $\mathrm{Cl}H_0 \neq \{1\}$.

Case 1. $\mathrm{Cl}H_0$ *is noncompact.* In this case $L \equiv L_\omega(\mathrm{Cl}H_0)$ is nonempty. Notice that $\mathrm{Cl}H_0$ is a normal subgroup of $\mathrm{Cl}H$. This implies that L is invariant by the action of $\mathrm{Cl}H$ and hence by H. Therefore by (5.22), we have $L(N) \subset L$. On the other hand, it is easy to show that

$$L = L_\omega(\mathrm{Cl}H_0) \subset L_\omega(\mathrm{Cl}H) = L_\omega(H) = L(N).$$

Therefore we have $L = L(N)$.

Also by (4.4), we obtain that $\mathrm{Card}L \geq 3$. In fact, if for example $\mathrm{Card}L = 2$, then (4.4) implies that N or its double covering is a Hopf manifold, contrary to our hypothesis.

Let us show next that there is no fixed point of $\mathrm{Cl}H_0$ in L. Suppose on the contrary that there exists one, say x. Then for any $h \in H$, hx is also a fixed point of $\mathrm{Cl}H_0$, since $\mathrm{Cl}H_0$ is a normal subgroup of $\mathrm{Cl}H$. On the other hand, the orbit Hx is dense in L and therefore has cardinality ≥ 3. That is, there exist at least three fixed points of $\mathrm{Cl}H_0$ in L. This implies by the argument of (2.22) that $\mathrm{Cl}H_0$ has a fixed point in D^{n+1}, contradicting the assumption that $\mathrm{Cl}H_0$ is noncompact.

By (5.6) this implies that any $\mathrm{Cl}H_0$-orbit K in L is dense in L. Notice that K is an injectively immersed submanifold in $\widehat{\mathbf{R}}^n$. By (5.11), there exists a loxodromic transformation $f \in H$. We may assume for simplicity that $f(x) = \lambda Px$, $(\lambda > 1,\ P \in O(n))$ and that $0 \in K$. Clearly K is kept invariant by f. Now the smoothness of K at 0 implies that $K = \widehat{\mathbf{R}}^k$ for some $1 \leq k \leq n$. ($K = \mathbf{R}^k$ implies that ∞ is a fixed point of $\mathrm{Cl}H_0$, contradicting the above observation.) This shows $L = \widehat{\mathbf{R}}^k$. Since the developing map D is a covering map onto its image, we have $D(\widetilde{N}) \cap L(N) = \phi$. In particular we obtain $k \neq n$. Finally we have $k = n-2$, since otherwise D is a homeomorphism onto a connected component of $\widehat{\mathbf{R}}^n \setminus \widehat{\mathbf{R}}^k$ and H must be discrete.

Case 2. $\mathrm{Cl}H_0$ *is compact.* Here the coordinates of S^n is convenient. First of all by (5.13), we may assume $\mathrm{Cl}H_0 \subset O(n+1)$. If 0 is the unique fixed point of $\mathrm{Cl}H_0$, then 0 is also a fixed point of H.

That is, $H \subset O(n+1)$. A contradiction. Therefore the fixed point set of $\mathrm{Cl}H_0$ in S^n is S^k $(0 \leq k \leq n)$. Since $\mathrm{Cl}H_0$ is nontrivial, we have $k \neq n$. Likewise we obtain $k \neq n-1$. In fact since $\mathrm{Cl}H_0$ is connected, we have $\mathrm{Cl}H_0 \subset SO(n+1)$. Therefore if $k = n-1$, then $\mathrm{Cl}H_0$ is trivial.

Notice that S^k is H-invariant and therefore $L(N) \subset S^k$. Let us show $L(N) = S^k$. Since $k \leq n-2$, we obtain as in the proof of (5.24) that $\widetilde{N} \setminus D^{-1}(S^k)$ is connected. In way of contradiction take a point $x \in S^k \setminus L(N)$. Consider an ε-neighbourhood V of x in S^n such that $V \cap L(N) = \phi$.

Then we have as well that $D^{-1}(V \setminus S^k)$ is connected. That is, $D^{-1}(V)$ is connected. This shows that D is a homeomorphism, contrary to our hypothesis. Therefore we have $L(N) = S^k$. As before we obtain $k = n-2$.

Step 2. We shall show the last part of (5.26). Since $L(N) = \widehat{\mathbf{R}}^{n-2}$, the DH pair (D, φ) lifts to $(\overline{D}, \overline{\varphi})$, where

$$\overline{D} : \widetilde{N} \to H^{n-1} \times \mathbf{R},$$
$$\overline{\varphi} : \pi_1(N) \to \mathcal{M}(\widehat{\mathbf{R}}^{n-2}) \times \mathbf{R}.$$

Denote by p_i the canonical projection to the i-th factor. Consider a small perturbation $\overline{\varphi}'$ of $\overline{\varphi}$ such that $p_1 \circ \overline{\varphi}' = p_1 \circ \overline{\varphi}$ and $p_2 \circ \overline{\varphi}'(\pi_1(N)) \subset \mathbf{Q}$. Let $\varphi' = \psi \circ \overline{\varphi}'$. Then there exists a submersion $D' : \widetilde{N} \to \widehat{\mathbf{R}}^n$ such that (D', φ') is a DH pair. (See Thurston [56] Chapt. 5 or Canary-Epstein-Green [9] Chapt. 1.) The limit set of the new DH pair (D', φ') is also $\widehat{\mathbf{R}}^{n-2}$, since we have altered $\overline{\varphi}$ only in the $\mathbf{R}$-direction. Therefore by (5.24), D' is a covering map onto $\widehat{\mathbf{R}}^n \setminus \widehat{\mathbf{R}}^{n-2}$. That is, D' lifts to a homeomorphism

$$\overline{D}' : \widetilde{N} \to H^{n-1} \times \mathbf{R}.$$

Since $p_2 \circ \overline{\varphi}'(\pi_1(N)) \subset \mathbf{Q}$, we have that $p_2 \circ \overline{\varphi}'(\pi_1(N))$ is infinite cyclic with a generator θ. Let $\Gamma = \mathrm{Ker}(p_2 \circ \overline{\varphi}')$. We have an exact sequence

$$1 \to \Gamma \to \pi_1(N) \to \theta\mathbf{Z} \to 1.$$

Correspondingly we have a bundle structure of N with fiber $H^{n-1}/p_1\overline{\varphi}'(\Gamma)$ over $\mathbf{R}/\theta\mathbf{Z} \cong S^1$. Clearly the monodromy map is an isometry of a hyperbolic manifold. Q.E.D.

§6. Elementary structure and C-structure

Also in this section flat conformal manifolds are to be connected and compact unless otherwise specified. The dimension is always ≥ 3.

Definition (6.1). A flat conformal manifold N is called *elementary* if and only if $\mathrm{Card}L(N) \leq 2$.

As applications of (5.20), we have the following characterizations of elementary flat conformal manifolds.

Proposition (6.2). *The following conditions are equivalent.*

(1) $L(N) = \phi$.
(2) *The holonomy group* H *consists purely of elliptic transformations.*
(3) N *is a spherical space form.*

Proposition (6.3). *The following conditions are equivalent.*

(1) $L(N)$ *is a singleton.*
(2) H *contains parabolic transformations and no loxodromic transformations.*
(3) N *is an Euclidean space form.*

Proof. All that need proof is (2) $\Longrightarrow$ (1). (2) implies $L(N) \neq \phi$. By (5.14), we have $H \subset \mathcal{M}(S^n)_a$, $a \in S^n$. This shows (1). Q.E.D.

Proposition (6.4). *CardL*$(N) = 2$ *if and only if* N *or its double covering is a Hopf manifold.*

Proposition (6.5). *If* $\mathrm{Card}L(N) \geq 3$, *then* $L(N)$ *is a perfect set.*

Proof. The assumption implies by (5.14) the existence of a loxodromic elememt $f \in H$. Then at least one point of $L = L(N)$ is not fixed by f. Since L is invariant by f, we obtain that L is an infinite set. Therefore the derived set L' is nonempty. By the minimality of L ((5.22)), we have $L' = L$. That is, L is perfect. Q.E.D.

Theorem (6.6). *If the holonomy group* H *of a connected compact flat conformal manifold* N *does not contain a free group of two generators, then* N *is elementary.*

Proof. Suppose on the contrary that N is nonelementary.
Then $\mathrm{Card}L(N) = \infty$ and therefore by (5.11), there exist two loxodromic transformations $f, g \in H$ with disjoint fixed points. Now it

is a well known fact that f^n and g^n generate a free group for large n. Q.E.D.

(6.6) was first proved under the hypothesis that H is virtually nilpotent by Goldman([15]) and then by Kamishima ([25]) when H is virtually solvable. It is well known that for matrix group virtual solvability is equivalent to the condition of (6.6). See Tits ([57]).

Corollary (6.7). *If $\pi_1(M)$ does not contain a free group of two generators and if a compact connected manifold M does not have as a finite covering S^n, $S^{n-1} \times S^1$ or T^n, then M does not admit a flat conformal structure.*

(6.7) forbids many manifolds to admit flat conformal structures, e.g., 3-manifolds with Nil or Solv geometry.

Given two flat conformal manifolds N_1 and N_2, a new flat conformal manifold, called connected sum, is obtained in the following way. This operation was first introduced by Kulkarni ([42]).

Inside a conformal atlas (U_i, q_i) of N_i, choose a closed ball B_i. Assume that there exists $f \in \mathcal{M}(S^n)$ such that $f(\mathrm{Int}q_1(B_1)) = S^1 \setminus q_2(B_2)$. See Figure (6.1). (This is always possible e.g., if we choose B_i so that $q_i(B_i)$ is a metric ball.) Define a homeomorphism $h : \partial B_1 \to \partial B_2$ so that $q_2 \circ h = f \circ q_1$. Then

$$(f \circ q_1) \cup q_2 : (U_1 \setminus \mathrm{Int}B_1) \cup (U_2 \setminus \mathrm{Int}B_2)/h \to S^n$$

is a well defined embedding. Using $(f \circ q_1) \cup q_2$ and other small charts in N_i, we can define a flat conformal structure on the connected sum

$$(N_1 \setminus \mathrm{Int}B_1) \cup (N_2 \setminus \mathrm{Int}B_2)/h.$$

Definition (6.8). The flat conformal structure constructed in this way is called a *connected sum* of N_1 and N_2 and is denoted by $N_1 \sharp N_2$.

Notice that $N_1 \sharp N_2$ is not uniquely determined. For example if we fix $B_2 \subset N_2$ and make $B_1 \subset N_1$ much smaller, then the resultant connected sum would be different as a flat conformal structure.

One can also define the operation of connected sum of more than two structures.

Definition (6.9). A flat conformal structure (manifold) is called a C-*structure* (C-*manifold*) if it is a connected sum of finitely many elementary structures and is not itself elementary.

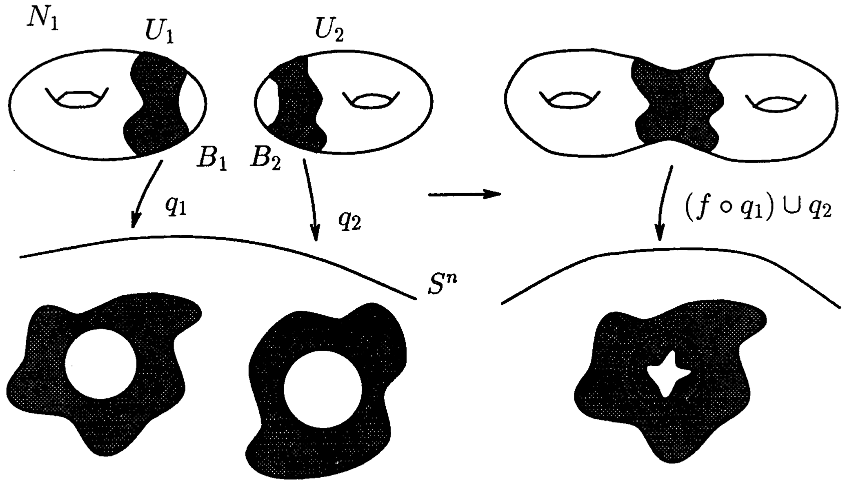

Figure (6.1)

It is easy to show that as a flat conformal manifold, $N \sharp S^n$ is isomorphic to N. This is called a trivial connected sum. There is only one case where a nontrivial connected sum of elementary flat conformal manifolds becomes again elementary, that is, when $N_1 = N_2 = \mathbf{R}P^n$, the real projective space. In this case $\pi_1(N_1 \sharp N_2)$ is isomorphic to the infinite dihedral group $\mathbf{Z}_2 * \mathbf{Z}_2$. One can show directly that $N_1 \sharp N_2$ has a Hopf manifold $S^{n-1} \times S^1$ as a double covering. In all the other cases the fundamental group of a connected sum contains a free group of two generators and therefore it cannot be elementary.

Definition (6.10). A Cantor set $\Upsilon \subset S^n$ is called *tame* if and only if there exists a homeomorphism $h : S^n \to S^n$ such that $h(\Upsilon) \subset S^1$. Otherwise Υ is called *wild.*

Proposition (6.11). *A C-structure N is of type 1. The limit set $L(N)$ is a tame Cantor set.*

Proof. For the first part of (6.11), it suffices to show the following; If the developing maps of the flat conformal structure N_1 and N_2 are injective, then the developing map of their connected sum is also injective. Let $S \subset N_1 \sharp N_2$ be the (n-1)-sphere on which the connected sum is made. S splits $N_1 \sharp N_2$ into two parts M_i such that $M_i \subset N_i$. Take a base point in $\mathrm{Int} M_1$. We have

$$\pi_1(N_1 \sharp N_2) \cong \pi_1(N_1) * \pi_1(N_2).$$

The element of $\pi_1(N_1 \sharp N_2)$ is represented uniquely as a reduced word of elements of $\pi_1(N_1)$ and $\pi_1(N_2)$. Consider the universal covering

$$\pi : \widetilde{N_1 \sharp N_2} \to N_1 \sharp N_2.$$

Choose one component of $\pi^{-1}(M_1)$ and denote it by $\widetilde{M}_1$. Choose one boundary component of $\widetilde{M}_1$ and denote it by $\widetilde{S}$. The component of $\pi^{-1}(M_2)$ which has $\widetilde{S}$ as a boundary component is denoted by $\widetilde{M}_2$. The boundary of $\widetilde{M}_1$ consists precisely of those components of $\pi^{-1}(S)$ which are of the form $\xi\widetilde{S}$ $(\xi \in \pi_1(M_1))$. $\xi\widetilde{M}_2$ is adjacent to $\widetilde{M}_1$ if and only if $\xi \in \pi_1(M_1)$. Now by the assumption the developing map D is injective on $\widetilde{M}_1$ and on $\xi\widetilde{M}_2$. $D(\widetilde{M}_1)$ and $D(\xi\widetilde{M}_2)$ are in the opposit sides of the sphere $D(\widetilde{S})$. This shows that D is injective on

$$\widetilde{M}_1 \cup (\cup_{\xi \in \pi_1(M_1)} \xi\widetilde{M}_2).$$

Boundary component of $\xi\widetilde{M}_2$ except $\xi\widetilde{S}$ are of the form $\xi\eta\widetilde{S}$ $(\eta \in \pi_1(M_2) \setminus \{1\})$. Again $D(\xi\widetilde{M}_2)$ and $D(\xi\eta\widetilde{M}_1)$ are in the opposite sides of the sphere $D(\xi\eta\widetilde{S})$. Therefore D is injective on the union of $\widetilde{M}_1$, $\xi\widetilde{M}_2$ and $\xi\eta\widetilde{M}^1$ ($\xi \in \pi_1(M_1)$, $\eta \in \pi_1(M_2) \setminus \{1\}$). An induction on the length of the word of $\pi_1(N_1 \sharp N_2)$ yields that the developing map D of a C-structure N is injective. We also have that Image(D) is contained in the complement of the limit set $L = L(N)$ and the holonomy homomorphism is an isomorphism onto a discrete group H.

Next we shall show that L is totally disconnected. Once this is established, we have by (6.5) that L is a Cantor set. For simplicity, we prove this only for the connected sum N of *two* elementary structures N_1 and N_2. We use the same notations as before. Choose a base point $x_0 \in D(\mathrm{Int}\widetilde{M}_1)$ and consider the family of disjoint topological spheres

$$\mathcal{S} = \{\varphi(\zeta)D(\widetilde{S}) \mid \zeta \in \pi_1(N)\}.$$

A point $x \in S^n \setminus \mathrm{Image}(D)$ is called *accessible* if there exists a path p in S^n combining x_0 and x such that p intersects finitely many spheres in $\mathcal{S}$. Accessible points consists precisely of the H-orbits of the points in $L(N_1) \cup L(N_2)$. See Figure (6.2). (We made the convension that $D|_{\widetilde{M}_i}$ coincides with the restriction of the developing map of N_i. This is always possible if we change the DH pairs of N_2 within the equivalence classes.)

Therefore accessible points are at most countable in number. Let $x \in S^n \setminus \mathrm{Image}(D)$ be a nonaccessible point. Then there are infinitely many nested spheres $\varphi(\zeta_i)D(\widetilde{S})$ $(i \geq 1)$ which separates x from x_0.

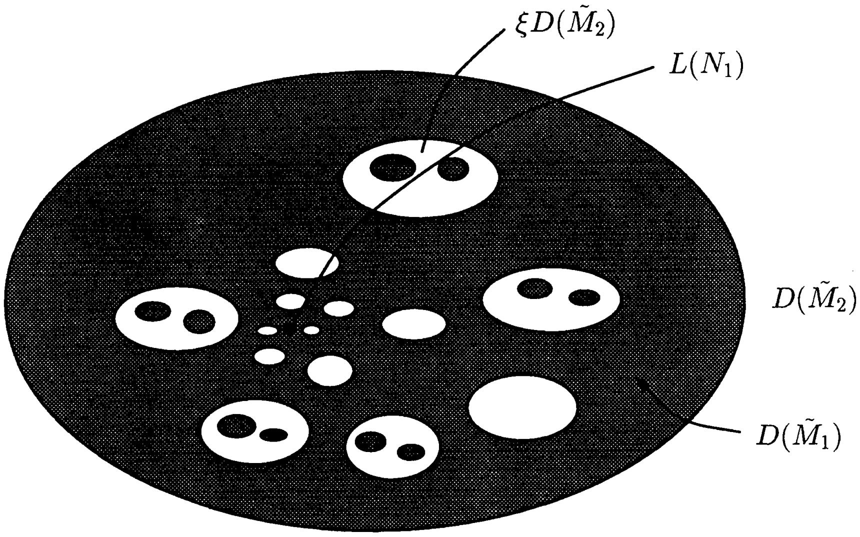

Figure (6.2)

Since ζ_i is distinct and H is discrete, we have $\varphi(\zeta_i) \to \infty$. Therefore by (5.8), we have

$$\mathrm{diam}\varphi(\zeta_i)D(\widetilde{S}) \to 0$$

since $D(\widetilde{S}) \cap L = \phi$. This shows that the component of $S^n \setminus \mathrm{Image}(D)$ at a nonaccessible point x is a singleton. Since accessible points are at most countable, this shows that $S^n \setminus \mathrm{Image}(D)$, hence L, is totally disconnected.

At this point we have obtained that $L = S^n \setminus \mathrm{Image}(D)$, since $S^n \setminus \mathrm{Image}(D)$, having no interior, is not evenly covered by D.

Finally to show the tameness of L, we have to define a homeomorphism $h : S^n \to S^n$ such that $h(L) \subset S^1$. First of all, define h on $\mathrm{Cl}D(\widetilde{M}_1)$ so that h carries all the boundary components to spheres intersecting S^1 and that h carries all the accessible points in $\mathrm{Cl}D(\widetilde{M}_1)$ into S^1. Next extend h to the adjacent components. Proceeding like this we can define the homomorphism h on the whole of S^n. Details are left to the reader. Q.E.D.

In dimension 3, we have the converse of (6.11).

Theorem (6.12). *Let N be a connected compact flat conformal manifold of dimension 3 such that the limit set $L(N)$ is a tame Cantor set. Then N is a C-manifold.*

Proof. We employ a method of Kulkarni ([43]) based upon the study of ends of a group. The necessary parts of the theory of ends are summarized in Appendix.

First of all notice that a tame Cantor set $L(N)$ satisfies the conditions of (5.24). Especially $\Omega = S^n \setminus L(N)$ is simply connected. Therefore the developing map $D : \widetilde{N} \to \Omega$ is a homeomorphism and the holonomy group H acts on Ω freely and discontinuously. That is, we have an isomorphism $N \cong \Omega/H$ of flat conformal structures. By Selberg's theorem (3.31), H has a torsion free subgroup H' of finite index. $N' = \Omega/H'$ is a finitely sheeted covering of N and therefore a compact manifold. Clearly we have $L(N') = L(N)$. By (A.4) of Appendix, we have

$$\mathrm{Card}\mathcal{E}(H') = \mathrm{Card}\mathcal{E}(\Omega) = \infty.$$

Therefore by Stalling's theorem (A.9), we obtain that H' is a nontrivial free product. Consequently N' decomposes as a nontrivial connected sum (as a manifold). See e.g. Hempel ([22]). Thus we have that $\pi_2(N) = \pi_2(N') \ncong 1$. By sphere theorem, this implies that N is reducible. It follows from (6.6) that N is not homeomorphic to $S^2 \times S^1$, since N is not an elementary structure. Therefore N is nonprime, that is, decomposes as a nontrivial connected sum $N = N_1 \sharp N_2$ as a manifold.

Let $S \subset N$ be the two sphere on which the connected sum is made. Let $N = M_1 \cup_S M_2$ and $N_i = M_i \cup_S B_i$, where B_i is homeomorphic to the closed 3-ball. Let $\pi : \widetilde{N} \to N$ be the universal covering and let $\widetilde{S}$ be a lift of S to $\widetilde{N}$. Denote by $\widetilde{M_i}$ the connected component of $\pi^{-1}(M_i)$ which has $\widetilde{S}$ as a boundary component. All the boundary components of $\widetilde{M_i}$ is of the form $\xi\widetilde{S}$ $(\xi \in \pi_1(N_i))$. Since $D|_{\widetilde{M_i}}$ is a homeomorphism, it extends in an equivariant way to $\widetilde{N_i} = \widetilde{M_i} \cup_{\xi\widetilde{S}} \xi\widetilde{B}_1$. From this we obtain a flat conformal structure on N_i, showing that the given structure on N is a connected sum of these two structures. Clearly we have $L(N_i) \subset L(N)$. Therefore either $\mathrm{Card}L(N_i) \leq 2$ or $L(N_i)$ is again a tame Cantor set. In the latter case, apply the whole argument once again to N_i. It is well known in 3-manifold theory that this process terminates. We obtain that N is a C-structure. Q.E.D.

It is unknown whether (6.12) holds in dimensin ≥ 4. In §8, we shall give an example of flat conformal 3-manifold whose limit set is a wild Cantor set. By (6.11), this is not a C-manifold.

§7. Poincaré polyhedron theorem

This section is devoted to the exposition of a fundamental theorem of Poincaré. It will be given in its simplest form, which is sufficient for our purpose of the next section. More general treatment is found e.g. in Maskit [47] in the framework of hyperbolic geometry.

Let T_i, T_i' $(1 \le i \le m)$ be metric $(n-1)$-spheres in S^n. Assume that any pair of them either intersect in an $(n-2)$-sphere or are disjoint and that any triple do not intersect at all. Let $\mathcal{E} = \{e_j\}$ be the family of $(n-2)$-spheres of the intersections. Let P be a component of the complement of the union of all T_i and T_i'. Assume that any element of $\mathcal{E}$ is contained in ∂P. See Figure (7.1).

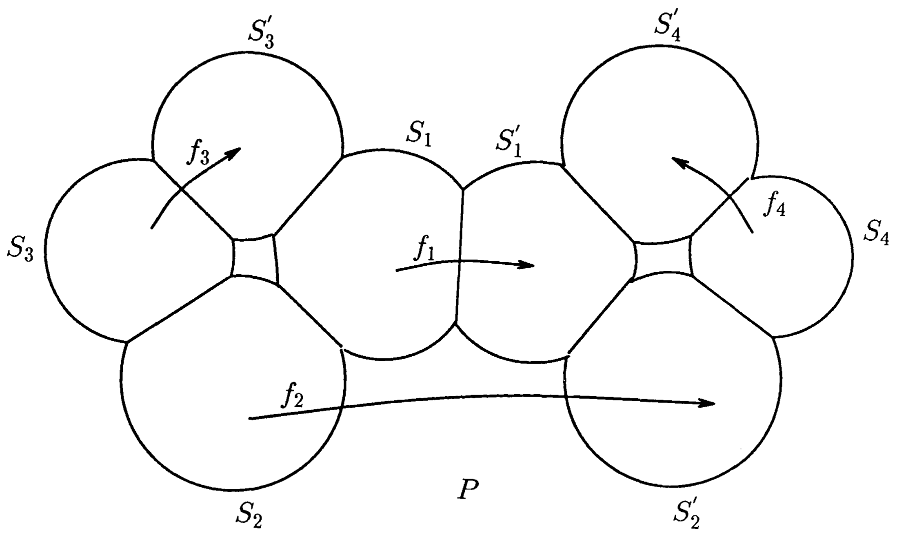

Figure (7.1)

Let $S_i = T_i \cap \partial P$ and $S_i' = T_i' \cap \partial P$. They are puctured $(n-1)$-spheres. Let $\mathcal{S} = \{S_i, S_i'\}$. An element of $\mathcal{S}$ or $\mathcal{E}$ is called respectively a *side* or an *edge* of P. Our first hypothesis is this.

(H.1). *For each* i, *there exists* $f_i \in \mathcal{M}(S^n)$ *such that* $f_i(S_i) = S_i'$ *and* $f_i(P) \cap P = \phi$.

Fix f_i once and for all and let $\mathcal{F} = \{f_i, f_i^{-1}\}$. An element of $\mathcal{F}$ is called a *side pairing transformation.* A side pairing transformation, say f_i, sends an edge e in ∂S_i to an edge e' in $\partial S_i'$. We call e and e' are related. This relation generates an equivalence relation in $\mathcal{E}$. $\mathcal{E}$ is partitioned into equivalence classes, called *cycles.* Each cycle $\mathcal{C}$ can be cyclically ordered as

$$\mathcal{C} = \{e_1 , \dots , e_{p-1}, e_p = e_0\}$$

in such a way that for each $1 \le \nu \le p$, there exists $f_\nu \in \mathcal{F}$ such that $f_\nu(e_{\nu-1}) = e_\nu$. Let

$$f_C = f_p \circ \cdots \cdots \circ f_1.$$

Clearly $f_C(e_0) = e_0$. For each cycle C, f_C is well defined up to inverse and conjugation. See Figure (7.2).

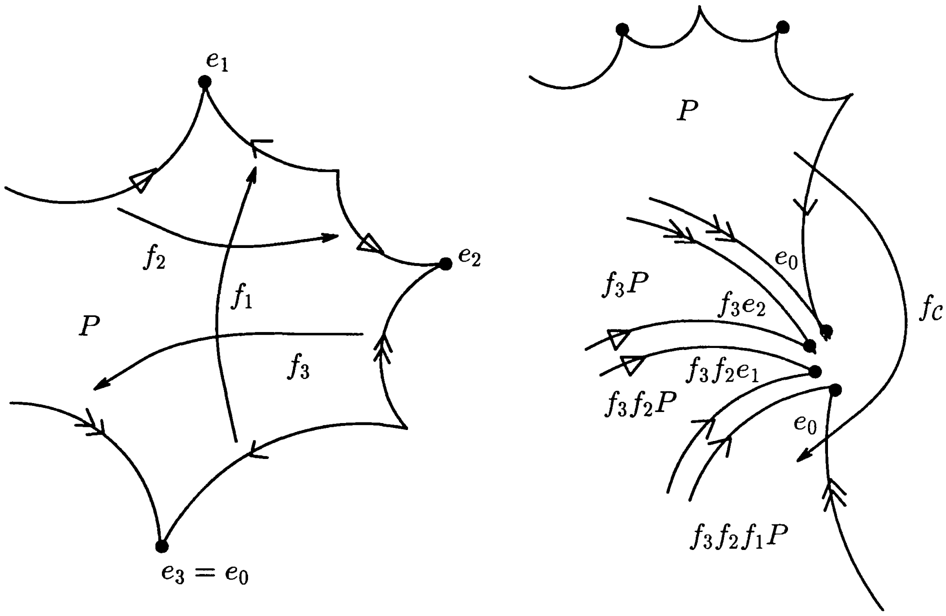

Figure (7.2)

For each edge $e \in \mathcal{E}$, the angle of P at e is denoted by $\theta(e)$. For a cycle C as above, define

$$\theta_C = \sum_{1 \le \nu \le p} \theta(e_\nu).$$

Our second hypothesis is;

(H.2). *For each cycle* C, *we have* $\theta_C = 2\pi/q$ *and* $f_C^q = \mathrm{id}$ *for some* $q \ge 1$.

The relation $f_C^q = \mathrm{id}$ is called a *cycle relation.* Denote by Γ the subgroup of $\mathcal{M}(S^n)$ generated by $\mathcal{F}$ and let Γ^* be the abstract group with generators the side pairing transformations and with relations the cycle relations. Clearly we have an epimorphism $\psi : \Gamma^* \to \Gamma$.

Definition (7.1). For a subgroup $G \subset \mathcal{M}(S^n)$, an open subset $R \subset S^n$ is called a *fundamental domain* of G if and only if the following two conditions are satisfied. (Ω_G denotes the domain of discontinuity of G.)

(FD.1) $$\Omega_G = \bigcup_{g \in G} g(\mathrm{Cl}R).$$

(FD.1) $$g(R) \cap R = \phi \text{ for any } g \in G \setminus \{1\}.$$

Theorem (7.2). *Assume* (H.1) *and* (H.2). *Then* $\psi : \Gamma^* \to \Gamma$ *is an isomorphism,* Γ *is a discrete subgroup of* $\mathcal{M}(S^n)$ *and* P *is a fundamental domain of* Γ.

Proof. Think of the family $\{\gamma P\}_{\gamma \in \Gamma}$ of domains. By a side pairing transformation f, fP is attached to P along a side of P. Next fgP $(g \in \mathcal{F},\, g \neq f^{-1})$ is attached to fP. If this process is continued, then around an edge $e_0 \in \mathcal{E}$ which belongs to a cycle

$$\mathcal{C} = \{e_1 \,, \ldots, e_{p-1}, e_p = e_0\}$$

such that $f_\nu(e_{\nu-1}) = e_\nu$, there is a sequence of domains

$$\begin{aligned} &P,\ f_pP,\ f_pf_{p-1}P,\ \ldots\ldots,\ f_p \cdots f_1P,\ \ldots \\ &\ldots\ ,(f_p \cdots f_1)f_pP,\ \ldots\ldots,\ (f_p \cdots f_1)^{q-1}f_p \cdots f_2P. \end{aligned}$$

They surround the edge e_0. By virture of (H.2), the sum of their angles at e_0 is just 2π and the last domain

$$(f_p \cdots f_1)^{q-1}f_p \cdots f_2P$$

is attached to P by f_1. The essential part of the proof is to show that in this way the family $\{\gamma P\}$ forms a "tesselation" of S^n. However in order to be precise, we must argue in a formal way as follows. Start with abstract copies of P and attach them one by one by side pairing transformations, thus constructing a replica of the domain of discontinuity of Γ. Next we show the existence of an embedding of the replica into S^n. The abstract group Γ^* is convenient for this development. Let us embark upon the proof.

Define an equivalence relation $\sim$ in $\Gamma^* \times \mathrm{Cl}P$ generated by the following.

$$(\gamma, x) \sim (\gamma', x') \quad \text{if } \gamma' = \gamma f,\ x = f(x') \text{ for some } f \in \mathcal{F}.$$

Let

$$\Omega^* = \Gamma^* \times \mathrm{Cl}P/\sim .$$

The action of Γ^* on Ω^* is defined by

$$\gamma'(\gamma, x) = (\gamma'\gamma, x).$$

Claim 1. *Ω^* is a flat conformal manifold on which Γ^* acts conformally.*

Choose $(\gamma, x) \in \Gamma^* \times \mathrm{Cl}P$. Suppose first of all that $x \in P$. Then by (H.1), there is no point $x' \in \mathrm{Cl}P$ such that $x = f(x')$ $(f \in \mathcal{F})$. That is, (γ, x) is equivalent to no other point and therefore it certainly has a neighbourhood homeomorphic to an n-ball. Suppose next that $x \in \mathrm{Int}S$, where S is a side of P, with a side pairing map $f : S \to S'$. Then as is shown easily, the only point in $\Gamma^* \times \mathrm{Cl}P$ which is equivalent to (λ, x) is $(\lambda f^{-1}, f(x))$. Clearly one can construct a neighbourhood of the identified point $[(\lambda, x)]$, homeomorphic to an n-ball, in

$$\{\lambda\} \times \mathrm{Cl}P \cup \{\lambda f^{-1}\} \times \mathrm{Cl}P/\sim .$$

Finally consider the case where $x \in e_0$ $(e_0 \in \mathcal{E})$. Let

$$\mathcal{C} = \{e_1, \ldots, e_{p-1}, e_p = e_0\}$$

be a cycle such that $f_\nu(e_{\nu-1}) = e_\nu$. Then we have

$$\begin{aligned}(\lambda, x) &\sim (\lambda f_p^{-1}, f_p x) \sim \cdots\cdots \\ &\sim (\lambda f_2^{-1} \cdots f_p^{-1}(f_1^{-1} \cdots f_p^{-1})^{q-1}, (f_p \cdots f_1)^{q-1} f_p \cdots f_2 x).\end{aligned}$$

By the definition of the cycle, these are shown to be all the points that are equivalent to (λ, x). By (H.2), we can construct a desired neighbourhood of $[(\lambda, x)]$. This shows that Ω^* is a manifold. Since the side pairing transformations are Moebius transformations, it is easy to endow Ω^* a flat conformal structure. Also one can show without difficulty that the action of Γ^* is conformal.

Next consider the conformal mapping

$$E : \Omega^* \to S^n$$

defined by $E(\lambda, x) = \psi(\lambda)x$ where $\psi : \Gamma^* \to \Gamma$ is the canonical projection. E is well defined and ψ-equivariant, that is,

$$E(\lambda'(\lambda, x)) = \psi(\lambda')E(\lambda, x).$$

Claim 2. *E is an embedding onto a connected open subet $\Omega \subset S^n$.*

For the proof we need hyperbolic geometry of D^{n+1}. Let us extend first of all $(n-1)$-spheres T_i and T_i' used to define P to half n-spheres $\overline{T}_i, \overline{T}_i' \subset D^{n+1}$ orthogonal to S^n. These are totally geodesic hyperplanes in (D^{n+1}, g_H). Using $\overline{T}_i$ and $\overline{T}_i'$ we can extend the domain P to a domain $\overline{P} \subset D^{n+1}$. Define as before

$$\overline{\Omega}^* = \Gamma^* \times \overline{P}/\sim,$$
$$\overline{E} : \overline{\Omega}^* \to D^{n+1}.$$

The argument of Claim 1 shows that $\overline{\Omega}^*$ is a hyperbolic manifold and that $\overline{E}$ is an isometric immersion. Furthermore one can show that there exists $\varepsilon > 0$ such that any point in $\overline{\Omega}^*$ has a neighbourhood isometric to hyperbolic ε-ball. Therefore $\overline{\Omega}^*$ is complete and thus $\overline{E}$ is a covering map. That is, $\overline{E}$ is a bijective isometry. Since

$$E \cup \overline{E} : \Omega^* \cup \overline{\Omega}^* \to S^n \cup D^{n+1}$$

is continuous, we obtain Claim 2.

Claim 2 implies that $\psi : \Gamma^* \to \Gamma$ is an isomorphism and that Γ is a discrete subgroup of $\mathcal{M}(S^n)$ which acts discontinuously on $\Omega = E(\Omega^*)$. What is left is to show that P is a fundamental domain. This is equivalent to the following.

Claim 3. *Ω is precisely the domain of discontinuity Ω_Γ of Γ.*

We already had $\Omega \subset \Omega_\Gamma$. To show the converse, it suffices by (5.15) to show that $S^n \setminus \Omega \subset L_\omega(\Gamma)$. Take a point $x \in S^n \setminus \Omega$. Then for any small neighbourhood U of x in $S^n \cup D^{n+1}$, we have $\gamma_k \overline{P} \cap U \neq \phi$ for infinitely many $\gamma_k \in \Gamma$. By (5.8), we have $CI(\gamma_k) \cap \overline{P} = \phi$ for large k, since $\mathrm{Cl}P \cap L_\omega(\Gamma) = \phi$. This implies

$$\mathrm{diam}\gamma_k \overline{P} \leq \mathrm{diam} CI(\gamma_k^{-1}) \to 0.$$

That is, U contains infinitely many $\gamma_k \overline{P}$, showing that $x \in L_\omega(\Gamma)$.
Q.E.D.

§8. Wild Cantor set as limit set

In this section we shall construct an example of type 1 compact flat conformal 3-manifold whose limit set is a wild Cantor set. Such an example was first obtained by Bestvina-Cooper [4] for an open 3-manifold. Our example is a variant of what they constructed. We shall follow [4] rather closely.

For a while we adopt the coordinates of $\widehat{\mathbf{R}}^3$ instead of S^3. First of all let K be a graph embedded in $\mathbf{R}^3$, depicted in Figure (8.1).

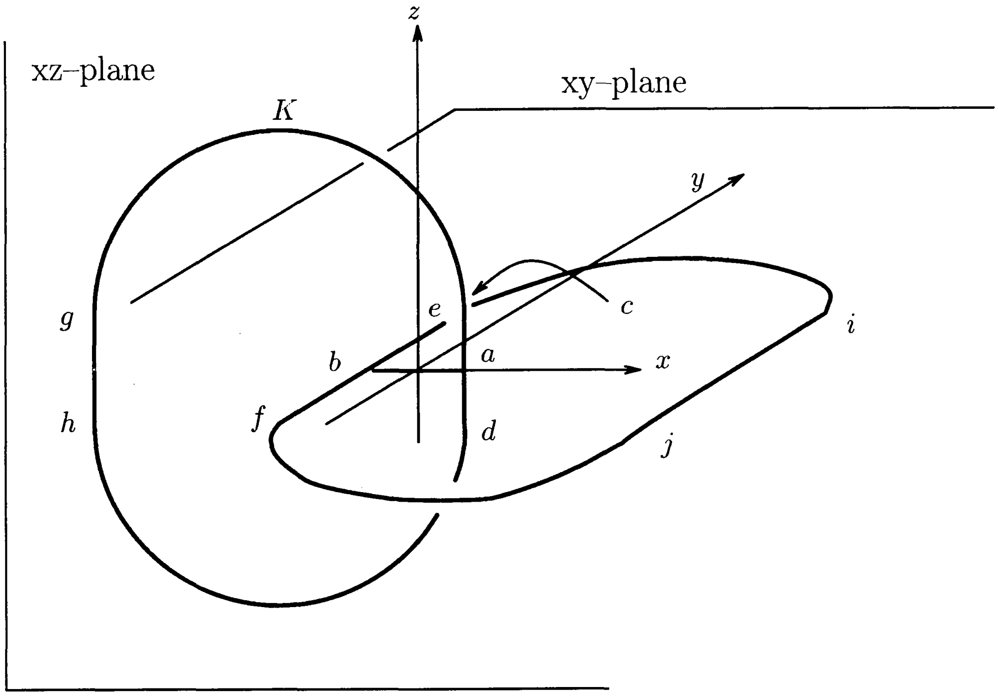

Figure (8.1)

The segment ab, cd, ef, gh and ij are straight lines and the other parts are circular arcs of the same radius. Choose a family of 2-spheres

$$A_1, \ldots, A_n,\ A'_1, \ldots, A'_n\ B_1, \ldots, B_n$$
$$B'_1, \ldots, B'_n\ C,\ C'\ D,\ D',\ E,\ E',$$

as in Figure (8.2).

We assume the followings.

(P.1) All the spheres have the same radius and have centers in K.

(P.2) The union of balls they bound covers K.

(P.3) The centers of C, E', D', C', E, D are in the x-axis. A_n, C, A'_n and A_1, E, A'_1 have centeres in straight lines parallel to z-axis. B_1, E', B'_1 and B_n, D, B'_n have centers in straight lines parallel to y-axis.

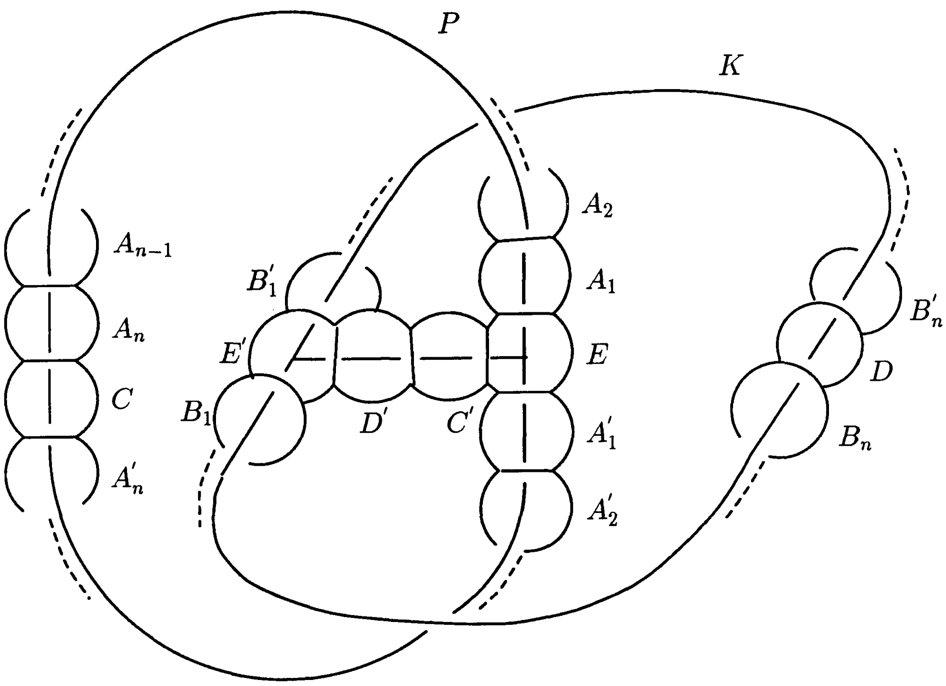

Figure (8.2)

(P.4) Adjacent two spheres intersect at angle $2\pi/28$.

Let P be the complement in $\widehat{\mathbf{R}}^3$ of the union of all the balls. Next we shall define side pairing transformations for P.

(S.1) $\alpha_j = I_{xy} \circ I_{A_j}$, where I_{xy} is the reflexion at the xy-plane and I_{A_j} is the inversion at the sphere A_j.

(S.2) $\beta_j = I_{xz} \circ I_{B_j}$.

(S.3) $\gamma = R_y^{C'} \circ I_{\pi(C,C')} \circ I_C$, where $R_y^{C'}$ is the rotation by $+90$ degrees around the oriented line through the center of C' parallel to the positive direction of y-axis and $\pi(C, C')$ is the bisector of the centers of C and C'.

(S.4) $\delta = R_z^{D'} \circ I_{\pi(D,D')} \circ I_D$.

(S.5) $\epsilon = R_x^{E'} \circ I_{\pi(E,E')} \circ I_E$.

Denote the side of P by $A_j^* = A_j \cap \mathrm{Cl}P$, $B_j^* = B_j \cap \mathrm{Cl}P$ and so forth. They satisfy the condition (H.1) of §7. That is,

(H.1) $\alpha_j(A_j^*) = A_j^{'*}$, $\beta_j(B_j^*) = B_j^{'*}$, $\gamma(C^*) = C^{'*}$, $\delta(D^*) = D^{'*}$, $\epsilon(E^*) = E^{'*}$ and $f(P) \cap P = \phi$ $(f = \alpha_1, \dots, \epsilon)$.

Next we shall verify the condition (H.2) of §7 by listing up the cycles.

First of all for any $1 \leq j \leq n-1$, we have the following cycle.

(C.1) $$A_j \cap A_{j+1} \xrightarrow{\alpha_j} A'_j \cap A'_{j+1} \xrightarrow{\alpha_{j+1}^{-1}} A_j \cap A_{j+1}$$

$\alpha_{j+1}^{-1}\alpha_j$ keeps points in $A_j \cap A_{j+1}$ fixed. By (P.4), it is a rotation by $2\pi/14$ around $A_j \cap A_{j+1}$. Therefore (H.2) is satisfied for $q = 14$. Likewise the following cycle satisfies (H.2).

(C.2) $$B_j \cap B_{j+1} \xrightarrow{\beta_j} B'_j \cap B'_{j+1} \xrightarrow{\beta_{j+1}^{-1}} B_j \cap B_{j+1}$$

There are two more cycles. The first one is;

(C.3) $$A_1 \cap E \xrightarrow{\alpha_1} A'_1 \cap E \xrightarrow{\epsilon} B'_1 \cap E' \xrightarrow{\beta_1^{-1}} B_1 \cap E' \xrightarrow{\epsilon^{-1}} A_1 \cap E$$

Computation shows that $\epsilon^{-1}\beta_1^{-1}\epsilon\alpha_1$ keeps $A_1 \cap E$ pointwise fixed. It is a rotation by $2\pi/7$ around $A_1 \cap E$ and thus (H.2) is satisfied for $q = 7$. Now the last cycle.

(C.4) $$E \cap C' \xrightarrow{\epsilon} E' \cap D' \xrightarrow{\delta^{-1}} B'_n \cap D \xrightarrow{\beta_n^{-1}} B_n \cap D \xrightarrow{\delta} C' \cap D'$$

$$\xrightarrow{\gamma^{-1}} A'_n \cap C \xrightarrow{\alpha_n^{-1}} A_n \cap C \xrightarrow{\gamma} E \cap C'$$

By studying Figure (8.2), we obtain that $\gamma\alpha_n^{-1}\gamma^{-1}\delta\beta_n^{-1}\delta^{-1}\epsilon$ yields the translation by $\pi/2$ on the circle $E \cap C'$. We also obtain (H.2) for $q = 4$.

Thus by (7.2), the group Γ generated by the side pairing transformations is discrete, with the domain of discontinuity Ω and P is the fundamental domain for Γ. That is, we have the followings.

(FD.1) $$\Omega = \bigcup_{\gamma \in \Gamma} \gamma(\mathrm{Cl}P).$$

(FD.2) $$\gamma(P) \cap P = \phi \quad \text{for any } \gamma \in \Gamma \setminus \{1\}.$$

Let Γ_0 be a torsion free subgroup of Γ of finite index. Then the quotient space $N = \Omega/\Gamma_0$ is a compact flat conformal manifold and we have

$$L = L(N) = L_P(\Gamma_0) = L_P(\Gamma) = \widehat{\mathbf{R}}^3 \setminus \Omega.$$

The rest of this section is devoted to the proof of the following theorem.

Theorem (8.1). *The limit set* L *is a wild Cantor set.*

The proof consists of a series of lemmas. The main part is to show that L is a Cantor set. First of all notice the following feature of our construction. See Figure (8.3). For any side T^* of P, let T be the 2-sphere which contains T^* and let $e \subset \partial T^*$ be an edge. Since all the translates of P which gather at e have angle $2\pi/28$ there, the part of T which is opposite to T^* w.r.t. e and near e is also a side of a translate of P. That is, the side "prolongs" in the tesselation.

So far in the construction of Γ, we have used the coordinates of $\widehat{\mathbf{R}}^3$. However in the rest, we change the coordinates from $\widehat{\mathbf{R}}^3$ to S^3. Thus, distance, radius, etc. are measured in the Euclidean metric of $\mathbf{R}^4$ which contains S^3 as a unit sphere $\{|x| = 1\}$.

Let $\{\gamma_k\} \subset \Gamma$ be an infinite sequence. Since Γ is discrete, we have $\gamma_k \to \infty$.

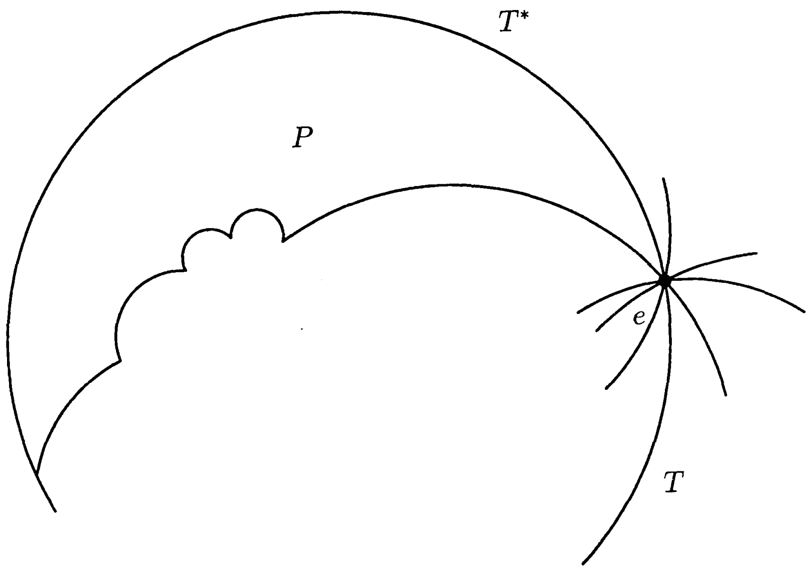

Figure (8.3)

Lemma (8.2). *For any edge* e *of* P, *we have* $\text{radius}\, \gamma_k(e) \to 0$.

Proof. This follows at once from (5.8), since we have $e \cap L = \phi$.

Q.E.D.

Let

$$\Sigma = \{\gamma(T) \mid \gamma \in \Gamma,\ T = A_1, \ldots, E'\},$$
$$\Lambda = \{\gamma(e) \mid \gamma \in \Gamma,\ e;\ \text{an edge of } P\}.$$

Notice that Λ consists of disjoint circles, while spheres in Σ may intersect. Furthermore at this point we do not know, for example, whether or not it so happens that two spheres in Σ are tangent. We have a control of Λ since any of its circle is contained in Ω. However this is not the case with Σ. We only have a rather weak grip on Σ.

For $S \in \Sigma$, let

$$\Lambda_S = \{l \in \Lambda \mid l \subset S\}.$$

Take a base point $x_0 \in S$ such that x_0 lies in a translate of the interior of a side of P.

Lemma (8.3). *For $x \in S$, x lies in Ω if and only if there exists only finitely many circles in Λ_S which separate x from x_0.*

Proof. Notice first of all that circles in Λ_S are mutually disjoint. To show the if part, let p be a path in S from x_0 to x which meets circles in Λ_S at finitely many points. An induction on the number of points shows that p is contained in the union of translates of sides of P. In particular, we have $x \in \Omega$.

For the converse, suppose that for a fixed edge e of P, $\gamma_k(e)$ $(1 \leq k < \infty)$ separates x from x_0. By (8.2), we have that

$$\operatorname{radius} \gamma_k(e) \to 0.$$

Notice that (FD.1) and (FD.2) implies that $\gamma_k(e)$ is disjoint from a small neighbourhood of x_0. Since $\gamma_k(e)$ are mutually disjoint, we obtain that $\gamma_k(e) \to x$. Therefore the family $\{\gamma_k^{-1}\}$ cannot be equicontinuous on any neighbourhood of x. That is, $x \in L_J(\Gamma) = L$. (See (5.15).) Q.E.D.

Corollary (8.4). *A connected component of $L \cap S$ is a singleton. In particular $\Omega \cap S$ is open and dense in S.*

Proof. This follows from the fact that $\gamma_k(e) \subset \Omega$. Q.E.D.

A word of caution. In the above corollary, we do not assert that the component of L at a point of S is a singleton.

In spirit we are going to show the total disconnectedness of L in a way similar to (8.4) using spheres in Σ instead of circles in Λ_S. However as we remarked earlier, we do not have yet a good grip on how Σ looks like. The main difficulty comes from the fact that $S \cap L \neq \phi$ for $S \in \Sigma$. In what follows we shall carry out study of Σ step by step.

Corollary (8.5). *We have* $S \cap \gamma(P) = \phi$ *for any* $\gamma \in \Gamma$ *and* $S \in \Sigma$. *(Recall that* P *is an open set.)*

Proof. For contradiction, take a point $x \in S \cap \gamma(P)$. By (FD.1), we have $x \in \Omega$. Therefore by (8.3), there exist finitely many circles in Λ_S separating x from x_0. Now the argument of the first part of (8.3) can be applied to show that x is contained in the union of sides of translates of P. That is, $x \notin \gamma(P)$. A contradiction. Q.E.D.

Lemma (8.6). *Let* $S,\ S' \in \Sigma$. *If* $l = S \cap S'$ *is a circle, then* $l \in \Lambda$. *In particular, we have* $l \subset \Omega$.

Proof. By (8.4), there exists a point $x \in l \setminus L$. By (8.5), we have that $x \notin \gamma(P)$ for any $\gamma \in \Gamma$. Likewise we obtain that x does not lie in a translate of the interior of a side of P, since otherwise either S or S' would intersect some $\gamma(P)$. Therefore we have $x \in l'$ for some $l' \in \Lambda$. Should l' not coincide with l, there would be another sphere $S'' \in \Sigma$ such that S, S' and S'' meet in general position at $x \in \gamma(P)$. Again one of the three spheres would intersect some $\gamma(P)$. A contradiction. Q.E.D.

Lemma (8.7). *Let* $S \in \Sigma$. *Suppose that for some* $\gamma_k \in \Gamma$, $\gamma_k(S)$ *are distinct spheres. Then we have* $\operatorname{radius} \gamma_k(S) \to 0$.

Proof. Suppose the contrary. We may assume further that $\gamma_k(S)$ converges to a 2-sphere S_0. Let us show first that $S_0 \cap L \neq \phi$. Take a point $a \in S$ and assume that $\gamma_k(a) \to b \in S_0$. For any neighbourhood U of b, we have $\gamma_j \gamma_i^{-1}(U) \cap U \neq \phi$ for arbitrary $j \gg i \gg 1$. That is, $b \in L_P(\Gamma) = L$. (See (5.15).) On the other hand, we have $S_0 \not\subset L$. In fact, $S_0 \subset L$ would imply that Ω is not connected. However this is impossible since the fundamental domain P is connected. (Notice that by the minimality (5.6) of L, we have $\operatorname{Int} L = \phi$. Compare (5.22).)

Consider a path in S_0 which combines a point of $\Omega \cap S_0$ to a point of $L \cap S_0$. As in the proof of (8.3), one finds a sphere $S' \in \Sigma$ which separates these two points. Clearly $S' \cap S_0 = l$ is a circle. Since $\gamma_k(S) \to S_0$, we have that $\gamma_k(S) \cap S' \to l$. By (8.6), we have $\gamma_k(S) \cap S' \in \Lambda$. Since $\gamma_k(S)$ are all distinct, we may assume (passing to a subsequence if necessary) that $\gamma_k \cap S'$ are all distinct. This contradicts (8.2). Q.E.D.

Lemma (8.8). *Fix once and for all* $x_0 \in P$. *A point* $x \in S^3$ *belongs to* Ω *if and only if there exist only finitely many spheres in* Σ *which separate* x *from* x_0.

Proof. Suppose there exist infinitely many $S_k \in \Sigma$. Then by (8.7), $\mathrm{diam}S_k \to 0$. As in the proof of (8.3), we have $x \in L$. (In fact this part will not be used in the sequel.)

Let us embark upon the proof of the converse. Define a closed subset $Y_j \subset S^3$ inductively as follows.

$$Y_0 = \mathrm{Cl}P.$$

$$Y_j = \bigcup_\gamma \gamma \mathrm{Cl}P, \quad \text{where} \quad \gamma\mathrm{Cl}P \cap Y_{j-1} \neq \phi, \quad \text{for} \quad j > 0.$$

Define an open subset X_j by

$$X_j = S^3 \setminus Y_j.$$

The set theoretic frontier ∂X_j is an angular surface (possibly with singularities) composed of the translates of sides of P, which we call *sides* of X_j. We have a filtration

$$(*) \qquad \mathrm{Cl}X_0 \supset X_0 \supset \mathrm{Cl}X_1 \supset X_1 \supset \mathrm{Cl}X_2 \supset X_2 \cdots .$$

See Figure (8.4).

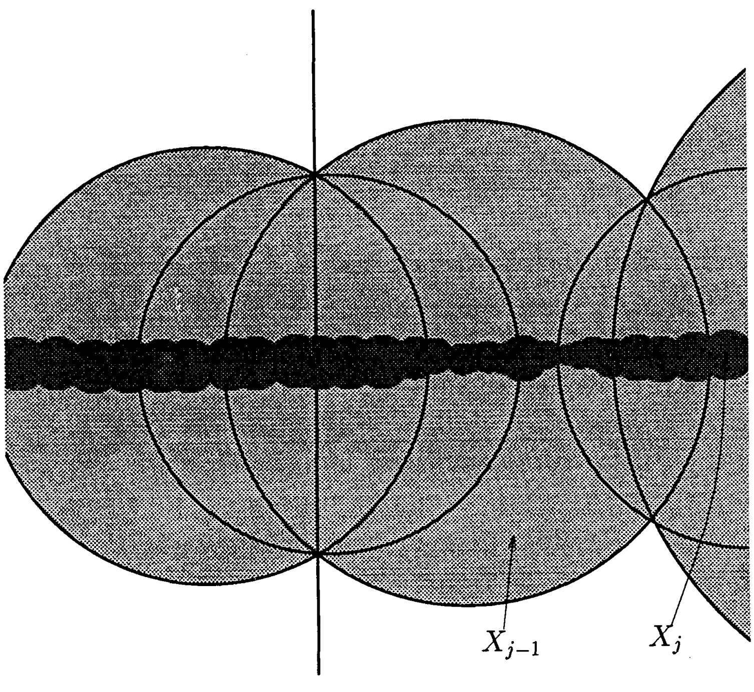

Figure (8.4)

Since P is the fundamental domain for Γ, we have

$$L = \bigcap_{j \geq 0} X_j.$$

For $S \in \Sigma$, the connected component of $S^3 \setminus S$ opposite to the base point x_0 is called the *inside* of S and is denoted by $\mathrm{Inside}(S)$. It is an open subset by definition. Let

$$\Sigma_j = \{S \in \Sigma \mid S \text{ contains a side of } X_j\}.$$

Once we establish the following sublemma, a point $x \in L$ can be shown to be inside infinitely many spheres in Σ, completing the proof of (8.8). Q.E.D.

Sublemma (8.8.1).

(1) *For any* $S \in \Sigma_j$, *we have* $S \subset \mathrm{Cl}X_j$.

(2) *We have*

$$X_j = \bigcup_{S \in \Sigma_j} \mathrm{Inside}(S).$$

Proof. The following properties of P, very easy to check, play a crucial part in th proof. Denote by T^*_α a side of P, by T_α the sphere containing T^*_α and by e_ν an edge of P.

(a) If $T^*_\alpha \cap T^*_\beta = \phi$, then we have $T_\alpha \cap T_\beta = \phi$.

(b) Suppose $T^*_\alpha \cap e_\nu = \phi$ and let $S \in \Sigma$ be an arbitrary sphere which passes through e_ν. If $S \cap T_\alpha \neq \phi$, then S contains a side S^* of P such that $S^* \cap T^*_\alpha \neq \phi$ and $e_\nu \subset \partial S^*$.

(c) If $T_\alpha \cap T_\beta \neq \phi$, $T_\beta \cap T_\gamma \neq \phi$ and $T_\gamma \cap T_\alpha \neq \phi$, then two of the three spheres T_α, T_β and T_γ must coincide.

The proof of (8.8.1) is by induction on j. For $j = 0$, this is clear by the construction of P. Let $j > 0$. Assume (8.8.1) for $j - 1$.

Proof of (1). For a given $S \in \Sigma_j$, let $S^* \subset S$ be a side of X_j. Choose a point x in the interior of S^*. By the filtration $(*)$, we have $x \in X_{j-1}$. The induction hypothesis implies that $x \in \mathrm{Inside}(T)$ for some $T \in \Sigma_{j-1}$. Since $S^* \subset \partial X_j$, there exists a translate $\gamma\mathrm{Cl}P$ having S^* as a side such that $\gamma\mathrm{Cl}P \cap X_j = \phi$. That is, $\gamma\mathrm{Cl}P \subset Y_j$. By the definition of Y_j, we have $\gamma\mathrm{Cl}P \cap \partial X_{j-1} \neq \phi$. Since $x \in \mathrm{Inside}(T)$, $\gamma\mathrm{Cl}P$ must lie in $T \cup \mathrm{Inside}(T)$. Therefore we have

$$\gamma\mathrm{Cl}P \cap T \cap \partial X_{j-1} \neq \phi.$$

Clearly $\gamma \mathrm{Cl}P \cap T \cap \partial X_{j-1}$ is either a side or an edge of $\gamma \mathrm{Cl}P$. Since $S^* \subset X_{j-1}$, we have $S^* \cap (\gamma \mathrm{Cl}P \cap T \cap \partial X_{j-1}) = \phi$.

If $\gamma \mathrm{Cl}P \cap T \cap \partial X_{j-1}$ is a side, then it follows from (a) that $S \cap T \neq \phi$. That is,

$$S \subset \mathrm{Inside}(T) \subset X_{j-1}.$$

Clearly this implies that $S \subset \mathrm{Cl}X_j$.

Suppose on the contrary that $\gamma \mathrm{Cl}P \cap T \cap \partial X_{j-1}$ is an edge, that is a circle $l \in \Lambda$. By (b), we obtain the same conclusion except in the case where T contains a side T^* of $\gamma \mathrm{Cl}P$ such that $T^* \cap S^* \neq \phi$ and $T^* \supset l$. See Figure (8.5). In this case choose a point $y \in S^* \cap T^*$. As before we obtain that $y \in \mathrm{Inside}(T')$ for some $T' \in \Sigma_{j-1}$ and that $l \subset T'$. If $S \cap T' \neq \phi$, then T' contains a side T'^* of $\gamma \mathrm{Cl}P$ such that $T'^* \cap S^* \neq \phi$ and $T'^* \supset l$. This contradicts (c). Therefore we have $S \cap T' = \phi$. As before we obtain $S \subset \mathrm{Cl}X_j$.

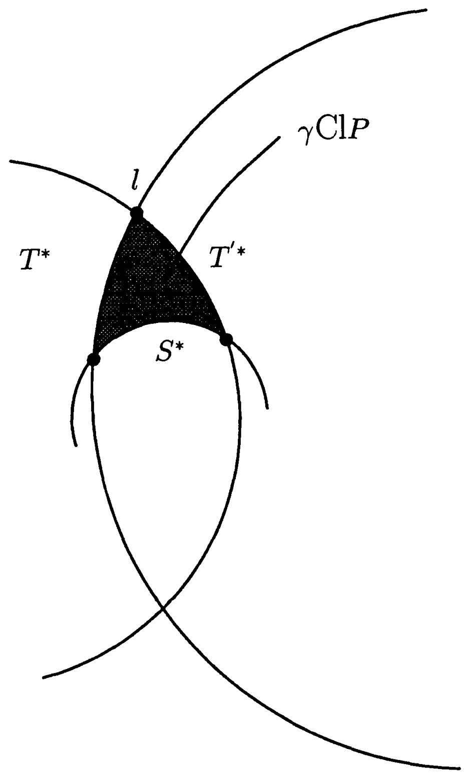

Figure (8.5)

Proof of (2). By the construction Y_j is connected. Therefore for

$S \in \Sigma_j$, $S \subset \mathrm{Cl}X_j$ implies that $\mathrm{Inside}(S) \subset X_j$. This shows

$$\bigcup_{S \in \Sigma_j} \mathrm{Inside}(S) \subset X_j.$$

For the converse, consider a path p combining the base point x_0 to a given point $x \in X_j$. Any such p must intersect ∂X_j and hence $\bigcap_{S \in \Sigma_j} S$. Choose p so that

(i) p does not pass through the intersection of two distinct spheres of Σ_j,

(ii) the sum

$$\sum_{S \in \Sigma_j} \mathrm{Card}(p \cap S)$$

is the minimal.

Then for each $S \in \Sigma_j$, we have $\mathrm{Card}(p \cap S) \leq 1$. In fact, if not, one can find a subarc q of p such that $\partial q \subset S$ and $q \setminus \partial q \subset \mathrm{Inside}(S)$. One can push q out of $\mathrm{Inside}(S)$ in such a way that the numbers of intersections of p with the other spheres do not change. This contradicts the minimality (ii).

We obtain that $\mathrm{Card}(p \cap S) = 1$ for some $S \in \Sigma_j$. That is, $x \in \mathrm{Inside}(S)$, as is required.

Q.E.D.

At this point we need a concrete picture how $\mathrm{Cl}X_1$ and $\mathrm{Cl}X_2$ look like. The picture of $\mathrm{Cl}X_1$ near $A_j \cap A_{j+1}$, $A_1 \cap E$ and $A_n \cap C$ are shown in Figures (8.6)$\sim$(8.8).

The point is that Figure (8.8) shows that there occurs a separation of components of $\mathrm{Cl}X_1$ near $A_n \cap C$. As a matter of fact, the same thing happens near any edge in the cycle of $A_n \cap C$. Furthermore we find a lot of separation of components of $\mathrm{Cl}X_2$. In particular in $\epsilon^{-1}(\mathrm{Cl}P)$ which is inside E, we observe that a component of $\mathrm{Cl}X_2 \cap \epsilon^{-1}(\mathrm{Cl}P)$ which intersets $\epsilon^{-1}B_1$ do not intersect $\epsilon^{-1}B_1'$. See Figure (8.9).

The same thing happens inside E'. In summary we have the following.

Let T, T' be any adjacent pair of 2-spheres chosen from $A_1, \ldots, E'$. Then a component of $\mathrm{Cl}X_2 \setminus (\mathrm{Inside}(T) \cup \mathrm{Inside}(T'))$ which intersects T does not intersect T'.

As a matter of fact, much more can be said concerning the smallness of components of $\mathrm{Cl}X_2$. However this is all that we need.

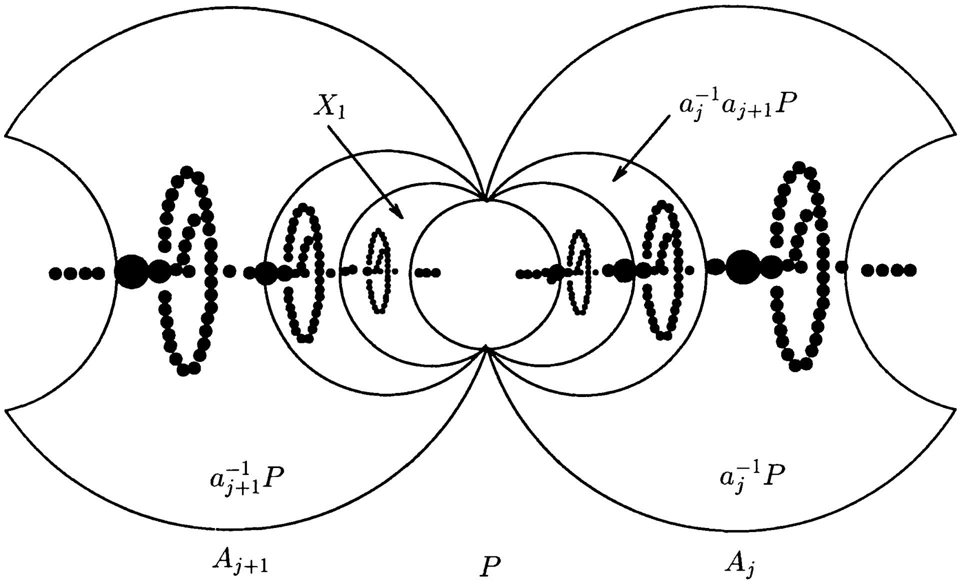

Figure (8.6)

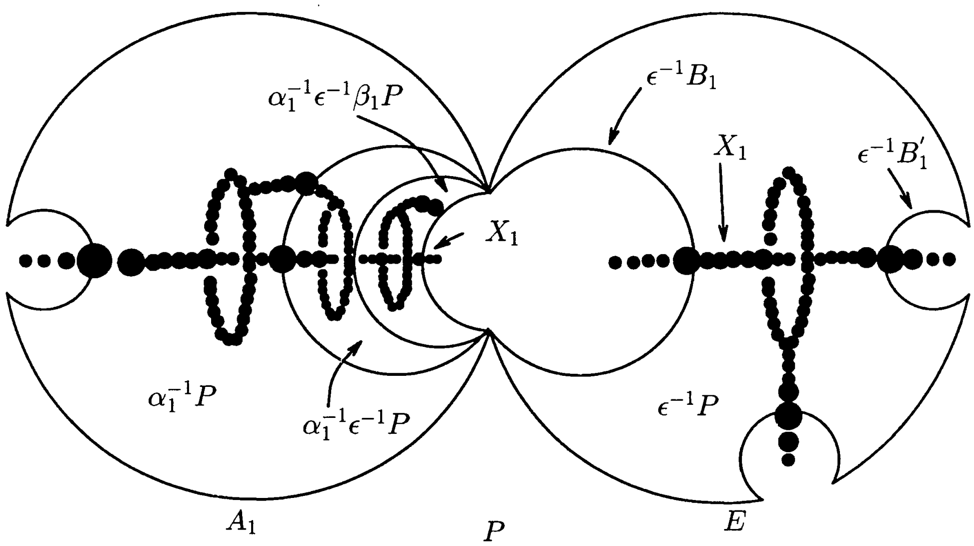

Figure (8.7)

Lemma (8.9). *For arbitrary spheres* S, $S' \in \Sigma$ *such that* $l = S \cap S'$ *is a circle, let* D *(resp.* D'*) be one of the disks in* S *(resp.* S'*) which is bounded by* l*. Suppose that the angle of* D *and* D' *at* l *is* $2\pi/28$*. Let* Q *be the closure of the component of* $S^3 \setminus (S \cup S')$ *bounded by* D *and* D'*. Then a component of* $L \cap Q$ *which intersects*

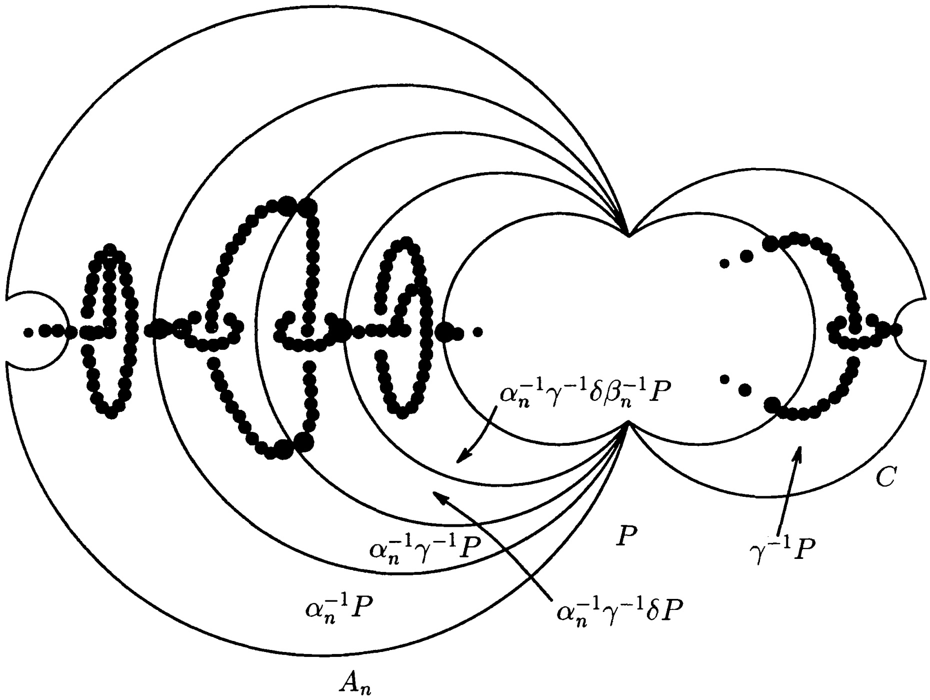

Figure (8.8)

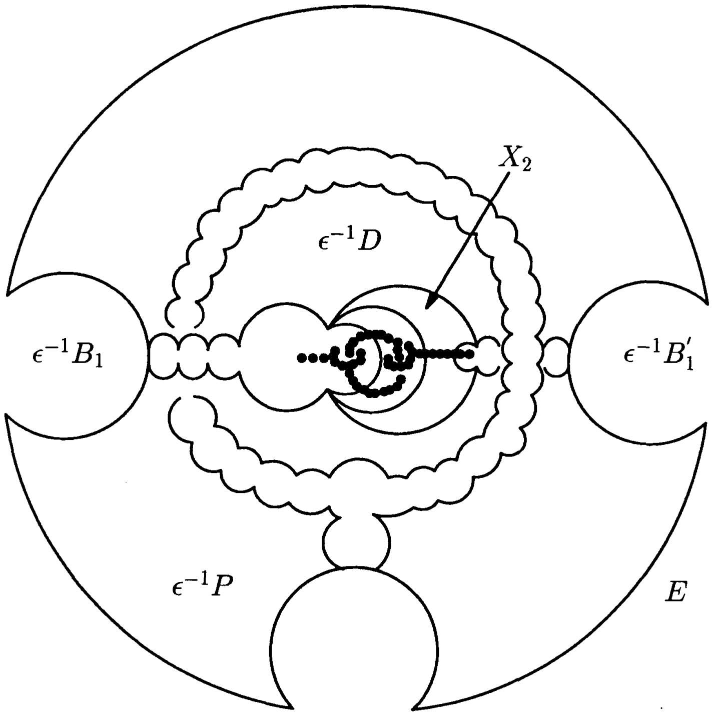

Figure (8.9)

D does not intersect D'.

Proof. Since we have $l \in \Lambda$ by (8.6), the proof reduces to the case where l is an edge of P and D and D' contains adjacent sides of P. Since $L \subset \mathrm{Cl}X_2$, (8.9) follows from the above observation. See Figure (8.10). Q.E.D.

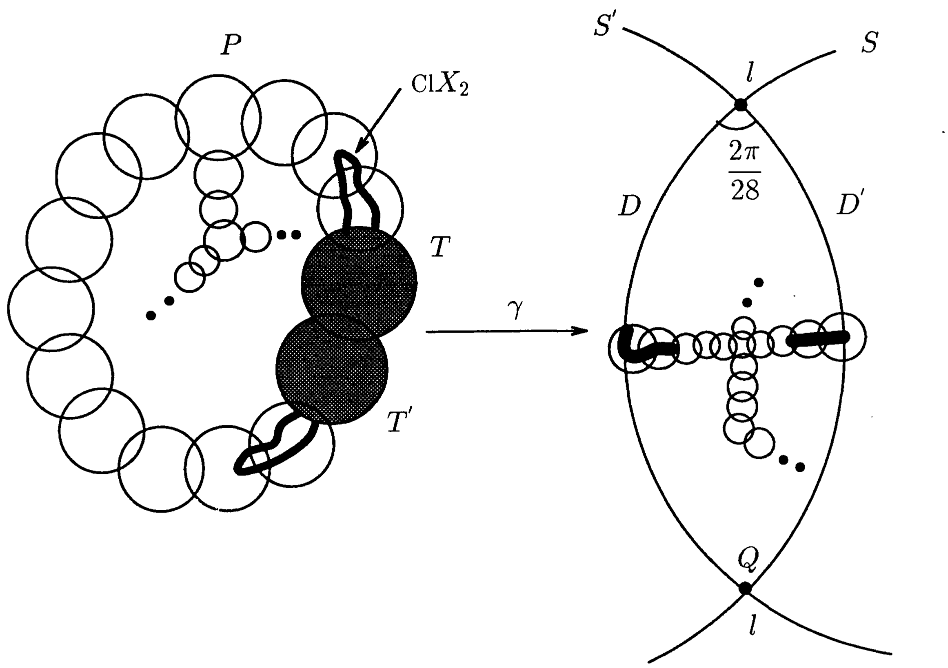

Figure (8.10)

Another way to put (8.9) is the following.

Corollary (8.10). *For arbitrary spheres S, $S' \in \Sigma$ such that $l = S \cap S'$ is a circle, the component of L at a point $x \in S' \setminus S$ does not intersect S.*

Proof. Let Δ be the component of $S' \setminus S$ at x and let Ξ be the closure of either of the components of $S^3 \setminus (S \cup S')$ which contains Δ in its boundary. Then by (8.9), we obtain that for any $y \in \Delta \cap L$, the component of $\Xi \cap L$ at y does not intersect S. See Figure (8.11). It is easy to show that (8.10) follows from this. Q.E.D.

Lemma (8.11). *Let S be an arbitrary sphere in Σ. For any $x \in S \cap L$, the component of L at x is $\{x\}$ itself.*

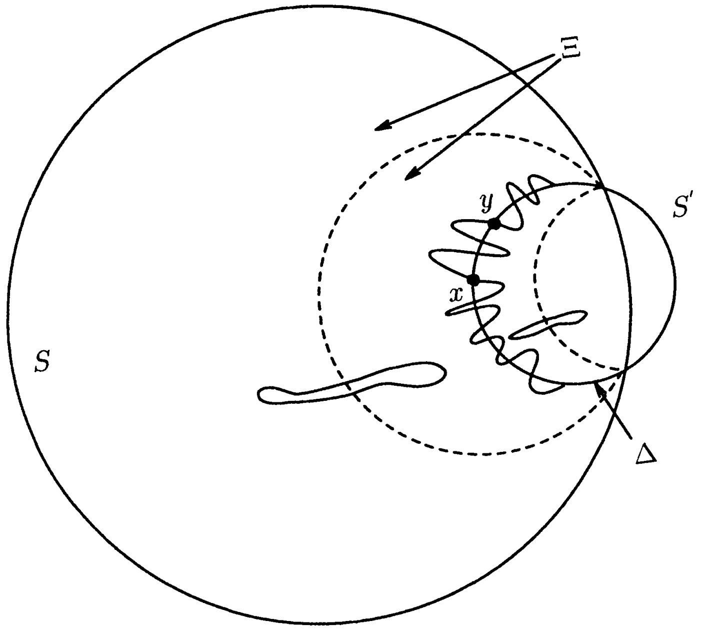

Figure (8.11)

Proof. By (8.8), there exists an infinite sequence $\{S_k\} \subset \Sigma$ such that S_k separates x from the base point $x_0 \in \Omega$. Note that $S_k \to x$. For large k, S_k intersects S at a circle. Therefore by (8.10), the component of L at x does not intersect S_k. This completes the proof. Q.E.D.

Corollary (8.12). *For any $S \in \Sigma$, the component of L at a point $x \in L \setminus S$ does not intersect S.*

Corollary (8.13). *L is totally disconnected.*

Proof. Let $x \in L$. If $x \in S$ for some $S \in \Sigma$, then we have already shown (See (8.11).) that the conponent of L at x is a singleton. So consider the other case. By (8.8), there exist infinitely many spheres $S_k \in \Sigma$ which separate x from a base point $x_0 \in P$. By (8.7), we have $S_k \to x$. Therefore (8.12) implies (8.13). Q.E.D.

By (6.5), this implies that L is a Cantor set. Thus we have finished the proof of the first part of Theorem (8.1). Let us show in the remainder that L is wild. First of all we have the following well known fact, which is easy to show.

Proposition (8.14). *If $\Upsilon \subset S^n$ is a tame Cantor set, then $S^3 \setminus \Upsilon$ is simply connected.*

Thus once we establish that the inclusion $i : P \to \Omega$ induces an injection on the fundamental groups, then the proof of Theorem (8.1) will be complete.

Some readers may have a feeling that this can be solved by looking at the homomorphism

$$\pi_1(P) \xrightarrow{i_*} \pi_1(\Omega) \xrightarrow{p_*} \pi_1(\Omega/\Gamma).$$

The computation of $\pi_1(\Omega/\Gamma)$, the fundamental group of an orbifold, is in fact easy. However in order to show the nontriviality of the homomorphism $p_* \circ i_*$, one is lead to the word problem of $\pi_1(\Omega/\Gamma)$, in which an approach geometric in nature is obviously indispensable. Instead of going to this direction, we employ the following argument which is totally geometric and is applicable also to the word problem of $\pi_1(\Omega/\Gamma)$.

The key fact is the following lemma.

Lemma (8.15). *Let T^* be a side of* ClP, *then the inclusion* $T^* \to \mathrm{Cl}P$ *induces an injection on* π_1.

Proof. By sliding handles of $S^3 \setminus P$, one obtains that ClP is a handlebody of genus 2. Therefore $\pi_1(\mathrm{Cl}P)$ is a free group freely generated by α and β. If $T^* \neq E^*$ or $E^{'*}$, then the lemma follows easily. For $T^* = E^*$, the image of $\pi_1(E^*)$ is generated by α and $\alpha\beta\alpha^{-1}\beta^{-1}$. It is well known, easy to show using the once puctured torus model, that they generate free subgroups. Q.E.D.

Now let us embark upon the proof. Let $\alpha : S^1 \to P$ be a loop such that $\alpha \not\simeq 1$ in P. Suppose on the contrary that $\alpha \simeq 1$ in Ω. Let $\beta : D^2 \to \Omega$ be the extension of α. By a small perturbation, one may assume that β is smooth and transverse to any circle in Λ and to any sphere in Σ. Their inverse images form a graph G in D^2. (G may contain smooth circles as connected components.) As a matter of fact, we have $G \neq \phi$ and $G \cap S^1 = \phi$. See Figure (8.12).

Let us choose β so

(M.1) the number of vertices of G is the minimal,
(M.2) the number of edges of G is the minimal among those which satisfy (M.1).

Let Δ be a connected component of $D^2 \setminus G$ which is homeomorphic to an open disk. Then $\beta(\Delta) \subset \gamma P$ for some $\gamma \in \Gamma$. Since β is transverse to $\gamma\partial P$, we have that $\partial\Delta$ is a simple closed curve.

Claim (8.16). *The number of vertices of $\partial\Delta$ is ≥ 3.*

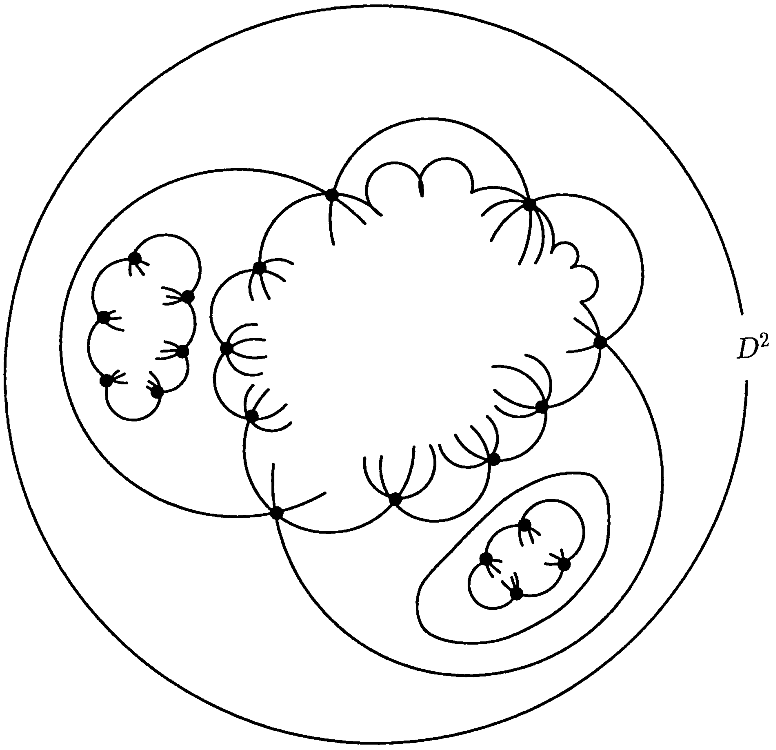

Figure (8.12)

Proof. If not, $\beta(\partial\Delta)$ is contained in the union $\mathcal{T}$ of adjacent two sides of $\gamma\partial P$ for some $\gamma \in \Gamma$. Clearly $\mathcal{T}$ is homotopic in $\gamma \mathrm{Cl} P$ to a single side. See Figure (8.2). Since $\beta(\partial\Delta)$ is null homotopic in $\gamma \mathrm{Cl} P$, we obtain by (8.15) that $\beta(\partial\Delta)$ is null homotopic in $\mathcal{T}$. But then we can alter the map β so that $\beta(\Delta) \subset \mathcal{T}$ and eventually push β out of $\gamma \mathrm{Cl} P$. This contradicts the minimality assumption (M.1) if $\partial\Delta$ has a vertex and (M.2) otherwise. Q.E.D.

Now consider the family of smooth circle components of G. Let l be the innermost one and let V be the open ball bounded by l. In case there is no smooth circles, let $V = \mathrm{Int} D^2$. By (8.16), there must exist components of G in V. Consider $G' = G \cap V$. G' has no longer a smooth circle component. Let $G'_1, \ldots, G'_r$ be the connected component of G'. Let E'_i be the component of $V \setminus G'_i$ which contains ∂V and let

$$H(G'_i) = V \setminus E'_i.$$

Notice that $\partial H(G'_i)$ is a simple closed curve since it is the inverse image by β of a surface $\gamma\partial P$ for some $\gamma \in \Gamma$. Therefore $H(G'_i)$ is a closed disk. Define a partial order $\prec$ in the set $\{G'_1, \ldots, G'_r\}$ by

$$G'_j \prec G'_i \Longleftrightarrow H(G'_j) \subset H(G'_i).$$

Let G'_j be the minimal element. Then any component of $H(G'_j)\backslash G'_j$ is an open disk. That is, G'_j gives a polyhedral decomposition of $H(G'_j)$. Let f, e and v be the number of faces, edges and vertices of the decomposition. By virtue of (8.16), we have

$$3f \leq 2e.$$

Notice that by (P.4), exactly 28 edges gather at each vertex of G'_j. Therefore we have

$$14v = e.$$

The computation of Euler number yields;

$$1 = f - e + v \leq \frac{2}{3}e - e + \frac{e}{14} < 0.$$

This contradiction shows that $i_* : \pi_1(P) \to \pi_1(\Omega)$ is an injection, as is requied.

Appendix End

The concept of end of a topological space and of a discrete group was first introduced in 1931 by Freudenthal ([11]) and was studied, among others, by Hopf ([23]). See also Freudenthal [12] and Epstein [8]. After almost 40 years, Stalling ([54],[55]) established a celebrated theorem concerning finitely generated groups with infinite ends. See Dunwoody [7] for related topics and a geometric proof of Stalling's theorem for finitely presented groups. All this has a wide range of applications. For the convenience of the reader, we collect here some parts of the theory, mostly without proof.

First of all we define the ends of a connected locally finite simplicial complex U.

Definition (A.1). A sequence $\{M_k\}$ of subsets of U is called *discrete* if for any compact subset C of U, we have $M_k \cap C = \phi$ for but finitely many k.

Definition (A.2). A point sequence $\{x_k\} \subset U$ is called *admissible* if for any $k > 0$, there exists a path $P_k \subset U$ combining x_k and x_{k+1} such that the family $\{P_k\}$ is discrete.

Definition (A.3). Two admissible sequence $\{x_k\}$ and $\{x'_k\}$ are said to be *equivalent,* (denoted by $\{x_k\} \sim \{x'_k\}$) if and only if there exists a path P_k $(k > 0)$ combining x_k and x'_k such that the family

$\{P_k\}$ is discrete. An equivalence class of admissible sequences are called an *end* of U. The set of the ends of U is denoted by $\mathcal{E}(U)$.

It is easy to show that the relation in (A.3) is in fact an equivalence relation. Notice that a subsequence of an admissible sequence is again admissible and in the same equivalence class.

For applications to flat conformal structures, we need the following.

Proposition (A.4). *For an open domain U of S^n $(n \geq 2)$, the set of ends $\mathcal{E}(U)$ is in one to one correspondence with the set of connected components of $Y = S^n \setminus U$.*

Proof. First of all let us define a correspondence of an end to a connected component. Let $\{x_k\}$ be an admissible sequence of U. Then we have $d(x_k, Y) \to 0$. Let us show furthermore that there exists a unique connected component Y_ν of Y such that $d(x_k, Y_\nu) \to 0$. Suppose the contrary. Then there exist subsequences $\{y_k\}$ and $\{z_k\}$ of $\{x_k\}$ such that $d(y_k, Y_\nu) \to 0$ and $d(z_k, Y_\mu) \to 0$ for disjoint components Y_ν and Y_μ of Y. Then there exists a compact neighbourhood B of Y_ν in S^n such that $B \cap Y_\mu = \phi$ and $\partial B \subset U$. Then any path P_k in U combining y_k and z_k must intersect the compact set ∂B. This contradicts the fact that $\{y_k\} \sim \{z_k\}$. The same argument shows that the component thus chosen is independent of the particular choice of an admissible sequence in the equivalence class. Thus an end corresponds to a connected component.

The converse correspondence is defined as follows. For any connected component Y_ν of Y, we can find a sequence $\{B_k\}$ of compact connected neighbourhoods of Y_ν in S^n such that $\partial B_k \subset U$ and that $\bigcap_k B_k = Y_\nu$. Furthermore one may assume that ∂B_k is a finite union of codimension one connected submanifold. Notice that any codimension one connected submanifold splits S^n $(n \geq 2)$ into two parts. Since U is connected, this shows that $\mathrm{Int}B_k \setminus Y$ is arcwise connected. Choose an arbitrary point $x_k \in \mathrm{Int}B_k$. Combine x_k and x_{k+1} by a path P_k in $\mathrm{Int}B_k \setminus Y$. This shows that $\{x_k\}$ is an admissible sequence. Q.E.D.

The ends of a group is defined by virtue of the following theorem.

Theorem (A.5). *Let Γ be a finitely generated group which acts on a connected locally finite simplicial complex U freely and discontinuously such that the quotient U/Γ is compact. Then the set of ends $\mathcal{E}(U)$ is determined (up to a bijection) only by the group Γ. It does not depends upon the particular choice of the space U.*

Definition (A.6). *The set of ends in (A.5) is called the* end set *of the group Γ and is denoted by $\mathcal{E}(\Gamma)$.*

Theorem (A.7). *The end set* $\mathcal{E}(\Gamma)$ *of a finitely generated group* Γ *is infinite (in fact uncountably infinite) if* $\mathrm{Card}\mathcal{E}(\Gamma) \geq 3$.

We have the following characterization of the group according to its end set.

Theorem (A.8). *Let* Γ *be a finitely generated group.*

(1) $\mathcal{E}(\Gamma) = \phi$ *if and only if* Γ *is a finite group.*
(2) $\mathcal{E}(\Gamma)$ *consists of two points if and only if* Γ *has the infinite cyclic group* $\mathbf{Z}$ *as a finite index subgroup.*

For a group with infinitely many ends, Stalling obtained a complete characterization. However for the sake of simplicity we only state the following partial result.

Theorem (A.9). *A finitely generated torsion free group* Γ *has infinite ends if and only if* Γ *has a nontrivial decomposition as a free product* $\Gamma = \Gamma_1 * \Gamma_2$.

As an application, if a torsion free group Γ in (A.9) acts on a domain $U \subset S^n$ freely and discontinuously and if the complement $S^n \setminus U$ has more than two components, then Γ is a nontrivial free product.

References

[1] B.N. Apanasov, On non-connectedness of the space of uniformized conformal structures on hyperbolic manifolds, Institut Mittag-Leffler, Djursholm Report No.1 (1989/90), p. 19.
[2] A.F. Beardon, "The geometry of discrete groups", Springer Verlag, 1983.
[3] L. Bers, On boundaries of Teichmüller spaces and on Kleinian groups, Ann. Math., **91** (1970), 570–600.
[4] M. Bestvina and D. Cooper, A wild Cantor set as the limit set of a conformal group action on S^3, Proc. A.M.S., **99** (1987), 623–626.
[5] L. Bieberbach, Über die Bewegungsgruppen der Euclidischer Räume I:, Math. Ann., **70** (1911), 297–336; II, Math. Ann., **72** (1912), 400–412.
[6] P. Buser, A geometric proof of Bieberbach's theorems on crystallographic groups, L'Enseignement Math., **31** (1985), 137–145.
[7] M.J. Dunwoody, The accessibility of finitely presented groups, Invent. math., **81** (1985), 449–457.
[8] D.B.A. Epstein, Ends, in "Topology of 3-Manifolds and Related Topics", Printice-Hall, 1962, pp. 110–117.
[9] D.B. A. Epstein, R. Canary and D. Green, Notes on notes of Thurston, in "Analytic and geometric aspects of hyperbolic spaces", London Math. Soc. Lecture Notes 111, Cambridge University Press, 1987.

[10] M.H. Freedman and R. Skora, Strange actions of groups on spheres, J. Differential Geom., **25** (1987), 75–98.

[11] H. Freudenthal, Über die Enden topologisher Räume und Gruppen, Math. Z., **33** (1931), 692–713.

[12] ———, Über die Enden diskreter Räume und Gruppen, Comment. Math. Helv., **17** (1945), 1–38.

[13] D. Fried, Closed similarity manifolds, Comment. Math. Helv., **55** (1980), 576–582.

[14] F.W. Gehring, Rings and quasiconformal mappings in spaces, Trans. A.M.S., **103** (1962), 353–393.

[15] W.M. Goldman, Conformally flat manifolds with nilpotent holonomy and the uniformization problem for 3-manifolds, Trans. A.M.S., **278** (1983), 573–583.

[16] ———, Projective structures with Fuchsian holonomy, J. Differential Geom., **25** (1987), 297–326.

[17] W.M. Goldman and Y. Kamishima, Conformal automorphisms and closed conformally flat manifolds, to appear in Trans. A.M.S. (1990).

[18] ———, Topological rigidity of developing maps with applications to flat conformal structures, Contemporary Math., **74**, 199–204.

[19] M. Gromov, H. B. Lawson and W. Thurston, Hyperbolic 4-manifolds and conformally flat 3-manifolds, Publ. Math. I.H.E.S., **68**, 27–45.

[20] N. A. Gusevskiǐ and M. E. Kapovich, Conformal structures on three dimensional manifolds, Soviet Math. Dokl., **34** (1987), 314–318.

[21] P. Hartman, On isometries and on a theorem of Liouville, Math. Z., **69** (1958), 202–210.

[22] J. Hempel, "3-manifolds", Ann. Math. Studies, Princeton University Press, 1978.

[23] H. Hopf, Enden offener Räume und unendliche diskontinuierliche Gruppen, Comment. Math. Helv., **15** (1943), 27–32.

[24] D. Johnson and J. J. Milson, Deformation spaces associated to compact hyperbolic manifolds, Progress in Mathematics, **67** (1987), 48–106.

[25] Y. Kamishima, Conformally flat manifolds whose developing maps are not surjective, Trans. A.M.S., **294** (1986), 607–623.

[26] M. E. Kapovich, Some properties of developments of conformal structures on 3-manifolds, Soviet Math. Dokl., **35** (1987), 146–149.

[27] M.E. Kapovich, Flat conformal structures on 3-manifolds.

[28] ———, Deformation of group representations and uniformization of 3-manifolds, to appear in Proceedings of International Conference "Automorphic forms and related topics", Khabarovsk.

[29] ———, On absence of Sullivan's cusp finiteness theorem in higher dimensions, in "Abstracts of International Algebraic Conference Dedicated to the Memory of A.I.Maltsev", Nobosibirsk, 1989.

[30] ———, Deformation space of flat conformal spaces, to appear in Answers and Questions in General Topology.

[31] ———, On absence of Sullivan's finiteness theorem, Preprint, Khabarovsk.

[32] M.E. Kapovich and L. Potyagailo, On absence of Ahlfors' finiteness theorem for Kleinian groups in dimension 3, To appear in Topology and Applications.

[33] C. Kourouniotis, Deformations of hyperbolic structures on manifolds of several dimensions, Dissertations, King's College London.

[34] S.L. Krushkal, B. N. Apanasov and N. A. Gusevskiĭ, "Kleinian groups and uniformization in examples and problems", Transl. Math. Mono. A.M.S. vol 62, 1986.

[35] N.H. Kuiper, On conformally flat spaces in the large, Ann. Math., **50** (1949), 916–924.

[36] ———, On compact conformally Euclidean spaces of dimension > 2, Ann. Math., **52** (1950), 478–490.

[37] ———, On convex locally projective surfaces, in "Convegno Internationale di Geometria Differenziale", Rome, 1953.

[38] ———, Hyperbolic manifolds and tesselations, IHES Publ. Math., **68** (1989), 76–97.

[39] ———, Fairly symmetric hyperbolic manifolds, Preprint IHES.

[40] R.S. Kulkarni, Conformally flat manifolds, Proc. Nat. Acad. Sci. USA, **69** (1972), 2675–2676.

[41] ———, Conformal geometry in higher dimensions I, Bull. AMS, **81** (1975), 736–738.

[42] ———, On the principle of uniformization, J. Differential Geom., **13** (1978), 109–138.

[43] ———, Groups with domains of discontinuity, Math. Ann., **237** (1978), 253–272.

[44] ———, Conjugacy classes in $M(n)$, in "Conformal Geometry (ed. R.S. Kulkarni and U. Pinkall)", Aspects of Math., Vieweg.

[45] R.S. Kulkarni and U. Pinkall, Uniformization of geometric structures with application to conformal geometry, Springer LNM, **1209**, 190–209.

[46] J. Liouville, Extension au cas des trois dimensions de la questions du tracé geographic, in "Application de l'Anlyse à la Geometrie", Paris, 1850, pp. 609–616.

[47] B. Maskit, "Kleinian groups", Springer-Verlag, 1987.

[48] S. Matsumoto, Topological aspects of conformally flat manifolds, Proc. Japan Acad., **65** (1989), 231–234.

[49] R. Nevanlinna, On differentiable mappings, in "Analytic Functions", Princeton University, 1960, pp. 3–9.

[50] L.D. Potyagailo, Schottky groups and three dimensional conformal manifolds with trivial automorphism groups, Soviet Math. Dokl., **31** (1985), 509–512.

[51] ———, "Kleinian groups in the space which are isomorphic to fundamental groups of Haken manifolds", (in Russian), Soviet Math. Dokl., 1988.

[52] R. Schoen and S.T. Yau, Conformally flat manifolds, Kleinian groups and scalar curvatures, Invent. Math., **92** (1988), 47–71.

[53] A. Selberg, On discontinuous groups in higher dimensional symmetric spaces, in "Contribution to Function Theory, Bombey", Tata Institute, 1960, pp. 147–164.

[54] J.R. Stallings, On torsion free groups with infinitely many ends, Ann. Math., **88** (1968), 312–334.

[55] ———, "Group theory and three dimensional manifolds", Yale Math. Monograph no.4, Yale University Press.

[56] W. Thurston, "Geometry and topology of 3-manifolds", Princeton University Lecture Notes, 1978.

[57] J. Tits, Free subgroups in linear groups, J. Algebra, **20** (1972), 250–270.

[58] R. Miner, Spherical CR manifolds with amenable holonomy, International J. Math., **1** (1990), 479–501.

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Deformation Spaces on Geometric Structures

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0. Introduction

In this note we shall study geometric structures on smooth manifolds and deformation spaces. In 1981 Thurston gave a lecture on projective structures on surfaces in which he has established the following structure theorems (unpublished):

I. There is a canonical decomposition by convex hulls on a hyperbolic surface S which admits a (one dimensional complex) projective structure.
II. There is an isomorphism between the deformation space $\mathbf{CP}^1(S_g)$ and the product $\mathcal{T}(S_g) \times \mathcal{ML}(S_g)$.

Here $\mathcal{T}(S_g)$ is the Teichmüller space of a closed orientable surface S_g of genus $g \geqq 2$ and $\mathcal{ML}(S_g)$ is the space of measured laminations.
Since there was considerable interest in the argument of proof and the key idea seems to be generalized in higher dimension, we have decided to write down an exposition of the above structure theorems (I), (II).

(Complex) projective structure on surfaces is equivalent to conformally flat structure on surfaces when we identify $\mathbf{C}P^1 = S^2$ and

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$\mathrm{PSL}_2(C) = PO(3,1)^0$. We shall generalize I to conformally flat structure on manifolds in arbitrary dimensions.
In Chapter 1, we show that there is a canonical decomposition of a conformally flat manifold. The above Thurston correspondence will be proved in Chapter 2. There is no originality concerning the argument of Chapter 2 except for a certain generalization. It is nothing but our interpretation of Thurston's lecture. In Chapter 3 we shall describe various projective structures by using Kleinian groups. In Chapter 4 we review (G, X)-structures and examine the properties of limit sets of (G, X)-manifolds. The deformation space for (G, X)-structures will be defined more generally. As an application, we study the deformation spaces of S^1 invariant geometric structures in Chapter 5. Particularly we treat spherical CR structures and conformally flat structures as such geometric structures.

The authors have been informed from Professor William Goldman, and Professor Sadayoshi Kojima that Kulkarni-Pinkall also showed the existence of canonical stratification of conformally flat manifolds (cf. [34]).

We would like to thank Professor William Goldman for showing us his note of the Thurston's lecture. And we also thank the referee to pointing out our mistakes in earlier draft.

1. Canonical decomposition of conformally flat manifolds

A conformally flat structure on a smooth n-manifold is a maximal collection of charts modelled on the standard n-sphere S^n whose coordinate changes lie in the group $\mathrm{Conf}(S^n)$ of conformal transformations of S^n. The group $\mathrm{Conf}(S^n)$ is isomorphic to the Lorentz group $PO(n+1, 1)$. If a smooth n-manifold M admits a conformally flat structure, then by the monodromy argument there exists a developing pair (ρ, dev), where $\mathrm{dev} : \tilde{M} \to S^n$ is a conformal immersion and $\rho : \pi_1(M) \to \mathrm{Conf}(S^n)$ is a homomorphism such that $\mathrm{dev} \cdot \gamma = \rho(\gamma) \cdot \mathrm{dev}$ $(\gamma \in \pi_1(M))$. Here $\tilde{M}$ is the universal covering space of M and $\pi_1(M)$ is the fundamental group. The map dev is called a developing map and ρ is called a holonomy homomorphism, both unique up to an element of $\mathrm{Conf}(S^n)$. Remark that the term "conformal" means "in the category $(\mathrm{Conf}(S^n), S^n)$", which is different from the usual terminology when $\dim = 2$.

1.1. Maximal balls

Definition 1.1.1. Let $\mathbf{H}^{n+1}$ be a (real) hyperbolic space with boundary $\partial\mathbf{H}^{n+1} = S^n$. The n dimensional sphere S^n is a conformally flat manifold by stereographic projection. A geometric k-sphere S^k is the boundary of a $(k+1)$ dimensional totally geodesic subspace of $\mathbf{H}^{n+1}$. A geometric ball is a domain of S^n bounded by a codimension one geometric sphere.

Definition 1.1.2. Let N be a conformally flat manifold. Given a conformal immersion $f : N \to S^n$, a geometric ball of N is an open subset U such that $f : U \to f(U)$ is a diffeomorphism onto a geometric ball of S^n. Then the set of geometric balls of N is partially ordered by inclusions. We call a maximal geometric ball a *maximal ball.*

The following is a generalization of the proposition due to Thurston.

Proposition 1.1.3. *Let $f : N \to S^n$ be a conformal immersion. Then either one of the following is true.*

(i) *N is conformally equivalent to the standard sphere S^n, a euclidean space $\mathbf{R}^n$, or a hyperbolic space $\mathbf{H}^n$.*
(ii) *Every point of N lies in a proper maximal ball.*

Proof. Suppose that (ii) is false. A point of N lies in some geometric ball but not in a maximal ball. And so there exists a sequence $U_1 \subset U_2 \subset \cdots \subset U_i \subset \cdots$ of geometric balls containing x. The union $W = \bigcup_{i=1}^{\infty} U_i$ is not a geometric ball. As f is injective on each U_i, f must map W isomorphically onto a euclidean space $\mathbf{R}^n(\approx S^n - \{\infty\})$. If $N \neq W$ then f maps the closure $\bar{W}$ isomorphically onto S^n. And thus it follows that $N = \bar{W}$. This proves (i). If some maximal ball U is not proper, then $N = U$. Since the image $f(U)$ is a geometric ball of S^n, it is conformally equivalent to a hyperbolic space $\mathbf{H}^n$. Q.E.D.

Let $f : N \to S^n$ be a conformal immersion. The spherical metric on S^n defines a Riemannian metric on N so that f is a local isometry. Let $\bar{N}$ be the metric completion of N. It is easy to see that f extends to a map $\bar{f} : \bar{N} \to S^n$. Recall that for a maximal ball U, $f : U \to B$ is a diffeomorphism onto an n dimensional ball B. Note that the closure of U in N is not compact by maximality. However we have

$$\bar{f} : \bar{U} \to \bar{B} \tag{1.1.4}$$

is a homeomorphism onto a closed ball of S^n. Denote by ∂U the boundary of $\bar{U}$ in $\bar{N}$. Let M^n be a closed conformally flat manifold and $(\phi, \mathrm{dev}) : (\pi_1(M), \tilde{M}) \to (\mathrm{Conf}(S^n), S^n)$ be the developing pair as in 1.1 of Chapter 1. Recall that dev is a conformal immersion. We have the following application.

Lemma 1.1.5. *Let M^n be a closed conformally flat manifold and $\mathcal{F}$ be the set of all maximal balls of the universal covering space $\tilde{M}$. If the boundaries of all the elements of $\mathcal{F}$ meet at a common point, then the developing map is a covering map.*

Proof. We put $\tilde{M} = N$, $\pi_1(M) = \Gamma$, $\mathrm{dev} = f$. Let $\bar{N}$ be the metric completion of N and $\bar{f} : \bar{N} \to S^n$ be the map extending f. Let x be a common point of ∂U for all $U \in \mathcal{F}$. Note that $x \notin N$, otherwise x would be an interior point of some maximal ball. Put $\bar{f}(x) = \infty$. We prove that f misses the point $\{\infty\}$. Suppose that there is $y \in N$ such that $f(y) = \infty$. The point y lies in some maximal ball U. Since $\bar{f} : \bar{U} \to \bar{B}$ is a homeomorphism and $x \in \partial U$, it is impossible. Therefore, as the developing map misses a point, it is a covering map onto its image (cf. [25],[34]). Q.E.D.

Proposition 1.1.6. *Let M^n be a closed conformally flat manifold. Suppose that $\mathcal{F}$ consists of finite elements (possibly empty) or the boundaries of all the elements of $\mathcal{F}$ meet at a finite number of common points. Then M is conformally equivalent to a spherical space form, a Hopf manifold, a euclidean space form, or a hyperbolic space form.*

Proof. Suppose the latter case. By the above lemma we have that $\mathrm{dev} : \tilde{M} \to \mathrm{dev}(\tilde{M}) \subset S^n - \{\infty\}$ is a covering map. Since the common points are finite, the fundamental group $\pi_1(M)$ has a subgroup π' of finite index those elements of which leave these points fixed. And so the holonomy subgroup $\phi(\pi')$ belongs to the similarity subgroup of $\mathrm{Conf}(S^n)$ which is the stabilizer at $\{\infty\}$ in S^n. Therefore dev is a homeomorphism of either $\mathbf{R}^n$ or $S^n - \{0, \infty\}$ (cf. [14]). In our case dev is a homeomorphism onto $S^n - \{0, \infty\}$ or M is a Hopf manifold.
For the remaining case, if $\mathcal{F}$ is empty then M is either a spherical space form or a euclidean space form. Suppose that $\mathcal{F}$ consists of finite elements. Then $\tilde{M}$ is covered by the union of those finite maximal balls U. It follows that the number of $\mathrm{dev}^{-1}(x)$ is finite for each $x \in S^n$. It is easy to see that $\mathrm{dev} : \tilde{M} \to \mathrm{dev}(\tilde{M})$ is a finite covering map. Passing to a subgroup of finite index in π if necessary we can assume that π leaves each element U of $\mathcal{F}$ invariant. Let Γ be the holonomy group to

π. Since the image of dev misses more than one point, we know that $\operatorname{dev}(\tilde{M}) \subset S^n - L(\Gamma)$. On the other hand, note that Γ is discrete because $\operatorname{dev} : U \to B$ is a homeomorphism. This implies that $L(\Gamma) \subset S^{n-1}$ where we view $\partial B = S^{n-1}$. Moreover Γ acts properly discontinuously on $S^n - L(\Gamma)$ by (1.1.5). In particular it follows that either $\operatorname{dev}(\tilde{M}) = \mathbf{H}^n$ or $\operatorname{dev}(\tilde{M}) = S^n - L(\Gamma)$. The former case implies that M is a hyperbolic space form. In the latter case the set of all maximal balls in $S^n - L(\Gamma)$ must be finite. However if we note that $S^{n-1} - L(\Gamma) \neq \emptyset$, it is easy to see that for any point $x \in S^{n-1} - L(\Gamma)$ there are infinitely many maximal balls containing x. This is impossible in this case. Q.E.D.

Note. *If M is a Hopf manifold, every maximal ball of $\tilde{M}$ meets at exactly two points.*

1.2. Decomposition of conformally flat manifolds

Let $f : N \to S^n$ be a conformal immersion and $\mathcal{F}$ the set of all maximal balls of N. Let U be an element of $\mathcal{F}$. Put

$$U_\infty = \bar{U} - N.$$

Then ∂U decomposes into a disjoint union of $\partial U \cap N$ and U_∞.

Definition 1.2.1. The set U_∞ is called an ideal set of U. The ideal set is a closed subset of $\bar{N}$. (For example, if $\bar{U}$ is a closed disk, then U_∞ may look like a Cantor set and $\partial U \cap N$ is a disjoint union of intervals. Since U has the natural Poincaré metric, U_∞ corresponds to a closed subset of points at infinity.)

Recall from (1.1.4) that $\bar{U}$ is conformally equivalent to a closed ball $\bar{B}$. We may form the convex hull $C(U_\infty)$ for U_∞ inside U. (Note that this can be defined when U_∞ contains more than one point.) Let $\mathbf{D}^{m+1} = \mathbf{H}^{m+1} \cup S^m$ be the compactification of a hyperbolic space $\mathbf{H}^{m+1}$. If K is a closed subset of S^m then we denote by $\mathcal{H}(K)$ the convex hull in $\mathbf{H}^{m+1}$. It is easy to check the following.

Lemma 1.2.2. *Let $P \subset \mathbf{H}^{m+1}$ be a totally geodesic hyperplane such that either one of the components of $S^m - \partial P$ does not meet K. Then $\mathcal{H}(K \cap \partial P) = \mathcal{H}(K) \cap P$.*

Using this lemma we define pleats on the boundary $\partial C(U_\infty)$ of $C(U_\infty)$. Given a closed convex set $\mathbf{C}$ of $\mathbf{D}^{n+1}$ there is a canonical retraction $\Phi_{\mathbf{C}} : \mathbf{D}^{n+1} \to \mathbf{C}$ called a closest point mapping. Recall that if $x \in S^n - \mathbf{C}$ there is a horoball centered at x disjoint from $\mathbf{C}$. Then

$\Phi_{\mathbf{C}}(x)$ is the point of the first contact when we increase the radius of this horoball continuously until it touches $\mathbf{C}$. See [8] for details. Denote by fU^c the complement of fU in S^n. If $C = \bar{\mathcal{H}}(fU^c)$ is the closure of the convex hull $\mathcal{H}(fU^c)$ in $\mathbf{H}^{n+1}$, then we have a map $\Phi_C : \mathbf{D}^{n+1} \to C$. Note that $\mathcal{H}(\partial(fU^c))$ is a totally geodesic subspace of $\mathbf{H}^{n+1}$, so we set $\mathcal{H}(\partial(fU^c)) = \mathbf{H}^n$. Since $\bar{\mathcal{H}}(\partial(fU^c)) \subset C$, the above map restricts to a map

$$\Phi_U : fU(= B) \to \mathbf{H}^n.$$

Note that it is a conformal diffeomorphism.

Definition 1.2.3. For a totally geodesic hyperplane $P \subset \mathbf{H}^n$ we call also $f^{-1}\Phi_U^{-1}(P)$ a totally geodesic hyperplane in U. Put $f^{-1}\Phi_U^{-1}(P) = Q$. If $P^1 < P^2 < \cdots < P^{n-2} < P^{n-1} = P$ is a chain of totally geodesic subspaces, then there exists a k dimensional totally geodesic subspace Q^k of U and similarly a chain $Q^1 < Q^2 < \cdots < Q^{n-2} < Q^{n-1} = Q$ and so on.

Since $\Phi_U(C(\bar{f}U_\infty \cap \partial P)) = \mathcal{H}(\bar{f}U_\infty \cap \partial P)$ and $\mathcal{H}(\bar{f}U_\infty \cap \partial P) = \mathcal{H}(\bar{f}U_\infty \cap P)$ by the above lemma, we have that $C(\bar{f}U_\infty \cap \partial P) = fC(U_\infty) \cap \Phi_U^{-1}(P)$. Noting that $fC(U_\infty \cap \partial Q) = C(\bar{f}U_\infty \cap \partial P)$, it is easy to check that $C(U_\infty \cap \partial Q) = C(U_\infty) \cap Q$. An iteration of this argument yields that

$$C(U_\infty \cap \partial Q^k) = C(U_\infty) \cap Q^k \ (k = 1, \cdots, n-1). \tag{1.2.4}$$

Definition 1.2.5. Let $Q^1 < Q^2 < \cdots < Q^{n-2} < Q^{n-1} = Q$ be a chain of totally geodesic subspaces in a maximal ball U. Suppose that either one of $\partial U - \partial Q$ does not meet U_∞. If $\operatorname{Int} C(U_\infty) \neq \emptyset$, (equivalently $\operatorname{Int} C(U_\infty) \cap U$ is open in U) and $C(U_\infty) \cap Q^k$ contains an open subset in Q^k then by (1.2.4) that $C(U_\infty \cap \partial Q^k)$ is said to be a k dimensional pleat of the boundary $\partial C(U_\infty)$ $(k = 1, 2, \cdots, n-1)$.
Put $\Lambda_U = \partial C(U_\infty)$. Choosing all possible geodesic hyperplanes Q in U and passing to all chains of geodesic subspaces $\{Q^k\}$, we obtain all pleats in Λ_U. The set Λ_U is composed of all possible pleats in dimension less than or equal to $n-1$. In the case that $\operatorname{Int} C(U_\infty) = \emptyset$, there exists a totally geodesic subspace Q' such that $C(U_\infty) = C(U_\infty) \cap Q'$ is open in Q'. We say that $C(U_\infty)$ is an m dimensional pleat if $\dim Q' = m$. Inductively we can define pleats of $\partial C(U_\infty)$ unless $m = 1$. Note that $C(U_\infty)$ is a one dimensional pleat if and only if U_∞ consists of a pair of points.

Let $f : N \to S^n$ be a conformal immersion as before. Using the spherical metric of S^n, N admits a Riemannian metric such that f is a local isometry. Recall that $\bar{N}$ is a metric completion and $\bar{f} : \bar{N} \to S^n$ is a map extending f. Choose a point x in N. Let $W(x)$ be the union of all maximal balls containing x.

Lemma 1.2.6. *If $\overline{W(x)}$ is the closure of $W(x)$ in $\bar{N}$ then $\overline{W(x)}$ is compact.*

Proof. Let ρ (resp. ρ_0) be the distance function of $\bar{N}$ (resp. S^n). Since f maps $W(x)$ injectively, f is a homeomorphism of $W(x)$ onto its image Ω. Let $\{p_i\}$ be an arbitrary sequence of $\overline{W(x)}$. Choose a sequence $\{q_i\}$ in $W(x)$ such that $\rho(p_i, q_i) < 1/i$. Since $\bar{\Omega}$ is compact, the sequence $\{f(q_i)\}$ has an accumulation point and so $\{f(q_i)\}$ is Cauchy. Given $\varepsilon > 0$, choose δ such that $0 < \delta < \varepsilon$. Suppose that $\rho_0(f(q_i), f(q_j)) < \delta$. If δ is sufficiently small, then there exists a maximal ball in Ω containing the points $f(q_i)$,$f(q_j)$. And so Ω contains a minimizing geodesic between $f(q_i)$ and $f(q_j)$. It implies that $\rho_0(f(q_i), f(q_j)) = \rho(q_i, q_j)$. In particular the sequence $\{q_i\}$ is Cauchy. Since $\bar{N}$ is complete, the sequence $\{q_i\}$ has a limit point q. And thus we have $\lim p_i = q$. Hence $\overline{W(x)}$ is compact.

Q.E.D.

Theorem 1.2.7. *Let $f : N \to S^n$ be a conformal immersion and $\mathcal{F}$ the nonempty set of all maximal balls. Then every point of N lies in the convex hull $C(U_\infty)$ for a unique element $U \in \mathcal{F}$.*

Proof. Choose a point x in N and let $W(x)$ be as above. Put $W(x)_\infty = \overline{W(x)} - N$. Note that it contains more than one point for otherwise there are no maximal balls containing x. Since $W(x)_\infty$ is a closed subset of $\overline{W(x)}$, $W(x)_\infty$ is compact by the above lemma. And so $\bar{f}(W(x)_\infty)$ is a closed subset of S^n. If $\mathcal{H} = \mathcal{H}(\bar{f}(W(x)_\infty))$ is the convex hull in $\mathbf{H}^{n+1}$, then we have a closest point mapping $\Phi_{\mathcal{H}} : \mathbf{D}^{n+1} \to \mathcal{H}$. Now there is a unique totally geodesic hyperplane P through $\Phi_{\mathcal{H}}(f(x))$ perpendicular to the geodesic from $f(x)$ to $\Phi_{\mathcal{H}}(f(x))$. Then we have from (1.2.2) that

$$\Phi_{\mathcal{H}}(f(x)) \in \mathcal{H}(\bar{f}(W(x)_\infty)) \cap P = \mathcal{H}(\bar{f}(W(x)_\infty) \cap \partial P).$$

Let B be a geometric ball containing $f(x)$ such that $\partial B = \partial P$.The set $U = f^{-1}(B)$ is a maximal ball containing x because $B \subset f(W(x))$. As $\bar{f}\partial U = \partial f U = \partial P$, it follows that

$$\begin{aligned}\mathcal{H}(\bar{f}(W(x)_\infty) \cap \partial P) &= \mathcal{H}(\bar{f}(W(x)_\infty \cap \partial U))\\ &= \mathcal{H}(\bar{f}(U_\infty)).\end{aligned}$$

Since $\Phi_U(fC(U_\infty)) = \Phi_U(C(\bar{f}U_\infty)) = \mathcal{H}(\bar{f}U_\infty)$ and $\Phi_{\mathcal{H}}|fU = \Phi_U$, it follows that $\Phi_U(f(x)) = \Phi_{\mathcal{H}}(f(x)) \in \Phi_U(f(C(U_\infty)))$ and hence $x \in C(U_\infty)$. The proof of the uniqueness is the converse of the above argument. Q.E.D.

Corollary 1.2.8.

(1) *The family* $\{C(U_\infty);\ U \in \mathcal{F}\}$ *consists of disjoint subsets.*
(2) *The set* $\cup\Lambda_U$ *is closed in* N.

Proof. There exists a unique element of $\mathcal{F}$ such that each point of N lies in its convex hull by the above theorem. This implies that $\{C(U_\infty);\ U \in \mathcal{F}\}$ are disjoint. For (2), it suffices to show that if a sequence $\{x_i\} \in \Lambda_{U_i}$ $(U_i \in \mathcal{F})$ converges to $x \in N$, then there exists an element $U \in \mathcal{F}$ such that $x \in \Lambda_U$. Recall that $\Lambda_U = C(U_\infty) - \operatorname{int} C(U_\infty)$. There exists $U \in \mathcal{F}$ such that $x \in C(U_\infty)$. If x is not contained in Λ_U, then $x \in \operatorname{int} C(U_\infty)$. It follows that for sufficiently large i, Λ_{U_i} meets with $C(U_\infty)$. This contradicts that $\{C(U_\infty);\ U \in \mathcal{F}\}$ are disjoint.
Q.E.D.

Let M^n be a closed conformally flat manifold and $\tilde{\mathcal{F}}$ the set of all maximal balls of the universal covering space $\tilde{M}$. It is obvious that $\tilde{\mathcal{F}}$, the family $\{C(U_\infty)\}$ and the set $\cup\Lambda_U$ are invariant under the fundamental group π.

Corollary 1.2.9. *Let* M^n *be a closed conformally flat manifold. Suppose that* $\tilde{\mathcal{F}}$ *is not empty. Then the universal covering space* $\tilde{M}$ *supports a* π *invariant canonical decomposition* $\{C(U_\infty); U \in \mathcal{F}\}$.

2. Thurston parametrization of projective structure on surfaces

In this chapter we shall prove the Thurston isomorphism II stated in Introduction. Recall that (complex) projective structure on surfaces is equivalent to conformally flat structure on surfaces when we identify $(S^2, \mathrm{Conf}(S^2)^0)$ with $(\mathbf{CP}^1, \mathrm{PSL}_2(C))$. As before, given a projective structure on a surface S we have a developing pair $(\phi, \mathrm{dev}) : (\pi_1(S), \tilde{S}) \to (\mathrm{PSL}_2(C), \mathbf{C}P^1)$ up to conjugation by elements of $\mathrm{PSL}_2(C)$.

2.1. Deformation spaces on surfaces

Suppose that S_g is a closed orientable surface of genus $g \geqq 1$. For the brevity we set $S = S_g$ and $\Gamma = \pi_1(S_g)$. A surface Σ is a hyperbolic surface if the universal covering space is conformally equivalent to a hyperbolic plane $\mathbf{H}^2$. Consider the subspace $\Omega^+(S)$ (cf. 4.3.4);

$$\Omega^+(S) = \{(\phi, \mathrm{dev}) : (\Gamma, \tilde{S}) \to (\mathrm{PSL}_2(\mathbf{C}), \mathbf{CP}^1)\}/\operatorname{Diff}^0(S),$$

where dev are orientation-preserving immersions.
The topology on $\Omega^+(S)$ is given by the following subbasis:

(1) $\mathcal{N}(U) = U/\sim$ where U is an open subset in $\operatorname{Map}(\tilde{S}, \mathbf{CP}^1)$ with the compact-open topology.
(2) $\mathcal{N}(K) = \{\mathrm{dev} \in \operatorname{Map}(\tilde{S}, \mathbf{CP}^1) |\ \mathrm{dev}\,|K$ is an embedding for a compact subset $K \subset \tilde{S}\}/\sim$.

Put

$$\mathbf{CP}^1(S)^+ = \mathrm{PSL}_2(\mathbf{C}) \setminus \Omega^+(S).$$

Definition 2.1.1. Let S_g be a hyperbolic surface. The space $\mathbf{CP}^1(S_g)^+$ is called the deformation space of projective structures or $\mathbf{CP}^1$-structures on S_g. $\mathcal{T}(S_g)$ is the usual Teichmüller space.

Thurston has introduced the notion of geodesic laminations on surfaces. (Cf. [9],[40],[8].) Namely, a geodesic lamination on a hyperbolic surface Σ is a closed subset consisting of a disjoint union of simple geodesics. Let Λ be a geodesic lamination on Σ. By a transversal we mean an embedding $\ell : [0,1] \to \Sigma$ such that at each t where $\ell(t) \in \Lambda$ the map ℓ is transverse to the leaf through $\ell(t)$.
A transverse measure on Λ is a function μ which assigns to each transversal ℓ a Radon measure $\mu(\ell)$ on $[0,1]$ supported by $\{t \in [0,1] \mid \ell(t) \in \Lambda\}$ which is compatible under the canonical homeomorphisms between nearby transversals. We call the pair (Λ, μ) a measured geodesic lamination of Σ.

Definition 2.1.2. $\mathcal{ML}(\Sigma)$ is the space of measured geodesic laminations on Σ, equipped with the weak $*$ topology.

If f is a homeomorphism of a closed hyperbolic surface Σ onto Σ' then f induces a homeomorphism $f : \mathcal{ML}(\Sigma) \to \mathcal{ML}(\Sigma')$. (See [8],[9].)

Let S_g be a hyperbolic surface. Note that $\mathcal{T}(S_g)$ is homeomorphic to $\mathbf{R}^{6g-6}$. In this case the space of measured laminations $\mathcal{ML}(S_g)$ is also homeomorphic to the real vector space of dimension $6g-6$. Moreover, $\mathbf{CP}^1(S_g)^+$ can be identified with the cotangent bundle of $\mathcal{T}(S_g)$. (Compare [5].) In contrast to this identification we have a new parametrization on $\mathbf{CP}^1(S_g)^+$.

Theorem (Thurston). *Let S_g be a hyperbolic surface. Then there exists a homeomorphism*

$$\Theta : \mathbf{CP}^1(S_g)^+ \longrightarrow \mathcal{T}(S_g) \times \mathcal{ML}(S_g).$$

See [16],[17],[15],[13] for the related topics. The rest of this section is devoted to the proof of this theorem.

2.2. Locally convex pleated maps

Let $f : N \to \mathbf{CP}^1$ be a conformal immersion. We have shown in Theorem 1.2.7 that every point x of N lies in the convex hull $C(U_\infty)$ of a unique maximal ball U. Let $\mathcal{F}$ be the set of maximal balls. For each U there is a closest point mapping $\Phi_U : f(U) \longrightarrow \mathbf{H}^2(\subset \mathbf{H}^3)$. (See Section 1.) Set $\Psi(x) = \Phi_U(f(x))$ if $x \in C(U_\infty)$. By uniqueness it defines a well defined map

$$\Psi : N \longrightarrow \mathbf{H}^3. \tag{2.2.1}$$

It is obvious that Ψ is a continuous map. Note that in our case each $C(U_\infty)$ is either a region or a (one dimensional) pleat, the image $\Psi(C(U_\infty))$ lies on a geodesic or in a totally geodesic hyperplane of $\mathbf{H}^3$.

Definition 2.2.2. Given an arbitrary conformal immersion $f : N \to \mathbf{CP}^1$, we have passed from it to a map $\Psi : N \longrightarrow \mathbf{H}^3$. The map Ψ is called a pleated map.

2.3. Assignment of $\mathbf{CP}^1(S_g)^+$ to $\mathcal{T}(S_g)$

In general a pleated map Ψ is not locally injective. By definition (2.2.2), Ψ is injective on each $C(U_\infty)$ for $U \in \mathcal{F}$. On the other hand, we consider a point $x \in N$ such that there is a sequence $\{x_i\}$ converging to x such that $\Psi(x_i) = \Psi(x)$. Since each x_i lies in a distinct $C(U^i_\infty)$ and Ψ is injective on $C(U^i_\infty)$ for sufficiently large i, all $C(U^i_\infty)$ have the same dimension equal to 1. The map Ψ fails to be injective on the union of those $C(U^i_\infty)$. Each ideal set U^i_∞ is a locally constant pair of points in $\bar{N}$.

Definition 2.3.1. Denote by $\mathcal{B}$ the set of those $C(U^i_\infty)$ on which Ψ fails to be injective.

Ψ is locally injective on $N - \mathcal{B}$. Let N' be the space obtained from $N - \mathcal{B}$ by identifying the boundaries of each component of $N - \mathcal{B}$ which have the same Ψ image. Let $\eta : N \to N'$ be the resulting collapse map which is clearly a homotopy equivalence. Since each component of $N - \mathcal{B}$ is isometric to a hyperbolic region with boundary composed of complete geodesics, the image N' supports a complete hyperbolic metric. Moreover if g is an conformal automorphism of N then it leaves $N - \mathcal{B}$ invariant. When g stabilizes a component, it acts as isometries with respect to a hyperbolic metric of that component. Otherwise g translates one component to another component preserving boundary geodesics. Therefore the map g induces a hyperbolic isometry $\theta(g) \in \mathrm{PSL}_2(\mathbf{R})$ on $\mathbf{H}^2$ where we put $N' = \mathbf{H}^2$. The map η satisfies that $\eta \circ g = \theta(g) \circ \eta$. Now given a projective structure (ϕ, dev) in $\mathbf{CP}^1(S_g)^+$, we apply the above argument to $(\phi, \mathrm{dev}) : (\Gamma, \tilde{S}) \to (\mathrm{PSL}_2(\mathbf{C}), \mathbf{CP}^1)$. Then it induces an equivariant homotopy equivalence $(\theta, \eta) : (\Gamma, \tilde{S}) \to (\mathrm{PSL}_2(\mathbf{R}), \mathbf{H}^2)$. The map η induces a homotopy equivalence of S onto $\mathbf{H}^2/\theta(\Gamma)$. Within the homotopy class of η, there is a diffeomorphism $h : S \to \mathbf{H}^2/\theta(\Gamma)$ up to an element of $\mathrm{Diff}^0(S)$. Hence a projective structure (ϕ, dev) defines a well defined element $[S, h]$ of $\mathcal{T}(S_g)$.

2.4. Canonical measure on circular lamination

Let $f : N \to \mathbf{CP}^1$ be as before. Recall from (1.3.2) that the subset $\tilde{\Lambda}_1 = \cup\{\Lambda_U |\ U \in \mathcal{F}\}$ is closed and consists of a disjoint union of (one dimensional) pleats. In order to define the canonical measure $\tilde{\mu}_1(\ell)$ on a transversal ℓ for $\tilde{\Lambda}_1$, it suffices to specify the nondecreasing function $\varphi(t) = \int_{[0,t]} \tilde{\mu}_1(\ell)dt$ whose derivative is equal to $\tilde{\mu}_1(\ell)$. For each $t \in [0,1]$, let $U^t \in \mathcal{F}$ be a unique maximal ball such that $\ell(t) \in C(U^t_\infty)$. If $s, t \in [0,1]$ are sufficiently close, the balls U^s and U^t must intersect. Let $\Theta(s,t)$ denote the *dihedral* angle of intersection of the circles ∂U^s, ∂U^t, measured inside one ball and outside the other. The function $\varphi(t)$ is then defined as the infimum of all Θ-sum

$$\Theta(0, t_1) + \Theta(t_1, t_2) + \cdots + \Theta(t_n, t)$$

over all subdivisions $0 < t_1 < t_2 < \cdots < t_n < t$ of $[0, t]$. The following is the elementary calculation of the trigonometry.

Lemma 2.4.1. *If $r < s < t$ are sufficiently close, then $\Theta(r, s)$, $\Theta(s, t)$ and $\Theta(r, t)$ are defined and $\Theta(r, s) + \Theta(s, t) \leqq \Theta(r, t)$.*

With this lemma a nondecreasing function can be defined as

$$\varphi(t) = \lim \sum \Theta(0, t_1) + \Theta(t_1, t_2) + \cdots + \Theta(t_n, t)$$

where $\sum$ runs over all subdivisions of $[0, t]$ and lim is taken as the mesh of subdivisions goes to zero. Therefore the derivative $\varphi\prime = \tilde{\mu}_1(\ell)$ is a measure on a transversal ℓ for $\tilde{\Lambda}_1$. Finally the compatiblity of the measure on the various transversals is deduced from the following remark. The leaf through $\ell(t)$ determines the ball U^t and the measure on ℓ is determined by the angles made by the ∂U^t. Thus corresponding transversals determine the same measure.

Let $\Psi : N \to \mathbf{H}^3$ be a pleated map for an immersion $f : N \to \mathbf{CP}^1$. Let $\mathcal{B}$ be the set as in (2.3.1). It follows that $\eta(\mathcal{B}) = \eta(\tilde{\Lambda}_1)$. If we put $\eta(\tilde{\Lambda}_1) = \tilde{\Lambda}_2$, then $\tilde{\Lambda}_2$ is a geodesic lamination on N' (=$\mathbf{H}^2$). Moreover let ℓ be a transversal to $\tilde{\Lambda}_2$, then $\eta^{-1}(\ell)$ is also a transversal to $\tilde{\Lambda}_1$. Set $\tilde{\mu}_2(\ell) = \tilde{\mu}_1(\eta^{-1}(\ell))$. We have a measure $\tilde{\mu}_2$ on $\tilde{\Lambda}_2$. And thus $(\tilde{\Lambda}_2, \tilde{\mu}_2)$ is a measured geodesic lamination on N'. As before suppose that $(\phi, \mathrm{dev}) : (\Gamma, \tilde{S}) \to (\mathrm{PSL}_2(\mathbf{C}), \mathbf{CP}^1)$ is a developing pair. The above argument implies that there is a measured geodesic lamination $(\tilde{\Lambda}_2, \tilde{\mu}_2)$ over $(\theta(\Gamma), \mathbf{H}^2)$. It is easy to see that $(\tilde{\Lambda}_2, \tilde{\mu}_2)$ is invariant under the group $\theta(\Gamma)$. That is, $\theta(\gamma)(\tilde{\Lambda}_2) = (\tilde{\Lambda}_2)$ and $\tilde{\mu}_2(\theta(\gamma)(\ell)) = \tilde{\mu}_2(\ell)$. It induces a measured geodesic lamination (Λ_2, μ_2) on $\mathbf{H}^2/\theta(\Gamma)$. There is a diffeomorphism $h : S \to \mathbf{H}^2/\theta(\Gamma)$ as above. Then we have a geodesic measured lamination (Λ, μ) on S such that $h(\Lambda) = \Lambda_2$ and $\mu(\ell) = \mu_2(h(\ell))$. Hence it defines an element $(\Lambda, \mu) \in \mathcal{ML}(S)$.

2.5. Thurston correspondence

We have a well defined map $\Theta : \mathbf{CP}^1(S_g) \to \mathcal{T}(S_g) \times \mathcal{ML}(S_g)$,

$$\Theta((\phi, \mathrm{dev})) = ([S, h], (\Lambda, \mu)).$$

It is easy to see that Θ is injective, because given two projective structures which have the same image in $\mathcal{T}(S_g) \times \mathcal{ML}(S_g)$. The coincidence on the first summand implies that each developing map coincides outside each $\mathcal{B}$. But the second summand measures the difference on $\mathcal{B}$ and so two developing maps coincide on the whole $\tilde{S}$.

Let $\mathcal{ML}(S_g, \mathcal{S})$ be the subspace of $\mathcal{ML}(S_g)$ such that every lamination consists of compact leaves. If $\overline{\mathcal{ML}_h(M, \mathcal{S}^{n-1})}$ is the closure in $\mathcal{ML}(S_g)$, then it is known that $\overline{\mathcal{ML}_h(S_g, \mathcal{S})} = \mathcal{ML}(S_g)$. We show that there is a map

$$\mathfrak{S} : \mathcal{T}(S_g) \times \mathcal{ML}(S_g) \longrightarrow \mathbf{CP}^1(S_g) \tag{2.5.1}$$

such that $\Theta \cdot \mathfrak{S} = \mathrm{id}$.

To prove this we need some preliminaries.

Definition 2.5.2 (cf. [16]). Let $\alpha > 0$ be any mumber and $W_\alpha = \{z \in \mathbf{C} | 0 \leqq \mathrm{Im}\, z \leqq \alpha\pi\}$.

Let s be the stereographic projection which maps $\mathbf{C}^*$ onto $S^2 - \{\infty\}$. Then we define a map $\xi : \mathbf{C} \to S^2$ to be the exponential map $\exp : \mathbf{C} \to \mathbf{C}^*$ followed by s.
Put $\xi(W_\alpha) = C_\alpha$. Both W_α and C_α are conformally flat manifolds with boundary. We call W_α an α-pile and C_α an α-crescent.

Proof of (2.5.1). Let $([S,h],(\Lambda,\mu))$ be a representative element of $\mathcal{T}(S_g) \times \mathcal{ML}(S_g)$. We suppose first that $(\Lambda,\mu) \in \mathcal{ML}(S_g,\mathcal{S})$. The map $(\theta,\tilde{h})$ maps a Γ invariant measured geodesic lamination $(\tilde{\Lambda},\tilde{\mu})$ onto a $\theta(\Gamma)$ invariant measured geodesic lamination $(\tilde{\Lambda}',\tilde{\mu}')$. The map $\tilde{h}$ is a homeomorphism of $\tilde{S}$ onto $\mathbf{H}^2$ where $\mathbf{H}^2$ is viewed as the upper hemisphere of S^2. Cut $\mathbf{H}^2$ along $\tilde{\Lambda}'$ and then insert the crescents C_α and glue them along the boundary components. Here these angles α come from those of the measure $\tilde{\mu}'$. Similarly cut $\tilde{S}$ along $\tilde{\Lambda}$ and insert the piles W_α and then glue along the corresponding boundary components by the map $\tilde{h}^{-1} \circ \xi$. The resulting manifold $\tilde{S}'$ is invariant under an action of Γ and thus the orbit space is still homeomorphic to S. Since both $\tilde{h}$ and ξ are projective immersions, combined with these maps, we have a well defined projective immersion $\mathrm{dev} : \tilde{S} \to S^2 = \mathbf{CP}^1$ and since the group Γ acts as projective transformations with respect to this structure on S', there is a holonomy homomorphism ϕ. If we set $\mathfrak{S}([S,h],(\Lambda,\mu)) = (\phi,\mathrm{dev})$, then the map is well defined on $\mathcal{T}(S_g) \times \mathcal{ML}(S_g,\mathcal{S})$ such that $\Theta \cdot \mathfrak{S} = \mathrm{id}$ on $\mathcal{T}(S_g)\times\mathcal{ML}(S_g,\mathcal{S})$. For an element $(\Lambda,\mu) \in \mathcal{ML}(S_g)$ there is a sequence $\{(\Lambda_i,\mu_i)\} \in \mathcal{ML}(S_g,\mathcal{S})$ that converges to (Λ,μ). Let $[S,h]$ be an arbitrary element of $\mathcal{T}(S_g)$ and fix it once. The map $\mathfrak{S}$ maps $([S,h],(\Lambda_i,\mu_i))$ to a sequence of projective structures $\{(\phi_i,\mathrm{dev}_i)\}$. Recalling the topology of $\mathbf{CP}^1(S_g)$ from (2.1) and by the fact that each dev_i coincides with the map $\tilde{h}$ outside $\tilde{\Lambda}_i$, the sequence of developing maps $\{\mathrm{dev}_i\}$ converges to a map on each compact set of $\tilde{S}$. And so it is easy to see that it converges to a map $\mathrm{dev} : \tilde{S} \to S^2 = \mathbf{CP}^1$ which is obviously a projective immersion. The projective immersion dev determines a holonomy homomorphism ϕ up to conjugation. Setting $\mathfrak{S}([S,h],(\Lambda,\mu)) = (\phi,\mathrm{dev})$, we obtain a continuous map $\mathfrak{S} : \mathcal{T}(S_g) \times \mathcal{ML}(S_g) \longrightarrow \mathbf{CP}^1(S_g)$ such that $\Theta \cdot \mathfrak{S} = \mathrm{id}$. Q.E.D.

2.6. Modular space of projective structures

Recall that $\mathbf{CP}^1(S)^+ = \mathrm{PSL}_2\,\mathbf{C} \setminus \Omega(S)^+ / \mathrm{Diff}^0(S)$. Then the space $\mathcal{M}\mathbf{CP}^1(S)^+ = \mathrm{PSL}_2\,\mathbf{C} \setminus \Omega(S)^+ / \mathrm{Diff}^+(S)$ is called the modular space of projective structures. On the other hand, the Teichmüller space $\mathcal{T}(S)$ is defined alternately to be $\mathbf{R}(\Gamma, \mathrm{PSL}_2\,\mathbf{R}) / \mathrm{PGL}_2\,\mathbf{R}$. And so $\mathcal{T}(S)$ is identified with the quotient space of sense-preserving discrete faithful representations, $\mathcal{T}(\Gamma) = \mathbf{R}^+(\Gamma, \mathrm{PSL}_2 \mathbf{R})/\mathrm{PSL}_2 \mathbf{R}$. There is also a similar identification $\mathcal{ML}(S) = \mathcal{ML}(\Gamma)$. Since each element of $\mathrm{Diff}^+(S)/\mathrm{Diff}^0(S)$ maps $\mathcal{ML}(S)$ onto itself, there is an action of $\mathrm{Out}^+(\Gamma)$ on $\mathcal{ML}(\Gamma)$.

Corollary 2.6.1. *There is a commutative diagram on which* $\mathrm{Out}^+(\Gamma)$ *acts diagonally.*

$$\begin{array}{ccc}
\mathrm{Diff}^+(S)/\mathrm{Diff}^0(S) & \longrightarrow & \mathrm{Out}^+(\Gamma) \\
\downarrow & & \downarrow \\
\mathbf{CP}^1(S)^+ & \xrightarrow{\Theta} & \mathcal{T}(\Gamma) \times \mathcal{ML}(\Gamma) \\
\downarrow & & \downarrow \\
\mathcal{M}\mathbf{CP}^1(S)^+ & \xrightarrow{\hat{\Theta}} & \mathcal{T}(\Gamma) \times \mathcal{ML}(\Gamma)/\mathrm{Out}^+(\Gamma).
\end{array}$$

If we recall that $\mathrm{Out}^+(\Gamma)$ acts properly discontinuously on $\mathcal{T}(\Gamma) \times \mathcal{ML}(\Gamma)$, it follows that

Corollary 2.6.2. $\mathrm{Diff}^+(S)/\mathrm{Diff}^0(S)$ *acts properly discontinuously on* $\mathbf{CP}^1(S)^+$.

3. Projective structures on surfaces and holonomy function groups

3.1 Subspaces of $\mathbf{CP}^1(S_g)^+$

As before S is a closed orientable surface S_g of genus $g \geqq 2$ and $\Gamma = \pi_1(S_g)$. Recall that

$$\mathbf{CP}^1(S)^+ = \mathrm{PSL}_2(\mathbf{C}) \setminus \Omega^+(S),$$

where $\Omega^+(S)$ is the deformation space of orientation-preserving developing maps. (See Chapter 3.)

Definition 3.1.1. Let $P : \Omega^+(S) \to \mathbf{CP}^1(S)^+$ be the canonical projection. Let $\mathbf{CP}^1(S)^+_0$ be the subspace of $\mathbf{CP}^1(S)^+$ consisting of injective developing maps. And $\mathbf{CP}^1(S)^+_1$ is the subspace of $\mathbf{CP}^1(S)^+$ consisting of nonsurjective developing maps. Let $\Omega^+(S)_i = P^{-1}(\mathbf{CP}^1(S)^+_i)$ $(i = 0, 1)$.

Proposition 3.1.2.

(i) $\mathbf{CP}^1(S)^+_0$ *is a closed subspace of* $\mathbf{CP}^1(S)^+$.

(ii) $\mathbf{CP}^1(S)^+_1$ *is a closed subspace of* $\mathbf{CP}^1(S)^+$

Proof. Suppose that a sequence $\{\mathrm{dev}_i\}$ in $\mathbf{CP}^1(S)^+_0$ converges to a developing map $\{\mathrm{dev}\}$.
Suppose that dev is not injective and $\mathrm{dev}(x) = \mathrm{dev}(y)$ for $x \neq y$ in $\tilde{S}$. There exists a compact neighborhood K of x which does not contain y and is mapped homeomorphically onto a closed ball $\mathrm{dev}(K)$. By the above topology on $\mathbf{CP}^1(S)^+$, dev_i is also an embedding on K for sufficiently large i. Let ρ be the spherical metric of $\mathbf{CP}^1$ and let ℓ_i be the shortest circular arc from $\mathrm{dev}_i(x)$ to $\mathrm{dev}_i(y)$. Since $\mathrm{dev}\,|K$ is an embedding for sufficiently large i, there is a sequence of points $\{p_i\} \in \mathbf{CP}^1$ each of which is the first contact of ℓ_i to $\partial\,\mathrm{dev}_i(K) = \mathrm{dev}_i(\partial K)$. There exists a sequence of points $\{z_i\} \in \partial K$ such that $\mathrm{dev}_i(z_i) = p_i$. Since ℓ_i gives the distance $\rho(\mathrm{dev}_i(x), \mathrm{dev}_i(y))$, it follows that $\rho(\mathrm{dev}_i(x), \mathrm{dev}_i(z_i)) \leqq \rho(\mathrm{dev}_i(x), \mathrm{dev}_i(y))$. The sequence $\{z_i\} \in \partial K$ converges to some point $z \in \partial K$. Then the above inequality yields that

$$\rho(\mathrm{dev}(x), \mathrm{dev}(z)) \leqq \rho(\mathrm{dev}(x), \mathrm{dev}(y)) = 0,$$

while x is an interior point of K, which is a contradiction. This proves (i).

Consider (ii). Suppose that a sequence $\{(\phi_i, d_i)\}$ converges to an element $\{(\phi, d)\}$ in $\mathbf{CP}^1(S)^+$. We can assume that the closure $\overline{\phi(\Gamma)}$ is neither a finite group nor a subgroup of the group of similarity transformations $\mathrm{Sim}(\mathbf{R}^2)$, for otherwise S would be covered by a sphere or a torus respectively. In particular $\phi(\Gamma)$ contains a loxodromic element. If $\phi(\gamma)$ is a loxodromic element for some $\gamma \in \Gamma$, there is a point x in $\mathbf{CP}^1$ such that $\phi(\gamma)x = x$. (Note that there exist exactly two points.) Let $L(\phi(\Gamma))$ be the limit set for $\phi(\Gamma)$. (See 4.1 or [11] for the definition.) It follows that $x \in L(\phi(\Gamma))$. The trace formula (cf. [4]) implies that g is either elliptic or parabolic if and only if $|\operatorname{tr}^2 g| \in [0, 4]$ for $g \in \mathrm{PSL}_2(\mathbf{C})$. Since $\phi_i(\gamma) \to \phi(\gamma)$ and $\phi(\gamma)$ is loxodromic, it follows that $\phi_i(\gamma)$ is also loxodromic for sufficiently large i. And so there exists a point x_i such that $\phi_i(\gamma)x_i = x_i$ for each i. Note that $\{x_i\} \in L(\phi_i(\Gamma))$ and $d_i(\tilde{S}) \cap L(\phi_i(\Gamma)) = \emptyset$. (See for example [25].)
The sequence $\{x_i\}$ has an accumulation point. Since $\phi_i(\gamma) \to \phi(\gamma)$ and $\phi_i(\gamma)x_i = x_i$, $\phi(\gamma)$ fixes that accumulation point. We can assume that $\lim x_i = x$.

We claim that d misses the point x. Suppose not. Let $d(p) = x$ for some $p \in \tilde{S}$. Choose a compact neighborhood C of p in $\tilde{S}$ and a closed ball $\bar{B}$ centered at x in $\mathbf{CP}^1$ so that $d : C \to \bar{B}$ is a diffeomorphism. We note that for sufficiently large i, $x_i \in \bar{B}$ and $d_i|C$ is an embedding.

Case 1. x lies inside $d_i(C)$. Since $x_i \notin d_i(C)$, it implies that $\rho(x_i, x) \geqq \operatorname{dist}(x_i, d_i(C))$ and let $\{a_i\} \in \partial C$ be the sequence of points which attains the distance $\operatorname{dist}(x_i, d_i(C))$, i.e., $\rho(x_i, d_i(a_i)) = \operatorname{dist}(x_i, d_i(C))$. The sequence $\{a_i\}$ has a limit point $a \in \partial C$. Since it follows that $d(a) \in \partial B$, we obtain that $\rho(x, d(a)) > 0$. On the other hand, the above inequality yields that $0 \geqq \rho(x, d(a))$, which is a contradiction.
Case 2. x lies outside $d_i(C)$. Since $p \in C$, it follows $\operatorname{dist}(x, d_i(C)) \leqq \rho(x, d_i(p))$. Note that $\lim \operatorname{dist}(x, d_i(C)) = 0$ because $\lim d_i(p) = d(p) = x$. Similarly as above we have a sequence of the points $\{b_i\} \in \partial C$ such that $\rho(x, d_i(b_i)) = \operatorname{dist}(x, d_i(C))$. As $\lim b_i = b$ for some point $b \in \partial C$, it follows that $\lim d_i(b_i) = d(b) \in \partial B$. And so $0 < \rho(x, d(b)) = \lim \operatorname{dist}(x, d_i(C))$, being a contradiction. Therefore d misses the point x.

By virtue of the theorem of [25] we have that d is a covering map. This shows (ii). Q.E.D.

3.2. Description of Kleinian groups by projective structures

Let G be a Kleinian group, i.e., a finitely generated discrete subgroup of $\mathrm{PSL}_2\,\mathbf{C}$. Put $\Omega = \Omega(G) = S^2 - L(G)$. Recall that
G is a function group if there is a component Ω_0 of Ω invariant under G.
G is a quasi-Fuchsian group if $\Omega = \Omega_0 \cup \Omega_1$ (i.e., consists of two components). As the special case if $\partial\Omega_0 (= \partial\Omega_1)$ is a round circle then G is a Fuchsian group.
G is a b-group if Ω has only one invariant simply connected component.
Let $S = \mathbf{H}^2/\Gamma$ be a closed orientable surface.

Definition 3.2.1.
$\mathcal{F}(\Gamma) = \{\theta \in \operatorname{Hom}(\Gamma, \mathrm{PSL}_2\,\mathbf{C}) \mid \theta(\Gamma)$ is a function group$\}$.
$\mathcal{B}(\Gamma) = \{\theta \in \operatorname{Hom}(\Gamma, \mathrm{PSL}_2\,\mathbf{C}) \mid \theta : \Gamma \to \theta(\Gamma)$ is an isomorphism, $\theta(\Gamma)$ is a Kleinian group and an invariant component is simply connected.$\}$
$\mathcal{R}_2(\Gamma) = \{\theta \in \operatorname{Hom}(\Gamma, \mathrm{PSL}_2\,\mathbf{C}) \mid \theta(\Gamma)$ is quasi-conformally equivalent to $\Gamma.\}$ (i.e., the set of quasi-Fuchsian groups).
Let

$$P : \operatorname{Hom}(\Gamma, \mathrm{PSL}_2\,\mathbf{C}) \to \operatorname{Hom}(\Gamma, \mathrm{PSL}_2\,\mathbf{C}) / \mathrm{PSL}_2\,\mathbf{C}$$

be the canonical projection and put

$$F(\Gamma) = P(\mathcal{F}(\Gamma)), \quad B(\Gamma) = P(\mathcal{B}(\Gamma)), \text{ and } \; T_2\Gamma = P(\mathcal{R}_2(\Gamma)).$$

Note that $\operatorname{Hom}(\Gamma, \mathrm{PSL}_2\,\mathbf{C}) / \mathrm{PSL}_2\,\mathbf{C}$ is connected but not a Hausdorff space.

Recall that $H : \Omega^+(S) \to \operatorname{Hom}(\Gamma, \mathrm{PSL}_2 \mathbf{C})$ is the map which assigns to an oriented projective structure its holonomy representation. The map H induces a holonomy map hol : $\mathbf{CP}^1(S)^+ \to \operatorname{Hom}(\Gamma, \mathrm{PSL}_2 \mathbf{C})/\mathrm{PSL}_2 \mathbf{C}$.

Corollary 3.2.2. *The holonomy map* hol *maps* $\mathbf{CP}^1(S)_1^+ \to F(\Gamma)$. *In particular the holonomy groups are discrete.*

Proposition 3.2.3. *The holonomy map* hol *defines a homeomorphism of* $\mathbf{CP}^1(S)_0^+$ *onto* $B(\Gamma)$.

Proof. Let $\theta \in \mathcal{B}(\Gamma)$ be a representative element of $B(\Gamma)$ with an invariant simply connected component Ω_0. Since $\theta \in \operatorname{Hom}(\Gamma, \mathrm{PSL}_2 \mathbf{C})$, there is an orientation-preserving conformal homeomorphism $f : \Omega_0 \to \mathbf{H}^2$ such that $f\theta(\Gamma)f^{-1}$ is Fuchsian. Let $\psi : \Gamma \to f\theta(\Gamma)f^{-1}$ be an isomorhism defined by $\psi(\gamma) = f\theta(\gamma)f^{-1}$. Then it is well known that there is a quasi-conformal homeomorphism $h : \mathbf{H}^2 \to \mathbf{H}^2$ which induces ψ. Put $\operatorname{dev} = f^{-1} \circ h$. It is easy to see that $[\theta, \operatorname{dev}]$ is an element of $\mathbf{CP}^1(S)_0^+$.

Let $P : \Omega_0^+(S) \to \mathbf{CP}^1(S)_0^+$ be the canonical projection of the deformation spaces (cf. (3.1.1)). We will show that H maps $\Omega_0^+(S)$ onto $\mathcal{B}(\Gamma)$. If $(\phi, \operatorname{dev})$ is an element of $\Omega_0^+(S)$, then it follows that $\phi \in \mathcal{F}$. If suffices to check that $\phi(\Gamma)$ has an invariant simply connected component of $\Omega = S^2 - L(\phi(\Gamma))$. Since dev is injective, $\phi(\Gamma)$ has a simply connected domain $\operatorname{dev}(\tilde{S})$ which sits in Ω. Let Ω_0 be an invariant maximal component in Ω containing $\operatorname{dev}(\tilde{S})$. Since $\phi(\Gamma)$ acts properly discontinuously and freely on Ω_0, we can choose a $\phi(\Gamma)$ invariant Riemannian metric on Ω_0. The map dev is a covering map because S is compact. Since $\phi : \Gamma \to \phi(\Gamma)$ is an isomorphism, dev must be an isometry. And thus $\operatorname{dev}(\tilde{S}) = \Omega_0$. We prove that H is injective. For $(\phi_i, \operatorname{dev}_i)$ $(i = 1, 2)$, suppose that $H(\phi_1, \operatorname{dev}_1) = H(\phi_2, \operatorname{dev}_2)$, i.e., $\phi_1 = \phi_2$. Then, $\operatorname{dev}_1(\tilde{S}) = \operatorname{dev}_2(\tilde{S})$. For this, if not then $\phi(\Gamma) = \phi_1(\Gamma) = \phi_2(\Gamma)$ has at least two invariant components, i.e., $\operatorname{dev}_1(\tilde{S})$, $\operatorname{dev}_2(\tilde{S})$ (cf. [5]). Hence $\phi(\Gamma)$ is quasi-Fuchsian. However since both dev_1 and dev_2 are orientation-preserving, it is impossible. Put $\tilde{f} = \operatorname{dev}_2^{-1} \circ \operatorname{dev}_1$. Then it follows that $\tilde{f} \circ \gamma = \gamma \circ \tilde{f}$ for $\gamma \in \Gamma$. Therefore $f \in \operatorname{Diff}^0(S)$ and $[\phi_2, \operatorname{dev}_2] \circ f = [\phi_1, \operatorname{dev}_1]$.
And hence H is a one-to-one continuous map. Since H is a local homeomorphism by the Holonomy theorem 4.3.9 (cf. Chapter 4), it follows that H is a homeomorphism of $\Omega_0(S)$ onto $\mathcal{B}(\Gamma)$. Since the action of $\mathrm{PSL}_2 \mathbf{C}$ on both $\Omega_0(S)$ and $\mathcal{B}(\Gamma)$ is free, it implies that hol is a homeomorphism. Q.E.D.

Let $\mathcal{R}_2(\Gamma)$ be the space of quasi-Fuchsian groups in $\mathrm{Hom}(\Gamma, \mathrm{PSL}_2\,\mathbf{C})$ as in (3.2.1). If $\overline{\mathcal{R}_2(\Gamma)}$ is the closure of $\mathcal{R}_2(\Gamma)$ in $\mathrm{Hom}(\Gamma, \mathrm{PSL}_2\,\mathbf{C})$, then we put $\partial\mathcal{R}_2(\Gamma) = \overline{\mathcal{R}_2(\Gamma)} - \mathcal{R}_2(\Gamma)$.

Definition 3.2.4. Define the following subspaces

$$\Omega^+(S, qf) = \{(\phi, \mathrm{dev}) \in \Omega^+(S) \mid \phi \in \mathcal{R}_2(\Gamma)\},$$
$$\mathbf{CP}^1(S, qf)^+ = P(\Omega^+(S, qf)),$$

and

$$\Omega^+(S, \partial) = \{(\phi,\ \mathrm{dev}) \in \Omega^+(S) \mid \phi \in \partial\mathcal{R}_2(\Gamma)\},$$
$$\mathbf{CP}^1(S, \partial)^+ = P(\Omega^+(S, \partial)).$$

$\mathbf{CP}^1(S, qf)^+$ (resp. $\mathbf{CP}^1(S, \partial)^+$) is called the deformation space of (oriented) projective structures with quasi-Fuchsian (resp. boundary) holonomy.

We have the following subspaces of $\mathbf{CP}^1(S, qf)^+$ (resp. $\mathbf{CP}^1(S, \partial)^+$) whose developing maps are injective;

$$\begin{aligned} \mathbf{CP}^1(S, qf)^+_0 &= \mathbf{CP}^1(S, qf)^+ \cap \mathbf{CP}^1(S)^+_0, \\ \mathbf{CP}^1(S, \partial)^+_0 &= \mathbf{CP}^1(S, \partial)^+ \cap \mathbf{CP}^1(S)^+_0. \end{aligned} \tag{3.2.5}$$

The simultaneous uniformization of Bers ([5]) is stated as follows.

Corollary 3.2.6 (Bers). $\mathbf{CP}^1(S, qf)^+_0 \approx \mathcal{T}(\Gamma) \times \Delta$.
Here Δ is an open cell contained in $\mathcal{ML}(S)$.

3.3. Insertion of annuli and operation on projective structures with boundary holonomy

An insertion of annuli (more generally, a grafting) produces a new structure from a given projective structure. (See Goldman [16].) Especially, let $\Omega^+_0(S, qf)$ be the space of projective structures with quasi-Fuchsian holonomy groups and with injective developing maps. Let $\mathcal{C}$ the set of all isotopy classes of a disjoint collection of homotopically nontrivial simple closed curves on S. Let $\mathcal{ML}(2\mathbf{Z})$ denote the set of measured geodesic laminations μ supported on a disjoint union of closed geodesics lying in $\mathcal{C}$ and together with 2π times positive integer weights. Then, each $\sigma \in \mathcal{ML}(2\mathbf{Z})$ defines an operation $\sharp$ which assigns to a structure of $\Omega_0(S, qf)$ a structure with surjective developing map.
Goldman ([16]) has shown that

Theorem 3.3.1. $\mathbf{CP}^1(S, qf)^+ \approx \mathbf{CP}^1(S, qf)_0^+ \times \mathcal{ML}(2\mathbf{Z})$.

It follows also that

$$\Omega^+(S, qf) \approx \Omega_0^+(S, qf) \times \mathcal{ML}(2\mathbf{Z}). \tag{3.3.2}$$

If $x \in \Omega_0^+(S, qf)$ and $\sigma \in \mathcal{ML}(2\mathbf{Z})$, then $x\sharp\sigma$ is a new structure with surjective developing map and with the same holonomy as that of x. It lies in one component of $\Omega^+(S, qf)$ different from $\Omega_0(S, qf)$. And so it follows that $\Omega^+(S, qf) = \bigcup_{\sigma \in \mathcal{ML}(2\mathbf{Z})} (\Omega_0^+(S, qf)\sharp\sigma)$ for which $\Omega_0^+(S, qf)\sharp\sigma$ is one component homomorphic to $\Omega_0^+(S, qf)$.

Let $\Omega_0^+(S, \partial)$ be the space of oriented projective structures with boundary holonomy and with injective developing maps (cf. (3.2.5)). The operation $\sharp$ can be defined on $\Omega_0^+(S, \partial)$. We shall prove the similar result for $\Omega^+(S, \partial)$.

Proposition 3.3.3.

$$\Omega_0^+(S, \partial) \times \mathcal{ML}(2\mathbf{Z}) \approx \Omega^+(S, \partial).$$

In order to prove this proposition, we need the following lemmata.

Lemma 3.3.4. *The holonomy map*

$$H : \overline{\Omega^+(S, qf)} \to \overline{\mathcal{R}_2(\Gamma)} \text{ is locally injective.}$$

Proof. If we note that $\mathcal{ML}(2\mathbf{Z})$ is discrete in $\mathcal{ML}(S)$, then $\overline{\Omega^+(S, qf)} \approx \overline{\Omega_0^+(S, qf)} \times \mathcal{ML}(2\mathbf{Z})$. We prove that $H : \overline{\Omega_0^+(S, qf)} \to \overline{\mathcal{R}_2(\Gamma)}$ is injective. Let x, $y \in \partial\Omega_0^+(S, qf) (= \overline{\Omega_0^+(S, qf)}(S) - \Omega_0^+(S, qf))$ and suppose $H(x) = H(y)$. First note that $H(x) \in \partial\mathcal{R}_2(\Gamma) = \overline{\mathcal{R}_2(\Gamma)} - \mathcal{R}_2(\Gamma)$ since H is a local homeomorphism and by the definition (3.2.4). There are neighborhoods U,V of x,y respectively such that $H : U \to W$, $H : V \to W$ are homeomorphisms where W is a neighborhood of $H(x)$. Since $W \cap \mathcal{R}_2(\Gamma) \neq \emptyset$ is open, there are $a \in U$, $b \in V$ so that $H(a) = H(b)$ in $W \cap \mathcal{R}_2(\Gamma)$. Since $H|\Omega_0^+(S, qf)$ is a homeomorphim, it follows that $a = b$, i.e., $U \cap V \neq \emptyset$. It implies that $x = y$.

We note that nearby structures outside $\overline{\Omega_0(S, qf)}$ do not have quasi-Fuchsian holonomy groups. Namely, for $x \in \partial\Omega_0^+(S, qf)$, there is a neighborfood U of x such that for any $z \in U - \overline{\Omega_0^+(S, qf)}$, $H(z) \notin \mathcal{R}_2(\Gamma)$. Q.E.D.

Lemma 3.3.5. $\partial\Omega^+(S,qf) = \Omega^+(S,\partial)$.

Proof. Using the above lemma, we have for $x \in \partial(\Omega^+(S,qf))$ that $H(x) \in \partial\mathcal{R}_2(\Gamma)$. By the definition 3.2.4 it follows that $\partial(\Omega^+(S,qf)) \subset \Omega^+(S,\partial)$. Let $x \in \Omega^+(S,\partial)$ so that $H(x) \in \partial\mathcal{R}_2(\Gamma)$. Let U be any neighborhood of x in $\Omega^+(S)$. Since H is a local homeomorphism, there exists a neighborhood W of x contained in U such that $H : W \to H(W)$ is a homeomorphism. As $H(x) \in \partial\mathcal{R}_2(\Gamma)$, we have $H(U) \cap \mathcal{R}_2(\Gamma) \neq \emptyset$. Choose $y \in U$ with $H(y) \in \mathcal{R}_2(\Gamma)$. Again by the definition 3.2.4, it follows that $y \in \Omega^+(S,qf)$, or $U \cap \Omega^+(S,qf) \neq \emptyset$. And hence $x \in \overline{\Omega^+(S,qf)}$. Since $H(x)$ is not a quasi-Fuchsian group, $x \in \partial(\Omega^+(S,qf))$. Q.E.D.

Proof of (3.3.3). Since $\overline{\Omega^+(S,qf)} \approx \overline{\Omega_0^+(S,qf)} \times \mathcal{ML}(2\mathbf{Z})$, it implies that

$$\partial\Omega^+(S,qf) \approx \partial\Omega_0^+(S,qf) \times \mathcal{ML}(2\mathbf{Z}). \tag{3.3.6}$$

On the other hand, $\Omega_0^+(S)$ is a closed subspace of $\Omega^+(S)$ by Proposition 3.1.2. It is noted that $\overline{\Omega_0^+(S,qf)} \subset \Omega_0^+(S)$. And so we have that $\partial\Omega_0^+(S,qf)) \subset \Omega_0^+(S)$. In view of (3.3.6), the subspace of $\partial\Omega^+(S,qf)$ consisting of injective developing maps, $(\partial\Omega^+(S,qf))_0 = \partial\Omega_0^+(S,qf)$. By Lemma 3.3.5 it follows that $\Omega_0^+(S,\partial) = (\partial\Omega^+(S,qf))_0$ and by (3.3.6) that $\Omega^+(S,\partial) \approx \Omega_0^+(S,\partial) \times \mathcal{ML}(2\mathbf{Z})$. Q.E.D.

4. (G,X)-structures

4.1. Limit sets in (G,X)

Recall that a geometric structure on a smooth n-manifold is a maximal collection of charts modelled on a simply connected n dimensional homogeneous space X of a Lie group G whose coordinate changes are restrictions of transformations from G. We call such a structure a (G,X)-structure. A manifold with this structure is called a (G,X)-manifold. Suppose that a smooth connected n-manifold M admits a (G,X)-structure. Then there exists a developing pair (ρ, dev), where $\mathrm{dev} : \tilde{M} \to X$ is a "structure-preserving" immersion and $\rho : \pi_1(M) \to G$ is a homomorphism (both unique up to elements of G). The group $\Gamma = \rho(\pi_1(M))$ is called the holonomy group for M.

In particular the developing pair (ρ, dev) is an invariant of the (G,X)-structure. In fact this developing map and holonomy give us a powerful tool in understanding the topology of (G,X)-manifolds. The first question arises when the developing map is a covering map onto its

image. In order to study this problem we introduce the notion of limit sets in (G,X) due to Kulkarni [33].
We consider the following sets. Let Γ be a subgroup of G.
(4.1.1)

Λ_0 = the closure of the set $\{x \in X \mid$ the stabilizer Γ_x is an infinite subgroup$\}$.
Λ_1 = the set of cluster points of $\{\gamma y \mid \gamma \in \Gamma\}$ where $y \in X - \Lambda_0$.
Λ_2 = the set of cluster points of $\{\gamma K \mid \gamma \in \Gamma\}$ where K is a compact subset of $X - \{\Lambda_0, \Lambda_1\}$.

Then the set $\Lambda = \Lambda(\Gamma) = \Lambda_0 \cup \Lambda_1 \cup \Lambda_2$ is said to be the limit set of Γ. And the set $\Omega = X - \Lambda$ is called the domain of discontinuity for Γ. (Compare also [39] for further results of limit sets.) It is the fundamental result that

Proposition 4.1.2. *If $\Omega \neq \emptyset$, then Γ acts properly discontinuously on Ω. In particular Γ is discrete in G.*

(4.1.3) We examine another limit sets to our use. Let Y be a complete simply connected Riemannian manifold of nonpositive sectional curvature. Then there is a compactification $\bar{Y} = \partial Y \cup Y$ of Y. The space $\bar{Y}$, equipped with the cone topology, is homeomorphic to the closed ball and the boundary ∂Y is the set of points at infinity consisting of the equivalence classes of asymptotic geodesics. The group of isometries $\mathrm{Iso}(Y)$ extends to a topological action on its boundary. For example, recall that the n-sphere S^n is viewed as the ideal boundary of the real hyperbolic space $\mathbf{H}^{n+1}$. Similarly S^{2n+1} is the ideal boundary of the complex hyperbolic space $\mathbf{H}_{\mathbf{C}}^{n+1}$. Moreover, when Y is a hyperbolic space $\mathbf{H}^n$ or $\mathbf{H}_{\mathbf{C}}^{n+1}$, $\mathrm{Iso}(Y)$ acts as conformal (resp. CR) automorphisms of the sphere. We write $\mathrm{Iso}(Y) = \mathrm{Conf}(S^n)$ or $\mathrm{Aut}_{CR}(S^{2n+1})$ respectively.

Definition 4.1.4. For a subgroup Γ of $\mathrm{Iso}(Y)$ the limit set $L(\Gamma) \subset \partial Y$ is defined to be the set of cluster points of the orbit $\Gamma \cdot x$ for $x \in Y$.

As to the relation between the above limit set Λ, we have (cf. [25])

Proposition 4.1.5. *Let Γ be a discrete subgroup of either $\mathrm{Conf}(S^n)$ or $\mathrm{Aut}_{CR}(S^{2n+1})$. Then it follows that*

$$\Lambda(\Gamma) = L(\Gamma).$$

4.2. Application to developing maps

Suppose that M is a closed connected (G,X)-manifold. Let us be given a Γ invariant closed subset F in X. Suppose that there exist a component Y of the complement $X-F$ and a component N of $\mathrm{dev}^{-1}(Y)$. We then have the restriction of the developing map $\mathrm{dev} : N \longrightarrow Y$. We have proved the following result in [18] (cf. also in [16],[19]).

Lemma 4.2.1. *Under the above hypothesis, suppose that Y admits a Γ invariant complete Riemannian metric. Then the developing map $\mathrm{dev} : N \longrightarrow Y$ is a covering map.*

As an application, we shall prove that;

Proposition 4.2.2. *Let Y be a Γ invariant closed subset of X with Hausdorff dimension $k < n-1$. Suppose that the complement $X - Y$ admits a Γ invariant complete Riemannian metric.*

(i) *if $k < n-2$, then $\mathrm{dev} : \tilde{M} \to X - \Lambda$ is a homeomorphism.*

(ii) *for $n - 2 \leqq k < n-1$, assume that either $\mathrm{dev}^{-1}(Y) = \emptyset$ or $\mathrm{dev}_* : \pi_1(\tilde{M} - \mathrm{dev}^{-1}(Y)) \to \pi_1(X - Y)$ is surjective. Then $\mathrm{dev} : \tilde{M} \to X - Y$ is a covering map, or $\mathrm{dev} : \tilde{M} \to X - \Lambda$ is a homeomorphism.*

Proof. Note first that $\tilde{M} - \mathrm{dev}^{-1}(Y)$ is connected since the Hausdorff dimension k is less than $n-1$ (cf. [18]). Moreover if $k < n-2$, $X - Y$ is 1-connected. We have from Lemma 4.2.1 that $\mathrm{dev} : \tilde{M} - \mathrm{dev}^{-1}(Y) \to X - Y$ is a covering map. As above if $k < n-2$, $\mathrm{dev} : \tilde{M} - \mathrm{dev}^{-1}(Y) \to X - Y$ is a homeomorphism. If $n-2 \leqq k < n-1$ then according to that $\mathrm{dev}^{-1}(Y) = \emptyset$ or $\mathrm{dev}^{-1}(Y) \neq \emptyset$ under the surjectivity assumption it follows that $\mathrm{dev} : \tilde{M} \to X - Y$ is a covering map or $\mathrm{dev} : \tilde{M} - \mathrm{dev}^{-1}(Y) \to X - Y$ is a homeomorphism. Since dev is an immersion and $k < n-1$, for any point x in $\tilde{M}$ there exists a neighborhood U of x in $\tilde{M}$ such that $\mathrm{dev}(U) \cap (X - Y) \neq \emptyset$. This implies that $\mathrm{dev} : \tilde{M} \to \mathrm{dev}(\tilde{M})$ is injective. Hence Γ acts properly discontinuously on $\mathrm{dev}(\tilde{M})$ which shows that $\mathrm{dev}(\tilde{M}) \cap \Lambda = \emptyset$. Since Γ acts properly discontinuously on $X - \Lambda$ by Proposition 4.1.2, it follows that $\mathrm{dev}(\tilde{M}) = X - \Lambda$. Q.E.D.

4.3. Deformation space of (G, X)-structures and the Holonomy theorem

In this section we shall examine the structure of the deformation space of (G, X)-structures invariant under Lie groups. Let H be a connected Lie group acting on a smooth closed $(2n+1)$-manifold M.

4.3.1. *The deformation space $\mathcal{T}(H, M)$ is a space of H invariant marked (G, X)-structures on manifolds homeomorphic to M. $\mathcal{T}(H, M)$*

consists of equivalence classes of equivariant diffeomorphisms $f: M \to M'$ from the action (H, M) to H invariant (G, X)-manifolds M'. Two such diffeomorphisms $f_i : M \to M_i$ $(i = 1, 2)$ are equivalent if there is an equivariant isomorphism (i.e., a (G, X)-structure preserving diffeomorphism) $h : M_1 \to M_2$ such that $h \circ f_1$ is isotopic to f_2.

$$\begin{tikzcd} M \arrow[r, "f_1"] \arrow[dr, "f_2"'] & M_1 \arrow[d, "h"] \arrow[dl, phantom, "\simeq"] \\ & M_2 \end{tikzcd}$$

Note that it is not necessarily assumed to be *equivariantly isotopic.* On the other hand if M is a (G, X)-manifold then there is a developing pair $(\rho, \mathrm{dev}) : (\mathrm{Aut}(\tilde{M}), \tilde{M}) \to (G, X)$ such that $\pi \subset \mathrm{Aut}(\tilde{M})$, where $\pi = \pi_1(M)$ and $\mathrm{Aut}(\tilde{M})$ is a group of (G, X)-isomorphisms of $\tilde{M}$.

4.3.2. *$\hat{\Omega}(H, M)$ is the space consisting of all possible developing pairs (ρ, dev) which satisfy that (ρ, dev) represents an H invariant (G, X)-structure on M and such that if one forgets the structure then the action (H, M) is smoothly equivalent to the original action (H, M). That is, the action of each element of $\hat{\Omega}(H, M)$ is topologically unique but geometrically distinct.*

The topology on $\hat{\Omega}(H, M)$ is given by the following subbasis (cf. [8]).

(1) $\mathcal{N}(U) = \{U\}$ *where U is an open subset of* $\mathrm{Maps}(\tilde{M}, X)$ *in the compact open topology of* $\mathrm{Maps}(\tilde{M}, X)$.
(2) $\mathcal{N}(K) = \{\mathrm{dev} \in \hat{\Omega}(H, M) \mid \mathrm{dev}|K$ *is an embedding for a compact subset* $K \subset \tilde{M}\}$.

4.3.3. *We introduce a subgroup* $\widehat{\mathrm{Diff}}(H, M)$ *of* $\mathrm{Diff}(\tilde{M})$. *Let* $\mathrm{Diff}(H, M)$ *be the group of equivariant diffeomorphisms of M onto itself. Denote by* $\mathrm{Diff}^0(H, M)$ *the subgroup of* $\mathrm{Diff}(H, M)$ *whose elements are isotopic to the identity map. Consider the following exact sequences of the diffeomorphism groups, where $N_{\mathrm{Diff}(\tilde{M})}(\pi)$ (resp. $C_{\mathrm{Diff}(\tilde{M})}(\pi)$) is the normalizer (resp. centralizer) of π in* $\mathrm{Diff}(\tilde{M})$;

$$\begin{tikzcd} 1 \arrow[r] & \pi \arrow[r] & N_{\mathrm{Diff}(\tilde{M})}(\pi) \arrow[r, "\eta"] & \mathrm{Diff}(M) \arrow[r] & 1 \\ & & C_{\mathrm{Diff}(\tilde{M})}(\pi) \arrow[r] \arrow[u] & \mathrm{Diff}^0(M) \arrow[u] & \end{tikzcd}$$

Put $\widehat{\mathrm{Diff}}(H,M)=\eta^{-1}(\mathrm{Diff}(H,M))$ *and let* $\widehat{\mathrm{Diff}}^0(H,M)$ *be the identity component. It follows easily that* $\eta(\widehat{\mathrm{Diff}}^0(H,M)) = \mathrm{Diff}^0(H,M)$ *and* $\widehat{\mathrm{Diff}}^0(H,M)\subset C_{\mathrm{Diff}(\tilde{M})}(\pi)$.

4.3.4. *The actions on* $\hat{\Omega}(H,M)$. *The natural right action of* $\widehat{\mathrm{Diff}}(H,M)$ *and the left action of* G *on* $\hat{\Omega}(H,M)$ *are defined by setting*

$$(\rho,\mathrm{dev})\circ\tilde{f}=(\rho\circ\mu(\tilde{f}),\mathrm{dev}\circ\tilde{f}),$$
$$g\circ(\rho,\mathrm{dev})=(g\circ\rho\circ g^{-1},g\circ\mathrm{dev}),$$

where $\mu(\tilde{f}):\pi\to\pi$ is an isomorphism defined by $\mu(\tilde{f})(\gamma)=\tilde{f}\circ\gamma\circ\tilde{f}^{-1}$. Obviously both actions commute.

It is noted that two developing pairs (ρ_i,dev_i) $(i=1,2)$ represent the same structure on M if and only if there exists an element $g\in G$ such that $g\circ\mathrm{dev}_1=\mathrm{dev}_2$. Put

$$\Omega(H,M)=\hat{\Omega}(H,M)/\widehat{\mathrm{Diff}}^0(H,M).$$

The action of G induces an action of $\Omega(H,M)$. Then it is easy to show that

Lemma 4.3.5. *The elements of* $\mathcal{T}(H,M)$ *are in one-to-one correspondence with the orbits of* $G\setminus\Omega(H,M)$.

Definition 4.3.6. The space $G\backslash\Omega(H,M)$ equipped with the quotient topology is called the deformation space $\mathcal{T}(H,M)$ of H invariant (G,X)-structures on M.

Note that if one choose the trivial group as H then $\mathcal{T}(M)=\mathcal{T}(\{1\},M)$ is the usual deformation space. If $f:M\to M'$ is a representative element of $\mathcal{T}(H,M)$ then there is a developing pair $(\rho,\mathrm{dev}):(\pi_1(M'),\tilde{M}')\to(G,X)$. We have the holonomy representation $\rho\circ f_\sharp:\pi\to G$ up to conjugate by an element of G. Let $\hat{H}\subset\widehat{\mathrm{Diff}}^0(H,M)$ be a closed connected subgroup such that $\eta(\hat{H})=H$. Note that the group $\hat{H}$ centralizes π and f is equivariant (cf. (4.3.1)). The group $\rho(\hat{H})$ centralizes $\rho\circ f_\sharp(\pi)$. Here we assume that

(4.3.7 $*$). *there exist a group* $K\subset G$ *and an isomorphism* $\phi:\hat{H}\to K$ *for which every representation* ρ *satisfies that* $g\circ\rho\circ g^{-1}=\phi$ *for some* $g\in G$.

It is noted that $\rho \circ f_\sharp(\pi)$ lies in the centralizer $C_G(K)$ up to conjugation. Let $\hat{\Omega}_0(H,M)$ be the subset of $\hat{\Omega}(H,M)$ whose holonomy representations ρ lie in $C_G(K)$ and $\rho|\hat{H} = \phi$. Put $\Omega_0(H,M) = \hat{\Omega}_0(H,M)/\widehat{\mathrm{Diff}}^0(H,M)$. The projection $\Omega_0(H,M) \to \mathcal{T}(H,M)$ is surjective by (4.3.7 $*$). Moreover we assume that

(4.3.7 $**$). *If two such representations of π are conjugate in G then they are conjugate by an element of $C_G(K)$.*

We then obtain a map $\mathrm{hol} : \mathcal{T}(H,M) \to \mathrm{Hom}(\pi, C_G(K))/C_G(K)$ which assigns to a marked structure its holonomy representation. By the definition hol lifts to a map $\widehat{\mathrm{hol}} : \Omega_0(H,M) \to \mathrm{Hom}(\pi, C_G(K))$ which makes the following diagram commute.

$$\begin{array}{ccc} \Omega_0(H,M) & \xrightarrow{\widehat{\mathrm{hol}}} & \mathrm{Hom}(\pi, C_G(K)) \\ \downarrow & & \downarrow \\ \mathcal{T}(H,M) & \xrightarrow{\mathrm{hol}} & \mathrm{Hom}(\pi, C_G(K))/C_G(K). \end{array} \tag{4.3.8}$$

If $H = \{1\}$ then it implies that $K = \{1\}$ and so $C_G(K) = G$. We have the usual holonomy map $\mathrm{hol} : \mathcal{T}(M) \to \mathrm{Hom}(\pi, G)/G$. It has been proved by Lok ([37]) (see also [24],[48]) that $\widehat{\mathrm{hol}} : \Omega(M) \to \mathrm{Hom}(\pi, G)$ is a local homeomorphism. We prove also that

Holonomy Theorem 4.3.9. $\widehat{\mathrm{hol}} : \Omega_0(H,M) \to \mathrm{Hom}(\pi, C_G(K))$ *is a local homeomorphism.*

Proof. If we prove that the canonical map $\Omega_0(H,M) = \hat{\Omega}_0(H,M)/\widehat{\mathrm{Diff}}^0(H,M) \to \Omega(M) = \hat{\Omega}(M)/\widehat{\mathrm{Diff}}^0(M)$ is injective, then the holonomy map $\widehat{\mathrm{hol}} : \Omega(M) \to \mathrm{Hom}(\pi, G)$ restricts to a holonomy map $\widehat{\mathrm{hol}} : \Omega_0(H,M) \to \mathrm{Hom}(\pi, C_G(K))$. And so it is a local homeomorphism. Now suppose that two elements (ρ, dev) and (ρ', dev') represent the same element in $\hat{\Omega}(M)/\widehat{\mathrm{Diff}}^0(M)$. There exists an element $\tilde{f} \in \widehat{\mathrm{Diff}}^0(M)$ such that $\mathrm{dev}' = \mathrm{dev} \circ \tilde{f}$. Since $\rho|\hat{H} = \rho'|\hat{H}$ by (4.3.7 $*$), it follows that $\mathrm{dev} = \mathrm{dev} \circ (h\tilde{f}h^{-1}\tilde{f}^{-1})$ for each $h \in \hat{H}$. As $\hat{H}$ is connected and the map dev is a local homeomorphism, this equality implies that $\tilde{f} \circ h = h \circ \tilde{f}$ for every $h \in \hat{H}$. It follows that $\tilde{f} \in \widehat{\mathrm{Diff}}^0(H,M)$ by the definition 4.3.3. Hence the canonical map is injective. Q.E.D.

Remark **4.3.10**. Two assumptions of (4.3.7) will be satisfied when we consider semifree circle actions of H over H invariant spherical CR structures and H invariant conformally flat structures. We shall see this in the next chapter.

5. S^1 invariant geometric structures

5.1. Description of deformation spaces $\mathcal{T}(S^1, M)$

In this section we examine deformation spaces of S^1 invariant spherical CR structures and S^1 invariant conformally flat structures. Namely,

5.1.1. *Let* $H = S^1$.

(1) $(G, X) = (PU(n+1,1), S^{2n+1})$. *The corresponding space* $\mathcal{T}(H, M) = \mathcal{CR}(S^1, M)$ *is the deformation space of* S^1 *invariant spherical* CR *structures on* M *by the definition* 4.3.1.
(2) $(G, X) = (PO(n+1,1), S^n)$. *As before, the corresponding space* $\mathcal{T}(H, M) = \mathcal{CO}(S^1, M)$ *is the deformation space of* S^1 *invariant conformally flat structures on* M.

5.1.2. Let M be a closed $(2n+1)$-manifold. We suppose that the action (S^1, M) has the following properties for the CR case.

(i) M has a fixed point.
(ii) The orbit space M^* is a complex Kleinian orbifold $\mathbf{D}^{2n} - L(\pi^*)/\pi^*$.

Recall that the complex hyperbolic group $PU(n,1)$ acts on $\mathbf{D}^{2n}$ by biholomorphic transformations of $\mathbf{H}^n_{\mathbf{C}}$ and CR transformations of S^{2n-1}. The group $\pi^* \subset PU(n,1)$ and recall that $L(\pi^*)$ is the limit set of π^* in S^{2n-1}. By (i) the fixed point set F is homeomorphic to the ideal boundary $S^{2n-1} - L(\pi^*)/\pi^*$. For the conformal case, the action (S^1, M) on a closed n-manifold M has the same property as (i), but instead of (ii) we suppose

(ii)$'$ the orbit space M^* is a Kleinian orbifold $\mathbf{D}^{n-1} - L(\pi^*)/\pi^*$.

Recall from (4.3.2) that every element of $\Omega(S^1, M)$ represents an H invariant $(PU(n+1,1), S^{2n+1})$-structure on M and the CR action (H, M) is topologically equivalent to the action (S^1, M) of (5.1.2). Since M has a fixed point, it is noted that a lift $\hat{H}$ of H to $\tilde{M}$ is isomorphic to $\hat{H} = H$ (cf. (4.3.6)).
We have shown the topological rigidity of developing maps (cf. [18],[27]).

Proposition 5.1.3. *Let M be a closed spherical CR (resp. conformally flat) manifold with CR (resp. conformal) circle actions. Suppose that the action (S^1, M) has the property of* (5.1.2). *Put $\pi = \pi_1(M)$. If (ρ, dev) is the developing pair of an S^1 invariant (G, X)-structure on M where $(G, X) = (PU(n+1,1), S^{2n+1})$, $(PO(n+1,1), S^n)$ respectively, then the developing map* dev *maps homeomorphically onto the following subset of X up to an element of G;*

(1) $(\rho, \mathrm{dev}) : (S^1, \pi, \tilde{M}) \to (U(1), U(n,1), S^{2n+1} - L(\rho(\pi)))$.

(2) $(\rho, \mathrm{dev}) : (S^1, \pi, \tilde{M}) \to (SO(2), SO(n-1,1)^0 \times SO(2), S^n - L(\rho(\pi)))$.

Here $\rho : \pi \to \rho(\pi) \subset U(n,1)$ (resp. $PO(n-1,1)^0 \times SO(2)$) is an isomorphism and $L(\rho(\pi))$ is the limit set of $\rho(\pi)$ lying in S^{2n-1} (resp. S^{n-2}).

Proof. Since M has a fixed point by the condition (i) of (5.1.2), we have a lift of action $(S^1, \tilde{M})$ such that $\tilde{M}$ has a fixed point (cf. [6]). Then it follows from Proposition 3 and Note 2 of [27] that

$$\mathrm{dev} : \tilde{M} \to S^{2n+1} - L(\rho(\pi)) \text{ is homeomorphic}$$

and

$$\rho : (S^1, \pi) \to (\rho(S^1), \rho(\pi)) \subset (U(n-m), P(U(m+1,1) \times U(n-m))).$$

is an isomorphism for some $m \leqq n-1$. Moreover the limit set $L(\rho(\pi)) \subset S^{2m+1}$ and $S^{2m+1} - L(\rho(\pi))$ is the fixed point set of $\rho(S^1)$. In particular we have that $M \approx S^{2n+1} - L(\rho(\pi))/\rho(\pi)$ and $\mathrm{Fix}(S^1, M) = S^{2m+1} - L(\rho(\pi))/\rho(\pi)$. On the other hand the CR action (S^1, M) is topologically equivalent to the action of (5.1.2) which implies that $\mathrm{Fix}(S^1, M) \approx S^{2n-1} - L(\rho(\pi^*))/\pi^*$. Hence $m = n-1$. It follows that $\rho(S^1) = U(1)$ and $P(U(n,1) \times U(1)) = U(n,1)$. The fixed point set of $U(1)$ is $S^{2n-1} - L(\rho(\pi))$ in this case. The similar result holds for the conformal case when we note the results of [19],[26]. Q.E.D.

We shall check that the conditions of (4.3.7) are satisfied for $\mathcal{CR}(S^1, M)$ and $\mathcal{CO}(S^1, M)$.

Remark **5.1.4.**
(1) Let (ρ, dev) be a spherical CR structure on M. Each $(g \circ \rho \circ g^{-1}, g \circ \mathrm{dev})$ for $g \in G(= PU(n,1))$ represents the same structure as (ρ, dev) by the definition. The structure on M does not depend on the choice of geometric $(2n-1)$-sphere S^{2n-1} such that $L(\rho(\pi)) \subset S^{2n-1}$ by

Proposition 5.1.3. When we choose $K = U(1)$ as K of (4.3.7 $*$), for every representation ρ there exists $g \in G$ such that $g \circ \rho \circ g^{-1} : S^1 \to U(1)$ is an isomorphism. And so the condition (4.3.7 $*$) is satisfied for $\mathcal{CR}(S^1, M)$, similarly for $\mathcal{CO}(S^1, M)$ if we choose $K = SO(2)$. Then it is easy to see that the centralizer

$$C_G(K) = \begin{cases} U(n,1) \text{ for } G = PU(n+1,1) \\ SO(n-1,1)^0 \times SO(2) \text{ for } G = PO(n+1,1). \end{cases}$$

Recall that $\Omega_0^{CR}(S^1, M)$ is the subspace of $\Omega^{CR}(S^1, M)$ whose holonomy represntations belong to $U(n,1)$ (cf. (4.3.7)). It is easy to see that two such pairs (ρ, dev), (ρ', dev') represent the same structure if and only if there is an element $h \in U(n,1)$ such that $\mathrm{dev}' = h \circ \mathrm{dev}$ and $h \circ \rho \circ h^{-1} = \rho'$. The condition (4.3.7 $**$) is satisfied by this fact. As in (4.3.7), $\Omega_0(S^1, M) \to \mathcal{CR}(S^1, M)$ is surjective. We have the commutative diagram from (4.3.9)

$$\text{(5.1.5)} \qquad \begin{array}{ccc} \Omega_0^{CR}(S^1, M) & \xrightarrow{\widehat{\mathrm{hol}}} & \mathrm{Hom}(\pi, U(n,1)) \\ \downarrow & & \downarrow \\ \mathcal{CR}(S^1, M) & \xrightarrow{\mathrm{hol}} & R(\pi, U(n,1))/U(n,1), \end{array}$$

similarly for $\mathcal{CO}(S^1, M)$.

(2) If $\tilde{M}^*$ is the orbit space of S^1 then the action $(\pi, \tilde{M})$ induces an action of π on $\tilde{M}^*$. Let $(\pi^*, \tilde{M}^*)$ be its action. The induced map $\pi \to \pi^*$ is an isomorphism. Let $U(1) \to U(n,1) \to PU(n,1)$ be the exact sequence for the CR case. The projection P maps $\rho(\pi)$ isomorphically onto its image $\rho(\pi)^*$. The homomorphism ρ induces an isomorphism $\rho^* : \pi^* \to \rho(\pi)^*$ such that the diagram is commutative:

$$\begin{array}{ccc} \pi & \xrightarrow{\rho} & \rho(\pi) \\ \downarrow & & \downarrow \\ \pi^* & \xrightarrow{\rho^*} & \rho(\pi)^* \end{array}.$$

Definition 5.1.6. $R_{CR}(\pi^*)$ is the subspace of $\mathrm{Hom}(\pi^*, PU(n,1))$ such that for each element ρ^* there exists a homeomorphism $f^* : \mathbf{D}^{2n} \to \mathbf{D}^{2n}$ such that $\rho^*(\alpha) = f^* \circ \alpha \circ f^{*-1}$ ($\alpha \in \pi^*$) and in addition the restriction $f^*|\mathbf{H}_{\mathbf{C}}^n$ is a smooth map. Note that $\rho^* : \pi^* \to \rho^*(\pi^*)$ is an isomorphism and $\rho^*(\pi^*)$ is discrete in $PU(n,1)$. $R_{CO}(\pi^*)$ is defined similarly to be the subspace of $\mathrm{Hom}(\pi^*, PO(n-1,1)^0)$.

Remark **5.1.7.** Given an isomorphism $\rho^* : \pi^* \to \rho^*(\pi^*) \subset \mathrm{Hom}(\pi^*, PU(n,1))$, it does not always exist such a homeomorphism $f^* : \mathbf{D}^{2n} \to \mathbf{D}^{2n}$. However, for example $n = 1$ ($PU(1,1) \approx PO(2,1) \approx \mathrm{PSL}_2(\mathbf{R})$), and ρ^* is type-preserving (cf. [p.302, 23]), then it is well known that there exists a quasiconformal homeomorphism $f^* : \mathbf{D}^2 \to \mathbf{D}^2$ which induces ρ^*. In this case the space $R_{CR}(\pi^*)$ is alternatively defined to be the set of those elements consisting of type-preserving discrete faithful representations of π^* into $PU(1,1)$. Note that $R_{CR}(\pi^*) \approx R_{CO}(\pi^*)$ in this case (cf. [26].)

Definition 5.1.8. Let $R_{CR}(\pi)$ be the subspace of $\mathrm{Hom}(\pi, U(n,1))$ whose elements project down to $R_{CR}(\pi^*)$. If we note the exact sequence, $\mathrm{Hom}(\pi, U(1)) \to \mathrm{Hom}(\pi, U(n,1)) \to \mathrm{Hom}(\pi^*, PU(n,1))$, then it follows that

$$R_{CR}(\pi) = R_{CR}(\pi^*) \times \mathrm{Hom}(\pi, U(1)). \tag{5.1.9}$$

Similarly,

$$R_{CO}(\pi) = R_{C0}(\pi^*) \times \mathrm{Hom}(\pi, SO(2)). \tag{5.1.10}$$

Lemma 5.1.11. $\widehat{\mathrm{hol}}$ *maps* $\Omega_0^{CR}(S^1, M)$ *into* $R_{CR}(\pi)$, *similarly for* $\Omega_0^{CO}(S^1, M)$.

Proof. Let (ρ, dev) be a representative element of $\Omega_0^{CR}(S^1, M)$. We know that $(\rho, \mathrm{dev}) : (S^1, \pi, \tilde{M}) \to (U(1), U(n,1), S^{2n+1} - L(\rho(\pi)))$ is homeomorphic. Then (ρ, dev) induces a homeomorphism

$$(\rho^*, \mathrm{dev}^*) : (\pi^*, \tilde{M}^*) \to (PU(n,1), \mathbf{D}^{2n} - L(\rho^*(\pi^*))).$$

Note from (ii) of (5.1.2) that $\tilde{M}^* = \mathbf{D}^{2n} - L(\rho^*(\pi^*))$. In particular $\mathrm{dev}^* : \mathbf{H}^n_{\mathbf{C}}$ ($= \mathrm{Int}\, \tilde{M}^*$) $\to \mathbf{H}^n_{\mathbf{C}}$ is homeomorphic. Since dev^* is still an immersion, the complete metric of $\mathbf{H}^n_{\mathbf{C}}$ with $\mathrm{Iso}(\mathbf{H}^n_{\mathbf{C}}) = PU(n,1)$ induces a Riemannian metric such that dev^* is a local isometry. And hence $\mathrm{dev}^* : \mathbf{H}^n_{\mathbf{C}} \to \mathbf{H}^n_{\mathbf{C}}$ is an isometry. The space $\tilde{M}^*$ has a compactification $\mathbf{D}^{2n} = \tilde{M}^* \cup L(\pi^*)$. The isometry dev^* extends to a homeomorphism $f^* : \mathbf{D}^{2n} \to \mathbf{D}^{2n}$ for which $f^*(L(\pi^*)) = L(\rho^*(\pi^*))$ and $\rho^*(\alpha) = f^* \circ \alpha \circ f^{*-1}$ ($\alpha \in \pi^*$). It follows by the definition 5.1.7 that $\rho^* \in R_{CR}(\pi^*)$ and thus $\rho \in R_{CR}(\pi)$. Q.E.D.

The diagram (5.1.5) reduces to the following commutative one.

$$\begin{array}{ccc} \Omega_0^{CR}(S^1,M) & \xrightarrow{\widehat{\mathrm{hol}}} & R_{CR}(\pi) \\ \downarrow & & \downarrow \\ \mathcal{CR}(S^1,M) & \xrightarrow{\mathrm{hol}} & R_{CR}(\pi)/U(n,1). \end{array} \tag{5.1.12}$$

Since $\mathrm{Hom}(\pi,U(1))$ is a k dimensional torus for some k, it follows from (5.1.9) that

(5.1.13) $R_{CR}(\pi)/U(n,1)=R_{CR}(\pi^*)/PU(n,1)\times T^k$.

(5.1.14) $R_{CO}(\pi)/SO(n-1,1)^0\times SO(2)=R_{CO}(\pi^*)/SO(n-1,1)^0\times T^k$.

5.2. Structure of deformation spaces $\mathcal{T}(S^1,M)$

There is the natural homomorphism $\varphi:\mathrm{Diff}(S^1,M)\to\mathrm{Out}(\Gamma)$. Note that Ker φ contains the subgroup $\mathrm{Diff}^0(S^1,M)$. Recall that there exists a right action of $\mathrm{Diff}(S^1,M)/\mathrm{Diff}^0(S^1,M)$ on $\mathcal{T}(S^1,M)$. We examine the structure of $\mathcal{T}(S^1,M)$ in terms of representation spaces, where $\mathcal{T}(S^1,M)=\mathcal{CR}(S^1,M)$ or $\mathcal{CO}(S^1,M)$.

Proposition 5.2.1. *Let*

$$\mathrm{hol}:\mathcal{CR}(S^1,M)\to R_{CR}(\pi^*)/PU(n,1)\times T^k$$

and

$$\mathrm{hol}:\mathcal{CO}(S^1,M)\to R_{CO}(\pi^*)/SO(n-1,1)^0)/SO(n-1,1)^0\times T^k$$

be the holonomy map respectively. Put $G=\mathrm{Ker}\,\varphi/\mathrm{Diff}^0(S^1,M)$. *If the fundamental group* π *is torsionfree, then*

(1) hol *is surjective.*
(2) *Each fiber of* hol *consists of the* G*-orbit.*
(3) *There exists a neighborhood* U *for each point of* $\mathcal{T}(S^1,M)$ *such that* $\mathrm{hol}(U)$ *is open.*

Proof. We prove for the CR case. (1) Given $\rho\in R_{CR}(\pi)$, $\rho(\pi)$ is discrete in $U(n,1)$ and $L(\rho(\pi))\subset S^{2n-1}$. Then the group $\rho(\pi)$ acts properly discontinuously on $S^{2n-1}-L(\rho(\pi))$. Since $\rho(\pi)$ is torsionfree, it acts freely. We obtain a spherical CR manifold $M(\rho)=S^{2n-1}-L(\rho(\pi))/\rho(\pi)$. It is noted that $U(1)$ acts on $M(\rho)$ by CR automorphisms. We then show that M is diffeomorphic to $M(\rho)$. For this, let ρ^* be an element of $R_{CR}(\pi^*)$ induced from ρ. There is a homeomorphism

$f^* : \mathbf{D}^{2n} \to \mathbf{D}^{2n}$ such that $\rho^*(\pi^*) = f^*\pi f^{*-1}$. If we note that $M(\rho)^* = \mathbf{D}^{2n} - L(\rho^*(\pi^*))/\rho(\pi^*)$ then the map f^* induces a homeomorphism $h^* : M^* \to M(\rho)^*$. Consider the following diagram (cf. (5.1.1));

$$\begin{tikzcd}
 & \pi \arrow[r, "\approx"] \arrow[d] & \pi^* \arrow[d] \\
S^1 \arrow[r] \arrow[d, equal] & \tilde{M} - \tilde{F} \arrow[r] \arrow[d] & \operatorname{Int} M^* (= \mathbf{H}^n_{\mathbf{C}}) \arrow[d] \\
S^1 \arrow[r] & M - F \arrow[r] & \mathbf{H}^n_{\mathbf{C}}/\pi^*,
\end{tikzcd}$$

where $\tilde{F} \approx F$ is the fixed point set of S^1. It follows that $M - F = \mathbf{H}^n_{\mathbf{C}}/\pi^* \times S^1$. The same is true for $M(\rho)$. Then we can find an equivariant homeomorphism $h_1 : M - F \to M(\rho) - F(\rho)$ which induces $h^*| \operatorname{Int} M^*$. Here $F(\rho)$ is the fixed point set of $U(1)$ in $M(\rho)$. Since $F \approx \partial M^* \xrightarrow{h^*} \partial M(\rho)^* \approx F(\rho)$, we can choose a homeomorphism $h_2 : F \to F(\rho)$ which covers h^*. Combining h_1 and h_2, it is easy to construct an equivariant homeomorphism $h : M \to M(\rho)$. Therefore M admits an S^1 invariant spherical CR structure which is mapped by hol to ρ. This proves (1).
(2) Suppose that hol $([\rho, \mathrm{dev}])$ = hol $([\rho', \mathrm{dev}'])$. Then it follows that $\rho' = g \circ \rho \circ g^{-1}$ for some $g \in U(n,1)$. Since dev : $\tilde{M} \to S^{2n+1} - L(\rho(\pi))$, and $\mathrm{dev}' : \tilde{M} \to S^{2n+1} - L(\rho'(\pi))$ are homeomorphisms, we can put $\tilde{f} = (\mathrm{dev}')^{-1} \circ g \circ \mathrm{dev}$. It is easy to see that $\tilde{f}$ induces an element $f \in \mathrm{Diff}(S^1, M)$ such that $\varphi(f) = 1$. Hence $(f) \in G$. By definition we have that $[\rho', \mathrm{dev}'] \circ (f) = [\rho, \mathrm{dev}]$.
(3) It follows from the Holonomy theorem 4.3.9 that there exists a neighborhood $\tilde{U}$ in $\Omega_0^{CR}(S^1, M)$ for which $\widehat{\mathrm{hol}}(\tilde{U})$ is open in $R(\pi, U(n,1))$. Let U be the image of $\tilde{U}$ in $\mathcal{CR}(S^1, M)$. Since vertical arrows are open maps in the diagram (5.1.6), we obtain that hol(U) is open. It can be shown similarly for $\mathcal{CO}(S^1, M)$. Q.E.D.

Corollary 5.2.2. *Suppose that* $\hat{\varphi} : \mathrm{Diff}(S^1, M)/\mathrm{Diff}^0(S^1, M) \to \mathrm{Out}(\pi)$ *is injective. Then* $\mathcal{CR}(S^1, M)$ *is homeomorphic to* $R_{CR}(\pi^*)/PU(n,1) \times T^k$ *(Similarly,* $\mathcal{CO}(S^1, M)$ *is homeomorphic to* $R_{CO}(\pi^*)/SO(n-1,1)^0 \times T^k$*).*

See [26] for examples of this Corollary. (Indeed, Ker $\varphi = \mathrm{Diff}^0(S^1, M)$ if $\dim M = 3$.) Recall that there exists a right action of $\mathrm{Diff}(S^1, M)/\mathrm{Diff}^0(S^1, M)$ on $\Omega(S^1, M)$ (cf. (4.3.4)). Let $G =$

$\operatorname{Ker}\varphi/\operatorname{Diff}^0(S^1,M)$ be as before. In order to study the action of G on $\Omega(S^1,M)$, we need the following lemma.

Lemma 5.2.3. *Suppose that π is not virtually solvable.*

(1) *$U(n,1)$ acts properly on $R_{CR}(\pi)$.*

(2) *$SO(n-1,1)^0 \times SO(2)$ acts properly on $R_{CO}(\pi)$.*

Proof. We prove (1). Recall that $R_{CR}(\pi) = R(\pi, U(n,1)) \approx R(\pi^*, PU(n,1)) \times T^k$. Let $P: R(\pi^*, PU(n,1)) \times T^k \to R(\pi^*, PU(n,1))$ be the projection. Given a compact subset K of $R(\pi^*, PU(n,1)) \times T^k$, put $K^* = P(K)$. Let $\zeta_{U(n,1)}(K) = \{g \in U(n,1) \mid g \cdot K \cap K \neq \emptyset\}$ where those elements of $U(n,1)$ act by conjugation on $R(\pi, U(n,1))$. Recall that $P: U(n,1) \to PU(n,1)$ is the projection with kernel isomorphic to $U(1)$. Then it follows that $\zeta_{U(n,1)}(K) \subset P^{-1}(\zeta_{PU(n,1)}(K^*)) \approx \zeta_{PU(n,1)}(K^*) \times U(1)$. Since $\zeta_{U(n,1)}(K)$ is a closed subset in $U(n,1)$, it suffices to show that $\zeta_{PU(n,1)}(K^*)$ is compact. By the hypothesis, $\pi \approx \pi^*$ is not virtually solvable. Then the set $R(\pi^*, PU(n,1))$ consists of stable representations in the sense of Johnson-Millson ([p.53, 24]). And so it follows from Proposition 1.1 ([24]) that $PU(n,1)$ acts properly on the subset $R(\pi^*, PU(n,1))$. Hence $\zeta_{PU(n,1)}(K^*)$ is compact.
(2) follows similarly when we note from Proposition 1.1 ([24]) that the set $R(\pi^*, SO(n-1,1)^0)$ consists of stable representations. Q.E.D.

Proposition 5.2.4. *Suppose that π is not virtually solvable. Let $G = \operatorname{Ker}\varphi/\operatorname{Diff}^0(S^1,M)$ be as before. Then G acts properly discontinuously on $\mathcal{T}(S^1,M)$ where $\mathcal{T}(S^1,M) = \mathcal{CR}(S^1,M)$ or $\mathcal{CO}(S^1,M)$.*

Proof. When K is a compact subset of $\mathcal{T}(S^1,M)$, it has only to be shown that $\zeta_G(K) = \{(f) \in G \mid K \circ (f) \cap K \neq \emptyset\}$ is compact. Suppose we have sequences $\{f_i\} \in G$ and $[\rho_i, \mathrm{dev}_i], [\rho'_i, \mathrm{dev}'_i] \in K$ such that $[\rho_i, \mathrm{dev}_i] \circ (f_i) = [\rho'_i, \mathrm{dev}'_i]$ where $\{[\rho_i, \mathrm{dev}_i]\}$ and $\{[\rho'_i, \mathrm{dev}'_i]\}$ converge to some $[\rho, \mathrm{dev}]$ and $[\rho', \mathrm{dev}']$ in K respectively. Then by the remark (1) of (5.1.5) there exists a sequence $\{g_i\} \in U(n,1)$ (resp. $SO(n-1,1)^0 \times SO(2)$) such that (i) $g_i \circ \mathrm{dev}'_i = \mathrm{dev}_i \circ \tilde{f}_i$, (ii) $g_i \circ \rho'_i \circ g_i^{-1} = \rho_i \circ \mu(\tilde{f}_i)$. Since each f_i lies in Ker φ, it follows that (ii)$'$ $g_i \circ \rho'_i \circ g_i^{-1} = \rho_i$. We note that $\{\rho_i\}, \{\rho'_i\} \in R_{CR}(\pi)$ (resp. $R_{CO}(\pi)$), and $\{\rho_i\}$ (resp. $\{\rho'_i\}$) $\to \rho$ (resp. ρ'). By Lemma 5.2.3, (ii)$'$ implies that the sequence $\{g_i\}$ converges to some $g \in U(n,1)$ (resp. $SO(n-1,1)^0 \times SO(2)$).

On the other hand, the maps dev_i, dev'_i induce homeomorphisms $\hat{\mathrm{dev}}_i : M \to S^m - L(\rho_i(\Gamma))/\rho_i(\Gamma)$, $\hat{\mathrm{dev}}'_i : M \to S^m - L(\rho'_i(\Gamma))/\rho'_i(\Gamma)$, where $m = 2n+1$ or n. Each g_i defines a homeomorphism $\hat{g}_i : S^m - L(\rho_i'(\Gamma))/\rho_i'(\Gamma) \to S^m - L(\rho_i(\Gamma))/\rho_i(\Gamma)$. Therefore we obtain from (i) that $f_i = (\hat{\text{dev}}_i)^{-1}(\hat{g}_i \circ \hat{\text{dev}}_i')$. Since M is compact, $(\hat{\text{dev}})^{-1}(\hat{g} \circ \hat{\text{dev}}')$ is also defined so that $\{f_i\} \to (\hat{\text{dev}})^{-1}(\hat{g} \circ \hat{\text{dev}}')$. Put $f = (\hat{\text{dev}})^{-1}(\hat{g} \circ \hat{\text{dev}}') : M \to M$. Since each $f_i \in \operatorname{Ker} \varphi$, it follows that f represents an element of G. Hence $\zeta_G(K)$ is compact. Q.E.D.

For example, $G = \operatorname{Ker} \varphi / \operatorname{Diff}^0(S^1, M)$ is trivial if dim $M = 3$ (cf. [26]). However in general there are examples in higher dimentions for which G is nontrivial. For them we have the following.

Proposition 5.2.5. *Suppose that π is not virtually solvable. Then G acts freely on $\mathcal{T}(S^1, M)$, where $\mathcal{T}(S^1, M) = \mathcal{CR}(S^1, M)$ or $\mathcal{CO}(S^1, M)$.*

Proof. We prove the case that $\mathcal{T}(S^1, M) = \mathcal{CR}(S^1, M)$. Suppose that $[\rho, \text{dev}] \circ (f) = [\rho, \text{dev}]$. Then there exists an element $g \in U(n,1)$ such that (1) $g \circ \text{dev} = \text{dev} \circ \tilde{f}$, (2) $g \circ \rho \circ g^{-1} = \rho \circ \mu(\tilde{f}) = \rho$. If ρ^* is the corresponding element in $R(\Gamma^*, PU(n,1))$ then (2) implies that (3) $g^* \circ \rho^* \circ g^{*-1} = \rho^*$ for $g^* \in PU(n,1)$. The group $\rho^*(\Gamma^*)$ acts invariantly in $\mathbf{H}^n_{\mathbf{C}}$. Suppose that $\rho^*(\Gamma^*)$ leaves invariant a totally geodesic subspace $\mathbf{H}^k_{\mathbf{C}}$ of $\mathbf{H}^n_{\mathbf{C}}$ for $1 \leqq k \leqq n$. Then $\rho^*(\Gamma^*)$ leaves S^{2k-1} invariant so that it belongs to the subgroup $\operatorname{Aut}_{CR}(S^{2n-1}, S^{2k-1}) = P(U(k,1) \times U(n-k))$. Let $Q : P(U(k,1) \times U(n-k)) \to PU(k,1)$ be the projection whose kernel is isomorphic to $U(n-k)$. We can assume that k is the smallest dimension. And so $Q(\rho^*(\Gamma^*))$ is Zariski-dense in $PU(k,1)$. The condition (3) implies that g^* leaves also S^{2k-1}. It implies that $g^* \in P(U(k,1) \times U(n-k))$. Then the element $Q(g^*)$ centralizes the group $Q(\rho^*(\Gamma^*))$ and so does its algebraic closure. Since the algebraic closure is $PU(k,1)$ by the above remark, $Q(g^*)$ must be the identity map.
In particular we obtain that $g^* \in U(n-k)$. As $U(n,1) = P(U(n,1) \dot{\times} U(1))$, it follows that $g \in U(n-k) \times U(1)$ $(= P(\mathcal{Z}(k,1) \times U(n-k) \times U(1))$ $)$ where $\mathcal{Z}(k,1)$ is the center of $U(k,1)$. On the other hand, $\text{dev} : \tilde{M} \to S^{2n+1} - L(\rho(\pi))$ is homeomorphic and by (1) it follows that $\tilde{f} = (\text{dev})^{-1} \circ g \circ \text{dev}$. It is noted that $L(\rho(\pi)) = L(\rho^*(\pi^*)) \subset S^{2k-1}$ and S^{2k-1} is the fixed point set of $U(n-k)$. We can choose a path c in $U(n-k)$ between g^* and the identity map. By the above remark there is a lift $\tilde{c}$ of the path c starting at g with its endpoint $\tilde{c}(1) \in U(1)$. Since dev is equivariant with respect to S^1 and $U(1)$ actions, we conclude that $\tilde{f}$ is isotopic to $\tilde{c}(1)$. It is easy to check that $\tilde{f}$ is isotopic to the identity map of $\tilde{M}$. Hence f belongs to $\operatorname{Diff}^0(S^1, M)$. That is, $(f) \equiv 1$ in G.

We can prove similary for the case that $\mathcal{T}(S^1, M) = \mathcal{CO}(SO(2), M)$.
Q.E.D.

Corollary 5.2.6. *Let M be a closed S^1 invariant spherical CR manifold of dimension $2n+1$ (resp. a closed S^1 invariant conformally flat n-manifold). Suppose that the orbit space M^* is a complex Kleinian orbifold $\mathbf{D}^{2n} - L(\pi^*)/\pi^*$ with nonempty boundary (resp. a Kleinian orbifold $\mathbf{D}^{n-1} - L(\pi^*)/\pi^*$ with nonempty boundary) and π^* is torsionfree.*
If $\pi_1(M)$ is not virtually solvable, then

(1) $\mathrm{hol} : \mathcal{CR}(S^1, M) \to R_{CR}(\pi^*)/PU(n,1) \times T^k$ *is a covering map whose fiber is isomorphic to G.*
(2) $\mathrm{hol} : \mathcal{CO}(S^1, M) \to R_{CO}(\pi^*)/SO(n-1,1)^0 \times T^k$ *is a covering map whose fiber is isomorphic to G.*

Proof. The group G acts properly discontinuously and freely on $\mathcal{T}(U(1), M)$ by Lemma 5.2.3 and Proposition 5.2.4. Thus there exists a neighborhood U in $\mathcal{T}(U(1), M)$ such that $U \circ g \cap U = \emptyset$ if and only if $g \neq 1$ for $g \in G$. Then the result follows from Proposition 5.2.1.
Q.E.D.

References

[1] B. Apanasov, Two kinds of deformation of conformal structure which are distinct from bending, Proceedings of the Thirteenth Rolf Nevalinna Colloquium and Teichmüller theory Conference, Helsinki (1987).

[2] ———, Deformations of conformal structures on hyperbolic manifolds, (1990), to appear.

[3] W. Ballmann, M. Gromov and V. Schroeder, "Manifolds of nonpositive curvature", Progress in Math. 61, Birkhäuser, 1985.

[4] A. Beardon, "The geometry of discrete subgoups", Springer, New York, 1983.

[5] L. Bers, On boundaries of Teichmüller spaces and on Kleinian groups I, Ann. of Math., **91** (1970), 570–600.

[6] G. Bredon, "Introduction to compact transformation groups", Academic Press, New York, 1972.

[7] D.M. Burns and S. Schnider, Spherical hypersurfaces in complex manifolds, Invent. Math., **33** (1976), 223–246.

[8] R. Canary, D.B.A. Epstein and P. Green, Notes on notes of Thurston, London Mathematical Society Lecture Notes, **111** (1987), 3–92, Cambridge University Press.

[9] A. Casson and S. Bleiler, "Automorphisms of surfaces after Nielsen and Thurston", London Mathematical Society Student Texts 9, Cambridge University Press, 1988.

[10] P.J. Cheeger and D. Ebin, "Comparison Theorems in Riemannian Geometry", North-Holland Publ. Co., Amsterdam, 1975.

[11] S. Chen and L. Greenberg, Hyperbolic Spaces, Contribution to Analysis (1974), 49–87, Academic Press, New York.

[12] P.E. Conner and F. Raymond, Injective operations of the toral groups, Topology, **10** (1971), 283–296.

[13] D.B.A. Epstein and A. Marden, Convex hulls in hyperbolic spaces, a theorem of Sullivan,and measured pleated surfaces, London Mathematical Society Lecture Notes, **111** (1987), 114–253, Cambridge University Press.

[14] D. Fried, Closed similarity manifolds, Comment. Math. Helv., **55** (1980), 576–582.

[15] D. Gallo, W. Goldman and R. Porter, Projective structures with monodromy in $\mathrm{PSL}(2, \mathbf{R})$, Preprint (1989).

[16] W. Goldman, Projective structures with Fuchsian holonomy, J. Diff. Geom., **25** (1987), 297–326.

[17] ———, Geometric Structures on manifolds and varieties of representations, Contemporary Math., **74** (1988), Proceedings of the Conference on Representation Theory.

[18] W. Goldman and Y. Kamishima, Topological rigidity of developing maps with applications to conformally flat structures, Contemporary Math. A.M.S., **74** (1988), 199–203.

[19] ———, Conformal automorphisms and conformally flat manifolds, Trans. A.M.S., **323** (1991), 797–810.

[20] W. Goldman and J. Millson, The deformation theory of representations of fundamental groups of compact Kähler Manifolds, Publ. I.H.E.S., to appear.

[21] W. Goldman and J. Millson, Local rigidity of discrete groups acting on complex hyperbolic space, Invent. Math., **88** (1987), 495–520.

[22] R. Gunning, Special coordinate coverings of Riemann surfaces, Math. Ann., **170** (1967), 67–85.

[23] W. Harvey, "Discrete groups and automorphic functions", Academic Press, 1972.

[24] D. Johnson and J. Millson, Deformation spaces associated to compact hyperbolic manifolds, Discrete groups in Geometry and Analysis, Progress in Math, **67** (1987), 48–106.

[25] Y. Kamishima, Conformally flat manifolds whose development maps are not surjective, Trans. A.M.S., **294** (1986), 607–623.

[26] ———, Conformally flat 3-manifolds with compact automorphism groups, J. London Math. Soc. 38(2) (1988), 367–378.

[27] Y. Kamishima and T. Tusboi, CR-structures on Seifert manifolds, Invent. Math., **104** (1991), 149–163.

[28] M. Kapovich, Flat conformal structures on 3-manifolds, Preprint (1989).

[29] ———, Deformation spaces of flat conformal structures, Preprint (1989).

[30] C. Korouniotis, Deformations of hyperbolic manifolds, Math. Proc. Cambridge Phil. Soc., **98** (1985), 247–261.

[31] I. Kra, A generalization of a theorem of Poincaré, Proc. of A.M.S., **27** (1971), 299–302.

[32] R. Kulkarni, On the principle of uniformization, J. Diff. Geom., **13** (1978), 109–138.

[33] ———, Groups with domains of discontinuity, Math. Ann., **237** (1978), 253–272.

[34] R. Kulkarni and U. Pinkall, Uniformization of geometric structures with applications to conformal geometry, Proceedings, Diff. Geom. Peñscola , Lecture notes in Math., **1209** (1986), 140–209, Springer-Verlag.

[35] ———, A canonical metric for Möbius structures and its applications, Preprint.

[36] R. Kulkarni and K.B. Lee and F. Raymond, Deformation spaces for Seifert manifolds, Proceedings, Special year in Geometry and Topology, Univ. of Maryland, Lecture notes in Math., **1167** (1985), 180–216, Springer-Verlag.

[37] L. Lok, Deformations of locally homogeneous spaces and Kleinian groups, Doctoral dissertation (1984), Columbia University.

[38] A. Marden, The geometry of finitely generated Kleinian groups, Ann. of Math., **99** (1974), 383–462.

[39] S. Matsumoto, Foudations of flat conformal structure, in these proceedings.

[40] J. Morgan and P. Shalen, Valuations, trees, and degenerations of hyperbolic structures, I, Ann. of Math, **120** (1984), 401–476.

[41] ———, "Degenerations of hyperbolic structures, II: Measured Laminations,Trees and 3-Manifolds", Preprint, 1986.

[42] P. Orlik, Seifert manifolds, Lecture Notes in Math., **291** (1972), Springer-Verlag.

[43] P. Orlik and F. Raymond, Actions of $SO(2)$ on 3-manifolds, Proceedings of the Conference on Transformation groups, Lecture Notes in Math., **299** (1968), 297–318, Springer-Verlag.

[44] F. Raymond, Classification of the actions of the circle on 3-manifolds, Trans. Amer. Math. Soc., **31** (1968), 51–78.

[45] G.P. Scott, The geometries of 3-manifolds, Bull. London Math., **15** (1983), 401–487.

[46] D. Sullivan and W. Thurston, Manifolds with canonical coordinates:Some exapmles, L'ens. Math., **29** (1983), 15–25.

[47] Ser P. Tan, Representations of surfaces into $\mathrm{PSL}(2, \mathbf{R})$ and Geometric structures, Doctoral dissertation (1988), University of Maryland.

[48] W. Thurston, "The geometry and topology of three manifolds", preprint.

[49] J. Wolf, "Spaces of constant curvature", McGraw-Hill, Inc., 1967.

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On 4-Manifolds Homotopy Equivalent to the 2-Sphere

Osamu Saeki

Dedicated to Professor Akio Hattori on his 60th birthday

§1. Introduction

Let V be a compact topological 4-manifold homotopy equivalent to the 2-sphere S^2. We say such a 4-manifold is a *homotopy* S^2. The boundary ∂V of V is always a closed connected 3-manifold with the same integral homology and the same linking pairing as those of the lens space $L(p,1)$ for some p (≥ 0). We say such a 3-manifold is a *homology* $L(p,1)$. For a fixed homology $L(p,1)$, M, the homotopy S^2's bounded by M are classified up to homeomorphism by certain equivalence classes of some elements of $H_1(M;\mathbf{Z})$ ([3]).

In this paper, concerning homotopy S^2's, we consider the following problems.

(A) For a fixed homology $L(p,1)$, M, how many homotopy S^2's does M bound? Furthermore, how many of them admit smooth structures?

(B) Give a lower bound for the genera of topologically locally flatly embedded surfaces in V representing the generator γ of $H_2(V;\mathbf{Z})$. If V is smooth, what is the necessary condition for γ to be represented by a smoothly embedded 2-sphere?

(C) Let K be a tame knot in the boundary of V. Under what condition does K bound a topologically embedded flat 2-disk in V? If V and K are smooth and K bounds such a topologically embedded 2-disk, does K also bound a smoothly embedded 2-disk in V?

(D) Does there exist a homotopy S^2 admitting more than one smooth structures?

In [28], we considered problem (A) and showed that if V is a smooth homotopy S^2 satisfying a certain condition on the order of $H_1(\partial V;\mathbf{Z})$

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such that the generator of $H_2(V;\mathbf{Z})$ is represented by a smoothly embedded 2-sphere, then every *smooth* homotopy S^2 with the same boundary as V is homeomorphic to V. Furthermore we gave the exact number of homotopy S^2's bounded by the 3-manifold ∂V. In this paper we consider smoothly immersed 2-spheres instead of embedded ones and give a similar result. More precisely, we show that if V is a smooth homotopy S^2 satisfying a certain condition on the order of $H_1(\partial V;\mathbf{Z})$ such that the generator of $H_2(V;\mathbf{Z})$ is represented by a smoothly immersed 2-sphere with relatively few double points, then the same results as above hold (§2). We note that, in some cases, this result can be applied to give a lower bound for the number of double points of smoothly immersed 2-spheres representing the generator of $H_2(V;\mathbf{Z})$.

In §3 we define some topological invariants for homotopy S^2's and use them to attack problem (B). First, we define a "Casson invariant" for a homotopy S^2 using the extension of the usual Casson invariant for homology 3-spheres ([6]) to marked homology lens spaces ([4, 15]). We show that if the generator of the second homology group of a smooth homotopy S^2 is represented by a smoothly embedded 2-sphere, then its Casson invariant modulo 2 must vanish. In fact, this is proved only using the well-known theorem of Rohlin. We do not know what essential properties of a homotopy S^2 this Casson invariant reflects. Next we define Casson-Gordon invariants for a homotopy S^2 (cf. [7]) and use them to give a lower bound for the genera of topologically locally flatly embedded surfaces representing the generator of the second homology group of a homotopy S^2. In our case these invariants are the p-signatures of a certain knot in a homology 3-sphere. If a smooth homotopy S^2 consists of one 0-handle and one 2-handle, then it has already been known that a lower bound for the genera of such smoothly embedded surfaces is given by the p-signatures of the knot along which the 2-handle is attached to the 0-handle ([31]). In this paper, we extend this to general homotopy S^2's employing a method similar to that in [19].

In §4, we give a sufficient condition for certain tame knots in the boundary of a homotopy S^2 to bound topologically embedded flat 2-disks. We see that almost all knots satisfying a certain homological condition bound such 2-disks in the homotopy S^2. Furthermore, we give an example of smooth knots in the boundary of a smooth homotopy S^2 which bound topologically embedded flat 2-disks in the homotopy S^2 but never bound smooth ones. In the case of knots in the boundary of the 4-ball, the same examples have been found first by Casson and, after that, by several authors [8, 9, 18]. Our technique is similar to theirs in that we use the celebrated theorem of Donaldson [10].

As to problem (D), Akbulut [1] has recently found a *compact* homotopy S^2 with more than one smooth structures. In this paper we give an example of infinitely many *open* 4-manifolds homotopy equivalent to S^2 each of which admits at least 3 smooth structures (§5).

Throughout the paper, all homology groups are with integral coefficients unless otherwise indicated.

§2. Smooth homotopy S^2 and immersed 2-spheres

Definition. We say that a non-zero integer p satisfies *property (#)* if -1 is not a quadratic residue modulo p; i.e., if $n^2 \not\equiv -1 \pmod p$ for every integer n.

Lemma 2.1. *Let* $|p| = 2^e p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r}$ $(e \geq 0, e_i \geq 1)$ *be the prime decomposition of* $|p|$. *Then* p *satisfies property (#) if and only if* $e \geq 2$ *or* $p_i \equiv 3 \pmod 4$ *for some* p_i.

For the proof of Lemma 2.1, see, for example, the proof of Corollary 3.11 in [28]. Note that, by Lemma 2.1, if $|p| \equiv 0, 3 \pmod 4$ then p satisfies property (#).

Let V be a smooth oriented homotopy S^2. Define $k_+(V)$ (resp. $k_-(V)$) to be the minimum number of positive (resp. negative) double points of smoothly immersed self-transverse 2-spheres representing the generator of $H_2(V)$. We call $k_+(V)$ (resp. $k_-(V)$) the *positive* (resp. *negative*) *kinkiness* of V. Then our main theorem of this section is the following.

Theorem 2.2. *Let* V *be a smooth oriented homotopy* S^2 *bounded by a homology* $L(p,1)$, M, *and let* $\gamma \in H_2(V)$ *be a generator. We assume* $p = |\gamma^2|$ *satisfies property (#). Let* ε *be the sign of* γ^2. *If*

$$k_\varepsilon(V) \leq \begin{cases} (p-6)/4 & (p\text{: even}) \\ (p-1)/4 & (p\text{: odd}, \neq 15, 21) \\ 2 & (p = 15) \\ 4 & (p = 21), \end{cases}$$

then the following holds.

(1) *Every smooth homotopy* S^2 *bounded by* M *is homeomorphic to* V.
(2) *Every homeomorphism* $h : M \to M$ *acts on* $H_1(M)$ *by the multiplication of* ± 1.
(3) *If* $p = 2^e p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r}$ $(e \geq 0, e_i \geq 1)$ *is the prime decomposition of* p, *then the number of homeomorphism types of homotopy* S^2*'s*

bounded by M is equal to

$$\begin{cases} 2^{r-1} & (e=1) \\ 2^{r} & (e=0,2) \\ 2^{r+1} & (e\geq 3). \end{cases}$$

Remark. Even when p does not satisfy property (#), part (2) of Theorem 2.2 does hold if the homeomorphism h is orientation preserving.

Before we prove Theorem 2.2, we describe some of its consequences. The proof will be given at the end of this section.

Let K be a smooth knot in the oriented 3-sphere S^3. For an integer p, we denote by $V(K;p)$ the smooth oriented homotopy S^2 obtained by attaching a 2-handle to the 4-ball D^4 along the knot K with the p-framing. Furthermore we denote by $M(K;p/1)$ the boundary 3-manifold of $V(K;p)$. Note that $M(K;p/1)$ is diffeomorphic to the 3-manifold obtained by performing the $p/1$-Dehn surgery on the knot K ([26]).

We denote by $u(K)$ the unknotting number of a knot K in S^3 (for example, see [16]). Note that K bounds in the 4-ball a smoothly immersed self-transverse 2-disk with $u(K)$ double points ([9]). Taking the union of this immersed 2-disk and the core disk of the 2-handle, we can represent the generator of $H_2(V(K;p))$, for any p, by a smoothly immersed 2-sphere with $u(K)$ double points; i.e., we always have

$$k_{\pm}(V(K;p)) \leq k_{+}(V(K;p)) + k_{-}(V(K;p)) \leq u(K).$$

Then we obtain the following immediately.

Corollary 2.3. *Let K be a smooth knot in S^3 and let p be an integer satisfying property (#). If $|p| \geq 4u(K)+6$, then the following holds.*

(1) *Every smooth homotopy S^2 bounded by $M(K;p/1)$ is homeomorphic to $V(K;p)$.*
(2) *Every homeomorphism $h : M(K;p/1) \to M(K;p/1)$ acts on $H_1(M(K;p/1))$ by the multiplication of ± 1.*
(3) *If $|p| = 2^e p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r}$ $(e \geq 0, e_i \geq 1)$ is the prime decomposition of $|p|$, then the number of homeomorphism types of homotopy S^2's bounded by $M(K;p/1)$ is equal to*

$$\begin{cases} 2^{r-1} & (e=1) \\ 2^{r} & (e=0,2) \\ 2^{r+1} & (e\geq 3). \end{cases}$$

This corollary shows that, *for any knot* K, if $|p|$ is sufficiently large and satisfies property (#), then $M(K;p/1)$ satisfies the above properties. This is the major difference between Theorem 2.2 and the result obtained in [28]; the latter is applicable only to slice knots.

Let a and b be relatively prime integers. Then we denote by $T(a,b)$ the left-hand ($ab > 0$) or the right-hand ($ab < 0$) torus knot of type (a,b). As a corollary of Theorem 2.2 we have the following.

Corollary 2.4. *Every smooth homotopy S^2 bounded by the lens space $L(4n+3,4)$ ($n \geq 0$) is homeomorphic to the handlebody $V(T(2,2n+1);-(4n+3))$.*

Proof. By Moser [25], $\partial V(T(2,2n+1);-(4n+3)) = M(T(2,2n+1);-(4n+3)/1)$ is homeomorphic to the lens space $L(4n+3,4)$. Note that the integer $p = 4n+3$ always satisfies property (#). On the other hand, it is well-known that $u(T(2,2n+1)) \leq n$. Thus by the previous remark, $k_-(V(T(2,2n+1);-p)) \leq n = (p-3)/4$. Since p is odd, the result follows from Theorem 2.2 unless $p = 15$. If $p = 15$, it can be shown that the number of homotopy S^2's bounded by $L(15,4)$ is 1 (see Example 2.8 of [28]). This completes the proof.

In [28], we remarked that there are exactly 1401 homotopy S^2's bounded by the lens spaces $L(p,q)$ with $2 \leq p \leq 100$ and that among them there exist at least 701 homotopy S^2's which cannot admit any smooth structures and at least 274 homotopy S^2's admitting smooth structures. Using the above Corollary 2.4 and the technique used in the proof of Theorem 2.2, we can find, among the 1401 homotopy S^2's, additional 16 non-smoothable homotopy S^2's. We also note that, using the result of Maruyama [23], we can find additional 14 homotopy S^2's admitting smooth structures.

Next we indicate how Theorem 2.2 can be applied to give lower bounds for the kinkinesses of a homotopy S^2.

Definition. Let V be a homotopy S^2 and let $\delta \in H_2(V,\partial V)$ be a generator. Then we define $\beta(V) = \partial\delta \in H_1(\partial V)$, where $\partial : H_2(V,\partial V) \to H_1(\partial V)$ is the boundary homomorphism. Note that $\beta(V)$ generates $H_1(\partial V)$ and is determined up to the multiplication of ± 1.

Remark. If $H_1(\partial V)$ is finite cyclic of order p, we always have $\mathrm{lk}(\beta(V),\beta(V)) = \pm 1/p$, where lk: $H_1(\partial V) \times H_1(\partial V) \to \mathbf{Q}/\mathbf{Z}$ is the linking pairing of ∂V.

Theorem 2.5. *Let V be a smooth homotopy S^2 with $H_1(\partial V)$ finite cyclic of order p. Suppose there exists a homeomorphism $h : \partial V \to \partial V$*

such that $h_*(\beta(V)) = r\beta(V)$ *with* $r \not\equiv \pm 1 \pmod p$ *and* $r^2 \equiv 1 \pmod p$. *Then we have*

$$k_\varepsilon(V) \geq \begin{cases} (p-4)/4 & (p: \textit{even}) \\ (p+1)/4 & (p: \textit{odd}, \neq 15, 21) \\ 3 & (p = 15) \\ 5 & (p = 21), \end{cases}$$

where ε $(= \pm 1)$ *is the signature of* V.

If p satisfies property (#), the above Theorem is a direct consequence of Theorem 2.2. The general case can be proved by the same method as in the proof of Theorem 2.2 below.

As a typical example, we can obtain a lower bound for the kinkinesses of the homotopy S^2's obtained by attaching a 2-handle to the 4-ball along some torus knots.

Corollary 2.6. *Let* s *and* r *be relatively prime integers greater than* 1. *Suppose that* $s^2 \not\equiv \pm 1 \pmod{rs + \varepsilon}$ *and that* $rs + \varepsilon$ *divides* $s^4 - 1$, *where* $\varepsilon = \pm 1$. *Then we have*

$$k_-(V(T(r,s); -(rs+\varepsilon))) \geq \begin{cases} (rs+\varepsilon-4)/4 & (rs+\varepsilon: \textit{even}) \\ (rs+\varepsilon+1)/4 & (rs+\varepsilon: \textit{odd} \neq 15, 21) \\ 3 & (rs+\varepsilon = 15) \\ 5 & (rs+\varepsilon = 21). \end{cases}$$

Proof. By Moser [25], $\partial V(T(r,s); -(rs+\varepsilon))$ is diffeomorphic to the lens space $L(rs+\varepsilon, s^2)$. Since $s^4 \equiv 1 \pmod{rs+\varepsilon}$, there exists a homeomorphism $h: L(rs+\varepsilon, s^2) \to L(rs+\varepsilon, s^2)$ which acts on the first homology group by the multiplication of s^2. Now the result follows from Theorem 2.5.

Example 2.7. There do exist r, s and ε satisfying the condition in Corollary 2.6. For example, we have

$$\begin{aligned} k_-(T(2,7); -15) &\geq 3, \\ k_-(T(3,5); -16) &\geq 3, \\ k_-(T(3,13); -40) &\geq 9, \\ k_-(T(4,13); -51) &\geq 13. \end{aligned}$$

We also note that $k_-(T(a,b); p) \leq (|a| - 1)(|b| - 1)/2$ for any p.

Using Corollary 2.6, we can obtain a lower bound for the unknotting numbers of certain torus knots. However, this lower bound is worse than the one obtained in [16]. There is a conjecture that the unknotting number of the torus knot $T(a,b)$ is equal to $(|a|-1)(|b|-1)/2$. We do not know whether there is a smoothly immersed 2-sphere in some $V(T(a,b);p)$ representing the generator of $H_2(V(T(a,b);p))$ with the number of double points strictly fewer than $(|a|-1)(|b|-1)/2$. Note that if $p=\pm 1$, then there always exists a topologically locally flatly embedded 2-sphere or torus representing the generator of the second homology group (see Proposition 3.6).

Now we proceed to the proof of Theorem 2.2. Our method is similar to that in [28].

Let X be a smooth closed 1-connected oriented 4-manifold. Given a homology class $\zeta \in H_2(X)$, one can represent ζ by an immersed 2-sphere whose self-intersections are transverse. Define d_ζ^+ (resp. d_ζ^-) to be the minimum number of positive (resp. negative) double points of such immersed 2-spheres representing ζ. The following is a theorem of Kuga and Suciu which plays a key role in the proof of Theorem 2.2. See also the remark after Theorem 4 in [16].

Theorem 2.8 (Kuga [22], Suciu [30]).

(1) *Let X be a smooth closed oriented 4-manifold homotopy equivalent to $S^2\times S^2$ and let ξ and η be generators of $H_2(X)$ such that $\xi^2=\eta^2=0$ and $\xi\cdot\eta=1$. If $\zeta=a\xi+b\eta\in H_2(X)$ satisfies $\zeta^2\neq 0$, then we have*

$$d_\zeta^\varepsilon \geq \min\left\{(|a|-1)(|b|-1), \left[\frac{|ab|+1}{2}\right]\right\},$$

where ε is the sign of ζ^2 and, for $x\in\mathbf{Q}$, $[x]$ denotes the largest integer not exceeding x.

(2) *Let X be a smooth closed oriented 4-manifold homotopy equivalent to $\mathbf{C}P^2\#\overline{\mathbf{C}P^2}$ and let ξ and η be generators of $H_2(X)$ such that $\xi^2=1, \eta^2=-1$ and $\xi\cdot\eta=0$. If $\zeta=a\xi+b\eta\in H_2(X)$ satisfies $\zeta^2\neq 0$, then we have*

$$d_\zeta^\varepsilon \geq \min\left\{\frac{(|a|+|b|-2)(\big||a|-|b|\big|-1)}{2}, \left[\frac{|a^2-b^2|+3}{4}\right]\right\},$$

where ε is the sign of ζ^2.

Proof of Theorem 2.2. Changing the orientation of V if necessary, we may assume V is positive definite (i.e., $\varepsilon=1$). Let V' be a smooth homotopy S^2 with $\partial V'$ diffeomorphic to $M = \partial V$ and let $h : \partial V' \to \partial V$ be a homeomorphism. Changing h by an isotopy, we assume h is a diffeomorphism. Furthermore, we orient V' so that h is an orientation preserving map. Thus V' is not necessarily positive definite. We have, for some $r \in \mathbf{Z}$, $h_*(\beta(V')) = r\beta(V)$. We will show $r \equiv \pm 1 \pmod p$. Then the part (1) of Theorem 2.2 follows from a result of Boyer [3]. Furthermore the part (2) is also proved if we set $V' = V$. Then the part (3) follows by the same argument as in [28].

Set $X = V \cup_h (-V')$, which is a smooth closed 1-connected 4-manifold with $H_2(X) \cong \mathbf{Z} \oplus \mathbf{Z}$. X has the orientation induced from those of V and $-V'$. By [28], there are generators θ and τ of $H_2(X)$ with the following properties.

(i) θ is represented by a smoothly immersed 2-sphere with $k_+(V)$ positive double points.

(ii) $\theta \cdot \theta = p$ and $\theta \cdot \tau = r$.

Remember that θ comes from the generator of $H_2(V)$ and τ is defined to be the "union" of the generator of $H_2(V', \partial V')$ and r times the generator of $H_2(V, \partial V)$.

Set $t = \tau \cdot \tau$. Then the intersection matrix of X with respect to θ and τ is

$$Q = \begin{pmatrix} p & r \\ r & t \end{pmatrix}.$$

Since Q is unimodular, $\det Q = pt - r^2 = \pm 1$. Hence $r^2 \equiv \mp 1 \pmod p$. Since p satisfies property (#), $r^2 \equiv 1 \pmod p$. Thus $\det Q = -1$ and the intersection form of X is indefinite. Hence, there are generators ξ and η of $H_2(X)$ with respect to which the intersection matrix of X is one of the following forms;

$$\text{(A)} \qquad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \text{or}$$

$$\text{(B)} \qquad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Case (A). Since Q must be of even type, p is even. Suppose $\theta = a\xi + b\eta$ $(a, b \in \mathbf{Z})$. Since $p = \theta \cdot \theta = 2ab$ is positive, we may assume $a > 0$ and $b > 0$, changing the orientations of ξ and η if necessary. By Theorem 2.8, we have

$$d_\theta^+ \geq \min\left\{(a-1)(b-1), \left[\frac{ab+1}{2}\right]\right\}. \tag{2.1}$$

On the other hand, by the hypothesis, we have

$$d_\theta^+ \leq k_+(V) \leq \frac{p-6}{4}. \tag{2.2}$$

Combining (2.2) with (2.1), we have

$$\frac{ab-3}{2} \geq \min\left\{(a-1)(b-1), \left[\frac{ab+1}{2}\right]\right\}. \tag{2.3}$$

Since $(ab-3)/2 < [(ab+1)/2]$, we have $(ab-3)/2 \geq (a-1)(b-1)$. This shows that $a \leq 1$ or $b \leq 1$. Then the same argument as in [28] shows $r \equiv \pm 1 \pmod p$.

Case (B). Suppose $\theta = a\xi + b\eta$ $(a, b \in \mathbf{Z})$. We may assume $a \geq 0$ and $b \geq 0$, changing the orientations of ξ and/or η if necessary. Note that $p = \theta \cdot \theta = a^2 - b^2 > 0$. By Theorem 2.8, we have

$$d_\theta^+ \geq \min\left\{\frac{(a+b-2)(a-b-1)}{2}, \left[\frac{a^2-b^2+3}{4}\right]\right\}. \tag{2.4}$$

If p is even, we have

$$d_\theta^+ \leq k_+(V) \leq \frac{p-6}{4}. \tag{2.5}$$

Combining this with (2.4), we obtain $a+b \leq 3$ or $a-b \leq 1$. Since $a \geq 0, b \geq 0$ and a is prime to b, we have $a-b=1$. This contradicts the fact that $p = a^2 - b^2$ is even.

If $p = (a+b)(a-b)$ is odd not equal to 15 or 21, we have, by the hypothesis,

$$d_\theta^+ \leq k_+(V) \leq \frac{p-1}{4}. \tag{2.6}$$

Combining this with (2.4), we obtain

$$(a+b-4)(a-b-2) \leq 3. \tag{2.7}$$

Note that $a+b$ and $a-b$ are odd and that a and b are relatively prime. Then we have $a-b=1$ or $(a,b) = (4,1)$ or $(5,2)$. Since $p = a^2 - b^2 \neq 15, 21$, we have $a-b=1$. When $p=15$, we obtain

$$2 \geq \min\left\{\frac{(a+b-2)(a-b-1)}{2}, 4\right\} \tag{2.8}$$

and when $p = 21$,

$$4 \geq \min \left\{ \frac{(a+b-2)(a-b-1)}{2}, 6 \right\}. \tag{2.9}$$

Both inequalities imply $a - b = 1$. Thus when p is odd we always have $a - b = 1$. Then the same argument as in [28] shows $r \equiv \pm 1 \pmod{p}$. This completes the proof.

Remark. When p is equal to 15 or 21, we cannot omit the special condition that $k_\varepsilon(V) \leq 2$ and $k_\varepsilon(V) \leq 4$ respectively. For example, consider $V = V(T(2,7); -15)$. Then $k_-(V) \leq 3$ (cf. Example 2.7). However $\partial V = L(15,4)$ admits a self-homeomorphism acting on $H_1(L(15,4))$ by the multiplication of ± 4. When $p = 21$, consider $V = V(T(2,11); -21)$. Then $k_-(V) \leq 5$. However $\partial V = L(21,4)$ bounds the smooth homotopy S^2, $V(T(4,5); -21)$, which is not homeomorphic to V.

§3. Topological invariants for homotopy S^2 and embedded surfaces

Let M be a homology $L(p,1)$ $(p > 0)$; i.e., M is a closed oriented 3-manifold such that $H_*(M) \cong H_*(L(p,1))$ and $\mathrm{lk}(\alpha, \alpha) \equiv -\varepsilon/p$ (mod $\mathbf{Z}$) for some generator α of $H_1(M)$ $(\varepsilon = \pm 1)$. We call such a pair (M, α) a *marked homology* $L(p,1)$. It is well-known that M is obtained by (εp)-surgery on a knot K in some homology 3-sphere Σ so that the core of the surgery torus in M represents α. Define

$$\lambda_0(M, \alpha) = p\lambda(\Sigma) + (\varepsilon/2)\Delta''_K(1), \tag{3.1}$$

where $\lambda(\Sigma)$ is the Casson invariant of the homology 3-sphere Σ ([6]) and $\Delta_K(t)$ is the normalized Alexander polynomial of K. For a fixed marked homology $L(p,1)$, Σ and K as above are not uniquely determined. However, by results of Boyer-Lines [4] and Fukuhara [15], $\lambda_0(M, \alpha)$ is an invariant of the marked homology $L(p,1)$, (M, α). We warn the reader that λ_0 here is p times the invariant defined in [4] or [15] and that λ_0 here is integer valued. Note also that if $p = 1$, i.e. if M is a homology 3-sphere, (3.1) with $\lambda_0(M, \alpha)$ replaced by $\lambda(M)$ is nothing but the surgery formula for the usual Casson invariant. Therefore, λ_0 agrees with λ for homology 3-spheres.

Definition. Let V be an oriented homotopy S^2 with $H_1(\partial V)$ finite cyclic of order p. Note that $(\partial V, \beta(V))$ is a marked homology $L(p,1)$.

Then we define $\overline{\lambda}(V) = \lambda_0(\partial V, \beta(V))$, which we call the *Casson invariant* of V. $\overline{\lambda}(V)$ is a topological invariant of V.

Remark. If ∂V is a homology 3-sphere, then $\overline{\lambda}(V) = \lambda(\partial V)$. In general, however, $\overline{\lambda}(V)$ is not an invariant of ∂V. For example, denoting by U the trivial knot in S^3, we have $\overline{\lambda}(V(U;5)) = 0$ and $\overline{\lambda}(V(T(2,3);-5)) = -1$, though $\partial V(U;5) \cong \partial V(T(2,3);-5) \cong L(5,1)$ ([25]).

The first result of this section is the following.

Theorem 3.1. *Let V be a smooth homotopy S^2 with $H_1(\partial V)$ finite. If the generator of $H_2(V)$ can be represented by a smoothly embedded 2-sphere, then $\overline{\lambda}(V) \equiv 0 \pmod 2$.*

Remark. If V is diffeomorphic to a handlebody $V(K;p)$ for some knot K in S^3, then $\overline{\lambda}(V)$ mod 2 agrees with the Arf invariant of K. In this case, the above result has already been known ([2]).

Proof of Theorem 3.1. Suppose S is a smoothly embedded 2-sphere in Int$V$ which represents the generator of $H_2(V)$. We denote by $N(S)$ the tubular neighborhood of S. Then it is easily seen that the 4-manifold $Y = V - \mathrm{Int}N(S)$ is a homology cobordism between ∂V and $L(p,1)$ ($\cong \partial N(S)$). Furthermore, if $\alpha \in H_1(L(p,1))$ is the homology class corresponding to $\beta(V) \in H_1(\partial V)$ through the homology cobordism Y, then α is represented by the core of the surgery torus of $M(U;p/1)$($\cong L(p,1)$), where U is the trivial knot in S^3; hence, $\lambda_0(L(p,1),\alpha) = 0$. Then Theorem 3.1 follows from the following Proposition 3.2.

Proposition 3.2. *The Casson invariant λ_0 modulo 2 for marked homology $L(p,1)$ is a homology cobordism invariant; i.e., if Y is a smooth homology cobordism between the homology $L(p,1)$'s M_0 and M_1 and if $\alpha_0 \in H_1(M_0)$ and $\alpha_1 \in H_1(M_1)$ correspond through this homology cobordism Y, then $\lambda_0(M_0,\alpha_0) \equiv \lambda_0(M_1,\alpha_1) \pmod 2$.*

Remember that, for homology 3-spheres, the Casson invariant modulo 2 is the Rohlin invariant, which is a homology cobordism invariant.

Proof of Proposition 3.2. We may assume $p \geq 2$. Let K_i $(i = 0,1)$ be a smooth knot in M_i representing $\alpha_i \in H_1(M_i)$. Let W be the 4-manifold obtained by attaching two 2-handles h_0 and h_1 to Y along K_0 and K_1 in such a way that ∂W consists of two disjoint homology 3-spheres Σ_0 and Σ_1. Denote by $K_i' \subset \Sigma_i$ the knot which is the boundary of the cocore of the 2-handle h_i (Figure 1). Then M_i is obtained by

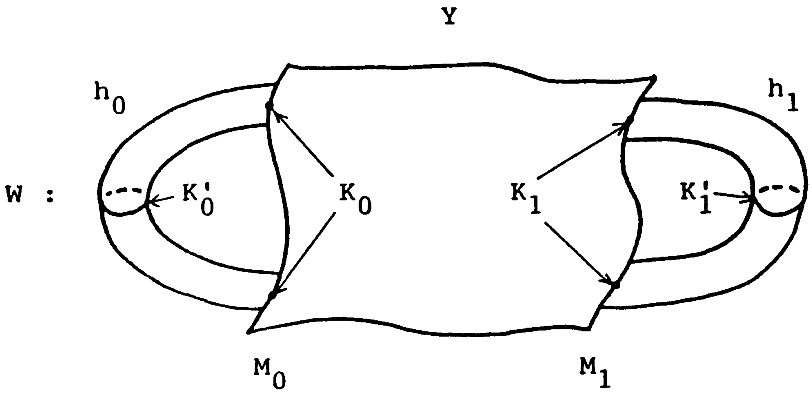

Figure 1

the $(\pm p)$-surgery on the knot K_i'. Furthermore the core of this surgery torus in M_i corresponds to K_i which represents $\alpha_i \in H_1(M_i)$. Hence $\lambda_0(M_i, \alpha_i) = p\lambda(\Sigma_i) \pm (1/2)\Delta''_{K_i'}(1)$. Let $f_i \in H_2(W, \partial W)(\cong H_2(W))$ be the homology class represented by the cocore of the 2-handle h_i. Since Y is a homology cobordism, there is a 2-chain c in Y with boundary K_0 and K_1. Denote by $e \in H_2(W)$ the homology class represented by the 2-cycle which consists of the cores of h_i and the 2-chain c. Then (f_i, e) are generators of $H_2(W, \partial W) \cong H_2(W) \cong \mathbf{Z} \oplus \mathbf{Z}$. Furthermore we have $f_i \cdot f_i = \pm p$, $f_0 \cdot f_1 = 0, f_i \cdot e = \pm 1$ and $\pm pe = \pm f_0 \pm f_1$. Hence,

$$\begin{aligned} e \cdot e &= \frac{1}{p^2}(\pm f_0 \pm f_1) \cdot (\pm f_0 \pm f_1) \\ &= \frac{1}{p^2}(\pm p \pm p). \end{aligned}$$

Since this must be an integer, we have

$$\begin{cases} \text{(A)} \quad e \cdot e = 0 \quad \text{or} \\ \text{(B)} \quad e \cdot e = \pm 1 \quad \text{and} \quad p = 2. \end{cases}$$

Case (A). The homology class $f_0 + f_1 \in H_2(W, \partial W) \cong H_2(W)$ is characteristic and we can represent it by a smoothly embedded annulus A by tubing the cocores of h_0 and h_1.

When p is even, W is spin and its signature is zero; hence, $\lambda(\Sigma_0) \equiv \lambda(\Sigma_1) \pmod 2$. Thus, for the proof of Proposition 3.2, it suffices to prove

$$\lambda(\Sigma_0) + \frac{1}{2}\Delta''_{K_0'}(1) \equiv \lambda(\Sigma_1) + \frac{1}{2}\Delta''_{K_1'}(1) \pmod 2. \tag{3.2}$$

Note that $\lambda(\Sigma_i) + (1/2)\Delta''_{K'_i}(1)$ is equal to the Casson invariant of the homology 3-sphere Σ'_i obtained by $(+1)$-surgery on K'_i in Σ_i. Let X be the 4-manifold obtained by attaching two 2-handles h'_0 and h'_1 to W along K'_i with the $(+1)$-framing. Note that ∂X consists of the homology 3-spheres Σ'_0 and Σ'_1. Denote by S' the smoothly embedded 2-sphere in X which consists of the annulus A and the cores of h'_0 and h'_1. Note that $[S'] \in H_2(X)$ is characteristic and that $[S']^2$ is equal to the signature of X. Then using the same method as in [21], one can deduce $\lambda(\Sigma'_0) \equiv \lambda(\Sigma'_1) \pmod 2$. This shows the equality (3.2) holds.

Case (B). The homology classes $f_0, f_1 \in H_2(W, \partial W) \cong H_2(W)$ are both characteristic. Let X_i be the 4-manifold obtained by attaching a 2-handle h'_i to W along K'_i with the $(+1)$-framing. Denote by S_i the smoothly embedded 2-sphere in X_i consisting of the cocore of the 2-handle h_i and the core of the 2-handle h'_i. Note that $[S_i] \in H_2(X_i)$ is characteristic and that $[S_i]^2$ is equal to the signature of X_i. Thus, by the same argument as in Case (A), we have

$$\lambda(\Sigma'_0) \equiv \lambda(\Sigma_1) \pmod 2 \quad \text{and}$$
$$\lambda(\Sigma_0) \equiv \lambda(\Sigma'_1) \pmod 2.$$

Hence,

$$\lambda(\Sigma_0) + \frac{1}{2}\Delta''_{K'_0}(1) \equiv \lambda(\Sigma_1) \pmod 2 \quad \text{and} \tag{3.3}$$

$$\lambda(\Sigma_0) \equiv \lambda(\Sigma_1) + \frac{1}{2}\Delta''_{K'_1}(1) \pmod 2. \tag{3.4}$$

Adding (3.3) and (3.4) shows

$$2\lambda(\Sigma_0) + \frac{1}{2}\Delta''_{K'_0}(1) \equiv 2\lambda(\Sigma_1) + \frac{1}{2}\Delta''_{K'_1}(1) \pmod 2.$$

This completes the proof of Proposition 3.2 and hence Theorem 3.1.

Remark. There is a smooth homotopy S^2, V, such that $\overline{\lambda}(V) \equiv 0 \pmod 2$ and yet the generator of $H_2(V)$ cannot be represented by a smoothly embedded 2-sphere. For example, consider $V = V(T(3,-11); 34)$. A computation shows $\overline{\lambda}(V) = 40$. However the generator of $H_2(V)$ cannot be represented by a smoothly embedded 2-sphere (see [28] or Example 3.5 below).

Remark. Let (M, α) be an oriented marked homology $L(p, 1)$. If p is odd, the Rohlin invariant $\mu(M)(\in \mathbf{Z}/16\mathbf{Z})$ of M is defined and it is a homology cobordism invariant. It can be shown, using the method similar to that in the proof of Proposition 3.2, that

$$8\lambda_0(M,\alpha) \equiv \mu(M) - (p-1)p\cdot \mathrm{lk}(\alpha,\alpha) \pmod{16},$$

where $\mathrm{lk}(\alpha,\alpha) = \pm 1/p$.

Next we define the Casson-Gordon invariants for a homotopy S^2. In our case, they are essentially the p-signatures of a certain knot in a homology 3-sphere.

Let K be a smooth knot in an oriented homology 3-sphere Σ and let L be a Seifert matrix of K. For a positive integer p, set $\omega_p = \exp(2\pi\sqrt{-1}/p)$. Then we define $\sigma_K(\omega_p^r)$ $(r = 1, 2, \cdots, p-1)$ to be the signature of the Hermitian matrix $(1-\omega_p^{-r})L + (1-\omega_p^r)L^T$, where L^T is the transpose of L. It is well-known that $\sigma_K(\omega_p^r)$ are invariants of K and they are called *p-signatures* of K. Note that they are independent of the orientation of K, while they depend on the orientation of Σ in general. If we change the orientation of K, its Seifert matrix becomes L^T and the signature of the corresponding Hermitian matrix does not change.

Lemma 3.3. *Let Δ_i $(i = 0, 1)$ be a compact oriented contractible topological 4-manifold and let K_i be a tame knot in the homology 3-sphere $\partial\Delta_i$. Let V_i be the oriented 4-manifold obtained by attaching a 2-handle to Δ_i along K_i with the $(\pm p)$-framing $(p \geq 0)$. If V_0 is orientation preservingly homeomorphic to V_1, then when $p > 0$,*

$$\sigma_{K_0}(\omega_p^r) = \sigma_{K_1}(\omega_p^r) \quad \textit{for} \quad 1 \leq r \leq p-1$$

and when $p = 0$,

$$\sigma_{K_0}(\omega_q^s) = \sigma_{K_1}(\omega_q^s) \quad \textit{for every} \quad q > 0 \quad \textit{and} \quad 1 \leq s \leq q-1.$$

Proof. We prove the case of $p > 0$. When $p = 0$, the proof is similar. We may assume V_i are positive definite. Orient K_i arbitrarily and let $\alpha_i \in H_1(\partial V_i)$ be the homology class represented by a meridian loop of K_i. Let $h : V_0 \to V_1$ be a homeomorphism. By the restriction $h|\partial V_0$, we identify ∂V_0 and ∂V_1 and denote it by M. We may assume that, by this identification, $\alpha_0 = \alpha_1$ in $H_1(M)$, changing orientations of K_i if necessary. Define the homomorphism $\varphi : H_1(M) \to \mathbf{Z}/p\mathbf{Z}$ by $\varphi(\alpha_i) = 1$ ($1 \in \mathbf{Z}/p\mathbf{Z}$ is the generator). Let L_i be a Seifert matrix for

K_i. Then by [7, Lemma 3.1],

$$\begin{aligned}(3.5)\qquad \sigma_r(M,\varphi) = \operatorname{sign}V_i - \operatorname{sign}((1-\omega_p^{-r})L_i + (1-\omega_p^r)L_i^T) \\ -\frac{2r(p-r)}{p} \quad (i=0,1),\end{aligned}$$

where $\sigma_r(M,\varphi)$ is the Casson-Gordon invariant of M associated with φ and r. Note that in [7] everything is assumed to be smooth. However, their method is easily extended to the topological category, since the G-signature theorem holds also in the topological case (see [32]). Since $\operatorname{sign}V_0 = \operatorname{sign}V_1$, we have $\sigma_{K_0}(\omega_p^r) = \sigma_{K_1}(\omega_p^r)$ by (3.5).

Definition. Let V be an oriented homotopy S^2 with $H_1(\partial V)$ isomorphic to $\mathbf{Z}/p\mathbf{Z}$ $(p \geq 0)$. Then by [3], V is obtained by attaching a 2-handle to a compact contractible 4-manifold Δ along some tame knot K in the homology 3-sphere $\partial\Delta$ with the $(\pm p)$-framing. If $p > 0$, we define

$$\sigma_V(\omega_p^r) = \sigma_K(\omega_p^r) \quad (1 \leq r \leq p-1),$$

which we call the *p-signatures* of V. Similarly if $p = 0$, we define, for every $q > 1$,

$$\sigma_V(\omega_q^s) = \sigma_K(\omega_q^s) \quad (1 \leq s \leq q-1).$$

By Lemma 3.3, this is well-defined.

As the equation (3.5) in the proof of Lemma 3.3 shows, the p-signatures of a homotopy S^2 are essentially the Casson-Gordon invariants of the boundary 3-manifold.

Next we use these invariants to attack problem (B) in §1.

Definition. Let V be a homotopy S^2. We define $g(V)$ to be the minimal genus of topologically locally flatly embedded surfaces representing the generator of $H_2(V)$.

Remark. Even if V itself is not smooth, $\operatorname{Int}V$ admits a smooth structure (see [14, §8.2]). Thus the generator of $H_2(V)(\cong H_2(\operatorname{Int}V))$ is always represented by a locally flatly embedded surface.

Theorem 3.4. *Let V be a homotopy S^2 with $H_1(\partial V)$ finite cyclic of order p $(p > 0)$. Then for every prime power d dividing p, we have*

$$2g(V) \geq |\, \sigma_V(\omega_d^s) \,| \quad (s = 1, 2, \cdots, d-1).$$

Remark. This lower bound has already been known for the case that V consists of one 0-handle and one 2-handle and the embedded surfaces considered are smooth ([31]). In this case, when $H_1(\partial V)$ is infinite cyclic, the above inequality holds for all prime power d. We do not know whether Theorem 3.4 also holds for V with $H_1(\partial V)$ infinite cyclic.

Proof of Theorem 3.4. We may assume V is positive definite. Let F be a topologically locally flatly embedded surface in $\mathrm{Int}V$ of genus $g = g(V)$ representing the generator of $H_2(V)$. There exists a d-fold cyclic branched covering $\pi : \widetilde{V} \to V$ branched along F such that the d-fold covering $\pi|\partial\widetilde{V} : \partial\widetilde{V} \to \partial V$ corresponds to the homomorphism $\varphi : H_1(\partial V) \to \mathbf{Z}/d\mathbf{Z}$ defined by $\varphi(\beta(V)) = 1$. Let $\tau : \widetilde{V} \to \widetilde{V}$ be the canonical covering translation. Define $E_s \subset H_2(\widetilde{V}) \otimes \mathbf{C}$ to be the ω_d^s-eigenspace of $\tau_* : H_2(\widetilde{V}) \otimes \mathbf{C} \to H_2(\widetilde{V}) \otimes \mathbf{C}$ (note that $\tau_*^d = \mathrm{id}$). Furthermore define $\varepsilon_s(\widetilde{V})$ to be the signature of the restriction to E_s of the intersection pairing on $H_2(\widetilde{V}) \otimes \mathbf{C}$. Then by the definition of the Casson-Gordon invariants [7],

$$\sigma_s(\partial V, \varphi) = \mathrm{sign}V - \varepsilon_s(\widetilde{V}) - \frac{2ps(d-s)}{d^2}.$$

Combining this with [7, Lemma 3.1] and the definition of p-signatures, we have

$$\varepsilon_s(\widetilde{V}) = \sigma_V(\omega_d^s).$$

Since $|\varepsilon_s(\widetilde{V})| \leq \dim_{\mathbf{C}} E_s$, it suffices to show that $\dim_{\mathbf{C}} E_s = 2g$.

It is easily verified, using a method similar to that in [19, §4], that

$$\dim_{\mathbf{C}} H_2(\widetilde{V}) \otimes \mathbf{C} = 2g(d-1) + 1$$

(note that d is a prime power by the assumption). It is well-known that $E_0 = \pi^*(H_2(V) \otimes \mathbf{C})$. Hence,

$$\sum_{s=1}^{d-1} \dim_{\mathbf{C}} E_s = 2g(d-1).$$

Using this equation and the linear algebra together with the assumption that d is a prime power, we easily deduce $\dim_{\mathbf{C}} E_s = 2g$ ($s = 1, 2, \cdots, d-1$). This completes the proof.

Example 3.5. Consider $V = V(T(a,b); p)$, where $|a|, |b| \geq 2$ and $p \neq 0, \pm 1$. Then the generator of $H_2(V)$ cannot be represented by a topologically locally flatly embedded 2-sphere since the p-signatures of the torus knot $T(a,b)$ do not vanish. Remember that if p is even, the

p-signature $\sigma_{T(a,b)}(-1)$ is the usual signature of the torus knot $T(a,b)$. Thus the lower bound for the 4-ball genera of torus knots given in [29] is also valid for $g(V(T(a,b);p))$ if p is even.

Definition. For a homotopy S^2, V, we denote by $KS(V)$ ($\in \mathbf{Z}/2\mathbf{Z}$) the Kirby-Siebenmann obstruction to extending the product smooth structure on $\partial V \times \mathbf{R}$ across $V \times \mathbf{R}$.

If ∂V is a homology 3-sphere, Theorem 3.4 gives no restrictions on $g(V)$. In that case, we have the following.

Proposition 3.6. *Let V be a homotopy S^2 with ∂V a homology 3-sphere. Then $g(V) = 0$ if $\mu(\partial V) = KS(V)$ and $g(V) = 1$ if $\mu(\partial V) \neq KS(V)$, where $\mu(\partial V)$ is the Rohlin invariant of ∂V.*

Proof. Let $P = \mathbf{C}P^2 -$ IntD^4 and $Q = Ch -$ IntD^4, where Ch is the Chern manifold ([12]). Furthermore let Δ be the contractible 4-manifold bounded by ∂V ([12]). Then by [3] V is homeomorphic to $P \natural \Delta$ if $\mu(\partial V) = KS(V)$ and $Q \natural \Delta$ if $\mu(\partial V) \neq KS(V)$, where $\natural$ denotes the boundary connected sum. Since P is homeomorphic to a D^2-bundle over S^2, the generator of $H_2(P \natural \Delta)$ is represented by a locally flatly embedded 2-sphere. Furthermore, Ch is homeomorphic to $V(T(2,3);1) \cup \Delta'$, where Δ' is the contractible 4-manifold bounded by $M(T(2,3);1/1)$. Thus the generator of $H_2(Q \natural \Delta)$ is represented by a locally flatly embedded torus. Hence, $g(V) = 0$ if $\mu(\partial V) = KS(V)$ and $g(V) \leq 1$ if $\mu(\partial V) \neq KS(V)$.

Next we show that $g(V) \neq 0$ if $\mu(\partial V) \neq KS(V)$. Suppose $g(V) = 0$ and let S be a locally flatly embedded 2-sphere in Int$V$ representing the generator of $H_2(V)$. By [13], S has a neighborhood N ($\subset$ Int$V$) which is a 2-disk bundle over S. Note that ∂N is homeomorphic to S^3. Set $\Delta'' = (V -$ Int$N) \cup_{\partial N} D^4$. Then $\mu(\partial V)$ is equal to the Kirby-Siebenmann obstruction of Δ'', which in turn is equal to $KS(V)$. This contradicts the assumption that $\mu(\partial V) \neq KS(V)$. This completes the proof.

In Theorem 3.4, only the p-signatures of the form $\sigma_V(\omega_d^s)$ with d a prime power are handled. For the general p-signatures, we have the following.

Proposition 3.7. *Let V be a smooth homotopy S^2 with $H_1(\partial V)$ finite cyclic of order p. For an integer d (> 0) dividing p, suppose $H_1(\widetilde{\partial V};\mathbf{Q}) = 0$, where $\widetilde{\partial V} \to \partial V$ is the d-fold cyclic covering associated with the homomorphism $\varphi : H_1(\partial V) \to \mathbf{Z}/d\mathbf{Z}$ defined by $\varphi(\beta(V)) = 1$. If* *the generator of* $H_2(V)$ *is represented by a smoothly embedded* 2*-sphere, then*

$$\sigma_V(\omega_d^s) = 0 \quad \textit{for} \quad 1 \le s \le d-1.$$

Remark. Suppose V is obtained (topologically) by attaching a 2-handle to a contractible 4-manifold Δ along a knot K in the homology 3-sphere $\partial\Delta$. Then $H_1(\widetilde{\partial V}; \mathbf{Q}) = 0$ if and only if $\Delta_K(\omega_d^s) \neq 0$ for $1 \le s \le d-1$, where $\Delta_K(t)$ is the Alexander polynomial of K. In particular, if d is a prime power, we always have $H_1(\widetilde{\partial V}; \mathbf{Q}) = 0$.

Proof of Proposition 3.7. Let S be a smoothly embedded 2-sphere in $\mathrm{Int}V$ representing the generator of $H_2(V)$ and let $Y = V - \mathrm{Int}N(S)$, where $N(S)$ is the tubular neighborhood of S. Then Y is a smooth homology cobordism between ∂V and $L(p,1)$. Let $\varphi' : H_1(L(p,1)) \to \mathbf{Z}/d\mathbf{Z}$ be the homomorphism defined by composing the isomorphism $H_1(L(p,1)) \to H_1(\partial V)$ induced by Y and the homomorphism $\varphi : H_1(\partial V) \to \mathbf{Z}/d\mathbf{Z}$. Then by Matic [24] and Ruberman[27],

$$\sigma_1(\partial V, \varphi^s) = \sigma_1(L(p,1), \varphi'^s) \quad (1 \le s \le d-1).$$

Since $\sigma_s(\partial V, \varphi) = \sigma_1(\partial V, \varphi^s)$ and $\sigma_s(L(p,1), \varphi') = \sigma_1(L(p,1), \varphi'^s)$, we have $\sigma_s(\partial V, \varphi) = \sigma_s(L(p,1), \varphi')$. Combining this with [7, Lemma 3.1], we obtain

$$\sigma_V(\omega_d^s) = 0 \quad (1 \le s \le d-1)$$

(see also the proof of Theorem 3.1 in this section). This completes the proof.

In [4], Boyer-Lines extended the Casson invariant for homology 3-spheres to homology lens spaces. Furthermore, they gave a relation of this invariant to the Casson-Gordon invariants. Using this result, Theorem 3.1, and Proposition 3.7, we have the following.

Proposition 3.8. *Let* V *be an* ε*-definite* ($\varepsilon = \pm 1$) *smooth homotopy* S^2 *satisfying the condition on* $H_1(\widetilde{\partial V}; \mathbf{Q})$ *as in Proposition* 3.7 *for* $d = p$. *If the generator of* $H_2(V)$ *is represented by a smoothly embedded* 2*-sphere, then*

$$12p\lambda(\partial V) \equiv -\varepsilon \frac{(p-1)(p-2)}{2} \pmod{24},$$

where $\lambda(\partial V)$ *is the Boyer-Lines' Casson invariant for the homology lens space* ∂V.

Remark. $12p\lambda(\partial V)$ is always an integer ([4, Theorem 2.8]).

Proof of Proposition 3.8. Suppose V is obtained (topologically) by attaching a 2-handle to a contractible 4-manifold Δ along a knot K in $\partial\Delta$ with (εp)-framing. Let $\alpha \in H_1(\partial V)$ be the homology class represented by a meridian of K. Then by [4, Proposition 2.23],

$$p\lambda(\partial V) = \lambda_0(\partial V, \alpha) + \frac{1}{8}\sigma(K,p) + \frac{1}{8}\tau(\partial V),$$

where

$$\sigma(K,p) = \sum_{r=1}^{p-1} \sigma_K(\omega_p^r)$$

and

$$\tau(\partial V) = \sum_{r=1}^{p-1} \sigma_r(\partial V, \varphi).$$

(Here, $\varphi : H_1(\partial V) \to \mathbf{Z}/p\mathbf{Z}$ is the homomorphism defined by $\varphi(\beta(V)) = 1$.) By Theorem 3.1, we have $\lambda_0(\partial V, \alpha) \equiv 0 \pmod 2$ and by Proposition 3.7, $\sigma(K,p) = 0$. Furthermore, by the proof of Proposition 3.7, we have $\sigma_r(\partial V, \varphi) = \sigma_r(L(p,\varepsilon), \varphi')$; hence,

$$\tau(\partial V) = \tau(L(p,\varepsilon)) = -\varepsilon\frac{(p-1)(p-2)}{3}.$$

Thus,

$$8p\lambda(\partial V) \equiv -\varepsilon\frac{(p-1)(p-2)}{3} \pmod{16}.$$

Multiplying 3/2 gives the result.

§4. Knots in the boundary of a homotopy S^2

Definition. Let V be a homotopy S^2 and let K be a tame knot in ∂V representing $\beta(V) \in H_1(\partial V)$. We say K is a *boundary knot* if there is a topologically embedded proper flat 2-disk D in V such that $\partial D = K$ and D represents the generator of $H_2(V, \partial V)$. Here, a properly embedded 2-disk D is *flat* if it is the core of an embedded open 2-handle $D \times \mathbf{R}^2$ in V with $(D \times \mathbf{R}^2) \cap \partial V = \partial D \times \mathbf{R}^2$.

Definition. Let K be a tame knot in a homology $L(p,1)$, M, with $\mathrm{lk}([K],[K]) = \pm 1/p$ $(p > 0, p \neq 2)$. Let Σ be the homology 3-sphere obtained by a surgery on K such that the coefficient of the corresponding surgery by which M is obtained from Σ is an integer. Note that Σ

depends only on M and K if $p \geq 3$. Then we denote by $\mu(K)(\in \mathbf{Z}/2\mathbf{Z})$ the Rohlin invariant $\mu(\Sigma)$ of Σ. Note that, when $p = 1$, $\mu(K)$ is still well-defined, though Σ is not. When $p = 2$, $\mu(K)$ cannot be defined.

Our first result of this section is the following.

Proposition 4.1. *Let V be a homotopy S^2 with $H_1(\partial V)$ finite cyclic of order p. Suppose K is a tame knot in ∂V representing $\beta(V) \in H_1(\partial V)$. Then we have the following.*
(1) *When p is even, K is always a boundary knot.*
(2) *When p is odd, K is a boundary knot if and only if $\mu(K) = KS(V)$.*

Proof. First, we construct an embedded flat 2-disk bounded by K. Set $M = \partial V$ and let X be the 4-manifold obtained by attaching a 2-handle h_0 to $M \times [0,1]$ along $K \times \{1\}$ in such a way that ∂X consists of M and a homology 3-sphere Σ. By [12], there exists a contractible 4-manifold $\varDelta$ with $\partial \varDelta = \Sigma$. Denote by V' the 4-manifold $X \cup_\Sigma \varDelta$. Note that V' is a homotopy S^2 with $\partial V' = M$. By [3] and the hypothesis on $\mu(K) = \mu(\Sigma)$, we see that there is a homeomorphism $h : V \to V'$ such that $h|\partial V = \mathrm{id}_M$. Since $h(K)$ bounds in V' an embedded flat 2-disk (the core of the 2-handle h_0), K also bounds one in V.

Conversely, suppose p is odd and K bounds in V a topologically embedded proper flat 2-disk D in V which represents the generator of $H_2(V, \partial V)$. Then there exists a closed neighborhood N of D in V homeomorphic to $D^2 \times D^2$. Denote by B the closure of $V - N$. Note that ∂B is a homology 3-sphere and that $\mu(K) = \mu(\partial B)$. Then it is easily shown that B is a homology 4-ball. Thus V is obtained by attaching a 2-handle N to B. By [3], $KS(V)$ is equal to $\mu(\partial B) = \mu(K)$. This completes the proof.

Remark. There exists a knot in the boundary of a homotopy S^2, V, which is not a boundary knot but bounds in V an embedded flat 2-disk. For example, consider the knot K in $\partial V(U;p)$ as in Figure 2, where U is the trivial knot and p is odd. Then it is easily seen $\mu(K) = 1$. (To see this, observe that $\mu(K)$ is equal to the Rohlin invariant of the homology 3-sphere obtained by the surgery along the framed link in S^3 as in Figure 3. Then we can apply a formula of Kaplan [20, Theorem 4.2].) Since the Kirby-Siebenmann obstruction of $V(U;p)(= V)$ vanishes, K is not a boundary knot by Proposition 4.1. However, K bounds in V a smoothly embedded 2-disk. This can be constructed as follows. There is a smoothly immersed 2-disk D' in D^4 with one self-intersection such that $\partial D' = K$. D' intersects in V the smoothly embedded 2-sphere S (the union of the core of the 2-handle and the cone over U in D^4)

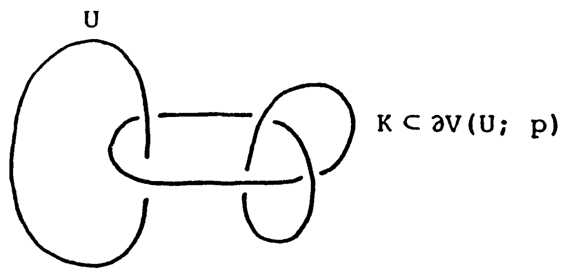

Figure 2

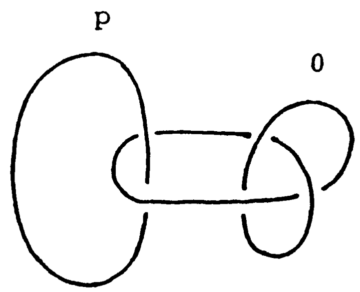

Figure 3

transversely in one point. Piping S and D' along an arc on D' which connects the self-intersection point of D' with the intersection point of D' and S as in Figure 4, we obtain the desired embedded 2-disk D. Of course, D does not represent the generator of $H_2(V, \partial V)$.

Next we give an example of knots in the boundary of a smooth homotopy S^2 which are boundary knots but never bound smoothly embedded 2-disks.

Definition. Let K be a smooth knot in the boundary of a smooth homotopy S^2, V. Then we denote by $k(K)$ the minimal number of double points of smooth properly immersed self-transverse 2-disks in V bounded by K. Following [17], we call $k(K)$ the *kinkiness* of K.

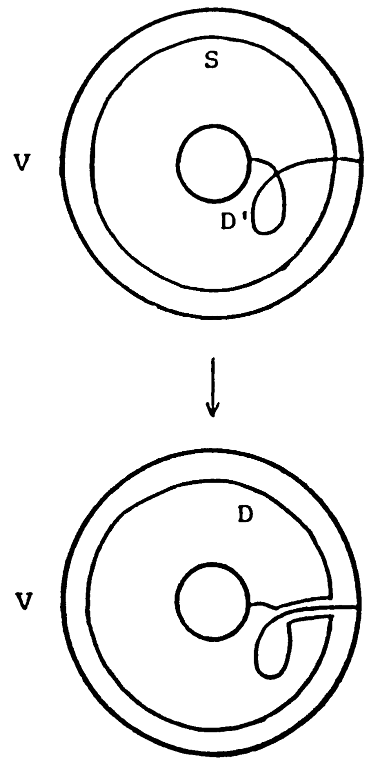

Figure 4

Let n and l be odd integers with $n, l \geq 3$ and set $p = l(nl+2)$ and $q = nl+1$. Note that p and q are relatively prime integers. Set $V = V(T(p,q); -pq)$. Note that by [25], ∂V is diffeomorphic to $L(p,q) \# L(q,p)$. Set $r = q^2 - q - 1$ and let $m_1, m_2, \cdots, m_r$ be the r oriented knots in ∂V represented by the meridians of $T(p,q)$ (see Figure 5). Here we make the orientation convention $[m_i] = -\beta(V)$ in $H_1(\partial V)$. Let K' be any knot in ∂V obtained by performing the oriented band connected sum operations to $m_1 \cup m_2 \cup \cdots \cup m_r$ between distinct components $(r-1)$ times. Note that $[K'] = -r\beta(V)$ in $H_1(\partial V)$. Since $q^2 \equiv 1 \pmod{p}$, there is a diffeomorphism $h' : L(p,q) \to L(p,q)$ which acts on $H_1(L(p,q))$ by the multiplication of q. Set $h = h' \# \mathrm{id} : L(p,q) \# L(q,p) \to L(p,q) \# L(q,p)$. It is easily seen that h acts on $H_1(\partial V)$ by the multiplication of $-r$ (note that $H_1(\partial V) \cong H_1(L(p,q) \# L(q,p)) \cong \mathbf{Z}/p\mathbf{Z} \oplus \mathbf{Z}/q\mathbf{Z} \cong \mathbf{Z}/pq\mathbf{Z}$). Note that $r^2 \equiv 1 \pmod{pq}$. Then we denote by K the smooth knot $h(K')$ in ∂V.

Note that K represents $\beta(V) \in H_1(\partial V)$.

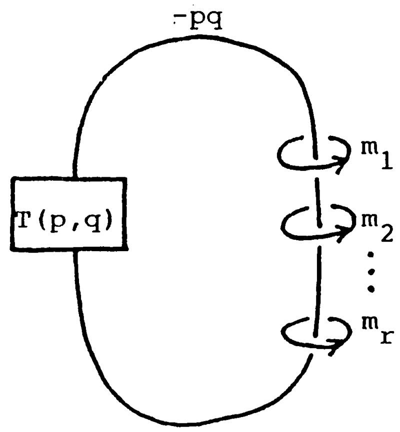

Figure 5

Proposition 4.2. *The knot K bounds in V a topologically embedded proper flat 2-disk. However,*

$$k(K) \geq \left\lceil \frac{n(nl-1)+2}{4} \right\rceil .$$

In particular, K cannot bound in V any smoothly embedded 2-disks.

Proof. Since pq is even, K is a boundary knot by Proposition 4.1.

Now set $X = V_0 \cup_h (-V_1)$ $(V_0 = V_1 = V, h : \partial V_1 \to \partial V_0)$, which is a smooth closed 1-connected 4-manifold with $H_2(X) \cong \mathbf{Z} \oplus \mathbf{Z}$. Let D be a smoothly immersed self-transverse 2-disk in $V_0 (\cong V)$ bounded by K with $k(K)$ double points. Let S_i be the topologically embedded (not locally flat) 2-sphere in V_i which consists of the core of the 2-handle of $V_i = V(T(p,q); -pq)$ and the cone over $T(p,q)$ in D^4. Furthermore let S_2 be the smoothly immersed 2-sphere in X which consists of the r cocores of the 2-handle in V_1 corresponding to $m_1 \cup m_2 \cup \cdots \cup m_r$, the $(r-1)$ bands used to make K', and the immersed 2-disk D. Note that S_2 is a smoothly immersed 2-sphere with $k(K)$ double points. Furthermore let $\tau \in H_2(X)$ be the homology class represented by the union of the cocore of the 2-handle of V_0, the r cocores of the 2-handle of V_1 and a 2-chain in $\partial V_0 = h(\partial V_1)$ connecting their boundaries (note that $h_*(r\beta(V_1)) = -\beta(V_0)$). Then by [28], $\theta = [S_1]$ and τ generate $H_2(X)$. Furthermore $[S_2] = \tau + j[S_0]$ for some integer j.

It is easily seen that the intersection matrix of X with respect to the basis θ and τ is

$$Q = \begin{pmatrix} -l(nl+1)(nl+2) & n^2l^2+nl-1 \\ n^2l^2+nl-1 & -n(nl-1) \end{pmatrix}.$$

($\tau \cdot \tau = -n(nl-1)$ is the consequence of the fact that $\det Q = \pm 1$ and that $|\, l(nl+1)(nl+2)\,| > 2$.) Furthermore Q is isomorphic to the form

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Thus there are generators ξ and η of $H_2(X)$ with $\xi \cdot \xi = \eta \cdot \eta = 0$ and $\xi \cdot \eta = 1$. Furthermore, we may assume

$$\theta = \frac{l(nl+1)}{2}\xi - (nl+2)\eta \quad \text{and}$$
$$\tau = \frac{1-nl}{2}\xi + n\eta.$$

Then we have

$$[S_0] = \frac{l(nl+1)}{2}\xi + (nl+2)\eta.$$

Thus

$$\begin{aligned} [S_2] &= \tau + j[S_0] \\ &= \left\{ \frac{1-nl}{2} + \frac{l(nl+1)}{2}j \right\} \xi + \{ n + (nl+2)j \} \eta. \end{aligned}$$

Set

$$\alpha = \frac{1-nl}{2} + \frac{l(nl+1)}{2}j \quad \text{and}$$
$$\beta = n + (nl+2)j.$$

By Theorem 2.8, the number of double points of S_2 ($= k(K)$) is greater than or equal to

$$\min \left\{ (|\,\alpha\,| - 1)(|\,\beta\,| - 1), \left[\frac{|\,\alpha\beta\,| + 1}{2} \right] \right\}.$$

Since both $|\,\alpha\,|$ and $|\,\beta\,|$ attains its minimum when $j = 0$, we have

$$\begin{aligned} k(K) &\geq \min \left\{ \left(\frac{nl-1}{2} - 1 \right)(n-1), \left[\frac{n(nl-1)+2}{4} \right] \right\} \\ &= \left[\frac{n(nl-1)+2}{4} \right]. \end{aligned}$$

This completes the proof.

Remark. The smooth knot $K' = h^{-1}(K)$ does bound in V a smoothly embedded 2-disk. Thus $h : \partial V \to \partial V$ does not extend to a self-diffeomorphism of V. In fact, it is easily seen that h does not extend even to a self-homeomorphism of V.

We end this section by posing a problem.

Problem. Is there a smooth homotopy S^2, V, such that some $l \in \pi_1(\partial V)$ cannot be represented by any knot which is the boundary of a smoothly embedded 2-disk in V?

If V admits a handlebody decomposition without 3-handles, every $l \in \pi_1(\partial V)$ is so represented ([11]). Thus if the problem above is affirmative, the homotopy S^2 needs a 3-handle in any of its handlebody decomposition. Note also that if l represents $\pm\beta(V)$ in $H_1(\partial V)$, then, by Proposition 4.1, l can be represented by a knot which bounds a topologically embedded flat 2-disk in V.

§5. Exotic open homotopy S^2

For an integer p, let $D(p)$ denote the D^2 bundle over S^2 with euler number p. $D(p)$ is a (compact) homotopy S^2. Our result of this section is the following.

Proposition 5.1. *Let l and m be relatively prime odd integers greater than* 2. *Then the open* 4*-manifold* $\mathrm{Int}D(4lm)$ *admits at least* 2 *smooth structures other than the canonical one.*

To prove Proposition 5.1, we need the following.

Lemma 5.2. *Let a and b be relatively prime integers and set $\zeta = a\xi + b\eta \in H_2(S^2 \times S^2)$, where $\xi = \{*\} \times [S^2]$ and $\eta = [S^2] \times \{*\}$ are the standard generators. Then ζ can be represented by a smoothly immersed* 2*-sphere in $S^2 \times S^2$ with simply connected complement and with $(|a| - 1)(|b| - 1)$ double points.*

Proof. We may assume $a \geq 0$ and $b \geq 0$. If $(a, b) = (1, 0)$ or $(0, 1)$, the assertion is trivial. Hence, we may assume $a \geq 1$ and $b \geq 1$. We construct a desired immersed 2-sphere by the "standard" method. Take distinct $(a + 1)$ points $x_0, x_1, \cdots, x_a \in S^2$ and distinct $(b + 1)$ points $y_0, y_1, \cdots, y_b \in S^2$. Set $R = (\cup_{i=1}^{a}\{x_i\} \times S^2) \cup (\cup_{j=1}^{b} S^2 \times \{y_j\})$ and $X = S^2 \times S^2 - R$. Here, we orient $\{x_i\} \times S^2$ and $S^2 \times \{y_j\}$ so that R

represents ζ. Note that $X = (S^2 - \{x_1, \cdots, x_a\}) \times (S^2 - \{y_1, \cdots, y_b\})$. Set $z = (x_0, y_0) \in X$ and define $\alpha_1, \alpha_2, \cdots, \alpha_a, \beta_1, \beta_2, \cdots, \beta_b \in \pi_1(X, z)$ as follows. Connect x_0 and a point near to x_i $(i = 1, 2, \cdots, a)$ by an arc in $(S^2 - \{x_1, \cdots, x_a\}) \times \{y_0\}$ as in Figure 6. Then α_i is represented by a loop in $(S^2 - \{x_1, \cdots x_a\}) \times \{y_0\} (\subset X)$ which starts at x_0, goes along the arc toward x_i, goes around x_i once counterclockwise, and goes back to x_0 along the same arc. β_i can be defined using an arc in $\{x_0\} \times (S^2 - \{y_1, \cdots, y_b\})$ in a similar way. Note that $\alpha_1 \alpha_2 \cdots \alpha_a = \beta_1 \beta_2 \cdots \beta_b = 1$ and that $\pi_1(X, z) \cong < \alpha_1, \cdots, \alpha_a \mid \alpha_1 \cdots \alpha_a = 1 > \times < \beta_1, \cdots, \beta_b \mid \beta_1 \cdots \beta_b = 1 >$.

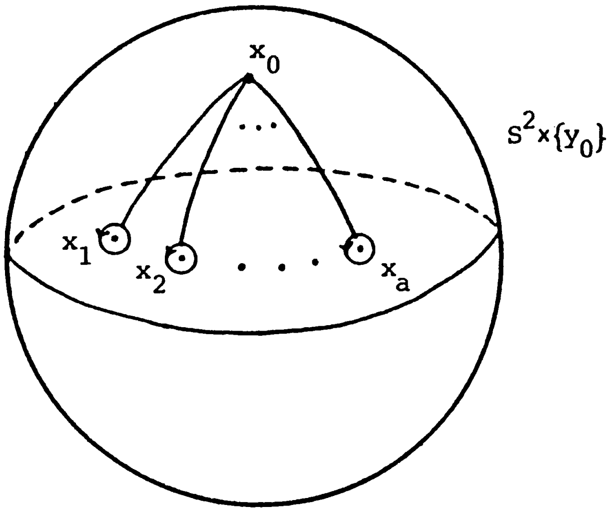

Figure 6

Next we do the "smoothing operations" to R at $(x_1, y_1), (x_1, y_2), \cdots, (x_1, y_b), (x_2, y_1), (x_3, y_1), \cdots, (x_a, y_1)$ as follows. Set $(D^4, B) = (D^2 \times D^2, D^2 \times \{0\} \cup \{0\} \times D^2)$ $(\partial(D^4, B) = (S^3, \text{Hopf link}))$. Furthermore let A' be the annulus embedded in S^3 as in Figure 7 and denote by A the properly embedded annulus in D^4 which is obtained by pushing $\text{Int}A'$ into $\text{Int}D^4$. Note that $\partial(D^4, A) \cong \partial(D^4, B)$ and that $\pi_1(D^4 - A) \cong \mathbf{Z}$. The smoothing operation at a double point q of R means that we replace $(D^4(q), D^4(q) \cap R) \cong (D^4, B)$ by (D^4, A) or $-(D^4, A)$, where $D^4(q)$ is a sufficiently small 4-ball in $S^2 \times S^2$ centered at q. Here we choose (D^4, A) or $-(D^4, A)$ so that the orientation is consistent with that of R. Denote by S the immersed oriented surface which results from the $(a + b - 1)$ smoothing operations. It is easily seen that S is an immersed 2-sphere representing ζ with $(a - 1)(b - 1)$ double points.

Furthermore, by van Kampen's Theorem we see that $\pi_1(S^2 \times S^2 - S, z)$ is isomorphic to $\pi_1(X, z)$ with additional relations $\alpha_i = \beta_j^{-1}$ $((i,j) = (1,1), (1,2), \cdots, (1,b), (2,1), (3,1), \cdots, (a,1))$. Thus $\pi_1(S^2 \times S^2 - S, z)$ is generated by α_1 and we have the relations $\alpha_1^a = \alpha_1^b = 1$. Since a and b are relatively prime, we see that the complement of S is simply connected. This completes the proof.

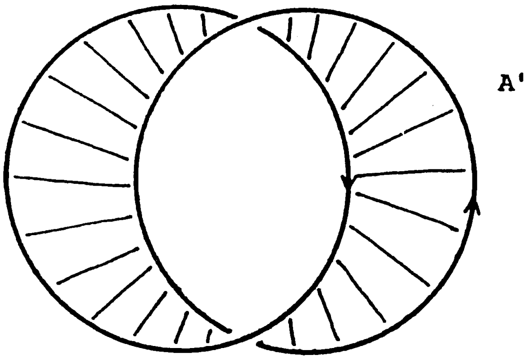

Figure 7

Proof of Proposition 5.1. Set

$$\begin{aligned}\zeta_1 &= \xi + 2lm\eta \\ \zeta_2 &= 2\xi + lm\eta \\ \zeta_3 &= 2l\xi + m\eta.\end{aligned}$$

By Lemma 5.2, ζ_i is represented by a smoothly immersed 2-sphere S_i in $S^2 \times S^2$ with simply connected complement with the number of double points equal to

$$\begin{cases} 0 & (i=1) \\ lm-1 & (i=2) \\ (2l-1)(m-1) & (i=3). \end{cases}$$

Let B_i^4 be a 4-ball in $S^2 \times S^2$ which avoids double points of S_i such that $(B_i^4, B_i^4 \cap S_i)$ is the standard disk pair. Then by [5], ζ_i is represented by a Casson handle which has $S_i -$ IntB_i^4 as its first stage core; i.e., ζ_i is represented by $V_i = B_i^4 \cup CH_i$, where CH_i is a Casson handle which is attached to B_i^4 along the trivial knot. Since CH_i is homeomorphic to $D^2 \times \mathbf{R}^2$ by [12], $V_i' =$ IntV_i is homeomorphic to Int$D(4lm)$ (note that

$\zeta_i^2 = 4lm$). Note that V_i' has the smooth structure as an open set of $S^2 \times S^2$ and that V_1' is diffeomorphic to $\mathrm{Int}D(4lm)$.

Next we show V_i' are mutually non-diffeomorphic. Let $k(V_i')$ be the minimum number of double points of smoothly immersed 2-spheres representing the generator of $H_2(V_i')$. Note that $k(V_i')$ is an invariant of the smooth manifold V_i'. By the construction of V_i', we have

$$\begin{aligned} k(V_1') &= 0 \quad \text{and} \\ k(V_2') &\leq lm - 1. \end{aligned}$$

On the other hand by Theorem 2.8,

$$\begin{aligned} k(V_2') &\geq lm - 1 \quad \text{and} \\ k(V_3') &\geq \min\left\{(2l-1)(m-1), \left[\frac{2lm+1}{2}\right]\right\} \\ &= lm, \end{aligned}$$

since V_i' are submanifolds of $S^2 \times S^2$ representing ζ_i. Hence, $k(V_1'), k(V_2')$ and $k(V_3')$ are all distinct. Thus V_i' are not diffeomorphic to each other. This completes the proof.

In [22], Kuga showed that $D^2 \times \mathbf{R}^2$ has infinitely many smooth structures. Our method above is similar to Kuga's.

We conjecture that for a given positive integer N, there exists an open homotopy S^2 admitting at least N smooth structures. This conjecture is true if, in Theorem 2.8 in §2, $d_\zeta^\varepsilon = (|a|-1)(|b|-1)$.

Remark. Akbulut [1] has recently found a compact homotopy S^2 with at least two smooth structures. The generator of the second homology group of this homotopy S^2 can be represented by an embedded 2-sphere which is smooth with respect to one of the smooth structures, while it can never be smooth with respect to the other smooth structure. Thus the interior of this homotopy S^2 also has at least two smooth structures. We note that this open homotopy S^2 is not homeomorphic to the interior of a D^2 bundle over S^2.

References

[1] S. Akbulut, An exotic 4-manifold, preprint.

[2] S. Boyer, Shake-slice knots and smooth contractible 4-manifolds, Math. Proc. Camb. Phil. Soc., **98** (1985), 93–106.

[3] ———, Simply-connected 4-manifolds with a given boundary, Trans. Amer. Math. Soc., **298** (1986), 331–357.

[4] S. Boyer and D. Lines, Surgery formulae for Casson's invariant and extensions to homology lens spaces, J. reine angew. Math., **405** (1990), 181–220.

[5] A.J. Casson, Three lectures on new infinite constructions in 4-dimensional manifolds, in "A la recherche de la topologie perdue", Birkhäuser, Boston-Basel-Stuttgart, 1986, pp. 201–244.

[6] ———, Lecture at MSRI, March 1985.

[7] A.J. Casson and C.McA. Gordon, On slice knots in dimension three, Proc. Symp. Pure Math., **32** (1978), 39–53.

[8] T.D. Cochran and R. Gompf, An application of Donaldson's theorem to classical knot concordance, homology 3-spheres and property P, Topology, **27** (1988), 495–512.

[9] T.D. Cochran and W.B.R. Lickorish, Unknotting information from 4-manifolds, Trans. Amer. Math. Soc., **297** (1986), 125–142.

[10] S.K. Donaldson, An application of gauge theory to the topology of 4-manifolds, J. Diff. Geom., **18** (1983), 269–316.

[11] R. Fenn, On Dehn's lemma in 4 dimensions, Bull. London Math. Soc., **3** (1971), 79–81.

[12] M. Freedman, The topology of 4-manifolds, J. Diff. Geom., **17** (1982), 357–453.

[13] ———, The disk theorem for four-dimensional manifolds, in "Proc. Internat. Cong. Math., Warszawa", 1983, pp. 647–663.

[14] M. Freedman and F. Quinn, "Topology of 4-manifolds", Princeton Mathematical Series 39, Princeton Univ. Press, Princeton, N.J., 1990.

[15] S. Fukuhara, Casson's invariant for homology lens spaces, Lecture at Tokyo Institute of Technology, January 1989.

[16] S. Fukuhara, Y. Matsumoto and O. Saeki, An estimate for the unknotting numbers of torus knots, Topology Appl., **38** (1991), 293–299.

[17] R. Gompf, Infinite families of Casson handles and topological disks , Topology, **23** (1984), 395–400.

[18] ———, Smooth concordance of topologically slice knots, Topology, **25** (1986), 353–373.

[19] W.C. Hsiang and R.H. Szczarba, On embedding surfaces in four-manifolds, Proc. Symp. Pure Math., **22** (1971), 97–103.

[20] S. Kaplan, Constructing framed 4-manifolds with given almost framed boundaries, Trans. Amer. Math. Soc., **254** (1979), 237–263.

[21] M.A. Kervaire and J.W. Milnor, On 2-spheres in 4-manifolds, Proc. Nat. Acd. Sci. U.S.A., **47** (1961), 1651–1657.

[22] K. Kuga, On immersed 2-spheres in $S^2 \times S^2$, in "A Fete of Topology (Papers dedicated to Itiro Tamura)", Academic Press, 1988, pp. 525–541.
[23] N. Maruyama, On Dehn surgery along a certain family of knots, J. Tsuda College, **19** (1987), 261–280.
[24] G. Matic, $SO(3)$-connections and rational homology cobordisms, J. Diff. Geom., **28** (1988), 277–307.
[25] L. Moser, Elementary surgery along a torus knot, Pacific J. Math., **38** (1971), 737–745.
[26] D. Rolfsen, "Knots and links", Publish or Perish Inc., Berkeley, 1976.
[27] D. Ruberman, Rational homology cobordisms of rational space forms, Topology, **27** (1988), 401–414.
[28] O. Saeki, Simply connected 4-manifolds of second betti number 1 bounded by homology lens spaces, J. Math. Soc. Japan, **41** (1989), 593–606.
[29] T. Shibuya, A lower bound of 4-genus for torus knots, Memoirs of the Osaka Inst. of Technology, Series A, **31** (1986), 11–17.
[30] A. Suciu, Immersed spheres in $\mathbf{C}P^2$ and $S^2 \times S^2$, Math. Z., **196** (1987), 51–57.
[31] A.G. Tristram, Some cobordism invariants for links, Proc. Camb. Phil. Soc., **66** (1969), 251–264.
[32] C.T.C. Wall, "Surgery on compact manifolds", Academic Press, New York, 1970.

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On the Deformations of the Geometric Structures on the Seifert 4-Manifolds

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We call a closed orientable 4-manifold S *a Seifert* 4-*manifold* if S has a structure of a fibered orbifold $\pi: S \to B$ over some 2-orbifold B with general fiber a 2-torus T^2 where the total space S is a nonsingular manifold. In [10], [11] we discussed the relations between them and certain eight geometries in dimension 4 in the sense of Thurston and also gave their topological classification. Here by a geometric structure of S we mean the structure of the form $\Gamma\backslash X$ diffeomorphic to S where X is a 1-connected complete Riemannian 4-manifold and Γ is a discrete subgroup of the group $\mathrm{Isom}^+ X$ of all orientation-preserving isomorphisms of X acting freely on X. The purpose of this paper is to determine the Teichmüller spaces for their geometric structures in the cases when the base orbifolds are either hyperbolic or euclidean (§1 and §2). Our results are parallel to [5], [6] in which the Teichmüller spaces for the geometric structures on the Seifert 3-manifolds were discussed. But a little more arguments are needed for our cases since we should take account of the nontrivial monodromies. In the meanwhile some of the Seifert 4-manifolds have complex structures compatible to their geometric structures ([12]). In these cases we will also give the relations between the Teichmüller spaces and the deformations of the associated complex structures via the Kodaira Spencer maps. In all cases we treat here these maps are surjective but not injective in general (and hence the Teichmüller spaces are not effectively parametrized as families of complex structures §3). Finally in §4 we also give a remark on the moduli spaces for the geometric structures when the base orbifolds are hyperbolic and show that they are defined as Hausdorff spaces whereas, as is well known, the moduli spaces for the complex structures can not be defined as Hausdorff spaces in general. For simplicity in this paper we only consider the Seifert 4-manifolds over the closed orientable base orbifolds. All the subjects will be considered in the smooth category.

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We will use the same notations for the geometries as in [12], [13], [10] and [11]. The 2-dimensional hyperbolic space will be denoted by H^2 and also by $\mathbf{H}$ (as the complex space). Furthermore $\mathbf{C}^*$ and $\mathbf{R}^+$ will be the set of nonzero complex numbers and the set of positive numbers respectively.

§1. Definitions of Teichmüller spaces and the cases when the bases are hyperbolic

Let S be a closed orientable Seifert 4-manifold over a 2-orbifold B and $\pi: S \to B$ be its fiber map with general fiber T^2. In [10] and [11] we proved that S has a geometric structure if B is either euclidean, spherical or bad and if B is hyperbolic S is geometric if and only if S has a complex structure (and is an elliptic surface). Let $G = \mathrm{Isom}^0 X$ be the identity component of $\mathrm{Isom}^+ X$. For simplicity we only consider the geometric Seifert 4-manifolds over either hyperbolic or euclidean orientable base orbifolds of the form $\Gamma\backslash X$ with $\Gamma \subset G$ and the (G, X) structures on them. We note that any geometric Seifert 4-manifold over the hyperbolic 2-orbifold (we have assumed that the base orbifold is orientable) is of the form $\Gamma\backslash X$ where X is either $H^2 \times E^2$ or $\widetilde{SL}_2 \times E$ and $\Gamma \subset G = \mathrm{Isom}^0 X$ ([11]). Here $\widetilde{SL}_2$ is the universal covering of $SL_2\mathbf{R}$.

Definition 1. Let $\mathcal{R}(\Gamma, G)$ be the set of all faithful discrete cocompact representations from Γ to G with compact open topology.

The group $\mathrm{Inn}\, G$ of the inner automorphisms of G and the group $\mathrm{Aut}\,\Gamma$ of the automorphisms of Γ act on $\mathcal{R}(\Gamma, G)$ by $g \cdot \rho(\gamma) \cdot g^{-1}$ and by $\rho \cdot \phi(\gamma) = \rho(\phi(\gamma))$ respectively where $g \in G$, $\gamma \in \Gamma$, $\phi \in \mathrm{Aut}\,\Gamma$ and $\rho \in \mathcal{R}(\Gamma, G)$. The second action commutes with the first one and induces the action of the group $\mathrm{Out}\,\Gamma$ of the outer automorphisms on the quotient $\mathrm{Inn}\, G\backslash\mathcal{R}(\Gamma, G)$.

Definition 2. We call the quotient space

$$\mathcal{T}(\Gamma, G) = \mathrm{Inn}\, G\backslash\mathcal{R}(\Gamma, G)$$

a Teichmüller space of $S = \Gamma\backslash X$, and the quotient

$$\mathcal{M}(\Gamma, G) = \mathrm{Inn}\, G\backslash\mathcal{R}(\Gamma, G)/\,\mathrm{Out}\,\Gamma$$

a moduli space of $S = \Gamma\backslash X$.

The fundamental group $\pi_1^{\mathrm{orb}} B$ of the base orbifold B has the representation of the form

$$\{\overline{\alpha}_1, \dots, \overline{\alpha}_{2g}, \overline{q}_1, \dots, \overline{q}_r \mid$$
$$\overline{q}_1^{m_1} = \dots = \overline{q}_r^{m_r} = \prod_{i=1}^{g} [\overline{\alpha}_{2i-1}, \overline{\alpha}_{2i}] \prod_{j=1}^{r} \overline{q}_j = 1\}$$

where $\overline{q}_i$ corresponds to a meridian circle around the i-th cone point of cone angle $2\pi/m_i$, and $\overline{\alpha}_1, \dots, \overline{\alpha}_{2g}$ form a symplectic base of the fundamental group of the underlying space $|B|$ of B of genus g. Here we define $[\alpha, \beta] = \alpha\beta\alpha^{-1}\beta^{-1}$. Note that $\pi^{\mathrm{orb}} B$ is isomorphic to a discrete subgroup $\overline{\Gamma}$ in $\mathrm{Isom}\,\overline{X}$ and $B = \overline{\Gamma} \backslash \overline{X}$ where $\overline{X} = \mathbf{H}$ if B is hyperbolic and $\overline{X} = \mathbf{E}^2$ if B is euclidean. Then $\pi_1 S$ has the following representation:

$$\begin{aligned}
&\{\alpha_1, \dots, \alpha_{2g}, q_1, \dots, q_r, \ell, h \mid \\
&\ [\ell, h] = 1, [q_j, \ell] = [q_j, h] = 1 \quad \text{for } j = 1, \dots, r, \\
&\ (\alpha_i \ell \alpha_i^{-1}, \alpha_i h \alpha_i^{-1}) = (\ell, h) A_i \quad \text{for } i = 1, \dots, 2g, \\
&\ q_s^{m_s} \ell^{a_s} h^{b_s} = 1 \quad \text{for } s = 1, \dots, r, \\
&\ \prod_{i=1}^{g} [\alpha_{2i-1}, \alpha_{2i}] \prod_{s=1}^{r} q_s = \ell^a h^b\}.
\end{aligned}$$

Here α_i, q_j are the lifts of $\overline{\alpha}_i$ and $\overline{q_j}$ respectively, ℓ and h form a base of the fundamental group $\mathbf{Z}^2$ of the general fiber T^2, $A_i \in SL_2\mathbf{Z}$ is the monodromy matrix corresponding to $\overline{\alpha}_i$ with respect to (ℓ, h), and (m_s, a_s, b_s) is the Seifert invariant of the s-th multiple fiber of multiplicity m_s over the s-th cone point. The Seifert invariants for such S are denoted by

$$\{A_1, \dots, A_{2g}, (a, b), (m_1, a_1, b_1), \dots, (m_r, a_r, b_r)\}.$$

First consider $S = \Gamma \backslash X$ when $B = \overline{\Gamma} \backslash \overline{X}$ is hyperbolic. Then by the results in [15] the fibration of S is unique up to fiber-preserving diffeomorphisms and $\pi_1^{\mathrm{orb}} B = \overline{\Gamma}$ is uniquely determined by Γ up to group automorphisms. Moreover all the monodromy matrices are the powers of some common periodic matrix Q ([11], Theorem B). If every monodromy is trivial then the pair $e = (a + \sum a_j/m_j, b + \sum b_j/m_j) \in \mathbf{Q}^2 \bmod GL_2\mathbf{Z}$ is well defined and is called the rational euler class of S. The type of X is $\widetilde{SL_2} \times E$ if every monodromy is trivial and $e \neq (0,0)$ and $X = H^2 \times E^2$ otherwise. Furthermore we can assume that $a + \sum a_j/m_j = 0$ in the first case by some coordinate change of the fiber ([11]).

Proposition 1. *Let S be a geometric Seifert 4-manifold over a hyperbolic orbifold B. Then for appropriate choices of the lattices of the general fiber and the symplectic basis of the curves on B generating $H_1(|B|, \mathbf{Z})$ we can assume that $A_i = I$ for $i \geq 2$ and A_1 is either* $\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\pm I$, *where I is the identity matrix.*

Proof. We can assume that all monodromy matrices A_i of S are powers of a periodic matrix Q. Then by choosing the lattice ℓ, h appropriately we can suppose that Q is either one of the five matrices given above since Q is periodic. In particular all A_i are mutually commutative. If we consider the pullback of S by a self-automorphism ϕ of the base B fixing all cone points and the base point of B then ϕ induces the symplectic isomorphism ϕ_* on $H_1(|B|, \mathbf{Z})$ and the monodromy matrices $A_1, \ldots, A_{2g}$ are transformed to $\phi_* A_1, \ldots, \phi_* A_{2g}$ where if $\phi_* \overline{\alpha}_i = \delta_1 \overline{\alpha}_1 + \delta_2 \overline{\alpha}_2 + \cdots \delta_{2g} \overline{\alpha}_{2g}$ for the symplectic bases $\overline{\alpha}_1, \ldots, \overline{\alpha}_{2g}$ of $H_1(|B|, \mathbf{Z})$ then $\phi_* A_i = A_1^{\delta_1} A_2^{\delta_2} \cdots A_{2g}^{\delta_{2g}}$. Since all A_i's are powers of Q Euclid algorithm shows that this process simplifies the monodromy matrices (which are still powers of Q after this process) of the Seifert fibration of S induced from the original one by ϕ if ϕ is chosen appropriately. In fact we can see that some automorphism ϕ of B fixing all the cone points induces the isomorphism ϕ_* such that $\phi_* A_i = I$ for $i \geq 2$ as follows. First for any $P_i \in SL_2\mathbf{Z}, (i = 1, \ldots, g)$ there is an automorphism ψ of B such that

$$(\psi_* \overline{\alpha}_{2i-1}, \psi_* \overline{\alpha}_{2i}) = (\overline{\alpha}_{2i-1}, \overline{\alpha}_{2i}) P_i$$

since every symplectic isomorphism can be realized by some orinetation preserving self-diffeomorphism of B fixing all the cone points. Using such ψ the monodromy matrices can be transformed so that $A_{2i} = I$ for $i = 1, \ldots, g$. Next consider the symplectic isomorphism ρ satisfying

$$\begin{aligned} \rho(\overline{\alpha}_2) &= \overline{\alpha}_2 - \overline{\alpha}_{2i} \\ \rho(\overline{\alpha}_{2i-1}) &= \overline{\alpha}_1 + \overline{\alpha}_{2i-1} - \overline{\alpha}_{2i} \\ \rho(\overline{\alpha}_j) &= \overline{\alpha}_j \quad \text{otherwise} \end{aligned}$$

which maps A_{2i-1} to $A_{2i-1} A_1$, leaves the other monodromies unchanged and can be realized by the Dehn twists along the curves representing $\overline{\alpha}_1 \overline{\alpha}_{2i}{}^{-1}$ and α_1^{-1}. Then applying the isomorphisms of this type or their inverses (or those obtained by exchanging the roles of α_1 and α_{2i-1}) and the isomorphism mapping (α_1, α_2) to $(\alpha_{2i-1}, \alpha_{2i})$ we can see (by Euclid algorithm) that A_{2i-1} $(i \geq 2)$ is reduced to I with all others except for A_1 left unchanged. The desired automorphism ϕ is obtained by the composition of the above automorphisms. Finally consider if necessary the symplectic isomorphism σ satisfying

$$\begin{aligned} \sigma(\overline{\alpha}_1) &= -\overline{\alpha}_1 \\ \sigma(\overline{\alpha}_2) &= -\overline{\alpha}_2 \\ \sigma(\overline{\alpha}_j) &= \overline{\alpha}_j \quad \text{for } i \geq 3 \end{aligned}$$

(which maps A_1 to A_1^{-1}, leaves $A_i = I$ ($i \geq 2$) unchanged and is also realized by an automorphism of B) and some further change of the lattice of the general fiber we obtain the desired representations of the monodromy matrices since A_1 is still a power of some periodic matrix. Q.E.D.

In the case of a geometric Seifert 4-manifold $S = \Gamma\backslash X$ over a hyperbolic base orbifold $B = \overline{\Gamma}\backslash\overline{X}$, X is complex analytically equivalent to $\mathbf{H} \times \mathbf{C}$ ([11], [12] and see the proof of Theorem A below) such that the lattice Γ_0 of the general fiber of S acts on the $\mathbf{C}$-factor as translations. Furthermore the lifts of the elements of $\overline{\Gamma}$ to Γ induce the orientation-preserving automorphisms of the $\mathbf{C}$-factor (up to translations) which do not depend on the choices of the lifts and which preserve the lattice in $\mathbf{C}$ defined by Γ_0. Then we have a homomorphism from $\overline{\Gamma}$ to $GL^+(1, \mathbf{C}) = \mathbf{C}^*$ which we call the monodromy representation of $\Gamma\backslash X$ and denote by ϕ. The relation between ϕ and the monodromy matrices of $S = \Gamma\backslash X$ is explained in Theorem A and its proof below. To describe the Teichmüller space of $\Gamma\backslash X$ we introduce the following extra notations.

$\mathcal{T}(\overline{\Gamma}, \mathrm{Isom}^+ \overline{X})$ the Teichmüller space of $B = \overline{\Gamma}\backslash\overline{X}$

$H^1(\overline{\Gamma}, \mathbf{C}^\phi)$ the 1st cohomology group of $\overline{\Gamma}$ with coefficients $\mathbf{C}^\phi$.

Here $\mathbf{C}^\phi$ is $\mathbf{C}$ twisted by ϕ, $\mathcal{T}(\overline{\Gamma}, \mathrm{Isom}^+ \overline{X}) = \mathbf{R}^{2(3g-3+r)} \times \mathbf{Z}_2$ where g is the genus of B and r is the number of the cone points of B. Let $\mathcal{T}_{g,r}$ be the identity component of $\mathcal{T}(\overline{\Gamma}, \mathrm{Isom}^+ \overline{X})$. (It is well known that $\mathcal{T}_{g,r}$ which is also the Teichmüller space of the orbifold B depends only on g and r.)

Theorem A. *$\mathcal{T}(\Gamma, G)$ for a Seifert 4-manifold $S = \Gamma\backslash X$ over a hyperbolic 2-orbifold $B = \overline{\Gamma}\backslash\overline{X}$ has a structure of a trivial fiber bundle of the following form.*

$$\mathcal{F} \to \mathcal{T}(\Gamma, G) \to \mathcal{T}(\overline{\Gamma}, \mathrm{Isom}^+\overline{X}).$$

Here the fiber $\mathcal{F}$ is isomorphic to $H^1(\overline{\Gamma}, \mathbf{C}^\phi) \times \mathbf{T}_1$ where

$$\mathbf{T}_1 = \begin{cases} \mathbf{R}^+ \times \mathbf{H} \times \mathbf{Z}_2, & \text{if } X = H^2 \times E^2 \text{ and } \phi \equiv \pm \mathrm{id} \\ \mathbf{R}^+ \times \mathbf{Z}_2, & \text{if } X = H^2 \times E^2 \text{ and } \phi \not\equiv \pm \mathrm{id} \\ \mathbf{H} \times \mathbf{Z}_2, & \text{if } X = \widetilde{SL}_2 \times E \end{cases}$$

and $\mathbf{T}_1$ corresponds to the deformations of the lattice of the general fiber generated by c and $c\lambda^{-1}$ with $c \in \mathbf{R}^+, \lambda \in \mathbf{C}$ and $\Im\lambda \neq 0$. $\mathbf{T}_1$ has two components according to the sign of $\Im\lambda$. The monodromy representation ϕ satisfies $\phi(\overline{\alpha}_i) = \pm 1$ if the monodromy matrix A_i corresponding to $\overline{\alpha}_i$ is $\pm I$. If $A_i \neq \pm I$ for some i, the lattice Γ_0 in $\mathbf{C}$ of the general fiber is uniquely determined up to scalar multiplication and ϕ is also uniquely determined once the sign of $\Im\lambda$ is fixed. $H^1(\overline{\Gamma}, \mathbf{C}^\phi)$ satisfies

$$H^1(\overline{\Gamma}, \mathbf{C}^\phi) = \begin{cases} \mathbf{C}^{2g} & \text{if } \phi \equiv \mathrm{id} \\ \mathbf{C}^{2g-2} & \text{otherwise.} \end{cases}$$

The identity component $\mathcal{T}_0$ of $\mathcal{T}(\Gamma, G)$ is $\mathcal{T}_{g,r} \times H^1(\overline{\Gamma}, \mathbf{C}^\phi) \times \mathbf{T}_0$ where $\mathbf{T}_0$ is the connected component of $\mathbf{T}_1$ and is homeomorphic to a euclidean space.

Proof. We may assume that S satisfies the conditions in Proposition 1. First suppose that the type X of the geometry of $S = \Gamma\backslash X$ is $H^2 \times E^2$ which is identified with $\mathbf{H} \times \mathbf{C}$. In this case $G = \mathrm{Isom}^0 H^2 \times \mathrm{Isom}^0 E^2$. Let $\rho \in \mathcal{R}(\Gamma, G)$ be any element for $S = \Gamma\backslash X$. Then ρ induces the representation $\overline{\rho} \in \mathcal{R}(\overline{\Gamma}, \overline{G})$ with $\overline{G} = \mathrm{Isom}^0 H^2 = PSL_2\mathbf{R}$. $\overline{\rho}$ gives the representation of the base B as the hyperbolic orbifold $B = \overline{\Gamma}\backslash\mathbf{H}$. Then for the coordinates $(z, w) \in \mathbf{H} \times \mathbf{C}$, we have

(0) $$\rho(\ell)(z, w) = (z, w + c)$$

(1) $$\rho(h)(z, w) = (z, w + d)$$

(2) $$\rho(\alpha_i)(z, w) = (\overline{\rho}(\overline{\alpha}_i)z, \lambda_i w + w_i) \quad (i = 1, \ldots, 2g)$$

(3) $$\rho(q_j)(z, w) = (\overline{\rho}(\overline{q}_i)z, w - (a_j c + b_j d)/m_j)$$

where $c, d \in \mathbf{C}$ are linearly independent over $\mathbf{R}$, $\lambda_i \in S^1 \subset \mathbf{C}, w_i \in \mathbf{C}$ which satisfy the following relations. We note that (3) comes from the relations ${q_j}^{m_j} \ell^{a_j} h^{b_j} = 1, [q_j, \ell] = [q_j, h] = 1$. Put $c = u + iv, d = u' + iv'$ and $P = \begin{pmatrix} u & v \\ u' & v' \end{pmatrix} \in GL_2\mathbf{R}$. Then we deduce from $\alpha_i(\ell, h)\alpha_i^{-1} = (\ell, h)A_i$ that

$$PA_iP^{-1} = \begin{pmatrix} \cos\theta_i & -\sin\theta_i \\ \sin\theta_i & \cos\theta_i \end{pmatrix}$$

where $\lambda_i = \exp(\sqrt{-1}\theta_i)$. Thus we have $\lambda_i = 1$ for $i \geq 2$ since $A_i = I$. If $A_1 = \pm I$, then $\lambda_1 = \pm 1$ and there is no further restriction on (c, d). For the remaining cases we have $(c, d) = (c, \lambda^{-1}c)$ with

$$\lambda_1 = \lambda = \exp(\pm 2\pi i/6) \quad \text{if } A_1 = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \tag{4}$$

$$\lambda = \exp(\pm 2\pi i/6), \lambda_1 = \lambda^2 \quad \text{if } A_1 = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \tag{5}$$

$$\lambda_1 = \lambda = \pm i \quad \text{if } A_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{6}$$

The monodromy representation $\phi\colon \overline{\Gamma} \to \mathbf{C}^*$ is defined by $\phi(\overline{\alpha}_i) = \lambda_i$, $\phi(\overline{q}_i) = 1$. We note that ϕ is uniquely determined if $A_i = \pm I$. If $A_1 \neq \pm I$ we have two choices of λ above according as $\Im\lambda > 0$ or $\Im\lambda < 0$. Hence the parameter space for the lattice of the general fiber (represented by c and λ) has two components and is homeomorphic to $\mathbf{C}^* \times \mathbf{Z}_2$. The monodromy representation ϕ depends only on the choice of λ. If $A_1 = \pm I$, (and if we write $(c, d) = (c, c\lambda^{-1})$) then (c, λ) ranges over $\mathbf{C}^* \times \mathbf{H} \times \mathbf{Z}_2$. In this case we have two components according to the sign of $\Im\lambda$. Finally from the relation $\prod_{j=1}^{g}[\alpha_{2j-1}, \alpha_{2j}] \prod_{j=1}^{r} q_j = \ell^a h^b$ we deduce

$$\begin{aligned} &\sum_{j=1}^{g}(\phi(\overline{\alpha}_{2j-1}) - 1)w_{2j} - \sum_{j=1}^{g}(\phi(\overline{\alpha}_{2j}) - 1)w_{2j-1} \\ &\qquad\qquad = (a + \sum_{j=1}^{r} a_j/m_j)c + (b + \sum_{j=1}^{r} b_j/m_j)d. \end{aligned} \tag{7}$$

Clearly we can find w_j satisfying (7) for any $\overline{\rho}$ since the right hand side on (7) is 0 if $\phi \equiv \mathrm{id}$. Hence ρ defined by (0)–(3) satisfying (4)–(7) defines a discrete faithful representation from Γ to G. Conversely every $\overline{\rho} \in \mathcal{R}(\overline{\Gamma}, \overline{G})$ has a lift $\rho \in \mathcal{R}(\Gamma, G)$. Hence the natural projection $p\colon \mathcal{R}(\Gamma, G) \to \mathcal{R}(\overline{\Gamma}, \overline{G})$ is surjective. Next we describe the fiber $\widetilde{\mathcal{F}}$ of p. Pick up $\overline{\rho}_0 \in \mathcal{R}(\overline{\Gamma}, \overline{G})$ and fix a lift $\rho_0 \in \mathcal{R}(\Gamma, G)$ of $\overline{\rho}$ satisfying (0)–(7) as a base point of the fiber over $\overline{\rho}_0$. Hereafter the parameters in (0)–(7) for ρ_0 are denoted by the same symbols with suffix 0. We can choose w_j^0 in (2) for ρ_0 so that $w_j^0 = 0$ if $\phi \equiv \mathrm{id}$ or $w_j^0 = 0$ for $j \neq 2$ and $w_2^0 = ((a + \sum a_j/m_j)c + (b + \sum b_j/m_j)d)/(\phi(\overline{\alpha_1}) - 1)$ if $\phi \not\equiv \mathrm{id}$ under the assumption in Proposition 1. Take $\rho \in \mathcal{R}(\Gamma, G)$ such that $\overline{\rho} = \overline{\rho}_0, (c, d) = (c_0, d_0)$ and the monodromy representations for ρ and

ρ_0 are the same. Put $m(\overline{\alpha}_j) = w_j - w_j^0$, $m(\overline{q}_j) = 0$. Then $m(\overline{\alpha}_j)$ satisfy

$$(8) \qquad \sum_{j=1}^{g}(\phi(\overline{\alpha}_{2j-1}) - 1)m(\overline{\alpha}_{2j}) - \sum_{j=1}^{g}(\phi(\overline{\alpha}_{2j}) - 1)m(\overline{\alpha}_{2j-1}) = 0$$

and m can be extended to a crossed homomorphism $m: \overline{\Gamma} \to \mathbf{C}$ satisfying $m(\overline{\alpha}\overline{\beta}) = m(\overline{\alpha}) + \phi(\overline{\alpha})m(\overline{\beta})$ for $\overline{\alpha}, \overline{\beta} \in \overline{\Gamma}$. Let $C^i(\overline{\Gamma}, \mathbf{C}^\phi)$ be the i-th cochain of $\overline{\Gamma}$ with coefficients $\mathbf{C}$ twisted by ϕ. Then m is contained in the kernel of the coboundary map $\delta: C^1(\overline{\Gamma}, \mathbf{C}^\phi) \to C^2(\overline{\Gamma}, \mathbf{C}^\phi)$. Conversely if $m \in \ker\delta$ then $m(\overline{q}_j) = 0$ since $\overline{q}_j^{m_j} = 1$ and $\phi(\overline{q}_j) = 1$. Furthermore for any such m, we can define a faithful discrete representation ρ satisfying $\overline{\rho} = \overline{\rho}_0, (c,d) = (c_0, d_0), w_j = w_j^0 + m(\overline{\alpha}_j)$. The choice of m does not depend on the choice of (c_0, d_0) if ϕ is fixed. Thus the fiber $\widetilde{\mathcal{F}}$ is homeomorphic to $(\ker\delta) \times \tilde{\mathbf{T}}_1$ where $\tilde{\mathbf{T}}_1 = \mathbf{C}^* \times \mathbf{Z}_2$ if $\phi \not\equiv \pm\,\mathrm{id}$ or $\mathbf{C}^* \times \mathbf{H} \times \mathbf{Z}_2$ if $\phi \equiv \pm\,\mathrm{id}$. We note that the choices of the parameters of $\tilde{\mathbf{T}}_1$ (and the choices of w_j^0 in (2) satisfying (7) for the fixed lift ρ_0 of $\overline{\rho}_0$) do not depend on $\overline{\rho}$. So the projection $p: \mathcal{R}(\Gamma, G) \to \mathcal{R}(\overline{\Gamma}, \overline{G})$ gives a product fibration and $\mathcal{R}(\Gamma, G) = \mathcal{R}(\overline{\Gamma}, \overline{G}) \times \ker\delta \times \tilde{\mathbf{T}}_1$. Next we check the action of Inn G on $\mathcal{R}(\Gamma, G)$. The action of Inn $PSL_2\mathbf{R}$ is nonrivial only on the first factor $\mathcal{R}(\overline{\Gamma}, \overline{G})$ of $\mathcal{R}(\Gamma, G)$ and yields a Teichmüller space $\mathcal{T}(\overline{\Gamma}, \overline{G})$ of the base orbifold B. $\mathcal{T}(\overline{\Gamma}, \overline{G})$ has just two components which correspond to the Teichmüller spaces of the hyperbolic structures on B and B with opposite orientation. Each one is identified with the Teichmüller space $\mathcal{T}_{g,r} = \mathbf{R}^{2(3g-3+r)}$ of r-pointed Riemann surface of genus g. Next we pick up $\mu \in \mathrm{Isom}^+ E^2$ defined by $\mu(z,w) = (z, \sigma w + w')$ with $\sigma \in S^1, w' \in \mathbf{C}$ (acting trivially on the first coordinate). If $\rho \in \mathcal{R}(\Gamma, G)$ satisfying (0)–(3), then we have

$$\begin{aligned}
&\mu\rho(\ell)\mu^{-1}(z,w) = (z, w + \sigma c),\\
&\mu\rho(h)\mu^{-1}(z,w) = (w, w + \sigma d),\\
&\mu\rho(\alpha_j)\mu^{-1}(z,w) = (\overline{\rho}(\overline{\alpha}_j)z, \lambda_j w + \sigma w_j + (1-\lambda_j)w'),\\
&\mu\rho(q_j)\mu^{-1}(z,w) = (\overline{\rho}(\overline{q_j})z, w - (a_i\sigma c + b_i \sigma d)/m_i).
\end{aligned}$$

where $\lambda_j = \phi(\overline{\alpha}_j)$, $d = c\lambda^{-1}$ as in (4)–(6). Then (c, λ) are transformed to $(\sigma c, \lambda)$ and w_j is transformed to $\sigma w_j + (1 - \phi(\overline{\alpha}_j))w'$. Thus if the representative of c in Inn $\mathrm{Isom}^+ E^2 \backslash \mathcal{F}$ is fixed so that $c \in \mathbf{R}^+$, then $m(\overline{\alpha}_j)$'s are defined modulo the image of $\delta^0: C^0(\overline{\Gamma}, \mathbf{C}^\phi) \to C^1(\overline{\Gamma}, \mathbf{C}^\phi)$ in Inn $\mathrm{Isom}^+ E^2 \backslash \widetilde{\mathcal{F}}$. Therefore we obtain

$$\mathcal{T}(\Gamma, G) = \mathcal{T}(\overline{\Gamma}, \overline{G}) \times (\mathbf{T}_1 \times H^1(\overline{\Gamma}, \mathbf{C}^\phi))$$

where $\mathcal{T}(\Gamma, G)$ is parametrized by $\overline{\rho}$ (mod Inn $\overline{G}$) $\in \mathcal{T}(\overline{\Gamma}, \overline{G})$, $(c, \lambda) \in \mathbf{T}_1 = \mathbf{R}^+ \times \mathbf{H} \times \mathbf{Z}_2$ or $\mathbf{R}^+ \times \mathbf{Z}_2$, and $m(\overline{\alpha}_j)$ (mod $\mathrm{Im}\delta^0$) $\in H^1(\overline{\Gamma}, \mathbf{C}^\phi)$ as desired. $H^1(\overline{\Gamma}, \mathbf{C}^\phi)$ is a vector space over $\mathbf{C}$ whose dimension is easily computed by (8) under the assumption of Proposition 1 as indicated in Theorem A. We note that $H^1(\overline{\Gamma}, \mathbf{C}^\phi)$ is the same as $H^1(\pi_1|B|, \mathbf{C}^\phi)$ since the coefficient of the cohomology is torsion free and the monodromies along the torsion elements $\overline{q}_j$ in $\overline{\Gamma}$ are trivial.

Next we consider the case when $S = \Gamma\backslash X$ with $X = \widetilde{SL_2} \times E$ and $G = \widetilde{SL_2} \times_{\mathbf{Z}} \mathbf{R} \times \mathbf{R}$. In this case X is identified with $\mathbf{H} \times \mathbf{C}$ with coordinates $(z, w), z \in \mathbf{H}, w \in \mathbf{C}$ so that w corresponds to $\log dz$ ([13]). Here the imaginary part of $\log dz$ corresponds to the lift of the unit tangent vector at $z \in H^2$ to the fiber of the natural projection $\pi\colon \widetilde{SL_2} \to H^2$. This projection is defined via the identification of $PSL_2\mathbf{R}$ with the unit tangent bundle T_1H^2 of H^2. The real part of $\log dz$ belongs to the E-factor of X. Then ρ in $\mathcal{R}(\Gamma, G)$ induces the element $\overline{\rho}$ in $\mathcal{R}(\overline{\Gamma}, \overline{G})$ where $\overline{\Gamma} = \pi_1^{\mathrm{orb}}B$ and $\overline{G} = PSL_2\mathbf{R}$. Moreover ρ must be of the following form:

$$\begin{aligned}
\rho(\ell)(z, w) &= (z, w + c)\\
\rho(h)(z, w) &= (z, w + d)\\
\rho(\alpha_j)(z, w) &= \tilde{\alpha}_j(z, w) + (0, w_j)\\
\rho(q_j)(z, w) &= \tilde{q}_j(z, w) + (0, y_j)
\end{aligned}$$

where $\tilde{\alpha}_j(z, w)$ is a lift of $\overline{\rho}(\overline{\alpha}_j)\colon z \to (a_jz + b_j)/(c_jz + d_j)$ defined by

$$\tilde{\alpha}_j(z, w) = (\overline{\rho}(\overline{\alpha}_j)z, w - 2\log(c_jz + d_j)).$$

Here the imaginary part of the second factor is chosen so that it is continuous and it coincides with the image of $\Im w$ by the parallel translation from z to $\overline{\rho}(\overline{\alpha_j})z$ along the axis of the hyperbolic element $\overline{\rho}(\overline{\alpha}_j)$ (which is defined as the lift of that on T_1H^2 via the projection $\widetilde{SL_2} \to T_1H^2$) if z lies in this axis. These conditions determine the choice of the branch of log in the image of $\tilde{\alpha}_j$. A lift $\tilde{q}_j$ of $\overline{\rho}(\overline{q}_j)$ is taken so that $\tilde{q}_j^{m_j} = 1$ in G (cf. [11], [13]). Note that $\tilde{q}_j$ is uniquely determined since the $\mathbf{R} \times \mathbf{R}$-factor of G lies in the center of G. Then

$$y_j = -(a_jc + b_jd)/m_j$$

from $q_j^{m_j}\ell^{a_j}h^{b_j} = 1$. We have chosen ℓ, h so that $a + \sum a_j/m_j = 0$, $b + \sum b_j/m_j \neq 0$ and hence we also deduce from $\prod[\alpha_{2j-1}, \alpha_{2j}]\prod q_j =$

$\ell^a h^b$ that

$$d = (2\pi i \chi^{\mathrm{orb}} B)/(b + \sum b_j/m_j) \neq 0$$

where χ^{orb} denotes the orbifold euler characteristic. Therefore the parameters y_j, d are fixed, $c = u + iv$ is an arbitrary complex number with $u \neq 0$ (since c and d must be linearly independent over $\mathbf{R}$), and w_j are arbitrary complex numbers. Then the natural projection $p: \mathcal{R}(\Gamma, G) \to \mathcal{R}(\overline{\Gamma}, \overline{G})$ defined by $p(\rho) = \overline{\rho}$ is surjective and the fiber $\widetilde{\mathcal{F}}$ of p is $C^1(\overline{\Gamma}, \mathbf{C}) \times \mathbf{T}_1$ where $\mathbf{T}_1 = \mathbf{H} \times \mathbf{Z}_2$ which corresponds to c (or equivalently ic). $\mathbf{T}_1$ has two components according to the sign of $\Re c$. Since every translation $(z, w) \to (z, w + s + ti)$ commutes with any element $g \in G$, the action of Inn G on $\mathcal{R}(\Gamma, G)$ yields the following isomorphism;

$$\mathcal{T}(\Gamma, G) = \mathcal{T}(\overline{\Gamma}, \overline{G}) \times H^1(\overline{\Gamma}, \mathbf{C}) \times \mathbf{T}_1$$

which proves Theorem A.

§2. The cases with euclidean base orbifolds

Suppose that S is a Seifert 4-manifold over a closed orientable euclidean 2-orbifold B. In this case S has always a geometric structure of type X where $X = E^4$, $\mathrm{Nil}^3 \times E$, Nil^4 or $\mathrm{Sol}^3 \times E$ ([10]). But in this paper we restrict our attention to the cases when $\pi_1 S = \Gamma$ is a subgroup of $G = \mathrm{Isom}^0 X$. Then by the results in [10] we have only to consider the cases when S is diffeomorphic to one of the followings (note that the fibration of S is not unique when B is not hyperbolic).

(I) $B = T^2$.

(1) $S = T^4$, $X = E^4$;
(2) $S = \{I, I, (a, b)\}$ with $(a, b) \neq (0, 0)$, $X = \mathrm{Nil}^3 \times E$;
(3) $S = \{\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}, I; (a, b)\}, \lambda \neq 0, b \neq 0$, $X = \mathrm{Nil}^4$;
(4) $S = \{A, I; (a, b)\}$ with $\mathrm{tr}\, A \geq 3$, $X = \mathrm{Sol}^3 \times E$.

(II) B has genus 0.

(1) The rational euler class e of S equals $(0, 0)$, $X = E^4$;
(2) $e \neq (0, 0)$, $X = \mathrm{Nil}^3 \times E$.

Here S is diffeomorphic to a hyperelliptic surface in case (II-1), a primary Kodaira surface in case (I-2), and a secondary Kodaira surface in case (II-2). We note that

$$G = \begin{cases} \mathrm{Isom}^0\, E^4 = \mathbf{R}^4 \rtimes SO_4 & \text{if } X = E^4, \\ \mathrm{Nil}^3 \rtimes SO_2 \times \mathbf{R} & \text{if } X = \mathrm{Nil}^3 \times E, \\ \mathrm{Sol}^3 \times \mathbf{R} & \text{if } X = \mathrm{Sol}^3 \times E, \\ \mathrm{Nil}^4 & \text{if } X = \mathrm{Nil}^4. \end{cases}$$

Theorem B. *For the Seifert* 4*-manifold* $S = \Gamma\backslash X$ *in the above list we have the list of* $\mathcal{T}(\Gamma, G)$ *as follows.*

(1) *The cases when* $B = T^2$.

$$\mathcal{T}(\Gamma, G) = \begin{cases} SO_4\backslash GL_4\mathbf{R}, & \textit{in case } (\text{I-1}), \\ \mathcal{T}_{1,0} \times \mathcal{F}, & \textit{in case } (\text{I-2}), \\ (\mathbf{R}^*)^2 \times \mathbf{R}^2, & \textit{in case } (\text{I-3}), \\ (\mathbf{R}^*)^2 \times (\mathbf{Z}_2)^2 \times \mathbf{R}, & \textit{in case } (\text{I-4}). \end{cases}$$

Here in case (I-2) $\mathcal{T}_{1,0} = SO_2\backslash GL_2\mathbf{R} = \mathbf{R}^+ \times \mathbf{H} \times \mathbf{Z}_2$, $\mathcal{F} = \mathbf{R}^2 \times \mathbf{H} \times \mathbf{Z}_2$.

(2) *The cases with the base orbifolds of genus 0. Let* r *be the number of the cone points of* B ($r = 3$ *or* 4)*. Then we have*

$$\mathcal{T}(\Gamma, G) = \mathcal{T}_{0,r} \times \mathbf{T}_1$$

where

$$\mathcal{T}_{0,r} = \begin{cases} \mathbf{R}^{2(r-3)} \times \mathbf{R}^+ \times \mathbf{Z}_2 & \textit{if } X = \mathrm{Nil}^3 \times E \\ \mathbf{R}^{2(r-3)} \times \mathbf{R}^+ & \textit{if } X = E^4 \end{cases}$$

and

$$\mathbf{T}_1 = \begin{cases} \mathbf{H} \times \mathbf{Z}_2 & \textit{if } X = \mathrm{Nil}^3 \times E \\ \mathbf{R}^+ \times \mathbf{H} \times \mathbf{Z}_2 & \textit{if } X = E^4. \end{cases}$$

Proof. In case (I-1) we have $\Gamma = \mathbf{Z}^4$ whose generators α_i are given by translations $\alpha_i x = x + \ell_i$ for $x, \ell_i \in E^4$. Here ℓ_i are mutually linearly independent and hence Γ is parametrized by $GL_4\mathbf{R}$. The action of Inn G is given by $(\ell_1, \ldots, \ell_4) = (\sigma\ell_1, \ldots, \sigma\ell_4)$ for $\sigma \in SO_4$. Hence $\mathcal{T}(\Gamma, G) = SO_4\backslash GL_4\mathbf{R}$ which has two components and each one is homeomorphic to $\mathbf{R}^{10}$.

Next consider case (I-2). In this case $X = \mathrm{Nil}^3 \times E$. Here we recall the structure of $\mathrm{Nil}^3 \times E$. The point of X is represented by $(w, z) \in \mathbf{C}^2$ such that the action of $G = (\mathrm{Nil}^3 \times \mathbf{R}) \rtimes S^1$ is defined by $(w', z')(w, z) = (w' + w - i\overline{z'}z, z' + z)$ for $(w', z') \in X$, and $t(w, z) = (w, tz)$ for $t \in S^1 \subset \mathbf{C}$. We can assume that $S = \{I, I; (a, 0)\}$ for $a \neq 0$ and $\Gamma = \{\alpha, \beta, \ell, h \mid$

$[\ell, h] = [\alpha, \ell] = [\alpha, h] = [\beta, \ell] = [\beta, h] = 1, [\alpha, \beta] = \ell^a\}$. The subgroup Γ_0 generated by ℓ, h is the center of Γ and the projection $\mathrm{Nil}^3 \times E \to \mathbf{R}^2$ defined by $(w, z) \to z$ induces the structure of a T^2-bundle over T^2 of the form $\Gamma_0 \backslash \mathbf{R} \times E \to \Gamma \backslash \mathrm{Nil}^3 \times E \to \mathbf{Z}^2 \backslash \mathbf{R}^2$ which gives the above fibration ([12]). Thus $\rho \in \mathcal{R}(\Gamma, G)$ must have the following form:

$$\begin{aligned} \rho(\ell)(w, z) &= (w + \ell_0, z) \\ \rho(h)(w, z) &= (w + h_0, z) \\ \rho(\alpha)(w, z) &= (a_0 + w - i\overline{b_0} z, b_0 + z) \\ \rho(\beta)(w, z) &= (a_1 + w - i\overline{b_1} z, b_1 + z) \end{aligned}$$

where ℓ_0 and h_0, b_0 and b_1 are linearly independent over $\mathbf{R}$. Since $\rho([\alpha, \beta])(w, z) = (w + i(b_0\overline{b_1} - \overline{b_0}b_1), z)$ must be equal to $\rho(\ell^a)(w, z)$, we have $\ell_0 = i(b_0\overline{b_1} - \overline{b_0}b_1)/a$ and since this is a nonzero real number h_0 is an arbitrary number with $\Im h_0 \neq 0$. Thus we have

$$\mathcal{R}(\Gamma, G) = \widetilde{\mathcal{F}} \times \mathcal{R}(\mathbf{Z}^2, \overline{G})$$

where $\widetilde{\mathcal{F}} = \mathbf{H} \times \mathbf{Z}_2 \times \mathbf{C}^2$ which is represented by h_0, a_0, a_1, $\overline{G} = \mathrm{Isom}^0 E^2$, and $\mathcal{R}(\mathbf{Z}^2, G) = GL_2\mathbf{R}$ represented by b_0 and b_1. Next taking the conjugation of ρ by $\gamma = (w_0, z_0)$ and $t \in S^1$ we can see that the parameters are transformed as follows:

$$\begin{aligned} a_0 &\to a_0 + i(\overline{b_0}z_0 - \overline{z_0}b_0) \\ a_1 &\to a_1 + i(\overline{b_1}z_0 - \overline{z_0}b_1) \\ b_0 &\to tb_0 \\ b_1 &\to tb_1. \end{aligned}$$

Thus to get the representation in $\mathcal{T}(\Gamma, G)$ we can assume that $b_0 \in \mathbf{R}^+$ (choose $t \in S^1$ appropriately). Then $i(\overline{b_0}z_0 - \overline{z_0}b_0) = -2b_0\Im z_0$ and hence choosing z_0 so that $\Im z_0 = \Re a_0/2b_0$ then $a_0 + i(\overline{b}_0 z_0 - \overline{z}_0 b_0)$ is pure imaginary. On the other hand $i(\overline{b_1}z_0 - \overline{z_0}b_1) = 2((\Im b_1)(\Re z_0) - (\Re b_1)(\Re a_0/2b_0))$. Since $\Im b_1 \neq 0$ by the assumption we can choose z_0 so that both a_0 and a_1 are transformed to pure imaginary numbers. Consequently we have $\mathcal{T}(\Gamma, G) = \mathcal{F} \times \mathcal{T}(\overline{\Gamma}, \overline{G})$ where $\mathcal{F} = \mathbf{R}^2 \times \mathbf{H} \times \mathbf{Z}_2$ represented by ia_0, ia_1, h_0 and $\mathcal{T}(\overline{\Gamma}, \overline{G}) = SO_2 \backslash GL_2\mathbf{R} = \mathbf{R}^+ \times \mathbf{H} \times \mathbf{Z}_2$ represented by b_0 and b_1.

In case (II) we can assume that

$$S = \{(0, 0), (m_1, a_1, b_1), \ldots, (m_r, a_r, b_r)\}$$

with $r = 3$ or 4 and

$$\Gamma = \{q_1, \dots, q_r, \ell, h \mid [\ell, h] = [q_i, \ell] = [q_i, h] = 1,$$

$$q_i^{m_i} \ell^{a_i} h^{b_i} = 1, \prod q_i = 1\}.$$

Here if $r = 3$, then $(m_1, m_2, m_3) = (2,4,4), (2,3,6)$, or $(3,3,3)$ and if $r = 4$ then $(m_1, m_2, m_3, m_4) = (2,2,2,2)$. Furthermore in this case $X = E^4$ or $\mathrm{Nil}^3 \times E$. In either case the subgroup Γ_0 generated by ℓ and h are the center of Γ and the exact sequence $1 \to \Gamma_0 \to \Gamma \to \overline{\Gamma} \to 1$ (where $\overline{\Gamma} = \pi_1^{\mathrm{orb}} B$) yields the original Seifert fibration. First suppose that $X = E^4$, $G = \mathbf{R}^4 \rtimes SO_4$. Pick up $\rho \in \mathcal{R}(\Gamma, G)$. Then the holonomy group of $\rho(\Gamma)$ which is the image $\overline{\rho(\Gamma)}$ of $\rho(\Gamma)$ under the natural map $G \to SO_4$ is cyclic (since S is diffeomorphic to a hyperelliptic surface). Since $\rho(\Gamma_0)$ must be contained in the translation parts of G we can assume that there is a decomposition $\mathbf{C} \times \mathbf{C}$ of E^4 such that $\rho(\Gamma_0)$ acts trivially on the first factor. Since $\rho(\Gamma_0)$ commutes with any element in $\rho(\Gamma)$ we can see that any element of $\overline{\rho(\Gamma)}$ is contained in $SO_2 \times 1 \subset SO_2 \times SO_2 \subset SO_4$. If we take another $\rho' \in \mathcal{R}(\Gamma, G)$, then there exists $\sigma \in SO_4$ such that $\sigma\rho'(\Gamma_0)\sigma^{-1}$ satisfies the above condition for the same decomposition of E^4 and hence the image of $\sigma\rho'(\Gamma)\sigma^{-1}$ under the above map is also contained in the same subgroup $SO_2 \times 1 \subset SO_4$. Therefore it suffices to consider the representation ρ satisfying the above conditions for the fixed decomposition of E^4. Thus ρ projects to $\overline{\rho} \in \mathcal{R}(\overline{\Gamma}, \overline{G})$ where $\overline{G} = \mathrm{Isom}^0 E^2$ and we must have

$$\begin{aligned}
\rho(\ell)(z, w) &= (z, w + \ell_0) \\
\rho(h)(z, w) &= (z, w + h_0) \\
\rho(q_i)(z, w) &= (\overline{\rho}(\overline{q_i})z, w + s_i), \quad (1 \le i \le r)
\end{aligned}$$

where ℓ_0 and h_0 are linearly independent over $\mathbf{R}$, $\overline{q_i} \in \overline{\Gamma}$ is the image of q_i under the projection $\Gamma \to \overline{\Gamma}$, $s_i \in \mathbf{C}$. Then from the relation $q_i^{m_i} \ell^{a_i} h^{b_i} = 1$ we deduce $s_i = -(a_i\ell_0 + b_i h_0)/m_i$. Next we must see exactly when the two representations ρ and ρ' of the above forms are in the same orbit under the action of $\mathrm{Inn}\, G$. Suppose that there exists $\sigma \in G$ such that $\sigma\rho\sigma^{-1} = \rho'$ with $\sigma\mathbf{x} = \overline{\sigma}\mathbf{x} + s_0$, $\overline{\sigma} \in SO_4$, $s_0 \in \mathbf{R}^4$ and $\mathbf{x} \in \mathbf{R}^4$. Then we can see that $\overline{\sigma}\mathbf{x} = (\sigma_1 z, \sigma_2 w)$ where $\sigma_1, \sigma_2 \in SO_2$ or $\sigma_1, \sigma_2 \in O_2 - SO_2$ and $\mathbf{x} = (z, w)$ with $z, w \in \mathbf{C}$. Suppose that $\sigma(z, w) = (\sigma_1 z + a_0, \sigma_2 w + b_0)$ with $\sigma_1, \sigma_2 \in SO_2$, $a_0, b_0 \in \mathbf{C}$. Then ρ'

satisfies

$$\rho'(\ell)(z,w) = (z, w+\sigma_2\ell_0)$$
$$\rho'(h)(z,w) = (z, w+\sigma_2 h_0)$$
$$\rho'(q_i)(z,w) = (\sigma_1\overline{\rho}(\overline{q_i})(\sigma_1^{-1}(z-a_0))+a_0, w+\sigma_2 s_i).$$

Next suppose that $\sigma(z,w) = (\sigma_1\overline{z}+a_0, \sigma_2\overline{w}+b_0)$ with $\sigma_1, \sigma_2 \in SO_2$, $a_0, b_0 \in \mathbf{C}$. Then

$$\rho'(\ell)(z,w) = (z, w+\sigma_2\overline{\ell_0})$$
$$\rho'(h)(z,w) = (z, w+\sigma_2\overline{h_0})$$
$$\rho'(q_i)(z,w) = (\overline{\rho'(q_i)}(z), w+\sigma_2\overline{s_i})$$

where $\overline{\rho'(q_i)}$ is in the component of $\mathcal{R}(\overline{\Gamma},\overline{G})$ different from that containing $\overline{\rho}$. On the other hand the identity component $\mathcal{R}^0(\overline{\Gamma},\overline{G})$ of $\mathcal{R}(\overline{\Gamma},\overline{G})$ is homeomorphic to $\mathbf{R}^{2(r-3)}\times\mathbf{R}^+$ where the first factor coincides with the Teichmüller space for the flat structures of area 1 of the base orbifold B and the second factor corresponds to the area of B. Thus we have $\mathcal{T}(\Gamma,G) = \mathbf{R}^{2(r-3)}\times\mathbf{R}^+\times\mathbf{R}^+\times\mathbf{H}\times\mathbf{Z}_2$ where $\mathbf{R}^+\times\mathbf{H}\times\mathbf{Z}_2 = SO_2\backslash GL_2\mathbf{R}$ corresponds to the deformations of the lattice of the general fiber.

On the other hand in case (II) with $X = \mathrm{Nil}^3\times E$ the natural projection $X\to\mathbf{C}$ represented by $(w,z)\to z$ for the coordinates defined above yields a given fibration for $S = \{(a,b),(m_1,a_1,b_1),\dots,(m_r,a_r,b_r)\}$. (In fact we can assume that $a=b=0$.) Here we can assume that $a+\sum a_i/m_i = 0$, $b+\sum b_i/m_i \neq 0$. Take an arbitrary representation $\rho\in\mathcal{R}(\Gamma,G)$. Then ρ induces a representation $\overline{\rho}\in\mathcal{R}(\overline{\Gamma},\overline{G})$ where $\overline{\Gamma} = \{\overline{q_1},\dots,\overline{q_r}\mid \overline{q}_1^{m_1}=\dots=\overline{q}_r^{m_r}=\overline{q}_1\cdots\overline{q}_r=1\}$ and $\rho(\overline{\Gamma})\subset\overline{G} = \mathrm{Isom}^0 E^2$. Here r and $(m_1,\dots,m_r)$ satisfy the same conditions as in the case with $X=E^4$. Each $\overline{\rho}(\overline{q_i})$ has the following representation:

$$\overline{\rho}(\overline{q_j})z = \rho_j(z-z_j)+z_j$$

for $\rho_j\in S^1$, $z_j\in\mathbf{C}$ such that the order of ρ_j is m_j and $\prod\rho_j = 1$. Thus ρ must be of the following form:

$$\rho(\ell)(w,z) = (w+\ell_0, z)$$
$$\rho(h)(w,z) = (w+h_0, z)$$
$$\rho(q_j)(w,z) = (w+w_j+i\overline{z_j}(z-\rho_j(z-z_j)), \overline{\rho}(\overline{q_j})z)$$

where ℓ_0 and h_0 are linearly independent over $\mathbf{R}$, $w_j\in\mathbf{C}$. Then we have

$$\rho(q_j^{m_j})(w,z) = (w+m_j(w_j+i|z_j|^2), z).$$

Thus form $q_j^{m_j}\ell^{a_j}h^{b_j} = 1$ we deduce

$$w_j = -i|z_j|^2 - a_j\ell_0/m_j - b_jh_0/m_j \tag{1}$$

Suppose that $r = 3$. Then from $\prod \overline{q_j} = 1$ we have

$$(1-\rho_1)z_1 + (\rho_1 - \rho_1\rho_2)z_2 + (\rho_1\rho_2 - \rho_1\rho_2\rho_3)z_3 = 0. \tag{2}$$

Thus we can see that

$$\rho(q_1q_2q_3)(w,z) = (w + \sum_{j=1}^{3} w_j + iU, z)$$

where

$$U = \sum \rho_j|z_j|^2 + \overline{z_2}z_3(1-\rho_2-\rho_3+\rho_2\rho_3) + \overline{z_1}z_2(1-\rho_1-\rho_2+\rho_1\rho_2)+$$
$$\overline{z_1}z_3(1-\overline{\rho}_1-\overline{\rho}_3+\overline{\rho}_1\overline{\rho}_3).$$

Thus from (1) ρ is well defined if and only if

$$iV = (a + \sum a_j/m_j)\ell_0 + (b + \sum b_j/m_j)h_0 \tag{3}$$

where $V = U - \sum |z_j|^2$.

Claim. *iV is a nonzero real number.*

Easy computation (using (2)) shows that iV is invariant under translations along the real line $(z_1, z_2, z_3) \to (z_1+\lambda, z_2+\lambda, z_3+\lambda)$ for $\lambda \in \mathbf{R}$ and the rotations in the origin. Therefore to prove Claim we can assume that $z_1 = 0$ and z_2 is a nonzero real number r. Then again by easy computation we can see that $V = r^2(\rho_1-1)(\rho_2-1)/(\rho_1\rho_2-1)$ and V is a nonzero pure imaginary number. Thus we deduce that $h_0 = iV/(b+\sum b_j/m_j) \neq 0$ from the normalization $a + \sum a_j/m_j = 0$. Then ℓ_0 is an arbitrary number with $\Im \ell_0 \neq 0$. The case with $r = 4$ $(m_i = 2, \rho_j = -1$ for all $j)$ can be treated by a similar computation and we can see that h_0 is some fixed real number and ℓ_0 is also an arbitrary number with $\Im \ell_0 \neq 0$. In any case $\overline{\rho} \in \mathcal{R}(\overline{\Gamma}, \overline{G})$ can be lifted to some $\rho \in \mathcal{R}(\Gamma, G)$ and we have $\mathcal{R}(\Gamma, G) = \mathcal{R}(\overline{\Gamma}, \overline{G}) \times \mathbf{H} \times \mathbf{Z}_2$ where the factor $\mathbf{H} \times \mathbf{Z}_2$ corresponds to ℓ_0. The action of Inn G on $\mathcal{R}(\Gamma, G)$ can be determined as in case (I-2). For any element $\gamma \in G$ the conjugation by γ acts on the factor $\overline{\rho}(\overline{q_i})$ as the inner automorphism of $\overline{G}$ and acts trivially on $\rho(\ell), \rho(h)$. Since these parameters determine the remaining ones uniquely and the natural map Inn $G \to$ Inn $\overline{G}$ is surjective we have

$$\mathcal{T}(\Gamma, G) = \mathcal{T}(\overline{\Gamma}, \overline{G}) \times \mathbf{H} \times \mathbf{Z}_2.$$

The proofs for the other cases are done by similar methods and hence are omitted.

§3. The Teichmüller spaces and the complex structures

In this section we consider the Seifert 4-manifolds over closed orientable hyperbolic or euclidean 2-orbifolds which admit complex structures. The arguments in [12, §7] show that any elliptic surface S with $c_2 = 0$ and with $\kappa = 0$ or 1 is biholomorphic to $\Gamma\backslash X$ where a geometry X has a complex structure such that any element of $G = \mathrm{Isom}^0 X$ acts on X as a biholomorphism and $\Gamma \subset G$. (We need to restrict G to $U(2)$ when $X = E^4$.) Here we start with such a Seifert 4-manifold $S = \Gamma\backslash X$ with a given compatible complex structure. Let g be the genus of B and r be the number of the cone points (with the prescribed cone angles) of $B = \overline{\Gamma}\backslash\overline{X}$. In this section let $\mathcal{R} = \mathcal{R}^0(\Gamma, G)$ and $\mathcal{T} = \mathcal{T}^0(\Gamma, G)$ be the connected components of $\mathcal{R}(\Gamma, G)$ and of $\mathcal{T}(\Gamma, G)$ containing $S = \Gamma\backslash X$ with given geometric structure $\rho_0 \in \mathcal{R}$ and its equivalence class $[\rho_0] \in \mathcal{T}$ respectively.

First suppose that B is hyperbolic. In this case $X = H^2 \times E^2$ or $\widetilde{SL_2} \times E$ each of which is identified with $\mathbf{H} \times \mathbf{C}$ as in §1. Hereafter we adopt the same notations for Γ, $\overline{\Gamma}$ and S as in §1 and we assume that the monodromies of S satisfy the conditions in Proposition 1. Now we will describe $\mathcal{T}^0(\Gamma, G)$ as a differentiable family of the complex structures on S. Let S_ρ be the Seifert 4-manifold S with the geometric structure corresponding to $\rho \in \mathcal{R}$ and $[\rho]$ be its equivalence class in $\mathcal{T}$ respectively. For simplicity put $S_0 = S_{\rho_0}$ and let Θ_0 be the sheaf of germs of holomorphic tangent vector fields on S_0. First recall that $\mathcal{T} = \overline{\mathcal{T}} \times \mathbf{T}_0 \times H^1(\overline{\Gamma}, \mathbf{C}^\phi)$. Here $\overline{\mathcal{T}}$ is (the connected component of) the Teichmüller space of $B = \overline{\Gamma}\backslash\mathbf{H}$ or equivalently of the Fuchsian group $\overline{\Gamma}$, ϕ is the monodromy representation for S, and $\mathbf{T}_0$ is the identity component of $\mathbf{T}_1$ in §1. Put

$$\widetilde{\mathbf{T}}_0 = \begin{cases} \mathbf{H} & \text{if } \phi \equiv \pm\mathrm{id} \\ 1 & \text{otherwise} \end{cases}$$

and let $\widetilde{\mathcal{T}} = \overline{\mathcal{T}} \times \widetilde{\mathbf{T}}_0 \times H^1(\overline{\Gamma}, \mathbf{C}^\phi)$. Then $\widetilde{\mathcal{T}}$ denotes the Teichmüller space of the geometric structures which fix the area of the general fiber (if $X = H^2 \times E^2$). On the other hand the Teichmüller space $\overline{\mathcal{T}}$ for the Fuchsian group $\overline{\Gamma}$ is realized as a bounded domain in $\mathbf{C}^{3g-3+r}$. Moreover there is a fiber space $\overline{\mathcal{C}} = \{(z, \tau) \in \mathbf{C} \times \overline{\mathcal{T}} \mid z \in D(\tau)\}$ over $\overline{\mathcal{T}}$ where $D(\tau)$ is a domain in $\mathbf{C}$ on which a quasi-Fuchsian group $\overline{\Gamma}^\tau$ corresponding to τ acts

([1]). $\overline{\mathcal{C}}$ has a complex structure such that the map $(z, \tau) \to (\tau(\alpha)z, \tau)$ for any $\alpha \in \overline{\Gamma}$ is holomorphic. Here $\tau(\alpha)$ acts on $z \in D(\tau)$ via $\overline{\Gamma}^{\tau}$. It follows that for any $\tau \in \overline{\mathcal{T}}$ we have a biholomorphism $h_\tau : \mathbf{H} \to D(\tau)$ and a representation $\overline{\rho} \in \overline{\mathcal{R}}$ with $[\overline{\rho}] = \tau$ such that $\tau(\alpha) \cdot h_\tau(z) = h_\tau(\overline{\rho}(\alpha)z)$ for any $z \in \mathbf{H}, \alpha \in \overline{\Gamma}$.

If $X = H^2 \times E^2$ then h_τ is lifted to $h_\tau \times \mathrm{id}: \mathbf{H} \times \mathbf{C} \to D(\tau) \times \mathbf{C}$ such that it commutes with the translations in the $\mathbf{C}$-factor. On the other hand we can choose the elements $m_1, \ldots, m_d$ of $C^1(\overline{\Gamma}, \mathbf{C}^\phi)$ which maps to the basis of $H^1(\overline{\Gamma}, \mathbf{C}^\phi)$ where $d = \dim_{\mathbf{C}} H^1(\overline{\Gamma}, \mathbf{C}^\phi)$. Thus we have the family of representations of Γ on $\overline{\mathcal{C}} \times \mathbf{C} \times \mathbf{C}^d \times \mathbf{T}_0$ as follows. Let $\tau_0 \in \overline{\mathcal{T}}$ be the equivalence class of the element $\overline{\rho_0} \in \overline{\mathcal{R}}$ determined by ρ_0. Then S_0 is biholomorphic to $\overline{\Gamma}^{\tau_0} \backslash (D(\tau_0) \times \mathbf{C})$ where ρ_0 is represented (via $\overline{\Gamma}^{\tau_0}$) as follows;

$$\begin{aligned}
\rho_0(\ell)(z, w) &= (z, w + r_0), \\
\rho_0(h)(z, w) &= (z, w + r_0 h_0), \\
\rho_0(\alpha_i)(z, w) &= (\tau_0(\overline{\alpha_i})z, \phi(\overline{\alpha_i})w + w_j^0), \\
\rho_0(q_i)(z, w) &= (\tau_0(\overline{q_i})z, w - a_i r_0/m_i - b_i r_0 h_0/m_i)
\end{aligned}$$

where $(z, w) \in D(\tau_0) \times \mathbf{C}$, $r_0 \in \mathbf{R}^+$, $h_0 \in \mathbf{H}$ and $w_j^0 \in \mathbf{C}$ is defined as in the proof of Theorem A. Then we have the following representations $\rho = \rho(\tau, r, h, s)$ from $\overline{\Gamma}$ to the group of the biholomorphisms of $D(\tau) \times \mathbf{C}$ where $s = (s_1, \ldots, s_d) \in \mathbf{C}^d$, $\tau \in \overline{\mathcal{T}}$, $(r, h) \in \mathbf{T}_0$ ($h = h_0$ if $\phi \not\equiv \pm\,\mathrm{id}$).

$$\begin{aligned}
\rho(\ell)(z, w) &= (z, w + r), \\
\rho(h)(z, w) &= (z, w + rh), \\
\rho(\alpha_i)(z, w) &= (\tau(\overline{\alpha_i})z, \phi(\overline{\alpha_i})w + w_j^0 + \sum_{j=1}^{d} s_j m_j(\overline{\alpha_i})), \\
\rho(q_i)(z, w) &= (\tau(\overline{q_i})z, w - a_i r/m_i - b_i rh/m_i).
\end{aligned}$$

Here $H^1(\overline{\Gamma}, \mathbf{C}^\phi)$ is identified with $\mathbf{C}^d$. Thus we get the fiber space $\mathcal{C}$ over $\mathcal{T}$ obtained from $\overline{\mathcal{C}} \times \mathbf{C} \times \mathbf{T}_0 \times H^1(\overline{\Gamma}, \mathbf{C}^\phi)$ by the actions of ρ defined above such that the fiber of $\mathcal{C}$ over $\tilde{\tau} \in \mathcal{T}$ is an elliptic surface corresponding to $\tilde{\tau}$. Also we have the fiber space $\widetilde{\mathcal{C}}$ over $\widetilde{\mathcal{T}}$ by restricting the above representations to the cases with $r = r_0$ (the constant). The above representation depends holomorphically on all the parameters except for $r \in \mathbf{R}^+$ (if $\mathbf{T}_0$ has the $\mathbf{R}^+$-factor). Therefore we have the differentiable family $\mathcal{C} \to \mathcal{T}$ and the complex analytic family $\widetilde{\mathcal{C}} \to \widetilde{\mathcal{T}}$ of the complex structures on S respectively.

If $X = \widetilde{SL_2} \times E$ then X is identified with $\mathbf{H} \times \mathbf{C}$ with coordinates (z, w) so that w corresponds to $\log dz$. Hence in this case $h_\tau: \mathbf{H} \to D(\tau)$ is lifted to the biholomorphism $\widetilde{h_\tau}: \mathbf{H} \times \mathbf{C} \to D(\tau) \times \mathbf{C}$ defined by $\widetilde{h_\tau}(z, w) = (h_\tau(z), w + \log(\partial h_\tau/\partial z)(z))$ where the branch of the log is chosen so that $\widetilde{h_\tau}$ depends holomorphically on $\tau \in \overline{\mathcal{T}}$. Again $\widetilde{h_\tau}$ commutes with the translation $(z, w) \to (z, w + c)$ with $c =$ a constant. On the other hand we can define $\tilde{\rho}(\alpha): \mathbf{H} \times \mathbf{C} \to \mathbf{H} \times \mathbf{C}$ for $\alpha \in \Gamma$ such that $\widetilde{\alpha}_i = \tilde{\rho}(\alpha_i)$ (where $\overline{\alpha}_i \in \overline{\Gamma}$ is hyperbolic), $\widetilde{q_i} = \tilde{\rho}(q_i)$ (where $\overline{q}_i \in \overline{\Gamma}$ is elliptic) satisfy the same conditions as $\widetilde{\alpha}_i$, $\widetilde{q_i}$ stated in the proof of Theorem A. Thus for $\overline{\rho} \in \overline{\mathcal{R}}$ with $[\overline{\rho}] = \tau \in \overline{\mathcal{T}}$, we can define $\tilde{\tau}(\alpha): D(\tau) \times \mathbf{C} \to D(\tau) \times \mathbf{C}$ for $\alpha = \alpha_i$ or q_i which covers $\tau(\overline{\alpha})$ such that $\widetilde{h_\tau}\tilde{\tau}(\alpha) = \tilde{\rho}(\alpha)\widetilde{h_\tau}$ where $\overline{\alpha}$ is the image of α in $\overline{\Gamma}$. Using this lift we can define the family of representations parametrized by $\mathcal{T}$ (in this case $\widetilde{\mathcal{T}} = \mathcal{T}$ since there is no $\mathbf{R}$-factor in $\mathcal{T}$) as in the case with $X = H^2 \times E^2$. Thus we also get the analogous family of complex structures on S.

Next we will describe the Kodaira-Spencer's infinitesimal deformation map

$$\Phi: \mathbf{T}_0\mathcal{T} \to H^1(S_0, \Theta_0)$$

where $\mathbf{T}_0\mathcal{T}$ is the tangent space of $\mathcal{T}$ at S_0 (or equivalently at $[\rho_0]$). Since the base orbifold B of S is hyperbolic in our case here $\mathcal{T}$ is homeomorphic to a euclidean space and is homeomorphic to $\mathbf{T}_0\mathcal{T}$. To describe $H^1(S_0, \Theta_0)$ we recall some results in [8]. Let $\mathbf{T}^1$ be the complex torus of dimension 1. For a holomorphic Seifert fibering $S = \Gamma_\rho \backslash X$ with $\Gamma_\rho = \rho(\Gamma) \subset G$, the base orbifold B is naturally a nonsingular curve B_ρ of the form $\overline{\Gamma}_\rho \backslash \overline{X}$ where $\overline{X} = \mathbf{H}$ or $\mathbf{C}$, $\overline{\Gamma}_\rho = \pi_1^{\mathrm{orb}} B$. (In [8], $S, \overline{X}, B, \overline{\Gamma}$ are denoted by M, W, V, N respectively.) Then $\widetilde{S} \to \overline{X}$ induced by the covering projection $\overline{X} \to B$ is a principal $\mathbf{T}^1$-bundle and $\widetilde{S} = \overline{X} \times \mathbf{T}^1$ ([8, §1,§7] in which $\widetilde{S}$ is denoted by B). Let $\mathcal{Z}^2$, $\mathcal{O}$, $\mathcal{T}^1$ be the sheaves over $\overline{X}$ of germs of local holomorphic maps from $\overline{X}$ into $\mathbf{Z}^2$, $\mathbf{C}$, $\mathbf{T}^1$ respectively. Then the action of $\overline{\Gamma}$ on $\widetilde{S}$ is defined by the element $m \in H^1(\overline{\Gamma}, \mathcal{T}^1)$. Here $H^1(\overline{\Gamma}, \mathcal{T}^1) = H^1(\overline{\Gamma}, H^0(\overline{X}, \mathcal{T}^1)^\phi)$ where the coefficients $H^0(\overline{X}, \mathcal{T}^1)$ on the right hand side is the space of global holomorphic maps on $\overline{X}$ and is twisted by the monodromy representation $\phi: \overline{\Gamma} \to GL_1\mathbf{C}$. The element m is represented by a crossed homomorphism $m: \overline{\Gamma} \to H^0(\overline{X}, \mathcal{T}^1)$ such that $m(x, \alpha)$ for a fixed $\alpha \in \overline{\Gamma}$ is a holomorphic map from $\overline{X}$ to $\mathbf{T}^1$ $(x \in \overline{X})$ satisfying

$$m(x, \alpha\beta) = \phi(\alpha)m(\alpha^{-1}x, \beta) + m(x, \alpha) \quad \text{for } \alpha, \beta \in \overline{\Gamma}$$

where $\overline{\Gamma}$ acts on $\overline{X}$ via $\overline{\rho}$ and $\phi(\alpha)$ gives the automorphism of $\mathbf{T}^1$ since it preserves the lattice of the fiber. The action of $\overline{\Gamma}$ on $\widetilde{S} = \overline{X} \times \mathbf{T}^1$ is given by

$$\alpha(x,t) = (\alpha x, \phi(\alpha)t + m(\alpha x, \alpha)) \quad \text{for } \alpha \in \overline{\Gamma}, x \in \overline{X}, t \in \mathbf{T}^1$$

(see [8, 7.2]). From the exact sequence

$$0 \to \mathcal{Z}^2 \to \mathcal{O} \to \mathcal{T}^1 \to 0$$

we have the exact sequence

$$H^1(\overline{\Gamma}, H^0(\overline{X}, \mathcal{O})^{\phi}) \xrightarrow{\eta} H^1(\overline{\Gamma}, H^0(\overline{X}, \mathcal{T}^1)^{\phi}) \xrightarrow{c} H^2(\overline{\Gamma}, \mathcal{Z}^{2\phi})$$

where $c(m)$ represents the extension $1 \to \mathbf{Z}^2 \to \Gamma \to \overline{\Gamma} \to 1$. On the other hand $H^1(S_0, \Theta_0)$ is described by the following exact sequences ([8, §2]).

$$0 \to D \to H^1(S_0, \Theta_0) \to G \to 0 \tag{1}$$

$$0 \to F \to G \to H \to 0 \tag{2}$$

$$0 \to C \to D \to E \to 0 \tag{3}$$

where the exact sequence (3) splits ([8, §4]). Here

$$E = \{a \in \mathbf{C}; a\overline{\phi(\alpha)} = \phi(\alpha)a \quad \text{for all } \alpha \in \overline{\Gamma}\} \tag{4}$$

corresponds to fiber deformations ([8, Theorem 7.10, §4]). Note that in our cases $\phi(\alpha)$ is a root of unity for any α when the base is hyperbolic (§1) and it suffices to consider the cases with trivial monodromies when the base is not hyperbolic (in the case of euclidean base orbifolds, we have only to consider the cases (I-1), (I-2), (II-1) and (II-2) in §2). Thus the kernel of ϕ has finite index in $\overline{\Gamma}$ and then the assumption in [8, §7] is automatically satisfied. Hence we have

$$E = \begin{cases} \mathbf{C}, & \text{if } \phi(\alpha) = \pm 1 \text{ for any } \alpha \in \overline{\Gamma} \\ 0, & \text{otherwise.} \end{cases}$$

The subspace C in (3) corresponds to the twist deformations coming from the the complex analytic family of the form $m + \eta(s\ell) \in H^1(\overline{\Gamma}, H^0(\overline{X}, \mathcal{T}^1)^{\phi})$ for $\ell \in H^1(\overline{\Gamma}, H^0(\overline{X}, \mathcal{O})^{\phi}), s \in \mathbf{C}$ ([8, §3]). In our case by [8, §7] we have

$$C = H^1(\overline{\Gamma}, H^0(\overline{X}, \mathcal{O})^{\phi}) \tag{5}$$

unless $g = 1, r = 0$. The subspace F in (2) corresponds to the base deformations and by [8, §7], we have

$$F = H^1(\overline{\Gamma}, \Theta_{\overline{X}}) = H^1(B, \Theta_{B|d}) \tag{6}$$

where $B = B_{\rho_0}$ is considered as a nonsingular curve, $d = \sum_{i=1}^{r} p_i$ is the divisor corresponding to the cone point p_i and $\Theta_{B|d}$ is the sheaf of germs of holomorphic tangent vector fields on B which vanish on d. Finally the element of the space H represents (if it is not obstructed) a deformation of S which destroys the fiber structure. By [8, Theorem 7.13] we have

$$H = 0 \quad \text{unless } g = 1, r = 0 \text{ or } g = 0, r < 3. \tag{7}$$

Now we consider the Kodaira Spencer map $\Phi\colon \mathbf{T}_0\mathcal{T} \to H^1(S_0, \Theta_0)$ for the case with $X = H^2 \times E^2$. The tangent space $\mathbf{T}_0\mathcal{T}$ is homeomorphic to $\mathbf{T}_0\overline{\mathcal{T}} \times \mathcal{F}_0$ where $\mathcal{F}_0 = \mathbf{T}_0 \times H^1(\overline{\Gamma}, \mathbf{C}^{\phi})$ and $\mathbf{T}_0 = \mathbf{R}^+ \times \mathbf{H}$ if $\phi \equiv \pm\,\mathrm{id}$, $\mathbf{T}_0 = \mathbf{R}^+$ if $\phi \not\equiv \pm\,\mathrm{id}$. The derivatives of the family of representations defined above span $\mathbf{T}_0\mathcal{T}$ and the discussions in §3–§5 in [8] show that the Kodaira Spencer map preserves the fiber structure as is indicated by the following commutative diagram.

$$\begin{array}{ccc} \mathbf{T}_0\mathcal{T} & \xrightarrow{\ \Phi\ } & H^1(S_0, \Theta_0) \\ \downarrow & & \downarrow \\ \mathbf{T}_0\overline{\mathcal{T}} & \xrightarrow{\ \overline{\Phi}\ } & F \end{array} \tag{8}$$

Here the vertical maps are the projections of the fibrations and $\overline{\Phi}$ gives the infinitesimal deformation map for the Teichmüller space of the r-pointed Riemann surface of genus g at $B = B_{\rho_0}$. We have $\mathbf{T}_0\overline{\mathcal{T}} = \mathbf{R}^{2(3g-3+r)}$ ([6]), $\dim_{\mathbf{C}} F = 3g - 3 + r$ ([8, Lemma 7.3]) and $\overline{\Phi}$ is a homeomorphism. Here we note that if two geometric Seifert 4-manifolds $S = \Gamma\backslash X, S' = \Gamma'\backslash X$ (with $X = \mathbf{H} \times \mathbf{C}$) over the hyperbolic 2-orbifolds B, B' are biholomorphic then B and B' are isometric. For, any biholomorphism $\varphi\colon S \to S'$ is lifted to a biholomorphism $\tilde{\varphi}$ from $\mathbf{H} \times \mathbf{C}$ to itself such that there is an automorphism $\psi\colon \Gamma \to \Gamma'$ satisfying $\tilde{\varphi}(\gamma(z, w)) = \psi(\gamma)(\tilde{\varphi}(z, w))$ for $(z, w) \in \mathbf{H} \times \mathbf{C}$, $\gamma \in \Gamma$. Here Γ and Γ' have the exact sequences $1 \to \Gamma_0 \to \Gamma \to \overline{\Gamma} \to 1$ and $1 \to \Gamma_0' \to \Gamma' \to \overline{\Gamma}' \to 1$ such that $\overline{\Gamma} = \pi_1^{\mathrm{orb}} B$, $\overline{\Gamma}' = \pi_1^{\mathrm{orb}} B'$ and Γ_0, Γ_0' correspond to the fundamental groups of the general fibers respectively. Moreover ψ induces the isomorphism between Γ_0 and Γ_0' and also induces the isomorphism $\overline{\psi}\colon \overline{\Gamma} \to \overline{\Gamma}'$. On the other hand Γ_0 and Γ_0' act on $\mathbf{H} \times \mathbf{C}$ by the translations in the $\mathbf{C}$-factor since S and S' are geometric (of type $H^2 \times E^2$ or $\widetilde{SL}_2 \times E$).

Hence if we write $\tilde{\varphi}(z,w) = (h(z,w), k(z,w)) \in \mathbf{H} \times \mathbf{C}$, then $h(z,w)$ is invariant under the action of Γ_0 in the w-coordinate. Since Γ_0 has rank 2, $h(z,w)$ depends only on z, i.e., $h(z,w) = h(z)$ which gives a biholomorphism from $\mathbf{H}$ to itself and $\tilde{\varphi}$ descends to an isometry $h: \mathbf{H} \to \mathbf{H}$. Since we have $h(\overline{\gamma}(z)) = \overline{\psi}(\overline{\gamma})h(z)$ for $z \in \mathbf{H}$, $\overline{\gamma} \in \overline{\Gamma}$ we can see that B and B' are isometric. Thus by the fact that the action of Aut $\overline{\Gamma}$ on the Teichmüller space of B is properly discontinuous we can see directly that the kernel of $\overline{\Phi}$ induced from Φ above is zero. Thus $\overline{\Phi}$ is an isomorphism (compare the dimensions of the spaces in (8)). Moreover Φ induces the map between the fibers of the projections in (8) of the form

$$H^1(\overline{\Gamma}, \mathbf{C}^{\phi}) \times \mathbf{T}_0 \xrightarrow{\Phi_1 \times \Phi_2} C \times E. \tag{9}$$

If $\phi \equiv \pm\,\mathrm{id}$ then $E = \mathbf{C}$ such that small $s \in \mathbf{C}$ determines the complex structure of the general fiber whose period matrix is given by $\Omega(s) = (1+s, h_0 + s\overline{h_0})$ where $\Omega(0)$ corresponds to that for the original S_0 ([8, Lemma 4.5]). We have the representations ρ parametrized by s in the above family whose $\mathbf{T}_0$-component $h(s)$ satisfies $h(s) = (h_0 + s\overline{h_0})/(1+s)$ and $\Phi(\partial/\partial s)$ corresponds to 1 ([8, §4]). Thus Φ_2 maps $\mathbf{T}_0$ onto E whose kernel is the $\mathbf{R}^+$-component of $\mathbf{T}_0$ represented by the parameter detecting the deformation of the area of the general fiber. On the other hand $C = H^1(\overline{\Gamma}, H^0(\overline{X}, \mathcal{O})^{\phi})$ and Φ_1 is the map $H^1(\overline{\Gamma}, \mathbf{C}^{\phi}) \to H^1(\overline{\Gamma}, H^0(\overline{X}, \mathcal{O})^{\phi})$ induced by the natural inclusion $\mathbf{C} \subset H^0(\overline{X}, \mathcal{O})$. Moreover by the naturality of the spectral sequences (used in [8]) we have the following commutative diagram.

$$\begin{array}{ccc} H^1(\overline{\Gamma}, \mathbf{C}^{\phi}) & \xrightarrow{\Phi_1} & H^1(\overline{\Gamma}, H^0(\overline{X}, \mathcal{O})^{\phi}) \\ \uparrow{\scriptstyle \varphi_1} & & \uparrow{\scriptstyle \varphi_2} \\ H^1(B, \mathbf{C}(Q)) & \xrightarrow{\Phi'} & H^1(B, \mathcal{O}(Q)) \end{array}$$

Here Q is the flat $\mathbf{C}$-bundle over $B = B_{\rho_0}$ determined by the monodromy representation ϕ which can be considered as the representation of $\pi_1(B_{\rho_0})$ (see §1). The coefficient $\mathbf{C}(Q)$ (resp. $\mathcal{O}(Q)$) is the sheaf of the germs of locally constant (resp. holomorphic) sections of Q and φ_1 and φ_2 are the isomorphisms ([8, §7]). The map Φ' is the part of the following exact sequence (in which the base B is omitted)

$$\begin{aligned} 0 \to H^0(\mathbf{C}(Q)) \to H^0(\mathcal{O}(Q)) \to H^0(\Omega^1(Q)) \\ \to H^1(\mathbf{C}(Q)) \xrightarrow{\Phi'} H^1(\mathcal{O}(Q)) \to H^1(\Omega^1(Q)) \end{aligned}$$

which comes from the exact sequence

$$0 \to \mathbf{C}(Q) \to \mathcal{O}(Q) \xrightarrow{\partial} \Omega^1(Q) \to 0.$$

Here Ω^1 is the sheaf of the germs of holomorphic 1 forms on the non-singular curve B. By the Riemann-Roch theorem (and since $c_1(Q) = 0$) $\dim_{\mathbf{C}} H^0(\Omega^1(Q)) = \dim_{\mathbf{C}} H^1(\mathcal{O}(Q))$ equals g if $\phi = \mathrm{id}$ and equals $g-1$ otherwise (cf. [8, §7]). Comparing this with $\dim_{\mathbf{C}} H^1(\mathbf{C}(Q)) = \dim_{\mathbf{C}} H^1(\overline{\Gamma}, \mathbf{C}^{\phi})$ which equals $2g$ if $\phi \equiv \mathrm{id}$ and $2g-2$ otherwise we deduce the exact sequence

$$0 \to H^0(\Omega^1(Q)) \to H^1(\mathbf{C}(Q)) \xrightarrow{\Phi'} H^1(\mathcal{O}(Q)) \to 0.$$

Hence the kernel of Φ_1 is isomorphic to $H^0(B, \Omega^1(Q))$. The same argument holds for the case with $X = \widetilde{SL_2} \times E$ except for the fact that the kernel of Φ_2 is trivial since there is no $\mathbf{R}^+$-component in $\mathbf{T}_0$. Thus we have

Theorem C-1. *The Kodaira Spencer map Φ for the Teichmüller space $\mathcal{T}$ for the Seifert 4-manifold S over the closed orientable hyperbolic 2-orbifold B with any given representation $\rho \in \mathcal{T}$ is surjective and the kernel of $\Phi\colon \mathbf{T}_0\mathcal{T} \to H^1(S, \Theta)$ at $S = S_\rho$ is homeomorphic to $H^0(B, \Omega^1(Q)) \times \mathbf{R}^+$ (if $X = H^2 \times E^2$) or $H^0(B, \Omega^1(Q))$ (if $X = \widetilde{SL_2} \times E$) for the base curve B determined by ρ. The subspace $\widetilde{\mathcal{T}}$ of $\mathcal{T}$ defined above gives a locally complete complex analytic family of the complex structures on S.*

The last statement comes from [4]. We can see directly that any deformation in the subspace $H^0(B, \Omega^1(Q))$ of $\mathbf{T}_0\mathcal{T}$ (which depends on the choice of $\rho \in \mathcal{T}$) does not change the complex structure as follows. Take any $w \in H^0(B, \Omega^1(Q))$. Lift w to the 1-form on $\mathbf{H}$ which is represented as $d\psi$ for some holomorphic function ψ on $\mathbf{H}$ satisfying $d\psi(\alpha z) = \phi(\alpha) d\psi(z)$ for any $\alpha \in \overline{\Gamma}, z \in \mathbf{H}$. Taking the integral we deduce that $b(\alpha) = \psi(\alpha z) - \phi(\alpha)\psi(z)$ is a constant. Furthermore we have $b(\alpha\beta) = b(\alpha) + \phi(\alpha) b(\beta)$. If α is a torsion then we can choose the fixed point of α as z and hence $b(\alpha) = 0$. The image of $b(\alpha)$ in $H^1(\overline{\Gamma}, \mathbf{C}^{\phi})$ maps to 0 in $H^1(\overline{\Gamma}, H^0(\mathbf{H}, \mathcal{O})^{\phi})$ since $b(\alpha) = \psi(z) - \phi(\alpha)\psi(\alpha^{-1}z)$ and conversely any element in the kernel of Φ_1 can be represented by the above way. Then the biholomorphic automorphism of $\mathbf{H} \times \mathbf{C}$ defined by $(z, w) \to (z, w + s\psi(z))$ for $s \in \mathbf{C}$ descends to the biholomorphism between S_ρ and $S_{\rho'}$ such that the difference $m_{\rho'} - m_\rho$ of the parameters in $H^1(\overline{\Gamma}, \mathbf{C}^{\phi})$ is sb and all the other parameters are the same.

Next consider the case when B is euclidean and S has a complex structure. If $g = 1$, $r = 0$ (B is a torus) then we can assume that S is either T^4 (a complex torus) or a primary Kodaira surface. In the first case $X = E^4$, $G = \mathrm{Isom}^0 E^4$. However since the complex structure of X is not preserved by G but is preserved by $G' = E^4 \rtimes U(2)$ we consider $\mathcal{T}' =$ the identity component of $\mathcal{T}(\Gamma, G')$ in this case. Then $\mathcal{T}' = U(2)\backslash GL_4^+\mathbf{R}$ and this is realized by the family of translations ρ defined by

$$\rho(\alpha_i)(z,w) = (z + w_{i1}, w + w_{i2})$$

for the generators α_i, $(i = 1, \dots, 4)$ in $\Gamma = \mathbf{Z}^4$ such that

$$w_{12} = 0, w_{11}, w_{22} \in \mathbf{R}^+, \det(\Re w_{ij}, \Im w_{ij}) > 0$$

where $(\Re w_{ij}, \Im w_{ij})$ is the matrix of rank 4 defined by $\Re w_{ij}$, $\Im w_{ij}$ for $i, j = 1, \dots, 4$. Thus we have a differentiable family $\mathcal{C}$ over $\mathcal{T}'$ of the complex structures on S. $\mathcal{T}'$ contains the subfamily (which is complex analytic) consisting of the representations with

$$w_{11} = w_{22} = 1, w_{12} = w_{21} = 0, \det(\Im(w_{ij})_{i,j=3,4}) > 0$$

which is complete and effectively parametrized ([3]). It follows that the Kodaira Spencer map for $\mathcal{T}'$ at any point is surjective. In the second case $X = \mathrm{Nil}^3 \times E$ and $\mathcal{T}$ is homeomorphic to $\mathbf{R}^+ \times \mathbf{H} \times \mathbf{R}^2 \times \mathbf{H}$. Here the $\mathbf{R}^+$-factor, the first $\mathbf{H}$-factor, the last $\mathbf{H}$-factor and the $\mathbf{R}^2$-factor correspond to the area of the base, the period of the base, the period of the fiber (the image of one of the lattices of the fiber is uniquely determined and not deformed) and the twisting parameters for the fibrations respectively (see §2). Hence we have a differentiable family $\mathcal{C} \to \mathcal{T}$ of the complex structures. On the other hand in the decomposition of $H^1(S, \Theta)$ we have $E = F = \mathbf{C}$ ([8]). Since the canonical divisor K of S is trivial there is an isomorphism $\Theta \cong \Omega^1$ and hence $\dim_{\mathbf{C}} H^1(S, \Theta) = h^{1,1} = 2$ since the Hodge numbers satisfy $h^{0,2} = h^{2,0} = 1$ and $b_2 = h^{2,0} + h^{0,2} + h^{1,1} = 4$. It follows that $C = H = 0$ and as in the cases when B is hyperbolic the Kodaira Spencer map is surjective with kernel $= \mathbf{R}^+ \times \mathbf{R}^2$.

Finally consider the case when B is euclidean of genus 0. In this case $\mathcal{T} = \mathcal{T}_{0,r} \times \mathbf{H} \times \mathbf{R}^+$ if $X = E^4$ and $\mathcal{T} = \mathcal{T}_{0,r} \times \mathbf{H}$ if $X = \mathrm{Nil}^3 \times E$. Here $\mathcal{T}_{0,r} = \mathbf{R}^{2(r-3)} \times \mathbf{R}^+$ (with $r = 3, 4$) denotes the Teichmüller space of the base orbifold B where the first factor corresponds to the Teichmüller space of an r-pointed Riemann surface of genus 0 (with fixed area) and the $\mathbf{R}^+$-factor corresponds to the area of the base B. The other factor in $\mathcal{T}$ corresponds to the deformations of the lattices of the fiber (in the case with $X = \mathrm{Nil}^3 \times E$ one of the lattices of the fiber has the fixed image and hence there is no $\mathbf{R}^+$-factor). If $r = 3$ then the base B is parametrized by the area only and if $r = 4$ then the $\mathbf{R}^{2(r-3)}$-factor is identified with the Teichmüller space of the double covering torus of B which is isomorphic to $\mathbf{H}$. In either case we have the differentiable family $\mathcal{C} \to \mathcal{T}$ of the complex structures of S as in the arguments in §2. (In the case with $X = E^4, g = 0$ we can choose the representatives ρ for $\mathcal{T}$ such that the image of ρ lies in $E^4 \rtimes U(1)$ and hence we do not need to restrict G to the subgroup $E^4 \rtimes U(2)$.) On the other hand in the decomposition of $H^1(S_0, \Theta_0)$ we have $C = H = 0, E = \mathbf{C}, F = \mathbf{C}^{r-3}$ and we can argue as in the case when B is hyperbolic. Thus we obtain

Theorem C-2. *Let S be a Seifert 4-manifold over some orientable hyperbolic or euclidean 2-orbifold B which admits a complex structure. Then S has a geometric structure of type (X, G) with $X = H^2 \times E^2$, $\widetilde{SL_2} \times E$, E^4 or $\mathrm{Nil}^3 \times E$ and $G = \mathrm{Isom}^0 X$. Let $\mathcal{T}$ be the identity component of the Teichmüller space $\mathcal{T}(\Gamma, G)$ where $\Gamma = \pi_1 X$. (In the case when $S = T^4$ restrict G to $E^4 \rtimes U(2)$.) Then $\mathcal{T}$ gives a differentiable family of complex structures on S such that the infinitesimal deformation map at any point in $\mathcal{T}$ is surjective.*

Remark. The statements in Theorem C-2 do not hold in general for a Seifert 4-manifold S over a closed orientable spherical or bad 2-orbifold B. In this case S is either a ruled surface of genus 1 (with $X = S^2 \times E^2$) or a Hopf surface (with $X = S^3 \times E$). In either case not every complex structure on S comes from the geometric one nor every differentiable family of the complex structures containing the geometric one comes from the Teichmüller space of the geometric structures. In general the dimension of $H^1(S, \Theta)$ (which is not constant) can be greater than that of the Teichmüller space (cf. [9], [14], [2], [12]).

§4. A remark on the moduli spaces

In this section we give a remark on the moduli space $\mathcal{M}(\Gamma, G)$ for a geometric Seifert 4-manifold $S = \Gamma \backslash X$ over a closed orientable hyperbolic base orbifold $B = \overline{\Gamma} \backslash \overline{X}$ with $\Gamma \subset G = \mathrm{Isom}^0 X$. We adopt the representations of Γ given in §1 and also assume that the monodromy matrices $A_1, \ldots, A_{2g}$ satisfy the conditions in Proposition 1. In this case the fibration of S is unique and then every element φ of $\mathrm{Aut}\,\Gamma$ induces the automorphism $\overline{\varphi}$ of $\overline{\Gamma}$ and also induces the automorphism of $\mathbf{Z}^2$ generated by ℓ, h. Put $\pi_* = \prod [\alpha_{2j-1}, \alpha_{2j}] \prod q_j$.

Proposition 2. *Any element $\varphi \in \mathrm{Aut}\,\Gamma$ must be of the following* *form.*

$$\varphi(\alpha_i) = \tilde{\varphi}(\alpha_i)\ell^{s_i}h^{t_i}$$
$$\varphi(q_j) = \tilde{\varphi}(q_j)\ell^{u_j}h^{v_j}$$
$$(\varphi(\ell), \varphi(h)) = (\ell, h)P.$$

Here $P \in GL_2\mathbf{Z}$ and $\tilde{\varphi}(\alpha_i), \tilde{\varphi}(q_j)$ are the words of $\alpha_1, \ldots, \alpha_{2g}$, $q_1, \ldots, q_r$ satisfying

$$\tilde{\varphi}(q_j) = \mu_j q^{\sigma}_{\nu(j)}\mu_j^{-1}$$
$$\tilde{\varphi}(\pi_*) = \mu\pi^{\sigma}_{*}\mu^{-1}$$

where $\sigma = \pm 1$, μ, μ_j are some words of $\alpha_1, \ldots, \alpha_{2g}$, $q_1, \ldots, q_r$ and $\nu\colon (1, \ldots, r) \to (\nu(1), \ldots, \nu(r))$ is a permutation. We have further conditions on the above parameters and the words as follows. Let ϵ_i, η_j, η be the exponent sums of α_1 in $\tilde{\varphi}(\alpha_i)$, μ_j, μ respectively.

(0) $\quad m_{\nu(i)} = m_i$

(1) $\quad P^{-1}A_1^{\epsilon_i}P = A_i$

(2) $$\begin{pmatrix} u_j \\ v_j \end{pmatrix} = \sigma A^{\eta_j}\begin{pmatrix} a_{\nu(j)}/m_{\nu(j)} \\ b_{\nu(j)}/m_{\nu(j)} \end{pmatrix} - P\begin{pmatrix} a_j/m_j \\ b_j/m_j \end{pmatrix}$$

(3) $$\sigma A_1^{\eta}\begin{pmatrix} a \\ b \end{pmatrix} + \sigma\sum A_1^{\eta_j}\begin{pmatrix} a_{\nu(j)}/m_{\nu(j)} \\ b_{\nu(j)}/m_{\nu(j)} \end{pmatrix} + (A_1 - I)\begin{pmatrix} s_2 \\ t_2 \end{pmatrix} = P\begin{pmatrix} a + \sum a_i/m_i \\ b + \sum b_i/m_i \end{pmatrix}$$

Sketch of Proof. The proof is similar to that of [7, §5, Lemma 4, Theorem 5] . (1) and (2) are derived from the relations $\alpha_i(\ell, h)\alpha_i^{-1} = (\ell, h)A_i$, $q_j^{m_j}\ell^{a_j}h^{b_j} = 1$. (3) comes from (1), (2) and the remaining relation $\prod[\alpha_{2j-1}, \alpha_{2j}]\prod q_j = \ell^a h^b$.

The map $\varphi \to \overline{\varphi}$ induces the homomorphism $q\colon \mathrm{Aut}\,\Gamma \to \mathrm{Aut}\,\overline{\Gamma}$ which descends to a homomorphism $\overline{q}\colon \mathrm{Out}\,\Gamma \to \mathrm{Out}\,\overline{\Gamma}$. Let $\mathrm{Aut}\,(\overline{\Gamma}, q)$ and $\mathrm{Out}\,(\overline{\Gamma}, \overline{q})$ be the images of q and $\overline{q}$ respectively. Also put $K = q^{-1}(\mathrm{Inn}\,\overline{\Gamma})$. Then since q maps $\mathrm{Inn}\,\Gamma$ onto $\mathrm{Inn}\,\overline{\Gamma}$ the natural projection $\pi\colon \mathrm{Aut}\,\Gamma \to \mathrm{Out}\,\Gamma$ maps K onto $\overline{K} = \ker\overline{q}$ and we have the following commutative diagram with exact rows and columns.

$$\begin{array}{ccccccccc}
 & & & & 1 & & 1 & & \\
 & & & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathrm{Inn}\,\Gamma & \longrightarrow & K & \xrightarrow{\pi} & \overline{K} & \longrightarrow & 1 \\
 & & \| & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathrm{Inn}\,\Gamma & \longrightarrow & \mathrm{Aut}\,\Gamma & \xrightarrow{\pi} & \mathrm{Out}\,\Gamma & \longrightarrow & 1 \\
 & & & & \downarrow & & \downarrow \overline{q} & & \\
 & & & & \mathrm{Out}\,(\overline{\Gamma},\overline{q}) & = & \mathrm{Out}\,(\overline{\Gamma},\overline{q}) & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 1 & & 1 & &
\end{array}$$

It is easy to see that the action of Aut Γ (resp. Out Γ) preserves the product fibration $\widetilde{\mathcal{F}} \to \mathcal{R} \to \overline{\mathcal{R}}$ for $\mathcal{R} = \mathcal{R}(\Gamma, G)$, $\overline{\mathcal{R}} = \overline{\mathcal{R}}(\overline{\Gamma}, \overline{G})$ (resp. $\mathcal{F} \to \mathcal{T} \to \overline{\mathcal{T}}$ for $\mathcal{T} = \mathcal{T}(\Gamma, G)$, $\overline{\mathcal{T}} = \overline{\mathcal{T}}(\overline{\Gamma}, \overline{G})$) (given in §1) and induces the natural action of Aut $(\overline{\Gamma}, q)$ (resp. Out $(\overline{\Gamma}, q)$) on $\overline{\mathcal{R}}$ (resp. $\overline{\mathcal{T}}$).

Now we will check the action of $\overline{K}$ on $\mathcal{F}$ (note that $\overline{K}$ acts trivially on $\overline{\mathcal{T}}$). First suppose that $X = H^2 \times E^2$. Define $\rho_0 = \rho_0(\ell_0, \lambda) \in \mathcal{R}$ by

$$\begin{aligned}
&\rho_0(\ell)(z,w) = (z, w + \ell_0) \\
&\rho_0(h)(z,w) = (z, w + \ell_0\lambda^{-1}) \\
&\rho_0(\alpha_1)(z,w) = (\overline{\rho}_0(\overline{\alpha}_1)z, \phi(\overline{\alpha}_1)w + w_1^0) \\
&\rho_0(\alpha_i)(z,w) = (\overline{\rho}_0(\overline{\alpha}_i)z, w + w_i^0) \quad (i \geq 2) \\
&\rho_0(q_j)(z,w) = (\overline{\rho}_0(\overline{q}_j)z, w - (a_j\ell_0 + b_i\ell_0\lambda^{-1})/m_j)
\end{aligned}$$

where

$$\ell_0 \in \mathbf{R}^+, \Im\lambda \neq 0,$$

$$w_j^0 = 0 \text{ for any } j \quad \text{if } A_1 = I$$

and if $A_1 \neq I$

$$w_j^0 = \begin{cases} \ell_0((a + \sum a_i/m_i) + (b + \sum b_i/m_i)\lambda^{-1})/(\phi(\overline{\alpha}_1) - 1) & \text{if } j = 2 \\ 0 & \text{if } j \neq 2. \end{cases}$$

(We can choose such w_j^0. See the proof of Theorem A in §1.) Furthermore by the assumption in Proposition 1 we have

$$\begin{aligned}
&\lambda = \phi(\overline{\alpha}_1) = \exp(\pm 2\pi i/6) \quad \text{if } A_1 = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \\
&\lambda = \phi(\overline{\alpha}_1) = \exp(\pm 2\pi i/4) \quad \text{if } A_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\
&\lambda = \exp(\pm 2\pi i/6), \phi(\overline{\alpha}_1) = \lambda^2 \quad \text{if } A_1 = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}.
\end{aligned}$$

There is no further restriction on λ if $A_1 = \pm I$. Then the image $[\rho_0] \in \mathcal{T}$ of ρ_0 belongs to the fiber ($\cong \mathcal{F} = \mathbf{T}_1 \times H^1(\overline{\Gamma}, \mathbf{C}^\phi)$) of $\mathcal{T}$ over the image $[\overline{\rho}_0] \in \overline{\mathcal{T}}$ of $\overline{\rho}_0$. Its $\mathbf{T}_1$-coordinates are detected by (ℓ_0, λ) and it corresponds to 0 in the $H^1(\overline{\Gamma}, \mathbf{C}^\phi)$-component. Now we take the subfamily $\mathcal{R}_0$ of $\mathcal{R}$ with a fixed image $\overline{\rho}_0$ in $\overline{\mathcal{R}}$ whose elements $\rho = \rho(\ell_0, \lambda, m)$ are defined by

$$\begin{aligned}
&\rho(\ell)(z, w) = (z, w + \ell_0) \\
&\rho(h)(z, w) = (z, w + \ell_0 \lambda^{-1}) \\
&\rho(\alpha_1)(z, w) = (\overline{\rho}_0(\overline{\alpha}_1)z, \phi(\overline{\alpha}_1)w + w_1) \\
&\rho(\alpha_i)(z, w) = (\overline{\rho}_0(\overline{\alpha}_i)z, w + w_i) \quad (i \geq 2) \\
&\rho(q_j)(z, w) = (\overline{\rho}_0(\overline{q}_j)z, w - (a_j \ell_0 + b_j \ell_0 \lambda^{-1})/m_j).
\end{aligned}$$

Here (ℓ_0, λ) satisfies the same conditions as before and

$$w_i = w_i^0 + m(\overline{\alpha}_i)$$

where m is a crossed homomorphism from $\overline{\Gamma}$ to $\mathbf{C}^\phi$ (with $m(\overline{q}_j) = 0$) satisfying

$$m(\overline{\alpha}_1) = m(\overline{\alpha}_2) = 0 \quad \text{if } A_1 \neq I.$$

(There are no restrictions on $m(\overline{\alpha}_j)$ if $A_1 = I$.) Hence we have

$$\begin{aligned}
&w_1 = 0, w_2 = w_2^0, w_j = m(\overline{\alpha}_j) \ (j \geq 3) \quad \text{if } \phi \not\equiv \mathrm{id} \\
&w_j = m(\overline{\alpha}_j) \text{ for any } j \quad \text{if } \phi \equiv \mathrm{id}.
\end{aligned}$$

We note that if $A_1 \neq I$ then $w_2 = w_2^0$ is fixed once (ℓ_0, λ) is fixed by the relations (7), (8) in the proof of Theorem A (§1) and any crossed homomorphism $n: \overline{\Gamma} \to \mathbf{C}^\phi$ with $n(\overline{\alpha}_j) = 0$ for $j \geq 2$ is contained in the image of $\delta: C^0(\overline{\Gamma}, \mathbf{C}^\phi) \to C^1(\overline{\Gamma}, \mathbf{C}^\phi)$. Therefore the m's satisfying the above conditions descend isomorphically onto $H^1(\overline{\Gamma}, \mathbf{C}^\phi)$. Taking these facts into account we can see that the family $\mathcal{R}_0$ generated by $\rho(\ell_0, \lambda, m)$ whose parameters satisfy the above conditions is the subfamily of the fiber over $\overline{\rho}_0 \in \mathcal{R}$ whose $\widetilde{\mathcal{F}}$-components give the representatives of $\mathcal{F}$. To check the action of $\overline{K}$ on $\overline{\mathcal{F}}$, it suffices to find the element $\mu \in \mathrm{Inn}\, G$ for given $\rho \in \mathcal{R}_0$, $\varphi \in K$ (μ may depend on ρ and φ) such that $\mu \cdot \rho \cdot \varphi \in \mathcal{R}_0$ and examine the action of $\mu \cdot \rho \cdot \varphi$ (which is independent of $\overline{\rho}_0$ in the w-coordinate). Since the action of K commutes with that of $\mathrm{Inn}\, G$, it suffices to consider the element $\varphi \in K$ of the following form.

$$\begin{aligned} \varphi(\alpha_i) &= \alpha_i \ell^{s_i} h^{t_i} \\ \varphi(q_j) &= q_j \ell^{u_j} h^{v_j} \\ (\varphi(\ell), \varphi(h)) &= (\ell, h) P \end{aligned} \tag{4}$$

where $s_i, t_i, u_i, v_i \in \mathbf{Z}, P \in GL_2\mathbf{Z}$ satisfy

$$PA_1P^{-1} = A_1 \tag{5}$$

$$\begin{pmatrix} u_j \\ v_j \end{pmatrix} = (I - P) \begin{pmatrix} a_j/m_j \\ b_j/m_j \end{pmatrix} \tag{6}$$

$$(A_1 - I) \begin{pmatrix} s_2 \\ t_2 \end{pmatrix} = (P - I) \begin{pmatrix} a + \sum a_i/m_i \\ b + \sum b_i/m_i \end{pmatrix}. \tag{7}$$

For such $\varphi \in K$ and $\rho \in \mathcal{R}_0$ denote the first and the second factors of $\rho \cdot \varphi(\alpha)(z, w)$ by $\rho \cdot \varphi(\alpha)(z, w)_i$ ($i = 1, 2$) respectively for $\alpha \in \Gamma$. Then we have

$$\rho \cdot \varphi(\alpha)(z, w)_1 = \begin{cases} \overline{\rho}_0(\overline{\alpha}) z & \text{if } \alpha = \alpha_i \text{ or } q_j \\ z & \text{if } \alpha = \ell \text{ or } h \end{cases} \tag{8}$$

and

$$\begin{aligned} \rho \cdot \varphi(\alpha_1)(z, w)_2 &= \phi(\overline{\alpha}_1)(w + \ell_0(s_1 + t_1\lambda^{-1})) + w_1 \\ \rho \cdot \varphi(\alpha_i)(z, w)_2 &= w + w_i + \ell_0(s_i + t_i\lambda^{-1}) \ (i \geq 2) \\ \rho \cdot \varphi(q_j)(z, w)_2 &= w - \ell_0(a_j + b_j\lambda^{-1})/m_j + \ell_0(u_j + v_j\lambda^{-1}) \\ \rho \cdot \varphi(\ell)(z, w)_2 &= w + \ell_0' \\ \rho \cdot \varphi(h)(z, w)_2 &= w + h_0' \end{aligned} \tag{9}$$

where $(\ell_0', h_0') = (\ell_0, \ell_0\lambda^{-1})P$. Let K_1 be the subgroup of K consisting of the elements satisfying (4)–(7) with $P = I$ and let $\overline{K}_1$ be its image in $\overline{K}$. For any $\varphi \in K_1$ we deduce $u_j = v_j = 0$ and if $A_1 \neq I$, $s_2 = t_2 = 0$. Also let K_0 be the subgroup of K_1 generated by the elements of the above forms with $s_j = t_j = 0$ for $j \geq 2$ and $\overline{K}_0$ be its image in $\overline{K}$. Note that $\overline{K}_1 \cong K_1/(K_1 \cap \mathrm{Inn}\ \Gamma)$, $\overline{K}_0 \cong K_0/(K_0 \cap \mathrm{Inn}\ \Gamma)$ and $K_1 \cap \mathrm{Inn}\ \Gamma = K_0 \cap \mathrm{Inn}\ \Gamma = \mathrm{Inn}\ \mathbf{Z}^2$ where $\mathbf{Z}^2$ is the subgroup of Γ generated by ℓ and h. (This comes from the fact that $\overline{\Gamma}$ is centerless.) The element $\varphi \in K_0$ comes from $\mathrm{Inn}\ \mathbf{Z}^2$ if $\begin{pmatrix} s_1 \\ t_1 \end{pmatrix} = (A_1^{-1} - I)\begin{pmatrix} s \\ t \end{pmatrix}$ and hence

$$\overline{K}_0 \text{ is finite if } A_1 \neq I \text{ and moreover } \overline{K}_0 = 1 \text{ if } A_1 = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}.$$

Given $\rho \in \mathcal{R}_0$ and $\varphi \in K_1$ we take an inner automorphism μ by the element $(z, w) \to (z, w + \phi(\overline{\alpha}_1)(s_1\ell_0 + t_1\ell_0\lambda^{-1})/(\phi(\overline{\alpha}_1) - 1))$ if $A_1 \neq I$. Then we can see by the conditions on w_i above for (8), (9) that the correspondence $\rho \to \rho \cdot \varphi$ (if $A_1 = I$) or $\rho \to \mu \cdot \rho \cdot \varphi$ (if $A_1 \neq I$) gives a map from $\mathcal{R}_0$ to itself such that the parameters are changed as follows.

$$\begin{aligned} &(\ell_0, \lambda) \to (\ell_0, \lambda) \\ &m(\overline{\alpha_i}) \to m(\overline{\alpha_i}) + s_i\ell_0 + t_i\ell_0\lambda^{-1} \end{aligned}$$

for $i \geq 1$ if $A_1 = I$ and for $i \geq 3$ if $A_1 \neq I$. (We can see from (7) that the w_2-parameter of $\mu \cdot \rho \cdot \varphi$ is the same as that for ρ if $A_1 \neq I$.) Since s_i, t_i ($i \geq 1$ if $A_1 = I$ and $i \geq 3$ if $A_1 \neq I$) are arbitrary integers this gives the action of $\overline{K}_1$ on $\mathcal{F}$. Hence

$$\mathcal{F}/\overline{K}_1 \cong \mathbf{T}_1 \times H^1(\overline{\Gamma}, \mathbf{C}^{\phi})/H^1(\overline{\Gamma}, \mathbf{Z}^{2^{\phi}})$$

with

$$H^1(\overline{\Gamma}, \mathbf{C}^{\phi})/H^1(\overline{\Gamma}, \mathbf{Z}^{2^{\phi}}) \cong \begin{cases} (\mathbf{T}^1)^{2g} & \text{if } A_1 = I \\ (\mathbf{T}^1)^{2g-2} & \text{if } A_1 \neq I \end{cases}$$

where $\mathbf{T}^1 \cong \mathbf{C}/\mathbf{Z}^2$ is the complex torus of dimension 1 whose lattice is generated by ℓ_0 and $\ell_0\lambda^{-1}$. Here we note that if $A_1 \neq I$ then $\overline{K}_0$ is the subgroup of $\overline{K}_1$ which acts trivially on $\mathcal{F}$. Hence $\overline{K}_1$ (or $\overline{K}_1/\overline{K}_0$ if $A_1 \neq I$) acts effectively and properly discontinuously on $\mathcal{F}$. Next consider the action of $\varphi \in K$ not decending to $\overline{K}_1$.

Case (1). $A_1 = \pm I$. Put $P = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \in GL_2\mathbf{Z}$ for $\varphi \in K$ defined in (4). Here u_j, v_j, s_2, t_2 for φ must be defined as elements in $\mathbf{Z}$ according to (6) and (7) (if $A_1 = I$ then (7) becomes obvious since the right hand side of (7) is 0 in case $X = H^2 \times E^2$). Then considering (8), (9) for $\rho \in \mathcal{R}_0$ and φ we can take an inner automorphism μ of the form $(z, w) \to (z, \sigma w + c)$ for some $c \in \mathbf{C}$ with $\sigma = |p + r\lambda^{-1}|/(p + r\lambda^{-1})$ such that the the correspondence $\rho \to \mu \cdot \rho \cdot \varphi$ gives a map from $\mathcal{R}_0$ to itself of the form

$$\begin{aligned}
&\lambda \to (p\lambda + r)/(q\lambda + s) \\
&\ell_0 \to |p + r\lambda^{-1}|\ell_0 \\
&m(\overline{\alpha}_i) \to \sigma(m(\overline{\alpha}_i) + s_i\ell_0 + t_i\ell_0\lambda^{-1})
\end{aligned}$$

for $i \geq 1$ if $A_1 = I$ and for $i \geq 3$ if $A_1 = -I$.

Case (2). $A_1 \neq \pm I$. In this case we deduce from the conditions on λ above that

$$(\ell_0, \ell_0\lambda^{-1})A_1' = (\lambda\ell_0, \ell_0) \quad \text{for } A_1' = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$$

if $A_1 = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$ and

$$(\ell_0, \ell_0\lambda^{-1})A_1 = (\lambda\ell_0, \ell_0)$$

otherwise. Furthermore for the presentation of φ in (4) we must have $P = A_1'^k$ (in case $A_1 = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$) or $P = A_1^k$ (otherwise) for some $k \in \mathbf{Z}$ by the condition (5) and

$$u_j, v_j, s_2, t_2 \in \mathbf{Z}$$

where these numbers are defined by (6) and (7) for the above P (we have assumed that $P \neq I$ since $\varphi \notin K_1$.) Then in the presentation (9) for $\rho \in \mathcal{R}_0$, $\varphi \in K$ we have

$$(\ell', h') = (\ell_0, \ell_0\lambda^{-1})P = (\lambda^k\ell_0, \lambda^{k-1}\ell_0)$$

for $P = A_1^k$ or $P = A_1'^k$ as above. Hence taking an inner automorphism μ by the element $(z, w) \to (z, \lambda^{-k}w + c)$ where

$$c = \lambda^{-k}\phi(\overline{\alpha}_1)(s_1\ell_0 + t_1\ell_0\lambda^{-1})/(\phi(\overline{\alpha}_1) - 1)$$

we can see that the correspondence $\rho \to \mu \cdot \rho \cdot \varphi$ gives a map from $\mathcal{R}_0$ to itself of the form

$$\begin{aligned}
&(\ell_0, \lambda) \to (\ell_0, \lambda) \\
&m(\overline{\alpha}_i) \to \lambda^{-k}(m(\overline{\alpha}_i) + s_i\ell_0 + t_i\ell_0\lambda^{-1}) \quad (i \geq 3).
\end{aligned}$$

In the cases when $A_1 = \pm I$ the above correspondence shows that the action of $\overline{K}$ on $\mathcal{F}$ preserves the product fibration of the form $H^1(\overline{\Gamma}, \mathbf{C}^\phi) \times \mathbf{R}^+ \to \mathcal{F} \to \mathbf{H} \times \mathbf{Z}_2$ (where $\mathbf{T}_1 = \mathbf{R}^+ \times \mathbf{H} \times \mathbf{Z}_2$) which induces the properly discontinuous action on $\mathbf{H} \times \mathbf{Z}_2$ (which is identified with $\{\lambda \in \mathbf{C} \mid \Im\lambda \neq 0\}$) of the form $\lambda \to (p\lambda + r)/(q\lambda + s)$ for some matrix $\begin{pmatrix} p & r \\ q & s \end{pmatrix} \in GL_2\mathbf{Z}$. In the cases when $A_1 \neq \pm I$ we have at most finite number of possible choices for P in the presentation of φ above since A_1 (or A_1') is periodic. Hence by the above correspondences and the action of $\overline{K}_1$ (and taking the fact that $\overline{K}_0$ is finite if $A_1 \neq I$ into account) we can easily see that $\overline{K}$ (or $\overline{K}/\overline{K}_0$ if $\phi \not\equiv \mathrm{id}$) acts properly discontinuously on $\mathcal{F}$ in either case. Finally consider the action of Out Γ on $\mathcal{T}$. The group Out Γ acts on $\mathcal{T}$ so that it preserves the product fibration $\mathcal{F} \to \mathcal{T} \to \overline{\mathcal{T}}$ and induces the action of Out $(\overline{\Gamma}, \overline{q})$ on $\overline{\mathcal{T}}$ which is properly discontinuous since the action of Out $(\overline{\Gamma})$ on $\overline{\mathcal{T}}$ has, as is well known, the same property. Since $\overline{K}$ (which is the subgoup of Out Γ which induces the identity on $\overline{\mathcal{T}}$) acts properly discontinuously on the fiber of $\mathcal{T}$ as above we can see that Out Γ acts also properly discontinuously on $\mathcal{T}$. The cases with $X = \widetilde{SL}_2 \times E$ (in this case $A_1 = I$) can be treated similarly and we omit the details. Thus we have

Proposition 3. *Let $S = \Gamma\backslash X$ be a geometric Seifert 4-manifold over a closed orientable hyperbolic orbifold B with $\Gamma \subset G = \mathrm{Isom}^0 X$. Then* Out Γ (*or* Out $\Gamma/\overline{K}_0$ *in case the monodromy representation ϕ of S is not trivial where $\overline{K}_0$ is a finite subgroup of $\overline{K}$ defined above*) *acts on $\mathcal{T}(\Gamma, G)$ properly discontinuously and the muduli space $\mathcal{M}(\Gamma, G)$ is Hausdorff.*

On the other hand if $\overline{K}_1 \neq \overline{K}$ then we must have $\varphi \in K$ of the form (4) satisfying (5)–(7) with $P \neq I$. In particular s_2, t_2, u_j, v_j defined by (5) and (6) for some appropriate $P \neq I$ satisfying (4) must be integers. From these conditions we can deduce some extra conditions on the Seifert invariants of S and derive the following proposition. Here we omit the details of the computations.

Proposition 4. *Let $S = \Gamma\backslash X$ be a geometric Seifert 4-manifold as in Proposition 3. Then if the monodromies of S satisfy the conditions in Proposition 1 and if the Seifert invariants of S do not satisfy the conditions below then $\overline{K} = \overline{K}_1$ and $\mathcal{M}(\Gamma, G)$ is a Seifert fibration over $\overline{\mathcal{T}}(\overline{\Gamma}, \overline{G})/$* Out $(\overline{\Gamma}, \overline{q})$ (*which is defined above*) *with general fiber* $\mathbf{T}_1 \times H^1(\overline{\Gamma}, \mathbf{C}^\phi)/H^1(\overline{\Gamma}, \mathbf{Z}^{2^\phi})$.

(I) $A_1 = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$.

(1) *There are no multiple fibers*;
(2) $(m_i, a_i, b_i) = (3, \epsilon_i, \epsilon_i)$, $\epsilon_i = \pm 1$ *for any* i;
(3) $m_i = 2$ *for any* i.

(II) $A_1 = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$.

(1) *There are no multiple fibers and* $a \equiv b \mod 3$;
(2) $(m_i, a_i, b_i) = (3, \epsilon_i, \epsilon_i)$ *with* $\epsilon_i = \pm 1$ *for any* i *and* $\sum \epsilon_i \equiv 0 \mod 3$;
(3) $m_i = 2$ *for any* i *and* $\sum a_i - 2a \equiv \sum b_i - 2b \mod 3$.

(III) $A_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

(1) *There are no multiple fibers*;
(2) $m_i = 2$ *for every* i *and* $\sum a_i \equiv \sum b_i \mod 2$.

(IV) $A_1 = \pm I$.

References

[1] L. Bers, Fiber spaces over Teichmüller spaces, Acta Math., **130** (1973), 89–126.

[2] K. Dabrowsky, Moduli spaces for Hopf surfaces, Math. Ann., **259** (1982), 201–226.

[3] K. Kodaira, D. C. Spencer, On deformations of complex analytic structures, I, II, Ann. Math., **67** (1958), 328–466.

[4] ———, A theorem of completeness for complex analytic fiber spaces, Acta. Math., **100** (1958), 281–294.

[5] R. Kulkarni, K. B. Lee, F. Raymond, Deformation spaces for Seifert manifolds, in "Geometry and Topology, Lect. Notes in Math. vol. 1167, Springer", 1985, pp. 180–216.

[6] K. Ohshika, Teichmüller spaces of Seifert fibered manifolds with infinite π_1, Topology and its appl., **27** (1987), 75–93.

[7] P. Orlik, "Seifert Manifolds", Springer, Berlin–Heidelberg–New York, 1972.

[8] T. Suwa, Deformations of Holomorphic Seifert Fiber Spaces, Invent. math., **51** (1979), 77–102.

[9] ———, On ruled surfaces of genus 1, J. Math. Soc. Japan, **21** (1969), 291–311.

[10] M. Ue, Geometric 4-manifolds in the sense of Thurston and Seifert 4-manifold I, J. Math. Soc. Japan, **42** (1990), 511–540.

[11] ———, Geometric 4-manifolds in the sense of Thurston and Seifert 4-manifolds II, J. Math. Soc. Japan, **43** (1991), 149–183.

[12] C. T. C. Wall, Geometric structures on compact complex surfaces, Topology, **25** (1986), 119–153.

[13] ———, Geometries and geometric structures in real dimension 4 and complex dimension 2, in "Geometry and Topology. Lect. Notes in Math., 1167, Springer", 1985, pp. 268–292.

[14] J. Wehler, Versal deformations for Hopf surfaces, J. Reine Angew. Math., **328** (1982), 22–32.

[15] H. Zieschang, On toric fiberings over surfaces, Math. Notes, **5** (1967), 341–345.

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Homologically Trivial Smooth Involutions on K3 Surfaces

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Dedicated to Professor Shôrô Araki on his 60th birthday

Abstract.

We will show that any smooth involution on a K3 surface induces a non-trivial action on its homology. In fact, a closed spin 4-manifold M with $H_1(M;\mathbf{Z}_2)=0$ and $\operatorname{sign} M \neq 0$ will be shown to admit no homologically trivial locally linear involutions. The proof uses only the G-signature theorem and the sublattices and branched coverings arguments.

§1. Introduction

Some complex surfaces including K3 surfaces admit no homologically trivial holomorphic involutions. There posed a question in [12;11.8] whether the same is true for the smooth involutions or not. This paper answers the question affirmatively at least for the smooth involutions on K3 surfaces. Note that a smooth involution is locally linear.

Theorem 1. *Let M be a closed connected oriented spin 4-manifold with $H_1(M;\mathbf{Z}_2)=0$. Suppose that there is an orientation preserving locally linear involution σ on M which operates as identity on $H_2(M;\mathbf{Q})$. Then,* $\operatorname{sign} M = 0$.

Since a K3 surface is a simply-connected spin 4-manifold with signature -16, it admits no homologically trivial locally linear involutions. According to Edmonds [5] Theorem 1 in the case that M is simply-connected is already proved by D. Ruberman.

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§2. Preliminary lemmas

We prepare some lemmas which will be used later and may be useful for the other purposes. We begin with a lemma to construct a double covering from two 2-sheet branched coverings.

Lemma 2.1. *Let σ be a locally linear involution on a connected manifold M with fixed point set F. Suppose there is a subunion of connected components $F' \subsetneq F$ with a non-trivial element e_τ of $H^1(M/\sigma - F'; \mathbf{Z}_2)$ which takes non-zero value on the image of $H_1(\partial N(x)/\sigma; \mathbf{Z})$ for any x of F', where $-/\sigma$ stands for the orbit space and $N(x)$ is a fiber at x of an equivariant normal disk bundle $N(U_x)$ for a neighborhood U_x of x in F. Then, there is a locally linear $\mathbf{Z}_2 \times \mathbf{Z}_2$-action with generators $\tilde{\sigma}$ and $\tilde{\tau}$ on a double (= connected 2-sheet unbranched) covering manifold $\widetilde{M}$ of M such that the orbit space $\widetilde{M}/\tilde{\tau}$ is canonically homeomorphic to M and $\tilde{\sigma}$ induces σ with this identification.*

$$\begin{array}{ccc}
\widetilde{M} & \xrightarrow[\text{covering}]{\text{unbranched}} & \widetilde{M}/\tilde{\tau} = M \\
\downarrow & & \downarrow \\
\widetilde{M}/\tilde{\sigma} = M' & \longrightarrow & M/\sigma
\end{array}$$

Proof. The projection $\pi : M - F \to M/\sigma - F$ is a covering map induced from a non-trivial element e_σ of $H^1(M/\sigma - F; \mathbf{Z}_2) = \mathrm{Hom}(H_1(M/\sigma - F; \mathbf{Z}), \mathbf{Z}_2) = \mathrm{Hom}(\pi_1(M/\sigma - F), \mathbf{Z}_2)$ which takes non-zero value on $H_1(\partial N(x)/\sigma; \mathbf{Z})$ for any x of F. Let $j : M/\sigma - F \to M/\sigma - F'$ be the inclusion. Then, we have $j^* e_\tau \neq e_\sigma$, since e_τ takes zero value on $H_1(\partial N(x)/\sigma; \mathbf{Z})$ for any x of $F - F'$. So, we get a $\mathbf{Z}_2 \times \mathbf{Z}_2$-covering of $M/\sigma - F$ associated to $(j^* e_\tau, e_\sigma) : H_1(M/\sigma - F; \mathbf{Z}) \to \mathbf{Z}_2 \times \mathbf{Z}_2$.

Consider the base change $(j^* e_\tau, j^* e_\tau + e_\sigma) : H_1(M/\sigma - F; \mathbf{Z}) \to \mathbf{Z}_2 \times \mathbf{Z}_2$. The completed 2-sheet branched coverings $\pi' : M' \to M/\sigma$ and $\pi'' : M'' \to M/\sigma$ (resp.) induced by $j^* e_\tau$ and $j^* e_\tau + e_\sigma$ (resp.) have the disjoint branch loci F' and $F - F'$ (resp.). So, the completed 2×2-sheet branched covering $\widetilde{\pi} : \widetilde{M} \to M/\sigma$, induced by $(j^* e_\tau, j^* e_\tau + e_\sigma) : H_1(M/\sigma - F; \mathbf{Z}) \to \mathbf{Z}_2 \times \mathbf{Z}_2$, has the locally linear involutions $\widetilde{\sigma}$ and $\widetilde{\sigma}'$ so that $\widetilde{\pi}' : \widetilde{M} \to \widetilde{M}/\widetilde{\sigma} = M'$ and $\widetilde{\pi}'' : \widetilde{M} \to \widetilde{M}/\widetilde{\sigma}' = M''$ are the 2-sheet branched coverings with branch loci $(\pi')^{-1}(F - F')$ and $(\pi'')^{-1}(F')$

respectively. By the definition $\widetilde{\sigma}$ and $\widetilde{\sigma}'$ commute outside $\widetilde{\pi}^{-1}(F)$. Since $\widetilde{M} - \widetilde{\pi}^{-1}(F)$ is dense in $\widetilde{M}$, $\widetilde{\sigma}$ and $\widetilde{\sigma}'$ commute also on whole $\widetilde{M}$.

Put $\widetilde{\tau} = \widetilde{\sigma} \circ \widetilde{\sigma}'$. Then, $\widetilde{\tau}$ has no fixed point either in $\widetilde{M} - \widetilde{\pi}^{-1}(F)$ or in $\widetilde{\pi}^{-1}(F) = (\widetilde{\pi}')^{-1}(\pi')^{-1}(F - F') \cup (\widetilde{\pi}'')^{-1}(\pi'')^{-1}(F')$ and hence in whole $\widetilde{M}$. Moreover, $\widetilde{M}/\widetilde{\tau} \to M/\sigma$ is the branched covering induced by $j^*e_\tau + j^*e_\tau + e_\sigma = e_\sigma$, that is, equivalent to $M \to M/\sigma$.

Since M is connected, M/σ is connected. If $F' = \emptyset$, the covering associated to the non-trivial element of $H^1(M/\sigma; \mathbf{Z}_2)$ is connected. Otherwise the branch locus of $M' \to M/\sigma$ is non-empty and M' is connected. Then, since the branch locus of $\widetilde{M} \to M'$ is non-empty, $\widetilde{M}$ is connected. Q.E.D.

We recall and define some notions about lattices now. A $\mathbf{Z}$-free module L of finite rank with non-degenerate symmetric bilinear form $\langle\ ,\ \rangle : L \times L \to \mathbf{Z}$ is called a lattice. Let L^* denote the dual module $\mathrm{Hom}_{\mathbf{Z}}(L, \mathbf{Z})$ and we have a canonical embedding $L \subset L^*$ defined by $x \mapsto \langle\ , x\rangle$. The factor group L^*/L is finite abelian and its order divides $|\operatorname{discr} L|$ where $\operatorname{discr} L = \det\langle e_i, e_j\rangle$ for some basis $\{e_i\}$. Let p be a prime. For a finite abelian group A we denote the minimal number of generators of A and $A \otimes \mathbf{Z}_p$ by $\ell(A)$ and $\ell_p(A)$ respectively. A lattice is called unimodular or p-unimodular if $L^*/L = 0$ or $\ell_p(L^*/L) = 0$ respectively. A submodule S of L is called primitive or p-primitive if L/S is $\mathbf{Z}$-free or contains no p-torsion respectively. Define the orthogonal complement $S^\perp = \{y \in L;\ \langle y, x\rangle = 0$ for any $x \in S\}$. If L is unimodular and S is a primitive sublattice, i.e., primitive and the pairing $\langle\ ,\ \rangle$ is non-degenerate not only on L but also on S, we have a natural isomorphism $S^*/S \cong S^{\perp *}/S^\perp$. (See [3;I.2.5] and [10] for example.) Moreover, we can prove

Lemma 2.2. *Let p be a prime. Let L be a p-unimodular lattice and S a p-primitive sublattice. Then, the orthogonal complement $K = S^\perp$ is also a sublattice and the p-torsion part $(S^*/S)_{(p)}$ of S^*/S is isomorphic to the p-torsion part of $(K^*/K)_{(p)}$ of K^*/K.*

Proof. Take an element ℓ of L. Then, $\ell^* = \langle\ , \ell\rangle$ can be considered as an element of S^*; $\ell_1^* = \ell_2^*$ in S^* if and only if $\ell_1 - \ell_2 \in K$. If we consider ℓ^* also as an element in K^*, we get a homomorphism $\mathrm{Im}(L \to S^*)/S \to K^*/K$. That S is p-primitive implies $(S^*/\mathrm{Im}(L^* \to S^*))_{(p)} = 0$. Since $(L^*/L)_{(p)} = 0$ by the assumption, we have $(S^*/S)_{(p)} = (\mathrm{Im}(L^* \to S^*)/S)_{(p)} = (\mathrm{Im}(L \to S^*)/S)_{(p)}$ and we get a correlation homomorphism $(S^*/S)_{(p)} \to (K^*/K)_{(p)}$. By the definition it is easy to see that K is a primitive sublattice of L and $K^\perp$ is a minimal primitive sublattice of L containing S. So, $(K^{\perp}/S)_{(p)} = 0$ by the assumption. Then, we get also a homomorphism $(K^*/K)_{(p)} \longrightarrow (K^{\perp *}/K^{\perp})_{(p)} = (S^*/S)_{(p)}$ which is an inverse of the homomorphism above. Q.E.D.

Next we give a sufficient and nearly necessary condition to get a branched covering in some cases.

Lemma 2.3. *Let p be a prime. Let $S_1^2, \ldots, S_\ell^2$ be disjointly embedded 2-spheres in a closed orientable 4-manifold M with normal disk bundles $N(S_1^2), \ldots, N(S_\ell^2)$.*

(1) *Suppose that the homology classes $[S_1^2], \ldots, [S_\ell^2]$ are linearly dependent in $H_2(M; \mathbf{Z}_p)$. Then, there is a non-trivial element of $H^1(M - \cup_{i=1}^{\ell} S_i^2; \mathbf{Z}_p)$ which takes non-zero value on $H_1(\partial N(S_i^2); \mathbf{Z})$ for some i.*

(2) *Suppose that $[S_1^2], \ldots, [S_\ell^2]$ are linearly independent in $H_2(M; \mathbf{Z})$ and generate a submodule S of $L = H_2(M; \mathbf{Z})/\operatorname{tor}$. Let $\overline{S}$ be the minimal primitive submodule of L containing S, that is, $L/\overline{S}$ is $\mathbf{Z}$-free. Then, $\overline{S}/S$ is a finite (possibly zero) abelian group and we have an isomorphism*

$$\overline{S}/S \cong \operatorname{Ker}(H_1(M - \cup_{i=1}^{\ell} S_i^2; \mathbf{Z}) \to H_1(M; \mathbf{Z})).$$

Note that the torsion part of L/S is $\overline{S}/S$. So, if L/S contains a non-trivial p-torsion, there is a non-trivial element of $H^1(M - \cup_{i=1}^{\ell} S_i^2; \mathbf{Z}_p)$ which takes non-zero value on $H_1(\partial N(S_i^2); \mathbf{Z})$ for some i. Moreover, when $H_1(M; \mathbf{Z}) \otimes \mathbf{Z}_p = 0$, the converse is also true, that is, if there is a non-trivial element of $H^1(M - \cup_{i=1}^{\ell} S_i^2; \mathbf{Z}_p)$ which takes non-zero value on $H_1(\partial N(S_i^2); \mathbf{Z})$ for some i, L/S contains a non-trivial p-torsion.

(3) *Suppose $[S_i^2]^2 \equiv 0 \mod p$ for every i and $2\ell > b_2(M)$. Then, either $[S_1^2], \ldots, [S_\ell^2]$ are linearly dependent in $H_2(M; \mathbf{Z}_p)$ or linearly independent in $H_2(M; \mathbf{Z}_p)$ and L/S contains a non-trivial p-torsion, where $L = H_2(M; \mathbf{Z})/\operatorname{tor}$ and S is a submodule generated by $[S_1^2], \ldots, [S_\ell^2]$ in L. Note that $b_2(M) = \dim H_2(M; \mathbf{Q}) = \operatorname{rank} L$.*

Proof. (1) Put $F = S_1^2 \cup \cdots \cup S_\ell^2$ and $N = M - \operatorname{Int} N(F)$. Under the hypothesis we have a non-zero element $a_1[S_1^2] + \cdots + a_\ell[S_\ell^2]$ of $H_2(F; \mathbf{Z}_p) = H_2(N(F); \mathbf{Z}_p)$ which sends to zero in $H_2(M; \mathbf{Z}_p)$ in the following commutative diagram:

$$
\begin{tikzcd}
H_3(M,N(F);\mathbf{Z}_p) \arrow[r, "\partial"] & H_2(N(F);\mathbf{Z}_p) \arrow[r] & H_2(M;\mathbf{Z}_p) \\
H^1(N;\mathbf{Z}_p) \arrow[u, "PD"', "\cong"] \arrow[r, "\delta"] \arrow[d] & H^2(M,N;\mathbf{Z}_p) \arrow[u, "PD"', "\cong"] \arrow[d, "\cong"] & \\
H^1(\partial N(F);\mathbf{Z}_p) \arrow[r, "\delta"] & H^2(N(F),\partial N(F);\mathbf{Z}_p) &
\end{tikzcd}
$$

Here the horizontal sequences are natural and exact. So, there is an element α' of $H_3(M,N(F);\mathbf{Z}_p)$ such that $\partial\alpha' \neq 0$. By the Poincaré duality we get an element $\alpha \in H^1(N;\mathbf{Z}_p) = H^1(M-F;\mathbf{Z}_p)$ such that $\delta\alpha \neq 0$. Since $\partial N(F) = \cup_{i=1}^{\ell}\partial N(S_i^2)$, α takes non-zero value on $H_1(\partial N(S_i^2);\mathbf{Z})$ for some i.

(2) Note first that there is an isomorphism $\overline{S}/S \cong S^*/\overline{S}^*$, where A^* stands for the dual $\mathrm{Hom}_{\mathbf{Z}}(A,\mathbf{Z})$. Consider the following commutative diagram whose horizontal sequences are exact and the coefficient is $\mathbf{Z}$:

$$
\begin{tikzcd}
 & H_2(M,N) \arrow[r, "\partial"] & H_1(N) \arrow[r, "j_*"] & H_1(M) \\
H^2(M) \arrow[r, "i^*"] \arrow[d, equal] & H^2(N(F)) \arrow[u, "PD"', "\cong"] \arrow[r, "\delta"] \arrow[d, equal] & H^3(M,N(F)) \arrow[u, "PD"', "\cong"] \arrow[r, "j^*"] & H^3(M) \arrow[u, "PD"', "\cong"] \\
L^* \oplus \mathrm{tor} \arrow[r] & S^* & &
\end{tikzcd}
$$

Since S^* is torsion free, $\mathrm{Im}\, i^* = \mathrm{Im}\, L^*$. Moreover since L is unimodular, $\mathrm{Im}\, L^*$ is $\overline{S}^*$ by the definition of $\overline{S}$. So,

$$\overline{S}/S \cong S^*/\overline{S}^* = \mathrm{Coker}\, i^* \cong \mathrm{Im}\,\delta = \mathrm{Ker}\, j^*$$

By the Poincaré duality we get $\mathrm{Ker}\, j^* \cong \mathrm{Ker}(j_* : H_1(N;\mathbf{Z}) = H_1(M-F;\mathbf{Z}) \to H_1(M;\mathbf{Z}))$.

(3) We may assume that the homology classes $[S_1^2],\dots,[S_\ell^2]$ are linearly independent in $H_2(M;\mathbf{Z}_p)$ and in particular linearly independent in $H_2(M;\mathbf{Z})$. We divide into two cases : (i) the case that $[S_i^2]^2 \neq 0$ for every i, and (ii) otherwise.

In case (i) the pairing $\langle\ ,\ \rangle$ on S is non-degenerate and $\ell_p(S^*/S) = \ell$. On the other hand rank $S^\perp = b_2(M)-\ell$ implies $\ell_p(S^{\perp *}/S^\perp) \leq b_2(M)-\ell$.

So, if S is p-primitive i.e., $\ell_p(\overline{S}/S) = 0$, then by Lemma 2.2 we have $\ell \leq b_2(M) - \ell$, which contradicts our hypothesis.

In case (ii) we may assume $[S_i^2]^2 = 0$ $(1 \leq i \leq k)$ and $\neq 0$ $(k+1 \leq i \leq \ell)$. Put $\xi_i = [S_i^2] \in H_2(M;\mathbf{Z})$ $(1 \leq i \leq \ell)$. Assume that S is p-primitive. Then, we have a homology class $\eta_1 \in H_2(M;\mathbf{Z})$ p-dual to ξ_1, that is, $\langle \xi_1, \eta_1 \rangle = mp + 1$. Now, we put $\xi_i' = (mp+1)\xi_i - \langle \xi_i, \eta_1 \rangle \xi_1$ for $2 \leq i \leq \ell$ so that $\langle \xi_i', \eta_1 \rangle = \langle \xi_i', \xi_1 \rangle = 0, \xi_i'^2 = 0$ $(2 \leq i \leq k)$ and $\neq 0$ $(k+1 \leq i \leq \ell)$ and $\xi_1, \xi_2', \ldots, \xi_\ell'$ are also linearly independent. Let U_1 be a sublattice generated by ξ_1 and η_1. Since $\ell_p(U_1^*/U_1) = 0$, $L_1 = \{x \in L : \langle x, \xi_1 \rangle = \langle x, \eta_1 \rangle = 0\}$ is a p-unimodular lattice by Lemma 2.2. Let S_1 be the submodule of L_1 generated by $\xi_2', \ldots, \xi_\ell'$. Recall we assume that L/S contains no p-torsion. Then, it is equivalent to say that L_1/S_1 contains no p-torsion, because $(U_1 \oplus L_1)/S \cong \mathbf{Z} \oplus L_1/S_1$ and $L/(U_1 \oplus L_1) \subset U_1^*/U_1 \oplus L_1^*/L_1$ in the exact sequence $0 \longrightarrow (U_1 \oplus L_1)/S \longrightarrow L/S \longrightarrow L/(U_1 \oplus L_1) \longrightarrow 0$.

By an induction argument we get a p-unimodular lattice L_k of rank $= \operatorname{rank} L - 2k$ containing modified linearly independent homology classes $\xi_{k+1}, \ldots, \xi_\ell$. If we define S_k by the submodule of L_k generated by these modified $\xi_{k+1}, \ldots, \xi_\ell$, then $\langle \ , \ \rangle$ on S_k is non-degenerate and L_k/S_k contains no p-torsion, that is, S_k is a p-primitive sublattice of the p-unimodular lattice L_k. Then, by Lemma 2.2 $\ell_p(S_k^*/S_k) = \ell_p(K_k^*/K_k)$, where K_k denotes the orthogonal complement of S_k in L_k. So, by an argument as in the case (i) $\ell - k \leq (b_2(M) - 2k) - (\ell - k)$ or equivalently $2\ell \leq b_2(M)$, which contradicts our hypothesis. This means that, if $[S_1^2], \ldots, [S_\ell^2]$ are linearly independent in $H_2(M;\mathbf{Z}_p)$, then L/S contains a non-trivial p-torsion. Q.E.D.

We want to estimate the first Betti number $b_1(\widetilde{M}) = \dim H_1(\widetilde{M};\mathbf{Q})$ of the 2-sheet branched covering $\widetilde{M}$ of M.

Lemma 2.4. *Let σ be a locally linear involution acting on a compact connected manifold $\widetilde{M}$ with fixed point set F and orbit space M. Suppose that $H_1(M;\mathbf{Q}) = 0$, F admits an equivariant normal disk bundle $\widetilde{N}(F)$ in $\widetilde{M}$ and one of the following three conditions is satisfied: (1) $F = \emptyset$, (2) F contains neither codimension one nor codimension two component, or (3) F contains no codimension one component and any connected component of codimension two part is simply-connected. Then,*

$$b_1(\widetilde{M}) \leq \ell_2(H_1(M - F;\mathbf{Z})) - 1.$$

Here $\ell_2(A)$ stands for the number of minimal generators of $A \otimes \mathbf{Z}_2$.

Proof. Sekine [13;§1] gives a proof in case $M = S^4$ and F has codimension two. Put $\widetilde{N} = \widetilde{M} - \operatorname{Int}\widetilde{N}(F)$. The natural projection $\pi : \widetilde{M} \to M$ induces a double covering $\pi : \widetilde{N} \to N$ of compact manifolds. We define a chain complex $\widehat{C}_*$ by the exact sequence:

$$0 \longrightarrow \widehat{C}_* \longrightarrow C_*(\widetilde{N};\mathbf{Z}) \xrightarrow{\pi_*} C_*(N;\mathbf{Z}) \longrightarrow 0.$$

Let t be a generator of $\mathbf{Z}_2$. Then, $\widehat{C}_* = (1-t)C_*(\widetilde{N};\mathbf{Z})$. So, $\widehat{C}_* \otimes \mathbf{Z}_2$ is isomorphic to $(1+t)C_*(\widetilde{N};\mathbf{Z}_2) \cong C_*(N;\mathbf{Z}_2)$ as chain complex.

Since $0 \to \widehat{C}_* \otimes \mathbf{Q} \to C_*(\widetilde{N};\mathbf{Q}) \to C_*(N;\mathbf{Q}) \to 0$ is also exact, we consider the exact sequence:

$$H_1(\widehat{C}_* \otimes \mathbf{Q}) \to H_1(\widetilde{N};\mathbf{Q}) \to H_1(N;\mathbf{Q}) \to H_0(\widehat{C}_* \otimes \mathbf{Q}) \to 0.$$

Put $d = \dim H_1(\widetilde{N};\mathbf{Q}) - \dim H_1(N;\mathbf{Q})$. Then, $d \leq \dim H_1(\widehat{C}_* \otimes \mathbf{Q}) - \dim H_0(\widehat{C}_* \otimes \mathbf{Q})$.

Because $H_0(\widehat{C}_* \otimes \mathbf{Z}_2) = \mathbf{Z}_2$ and $H_0(\widehat{C}_*)$ is finitely generated, we have two cases: (i) $H_0(\widehat{C}_*)$ is finite and $\ell_2(H_0(\widehat{C}_*)) = 1$ and (ii) $H_0(\widehat{C}_*) \cong \mathbf{Z} \oplus (\text{odd torsion})$. In case (i) we have $H_0(\widehat{C}_*) * \mathbf{Z}_2 = \mathbf{Z}_2$ and $H_1(\widehat{C}_* \otimes \mathbf{Z}_2) = (H_1(\widehat{C}_*) \otimes \mathbf{Z}_2) \oplus \mathbf{Z}_2$ by the universal coefficient theorem. So,

$$d \leq \dim H_1(\widehat{C}_* \otimes \mathbf{Q}) \leq \dim_{\mathbf{Z}_2} H_1(\widehat{C}_*) \otimes \mathbf{Z}_2 = \dim_{\mathbf{Z}_2} H_1(\widehat{C}_* \otimes \mathbf{Z}_2) - 1.$$

In case (ii) we have $H_0(\widehat{C}_*) * \mathbf{Z}_2 = 0$. So,

$$d \leq \dim H_1(\widehat{C}_* \otimes \mathbf{Q}) - 1 \leq \dim_{\mathbf{Z}_2} H_1(\widehat{C}_*) \otimes \mathbf{Z}_2 - 1 = \dim_{\mathbf{Z}_2} H_1(\widehat{C}_* \otimes \mathbf{Z}_2) - 1.$$

Note that $H_1(\widehat{C}_* \otimes \mathbf{Z}_2) \cong H_1(N;\mathbf{Z}_2) = H_1(M-F;\mathbf{Z}_2) = H_1(M-F;\mathbf{Z}) \otimes \mathbf{Z}_2$. If $F = \emptyset$, then $H_1(N;\mathbf{Q}) = H_1(M;\mathbf{Q}) = 0$. Hence, the result follows from the condition (1).

Under the condition (2) or (3) the natural maps $H_0(\partial\widetilde{N}(F)) \to H_0(\widetilde{N}) \oplus H_0(\widetilde{N}(F))$ and $H_0(\partial N(F)) \to H_0(N) \oplus H_0(N(F))$ are injective with coefficient in $\mathbf{Q}$ due to the condition that F has no codimension one component. Hence, we have the following commutative diagram of Mayer-Vietoris exact sequences with coefficient in $\mathbf{Q}$:

$$\begin{array}{ccccccc}
H_1(\partial\widetilde{N}(F)) & \xrightarrow{(\tilde{\jmath}_*,\tilde{\imath}_*)} & H_1(\widetilde{N}) \oplus H_1(\widetilde{N}(F)) & \longrightarrow & H_1(\widetilde{M}) & \longrightarrow & 0 \\
{\scriptstyle \pi_*}\downarrow & & {\scriptstyle \pi_* \oplus}\downarrow{\scriptstyle \pi_*} & & {\scriptstyle \pi_*}\downarrow & & \\
H_1(\partial N(F)) & \xrightarrow{(j_*,i_*)} & H_1(N) \oplus H_1(N(F)) & \longrightarrow & H_1(M) & \longrightarrow & 0.
\end{array}$$

Note that $\pi_* : H_1(\widetilde{N}(F)) \longrightarrow H_1(N(F))$ is an isomorphism in any coefficient because they are canonically equal to $H_1(F)$. If F has no codimension two component, we have an exact sequence of groups $\mathbf{Z}_2 \longrightarrow \pi_1(\partial N(F)) \xrightarrow{i_*} \pi_1(N(F)) \longrightarrow 0$. So, $i_* : H_1(\partial N(F); \mathbf{Q}) \longrightarrow H_1(N(F); \mathbf{Q})$ is onto. Since $\tilde{\imath}_* : \pi_1(\partial \widetilde{N}(F)) \cong \pi_1(\widetilde{N}(F))$, $i_* : H_1(\partial N(F); \mathbf{Q}) = H_1(\partial \widetilde{N}(F); \mathbf{Q})^{\sigma_*} \hookrightarrow H_1(\partial \widetilde{N}(F); \mathbf{Q}) \cong H_1(\widetilde{N}(F); \mathbf{Q}) = H_1(N(F); \mathbf{Q})$ is injective. Hence, $i_* : H_1(\partial N(F); \mathbf{Q}) \longrightarrow H_1(N(F); \mathbf{Q})$ is also an isomorphism. So, the condition (2) implies $\dim H_1(\widetilde{M}; \mathbf{Q}) - \dim H_1(M; \mathbf{Q}) = \dim H_1(\widetilde{N}; \mathbf{Q}) - \dim H_1(N; \mathbf{Q}) = d$, which implies the result as before.

Let F_2 be a connected component of codimension two. Assume the condition (3). Then, there is an exact sequence $\mathbf{Z} \to \pi_1(\partial N(F_2)) \to 0$. If $\pi_1(\partial N(F_2))$ is finite, then $H_1(\partial \widetilde{N}(F_2); \mathbf{Q}) = H_1(\partial N(F_2); \mathbf{Q}) = 0$. Otherwise $\tilde{\jmath}_* : H_1(\partial \widetilde{N}(F_2); \mathbf{Q}) \to H_1(\widetilde{N}; \mathbf{Q})$ is injective or zero if and only if $j_* : H_1(\partial N(F_2); \mathbf{Q}) \to H_1(N; \mathbf{Q})$ is injective or zero respectively. So, the condition (3) also implies $\dim H_1(\widetilde{M}; \mathbf{Q}) - \dim H_1(M; \mathbf{Q}) = \dim H_1(\widetilde{N}; \mathbf{Q}) - \dim H_1(N; \mathbf{Q}) = d$, which completes a proof. Q.E.D.

Remark. Probably we need not to assume the existence of equivariant normal disk bundle; it suffices that $F \times CP^2$ has a compact invariant manifold neighborhood $\widetilde{N}'(F \times CP^2)$ in $\widetilde{M} \times CP^2$ so that $F \times CP^2 \hookrightarrow \widetilde{N}'(F \times CP^2)$ is a homotopy equivalence and $\partial \widetilde{N}'(F \times CP^2) \to \widetilde{N}'(F \times CP^2)$ is a spherical homotopy fibration.

The following lemmas are not new but we list them up to quote in the proof of Theorem.

Lemma 2.5. *Let σ be an orientation preserving locally linear involution on an oriented closed 4-manifold M with fixed point set F. Let F^2 denote the 2-dimensional part of F.*

(1) *Any isolated point x of F can be blow up, that is, there is a locally linear involution σ' on $M^* = M \# \overline{CP}^2 = (M - x) \cup CP^1$ such that $\sigma' | M^* - CP^1 = \sigma | M - x$ and $\sigma' | CP^1 = \mathrm{id}$. In particular, σ' operates as identity on the newly introduced homology class represented by CP^1 and $\pi_1(M^*/\sigma') = \pi_1(M/\sigma)$. We may take also $M \# CP^2$ instead of $M \# \overline{CP}^2$; this comes from that we have an orientation reversing diffeomorphism of RP^3.*

(2) (Freedman-Quinn) *F^2 admits an equivariant normal disk bundle $N(F^2)$ in M.*

(3) (G-signature theorem)

$$\operatorname{sign}(-1, M) = e(F^2),$$

where $e(F^2)$ denotes the total Euler number of the normal bundle of F^2 and -1 stands for the involution concerned.

Proof. (1) Since σ is locally linear, we have a local complex coordinate (z_1, z_2) in a disk neighborhood U of x so that $x = (0,0)$ and $\sigma(z_1, z_2) = (-z_1, -z_2)$. Take a homogeneous coordinate $[\zeta_1, \zeta_2]$ of CP^1 and consider on the product space $U \times CP^1$ the subset U^* defined by $z_1\zeta_2 - z_2\zeta_1 = 0$. It is easy to see that U^* is a complex surface in $U \times CP^1$, the projection $\pi : U^* \to U$ gives an identification of $U^* - \pi^{-1}(0,0)$ with $U - (0,0)$, the preimage $(0,0) \times CP^1$ of $(0,0)$ is isomorphic to CP^1. Consider a holomorphic involution $(\sigma|U) \times \mathrm{id}$ on $U \times CP^1$. Then, we get a holomorphic involution $\sigma'|U^*$ on U^* such that $\sigma'|U^* - \pi^{-1}(0,0) = \sigma|U - (0,0)$ and $\sigma'|(0,0) \times CP^1 = \mathrm{id}$. Define $M^* = (M - U) \cup U^*$ and $\sigma'|M^* - U = \sigma|M - U$. Then, $M^* - CP^1 = M - x$ and M^* is diffeomorphic to $M \# \overline{CP}^2$ because $[CP^1]^2 = -1$. Since $\partial U^*/\sigma' = \partial U/\sigma = RP^3$ and $\pi_1(U^*/\sigma') = \pi_1(U/\sigma) = 0$, we have $\pi_1(M^*/\sigma') = \pi_1(M/\sigma)$ by the van Kampen theorem.

(2) Since M/σ is a manifold near F^2 and F^2 is a locally flat submanifold, F^2 admits a normal disk bundle due to Freedman-Quinn [6;9.3]. So, a lifting gives an equivariant normal disk bundle.

(3) In the smooth case G-signature theorem is due to Atiyah-Singer [2] but has many elementary proofs at least in our case of dimension 4 and semi-free, for example, in Gordon [8]. These elementary proofs can apply also to a locally linear involution, because it admits an equivariant tubular neighborhood of F^2 by (2). See also the comments in Edmonds [5;§4]. Q.E.D.

Lemma 2.6 (Edmonds [5;Prop. 3.1&3.2]). *Let M be a connected oriented spin 4-manifold and σ a locally linear involution that preserves orientation and some spin structure. Then, the fixed point set F, if non-empty, consists either of isolated points or of orientable surfaces.*

In the smooth case the codimension homogeneity modulo 4 is proved by Atiyah-Bott [1] and the orientability of surfaces has many proofs including Edmonds [4]. The proof in the locally linear case is given in Edmonds [5].

§3. Proof of Theorem 1

Since $H_1(M; \mathbf{Z}_2) = 0$, the spin structure on M is unique and we may assume that σ preserves the spin structure. Lemma 2.6 implies that the fixed point set F consists either of isolated points or of orientable surfaces. If F consists of isolated points, then by the G-signature theorem described as Lemma 2.5 (3) $\operatorname{sign}(-1, M) = 0$. Hence, $\operatorname{sign} M = 0$ because σ operates as identity on $H_2(M; \mathbf{Q})$. So, we may assume that F consists of orientable surfaces. In particular, M/σ is also a manifold. Note that F has an equivariant normal disk bundle $N(F)$ in M by Lemma 2.5 (2).

Since $H_*(M/\sigma; \mathbf{Q}) = H_*(M; \mathbf{Q})^{\sigma_*}$, $H_1(M; \mathbf{Q}) = 0$ and $\sigma_* | H_2(M; \mathbf{Q}) = \mathrm{id}$, we have the equality $\chi(M/\sigma) = \chi(M)$ of Euler numbers. Put $\chi = \chi(M)$. Then, from the formula $\chi(M) = 2\chi(M/\sigma) - \chi(F)$ we get also $\chi(F) = \chi$. So, F contains at least $\chi/2$ numbers of components of S^2. Note that M has an even intersection form $q_M : H_2(M; \mathbf{Z})/\operatorname{tor} \times H_2(M; \mathbf{Z})/\operatorname{tor} \to \mathbf{Z}$ and hence $\chi = \chi(M)$ is even. Let $F' = S_1^2, \ldots, S_{\chi/2}^2$ be the subset of F consisting of $\chi/2$ numbers of S^2. Since $H_1(M/\sigma; \mathbf{Q}) = H_1(M; \mathbf{Q})^{\sigma_*} = 0$, we have $\chi = 2 + b_2(M/\sigma) > b_2(M/\sigma)$. Taking account of $[S_i^2]^2_{M/\sigma} = 2[S_i^2]^2_M$ and Lemma 2.5 (2), we can apply Lemma 2.3 (3) for $p = 2$ and $F' \subset M/\sigma$. So, by Lemma 2.3 (1) and (2) there is a subunion F'' of connected components of F' such that we have a branched covering of M/σ with branch locus F'', that is, $(M, \sigma, F'' \subset F)$ satisfies the condition of Lemma 2.1 except $F'' \neq F$. Note here that $H_1(\partial N(x); \mathbf{Z}) \to H_1(\partial N(S_i^2); \mathbf{Z})$ is a surjection for any x of S_i^2. If $F'' \neq F$, then Lemma 2.1 implies that there is a connected 2-sheet unbranched covering of M. But this contradicts the condition that $H^1(M; \mathbf{Z}_2) = \operatorname{Hom}(H_1(M; \mathbf{Z}), \mathbf{Z}_2) = \operatorname{Hom}(\pi_1(M), \mathbf{Z}_2) = 0$. This means $F'' = F$. Hence, $F' = F$, that is, F consists of $\chi/2$ numbers of S^2.

Since the intersection form q_M of M is even, we can also apply Lemma 2.3 (3) for $p = 2$ and $F \subset M$. By Lemma 2.3 (1) and (2) there is a non-trivial element of $H^1(M - F; \mathbf{Z}_2)$ which takes non-zero value on $H_1(\partial N(S_i^2); \mathbf{Z})$ for some i. This means that there is a branched covering $\tilde{\pi} : \widetilde{M} \to M$ with branch locus $F_1 \subset F$; a locally linear involution τ on $\widetilde{M}$ with fixed point set F_1. So, there is a non-trivial element of $H^1(M - F_1; \mathbf{Z}_2)$ which takes non-zero value on $H_1(\partial N(S_i^2); \mathbf{Z})$ for every $S_i^2 \subset F_1$. Because $H^1(M; \mathbf{Z}_2) = 0$, this implies that (i) the homology classes of the connected components of F_1 are linearly dependent in $H_2(M; \mathbf{Z}_2)$ or (ii) they are independent and generate a submodule S of $L = H_2(M; \mathbf{Z})/\operatorname{tor}$ so that $\overline{S}/S$ contains a non-trivial 2-torsion according to the last part of Lemma 2.3 (2). Assume that $F_1 \neq F$. In case (i) the homology classes of the connected components of F_1 are also linearly dependent in $H_2(M/\sigma; \mathbf{Z}_2)$ and this leads to a contradiction with $H^1(M; \mathbf{Z}_2) = 0$

through Lemma 2.3 (1) and Lemma 2.1 as before. In case (ii) notice that $\pi_* S$ is the submodule generated by the homology classes of the connected components of F_1 in $H_2(M/\sigma;\mathbf{Z})/\operatorname{tor}$ for the projection $\pi : M \to M/\sigma$. Since $\pi_*|S$ is an isomorphism, $\pi_*\overline{S}/\pi_*S$ is isomorphic to $\overline{S}/S$. Note also that $\pi_*\overline{S}/\pi_*S \subset \overline{\pi_*S}/\pi_*S$. Then, $\overline{\pi_*S}/\pi_*S$ contains a non-trivial 2-torsion. We can apply Lemma 2.3 (2) for $p = 2$ and $F_1 \subset M/\sigma$ and we get the same contradiction with $H^1(M;\mathbf{Z}_2) = 0$ by applying Lemma 2.1 for $(M, \sigma, F_1 \subset F)$ since we have assumed $F_1 \neq F$. Hence, $F_1 = F$, that is, the branch locus for $\tilde{\pi} : \widetilde{M} \to M$ is also F and $\chi(\widetilde{M}) = \chi(M)$.

We will show that $\ell_2(H_1(M - F;\mathbf{Z})) = 1$. Since $H^1(M;\mathbf{Z}_2) = 0$, it is equivalent to say $\ell_2(\operatorname{Ker}(H_1(M - F;\mathbf{Z}) \to H_1(M;\mathbf{Z}))) = 1$. Put $N = M - \operatorname{Int} N(F)$ and consider the following commutative diagram:

$$\begin{array}{ccccc}
H_1(\partial N(F);\mathbf{Z}) & \longrightarrow & H_1(N;\mathbf{Z}) & \longrightarrow & H_1(N, \partial N(F);\mathbf{Z}) \\
 & & \downarrow & & \downarrow\cong \\
 & & H_1(M;\mathbf{Z}) & \longrightarrow & H_1(M, N(F);\mathbf{Z})
\end{array}$$

Since the horizontal sequence is exact, any element of $\operatorname{Ker}(H_1(N;\mathbf{Z}) = H_1(M - F;\mathbf{Z}) \to H_1(M;\mathbf{Z}))$ comes from $H_1(\partial N(F);\mathbf{Z})$. We know that there is an element α of $\operatorname{Hom}(H_1(M - F;\mathbf{Z}), \mathbf{Z}_2)$ which takes non-zero value on $H_1(\partial N(S_i^2);\mathbf{Z})$ for every S_i^2 in F. Now we assume that $\ell_2(\operatorname{Ker}(H_1(M - F;\mathbf{Z}) \to H_1(M;\mathbf{Z}))) \geq 2$. Then, we have some element β of $\operatorname{Hom}(H_1(M - F;\mathbf{Z}), \mathbf{Z}_2)$ which is different from α, that is, takes zero value on $H_1(\partial N(S_i^2);\mathbf{Z})$ for at least one i. Note that we used here the special property of $\mathbf{Z}_2$. Let F' be the subset of F removed such S_i^2 off. Since $F' \neq F$, the same argument as the above paragraph can be applied again and get a contradiction with the condition $H^1(M;\mathbf{Z}_2) = 0$.

Now since $H_1(M;\mathbf{Q}) = 0$ and F consists of $\chi/2$ numbers of S^2, $\ell_2(H_1(M - F;\mathbf{Z})) = 1$ implies $b_1(\widetilde{M}) = 0$ by Lemma 2.4. So, $\chi(\widetilde{M}) = \chi(M)$ implies $b_2(\widetilde{M}) = b_2(M)$. Hence, $H_2(M;\mathbf{Q}) = H_2(\widetilde{M};\mathbf{Q})^{\tau_*}$ implies $H_2(\widetilde{M};\mathbf{Q})^{\tau_*} = H_2(\widetilde{M};\mathbf{Q})$, that is, $\tau_* = \mathrm{id}$ on $H_2(\widetilde{M};\mathbf{Q})$. Therefore, $\operatorname{sign}(-1, \widetilde{M}) = \operatorname{sign} \widetilde{M}$. Recall that $\operatorname{sign}(-1, M) = \operatorname{sign} M$ and the G-signature theorem says that

$$\operatorname{sign}(-1, M) = \sum_{i=1}^{\chi/2} [S_i^2]_M^2 = \sum_{i=1}^{\chi/2} 2[S_i^2]_{\widetilde{M}}^2 = 2 \operatorname{sign}(-1, \widetilde{M}).$$

On the other hand $\operatorname{sign} M = \operatorname{sign} \widetilde{M}$ because $H_2(M;\mathbf{Q}) = H_2(\widetilde{M};\mathbf{Q})^{\tilde{\tau}_*} = H_2(\widetilde{M};\mathbf{Q})$. Hence, $\operatorname{sign} M = 0$. This completes a proof of Theorem 1.

References

[1] M. F. Atiyah and R. Bott, A Lefshetz fixed point formula for elliptic complexes II. Applications, Ann. Math., **88** (1968), 451–491.

[2] M. F. Atiyah and I. M. Singer, The index of elliptic operators: III, Ann. Math., **87** (1968), 546–604.

[3] W. Barth, C. Peters and A. Van de Ven, "Compact Complex Surfaces", Springer-Verlag, Berlin Heiderberg New York Tokyo, 1984.

[4] A. Edmonds, Orientability of fixed point sets, Proc. Amer. Math. Soc., **82** (1981), 120–124.

[5] ———, Aspects of group actions on four-manifolds, Topology Appl., **31** (1989), 109–124.

[6] M. Freedman and F. Quinn, "Topology of 4-manifolds", Princeton Math. Ser. 39, Princeton Univ. Press, Princeton, 1990.

[7] R. H. Fox, Covering space with singularities, in "Algebraic Geometry and Topology", Princeton Univ. Press, Princeton, 1957, pp. 243–257.

[8] C. McA. Gordon, On the G-signature theorem in dimension four, in "A la Recherche de la Topologie Perdue", Birkhaüser, Boston, 1986, pp. 159–180.

[9] R. C. Kirby and L. C. Siebenmann, Normal bundles for codimension 2 locally flat imbeddings, in "Geometric Topology", Lecture Notes in Math. 438, Springer-Verlag, Berlin Heidelberg New York, 1975, pp. 310–324.

[10] V. V. Nikulin, Integral symmetric bilinear forms and some of their applications, Math. USSR Izvestia, **14** (1980), 103–167.

[11] V. A. Rokhlin, Two-dimensional submanifolds of four-dimensional manifolds, Functional Anal. Appl., **5** (1971), 39–48.

[12] R. Schultz, Problems submitted to the A.M.S. summer research conference on group actions, in "Group Actions on Manifolds", Contemp. Math. 36, Amer. Math. Soc., Providence, 1984, pp. 513–568.

[13] M. Sekine, On homology of the double covering over the exterior of a surface in 4-sphere, Hiroshima Math. J., **21** (1991), 419–426.

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