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Compactification of Submanifolds in Euclidean Space by the Inversion

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Dedicated to Professor Tominosuke Otsuki

Introduction

Let $\bar{M}$ be an n-dimensional compact connected C^2 submanifold in the N-dimensional Euclidean space R^N. Let Ψ be the inversion of R^N, which is defined by $\Psi(x) = x/|x|^2$ for x in $R^N \cup \{\infty\}$. If the origin O is contained in $\bar{M}$, $\Psi(\bar{M})$ becomes a noncompact, complete, connected C^2 submanifold properly immersed into R^N. If we denote the second fundamental form of $\Psi(\bar{M})$ by B, $|x|^2|B|$ $(x \in \Psi(\bar{M}))$ is bounded on $\Psi(\bar{M})$. In this paper we study the image by the inversion of a noncompact, complete, connected C^2 submanifold M of dimension $n \geq 2$ which is properly immersed into R^N. We are particularly interested in the smoothness of $\Psi(M)$ at the origin O. We say that M satisfies the condition $\mathrm{P}(\alpha)$ if $|x|^\alpha|B|$ $(x \in M)$ is bounded on M. We prove that if M satisfies $\mathrm{P}(2+\varepsilon)$ for some positive constant ε, then the image of each end of M by Ψ is C^2 at O (Theorem 2). Boundedness of $|x|^2|B|$ (i.e., $\mathrm{P}(2)$) is not sufficient to assure that $\Psi(M)$ is C^2 at O, while $\Psi(M)$ is C^1 at O if $\mathrm{P}(1+\varepsilon)$ is satisfied for some $\varepsilon > 0$ (Theorem 1).

Noncompact submanifolds satisfying $\mathrm{P}(1+\varepsilon)$ are studied by Kasue and Sugahara ([4], [5]). They show that those submanifolds become totally geodesic under certain additional conditions on the mean curvature or the sectional curvature. We will make use of some of their results in our proof. As a direct consequence of our theorems, we see that if M satisfies $\mathrm{P}(1+\varepsilon)$, the Gauss map is continuous at infinity, and if M satisfies $\mathrm{P}(2+\varepsilon)$, then M is conformally equivalent to a compact C^2 Riemannian manifold punctured at a finite number of points. We also show that the total integral of the Lipschitz-Killing curvature over the unit normal bundle is an integer if M satisfies $\mathrm{P}(2+\varepsilon)$ (Theorem 3).

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These properties have been studied for submanifolds with $\int_M |B|^n < \infty$ in [8] when $\dim M = 2$, and in [1] when M is minimal. We note that if M satisfies $\mathrm{P}(1+\varepsilon)$, $\int_M |B|^n$ is finite (Proposition 4.1), and if M is minimal, $\dim M \geq 3$ and $\int_M |B|^n$ is finite, then M satisfies $\mathrm{P}(n)$ ([1]).

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§1. Asymptotic behavior of submanifolds

Let $\langle\ ,\ \rangle$ denote the standard inner product of R^N. We denote the covariant differentiation of R^N by D. For x in R^N let $|x| = \langle x, x\rangle^{1/2}$. Let $B(R) = \{x \in R^N : |x| < R\}$ and $S(R) = \{x \in R^N : |x| = R\}$.

Throughout this paper, M will denote a noncompact, complete, connected C^2 submanifold of dimension $n \geq 2$ properly immersed into R^N. For x in M let T_xM and $T_x^\perp M$ denote the tangent and the normal space of M at x respectively. The second fundamental form $B : T_xM \times T_xM \to T_x^\perp M$ is defined by $B(X,Y) = (D_XY)^\perp$, where $(D_XY)^\perp$ is the normal component of D_XY. We also define the shape operator $A_\xi : T_xM \to T_xM$ with respect to a unit normal vector ξ by $A_\xi X = -(D_X\xi)^\top$, where $(D_X\xi)^\top$ is the tangential component of $D_X\xi$. We denote by ∇ the covariant differentiation of M with respect to the induced metric. Let $r(x) = |x|$ for x in M.

Definition. We say that M satisfies the condition $\mathrm{P}(\alpha)$ if there exists a constant K such that

$$r^\alpha |B| \leq K$$

holds at every point of M.

We set $M(R) = M \setminus B(R)$. Since M is properly immersed, $M(R)$ is a union of a finite number of submanifolds $M_1(R), \ldots, M_q(R)$ and $\partial M_\lambda(R) = M_\lambda(R) \cap S(R)$ is compact for each $\lambda = 1, \ldots, q$. The following lemma is due to Kasue ([4, Lemma 2]).

Lemma 1.1. *Suppose M satisfies $P(1+\varepsilon)$ for some positive constant ε. Then there exist positive constants C_1 and R_1 such that*

(1) $|\nabla r| \geq C_1^{-1}$ *for all x in M with $r \geq R_1$,*

(2) $M_\lambda(R_1)$ *is diffeomorphic to* $\partial M_\lambda(R_1) \times [R_1, \infty)$ *for each* $\lambda = 1, \ldots, q$.

$M_\lambda(R)$ ($R \geq R_1$) is called an end of M. In the following argument, we assume that the position vector of a point x in R^N is denoted by the same letter x. For x in M, regarding the vector x as a tangent vector to R^N at the point x, we denote by $x^\top$ (resp. $x^\perp$) the image of x by the orthogonal projection from R^N onto the tangent space T_xM (resp. the normal space $T_x^\perp M$) of M at x.

Lemma 1.2. $x^\top = r\nabla r$.

Proof. The gradient vector of $\langle x, x\rangle^{1/2}$ as a function on R^N is given by $r^{-1}x$. For x in M we take its tangential component to see that $\nabla r = r^{-1}x^\top$.

We will use several results from [5] to prove our theorems.

Lemma 1.3. *Suppose M satisfies $P(1+\varepsilon)$ with $\varepsilon > 0$. Then:*

(1) *For any constant δ satisfying $\delta < \min\{\varepsilon, 1\}$, $r^{-1+\delta}|x^\perp|$ tends to zero as $r \to \infty$.* ([5, Lemma 5 (ii)])

(2) *For every $t \geq R_1$ any two points on $\partial M_\lambda(t)$ can be joined by a curve on $\partial M_\lambda(t)$ whose length is less than $C_2 t$, where C_2 is a constant which does not depend on t.* ([5, Lemma 6])

(3) *The second fundamental form of $t^{-1}\partial M_\lambda(t)$ as a submanifold of $S(1)$ tends to zero as $t \to \infty$.* ([5, Lemma 7])

For a submanifold satisfying P$(2+\varepsilon)$ ($\varepsilon > 0$) we have the following lemma.

Lemma 1.4. *Suppose M satisfies $P(2+\varepsilon)$ with $\varepsilon > 0$. Then $x^\perp$ is continuous at infinity on each end $M_\lambda(R_1)$.*

Proof. Let x be a point in $M(R_1)$. We first observe that, for X, Y in T_xM and N in $T_x^\perp M$,

$$\begin{aligned} \left\langle D_X x^\perp, Y\right\rangle &= -\left\langle D_X Y, x^\perp\right\rangle \\ &= -\left\langle B(X,Y), x^\perp\right\rangle \end{aligned} \tag{1.1}$$

and

$$\begin{aligned} \left\langle D_X x^\perp, N\right\rangle &= \left\langle D_X(x - x^\top), N\right\rangle \\ &= \left\langle X - D_X x^\top, N\right\rangle \\ &= -\left\langle B(X, x^\top), N\right\rangle . \end{aligned} \tag{1.2}$$

Hence there exists a constant C_3 such that

$$\begin{aligned} |D_X x^\perp| &\le C_3|B|\ |X|\ |x| \\ &\le KC_3 r^{-1-\varepsilon}|X|\,. \end{aligned} \tag{1.3}$$

Now we fix an end $M_\lambda(R_1)$. Let y be a point in $\partial M_\lambda(R_1)$ and let γ_y be the integral curve of $|\nabla r|^{-2}\nabla r$ on $M_\lambda(R_1)$ which starts at y. γ_y is parametrized by r. Set $N(r) = (\gamma_y(r))^\perp$. $N(r)$ is the restriction of the vector field $x^\perp$ to γ_y. By (1.3), we have

$$\begin{aligned} \left|\frac{dN}{dr}\right| &\le KC_3 r^{-1-\varepsilon}|\nabla r|^{-1} \\ &\le KC_1C_3 r^{-1-\varepsilon}\,. \end{aligned} \tag{1.4}$$

(1.4) implies that $N(r)$ converges to a constant vector N_y as $r \to \infty$.

To prove that N_y does not depend on y we will show that for any y_1 and y_2 in $\partial M_\lambda(R_1)$ and any positive number η we have $|N_{y_1} - N_{y_2}| < \eta$. We first take $R_2 \ge R_1$ such that

$$|(\gamma_{y_j}(R_2))^\perp - N_{y_j}| < \frac{\eta}{3} \tag{1.5}$$

for $j = 1, 2$. By Lemma 1.3 (2), there exists a curve σ on $\partial M_\lambda(R_2)$ which joins $\gamma_{y_1}(R_2)$ and $\gamma_{y_2}(R_2)$ and has length less than C_2R_2, where C_2 is a constant independent of R_2. We parametrize σ by its arclength s. Let $N(s) = (\sigma(s))^\perp$. By (1.3), we have

$$\left|\frac{dN}{ds}\right| \le KC_3R_2^{-1-\varepsilon}\,.$$

Hence

$$\begin{aligned} |(\gamma_{y_2}(R_2))^\perp - (\gamma_{y_1}(R_2))^\perp| &= \left|\int_\sigma \frac{dN}{ds}ds\right| \\ &\le \int_\sigma \left|\frac{dN}{ds}\right| ds \\ &\le KC_2C_3R_2^{-\varepsilon}\,. \end{aligned}$$

If we take R_2 sufficiently large, it is possible to have

$$|(\gamma_{y_2}(R_2))^\perp - (\gamma_{y_1}(R_2))^\perp| < \frac{\eta}{3}\,. \tag{1.6}$$

It follows from (1.5) and (1.6) that

$$\begin{aligned}|N_{y_2} - N_{y_1}| &\le |N_{y_2} - (\gamma_{y_2}(R_2))^{\perp}| + |(\gamma_{y_2}(R_2))^{\perp} - (\gamma_{y_1}(R_2))^{\perp}| \\ &\quad + |(\gamma_{y_1}(R_2))^{\perp} - N_{y_1}| \\ &< \eta .\end{aligned}$$

This completes the proof of Lemma 1.4.

§2. C^1 compactification by the inversion

Let $R^N \cup \{\infty\}$ be the union of R^N and the point of infinity. The inversion Ψ is a map from $R^N \cup \{\infty\}$ onto $R^N \cup \{\infty\}$ which is defined by $\Psi(x) = \langle x, x\rangle^{-1} x$ for all x in $R^N \setminus \{O\}$, $\Psi(O) = \infty$ and $\Psi(\infty) = O$. If X and Y are tangent vectors of R^N at x, then

$$d\Psi(X) = \langle x, x\rangle^{-1} X - 2\langle x, x\rangle^{-2} \langle x, X\rangle x$$

and we have

$$\langle d\Psi(X), d\Psi(Y)\rangle = \langle x, x\rangle^{-2} \langle X, Y\rangle .$$

Let $\bar{M} = \Psi(M)$. We denote the second fundamental form of $\bar{M}$ by $\bar{B}$. Let $\bar{x} = \Psi(x)$ and $\bar{r} = \langle \bar{x}, \bar{x}\rangle^{1/2}$. We have $\bar{r} = r^{-1}$, where $r = \langle x, x\rangle^{1/2}$. For a unit tangent vector X and a unit normal vector ξ of M at x we set $\bar{X} = r^2 d\Psi(X)$ and $\bar{\xi} = r^2 d\Psi(\xi)$. $\bar{X}$ (resp. $\bar{\xi}$) is a unit tangent (resp. normal) vector of $\bar{M}$ at $\bar{x}$.

Lemma 2.1. *For any tangent vectors X and Y of M at x, we have*

$$\bar{B}(\bar{X}, \bar{Y}) = r^4 d\Psi(B(X, Y)) + 2r^2 \langle X, Y\rangle d\Psi(x^{\perp}) .$$

Proof. We have

$$\begin{aligned}D_{\bar{X}}\bar{Y} &= r^2 D_X(Y - 2r^{-2}\langle x, Y\rangle x) \\ &= r^2 D_X Y + (4r^{-2}\langle x, X\rangle \langle x, Y\rangle - 2\langle X, Y\rangle \\ &\quad - 2\langle x, D_X Y\rangle) x - 2\langle x, Y\rangle X \\ &= r^4 d\Psi(D_X Y) - 2r^2 \langle x, Y\rangle d\Psi(X) + 2r^2 \langle X, Y\rangle d\Psi(x) .\end{aligned} \tag{2.1}$$

In the last equality, we note that $d\Psi(x) = -r^{-2}x$. Since $d\Psi$ maps tangent spaces and normal spaces of M onto tangent spaces and normal spaces of $\bar{M}$ respectively, the lemma follows from (2.1).

Lemma 2.2. *Suppose M satisfies $P(1+\varepsilon)$ with $\varepsilon > 0$. Then there exists a positive constant δ such that $\bar{r}^{1-\delta}|\bar{B}|$ is bounded in a neighborhood of O in $\bar{M}$.*

Proof. For any unit tangent vectors X and Y of M at x it follows from Lemma 2.1 that

$$\begin{aligned} |\bar{B}(\bar{X},\bar{Y})| &= |r^2 B(X,Y) + 2\langle X, Y\rangle x^{\perp}| \\ &\le r^2|B| + 2|x^{\perp}|\,. \end{aligned} \tag{2.2}$$

Let δ be any constant such that $0 < \delta < \min\{\varepsilon, 1\}$. Then, by Lemma 1.3 (1) and the condition $\mathrm{P}(1+\varepsilon)$, there exists a constant C_4 such that

$$|\bar{B}| \le C_4 r^{1-\delta}\,. \tag{2.3}$$

Now we have the lemma since $\bar{r} = r^{-1}$.

We write $M_\lambda = M_\lambda(R_1)$ $(\lambda = 1, \ldots, q)$ and $\bar{M}_\lambda = \Psi(M_\lambda)$.

Lemma 2.3. *Suppose M satisfies $P(1+\varepsilon)$ with $\varepsilon > 0$. Let $R \ge R_1$. Then any two points $\bar{x}_1$, $\bar{x}_2$ in $B(1/R) \cap \bar{M}_\lambda$ can be joined by a curve on $\bar{M}$ whose length is less than C_5/R, where C_5 is a constant which does not depend on R.*

Proof. Let $\bar{\gamma}_i$ be the integral curve of $|\nabla\bar{r}|^{-2}\nabla\bar{r}$ on $\bar{M}_\lambda$ which passes through $\bar{x}_i$ $(i = 1, 2)$. $\bar{\gamma}_i$ is parametrized by $\bar{r}$. Let $\bar{y}_i = \bar{\gamma}_i(1/R)$. Since $\nabla\bar{r} = -r^2 d\Psi(\nabla r)$, it follows from Lemma 1.1 that

$$|\nabla\bar{r}| = r^2|d\Psi(\nabla r)| = |\nabla r| \ge C_1^{-1}$$

for all $\bar{x}$ in $\bar{M}$ with $\bar{r} \le 1/R_1$. Hence the length of $\bar{\gamma}_i$ between $\bar{x}_i$ and $\bar{y}_i$ is less than C_1/R. By Lemma 1.3 (2), there exists a curve σ on $\partial M_\lambda(R)$ which joins $\Psi(y_1)$ and $\Psi(y_2)$ and has length less than $C_2 R$, where C_2 is a constant which does not depend on R. Let $\bar{\sigma} = \Psi(\sigma)$. Then $\bar{\sigma}$ joins $\bar{y}_1$ and $\bar{y}_2$ and has length less than C_2/R. Connecting $\bar{\gamma}_1$, $\bar{\sigma}$ and $\bar{\gamma}_2$, we obtain a curve in $\bar{M}_\lambda$ which joins $\bar{x}_1$ and $\bar{x}_2$ and has length less than $(2C_1 + C_2)/R$.

Theorem 1. *Let M be a noncompact, complete, connected C^2 submanifold of dimension $n \ge 2$ properly immersed into R^N. Suppose M satisfies the condition $P(1+\varepsilon)$ for some positive constant ε. Then the image of each end M_λ by the inversion is C^1 at the origin O.*

Proof. We will use the generalized Gauss map $\bar{G}$ which maps each point $\bar{x}$ of $\bar{M}$ to the n-dimensional linear subspace parallel to the tangent space of $\bar{M}$ at $\bar{x}$. $\bar{G}$ defines a map from $\bar{M}$ into the Grassmannian manifold $G_n(R^N)$ which consists of n-dimensional (unoriented) linear subspaces of R^N. $G_n(R^N)$ has the standard invariant metric g as a symmetric space. If $\bar{Y}$ and $\bar{Z}$ are tangent vectors of $\bar{M}$ at $\bar{x}$, we have

$$(2.4) \qquad g\,(\,d\bar{G}(\bar{Y}), d\bar{G}(\bar{Z})\,) = \sum_{i=1}^{n} \langle \bar{B}(\bar{Y}, \bar{X}_i), \bar{B}(\bar{Z}, \bar{X}_i) \rangle \ ,$$

where $\{\bar{X}_1, \ldots, \bar{X}_n\}$ is an orthonormal base of $T_{\bar{x}}\bar{M}$ ([6]). By Lemma 2.3, any two points $\bar{x}_1$, $\bar{x}_2$ in $B(1/R) \cap \bar{M}_\lambda$ $(R \geq R_1)$ is joined by a curve $\bar{\tau}$ in $\bar{M}_\lambda$ whose length is less than C_5/R, where C_5 is a constant which does not depend on R. Now we use (2.4) to see that the length of $\bar{G}(\bar{\tau})$ is less than $C_5\sqrt{n}|\bar{B}|R^{-1}$. Hence, by Lemma 2.2, the length of $\bar{G}(\bar{\tau})$ is less than $C_6R^{-\delta}$ for some positive constants δ and C_6. This implies that, for an open subset $\bar{U}$ of $\bar{M}_\lambda$ containing O, $\bar{G}(\bar{U})$ converges to a certain point in $G_n(R^N)$ when $\bar{U}$ shrinks to the point O. Hence the tangent space $T_{\bar{x}}\bar{M}_\lambda$ converges to an n-dimensional linear subspace P as $\bar{x}$ in $\bar{M}_\lambda$ approaches O, which means that $\bar{M}_\lambda$ is C^1 at O.

Corollary. *Let M be as in Theorem 1. Then the Gauss map $G\colon M \to G_n(R^N)$ is continuous at infinity on each end.*

Remarks. (1) Theorem 1 does not hold if $n = 1$.

(2) The condition P(1) is not sufficient for $\bar{M}$ to be C^1 at O. Such an example is given in [8]; If M is the graph of a smooth function $z = u(x, y)$ which away from the origin is given by $u(x, y) = x\sin(\log(\log\rho))$ $(\rho = \sqrt{x^2 + y^2})$, then M satisfies P(1) but the Gauss map is not continuous at infinity.

§3. C^2 Compactification by the inversion

In this section we study the image by the inversion of a submanifold which satisfies P$(2+\varepsilon)$. As in §2, let $M_\lambda = M_\lambda(R_1)$ and $\bar{M}_\lambda = \Psi(M_\lambda)$ for $\lambda = 1, \ldots, q$.

Lemma 3.1. *Suppose M satisfies P$(2+\varepsilon)$ with $\varepsilon > 0$. Then for each $\lambda = 1, \ldots, q$ there exist a constant a_λ and a constant unit vector A_λ in R^N such that $\bar{B}(\cdot, \cdot)$ converges to $2a_\lambda \langle \cdot, \cdot \rangle A_\lambda$ as $\bar{x}$ in $\bar{M}_\lambda$ approaches O.*

Proof. By Lemma 1.4, there exist a constant a_λ and a constant unit vector A_λ for each λ such that $x^\perp$ converges to $a_\lambda A_\lambda$ when x lies in M_λ and $|x|$ tends to ∞. For any tangent vectors X and Y of M it follows from Lemma 2.1 that

$$\bar{B}(\bar{X},\bar{Y}) = r^4 d\Psi(B(X,Y)) + 2\langle X,Y\rangle x^\perp - 4r^{-2}\langle X,Y\rangle \langle x, x^\perp\rangle x \,. \tag{3.1}$$

When $\bar{x}$ approaches O, we have $r\to\infty$ and hence

$$|r^4 d\Psi(B(X,Y))| \le r^2|B| \le Kr^{-\varepsilon} \quad \to 0$$

and

$$|r^{-2}\langle x, x^\perp\rangle x| = r^{-1}|x^\perp|^2 \quad \to 0.$$

Thus $\bar{B}(\bar{X},\bar{Y})$ converges to $2\langle X,Y\rangle a_\lambda A_\lambda = 2\langle \bar{X},\bar{Y}\rangle a_\lambda A_\lambda$ when $\bar{x}$ lies in $\bar{M}_\lambda$ approaches O.

Theorem 2. *Let M be a noncompact, complete, connected C^2 submanifold of dimension $n\ge 2$ properly immersed into R^N. Suppose M satisfies the condition $P(2+\varepsilon)$ for some positive constant ε. Then the image of each end M_λ by the inversion is C^2 at the origin O.*

Proof. Since $\bar{M}_\lambda$ is C^1 at O by Theorem 1, we may express a neighborhood $\bar{U}$ of O in $\bar{M}_\lambda$ as a graph

$$(x_1,\ldots,x_n)\mapsto (x_1,\ldots,x_n, f_{n+1}(x_1,\ldots,x_n),\ldots, f_N(x_1,\ldots,x_n))$$

with $f_\alpha(0,\ldots,0) = 0$ and $\dfrac{\partial f_\alpha}{\partial x_i}(0,\ldots,0) = 0$ for all $i=1,\ldots,n$ and $\alpha = n+1,\ldots,N$. The normal space of $\bar{M}_\lambda$ at O is spanned by $\{E_\alpha : \alpha = n+1,\ldots,N\}$, where $E_\alpha = (\xi_1,\ldots,\xi_N)$ with $\xi_\alpha = 1$ and $\xi_s = 0$ for $s\ne\alpha$. Then we have

$$\lim_{\bar{x}\to O}\frac{\partial^2 f_\alpha}{\partial x_i\partial x_j} = \left\langle \lim_{\bar{x}\to O}\bar{B}\Big(\frac{\partial}{\partial x_i},\frac{\partial}{\partial x_j}\Big), E_\alpha\right\rangle . \tag{3.2}$$

Since $\bar{B}$ is continuous at O by Lemma 3.1, (3.2) shows that all f_α's have continuous second derivatives at O. Hence $\bar{M}_\lambda$ is C^2 at O.

Corollary. *Let M be as in Theorem 2. Then M is conformally equivalent to a compact C^2 Riemannian manifold $\bar{M}$ punctured at a finite number of points.*

Remarks. (1) The origin O is an umbilic point on each $\bar{M}_\lambda$. (cf. Lemma 3.1)

(2) If $\bar{N}$ is a compact C^2 submanifold of R^N containing O, $\Psi(\bar{N})$ satisfies P(2). But if we replace the condition P$(2+\varepsilon)$ in Theorem 2 by P(2), $\Psi(M)$ is not necessarily C^2 at O.

§4. Total curvatures

We mean by a total curvature the total integral of a geometric quantity defined through the second fundamental form of the submanifold. We will define two types of total curvatures. To do this let $\nu(M)$ denote the unit normal bundle of an n-dimensional submanifold M in R^N. Let $G(\xi)$ be the Lipshitz-Killing curvature of M with respect to a unit normal vector ξ, i.e., $G(\xi) = \det A_\xi$. We denote the volume of the k-dimensional unit sphere by c_k. We define $\sigma(M)$ and $\kappa(M)$ by

$$\sigma(M) = \int_M |B|^n$$

$$\kappa(M) = \frac{1}{c_{N-1}} \int_\nu G\,.$$

Proposition 4.1. *Let M be as in Theorem* 1. *Then $\sigma(M) < \infty$.*

Proof. Let K_t and β_t denote the sectional curvature of and the second fundamental form of $\partial M_\lambda(t)$ as a submanifold of $S(t)$, respectively. Since, by Lemma 1.3 (3), there exists a positive continuous function $\eta(t)$ which satisfies $t|\beta_t| \leq \eta(t)$ and $\eta(t) \to 0$ as $t \to \infty$, it follows from the Gauss equation that there exists a positive constant C_7 independent of t such that $K_t \geq C_7 t^{-2}$. By the standard comparison argument, we see that $\mathrm{Vol}(\partial M(t)) \leq C_8 t^{n-1}$ for some constant C_8 independent of t. If R is sufficiently large, we have

$$\begin{aligned}\int_{M_\lambda(R)} |B|^n &= \int_R^\infty \left(\int_{\partial M_\lambda(t)} |B|^n \right) dt \\ &\leq \int_R^\infty (K^n t^{-n(1+\varepsilon)} \mathrm{Vol}(\partial M_\lambda(t)))\, dt \\ &\leq \int_R^\infty K^n C_8 t^{-1-n\varepsilon}\, dt \\ &< \infty\,.\end{aligned}$$

This yields $\sigma(M) < \infty$.

Remark. If M is a complete, connected, minimal submanifold of dimension $n \geq 3$ in R^N with $\sigma(M) < \infty$, then M satisfies P(n) (and hence P$(2+\varepsilon)$). ([1]. See also [4].)

In order to apply results in [7], we imbed R^N into R^{N+1} as an N-dimensional linear subspace. Let p be a unit normal vector of R^N in

R^{N+1}. Let S^N be the unit sphere in R^{N+1}. The stereographic projection $\pi_p : S^N \setminus \{p\} \to R^N$ is given by

$$\pi_p(z) = p + \frac{1}{1 - \langle z, p \rangle}(z - p)$$

for z in $S^N \setminus \{p\}$. The stereographic projections are related to the inversion by $\Psi = \pi_{-p} \circ \pi_p^{-1}$. If M is a submanifold of R^N as in Theorem 2, there exists a compact C^2 manifold $\widetilde{M}$ in S^N such that $\pi_p(\widetilde{M}) = M$ and $\pi_{-p}(\widetilde{M}) = \bar{M}$.

Lemma 4.1 ([7]). *If $n = \dim M$ is even, then*

$$\kappa(M) = \chi(\widetilde{M}) - 2q,$$

where $\chi(\widetilde{M})$ is the Euler characteristic of $\widetilde{M}$ and q is the number of the ends of M. If n is odd, then $\kappa(M) = 0$.

Since $\chi(\widetilde{M}) = \chi(\bar{M}) = \chi(M) + q$, we obtain the following theorem.

Theorem 3. *Let M be as in Theorem 2. If $n = \dim M$ is even, then*

$$\kappa(M) = \chi(M) - q,$$

where $\chi(M)$ is the Euler characteristic of M and q is the number of the ends of M. If n is odd, then $\kappa(M) = 0$.

Corollary. *Let M be as in Theorem 2 and $\dim M = 2$. Then*

$$\int_M K = 2\pi(\chi(M) - q),$$

where K denotes the Gaussian curvature of M.

Proof. Let $\nu(M, x)$ be the unit normal space of M at x. Then we have

$$\int_{\nu(M,x)} G = \frac{c_{N-3}}{N-2} K(x).$$

Hence

$$\begin{aligned}\kappa(M) &= \frac{1}{c_{N-1}} \int_M \left(\int_{\nu(M,x)} G \right) \\ &= \frac{1}{2\pi} \int_M K\,.\end{aligned}$$

Remark. White ([8]) proved that if $\sigma(M)$ is finite for an oriented surface M in R^N, then $\int_M K = 2\pi m$ for some integer m.

Remark. Another popular total curvature is the total mean curvature, which is defined by $\mu(M) = \int_M |H|^n$. Here H denotes the mean curvature vector of M. When $n = 2$, it is known that the total mean curvature is invariant under the inversion if both M and $\Psi(M)$ are compact ([3]). For a surface M satisfying the conditions in Theorem 2, one can show that $\mu(M) = \mu(\bar{M}) - 4\pi q$, where $\bar{M} = \Psi(M)$. If M is minimal and $q = 1$, one has $\mu(\bar{M}) = 4\pi$. Then a theorem by B.Y. Chen ([2]) implies that $\bar{M}$ is a round sphere. Therefore M must be totally geodesic. This is a special case of a theorem in [4], which says that if a complete minimal submanifold M properly immersed into R^N has one end and satisfies P(2) (or P$(1+\varepsilon)$ if $n \geq 3$), then M must be totally geodesic.

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Gauss Maps of Complete Minimal Surfaces

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§1. Introduction

In 1961, R. Osserman showed that the Gauss map of a complete nonflat minimal surface immersed in $\mathbf{R}^3$ cannot omit a set of positive logarithmic capacity ([16]). Afterwards, F. Xavier proved that the Gauss map of such a surface can omit at most six points ([25]). In 1988, the author has shown that the number of exceptional values of the Gauss map of such a surface is at most four ([6]). Here, the number four is best-possible. Indeed, there are many examples of nonflat complete minimal surfaces in $\mathbf{R}^3$ whose Gauss maps omit four values. Moreover, he revealed some relations between these results and the defect relation in Nevanlinna theory on value distribution of meromorphic functions, and gave some modified defect relation for the Gauss map of such a surface in [8]. Recently, as an improvement of these results, X. Mo and R. Osserman showed that, if the Gauss map of a nonflat complete minimal surface M immersed in $\mathbf{R}^3$ takes on five distinct values only a finite number of times, then M has finite total curvature ([14]).

The author gave also modified defect relations for the Gauss map G of a complete minimal surface immersed in $\mathbf{R}^m$ for the case where G is nondegenerate as a map into $P^{m-1}(\mathbf{C})$ and, as its application, he showed that G can omit at most $m(m+1)/2$ hyperplanes in general position ([9]). Here, the number $m(m+1)/2$ is best-possible for arbitrary odd numbers and some small even numbers ([7]). Recently, M. Ru showed that the "nondegenerate" assumption of the above result can be dropped ([20]). In [10], the author introduced a new definition of modified defect and proved a refined modified defect relation for the Gauss map of complete minimal surfaces possibly with branch points and gave some improvements of the above-mentioned results in [9], [14] and [20].

The purpose of this lecture is to survey the above-mentioned results more precisely and to give the outline of their proofs. We first give

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the definition of modified defect and some fundamental properties in §2. We next explain a modified defect relation for a holomorphic map of an open Riemann surface with a complete pseudo-metric into the complex projective space $P^n(\mathbf{C})$ and some consequences of it in §3. The outline of its proof is given in §4. After these expositions, we discuss the value distributions of the Gauss maps of complete minimal surfaces in $\mathbf{R}^m$ in the last two sections.

§2. Modified defect for a holomorphic curve in $P^n(\mathbf{C})$

Let M be an open Riemann surface. We consider a function u on M possibly with singularities in a discrete subset of M.

Definition 2.1. We call u to be a function with *mild singularities* on an open set D in M if u is a C^∞ function on D except a discrete set and around each point $a \in D$ we can write

$$|u| = |z-a|^\sigma |\log|z-a||^\tau u^* \tag{2.2}$$

with a holomorphic local coordinate z, a positive continuous function u^* and real numbers σ and τ.

For a function u with mild singularities on D, we define by

$$\nu_u(a) := \text{the number } \sigma \text{ in the expression (2.2) for some } \tau \text{ and } u^*$$

the divisor $\nu_u : D \to \mathbf{R}$. Here, a divisor on D means a map $\nu : D \to \mathbf{R}$ such that the support $|\nu| := \{z; \nu(z) \neq 0\}$ is discrete. For a nonzero meromorphic function ψ, $\nu_\psi(a)$ is nothing but the order of ψ at a.

Let ν be a divisor on M. We denote by $[\nu]$ the (1, 1)-current corresponding to ν, namely, the map $[\nu] : \mathcal{D} \to \mathbf{C}$ defined by

$$[\nu](\varphi) := \int_M \nu\varphi = \sum_{z\in M} \nu(z)\varphi(z) \qquad (\varphi \in \mathcal{D}),$$

where $\mathcal{D}$ denotes the space of all C^∞ differentiable functions on M with compact supports. In some cases, a $(1,1)$-form Ω on M is regarded as a current on M defined by $\Omega(\varphi) := \int_M \varphi\Omega$ for each $\varphi \in \mathcal{D}$.

For two $(1,1)$-currents Ω_1, Ω_2 and a positive constant c, by $\Omega_1 \prec_c \Omega_2$ we mean that there are a divisor ν and a bounded continuous nonnegative function k with mild singularities such that $\nu \geq c$ on $|\nu|$ and

$$\Omega_1 + [\nu] = \Omega_2 + dd^c \log k^2,$$

where $d^c = \dfrac{\sqrt{-1}}{4\pi}(\bar{\partial} - \partial)$. We write $\Omega_1 \prec \Omega_2$ if $\Omega_1 \prec_c \Omega_2$ for some $c > 0$.

Let $f : M \to P^n(\mathbf{C})$ be a holomorphic map which is nondegenerate, namely, whose image is not included in any hyperplane, and let

$$H : a_0 w_0 + \cdots + a_n w_n = 0$$

be a hyperplane in $P^n(\mathbf{C})$. Take a representation $f = (f_0 : \cdots : f_n)$ on M which is reduced, namely, whose components f_i are holomorphic functions without common zero. Set $F(H) := a_0 f_0 + \cdots + a_n f_n$ and define $\nu(f, H) := \nu_{F(H)}$. The n-truncated pull-back $f^*(H)^{[n]}$ of H as divisor is defined by $f^*(H)^{[n]} := [\min(\nu(f,H), n)]$. We see easily $f^*(H)^{[n]} \prec \Omega_f$, where Ω_f denotes the pull-back of the Fubini-Study metric on $P^n(\mathbf{C})$ by f, namely, $\Omega_f = dd^c \log \|f\|^2$ for $\|f\| := (\sum_{i=0}^n |f_i|^2)^{1/2}$.

Definition 2.3. We define the *modified H-defect* of H for f by

$$D_f(H) := 1 - \inf\{\eta; f^*(H)^{[n]} \prec \eta\, \Omega_f \text{ on } M - K \text{ for a compact set } K\}.$$

For a not necessarily nondegenerate holomorphic map f of M into $P^n(\mathbf{C})$, if $f(M) \subseteq H$, we set $D_f(H) = 0$, and otherwise we define H-defect for f by H-defect for the map f considered as a map into the smallest projective linear subspace of $P^n(\mathbf{C})$ including $f(M)$.

The modified H-defect has the following properties.

Proposition 2.4. (i) $0 \le D_f(H) \le 1$.

(ii) *If there exists a bounded nonzero holomorphic function g on $M - K$ for a compact set K such that $\nu_g \ge \min(\nu(f,H), n)$ on $M - K$, or particularly, if $\# f^{-1}(H) < \infty$, then $D_f(H) = 1$.*

(iii) *If $\nu(f,H) \ge m$ at every $a \in f^{-1}(H) - K$ for some compact set K, then $D_f(H) \ge 1 - n/m$.*

Proof. The assertion (i) is trivial and (ii) is also obvious because

$$f^*(H)^{[n]} + [\nu_g - \min(\nu(f,H), n)] = [\nu_g] = dd^c \log |g|^2$$

on $M - K$ by Poincaré-Lelong formula. Moreover, (iii) is true because

$$f^*(H)^{[n]} + \left[\frac{n}{m}\nu(f,H) - \min(\nu(f,H), n)\right] = \frac{n}{m} dd^c \log \|f\|^2 + dd^c \log k^2$$

on $M - K$ for the bounded function $k := (|F(H)|/\|f\|)^{n/m}$.

We recall the classical defect for a nondegenerate holomorphic map of (an open neighborhood) of $\Delta_{R,\infty} := \{z; R \le |z| < +\infty\}$ into $P^n(\mathbf{C})$.

The order function of f and the counting function (truncated by n) of a hyperplane H for f are defined by

$$T_f(r) = \int_R^r \frac{dt}{t} \int_{R \le |z| < t} \Omega_f \qquad (R < r < +\infty),$$

$$N_f(r)^{[n]} = \int_R^r \frac{dt}{t} \int_{R \le |z| < t} f^*(H)^{[n]} \qquad (R < r < +\infty),$$

respectively. The classical defect (truncated by n) is defined by

$$\delta_f(H)^{[n]} = 1 - \limsup_{r \to \infty} \frac{N_f(r)^{[n]}}{T_f(r)}.$$

We can prove the following relation.

Proposition 2.5. *Let f be a nondegenerate holomorphic map of an open Riemann surface M into $P^n(\mathbf{C})$. Assume that there is a biholomorphic map Φ of an open neighborhood of $\Delta_{R,\infty}$ onto an open set in M such that $\tilde{f} := f \cdot \Phi$ has an essential singularity at ∞. Then, for every hyperplane H*

$$0 \le D_f(H) \le \delta_{\tilde{f}}(H)^{[n]} \le 1.$$

Proof. We take a nonnegative constant η such that

$$f^*(H)^{[n]} + [\nu] = \eta \Omega_f + dd^c \log k^2$$

on $M - K$ for a compact set K, a bounded continuous function k with mild singularities and a divisor ν satisfying the condition that $\nu \ge c$ on $|\nu|$ for some $c > 0$. Then, by the monotonicity of integral, we see

$$N_{\tilde{f}}(r)^{[n]} \le \eta T_{\tilde{f}}(r) + O(\log r) \qquad (R < r < +\infty)$$

and so $1 - \eta \le 1 - N_{\tilde{f}}(r)^{[n]}/T_{\tilde{f}}(r) + O(\log r)/T_{\tilde{f}}(r)$. This concludes the desired inequality.

§3. Modified defect relation

Let $N \ge n$ and $q > 2N - n + 1$ and consider q hyperplanes $H_1, \dots, H_q$ in $P^n(\mathbf{C})$. After W. Chen ([2]), we give the following :

Definition 3.1. We say that $H_1, \ldots, H_q$ are in *N-subgeneral position* if, for every $1 \leq j_0 < \cdots < j_N \leq q$,

$$H_{j_0} \cap \ldots \cap H_{j_N} = \emptyset.$$

In [15], E. I. Nochka has given the following theorem :

Theorem 3.2. *For given hyperplanes $H_1, \ldots, H_q$ in N-subgeneral position, there are some constants $\omega(1), \ldots, \omega(q)$ and θ such that*

(i) $0 < \omega(j) \leq \theta \ (1 \leq j \leq q)$ *and* $\dfrac{n+1}{2N-n+1} \leq \theta \leq \dfrac{n+1}{N+1}$,

(ii) $\displaystyle\sum_{j=1}^{q} \omega(j) = n + 1 + \theta(q - 2N + n - 1)$,

(iii) *if* $R \subset Q$ *and* $0 < \# R \leq N+1$, *then* $\displaystyle\sum_{j \in R} \omega(j) \leq d(R)$.

For the proof, see [2].

Definition 3.3. We call constants $\omega(j)$ and θ with the properties (i) $\sim$ (iii) *Nochka weights* and *a Nochka constant* for H_j's respectively.

By definition, $H_j (1 \leq j \leq q)$ are in general position if and only if they are in n-subgeneral position. If $H_1, \ldots, H_q$ are in general position, then we have necessarily $\omega(1) = \cdots = \omega(q) = \theta = 1$.

We give here the classical defect relation improved by E. I. Nochka.

Theorem 3.4. *Let $f : \Delta_{R,\infty} \to P^n(\mathbf{C})$ be a nondegenerate holomorphic map with an essential singularity at ∞. Then, for arbitrary hyperplanes $H_j (1 \leq j \leq q)$ in N-subgeneral position with Nochka constants $\omega(j)$, it holds that*

$$\sum_{j=1}^{q} \omega(j) \delta_f(H_j)^{[n]} \leq n + 1.$$

For the proof, see [15] or [2].

Definition 3.5. We call ds^2 a *pseudo-metric* on M if, for each holomorphic local coordinate z, it is written as $ds^2 = \lambda_z^2 |dz|^2$ with a nonnegative function λ_z which has mild singularities. A continuous pseudo-metric ds^2 means a pseudo-metric such that λ_z is continuous.

We define the divisor of a pseudo-metric $ds^2 = \lambda_z^2 |dz|^2$ by $\nu_{ds} := \nu_{\lambda_z}$.

For a pseudo-metric $ds^2 = \lambda_z^2|dz|^2$ the Ricci form is defined by

$$\mathrm{Ric}_{ds^2} := -dd^c \log \lambda_z^2$$

as a current, and the Gaussian curvature of ds^2 is given by $K_{ds^2} = \Delta \log \lambda_z/\lambda_z^2$ only on the set $M_1 := \{ds^2 \neq 0\}$, which is called to be strictly negative if $K_{ds^2} \leq -Cds^2$ on M_1 for some $C > 0$. A Riemann surface M whose universal covering is biholomorphic with the unit disc has the unique complete conformal metric with constant curvature -1, which we call Poincaré metric of M and denote by $d\sigma_M^2$ in the following.

To state the modified defect relation, we give two more definitions.

Definition 3.6. We define the *H-order* of f by

$$\rho_f := \inf\{\rho; -\mathrm{Ric}_{ds^2} \prec \rho\Omega_f \text{ on } M - K \text{ for some compact set } K\}.$$

Definition 3.7. Let M be an open Riemann surface of finite type, namely, M is biholomorphic with a compact Riemann surface $\bar{M}$ with finitely many points removed. A holomorphic map f of M into $P^n(\mathbf{C})$ is said to be *transcendental* if f has no holomorphic extension to $\bar{M}$.

The modified defect relation is stated as follows :

Theorem 3.8. *Let M be an open Riemann surface with a complete continuous pseudo-metric ds^2 and f a nondegenerate holomorphic map of M into $P^n(\mathbf{C})$. Take hyperplanes $H_1, \ldots, H_q$ in N-subgeneral position with Nochka constants $\omega(j)$. If M is not of finite type or else f is transcendental, then*

$$\sum_{j=1}^{q} \omega(j) D_f(H_j) \leq n + 1 + \frac{\rho_f n(n+1)}{2}. \tag{3.9}$$

The outline of the proof of Theorem 3.8 will be given in §4. We give here the following corollary to this theorem.

Corollary 3.10. *Let M be an open Riemann surface with a complete pseudo-metric and $f : M \to P^n(\mathbf{C})$ a nondegenerate holomorphic map. If M is not of finite type, then for arbitrary hyperplanes $H_1, \ldots, H_q$ in N-subgeneral position,*

$$\sum_{j=1}^{q} D_f(H_j) \leq (2N - n + 1) + \frac{\rho_f n(2N - n + 1)}{2}. \tag{3.11}$$

Proof. Let A denote the right hand side of (3.11). According to Theorems 3.2 and 3.8, we have

$$A \geq 2N - n + 1 + \frac{\rho_f \sigma_n}{\theta} \geq q + \frac{1}{\theta}\left(n + 1 + \rho_f \sigma_n - \sum_{j=1}^{q} \omega(j)\right)$$

$$\geq q + \sum_{j=1}^{q} \frac{\omega(j)}{\theta}(D_f(H_j) - 1) \geq q + \sum_{j=1}^{q}(D_f(H_j) - 1) = \sum_{j=1}^{q} D_f(H_j),$$

where $\sigma_n := n(n+1)/2$. This gives Corollary 3.10.

§4. The proof of the modified defect relation

In this section, we shall give the outline of the proof of Theorem 3.8. For this purpose, we first give the following theorem.

Theorem 4.1. *Let M be an open Riemann surface with a complete continuous pseudo-metric ds^2 and let $d\tau^2$ be a continuous pseudo-metric on $M - K$ with strictly negative curvature for some compact set K. Assume that there exists a constant p with $0 < p < 1$*

$$-\operatorname{Ric}_{ds^2} \prec_{1-p} p\,(-\operatorname{Ric}_{d\tau^2}) \tag{4.2}$$

on $M - K$. Then M is of finite type.

Proof. Taking a nowhere zero holomorphic 1-form ω, we write $ds^2 = \lambda^2|\omega|^2$ and $d\tau^2 = \eta^2|\omega|^2$. By assumption, we can take a divisor ν and a continuous nonnegative bounded function k on $M - K$ with mild singularities such that $\nu(z) \geq 1 - p$ for every $z \in |\nu|$ and

$$dd^c \log \lambda^2 + [\nu] = p\, dd^c \log \eta^2 + dd^c \log k^2.$$

Here, we may assume that ν and k are defined on M and $0 \leq k \leq 1$. Set $u := k\eta^p/\lambda$. Then, $\log u$ is harmonic outside $K \cup |\nu|$, $\nu_u \geq 1 - p$ on $|\nu| - K$ and $\lambda = k\eta^p/u \leq \eta^p/u$. Define a new pseudo-metric

$$d\rho^2 := u^{-2/(1-p)}|\omega|^2$$

on M and set $M_1 := \{a \in M; \nu_{d\rho}(a) = 0\}$. Then, $d\rho^2$ is a flat metric on $M_1 - K$ and, since $\nu_u \geq 1 - p$ on $|\nu_u| - K$, $\nu_{d\rho} \leq -1$ on $M - (K \cup M_1)$.

We recall here the following theorem of A. Huber ([13]).

Theorem 4.3. *For an open Riemann surface M, if there is a complete metric $d\rho^2$ on M such that*

$$\int_M \max(-K_{d\rho^2}, 0)\Omega_{d\rho^2} < +\infty,$$

then M is of finite type, where $\Omega_{d\rho^2}$ denotes the area form associated with $d\rho^2$.

To prove Theorem 4.1, it suffices to show the following fact.

(4.4). *The surface M_1 is complete with respect to the metric $d\rho^2$.*

In fact, by the aid of Theorem 4.3 we can conclude from (4.4) that M_1, and so M, are of finite type because

$$\int_{M_1} \max(-K_{d\rho^2}, 0)\Omega_{d\rho^2} = \int_K \max(-K_{d\rho^2}, 0)\Omega_{d\rho^2} < \infty.$$

Assume that M_1 is not complete, and so $d_0 := \mathrm{dist}_{d\rho}(K, \partial M_1)$ is finite, where $\mathrm{dist}_{d\rho}(K, \partial M_1)$ denotes the distance between K and ∂M_1. Then we can find a continuous curve $\gamma_0(t)\,(0 \le t < 1)$ such that $\gamma_0(0) \in K$, $\gamma_0(t)$ tends to ∂M_1 as $t \to 1$ and the length $L_{d\rho}(\gamma_0)$ of γ_0 is smaller than $2d_0$ and take a point $p_0 := \gamma_0(t_0)\,(0 \le t_0 < 1)$ such that $\mathrm{dist}_{d\rho}(K, p_0) > d_0/2$ and $L_{d\rho}(\gamma_0|[t_0, 1)) < d_0/2$, where $\gamma|[\alpha, \beta)$ denotes the part of γ from $t = \alpha$ to $t = \beta$.

Since $d\rho^2$ is flat on $M_1 - K$, there is an isometry Φ of a disc $\Delta_R := \{w \in \mathbf{C}; |w| < R\}$ with the standard metric onto an open neighborhood of p_0 in $M_1 - K$ with the metric $d\rho^2$ such that $\Phi(0) = p_0$. Take the largest $R(\le +\infty)$ such that there is a local isometry Φ of Δ_R onto some open set in $M_1 - K$ with $\Phi(0) = p_0$. Then, $R \le L_{d\rho}(\gamma_0|[t_0, 1)) < d_0/2$ and there is a line segment Γ joining the origin and a point in $\partial\Delta_R$ such that $\gamma := \Phi(\Gamma)$ tends to the boundary of $M_1 - K$. Then, if γ tends to K or to the set $M - (K \cup M_1)$, then we have an absurd conclusion $R \ge \mathrm{dist}_{d\rho}(K, p_0) > d_0/2$ or $R = L_{d\rho}(\gamma) = +\infty$ respectively, because $\nu_{d\rho} \le -1$ on $M - (K \cup M_1)$. Therefore, γ tends to the boundary of M.

Now, we shall estimate the length $L_{ds}(\gamma)$ of γ. To this end, we set $\tilde{\eta} := d\tau/d\rho$. Then, we have $\eta = \tilde{\eta} u^{-1/(1-p)}$. So,

$$L_{ds}(\gamma) \le \int_\gamma u^{-1}\eta^p|\omega| \le \int_\gamma u^{-1}\tilde{\eta}^p u^{-\frac{p}{1-p}}|\omega|$$
$$\le \int_\gamma \tilde{\eta}^p d\rho = \int_\Gamma (\tilde{\eta}\cdot\Phi)^p \Phi^*(d\rho) = \int_\Gamma (\tilde{\eta}\cdot\Phi)^p |dw|.$$

On the other hand, the curvature of $\Phi^*(d\tau)$ is strictly negative on Δ_R by assumption. By the generalized Schwarz lemma we obtain

$$\Phi^*(d\tau) = (\tilde{\eta}\cdot\Phi)|dw| \le C_0\, d\sigma_{\Delta_R} \le C_1 \frac{R}{R^2 - |w|^2}|dw|$$

for some constants C_0 and C_1. Therefore, we have

$$L_{ds}(\gamma) \leq C_2 \int_\Gamma \left(\frac{R}{R^2 - |w|^2}\right)^p |dw| \leq \frac{C_3}{1-p} R^{1-p} < \infty$$

for some constants C_2 and C_3. This contradicts the completeness of M with respect to ds^2. Thus, we conclude (4.4) and so Theorem 4.1.

Now, we start to prove Theorem 3.8. To this end, we may assume $\rho_f < +\infty$ and M is not of finite type because Theorem 3.8 is obvious from Proposition 2.5 and Theorem 3.4 for the other cases.

Take arbitrary constants $\rho > 0$ and η_j $(1 \leq j \leq q)$ such that

$$-\mathrm{Ric}_{ds^2} \prec \rho\, \Omega_f, \qquad f^*(H_j)^{[n]} \prec \eta_j\, \Omega_f \tag{4.5}$$

on $M' := M - K$ for a compact set K. By definition, there are divisors ν_j and bounded continuous nonnegative functions k_j with mild singularities such that $\nu_j \geq c_j$ on $|\nu_j|$ for some $c_j > 0$ and

$$f^*(H_j)^{[n]} + [\nu_j] = \eta_j\, \Omega_f + dd^c \log k_j^2$$

on M'. Set $h_j := k_j \|f\|^{\eta_j}$. Then, $\log h_j$ are harmonic on $M' - |\nu_{h_j}|$.

For our purpose, we have only to show that

$$\gamma := \theta(q - 2N + n - 1) - \sum_{j=1}^{q} \omega(j)\eta_j \leq \rho\sigma_n. \tag{4.6}$$

In fact, if this is true, then we easily obtain (3.9) from the definitions of the modified defect and ρ_f because (4.6) can be rewritten

$$\sum_{j=1}^{q} \omega(j)(1 - \eta_j) \leq n + 1 + \rho\sigma_n$$

by the use of Theorem 3.2, (ii).

Assume that $\gamma > \rho\sigma_n$. We shall show that there exists a pseudo-metric on M' with strictly negative curvature which satisfies (4.2) for a suitable constant p with $0 < p < 1$, which leads to a contradiction by Theorem 4.1 and concludes (4.6). To this end, we represent each H_j as

$$H_j : a_{j0}w_0 + \cdots + a_{jn}w_n = 0 \qquad (1 \leq j \leq q).$$

Take a reduced representation $f = (f_0 : \cdots : f_n)$ and set $F(H_j) := a_{j0}f_0 + \cdots + a_{jn}f_n$. Moreover, for an arbitrarily fixed holomorphic local coordinate z we consider the functions

$$|F_k| := \left(\sum_{0 \le j_0 < \cdots < j_k \le n} |W(f_{j_0}, \ldots, f_{j_k})|^2 \right)^{1/2},$$

$$|F_k(H_j)| := \left(\sum_{0 \le i_1 < \cdots < i_k \le n} \left| \sum_{j \ne i_1, \ldots, i_k} a_j W(f_j, f_{i_1}, \ldots, f_{i_k}) \right|^2 \right)^{1/2},$$

$$\varphi_k(H_j) := \frac{|F_k(H_j)|^2}{|F_k|^2},$$

where $W(g_0, \ldots, g_k)$ denotes the Wronskian of holomorphic functions $g_0, \ldots, g_k$. Now, choosing some ε with $\gamma > \varepsilon\sigma_{n+1}$, we set

$$\eta_z := \left(\frac{\|f\|^{\gamma - \varepsilon\sigma_{n+1}} |F_n| \prod_{j=1}^q |h_j|^{\omega(j)} \prod_{k=0}^n |F_k|^\epsilon}{\prod_{j=1}^q (|F(H_j)| \prod_{k=0}^{n-1} \log(a/\varphi_k(H_j)))^{\omega(j)}} \right)^{\frac{1}{\sigma_n + \epsilon\tau_n}}$$

and define the pseudo-metric $d\tau^2 := \eta_z^2 |dz|^2$, which is well-defined on $M - K$. Set $\varphi := |F_n| / \prod_{j=1}^q |F(H_j)|^{\omega(j)}$. Then we can prove that $\nu_\varphi + \sum_{j=1}^q \omega(j) \min(\nu(f, H_j), n) \ge 0$ (cf., [10, §2]). This implies that $\nu_0 \ge c'$ on $|\nu_0|$ for some $c' > 0$ and $d\tau^2$ is a continuous pseudo-metric on M'. Moreover, we can prove that $d\tau^2$ has strictly negative curvature on M'(cf., [10, §5]).

For some holomorphic local coordinate z and each pair of indices j, k, we choose indices $i_1, \ldots, i_k$ with $1 \le i_1 < \cdots < i_k \le q$ such that

$$\psi_{jk}^z := \sum_{\ell \ne i_1, \ldots, i_k} a_{j\ell} W(f_\ell, f_{i_1}, \ldots, f_{i_k}) \not\equiv 0.$$

For convenience' sake, we set $\psi_{j0}^z = F(H_j)$. By the theorem of identity, $\psi_{jk}^z \not\equiv 0$ for every holomorphic local coordinate z. We now define

$$k := \left(\frac{\prod_{1 \le j \le q, 0 \le k \le n-1} |\psi_{jk}^z|^{\epsilon/q} \log^{\omega(j)}(a/\varphi_k(H_j))}{\prod_{0 \le k \le n-1} |F_k|^\epsilon} \right)^{1/(\sigma_n + \epsilon\tau_n)}.$$

Then, k is bounded because

$$\frac{|\psi_{jk}^z|^{\epsilon/q} \log^{\omega(j)}(a/\varphi_k(H_j))}{|F_k|^{\epsilon/q}} \le \left(\frac{|F_k(H_j)|}{|F_k|} \right)^{\epsilon/q} \log^{\omega(j)}(a/\varphi_k(H_j))$$
$$\le \sup_{0 < x \le 1} x^{\epsilon/q} \log^{\omega(j)}\left(\frac{a}{x}\right) < +\infty.$$

Set $v := |\varphi| \prod_{j=1}^{q} |h_j|^{\omega(j)} \prod_{1\le j\le q, 0\le k\le n-1} |\psi_{jk}^z|^{\epsilon/q}$. Then $\log v$ is harmonic outside $|\nu_v|$. If we choose a constant ε which is smaller than and sufficiently near to the number $(\gamma - \rho\sigma_n)/(\sigma_{n+1} + \rho\tau_n)$ and set $p := \rho(\sigma_n + \varepsilon\tau_n)/(\gamma - \varepsilon\sigma_{n+1})$, it holds that $-\mathrm{Ric}_{ds^2} \prec_{1-p} \rho\,\Omega_f$, $0 < p < 1$ and $\nu_v \geq (\gamma - \varepsilon\sigma_{n+1})(1-p)/\rho$. on $|\nu_v|$. Moreover, we see easily

$$\rho\,\Omega_f + \left[\frac{\rho}{\gamma - \varepsilon\sigma_{n+1}}\nu_v\right] = p\, dd^c \log \eta_z^2 + dd^c \log k^{2p}.$$

This shows that (4.2) holds on M'. Thus, we have proved Theorem 3.8.

§5. Gauss maps of complete minimal surfaces in $\mathbf{R}^m$

Let $x : M \to \mathbf{R}^m$ be a (possibly branched) minimal surface. By defintion, M is an open Riemann surface, $x = (x_1, \ldots, x_m)$ is a nonconstant map whose components x_i are harmonic and satisfy the condition

$$f_1^2 + \cdots + f_m^2 = 0$$

for holomorphic functions $f_i := \partial x_i/\partial z$ locally defined with a holomorphic local coordinate z on M. The pseudo-metric ds^2 on M induced from $\mathbf{R}^m$ are locally written as $ds^2 = 2\|f\|^2|dz|^2$ (cf., [18]). The set S of all branch points coincides with the set of common zeros of the functions $f_i (1 \le i \le m)$ and we have $\nu_{ds} = \min\{\nu_{f_i}; 1 \le i \le m\}$.

As is well-known, the set of all oriented 2-planes in $\mathbf{R}^m$ may be identified with the quadric $Q_{m-2}(\mathbf{C})$ in $P^{m-1}(\mathbf{C})$. By definition, the Gauss map G of M maps each $p \in M - S$ to the point in $Q_{m-2}(\mathbf{C})$ corresponding to the oriented tangent plane of M at p and it is locally given by $G = (f_0 : \cdots : f_n)$ on $M - S$. Take a nonzero holomorphic function h on M with $\nu_h = \nu_{ds}$. If we set $g_i := f_i/h (1 \le i \le m)$, we have $G = (g_1 : g_2 : \cdots : g_m)$ outside S, which is holomorphically extended across S. So, we may consider the Gauss map G as a holomorphic map of M into $P^{m-1}(\mathbf{C})$.

A surface with a pseudo-metric is called to be flat if the Gaussian curvature identically vanishes. It is easily seen that a minimal surface is flat if and only if the Gauss map is a constant.

Definition 5.1. The *total curvature* of M is defined by $-\int_M \mathrm{Ric}_{ds^2}$.

Proposition 5.2. *A complete minimal surface $x : M \to \mathbf{R}^m$ has finite total curvature if and only if M is of finite type and the Gauss map of M is not transcendental.*

For the proof, see [3].

Definition 5.3. We define the *branching H-order* of M by

$$\rho_{ds} := \inf\{\rho; [\nu_{ds}] \prec \rho\,\Omega_G \text{ on } M - K \text{ for some compact set } K\}.$$

Obviously, if $x : M \to \mathbf{R}^m$ is an immersion, then $\rho_{ds} = 0$.

Theorem 5.4. *Let $x : M \to \mathbf{R}^m$ be a complete nonflat minimal surface with infinite total curvature and $G : M \to P^N(\mathbf{C})$ the Gauss map of M, where $N = m - 1$. Consider the smallest linear subspace $P^n(\mathbf{C})$ of $P^N(\mathbf{C})$ which includes $G(M)$. Then, for arbitrary hyperplanes $H_1, \dots, H_q (1 \le j \le q)$ in $P^N(\mathbf{C})$ located in general position,*

$$\sum_{j=1}^{q} D_G(H_j) \le 2N - n + 1 + \frac{(1 + \rho_{ds})n(2N - n + 1)}{2}.$$

Proof. By assumption, the Gauss map G is nondegenerate as the map into $P^n(\mathbf{C})$. On the other hand, the metric of M is given by $ds^2 = 2\|f\|^2|dz|^2 = 2|h|^2\|g\|^2|dz|^2$ for a holomorphic function h and a reduced representation $g = (g_1 : \cdots : g_m)$. For each $\rho \ge 0$ such that $[\nu_{ds}] \prec \rho\,\Omega_G$ on M outside a compact set K, we have $-\operatorname{Ric}_{ds^2} \prec (\rho + 1)\Omega_G$ on $K - M$. Taking the infimum of the right hand side for various ρ, we obtain $\rho_G \le \rho_{ds} + 1$. Since $H_1, \dots, H_q$ considered as hyperplanes in $P^n(\mathbf{C})$ are located in N-subgeneral position, Theorem 5.4 is now an immediate consequence of Corollary 3.10.

Theorem 5.5. *Let G be the Gauss map of a nonflat complete minimal surface immersed in $\mathbf{R}^m$ with infinite total curvature. Then, for arbitrary hyperplanes $H_1, \dots, H_q$ in $P^{m-1}(\mathbf{C})$ located in general position,*

$$\sum_{j=1}^{q} D_G(H_j) \le \frac{m(m+1)}{2}.$$

Proof. By assumption, Theorem 5.4 is valid for some n with $1 \le n \le N$. Therefore, we have

$$\sum_{j=1}^{q} D_G(H_j) \le 2N - n + 1 + \frac{n(2N - n + 1)}{2}$$
$$= \frac{(N+1)(N+2) - (N-n)(N-n-1)}{2} \le \frac{m(m+1)}{2}.$$

This gives Theorem 5.5.

In view of Proposition 2.4, Theorem 5.5 yields the following :

Corollary 5.6. *Let M be a nonflat complete minimal surface immersed in $\mathbf{R}^m$ with infinite total curvature, and let G be the Gauss map of M. If $G^{-1}(H_j)$ are finite for q hyperplanes $H_1, \dots, H_q$ in $P^{m-1}(\mathbf{C})$ located in general position, then $q \leq m(m+1)/2$.*

We have also the following result by Ru ([20]).

Corollary 5.7. *The Gauss map of a nonflat complete minimal surface immersed in $\mathbf{R}^m$ can omit at most $m(m+1)/2$ hyperplanes in general position.*

Proof. If M has infinite total curvature, then this is a direct result of Corollary 5.6. Otherwise, take the smallest projective linear subspace $P^n(\mathbf{C})$ of $P^{m-1}(\mathbf{C})$. If given hyperplanes are in general position in $P^n(\mathbf{C})$, then by the result of Chern and Osserman ([3]) G can omit at most $n(n+3)/2(< m(m+1)/2)$ hyperplanes in general position. By the use of Theorem 3.2, the arguments in [3] is available for the case where given hyperplanes are in general position in $P^{m-1}(\mathbf{C})$(cf., [21]).

Here, the number $m(m+1)/2$ is best-possible for an arbitrary odd numbers and some small even numbers m. In fact, we can construct some complete minimal surfaces in $\mathbf{R}^m$ whose Gauss maps are non-degenerate and omit $m(m+1)/2$ hyperplanes in general position for such numbers. For the details, see [7].

Now, we consider a holomorphic curve in $\mathbf{C}^m$ given by a nonconstant holomorphic map $w = (w_1, w_2, \dots, w_m) : M \to \mathbf{C}^m$. The space $\mathbf{C}^m$ is identified with $\mathbf{R}^{2m}$ by associating $(x_1+\sqrt{-1}y_1, \dots, x_m+\sqrt{-1}y_m) \in \mathbf{C}^m$ with $(x_1, y_1, \dots, x_m, y_m)$. The curve $w : M \to \mathbf{C}^m$ is considered as a minimal surface $w = (x_1, y_1, \dots, x_m, y_m) : M \to \mathbf{R}^{2m}$. By Cauchy-Riemann's equations, $f_i := \partial x_i/\partial z = \sqrt{-1}\partial y_i/\partial z (1 \leq i \leq m)$. So, the Gauss map of M is given by $G = (f_1 : -\sqrt{-1}f_1 : \cdots : f_m : -\sqrt{-1}f_m)$, and therefore the image $G(M)$ of G is included in the projective subspace

$$P^{m-1}(\mathbf{C}) := \{(u_1 : v_1 : \cdots : u_m : v_m); u_i = -\sqrt{-1}v_i (1 \leq i \leq m)\}$$

of $P^{2m-1}(\mathbf{C})$. As a consequence of Theorem 3.8, we have the following :

Corollary 5.8. *Let $w : M \to \mathbf{C}^m$ be a holomorphic curve in $\mathbf{C}^m$ which is complete and not included in any affine hyperplane, and let G be* *the Gauss map of M considered as a map of M into the above-mentioned space $P^{m-1}(\mathbf{C})$. If M is not of finite type, then*

$$\sum_{j=1}^{q} D_G(H_j) \leq m + \frac{(\rho_{ds}+1)m(m-1)}{2}$$

for arbitrary hyperplanes $H_1, \ldots, H_q$ in $P^{m-1}(\mathbf{C})$ in general position.

For the proof, see [10, §6].

§6. The Gauss maps of minimal surfaces in $\mathbf{R}^3$ or $\mathbf{R}^4$

We next consider a minimal surface $x = (x_1, x_2, x_3) : M \to \mathbf{R}^3$. In this case, the quadric $Q_1(\mathbf{C})$ is canonically biholomorphic with $P^1(\mathbf{C})$. Instead of the Gauss map $G : M \to Q_1(\mathbf{C})$ we may study the classical Gauss map $g : M \to P^1(\mathbf{C})$ defined by $g = (f_3 : f_1 - \sqrt{-1}f_2)$, where $f_i := \partial x_i/\partial z (i = 1, 2, 3)$. Then, the metric of M is given by $ds^2 = |h|^2(|g_0|^2 + |g_1|^2)^2|dz|^2$ for a reduced representation $g = (g_0 : g_1)$ and a nonzero holomorphic function h with $\nu_h = \min(\nu_{f_1}, \nu_{f_2}, \nu_{f_3})$. Since $\nu_{ds} = \nu_h$, we have $-\operatorname{Ric}_{ds} \leq \rho + 2$ whenever $[\nu_{ds}] \prec \rho\Omega_g$. This yields $\rho_g \leq \rho_{ds} + 2$. From Theorem 3.8, we can easily conclude the following :

Theorem 6.1. *Let $x : M \to \mathbf{R}^3$ be a nonflat complete minimal surface with infinite total curvature and let g be the classical Gauss map. Then, for arbitrary distinct points $\alpha_1, \ldots, \alpha_q$ in $P^1(\mathbf{C})$,*

$$\sum_{j=1}^{q} D_g(\alpha_j) \leq 4 + \rho_{ds}.$$

Here, we can construct an example of a nonflat complete minimal surface in $\mathbf{R}^3$ with $\rho_{ds} = 2$ whose Gauss map omit six distinct points in $P^1(\mathbf{C})$ (cf., [10, §6]).

Relating to Theorem 6.1, we can prove the following theorem for noncomplete minimal surfaces in $\mathbf{R}^3$.

Theorem 6.2. *Let $x : M \to \mathbf{R}^3$ be a nonflat minimal surface and $g : M \to P^1(\mathbf{C})$ the classical Gauss map. If there exist distinct points $\alpha_1, \ldots, \alpha_q \in P^1(\mathbf{C})$ and positive integers $m_1, \ldots, m_q$ satisfying the condition that each $g - \alpha_j$ has no zeros with multiplicity $< m_j$ and*

$$\sum_{j=1}^{q} \left(1 - \frac{1}{m_j}\right) > 4,$$

then there is a constant $C > 0$ depending only on α_j and m_j such that

$$|K(p)| \leq \frac{C}{\mathrm{dist}_{ds^2}(p, \partial M)^2} \qquad (p \in M).$$

For the proof, see [6] and [19].

We next consider a complete minimal surface $x : M \to \mathbf{R}^4$. In this case, the Gauss map G of M is a map into $Q_2(\mathbf{C})$, which is canonically identified with $P^1(\mathbf{C}) \times P^1(\mathbf{C})$. Instead of the Gauss map $G : M \to Q_2(\mathbf{C})(\subset P^3(\mathbf{C}))$ we consider the map $g : M \to P^1(\mathbf{C}) \times P^1(\mathbf{C})$, which we call the classical Gauss map of M.

We can prove the following defect relation.

Theorem 6.3. *Let $x : M \to \mathbf{R}^4$ be a complete minimal surface not of finite type and $g = (g_1, g_2) : M \to P^1(\mathbf{C}) \times P^1(\mathbf{C})$ the classical Gauss map of M. Take two systems of distinct points $\{\alpha_1, ..., \alpha_{q_1}\}$ and $\{\beta_1, ..., \beta_{q_2}\}$.*

(i) *If g_1 and g_2 are nonconstant and, moreover, $\sum_{i=1}^{q_1} D_{g_1}(\alpha_i) > 2$ and $\sum_{j=1}^{q_2} D_{g_2}(\beta_j) > 2$, then*

$$\frac{1}{\sum_{i=1}^{q_1} D_{g_1}^H(\alpha_i) - 2} + \frac{1}{\sum_{j=1}^{q_2} D_{g_2}^H(\beta_j) - 2} \geq 1.$$

(ii) *If g_1 is nonconstant and g_2 is a constant, then*

$$\sum_{j=1}^{q_1} D_{g_1}(\alpha_j) \leq 3.$$

The proof is omitted. For the details, see [8].

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Applications of Jacobi and Riccati Equations along Flows to Riemannian Geometry

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Abstract.

In the present paper we show a model for geodesic flows on the unit tangent bundles of complete Riemannian manifolds. By treating it as in the study of manifolds without conjugate points we have two theorems of the same type as E. Hopf and L. Green proved. One is for spaces of constant curvature instead of flat manifolds. The other is for differentiable flows without conjugate points, and in particular, gradient flows. In addition, we give the formula of the same type as R. Ossermann and P. Sarnak did. As its application, we get the simpler proof of the extension due to W. Ballmann and W. P. Wojtkowski.

§1. Introduction

Let N be a manifold with volume form ω and let $f^t : N \longrightarrow N$ be a (complete) flow preserving ω. Let $\pi : E \longrightarrow N$ be a vector bundle over N with inner product $\langle \cdot, \cdot \rangle$. Let $L(E) = \{D | D(p) : E_p \longrightarrow E_p$ is a linear map for each $p \in N\}$ and $S(E) = \{A | A(p) : E_p \longrightarrow E_p$ is a symmetric linear map for each $p \in N\}$. We assume that there is a connection ∇ along the flow f^t such that

$$X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$$

for any sections Y, Z on N into E where X is the vector field on N generating the flow f^t. For $A \in S(E)$ we consider the E-valued differential equation of Jacobi type along the flow.

$$(J_A) \qquad \nabla_X \nabla_X Y + AY = 0.$$

And also, the $L(E)$-valued differential equation of Riccati type.

$$(R_A) \qquad \nabla_X U + U^2 + A = 0.$$

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We say that (J_A) is *disconjugate on* N if any nontrivial E-valued solution of (J_A) has at most one zero on each trajectory $\{f^t p| -\infty < t < \infty\}$, or equivalently (R_A) has a symmetric solution defined on each trajectory $\{f^t p| -\infty < t < \infty\}$.

To state the Theorems we need some definitions. We say that a point $p \in N$ is *nonwandering* if there exist sequences $\{p_n\} \subset N$ and $\{t_n\} \subset \mathbf{R}$ such that $t_n \longrightarrow \infty$, $p_n \longrightarrow p$ and $f^{t_n} p_n \longrightarrow p$ as $n \longrightarrow \infty$. We denote by Ω_0 the set of all nonwandering points such that their trajectories are not bounded in N with respect to a complete Riemannian metric on N and by Ω the set of all nonwandering points. A function $F : N \longrightarrow \mathbf{R}$ is by definition *summable* if $|F|$ is integrable over N.

Theorem A. *Suppose $A \in S(E)$ and the equation (J_A) is disconjugate on N. Suppose Ω_0 is decomposed into at most countably many f^t-invariant sets with finite measure. If the trace of A is summable on N, then*

$$\int_N \operatorname{tr} A \, d\omega \leq 0$$

and equality holds only if $A = 0$ identically.

Theorem A can be proven by almost the same way as in [In4]. The examples and applications are seen in Subsections 1.1 and 1.2.

In the following theorem let U denote a symmetric solution of (R_A) defined on N, i.e., $U(f^t p)$ satisfies the equation (R_A) along each trajectory $\{f^t p| -\infty < t < \infty\}$.

Theorem B. *Suppose $A \in S(E)$ and $A(p) \leq 0$ for all $p \in N$. If N is compact, then*

$$-\int_N \operatorname{tr} U \, d\omega \geq \int_N \operatorname{tr} \sqrt{-A} \, d\omega$$

and equality holds only if A is parallel along the flow f^t, i.e., $\nabla_X A = 0$ identically.

The application of Theorem B can be seen in Subsection 1.3.

1.1. Geodesic flows

Let M be a complete Riemannian manifold with Riemannian metric $\langle \cdot, \cdot \rangle^0$ and let $N = SM$ be the unit tangent bundle with volume form $\omega = \sigma \wedge \theta$ where σ is the volume form induced from the Riemannian metric of M and θ is the canonical volume form of the unit sphere S^{n-1}, $n = \dim M$. Let $f^t : N \longrightarrow N$ be the geodesic flow, namely $f^t v = \dot{\gamma}_v(t)$

for any $t \in \mathbf{R}$ where $v \in N$ and $\gamma_v : \mathbf{R} \longrightarrow M$ is the geodesic with $\dot{\gamma}_v(0) = v$. Then, f^t preserves ω. Let $E = \cup_{v \in N} v^{\perp}$ where $v^{\perp}$ is the subspace of $T_{\pi(v)}M$ orthogonal to v with respect to the Riemannian metric of M. The inner product $\langle \cdot, \cdot \rangle$ on E and connection ∇ along the geodesic flow is defined as follows:

Let $Y, Z \in E_v = v^{\perp}, v \in N$. Then, $\langle Y, Z \rangle = \langle \bar{Y}, \bar{Z} \rangle^0(\pi(v))$ where we think as $\bar{Y} = Y, \bar{Z} = Z \in v^{\perp} \subset T_{\pi(v)}M$. Let X be the vector field on N generating the geodesic flow f^t, i.e., $X(v) = \left.\frac{df^t v}{dt}\right|_{t=0}$ for any $v \in N$. Define ∇ by $\nabla_{X(v)}Y := \nabla_v \bar{Y}$ for any E-valued vector field Y along $f^t v$. Then, ∇ satisfies the condition of compatibility.

Let R be the Riemannian curvature tensor given by

$$R(Y, Z)W = \nabla_Y \nabla_Z W - \nabla_Z \nabla_Y W - \nabla_{[Y,Z]} W$$

for any vector fields Y, Z, W on M. We put $A(v) = R(\cdot, v)v$ for any $v \in N$. Then, we have $A \in S(E)$. In this situation, E. Hopf ([Ho]) and L. Green ([Gr]) proved that if M is compact and without conjugate points, then the integral of $\operatorname{tr} A$ on N is nonpositive, and it vanishes only if M is flat. Here we note that the integral of $\operatorname{tr} A$ over N is that of the scalar curvature S of M over M with constant multiple. This theorem is extended by N. Innami ([In4]). However, Theorem A shows that this integral inequality is not Riemannian. Combined with Schur's lemma we have the following.

Theorem C. *Suppose Ω_0 is decomposed into at most countably many f^t-invariant sets with finite measure. If there is a function F on M such that $(J_{A-(F\circ\pi)I})$ is disconjugate on N and* $\operatorname{tr}(A - (F \circ \pi)I)$ *is summable over N, then*

$$\int_M (S - n(n-1)F)\, d\sigma \leq 0.$$

Equality holds only if $A = (F \circ \pi)I$, and in particular, M is a space of constant curvature if $\dim M \geq 3$.

It should be noted that a space of positive constant curvature is compact ([CE]). In Section 4 we discuss the assumption that Ω_0 has finite measure. There we have the following.

(1) M has finite volume. In this case $\operatorname{vol}(\Omega) = \operatorname{vol}(N)$.
(2) M is simply connected and without conjugate points. In this case $\Omega = \phi$.
(3) M has nonnegative sectional curvature outside a compact set. In this case there exists an exhaustion Lipschitz continuous convex

function B on M, and $\mathrm{vol}(\Omega)$ is less than or equal to the measure of the unit tangent bundle of the minimum level set of B.

(4) There exists an increasing sequence $\{C_i\}$ of compact totally convex sets such that $\cup_{i=1}^{\infty} C_i = M$ and any sequence $\{p_i\}, p_i \in C_{m_{i+1}} - C_{m_i}$, has no accumulation point. In this case $\Omega_0 = \phi$. We find such surfaces in Section 4.

Here we say that $C \subset M$ is *totally convex* in M if any geodesic segment with endpoints of both sides in C is contained in C ([CG]). It is noted that if we want the goal in the equality case of Theorem C to be symmetric spaces of rank one, we have only to use a symmetric linear map that comes from a suitable curvature tensor satisfying the second Bianchi identity as A does, instead of using the identity map in $A - (F \circ \pi)I$ (see [In6], Lemma).

1.2. Gradient flows

Let N be a manifold and let $f^t : N \longrightarrow N$ be a flow which is generated by the vector field X on N. Assume that f^t preserves a volume form ω on N. Let $\pi : E = TN \longrightarrow N$ be the tangent bundle over N. Let $\langle \cdot, \cdot \rangle$ be a complete Riemannian metric on N and $P(t,p)$ the parallel translation from p to $f^t p$ along the curve $c_p : [0,t] \longrightarrow N$ with $c_p(s) = f^s p$ by Riemannian connection ∇. Define $D(t,p) = df_p^t \circ P(t,p)^{-1}$ for all $t \in \mathbf{R}$ and $p \in N$, and $D(t,p)$ is a $(1,1)$-tensor along c_p for each $p \in N$. Then, $D(t,p)$ are linear isomorphisms of $T_{f^t p}N = E_{f^t p}$ for all $t \in \mathbf{R}$ and $p \in N$. We define $(1,1)$-tensors U and A on N by

$$U(p) = \nabla_X D(0,p) \left(= \frac{d}{dt} D(t,p) \bigg|_{t=0} \right)$$
$$A(p) = -\nabla_X \nabla_X D(0,p)$$

for each $p \in N$. The $D(t,p)$ satisfies the following differential equations:

$$(L_U) \qquad \nabla_X D(t,p) = U(f^t p) D(t,p),$$

$$(J_A) \qquad \nabla_X \nabla_X D(t,p) + A(f^t p) D(t,p) = 0,$$

$$(R_A) \qquad \nabla_X U(f^t p) + U(f^t p)^2 + A(f^t p) = 0,$$

If U is symmetric on N, then there exists a function $F : \widetilde{N} \longrightarrow \mathbf{R}$ such that $\mathrm{grad} F = \widetilde{X}$ where $\widetilde{N}$ is the universal covering space of N and $\widetilde{X}$ is the lift of X to $\widetilde{N}$ (see Section 5). If $\langle X, X \rangle$ is constant on N in addition to symmetry of U, then the trajectories of the flow f^t are geodesics and A is therefore the curvature tensor of N (Section 5). We say that the flow f^t is *disconjugate on* N if the differential equation (J_A) on all trajectories $\{f^t p \mid -\infty < t < \infty\}$ are disconjugate. It is known that f^t is disconjugate on N if U is symmetric on N ([Ha], Theorem 10.2 and [In5]). As an application of Theorem A we have the following.

Theorem D. *Suppose Ω_0 is decomposed into at most countably many f^t-invariant sets with finite measure. If U is symmetric on N and the trace of A is summable over N, then*

$$\int_N \operatorname{tr} A \, d\omega \leq 0.$$

Equality holds only if either f^t are the identity map of N for all $t \in \mathbf{R}$, or N and f^t are such that

(1) *the universal covering space $\widetilde{N}$ of N is isometric to the Riemannian product $M \times \mathbf{R}$,*
(2) *there exists a constant a such that the lift $\tilde{f}^t$ of f^t to $\widetilde{N}$ is given by $\tilde{f}^t(p, s) = (p, s + at)$ for any $(p, s) \in M \times \mathbf{R}$ and $t \in \mathbf{R}$,*
(3) *if $N = \widetilde{N}/\Gamma$ where Γ is an isometry group of $\widetilde{N}$, then each element of Γ splits, namely for any $\widetilde{\varphi} \in \Gamma$ there exists an isometry φ of M and a constant b such that $\widetilde{\varphi}(p, s) = (\varphi(p), s + b)$ for any $(p, s) \in M \times \mathbf{R}$.*

In particular, if $\langle X, X \rangle$ is in addition constant on N, the integral inequality is written as

$$\int_N \operatorname{Ric}(X) \, d\omega \leq 0$$

where $\operatorname{Ric}(X)$ *is the Ricci curvature of* X.

It should be noted that if f^t is disconjugate and A is symmetric on N, then the integral inequality holds without symmetric hypothesis of U, but we cannot determine what the flow is isometrically as seen in the example of Section 5. The most interesting cases having symmetric A are geodesic flows on the unit tangent bundles. However, in this case, the disconjugacy of the geodesic flow on the unit tangent bundle is equivalent to that of the underlying manifold ([In5]). The manifolds without conjugate points have already been studied in [In4].

Corollary E. *If there exists a function $F : N \longrightarrow \mathbf{R}$ such that $X = \operatorname{grad} F$, and if* $\operatorname{tr} A$ *is summable over N, then*

$$\int_N \operatorname{tr} A \, d\omega \leq 0$$

and equality holds only if either F is constant on N, or F is a non-trivial affine function on N, i.e., for any geodesic $\alpha : \mathbf{R} \longrightarrow N$ *there exists constants a and b such that* $F \circ \alpha(t) = at + b$*. In the latter case N is isometric to the Riemannian product* $F^{-1}(0) \times \mathbf{R}$.

In Section 5 we discuss what the symmetry of U implies and what relation there is between A and the Riemannian curvature tensor. We give the proofs of Theorem D and Corollary E in Section 5.

1.3. Measure theoretic entropy of geodesic flows

We use the same notation here as in Subsection 1.1. R. Ossermann and P. Sarnak ([OS]) give the lower estimate of measure entropy for geodesic flows on compact negatively curved manifold in terms of the curvature invariant as an application of Pesin's formula ([Pe]). And, W. Ballmann and M. P. Wojtkowski ([BW]) extend it to the case of nonpositive curvature. The main part of their proofs is to show that the integral inequality as in Theorem B. We write their estimate of measure entropy here:

Theorem F. *Suppose M is compact and has nonpositive curvature. Then,*

$$h_\omega(f^1) \geq \int_{SM} \operatorname{tr} \sqrt{-A}\, d\omega,$$

where $h_\omega(f^1)$ *is the measure theoretic entropy of the geodesic flow* f^t *on the unit tangent bundle SM. Equality holds if and only if M is a locally symmetric space.*

W. Ballmann ([Ba]) give other information concerning this result. We give the proof of Theorem F in Section 3. The proof of ours will be achived by approximation of the Riemannian curvature tensor A from below, which does not mean that of the Riemannian metric on M.

§2. Proofs of Theorems A and C

The proof of Theorem A is the same as in [In4] which is the version of the geodesic flows on the unit tangent bundles of Riemannian manifolds without conjugate points. We start discussing the assumption of decomposition of Ω_0.

2.1. The decomposition of Ω_0 and Ω

We can prove the following.

Lemma 2.1. *If Ω_0 is decomposed into at most countably many f^t-invariant sets with finite measure, then so is Ω.*

Proof. Let $\{D_i\}$ be an increasing sequence of compact sets D_i such that $\cup_{i=1}^{\infty} D_i = N$ and any sequence $\{p_i\}$, $p_i \in D_{m_i+1} - D_{m_i}$, has no accumulation point. Such a sequence always exists. Let $\Omega'_i = \{p \in \Omega | f^t p \in D_i \text{ for any } t \in (-\infty, \infty)\}$, and let $\Omega_i = \Omega'_i - \Omega'_{i-1}$ for each $i \geq 2$ and $\Omega_1 = \Omega'_1$. Then, Ω_i is an f^t-invariant measurable set with finite measure for each i. Obviously, $\cup_{i=1}^{\infty} \Omega_i = \Omega - \Omega_0$. Therefore, we can get a decomposition of at most countably many f^t-invariant sets of Ω with finite measure. Q.E.D.

The condition $\mathrm{vol}_\omega(\Omega_0) = 0$ will be discussed in Section 4. For the proof of Theorem A we need some preliminalies.

2.2. The trajectories of the flow

We introduce an equivalence relation $\sim$ in $N - \Omega$ in such a way that $p \sim q$ if $p = f^t q$ for some $t \in (-\infty, \infty)$, where $p, q \in N - \Omega$. Let M be the set of all equivalence classes $[p]$, $p \in N - \Omega$. Since $N - \Omega$ is open and f^t-invariant, for any $p \in N$ there exists locally a hypersurface H in $N - \Omega$ containing p and diffeomorphic to an open subset in $\mathbf{R}^{n-1}$ such that $[q] \cap H = \{q\}$ and H intersects $[q]$ transversely for any $q \in H$. The collection of such hypersurfaces H yields a differentiable structure of M with dimension $n - 1$. We define the volume form η on M such that $\eta_{[p]} \wedge dt = \omega_p$ for any $[p] \in M$. Then we have, for any summable function F on $N - \Omega$,

$$\int_{N-\Omega} F\, d\omega = \int_{[p]\in M} d\eta \int_{-\infty}^{\infty} F_{[p]}(f^t p)\, dt, \tag{2.1}$$

where $F_{[p]} : [p] \longrightarrow \mathbf{R}$ is given by $F_{[p]}(q) = F(q)$ for any $q \in [p]$.

2.3. The Birkhoff ergodic theorem

Let D be a f^t-invariant subset of N with finite measure. The Birkhoff ergodic theorem (see [AA]) says that for any summable function F on D

1) $$F_*(p) = \lim_{T\to\infty} \frac{1}{T} \int_0^T F(f^t p)\, dt$$

exist and are f^t-invariant for almost all $p \in D$,

2) $$\int_B F_*\, d\omega = \int_B F\, d\omega$$

for any f^t-invariant measurable subset $B \subset D$.

We say that a $p \in D$ is *uniformly recurrent* if for any neighborhood U of p, we have

$$\liminf_{T\to\infty} \frac{1}{T}\int_0^T \chi_U(f^t p)\,dt > 0,$$

where $\chi_U : D \longrightarrow \mathbf{R}$ is the characteristic function of U. We denote by $W(D)$ the set of all uniformly recurrent points in D. It follows from the Birkhoff ergodic theorem that $W(D)$ has full measure in D (see [BBE]).

2.4. The integral of $\operatorname{tr} A$ on $N - \Omega$

We will prove the following.

Lemma 2.2.

$$\int_{N-\Omega} \operatorname{tr} A\,d\omega \le 0$$

and equality holds only if $A = 0$ identically on $N - \Omega$.

Proof. Since $\operatorname{tr} A$ is summable and by the formula (2.1), the integral of the absolute of $\operatorname{tr} A$ is finite along the trajectory $\{f^t p; -\infty < t < \infty\}$ for almost all $p \in N - \Omega$. It follows (cf. [Ha], [In4]) that

$$\int_{-\infty}^{\infty} \operatorname{tr} A(f^t p)\,dt \le 0$$

for almost all $p \in N - \Omega$. Integrating it on M as in Subsection 2.2, we obtain

$$\int_{N-\Omega} \operatorname{tr} A\,d\omega = \int_{[p]\in M} d\eta \int_{-\infty}^{\infty} \operatorname{tr} A(f^t p)\,dt \le 0.$$

Equality means (cf. [In4]) that $A(p) = 0$ for almost all $p \in N - \Omega$. Since $A(p)$ depend continuously on the points $p \in N$, we see that A is identically zero on $N - \Omega$. Q.E.D.

2.5. The integral of $\operatorname{tr} A$ on Ω

Let Ω_1 be an f^t-invariant subset of Ω with finite measure. We work in Ω_1 and prove the following.

Lemma 2.3.

$$\int_{\Omega_1} \operatorname{tr} A\,d\omega \le 0,$$

and equality holds only if $A(p) = 0$ for any $p \in \Omega_1$.

Proof. Let $X(\Omega_1)$ be the set of all points p such that $(\operatorname{tr} A)_*(p)$ exists as in 1) of Subsection 2.3. Then, $X(\Omega_1) \cap W(\Omega_1)$ has full measure in Ω_1. Let a point $p \in X(\Omega_1) \cap W(\Omega_1)$ and let K be a compact neighborhood of p in Ω_1. It follows (cf. [In4], [Go]) that there exists a constant $C(K) > 0$ such that $\|U(q)\| < C(K)$ for any $q \in K$, where $U(f^t q)$ is the minimal symmetric solution of (R_A) along the trajectory $\{f^t q; -\infty < t < \infty\}$. Since p is uniformly recurrent, there exists a sequence $\{T_n\} \subset \mathbf{R}$ such that $T_n \longrightarrow \infty$, $f^{T_n} p \longrightarrow p$ as $n \longrightarrow \infty$ and $f^{T_n} p \in K$ for all n. Since $U(f^t p)$ satisfies the equation (R_A), we have

$$\frac{1}{T_n}(\operatorname{tr} U(f^{T_n} p) - \operatorname{tr} U(p)) + \frac{1}{T_n}\int_0^{T_n} \operatorname{tr} U(f^t p)^2\, dt + \frac{1}{T_n}\int_0^{T_n} \operatorname{tr} A(f^t p)\, dt = 0.$$

Taking $n \longrightarrow \infty$ we obtain

$$(\operatorname{tr} A)_*(p) = -\lim_{n\to\infty} \frac{1}{T_n}\int_0^{T_n} \operatorname{tr} U(f^t p)^2\, dt \le 0.$$

Hence, by the Birkhoff ergodic theorem, i.e., 2) of Subsection 2.3, we get

$$\int_{\Omega_1} \operatorname{tr} A d\omega = \int_{\Omega_1} (\operatorname{tr} A)_* d\omega \le 0.$$

Suppose equality holds. Then, $X_0(\Omega_1) = \{p \in \Omega | (\operatorname{tr} A)_*(p) = 0\}$ has full measure in Ω_1, and, hence, $X_0(\Omega_1) \cap W(\Omega_1)$ has full measure in Ω_1. We will prove that $A(p) = 0$ for any $p \in X_0(\Omega_1) \cap W(\Omega_1)$. The idea of the proof is seen in [In3]. Let a $p \in X_0(\Omega_1) \cap W(\Omega_1)$. Suppose $X(p) \neq 0$. Let $\gamma(t) = f^t p$ for any $t \in (-\infty, \infty)$. We put $U(t) = U(f^t v)$ and $\operatorname{tr} A(t) = \operatorname{tr} A(f^t v)$ for all $t \in (-\infty, \infty)$. Let B be the neighborhood of p in N such that $B = \{f^t q | -\ell \le t \le \ell,\, q \in H\}$ for some $\ell > 0$ and some hypersurface H containing p diffeomorphic to the closed disk in $\mathbf{R}^{n-1}$ and B is differmorphic to $H \times [-\ell, \ell]$. Since $(\operatorname{tr} A)_*(p) = 0$ and $p \in W(\Omega_1)$, it follows from the argument above that

$$\lim_{n\to\infty} \frac{1}{T_n}\int_0^{T_n} \operatorname{tr} U(t)^2\, dt = 0,$$

if a sequence $\{T_n\} \subset \mathbf{R}$ is such that $T_n \longrightarrow \infty$ as $n \longrightarrow \infty$ and $\gamma(T_n)$ lie in the boundary of B for all n.

Assertion. *There exists a sequence $\{t_n\} \subset [0, \infty)$ such that*

(1) $t_n \longrightarrow \infty$ *as* $n \longrightarrow \infty$,

(2) *if* $U_n(t)$ *is the matrix given by* $U_n(t) = U(t_n + t)$ *for any* $t \in [-\ell, \ell]$*, then*

$$\int_{-\ell}^{\ell} \operatorname{tr} U_n(t)^2\, dt \longrightarrow 0 \qquad as \qquad n \longrightarrow \infty,$$

and $\operatorname{tr} U_n(t) \longrightarrow 0$ *for almost all* $t \in [-\ell, \ell]$ *as* $n \longrightarrow \infty$,

(3) *if* $\gamma_n : [-\ell, \ell] \longrightarrow N$ *is given by* $\gamma_n(t) = f^{t_n+t}p$ *for any* $t \in [-\ell, \ell]$*, then* γ_n *converges to the curve* $\gamma_0 : [-\ell, \ell] \longrightarrow N$ *with* $\gamma_0(t) = f^t p$ *for any* $t \in [-\ell, \ell]$ *as* $n \longrightarrow \infty$.

Proof. We denote the set $\gamma^{-1}(H)$ by $\{t_i\}$. We assume that if $i < j$ then $t_i < t_j$ and $t_1 = 0$. Put $a_i = t_i - \ell$ and $b_i = t_i + \ell$ for each $i = 1, 2, \cdots$. Then, $\gamma([a_i, b_i]) \subset B$. Suppose

$$\liminf_{i \to \infty} \int_{a_i}^{b_i} \operatorname{tr} U(t)^2\, dt > \alpha > 0.$$

For any n, we have

$$\frac{1}{T_n}\int_0^{T_n} \operatorname{tr} U(t)^2\, dt \geq \frac{1}{T_n}\sum_{i=1}^{m_n}\int_{a_i}^{b_i} \operatorname{tr} U(t)^2\, dt$$

$$\geq \frac{1}{T_n}\sum_{i=1}^{m}\int_{a_i}^{b_i} \operatorname{tr} U(t)^2\, dt + \frac{\alpha}{2\ell T_n}\sum_{i=m+1}^{m_n}(b_i - a_i)$$

$$\geq \frac{\alpha}{2\ell T_n}\sum_{i=m+1}^{m_n}(b_i - a_i)$$

$$\geq \frac{\alpha}{2\ell T_n}\int_0^{T_n} \chi_{B \cap B(\ell/k)}(\gamma(t))\, dt - \frac{\alpha}{2\ell T_n}\sum_{i=1}^{m}(b_i - a_i),$$

where m_n and m are chosen so that

$$b_{m_n} < T_n < a_{m_n+1} \qquad \text{and} \qquad \inf_{i \geq m}\int_{a_i}^{b_i} \operatorname{tr} U(t)^2\, dt > \alpha,$$

and $B(\ell/k)$ is the ball with center p radius ℓ/k for a Riemannian metric on N. This implies that

$$0 = \lim_{n \to \infty} \frac{1}{T_n}\int_0^{T_n} \operatorname{tr} U(t)^2\, dt$$

$$\geq \frac{\alpha}{2\ell} \liminf_{T\to\infty} \frac{1}{T} \int_0^T \chi_{B\cap B(\ell/k)}(f^t p)dt > 0,$$

a contradiction. Thus we can find an integer $i(k) \geq k$ such that

$$\gamma(t_{i(k)}) \in H \cap B(\ell/k) \qquad \text{and} \qquad \int_{a_{i(k)}}^{b_{i(k)}} \operatorname{tr} U(t)^2\, dt \leq \frac{1}{k},$$

If we change the notation by $t_k = t_{i(k)}$, then the sequence $\{t_k\}$ satisfies the condition (1) and the first part of (2). For the second part of (2) and (3) we have only to choose a suitable subsequence $\{t_n\}$ of $\{t_k\}$ if necessary. This completes the proof of Assertion. Q.E.D.

We return to the proof of $\operatorname{tr} A(p) = 0$. Rewriting the equation (R_A) for $U(f^t p)$ in terms of (2), we get for each n

$$\operatorname{tr} U_n'(t) + \operatorname{tr} U_n(t)^2 + \operatorname{tr} A_n(t) = 0 \tag{2.2}$$

for any $t \in [-\ell, \ell]$, where $\operatorname{tr} A_n(t) = \operatorname{tr} A(t_n + t)$. It should be noted that $\operatorname{tr} A_n(t)$ converges to $\operatorname{tr} A(t)$ uniformly in $t \in [-\ell, \ell]$ as $n \longrightarrow \infty$. Suppose $\operatorname{tr} A(0) = \operatorname{tr} A(p) \neq 0$, say $\operatorname{tr} A(p) > 0$. Then, there exist a and $b \in [-\ell, \ell]$, $a < 0 < b$ such that $\operatorname{tr} A(t) > 0$ for any $t \in [a, b]$ and $\operatorname{tr} U_n(a)$, $\operatorname{tr} U_n(b) \longrightarrow 0$ as $n \longrightarrow \infty$. On the other hand, by integrating (2.2) on the interval $[a, b]$ and taking n to infinity, we have

$$\int_a^b \operatorname{tr} A(t)\, dt = 0,$$

a contradiction. Therefore, $\operatorname{tr} A(p) = 0$ for any $p \in X_0(\Omega_1) \cap W(\Omega_1)$ with $X(p) \neq 0$, namely p is not a fixed point of the flow. Next we assume that $X(p) = 0$, namely $f^t p = p$ for any $t \in (-\infty, \infty)$. Then, we have that $\operatorname{tr} A(p) = (\operatorname{tr} A)_*(p) = 0$. Thus, it follows ([In4]) that $A(p) = 0$ for any $p \in X_0(\Omega_1) \cap W(\Omega_1)$. Since $A(p)$ depends continuously on the points $p \in N$, we see that A is identically zero on Ω_1. Lemma 2.3 is proved. Q.E.D.

Now we can prove Theorem A.

Proof of Theorem A. Let Ω_1, Ω_2, $\cdots$ be the decomposition of f^t-invariant sets of Ω with finite measure. Then,

$$\int_N \operatorname{tr} A\, d\omega = \int_{N-\Omega} \operatorname{tr} A\, d\omega + \sum_{i=1}^{\infty} \int_{\Omega_i} \operatorname{tr} A\, d\omega \leq 0.$$

Equality holds only if $\int_{N-\Omega} \operatorname{tr} A \, d\omega = 0$ and $\int_{\Omega_i} \operatorname{tr} A \, d\omega = 0$ for all $i = 1, 2, \cdots$. Lemmas 2.2 and 2.3 state that $A = 0$ identically. This completes the proof of Theorem A.

Proof of Theorem C. We use the notation in Subsection 1.1. Theorem A states that

$$0 \geq \int_{SM} \operatorname{tr}(A - F \circ \pi I) \, d\omega = \int_{SM} \operatorname{Ric} \, d\omega - (n-1) F \circ \pi \, d\omega$$

$$= \frac{\theta_{n-1}}{n} \int_M S - n(n-1) F \, d\sigma,$$

where θ_{n-1} is the volume of the unit sphere in the n-dimensional Euclidean space, and therefore

$$\int_M S - n(n-1) F \, d\sigma \leq 0.$$

Equality holds only if $A(v) = F(\pi(v)) I$ for any $v \in SM$. In that case Schur's lemma (cf. [Ch], [Sc]) shows that M is a space of constant curvature. This completes the proof of Theorem C.

§3. Proofs of Theorems B and F

We shall use the following fact without proof, since J.-H. Eschenburg stated it in [Es]. In the statement $S(n)$ denotes the set of all (n, n) symmetric matrices.

Lemma 3.1. *Let $A_0 : [a, b] \longrightarrow S(n)$ be such that (J_{A_0}) is disconjugate on $[a, b]$. Suppose a sequence of $A_i : [a, b] \longrightarrow S(n)$ satisfies that $A_i(t) \leq A_0(t)$ for any $t \in [a, b]$ and each $i = 1, 2, \cdots$. If the sequence $\{A_i\}$ converges to A_0, then the sequence of minimal symmetric solutions $U_{is} : [a, s) \longrightarrow S(n)$ of (R_{A_i}) converges to that of (R_{A_0}), $U_{0s} : [a, s) \longrightarrow S(n)$, where $s \in [a, b]$.*

We need the following to prove Theorem B according to R. Ossermann and P. Sarnak ([OS]). Hereafter we assume that N is compact.

Lemma 3.2. *If (J_A) is disconjugate on N and $U(p)$ is invertible for all $p \in N$, then*

$$-\int_N \operatorname{tr} U \, d\omega = \int_N \operatorname{tr} A U^{-1} \, d\omega.$$

Proof. Since $U'U^{-1} + U + AU^{-1} = 0$ on N, we have

$$(\log|\det U(f^tp)|)' + \operatorname{tr} U(f^tp) + \operatorname{tr} A(f^tp)U^{-1}(f^tp) = 0.$$

Integrating it over $[0,1] \times N$ we get

$$\int_N \operatorname{tr} U\, d\omega + \int_N \operatorname{tr} AU^{-1}\, d\omega = 0.$$

Q.E.D.

Now we prove Theorem B. Put $A_\varepsilon = A - \varepsilon I$ for $\varepsilon > 0$. Since $A_\varepsilon \leq -\varepsilon I$, the minimal symmetric solution U_ε of (R_{A_ε}) is invertible and $U_\varepsilon \longrightarrow U$ as $\varepsilon \longrightarrow 0$ as seen in Lemma 3.1. Since

$$0 \leq \operatorname{tr}\left(\sqrt{-U_\varepsilon} - \sqrt{-A_\varepsilon}\sqrt{-U_\varepsilon}^{-1}\right)\left(\sqrt{-U_\varepsilon} - \sqrt{-A_\varepsilon}\sqrt{-U_\varepsilon}^{-1}\right)^*$$

$$= -\operatorname{tr} U_\varepsilon - 2\operatorname{tr}\sqrt{-A_\varepsilon} + \operatorname{tr} A_\varepsilon U_\varepsilon^{-1},$$

(This formula is due to W. Ballmann ([Ba]).) and Lemma 3.2 is true, we have

$$0 \leq -2\int_N \operatorname{tr} U_\varepsilon\, d\omega - 2\int_N \operatorname{tr}\sqrt{-A_\varepsilon}\, d\omega.$$

Taking ε to zero we have that

$$0 \leq -2\int_N \operatorname{tr} U\, d\omega - 2\int_N \operatorname{tr}\sqrt{-A}\, d\omega.$$

If equality holds, then the convergence of $\sqrt{-A_\varepsilon}\sqrt{-U_\varepsilon}^{-1}$ to $\sqrt{-U}$ is in L^2. Hence, if $X_\varepsilon = \sqrt{-A_\varepsilon}\sqrt{-U_\varepsilon}^{-1}$, then $X_\varepsilon\sqrt{-U_\varepsilon} = \sqrt{-A_\varepsilon} \longrightarrow (\sqrt{-U})^2$ as $\varepsilon \longrightarrow 0$ in L^2. Therefore, $\sqrt{-A(p)} = -U(p)$ i.e., $U(p)^2 + A(p) = 0$ for almost all $p \in N$. Continuity of U on each trajectory of the flow implies that $U'(f^tp) = 0$ for almost all p and all $t \in (-\infty, \infty)$. It follows from this that $A'(f^tp) = -U'(f^tp)U(f^tp) - U(f^tp)U'(f^tp) = 0$ for almost all $p \in N$. Since A is continuous on N, we conclude that A is parallel along the flow. This completes the proof of Theorem B.

Proof of Theorem F. Ja. Pesin proved in [Pe] that the measure theoretic entropy of the geodesic flow of the unit tangent bundle SM of a compact manifold M without conjugate points is $-\int_{SM} \operatorname{tr} U\, d\omega$ where U is the minimal symmetric solution of (R_A), A is the Riemannian curvature tensor of M. If we use the Riemannian curvature tensor of M as A in Theorem A, we immediately have Theorem F.

§4. Decomposition of the set of nonwandering points

In this section we study the condition which decomposes the set of all nonwandering points into at most countably many f^t-invariant sets with finite measure.

Let N be a manifold and let $f^t : N \longrightarrow N$ be a flow. We say that a subset $C \subset N$ is *a pencil of* f^t*-segments* if $f^s p \in C$ and $f^u p \in C$ implies that $f^t p \in C$ for any t, $s < t < u$. The condition shows that $f^t p \notin C$ for all $t > s$ (or $t < s$) if $p \in C$ and $f^s p \notin C$ for some $s > 0$ (or $s < 0$, resp.).

Example. Let M be a complete Riemannian manifold and let $N = SM$ be the unit tangent bundle. We denote by f^t the geodesic flow. A subset $C_0 \subset M$ is called *totally convex* if $\gamma([a, b]) \subset C_0$ for any geodesic $\gamma : [a, b] \longrightarrow M$ with $\gamma(a) \in C_0$ and $\gamma(b) \in C_0$. Let $C_0 \subset M$ be a totally convex set in M and let $C = \{v \in SM\ ;\ \pi(v) \in C_0\ \}$ where π is the canonical projection of SM to M. Then, C is a pencil of f^t-segments.

We denote by Ω the set of all non-wandering points.

Lemma 4.1. *Suppose there exists an increasing sequence* $\{D_i\}$ *of compact pencils of* f^t*-segments such that* $\cup_{i=1}^{\infty} D_i = N$ *and any sequence* $\{p_i\}, p_i \in D_{m_{i+1}} - D_{m_i}$*, has no accumulation point. Then,* Ω *is decomposed into at most countably many* f^t*-invariant sets with finite measure. More precisely, there exists no nonwandering point* p *such that the trajectory through* p *is not contained in any compact set.*

Proof. Let $\Omega'_i = \{p \in \Omega | f^t p \in D_i$ for all $t \in (-\infty, \infty)\}$ for each i and let $\Omega_i = \Omega'_i - \Omega'_{i-1}$ for each $i \geq 2$ and $\Omega_1 = \Omega'_1$. It is clear that $\Omega_i \cap \Omega_j = \phi$ for any $i \neq j$ and Ω_i is f^t-invariant for each i. Since Ω'_i is a closed set in a compact set D_i we see that Ω'_i is compact and measurable, and hence, Ω_i has finite measure. We must prove that $\cup_{i=1}^{\infty} \Omega_i = \Omega$. Obviousely, $\cup_{i=1}^{\infty} \Omega_i \subset \Omega$. Let $p \in \Omega$. Then, there exists an i_0 with $p \in \mathrm{Int} D_{i_0}$ from the property of $\{D_i\}$ that $\{p_i\}$, $p_i \in D_{m_{i+1}} - D_{m_i}$, has no accumulation point. If $f^t p \in D_{i_0}$ for all $t \in (-\infty, \infty)$, then $p \in \Omega'_{i_0}$, and hence, $p \in \cup_{i=1}^{i_0} \Omega_i$. Suppose $f^s p \notin D_{i_0}$ for some s. We may assume that $s > 0$ because the same argument is valid in the case $s < 0$. Since D_{i_0} is compact, we have that $f^s p \in \mathrm{Int}(N - D_{i_0}) = N - D_{i_0}$. By definition of nonwandering point, there exist a sequence $p_n \longrightarrow p$ and a sequence $t_n \longrightarrow \infty$ such that $f^{t_n} p_n \longrightarrow p$ as $n \longrightarrow \infty$. Hence, we can find by continuity of $f^t : N \longrightarrow N$ an m such that $t_m > s$, $p_m \in D_{i_0}$, $f^{t_m} p_m \in D_{i_0}$ and $f^s p_m \notin D_{i_0}$, contradicting that D_{i_0} is a pencil of f^t-segments. Thus, $f^t p \in D_{i_0}$ for all $t \in (-\infty, \infty)$. This completes the proof. Q.E.D.

We shall describe the assumption in Lemma 4.1 by using a nealy f^t-peakless function. We say that a continuous function $F : [a, b] \longrightarrow \mathbf{R}$ is *nearly peakless* if $a \le t_1 < t_2 < t_3 \le b$ implies that $F(t_2) \le \max\{F(t_1), F(t_3)\}$. The function was defined by H. Busemann and B. B. Phadke ([BP]) as convexities of functions degenerate. We say that a continuous function $F : N \longrightarrow \mathbf{R}$ is *nearly f^t-peakless* if for each $p \in N$ the function $F(f^t p)$ in $t \in (-\infty, \infty)$ is nearly peakless. We denote a sublevel set of F by $[F \le b] = \{p \in N | F(p) \le b\}$. If $F : N \longrightarrow \mathbf{R}$ is nearly f^t-peakless, then all sublevel sets are pencils of f^t-segments. A function $F : N \longrightarrow \mathbf{R}$ is said to be *exhaustive* if the sublevel sets $[F \le b]$ are compact for all $b < \sup F(N)$. In these words we rewrite Lemma 4.1.

Proposition 4.2. *Suppose there exists a nearly f^t-peakless exhaustion function $F : N \longrightarrow \mathbf{R}$. Then, Ω is decomposed into at most countably many f^t-invariant sets with finite measure.*

We shall discuss some examples.

4.1. Example

Let M be a complete noncompact Riemannian manifold of nonnegative sectional curvature outside some compact set C. We construct an f^t-convex function on SM as follows: Let $\gamma : [0, \infty) \longrightarrow M$ be a ray with $\gamma(0) = o \in C$, namely a minimizing geodesic from o. We denote the Busemann function of γ by $B_\gamma : M \longrightarrow \mathbf{R}$ ([Bu]), $B_\gamma(p) = \lim_{t \to \infty}\{t - d(p, \gamma(t))\}$, where $d(\cdot, \cdot)$ is the distance function induced from the Riemannian metric. This function satisfies that $|B_\gamma(p) - B_\gamma(q)| \le d(p, q)$. H. Wu ([Wu]) shows that B_γ is convex in the case $C = \phi$, namely $B_\gamma \circ \alpha(t)$ is a convex function in t for any geodesic $\alpha : (a, b) \longrightarrow M$. The theorem is still true in the case $C \ne \phi$ with slight modification. We say that a function $F : M \longrightarrow \mathbf{R}$ is *convex outside a set D* if $F \circ \alpha(t)$ is convex in t for any geodesic $\alpha : (a, b) \longrightarrow M - D$. In this words Wu's theorem is stated such as B_γ is convex outside a set $D = [B_\gamma \le \sup B_\gamma(C)]$. Define a function $B : N \longrightarrow \mathbf{R}$ by $B(p) = \sup\{B_\gamma(p) | \gamma \text{ is a ray with } \gamma(0) = o\}$. This function satisfies that $|B(p) - B(q)| \le d(p, q)$ for any p and $q \in M$, and therefore B is continuous on M. Furthermore, B is a convex exhaustion function outside some compact set. The convexity of the function is proved as follows: It should be noted that the convexity of functions is a local property. Let $b = \sup B(C)$ and let $p \notin [B \le b]$. For any $\varepsilon > 0$ with $b < B(p) - \varepsilon$ there exists a ray $\gamma : [0, \infty) \longrightarrow M$ with $\gamma(0) = o$ such that $B(p) - \varepsilon \le B_\gamma(p)$. Let $\alpha : (-a, a) \longrightarrow M$ be a geodesic with $\alpha(0) = p$ and $\alpha((-a, a)) \cap [B_\gamma \le \sup B_\gamma(C)] = \phi$. By the above theorem of Wu we have that

$$(B_\gamma \circ \alpha)(\lambda t_1 + (1-\lambda)t_2) \leq \lambda(B_\gamma \circ \alpha)(t_1) + (1-\lambda)(B_\gamma \circ \alpha)(t_2)$$

for any t_1, $t_2 \in (-a, a)$ and $0 < \lambda < 1$. Therefore, we get

$$(B \circ \alpha)(\lambda t_1 + (1-\lambda)t_2) \leq \lambda(B \circ \alpha)(t_1) + (1-\lambda)(B \circ \alpha)(t_2)$$

for any t_1, $t_2 \in (-a, a)$ and $0 < \lambda < 1$. This proves that B is convex outside $[B \leq b]$, and therefore any sublevel set $[B \leq c]$ is totally convex for any $c > b$. Next we prove that B is an exhaustion function. Suppose for indirect proof that there exists a $c > b$ such that $[B \leq c]$ is noncompact. Since $[B \leq c]$ is totally convex, we can find a ray $\gamma : [0, \infty) \longrightarrow M$ with $\gamma(0) = o$ and $\gamma([0, \infty)) \subset [B \leq c]$. Since $B_\gamma(\gamma(t)) = t \leq B(\gamma(t))$, we have a contradiction.

Define a function $B_1 : M \longrightarrow \mathbf{R}$ as follows:

$$B_1(p) = \begin{cases} B(p) & \text{if } p \notin [B \leq b] \\ b & \text{if } p \in [B \leq b] \end{cases}$$

Then, B_1 is a convex function on M. Let $\widetilde{B} : SM \longrightarrow \mathbf{R}$ be a function given by $\widetilde{B} = B_1 \circ \pi$. We finally get an f^t-convex exhaustion function $\widetilde{B}$ on SM. The weaker version of the existence of a filtration of M by compact totally convex sets has been proved by G. Thorbergsson ([Th]) by an analogous way as in Cheeger and Gromoll's paper ([CG]).

4.2. Example

We shall get a surface such that the set of all nonwandering points of the geodesic flow is decomposed into countably many f^t-invariant sets with finite measure.

Let $\mathbf{R}^2$ be the affine plane and let $F : \mathbf{R}^2 \longrightarrow \mathbf{R}$ be a positive function which depends only on the x-coordinate. We define a metric on $\mathbf{R}^2$ by $ds^2 = dx^2 + F(x)^2 dy^2$. Let $\theta_\alpha(t)$ is the angle of the velocity vector $\dot{\alpha}(t)$ of a curve $\alpha(t) = (x(t), y(t))$ and the curve $c(s) = (x(t), s)$, $-\infty < s < \infty$. Clairaut's theorem states that if $\alpha : (-\infty, \infty) \longrightarrow (\mathbf{R}^2, ds^2)$ is a geodesic with $\alpha(t) = (x(t), y(t))$, then $F(x(t)) \cos \theta_\alpha(t) = \text{const.}$ for any $t \in (-\infty, \infty)$ (cf. [In2]).

Lemma. *Let (x_0, y_0) be a point in $(\mathbf{R}^2, ds^2)$ such that $F(x_0) = \min\{F(x) | x \geq x_0\}$ and let $\alpha : [0, \infty) \longrightarrow (\mathbf{R}^2, ds^2)$ be a geodesic emanating from (x_0, y_0) with $\alpha(t) = (x(t), y(t))$ and $\dot{x}(0) \geq 0$. Then, $x(t)$ is either constant or strictly increasing in $t \in [0, \infty)$.*

Proof. Suppose that $\dot{x}(0) > 0$. If this lemma is not true, then there exists an $s > 0$ such that $\theta_\alpha(s) = 0$. By Clairaut's theorem we have that

$F(x_0)\cos\theta_\alpha(0) = F(x(s))$. However, since $\dot{x}(0) > 0$, namely $\theta_\alpha(0) \neq 0$, and by the choice of x_0, this is a contradiction. Suppose that $\dot{x}(0) = 0$. By taking the limit of a sequence of geodesics $\alpha_n = (x_n, y_n)$ with $\dot{x}_n(0) > 0$ we see that $\alpha([0,\infty)) \subset \{(x,y)|x \geq x_0, -\infty < y < \infty\}$. Assume in addition that $x(t)$ is not constant in $t \in [0,\infty)$. Let $t_0 > 0$ such that $x(t_0) > x_0$. If $x(t)$ is not strictly increasing in $t \geq t_0$, then there exists a $t_1 \geq t_0$ such that $\theta_\alpha(t_1) = 0$. By Clairaut's theorem again we have that $F(x(t_1)) = F(x_0)$. Hence, $F(x(t_1))$ is a minimum in a neighborhood of $x(t_1)$, so we have that $F'(x(t_1)) = 0$. Then, $\beta(t) = (x(t_1), \frac{t}{F(x(t_1))})$ is a geodesic other than α, contradicting the uniqueness of the geodesic with initial point and velocity. Let $t_2 = \inf\{t > 0|x(t) > x_0\}$. If $t_2 = 0$, then we have nothing to prove. Suppose $t_2 > 0$. Then, $\alpha : [0, t_2] \longrightarrow (\mathbf{R}^2, ds^2)$ is a geodesic contained in $\{(x_0, y)| -\infty < y < \infty\}$. Since the family of maps $\varphi_\lambda : (\mathbf{R}^2, ds^2) \longrightarrow (\mathbf{R}^2, ds^2)$ given by $\varphi_\lambda(x,y) = (x, y+\lambda)$ for all $(x,y) \in \mathbf{R}^2$ is a one-parameter group of isometries on $(\mathbf{R}^2, ds^2)$, this implies that $\beta(t) = (x_0, \frac{t}{F(x_0)} + y_0)$ is the geodesic with $\beta(t) = \alpha(t)$ for any $t \in [0, t_2]$, contradicting that $x(t)$ is not constant in $t \in [0,\infty)$. This completes the proof of Lemma. Q.E.D.

Let $\varphi_\lambda : (\mathbf{R}^2, ds^2) \longrightarrow (\mathbf{R}^2, ds^2)$ be given by $\varphi(x,y) = (x, y+\lambda)$. We consider the tube $T = \{(x,y)|x \geq x_0, -\infty < y < \infty\}/\{\varphi_\lambda^n\}_{n\in\mathbf{Z}}$, and the condition:

(C) *There exists a sequence* $\{x_i\}$, $x_i < x_{i+1}$, *such that* $x_i \longrightarrow \infty$ *as* $i \longrightarrow \infty$ *and* $F(x_i) = \min\{F(x)|x \geq x_i\}$ *for each* i.

Let M be a surface on which there exists a compact set K such that each connected component of $M - K$ is one of the following types:

(1) It is a tube constructed above and satisfying the condition (C).
(2) It has finite volume.

We denote by $T_1, \cdots, T_m$ the tubes of type (1) and by $\{x_{j,i}\}_{i=1,2,\cdots}$ a sequence in the condition (C) for the tube T_j, $j = 1, 2, \cdots, m$. Let $T_{j,i} = \{(x,y) \in T_j | x \geq x_{j,i}\}$. Then, it follows from Lemma that $D'_i = M - (\cup_{j=1}^{\infty} T_{j,i})$ is a totally convex set with finite measure for each i. Put $D_i = \{v \in SM|\pi(v) \in D'_i\}$. Although D_i are not compact, by using the sequence $\{D_i\}$ in the same way as Lemma 4.1, the set of all nonwandering points can be decomposed into countably many f^t-invariant sets with finite measure.

§5. Jacobi and Riccati equations from flows

In this section we use the same notation as in Subsection 1.2. We begin with the study of symmetry of U.

5.1. Flows with symmetric U

First of all we note that if $y \in T_pN$ and $Y(t) = P(t,p)y$, then $(DY)(t) := D(t,p)Y(t) = df_p^t y$, and hence

$$\begin{aligned}\nabla_X(DY)(t) &= (\nabla_X D)Y(t) = ((\nabla_X D)D^{-1})(DY)(t)\\ &= U(f^t p)(DY)(t) = U(f^t p)(df_p^t y).\end{aligned}$$

Let $\widetilde{N}$ be the universal covering space of N and $\widetilde{X}$ the lift of X to $\widetilde{N}$. Let $\widetilde{U}$ be the lift of U to $\widetilde{N}$.

Lemma 5.1. *If $\widetilde{U}$ is symmetric, then there exists a function $\widetilde{F} : \widetilde{N} \longrightarrow \mathbf{R}$ with $\widetilde{X} = \operatorname{grad} \widetilde{F}$.*

Proof. Define a 1-form η on $\widetilde{N}$ by $\eta(\widetilde{Y}) = \langle \widetilde{X}, \widetilde{Y} \rangle$ for any vector field $\widetilde{Y}$ on $\widetilde{N}$. Since $\widetilde{N}$ is simply connected, the Poincaré Lemma states Lemma 5.1 if $d\eta = 0$ on $\widetilde{N}$ (cf. [Wa]). Let y, $z \in T_{\tilde{p}}\widetilde{N}$ and let $\varphi : (-\varepsilon, \varepsilon)^3 \longrightarrow \widetilde{N}$ be a variation such that

(1) $\varphi(0,0,0) = \tilde{p}$,
(2) $\varphi(t,s,u) = f^t\varphi(0,s,u)$,
(3) $d\varphi_{(0,0,0)}\left(\frac{\partial}{\partial s}\right) = y$, $d\varphi_{(0,0,0)}\left(\frac{\partial}{\partial u}\right) = z$.

If we put

$$\widetilde{Y}(t,s,u) := d\varphi_{(t,s,u)}\left(\frac{\partial}{\partial s}\right) = df^t_{\varphi(0,s,u)} d\varphi_{(0,s,u)}\left(\frac{\partial}{\partial s}\right),$$

and

$$\widetilde{Z}(t,s,u) := d\varphi_{(t,s,u)}\left(\frac{\partial}{\partial u}\right) = df^t_{\varphi(0,s,u)} d\varphi_{(0,s,u)}\left(\frac{\partial}{\partial u}\right),$$

then we have

$$\begin{aligned} d\eta(\widetilde{Y}, \widetilde{Z}) &= \widetilde{Y}\eta(\widetilde{Z}) - \widetilde{Z}\eta(\widetilde{Y}) - \eta([\widetilde{Y}, \widetilde{Z}])\\ &= \widetilde{Y}\langle \widetilde{X}, \widetilde{Z}\rangle - \widetilde{Z}\langle \widetilde{X}, \widetilde{Y}\rangle - \langle \widetilde{X}, [\widetilde{Y}, \widetilde{Z}]\rangle\\ &= \langle \nabla_{\widetilde{Y}}\widetilde{X}, \widetilde{Z}\rangle - \langle \nabla_{\widetilde{Z}}\widetilde{X}, \widetilde{Y}\rangle\\ &= \langle \nabla_{\widetilde{X}}\widetilde{Y}, \widetilde{Z}\rangle - \langle \nabla_{\widetilde{X}}\widetilde{Z}, \widetilde{Y}\rangle\\ &= \langle \widetilde{U}\widetilde{Y}, \widetilde{Z}\rangle - \langle \widetilde{U}\widetilde{Z}, \widetilde{Y}\rangle = 0,\end{aligned}$$

since $\widetilde{X}(\varphi(t,s,u)) = df^t_{\varphi(t,s,u)}\widetilde{X}(\varphi(0,s,u))$. Q.E.D.

Conversely we have the following.

Lemma 5.2. *If* $X = \operatorname{grad} F$ *for some function* $F : N \longrightarrow \mathbf{R}$, *then* U *is symmetric.*

Proof. Let y, $z \in T_pN$ and let $\varphi : (-\varepsilon, \varepsilon)^3 \longrightarrow N$ be a variation as in the proof of Lemma 5.1. Then

$$\begin{aligned}&\langle UY, Z\rangle = \langle \nabla_X Y, Z\rangle = \langle \nabla_Y X, Z\rangle \\ =&Y\langle X, Z\rangle - \langle X, \nabla_Y Z\rangle = Y(ZF) - (\nabla_Y Z)F \\ =&Z(YF) - (\nabla_Z Y)F = \langle UZ, Y\rangle.\end{aligned}$$

This implies that $\langle Uy, z\rangle = \langle y, Uz\rangle$ at p. Q.E.D.

5.2. The relation between A and the curvature tensor

Let R be the curvature tensor of N. Let $p \in N$ and $y \in T_pN$. Put $Y(t) = P(t,p)y$ for any t. Then,

$$\begin{aligned}\nabla_X \nabla_X DY - \nabla_{DY} \nabla_X X &= \nabla_X \nabla_{DY} X - \nabla_{DY} \nabla_X X \\ &= R(X, DY)X.\end{aligned}$$

Hence, we have

$$A(y) = R(y, X(p))X(p) - \nabla_y \nabla_X X.$$

We prove the following.

Lemma 5.3. *If* $\langle X, X\rangle$ *is constant on* N *and* U *is symmetric on* N, *then the trajectories of* f^t *are geodesics in* N *and therefore* $A(y) = R(y, X(p))X(p)$ *for any* $p \in N$ *and* $y \in T_pN$.

Proof. To prove Lemma 5.3 it suffices that the trajectories of f^t are geodesics in N. Since $\langle X, X\rangle$ is constant on N, we see that

$$0 = \langle \nabla_y X, X(p)\rangle = \langle \nabla_{X(p)} DY, X(p)\rangle$$

$$= \langle U(p)y, X(p)\rangle = \langle y, U(p)X(p)\rangle = \langle y, \nabla_{X(p)} X\rangle.$$

Therefore, $\nabla_X X$ is identically zero on N. Q.E.D.

5.3. Proofs of Theorem D and Corollary E

Since the integral inequality immediately follows from the disconjugacy of the flow which comes from the symmetric property of U, we have only to show what N is when $U = 0$ identically on N. The lift $\widetilde{U}$ of U to $\widetilde{N}$ is identically zero also. Let $\widetilde{F} : \widetilde{N} \longrightarrow \mathbf{R}$ be a function as in

Lemma 5.1. Let $y, z \in T_{\tilde{p}}\widetilde{N}$ and let $\varphi : (-\varepsilon, \varepsilon)^3 \longrightarrow \widetilde{N}$ be a variation as in the proof of Lemma 5.1. We have

$$\langle \nabla_{\widetilde{Y}} \widetilde{X}, \widetilde{Z} \rangle = \langle \nabla_{\widetilde{X}} \widetilde{Y}, \widetilde{Z} \rangle = \langle \widetilde{U} \widetilde{Y}, \widetilde{Z} \rangle = 0,$$

and, in particular, $\widetilde{X}$ is parallel on $\widetilde{N}$. If $\alpha : (-\infty, \infty) \longrightarrow \widetilde{N}$ be a geodesic in $\widetilde{N}$, then

$$(\widetilde{F} \circ \alpha)''(t) = \dot{\alpha}(t)\langle \dot{\alpha}(t), \widetilde{X} \rangle = 0$$

for all $t \in (-\infty, \infty)$. Therefore, $\widetilde{F}$ is an affine function on $\widetilde{N}$. Theorem D follows from Main Theorem in [In1]. It should be noted that the nonwandering points p of gradient flows are fixed ones, and therefore $\Omega_0 = \phi$. Since the assumption of Main Theorem in [In1] does not require that the manifold is simply connected, Corollary E is true also by Lemma 5.2.

5.4. Example

We give an example such that $A = 0$ but U is not symmetric. Let $\widetilde{X}$ be the vector field on $\mathbf{E}^3$ given by

$$\widetilde{X}(x, y, z) = (\cos 2\pi z, \sin 2\pi z, 0),$$

where (x, y, z) are canonical coordinates on $\mathbf{E}^3$. Let

$$\Gamma = \left\{ \varphi_{\ell,m,n} | \varphi_{\ell,m,n}(x, y, z) = (x + \ell, y + m, z + n), \ell, m, n \in \mathbf{Z} \right\}$$

and let $T^3 = \mathbf{E}^3/\Gamma$. Then, there exists the vector field X on T^3 whose lift is $\widetilde{X}$. The flow generated from X preserves canonical volume form of T^3 and its trajectories are geodesics in the flat torus T^3. Thus we have a desired example.

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Maslov Class of an Isotropic Map-Germ Arising from One-Dimensional Symplectic Reduction

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Dedicated to Professor Noboru Tanaka on his 60th birthday

§0. Introduction

Let (M^{2n}, ω) be a symplectic manifold of dimension $2n$ and $N^n \subset M^{2n}$ be a Lagrangian submanifold with singularities. For each regular point x of N, T_xN is a Lagrangian subspace of the sympletic vector space T_xM.

To investigate the local structure of N near a singular point x_0 of N, it is natural to study the behavior of the distribution $\{T_xN \mid x \text{ is a regular point of } N\}$ near x_0. Then we can grasp an invariant of the singularity, which is called the Maslov class in this paper.

In studying the problem of Lagrangian immersion of surfaces to four dimensional symplectic manifolds, Givental' [G] introduced a Lagrangian variety, so called an open Whitney umbrella or an unfolded Whitney umbrella, and investigated some local and global problems. In particular, he calculated the "Maslov index" of an open Whitney umbrella. The main purpose of this paper is to generalize the result of Givental'.

Singular Lagrangian varieties appear typically in the process of symplectic reduction (see §5, [A2] and [I1]).

Note that singular Lagrangian varieties obtained by reduction are parametrized by isotropic mappings.

Originally, the notion of Maslov class (Keller-Maslov-Arnol'd class) stemed from the asymptotic method of linear partial differential equation, representation theory, and symplectic topology ([A1], [GS], [Gr], [Hö], [M], [V], [W]).

Maslov classes represent obstruction for transversality of two Lagrangian subbundles (see §1). Applying this understanding, we define

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the Maslov class of isotopic mapping as an obstruction for extendability of a partially defined Lagrangian subbundle. Further, we show several results on Maslov classes of isotropic mappings obtained by one-dimensional reduction process.

The first result is on vanishing of Maslov classes: The Maslov class of an isotropic map-germ obtained by one-dimensional reduction of a Lagrangian manifold is zero (Theorem 6.1).

Thus, for a singularity of one-dimensional symplectic reduction of an isotropic manifold, the Maslov class has a meaning of obstruction for representability as an intersection of a Lagrangian submanifold and a hypersurface.

In general, Maslov classes do not vanish. We give local models of singularities of isotropic mappings generically obtained by one-dimensional reduction of isotropic submanifolds, up to local symplectic diffeomorphisms of the reduced symplectic manifold (Theorem 7.1). These models are open Whitney umbrellas of arbitrary dimension and their suspensions, and their Maslov classes proves not to vanish (Theorem 8.3). Therefore, we see that a generic isotropic submanifold in a hypersurface of a symplectic manifold is not an intersection of a Lagrangian submanifold and the hypersurface, locally at each point where the characteristic direction is tangent to the hypersurface (Corollary 8.4).

In §1, we recall the notion of classical Maslov class. In the next section, we define the Maslov class of an isotropic mapping. In §3, the notion of symplectic equivalence of isotropic mappings is introduced. The Maslov class of an isotropic map-germ is defined in §4.

After a preliminary on the symplectic reduction in §5, Theorem 6.1 is stated in §6 and proved in §§9–10. Theorem 7.1 is stated in §7. The proof is given in [I2]. Theorem 8.3 is stated in §8 and proved in §§11–14.

Throughout this paper, all manifolds and maps are of class C^∞.

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§1. Classical Maslov class

Let (M^{2n}, ω) be a symplectic manifold, $N^n \subset M$ a Lagrangian submanifold ($\omega|N = 0$), and $\pi : M \longrightarrow B$ a Lagrangian fibration. Then the symplectic vector bundle $\mathbf{E} = TM|N$ has two Lagrangian subbundles $\mathbf{L} = \operatorname{Ker} \pi_*|N$ and $\mathbf{L}' = TN$.

In general, for a symplectic vector bundle $\mathbf{E}$ of rank $2n$ over N and Lagrangian subbundles $\mathbf{L}$ and $\mathbf{L}'$ of $\mathbf{E}$, the Maslov class $m(\mathbf{E};\mathbf{L},\mathbf{L}') \in H^1(N,\mathbf{Z})$ is defined as follows.

Consider the bundle $\Lambda(\mathbf{E})$ over N of Lagrangian subspaces of fibers of $\mathbf{E}$. The Lagrangian subbundle $\mathbf{L}'$ defines a section $s(\mathbf{L}') : N \longrightarrow \Lambda(\mathbf{E})$ by $s(\mathbf{L}')(x) = \mathbf{L}'_x \in \Lambda(\mathbf{E}_x) \subset \Lambda(\mathbf{E}), x \in N$.

Let Ω denote the symplectic form of $\mathbf{E}$. Then there exist a complex structure J and a Hermitian form G on $\mathbf{E}$, unique up to homotopy, such that Ω is the imaginary part of G. Denote by g the real part of G.

Let $S = \{e_1, \dots, e_n\}$ be an orthonormal frame of $\mathbf{L}$ over an open subset U of N, with respect to g. Then S turns out to be a unitary frame of the Hermitian vector bundle $(\mathbf{E}; J, G)$ over U. Then we have an isomorphism $\mathbf{E}|U \cong \mathbf{C}^n \times U$ as Hermitian vector bundles, which maps $\mathbf{L}|U$ to $\mathbf{R}^n \times U$. Since $U(n)$ acts on the space $\Lambda(\mathbf{C}^n)$ of Lagrangian subspaces of $\mathbf{C}^n$ transitively, $\Lambda(\mathbf{C}^n)$ is identified with the homogeneous space $\Lambda(n) = U(n)/O(n)$ (see [A1]). Thus we have $\Lambda(\mathbf{E})|U \longrightarrow \Lambda(n)$, which are glued into a C^∞ mapping $\Phi(\mathbf{L}) : \Lambda(\mathbf{E}) \longrightarrow \Lambda(n)$ (cf.[F]).

Set $\Phi = \Phi(\mathbf{L}) \circ s(\mathbf{L}') : N \longrightarrow \Lambda(n) = U(n)/O(n)$. Then the homotopy type of Φ is independent of the choice of (J, G).

If $S' = \{e'_1, \dots, e'_n\}$ is an orthonormal frame of $\mathbf{L}'$ over U, then, at $x \in U$, $e'_j = \sum_{i=1}^n a_{ij} e_i$ for some $A = (a_{ij}) \in U(n)$. Then $\Phi(x) = [A] \in \Lambda(n)$. Remark that $G(e'_j, e_i) = a_{ij}$.

Define the Maslov class $m(\mathbf{E};\mathbf{L},\mathbf{L}')$ of triplet $(\mathbf{E};\mathbf{L},\mathbf{L}')$ to be the image of the generator of $H^1(S^1,\mathbf{Z}) \cong \mathbf{Z}$, relatively to counterclockwise orientation, under

$$(\det{}^2 \circ \Phi)^* : H^1(S^1, \mathbf{Z}) \longrightarrow H^1(N, \mathbf{Z}),$$

where $\det^2 : U(n)/O(n) \longrightarrow S^1$ is defined by $\det^2([A]) = (\det A)^2$, $A \in U(n)$.

The following is well known (cf.[V]) or easily proved.

Lemma 1.1. *The Maslov class satisfies following properties:*

(0) $m(\mathbf{E};\mathbf{L},\mathbf{L}) = 0$.

(1) $m(\mathbf{E};\mathbf{L},\mathbf{L}') + m(\mathbf{E};\mathbf{L}',\mathbf{L}'') = m(\mathbf{E};\mathbf{L},\mathbf{L}'')$.

(2) $\mathbf{L}$ *and* $\mathbf{L}'$ *are transverse in* $\mathbf{E}$, *then* $m(\mathbf{E};\mathbf{L},\mathbf{L}') = 0$.

(3) *If there is an isomorphism between symplectic vector bundles* $\mathbf{E}_1$ *and* $\mathbf{E}_2$ *mapping* $\mathbf{L}_1, \mathbf{L}'_1$ *to* $\mathbf{L}_2, \mathbf{L}'_2$, *then* $m(\mathbf{E}_1;\mathbf{L}_1,\mathbf{L}'_1) = m(\mathbf{E}_2;\mathbf{L}_2,\mathbf{L}'_2)$.

(4) $m(h^*\mathbf{E}; h^*\mathbf{L}, h^*\mathbf{L}') = h^*m(\mathbf{E};\mathbf{L},\mathbf{L}') \in H^1(P,\mathbf{Z})$.

Here $\mathbf{L}, \mathbf{L}', \mathbf{L}''$ *are Lagrangian subbundles of* $\mathbf{E}$ *and* h *is a mapping from a manifold* P *to* N.

Remark 1.2. Vaisman [V] defines the Maslov classes $\mu_h(\mathbf{E};\mathbf{L},\mathbf{L}') \in H^{4h-3}(N,\mathbf{R}), h = 1,2,3,\dots,$ for two Lagrangian subbundles $\mathbf{L},\mathbf{L}'$ of a symplectic vector bundle $\mathbf{E}$ over N, such that

(i) $\mu_1(\mathbf{E};\mathbf{L},\mathbf{L}') = (1/2)m(\mathbf{E};\mathbf{L},\mathbf{L}') \in H^1(N,\mathbf{R})$;

(ii) μ_h satisfies the properties of Lemma 1.1.

Returning to the situation we begun with, we define the Maslov class $m(N)$ by $m(N) = m(TM|N;\ \mathrm{Ker}\ \pi_*|N, TN) \in H^1(N,\mathbf{Z})$.

§2. Maslov class of an isotropic mapping

Let (M^{2n},ω) be a symplectic manifold of dimension $2n$, and N^n be a C^∞ manifold of dimension n.

A C^∞ mapping $f : N \longrightarrow M$ is called an isotropic mapping if, for each $x \in N$, the image of $T_xf : T_xN \longrightarrow T_xM$ is an isotropic subspace of the symplectic vector space T_xM, that is, if $f^*\omega = 0$.

For an isotropic mapping f, set

$$\Sigma = \Sigma(f) = \{x \in N \mid T_xf \text{ is not injective}\}.$$

Then the restriction $f|N-\Sigma : N-\Sigma \longrightarrow M$ is a Lagrangian immersion.

Set $\Lambda(M) = \Lambda(TM)$, and denote by $\pi : \Lambda(M) \longrightarrow M$ the canonical projection. In the symplectic vector bundle π^*TM over $\Lambda(M)$, define the tautological bundle $\mathcal{L}$ by $\mathcal{L}_{(y,\lambda)} = \lambda \subset T_yM$, $(y,\lambda) \in \Lambda(M)$.

Associated to f, $\varphi(f) : N-\Sigma \longrightarrow \Lambda(M)$ is defined by $\varphi(f)(x) = (f(x),\ \mathrm{Im}(T_xf))$. Then $\pi \circ \varphi(f) = f$.

Set $\mathbf{L}_f = \varphi(f)^*\mathcal{L}$. Then $\mathbf{L}_f$ is a Lagrangian subbundle of $f^*TM = \varphi(f)^*\pi^*TM$ over $N-\Sigma$.

Definition 2.1. Assume f^*TM has a Lagrangian subbundle $\mathbf{L}$ (over N). Then define the Maslov class of f by

$$m(f) = \delta(m(f^*TM;\mathbf{L},\mathbf{L}_f)) \in H^2(N,N-\Sigma;\mathbf{Z}),$$

where $\delta : H^1(N-\Sigma;\mathbf{Z}) \longrightarrow H^2(N,N-\Sigma;\mathbf{Z})$ is the coboundary map.

Remark that $m(f^*TM;\mathbf{L},\mathbf{L}_f) \in H^1(N-\Sigma;\mathbf{Z})$. For another Lagrangian subbundle $\mathbf{L}'$ of f^*TM over N, we have

$$m(f^*TM;\mathbf{L}',\mathbf{L}_f) = m(f^*TM;\mathbf{L}',\mathbf{L}) + m(f^*TM;\mathbf{L},\mathbf{L}_f),$$

in $H^1(N-\Sigma;\mathbf{Z})$, by Lemma 1.1.(2).

Since $m(f^*TM; \mathbf{L}', \mathbf{L})$ comes from an element of $H^1(N; \mathbf{Z})$, we have

$$\delta(m(f^*TM; \mathbf{L}', \mathbf{L}_f)) = \delta(m(f^*TM; \mathbf{L}, \mathbf{L}_f)).$$

Therefore $m(f)$ is independent of the choise of $\mathbf{L}$.

Remark 2.2. By Remark 1.2 and the same argument as above, we can define a class $\mu_h(f) \in H^{4h-2}(N, N - \Sigma; \mathbf{R})$ for an isotropic mapping $f : N \longrightarrow M$ by

$$\mu_h(f) = \delta(\mu_h(f^*TM; \mathbf{L}, \mathbf{L}_f)),$$

$h = 1, 2, \ldots$, provided f^*TM has a Lagrangian subbundle $\mathbf{L}$.

§3. Symplectic equivalence

Let $f : N^n \longrightarrow (M^{2n}, \omega)$ and $f' : N'^n \longrightarrow (M'^{2n}, \omega')$ be isotropic mappings.

Definition 3.1. A pair (σ, τ) of a diffeomorphism $\sigma : N \longrightarrow N'$ and a symplectic diffeomorphism $\tau : M \longrightarrow M', \tau^*\omega' = \omega$, is called a symplectic equivalance between f and f' if $\tau \circ f = f' \circ \sigma$. Then we call f is symplectically equivalent to f, and write $f \sim f'$.

If (σ, τ) is a symplectic equivalence between f and f', then σ induces an isomorphism $\sigma^* : H^2(N', N' - \Sigma'; \mathbf{Z}) \longrightarrow H^2(N, N - \Sigma; \mathbf{Z})$, where $\Sigma' = \Sigma(f')$, and $\sigma^* m(f') = m(f)$, if f^*TM has a Lagrangian subbundle $\mathbf{L}$. In fact, τ induces isomorphisms

$$\tau' : \tau^*TM' \longrightarrow TM \quad \text{and } \tau'' : \sigma^* f'^* TM' = f^* \tau^* TM' \longrightarrow f^*TM$$

of symplectic vector bundles over M and over N respectively, and τ'' maps $\sigma^* \mathbf{L}_{f'}$ to $\mathbf{L}_f$. Thus

$$\begin{aligned} \sigma^* m(f'^*TM'; \tau''^{-1}\mathbf{L}, \mathbf{L}_{f'}) &= m(\sigma^* f'^* TM'; \sigma^* \tau''^{-1} \mathbf{L}, \sigma^* \mathbf{L}_{f'}) \\ &= m(f^*TM; \mathbf{L}_1, \mathbf{L}_f). \end{aligned}$$

for some Lagrangian subbundle $\mathbf{L}_1$ of f^*TM, by Lemma 1.1.(3) and (4). Therefore

$$\begin{aligned} \sigma^* m(f') &= \sigma^* \delta m(f'^*TM'; \tau''^{-1}\mathbf{L}, \mathbf{L}_{f'}) \\ &= \delta(m(f^*TM; \mathbf{L}_1, \mathbf{L}_f)) \\ &= m(f) \end{aligned}$$

§4. Maslov class of an isotropic map-germ

Let $f : N^n, x \longrightarrow (M^{2n}, \omega)$ be a germ of an isotropic mapping. For each representative (f, U) such that $f : U \longrightarrow M$ is isotropic and U is a contractible neighborhood of x, we have $m(f, U) \in H^2(U, U - \Sigma; \mathbf{Z})$, since f^*TM is trivial over U. If V is a contractible neighborhood of x, with $V \subset U$ and $\iota^* : H^2(U, U - \Sigma; \mathbf{Z}) \longrightarrow H^2(V, V - \Sigma; \mathbf{Z})$ is the restriction, then $\iota^*(m(f, U)) = m(f, V)$ by Lemma 1.1.(4). Set

$$H^2(N, N - \Sigma; \mathbf{Z})_x = \varinjlim H^2(U, U - \Sigma; \mathbf{Z}),$$

where U runs over contractible neighborhoods of x. Then we have an element

$$m(f) \in H^2(N, N - \Sigma; \mathbf{Z})_x.$$

We call it the Maslov class of the isotropic map-germ f.

We can define the notion of symplectic equivalence between two isotropic map-germs in a similar manner to that in §3.

If (σ, τ) is a symplectic equivalance between f and $f' : N', x' \longrightarrow (M', \omega')$, then $\sigma^* : H^2(N', N' - \Sigma'; \mathbf{Z}) \longrightarrow H^2(N, N - \Sigma; \mathbf{Z})$ maps $m(f')$ to $m(f)$.

§5. Symplectic reduction

Let $(M^{2(n+k)}, \omega)$ be a symplectic manifold of dimension $2(n + k)$, and $K^{2n+k} \subset M$ be a coisotropic submanifold of codimension k. We denote by $(TK)^\perp$ the skew-orthogonal complement to TK in $TM|K$.

Remark that the rank of $(TK)^\perp$ is equal to k. Since K is coisotropic, $(TK)^\perp \subset TK$, and hence $(TK)^\perp$ is integrable ([AM]). We call $(TK)^\perp$ (resp. induced foliation on K) the characteristic distribution (resp. foliation) relative to K.

Let $x \in K$. Then, in an open neighborhood U of x in K, a submersion $\pi : U \longrightarrow {M'}^{2n}$ is induced, where M' is the leaf space. Then M' has a unique symplectic structure ω', up to symplectic diffeomorphisms of M', such that $\pi^*\omega' = \omega|K$ ([AM]). M' is called the reduction of M by K.

(1) By this reduction procedure, Lagrangian submanifolds of M also reduces to "Lagrangian varieties".

Now, let $L^{n+k} \subset M$ be a Lagrangian submanifold and $x \in L$. If $N = L \cap K$ is an n-dimensional submanifold of K in a neighborhood of x, then $f = \pi|N : N, x \longrightarrow M'$ is an isotropic map-germ. In fact, $f^*\omega' = \pi^*\omega'|N = \omega \mid N = 0$.

Remark that f is an immersion at x if and only if $T_xL \cap (T_xK)^\perp = 0$. In particular, if L is transverse to K, then we have an immersed Lagrangian submanifold in the reduced symplectic manifold M'.

In fact, in this case, $T_xL \cap (T_xK)^\perp = T_xL \cap (T_xL + T_xK)^\perp = T_xL \cap (T_xM)^\perp = T_xL \cap \{0\} = \{0\}$.

In general, f is not an immersion and has a singularity.

Definition 5.1. Let f be the same as in the above. Then f is called an isotropic map-germ arising from a k-dimensional reduction of a Lagrangian manifold.

(2) Moreover, let N^n be an isotropic submanifold of $M^{2(n+k)}$ contained in K^{2n+k} and containing x. Then $f = \pi|N : N, x \longrightarrow M'$ is isotropic, and f is immersive if and only if $T_xN \cap (T_xK)^\perp = 0$.

Definition 5.2. Such a germ f is simply called an isotropic map-germ arising from k-dimensional reduction.

§6. Reduction of a Lagrangian manifold and Maslov class

In §4, we have defined the Maslov class $m(f) \in H^2(N, N - \Sigma; \mathbf{Z})_x$ for an isotropic map germ $f : N^n, x \longrightarrow M^{2n}$, where Σ is the singular set of f.

Theorem 6.1. *Let $f : N, x \longrightarrow M$ be an isotropic map-germ. If f is symplectically equivalent to an isotropic map-germ arising from a one-dimensional reduction of a Lagrangian manifold, then $m(f) = 0 \in H^2(N, N - \Sigma; \mathbf{Z})_x$.*

Precisely, for any open neighborhood U of x, and for any respersentative $f : U \longrightarrow M$ of f, there exist a contractible open neighborhood V such that $x \in V \subset U$ and $m(f|V) = 0$ in $H^2(V, V - \Sigma; \mathbf{Z})$.

§7. Local models for generic one-dimensional reductions

We consider, by symplectic equivalences, a generic local classification of isotropic mappings arising from symplectic reduction relative to a hypersurface (i.e., one-dimensional reduction) (see Definition 5.2).

Let (M^{2n+2}, ω) be a symplectic manifold, K^{2n+1} be a hypersurface of M, and N^n be an n-dimensional manifold.

Denote by $\mathcal{I}$ the set of isotropic embeddings $i : N \longrightarrow M$ with $i(N) \subset K$, endowed with the Whitney C^∞ topology.

Next we introduce special isotropic map-germs $f_{n,k}$ as local models for singularities of isotropic mappings. Consider the cotangent bundle $T^*\mathbf{R}^n$ with canonical coordinates $q_1, \dots, q_n; p_1 \dots, p_n$ and with the symplectic form $\omega = \sum_i dp_i \wedge dq_i$. Besides, consider the space $\mathbf{R}^n$ with coordinates $x_1, \dots, x_n$. Then

$$f_{n,k} : \mathbf{R}^n, 0 \longrightarrow T^*\mathbf{R}^n, \ 0 \leq k \leq \left[\frac{n}{2}\right],$$

is defined by

$$q_i \circ f_{n,k} = x_i, \ 1 \leq i \leq n-1,$$

$$u = q_n \circ f_{n,k} = \frac{x_n^{k+1}}{(k+1)!} + \sum_{i=1}^{k-1} x_i \frac{x_n^{k-i}}{(k-i)!},$$

$$v = p_n \circ f_{n,k} = \sum_{i=0}^{k-1} x_{k+i} \frac{x_n^{k-i}}{(k-i)!},$$

and

$$p_j \circ f_{n,k} = \int_0^{x_n} \left(\frac{\partial v}{\partial x_j} \frac{\partial u}{\partial x_n} - \frac{\partial v}{\partial x_n} \frac{\partial u}{\partial x_j} \right) dx_n, \ 1 \leq j \leq n-1.$$

Remark that each $f_{n,k}$ is a polynomial mapping of kernel rank one and of very simple form.

Theorem 7.1. *There exists an open dense subset $\mathcal{G}$ in $\mathcal{I}$ such that, for each $i \in \mathcal{G}$ and for each $x \in N$, the isotropic map-germ $f : N^n, x \longrightarrow M'^{2n}$ arising from one-dimensional reduction relative to K is symplectically equivalent to some $f_{n,k}, 0 \leq k \leq [n/2]$.*

§8. Maslov class of an open Whitney umbrella

Let us study properties of local models $f_{n,k} : \mathbf{R}^n, o \longrightarrow T^*\mathbf{R}^n, 0 \leq k \leq [n/2]$.

For $k = 0$, $f_{n,0}$ is just the zero-section $\zeta_n : \mathbf{R}^n, 0 \longrightarrow T^*\mathbf{R}^n$ and is an immersion.

For $k \neq 0$, we easily verify that

$$\Sigma = \Sigma(f_{n,k}) = \{\partial u / \partial x_n = \partial v / \partial x_n = 0\}$$

is a submanifold of codimension 2 in $\mathbf{R}^n$. Thus we have

$$(*) : H^2(\mathbf{R}^n, \mathbf{R}^n - \Sigma; \mathbf{Z})_0 \cong H^1(\mathbf{R}^n - \Sigma; \mathbf{Z})_0 \cong \mathbf{Z}.$$

By defnition, we can write $f_{n,k} = f_{2k,k} \times \zeta_{n-2k}$. Then $f_{n,k}$ is a "suspension" of $f_{2k,k}$.

Definition 8.1. $f_{2n,n}$ is called a $2n$-dimensional open Whitney umbrella.

Remark 8.2. $f_{2,1}$ is just the (2-dimensional) open Whitney umbrella introduced by Givental' [G].

For Maslov classes, we have

Theorem 8.3. *Under the identification* $(*)$,

$$m(f_{n,k}) = \begin{cases} 0 & \text{for} \quad k = 0, \\ \pm 2 & \text{for} \quad 0 < k \leq \left[\dfrac{n}{2}\right]. \end{cases}$$

Corollary 8.4. *For a generic, that is, belonging to $\mathcal{G}$ in Theorem* 7.1, *isotropic submanifold $i : N^n \longrightarrow K^{2n+1} \subset M^{2n+2}$, if T_xN contains the characteristic direction of K at a point $x \in N$, then N is never representable as an intersection of any Lagrangian submanifold and K, as germ at x*

Proof. If N were an intersection of a Lagrangian submanifold and K, then the Maslov class of isotropic map-germ arising from the reduction relative to K would vanish by Theorem 6.1.

By Theorem 7.1, that map-germ is symplectically equivalent to some $f_{n,k}, k \neq 0$. By Theorem 8.3, $m(f_{n,k}), k \neq 0$, does not vanish. Combined with the argument in §4, this leads to a contradiction. Q.E.D.

§9. Reduction of symplectic vector bundles

(1) Let $\mathbf{E}$ be a symplectic vector bundle over a manifold X, and $\mathbf{K}$ be a coisotropic subbundle. Then the bundle $\mathbf{K}/\mathbf{K}^\perp$ has the induced symplectic structure, where $\mathbf{K}^\perp$ is the skew-orthogonal complement of $\mathbf{K}$ in $\mathbf{E}$ (see [AM],[W]).

Let $\mathbf{L}$ be a Lagrangian subbundle of $\mathbf{E}$. If $\mathbf{L} \subset \mathbf{K}$, then $\mathbf{K}^\perp \subset \mathbf{L}^\perp = \mathbf{L} \subset \mathbf{K}$, and $\mathbf{L}/\mathbf{K}^\perp \subset \mathbf{K}/\mathbf{K}^\perp$ is a Lagrangian subbundle.

Lemma 9.1. *Let* $\mathbf{E}$ *be a symplectic vector bundle over* X, $\mathbf{K}$ *a coisotropic subbundle of* $\mathbf{E}$, *and* $\mathbf{L}$ *(resp.* $\mathbf{L}'$*) a Lagrangian subbundle contained in* $\mathbf{K}$. *Then* $m(\mathbf{K}/\mathbf{K}^{\perp}; \mathbf{L}/\mathbf{K}^{\perp}, \mathbf{L}'/\mathbf{K}^{\perp}) = m(\mathbf{E}; \mathbf{L}, \mathbf{L}')$ *in* $H^1(X; \mathbf{Z})$ *(cf.*§1*).*

Proof. Set $\operatorname{rank}\mathbf{E} = 2(n+k)$ and $\operatorname{rank}\mathbf{K} = 2n+k$. Then $\operatorname{rank}\mathbf{K}^{\perp} = k$. Compare $\Phi_1 = \Phi(\mathbf{L}) \circ s(\mathbf{L}') : X \longrightarrow \Lambda(n+k)$ and $\Phi_2 = \Phi(\mathbf{L}/\mathbf{K}^{\perp}) \circ s(\mathbf{L}'/\mathbf{K}^{\perp}) : X \longrightarrow \Lambda(n)$.

Set $\Lambda(n+k, k) = \{\lambda \in \Lambda(\mathbf{C}^{n+k}) \mid \lambda \subset \mathbf{C}^n \times \mathbf{R}^k\}$. Then we can choose a Hermitian structure on $\mathbf{E}$ such that $\Phi_1(X) \subset \Lambda(n+k, k)$ and $\tilde{\pi} \circ \Phi_1 = \Phi_2$, where $\pi : \mathbf{C}^n \times \mathbf{C}^k \longrightarrow \mathbf{C}^n$ is the projection and $\tilde{\pi} : \Lambda(n+k,k) \longrightarrow \Lambda(n)$ is defined by $\tilde{\pi}(\lambda) = \pi(\lambda) \subset \mathbf{C}^n$ $(\lambda \in \Lambda(n+k,k))$. Remark that $\det^2 \circ \tilde{\pi} = \det^2 : \Lambda(n+k,k) \longrightarrow S^1$. Then $\det^2 \circ \Phi_2 = \det^2 \circ \tilde{\pi} \circ \Phi_1 = \det^2 \circ \Phi_1$. Thus we have the required result.

(2) We apply Lemma 9.1 to the situation of §5,(1).

Shrinking K around x if necessary, we assume that the characteristic foliation of K comes from a submersion $\pi : K \longrightarrow M'$, $N = L \cap K$ is an n-dimensional submanifold in K, and that K and N are contractible.

Set $\mathbf{E} = TM|N - \Sigma, \mathbf{K} = TK|N - \Sigma, \mathbf{K}' = (TN)^{\perp}|N - \Sigma, \mathbf{L} = TL|N - \Sigma$, and $\mathbf{L}' = TN + (TK)^{\perp}|N - \Sigma$.

Note that $TN + (TK)^{\perp}$ is a direct sum in TM over $N - \Sigma$. Therefore, $\mathbf{L}'$ is a subbundle of $\mathbf{E}$ of rank $n+k$. Furthermore, $\mathbf{L}'^{\perp} = \mathbf{K}' \cap \mathbf{K} \supset \mathbf{L}'$. Hence $\mathbf{L}'$ is Lagrangian.

Thus we have a symplectic vector bundle $\mathbf{E}$, coisotropic subbundles $\mathbf{K}$ and $\mathbf{K}'$ of rank $2n+k$ and $n+2k$, respectively, and Lagrangian subbundles $\mathbf{L}$ and $\mathbf{L}'$ with $\mathbf{L} \subset \mathbf{K}', \mathbf{L}' \subset \mathbf{K}$, and $\mathbf{L}' \subset \mathbf{K}'$.

Since M' is a symplectic reduction of M relative to K, we have an isomorphism $\alpha : TK/(TK)^{\perp} \longrightarrow \pi^* TM'$, which induces an isomorphism $\beta : TK/(TK)^{\perp}|N \longrightarrow f^*TM'$.

For each $y \in N - \Sigma$, $\beta(\mathbf{L}'_y/\mathbf{K}^{\perp}_y) = T_y f(T_y N) = (\mathbf{L}_f)_y$ in the fiber $(f^*TM')_y$ over y. By restriction, β induces an isomorphism

$$\gamma : \mathbf{K}/\mathbf{K}^{\perp} \longrightarrow f^*TM'|N - \Sigma$$

such that $\gamma(\mathbf{L}'/\mathbf{K}^{\perp}) = \mathbf{L}_f$.

Therefore, for a Lagrangian subbundle $\mathbf{L}_1$ of f^*TM' over N,

$$m(f^*TM'; \mathbf{L}_1, \mathbf{L}_f) = m(\mathbf{K}/\mathbf{K}^{\perp}; \beta^{-1}(\mathbf{L}_1), \mathbf{L}'/\mathbf{K}^{\perp})$$

in $H^1(N - \Sigma; \mathbf{Z})$, by Lemma 1.1.(3).

Take the Lagrangian subbundle $\mathbf{L}_2$ of $TM|N$ contained in $TK|N$ which projects to $\beta^{-1}(\mathbf{L}_1) \subset (TK/(TK)^{\perp})|N$. Then, by Lemma 9.1,

$$m(\mathbf{K}/\mathbf{K}^{\perp}; \beta^{-1}(\mathbf{L}_1), \mathbf{L}'/\mathbf{K}^{\perp}) = m(\mathbf{E}; \mathbf{L}_2, \mathbf{L}').$$

By Lemma 1.1.(1),

$$m(\mathbf{E}; \mathbf{L}_2, \mathbf{L}') = m(\mathbf{E}; \mathbf{L}_2, \mathbf{L}) + m(\mathbf{E}; \mathbf{L}, \mathbf{L}').$$

Since $\mathbf{L} = TL|N - \Sigma$ is a restriction of the Lagrangian subbundle $TL|N$ over N, $m(\mathbf{E}; \mathbf{L}_2, \mathbf{L})$ is the restriction of an element in $H^1(N; \mathbf{Z})$.

Since these arguments are valid over any contractible neighborhood V of x in N, we have Theorem 6.1 if $m(\mathbf{E}; \mathbf{L}, \mathbf{L}') = 0$ in $H^1(N - \Sigma, \mathbf{Z})_x$.

Furthermore, using Lemma 9.1 again, we see

$$m(\mathbf{E}; \mathbf{L}, \mathbf{L}') = m(\mathbf{K}'/\mathbf{K}'^{\perp}; \mathbf{L}/\mathbf{K}'^{\perp}, \mathbf{L}'/\mathbf{K}'^{\perp}).$$

In the next section, we will show that the right hand side is equal to zero in $H^1(N - \Sigma, \mathbf{Z})_x$, at least if $k = 1$.

§10. Proof of Theorem 6.1

It is sufficient to show $m(\mathbf{K}'/\mathbf{K}'^{\perp}; \mathbf{L}/\mathbf{K}'^{\perp}, \mathbf{L}'/\mathbf{K}'^{\perp}) = 0$ in $H^1(V - \Sigma; \mathbf{Z})$ for any sufficiently small contractible neighborhood V of x, in the notation of §9,(2).

Let $h = 0$ be a local equation of K in M, where $h \in C^{\infty}(M)$. By the sign of h, $L - N$ is devided into two: $L - N = L_+ \cup L_-, L_{\pm} = \{y \in N \mid \pm h(y) > 0\}$.

Take a vector field v tangent to L toward L_+ at $x \in N$. Then $dh(v) \geq 0$.

Let w be the Hamiltonian vector field with Hamiltonian h. Then the imaginary part of $G(w, v)$ is equal to $\Omega(w, v) = (w \rfloor \Omega)(v) = -dh(v) \leq 0$.

Remark that normalized v (resp. w) turns into an orthonormal frame of $\mathbf{L}/\mathbf{K}'^{\perp}$ (resp. $\mathbf{L}'/\mathbf{K}'^{\perp}$). Therefore, for $\Phi : V - \Sigma \longrightarrow \Lambda(1) = U(1)/O(1)$ in the definition of Maslov class, we see that $\det \circ \Phi : V - \Sigma \longrightarrow S^1 \subset \mathbf{C}$ has non-positive imaginary part. Therefore, $\det \circ \Phi$ is homotopically zero, and so is $\det^2 \circ \Phi$. Thus

$$m(\mathbf{K}'/\mathbf{K}'^{\perp}; \mathbf{L}/\mathbf{K}'^{\perp}, \mathbf{L}'/\mathbf{K}'^{\perp}) = (\det^2 \circ \Phi)^* 1 = 0$$

in $H^1(V - \Sigma; \mathbf{Z})$ for any sufficiently small contractible neighborhood V of x in N. Q.E.D.

§11. Variety of singular isotropic jets

Let N be a manifold of dimension n and M be a symplectic manifold of dimension $2n$. In the k-jet bundle $J^k(N, M)$, we set

$$J_I^k(N, M) = \{j^k f(x) \in J^k(N, M) \mid f : N, x \longrightarrow M \text{ is isotropic}\},$$

and

$$\widetilde{\Sigma} = \{j^1 f(x) \in J_I^1(N, M) \mid f : N, x \longrightarrow M \text{ is not an immersion}\}.$$

Further, set

$$\widetilde{\Sigma}^i = \{j^1 f(x) \in J_I^1(N, M) \mid \text{ dim Ker } T_x f = i\}.$$

Then we have

$$\widetilde{\Sigma} = \bigcup_{i=1}^{n} \widetilde{\Sigma}^i.$$

Set

$$\bar{\Sigma}^j = \bigcup_{i=j}^{n} \widetilde{\Sigma}^i.$$

Proposition 11.1. *The set $J_I^1(N, M) - \bar{\Sigma}^2$ of isotropic 1-jets with kernel dimension ≤ 1 is a submanifold of $J^1(N, M)$. Further, $\widetilde{\Sigma}^1$ is a submanifold of $J_I^1(N, M) - \bar{\Sigma}^2$ of codimension 2.*

Set $V = \mathrm{Hom}_{\mathbf{R}}(\mathbf{R}^n, \mathbf{C}^n) \cong M_n(\mathbf{C})$.

Let $\langle \cdot, \cdot \rangle$ denote the standard Hermitian structure on $\mathbf{C}^n$. Define the symplectic structure $[\cdot, \cdot]$ on $\mathbf{C}^n$ by $[u, v] = \mathrm{Im}\, \langle u, v \rangle, u, v \in \mathbf{C}^n$. Let $X \subset V$ be the set of isotropic linear maps $\mathbf{R}^n \longrightarrow \mathbf{C}^n$, and $\Sigma^i \subset X$ be the set of isotropic linear maps $\mathbf{R}^n \longrightarrow \mathbf{C}^n$ with kernel dimension i. Set $S^j = \bigcup_{i=j}^{n} \Sigma^i$.

To prove Proposition 11.1, it is sufficient to show

Lemma 11.2. *X is a real algebraic variety in V, with $\mathrm{Sing}(X) \subset S^2$. Further, Σ^i is a submanifold of V of dimension $(1/2)\{n(3n+1) - i(3i+1)\}$. In particular, Σ^1 is a submanifold of codimension 2 in $X - S^2$.*

Proof. Denote by $\mathrm{Alt}(n)$ the set of skewsymmetric bilinear forms $a : \mathbf{R}^n \times \mathbf{R}^n \longrightarrow \mathbf{R}$ on $\mathbf{R}^n$ and by $Sp(n)$ the group of symplectic linear isomorphisms on $(\mathbf{C}^n, [\,,\,])$.

Set $G = GL(n, \mathbf{R}) \times Sp(n)$. Define G-actions on $V = \mathrm{Hom}_{\mathbf{R}}(\mathbf{R}^n, \mathbf{C}^n)$ and $\mathrm{Alt}(n)$ by

$$(\sigma, \tau)\ell = \tau \circ \ell \circ \sigma^{-1},$$

$$(\sigma, \tau)a = a \circ (\sigma^{-1} \times \sigma^{-1})$$

for $(\sigma, \tau) \in G, \ell \in V, a \in \mathrm{Alt}(n)$, respectively.

Consider the map $\rho : V \longrightarrow \mathrm{Alt}(n)$ defined by $\rho(\ell)(u, v) = [\ell u, \ell v]$, $\ell \in V, u, v \in \mathbf{R}^n$. Then ρ is a G-equivariant polynomial map and $X = \rho^{-1}(O)$. In particular, X is a real algebraic variety.

Let $\ell \in X$. Then $\mathrm{rank}\,(\ell) = i, 0 \leq i \leq n$ if and only if there exists $g \in G$ such that

$$g \cdot \ell = \begin{pmatrix} E_i & O \\ O & O \end{pmatrix}.$$

In fact, if $\mathrm{rank}\,(\ell) = i$, then there exists $\tau \in U(n) \subset Sp(n)$ such that $\tau(\mathrm{image}\,\ell) = \mathbf{R}^i \times 0 \subset \mathbf{C}^n$. Thus, for some $\sigma \in GL(n, \mathbf{R})$, $\tau \circ \ell \circ \sigma^{-1} : \mathbf{R}^n \longrightarrow \mathbf{C}^n$ is the projection to $\mathbf{R}^i \times 0 \subset \mathbf{C}^n$. The converse is clear.

Remark that the matrix representation of ρ is

$$A + \sqrt{-1}B \mapsto {}^tBA - {}^tAB \in \mathrm{Alt}(n), \quad A, B \in M_n(\mathbf{R}).$$

Let ℓ be isotropic. Then ρ is a submersion at ℓ if and only if $\mathrm{rank}\,(\ell) \geq n-1$. To see this, we may assume

$$\ell = \begin{pmatrix} E_i & O \\ O & O \end{pmatrix}$$

without loss of generality. The tangent map of ρ at ℓ,

$$T_\ell(\rho) : T_\ell V \longrightarrow T_{\rho(\ell)}\,\mathrm{Alt}(n),$$

is described by

$$A' + \sqrt{-1}B' \mapsto {}^tB'\ell - {}^t\ell B' = \begin{pmatrix} {}^tB_{11} - B_{11} & -B_{12} \\ {}^tB_{12} & O \end{pmatrix},$$

where

$$B' = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix},$$

B_{11}, B_{12}, B_{21} and B_{22} are real matrices of type $(i,i), (i, n-i), (n-i, i)$ and $(n-i, n-i)$, respectively. Therefore, $T_\ell(\rho)$ is surjective if and only if $i = n$ or $i = n-1$. Thus $\mathrm{Sing}(X) \subset S^2$.

Define a subset Y of $X \times \Lambda(n)$ to be the totality of pairs (ℓ, λ) such that the image of ℓ is contained in λ. The projection $Y \to \Lambda(n)$ is a fibration with fiber diffeomorphic to $M_n(\mathbf{R}) = \mathrm{Hom}_{\mathbf{R}}(\mathbf{R}^n, \mathbf{R}^n)$. Set $Y^i = Y \cap (\Sigma^i \times \Lambda(n))$. Then Y^i is a submanifold of Y of codimension i^2. On the other hand, the projection $Y^i \to X \subset V$ is a fibration with fiber

$\Lambda(i)$, that is, of constant kernel dimension $(1/2)i(i+1)$. Therefore, Σ^i is a submanifold of V and $\dim \Sigma^i = \dim Y^i - (1/2)i(i+1) = \dim Y - (1/2)i(3i+1) = (1/2)n(3n+1) - (1/2)i(3i+1)$. Q.E.D.

§12. Universal Maslov class

The calculation of Maslov classes of isotropic map-germs can be reduced to that in jet spaces.

Define $\Psi : J^1_I(N, M) - \widetilde{\Sigma} \longrightarrow \Lambda(M)$ by

$$\Psi(j^1 f(x)) = T_x f(T_x N) \subset T_{f(x)} M$$

for $j^1 f(x) \in J^1_I(N, M) - \widetilde{\Sigma}$.

Remark that $\Psi \circ j^1 f = \varphi(f)$ (see §2).

Definition 12.1. The universal Maslov class of an isotropic 1-jet $z = j^1 f(x)$ is defined by

$$m(z) = \delta(m(\Psi^* \pi^* TM; \mathbf{L}, \Psi^* \mathcal{L})) \in H^2(J^1_I(N, M), J^1_I(N, M) - \widetilde{\Sigma}; \mathbf{Z})_z,$$

where $\mathbf{L}$ is a Lagrangian subbundle of $\Psi^* \pi^* TM | U$ over a contractible neighborhood of z in $J^1_I(N, M)$ and $\mathcal{L} \subset \pi^* TM$ is the tautological Lagrangian subbundle over $\Lambda(M)$.

Lemma 12.2. *Let $f : N, x \longrightarrow M$ be an isotropic map-germ. Then $j^1 f : N, x \longrightarrow J^1_I(N, M)$ induces*

$$(j^1 f)^* : H^2(J^1_I(N, M), J^1_I(N, M) - \widetilde{\Sigma}; \mathbf{Z})_{j^1 f(x)} \longrightarrow H^2(N, N - \Sigma; \mathbf{Z})_x,$$

which maps $m(j^1 f(x))$ to $m(f)$.

Proof. We have

$$\begin{aligned}
(j^1 f)^* m(j^1 f(x)) &= (j^1 f)^* \delta m(\mathbf{L}, \Psi^* \mathcal{L}) \\
&= \delta m((j^1 f)^* \mathbf{L}, (\Psi \circ j^1 f)^* \mathcal{L}) \\
&= \delta m((j^1 f)^* \mathbf{L}, (\varphi(f))^* \mathcal{L}) \\
&= \delta m((j^1 f)^* \mathbf{L}, \mathbf{L}_f) \\
&= m(f).
\end{aligned}$$

§13. Calculation of an universal Maslov class

Proposition 13.1. *Let $z \in \widetilde{\Sigma}^1$. Then*

$$H^2(J^1_I(N,M), J^1_I(N,M) - \widetilde{\Sigma}; \mathbf{Z})_z \cong \mathbf{Z},$$

and $m(z) = \pm 2$.

Proof. The first half is clear from Proposition 11.1.

To see the second half, without loss of generality, we may assume that $z = j^1 f(0) \in J^1_I(\mathbf{R}^n, T^*\mathbf{R}^n)$ with $q_i \circ f = x_i, 1 \le i \le n-1, q_n \circ f = 0, p_i \circ f = 0, 1 \le i \le n$. Define $c : \mathbf{R}^2 \longrightarrow J^1_I(\mathbf{R}^n, T^*\mathbf{R}^n)$ by

$$c(t,s) = j^1(x_1, \ldots, x_{n-1}, tx_n; 0, \ldots, 0, sx_n)(0)$$

for $(t,s) \in \mathbf{R}^2$. Then $c(0) = z$ and c is transverse to $\widetilde{\Sigma}^1$.

Take a small loop $\ell_\varepsilon : S^1 \longrightarrow J^1_I(\mathbf{R}^n, T^*\mathbf{R}^n)$, where $\ell_\varepsilon(e^{i\theta}) = c(\varepsilon\cos\theta, \varepsilon\sin\theta)$. Then ℓ_ε is a generator of $H_1(J^1_I(\mathbf{R}^n, T^*\mathbf{R}^n) - \widetilde{\Sigma}; \mathbf{Z})_z$. Thus $|m(z)|$ is determined by the evaluation to ℓ_ε. Remark that $\Psi \circ \ell_\varepsilon$ is represented by

$$e^{i\theta} \mapsto \begin{pmatrix} E_{n-1} & O \\ O & e^{i\theta} \end{pmatrix} \in U(n).$$

Thus $\det^2 \circ \Psi \circ \ell_\varepsilon : S^1 \longrightarrow S^1$ is of degree 2. Therefore, $|m(z)| = 2$.

§14. Proof of Theorem 8.3

Lemma 14.1. *Let $f : N, x \longrightarrow M$ be isotropic. If $j^1 f(x) \in \widetilde{\Sigma}^1$ and $j^1 f$ is transverse to $\widetilde{\Sigma}^1$ in $J^1_I(N,M)$. Then*

$$H^2(N, N - \Sigma; \mathbf{Z})_x \cong \mathbf{Z},$$

and $m(f) = \pm 2$, where $\Sigma = \Sigma(f) = (j^1 f)^{-1}(\widetilde{\Sigma})$.

Proof. Since $j^1 f$ is transverse to $\widetilde{\Sigma}^1$, and $\Sigma = (j^1 f)^{-1}(\widetilde{\Sigma}^1)$, we see that Σ is a submanifold of codimension 2 in N near x, and

$$H^2(N, N-\Sigma; \mathbf{Z})_x \cong H^2(J^1_I(N,M), J^1_I(N,M) - \widetilde{\Sigma}; \mathbf{Z})_{j^1 f(x)} \cong \mathbf{Z}.$$

Since $m(j^1 f(x)) = \pm 2$ by Proposition 13.1, we have

$$m(f) = (j^1 f)^* m(j^1 f(x)) = \pm 2,$$

relatively to the above isomorphism.

By checking the 2-jets of $f_{n,k}$ at 0, we easily verify the following

Lemma 14.2. *The 1-jet extension $j^1 f_{n,k}, k \neq 0$, is transverse to $\widetilde{\Sigma}^1$ in $J^1_I(\mathbf{R}^n, T^*\mathbf{R}^n)$ at $j^1 f_{n,k}(0)$.*

Now, Theorem 8.3 follows from Lemmata 14.1 and 2.

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On Symmetry Groups of the MIC-Kepler Problem and Their Unitary Irreducible Representations

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It is well known that the quantized Kepler problem (i.e., the hydrogen atom) admits the symmetry groups, $SO(4)$, $E(3)$ (the Euclidean motion group), or $SO^+(1,3)$ (the proper Lorentz group), according as the energy is negative, zero, or positive (cf. [B-I]). The symmetry groups here stand for Lie groups which act unitarily irreducibly on the Hilbert spaces associated with the energy-spectrum for the Kepler problem. However, only a part of the unitary irreducible representations are realized as the symmetry group for the Kepler problem. A question now arises: Are the other unitary irreducible representations realizable as symmetry groups for a "modified" Kepler problem ?

This question is worked out in this article. Both in classical and quantum mechanics, the Kepler problem is generalized to the MIC-Kepler problem, the Kepler problem along with a centrifugal potential and Dirac's monopole field, which is named after McIntosh and Cisneros [MI-C]. It will be shown that the quantized MIC-Kepler problem exhausts almost all the unitary irreducible representations of $SU(2) \times SU(2)$, $\mathbf{R}^3 \ltimes SU(2)$, or $SL(2,\mathbf{C})$ as the symmetry group, according as the energy is negative, zero, or positive, which groups are the double covers of $SO(4)$, $E(3)$, and $SO^+(1,3)$, respectively. For $SL(2,\mathbf{C})$, the principal series representations are all realizable, but not the others.

§1. Setting up the quantized MIC-Kepler problem

The MIC-Kepler problem is to be defined as a reduced system of the conformal Kepler problem. Consider the principal $U(1)$ bundle $\pi : \mathbf{R}^4 - \{0\} \to \mathbf{R}^3 - \{0\}$ whose projection π and $U(1)$ action Φ_t are given, respectively, by

$$(1.1) \qquad \pi(q) = (2(q_1q_3 + q_2q_4),\ 2(-q_1q_4 + q_2q_3),\ q_1^2 + q_2^2 - q_3^2 - q_4^2),$$

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and

$$\Phi_t : q \longmapsto T(t)q \tag{1.2}$$

with

$$T(t) = \begin{pmatrix} N & \\ & N \end{pmatrix} \quad \text{with} \quad N = \begin{pmatrix} \cos\frac{t}{2} & -\sin\frac{t}{2} \\ \sin\frac{t}{2} & \cos\frac{t}{2} \end{pmatrix} \quad t \in [0, 4\pi], \tag{1.3}$$

where $(q_j)_{j=1,2,3,4}$ are the Cartesian coordinates in $\mathbf{R}^4$. The missing matrix entries are all zero, here and henceforth.

For any fixed integer m, let ρ_m be the unitary irreducible representation of $U(1)$ on $\mathbf{C}$,

$$\rho_m : T(t) \longmapsto e^{imt/2}, \quad t \in [0, 4\pi]. \tag{1.4}$$

Then the associated complex line bundle $L_m = (\mathbf{R}^4 - \{0\} \times_m \mathbf{C},\ \pi_m,\ \mathbf{R}^3 - \{0\})$ is formed through the representation ρ_m. Note that, contrary to the literature [K-N], the left action is under consideration.

The standard connection on $\mathbf{R}^4 - \{0\}$ gives rise to the linear connection ∇ for L_m, the curvature of which, Ω_m, takes the form

$$\Omega_m = \frac{im}{2r^3}(x_1 dx_2 \wedge dx_3 + x_2 dx_3 \wedge dx_1 + x_3 dx_1 \wedge dx_2), \tag{1.5}$$

where $\pi(q) = (x_j)_{j=1,2,3}$ are the Cartesian coordinates in $\mathbf{R}^3 - \{0\}$ and $r^2 = \sum_{j=1}^{j=3} x_j^2$. The Ω_m describes Dirac's monopole field of strength $-m/2$. The *MIC-Kepler problem* is then defined on the complex line bundle L_m.

Definition. The MIC-Kepler problem is a quantum system defined on L_m together with the Hamiltonian operator

$$\widehat{H}_m = -\frac{1}{2}\sum_{j=1}^{3} {\nabla_j}^2 + \frac{(m/2)^2}{2r^2} - \frac{k}{r} \tag{1.6}$$

acting on cross sections in L_m, where ∇_j stands for the covariant derivation, $\nabla_{\partial/\partial x_j}$, and k is a positive constant.

The reduction process giving this definition proceeds as follows: The *conformal Kepler problem* is defined as a quantum system with the Hamiltonian operator

$$\widehat{H} = -\frac{1}{2}\left(\frac{1}{4r}\sum_{\ell=1}^{4} \frac{\partial^2}{\partial q_\ell^2}\right) - \frac{k}{r} \tag{1.7}$$

acting on the functions on $\mathbf{R}^4 - \{0\}$, where $r = \sum_{\ell=1}^{\ell=4} q_\ell^2$.

A function $f(q)$ on $\mathbf{R}^4 - \{0\}$ is referred to as *ρ_m-equivariant*, if it satisfies

$$f(T(t)q) = e^{imt/2} f(q), \quad t \in [0, 4\pi]. \tag{1.8}$$

The ρ_m-equivariant functions are in one-to-one correspondence with the cross sections in L_m. Then, on denoting by q_m the correspondence of the ρ_m-equivariant functions to the cross sections in L_m, one has

$$\widehat{H}_m = q_m \circ \widehat{H} \circ q_m^{-1}, \tag{1.9}$$

which turns out to be expressed as (1.6).

Since our interest centers on quantum systems only, the adjective "quantized" is to be omitted. Further, for convenience' sake, we will often abbreviate the MIC-Kepler problem and the conformal Kepler problem to MICK and CK, respectively.

Equation (1.9) is the relation on the base of which we study symmetry groups for the MIC-Kepler problem in each case of energy, negative, zero, or positive. The procedure is as follows:

(1) Find symmetry groups of the conformal Kepler problem. As the results, the harmonic oscillator, a free particle, or the repulsive oscillator are associated with CK, according respectively as the energy of CK is negative, zero, or positive. These symmetry groups are represented in Hilbert spaces labeled with the energies of CK.
(2) Equation (1.9) shows that the subspace of ρ_m-equivariant functions in the representation space for the symmetry group of CK reduces to the Hilbert space of cross-sections in L_m associated with each of the spectra of the MIC-Kepler problem. Through this reduction, a symmetry group of MICK turns out to be given by a subgroup of the symmetry group of CK that leaves invariant each subspace of ρ_m-equivariant functions.
(3) Prove the irreducibility of the representations of the symmetry groups of MICK.

There is another way to study the quantized Kepler and MIC-Kepler problem. For negative energies, the geometric quantization method provides the negative energy eigenvalues [S, Ml-T, Ml]. However, the geometric quantization turns no attention to zero or positive energy, nor to the relation with representation of symmetry groups.

§2. The negative energy case and $SU(2) \times SU(2)$

2.1. A symmetry group of the conformal Kepler problem with negative energy

Following Procedure (1)–(3) presented in Section 1, we start with a symmetry group of CK with negative energy. It is of great help to associate CK with the four-dimensional harmonic oscillator, which is the quantum system with the Hamiltonian operator

$$\widehat{J}_\lambda = -\frac{1}{2}\sum_{j=1}^{4}\frac{\partial^2}{\partial q_j{}^2} + \frac{\lambda^2}{2}\sum_{j=1}^{4} q_j^2, \tag{2.1}$$

where λ is a positive parameter. The harmonic oscillator will be often abbreviated to HO, henceforth. $\widehat{H}$ and $\widehat{J}_\lambda$ satisfy

$$4r\left(\widehat{H} + \frac{\lambda^2}{8}\right) = \widehat{J}_\lambda - 4k. \tag{2.2}$$

This means that the eigenfunctions of CK with negative eigenvalue $-\lambda^2/8$ are obtained from eigenfunctions of HO with positive eigenvalue $4k$. Thus to find the symmetry group of CK for the eigenvalue $-\lambda^2/8$ is to find that of HO for the eigenvalue $4k$. Let us define the creation operator $(a_j^\dagger)_{j=1,2,3,4}$ for the harmonic oscillator by

$$\begin{aligned}
a_1^\dagger &= \frac{1}{2\sqrt{\lambda}}\left(\lambda q_1 - i\lambda q_2 - \frac{\partial}{\partial q_1} + i\frac{\partial}{\partial q_2}\right),\\
a_2^\dagger &= \frac{1}{2\sqrt{\lambda}}\left(\lambda q_3 - i\lambda q_4 - \frac{\partial}{\partial q_3} + i\frac{\partial}{\partial q_4}\right),\\
a_3^\dagger &= \frac{1}{2\sqrt{\lambda}}\left(\lambda q_1 + i\lambda q_2 - \frac{\partial}{\partial q_1} - i\frac{\partial}{\partial q_2}\right),\\
a_4^\dagger &= \frac{1}{2\sqrt{\lambda}}\left(\lambda q_3 + i\lambda q_4 - \frac{\partial}{\partial q_3} - i\frac{\partial}{\partial q_4}\right).
\end{aligned} \tag{2.3}$$

Then, by using multi-indices $\mathbf{k}$ denoting $k_1k_2k_3k_4$ ($k_j \geq 0$: integers, $j = 1, \cdots, 4$), the normalized eigenfunctions for HO, a basis in $L^2(\mathbf{R}^4)$, are expressed in the form

$$\psi_\mathbf{k}(q) = (\mathbf{k}!)^{-1/2}(a_1^\dagger)^{k_1}(a_2^\dagger)^{k_2}(a_3^\dagger)^{k_3}(a_4^\dagger)^{k_4}\psi_\mathbf{0}(q) \tag{2.4a}$$

with

$$\psi_0(q) = \sqrt{\lambda/\pi}\exp(-\lambda r/2), \tag{2.4b}$$

where $\mathbf{k}! = k_1!k_2!k_3!k_4!$. Equation (2.2) and the fact that $\psi_{\mathbf{k}}$ is associated with the eigenvalue $\lambda(n+2)$ with $k_1 + \cdots + k_4 = n$ are put together to provide the following.

Proposition 2.1. *The negative eigenvalues of the conformal Kepler problem are given by $E_n = -2k^2/(n+2)^2$ ($n \geq 0$: integer), and their associated eigenspaces denoted by S_n are spanned by the functions $\psi_{\mathbf{k}}$ given by (2.4) subject to the conditions*

$$(2.5) \qquad k_1 + k_2 + k_3 + k_4 = n \quad \textit{and} \quad \lambda = 4k/(n+2).$$

We mention here that the Hilbert space structure of each S_n is determined by restricting the inner product

$$(2.6) \qquad \langle f, g \rangle = \int_{\mathbf{R}^4} \overline{f(q)} g(q) 4r dq_1 dq_2 dq_3 dq_4.$$

to S_n. The restricted inner product is denoted by $\langle\ ,\ \rangle_n$. Note that with respect to $\langle\ ,\ \rangle$ the $\widehat{H}$ becomes a symmetric operator in $C_0^\infty(\mathbf{R}^4)$ (see [I]).

We note here that the relation similar to (2.2) holds also in classical theory, so that the well-known symmetry group $SU(4)$ of the harmonic oscillator turns out to be the symmetry group of CK (see [I, I-U1]). Hence, on "quantizing" the action of the classical symmetry group $SU(4)$ of CK, a symmetry group of CK with negative energy is to be derived so as to act on S_n. We thus obtain the following.

Proposition 2.2. *The conformal Kepler problem with negative energy admits $SU(4)$ as a symmetry group which acts unitarily irreducibly on $(S_n, \langle\ ,\ \rangle_n)$ in the manner*

$$(2.7) \quad (U_C^{(n)} \psi_{\mathbf{k}})(q) = (\mathbf{k}!)^{-1/2} (C^T a^\dagger)_1^{k_1} (C^T a^\dagger)_2^{k_2} (C^T a^\dagger)_3^{k_3} (C^T a^\dagger)_4^{k_4} \psi_0(q)$$

for $C \in SU(4)$ and $\psi_{\mathbf{k}} \in S_n$, where $a^\dagger$ stands for the column vector of operators and $(C^T a^\dagger)_j$ ($j = 1, 2, 3, 4$) is the j-th component.

2.2. A symmetry group of the MIC-Kepler problem with negative energy

We proceed to a symmetry group of MIC-Kepler problem with negative energy. As was stated in Procedure (2) in Section 1, the subgroup of the symmetry group $SU(4)$ leaving invariant the subspace $S_{n,m}$ of ρ_m-equivariant functions in S_n will become a symmetry group of MICK. We shall first look into $S_{n,m}$. From (2.3) and (2.7), the $U(1)$ action given by (1.2) proves to be expressed as

$$\psi_{\mathbf{k}}(T(t)q) = (U^{(n)}_{\widetilde{T}(t)}\psi_{\mathbf{k}})(q) = e^{i(-k_1-k_2+k_3+k_4)t/2}\psi_{\mathbf{k}}(q) \tag{2.8a}$$

with

$$\widetilde{T}(t) = \begin{pmatrix} e^{-it/2}I_2 & \\ & e^{it/2}I_2 \end{pmatrix} \quad (I_2 :\ 2\times 2 \text{ identity matrix}), \tag{2.8b}$$

This equation yields the following.

Lemma 2.3. *The subspace $S_{n,m}$ of ρ_m-equivariant functions in S_n is spanned by the functions $\psi_{\mathbf{k}} \in S_n$ subject to*

$$k_1 + k_2 = \frac{n-m}{2}, \quad k_3 + k_4 = \frac{n+m}{2}, \tag{2.9}$$

where the integers n and m are simultaneously even or odd with $|m| \le n$. The $S_{n,m}$ is of dimension $(n-m+2)(n+m+2)/4$.

From the relation (1.9), we see that the $S_{n,m}$ reduces to the space of eigen-cross sections of $\widehat{H}_m$ with negative eigenvalue $E_n = -2k^2/(n+2)^2$. Indeed, for any $f \in S_{n,m}$, one has

$$\widehat{H}_m(q_m f) = q_m(\widehat{H} f) = E_n(q_m f). \tag{2.10}$$

We hence denote by $q_m(S_{n,m})$ the space of eigen-cross sections of $\widehat{H}_m$ derived from $S_{n,m}$. The Hilbert space structure $\langle\ ,\rangle_{n,m}$ is, of course, induced from the inner product $\langle\ ,\ \rangle_n$ as

$$\langle \gamma_1, \gamma_2\rangle_{n,m} = \langle q_m^{-1}\gamma_1, q_m^{-1}\gamma_2\rangle_n \tag{2.11}$$

for $\gamma_1, \gamma_2 \in q_m(S_{n,m})$. We now have the following.

Proposition 2.4. *The subspace $S_{n,m}$ of ρ_m-equivariant functions in S_n is mapped, through q_m, bijectively to the space of eigen-cross sections, $q_m(S_{n,m})$, for the MIC-Kepler problem with negative eigenvalue $E_n = -2k^2/(n+2)^2$, where n and m are simultaneously even or odd with $|m| \le n$.*

Now that we have the eigenspace $q_m(S_{n,m})$, we are ready to discuss a symmetry group of MICK with negative energy. In view of the course of obtaining $q_m(S_{n,m})$, we see that a subgroup of $SU(4)$ that leaves $S_{n,m}$ invariant turns into the symmetry group of MICK with negative energy. The largest subgroup of $SU(4)$ that leaves $S_{n,m}$ invariant is $S(U(2)\times U(2))$, which includes the $U(1)$ with the action (2.8). However, since the $U(1)$ action (2.8) is the identity in $q_m(S_{n,m})$ we had better get rid of this $U(1)$ action, so that we treat $SU(2) \times SU(2)$. For any $(C_1, C_2) \in SU(2) \times SU(2)$, we have, of course, the inclusion

$$(C_1, C_2) \in SU(2) \times SU(2) \longmapsto \begin{pmatrix} C_1 & \\ & C_2 \end{pmatrix} \in SU(4). \tag{2.12}$$

In order to express the action of $SU(2) \times SU(2)$ in a concrete form, we rewrite the creation operators $a_1^\dagger$, $a_2^\dagger$, $a_3^\dagger$, and $a_4^\dagger$ as $A_1^\dagger$, $A_2^\dagger$, $B_1^\dagger$, and $B_2^\dagger$, respectively. Then Proposition 2.2 reduces to the following (cf. [I-U1]).

Proposition 2.5. *A subgroup $SU(2) \times SU(2)$ of $SU(4)$ acts unitarily on $S_{n,m}$, whose action is expressed, for $(C_1, C_2) \in SU(2) \times SU(2)$ and $\psi_{\mathbf{k}} \in S_{n,m}$, as*

$$\begin{aligned}(U^{(n,m)}_{(C_1,C_2)}\, \psi_{\mathbf{k}})(q) \\ = (\mathbf{k}!)^{-1/2} (C_1^T A^\dagger)_1^{k_1} (C_1^T A^\dagger)_2^{k_2} (C_2^T B^\dagger)_1^{k_3} (C_2^T B^\dagger)_2^{k_4} \psi_{\mathbf{0}}(q),\end{aligned} \tag{2.13}$$

where $(C_1^T A^\dagger)_j$ and $(C_2^T B^\dagger)_j$ $(j = 1, 2)$ are the j-th components of the column vectors of operators $C_1^T A^\dagger$ and $C_2^T B^\dagger$, respectively.

Owing to Proposition 2.5, we can define well a unitary $SU(2) \times SU(2)$ action, $W^{(n,m)}$, on the eigenspace $q_m(S_{n,m})$ of the MIC-Kepler problem; for $\gamma \in q_m(S_{n,m})$ and $(C_1, C_2) \in SU(2) \times SU(2)$,

$$W^{(n,m)}_{(C_1,C_2)}\gamma := (q_m \circ U^{(n,m)}_{(C_1,C_2)} \circ q_m^{-1})\gamma. \tag{2.14}$$

Proposition 2.6. *The group $SU(2) \times SU(2)$ has unitary representation on each of the eigenspace $q_m(S_{n,m})$ of the MIC-Kepler problem with the eigenvalue E_n, where n and m are simultaneously even or odd with $|m| \le n$.*

2.3. The irreducibility of the $SU(2) \times SU(2)$ representation

On recalling Lemma 2.3, the condition (2.9) enables us to identify $S_{n,m}$ with the tensor product of the space of homogeneous polynomials in $(A_j^\dagger)$ of degree $(n-m)/2$ and that in $(B_j^\dagger)$ of degree $(n+m)/2$. Then, it follows from (2.13) that each of the factor groups of $SU(2) \times SU(2)$ is represented irreducibly in homogeneous polynomial space of degree

$(n-m)/2$ in $A^\dagger$ and in that of degree $(n+m)/2$ in $B^\dagger$, so that the representation $U^{(n,m)}$ proves to be irreducible. Further, according to Wigner [W], the tensor product representations exhaust all the unitary irreducible representations of $SU(2)\times SU(2)$, up to isomorphisms. Owing to the unitary equivalence, the representations $W^{(n,m)}$ turn out to exhaust all the unitary irreducible representations of $SU(2)\times SU(2)$, up to isomorphisms. The results in this section is summarized as follows:

Theorem 2.7 [I-U1]. *The MIC-Kepler problem with negative energies admits $SU(2)\times SU(2)$ as a symmetry group. The representation of $SU(2)\times SU(2)$ on each of the eigenspaces $(q_m(S_{n,m}), \langle\ ,\ \rangle_{n,m})$ covers all the unitary irreducible representations of $SU(2)\times SU(2)$, up to isomorphisms, if n and m vary under the condition stated in Proposition 2.6.*

§3. The zero-energy case and $\mathbf{R}^3 \ltimes SU(2)$

3.1. A symmetry group of the conformal Kepler problem with zero-energy

In the case of zero-energy, we associate CK with a *four-dimensional free particle*; a quantum system with the Hamiltonian operator

$$\widehat{F} = -\frac{1}{2}\sum_{j=1}^{4}\frac{\partial^2}{\partial {q_j}^2}, \tag{3.1}$$

which can be extended to a self-adjoint operator in $L^2(\mathbf{R}^4)$. The free particle will be often abbreviated to FP, in what follows. $\widehat{H}$ and $\widehat{F}$ satisfy

$$4r\,\widehat{H} = \widehat{F} - 4k, \tag{3.2}$$

which implies that to study the $\widehat{H}$ with zero-spectrum is to study the $\widehat{F}$ with spectrum $4k$.

We start with a review of $\widehat{F}$ with positive spectra. Let us denote by $\mathcal{S}(\mathbf{R}^4_q)$ and $\mathcal{S}(\mathbf{R}^4_u)$ the spaces of smooth rapidly decreasing functions on $\mathbf{R}^4_q$ and $\mathbf{R}^4_u$, respectively, where the subscripts q and u indicate the variables used in $\mathbf{R}^4$'s. For $\phi \in \mathcal{S}(\mathbf{R}^4_q)$, we denote its Fourier transform by $\widetilde{\phi} \in \mathcal{S}(\mathbf{R}^4_u)$. On using the polar coordinates, $u = \sqrt{2s}\omega$, with $\omega \in S^3$ and $s \geq 0$, the Fourier integral formula is put in the form

$$\phi(q) = \frac{1}{(2\pi)^2}\int_0^\infty 2s\,ds \int_{S^3} e^{iq\cdot\sqrt{2s}\omega}\,\widetilde{\phi}(\sqrt{2s}\omega)\,dS^3, \tag{3.3}$$

where dS^3 denotes the standard volume element of the three-sphere S^3. Further, Plancherel's theorem takes the form

$$\|\phi\|^2_{L^2(\mathbf{R}^4_q)} = \|\widetilde{\phi}\|^2_{L^2(\mathbf{R}^4_u)} = \int_0^\infty 2s\,ds \int_{S^3} |\widetilde{\phi}(\sqrt{2s}\omega)|^2\,dS^3. \tag{3.4}$$

Thus, if we define the function space

$$\mathcal{H}_s := \left\{ f(q) = \int_{S^3} e^{iq\cdot\sqrt{2s}\omega}\, F(\omega)dS^3;\ F \in L^2(S^3) \right\}, \tag{3.5}$$

$L^2(\mathbf{R}^4_q)$ turns out to be decomposed into a direct integral

$$L^2(\mathbf{R}^4_q) = \int_0^\infty \oplus \mathcal{H}_s\, 2s\,ds. \tag{3.6}$$

It is worth pointing out that any $f \in \mathcal{H}_s$ satisfies the Schrödinger equation; $\widehat{F}f = sf$ (see Helgason [H]). Moreover, the map $\kappa_s : L^2(S^3) \to \mathcal{H}_s$ given by $F(\omega) \mapsto f(q)$ through (3.5) makes $\mathcal{H}_s$ into a Hilbert space, so that one has the isomorphism, for all positive s,

$$\mathcal{H}_s \cong L^2(S^3). \tag{3.7}$$

Summarizing the above, we have the following.

Proposition 3.1. *$L^2(\mathbf{R}^4_q)$ is decomposed into the direct integral of Hilbert spaces $\{\mathcal{H}_s\}$ each of which is isomorphic to $L^2(S^3)$ and associated with the spectrum s of $\widehat{F}$. Hence, $\mathcal{H}_{4k}$ is a Hilbert space associated with the zero-energy of the conformal Kepler problem.*

We proceed to study a symmetry group of FP to get a symmetry group of CK with zero-energy. In classical theory, the symmetry group of FP is known to be $\mathbf{R}^9 \ltimes SO(4)$, where $\mathbf{R}^9$ and $\ltimes$ denote the additive group of 4×4 traceless real symmetric matrices and the semi-direct product, respectively [I-U2]. In quantum theory, the action of $\mathbf{R}^9 \ltimes SO(4)$ is "quantized" to give a unitary representation in $L^2(\mathbf{R}^4_q)$ as

$$(X_{(M,g)}\phi)(q) = \frac{1}{(2\pi)^2} \int_{\mathbf{R}^4_u} e^{iq\cdot u} \exp\left(-\frac{i}{2} u \cdot Mu\right) \widetilde{\phi}(g^{-1}u)\,du, \tag{3.8}$$

where $\widetilde{\phi}$ is the Fourier transform of $\phi \in L^2(\mathbf{R}^4_q)$, and $(M, g) \in \mathbf{R}^9 \ltimes SO(4)$ (see [I-U2]). On applying (3.6) to (3.8), the representation $X_{(M,g)}$ gives rise to unitary representations $U^s_{(M,g)}$ of $\mathbf{R}^9 \ltimes SO(4)$ in each $\mathcal{H}_s$, which takes the form

$$(3.9) \qquad (U^s_{(M,g)}f)(q) = \int_{S^3} e^{iq\cdot\sqrt{2s}\omega} \exp\left(-\frac{i}{2}\omega\cdot M\omega\right) F(g^{-1}\omega)\, dS^3,$$

where $f \in \mathcal{H}_s$ is of the form (3.5).

Proposition 3.2. *The free particle admits $\mathbf{R}^9 \ltimes SO(4)$ as a symmetry group, which is represented unitarily as* (3.9) *in each Hilbert spaces $\mathcal{H}_s$ given by* (3.5).

Propositions 3.1 and 3.2 are put together to give the following.

Theorem 3.3. *The conformal Kepler problem with zero-energy admits $\mathbf{R}^9 \ltimes SO(4)$ as a symmetry group whose action is represented unitarily on the Hilbert space $\mathcal{H}_{4k}$.*

Remark. Because of (3.7), a unitary representation, denoted by V^s, of $\mathbf{R}^9 \ltimes SO(4)$ in $L^2(S^3)$ is also defined by

$$(3.10) \qquad U^s_{(M,g)} \circ \kappa_s = \kappa_s \circ V^s_{(M,g)} \quad ((M,g) \in \mathbf{R}^9 \ltimes SO(4)).$$

3.2. A symmetry group of the MIC-Kepler problem with zero-energy

We derive a symmetry group of MICK with zero-energy from the symmetry group of CK obtained in Theorem 3.3. A subgroup of the symmetry group, $\mathbf{R}^9 \ltimes SO(4)$, of CK that leaves the subspace of ρ_m-equivariant functions in $\mathcal{H}_{4k}$ is shown to be a symmetry group of MICK with zero-energy.

We study first the subspace of ρ_m-equivariant functions in $\mathcal{H}_s$, which subspace is denoted by $\mathcal{H}_{s,m}$. On carrying out a similar argument to that of negative energy case, we have the following.

Proposition 3.4. *The subspace, $\mathcal{H}_{4k,m}$, of ρ_m-equivariant functions in $\mathcal{H}_{4k}$ is mapped, through q_m, to the space of cross sections associated with the MIC-Kepler problem with zero-energy.*

We then understand that $q_m(\mathcal{H}_{4k,m})$ is the space associated with $\widehat{H}_m = 0$. The Hilbert space structure is defined on $q_m(\mathcal{H}_{s,m})$ by the inner product $\langle\ ,\ \rangle_{s,m}$ which is naturally induced from the inner product, say, $\langle\ ,\ \rangle_s$, in $\mathcal{H}_s$ as

$$(3.11) \qquad \langle\gamma_1,\gamma_2\rangle_{s,m} = \langle q_m^{-1}\gamma_1, q_m^{-1}\gamma_2\rangle_s,$$

where $\gamma_1, \gamma_2 \in q_m(\mathcal{H}_{s,m})$. We study further the structure of $\mathcal{H}_{s,m}$ and of $q_m(\mathcal{H}_{s,m})$. Specializing (3.9) in a subgroup $\{(0, T(t));\ t \in [0, 4\pi]\}$ of $\mathbf{R}^9 \ltimes SO(4)$, one has, for $f \in \mathcal{H}_s$,

$$f(T(t)q) = (U^s_{(0,T(-t))}f)(q) = \int_{S^3} e^{iq\cdot\sqrt{2s}\omega}\, F(T(t)\omega)dS^3, \tag{3.12}$$

where $f = \kappa_s(F)$ for $F \in L^2(S^3)$. This implies that f is ρ_m-equivariant in $\mathcal{H}_s$ if and only if F is ρ_m-equivariant in $L^2(S^3)$. We denote by $L^2(S^3)_m$ the space of ρ_m-equivariant functions in $L^2(S^3)$, which has a Hilbert space structure as a closed subspace of $L^2(S^3)$. Thus we have the isomorphisms,

$$q_m(\mathcal{H}_{s,m}) \cong \mathcal{H}_{s,m} \cong L^2(S^3)_m. \tag{3.13}$$

We are now in a position to find a symmetry group of MICK with zero-energy. Like in the negative energy case, a subgroup of $\mathbf{R}^9 \ltimes SO(4)$ that leaves $\mathcal{H}_{s,m}$ invariant proves to be a subgroup which is commutative with the $U(1)$ action (3.12). We can show that the largest one of such subgroups is isomorphic to $\mathbf{R}^3 \ltimes U(2)$. However, since this subgroup includes the $U(1) \cong \{(0, T(t));\ t \in [0, 4\pi]\}$, we choose to take $\mathbf{R}^3 \ltimes SU(2)$ after eliminating the $U(1)$ from $\mathbf{R}^3 \ltimes U(2)$. We have to notice here how the $\mathbf{R}^3 \ltimes SU(2)$ is represented as pairs of 4×4 matrices in $\mathbf{R}^9 \ltimes SO(4)$: Let a map β be defined as

$$\beta : \begin{pmatrix} a_1 + ib_1 & a_2 + ib_2 \\ a_3 + ib_3 & a_4 + ib_4 \end{pmatrix} \longmapsto Z = \begin{pmatrix} Z_1 & Z_2 \\ Z_3 & Z_4 \end{pmatrix} \tag{3.14}$$

$$\text{with} \quad Z_j = \begin{pmatrix} a_j & -b_j \\ b_j & a_j \end{pmatrix} \quad (j = 1, 2, 3, 4).$$

Then $\mathbf{R}^3 \ltimes SU(2)$ is represented as a subgroup of $\mathbf{R}^9 \ltimes SO(4)$, $\beta(\mathbf{R}^3) \ltimes \beta(SU(2))$. We thus obtain the following.

Proposition 3.5. *The semi-direct product group $\mathbf{R}^3 \ltimes SU(2)$ acts unitarily on $\mathcal{H}_{s,m}$, where $\mathbf{R}^3$ indicates the additive group of 2×2 traceless Hermitian matrices.*

By $U^{(s,m)}$ we denote the induced action of $\mathbf{R}^3 \ltimes SU(2)$ on $\mathcal{H}_{s,m}$. Then, the action of $\mathbf{R}^3 \ltimes SU(2)$ on the Hilbert space $q_m(\mathcal{H}_{s,m})$ is defined, for $\gamma \in q_m(\mathcal{H}_{s,m})$, by

$$W^{(s,m)}_{(M,g)}\gamma = (q_m \circ U^{(s,m)}_{(M,g)} \circ q_m^{-1})\gamma, \tag{3.15}$$

where $(M, g) \in \mathbf{R}^3 \ltimes SU(2)$ is represented as a pair of 4×4 real matrices (cf. (3.14)). This action is unitary, of course.

Proposition 3.6. *The MIC-Kepler problem with zero-energy admits the semi-direct product group $\mathbf{R}^3 \ltimes SU(2)$ as a symmetry group, which is unitarily represented in $q_m(\mathcal{H}_{s,m})$ in the manner of* (3.15) *together with* (3.9).

In closing Section 3.2, we make a mention of the unitary representation V^s of $\mathbf{R}^9 \ltimes SO(4)$ in $L^2(S^3)$ (see (3.10)). From (3.10) and (3.13), we can define a unitary representation $V^{(s,m)}$ of $\mathbf{R}^3 \ltimes SU(2)$ on $L^2(S^3)_m$ by

$$U^{(s,m)}_{(M,g)} \circ \kappa_s = \kappa_s \circ V^{(s,m)}_{(M,g)}, \tag{3.16}$$

where $(M,g) \in \mathbf{R}^3 \ltimes SU(2)$ is represented as a pair of 4×4 real matrices. The representations $W^{(s,m)}$ and $V^{(s,m)}$ are unitarily equivalent on account of (3.15) and (3.16).

3.3. Relation to the Mackey's induced representation

In this section, the representation $W^{(s,m)}$ of $\mathbf{R}^3 \ltimes SU(2)$ is shown to be equivalent to the *Mackey*'s induced representation. Since $W^{(s,m)}$ is equivalent to $V^{(m,s)}$, we choose to show the equivalence between $V^{(s,m)}$ and Mackey's representation. According to Mackey [Mk], the induced representation of $\mathbf{R}^3 \ltimes SU(2)$ is realized on the Hilbert space of functions on the group manifold $\mathbf{R}^3 \ltimes SU(2)$.

Let α be a bijection of S^3 to $SU(2)$ given by

$$\alpha : \omega \in S^3 \longmapsto \begin{pmatrix} \omega_1 + i\omega_2 & -\omega_3 + i\omega_4 \\ \omega_3 + i\omega_4 & \omega_1 - i\omega_2 \end{pmatrix}. \tag{3.17}$$

Using α, we can define an injection $A^{(s,m)}$ of $L^2(S^3)_m$ to a space of functions on $\mathbf{R}^3 \ltimes SU(2)$; for $F \in L^2(S^3)_m$, $A^{(s,m)}$ is given by

$$(A^{(s,m)}\, F)(M,\, g) = \exp(isv \cdot g^{-1} M g\, v)\, F(\alpha^{-1}(g)) \tag{3.18}$$

with $v = (1,0)^T$, where $(M,g) \in \mathbf{R}^3 \ltimes SU(2)$ indicates a pair of 2×2 complex matrices. It is easy to prove that for a subgroup $\mathbf{R}^3 \ltimes U(1)$ acting on $\mathbf{R}^3 \ltimes SU(2)$ to the right, functions $A^{(s,m)}F$ are subject to the transformation

$$A^{(s,m)}F((M,g)(N,u(t)) = \chi_{s,m}(N,\, u(t))^{-1}(A^{(s,m)}F)(M,g) \quad \text{with} \quad \chi_{s,m}(N,\, u(t)) = \exp(-isv \cdot N\, v)\, e^{-imt/2}, \tag{3.19}$$

where $u(t)$ is the 2×2 matrix given by $u(t) = \mathrm{diag}(e^{it/2}, e^{-it/2})$, and the $\chi_{s,m}$ is known as an irreducible representation of $\mathbf{R}^3 \ltimes U(1)$ on

C. Equations (3.18) and (3.19) imply that $L^2(S^3)_m$ is mapped to the space of $\chi_{s,m}$-equivariant functions on $\mathbf{R}^3 \ltimes SU(2)$, which space can be made into a Hilbert space, and is isomorphic to the Hilbert space of L^2-cross sections in a complex line bundle over the quotient space $(\mathbf{R}^3 \ltimes SU(2))/(\mathbf{R}^3 \ltimes U(1)) \cong S^2$. We denote by $L^2(\mathbf{R}^3 \ltimes SU(2))_{s,m}$ the Hilbert space of $\chi_{s,m}$-equivariant functions on $\mathbf{R}^3 \ltimes SU(2)$.

Let $T^{(s,m)}$ be the Mackey's induced representation of $\mathbf{R}^3 \ltimes SU(2)$ in $L^2(\mathbf{R}^3 \ltimes SU(2))_{s,m}$. Then a straightforward calculation shows that $A^{(s,m)}$ gives an intertwining operator between $T^{(s,m)}$ and $V^{(s,m)}$;

$$(3.20) \quad T^{(s,m)}_{(M,g)} \circ A^{(s,m)} = A^{(s,m)} \circ V^{(s,m)}_{\mathrm{Re}(M,g)} \quad ((M,g) \in \mathbf{R}^3 \ltimes SU(2)),$$

where $\mathrm{Re}(M,g)$ denotes the real matrix representation (see (3.14)). Since $T^{(s,m)}$ is known to be irreducible and to exhaust all the unitary irreducible representations, up to isomorphisms, we have the following.

Theorem 3.7. *The unitary representation* $(W^{(4k,m)}, q_m(\mathcal{H}_{4k,m}))$ *of the symmetry group* $\mathbf{R}^3 \ltimes SU(2)$ *of the MICK is irreducible. The* $W^{(4k,m)}$ *exhausts all the unitary irreducible representations of* $\mathbf{R}^3 \ltimes SU(2)$, *up to isomorphisms, if the parameter* k *and* m *range over all the positive real numbers and the integers, respectively.*

§4. The positive energy case and $SL(2, \mathbf{C})$

4.1. A symmetry group of the conformal Kepler problem with positive energies

In the positive energy case, the four-dimensional repulsive oscillator is associated with the conformal Kepler problem. The repulsive oscillator is a quantum system with the Hamiltonian operator

$$(4.1) \qquad \widehat{R}_\lambda = -\frac{1}{2}\sum_{j=1}^{4} \frac{\partial^2}{\partial {q_j}^2} - \frac{\lambda^2}{2}\sum_{j=1}^{4} q_j^2,$$

where λ is a positive parameter. From now on, the repulsive oscillator will be often abbreviated to RO. The $\widehat{H}$ and $\widehat{R}_\lambda$ satisfy the relation

$$(4.2) \qquad 4r\left(\widehat{H} - \frac{\lambda^2}{8}\right) = \widehat{R}_\lambda - 4k.$$

Like in the zero-energy case, studying $\widehat{H}$ with positive spectrum $\lambda^2/8$ amounts to studying $\widehat{R}_\lambda$ with positive spectrum $4k$.

We start by studying the repulsive oscillator. On physical grounds, it is better for us to introduce a unitary operator ξ of $L^2(\mathbf{R}^4_q)$ to $L^2(\mathbf{R}^4_u)$, which is expressed as the integral transform (see [I-U3])

$$(4.3)\quad (\xi\phi)(u) = \frac{\lambda}{2\pi^2}\int_{\mathbf{R}^4_q} \exp\left\{\frac{i}{2}(u\cdot u - 2\sqrt{2\lambda}\,u\cdot q + \lambda q\cdot q)\right\}\phi(q)\,dq.$$

Then the ξ maps $\widehat{R}_\lambda$ into

$$(4.4)\qquad \widehat{L}_\lambda := \xi\circ\widehat{R}_\lambda\circ\xi^{-1} = \frac{\lambda}{i}\left(\sum_{j=1}^4 u_j\frac{\partial}{\partial u_j} + 2\right),$$

where the domains of $\widehat{R}_\lambda$ and $\widehat{L}_\lambda$ are considered, say, as the spaces of smooth functions of rapid descent on $\mathbf{R}^4_q$ and $\mathbf{R}^4_u$, respectively. The $\widehat{L}_\lambda$ is easy to treat. In fact, $\widehat{L}_\lambda/\lambda$ is immediately integrated to give a one-parameter group of unitary transformations D_t on $L^2(\mathbf{R}^4_u)$;

$$(4.5)\qquad (D_t\phi)(u) = e^{2t}\phi(e^t u).$$

The generator of D_t should be a self-adjoint extension of $\widehat{L}_\lambda/\lambda$, which we denote by the same letter. The unitary operator F_t defined by $F_t = \xi^{-1}\circ D_t\circ\xi$ then have the generator that is a self-adjoint extension of $\widehat{R}_\lambda/\lambda$, which we denote by the same letter. Hence we have the following.

Lemma 4.1. *The repulsive oscillator $(\widehat{R}_\lambda, L^2(\mathbf{R}^4_q))$ is unitarily equivalent to the quantum system $(\widehat{L}_\lambda, L^2(\mathbf{R}^4_u))$.*

We may choose to study the system $(D_t, L^2(\mathbf{R}^4_u))$, instead of $(\widehat{L}_\lambda, L^2(\mathbf{R}^4_u))$. For $\phi\in\mathcal{S}(\mathbf{R}^4_u)$, the space of smooth functions of rapid descent on $\mathbf{R}^4_u$, set

$$(4.6)\qquad (P_s\phi)(u) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ist}e^{2t}\phi(e^t u)\,dt.$$

Then, from the Fourier integral formula, one obtains a decomposition

$$(4.7)\qquad \phi(u) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(P_s\phi)(u)\,ds.$$

It is worth noting that $P_s\phi$ satisfies a homogeneity condition

$$(4.8)\qquad (P_s\phi)(\epsilon u) = \epsilon^{is-2}\,(P_s\phi)(u)\quad (u\in\mathbf{R}^4_u,\ \epsilon>0).$$

Equation (4.7) together with Plancherel's theorem results in a direct integral decomposition

$$L^2(\mathbf{R}^4_u) = \int_{-\infty}^{\infty} \oplus \mathcal{H}_s(\mathbf{R}^4_u)\, ds, \tag{4.9}$$

where $\mathcal{H}_s(\mathbf{R}^4_u)$ are a one-parameter family of Hilbert spaces defined by

$$\mathcal{H}_s(\mathbf{R}^4_u) = \{f \in L^2_{loc}(\mathbf{R}^4_u - \{0\});\quad f(\epsilon u) = \epsilon^{is-2} f(u), \quad \epsilon > 0, \quad \text{and} \quad \int_{S^3} |f|^2\, dS^3 < +\infty\}. \tag{4.10}$$

The Hilbert space structure of $\mathcal{H}_s(\mathbf{R}^4_u)$ is, of course, defined, for $f \in \mathcal{H}_s(\mathbf{R}^4_u)$, by

$$\|f\|^2_{\mathcal{H}_s(\mathbf{R}^4_u)} = \int_{S^3} |f(\omega)|^2\, dS^3. \tag{4.11}$$

Further, the homogeneity condition in (4.10) makes any $f \in \mathcal{H}_s(\mathbf{R}^4_u)$ be determined by its restriction to S^3, so that one has the isomorphism

$$\mathcal{H}_s(\mathbf{R}^4_u) \cong L^2(S^3). \tag{4.12}$$

Further, that homogeneity condition along with (4.8) and $\epsilon = e^t$ gives rise to the equation, $\widehat{L}_\lambda f = \lambda s f$, for a smooth function f in $\mathcal{H}_s(\mathbf{R}^4_u)$. We thus have the following.

Proposition 4.2. *The Hilbert space $\mathcal{H}_s(\mathbf{R}^4_u)$ which is isomorphic to $L^2(S^3)$ is associated with the spectrum λs of $\widehat{L}_\lambda$.*

We turn to a symmetry group of $(D_t,\, L^2(\mathbf{R}^4_u))$. On "quantizing" an $SL(4,\mathbf{R})$ action on the phase space in classical theory (see [I-U3]), a unitary action of $SL(4,\mathbf{R})$ is derived on $L^2(\mathbf{R}^4_u)$, which is given, for $\phi \in L^2(\mathbf{R}^4_u)$, by

$$Y_g : \phi(u) \longmapsto \phi(g^{-1}u) \quad (g \in SL(4,\mathbf{R})). \tag{4.13}$$

Since $P_s \circ Y_g = Y_g \circ P_s$, one can restrict Y to $\mathcal{H}_s(\mathbf{R}^4_u)$ to define a unitary representation U^s in $\mathcal{H}_s(\mathbf{R}^4_u)$ by

$$(U^s_g f)(u) = f(g^{-1}u) = |g^{-1}u|^{is-2} f\left(\frac{g^{-1}u}{|g^{-1}u|}\right) \quad (g \in SL(4,\mathbf{R})). \tag{4.14}$$

We thus have the following.

Proposition 4.3. *$SL(4,\mathbf{R})$ is a symmetry group of the system $(D_t,\ L^2(\mathbf{R}^4_u))$, which group is unitarily represented in $\mathcal{H}_s(\mathbf{R}^4_u)$ as* (4.14).

The $SL(4,\mathbf{R})$ turns into a symmetry group of RO on the unitary equivalence between RO and $(\widehat{L}_\lambda,\ L^2(\mathbf{R}^4_u))$ (see Lemma 4.1). The representation space of $SL(4,\mathbf{R})$ for RO is, however, not easy to identify, since we cannot apply the unitary operator ξ^{-1} (cf. (4.3)) directly to $\mathcal{H}_s(\mathbf{R}^4_u)$ because of $\mathcal{H}_s(\mathbf{R}^4_u) \not\subseteq L^2(\mathbf{R}^4_u)$. An alternative way to get that space is to view $f \in \mathcal{H}_s(\mathbf{R}^4_u)$ as a tempered distribution. Then, it can be shown by calculation that, for $f \in \mathcal{H}_s(\mathbf{R}^4_u)$, there exists a unique function $h(q)$ on $\mathbf{R}^4_q$ which satisfies

$$T_f(\xi\phi) = T_h(\phi) \qquad (\phi \in \mathcal{S}(\mathbf{R}^4_q)) \tag{4.15}$$

(see [I-U3]), where T_f and T_h stand for the tempered distributions associated with f and h, respectively. Moreover, $h(q)$ proves to satisfy $\widehat{R}_\lambda h = \lambda s\, h$. Therefore, on denoting by η_s the map, determined by (4.15), of f to $h(q)$, the space $\eta_s(\mathcal{H}_s(\mathbf{R}^4_u))$ of functions on $\mathbf{R}^4_q$ is what we have looked for as a representation space of $SL(4,\mathbf{R})$, which space will be denoted by $\mathcal{K}_s(\mathbf{R}^4_q)$ henceforth. The Hilbert space structure for $\mathcal{K}_s(\mathbf{R}^4_q)$ is defined from that for $\mathcal{H}_s(\mathbf{R}^4_u)$ through η_s. Then it follows that

$$\mathcal{K}_s(\mathbf{R}^4_q) \cong \mathcal{H}_s(\mathbf{R}^4_u) \cong L^2(S^3). \tag{4.16}$$

A unitary representation V^s of $SL(4,\mathbf{R})$ in $\mathcal{K}_s(\mathbf{R}^4_q)$ is now induced from the representation U^s in $\mathcal{H}_s(\mathbf{R}^4_u)$;

$$V^s = \eta_s \circ U^s \circ \eta_s^{-1}. \tag{4.17}$$

Proposition 4.4. *The repulsive oscillator admits $SL(4,\mathbf{R})$ as a symmetry group, which has a unitary representation in the Hilbert space $\mathcal{K}_s(\mathbf{R}^4_q)$ associated the spectrum λs of $\widehat{R}_\lambda$.*

Proposition 4.4, in turn, provides the symmetry group of CK. If we set $\lambda s = 4k$ in accordance with (4.2), the $\mathcal{K}_{4k/\lambda}(\mathbf{R}^4_q)$ turns into the carrier space of the unitary representation of the symmetry group $SL(4,\mathbf{R})$ of CK.

Theorem 4.5. *The conformal Kepler problem with positive energy $\lambda^2/8$ admits $SL(4,\mathbf{R})$ as a symmetry group, which is unitarily represented in the Hilbert space $\mathcal{K}_{4k/\lambda}(\mathbf{R}^4_q)$.*

4.2. A symmetry group of the MIC-Kepler problem with positive energy

Like in the negative and the zero-energy cases, we study first a Hilbert space of cross sections in L_m associated with MICK with positive energy. Let $\mathcal{H}_{s,m}(\mathbf{R}^4_u)$ and $\mathcal{K}_{s,m}(\mathbf{R}^4_q)$ be the spaces of ρ_m-equivariant functions in $\mathcal{H}_s(\mathbf{R}^4_u)$ and in $\mathcal{K}_s(\mathbf{R}^4_q)$, respectively. In the positive energy case, we obtain the following proposition similar to Propositions 2.4 and 3.4.

Proposition 4.6. *The space $\mathcal{K}_{4k/\lambda,\, m}(\mathbf{R}^4_q)$ is mapped, through q_m, injectively to a space of cross sections in L_m in association with the positive spectrum $\lambda^2/8$ of the MIC-Kepler problem.*

In view of Proposition 4.6, we will denote by $q_m(\mathcal{K}_{s,m})$ the space of cross sections in L_m mapped from $\mathcal{K}_{s,m}(\mathbf{R}^4_q)$. The Hilbert space structure of $q_m(\mathcal{K}_{s,m})$, of course, comes from that of $\mathcal{K}_{s,m}(\mathbf{R}^4_q)$. We study the structure of $\mathcal{K}_{s,m}(\mathbf{R}^4_q)$ and $q_m(\mathcal{K}_{s,m})$ further. From (4.15) it follows that f in $\mathcal{H}_s(\mathbf{R}^4_u)$ is ρ_m-equivariant if and only if h in $\mathcal{K}_s(\mathbf{R}^4_q)$ is ρ_m-equivariant, hence $\mathcal{H}_{s,m}(\mathbf{R}^4_u) \cong \mathcal{K}_{s,m}(\mathbf{R}^4_q)$. Combined with (4.12), this fact yields

$$q_m(\mathcal{K}_{s,m}) \cong \mathcal{K}_{s,m}(\mathbf{R}^4_q) \cong \mathcal{H}_{s,m}(\mathbf{R}^4_u) \cong L^2(S^3)_m. \tag{4.18}$$

We proceed to a symmetry group of MICK with positive energy. A subgroup of the symmetry group $SL(4,\mathbf{R})$ of CK that leaves $\mathcal{K}_{4k/\lambda,m}$ $(\mathbf{R}^4_q)$ invariant turns into a symmetry group of MICK with energy $\lambda^2/8$. Like in the cases of negative and zero-energies, the subgroup to be looked for is a subgroup commutative with the $U(1) \cong \{T(t),\ t \in [0,\, 4\pi]\}$ $\subset SL(4,\mathbf{R})$. As a result, we have a real representation of $SL(2,\mathbf{C})$ in $SL(4,\mathbf{R})$, which is given by $\beta(g)$ for $g \in SL(2,\mathbf{C})$ and β in (3.14).

Proposition 4.7. *A real representation of $SL(2,\mathbf{C})$ in $SL(4,\mathbf{R})$ acts unitarily on $\mathcal{K}_{s,m}(\mathbf{R}^4_q)$.*

By $V^{(s,m)}$ we denote the restriction of V^s to $\mathcal{K}_{s,m}(\mathbf{R}^4_q)$. We can then define the unitary representation of $SL(2,\mathbf{C})$, denoted by $W^{(s,m)}$, in the Hilbert space $q_m(\mathcal{K}_{s,m})$ by

$$W_g^{(s,m)} = q_m \circ V_g^{(s,m)} \circ q_m^{-1}, \tag{4.19}$$

where $g \in SL(2,\mathbf{C})$ is represented in a 4×4 real matrix form. Thus we have the following.

Theorem 4.8. *The MIC-Kepler problem with positive energy admits $SL(2,\mathbf{C})$ as a symmetry group, which is unitarily represented in the Hilbert space $q_m(\mathcal{K}_{4k/\lambda,\,m})$ associated with $\widehat{H}_m = \lambda^2/8$.*

In conclusion, we give another unitary representation of $SL(2,\mathbf{C})$ in $L^2(S^3)_m$, which representation is unitarily equivalent to the representation $W^{(s,m)}$ in $q_m(\mathcal{K}_{s,m})$. The isomorphism (4.18) enables us to define $D^{(s,m)}$ through (4.14) together with the restriction map $f \mapsto F := f\,|_{S^3}$;

$$(D_g^{(s,m)}F)(\omega) = |g^{-1}\omega|^{is-2}F(\frac{g^{-1}\omega}{|g^{-1}\omega|}), \tag{4.20}$$

where $F \in L^2(S^3)_m$, and $g \in SL(2,\mathbf{C})$ is represented in the 4×4 real matrix form.

4.3. Relation to principal series representations of $SL(2,\mathbf{C})$

We show that the $W^{(s,m)}$ is unitarily equivalent to the so-called principal series representation of $SL(2,\mathbf{C})$. On account of the equivalence between $D^{(s,m)}$ and $W^{(s,m)}$, we choose to deal with $D^{(s,m)}$. If we introduce the complex vector space structure $\mathbf{C}^2$ into $\mathbf{R}^4_u$ by $u_1 + iu_2 = z_1$, and $u_3 + iu_4 = z_2$, the defining condition for $f \in \mathcal{H}_{s,m}(\mathbf{R}^4_u)$ is put in the form

$$f(\alpha z) = \alpha^{n_1}\overline{\alpha}^{n_2} f(z) \quad (\alpha \in \mathbf{C}-\{0\}) \tag{4.21a}$$

with

$$n_1 = \frac{1}{2}(is-2+m), \quad n_2 = \frac{1}{2}(is-2-m). \tag{4.21b}$$

This is identical with the condition required for the principal series representation due to Gel'fand et. al., which is denoted by $T_{(n_1+1,n_2+1)}$ ([G-G-V]). Indeed, one easily gets the equivalence

$$D_g^{(s,m)} = T_{(n_1+1,n_2+1)}((g^{-1})^T) \qquad \text{for} \qquad g \in SL(2,\mathbf{C}).$$

Since this principal series representation is irreducible, we have the following.

Theorem 4.9. *The unitary representation $W^{(4k/\lambda,m)}$ of $SL(2,\mathbf{C})$ in $q_m(\mathcal{K}_{4k/\lambda,m})$ is irreducible and exhausts all the principal series of unitary irreducible representations of $SL(2,\mathbf{C})$, up to isomorphisms, as λ and m range over all the positive real numbers and the integers, respectively.*

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Geometric Singularities for Hamilton-Jacobi Equation

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§1. Introduction

In this paper we will consider the Cauchy problem for Hamilton-Jacobi equation :

$$(1) \qquad \frac{\partial y}{\partial t} + H(t, x_1, \dots, x_n, \frac{\partial y}{\partial x_1}, \dots, \frac{\partial y}{\partial x_n}) = 0$$

$$(2) \qquad y(0, x_1, \dots, x_n) = \phi(x_1, \dots, x_n),$$

where H and ϕ are C^∞-functions.

By the method of characteristics, the solution of this problem is explicitly constructed. It is well-known that, even for smooth initial data, the solution becomes multi-valued in finite time. That is, singularities appear.

Recently Tsuji ([6] [7]) and Nakane [5] studied the behavior of solutions near the singular point. They assumed that singularities of the projection to the base space from the multi-valued solution are fold or cusp type singularities. But, other type of singularities may be appeared in generic.

Our purpose is to describe bifurcations of singularities of solutions along the time parameters geometrically. We will study this problem in the framework of the theory of Legendrian unfoldings. In §2 we will introduce the theory of one-parameter Legendrian unfoldings for preparations. In §3 the geometric treatment of Hamilton-Jacobi equation will be given and we will formulate a generalized Cauchy problem associated with the time parameter. Theorem 3.2 is the base of our theory. By this theorem we can apply Arnol'd-Zakalyukin's classifications of one-parameter perestroika of wave front sets and caustics to our situations ([1],[2],[8]).

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All map germs and diffeomorphisms considered here are class C^∞, unless stated otherwise.

§2. One parameter Legendrian unfoldings

In this section we will briefly introduce the theory of one-parameter Legendrian unfoldings. In [3] we will develop the general theory and its' applications. Thus detailed proof will be appeared in there.

Let $J^1(\mathbb{R}^n, \mathbb{R})$be the 1-jet bundle of functions of n-variables. Since we only consider the local situation, the 1-jet bundle $J^1(\mathbb{R}^n, \mathbb{R})$ may be considered as $\mathbb{R}^{2n+1}$ with a natural coordinate system

$$(x_1, \ldots, x_n, y, p_1, \ldots, p_n),$$

where $(x_1, \ldots, x_n)$ is a coordinate system of $\mathbb{R}^n$. We also have natural projections. Namely,

$$\pi : J^1(\mathbb{R}^n, \mathbb{R}) \to \mathbb{R}^n \times \mathbb{R} \quad ; \quad \pi(x, y, p) = (x, y)$$

$$\pi' : J^1(\mathbb{R}^n, \mathbb{R}) \to \mathbb{R}^n \quad ; \quad \pi'(x, y, p) = x.$$

An immersion germ

$$i : (L, q) \to J^1(\mathbb{R}^n, \mathbb{R})$$

is said to be *a Legendrian immersion germ* if

$$\dim L = n \qquad \text{and} \qquad i^*\theta = 0,$$

where $\theta = dy - \sum_{i=1}^n p_i dx_i$. The image of $\pi \circ i$ is called *a wave front set* of i. It is denoted by $W(i)$. We say that $q \in L$ is *a Lagrangian singular point* if

$$\operatorname{rank} d(\pi' \circ i)_q < n.$$

The critical value set of $\pi' \circ i$ is called *a caustics* of i. It is denoted by $C(i)$.

Let $i : (L, q) \to J^1(\mathbb{R}^n, \mathbb{R})$ be a Legendrian immersion germ, then $\tilde{\pi} \circ i : (L, q) \to T^*\mathbb{R}^n$ is a Lagrangian immersion germ, where $\tilde{\pi}$ is the canonical projection onto the cotangent bundle $T^*\mathbb{R}^n$. Hence the above definition is reasonable.

We now describe the notion of one-parameter Legendrian unfoldings.

Let R be an $(n+1)$-dimensional smooth manifold and

$$\mu : (R, u_0) \to (\mathbb{R}, t_0)$$

be a submersion germ and

$$\ell : (R, u_0) \to J^1(\mathbb{R}^n, \mathbb{R})$$

be a smooth map germ. We say that the pair (μ, ℓ) is *a Legendrian family* if $\ell_t = \ell|\mu^{-1}(t)$ is a Legendrian immersion germ for any $t \in (\mathbb{R}, t_0)$. Then we have the following simple but very important lemma.

Lemma 2.1. *Let (μ, ℓ) be a Legendrian family. Then there exists a unique element $h \in C^\infty_{u_0}(R)$ such that*

$$\ell^*\theta = h \cdot d\mu,$$

where $C^\infty_{u_0}(R)$ is the ring of smooth function germs at u_0.

We now consider the 1-jet bundle $J^1(\mathbb{R}\times\mathbb{R}^n, \mathbb{R})$ and the canonical 1-form Θ on the space. Let $(t, x_1, \ldots, x_n)$ be canonical coordinate system on $\mathbb{R} \times \mathbb{R}^n$ and

$$(t, x_1, \ldots, x_n, y, q, p_1, \ldots, p_n)$$

be corresponding coordinate system on $J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$. Then the canonical 1-form is given by

$$\Theta = dy - \sum_{i=1}^{n} p_i \cdot dx_i - q \cdot dt = \theta - q \cdot dt.$$

We define two natural projections. Namely,

$$\Pi : J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}) \to (\mathbb{R} \times \mathbb{R}^n) \times \mathbb{R}$$

by

$$\Pi(t, x, y, q, p) = (t, x, y)$$

and

$$\Pi' : J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}) \to \mathbb{R} \times \mathbb{R}^n$$

by

$$\Pi'(t, x, y) = (t, x).$$

We call the above 1-jet bundle *an unfolded 1-jet bundle.*

Define a map germ

$$\mathcal{L} : (R, u_0) \to J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$$

by

$$\mathcal{L}(u) = (\mu(u), x \circ \ell(u), y \circ \ell(u), h(u), p \circ \ell(u)).$$

Then we can easily show that $\mathcal{L}$ is a Legendrian immersion germ. If we fix 1-forms Θ and θ, the Legendrian immersion germ $\mathcal{L}$ is uniquely determined by the Legendrian family (μ, ℓ). We call $\mathcal{L}$ *a Legendrian unfolding associated with the Legendrian family* (μ, ℓ).

Since $\mathcal{L}$ is a Legendrian immersion germ, there exists a generating family of $\mathcal{L}$ by the Arnol'd-Zakalyukin's theory ([1],[2],[8]). In this case the generating family is naturally constructed by the one-parameter family of generating families associated with (μ, ℓ).

Let

$$F : ((\mathbb{R} \times \mathbb{R}^n) \times \mathbb{R}^k, 0) \to (\mathbb{R}, 0)$$

be a function germ such that

$$d_2F|0 \times \mathbb{R}^n \times \mathbb{R}^k$$

is non-singular, where

$$d_2F(t,x,q) = (\frac{\partial F}{\partial q_1}(t,x,q), \dots, \frac{\partial F}{\partial q_k}(t,x,q)).$$

We call F *a generalized phase function germ.* Then $C(F) = d_2F^{-1}(0)$ is a smooth $(n+1)$-manifold germ and

$$\pi_F : (C(F), 0) \to \mathbb{R}$$

is a submersion germ, where

$$\pi_F(t,x,q) = t.$$

Define map germs

$$\tilde{\Phi}_F : (C(F), 0) \to J^1(\mathbb{R}^n, \mathbb{R})$$

by

$$\tilde{\Phi}_F(t,x,q) = (x, F(t,x,q), \frac{\partial F}{\partial x}(t,x,q))$$

and

$$\Phi_F : (C(F), 0) \to J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$$

by

$$\Phi_F(t,x,q) = (t, x, F(t,x,q), \frac{\partial F}{\partial t}(t,x,q), \frac{\partial F}{\partial x}(t,x,q)).$$

Since $\frac{\partial F}{\partial q_i} = 0$ on $C(F)$, we can easily show that

$$(\tilde{\Phi}_F)^*\theta = \frac{\partial F}{\partial t}|C(F) \cdot dt|C(F).$$

By the definition, Φ_F is a Legendrian unfolding associated with the Legendrian family $(\pi_F, \tilde{\Phi}_F)$. By the same method of the theory of Arnol'd-Zakalyukin ([1],[2],[8]), we can show the following proposition.

Proposition 2.2. *Every Legendrian unfolding germs are constructed by the above method.*

§3. Geometry of Hamilton-Jacobi equation

In this section we will treat Hamilton-Jacobi equation in the framework of the geometric theory of first order partial differential equations [4]. *Hamilton-Jacobi equation* is defined to be a hypersurface

$$E(H) = \left\{(t, x, y, q, p) \in J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}) | q + H(t, x, p) = 0\right\}$$

in $J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$. *A geometric solution* of $E(H)$ is a Legendrian submanifold $\mathcal{L}$ lying in $E(H)$.

Since the equation is contact regular at every points (i.e. $\Theta | E(H) \neq 0$), a generalized Cauchy problem (GCP) has a unique solution: It is solved by the method of characteristic equations. In this case the characteristic vector field is given by

$$X_H = -\frac{\partial}{\partial t} - \sum_{i=1}^{n} \frac{\partial H}{\partial p_i} \frac{\partial}{\partial x_i} - \left(\sum_{i=1}^{n} p_i \frac{\partial H}{\partial p_i} - H\right) \frac{\partial}{\partial y} + \frac{\partial H}{\partial t} \frac{\partial}{\partial q} + \sum_{i=1}^{n} \frac{\partial H}{\partial x_i} \frac{\partial}{\partial p_i}.$$

We say that *a generalized Cauchy problem* (GCP)*is given for an equation* $E(H)$ if there is given an n-dimensional submanifold $i : L' \subset E(H)$ such that $i^* \Theta = 0$ and $X_{H,x} \notin T_x(L')$ for any $x \in L'$.

Theorem 3.1 (Classical existence theorem [4]). *A GCP* $i : L' \subset E(H)$ *has a unique solution, that is, there is a Legendrian submanifold* $\mathcal{L} \subset E(H)$, $L' \subset \mathcal{L}$ *and any two such Legendrian submanifolds coincide in a neighbourhood of* L'.

But GCP is not enough to serve our purpose. We need a more restricted framework. For any $c \in (\mathbb{R}, 0)$, we set

$$E(H)_c = \left\{(c, x, y, -H(c, x, p), p) | (x, y, p) \in J^1(\mathbb{R}^n, \mathbb{R})\right\}.$$

Then it is a $(2n+1)$-dimensional submanifold of $J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$ and $\Theta_c = \Theta | E(H)_c = dz - \sum_{i=1}^{n} p_i dx_i$ gives a contact structure on $E(H)_c$.

We define a mapping

$$\iota_c : J^1(\mathbb{R}^n, \mathbb{R}) \to E(H)_c$$

by

$$\iota_c(x,y,p) = (c,x,y,-H(c,x,p),p).$$

Then it is a contact diffeomorphism and the following diagram is commutative:

$$\begin{array}{ccc} J^1(\mathbb{R}^n,\mathbb{R}) & \xrightarrow{\iota_c} & E(H)_c \\ \pi\downarrow & & \downarrow\pi_c \\ \mathbb{R}^n\times\mathbb{R} & = & \mathbb{R}^n\times\mathbb{R}. \end{array}$$

We say that *a generalized Cauchy problem associated with the time parameter*(GCPT) *is given for an equation* $E(H)$ if a GCP $i : L' \subset E(H)$ with $i(L') \subset E(H)_c$ for some $c \in (\mathbb{R},0)$ is given.

Remark. The Cauchy problem $y(0,x_1,\ldots,x_n) = \phi(x_1,\ldots,x_n)$ is a GCPT. The initial submanifold is given by

$$L_{\phi,0} = \left\{(0,x,\phi(x),-H(0,x,\frac{\partial\phi}{\partial x}),\frac{\partial\phi}{\partial x})|x\in\mathbb{R}^n\right\} \subset E(H)_0.$$

Our purpose is formulated as follows:

Problem. Classify the generic bifurcations of singularities of

$$\pi_t| : \mathcal{L}\cap E(H)_t \to \mathbb{R}^n\times\mathbb{R}$$

and

$$\pi'_t| : \mathcal{L}\cap E(H)_t \to \mathbb{R}^n$$

with respect to the parameter t.

Remark. In their papers Tsuji ([6],[7]) and Nakane [5] assumed that singularities of $\pi'_t|$ are fold or cusp type singularities. But these singularities are stable in the sense of Thom, then these do not bifurcate along the time parameter. Since the initial condition of the Cauchy problem is non-singular, then other types of singularities must be appeared in generic.

Let $i : L' \subset E(H)_0 \subset E(H)$ be a GCPT and $\mathcal{L}$ be the unique solution of L'. Since $X_{H,x} \notin T_xE(H)_c$, then $\mathcal{L}$ is transverse to $E(H)_c$ in $E(H)$ for any $c \in (\mathbb{R},0)$. It follows that $\mathcal{L}_c = \mathcal{L}\cap E(H)_c$ is an n-dimensional submanifold of $E(H)_c$ and it satisfies $\Theta_c|\mathcal{L}_c = 0$ (i.e. $\mathcal{L}_c$ is a Legendrian submanifold of $E(H)_c$). If we consider the local parametrization of $\mathcal{L}$, we may assume that $\mathcal{L}$ is a image of an immersion germ

$$\mathcal{L} : (\mathbb{R}\times\mathbb{R}^n,0) \to E(H)$$

such that $\mathcal{L}|(c \times \mathbb{R}^n)$ is a Legendrian immersion germ of $E(H)_c$. Hence the coordinate representation of $\mathcal{L}$ is given by

$$\mathcal{L}(t,u) = (t, x(t,u), y(t,u), -H(t, x(t,u), p(t,u)), p(t,u)).$$

We now define the projection

$$\tilde{\pi} : J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}) \to J^1(\mathbb{R}^n, \mathbb{R})$$

by

$$\tilde{\pi}(t, x, y, q, p) = (x, y, p).$$

It follows from the above arguments that $(\pi_1, \tilde{\pi} \circ \mathcal{L})$ is a Legendrian family, where

$$\pi_1 : (\mathbb{R} \times \mathbb{R}^n, 0) \to (\mathbb{R}, 0)$$

is the canonical projection. Hence $\mathcal{L}$ is a Legendrian unfolding associated with $(\pi_1, \tilde{\pi} \circ \mathcal{L})$.

The following theorem is fundamental in our theory.

Theorem 3.2. (1) *The local solution of the generalized Cauchy problem associated with the time parameter for Hamilton-Jacobi equation*

$$q + H(t, x, p) = 0$$

is a Legendrian unfolding

$$\mathcal{L} : (\mathbb{R} \times \mathbb{R}^n, 0) \to J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}).$$

(2) *Let* $\mathcal{L} : (\mathbb{R} \times \mathbb{R}^n, 0) \to J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$ *be a Legendrian unfolding associated with* (π_1, ℓ). *Then there exists a* C^∞*-function germ* $H(t, x_1, \ldots, x_n, p_1, \ldots, p_n)$ *such that* $\mathcal{L}$ *is a local solution of the generalized Cauchy problem associated with the time parameter for Hamilton-Jacobi equation*

$$q + H(t, x, p) = 0,$$

where the initial condition is given by $\ell(0, u)$.

Proof. The assertion (1) is already proved by the above arguments. We now prove the assertion (2). Taking a coordinate representation of ℓ, we have

$$\ell(t,u) = (x(t,u), y(t,u), p(t,u)).$$

Since (π_1, ℓ) is a Legendrian family, we have

$$\langle dt \rangle_{\mathcal{E}_{n+1}} \supset \langle \ell^* \theta \rangle_{\mathcal{E}_{n+1}}.$$

Hence, there exists a C^∞-function germ $h(t,u)$ such that

$$dy(t,u) - \sum_{i=1}^{n} p_i(t,u)dx_i(t,u) = h(t,u)dt.$$

By the definition of the Legendrian unfolding, we have

$$\mathcal{L}(t,u) = (t, x(t,u), y(t,u), h(t,u), p(t,u)).$$

We now define a C^∞-map germ

$$\tilde{\ell} : (\mathbb{R}\times\mathbb{R}^n, 0) \to T^*\mathbb{R}^n$$

by

$$\tilde{\ell}(t,u) = (x(t,u), p(t,u)).$$

Since (π_1, ℓ) is a Legendrian family, $\tilde{\ell}_t$ is a Lagrangian immersion germ with respect to the canonical symplectic structure on $T^*\mathbb{R}^n$ for any $t \in (\mathbb{R}, 0)$.

We also define a C^∞-map germ

$$\ell' : (\mathbb{R}\times\mathbb{R}^n, 0) \to \mathbb{R}\times T^*\mathbb{R}^n$$

by

$$\ell'(t,u) = (t, x(t,u), p(t,u)).$$

By the above argument, ℓ' is an immersion germ. Then

$$\ell'^* : C^\infty_{\ell'(0)}(\mathbb{R}\times T^*\mathbb{R}^n) \to C^\infty_0(\mathbb{R}\times\mathbb{R}^n)$$

is an epimorphism. Since $h \in C^\infty_0(\mathbb{R}\times\mathbb{R}^n)$, there exists $H \in C^\infty_{\ell'(0)}(\mathbb{R}\times T^*\mathbb{R}^n)$ such that $\ell'^*(H) = -h$. That is,

$$-H(t, x(t,u), p(t,u)) = h(t,u).$$

Thus the Legendrian unfolding

$$\mathcal{L} : (\mathbb{R}\times\mathbb{R}^n, 0) \to J^1(\mathbb{R}\times\mathbb{R}^n, \mathbb{R})$$

is a geometric solution of the Hamilton-Jacobi equation

$$q + H(t,x,p) = 0.$$

By the uniqueness of the solution, it is also a local solution of GCPT of the Hamilton-Jacobi equation whose initial condition is $\ell(0,u)$.

§4. Classifications

We now give generic classifications for bifurcations of singularities of solutions along the time parameter. By Theorem 3.2, we can apply the classification theory of bifurcations of singularities of one-parameter Legendrian unfoldings. This section depends heavily on Arnol'd and Zakalyukin's results ([1],[2],[8]). Let

$$\mathcal{L} : (R, u_0) \to (J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), z_0)$$

and

$$\mathcal{L}' : (R, u_1) \to (J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), z_1)$$

be Legendrian unfoldings. We say that *two wave front sets $W(\mathcal{L})$ and $W(\mathcal{L}')$ have diffeomorphic bifurcations* if there exists a diffeomorphism germ

$$\Phi : ((\mathbb{R} \times \mathbb{R}^n) \times \mathbb{R}, \Pi(z_0)) \to ((\mathbb{R} \times \mathbb{R}^n) \times \mathbb{R}, \Pi(z_1))$$

of the form $\Phi(t, x, y) = (\phi(t), \Phi_2(t, x, y))$ such that

$$\Phi(W(\mathcal{L})) = W(\mathcal{L}').$$

We also say that *two caustics $C(\mathcal{L})$ and $C(\mathcal{L}')$ have diffeomorphic bifurcations* if there exists a diffeomorphism germ

$$\Psi : (\mathbb{R} \times \mathbb{R}^n, \Pi'(z_0)) \to (\mathbb{R} \times \mathbb{R}^n, \Pi'(z_1))$$

of the form $\Psi(t, x) = (\psi(t), \Psi_2(t, x))$ such that

$$\Psi(C(\mathcal{L})) = C(\mathcal{L}').$$

In their papers ([1],[2],[8]) Arnol'd and Zakalyukin gave generic classifications of one-parameter perestroika of wave front sets and caustics in the case $n \leq 4$. As an application of their classifications, we have classifications as follows.

Theorem 4.1. (1) *Bifurcations of wave front sets in generic one-parameter Legendrian unfoldings for $n \leq 2$ are diffeomorphic to one of the bifurcation of wave front sets defined by generalized phase function germs from the following list :*
$n = 1$:
${}^0A_1$: q_1^2 ;
${}^0A_2$: $q_1^3 + x_1 q_1$;
${}^1A_3$: $q_1^4 + q_1^2 t + x_1 q_1$.
$n = 2$:
${}^0A_1$: q_1^2 ;

${}^0A_2$: $q_1^3 + x_1q_1$;
${}^0A_3$: $q_1^4 + x_1q_1 + x_2q_1^2$;
${}^1A_3$: $q_1^4 + q_1^2(t \pm x_2^2) + x_1q_1$;
${}^1A_4$: $q_1^5 + q_1^3t + x_1q_1 + x_2q_1^2$;
${}^1D_4$: $q_2q_1^2 \pm q_2^3 + q_2^2t + x_2q_2 + x_1q_1$.

(2) *Bifurcations of caustics in generic one-parameter Legendrian unfoldings for* $n \leq 2$ *are diffeomorphic to one of the bifurcation of caustics defined by generalized phase function germs from the following list :*
$n = 1$:
A_2 : $q_1^3 + x_1q_1$;
A_3 : $q_1^4 + tq_1^2 + x_1q_1$;
$n = 2$:
A_2 : $q_1^3 + x_1q_1$;
A_3 : $q_1^4 + x_2q_1^2 + x_1q_1$;
A_3^1 : $q_1^4 + (t \pm x_2^2)q_1^2 + x_1q_1$;
A_4 : $q_1^5 + tq_1^3 + x_2q_1^2 + x_1q_1$;
$D_4^\pm$: $q_2q_1^2 \pm q_2^3 + q_2^2(x_1 + ax_2 \pm t) + x_1q_1 + x_2q_2, a \in \mathbb{R}$, *where* a *is a moduli parameter.*

In their classifications ([1],[2],[8]), ${}^1A_1$-type (i.e., $q_1^2 + t^2 \pm x_1^2 \pm y^2$, $q_1^2 + t \pm x_1^2 \pm x_2^2 \pm y^2$) and ${}^1A_2$-type (i.e., $q_1^3 + q_1(t \pm x_1^2)$, $q_1^3 + q_1(t \pm x_1^2 \pm x_2^2)$) bifurcations are contained in the list of perestroikas of wave front sets. Because the notion of extended Legendrian manifolds in [8] is slightly different from our notion of Legendrian unfoldings, thus ${}^1A_1$-type and ${}^1A_2$-type bifurcations do not appear in our list. We can also list up normal forms of bifurcations in the case when $n = 3$ or $n = 4$. But it is too complicated to explain here.

We now represent the list of bifurcations in the above theorem by using the coordinate representation of map germs. Let

$$f, g : (\mathbb{R} \times \mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$$

be smooth map germs. We define map germs

$$F, G : (\mathbb{R} \times \mathbb{R}^n, 0) \to (\mathbb{R} \times \mathbb{R}^p, 0)$$

by

$$F(t, x) = (t, f(t, x)) \quad \text{and} \quad G(t, x) = (t, g(t, x)).$$

We say that *two images* Image(F) *and* Image(G) *have diffeomorphic bifurcations* if there exists a diffeomorphism germ

$$\Phi : (\mathbb{R} \times \mathbb{R}^p, 0) \to (\mathbb{R} \times \mathbb{R}^p, 0)$$

of the form $\Phi(t,y) = (\phi(t), \Phi_2(t,y))$ such that

$$\Phi(\mathrm{Image}(F)) = \mathrm{Image}(G).$$

We also say that *two critical value sets $C(F)$ and $C(G)$ have diffeomorphic bifurcations* if there exists diffeomorphism germ

$$\Psi : (\mathbb{R} \times \mathbb{R}^p, 0) \to (\mathbb{R} \times \mathbb{R}^p, 0)$$

of the form $\Psi(t,y) = (\psi(t), \Psi_2(t,y))$ such that

$$\Psi(C(F)) = C(G).$$

Let $\mathcal{L} : (\mathbb{R} \times \mathbb{R}^n, 0) \to J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$ be Legendrian unfolding associated with (π_1, ℓ). Then the wave front set $W(\mathcal{L})$ is the image of $\Pi \circ \mathcal{L}$ and the caustics $C(\mathcal{L})$ is the critical value set of $\Pi' \circ \mathcal{L}$. Thus we have the following corollary of Theorem 4.1.

Corollary 4.2. (1) *Bifurcations of wave front sets in generic one-parameter Legendrian unfoldings for $n \leq 2$ are diffeomorphic to one of the bifurcation of images defined by map germs from the following list :*
$n = 1$:
${}^0A_1$: $(x, 0)$;
${}^0A_2$: (x^2, x^3) ;
${}^1A_3$: $(2x^3 + tx, 3x^4 + tx^2)$.
$n = 2$:
${}^0A_1$: $(x_1, x_2, 0)$;
${}^0A_2$: (x_1, x_2^2, x_2^3) ;
${}^0A_3$: $(x_1, x_2^3 + tx_2, 3x_2^4 + x_1x_2^2)$;
${}^1A_3$: $(x_1, 2x_2^3 + x_2(t \pm x_1^2), 3x_2^4 + x_2^2(t \pm x_1^2))$;
${}^1A_4$: $(x_1, 5x_2^4 + 3x_2^2t + 2x_2x_1, 4x_2^5 + 2x_2^3t + x_2^2x_1)$;
${}^1D_4$: $(x_1x_2, (x_1^2 \pm 3x_2^2) + 2x_2t, 2(x_1^2x_2 \pm x_2^3) + x_2^2t)$.

(2) *Bifurcations of caustics in generic one-parameter Legendrian unfoldings for $n \leq 2$ are diffeomorphic to one of the bifurcation of critical value sets defined by map germs from the following list :*
$n = 1$:
A_2 : x^2 ;
A_3 : $x^3 + tx$.
$n = 2$:
A_2 : (x_1, x_2^2) ;
A_3 : $(x_1, x_2^3 + x_1x_2)$;
A_3^1 : $(x_1, x_2^3 + x_2(t \pm x_1^2))$;
A_4 : $(x_1, x_2^4 + x_2^2t + x_2x_1)$;

$D_4^{\pm}$: $(x_1x_2, \frac{1}{1+2ax_2}(4x_1x_2^2 \mp 2x_2t - x_1^2 \mp 3x_2^2)), a \in \mathbb{R}$, *where* a *is a moduli parameter.*

Remark. In the above lists, bifurcations of caustics given by A_3 for $n = 1$ and A_3^1 for $n = 2$ describe the process of birth of the caustics from the empty. If the initial condition of the Cauchy problem is smooth, these process must exist for a neighbourhood of some t_0.

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Characterization of Images of Radon Transforms

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§0. Introduction

Since F. John [7], the characterization of images of Radon transforms has been one of the main subjects of the theory of Radon transforms. When we recall that the origin of Radon transform was the transform of functions on the 2-sphere by averaging over the great circles, it is rather surprising to find that the characterization of images of Radon transforms on compact symmetric spaces had not been treated until E. Grinberg [4]. There Grinberg showed that the image of Radon transform concerning real or complex Grassmann manifolds can be characterized by an invariant system of differential operators, using the representation theoretical argument. We can see easily that the characterization may also be done by an invariant differential operator of higher order, though Grinberg did not mention it explicitly.

The purpose of this paper is to give another type of characterization, that is, the characterization by an invariant differential operator that takes values in the sections of a vector bundle. The approach by Grinberg used the left action of a group, and ours uses the right action, which lies, in a sense, on the other side with respect to the bi-sided invariant differential operator. We hope our approach will be the first step to fill some vacancy in the theory of invariant differential operators on compact symmetric spaces.

§1. The Radon transform on the sphere

We first consider the case of the standard sphere S^n of radius 1 in the Euclidean space $\mathbf{R}^{n+1}$. A geodesic γ of the sphere S^n is nothing but a great circle, which is determined by a 2-dimensional vector subspace of $\mathbf{R}^{n+1}$. We shall treat the geodesics with their orientation for convenience' sake. The set of oriented geodesics, which we denote by $\operatorname{Geod} S^n$, is the oriented real Grassmann manifold $G_{n+1,2}(\mathbf{R})$.

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For a function f on S^n, we define its Radon transform $\mathcal{R}(f)$ to be a function on $\operatorname{Geod} S^n$, the value of which at a point $\gamma \in \operatorname{Geod} S^n$ is given by the average of f over γ. More specifically speaking, we set

$$(\mathcal{R}(f))(\gamma) = \frac{1}{2\pi} \int_0^{2\pi} f(\gamma(s))\, ds,$$

where $\gamma(s)$ is the parametrization of γ by its arclength. We will always concern with smooth functions and denote the space of smooth functions by $\mathcal{F}$. The Radon transform $\mathcal{R}$ is a mapping from the space $\mathcal{F}(S^n)$ to the space $\mathcal{F}(\operatorname{Geod} S^n)$.

The antipodal mapping σ on the sphere is a smooth involution given by $\sigma(x) = -x$ for $x \in S^n \subset \mathbf{R}^{n+1}$. A function f on the sphere is called even when $f \circ \sigma = f$, and odd when $f \circ \sigma = -f$. We denote by $\mathcal{F}_{\text{even}}(S^n)$ and $\mathcal{F}_{\text{odd}}(S^n)$ the spaces of smooth even functions and smooth odd functions, respectively. It is obvious that the space $\mathcal{F}_{\text{odd}}(S^n)$ is included in the kernel of the Radon transform $\mathcal{R}$.

In the case of $n = 2$, $\operatorname{Geod} S^2$ is isomorphic to S^2, for an oriented geodesic has one-to-one correspondence with the *north pole* that makes that geodesic the *equator* with the suitable orientaion of longitude. The Radon transform $\mathcal{R}$ on S^2 is considered to be a mapping from the space $\mathcal{F}(S^2)$ to itself. It is also obvious that the image of $\mathcal{R}$ is included in the space $\mathcal{F}_{\text{even}}(S^2)$.

The following theorem by P. Funk [2] was the starting point of the theory of Radon transform.

Theorem 1.1. *The kernel of the Radon transform $\mathcal{R}$ on S^2 is equal to the space $\mathcal{F}_{\text{odd}}(S^2)$. As the mapping from $\mathcal{F}_{\text{even}}(S^2)$ to itself, the Radon transform $\mathcal{R}$ is an isomorphism.*

We can generalize this theorem to higher dimensions in the same form if we consider not the geodesic, that is, the great circle, but the great sphere of codimension 1. See, for example, S. Helgason [6]. Since the average of a function $f \in \mathcal{F}(S^n)$ over a great sphere of codimension 1 can be calculated by averaging the values of $(\mathcal{R}(f))(\gamma)$ for all the γ included in the great sphere, we can deduce the following theorem.

Theorem 1.2. *The kernel of the Radon transform $\mathcal{R}$ on S^n is equal to the space $\mathcal{F}_{\text{odd}}(S^n)$. The image $\operatorname{Im}\mathcal{R}$ is a closed subspace of $\mathcal{F}(\operatorname{Geod} S^n)$ in the C^∞-topology.*

We notice that the latter part of Theorem 1.2 is a consequence of the inversion formula of the Radon transform concerning the great sphere of codimension 1.

Since the dimension of $\operatorname{Geod} S^n$ is greater than n for $n \geq 3$, we cannot expect the Radon transform $\mathcal{R}$ to be surjective. In the next section, we try to find a good characterization of the image of $\mathcal{R}$.

§2. The differential operator $\mathcal{L}$ on $\operatorname{Geod} S^n$

We fix an orthonomal basis $\{e_1, e_2, \dots, e_{n+1}\}$ of $\mathbf{R}^{n+1}$. The special orthogonal group $SO(n+1)$ acts on S^n transitively and isometrically. We set $G = SO(n+1)$, and denote the isotropy group at $e_1 \in S^n$ by $H \cong SO(n)$. The group G acts transitively on the set of all oriented geodesics $\operatorname{Geod} S^n$, too. We take the oriented geodesic γ_0 that passes through e_1 and is pointing e_2 there as the origin of $\operatorname{Geod} S^n$ and denote the isotropy group at γ_0 by $K \cong SO(2) \times SO(n-1)$. We consider $\operatorname{Geod} S^n$ as a symmetric space G/K with the standard invariant metric.

We take $\{X_{ij}\}_{1 \leq j < i \leq n+1}$ as a basis of the Lie algebra $\mathfrak{g}$ of G, where X_{ij} is a matrix whose (k, l)-element is given by $\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}$. As usual, the orthogonal complement of the Lie algebra $\mathfrak{k}$ of K in $\mathfrak{g}$ is denoted by $\mathfrak{m}$.

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}, \qquad \mathfrak{m} = \bigoplus_{\substack{3 \leq a \leq n+1 \\ l=1,2}} \mathbf{R} X_{a,l}.$$

We always consider the action of G on the functions of a G-space to be the contravariant action of the left action.

$$(g \cdot F)(x) = F(g^{-1}x), \quad \text{for } g \in G,\ x \text{ a point of a } G\text{-space}.$$

We consider the group G to be a G-space by multiplication from the left. The element X_{ij} is considered to be an invariant differential operator acting on the space $\mathcal{F}(G)$ as follows.

$$(X_{ij}F)(g) = \frac{d}{dt} F(g \exp tX_{ij})|_{t=0}, \quad \text{for } F \in \mathcal{F}(G),\ g \in G.$$

For each pair of integers a and b satisfying $3 \leq a < b \leq n+1$, we define a second order differential operator L_{ab} acting on $\mathcal{F}(G)$ by

$$(L_{ab}F)(g) = (X_{a1}(X_{b2}F))(g) - (X_{a2}(X_{b1}F))(g).$$

The commutation relations $[X_{a1}, X_{b2}] = [X_{a2}, X_{b1}] = 0$ enable us to rewrite it as

$$(L_{ab}F)(g) = \left(\frac{\partial^2}{\partial t_{a1} \partial t_{b2}} - \frac{\partial^2}{\partial t_{a2} \partial t_{b1}} \right) F(g \exp X(t)) \Big|_{t=0},$$

where an element $X(t)$ of $\mathfrak{m}$ depending on $t = \{t_{al}\}_{\substack{3\le a\le n+1\\ l=1,2}}$ is given by $X(t) = \sum_{\substack{3\le a\le n+1\\ l=1,2}} t_{al}X_{al}$.

The space $\mathcal{F}(\mathrm{Geod}\, S^n)$ is regarded as a subspace of $\mathcal{F}(G)$ consisting of the elements F that satisfy $F(gk) = F(g)$ for all $k \in K$, $g \in G$. For these elements F, we have

$$\begin{aligned}(L_{ab}F)(gk) &= \left(\frac{\partial^2}{\partial t_{a1}\partial t_{b2}} - \frac{\partial^2}{\partial t_{a2}\partial t_{b1}}\right) F(gk\exp X(t)k^{-1}k)\Big|_{t=0}\\ &= \left(\frac{\partial^2}{\partial t_{a1}\partial t_{b2}} - \frac{\partial^2}{\partial t_{a2}\partial t_{b1}}\right) F(g\exp \mathrm{Ad}(k)X(t))\Big|_{t=0}.\end{aligned}$$

Notice that $\mathrm{Ad}(k)X(t)$ is written as $X(t')$, where t' is a linear combination of t determined by k. For an element $k \in K$ of the form

$$k = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix} \times (k_{cd})_{3\le c,d\le n+1} \qquad ((k_{cd}) \in SO(n-1)),$$

an easy calculation gives

$$(L_{ab}F)(gk) = \sum_{3\le c<d\le n+1} (k_{ca}k_{db} - k_{da}k_{cb})(L_{cd}F)(g).$$

Now we consider a vector space V of dimension $(n-1)(n-2)/2$, with a fixed basis $\{u_a \wedge u_b\}$ $(3 \le a < b \le n+1)$ and an action ρ of K given by

$$\rho(k)(u_a \wedge u_b) = \sum_{3\le c<d\le n+1} (k_{ca}k_{db} - k_{da}k_{cb})\, u_c \wedge u_d,$$

and define a V-valued function $\mathcal{L}F$ on G by

$$\mathcal{L}F = \sum_{3\le a<b\le n+1} (L_{ab}F)\, u_a \wedge u_b.$$

Then we have

$$\begin{aligned}(\mathcal{L}F)(gk) &= \sum_{3\le a<b\le n+1} (L_{ab}F)(gk)\, u_a \wedge u_b\\ &= \rho(k^{-1})(\mathcal{L}F)(g),\end{aligned}$$

which means that $\mathcal{L}F$ is a section of the vector bundle $E = G \times_K V$ on $G/K \cong \mathrm{Geod}\, S^n$ of rank $(n-1)(n-2)/2$, associated with the principal bundle $G \to G/K$ under the representation ρ.

We take the formal adjoint operator $\mathcal{L}^*$ of $\mathcal{L}$ with respect to the invariant inner products on $\mathcal{F}(\mathrm{Geod})$ and $C^\infty(E)$ induced by the invariant measure on G, and set $\mathcal{D} = \mathcal{L}^*\mathcal{L}$. In fact, the differential operator $\mathcal{D}$ is given by

$$(\mathcal{D}F)(g) = \sum_{3 \leq a < b \leq n+1} (L_{ab}(L_{ab}F))(g).$$

By construction, it is obvious that $\mathcal{L}$ and $\mathcal{D}$ are invariant differential operators.

Proposition 2.1. *The image of the Radon transform $\mathcal{R}$ is included in the kernel of the differential operator $\mathcal{D}$.*

Proof. Since $\operatorname{Ker}\mathcal{D}$ is equal to $\operatorname{Ker}\mathcal{L}$, it is enough to show that $\mathcal{L}(\mathcal{R}(f))$ vanishes for any function f on S^n. Since $\mathcal{R}$ and $\mathcal{L}$ are invariant operators, it is enough to show $\mathcal{L}(\mathcal{R}(f))(\gamma_0) = 0$ for any $f \in \mathcal{F}(S^n)$. (Notice that, for any $\gamma \in \operatorname{Geod} S^n$, there exists an element $g \in G$ that satisfies $\gamma = g\gamma_0$, and that we have $\mathcal{L}(\mathcal{R}(f))(\gamma) = \mathcal{L}(\mathcal{R}(g^{-1} \cdot f))(\gamma_0)$.)

Let us fix the indices a and b and show that $L_{ab}(\mathcal{R}(f))(e) = 0$ for any $f \in \mathcal{F}(S^n)$. We recall that, for $f \in \mathcal{F}(G/H)$, our definition of the Radon transform $\mathcal{R}$ is rewritten as

$$\mathcal{R}(f)(g) = \frac{1}{2\pi}\int_0^{2\pi} f(gk(\theta))\, d\theta,$$

where

$$k(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \times \mathrm{Id}.$$

Therefore we have

$$\begin{aligned} &L_{ab}(\mathcal{R}(f))(e) \\ &= \frac{1}{2\pi}\left(\frac{\partial^2}{\partial t_{a1}\partial t_{b2}} - \frac{\partial^2}{\partial t_{a2}\partial t_{b1}}\right)\int_0^{2\pi} f(\exp X(t)k(\theta))\, d\theta\bigg|_{t=0} \\ &= \frac{1}{2\pi}\int_0^{2\pi} \frac{\partial^2}{\partial t_{a1}\partial t_{b2}} f(\exp X(t)k(\theta))\bigg|_{t=0} d\theta \\ &\quad - \frac{1}{2\pi}\int_0^{2\pi} \frac{\partial^2}{\partial t_{a2}\partial t_{b1}} f(\exp X(t)k(\theta))\bigg|_{t=0} d\theta. \end{aligned}$$

The value of $f \in \mathcal{F}(G)$ at the point $\exp X(t)k(\theta) \in G$ where the components of t vanish except for $t_{a1} = r$ and $t_{b2} = s$ is given by the value of $f \in \mathcal{F}(S^n)$ at the point $\cos\theta(\cos r\, e_1 + \sin r\, e_a) + \sin\theta(\cos s\, e_2 +$

$\sin s\, e_b) \in S^n$. Therefore the former integral in the last expression is equal to

$$\frac{1}{2\pi}\int_0^{2\pi} \cos\theta\sin\theta(\nabla_{e_a}\nabla_{e_b}f)(\cos\theta\, e_1 + \sin\theta\, e_2)\, d\theta.$$

Since the point $\exp X(t)k(\theta)$ where the components of t vanish except for $t_{a2} = r$ and $t_{b1} = s$ corresponds to the point $\cos\theta(\cos s\, e_1 + \sin s\, e_b) + \sin\theta(\cos r\, e_2 + \sin r\, e_a)$, the latter integral in the last expression has the same value, and hence $L_{ab}(\mathcal{R}(f))(e)$ vanishes. Q.E.D.

Remark 2.2. The vanishing of $L_{ab}(\mathcal{R}(f))(e)$ is deduced from the geometric observation related to two 2-parameter families of geodesics, which is the same argument as given in F. John [7].

Remark 2.3. The ring of invariant differential operators on the rank 2 symmetric space $\mathrm{Geod}\, S^n$ $(n > 3)$ is generated by the Laplace operator Δ and the 4-th order differential operator $\mathcal{D}$. For the case $n = 3$, see the next section.

§3. The case $n = 3$

Let us recall the elementary facts on $\mathrm{Geod}\, S^3$. An oriented great circle on S^3 is specified by its point e_1 and its unit tangent vector e_2 at e_1, and corresponds one-to-one to the exterior product $\omega = e_1 \wedge e_2$ with unit norm. A 2-vector $\omega \in \bigwedge^2 \mathbf{R}^4$ with unit norm corresponds to a great circle if and only if it is decomposable, that is, $\omega \wedge \omega$ vanishes. In view of the Hodge star operator $*$ on $\bigwedge^2 \mathbf{R}^4$, the latter condition is the same as saying the norm of the self-dual part $\omega_+ = (\omega + *\omega)/2$ is equal to the norm of th anti-self-dual part $\omega_- = (\omega - *\omega)/2$. Since the self-dual 2-vectors and the anti-self-dual 2-vectors both form the 3-dimensional vector spaces V_+ and V_-, a decomposable 2-vector with unit norm has one-to-one correspondence with the product of two 2-spheres, $S^2_+ \subset V_+$ and $S^2_- \subset V_-$, with radius $1/\sqrt{2}$. We thus have the isomorphism $\mathrm{Geod}\, S^3 \cong S^2_+ \times S^2_-$.

In the case $n = 3$, since the representation ρ in the last section is trivial, the vector bundle E of rank 1 is also trivial. We have only to consider the invariant differential operator $\mathcal{L} = L_{34}$. In view of the above isomorphism, $\mathcal{L}$ is shown to be the differential operator $\Delta_+ - \Delta_-$, where $\Delta_\pm$ is the Laplace operator on $S^2_\pm$.

Notice that the ring of invariant differential operators on the rank 2 (but not irreducible) symmetric space $\mathrm{Geod}\, S^3$ is generated by the Laplacian $\Delta = \Delta_+ + \Delta_-$ and the second order differential operator $\mathcal{L}$.

We shall show the main theorem for S^3 by means of the representation theory of $SO(4)$.

We denote by E^n_k the space of fucntions on S^n that are the restrictions of the harmonic polynomials on $\mathbf{R}^{n+1}$ of degree k. The following decompositions of the function spaces are well-known.

Lemma 3.1. *We have the direct sum decompositions*

$$\mathcal{F}(S^3) \approx \sum_{k=0}^{\infty} E^3_k,$$

$$\mathcal{F}_{\text{even}}(S^3) \approx \sum_{k=0}^{\infty} E^3_{2k},$$

$$\mathcal{F}(S^2_+ \times S^2_-) \approx \sum_{k,l=0}^{\infty} E^2_k \boxtimes E^2_l,$$

where the symbol $\approx$ means that the right-hand side is densely included in the left-hand side.

The above decompositions are in fact the decompositions as $SO(4)$-modules. We fix the Lie algebra $\mathfrak{t} \subset \mathfrak{g}$ corresponding to $SO(2)\times SO(2) \subset SO(4)$, and the basis $\{\lambda_1, \lambda_2\}$ of the complexified dual space $\mathfrak{t}^*_{\mathbf{C}}$ of $\mathfrak{t}$ as follows.

$$\lambda_1\left(\begin{pmatrix} 0 & -a \\ a & 0 \end{pmatrix}, \begin{pmatrix} 0 & -b \\ b & 0 \end{pmatrix}\right) = \sqrt{-1}a,$$

$$\lambda_2\left(\begin{pmatrix} 0 & -a \\ a & 0 \end{pmatrix}, \begin{pmatrix} 0 & -b \\ b & 0 \end{pmatrix}\right) = \sqrt{-1}b.$$

We order them as $\lambda_1 > \lambda_2$. The following lemma is easy to verify.

Lemma 3.2. *The space E^3_k is an irreducible $SO(4)$-module with the highest weight $k\lambda_1$. The space $E^2_k \boxtimes E^2_l$ is an irreducible $SO(4)$-module with the highest weight $(k+l)\lambda_1 + (k-l)\lambda_2$.*

Theorem 3.3. *The image of the Radon transform $\mathcal{R}$ is equal to the kernel of the differential operator $\mathcal{L}$.*

Proof. Since the operator $\mathcal{R}$ is injective on $\mathcal{F}_{\text{even}}(S^3)$ and commutes with the action of $SO(4)$, it isomorphically maps the space E^3_{2k} with the highest weight $2k\lambda_1$ to the space of the same highest weight, which must be the space $E^2_k \boxtimes E^2_k$. Therefore we have $\operatorname{Im}\mathcal{R} \approx \sum_k E^2_k \boxtimes E^2_k$.

On the other hand, since Δ_+ acts on $E_k^2 \boxtimes E_l^2$ as $k(k+1)\mathrm{Id}$ and Δ_- acts on it as $l(l+1)\mathrm{Id}$,

$$\mathrm{Ker}(\Delta_+ - \Delta_-) \approx \sum_{\substack{k,l \\ k(k+1)=l(l+1)}} E_k^2 \boxtimes E_l^2 = \sum_k E_k^2 \boxtimes E_k^2.$$

Since we have $\mathrm{Im}\,\mathcal{R} \subset \mathrm{Ker}\,\mathcal{L}$ and these closed subspaces include the same dense subspace in common, they must coincide. Q.E.D.

§4. Reduction to the case $n = 3$

We shall prove the main theorem for a general case by reducing it to the case $n = 3$.

We denote by S_0^3 the totally geodesic 3-sphere in S^n that is included in the subspace spanned by e_1, e_2, e_3, and e_4. All the other totally geodesic 3-sphere in S^n is written as gS_0^3 for some element $g \in G = SO(n+1)$. The manifold $N = \mathrm{Geod}\,S_0^3$ of the oriented great circles included in S_0^3 is a homogeneous manifold G'/K', where G' is $SO(4)$ considered as a subgroup of G and K' is $G' \cap K \cong SO(2) \times SO(2)$.

Now let us consider what happens when the vector bundle E is restricted to N. Since E is an associated vector bundle $G \underset{K}{\times} V$ and N is a homogeneous manifold G'/K', we have $E|_N = G' \underset{K'}{\times} V$, where the action of K' on V is that of K restricted. When the representation ρ of K in V is restricted to the subgroup K', it decomposes to a sum of irreducible components and has the subspace spanned by the vector $u_3 \wedge u_4$ as its irreducible component with trivial action. Therefore the vector bundle $E|_N$ splits to a sum of subbundles, one of which is the trivial subbundle of rank 1 corresponding to $u_3 \wedge u_4$.

When a section $\mathcal{L}(F)$ of E for $F \in \mathcal{F}(\mathrm{Geod}\,S^n)$ is restricted to N, its $u_3 \wedge u_4$-component is just $L_{34}(F)$, and, by construction, is equal to $\mathcal{L}(F|_N)$. The vanishing of $\mathcal{L}(F)$ implies the vanishing of $\mathcal{L}(F|_N)$, and $F|_N$ is in the image of the Radon transform on S_0^3 by Theorem 2.1. Taking account of the equivariance of our construction, we have the following lemma.

Lemma 4.1. *If $F \in \mathcal{F}(\mathrm{Geod}\,S^n)$ is in the kernel of $\mathcal{L}$, its restriction to the submanifold $gN = \mathrm{Geod}(gS_0^3)$ is in the image of the Radon transform on $S^3 = gS_0^3$ for every $g \in G$.*

We notice that this lemma implys that $F \in \mathrm{Ker}\,\mathcal{L}$ is an even function in the sense that, for any totally geodesic $S^2 \subset S^n$, the restriction of

F to Geod S^2 is an even function; there exists a totally geodesic S^3 statisfying $S^2 \subset S^3 \subset S^n$ and the restriction of F to Geod S^3, and hence to Geod S^2, is in the image of the Radon transform.

Theorem 4.2. *For $n \geq 3$, if $F \in \mathcal{F}(\mathrm{Geod}\, S^n)$ is in the kernel of $\mathcal{L}$, it is in the image of the Radon transform $\mathcal{R}$ on S^n. Therefore we have* $\mathrm{Ker}\, \mathcal{D} = \mathrm{Ker}\, \mathcal{L} = \mathrm{Im}\, \mathcal{R}$.

Proof. We take a point $x \in S^n$ and shall fix a value $f(x)$ of a function $f \in \mathcal{F}(S^n)$ for which we should have $\mathcal{R}(f) = F$.

If we choose a totally geodesic 2-sphere S^2 containing x, we can uniquely determine an even function f on S^2 with the property that the image of the Radon transform on S^2 of f is equal to the restriction of F, since the Radon transform on S^2 is an isomorphism on the even functions. We claim that the value $f(x)$ does not depend on the totally geodesic S^2 chosen.

For any two totally geodesic 2-spheres S_a^2 and S_b^2 containing x, there exist the third totally geodesic 2-sphere S_c^2 containing x and two totally geodesic 3-spheres S_{ac}^3 and S_{bc}^3 that satisfy S_a^2, $S_c^2 \subset S_{ac}^3$ and S_b^2, $S_c^2 \subset S_{bc}^3$. We denote by f_r ($r = a$, b, or c) the even functions on S_r^2 with the property that the image of the Radon transform on S_r^2 of f_r is equal to the restriction of F to Geod S_r^2. By the last lemma, the restriction of F to Geod S_{ac}^3 is in the image of the Radon transform on S_{ac}^3 of a function on S_{ac}^3, say, f_{ac}. Taking the even part of f_{ac} if needed, we may assume that f_{ac} is an even function. Since the Radon transform is injective on the even functions, the restriction of f_{ac} to S_a^2 is equal to f_a and that to S_c^2 is equal to f_c. Therefore we have $f_a(x) = f_c(x)$ and, by the same reasoning, $f_c(x) = f_b(x)$, which assures our claim.

We see easily that the function f on S^n thus constructed is continuous and has the property $\mathcal{R}(f) = F$. By the inversion formula, f is shown to be smooth. Q.E.D.

§5. The case of the complex projective space

In the case of the complex projective space $P^n(\mathbf{C})$, we consider the projective line as its counter part of the oriented geodesic in the sphere. Since a projective line $C \subset P^n(\mathbf{C})$ corresponds to a 2-dimentional complex vector subspace of $\mathbf{C}^{n+1}$, the set of projective lines is the complex Grassmann manifold $G_{n+1,2}(\mathbf{C})$. We define the Radon transform $\mathcal{R}(f)$ of a function f on the complex projective space $P^n(\mathbf{C})$ by assigning to each point C of $G_{n+1,2}(\mathbf{C})$ the averaged value of f over C.

By the same argument as in Theorem 1.2, we have the following theorem.

Theorem 5.1. *The Radon transform $\mathcal{R}$ on $P^n(\mathbf{C})$ is an injective mapping from $\mathcal{F}(P^n(\mathbf{C}))$ to $\mathcal{F}(G_{n+1,2}(\mathbf{C}))$. The image* $\operatorname{Im}\mathcal{R}$ *is closed in the C^∞-topology.*

For $n = 2$, the complex Grassmann manifold $G_{3,2}(\mathbf{C})$ is isomorphic to the complex projective space $P^2(\mathbf{C})$ and the Radon transform on $P^2(\mathbf{C})$ is an isomorphism. For $n \geq 3$, the dimension of $G_{n+1,2}(\mathbf{C})$ is greater than that of $P^n(\mathbf{C})$ and the Radon transform is not surjective.

We fix an orthonomal basis $\{e_1, e_2, \ldots, e_{n+1}\}$ of $\mathbf{C}^{n+1}$. The unitary group $U(n+1)$ acts on $P^n(\mathbf{C})$ transitively and isometrically. We here set $G = U(n+1)$, and denote the isotropy group at $[e_1] \in P^n(\mathbf{C})$ by $H \cong U(1) \times U(n)$. The group G acts transitively on the complex Grassmann manifold $G_{n+1,2}(\mathbf{C})$, too. We take the vector subspace spanned by e_1 and e_2 as the origin C_0 of $G_{n+1,2}(\mathbf{C})$ and denote the isotropy group at C_0 by $K = U(2) \times U(n-1)$. We consider $G_{n+1,2}(\mathbf{C})$ as a symmetric space G/K with the standard invariant metric.

We denote by $\mathfrak{g}$, $\mathfrak{k}$, and $\mathfrak{h}$ the Lie algebras of G, K, and H, respectively. The orthogonal complement $\mathfrak{m}$ of $\mathfrak{k}$ in $\mathfrak{g}$ is the subspace given by

$$\mathfrak{m} = \{\, Z(z_1, z_2) \in M(n+1; \mathbf{C}) \mid z_1, z_2 \in \mathbf{C}^{n-1} \,\},$$

where, for each two elements $z_l = (z_{al}) \in \mathbf{C}^{n-1}$ $(3 \leq a \leq n+1,\ l = 1$ or $2)$, the (i,j)-element $(Z)_{ij}$ of an $(n+1) \times (n+1)$-matrix $Z = Z(z_1, z_2)$ is given by

$$(Z)_{ij} = \begin{cases} z_{ij}, & \text{for } 3 \leq i \leq n+1,\ j = 1 \text{ or } 2, \\ -\overline{z_{ji}}, & \text{for } i = 1 \text{ or } 2,\ 3 \leq j \leq n+1, \\ 0, & \text{otherwise.} \end{cases}$$

In the following we always treat the $\mathbf{C}$-valued functions and denote by $\mathcal{F}(G)$ the space of $\mathbf{C}$-valued smooth functions on G. For each pair (a, l) of indices with $3 \leq a \leq n+1$ and $l = 1$ or 2, we define differential operators Z_{al} and $\overline{Z}_{al}$ on $\mathcal{F}(G)$ by

$$(Z_{al}F)(g) = \frac{\partial}{\partial z_{al}} F(g \exp Z)\Big|_{z_1 = z_2 = 0},$$
$$(\overline{Z}_{al}F)(g) = \frac{\partial}{\partial \bar{z}_{al}} F(g \exp Z)\Big|_{z_1 = z_2 = 0},$$
$$(F \in \mathcal{F}(G),\ g \in G).$$

The formal adjoint operator $(Z_{al})^*$, with respect to the invariant hermitian inner product on $\mathcal{F}(G)$ induced by the invariant measure on G, is equal to $-\overline{Z}_{al}$. For $3 \le a < b \le n+1$, we define differential operators L_{ab} and L_{ab}^* on $\mathcal{F}(G)$ by

$$\begin{aligned}(L_{ab}F)(g) &= \left(\frac{\partial^2}{\partial z_{a1}\partial z_{b2}} - \frac{\partial^2}{\partial z_{a2}\partial z_{b1}}\right) F(g \exp Z)\Big|_{z_1=z_2=0} \\ &= (Z_{a1}(Z_{b2}F))(g) - (Z_{a2}(Z_{b1}F))(g), \\ (L_{ab}^*F)(g) &= \left(\frac{\partial^2}{\partial \bar{z}_{a1}\partial \bar{z}_{b2}} - \frac{\partial^2}{\partial \bar{z}_{a2}\partial \bar{z}_{b1}}\right) F(g \exp Z)\Big|_{z_1=z_2=0} \\ &= (\overline{Z}_{a1}(\overline{Z}_{b2}F))(g) - (\overline{Z}_{a2}(\overline{Z}_{b1}F))(g), \\ &\qquad\qquad (F \in \mathcal{F}(G),\ g \in G).\end{aligned}$$

Let F be a smooth function on G/K, that is, a function $F \in \mathcal{F}(G)$ statisfying $F(gk) = F(g)$ $(k \in K)$. For an element $k = (k_{ij})$ of K, we have

$$(L_{ab}F)(gk) = \overline{(k_{11}k_{22} - k_{12}k_{21})} \sum_{3 \le c < d \le n+1} (k_{ca}k_{db} - k_{da}k_{cb})(L_{cd}F)(g),$$

$$(L_{ab}^*F)(gk) = (k_{11}k_{22} - k_{12}k_{21}) \sum_{3 \le c < d \le n+1} \overline{(k_{ca}k_{db} - k_{da}k_{cb})}(L_{cd}^*F)(g).$$

Now we consider a complex vector space V of dimension $(n-1)(n-2)/2$, with a fixed basis $\{u_a \wedge u_b\}$ $(3 \le a < b \le n+1)$ and an action ρ of K given by

$$\rho(k)(u_a \wedge u_b) = (k_{11}k_{22} - k_{12}k_{21}) \sum_{3 \le c < d \le n+1} \overline{(k_{ca}k_{db} - k_{da}k_{cb})}\, u_c \wedge u_d.$$

For a function $F \in \mathcal{F}(G/K)$, a V-valued function $\mathcal{L}F$ on G defined by $\mathcal{L}F = \sum_{3 \le a < b \le n+1} (L_{ab}F)\, u_a \wedge u_b$ satisfies

$$\begin{aligned}&(\mathcal{L}F)(gk) \\ &\quad = \sum (L_{ab}F)(gk)\, u_a \wedge u_b \\ &\quad = \overline{(k_{11}k_{22} - k_{12}k_{21})} \sum (k_{ca}k_{db} - k_{da}k_{cb})(L_{cd}F)(g)\, u_a \wedge u_b \\ &\quad = (k_{11}^{-1}k_{22}^{-1} - k_{21}^{-1}k_{12}^{-1}) \sum \overline{(k_{ac}^{-1}k_{bd}^{-1} - k_{ad}^{-1}k_{bc}^{-1})}(L_{cd}F)(g)\, u_a \wedge u_b \\ &\quad = \sum (L_{cd}F)(g)\, \rho(k^{-1})(u_c \wedge u_d), \\ &\quad = \rho(k^{-1})((\mathcal{L}F)(g)).\end{aligned}$$

It means that $\mathcal{L}F$ can be considered as a section of the vector bundle $E = G \underset{K}{\times} V$ over G/K.

We define a differential operator $\mathcal{D}$ on $\mathcal{F}(G/K)$ by $\mathcal{D} = \mathcal{L}^*\mathcal{L}$, where $\mathcal{L}^*$ is the formal adjoint operator of $\mathcal{L}$. It can be explicitly written as follows.

$$(\mathcal{D}F)(g) = \sum_{3\leq a<b\leq n+1} (L_{ab}^*(L_{ab}F))(g).$$

By construction, it is obvious that both $\mathcal{L}$ and $\mathcal{D}$ are invariant differential operators. In fact, it can be shown that the ring of invariant differential operators on the rank 2 symmetric space G/K is generated by the differential operator $\mathcal{D}$ and the Laplacian Δ.

Theorem 5.2. *The image of the Radon transform $\mathcal{R}$ on the complex projective space $P^n(\mathbf{C})$ is equal to the kernel of the differential operator $\mathcal{D}$ on the complex Grassmann manifold $G_{n+1,2}(\mathbf{C})$.*

We prove this theorem in the next section following the same steps as the sphere case.

§6. The proof of Theorem 5.2

We first fix a maximal abelian subalgebra $\mathfrak{t}$ of $\mathfrak{g}$, which is included in both $\mathfrak{k}$ and $\mathfrak{h}$, by

$$\mathfrak{t} = \{\, \mathrm{diag}(\mu_1, \ldots, \mu_{n+1}) \mid \mu_i \in \sqrt{-1}\mathbf{R}, \quad \text{for } 1 \leq i \leq n+1 \,\},$$

where $\mathrm{diag}(\mu_1, \ldots, \mu_{n+1})$ is a diagonal matrix with the diagonal elements $\mu_1, \ldots, \mu_{n+1}$. We take as the basis of the complexified dual vector space $\mathfrak{t}_{\mathbf{C}}^*$ of $\mathfrak{t}$ the following elements $\lambda_1, \ldots, \lambda_{n+1}$.

$$\lambda_i(\mathrm{diag}(\mu_1, \ldots, \mu_{n+1})) = \mu_i, \qquad \text{for } 1 \leq i \leq n+1.$$

We introduce an order on the real vector subspace of $\mathfrak{t}_{\mathbf{C}}^*$ spanned by λ_1, $\ldots$, λ_{n+1} such as $\lambda_1 > \cdots > \lambda_{n+1}$.

An irreducible G-module is specified by the highest weight, which has the form $l_1\lambda_1 + \cdots + l_{n+1}\lambda_{n+1}$, where $l_1, \ldots, l_{n+1}$ are integers satisfying $l_1 \geq \cdots \geq l_{n+1}$. The same is true for an irreducible H-module or an irreducible K-module, and its highest weight has the form $h_1\lambda_1 + \cdots + h_{n+1}\lambda_{n+1}$, where $h_1, \ldots, h_{n+1}$ are integers satisfying $h_2 \geq \cdots \geq h_{n+1}$, for the former, or the form $k_1\lambda_1 + \cdots + h_{n+1}\lambda_{n+1}$, where k_1, $\ldots$, k_{n+1} are integers satisfying $k_1 \geq k_2$ and $k_3 \geq \cdots \geq k_{n+1}$, for the latter. We shall denote an irreducible module with the highest weight Λ by $V(\Lambda)$.

When an irreducible G-module is considered as an H-module (resp. a K-module) by restricting the action, it decomposes into the sum of irreducible H-modules (resp. K-modules). The following two branching laws specify which irreducible module appears in the decomposition.

Theorem 6.1. *In the decomposition of the irreducible G-module with the highest weight $l_1\lambda_1 + \cdots + l_{n+1}\lambda_{n+1}$, an irreducible H-module with the highest weight $h_1\lambda_1 + \cdots + h_{n+1}\lambda_{n+1}$ appears if and only if $l_1 \geq h_2 \geq l_2 \geq \cdots \geq h_{n+1} \geq l_{n+1}$ and $\sum_{i=1}^{n+1} l_i = \sum_{i=1}^{n+1} h_i$. And then it appears only once.*

Theorem 6.2. *In the decomposition of the irreducible G-module with the highest weight $l_1\lambda_1 + \cdots + l_{n+1}\lambda_{n+1}$, an irreducible K-module with the highest weight $k_1\lambda_1 + \cdots + k_{n+1}\lambda_{n+1}$ appears if and only if $l_i \geq k_{i+2} \geq l_{i+2}$ for $1 \leq i \leq n-1$, $\sum_{i=1}^{n+1} l_i = \sum_{i=1}^{n+1} k_i$ $(= p)$, and the following condition is satisfied: Let the integers $m_1, \ldots, m_{2n}$ be the descending reordering of $l_1, \ldots, l_{n+1}$ and $k_3, \ldots, k_{n+1}$. The irreducible $U(2)$-module $V((p-k_2)\lambda_1 + (p-k_1)\lambda_2)$ appears in the decomposition of the tensor product of irreducible $U(2)$-modules $V(m_1\lambda_1 + m_2\lambda_2) \otimes \cdots \otimes V(m_{2n-1}\lambda_1 + m_{2n}\lambda_2)$.*

An irreducible K-module appears in the same times as the corresponding irreducible $U(2)$-module.

For their proofs, see H. Boerner [1] and J. Mikelsson [7].

Frobenius' reciprocity law enables us to determine the irreducible decomposition of the spaces $\mathcal{F}(G/H)$, $\mathcal{F}(G/K)$, and $C^\infty(E)$ as G-modules. For example, a G-module appears in the decomposition of $\mathcal{F}(G/H)$ if and only if its irreducible decomposition as an H-module includes a trivial H-module. An easy calculation shows the following proposition.

Proposition 6.3. *For $n \geq 3$, we have the direct sum decompositions*

$$\mathcal{F}(G/H) \approx \sum_{l=0}^{\infty} V(l\lambda_1 - l\lambda_{n+1}),$$

$$\mathcal{F}(G/K) \approx \sum_{l,m=0}^{\infty} V((l+m)\lambda_1 + m\lambda_2 - m\lambda_n - (l+m)\lambda_{n+1}).$$

In the same way, we can compute the decomposition of $C^\infty(E)$. A G-module appears in the decomposition of $C^\infty(E)$ if and only if its irreducible decomposition as a K-module includes a K-module isomorphic to (V, ρ), which is an irreducible K-module with the highest weight $\lambda_1 + \lambda_2 - \lambda_n - \lambda_{n+1}$. The result varies depending on n and is somewhat cumbersome. Anyway, what we need is the following proposition, which can be shown easily.

Proposition 6.4. *An irreducible G-module with the highest weight $l\lambda_1 - l\lambda_{n+1}$ $(l \geq 0)$ never appears in the decomposition of $C^\infty(E)$.*

Theorem 6.5. *The image of the Radon transform $\mathcal{R}$ on $P^n(\mathbf{C})$ is included in the kernel of the differential operator $\mathcal{L}$.*

Proof. Let us denote by W_l the irreducible G-submodule of $\mathcal{F}(P^n(\mathbf{C})) = \mathcal{F}(G/H)$ with the highest weight $l\lambda_1 - l\lambda_{n+1}$ $(l \geq 0)$. Since the Radon transform $\mathcal{R}$ is an injective G-homomorphism, $\mathcal{R}(W_l)$ is an irreducible G-submodule of $\mathcal{F}(G/K)$ with the same highest weight, by Schur's lemma. The differential operator $\mathcal{L}$ is also an G-homomorphism, and therefore $\mathcal{L}(\mathcal{R}(W_l))$ is an irreducible G-submodule of $C^\infty(E)$ with the same highest weight or vanishes totally. But, by Proposition 6.4, an irreducible G-module with the highest weight $l\lambda_1 - l\lambda_{n+1}$ $(l \geq 0)$ cannot be a G-submodule of $C^\infty(E)$. Thus we have $\mathcal{L}(\mathcal{R}(W_l)) = \{0\}$.

Since the direct sum $\sum \mathcal{R}(W_l)$ is dense in $\operatorname{Im}\mathcal{R}$, the image $\operatorname{Im}\mathcal{R}$ itself is included in the kernel $\operatorname{Ker}\mathcal{L}$. Q.E.D.

To prove the other inclusion, it is enough to show $\operatorname{Im}\mathcal{R} = \operatorname{Ker}\mathcal{D}$ for $n = 3$, because the same argument as in §4 holds for $P^n(\mathbf{C})$. In the case $n = 3$, we can explicitly compute how $\mathcal{D}$ acts on each irreducible G-sumodule of $\mathcal{F}(G/K)$. $(G = U(4),\ K = U(2) \times U(2).)$

A G-module U_{lm} with the highest weight $(l+m)\lambda_1 + m\lambda_2 - m\lambda_3 - (l+m)\lambda_4$ can be endowed with an invariant hermitian inner product, which is unique up to a constant factor. We fix one and denote it by $\langle\ ,\ \rangle$. By Theorem 6.2, the K-invariant elements in U_{lm} forms a 1-dimensional subspace, and we fix a K-invariant element v_K with unit norm. A G-isomorphism from U_{lm} into $\mathcal{F}(G/K)$ is given by

$$U_{lm} \ni v \mapsto f_v(g) = \langle \rho(g)v_K, v\rangle \in \mathcal{F}(G/K),$$

where ρ denotes the action of G on U_{lm}.

The computation can be simplified by studying the relations in the universal enveloping algebra $\mathcal{U}(\mathfrak{g}^{\mathbf{C}})$ of the complexification $\mathfrak{g}^{\mathbf{C}} = M(4,\mathbf{C})$ of the Lie algebra $\mathfrak{g}$. Notice that the action ρ of G can be extended to the action of $\mathcal{U}(\mathfrak{g}^{\mathbf{C}})$, denoted by the same letter ρ.

The differential operator $\mathcal{D}$ corresponds to an element D in $\mathcal{U}(\mathfrak{g}^{\mathbf{C}})$ by the following formula.

$$(\mathcal{D}f_v)(g) = \langle \rho(g)\rho(D)v_K, v\rangle.$$

We denote by E_{ij} a matrix whose (k,l)-element is given by $\delta_{ik}\delta_{jl}$. Then the element D in $\mathcal{U}(\mathfrak{g}^{\mathbf{C}})$ is written explicitly as

$$D = (E_{13}E_{24} - E_{14}E_{23})(E_{31}E_{42} - E_{32}E_{41}).$$

This element D commutes with the elements of $\mathfrak{k}^{\mathbf{C}}$ in $\mathcal{U}(\mathfrak{g}^{\mathbf{C}})$, but it does not belong to the center of $\mathcal{U}(\mathfrak{g}^{\mathbf{C}})$.

We introduce two elements D_1 and D_2 in the center of $\mathcal{U}(\mathfrak{g}^{\mathbf{C}})$.

$$D_1 = \sum_{i,j=1}^{4} E_{ij}E_{ji},$$
$$D_2 = \sum_{\sigma,\tau\in\mathfrak{S}_4} \operatorname{sgn}(\sigma\tau)E_{\sigma(1)\tau(1)}E_{\sigma(2)\tau(2)}E_{\sigma(3)\tau(3)}E_{\sigma(4)\tau(4)}.$$

Then a straightforward computation yields

$$24D \equiv D_2 + 2D_1 \mod \mathcal{U}(\mathfrak{g}^{\mathbf{C}})\mathfrak{k}^{\mathbf{C}}.$$

Therefore we have $\langle \rho(g)\rho(D)v_K, v\rangle = (1/24)\langle \rho(g)\rho(D_2+2D_1)v_K, v\rangle$.

Since D_2+2D_1 is in the center of $\mathcal{U}(\mathfrak{g}^{\mathbf{C}})$, its action on the irreducible G-module U_{lm} is nothing but multiplication by a constant. The constant can be computed by its action on the maximal vector v_Λ, i.e., the vector of the highest weight. Let us denote by $\mathfrak{b}^+$ the subalgebra of $\mathfrak{g}^{\mathbf{C}}$ spanned by $\{E_{ij}\}_{i<j}$. Then a straightforward computation yields

$$\begin{aligned}
D_1 \equiv{}& E_{11}^2 + E_{22}^2 + E_{33}^2 + E_{44}^2 \\
&+ 3(E_{11} - E_{44}) + E_{22} - E_{33} \mod \mathcal{U}(\mathfrak{g}^{\mathbf{C}})\mathfrak{b}^+, \\
D_2 \equiv{}& 24E_{11}E_{22}E_{33}E_{44} \\
&- 36E_{22}E_{33}(E_{11} - E_{44}) - 12E_{11}E_{44}(E_{22} - E_{33}) \\
&+ 28(E_{11}E_{22} + E_{33}E_{44}) - 8(E_{11}E_{33} + E_{22}E_{44}) \\
&+ 4E_{11}E_{44} - 44E_{22}E_{33} \\
&- 6(E_{11} - E_{44}) + 22(E_{22} - E_{33}) \mod \mathcal{U}(\mathfrak{g}^{\mathbf{C}})\mathfrak{b}^+.
\end{aligned}$$

Since we have $\rho(E_{ij})v_\Lambda = 0$ for $i < j$ and $\rho(E_{ii})v_\Lambda = \Lambda(E_{ii})v_\Lambda$, the following proposition can be easily deduced.

Proposition 6.6. *The action of the differential operator $\mathcal{D}$ on the irreducible G-submodule of $\mathcal{F}(G/K)$ isomorphic to U_{lm} is multiplication by the constant $m(m+1)(l+m+1)(l+m+2)$.*

Therefore the irreducible G-submodule of $\mathcal{F}(G/K)$ isomorphic to U_{lm} is in the kernel of $\mathcal{D}$ if and only if m vanishes. Since the irreducible G-submodule of $\mathcal{F}(G/K)$ with the highest weight $l\lambda_1 - l\lambda_4$ is unique, the module then coincides with the image of W_l by $\mathcal{R}$. By the same argument as in §3, we can prove $\operatorname{Ker}\mathcal{D} = \operatorname{Im}\mathcal{R}$, and thus our proof of Theorem 5.2 is completed.

Remark 6.7. The eigenvalue of the differential operator $\mathcal{D}$ can be computed also by using the formula that gives the radial part of $\mathcal{D}$. The first author has exploited this approach and the further results will be shown in the forthcoming papers.

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A Uniqueness Result for Minimal Surfaces in S^3

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§1. Introduction

In the study of minimal surfaces, the uniqueness for minimal surfaces bounded by a given contour is an important problem which is not yet solved completely.

The first uniqueness result was proved by Radó [4] for minimal surfaces in $\mathbf{R}^3$. He proved that if a Jordan curve Γ has a one-to-one parallel or central projection onto a convex plane Jordan curve, then Γ bounds a unique minimal disk. The second result is due to Nitsche [3] and states that if the total curvature of an analytic Jordan curve Γ does not exceed 4π, then Γ bounds a unique minimal disk. The third result is due to Tromba [6] and states that if a C^2-Jordan curve Γ is sufficiently closed to a C^2-plane Jordan curve in the C^2-topology, then Γ bounds a unique minimal disk.

For minimal surfaces in other Riemannian manifolds, uniqueness theorems in the three dimensional hemisphere of S^3 were proved by Sakaki [5] and Koiso [2]. Sakaki's result is an analogy of Tromba's uniqueness theorem, and Koiso's is an analogy of Radó's theorem.

In this paper we restrict ourselves to minimal surfaces in S^3 which are "graphs" in some sense (Definition 1.1).

Set $S^3 = \{\mathbf{x} \in \mathbf{R}^4; |\mathbf{x}| = 1\}$. Let Σ be a 2-plane in $\mathbf{R}^4$ containing the origin of $\mathbf{R}^4$. We denote by B the two dimensional unit open disk in Σ which is bounded by $\Sigma \cap S^3$.

Definition 1.1. Let D be a subset of the closed disk $\overline{B}$. A subset M of S^3 is called a "graph" over D if M intersects with each 2-plane containing a point of D which is orthogonal to Σ in $\mathbf{R}^4$ at precisely one point.

Definition 1.2. (1) A minimal surface M in S^3 is a continuous mapping Φ of a two dimensional compact C^∞-manifold R with boundary

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∂R into S^3 which is of class C^2 in the interior of R and which is a critical point of the area functional for every variation preserving the boundary values $\Phi|_{\partial R}$.

(2) We sometimes call the image $\Phi(R)$ of a minimal surface $\Phi : R \longrightarrow S^3$ to be a minimal surface. On such an occasion we call $\Phi(\partial R)$ to be the boundary of the minimal surface $\Phi(R)$, and denote $\Phi(\partial R)$ by $\partial\Phi(R)$.

(3) When we mention the uniqueness for minimal surfaces, we mean the uniqueness for the images of minimal surfaces.

Now we can state our uniqueness result:

Theorem 1.3. *Let D be a simply-connected domain whose closure $\overline{D}$ is contained in B. If M is a minimal surface which is a "C^2-graph" over $\overline{D}$, then M is the unique minimal surface bounded by ∂M which is a "C^2-graph" over $\overline{D}$.*

For the proof, we represent each "graph" over $\overline{D}$ in terms of a single real-valued function φ defined on $\overline{D}$. We prove that the considered "graph" is a minimal surface if and only if the function φ satisfies a certain quasilinear elliptic partial differential equation (Lemma 2.4). A uniqueness theorem for the Dirichlet problem for quasilinear elliptic operators assures the uniqueness of our minimal surface.

We conjecture that under the assumption of Theorem 1.3, the uniqueness of the area-minimizing surface bounded by ∂M is valid.

§2. Proof of Theorem 1.3

Throughout this section, we assume that D is a simply-connected domain whose closure is contained in B.

We introduce the orthogonal coordinates (x, y, z, w) in $\mathbf{R}^4$. Without loss of generality, we set Σ the (x, y)-plane. For simplicity we denote a point $(x, y, 0, 0)$ in Σ by (x, y). If f is a differentiable function of x and y, we denote $\partial f/\partial x, \partial f/\partial y, \partial^2 f/\partial x^2$ by f_x, f_y, f_{xx}, etc.

A "graph" over $\overline{D}$ is represented as follows:

$$\left(x, y, \sqrt{1 - x^2 - y^2}\cos\varphi(x, y), \sqrt{1 - x^2 - y^2}\sin\varphi(x, y)\right), \tag{2-1}$$

$$(x, y) \in \overline{D},$$

where $\varphi(x, y)$ is a real-valued function defined on $\overline{D}$.

Definition 2.1. A "graph" over $\overline{D}$ represented by (2-1) is called a "C^n-graph" over $\overline{D}$ if φ can be chosen to be of class C^n on $\overline{D}$.

Remark 2.2. If M is a "C^n-graph" represented by (2-1), then

$$\partial M = \left\{ \left(x, y, \sqrt{1-x^2-y^2}\cos\varphi, \sqrt{1-x^2-y^2}\sin\varphi \right); (x,y) \in \partial D \right\}$$

is a Jordan curve of class C^n.

Remark 2.3. Since $\overline{D}$ is contained in B, $x^2+y^2<1$ for any point (x,y) in $\overline{D}$.

Lemma 2.4. *Let φ be of class $C^2(\overline{D}, \mathbf{R})$.*

$$M = \{ \left(x, y, \sqrt{1-x^2-y^2}\cos\varphi(x,y), \sqrt{1-x^2-y^2}\sin\varphi(x,y) \right); \quad (x,y) \in \overline{D}\}$$

is a minimal surface if and only if

$$L\varphi = 0 \qquad inD,$$

where L is a quasilinear elliptic operator of the form

(2-2)
$$\begin{aligned} L\varphi = &\left\{1-x^2+(1-x^2-y^2)^2{\varphi_y}^2\right\}\varphi_{xx} \\ &-2\left\{xy+(1-x^2-y^2)^2\varphi_x\varphi_y\right\}\varphi_{xy} \\ &+\left\{1-y^2+(1-x^2-y^2)^2{\varphi_x}^2\right\}\varphi_{yy} \\ &-4x\varphi_x-4y\varphi_y \\ &+2(1-x^2-y^2)(-x+x^3){\varphi_x}^3 \\ &+(1-x^2-y^2)(-2y+6x^2y){\varphi_x}^2\varphi_y \\ &+(1-x^2-y^2)(-2x+6xy^2)\varphi_x{\varphi_y}^2 \\ &+2(1-x^2-y^2)(-y+y^3){\varphi_y}^3, \qquad (x,y) \in D. \end{aligned}$$

Proof. Set

$$\Phi(x,y) = \left(x, y, \sqrt{1-x^2-y^2}\cos\varphi(x,y), \sqrt{1-x^2-y^2}\sin\varphi(x,y) \right), \quad (x,y) \in \overline{D}.$$

Then $\Phi \in C^2(\overline{D}, S^3)$. The area A of M is represented as

$$A = \iint_D \left\{ |\Phi_x|^2 \, |\Phi_y|^2 - (\Phi_x, \Phi_y)^2 \right\}^{1/2} dx\, dy,$$

where (Φ_x, Φ_y) is the usual inner product in $\mathbf{R}^4$ and $|\Phi_x|^2 = (\Phi_x, \Phi_x)$, $|\Phi_y|^2 = (\Phi_y, \Phi_y)$. By easy calculations we get

$$A = \iint_D \left\{ \left(1 - x^2 - y^2\right)^{-1} + \left(1 - x^2 - y^2\right)\left({\varphi_x}^2 + {\varphi_y}^2\right) \right.$$
$$\left. + \left(x\varphi_y - y\varphi_x\right)^2 \right\}^{1/2} dx\, dy.$$

Let $f = f(x, y)$ be a real-valued C^2-function on $\overline{D}$ which vanishes on the boundary ∂D. Then we get 1-parameter family of surfaces M_t represented as follows.

$$\left(x, y, \sqrt{1 - x^2 - y^2}\cos(\varphi + t\, f), \sqrt{1 - x^2 - y^2}\sin(\varphi + t\, f)\right),$$

$$(x, y) \in \overline{D}, \qquad t \in \mathbf{R}.$$

Denote the area of M_t by $A(t)$. Then $M = M_0$ is a minimal surface if and only if

$$\left.\frac{d}{dt} A(t)\right|_{t=0} = 0$$

for any f.

We observe that

$$\left.\frac{d}{dt} A(t)\right|_{t=0}$$
$$= \iint_D \frac{\left\{\left(1 - x^2\right)\varphi_x - xy\varphi_y\right\} f_x + \left\{\left(1 - y^2\right)\varphi_y - xy\varphi_x\right\} f_y}{Q} \, dx\, dy,$$

where

$$Q = \left\{ \frac{1}{1 - x^2 - y^2} + \left(1 - x^2 - y^2\right)\left({\varphi_x}^2 + {\varphi_y}^2\right) + \left(x\varphi_y - y\varphi_x\right)^2 \right\}^{\frac{1}{2}}.$$

By virtue of the Stokes' formula and the assumption $f|_{\partial D} = 0$, we see that

$$\frac{d}{dt}A(t)\bigg|_{t=0} =$$

$$-\iint_D f\left[\left\{\frac{\left(1-x^2\right)\varphi_x - xy\varphi_y}{Q}\right\}_x + \left\{\frac{\left(1-y^2\right)\varphi_y - xy\varphi_x}{Q}\right\}_y\right] dxdy.$$

By lengthy but easy calculations we get

$$\frac{d}{dt}A(t)\bigg|_{t=0} = -\iint_D f\left(1-x^2-y^2\right)^{-1} Q^{-3} L\varphi \, dx\, dy,$$

where $L\varphi$ is given by the equality (2-2) in the statement of Lemma 2.4. If $(d/dt)A(t)|_{t=0} = 0$ for any $f \in C^2(\overline{D}, \mathbf{R})$ with $f|_{\partial D} = 0$, then $L\varphi$ must vanish in D, and vice versa.

To see the ellipticity of L, we regard $L\varphi$ as a function of x, y, φ, φ_x, φ_y, φ_{xx}, φ_{xy}, φ_{yy}, and we set $p = \varphi_x$, $q = \varphi_y$, $r = \varphi_{xx}$, $s = \varphi_{xy}$, and $t = \varphi_{yy}$. Then

$$\begin{aligned} L_\varphi &= 0, \\ L_r L_t - (L_s/2)^2 & \\ &= 1 - x^2 - y^2 \\ &\quad + \left(1-x^2-y^2\right)^2 \left\{\left(1-x^2-y^2\right)\left(p^2+q^2\right) + (yp-xq)^2\right\} \\ &> 0 \end{aligned}$$

for any point $(x, y) \in \overline{D}$, which implies that L is elliptic Q.E.D.

Proof of Theorem 1.3. If two functions $\varphi \in C^2(\overline{D}, \mathbf{R})$ and $\psi \in C^2(\overline{D}, \mathbf{R})$ define minimal surfaces

$$\Phi(x,y) = \left(x, y, \sqrt{1-x^2-y^2}\cos\varphi(x,y), \sqrt{1-x^2-y^2}\sin\varphi(x,y)\right),$$

$$(x,y) \in \overline{D},$$

and

$$\Psi(x,y) = \left(x, y, \sqrt{1-x^2-y^2}\cos\psi(x,y), \sqrt{1-x^2-y^2}\sin\psi(x,y)\right),$$

$$(x,y) \in \overline{D},$$

and if these two minimal surfaces have the same boundary, then we can assume that $\varphi = \psi$ on ∂D. Moreover, by Lemma 2.4, we see that $L\varphi = 0$

and $L\psi = 0$ in D. Therefore by virtue of the uniqueness theorem for the Dirichlet problem for quasilinear elliptic operators ([1, p.208, Theorem 9.3]), φ and ψ must coincide in D. Q.E.D.

§3. The final remark

Remark 3.1. The assumption that $\overline{D}$ is contained in B is essential in the following sense. Set

$$D = B = \left\{(x, y, 0, 0) \in \mathbf{R}^4; x^2 + y^2 < 1\right\}.$$

Then the uniqueness result does not hold. In fact,

$$\Phi(x, y) = \left(x, y, \frac{a\sqrt{1 - x^2 - y^2}}{\sqrt{a^2 + b^2}}, \frac{b\sqrt{1 - x^2 - y^2}}{\sqrt{a^2 + b^2}}\right),$$

$$(a, b) \in \mathbf{R}^2 - (0, 0), \qquad (x, y) \in \overline{D}$$

is a half of a geodesic 2-sphere bounded by the geodesic circle ∂D, hence Φ is a minimal surface bounded by ∂D. Therefore we obtain 2-parameter family of minimal surfaces bounded by the same contour ∂D which are "C^∞-graphs" over $\overline{D}$ and all of which are area-minimizing.

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Self-dual Einstein Hermitian Surfaces

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§1. Introduction

N. Hitchin [4] has proved that a 4-dimensional compact half conformally flat Einstein space of positive scalar curvature is isometric to a 4-dimensional sphere or a complex projective surface with the respective standard metric.

A 4-dimensional almost Hermitian manifold $M = (M, J, g)$ with integrable almost complex structure J is called a Hermitian surface. In the present paper, concerning the above result by Hitchin, we shall prove the following

Theorem A. *Let $M = (M, J, g)$ be a compact self-dual Einstein Hermitian surface. Then M is a Kähler surface of constant holomorphic sectional curvature, i.e., M is one of the following*
(1) *flat,*
(2) *$P^2(\mathbf{C})$ with its standard Fubini-Study metric and*
(3) *a compact quotient of unit disk D^2 with the Bergman metric.*

Remark. C.P. Boyer [2] has asserted the above result without detailed proof. In the present paper, we shall give another explicit proof.

In the sequel, unless otherwise stated, we assume the manifold under consideration to be connected.

§2. Preliminaries

Let $M = (M, J, g)$ be a Hermitian surface and Ω the Kähler form of M given by $\Omega(X, Y) = g(X, JY)$, $X, Y \in \mathfrak{X}(M)$. ($\mathfrak{X}(M)$ denotes the Lie algebra of all differentiable vector fields on M). We assume that M is oriented by the volume form $dM = \frac{1}{2}\Omega^2$. We have

$$(2.1) \qquad d\Omega = \omega \wedge \Omega, \quad \omega = \delta\Omega \circ J.$$

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The 1-form $\omega = (\omega_i)$ is called the Lee form of M. We denote by $\nabla, R = (R_{ijk}{}^l), \rho = (\rho_{ij})$ and τ the Riemannian connection, the Riemannian curvature tensor, the Ricci tensor and the scalar curvature of M, respectively. The Ricci $*$-tensor $\rho^* = ({\rho^*}_{ij})$ and the $*$-scalar curvature τ^* are defined respectively by

$$(2.2) \qquad {\rho^*}_{ij} = \frac{1}{2} J_j{}^s R_{isa}{}^b J_b{}^a,$$

$$(2.3) \qquad \tau^* = g^{ij} {\rho^*}_{ij}.$$

The generalized Chern form $\gamma = (\gamma_{ij})$ is given by

$$(2.4) \qquad 8\pi\gamma_{ij} = -4J_j{}^k {\rho^*}_{ik} - J^{kl}(\nabla_j J_k{}^h)\nabla_i J_{lh}.$$

It is well-known that the 2-form γ represents the first Chern class of M in the de Rham cohomology group. The Lee form $\omega = (\omega_j)$ satisfies the following:

$$(2.5) \qquad J^{ij}\nabla_i \omega_j = 0,$$

$$(2.6) \qquad 2\nabla_i J_j{}^k = \omega_a J_j{}^a \delta_i^k - \omega_a J^{ka} g_{ij} - \omega_j J_i{}^k + \omega^k J_{ij},$$

$$(2.7) \qquad \tau - \tau^* = 2\delta\omega + \|\omega\|^2,$$

(cf. [7], [9], [10]).

We denote by $\chi(M), c_1(M), c_2(M)$ and $p_1(M)$ the Euler class, the first Chern class, the second Chern class and the first Pontrjagin class of M, respectively. We note that $c_2(M)$ is equal to $\chi(M)$ when M is compact. Now, we assume that $M = (M, J, g)$ is of pointwise constant holomorphic sectional curvature $c = c(p)$ $(p \in M)$. Then we have ([7])

$$(2.8) \qquad \begin{aligned} R_{ijkl} =& \frac{1}{4}\|\omega\|^2 C_{ijkl} + \left(\frac{c}{4} - \frac{\|\omega\|^2}{16}\right) H_{ijkl} \\ &+ \frac{1}{96}\{g_{ik}A_{jl} - g_{il}A_{jk} + g_{jl}A_{ik} - g_{jk}A_{il} \\ &+ J_{ik}B_{jl} - J_{il}B_{jk} + J_{jl}B_{ik} - J_{jk}B_{il} \\ &+ 2J_{ij}B_{kl} + 2J_{kl}B_{ij}\}, \end{aligned}$$

where

$$\begin{aligned} C_{ijkl} =& g_{il}g_{jk} - g_{ik}g_{jl}, \\ H_{ijkl} =& g_{il}g_{jk} - g_{ik}g_{jl} \\ &+ J_{il}J_{jk} - J_{ik}J_{jl} - 2J_{ij}J_{kl}, \end{aligned}$$

$$\begin{aligned} A_{ij} =& 21(\nabla_i\omega_j + \nabla_j\omega_i + \omega_i\omega_j) \\ & - 3J_i{}^aJ_j{}^b(\nabla_a\omega_b + \nabla_b\omega_a + \omega_a\omega_b), \\ B_{ij} =& 7(J_j{}^a\nabla_i\omega_a - J_i{}^a\nabla_j\omega_a) \\ & - (J_j{}^a\nabla_a\omega_i - J_i{}^a\nabla_a\omega_j) \\ & + 3(J_j{}^a\omega_i\omega_a - J_i{}^a\omega_j\omega_a). \end{aligned}$$

By (2.7) and (2.8), we have

$$\rho_{ij} = \{\frac{3}{2}c + \frac{3}{16}(\tau - \tau^*)\}g_{ij} - \frac{1}{4}T_{ij}, \tag{2.9}$$

$$\rho^*{}_{ij} = \{\frac{3}{2}c - \frac{1}{16}(\tau - \tau^*)\}g_{ij} + \frac{1}{4}T^*{}_{ij}, \tag{2.10}$$

where

$$\begin{aligned} T_{ij} =& \nabla_i\omega_j + \nabla_j\omega_i + \omega_i\omega_j \\ & - J_i{}^aJ_j{}^b(\nabla_a\omega_b + \nabla_b\omega_a + \omega_a\omega_b), \end{aligned} \tag{2.11}$$

$$T^*{}_{ij} = \nabla_i\omega_j - \nabla_j\omega_i - J_i{}^aJ_j{}^b(\nabla_a\omega_b - \nabla_b\omega_a). \tag{2.12}$$

By (2.9), we get

$$\tau + 3\tau^* = 24c. \tag{2.13}$$

By (2.13), (2.9) and (2.10) are rewritten by

$$\rho_{ij} = \frac{\tau}{4}g_{ij} - \frac{1}{4}T_{ij}, \tag{2.9$'$}$$

$$\rho^*{}_{ij} = \frac{\tau^*}{4}g_{ij} + \frac{1}{4}T^*{}_{ij}. \tag{2.10$'$}$$

We assume that the manifold M under consideration is compact. We shall recall several integral formulas which will be needed in the proof of Theorem A.

$$\begin{aligned} &\int_M \omega^i\omega^j J_i{}^aJ_j{}^b\nabla_a\omega_b dM \\ &= \int_M \{\tau\delta\omega + \frac{1}{4}\|\omega\|^4 - \frac{1}{2}(\tau - \tau^*)^2 \\ &\quad + 6c\|\omega\|^2 - \|d\omega\|^2\}dM, \end{aligned} \tag{2.14}$$

$$\chi(M) = \frac{1}{32\pi^2}\int_M \{12c^2 - \frac{1}{16}(\tau - \tau^*)^2 + \frac{1}{2}\tau^*\|\omega\|^2\}dM, \tag{2.15}$$

$$(2.16)\qquad p_1(M) = \frac{1}{32\pi^2}\int_M \{\frac{1}{12}(\tau - 3\tau^*)^2 + \|d\omega\|^2\}dM,$$

$$(2.17)\qquad c_1(M)^2 = \frac{1}{32\pi^2}\int_M \{(\tau^*)^2 + \tau^*\|\omega\|^2 + \|d\omega\|^2\}dM,$$

(see [7]). We define a tensor field $S = (S_{ij})$ of type (0,2) by

$$(2.18)\qquad \begin{aligned} S_{ij} =& \nabla_i\omega_j - J_i{}^a J_j{}^b \nabla_a\omega_b \\ &+ \frac{1}{2}(\omega_i\omega_j - J_i{}^a J_j{}^b\omega_a\omega_b). \end{aligned}$$

Then we have

$$(2.19)\qquad \int_M \|S\|^2 dM = \int_M \{\frac{1}{2}(\tau - \tau^*)^2 - \tau^*\|\omega\|^2\}dM.$$

We assume furthermore that the manifold M under consideration is Einsteinian. Then, by (2.9)$'$, we get $T_{ij} = 0$. Thus, taking account of (2.7), (2.13), (2.14) and (2.18), we get

$$\begin{aligned} 0 =& \int_M T_{ij}\omega^i\omega^j dM \\ =& \int_M \{\|\omega\|^2\delta\omega + \|\omega\|^4 - 2\omega^i\omega^j J_i{}^a J_j{}^b\nabla_a\omega_b\}dM \\ =& \int_M \{\|\omega\|^2\delta\omega + \frac{1}{2}\|\omega\|^4 + (\tau - \tau^*)^2 \\ &- \frac{\tau + 3\tau^*}{2}\|\omega\|^2 + 2\|d\omega\|^2\}dM \\ =& \int_M \{\frac{1}{2}\|\omega\|^2(\tau - \tau^*) + (\tau - \tau^*)^2 \\ &- \frac{\tau + 3\tau^*}{2}\|\omega\|^2 + 2\|d\omega\|^2\}dM \\ =& \int_M \{(\tau - \tau^*)^2 - 2\tau^*\|\omega\|^2\}dM \\ &+ 2\int_M \|d\omega\|^2 dM \\ =& 2\int_M \{\|S\|^2 + \|d\omega\|^2\}dM. \end{aligned}$$

Thus, we have

Proposition 2.1. *Let $M = (M, J, g)$ be a compact Einstein Hermitian surface of pointwise constant holomorphic sectional curvature.*

Then M is a locally conformal Kähler surface and the tensor field S vanishes.

By (2.6), (2.18) and Proposition 2.1, we get

$$
\begin{aligned}
(2.20)\qquad 0 =& 2\nabla^i\nabla_i\omega_j - 2(\nabla^i J_i{}^a)J_j{}^b\nabla_a\omega_b \\
&- 2J_i{}^a(\nabla^i J_j{}^b)\nabla_a\omega_b - 2J_i{}^a J_j{}^b\nabla^i\nabla_a\omega_b \\
&+ \{(\nabla^i\omega_i)\omega_j + \omega^i\nabla_i\omega_j \\
&- (\nabla^i J_i{}^a)J_j{}^b\omega_a\omega_b - J_i{}^a(\nabla^i J_j{}^b)\omega_a\omega_b \\
&- J_i{}^a J_j{}^b\omega_a\nabla^i\omega_b\} \\
=& 2\nabla^i\nabla_j\omega_i - 2(\omega^i J_i{}^a)J_j{}^b\nabla_a\omega_b \\
&- J^{ia}(\omega_c J_j{}^c\delta_i^b - \omega_c J^{bc}g_{ij} - \omega_j J_i{}^b + \omega^b J_{ij})\nabla_a\omega_b \\
&+ J^{ia}J_j{}^b R_{iab}{}^c\omega_c \\
&+ \{-(\delta\omega)\omega_j + \frac{1}{2}\nabla_j\|\omega\|^2 - J^{ia}J_j{}^b\omega_a\nabla_i\omega_b\} \\
=& -2\nabla_j\delta\omega + \frac{\tau}{2}\omega_j - 2(\omega^i J_i{}^a)J_j{}^b\nabla_a\omega_b \\
&- \omega_j\delta\omega - \frac{1}{2}\nabla_j\|\omega\|^2 + 2\rho^*{}_{ij}\omega^i - \omega_j\delta\omega + \frac{1}{2}\nabla_j\|\omega\|^2 \\
=& -2\nabla_j\delta\omega - 2\omega_j\delta\omega + 2\rho_{ji}\omega^i + 2\rho^*{}_{ji}\omega^i \\
&- 2(\omega^i J_i{}^a)J_j{}^b\nabla_a\omega_b.
\end{aligned}
$$

Taking account of $S = 0$, we get

$$
\begin{aligned}
(2.21)\qquad &(\omega^c J_c{}^a)J_j{}^b\nabla_a\omega_b \\
&= \omega^c\{\nabla_c\omega_j + \frac{1}{2}(\omega_c\omega_j - J_c{}^b J_j{}^d\omega_b\omega_d)\} \\
&= \frac{1}{2}\nabla_j\|\omega\|^2 + \frac{1}{2}\omega_j\|\omega\|^2.
\end{aligned}
$$

Thus, by (2.7), (2.19) and (2.20), we have

$$
-2\nabla_j\delta\omega + \frac{\tau+\tau^*}{2}\omega_j - \nabla_j\|\omega\|^2 - \omega_j\|\omega\|^2 - 2\omega_j\delta\omega = 0,
$$

and hence

$$
(2.22)\qquad \nabla_j(\tau - 3\tau^*) + \frac{3}{2}(\tau - 3\tau^*)\omega_j = 0.
$$

§3. Proof of Theorem A

First, we shall recall the following results by the first author of the present paper.

Proposition 3.1 ([5]). *Let $M = (M, J, g)$ be a self-dual Einstein almost Hermitian 4-manifold. Then M is of pointwise constant holomorphic sectional curvature.*

Remark. Conversely, we may see that if M is an almost Hermitian 4-manifold of pointwise constant holomorphic sectional curvature, then M is self-dual.

Proposition 3.2 ([5]). *Let $M = (M, J, g)$ be a compact Hermitian surface. Then M is anti-self-dual if and only if M is a locally conformal Kähler surface with $\tau = 3\tau^*$.*

On one hand, the second author has proved the following.

Proposition 3.3 ([9]). *Let $M = (M, J, g)$ be a compact Einstein Hermitian surface. If $\tau^* < 0$ on M, then M is a Kähler surface.*

Let $M = (M, J, g)$ be a compact self-dual Einstein Hermitian surface. Then by Proposition 3.1, M is of pointwise constant holomorphic sectional curvature, say, c. Hence, by Proposition 2.1, M is also a locally conformal Kähler surface.

We suppose that $\tau = 3\tau^*$ at some point of M. Then, taking account of (2.22), we may observe that $\tau = 3\tau^*$ holds everywhere on M. Thus, by Proposition 3.2, M is anti-self-dual and hence conformally flat. Since M is Einsteinian, M is thus a compact Hermitian surface of non-positive constant sectional curvature. If M is of negative constant sectional curvature c, then $\tau = 12c, \tau^* = 4c$, and hence from Proposition 3.3, it follows that M is a Kähler surface of negative constant curvature. But this is impossible. If M is locally flat, then $\tau = \tau^* = 0$ and hence from (2.7), it follows immediately that M is a locally flat Kähler surface.

Next, we assume that $\tau - 3\tau^* \neq 0$ at every point of M. Let $\widetilde{M} = (\widetilde{M}, \widetilde{J}, \widetilde{g})$ be the universal Hermitian covering of M and $\pi : \widetilde{M} \longrightarrow M$ be the covering projection. Then, by Proposition 2.1, $\widetilde{M}$ is a globally conformal Kähler surface with $\widetilde{\omega} = d\widetilde{f}$, for some differentiable function $\widetilde{f}$ on $\widetilde{M}$, where $\widetilde{\omega}$ is the Lee form of $\widetilde{M}$. We denote by $\widetilde{\tau}, \widetilde{\tau}^*$ the scalar curvature, the $*$-scalar curvature of $\widetilde{M}$, respectively. Then $\widetilde{\tau} = \tau \circ \pi, \widetilde{\tau}^* = \tau^* \circ \pi, \widetilde{\omega} = \pi^*\omega$. Solving the system of partial differential equations corresponding to (2.22), we have

$$\widetilde{\tau} - 3\widetilde{\tau}^* = \widetilde{c}e^{-\frac{3}{2}\widetilde{f}}, \tag{3.1}$$

$\widetilde{c}$ is a non-zero constant. By (3.1), we see that the function $\widetilde{f}$ is projectable, i.e., there exists a differentiable function f on M such that $\widetilde{f} = f \circ \pi$. Thus, $\widetilde{\omega} = d\widetilde{f} = \pi^* df$ and have $\pi^*(\omega - df) = 0$. Therefore, M is a globally conformal Kähler surface with $\omega = df$. Taking account of (2.7), we have

$$\tau - \tau^* = -2\Delta f + \|df\|^2, \tag{3.2}$$

where $\Delta = -\delta d$ is the Laplace-Beltrami operator acting differentiable functions on M.

First, we suppose that $\tau > 3\tau^*$ on M. Let $f(p_0) = \min_{p\in M} f(p)$. Then we have $\Delta f(p_0) \geq 0$. Thus, $\tau - \tau^* \leq 0$ at p_0 and hence $2\tau^* < \tau \leq \tau^*$ at p_0. Thus, $\tau^* < 0$ at p_0 (and hence $\tau < 0$). Since $\tau > 3\tau^*$, we see therefore that $\tau^* < 0$ on M. Thus, by Proposition 3.3, M is a Kähler surface of negative constant holomorphic sectional curvature.

Next, we assume that $\tau < 3\tau^*$ on M. Let $f(p_0) = \max_{p\in M} f(p)$. Then $\tau - \tau^* \geq 0$ at p_0. Thus, $\tau^* \leq \tau < 3\tau^*$ at p_0 and hence $\tau^* > 0$ at p_0. Thus, in this case, we see that $\tau > 0$ (and hence $\tau^* > 0$ on M). By (2.17), $c_1(M)^2 > 0$ and hence M is algebraic. Since $\tau > 0$ and $\tau^* > 0$ on M, taking account of the arguments in [8] and [12], we may see that the plurigenera of M all vanish, that is, the Kodaira dimension of M is equel to -1. Thus, the Noether's formula ([6]) is of the form

$$c_1(M)^2 + c_2(M) = 12(1 - q), \tag{3.3}$$

where $q = q(M)$ is the irregularity of M. Since $c_1(M)^2 > 0, c_2(M) = \chi(M) > 0$, from (3.3), we have $q = 0$. This reduces to

$$c_1(M)^2 + c_2(M) = 12. \tag{3.4}$$

Referring to the well-known classification of compact complex surfaces (see, e.g., [1] p.415), we may see that M is rational, equivalently, obtained by successive blowing up's from a complex projective plane $P^2(\mathbf{C})$ or a (geometrically) ruled surface over a complex projective line $P^1(\mathbf{C})$. Since $c_2(M) = \chi(M) > 0$, Miyaoka's inequality is of the form

$$c_1(M)^2 \leq 3c_2(M). \tag{3.5}$$

By (3.4) and (3.5), we have

$$c_2(M) \geq 3. \tag{3.6}$$

Furthermore, by Wu's theorem and $p_1(M) \geq 0$,

$$c_1(M)^2 \geq 2c_2(M). \tag{3.7}$$

By (3.4), (3.7) and (3.6), we have

$$c_2(M) = 3 \text{ or } 4. \tag{3.8}$$

We assume $c_2(M) = 4$. Then by (3.4), we have $c_1(M)^2 = 8$. Hence, by Wu's theorem, $p_1(M) = 0$. Thus, by (2.17) and Proposition 2.1, we have $\tau = 3\tau^*$. But, this is a contradiction. So, we see that $c_2(M) = 3$. Then, we have $c_1(M)^2 = 9$ and $p_1(M) = 3$. Thus, we may conclude that M is biholomorphically equivalent to a complex projective plane $P^2(\mathbf{C})$.

The new metric $\bar{g} = e^{-f}g$ on M is a self-dual Kähler metric. By the classification of self-dual Kähler surfaces [3], we see that $\bar{g}$ is the Fubini-Study metric on $P^2(\mathbf{C})$. Taking account of (3.2), the scalar curvature $\bar{\tau}$ of $\bar{g}$ is given by

$$\begin{aligned}\bar{\tau} =& e^f(\tau + 3\Delta f - \frac{3}{2}\|\text{grad } f\|^2) \\ =& e^f\{\tau - \frac{3}{2}(-2\Delta f + \|\text{grad } f\|^2)\} \\ =& e^f\{\tau - \frac{3}{2}(\tau - \tau^*)\} \\ =& e^f\left(\frac{-\tau + 3\tau^*}{2}\right).\end{aligned}$$

Here, by (2.22), we have

$$e^f = \left(\frac{\tau - 3\tau^*}{C}\right)^{-\frac{2}{3}}, \tag{3.9}$$

where C is a negative constant. Hence

$$\bar{\tau} = -\frac{(\tau - 3\tau^*)^{\frac{1}{3}}}{2C^{-\frac{2}{3}}}.$$

Since $\bar{\tau}$ is constant, so is $\tau - 3\tau^*$. Hence, by (3.9), we see that f is constant. Therefore, g is homothetic to the Fubini-Study metric. This completes the proof of Theorem A.

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Non-Commutative Complex Projective Space

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§0. Introduction

The concept of quantized manifolds has much interest from a geometrical point of view. In fact, quantum groups [6] and non-commutative tori [4] [12] are typical examples in this spirit. One approach to constructing quantized manifolds is based on the deformation quantization introduced by Bayen et al [1]. This is the deformation of the Poisson algebra of functions on a symplectic manifold via a star product.

However, deformation quantization providing only an algebraic description does not seem to describe the "underlying space" adequately. From the geometric point of view, we want to construct something like non-commutative manifolds which just represent the quantum state space.

For this purpose, we introduced the notion of Weyl manifolds [10], [11] as a prototype of non-commutative manifolds. A Weyl manifold W_M is defined as a certain algebra bundle over a symplectic manifold M with the formal Weyl algebra as the fiber. The star product given by the deformation quantization is realized on a certain class of sections on W_M, called *Weyl functions.* We present in this paper a *non-commutative complex projective space* $W_{P_n(\mathbf{C})}$ as an example of a Weyl manifold.

There are two ways of constructing star products on $P_n(\mathbf{C})$. The first is *intrinsic*, and was initiated by Berezin [2], who gave a covariant symbol calculus for certain operators acting on local holomorphic functions on the 2-sphere and on the Lobachevskii plane, and defined the star product on these spaces by using the symbol calculus. Moreno [9] and Cahen-Gutt-Rawnsley [3] extended these ideas to Kaehler symmetric spaces.

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The second construction, which is in fact the aim of this paper, is *extrinsic.* We shall regard the ring of Weyl functions on $P_n(\mathbf{C})$ as the subalgebra of all $\mathbf{C}^*$-invariant Weyl functions on $\mathbf{C}^{n+1}-\{0\}$, where one can define the star product and the Weyl manifold structure naturally. In a forthcoming paper, we shall show that the two star products are isomorphic by using the fact that $\dim H^2(P_n(\mathbf{C}))=1$. However, in this paper we shall concentrate our attention to the extrinsic construction of star products and Weyl manifolds.

Throughout this paper, we use the following convention on multi-indices, unless otherwise stated: $\alpha,\beta,\gamma\cdots\in\mathbf{N}^{n+1};\alpha=(\alpha_1,\cdots,\alpha_{n+1})$. Denote ∂_{z_i} by ∂_i and $\partial_{\bar{z}_i}$ by $\bar{\partial}_i$, and for $\alpha\in\mathbf{N}^{n+1}$, set $\partial^\alpha=\partial_1^{\alpha_1}\cdots\partial_{n+1}^{\alpha_{n+1}}$ and $\bar{\partial}^\alpha=\bar{\partial}_1^{\alpha_1}\cdots\bar{\partial}_{n+1}^{\alpha_{n+1}}$, etc.

§1. Deformation quantization on $P_n(\mathbf{C})$

1.1. Deformation quantization

Let (M,ω) be a symplectic manifold, where ω is the symplectic 2-form on M. Let ν be a (formal) parameter and let $\mathbf{C}[[\nu]]$ denote the formal power series ring in ν. Let $C^\infty(M;\mathbf{C}[[\nu]])$ be the set of the $\mathbf{C}[[\nu]]$-valued smooth functions on M. Any $a\in C^\infty(M;\mathbf{C}[[\nu]])$ has a formal sum expansion

$$a=\sum_{l=0}^{\infty}a_l(p)\nu^l \tag{1.1}$$

where $a_l\in C^\infty(M;\mathbf{C})$. $a\in C^\infty(M;\mathbf{C}[[\nu]])$ of the form (1.1) will be denoted by $a=a(p;\nu)$. ν is called a deformation parameter. Following to Bayen et al [1], we introduce the star product $*$:

(D 1) $*$ is an associative product on $C^\infty(M;\mathbf{C}[[\nu]])$.

(D 2) $\quad a*b=ab+\frac{\nu}{2}\{a,b\} \qquad (\mathrm{mod}\,\nu^2)$.

where $\{\,,\,\}$ is the Poisson bracket given by ω.

(M,ω) is called to be *deformation quantizable* if there exists a star product on $C^\infty(M;\mathbf{C}[[\nu]])$. It is known that there exists a star product for any symplectic manifold (M,ω) (cf. [10] and [5]), i.e. it is deformation quantizable.

1.2. The star product on $\mathbf{C}^{n+1}$

Let $\omega_0=\frac{1}{2\sqrt{-1}}\sum_{l=1}^{n+1}dz_l\wedge d\bar{z}_l$ be the canonical symplectic structure on $\mathbf{C}^{n+1}$. To give a star product on $\mathbf{C}^{n+1}$, we introduce a following integral transformation involving a real parameter $h > 0$ acting on holomorphic functions $\tilde{s}(z)$ of $\mathbf{C}^{n+1}$ (cf. [2], [9]):

$$(1.2) \quad (H_{\tilde{a}}\tilde{s})(z) = \left(\frac{1}{4\pi h}\right)^{n+1} \int_{\mathbf{C}^{n+1}} \tilde{a}(z, \bar{z}')e^{\frac{1}{2h}(z-z')\bar{z}'}\tilde{s}(z')d\mu(z', \bar{z}'),$$

where $d\mu(z', \bar{z}')$ is the volume element on $\mathbf{C}^{n+1}$, and $\tilde{a}(z, \bar{z}) \in C^{\omega}(\mathbf{C}^{n+1})$ must be chosen so that (1.2) makes sense (e.g., $\tilde{a}$ is a polynomial) and $\tilde{a}(z, \bar{v})$ is the analytic continuation of $\tilde{a}$ from the diagonal of $\mathbf{C}^{n+1} \times \bar{\mathbf{C}}^{n+1}$.

The operator in (1.2) has various expressions via non-holomorphic coordinate transformations. For instance, (1.2) can be rewritten as

$$(H_{\tilde{a}}\tilde{s})(z) = \left(\frac{1}{4\pi h}\right)^{n+1} \int_{\mathbf{C}^{n+1}} \tilde{a}(z, \bar{z}')e^{\frac{-1}{2h}z'\bar{z}'}\tilde{s}(z + z')d\mu(z', \bar{z}').$$

To compute asymptotic expansions, the class of admissible symbol functions $\tilde{a} = \tilde{a}(z, \bar{z})$ should be enlarged to the so-called class of admissible symbols of the form $\tilde{a}(z, \bar{z}; h) = \sum \tilde{a}_l(z, \bar{z})h^l$ (formal sum).

As in the computation of Ψ.D.Ops, we have the product formula:

$$(1.3) \qquad H_{\tilde{a}}H_{\tilde{b}} = H_{\tilde{e}(\tilde{a}, \tilde{b})}$$

where

$$(1.4) \qquad \tilde{e}(\tilde{a}, \tilde{b})(z, \bar{z}) = \left(\frac{1}{4\pi h}\right)^{n+1} \int_{\mathbf{C}^{n+1}} \tilde{a}(z, \bar{z}')\tilde{b}(z', \bar{z})e^{\frac{-1}{2h}|z-z'|^2}d\mu(z', \bar{z}').$$

Moreover, we may modify (1.2) to a so-called Weyl type integral transformation of $\tilde{s}(z)$:

$$(1.5) \qquad (H_{\tilde{a}}^{w}\tilde{s})(z) = \left(\frac{\sqrt{-1}}{4\pi\tilde{\nu}}\right)^{n+1} \int_{\mathbf{C}^{n+1}} \tilde{a}(\frac{z + z'}{2}, \bar{z}')e^{\frac{\sqrt{-1}}{2\tilde{\nu}}(z-z')\bar{z}'}\tilde{s}(z')d\mu(z', \bar{z}'),$$

where $\tilde{\nu} = \sqrt{-1}h$. By a computation similar to (1.3), we have for suitable $\tilde{a}, \tilde{b} \in C^{\infty}(\mathbf{C}^{n+1}; \mathbf{C}[[\nu]])$,

$$(1.6) \qquad H_{\tilde{a}}^{w}H_{\tilde{b}}^{w} = H_{\tilde{e}^{w}(\tilde{a}, \tilde{b})}^{w},$$

where after a non-holomorphic coordinate transformation (cf. Hörmander [7], p.374), we have

$$(1.7)\quad \tilde{e}^w(\tilde{a},\tilde{b})(z,\bar{z}) = \left(\frac{\sqrt{-1}}{2\pi\tilde{\nu}}\right)^{2(n+1)} \int_{\mathbf{C}^{2(n+1)}} \tilde{a}(z+u,\bar{z}+\bar{v})\tilde{b}(z+v,\bar{z}-\bar{u}) \times e^{-\frac{\sqrt{-1}}{\tilde{\nu}}(u\bar{u}+v\bar{v})} d\mu(u,\bar{u})d\mu(v,\bar{v}).$$

Note that (1.7) has the asymptotic expansion

$$(1.8)\qquad \tilde{e}^w(\tilde{a},\tilde{b}) \sim \sum_l \tilde{c}_l(\tilde{a},\tilde{b})\tilde{\nu}^l,$$

where

$$(1.9)\qquad \tilde{c}_l(\tilde{a},\tilde{b}) = \sum_{|\alpha|+|\beta|=l} \frac{(\sqrt{-1})^l}{\alpha!\beta!}\partial_z^\alpha\partial_{\bar{z}}^\beta\tilde{a}\cdot\partial_{\bar{z}}^\alpha(-\partial_z)^\beta\tilde{b},$$

so that $\tilde{e}^w(\tilde{a},\tilde{b})$ can be viewed as an element of $C^\infty(\mathbf{C}^{n+1};\mathbf{C}[[\tilde{\nu}]])$.

We now define a star product $\tilde{*}$ on $C^\infty(\mathbf{C}^{n+1};\mathbf{C}[[\tilde{\nu}]])$ as follows: For $\tilde{a},\tilde{b}\in C^\infty(\mathbf{C}^{n+1};\mathbf{C}[[\tilde{\nu}]])$, we put

$$(1.10)\qquad \tilde{a}\tilde{*}\tilde{b} = \sum \tilde{c}_l(\tilde{a},\tilde{b})\tilde{\nu}^l,$$

where $\tilde{c}_l(\tilde{a},\tilde{b})$ is given by (1.9). In fact, the formula (1.9) can be applied for any C^∞ functions $\tilde{a},\tilde{b}$ with the parameter $\tilde{\nu}$ viewed as a complex parameter. The restriction of $\tilde{*}$ to $C^\infty(\mathbf{C}^{n+1}-\{0\};\mathbf{C}[[\tilde{\nu}]])$ is denoted by the same symbol. In the following, we denote by $\tilde{\mathfrak{a}}[[\tilde{\nu}]]$ the topological vector space $C^\infty(\mathbf{C}^{n+1}-\{0\};\mathbf{C}[[\tilde{\nu}]])$ with the C^∞ topology. It has two products; one is the natural commutative product, and the other is the star product given above. It is a remarkable fact that the former $\cdot$ can be expressed in terms of the star product:

$$(1.11)\qquad \tilde{a}\cdot\tilde{b} = \sum_{l=0}^\infty \tilde{\nu}^l \sum_{|\alpha|+|\beta|=l} \frac{(\sqrt{-1})^l}{\alpha!\beta!}(-\partial_z^\alpha)(\partial_{\bar{z}}^\beta)\tilde{a}\tilde{*}(\partial_{\bar{z}}^\alpha)(\partial_z^\beta)\tilde{b}.$$

By (1.7), the both products on $\mathbf{C}^{n+1}$ are invariant under the parallel displacement and under the unitary group $U(n+1)$.

1.3. $\mathbf{C}^*$-action on $\tilde{\mathfrak{a}}[[\tilde{\nu}]]$

For $\lambda \in \mathbf{C}^* = \mathbf{C} - \{0\}$, we define an action $\rho(\lambda)$ on $\tilde{\mathfrak{a}}[[\tilde{\nu}]]$ as follows:

Definition 1.1. For $\lambda \in \mathbf{C}^*$, and $\tilde{a} \in \tilde{\mathfrak{a}}[[\tilde{\nu}]]$,

$$(\rho(\lambda)\tilde{a})(z, \bar{z}; \tilde{\nu}) = \tilde{a}(\lambda z, \bar{\lambda}\bar{z}; |\lambda|^2 \tilde{\nu}). \tag{1.12}$$

Set

$$\tilde{\mathfrak{a}}[[\tilde{\nu}]]^\rho = \{\tilde{a} \in \tilde{\mathfrak{a}}[[\tilde{\nu}]] \mid \rho(\lambda)\tilde{a} = \tilde{a} \text{ for all } \lambda \in \mathbf{C}^*\}. \tag{1.13}$$

It is obvious that $\rho(\lambda)$, $\lambda \in \mathbf{C}^*$, commutes with any $T \in U(n+1)$.

By (1.7), we have

Lemma 1.2. *For any $\tilde{a}, \tilde{b} \in \tilde{\mathfrak{a}}[[\tilde{\nu}]]$ and $\lambda \in \mathbf{C}^*$, we have*

$$\rho(\lambda)(\tilde{a} \tilde{*} \tilde{b}) = (\rho(\lambda)\tilde{a}) \tilde{*} (\rho(\lambda)\tilde{b}). \tag{1.14}$$

1.4. A deformation quantization on $P_n(\mathbf{C})$

In this section, using the product $\tilde{*}$ in 1.2, we construct a star product on $P_n(\mathbf{C})$ with the deformation parameter replaced by ν.

Let $P_n(\mathbf{C})$ be the n-dimensional complex projective space equipped with the standard symplectic structure ω (cf. [8], p. 160) and let $\pi : \mathbf{C}^{n+1} - \{0\} \to P_n(\mathbf{C})$ be the natural projection. Taking the deformation parameter ν, we put $\mathfrak{a}[[\nu]] = C^\infty(P_n(\mathbf{C}); \mathbf{C}[[\nu]])$. For $a \in \mathfrak{a}[[\nu]]$, we define a lift of a, denoting by $\pi^* a$ as an element of $\tilde{\mathfrak{a}}[[\tilde{\nu}]]$ by

$$(\pi^* a)(z, \bar{z}; \tilde{\nu}) = a(p; |z|^{-2}\tilde{\nu}), \qquad \pi(z) = p. \tag{1.15}$$

From Definition 1.1, we easily see that $\pi^* a \in \tilde{\mathfrak{a}}[[\tilde{\nu}]]^\rho$.

For any $\tilde{a} \in \tilde{\mathfrak{a}}[[\tilde{\nu}]]^\rho$, we put

$$(\iota\tilde{a})(p; \nu) = \tilde{a}(z, \bar{z}; |z|^2 \nu), \quad \pi(z) = p. \tag{1.16}$$

(1.16) is independent of the choice of z.

Lemma 1.3.

$$\iota : \tilde{\mathfrak{a}}[[\tilde{\nu}]]^{\rho} \to \mathfrak{a}[[\nu]]$$

is an isomorphism with $\iota\pi^* = \mathrm{id}$.

By this lemma, we can identify $\tilde{\mathfrak{a}}[[\tilde{\nu}]]^{\rho}$ with $\mathfrak{a}[[\nu]]$. By Lemma 1.2 and Lemma 1.3, we can project the product $\tilde{*}$ onto $P_n(\mathbf{C})$. Namely, for any $a, b \in \mathfrak{a}[[\nu]]$, we put

$$a * b = \iota(\pi^* a \tilde{*} \pi^* b). \tag{1.17}$$

Consider the chart $U_{n+1} = \{p = \pi(z) \mid z_{n+1} \neq 0\}$ and the coordinate map $\phi_{n+1} : U_{n+1} \to \phi_{n+1}(U_{n+1}) = \mathbf{C}^n$, $\phi_{n+1}(p) = w = (w_1, \cdots, w_n)$, where $w_j = \frac{z_j}{z_{n+1}}$ $(j = 1, \cdots, n)$. Using these coordinates, the symplectic structure ω on $P_n(\mathbf{C})$ becomes (cf. [8] p. 160):

$$\begin{aligned} \omega \mid_{U_{n+1}} = \frac{1}{2\sqrt{-1}(1+|w|^2)^2}((1+|w|^2)\sum_{l=1}^{n} dw_l \wedge d\bar{w}_l \\ - \sum_{l,m=1}^{n} \bar{w}_l dw_l \wedge w_m d\bar{w}_m). \end{aligned} \tag{1.18}$$

By (1.18), in these coordinates, the Poisson bracket $\{a, b\}$ on $P_n(\mathbf{C})$ is

(1.19)

$$\begin{aligned} &\{a, b\}(w_1, \cdots, w_n) \\ &\quad = 2\sqrt{-1}(1+|w|^2)[\textstyle\sum_{l=1}^{n}(\partial_{w_l} a \cdot \partial_{\bar{w}_l} b - \partial_{\bar{w}_l} a \cdot \partial_{w_l} b) \\ &\qquad + \displaystyle\sum_{k,l}(w_k \partial_{w_k} a \cdot \bar{w}_l \partial_{\bar{w}_l} b - \bar{w}_k \partial_{\bar{w}_k} a \cdot w_l \partial_{w_l} b)] \end{aligned}$$

On the other hand, since $w_j = w_j(z_1, \cdots, z_{n+1})$, we have

(1.20)

$$\begin{aligned} \partial_{z_{n+1}} &= -\frac{1}{z_{n+1}}\sum_{i=1}^{n} w_l \partial_{w_l}, & \partial_{z_m} &= \frac{1}{z_{n+1}}\partial_{w_m} \quad (m = 1, \cdots, n), \\ \partial_{\bar{z}_{n+1}} &= -\frac{1}{\bar{z}_{n+1}}\sum_{l=1}^{n} \bar{w}_l \partial_{\bar{w}_l}, & \partial_{\bar{z}_m} &= \frac{1}{\bar{z}_{n+1}}\partial_{\bar{w}_m} \quad (m = 1, \cdots, n). \end{aligned}$$

By a direct computation using (1.20) and (1.10) and putting $z_{n+1} = 1$, $z_l = w_l (l = 1, \cdots, n)$, we have

Proposition 1.4. (1.17) *gives a star product* $*$ *on* $P_n(\mathbf{C})$, *i.e. for any* $a, b \in C^\infty(P_n(\mathbf{C}))$ *we have*

$$a * b = ab + \frac{\nu}{2}\{a, b\} \qquad (\text{mod } \nu^2). \tag{1.21}$$

§2. A Weyl manifold over $P_n(\mathbf{C})$

Using the notion of Weyl manifolds given in [10, 11], we describe the algebra $\mathfrak{a}[[\nu]]$ more geometrically.

2.1. The formal Weyl algebra

Let $\tilde{\mathbf{W}}'$ denote the algebra with $2n+3$ generators $\{\tilde{\nu}, Z_1, \cdots, Z_{n+1}, \bar{Z}_1, \cdots, \bar{Z}_{n+1}\}$ over $\mathbf{C}$ with the relations:

$$\begin{cases} [\tilde{\nu}, Z_i] = 0 \quad , \quad [\tilde{\nu}, \bar{Z}_i] = 0, \\ [Z_i, Z_j] = 0 \quad , \quad [\bar{Z}_i, \bar{Z}_j] = 0 \\ [Z_i, \bar{Z}_j] = 2\sqrt{-1}\nu\delta_{ij} \quad (1 \le i, j \le n+1), \end{cases} \tag{2.1}$$

where $[\ ,\ ]$ denotes the commutator $[a, b]=ab - ba$. For any $a, b \in \tilde{\mathbf{W}}'$, the product is denoted by $a * b$; for any α, $\beta \in \mathbf{N}^{n+1}$, we denote $Z_1^{\alpha_1} * \cdots * Z_{n+1}^{\alpha_{n+1}} * \bar{Z}_1^{\beta_1} * \cdots * \bar{Z}_{n+1}^{\beta_{n+1}}$, by $Z^\alpha * \bar{Z}^\beta$ where $Z_i^{\alpha_i} = \underbrace{Z_i * \cdots * Z_i}_{\alpha_i}$, $\bar{Z}_i^{\beta_i} = \underbrace{\bar{Z}_i * \cdots * \bar{Z}_i}_{\beta_i}$.

Define the degree of the generators by $d(\tilde{\nu})=2$, $d(Z_i) = d(\bar{Z}_i)=1$ $(1 \le i \le n+1)$. For $l \ge 0$, let $\tilde{\mathbf{W}}(l)$ be the set of polynomials of degree l and $\tilde{\mathbf{W}}(0) = \mathbf{C}$. Then

$$\tilde{\mathbf{W}}' = \oplus_{l \ge 0} \tilde{\mathbf{W}}(l), \qquad (\text{direct sum}). \tag{2.2}$$

Any element $a \in \tilde{\mathbf{W}}'$ can be written as a finite sum $\sum a_l$, $a_l \in \tilde{\mathbf{W}}(l)$; a_l is called the l*-th component* of a.

Give $\tilde{\mathbf{W}}' = \oplus_l \tilde{\mathbf{W}}(l)$ the direct product topology. Denote by $\tilde{\mathbf{W}}$ the completion of $\tilde{\mathbf{W}}'$; $\tilde{\mathbf{W}}$ is called the *formal Weyl algebra* with generators

$\{\tilde{\nu}, Z_1, \cdots, Z_{n+1}, \bar{Z}_1, \cdots, \bar{Z}_{n+1}\}$. The formal Weyl algebra $\tilde{\mathbf{W}}$ is isomorphic (as a vector space) to the formal power series ring $\mathbf{C}[[\tilde{\nu}, Z_1, \cdots, Z_{n+1}, \bar{Z}_1, \cdots, \bar{Z}_{n+1}]]$. If we replace $Z_i, \bar{Z}_i$ by $(X_i + \sqrt{-1}Y_i)$ and $(X_i - \sqrt{-1}Y_i)$ respectively, then the algebra $\tilde{\mathbf{W}}$ is exactly the same as in [10]. We also use the formal Weyl algebra $\mathbf{W}$ with $2n+1$ generators $\{\nu, Z_1, \cdots, Z_n, \bar{Z}_1, \cdots, \bar{Z}_n\}$.

2.2. Symmetric product

For $a, b \in \tilde{\mathbf{W}}$, define the symmetric product by

$$a \circ b = \frac{1}{2}(a * b + b * a).$$

The above product is *not* associative but $(\tilde{\mathbf{W}}, \circ)$ is a Jordan algebra. However, by the general formula

$$(a \circ b) \circ c - a \circ (b \circ c) = \frac{1}{4}[b, [a, c]], \tag{2.3}$$

and the fact that $[Z_i, \bar{Z}_j]$ is in the center of $\tilde{\mathbf{W}}$, we have

$$\hat{Z}_i \circ (\hat{Z}_j \circ a) = \hat{Z}_j \circ (\hat{Z}_i \circ a) \quad (1 \le i, j \le n+1), \tag{2.4}$$

where $\hat{Z}_i = Z_i$ or $\bar{Z}_i$. Thus, we may set

$$(\hat{Z}_i \circ)^l \cdot a = \underbrace{\hat{Z}_i \circ (\hat{Z}_i \circ \cdots (\hat{Z}_i \circ a) \cdots)}_{l \text{ times}},$$

and

$$\begin{aligned} &(Z\circ)^\alpha (\bar{Z}\circ)^\beta \cdot a \\ &= (Z_1 \circ)^{\alpha_1} \cdots (Z_{n+1}\circ)^{\alpha_{n+1}} (\bar{Z}_1 \circ)^{\beta_1} \cdots (\bar{Z}_{n+1}\circ)^{\beta_{n+1}} \cdot a, \end{aligned} \tag{2.5}$$

where the right hand side of (2.5) is independent of the order of the $Z_i\circ$'s, and $\bar{Z}_i\circ$'s. Obviously, $\{\tilde{\nu}^l (Z\circ)^\alpha (\bar{Z}\circ)^\beta \cdot 1\ ; \alpha, \beta \in \mathbf{N}^{n+1}\}$ forms a linear basis of $\tilde{\mathbf{W}}$. $\tilde{\mathbf{W}}(k)$ is spanned by $\{\tilde{\nu}^l (Z\circ)^\alpha (\bar{Z}\circ)^\beta \cdot 1 : 2l + |\alpha| + |\beta| = k\}$ (cf. [10], Lemma 1.2).

By the above fact, we may introduce a new product $\odot$ defined by

$$(\hat{Z}\circ)^\alpha \cdot 1 \odot (\hat{Z}\circ)^\beta \cdot 1 = (\hat{Z}\circ)^{\alpha+\beta} \cdot 1, \qquad \alpha, \beta \in \mathbf{N}^{n+1}.$$

We denote $\hat{Z}_i \circ \hat{Z}_j$ and $(\hat{Z}\circ)^\alpha \cdot 1$ by $\hat{Z}_i \odot \hat{Z}_j$ and $(\hat{Z}\odot)^\alpha$ respectively. The following are easily seen:

(a) $(\tilde{\mathbf{W}}, \odot)$ is a commutative, associative topological algebra over $\mathbf{C}$.

(b) $(\tilde{\mathbf{W}}, \odot)$ is isomorphic to the algebra $\mathbf{C}[[\tilde{\nu}, Z_1, \cdots, Z_{n+1}, \bar{Z}_1, \cdots, \bar{Z}_{n+1}]]$.

2.3. Localization of the algebras $\tilde{\mathfrak{a}}[[\tilde{\nu}]]$ and $\mathfrak{a}[[\nu]]$

Let $\tilde{U}$ and U be open sets of $\mathbf{C}^{n+1} - \{0\}$ and $P_n(\mathbf{C})$ respectively. By formula (1.8) and Definition (1.17), the $\tilde{*}$(resp. $*$)-product can be restricted on $\tilde{U}$ (resp. U) and then extended to $C^\infty(\tilde{U}; \mathbf{C}[[\tilde{\nu}]])$ (resp. $C^\infty(U; \mathbf{C}[[\nu]])$). If $\pi(\tilde{U}) = U$, then π^* and ι given in (1.15) and (1.16) can be also restricted on U and $\tilde{U}$, wich are denoted by $\pi_U^*, \iota_{\tilde{U}}$ respectively. In particular, for any $a, b \in \mathfrak{a}_U[[\nu]]$,

$$a * b = \iota_{\tilde{U}}(\pi_U^* a \tilde{*} \pi_U^* b). \tag{2.6}$$

The algebra $(C^\infty(\tilde{U}; \mathbf{C}[[\tilde{\nu}]]), \tilde{*})$ (resp. $(C^\infty(U; \mathbf{C}[[\nu]]), *)$) with the C^∞-topology is denoted by $\tilde{\mathfrak{a}}_{\tilde{U}}[[\tilde{\nu}]]$ (resp. $\mathfrak{a}_U[[\nu]]$).

Given an open set $\tilde{U} \subset \mathbf{C}^{n+1} - \{0\}$, we consider the trivial bundle $W_{\tilde{U}} = \tilde{U} \times \tilde{\mathbf{W}} \xrightarrow{\pi} \tilde{U}$. Define $2n+2$ smooth sections ζ_i, $\bar{\zeta}_i$ on $W_{\tilde{U}}$ by:

$$\zeta_i(z, \bar{z}) = z_i + Z_i, \quad \bar{\zeta}_i(z, \bar{z}) = \bar{z}_i + \bar{Z}_i, \quad (i = 1, \cdots, n+1). \tag{2.7}$$

For $f \in \tilde{\mathfrak{a}}_{\tilde{U}}[[\tilde{\nu}]]$, we define a section $f^\#(\zeta, \bar{\zeta}) \in \Gamma(W_{\tilde{U}})$ by

$$f^\#(\zeta, \bar{\zeta})(z, \bar{z}) = \sum \frac{1}{\alpha!\beta!}(\partial^\alpha \bar{\partial}^\beta f)(z, \bar{z}) \cdot Z^\alpha \odot \bar{Z}^\beta, \quad \alpha, \beta \in \mathbf{N}^{n+1}. \tag{2.8}$$

$f^\#$ is called the *Weyl continuation* of $f \in \tilde{\mathfrak{a}}_{\tilde{U}}[[\tilde{\nu}]]$. Let $\mathcal{F}(W_{\tilde{U}})$ be the algebra of $f^\#$ for $f \in \tilde{\mathfrak{a}}_{\tilde{U}}[[\tilde{\nu}]]$ where the product is defined pointwisely on $\tilde{\mathbf{W}}$.

We have shown in [10]:

Proposition 2.1. *$\mathcal{F}(W_{\tilde{U}})$ is naturally isomorphic to $\tilde{\mathfrak{a}}_{\tilde{U}}[[\tilde{\nu}]]$ as an algebra.*

2.4. Main results

We now introduce systems of local generators:

Definition 2.2. Let $\tilde{U}$ and $U = \pi(\tilde{U})$ be open sets of $\mathbf{C}^{n+1} - \{0\}$ and $P_n(\mathbf{C})$ respectively. A $(2n+3)$-tuple $\{\tilde{w}_0; \tilde{w}_1, \cdots, \tilde{w}_{2n+2}\}$ of $\tilde{\mathfrak{a}}_{\tilde{U}}[[\tilde{\nu}]]$

(resp. $(2n+1)$-tuple $\{w_0; w_1, \cdots, w_{2n}\}$ of $\mathfrak{a}_U[[\nu]]$) is called a *system of local generators* for $\tilde{\mathfrak{a}}_{\tilde{U}}[[\tilde{\nu}]]$ (resp. $\mathfrak{a}_U[[\nu]]$) if they satisfy

(L 1) $\tilde{w}_0$ (resp. w_0) is in the center of $\tilde{\mathfrak{a}}_{\tilde{U}}[[\tilde{\nu}]]$ (resp. $\mathfrak{a}_U[[\nu]]$).

(L 2) The closure of the algebra generated by $\{\tilde{w}_0; \tilde{w}_1, \cdots, \tilde{w}_{2n+2}\}$ (resp. $\{w_0; w_1, \cdots, w_{2n}\}$) coincides with $\tilde{\mathfrak{a}}_{\tilde{U}}[[\tilde{\nu}]]$ (resp. $\mathfrak{a}_U[[\nu]]$).

We now consider this definition on each chart (U_l, ϕ_l) of $P_n(\mathbf{C})$. Namely, for each $l = 1, 2, \cdots, n+1$, let $\tilde{U}_l = \{z = (z_1, \cdots, z_{n+1}) \in \mathbf{C}^{n+1} - \{0\} \mid z_l \neq 0\}$, $U_l = \pi(\tilde{U}_l)$, and $\phi_l : U_l \to \phi_l(U_l) = \mathbf{C}^n$. Then, $\phi_l(p) = (\frac{z_1}{z_l}, \cdots, \hat{\frac{z_l}{z_l}}, \cdots, \frac{z_{n+1}}{z_l})$ with $p = \pi(z)$ gives the local coordinate of $P_n(\mathbf{C})$. For simplicity, we set $\pi_l^* = \pi_{U_l}^*$ and $\iota_l = \iota_{\tilde{U}_l}$.

Definition 2.3. A collection of systems of local generators $\{w_0^{(l)}; u_1^{(l)}, \cdots, u_n^{(l)}, v_1^{(l)}, \cdots, v_n^{(l)}\}$ for $\mathfrak{a}_{U_l}[[\nu]]$ for each $l = 1, \cdots, n+1$ is called a (*system of*) *Weyl coordinates on* $P_n(\mathbf{C})$ *associated with* $\{(U_l, \phi_l)\}$ if for any $l, m = 1, \cdots, n+1$

(C 1) $\pi_l^* w_0^{(l)} = \pi_m^* w_0^{(m)}$ on $\tilde{\mathfrak{a}}_{\tilde{U}_l \cap \tilde{U}_m}[[\tilde{\nu}]]$ if $U_l \cap U_m \neq \emptyset$

(C 2)

$$\begin{cases} [w_0^{(l)}, u_i^{(l)}] = 0, \quad [w_0^{(l)}, v_i^{(l)}] = 0, \quad [u_i^{(l)}, u_j^{(l)}] = 0, \\ [v_i^{(l)}, v_j^{(l)}] = 0, \quad [u_i^{(l)}, v_j^{(l)}] = -w_0^{(l)} \delta_{ij}. \end{cases}$$

(C 3) On each $U_k \cap U_l (\neq \emptyset)$, $u_1^{(k)}, \cdots, u_n^{(k)}, v_1^{(k)}, \cdots, v_n^{(k)} \mod \nu$ are $\mathbf{R}$-valued C^∞ functions of $(u_1^{(l)}, \cdots, u_n^{(l)}, v_1^{(l)}, \cdots, v_n^{(l)})$.

In §3-4, we shall prove the following:

Theorem 2.4. *There exists a system of Weyl coordinates on* $P_n(\mathbf{C})$ *associated with* $\{(U_l, \phi_l)\}$. (cf. Theorem 4.5.)

By this theorem, we can construct an algebra bundle over $P_n(\mathbf{C})$ with the formal Weyl algebra $\mathbf{W}$ of $2n+1$ generators as fiber. Namely, on each U_l we consider a trivial algebra bundle $\pi_l : U_l \times \mathbf{W} \to U_l$. Since $\{w_0^{(l)}; u_1^{(l)}, \cdots, u_n^{(l)}, v_1^{(l)}, \cdots, v_n^{(l)}, \}$ can be viewed as C^∞-sections of W_{U_l}, this trivializes the bundle W_{U_l}. Moreover, we can patch the W_{U_j} together. This gives a Weyl manifold over $P_n(\mathbf{C})$ introduced in [9, 10]. Using the notation of [9, 10] on Weyl manifolds, we have

Theorem 2.5. *The algebra* $(\mathfrak{a}[[\nu]], *) = (C^\infty(P_n(\mathbf{C}); \mathbf{C}[[\nu]]), *)$ *gives a Weyl manifold* $W_{P_n(\mathbf{C})}$ *over* $P_n(\mathbf{C})$*. In particular,* $\mathfrak{a}[[\nu]]$ *is isomorphic to* $\mathcal{F}(W_{P_n(\mathbf{C})})$*, where* $\mathcal{F}(W_{P_n(\mathbf{C})})$ *is the set of all Weyl functions on* $P_n(\mathbf{C})$*.*

§3. Properties for $\tilde{\mathfrak{a}}[[\tilde{\nu}]]^\rho$

3.1. Several operations on $\tilde{\mathfrak{a}}[[\tilde{\nu}]]$

Note that the natural product $\cdot$ can be defined on $\tilde{\mathfrak{a}}_{\tilde{U}}[[\tilde{\nu}]]$ for any open set $\tilde{U} \subset \mathbf{C}^{n+1} - \{0\}$. We use the notation $(\tilde{\mathfrak{a}}_{\tilde{U}}[[\tilde{\nu}]], \cdot)$ when we consider $\tilde{\mathfrak{a}}_{\tilde{U}}[[\tilde{\nu}]]$ as a commutative algebra. We can introduce a partial derivative $\partial_{\tilde{\nu}}$ on $\tilde{\mathfrak{a}}[[\tilde{\nu}]]$ and $\tilde{\mathfrak{a}}_{\tilde{U}}[[\tilde{\nu}]]$ as follows: for any element $a \in \tilde{\mathfrak{a}}_{\tilde{U}}[[\tilde{\nu}]]$ with the form $a = \sum a_l \tilde{\nu}^l$ where $a_l = a_l(z, \bar{z})$ is C^∞,

$$\partial_i a = \sum (\partial_i a_l)\tilde{\nu}^l, \quad \bar{\partial}_i a = \sum (\bar{\partial}_i a_l)\tilde{\nu}^l, \quad \partial_{\tilde{\nu}} a = \sum l a_l \tilde{\nu}^{l-1}. \tag{3.1}$$

We introduce the differential operators L_0 and L_1 on $\tilde{\mathfrak{a}}_{\tilde{U}}[[\tilde{\nu}]]$ by

$$L_0 \tilde{a} = 2\tilde{\nu}\partial_{\tilde{\nu}}\tilde{a} + \sum_i (z_i \cdot \partial_i + \bar{z}_i \cdot \bar{\partial}_i)\tilde{a} \tag{3.2}$$

and

$$L_1 \tilde{a} = \sum \sqrt{-1}(\bar{z}_i \cdot \partial_i - z_i \cdot \bar{\partial}_i)\tilde{a}, \tag{3.3}$$

for $\tilde{a} \in \tilde{\mathfrak{a}}_{\tilde{U}}[[\tilde{\nu}]]$.

Lemma 3.1. L_0 *and* L_1 *are derivations of* $(\tilde{\mathfrak{a}}_{\tilde{U}}[[\tilde{\nu}]], \cdot)$: *i.e. for any* $\tilde{a}, \tilde{b} \in (\tilde{\mathfrak{a}}_{\tilde{U}}[[\tilde{\nu}]], \cdot)$,

$$L_k(\tilde{a} \cdot \tilde{b}) = L_k(\tilde{a}) \cdot \tilde{b} + \tilde{a} \cdot L_k(\tilde{b}) \quad (k = 0, 1). \tag{3.4}$$

Note that L_1 can be rewritten as

$$L_1 \tilde{a} = -\frac{1}{\nu}[r, \tilde{a}] \; (= -\frac{1}{\nu} ad(r)\tilde{a}), \tag{3.5}$$

where $r = \frac{1}{2}|z|^2 = \frac{1}{2}\sum_{i=1}^{n+1} z_i \bar{z}_i$.

Remark. In general, for $\tilde{a}, \tilde{b}, \tilde{c} \in \tilde{\mathfrak{a}}_{\tilde{U}}[[\tilde{\nu}]]$, the equality

$$[\tilde{a}, \tilde{b} \cdot \tilde{c}] = [\tilde{a}, \tilde{b}] \cdot \tilde{c} + \tilde{b} \cdot [\tilde{a}, \tilde{c}]$$

does not hold.

Let $\tilde{U}$ be a conic open set in $\mathbf{C}^{n+1} - \{0\}$ and put $\tilde{\mathfrak{a}}_{\tilde{U}}[[\tilde{\nu}]]^{\rho} = \tilde{\mathfrak{a}}[[\tilde{\nu}]]^{\rho} \cap \tilde{\mathfrak{a}}_{\tilde{U}}[[\tilde{\nu}]]$. A characterization of $\tilde{\mathfrak{a}}_{\tilde{U}}[[\tilde{\nu}]]^{\rho}$ by L_0 and r is given as follows:

Proposition 3.2. $\tilde{\mathfrak{a}}_{\tilde{U}}[[\tilde{\nu}]]^{\rho} = \{\tilde{a} \in \tilde{\mathfrak{a}}_{\tilde{U}}[[\tilde{\nu}]] \mid L_0\tilde{a} = 0, \quad [r, \tilde{a}] = 0\}.$

Proof. For a real parameter t and $\tilde{a} \in \tilde{\mathfrak{a}}_{\tilde{U}}[[\tilde{\nu}]]$, consider curves $t \mapsto \rho(e^t)\tilde{a}$, $\rho(e^{\sqrt{-1}t})\tilde{a}$. Taking the derivatives at $t = 0$, we get

$$\frac{d}{dt}\rho(e^t)\tilde{a} \mid_{t=0} = L_0\tilde{a}, \tag{3.6}$$

$$\frac{d}{dt}\rho(e^{\sqrt{-1}t})\tilde{a} \mid_{t=0} = L_1\tilde{a}. \tag{3.7}$$

Since $L_0 r = 2r$ and $L_0\tilde{\nu} = 2\tilde{\nu}$, we have formally $L_0(\frac{1}{\tilde{\nu}}r) = 0$. This implies $[L_0, L_1] = 0$, which gives Proposition 3.2. Q.E.D.

Using Lemma 3.1 and Proposition 3.2, we have

Corollary 3.3. *Let $\tilde{U}$ be a conic open set in $\mathbf{C}^{n+1} - \{0\}$.*

(1) *$\tilde{\mathfrak{a}}_{\tilde{U}}[[\tilde{\nu}]]^{\rho}$ is closed under the $\cdot$-product.*

(2) *For any $T \in U(n+1)$, we have*

(a) $T(r) = r, \qquad [T, L_0] = 0,$

(b) $T\tilde{\mathfrak{a}}_{T\tilde{U}}[[\tilde{\nu}]]^{\rho} = \tilde{\mathfrak{a}}_{\tilde{U}}[[\tilde{\nu}]]^{\rho}$.

3.2. Inverse of r

Since $r \neq 0$ on $\mathbf{C}^{n+1} - \{0\}$, it has the inverse $\frac{1}{r}$ for the $\cdot$-product. To obtain the inverse r^{-1} for the $\tilde{*}$-product, we first assume that r^{-1} is a function $f(r)$ of r and solve the equation $r\tilde{*}f(r) = 1$. By the product formulas (1.9) (1.10), we have

$$r\tilde{*}f(r) = rf(r) + \tilde{\nu}^2(\frac{n+1}{2}f'(r) + \frac{1}{2}f''(r)r) = 1.$$

Setting $f = \sum_{l=0}^{\infty} f_l \tilde{\nu}^l$, we have

$$\text{(3.8)} \qquad \begin{cases} f_{2l}(t) = \left(-\frac{1}{2}\right)^l \left(\frac{d^2}{dt^2} + \frac{n+1}{t}\frac{d}{dt}\right)^l \left(\frac{1}{t}\right), \\ f_{2l+1} = 0. \end{cases}$$

By (3.8), r^{-1} has the form

$$\text{(3.9)} \qquad \begin{aligned} r^{-1} = \frac{1}{r}\{1 + \frac{n-1}{2}(\frac{\tilde{\nu}}{r})^2 - \frac{(n-1)}{2}\frac{3(n-3)}{2}(\frac{\tilde{\nu}}{r})^4 \\ + \frac{(n-1)}{2}\frac{3(n-3)}{2}\frac{5(n-5)}{2}(\frac{\tilde{\nu}}{r})^6 + \cdots\}. \end{aligned}$$

On the other hand, $e_{\tilde{*}}^{t\tilde{\nu}r^{-1}} = \sum \frac{t^m}{m!}(\tilde{\nu}r^{-1}\tilde{*})^m$, $t \in \mathbf{R}$, in the $\tilde{*}$-product, satisfies the differential equation

$$\text{(3.10)} \qquad \frac{d}{dt}g_t(r) = \tilde{\nu}r^{-1}\tilde{*}g_t(r), \quad g_0(r) = 1.$$

Multiplying both sides of (3.10) by r, we have

$$\frac{d}{dt}\{r \cdot g_t(r) + \tilde{\nu}^2(\frac{n+1}{2}g_t'(r) + \frac{1}{2}g_t''(r) \cdot r)\} = \tilde{\nu}g_t(r).$$

By setting $g_t = \sum_{l=0}^{\infty} \tilde{\nu}^l g_t^{(l)}(r)$, we can compute $e_{\tilde{*}}^{t\tilde{\nu}r^{-1}}$ in the form $\sum_{l \geq k} a_{k,l} t^k (\frac{\tilde{\nu}}{r})^l$, where $a_{kk} = \frac{1}{k!}$. Comparing coefficients of t^k, we see that

$$\text{(3.11)} \qquad (\tilde{\nu}r^{-1}\tilde{*})^m = \sum_{l=m}^{\infty} a_{m,l}(\frac{\tilde{\nu}}{r})^l \quad (m = 1, 2, \cdots).$$

Since (3.11) can be solved conversely with respect to $(\frac{\tilde{\nu}}{r})^l$, we see that $\frac{\tilde{\nu}}{r}$ is written as a function of $\tilde{\nu}r^{-1}$.

3.3. The center of $\tilde{\mathfrak{a}}[[\tilde{\nu}]]^\rho$.

Put $\nu = \frac{\tilde{\nu}}{r} \in \tilde{\mathfrak{a}}[[\tilde{\nu}]]$. Then we have:

Proposition 3.4. *$\nu = \frac{\tilde{\nu}}{r}$ satisfies the following:*

(a) $\nu \in \tilde{\mathfrak{a}}[[\tilde{\nu}]]^{\rho}$,

(b) $[\nu, f] = 0$ *for any* $f \in \tilde{\mathfrak{a}}[[\tilde{\nu}]]^{\rho}$.

Proof. Since $[r, \tilde{\mathfrak{a}}[[\tilde{\nu}]]^{\rho}] = \{0\}$ by Proposition 3.2, we have $[r^{-1}, \tilde{\mathfrak{a}}[[\tilde{\nu}]]^{\rho}] = \{0\}$. Thus $[f(r^{-1}), \tilde{\mathfrak{a}}[[\tilde{\nu}]]^{\rho}] = \{0\}$. By Proposition 3.2, we obtain (b). Moreover, since $[\frac{\tilde{\nu}}{r}, r] = 0$ and $L_0 r = 2r$, we have $\frac{\tilde{\nu}}{r} \in \tilde{\mathfrak{a}}[[\tilde{\nu}]]^{\rho}$. Q.E.D.

By Proposition 3.4, we may use $\nu = \frac{\tilde{\nu}}{r}$ as a deformation parameter of $\mathfrak{a}[[\nu]]$. However, note that there is no general rule for determining deformation parameters as one may replace $\frac{\tilde{\nu}}{r}$ by $\tilde{\nu} r^{-1}$. If we choose $\tilde{\nu} r^{-1}$ as a deformation parameter, then the expression of $*$-product on $\mathfrak{a}[[\nu]]$ is changed.

§4. Manifold structures on $\mathfrak{a}[[\nu]]$

4.1. Local generators of $\mathfrak{a}[[\nu]]$

It is impossible to find generators of $\mathfrak{a}[[\nu]]$ with respect to which any element of $\mathfrak{a}[[\nu]]$ has a unique expression. Instead, we can localize $\mathfrak{a}[[\nu]]$ on open subsets to have convenient expressions for its elements. On the open set $\tilde{U}_{n+1} = \{z \in \mathbf{C}^{n+1} - \{0\} \mid z_{n+1} \neq 0\}$, consider

$$\tilde{\mathfrak{a}}_{\tilde{U}_{n+1}}[[\tilde{\nu}]]^{\rho} = \{\tilde{a} \in \tilde{\mathfrak{a}}_{\tilde{U}_{n+1}}[[\tilde{\nu}]] \mid \rho(\lambda)\tilde{a} = \tilde{a}, \lambda \in \mathbf{C}^*\}. \tag{4.1}$$

Note that on $\tilde{U}_{n+1}$, $\frac{1}{z_{n+1}}$ and $\frac{1}{\bar{z}_{n+1}}$ are well-defined. Thus, setting

$$\nu = \frac{\tilde{\nu}}{r}, \quad w_i = \frac{z_i}{z_{n+1}}, \quad \bar{w}_i = \frac{\bar{z}_i}{\bar{z}_{n+1}} \qquad (i = 1, \cdots, n), \tag{4.2}$$

we have $\nu, w_i, \bar{w}_i \in \tilde{\mathfrak{a}}_{\tilde{U}_{n+1}}[[\tilde{\nu}]]^{\rho}$. By Lemma 1.3, we can identify $\nu, w_i, \bar{w}_i$ with elements of $\mathfrak{a}_{U_{n+1}}[[\nu]]$.

For $\tilde{f} \in \tilde{\mathfrak{a}}_{\tilde{U}_{n+1}}[[\tilde{\nu}]]^{\rho}$, we may write $\tilde{f} = \sum_{l \geq 0} \tilde{f}_l(z, \bar{z})\tilde{\nu}^l$. Since $\tilde{f}$ is invariant under $\rho(\frac{1}{z_{n+1}})$, we have

$$\begin{aligned}\tilde{f}(z,\bar{z};\tilde{\nu}) &= \left(\rho\left(\frac{1}{z_{n+1}}\right)\tilde{f}\right)(z,\bar{z};\tilde{\nu}) \\ &= \tilde{f}\left(\frac{z}{z_{n+1}},\frac{\bar{z}}{\bar{z}_{n+1}};\frac{\tilde{\nu}}{|z_{n+1}|^2}\right) \\ &= \sum_l \tilde{f}_l\left(\frac{z}{z_{n+1}},\frac{\bar{z}}{\bar{z}_{n+1}}\right)\left(\frac{r}{|z_{n+1}|^2}\right)^l\left(\frac{\tilde{\nu}}{r}\right)^l \\ &= \sum_l f_l(w,\bar{w})\nu^l\end{aligned} \tag{4.3}$$

where $f_l(w,\bar{w}) = \tilde{f}_l(w,\bar{w})(\frac{1}{2}(1+|w|^2))^l$. This gives:

Theorem 4.1. *$\tilde{f} \in \tilde{\mathfrak{a}}_{\tilde{U}_{n+1}}[[\tilde{\nu}]]^\rho$ if and only if there exists $f \in C^\infty(U_{n+1};\mathbf{C}[[\nu]])$ such that $\tilde{f} = \pi^*_{\tilde{U}_{n+1}}f$.*

4.2. Commutation relations for Weyl coordinates

We compute the commutation relations for $\{\tilde{\nu}, w_1, \cdots, w_n, \bar{w}_1,\cdots,\bar{w}_n\}$ on $\phi_{n+1}(U_{n+1})$. Using (1.9) and Proposition 3.4 (b), we easily have

Lemma 4.2. *For any $i,j = 1,\cdots,n$,*

$$\begin{cases} [\nu,w_i] = [\nu,\bar{w}_i] = 0, \\ [w_i,w_j] = [\bar{w}_i,\bar{w}_j] = 0. \end{cases} \tag{4.4}$$

By Lemma 4.2 and the polynomial approximation theorem, the commutative algebra of the $\mathbf{C}[[\nu]]$-valued holomorphic functions on $\phi_{n+1}(U_{n+1})$ (resp. anti-holomorphic functions on $\phi_{n+1}(U_{n+1})$) is isomorphic to the subalgebra of $\mathcal{F}(W_{\phi_{n+1}(U_{n+1})})$ whose element has the form $f^\# = f(\nu,w_1,\cdots,w_n)^\#$ (resp. $f^\# = f(\nu,\bar{w}_1,\cdots,\bar{w}_n)^\#$).

By Theorem 4.1, we may call $\{\nu,w_1,\cdots,w_n,\bar{w}_1,\cdots,\bar{w}_n\}$ *the homogeneous complex Weyl coordinates* on $W_{\phi_{n+1}(U_{n+1})}$. By a careful computation, we have the following commutation relation:

Proposition 4.3.

(4.5)
$$\begin{aligned}[w_i, \bar{w}_j] =& \nu(1+\sum_{l=1}^{n} w_l\bar{w}_l)\cdot(\delta_{jk}+w_j\bar{w}_k) \\ & -(\nu(1+\sum_{l=1}^{n} w_l\bar{w}_l))^3\cdot(2!\delta_{jk}+3!w_j\bar{w}_k) \\ & +(\nu(1+\sum_{l=1}^{n} w_l\bar{w}_l))^5(4!\delta_{jk}+5!w_j\bar{w}_k)-\cdots.\end{aligned}$$

4.3. Local trivialization on $\mathfrak{a}_{U_{n+1}}[[\nu]]$.

As seen in 4.2, it seems not so simple to write the commutation relations for $\{\nu, w_1, \cdots, w_n, \bar{w}_1, \cdots, \bar{w}_n\}$. By a change of generators, we can give a structure on $\mathfrak{a}_{U_{n+1}}[[\nu]]$ simpler than (4.5). However, we have to use a non-holomorphic transformation here.

Let $H = \frac{1}{\sqrt{1+\sum w_l\cdot\bar{w}_l}} \in \mathfrak{a}_{U_{n+1}}[[\nu]]$, where the square root is given in the $\cdot$-product.

Lemma 4.4. *For any $j, k = 1, \cdots, n$,*

(4.6)
$$\begin{cases} [H\cdot w_j, H\cdot w_k] = [H\cdot\bar{w}_j, H\cdot\bar{w}_k] = 0 \qquad (\mathrm{mod}\ \nu^2), \\ [H\cdot w_j, H\cdot\bar{w}_k] = 2\sqrt{-1}\nu\delta_{ik} \qquad (\mathrm{mod}\ \nu^2). \end{cases}$$

Proof. By the product formula (1.9),

$$H\cdot w_j = H * w_j \quad (\mathrm{mod}\,\nu) \quad (\nu = \frac{\tilde{\nu}}{r}).$$

Hence

$$[H\cdot w_j, H\cdot\bar{w}_k] = [H * w_j, H * \bar{w}_k] \quad (\mathrm{mod}\,\nu^2), etc.$$

Thus

$$[H\cdot w_j, H\cdot\bar{w}_k] = H^2[w_j,\bar{w}_k] + H\cdot[w_j, H]\bar{w}_k + [H,\bar{w}_k]\cdot H\cdot w_j \quad (\mathrm{mod}\,\nu^2).$$

By these equalities and (1.11), we obtain the formulas (4.6). Q.E.D.

Setting

$$\xi''_j = \frac{1}{2}(H \cdot w_j + H \cdot \bar{w}_j), \quad \eta''_j = \frac{1}{2\sqrt{-1}}(H \cdot w_j - H \cdot \bar{w}_j) \quad (1 \leq j \leq n),$$

and using the last lemma yields

$$\text{(4.7)} \qquad \begin{cases} [\xi''_j, \xi''_k] & = [\eta''_j, \eta''_k] = 0 \pmod{\nu^2} \\ [\xi''_j, \eta''_k] & = -\nu\delta_{jk} \pmod{\nu^2}. \end{cases}$$

In particular, $\{\xi''_j, \xi''_k\}= \{\eta''_j, \eta''_k\}=0$, and $\{\xi''_j, \eta''_k\}=-\delta_{jk}$. The following theorem may be called a quantized Darboux theorem:

Theorem 4.5. *There exist $\xi_1, \cdots, \xi_n, \eta_1, \cdots, \eta_n \in \mathfrak{a}_{U_{n+1}}[[\nu]]$ such that*

$$[\xi_i, \xi_j] = [\eta_i, \eta_j] = 0$$
$$[\xi_i, \eta_j] = -\nu\delta_{ij}, \quad \textit{where } \nu = \frac{\tilde{\nu}}{r}.$$

Proof. (cf. [11], 3.4 Lemma) Set

$$[\xi''_i, \xi''_j] = \nu^2 a^{(2)}_{ij} + \nu^3 a^{(3)}_{ij} + \cdots,$$
$$[\eta''_i, \eta''_j] = \nu^2 a^{(2)}_{n+i,n+j} + \nu^3 a^{(3)}_{n+i,n+j} + \cdots,$$
$$[\xi''_i, \eta''_j] = -\nu\delta_{ij} + \nu^2 a^{(2)}_{i,n+j} + \cdots.$$

By the Jacobi identity, we have

$$\text{(4.8)} \qquad \sum_{(i,j,k):cyclic} \{\zeta_i, a^{(2)}_{jk}\} = 0 \quad (1 \leq i, j, k \leq 2n),$$

where $(\zeta_1, \cdots, \zeta_{2n}) =(\xi''_1, \cdots, \xi''_n,\eta''_1, \cdots, \eta''_n)$. Define a 2-form ω' on ϕU_{n+1} as

$$\omega' = \frac{1}{2} \sum_{1\leq i,j\leq n} (a^{(2)}_{n+i,n+j} dx_i \wedge dx_j - 2a^{(2)}_{n+i,j} dx_i \wedge dy_j + a^{(2)}_{ij} dy_i \wedge dy_j),$$

where $\xi''_i = x_i + O(\nu)$, $\eta''_i = y_i + O(\nu)$ and $x_1, \cdots, x_n, y_1, \cdots, y_n$ is a symplectic coordinate system on $\phi_{n+1}(U_{n+1})$. Then (4.8) implies $d\omega'=0$. Since $\phi_{n+1}(U_{n+1})= \mathbf{C}^n$ is 2-connected, there exists $\theta' = \sum_{s=1}^n (b_s dx_s + b_{n+s} dy_s)$ such that $\omega'=d\theta'$.

Consider

$$\begin{cases} \xi'_i &= \xi''_i + \nu b_{n+i} \\ \eta'_i &= \eta''_i - \nu b_i. \end{cases}$$

Replacing $(\xi''_i, \cdots, \xi''_n, \eta''_1, \cdots, \eta''_n)$ by $(\xi'_1, \cdots, \xi'_n, \eta'_1, \cdots, \eta'_n)$, we see that

$$\begin{cases} [\xi'_i, \xi'_j] &= [\eta'_i, \eta'_j] = 0 \mod \nu^3 \\ [\xi'_i, \eta'_j] &= -\nu\delta_{ij} \mod \nu^3. \end{cases}$$

Repeating this procedure for ν^3,ν^4, $\cdots$ finishes the proof. Q.E.D.

Note that $(w_1, \cdots, w_n)$ in 4.3 is a complex local coordinate system of $P_n(\mathbf{C})$ and hence $(\xi''_1, \cdots, \xi''_n, \eta''_1, \cdots, \eta''_n)$ is a real local coordinate system of $P_n(\mathbf{C})$. Since $\xi_i = \xi''_i$, $\eta_i = \eta''_i$ mod ν in the above proof, Theorem 4.5 implies also Theorem 2.4.

Using $\nu,\xi_1,\cdots,\xi_n$ $\eta_1,\cdots,\eta_n$ obtained above, we may define the $\odot$-product on $\mathfrak{a}_{U_{n+1}}[[\nu]]$ by the same manner as in 2.2. Let $B_{\xi,\eta}$ be the closure of the space of all polynomials of the form $\sum a_{\alpha\beta}\xi^\alpha \odot \eta^\beta$, $a_{\alpha\beta} \in \mathbf{R}$. It is a $\odot$-subalgebra over $\mathbf{R}$ of $(\mathfrak{a}_{U_{n+1}}[[\nu]], \odot)$, and $(B_{\xi,\eta}, \odot)$ is isomorphic to the algebra $(C^\infty(U_{n+1}; \mathbf{R}), \cdot)$. Via this isomorphism, we can regard $\xi_1,\cdots,\xi_n$, $\eta_1,\cdots,\eta_n$ as coordinate functions on U_{n+1}.

Since $\phi_{n+1}(U_{n+1})=\mathbf{C}^n$, we have

Corollary 4.6. $(\mathfrak{a}_{U_{n+1}}[[\nu]], *) \cong \mathcal{F}(W_{\mathbf{C}^n})$

Since U_{n+1} can be replaced by any U_l, this result shows that $\mathfrak{a}[[\nu]]$ is obtained by patching $\mathcal{F}(W_{\mathbf{C}^n})$'s, and hence $\mathfrak{a}[[\nu]]$ can be regarded as the space of certain sections of a Weyl algebra bundle $W_{P_n(\mathbf{C})}$ over $P_n(\mathbf{C})$. The coordinate transformations are given by isomorphisms

$$\Psi_{k,l} : \mathcal{F}(W_{\mathbf{C}^n - \{k\}}) \longrightarrow \mathcal{F}(W_{\mathbf{C}^n - \{l\}})$$

with $\Psi_{k,l}(\nu)=\nu$, where $\mathbf{C}^n - \{k\}=\mathbf{C}^n - \{\xi_k = 0\}$.

Remark 1. The $\odot$-product defined on $\mathfrak{a}_{U_{n+1}}$ may *not equal* the usual $\cdot$-product.

Remark 2. By Lemma 3.2 of [10], $\Psi_{k,l}$ are given as the pull back of pre-Weyl diffeomorphisms $\Phi_{k,l}$: $W_{\mathbf{C}^n - \{l\}} \longrightarrow W_{\mathbf{C}^n - \{k\}}$, where $W_{\mathbf{C}^n - \{k\}}$

$=(\mathbf{C}^n - \{k\}) \times \mathbf{W}$. Thus, strictly speaking, we should call the obtained Weyl algebra bundle $W_{P_n(\mathbf{C})}$ a pre-Weyl manifold.

It is, however, possible to correct $W_{P_n(\mathbf{C})}$ to a genuine Weyl manifold defined in [10] by the same procedure discussed in [10, §5]. This proves Theorem 2.5.

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Some Remarks on Fields of 2-Planes on Compact Smooth 4-Manifolds

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§1. Introduction

Throughout this paper by a manifold we mean a compact oriented smooth manifold, and by a field of q-planes on a manifold a nonsingular field of oriented tangent q-planes on it.

It is interesting to observe, as a specific feature in four dimension, that same are the conditions for a 4-manifold to admit the following three different structures:

(A) a field of 2-planes

(B) a pseudo-riemannian metric of signature $(++--)$ with the structure group $SO_o(2,2)$

(C) a pair of an almost complex structure and an opposite almost complex structure.

On the basis of such an observation, the purpose of this paper is to discuss some particular aspects of geometry of 4-manifolds which admit fields of 2-planes.

The paper contains two main results: Theorems 4-1 and 7-2, and is organized as follows. §2 is a quick survay on the problem of fields of 2-planes on 4-manifolds. In §3, we consider the reduction of the structure group of the tangent bundle of a 4-manifold which admits a field of 2-planes in connection with twistor spaces. One part of a couple of the main results is given in §4, which is concerned with the existence of a riemannian metric invariant both by two kinds of almost complex structures on a 4-manifold with a field of 2-planes. In §5, we review the irreducible decomposition of the curvature tensors on an almost Hermitian 4-manifold. In §6, we give an analogue of the irreducible decomposition of the curvature tensors for an opposite almost Hermitian 4-manifold. In the last section (§7), the other part of our main results is stated, which is concerned with the irreducible decomposition of the curvature

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tensors for an almost Hermitian 4-manifold with two kinds of almost complex structures. We shall see there that such a curvature tensor can be decomposed into eleven irreducible factors under the action of the structure group $G = SO(2) \times SO(2) \cong U(1) \times U(1)$.

§2. General aspects

We shall give in this section a brief survay of the problem of fields of 2-planes on 4-manifolds. The detailed arguments and results here should be referred to the author's papers [M1], [M2].

It follows from Steenrod's theorem [S, 40.11] that an n-manifold admits a pseudo-riemannian metric of signature $(n-q, q)$ with the structure group $SO_o(n-q, q)$ if and only if the manifold carries a field of q-planes (i.e., the structure group G is reduced to $SO(n-q) \times SO(q)$). The correspondence between (A) and (B) is therefore a special case in dimension $n = 4$ with $q = 2$, i.e., G is reduced to $SO(2) \times SO(2)$. However, the correspondence between (A) and (C) (or (B) and (C)) is a particular feature in four dimension.

Let M be a 4-manifold, and μ_M be the intersection form on $H^2(M, \mathbb{Z})/\operatorname{Tor}$. The condition for M to admit a field of 2-planes has been established by Hirzebruch and Hopf [HH, 4.5] as follows: M admits a field of 2-planes if and only if the Hirzebruch index $\tau[M]$ (or the signature) of M and the Euler characteristic $\chi[M]$ of M satisfy a pair of conditions

$$3\tau[M] + 2\chi[M] \in \Omega(M) \tag{2-1}$$

$$3\tau[M] - 2\chi[M] \in \Omega(M), \tag{2-2}$$

where

$$\Omega(M) = \{\mu_M(w, w) \in \mathbb{Z} | w \text{ are arbitrary characteristic elements in } H^2(M, \mathbb{Z})/\operatorname{Tor}\}.$$

As Hirzebruch and Hopf also pointed out [HH, 4.6], the first condition (2-1) is equivalent to the condition (due to Wu [W, p.74]) for M to admit an almost complex structure. The second condition (2-2) has a similar meaning.

Proposition 2-1. *A 4-manifold M admits an almost complex structure whose preferred orientation is opposite to the orientation of M if and only if M satisfies the condition* (2-2).

Let $-M$ be M with the orientation reversed. We know that $\tau[-M] = -\tau[M]$, $\chi[-M] = \chi[M]$, and $\Omega(-M) = -\Omega(M)$ (since $\mu_{-M} = -\mu_M$).

Thus the condition (2-2) is written in the form

$$3\tau[-M] + 2\chi[-M] \in \Omega(-M), \tag{2-3}$$

which is just the first condition (2-1) for $-M$. This implies that $-M$ admits an almost complex structure if and only if (2-2) holds.

Definition 2-2. We call an almost complex structure on $-M$ an *opposite almost complex structure* on M.

We thus have a theorem concerning the correspondence between (A) and (C) as follows.

Theorem 2-3. *A 4-manifold M admits a field of 2-planes if and only if it admits a pair of an almost complex structure and an opposite almost complex structure.*

It is known (Atiyah [A, Theorem 3.1]) that if a $4k$-dimensional manifold admits a field of q-planes ($q \equiv 2 \mod 4$), then its Euler characteristic must be even and is congruent to the Hirzebruch index modulo 4. Thus the problem of fields of 2-planes on 4-manifolds is the lowest dimensional examples of Atiyah's theorem. A 4-manifold M with a field of 2-planes must therefore satisfy

$$\chi[M] \equiv 0 \mod 2, \qquad \chi[M] \equiv \tau[M] \mod 4, \tag{2-4}$$

which will be referred to as Atiyah's condition. In the author's earlier paper [M1], it is shown on the basis of the work of Hirzebruch and Hopf that Aitiyah's condition is sufficient for a simply-connected 4-manifold to admit a field of 2-planes, and hence also the existence condition of (opposite) almost complex structures on such a 4-manifold is established. Recently, Saeki (see [M2, Theorem 2]) obtained a refined and full general version of the theorem of Hirzebruch and Hopf. In fact, he showed that if a 4-manifold has an indefinite intersection form, then Atiyah's condition is also sufficient for the 4-manifold to admit a field of 2-planes, and moreover that if a 4-manifold M has a definite intersection form, then it admits a field of 2-planes if and only if M satisfies Atiyah's condition and

$$\chi[M] + |\tau[M]| \geq 0. \tag{2-5}$$

It should be noted that he also established the condition for an arbitrary 4-manifold to admit (opposite) almost complex structures [M2, Theorems 8 and 10].

We now give a brief discussion on Chern classes. Suppose now that a 4-manifold M admits a field of 2-planes. Then M also admits a pair of an almost complex structure, denoted by J, and an opposite almost complex structure, denoted by J'. The first Chern class $c_1(J) \in H^2(M, \mathbb{Z})$ determined by J satisfies two conditions ([W], [BPV, Proposition 7.3], [DK, 1.1.7]):

$$c_1(J) \equiv w_2(M) \mod 2 \tag{2-6a}$$

$$c_1^2(J)[M] = 3\tau[M] + 2\chi[M], \tag{2-6b}$$

which correspond to (2-1). Similarly, the opposite almost complex structure J' defines a first Chern class $c_1(J')$ which satisfies the conditions

$$c_1(J') \equiv w_2(M) \mod 2 \tag{2-7a}$$

$$c_1^2(J')[-M] = -3\tau[M] + 2\chi[M], \tag{2-7b}$$

which correspond to (2-2).

The Chern numbers $c_2(J)[M]$ and $c_2(J')[-M]$ coincides with each other and also with the Euler characteristic of the manifold:

$$c_2(J)[M] = c_2(J')[-M] = \chi[M]. \tag{2-8}$$

It is worthwhile to note some relations

$$c_1^2(J')[-M] = 4c_2(J)[M] - c_1^2(J)[M] \tag{2-9a}$$

$$c_1^2(J')[-M] + c_2(J')[-M] \equiv 0 \mod 12. \tag{2-9b}$$

The second formula is an analogue of a fundamental relation $c_1^2(J)[M] + c_2(J)[M] \equiv 0 \mod 12$.

There are many examples of 4-manifolds which admit fields of 2-planes. We now restrict, however, our attention to the underlying real 4-manifolds of compact complex surfaces. For such a 4-manifold, it turns out from the above arguments that the pair of conditions (2-1) and (2-2) can be stated in terms of its Chern numbers. In fact, the underlying real 4-manifold of a surface admits a field of 2-planes if and only if $c_1^2(J)[M] - 5c_2(J)[M] \equiv 0 \mod 12$ ($\Leftrightarrow$ (2-7b) $\Leftrightarrow$ (2-9b)).

Therefore, the underlying real 4-manifolds of the following (minimal) surfaces [BPV, VI Table 10] admit fields of 2-planes: minimal rational surfaces with $(c_1^2, c_2) = (8, 4)$, Hopf surfaces and Inoue surfaces in the Class 2) of minimal surfaces of class VII, ruled surfaces of genus $g \geq 1$, Enriques surfaces, hyperelliptic surfaces, Kodaira surfaces, K3 surfaces, tori, minimal properly elliptic surfaces, and the surfaces of general type which satisfy (2-7b).

§3. Grassmann bundles and twistor spaces

Let M be a 4-manifold. We assume that the structure group G of the tangent bundle of M is already reduced to $SO(4)$, or equivalently there exits a riemmanian metric on M. Let S^3 (the 3-sphere) denote $SU(2)$, the 2-dimensional special unitary group. Then we have an exact sequence

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow S^3 \times S^3 \xrightarrow{\alpha} SO(4) \longrightarrow 1, \tag{3-1}$$

where α is a covering map. The product $S^3 \times S^3$ is the spin group of $SO(4)$. If we denote by S^1 a maximal torus of S^3, then $\alpha(S^1 \times S^3)$ is the 2-dimensional unitary group $U(2)$. If we take a maximal torus S^1 in the second component S^3 of the spin group $S^3 \times S^3$, then the subgroup $\alpha(S^3 \times S^1)$ is also a 2-dimensional unitary group, and in this case, is denoted by $U'(2)$. The quotient space

$$SO(4)/U(2) = \alpha(S^3 \times S^3)/\alpha(S^1 \times S^3) \cong S^2 \tag{3-2}$$

is the space of almost complex structures on TM_x at each point $x \in M$, which are orthogonal with respect to the riemannian metric. Similarly the quotient space

$$SO(4)/U'(2) = \alpha(S^3 \times S^3)/\alpha(S^3 \times S^1) \cong S^2 \tag{3-3}$$

is the space of opposite almost complex structures on TM_x at each point $x \in M$, which are orthogonal with respect to the riemannian metric.

It is known that the space of oriented tangent 2-planes in TM_x at each point $x \in M$ is the Grassmann manifold

$$G_2(\mathbb{R}^4) \cong SO(4)/SO(2) \times SO(2) \cong S^2 \times S^2. \tag{3-4}$$

In [HH], Hirzebruch and Hopf obeserved that the first component S^2 of the above product $S^2 \times S^2$ can be identified with the quotient space $SO(4)/U(2)$, and similarly the second component S^2 corresponds to the quotient space $S(4)/U'(2)$. Thus,

$$G_2(\mathbb{R}^4) \cong SO(4)/U(2) \times SO(4)/U'(2). \tag{3-5}$$

Therefore, at each point $x \in M$ the oriented tangent 2-planes are in one-to-one correspondence with the pairs (J_x, J'_x) of almost complex structures and an opposite almost complex structures.

The bundle over M with the space of such almost complex structures as fibre is the $SO(4)/U(2)$-bundle over M, and its total space is known as the *twistor space* of M, denoted by $Z^+(M)$ (see Atiyah, Hitchin and Singer [AHS]). Similarly, the bundle over M with the space of opposite almost complex structures as fibre is the $SO(4)/U'(2)$-bundle over M, and its total space is denoted by $Z^-(M)$. Thus the Grassmann manifold bundle $G_2(M)$ over M with $G_2(\mathbb{R}^4)$ as fibre can be written as follows:

$$G_2(M) = Z^+(M) \times_M Z^-(M). \tag{3-6}$$

With this identification, we see that the sections of $G_2(M)$ (fields of 2-planes) are in one-to-one correspondence with the pairs of sections of $Z^+(M)$ and $Z^-(M)$. The conditions (2-1) and (2-2) thus can be interpreted as the conditions for the $SO(4)/U(2)$-bundle $Z^+(M)$ and $SO(4)/U'(2)$-bundle $Z^-(M)$ both to admit sections.

If M admits a field of 2-planes, then the the structure group G is reduced from $SO(4)$ to

$$\begin{aligned} \alpha(S^1 \times S^3) \cap \alpha(S^3 \times S^1) &= \alpha(S^1 \times S^1) \\ &= SO(2) \times SO(2) \cong U(1) \times U(1), \end{aligned} \tag{3-7}$$

and moreover the tangent bundle of M admits also a reduction of G to $SO_o(2,2)$ whose maximal compact subgroup is $SO(2) \times SO(2)$, or M carries a pseudo-riemannian metric of signature $(+ + - -)$ with the structure group $SO_o(2,2)$. Thus we see again the coincidence (A) $\Leftrightarrow$ (B) $\Leftrightarrow$ (C) from a group theoretical point of view.

§4. Two kinds of almost complex structures

Let M be a 4-manifold which admits a field of 2-planes. As we have seen, M admits a pair (J,J') of an almost complex structure and an opposite almost complex structure.

Consider M to be a riemannian 4-manifold (M,g_o), and choose a pair (J,J') of two kinds of almost complex structures on M, where g_o and (J,J') are arbitrarily chosen. Then we have a riemannian 4-manifold (M,g_o), together with a pair (J,J'), where J, J' have at this stage no relation to the metric g_o. It is well-known, however, that for any almost complex structure J on (M,g_o), we can construct a J-invariant metric g_1 on M as follows:

$$g_1(X,Y) = g_o(X,Y) + g_o(JX,JY), \qquad \text{for } X,Y \in \mathfrak{X}(M), \tag{4-1}$$

where $\mathfrak{X}(M)$ is the algebra of C^∞ vector fields on M. Similary, there is a J'-invariant metric g_2 on M defined by

$$g_2(X,Y) = g_o(X,Y) + g_o(J'X,J'Y), \qquad \text{for } X,Y \in \mathfrak{X}(M). \tag{4-2}$$

Here by J-invariance (resp. J'-invariance), we mean that $g_1(JX, JY) = g_1(X, Y)$ (resp. $g_2(J'X, J'Y) = g_2(X, Y)$). We thus obtain two kinds of almost Hermitian 4-manifolds (M, g_1, J) and (M, g_2, J'). Such a metric g_1 (resp. g_2) is not in general invariant by J' (resp. J).

For such a 4-manifold with two kinds of almost complex structures, we are led to consider a question: *Does M admit a riemannian metric which is invariant by both almost complex structures J and J'?* We have one of the main results as follows.

Theorem 4-1. *Let M be a 4-manifold which admits a field of 2-planes. Associated with each field τ of 2-planes on M, there exits on M a pair of an almost complex structure J_τ and an opposite almost complex structure J'_τ, together with an invariant riemannian metric g such that*
(i) *J_τ and J'_τ commute with each other,*
(ii) *g is invariant by both J_τ and J'_τ.*

We shall prove this theorem by constructing such two kinds of almost complex structures.

Let τ be a field of 2-planes on M, and g_o a riemannian metric on M, both of which are arbitrarily chosen. Associated with the field τ of 2-planes, we can choose a local orthonormal frame $\{e_1,\ e_2,\ e_3,\ e_4\}$ at each point x of M with respect to g_o such that

$$e_1,\ e_2 \in \tau_x, \qquad e_3,\ e_4 \in \nu_x, \tag{4-3}$$

where ν is the 2-dimensional subbundle of the tangent bundle consisting of normal vectors to τ. Relative to such a frame, we can always construct locally a couple of rank 2 tensor fields J_τ and J'_τ of type (1,1) at each point $x \in M$ as follows:

$$(J_\tau)_x = e^1 \otimes e_2 - e^2 \otimes e_1 + e^3 \otimes e_4 - e^4 \otimes e_3, \tag{4-4a}$$

$$(J'_\tau)_x = e^1 \otimes e_2 - e^2 \otimes e_1 - e^3 \otimes e_4 + e^4 \otimes e_3, \tag{4-4b}$$

where $\{e^1,\ e^2,\ e^3,\ e^4\}$ is the local dual orthonormal basis of 1-forms such that $e^i(e_j) = \delta_{ij}$. Note that

(4-5a)
$$(J_\tau)_x(e_1) = e_2,\ \ (J_\tau)_x(e_2) = -e_1,\ \ (J_\tau)_x(e_3) = e_4,\ \ (J_\tau)_x(e_4) = -e_3,$$
(4-5b)
$$(J'_\tau)_x(e_1) = e_2,\ \ (J'_\tau)_x(e_2) = -e_1,\ \ (J'_\tau)_x(e_3) = -e_4,\ \ (J'_\tau)_x(e_4) = e_3.$$

Lemma 4-2. *J_τ and J'_τ are globally defined nonsingular tensor fields of type $(1,1)$ on M.*

Proof. If we choose another orthonormal frame $\{\hat{e}_1, \hat{e}_2, \hat{e}_3, \hat{e}_4\}$ with the same property (4-3) ($\hat{e}_1, \hat{e}_2 \in \tau_x$, $\hat{e}_3, \hat{e}_4 \in \nu_x$), then it is locally related to the frame $\{e_1, e_2, e_3, e_4\}$ by

$$\text{(4-6)} \qquad \begin{bmatrix} \hat{e}_1 \\ \hat{e}_2 \\ \hat{e}_3 \\ \hat{e}_4 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\phi & -\sin\phi \\ 0 & 0 & \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{bmatrix}$$

for some $\theta, \phi \in \mathbb{R}$, since the structure group G of the tangent bundle is reduced to $SO(2) \times SO(2)$.

Associated with the new frame $\{\hat{e}_1, \hat{e}_2, \hat{e}_3, \hat{e}_4\}$, we take the tensors $(\hat{J}_\tau)_x$ and $(\hat{J}'_\tau)_x$ as defined by the similar forms to $(J_\tau)_x$ and $(J'_\tau)_x$ as follows

$$\text{(4-7a)} \qquad (\hat{J}_\tau)_x = \hat{e}^1 \otimes \hat{e}_2 - \hat{e}^2 \otimes \hat{e}_1 + \hat{e}^3 \otimes \hat{e}_4 - \hat{e}^4 \otimes \hat{e}_3,$$

$$\text{(4-7b)} \qquad (\hat{J}'_\tau)_x = \hat{e}^1 \otimes \hat{e}_2 - \hat{e}^2 \otimes \hat{e}_1 - \hat{e}^3 \otimes \hat{e}_4 + \hat{e}^4 \otimes \hat{e}_3.$$

Using the transformation law (4-6) of the frames, we can easily verify the following coincidences in the overlap region where both frames are defined:

$$\text{(4-8)} \qquad (\hat{J}_\tau)_x = (J_\tau)_x, \qquad (\hat{J}'_\tau)_x = (J'_\tau)_x.$$

This implies that J_τ and J'_τ are both globally defined nonsingular tensor fields of type (1,1) due to the existence of the field τ of 2-planes.

Q.E.D.

It is clear that as the endomorphisms of the tangent bundle, J_τ and J'_τ have the property

$$\text{(4-9)} \qquad J_\tau^2 = -1, \qquad {J'}_\tau^2 = -1.$$

Lemma 4-3. $J_\tau J'_\tau = J'_\tau J_\tau$.

Proof. Such a commutativity is easily verified from (4-5a) and (4-5b). Q.E.D.

Proof of Theorem 4-1. In the same spirit of constructing invariant metrics g_1 and g_2, we define a new metric g as follows: for $X, Y \in \mathfrak{X}(M)$

$$\text{(4-10)} \qquad \begin{aligned} g(X,Y) = g_o(X,Y) &+ g_o(J_\tau X, J_\tau Y) \\ &+ g_o(J'_\tau X, J'_\tau Y) + g_o(J_\tau J'_\tau X, J_\tau J'_\tau Y). \end{aligned}$$

It is easy to see that g is invariant by both J_τ and J'_τ, since J_τ and J'_τ cummute with each other. Such a metric g is the desired one. Q.E.D.

We thus obtain a quadruple (M, g, J_τ, J'_τ), which will be the main concern of the remainder of the paper.

§5. Almost Hermitian 4-manifolds $(M,\ g,\ J_\tau)$

The contents of this section are all known facts, but they are important and are the prototypes of our results treated in the subsequent two sections.

An almost Hermitian 4-manifold $(M,\ g,\ J_\tau,\ J'_\tau)$ with two kinds of almost complex structures clearly carries two kinds of Kähler forms F_τ and F'_τ, defined as follows: for $X,\ Y \in \mathfrak{X}(M)$

$$F_\tau(X,Y) = g(J_\tau X, Y), \qquad F'_\tau(X,Y) = g(J'_\tau X, Y). \tag{5-1}$$

As stated before, the tangent bundle of M admits a reduction of the structure group G to $U(1) \times U(1)$.

If we forget two kinds of almost complex structures J_τ, J'_τ from the quadruple $(M,\ g,\ J_\tau,\ J'_\tau)$, then the bundle of 2-forms over the riemannian 4-manifold $(M,\ g)$ with the structure group $G = SO(4)$ splits into two subbundles

$$\Lambda^2(M) = \Lambda_+ \oplus \Lambda_-, \tag{5-2}$$

where $\Lambda_\pm$ are ± 1 eigenspaces of the Hodge star operator.

Although M admits two kinds of almost complex structures J_τ and J'_τ, we forget J'_τ for a moment and cosider the triple $(M,\ g,\ J_\tau)$, an almost Hermitian 4-manifold. The structure group G is the unitary group $U(2)$ $(= \alpha(S^1 \times S^3))$. It is well-known that under the action of $U(2)$, the bundle $\Lambda^2(M)$ of 2-forms over the triple $(M,\ g,\ J_\tau)$ splits further as follows:

$$\Lambda^2(M) = \mathbb{R}F_\tau \oplus L_\tau \oplus \Lambda_-, \tag{5-3}$$

where $\mathbb{R}F_\tau$ is the 1-dimensional subbundle spanned by the Kähler form F_τ, L_τ the 2-dimensional subbundle of J_τ-skew invariant 2-forms

$$L_\tau = \{\Phi \in \Lambda^2(M) |\ J_\tau \cdot \Phi = -\Phi\}. \tag{5-4}$$

Moreover, it should be noted that the sum $\mathbb{R}F_\tau \oplus \Lambda_-$ consists of J_τ-invariant 2-forms, and that the sum $\mathbb{R}F_\tau \oplus L_\tau$ coincides with Λ_+ .

Associated with the splitting (5-3), the bundle $\mathcal{R}(M)$ of the curvature tensors over $(M,\ g,\ J_\tau)$ splits into a direct sum of seven irreducible factors [TV, Theorem 14.3]:

$$\begin{aligned}\mathcal{R}(M) = \mathcal{L}(\pi_1) \oplus \mathcal{L}(\pi_1 - \pi_2(J_\tau)) \oplus \mathcal{W}_7(J_\tau) \oplus \mathcal{W}_9(J_\tau) \\ \oplus \mathcal{R}_W^- \oplus \mathcal{W}_2(J_\tau) \oplus \mathcal{W}_8(J_\tau),\end{aligned} \tag{5-5}$$

where with the aid of notation: $\pi_2(J_\tau) = -2* - 3J_\tau + 2$, we have put

$$\begin{aligned}
\mathcal{L}(\pi_1-\pi_2(J_\tau)) &= \ \text{span}\{2*+3J_\tau - 1\} \\
\mathcal{W}_7(J_\tau) &= \{R \in \mathcal{R}(M)|\ R* = R\} \cap \mathcal{L}^\perp(\pi_1 - \pi_2(J_\tau)) \\
\mathcal{W}_9(J_\tau) &= \{R \in \mathcal{R}(M)|\ R* = R,\ \ RJ_\tau = -J_\tau R\} \\
\mathcal{R}_W^- &= \{R \in \mathcal{R}(M)|\ R* = -R\} \\
\mathcal{W}_2(J_\tau) &= \{R \in \mathcal{R}(M)|\ R* = -*R,\ \ RJ_\tau = R\} \\
\mathcal{W}_8(J_\tau) &= \{R \in \mathcal{R}(M)|\ R* = -*R,\ \ RJ_\tau = -J_\tau R\}.
\end{aligned}$$

Further, we have

$$\dim \mathcal{L}(\pi_1) = \dim \mathcal{L}(\pi_1 - \pi_2(J_\tau)) = 1, \quad \dim \mathcal{W}_7(J_\tau) = \mathcal{W}_9(J_\tau) = 2,$$
$$\dim \mathcal{R}_W^- = 5, \quad \dim \mathcal{W}_2(J_\tau) = 3, \quad \dim \mathcal{W}_8(J_\tau) = 6.$$

Moreover, the sum $\mathcal{L}(\pi_1 - \pi_2(J_\tau)) \oplus \mathcal{W}_7(J_\tau) \oplus \mathcal{W}_9(J_\tau)$ coincides with the bundle $\mathcal{R}_W^+$ of self-dual Weyl curvature tensors, and the sum $\mathcal{W}_2(J_\tau) \oplus \mathcal{W}_8(J_\tau)$ is the bundle $\mathcal{R}_o$ of traceless Ricci tensors. Thus, the existence of J_τ induces the splitting $\mathcal{R}_W^+$ into three factors and $\mathcal{R}_o$ into two factors.

§6. Opposite almost Hermitian 4-manifolds (M, g, J'_τ)

In this section, we consider the irreducible decomposition of the curvature tensors of an opposite almost Hermitian 4-manifold (M, g, J'_τ), which is obtained by deleting J_τ from the quadruple (M, g, J_τ, J'_τ).

The structure group G is now the unitary group $U'(2)$ $(= \alpha(S^3 \times S^1))$. For the bundle of 2-forms over (M, g, J'_τ), we have a decomposition similar to (5-3) for (M, g, J_τ).

Proposition 6-1. *For an opposite almost Hermitian* 4*-manifold* (M, g, J'_τ), *the bundle* $\Lambda^2(M)$ *of* 2*-forms over* M *splits into a direct sum*

$$\Lambda^2(M) = \Lambda_+ \oplus \mathbb{R}F'_\tau \oplus L'_\tau, \tag{6-1}$$

where $\mathbb{R}F'_\tau$ *is the* 1*-dimensional subbundle spanned by the Kähler form* F'_τ, L'_τ *the* 2*-dimensional subbundle of* J'_τ*-skew invariant* 2*-forms*

$$L'_\tau = \{\Phi \in \Lambda^2(M)|\ J'_\tau \cdot \Phi = -\Phi\}. \tag{6-2}$$

Moreover, the sum $\mathbb{R}F'_\tau \oplus \Lambda_+$ *consistes of* J'_τ*-invariant* 2*-forms, and the sum* $\mathbb{R}F'_\tau \oplus L'_\tau$ *coincides with* Λ_-.

Associated with the above splitting of $\Lambda^2(M)$, we have the following (cf. [TV, Theorem 14.3]).

Proposition 6-2. *For an opposite almost Hermitian 4-manifold $(M,\, g,\, J'_\tau)$, the bundle $\mathcal{R}(M)$ of the curvature tensors over M splits into a direct sum of seven irreducible factors under the action of the structure group $U'(2)$:*

$$\begin{aligned}\mathcal{R}(M) = \mathcal{L}(\pi_1) \oplus \mathcal{L}(\pi_1 - \pi_2(J'_\tau)) \oplus \mathcal{W}_7(J'_\tau) \oplus \mathcal{W}_9(J'_\tau)\\ \oplus \mathcal{R}_W^+ \oplus \mathcal{W}_2(J'_\tau) \oplus \mathcal{W}_8(J'_\tau),\end{aligned} \tag{6-3}$$

where

$$\begin{aligned}
\mathcal{L}(\pi_1 - \pi_2(J'_\tau)) &= \operatorname{span}\{2 * + 3J'_\tau - 1\},\\
\mathcal{W}_7(J'_\tau) &= \{R \in \mathcal{R}(M) |\ R* = -R\} \cap \mathcal{L}^\perp(\pi_1 - \pi_2(J'_\tau))\\
\mathcal{W}_9(J'_\tau) &= \{R \in \mathcal{R}(M) |\ R* = -R,\ RJ'_\tau = -J'_\tau R\}\\
\mathcal{R}_W^+ &= \{R \in \mathcal{R}(M) |\ R* = R\}\\
\mathcal{W}_2(J'_\tau) &= \{R \in \mathcal{R}(M) |\ R* = - * R,\ RJ'_\tau = R\}\\
\mathcal{W}_8(J'_\tau) &= \{R \in \mathcal{R}(M) |\ R* = - * R,\ RJ'_\tau = -J'_\tau R\}.
\end{aligned}$$

Further, we have

$$\begin{aligned}\dim \mathcal{L}(\pi_1) = \dim \mathcal{L}(\pi_1 - \pi_2(J'_\tau)) = 1, \quad \dim \mathcal{W}_7(J'_\tau) = \mathcal{W}_9(J'_\tau) = 2,\\ \dim \mathcal{R}_W^- = 5, \quad \dim \mathcal{W}_2(J'_\tau) = 3, \quad \dim \mathcal{W}_8(J'_\tau) = 6.\end{aligned}$$

Moreover, the sum $\mathcal{L}(\pi_1 - \pi_2(J'_\tau)) \oplus \mathcal{W}_7(J'_\tau) \oplus \mathcal{W}_9(J'_\tau)$ coincides with the bundle $\mathcal{R}_W^-$ of anti-self-dual Weyl curvature tensors, and the sum $\mathcal{W}_2(J'_\tau) \oplus \mathcal{W}_8(J'_\tau)$ is the bundle $\mathcal{R}_o$ of traceless Ricci tensors.

Thus, the existence of J'_τ induces the splitting $\mathcal{R}_W^-$ into three factors and $\mathcal{R}_o$ into two factors.

§7. Almost Hermitian 4-manifolds $(M,\, g,\, J_\tau,\, J'_\tau)$ with two kinds of almost complex structures

In this section we shall state the second part of our main results concerning the irreducible decomposition of the curvature tensor on an almost Hermitian 4-manifold $(M,\, g,\, J_\tau,\, J'_\tau)$ with two kinds of almost complex structures.

Now the structure group G is reduced to $U(1) \times U(1)$. Such a reduction induces a further splitting of the bundle $\Lambda^2(M)$.

Proposition 7-1. *The bundle $\Lambda^2(M)$ of 2-forms over an almost Hermitian 4-manifold $(M,\, g,\, J_\tau,\, J'_\tau)$ with two kinds of almost complex* *structures splits into a direct sum of four subbundles under the action of* $G = U(1) \times U(1)$ *as follows*

$$\Lambda^2(M) = \mathbb{R}F_\tau \oplus L_\tau \oplus \mathbb{R}F'_\tau \oplus L'_\tau, \tag{7-1}$$

where $\mathbb{R}F_\tau$, $\mathbb{R}F'_\tau$ *are respectively the* 1*-dimensional subbundles spanned by the Kähler forms* F_τ, F'_τ, *and* L_τ, L'_τ *are the* 2*-dimensional subbundles of* J_τ, J'_τ *skew invariant* 2*-forms:*

$$L_\tau = \{\Phi \in \Lambda^2(M) |\ J_\tau \cdot \Phi = -\Phi\} \tag{7-2a}$$

$$L'_\tau = \{\Phi \in \Lambda^2(M) |\ J'_\tau \cdot \Phi = -\Phi\}. \tag{7-2b}$$

Proof. Since $U(1) \times U(1) \subset \alpha(S^1 \times S^3)$, the action of the structure group induces a splitting: $\Lambda^2_+ = \mathbb{R}F_\tau \oplus L_\tau$. Similarly, $U(1) \times U(1) \subset \alpha(S^3 \times S^1)$, and hence we have $\Lambda^2_- = \mathbb{R}F'_\tau \oplus L'_\tau$. Q.E.D.

Note that $\mathbb{R}F_\tau \oplus \mathbb{R}F'_\tau = \{\Phi \in \Lambda^2(M) | J_\tau \cdot \Phi = \Phi,\ J'_\tau \cdot \Phi = \Phi\}$.

At this stage, based on the splitting of $\Lambda^2(M)$ we can state the second part of our main results on the decomposition of $\mathcal{R}(M)$ into irreducible factors.

Theorem 7-2. *For an almost Hermitian* 4*-manifold* (M, g, J_τ, J'_τ) *with two kinds of almost complex structures, the bundle* $\mathcal{R}(M)$ *of the curvature tensors over* M *splits into a direct sum of eleven irreducible factors under the action of* $G = U(1) \times U(1)$*:*

$$\begin{aligned} \mathcal{R}(M) = \mathcal{L}(\pi_1) &\oplus \mathcal{L}(\pi_1 - \pi_2(J_\tau)) \oplus \mathcal{W}_7(J_\tau) \oplus \mathcal{W}_9(J_\tau) \\ &\oplus \mathcal{L}(\pi_1 - \pi_2(J'_\tau)) \oplus \mathcal{W}_7(J'_\tau) \oplus \mathcal{W}_9(J'_\tau) \\ &\oplus \mathcal{W}_A \oplus \mathcal{W}_B \oplus \mathcal{W}_C \oplus \mathcal{W}_D, \end{aligned} \tag{7-3}$$

where

$$\begin{aligned} \mathcal{W}_A &= \{R \in \mathcal{R}(M) |\ R* = -*R,\ RJ_\tau = R,\ RJ'_\tau = R\} \\ \mathcal{W}_B &= \{R \in \mathcal{R}(M) |\ R* = -*R,\ RJ_\tau = R,\ RJ'_\tau = -J'_\tau R\} \\ \mathcal{W}_C &= \{R \in \mathcal{R}(M) |\ R* = -*R,\ RJ_\tau = -J_\tau R,\ RJ'_\tau = R\} \\ \mathcal{W}_D &= \{R \in \mathcal{R}(M) |\ R* = -*R,\ RJ_\tau = -J_\tau R,\ RJ'_\tau = -J'_\tau R\}. \end{aligned}$$

Further, we have

$$\begin{gathered} \dim \mathcal{L}(\pi_1) = \dim \mathcal{L}(\pi_1 - \pi_2(J_\tau)) = \dim \mathcal{L}(\pi_1 - \pi_2(J'_\tau)) = \dim \mathcal{W}_A = 1, \\ \dim \mathcal{W}_7(J_\tau) = \dim \mathcal{W}_7(J'_\tau) = \dim \mathcal{W}_9(J_\tau) = \dim \mathcal{W}_9(J'_\tau) \\ = \dim \mathcal{W}_B = \dim \mathcal{W}_C = 2, \quad \dim \mathcal{W}_D = 4. \end{gathered}$$

Proof. We may assume that the bundle $\mathcal{R}(M)$ has been decomposed as (5-5) for the triple $(M,\ g,\ J_\tau)$. We shall show, due to the existence J'_τ, the following three: (i) the bundle $\mathcal{R}^-_W$ of anti-self-dual Weyl curvatures splits into a sum of three irreducible factors:

$$\mathcal{R}^-_W(M) = \mathcal{L}(\pi_1 - \pi_2(J'_\tau)) \oplus \mathcal{W}_7(J'_\tau) \oplus \mathcal{W}_9(J'_\tau),$$

(ii) $\mathcal{W}_2(J_\tau)$ into two irreducible factors:

$$\mathcal{W}_2(J_\tau) = \mathcal{W}_A \oplus \mathcal{W}_B,$$

and (iii) $\mathcal{W}_8(J_\tau)$ into two irreducible factors:

$$\mathcal{W}_8(J_\tau) = \mathcal{W}_C \oplus \mathcal{W}_D.$$

(Note that $\mathcal{W}_2(J'_\tau) = \mathcal{W}_A \oplus \mathcal{W}_C$, and $\mathcal{W}_8(J'_\tau) = \mathcal{W}_B \oplus \mathcal{W}_D$.)

Concerning (i), we must recognize first that $\mathcal{R}^-_W(M) \subset \mathrm{End}(\Lambda_-)$. From Proposition 6-1, $\mathrm{End}(\Lambda_-)$ splits as follows:

$$\begin{aligned}\mathrm{End}(\Lambda_-) &= \mathrm{End}(\mathbb{R}F'_\tau \oplus L'_\tau)\\ &= \mathrm{End}(\mathbb{R}F'_\tau) \oplus \mathrm{Hom}(\mathbb{R}F'_\tau, L'_\tau) \oplus \mathrm{Hom}(L'_\tau, \mathbb{R}F'_\tau) \oplus \mathrm{End}(L'_\tau).\end{aligned}$$

Thus, it is easy to see that $\mathcal{R}^-_W(M)$ consists of three factors:

$$\begin{aligned}\mathcal{L}(\pi_1 - \pi_2(J'_\tau)) &\subset \mathrm{End}(\mathbb{R}F'_\tau)\\ \mathcal{W}_7(J'_\tau) &\subset \mathrm{Hom}(\mathbb{R}F'_\tau, L'_\tau) \oplus \mathrm{Hom}(L'_\tau, \mathbb{R}F'_\tau)\\ \mathcal{W}_9(J'_\tau) &\subset \mathrm{End}(L'_\tau).\end{aligned}$$

For (ii), we know that $\mathcal{W}_2(J_\tau) \subset \mathrm{Hom}(\mathbb{R}F_\tau, \Lambda_-) \oplus \mathrm{Hom}(\Lambda_-, \mathbb{R}F_\tau)$. Since $\Lambda_- = \mathbb{R}F'_\tau \oplus L'_\tau$, we see that $\mathcal{W}_2(J_\tau)$ splits into two factors

$$\begin{aligned}\mathcal{W}_A &\subset \mathrm{Hom}(\mathbb{R}F_\tau, \mathbb{R}F'_\tau) \oplus \mathrm{Hom}(\mathbb{R}F'_\tau, \mathbb{R}F_\tau)\\ \mathcal{W}_B &\subset \mathrm{Hom}(\mathbb{R}F_\tau, L'_\tau) \oplus \mathrm{Hom}(L'_\tau, \mathbb{R}F_\tau).\end{aligned}$$

For the last case (iii), we know that $\mathcal{W}_8(J_\tau) \subset \mathrm{Hom}(L_\tau, \Lambda_-) \oplus \mathrm{Hom}(\Lambda_-, L_\tau)$. Due to the splitting $\Lambda_- = \mathbb{R}F'_\tau \oplus L'_\tau$, we see that $\mathcal{W}_8(J_\tau)$ splits into two factors:

$$\begin{aligned}\mathcal{W}_C &\subset \mathrm{Hom}(L_\tau, \mathbb{R}F'_\tau) \oplus \mathrm{Hom}(\mathbb{R}F'_\tau, L_\tau)\\ \mathcal{W}_D &\subset \mathrm{Hom}(L_\tau, L'_\tau) \oplus \mathrm{Hom}(L'_\tau, L_\tau).\end{aligned}$$

It is elementary to know (cf. [TV]) that the action of the structure group $U(1) \times U(1)$ is irreducible on each of these factors

$$\mathcal{L}(\pi_1 - \pi_2(J_\tau)),\ \ \mathcal{W}_7(J'_\tau),\ \ \mathcal{W}_9(J'_\tau),\ \ \mathcal{W}_A,\ \ \mathcal{W}_B,\ \ \mathcal{W}_C,\ \ \mathcal{W}_D.$$

Q.E.D.

We end this paper with some remarks.

Remarks. (A) Let M be a 4-manifold which carries a field τ of 2-planes and a riemannian metric g_o. Associated with τ, choose a local orthonormal frame $\{e_1, e_2, e_3, e_4\}$ at each point x of M with respect to g_o, which satisfies (4-3): $e_1, e_2 \in \tau_x$, $e_3, e_4 \in \nu_x$. In terms of the riemannian metric g_o and the globally defined field τ, we can construct a pseudo-riemannian metric h of signature $(+ + - -)$ as follows: for X, $Y \in \mathfrak{X}(M)$

$$\begin{aligned} h_o(X, Y) = g_o(X, Y) &- 2g_o(X, e_3) \cdot g_o(X, e_3) \\ &- 2g_o(X, e_4) \cdot g_o(X, e_4). \end{aligned}$$

Such a metric h_o does not depent on a particular frame with property (4-3), but only on the field τ of 2-planes, i.e., invariant by (4-6). In the above formula, if we take the (J_τ, J'_τ)-invariant metric g instead of g_o, then the following metric

$$\begin{aligned} h(X, Y) = g(X, Y) &- 2g(X, e_3) \cdot g(X, e_3) \\ &- 2g(X, e_4) \cdot g(X, e_4) \end{aligned}$$

is also (J_τ, J'_τ)-invariant, and therefore we obtain a quadruple (M, h, J_τ, J'_τ), a pseudo-riemannian version of the quadruple (M, g, J_τ, J'_τ). For such a pseudo-riemannian 4-manifold (M, h, J_τ, J'_τ) with the structure group $G = U(1) \times U(1)$, the bundle of pseudo-curvature tensors also splits into eleven irreducible factors in a similar way to that of the riemannian case in Theorem 7-2.

(B) It is highly expected that there may exist some intimate relations among the integrability conditions of fields of 2-planes (giving rise to 2-dimensional *foliations*), the integrability of two kinds of almost complex structures (giving rise to two kinds of *complex structures* with opposite orientations), and the *parallelizability* of the two kinds of almost complex structures on 4-manifolds with fields of 2-planes. We shall discuss this issue elsewhere.

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A Note on Lie Contact Manifolds

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Dedicated to Professor T. Otsuki on his 75th birthday

§1. Introduction

The classical projective and conformal connections of H.Weyl fit into harmonic theory in the Spencer cohomology of graded Lie algebras in the sense that the curvature forms of such connections are harmonic. These structures are treated systematically by N. Tanaka [T1] as special cases of a structure associated with an $\mathfrak{l}$-system, called later a graded Lie algebra of the first kind. A lucid explanation of this theory is given by T. Ochiai [O] where he rebuilds Tanaka theory using semisimple flat homogeneous spaces as model spaces.

To deal with more general structures such as CR-structure, Tanaka developed the theory to simple graded Lie algebras of contact type [T2] and then to the full class of simple graded Lie algebras [T3]. The argument essentially depends on the generalized prolongation scheme and on the harmonic theory in the refined Spencer cohomology of Lie algebras. The vanishing of certain cohomology group guarantees the existence and uniqueness of *the normal Cartan connection* (= *Tanaka connection*, for short), attached to the equivalence class of the structure. Though the curvature form of Tanaka connection is no more harmonic in general, its harmonic part gives a fundamental system of invariants of the structure.

Going back to the starting point, we know that the study of projective and conformal structures on a manifold has a background of the classical projective and conformal geometry. This reminds us of another classical geometry, Lie's sphere geometry. Then what kind of structure corresponds to this geometry? Why has this object not yet been investigated? H. Sato [S, SY] is probably the first to consider this problem and finds a Lie contact structure, which is a structure on a contact manifold with model space T_1S^n of which transformation group is the

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Lie transformation group $PO(n+1,2)$. Noting that the Lie algebra of $PO(n+1,2)$ is a simple graded Lie algebra of contact type, Sato claims that the developed Tanaka theory plays a main role on this structure. A typical and important example of Lie contact structure is found on the unit tangent bundle T_1M of a riemannian manifold M. Now, how can we express Tanaka connection on this structure? The author answered this question in [M2] and obtained a close relation between the Lie contact structure on T_1M and the conformal structure on M.

The purpose of this note is to give a survey of these results, and add some explanations. In particular, Theorem 1 in §4, which clarifies the relation between Tanaka connection and the normal conformal connection, is due to Sato, who suggested it to the author, observing the result in [M2]. Recently, this relation is also investigated by K. Yamaguchi in a different way.

For these valuable suggestions as well as criticisms, the author would like to express her hearty thanks to Professors H. Sato and K. Yamaguchi.

§2. $\tilde{G}$-structure and Cartan connection

Let M be an n-dimensional differentiable manifold and let $F(M)$ be the linear frame bundle over M. For a Lie subgroup $\tilde{G}$ of $GL(n,\mathbf{R})$, a $\tilde{G}$-reduction $\tilde{P}$ of $F(M)$ is called a $\tilde{G}$-structure on M. When $\pi\colon\tilde{P}\to M$ is a principal $\tilde{G}$-bundle over M, an $\mathbf{R}^n$-valued 1-form $\tilde{\theta}$ defined by

$$\tilde{\theta}(X)=u^{-1}(\pi_*X), \qquad u\in P, \quad X\in T_u\tilde{P}$$

is called the basic form, which satisfies

(1) $\tilde{\theta}(X)=0$ if and only if X is a vertical vector.

(2) $R_a^*\tilde{\theta}=a^{-1}\tilde{\theta}, \quad a\in\tilde{G}$.

Sometimes, we define a $\tilde{G}$-structure by a pair $(\tilde{P},\tilde{\theta})$, satisfying (1) and (2).

Let G/G' be a homogeneous space of dimension n and let $\mathfrak{g}$ and $\mathfrak{g}'$ be the Lie algebra of G and G', respectively.

Definition. *A Cartan connection* (P,ω) *of type* G/G' is by definition

C1 P is a principal fiber bundle over M with the structure group G'.

C2 ω is a $\mathfrak{g}$-valued 1-form on P satisfying

(a) $\omega(X)=0$ implies $X=0$, $\quad X\in TP$.

(b) $R_a^*\omega = \mathrm{Ad}(a^{-1})\omega, \quad a \in G'.$
(c) $\omega(A^*) = A, \quad A \in \mathfrak{g}'.$

When (P, ω) is a Cartan connection of type G/G', let θ be the $\mathfrak{m}$-component of ω, where $\mathfrak{m} = T_0(G/G')$. Putting $\tilde{P} = P/\mathrm{Ker}\,\rho$ where ρ is the isotropy representation, we denote the projection $P \to \tilde{P}$ by $\tilde{\rho}$. Then $(\tilde{P}, \tilde{\theta})$, where $\theta = \tilde{\rho}^*\tilde{\theta}$, is a $\tilde{G}$-structure on $M, \tilde{G} = \rho(G')$. To give a standard choice of connection which induces a given $\tilde{G}$-structure, Tanaka defined normal Cartan connections as follows.

Let $\mathfrak{g} = \sum_{p \in \mathbf{Z}} \mathfrak{g}_p$ be a simple graded Lie algebra of non-compact type with subalgebra $\mathfrak{m} = \sum_{p<0} \mathfrak{g}_p$. A cochain complex $(C^q(\mathfrak{m}, \mathfrak{g}), \partial)$ is given where $C^q(\mathfrak{m}, \mathfrak{g}) = \mathfrak{g} \otimes \wedge^q(\mathfrak{m}^*)$, and $\partial : C^q \to C^{q+1}$ is the coboundary operator [T3, K]. Let $\partial^* : C^{q+1} \to C^q$ be the adjoint operator with respect to the metric $(X, Y) = -B(X, \sigma Y)$ defined by the Killing form B and the involution σ of $\mathfrak{g}$. Explicitly, they are given by

$$\begin{aligned}(\partial c)(X_1 \wedge \cdots \wedge X_{q+1}) &= \sum_i (-1)^{i+1}[X_i, c(X_1 \wedge \cdots \wedge \hat{X}_i \wedge \cdots \wedge X_{q+1})] \\ &+ \sum_{i<j} (-1)^{i+j} c([X_i, X_j] \wedge X_1 \wedge \cdots \wedge \hat{X}_i \wedge \cdots \wedge \hat{X}_j \wedge \cdots \wedge X_{q+1}), \\ (\partial^* c)(X_1 \wedge \ldots \wedge X_{q-1}) &= \sum_j [e_j^*, c(e_j \wedge X_1 \wedge \cdots \wedge X_{q-1})] \\ &+ \frac{1}{2} \sum_{i,j} (-1)^{i+1} c([e_j^*, X_i]_- \wedge e_j \wedge X_1 \wedge \cdots \wedge \hat{X}_i \wedge \cdots \wedge X_{q-1}),\end{aligned}$$

where $c \in C^q, X_1, \ldots, X_{q+1} \in \mathfrak{m}$ and $[e_j^*, X_i]_-$ denotes the $\mathfrak{m}$-component of $[e_j^*, X_i]$ with respect to the decomposition $\mathfrak{g} = \mathfrak{m} + \mathfrak{g}'$, $\{e_j^*\}$ is the base of $\mathfrak{m}^* = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ dual to a base $\{e_j\}$ of $\mathfrak{m}$ with respect to B. Define

$$\wedge_i^q = \sum \mathfrak{g}_{r_1}^* \wedge \cdots \wedge \mathfrak{g}_{r_q}^*,$$

where the summation is taken over $r_1, \ldots, r_q < 0, \sum_{k=1}^q r_k = i$. Then we have $\wedge^q(\mathfrak{m}^*) = \sum_i \wedge_i^q$. Put

$$C^{p,q} = \sum_j \mathfrak{g}_j \otimes \wedge_{j-p-q+1}^q.$$

When (P, ω) is a Cartan connection of type G/G' where the Lie subalgebra of G (G', resp.) is $\mathfrak{g}$ ($\mathfrak{g}'$, resp.), the coefficient $K(z)$ of the curvature form $\Omega = \frac{1}{2}K\theta \wedge \theta$ (P, ω) belongs to $C^2(\mathfrak{m}, g), z \in P$. Let K^p be the $C^{p,2}$-component of K.

Definition. (P, ω) is *a normal Cartan connection of type* G/G' if the curvature form $\Omega = \frac{1}{2} K\theta \wedge \theta$ satisfies

N1 $K^p = 0 \quad (p < 0)$
N2 $\partial^* K^p = 0 \quad (p \geq 0)$.

Definition. A simple graded Lie algebra $\mathfrak{g}$ is called *of the* μ*-th kind* if $\mathfrak{g}_p = 0, (p < -\mu)$ and $\mathfrak{g}_{-\mu} \neq 0$. When $\mathfrak{g}$ is of the second kind and $\dim \mathfrak{g}_{-2} = 1$, $\mathfrak{g}$ is called *of contact type.*

Remark 1. When $\mathfrak{g}$ is of the first kind, **N1** means ω is torsion free while **N2** means that the curvature form is harmonic. When $\mathfrak{g}$ is of contact type, **N1** is satisfied if and only if the associated $\tilde{G}$-structure is of type $\mathfrak{m}$ (see [T3] for definition. Here, for simplicity, we adopt **N1** as a definition of $\tilde{G}$*-structure of type* $\mathfrak{m}$, when the $\tilde{G}$-structure is induced from (P, ω)). As we see later, this is the case for conformal contact and Lie contact structures. In the following, to avoid more definitions, let $\mathfrak{g}$ be of the first kind or of contact type. The following is important:

Fact 1 [T3]. *When* $H^{q,1}(\mathfrak{m}, g) = 0$ *for* $q \geq 1$*, there exists a unique normal Cartan connection of type* G/G' *attached to the isomorphism class of* $\tilde{G}$*-structures of type* $\mathfrak{m}$.

§3. Definitions and basic facts of Lie contact structures

Let $\mathbf{R}^N_k$ be the N-dimensional real vector space equipped with the scalar product $\langle\ ,\ \rangle_k$ of signature $(+, \ldots, +, -, \ldots, -)$, where $-$ appear k-times, $0 \leq k \leq N$. The projective space associated with $\mathbf{R}^N_k$ is denoted by $P^{N-1}_k\mathbf{R}$. We identify $S^n = \{ x \in \mathbf{R}^{n+1}_0 \mid \langle x, x\rangle_0 = 1 \}$ with $Q^n = \{ [y] \in P^{n+1}_1\mathbf{R} \mid \langle y, y\rangle_1 = 0 \}$, by the correspondence

$$S^n \ni x \mapsto y = (x, 1) \in \mathbf{R}^{n+2}_1.$$

Then the projective transformation group of $P^{n+1}_1\mathbf{R}$ fixing Q^n is $L = PO(n+1, 1)$, *the Möbius group.*

Fact 2. L *acts on* S^n *transitively and* $S^n = L/L'$, *for an isotropy subgroup* L'.

Let Σ be the set of all oriented hyperspheres in S^n (including point spheres). An element of Σ is given by (m, θ), where $m \in S^n$ is the center of the hypersphere and $0 \leq \theta < \pi$ is the oriented radius. Identify Σ with $Q^{n+1} = \{ [k] \in P^{n+2}_2\mathbf{R} \mid \langle k, k\rangle_2 = 0 \}$ by

$$\Sigma \ni (m, \theta) \mapsto k = (m, \cos\theta, \sin\theta) \in \mathbf{R}^{n+3}_2.$$

The projective transformation group of $P_2^{n+2}\mathbf{R}$ fixing Q^{n+1} is $G = PO(n+1,2)$, the so called *Lie transformation group*. $[k_1] \in Q^{n+1}$ is in oriented contact with $[k_2] \in Q^{n+1}$ if and only if $\langle k_1, k_2\rangle_2 = 0$. A pair (k_1, k_2) in Q^{n+1} satisfying $\langle k_1, k_2\rangle_2 = 0$, defines a line l in Q^{n+1}, which consists of points $[ak_1 + bk_2] \in Q^{n+1}$, $a, b \in \mathbf{R}$. Let Λ^{2n-1} be the set of all lines in Q^{n+1}:

$$\Lambda^{2n-1} = \{\, (k_1, k_2) \mid \langle k_i, k_j\rangle_2 = 0, i, j = 1, 2 \,\}.$$

G acts on Λ^{2n-1} since G preserves $\langle\,,\,\rangle_2$. We identify $T_1S^n = \{\, (u, v) \in S^n \times S^n \mid \langle u, v\rangle_0 = 0 \,\}$ with Λ^{2n-1} by

$$T_1S^n \ni (u, v) \mapsto (k_1, k_2) \in \Lambda^{2n-1}$$

where $k_1 = (u, 1, 0)$ and $k_2 = (v, 0, 1)$. It is now clear that the line (k_1, k_2) is identified with the family of oriented hyperspheres through u normal to v.

Fact 3. *G acts on T_1S^n transitively and $T_1S^n = G/G'$, for an isotropy subgroup G'.*

Lemma 1. *An element $f \in L$ is lifted to Lie transformations $f^\pm \in G$ by*

$$f^\pm(v) = \pm f_* v/\|f_* v\|, \qquad v \in T_1S^n.$$

Proof. Let $e_0, \dots, e_{n+2}$ be the standard base of $\mathbf{R}_2^{n+3}$, i.e. such that

$$(\langle e_\alpha, e_\beta\rangle_2)_{0\le\alpha,\beta\le n+2} = \begin{pmatrix} I_{n+1} & 0 \\ 0 & -I_2 \end{pmatrix}.$$

Then L is embedded in G by

$$L \ni f \overset{\iota}{\mapsto} \begin{pmatrix} \pm f & 0 \\ 0 & 1 \end{pmatrix} \in G.$$

Now, recall the meaning of $S\colon T_1S^n \to T_1S^n$, $S \in G$. If we identify $(u, v) \in T_1S^n$, $\langle u, v\rangle_0 = 0$, with a family of hyperspheres through u normal to v, $S(u, v)$ corresponds to a family of hyperspheres through some point $\tilde{u}$ normal to some fixed vector $\tilde{v}$ at $\tilde{u}$. When $S = \iota f \in G$, we have in particular $\tilde{u} = f(u)$ (here we identify $S^n \cong Q^n$) and $\tilde{v}$ is normal to the image of every hypersphere through u normal to v, under f. But since f is conformal, $f_* v$ is also normal to these image hyperspheres, and hence $\tilde{v}$ is parallel with $f_* v$. Q.E.D.

Let G_M be the image of L in G via the map $\iota(f) = f^+$.

Fact 4. *G_M acts on T_1S^n transitively and $T_1S^n = G_M/G'_M$ for an isotropy subgroup G'_M.*

Let $\mathfrak{l}, \mathfrak{l}', \mathfrak{g}, \mathfrak{g}', \mathfrak{g}_M$ and $\mathfrak{g}'_M$ be the Lie algebras of L, L', G, G', G_M and G'_M, respectively. The following expression of these Lie algebras in certain bases is significant and is used in the last section. A base of $\mathbf{R}_1^{n+2}$ is given by $e_0, \dots, e_{n+1}$, and we change it by

$$\begin{cases} \tilde{e}_0 = \dfrac{-e_0 + e_{n+1}}{2}, \\ \tilde{e}_i = e_i, \qquad 1 \leq i \leq n, \\ \tilde{e}_{n+1} = e_0 + e_{n+1}. \end{cases}$$

With respect to this base, we have

$$\varepsilon = (\langle \tilde{e}_\alpha, \tilde{e}_\beta \rangle_1)_{0 \leq \alpha, \beta \leq n+1} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & I_n & 0 \\ -1 & 0 & 0 \end{pmatrix},$$

$$L = \{A \in GL(n+2, \mathbf{R}) \mid {}^tA\varepsilon A = \varepsilon\},$$

$$\mathfrak{l} = \{X \in \mathfrak{gl}(n+2, \mathbf{R}) \mid {}^tX\varepsilon + \varepsilon X = 0\}.$$

Fact 5. *The Lie algebra $\mathfrak{l}$ associated with the homogeneous space $S^n = L/L'$ is a simple graded Lie algebra of the first kind, i.e.*

$$\mathfrak{l} = \mathfrak{l}_{-1} \oplus \mathfrak{l}_0 \oplus \mathfrak{l}_1, \quad [\mathfrak{l}_p, \mathfrak{l}_q] = \mathfrak{l}_{p+q},$$

where $\mathfrak{l}_{-1} = T_0(L/L')$, $\mathfrak{l}' = \mathfrak{l}_0 \oplus \mathfrak{l}_1$,

$$\mathfrak{l}_0 = \left\{ \begin{pmatrix} r & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & -r \end{pmatrix} \mid r \in \mathbf{R}, a \in \mathfrak{o}(n) \right\} \simeq \mathfrak{co}(n),$$

$$\mathfrak{l}_{-1} = {}^t\mathfrak{l}_1 = \left\{ \begin{pmatrix} 0 & {}^tb & 0 \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix} \mid b \in \mathbf{R}^n \right\} \simeq \mathbf{R}^n.$$

Next, change $e_0, \dots, e_{n+2}$ by

$$\begin{cases} f_0 = \dfrac{-e_0 + e_{n+1}}{2}, & f_{n+1} = e_0 + e_{n+1}, \\ f_1 = \dfrac{-e_1 + e_{n+2}}{2}, & f_{n+2} = e_1 + e_{n+2}, \\ f_i = e_i, \quad 2 \leq i \leq n \end{cases}$$

Then we have

$$\varepsilon' = (\langle f_\alpha, f_\beta\rangle_2)_{0\le\alpha,\beta\le n+2} = \begin{pmatrix} 0 & 0 & -I_2 \\ 0 & I_{n-1} & 0 \\ -I_2 & 0 & 0 \end{pmatrix}$$
$$G = \{S \in GL(n+3,\mathbf{R}) \mid {}^tS\varepsilon' S = \varepsilon'\}$$
$$\mathfrak{g} = \{X \in \mathfrak{gl}(n+3,\mathbf{R}) \mid {}^tX\varepsilon' + \varepsilon' X = 0\}.$$

Fact 6. *The Lie algebra* $\mathfrak{g}$ *associated with the homogeneous space* $T_1S^n = G/G'$ *is a simple graded Lie algebra of the second kind, or more precisely, of contact type* [S], *i.e.*

$$\mathfrak{g} = \sum_{p=-2}^{2} \mathfrak{g}_p, \quad [\mathfrak{g}_p, \mathfrak{g}_q] \subset \mathfrak{g}_{p+q}, \quad \dim \mathfrak{g}_{-2} = 1,$$

$$\mathfrak{g}_{-2} = {}^t\mathfrak{g}_2 = \left\{ \begin{pmatrix} 0 & 0 & c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mid c = \begin{pmatrix} 0 & p \\ -p & 0 \end{pmatrix} \right\},$$

$$\mathfrak{g}_{-1} = {}^t\mathfrak{g}_1 = \left\{ \begin{pmatrix} 0 & {}^tb & 0 \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix} \mid b = (b_1, b_2), b_i \in \mathbf{R}^{n-1} \right\},$$

$$\mathfrak{g}_0 = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & -{}^ta \end{pmatrix} \mid a \in \mathfrak{gl}(2,\mathbf{R}), e \in \mathfrak{o}(n-1) \right\}.$$

Putting $\mathfrak{m} = T_0(G/G')$, *we have* $\mathfrak{m} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$, *and* $\mathfrak{g}' = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$.

Since G_M preserves $\mathbf{R}_1^{n+2} = \{\sum_{\alpha=0}^{n+2} x_\alpha f_\alpha \mid x_1 + 2x_{n+2} = 0\}$, we have easily

Fact 7. *As a Lie algebra associated with* $T_1S^n = G_M/G'_M$, $\mathfrak{g}_M$ *is given by*

(3.1)

$$\mathfrak{g}_M = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \{\mathfrak{g}_M \cap (\mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2)\}$$

$$= \left\{ \begin{pmatrix} r & -\frac{p}{2} & {}^t b_1 & 0 & p \\ -q & 0 & -{}^t b_2 & -p & 0 \\ d_1 & \frac{b_2}{2} & e & b_1 & -b_2 \\ 0 & -\frac{q}{2} & {}^t d_1 & -r & q \\ \frac{q}{2} & 0 & \frac{{}^t b_2}{2} & \frac{p}{2} & 0 \end{pmatrix} \;\middle|\; \begin{array}{c} p, q, r \in \mathbf{R}, \\ b_1, b_2, d_1 \in \mathbf{R}^{n-1}, \\ e \in \mathfrak{o}(n-1) \end{array} \right\}.$$

In particular,

$$\mathfrak{g}'_M = \mathfrak{g}_M \cap \mathfrak{g}'$$

(3.2)

$$= \left\{ \begin{pmatrix} r & 0 & 0 & 0 & 0 \\ -q & 0 & 0 & 0 & 0 \\ d_1 & 0 & e & 0 & 0 \\ 0 & -\frac{q}{2} & {}^t d_1 & -r & q \\ \frac{q}{2} & 0 & 0 & 0 & 0 \end{pmatrix} \;\middle|\; \begin{array}{c} p, q, r \in \mathbf{R}, \\ d_1 \in \mathbf{R}^{n-1}, \\ e \in \mathfrak{o}(n-1) \end{array} \right\}$$

$$\simeq \mathbf{R}^n \oplus \mathfrak{co}(n-1).$$

In fact, with respect to the base $\tilde{e}_0, \ldots, \tilde{e}_{n+2}$ of $\mathbf{R}^{n+3}_2$, where $\tilde{e}_{n+2} = e_{n+2}$, $\iota_* \mathfrak{l}$ is given by

$$\iota_* \mathfrak{l} = \{X = \begin{pmatrix} r & {}^t b & 0 & 0 \\ d & a & b & 0 \\ 0 & {}^t d & -r & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \mid r \in \mathbf{R}, b, d \in \mathbf{R}^n, a \in \mathfrak{o}(n)\},$$

and if we put

$$(3.3) \qquad b = \begin{pmatrix} p \\ b_1 \end{pmatrix}, \qquad d = \begin{pmatrix} q \\ d_1 \end{pmatrix}, \qquad a = \begin{pmatrix} 0 & {}^t b_2 \\ -b_2 & e \end{pmatrix},$$

where $p, q \in \mathbf{R}$, $b_1, b_2, d_1 \in \mathbf{R}^{n-1}$ and $e \in \mathfrak{o}(n-1)$, the expression of X in the base $\{f_0, \ldots, f_{n+2}\}$ is exactly as in (3.1).

We denote by ρ the isotropy representation of each homogeneous space above. Let M and $F(M)$ be as in §2 and put $\tilde{L} = \rho(L') \subset GL(n, \mathbf{R})$. It is easy to see that

$$(3.4)\quad \rho(A) = \alpha k, \qquad A = \begin{pmatrix} \alpha & 0 & 0 \\ * & k & 0 \\ * & * & \alpha^{-1} \end{pmatrix} \in L', \quad \alpha \neq 0, \quad k \in O(n),$$

and $\tilde{L} = CO(n)$.

Definition. An $\tilde{L}$-reduction of $F(M)$ is called *a conformal structure on M.*

Let $F(N)$ be the frame bundle of a $(2n-1)$-dimensional contact manifold N. It is well known that $F(N)$ is reduced to a $G_0^\sharp$-bundle $L^\sharp(N)$, where

$$G_0^\sharp = \left\{ \begin{pmatrix} \alpha & 0 \\ * & CSp(n-1) \end{pmatrix} \mid \alpha \neq 0 \right\}.$$

Fact 8 [SY,M1]. *$\tilde{G}_M = \rho(G'_M)$ and $\tilde{G} = \rho(G')$ are subgroups of $G_0^\sharp$. In fact, we have*

$$\tilde{G} = \{ \begin{pmatrix} \det A & 0 & 0 \\ * & & \\ & & h \otimes A \\ * & & \end{pmatrix} \mid h \in O(n-1),\ A \in GL(2, \mathbf{R}) \}$$

$$\tilde{G}_M = \{ \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha h & 0 \\ * & \gamma h & h \end{pmatrix} \mid \alpha \neq 0,\ h \in O(n-1) \}.$$

Denoting by $\tilde{O}(n-1)$ the subgroup of $\tilde{G}$ given by $A = I_2$, we obtain

$$(3.5)\qquad \tilde{O}(n-1) \subset \tilde{G}_M \subset \tilde{G}.$$

Definition. A $\tilde{G}_M$-reduction of $L^\sharp(N)$ is called *a conformal contact structure on N.*

Definition. A $\tilde{G}$-reduction of $L^\sharp(N)$ is called *a Lie contact structure on N.*

Let (M, g) be an n-dimensional riemannian manifold and let

$$(3.6)\qquad O(n) \to Q_g \xrightarrow{\pi_g} M$$

be the associated principal $O(n)$-bundle over M. According to [KN, Proposition 5.5 in Chapter I], we define *the lifted riemannian structure on* T_1M by

$$O(n-1) \to Q_g \overset{\tilde{\pi}_g}{\to} Q_g/O(n-1) = T_1M, \tag{3.7}$$

where

$$\tilde{\pi}_g^{-1}(z_1) = \{e(z)\ |z = (z_1, \dots z_n) \in Q_g,$$
$$e(z) = (z_i^h, z_j^v), 1 \leq i \leq n, 2 \leq j \leq n\},$$

using the horizontal (resp. vertical) lift z_i^h (resp. z_i^v) of $z_i \in T_1M$ with respect to the Levi-Civita connection on M. We distinguish the total space of (3.6) and (3.7) by Q_g and P_g, respectively, where Q_g is diffeomorphic to P_g via the map

$$\psi: Q_g \ni z \mapsto e(z) \in P_g.$$

Since the $O(n-1)$-action on P_g is given by $e(z)h = e(zh'), h' = \begin{pmatrix} 1 & 0 \\ 0 & h \end{pmatrix}$ and easily seen to coincide with $\tilde{O}(n-1)$, we obtain

Lemma 2 [SY,M1]. *Let M be an n-dimensional riemannian manifold. Then on the unit tangent bundle T_1M of M exist a conformal contact structure and a Lie contact structure.*

Proof. They are given by

$$\tilde{P}_M = P_g \times_{\tilde{O}(n-1)} \tilde{G}_M, \qquad \tilde{P} = P_g \times_{\tilde{O}(n-1)} \tilde{G},$$

respectively, by virtue of (3.5). Q.E.D.

Remark 2. P_g is an $O(n-1)$-reduction of the principal $O(2n-1)$-bundle over the riemannian manifold (T_1M, s_g) where s_g is the metric induced from the Sasakian metric on TM.

§4. Geometry of unit tangent bundles

4.1. Riemannian case

Let (M, g) be an n-dimensional riemannian manifold and let Q_g, P_g and ψ be as in the last section. Let $A = \left\{ \begin{pmatrix} k & \xi \\ 0 & 1 \end{pmatrix} \mid k \in O(n), \xi \in \mathbf{R}^n \right\}$.

When (Q_g, χ) is a Cartan connection of type $A/O(n)$, define a 1-form $\tilde{\chi}$ on P_g by

$$\tilde{\chi}(X) = \chi(\psi_*^{-1}X), \qquad X \in TP_g.$$

Putting $B = \left\{ \begin{pmatrix} h' & 0 & \eta_1 \\ 0 & h & \eta_2 \\ 0 & 0 & 1 \end{pmatrix} \mid h \in O(n-1), \eta_1 \in \mathbf{R}^n, \eta_2 \in \mathbf{R}^{n-1} \right\}$, we show that $(P_g, \tilde{\chi})$ is a Cartan connection of type $B/O(n-1)$. In fact, the Lie algebra $\mathfrak{a}$ of A and $\mathfrak{b}$ of B are isomorphic (as a vector space) by

$$\text{(4.1)} \qquad \mathfrak{a} \ni \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & {}^t b_2 & p \\ -b_2 & e & b_1 \\ 0 & 0 & 0 \end{pmatrix} \leftrightarrow \begin{pmatrix} \tilde{e} & 0 & b \\ 0 & e & -b_2 \\ 0 & 0 & 0 \end{pmatrix} \in \mathfrak{b},$$

where $a = \begin{pmatrix} 0 & {}^t b_2 \\ -b_2 & e \end{pmatrix} \in \mathfrak{o}(n)$, $b = \begin{pmatrix} p \\ b_1 \end{pmatrix} \in \mathbf{R}^n$, $e \in \mathfrak{o}(n-1)$ and $\tilde{e} = \begin{pmatrix} 0 & 0 \\ 0 & e \end{pmatrix}$. Moreover, **C2** follows from the commutative diagram

$$\begin{array}{ccc} P_g & \xrightarrow{\psi^{-1}} & Q_g \\ {\scriptstyle R_h}\downarrow & & \downarrow{\scriptstyle R_{h'}} \\ P_g & \xrightarrow{\psi^{-1}} & Q_g \end{array} \qquad h \in O(n-1),$$

and from

$$\tilde{\chi}(E^*) = \chi(\psi_*^{-1}E^*) = \chi(E^*) = E, \qquad E \in \mathfrak{o}(n-1),$$

where E^* denotes the fundamantal vector field on P_g and Q_g. The decomposition $\chi = \theta + \chi_0 + \chi_1$ with respect to $\mathfrak{a} \ni \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \leftrightarrow (b, e, -b_2) \in \mathbf{R}^n \oplus \mathfrak{o}(n-1) \oplus \mathbf{R}^{n-1}$ determines the basic form θ of χ. On the other hand, (4.1) implies that the basic form $\tilde{\theta}$ of $\tilde{\chi}$ is given by

$$\tilde{\theta}(X) = (\theta + \chi_1)(\psi_*^{-1}X).$$

Remark 3. When χ is the Levi-Civita connection on M, $\tilde{\chi}$ is *not* in general the Levi-Civita connection on (T_1M, s_g), since the torsion appears whenever (M, g) is not riemannian flat.

Definition. When χ is the Levi-Civita connection on M, $\tilde{\chi}$ is called *the lifted riemannian connection on* T_1M.

4.2. Conformal case

In the following we assume that $\dim M \geq 3$. Let M be a manifold with conformal structure. Recall that a conformal structure corresponds uniquely to the normal Cartan connection (Q_L, ω) of type L/L', called the normal conformal connection, where

$$(4.2) \qquad L' \to Q_L \xrightarrow{\pi} M$$

is the associated principal L'-bundle [OG]. By Fact 5, we have

$$\mathfrak{l}' = \mathfrak{l}_0 \oplus \mathfrak{l}_1 = \{ \begin{pmatrix} r & 0 & 0 \\ d & a & 0 \\ 0 & {}^t d & -r \end{pmatrix} \mid r \in \mathbf{R}, a \in \mathfrak{o}(n), d \in \mathbf{R}^n \} = \mathfrak{co}(n) \oplus \mathbf{R}^n.$$

Define a subalgebra $\mathfrak{h}'$ of $\mathfrak{l}'$ by

$$\mathfrak{h}' = \{ \begin{pmatrix} r & 0 & 0 \\ d & a & 0 \\ 0 & {}^t d & -r \end{pmatrix} \in \mathfrak{l}' \mid a = \begin{pmatrix} 0 & 0 \\ 0 & e \end{pmatrix} \in \mathfrak{o}(n-1) \} = \mathfrak{co}(n-1) \oplus \mathbf{R}^n,$$

and let H' be the corresponding connected Lie subgroup. Then we obtain a principal H'-bundle

$$(4.3) \qquad H' \to Q_L \xrightarrow{\tilde{\pi}} Q_L/H',$$

the total space of which is denoted by P_L to distinguish from (4.2). We call P_L *the lifted conformal structure.* Let g be a riemannian metric on M belonging to the conformal class. It is easy to see that Q_L/H' is identified with T_1M. Noting that for any $l' \in L'$, there exists $k \in O(n)$ such that $k^{-1}l' \in H'$, and $H' \overset{\iota}{\cong} G'_M$, define $\tilde{\psi}: Q_L = Q_g \times_{O(n)} L' \to P_g \times_{O(n-1)} G'_M$ by

$$Q_L \ni (z, l') \mapsto (e(zk), \iota(k^{-1}l')) \in P_g \times_{O(n-1)} G'_M, \quad z \in Q_g,\ l' \in L',$$

which is well-defined since if $k_1, k_2 \in O(n)$ are such that $k_1^{-1}l'$, $k_2^{-1}l' \in H'$, it follows $k_1^{-1}l'(k_2^{-1}l')^{-1} = k_1^{-1}k_2 \in H' \cap O(n) = O(n-1)$, and we get

$$\begin{aligned}(e(zk_1), \iota(k_1^{-1}l')) &= (e(zk_1)k_1^{-1}k_2, (k_1^{-1}k_2)^{-1}\iota(k_1^{-1}l')) \\ &= (e(zk_2), \iota(k_2^{-1}l')).\end{aligned}$$

Moreover, it is easy to see that $\tilde{\psi}$ is a diffeomorphism. Thus identifying P_L with $P_g \times_{O(n-1)} G'_M$ and using $Q_L \overset{\tilde{\psi}}{\cong} P_L$, we can define a 1-form $\tilde{\omega}$

on P_L by

$$\tilde{\omega}(X) = \omega(\tilde{\psi}_*^{-1}X), \qquad X \in TP_L.$$

As before, $\tilde{\omega}$ is a Cartan connection of type G_M/G'_M on P_L, since $\mathfrak{l}$ is isomorphic to $\mathfrak{g}_M$ by

(4.4)

$$\mathfrak{l} \ni \begin{pmatrix} r & {}^tb & 0 \\ d & a & b \\ 0 & {}^td & -r \end{pmatrix} \overset{\iota_*}{\mapsto} \begin{pmatrix} r & -\frac{p}{2} & {}^tb_1 & 0 & p \\ -q & 0 & -{}^tb_2 & -p & 0 \\ d_1 & \frac{b_2}{2} & e & b_1 & -b_2 \\ 0 & -\frac{q}{2} & {}^td_1 & -r & q \\ \frac{q}{2} & 0 & \frac{{}^tb_2}{2} & \frac{p}{2} & 0 \end{pmatrix} \in \mathfrak{g}_M,$$

where we use (3.3), and since the following diagram is commutative.

$$\begin{array}{ccc} P_L & \xrightarrow{\tilde{\psi}^{-1}} & Q_L \\ {\scriptstyle R_{\iota(s)}}\downarrow & & \downarrow{\scriptstyle R_s} \\ P_L & \xrightarrow{\tilde{\psi}^{-1}} & Q_L \end{array} \qquad s \in H'.$$

A decomposition $\omega = \theta + \omega_c + \omega'$ with respect to $\mathfrak{l} \ni \begin{pmatrix} r & {}^tb & 0 \\ d & a & b \\ 0 & {}^td & -r \end{pmatrix} \leftrightarrow$ $(b, (r,a), d) \in \mathbf{R}^n \oplus \mathfrak{co}(n) \oplus \mathbf{R}^n$, and $\omega_c = \omega_c^1 + \omega_c^2$ with respect to $\mathfrak{co}(n) \ni (r,a) \leftrightarrow (-b_2, (r,e)) \in \mathbf{R}^{n-1} \oplus \mathfrak{co}(n-1)$ determines the basic form $\tilde{\theta}$ of $\tilde{\omega}$ by $\tilde{\theta} = \theta + \omega_c^1$.

Definition. When ω is the normal conformal connection on M, we call $\tilde{\omega}$ *the lifted conformal connection on* T_1M.

Remark 4. Note the difference between the conformal connection and the lifted conformal connection on (T_1M, s_g). The former is a Cartan connection of type $PO(2n,1)/K = S^{2n-1}$ (K is an isotropy subgroup), while the latter is of type $G_M/G'_M = T_1S^n$.

4.3. Lie contact case

Let $P = P_L \times_{G'_M} G'$ and extend the lifted conformal connection

$(P_L, \tilde{\omega})$ flattly to $(P, \tilde{\omega})$, i.e.

$$\begin{aligned}\tilde{\omega}(X) &= \mathrm{Ad}(a^{-1})\tilde{\omega}(Y) + A \\ &= \mathrm{Ad}(a^{-1})\omega(\psi_*^{-1}Y) + A,\end{aligned}$$

where $X \in T_uP, a \in G', ua \in P_L, Y \in T_{ua}P_L, A \in \mathfrak{g}'$ and

$$X = R_{a_*}Y + A^*.$$

Now, compare this connection with Tanaka connection τ obtained in [M2]. The latter is given, after a long calculation, as follows : Let $p \in M$ and let (x^i) be a geodesic normal coordinate in a neighborhood U of p, such that

$$p = (0, \dots, 0), \qquad g_{ij}(0) = \delta_{ij}, \qquad \{{}^{\,i}_{jk}\}(0) = 0,$$

where g_{ij} and $\{{}^{\,i}_{jk}\}$ are, respectively, the coefficients of the riemannian metric g of M and the Christoffel's symbols of its riemannian connection. Take a local coordinate (x^i, z^i_j) of Q_g so that

$$z_i = z_i^j \frac{\partial}{\partial x^j}, \qquad g_{ij} z^i_k z^j_l = \delta_{kl}.$$

Let $(s^\alpha_\beta)_{0 \le \alpha, \beta \le n+2} \in G'$. It is shown in [M2, §2] that $(x^i, z^i_j, s^a_b, s_i, s_{\bar{i}}, s_1)$ is a local coordinate of P around $e(z(0))$ where $0 \le a, b \le 1, s_i = s_i^{n+1}, s_{\bar{i}} = s_i^{n+2}, 2 \le i \le n$ and $s_1 = s_0^{n+2}$.

Fact 9 [M2]. *Let $(x^i, z^i_j, s^a_b, s_i, s_{\bar{i}}, s_1)$ be the local coordinate chosen as above. Then at $e = e(z) = (x^i, z^i_j, \delta^a_b, \mathbf{0}, \mathbf{0}, 0) \in P_g$, Tanaka connection (P, τ) is given by*

$$\tau = \begin{pmatrix} \tau^0_0 & \tau^0_1 & \theta^i & 0 & \theta^1 \\ \tau^1_0 & \tau^1_1 & \theta^{\bar{i}} & -\theta^1 & 0 \\ & & & & \\ \tau_i & \tau_{\bar{i}} & \tau^i_j & \theta^i & \theta^{\bar{i}} \\ & & & & \\ 0 & -\tau_1 & \tau_i & -\tau^0_0 & -\tau^1_0 \\ \tau_1 & 0 & \tau_{\bar{i}} & -\tau^0_1 & -\tau^1_1 \end{pmatrix},$$

$$
(4.5)\quad
\begin{aligned}
&\theta^i = g_{jk} z_i^k dx^j, \qquad 1 \le i \le n,\\
&\theta^{\bar{i}} = g_{jk} z_i^k (dz_1^j + \{{}^{j}_{st}\} z_1^s dx^t), \qquad 2 \le i \le n,\\
&\tau_j^i = g_{uv} z_i^v (dz_j^u + \{{}^{u}_{st}\} z_j^s dx^t), \qquad 2 \le i, j \le n,\\
&\tau_0^0 = ds_0^0, \qquad \tau_1^0 = ds_1^0 + A_{11}^0 \theta^1,\\
&\tau_0^1 = ds_0^1 + \sum_{j=1}^{n} A_{0j}^1 \theta^j, \qquad \tau_1^1 = ds_1^1,\\
&\tau_i = ds_i + \sum_{j=1}^{n} A_{ij} \theta^j, \qquad 2 \le i \le n,\\
&\tau_{\bar{i}} = ds_{\bar{i}} + A_{\bar{i}\bar{i}} \theta^{\bar{i}}, \qquad 2 \le i \le n,\\
&\tau_1 = ds_1 + \sum_{j=1}^{n} A_{1j} \theta^j,
\end{aligned}
$$

where

(4.6)

$$
\begin{aligned}
A_{11}^0 &= A_{\bar{i}\bar{i}} = -\frac{1}{2},\\
A_{0j}^1 &= \frac{1}{n-2} R_{1j}, \quad A_{01}^1 = \frac{1}{n-2} R_{11} - \frac{R}{2(n-1)(n-2)},\\
A_{ij} &= -\frac{1}{n-2} R_{ij} + \frac{R}{2(n-1)(n-2)} \delta_{ij}, \quad A_{i1} = -\frac{1}{n-2} R_{1i},\\
A_{1i} &= -\frac{1}{2(n-2)} R_{1i}, \quad A_{11} = -\frac{1}{2(n-2)} R_{11} + \frac{R}{4(n-1)(n-2)},
\end{aligned}
$$

using the Ricci curvature R_{ij} and the scalar curvature R of M at (x^i). Denoting the component of the curvature by $K^i, K^{\bar{i}}, K_j^i, K_b^a, K_i, K_{\bar{i}}, K_1$, respectively, we obtain

$$
\begin{aligned}
&K_{1j}^{\bar{i}} = C_{11j}^i,\ K_{jk}^{\bar{i}} = C_{1jk}^i,\ K_{01i}^1 = C_{11i},\ K_{0ij}^1 = C_{1ij},\\
&K_{jk1}^i = C_{jk1}^i,\ K_{jkl}^i = C_{jkl}^i,\ K_{i1j} = -C_{i1j},\ K_{ijk} = -C_{ijk},\\
&K_{\bar{i}1j} = -\frac{1}{2} C_{11j}^i,\ K_{\bar{i}jk} = -\frac{1}{2} C_{1jk}^i,\ K_{11j} = -\frac{1}{2} C_{11j},\ K_{1jk} = -\frac{1}{2} C_{1jk}
\end{aligned}
$$

for $2 \le i, j, k, l \le n$, and all other components vanish, where C_{jkl}^i and C_{ijk} are the coefficients of Weyl's confomal curvature.

Theorem 1. *Let M be an n-dimensional manifold with conformal* *structure*

$$L' \to Q_L \to M,$$

and let (Q_L, ω) *be the normal conformal connection. Moreover, let* P_L *be the lifted conformal structure*

$$H' \to Q_L \to Q_L/H',$$

and let $\tilde{\omega}$ *be the lifted conformal connection. Then on the Lie contact structure* $P = P_L \times_{G'_M} G'$, *Tanaka connection* (P, τ) *coincides with the connection* $(P, \tilde{\omega})$ *which is flatly extended from* $(P_L, \tilde{\omega})$.

Proof. First, let $j : P_L \to P$ be the natural inclusion. Noting $P_L \cong P_g \times_{O(n-1)} G'_M$ and (3.2), we have

$$j^* ds^0_1 = 0, \qquad j^* ds^1_0 = -2j^* ds_1, \qquad j^* ds^1_1 = 0, \qquad j^* ds_{\bar{i}} = 0.$$

Thus we obtain

$$\begin{aligned}
&j^*\theta^i = \theta^i, \qquad 1 \le i \le n,\\
&j^*\theta^{\bar{i}} = \theta^{\bar{i}}, \qquad 2 \le i \le n,\\
&j^*\tau^i_j = \tau^i_j, \qquad 2 \le i, j \le n,\\
&j^*\tau^0_0 = ds^0_0,\\
&j^*\tau^0_1 = j^* ds^0_1 + A^0_{11} j^*\theta^1 = -\frac{\theta^1}{2},\\
&j^*\tau^1_0 = \tau^1_0,\\
&j^*\tau^1_1 = j^* ds^1_1 = 0,\\
&j^*\tau_i = \tau_i, \qquad 2 \le i \le n,\\
&j^*\tau_{\bar{i}} = j^* ds_{\bar{i}} + A_{\bar{i}\bar{i}} j^*\theta^{\bar{i}} = -\frac{\theta^{\bar{i}}}{2}, \qquad 2 \le i \le n,\\
&j^*\tau_1 = j^* ds_1 + \sum_{j=1}^{n} A_{1j} j^*\theta^j = -\frac{1}{2} j^* ds^1_0 - \frac{1}{2(n-1)} R_{1j}\theta^j = -\frac{1}{2}\tau^1_0,
\end{aligned}$$

and hence $j^*\tau$ is a $\mathfrak{g}_M$-valued 1-form on P_L satisfying **C2**. Now, by $Q_L \overset{\tilde{\psi}}{\cong} P_L$, and $\mathfrak{l} \cong \mathfrak{g}_M$ using (3.3), we may consider $\nu = \tilde{\psi}^* j^* \tau$ as a Cartan connection of type L/L' on Q_L. In fact, putting

$$\nu = \begin{pmatrix} \nu^0_0 & \nu^i & 0 \\ \nu_i & \nu^i_j & \nu^i \\ 0 & \nu_i & -\nu^0_0 \end{pmatrix} \in \mathfrak{l}, \qquad 1 \le i, j \le n,$$

and noting (4.4), we have,

$$\begin{aligned}
&\nu^1 = \theta^1, \\
&\nu^i = \theta^i, \qquad 2 \le i \le n, \\
&\nu_i^1 = -\theta^{\bar{i}} = -\nu_1^i, \qquad 2 \le i \le n, \\
(4.7) \qquad &\nu_0^0 = ds_0^0, \\
&\nu_j^i = \tau_j^i, \qquad 2 \le i, j \le n, \\
&\nu_1 = -\tau_0^1 \\
&\nu_i = \tau_i, \qquad 2 \le i \le n.
\end{aligned}$$

When $\Phi = \frac{1}{2} N\theta \wedge \theta$ is its curvature form, we obtain

$$\begin{aligned}
N^i &= 0, && 1 \le i \le n, \\
N_0^0 &= 0, && \\
N_{jkl}^i &= C_{jkl}^i, && 1 \le i, j, k, l \le n, \\
N_{ijk} &= -C_{ijk}, && 1 \le i, j, k \le n,
\end{aligned}$$

since the calculation is carried out in parallel with the calculation of K, because the structure equations of $\mathfrak{l}, \mathfrak{g}_M$ and $\mathfrak{g}$ correspond each other in the relation of (3.1) and (3.3). Thus N satisfies the normality condition of a Cartan connection (Q_L, ν) of type L/L', and by the uniqueness of such connection, we conclude $\nu = \omega$, and hence $\tau = \tilde{\omega}$. Q.E.D.

Remark 5. A local expression of the normal conformal connection ω is given, for instance, in [OG,§11]. Noting that ω_j^i there corresponds to $\nu_j^i - \nu_0^0 \delta_j^i$, and that the sign of C_{ijk} is opposite, we can prove $\nu = \omega$ directly by (4.5) $\sim$ (4.7).

Finally, by the argument in [M2, §4], we obtain

Theorem 2. *$(P, \tilde{\omega})$ is the normal Cartan connection of type G/G', which induces the Lie contact structure $\tilde{P}$ on T_1M of an n-dimensional riemannian manifold M, if $n \ge 3$. The fundamental system of invariants of the structure is given by the torsion part $K^0 = (K_{jk}^{\bar{i}})$ of the curvature form of $\tilde{\omega}$, when $n \ge 4$, and by $K^1 = (K_{0ij}^1)$, when $n = 3$. In both cases, they are written down in terms of all the coefficients of Weyl's conformal curvature tensor of M.*

Corollary. *T_1M is Lie flat if and only if M is conformally flat.*

Remark 6. We may view a conformal structure $Q_L \to M$ as an enlarged bundle structure

$$Q_L = Q_g \times_{O(n)} L'$$

or

$$\tilde{Q}_L = Q_g \times_{O(n)} CO(n).$$

In this case, riemannnian flatness and conformal flatness are not, of course, equivalent. The Lie contact structure $\tilde{P}$, is also regarded as an enlarged bundle structure

$$P = P_L \times_{G'_M} G'$$

or

$$\tilde{P} = \tilde{P}_L \times_{\tilde{G}_M} \tilde{G}.$$

Thus, Corollary is a non-trivial fact, indeed, though it may be trivial that the conformally flatness is equivalent with the flatness of P_L.

Remark 7. When $n = 3$, we have $K^0 \equiv 0$, which means that the Lie contact structure is integrable. This structure is shown [SY] to be equivalent with a CR-structure with indefinite Levi form, discovered independently by H. Sato and LeBrun [LB].

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Rational Solutions of the Ernst Equation

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Abstract.

We find infinitely many rational solutions of the Ernst equation in general relativity. These are constructed by solving a formal version of Hilbert's homogeneous problem and are expressed in terms of determinants of Toeplitz type matrices.

§1. Introduction

In this note we consider a family of rational solutions of the Ernst equation

$$(1)\qquad \begin{cases} f\nabla^2 f - (\partial_z f)^2 - (\partial_\rho f)^2 + (\partial_z e)^2 + (\partial_\rho e)^2 = 0, \\ f\nabla^2 e - 2(\partial_z f \partial_z e + \partial_\rho f \partial_\rho e) = 0, \end{cases}$$

where ∇^2 is the 3-dimensional Laplace operator acting on axially symmetric functions ; $\nabla^2 = \partial_\rho^2 + (1/\rho)\partial_\rho + \partial_z^2$. In the previous paper [1] we have discussed the following initial value problem for Equation (1) with an initial value at $\rho = 0$:

$$(2)\qquad \begin{aligned} f(z,\rho)\ |_{\rho=0} = f(z), \qquad e(z,\rho)\ |_{\rho=0} = e(z) \\ f(z) \in \mathbf{R}[[z]]^\times, \qquad e(z) \in \mathbf{R}[[z]] \end{aligned}$$

where $\mathbf{R}[[z]]$ denotes the set of the formal power series in z and $\mathbf{R}[[z]]^\times$ is the set of invertible elements in $\mathbf{R}[[z]]$. We have proven that the above initial value problem is uniquely soluble in the category $\mathbf{R}[[z,\rho]]$ and have found several special solutions which are rational with respect to the variables z and ρ. The aim of this note is to clarify the reason why these solutions are rational. Recall that the initial values of all these solutions have the following algebraic properties.

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Rationality. *The initial values $f(z)$ and $e(z)$ are rational functions of z of the form*

$$f(z) = \frac{1}{a(z)}, \qquad e(z) = \frac{b(z)}{a(z)} \tag{3}$$

where $a(z)$ and $b(z)$ are polynomials of z such that $a(0) \neq 0$ and $a|1+b^2$.

We will show that this rationality is not accidental, that is, any solution whose initial value has the above property is a rational function.

Theorem. *Let a and b be polynomials of z such that $a(0) \neq 0$ and $a|1+b^2$. Then the solution of the Ernst equation with the initial value $f(z,\rho)\,|_{\rho=0} = 1/a(z)$, $e((z,\rho)\,|_{\rho=0} = b(z)/a(z)$ is a rational function of z and ρ.*

We prove this theorem by solving a formal Hilbert's homogeneous problem associated with our initial value problem. In this proof we determine the explicit form of the solution from the prescribed initial value; if the initial value has the property (3), then corresponding solution is expressed in terms of determinants of Toeplitz type matrices whose components are polynomials of the variables z and ρ. Hence the rationality is immediately proven.

To reduce our problem to Hilbert's homogeneous problem , we need another expression of the Ernst equation. It is well known [2] that the Ernst equation is equivalent to the following 2nd order differential equation for a 2×2 matrix τ with supplementary conditions:

$$\partial_z(\rho\partial_z\tau \cdot \tau^{-1}) + \partial_\rho(\rho\partial_\rho\tau \cdot \tau^{-1}) = 0 \tag{4a}$$

$$\det(\tau) = 1, \qquad {}^t\tau = \tau \tag{4b}$$

This equivalence is given by

$$\tau = \begin{bmatrix} -\frac{f^2+e^2}{f} & \frac{e}{f} \\ \frac{e}{f} & -\frac{1}{f} \end{bmatrix}. \tag{5}$$

In this expression, Condition (3) is replaced by Condition (4b) and "every component of $\tau(z)$ is a polynomial of z". Hence it suffices to prove that if an initial value $\tau(z)$ satisfies the conditions mentioned above, then the corresponding solution of Equation (4) surely exists and every component of this is a rational function of the variables z and ρ.

§2. Linear problem and Hilbert's homogeneous problem

Let $\mathcal{C}$ be the set of all 2×2 matrices with components in $\mathbf{R}$ (real number). Hereafter, $\mathcal{C}[[z]]$ and $\mathcal{C}[[z, \rho]]$ denote respectively the set of all formal power series in z and (z, ρ) with coefficients in $\mathcal{C}$.

Let us consider a (formal) initial value problem

$$\begin{gathered} \partial_z(\rho \partial_z \tau \cdot \tau^{-1}) + \partial_\rho(\rho \partial_\rho \tau \cdot \tau^{-1}) = 0 \\ \tau(z, \rho)\,|_{\rho=0} = \tau(z) \end{gathered} \tag{6}$$

where $\tau(z, \rho) \in \mathcal{C}[[z, \rho]]^\times$ and $\tau(z) \in \mathcal{C}[[z]]^\times$. In [1] we have shown that the initial value problem (6) is uniquely soluble and that if an initial value $\tau(z)$ satisfies Condition (4b), then this property is preserved for any value of ρ.

The key to analyze the above initial value problem is that Equation (4a) implicitly involves infinitely many conservation laws. The collection of these conservation laws is simply expressed by using "wave function" as follows. Let $P = \partial_z \tau \cdot \tau^{-1}$ and $Q = \partial_\rho \tau \cdot \tau^{-1}$ and introduce a new variable λ (so-called spectral parameter). A solution of the following system of linear differential equations is called a wave function:

$$D_1 W = PW, \qquad D_2 W = QW \tag{7}$$

where $D_1 = \partial_z - \lambda \rho \partial_\rho + 2\lambda^2 \partial_\lambda$ and $D_2 = \lambda \rho \partial_z + \partial_\rho$. Here λ is regarded as a formal variable. However, in analytic category, λ is to be considered as a homogeneous coordinates of the Riemann sphere $\mathbb{P}^1(\mathbf{C})$ and the above linear system admits many kinds of solutions corresponding to the specification of the variable λ. There are two important solutions, one is analytic at $\lambda = \infty$ and the other is analytic at $\lambda = 0$; we use notations W and V respectively. We can also define these two class of the solutions in our formal category. We first give fundamental properties of W.

Lemma 1. *Let $\tau(z, \rho)$ be a solution of the Ernst equation. Then there exists uniquely a solution of Equation* (7) *of the form*

$$W = 1_2 + \sum_{j=1}^{\infty} w_j(z, \rho)\lambda^{-j}, \qquad w_j(z, \rho) \in \mathcal{C}[[z, \rho]]. \tag{8}$$

The value at $\rho = 0$ of this unique solution is evaluated by

$$W(z, \rho, \lambda)\,|_{\rho=0} = \tau(z) \cdot [\tau(z + 1/2\lambda)]^{-1}.$$

Furthermore if $\tau(z)^{-1}$ is a polynomial of z with degree m, then W is a polynomial of λ^{-1} with degree at most m.

Proof. The first two statements have been proven in [1] (Proposition 2.1). We have to prove the last statement. Using the expression of W (see Equation (8)), Equation (7) is equivalent to the following infinite series of the differential equations

$$\text{(9-j)} \qquad \begin{aligned} \rho\partial_\rho w_j + 2jw_j &= \partial_z w_{j-1} - Pw_{j-1}, \\ \rho\partial_z w_j &= -\partial_\rho w_{j-1} + Qw_{j-1}. \end{aligned}$$

Eliminating w_{j+1} by using Equation (9-j+1), we have

(10-j)

$$\begin{aligned} \rho\partial_\rho^2 w_j + &(2j+1-\rho Q)\partial_\rho w_j \\ &= -\rho\partial_z^2 w_j + \rho P\partial_z w_j + (\rho\partial_z P + \rho\partial_\rho Q + 2j+1)w_j. \end{aligned}$$

We first note that $w_j(z,0) = 0$ for $j \geqq m+1$. Hence it is sufficient to prove that any solution of Equation (10-j) such that $w_j(z,0) = 0$ is trivial. To do this, differentiate r times both sides of Equation (10-j) by ρ and set $\rho = 0$. Then we find

$$\begin{aligned} (r+2j+1)c_j^{r+1}(z) = {} & (2j+1)c_j^r(z) - r\partial_z^2 c_j^{r-1}(z) \\ & + r\sum_{k=0}^{r-1}\binom{r-1}{k}\partial_\rho^k Q\,|_{\rho=0} c_j^{r-k}(z) \\ & + r\sum_{k=0}^{r-1}\binom{r-1}{k}\partial_\rho^k P\,|_{\rho=0}\partial_z c_j^{r-1-k}(z) \\ & + r\sum_{k=0}^{r-1}\binom{r-1}{k}(\partial_\rho^k Q + \partial_\rho^k\partial_z P)\,|_{\rho=0} c_j^{r-1-k}(z). \end{aligned}$$

where we set

$$c_j^r(z) = \partial_\rho^r w_j(z,\rho)\,|_{\rho=0}.$$

Since $c_j^0(z) = 0$ for any $j \geqq m+1$, we have $c_j^r(z) = 0, \quad r \geqq 0, \quad j \geqq m+1$. Q.E.D.

Secondly we consider another important solution, the formal version of a locally analytic solution at $\lambda = 0$.

Lemma 2. *Let $\tau(z,\rho)$ be a solution of the Ernst equation. Then there exists uniquely a solution of Equation (7) of the form*

$$V = \sum_{j=0}^{\infty} v_j(z,\rho)\lambda^j, \qquad v_j(z,\rho) \in \mathcal{C}[[z,\rho]].$$

which satisfies $V(z,\rho,\lambda)\,|_{\rho=0} = \tau(z)$.

Proof. Substituting the above expression of V into Equation (7), we have

$$\partial_z v_j - \rho\partial_\rho v_{j-1} + 2(j-1)v_{j-1} = Pv_j, \tag{11-j}$$

$$\partial_\rho v_j + \rho\partial_z v_{j-1} = Qv_j. \tag{12-j}$$

where we set $v_{-1} = 0$. We now solve Equation (12-j) with an initial value $v_0(z,0) = \tau(z)$ and $v_j(z,0) = 0, \quad j \geqq 1$. We can easily find a unique solution of Equation (12-j);

$$v_0 = \tau(z,\rho), \qquad v_j(z,\rho) = -\tau(z,\rho)\int_0^\rho r[\tau(z,r)]^{-1}\partial_z v_{j-1}(z,r)\,dr.$$

Then we show that $v_j(z,\rho)$ defined by the above equation also satisfies Equation (11-j). Clearly v_0 satisfies Equation (11-0). Assume Equation (11-j) is satisfied for v_j. Eliminating v_{j-1} by using Equations (11-j) and (12-j), we have

$$\rho(\partial_z^2 v_j + \partial_\rho^2 v_j) - (2j-1)\partial_\rho v_j + (2j-1)Qv_j - \rho P\partial_z v_j - \rho Q\partial_\rho v_j = 0.$$

Substituting this into

$$\partial_z v_{j+1} = -\partial_z\tau\int_0^\rho r\tau^{-1}\partial_z v_j\,dr - \tau\int_0^\rho r\partial_z(\tau^{-1})\partial_z v_j\,dr - \tau\int_0^\rho r\tau^{-1}\partial_z^2 v_j\,dr,$$

we have the desired result. Q.E.D.

By using W and V we can define a kind of transition function $u(z,\rho,\lambda)$ by

$$u(z,\rho,\lambda) = W^{-1}\cdot V.$$

Noticing the recursive definition of v_j in the proof of Lemma 2, we can easily show that $v_j(z,\rho) = \rho^{2j}\widehat{v}_j, \widehat{v}_j \in \mathcal{C}[[z,\rho]]$, and hence $u(z,\rho,\lambda)$ is well defined as an element of $\mathcal{C}[[z,\rho,\lambda,\lambda^{-1}]]$. Clearly $u(z,\rho,\lambda)$ satisfies the linear differential equation $D_1u = D_2u = 0$. By virtue of this equation and our choice of an initial value for V, we can prove an important relation

$$u(z,\rho,\lambda) = \tau(-\rho^2\lambda/2 + z + 1/2\lambda) \tag{13}$$

as follows. First we note $u(z,0,\lambda)=\tau(z+1/2\lambda)$. The right hand side of (13) clearly satisfies $D_1u=D_2u=0$. It is sufficient to prove uniqueness of the solution of the equation $D_1u=D_2u=0$ under the prescribed initial value. Express u as $u=\sum_{j\in\mathbb{Z}}u_j(z,\rho)\lambda^j$. Similarly as the proof of Lemma 2 we have

$$\begin{aligned}\partial_\rho u_{j+1}&=-\rho\partial_z u_j,\\ \partial_z u_{j+1}&=\rho\partial_\rho u_j-2ju_j\end{aligned}\tag{14}$$

and hence

$$\rho(\partial_z^2+\partial_\rho^2)u_j+(1-2j)\partial_\rho u_j=0.$$

Therefore $u_j(z,\rho), j\leqq 0$ is uniquely determined from an initial value $u_j(z,0)$. For $j>0$, $u_j(z,\rho)$ is uniquely determined by Equation (14) and an initial value.

Proposition 3.

$$W^{-1}\cdot V=\tau(-\rho^2\lambda/2+z+1/2\lambda)\tag{15}$$

If $\tau(z)$ and $\tau(z)^{-1}$ are polynomials of z, then both W and V are polynomials of λ^{-1} and λ respectively with coefficients in $\mathcal{C}[[z,\rho]]$.

Proof. Since $\tau(z)^{-1}$ is a polynomial of z, Lemma 1 asserts that W is a polynomial of λ^{-1} with coefficients in $\mathcal{C}[[z,\rho]]$. On the other hand, the relation $V=W\tau(-\rho^2\lambda/2+z+1/2\lambda)$ shows that V is a polynomial of λ, since W involves no positive power of λ. Q.E.D.

In the proof of Lemma 2 we have shown $v_0=\tau$. Hence by using Equation (15) we get

$$\tau(z,\rho)=\sum_{j=0}^{\infty}w_j(z,\rho)\chi_j(z,\rho)\tag{16}$$

where we set $\tau(-\rho^2\lambda/2+z+1/2\lambda)=\sum_{j\in\mathbb{Z}}\chi_j(z,\rho)\lambda^j$. Here we remark that

$$\chi_j(z,\rho)=\rho^{2j}\widehat{\chi}_j(z,\rho),\quad \widehat{\chi}_j\in\mathcal{C}[[z,\rho]],$$

hence the right hand side of Equation (16) is well defined as an element of $\mathcal{C}[[z,\rho]]$.

§3. Construction of rational solutions

In this section we assume that the initial value $\tau(z)$ is a polynomial of z with degree m and $\det\tau(z)=1$. In this case $\tau(z)^{-1}$ is also a polynomial of z with degree m. From Proposition 3, both W and V are polynomials of λ^{-1} and λ respectively with degree at most m. Substituting the expression $W = 1_2 + \sum_{j=1}^{m} w_j(z,\rho)\lambda^{-j}$ into Equation (15), since V involves no negative power of λ we have

$$\chi_k + \sum_{j=1}^{m} w_j \chi_{j+k} = 0, \qquad -1 \leqq k \leqq -m. \tag{17}$$

The important point is that χ_j is determined exactly from the initial value only. Hence if we can seek $W_j, 1 \leqq j \leqq m$ from Equation (17), by virtue of Equation (16) we have an expression of a solution of the Ernst equation in terms of the initial value. We now introduce the following three matricies $X = (w_1, w_2, \ldots, w_m)$, $A = (\chi_{j+k})_{-1\leqq k\leqq -m, 1\leqq j\leqq m}$, $b = {}^t(-\chi_{-1}, \ldots, -\chi_{-m})$. Then Equation (17) is simply expressed as $XA = b$. Since

$$A\,|_{z=\rho=0} = \begin{bmatrix} \tau(0) & & & \\ & \cdot & * & \\ & & \cdot & \\ & 0 & \cdot & \\ & & & \tau(0) \end{bmatrix},$$

the matrix A is invertible in a neighborhood of $(z, \rho) = (0, 0)$. Further every entry of A^{-1} is a rational function of the variables z and ρ.

Proposition 4. *Let $\tau(z) \in \mathcal{C}[[z, \rho]]$ be a polynomial of z with degree m such that* $\det \tau(z) = 1$. *Then the unique solution of the initial value problem (6) is given by*

$$\tau(z, \rho) = \sum_{j=0}^{m} w_j(z, \rho)\chi_j(z, \rho)$$

where $\tau(-\rho^2\lambda/2 + z + 1/2\lambda) = \sum_{j\in\mathbb{Z}} \chi_j(z, \rho)\lambda^j$ and $w_j(z, \rho)$ is a unique solution of the linear equations (17).

Using the above proposition and equivalence of the Ernst equation and Equation (4) we have

Theorem. *Let a and b be polynomials of z such that $a(0) \neq 0$ and $a|1 + b^2$. Then the solution of the Ernst equation with the initial value $f(z, 0) = 1/a(z)$, $e(z, 0) = b(z)/a(z)$ is a rational function of z and ρ.*

Proof. Let us define an initial value by

$$\tau(z)=\begin{bmatrix} -\frac{1+b^2}{a} & b \\ b & -a \end{bmatrix}.$$

Then the theorem is immediately derived by Proposition 4. Q.E.D.

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Submanifolds of Symmetric Spaces and Gauss Maps

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Dedicated to Professor Tadashi Nagano on his sixtieth birthday

Abstract.

We study Gauss maps for submanifolds of riemannian symmetric spaces and show that they have the same properties as the Gauss maps for submanifolds of euclidean spaces.

Let (M, g) be a simply connected riemannian symmetric space without Euclidean factor and denote by R the curvature tensor. A linear subspace V of a tangent space T_pM is called *strongly curvature invariant* if it satisfies that

$$(0.1) \qquad R_p(V, V)V \subset V \qquad \text{and} \qquad R_p(V^\perp, V^\perp)V^\perp \subset V^\perp,$$

where $V^\perp$ denotes the orthogonal complement of V in T_pM. Strongly curvature invariant subspaces V of T_pM and W of T_qM are said to be *equivalent* to each other if there exists an isometry φ of (M, g) such that $\varphi(p) = q$, $\varphi_{*p}(V) = W$. Denote by $[V]$ the equivalence class of V and by $\mathcal{S}(M, g)$ the set of all the equivalence classes. For $\mathcal{V} \in \mathcal{S}(M, g)$ a connected submanifold S of M is called a *$\mathcal{V}$-submanifold* if it holds that $[T_pS] = \mathcal{V}$ for any $p \in S$. For each $\mathcal{V}$ there exists a unique complete totally geodesic $\mathcal{V}$-submanifold except the congruence by isometries, and for any $\mathcal{V}$-submanifold we can construct "Gauss map" (Naitoh [5]).

In this paper we first show that the target space of this Gauss map is a connected component of the space of all the complete totally geodesic $\mathcal{V}^\perp$-submanifolds. Here $\mathcal{V}^\perp$ is the equivalence class of the orthogonal complement of a subspace representing $\mathcal{V}$. We next show that the following two properties hold for our Gauss map. These properties seem to be fundamental for "Gauss map". One is that a $\mathcal{V}$-submanifold has

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the parallel mean curvature vectors if and only if the Gauss map is harmonic, and another is that a $\mathcal{V}$-submanifold has the parallel second fundamental form if and only if the Gauss map is totally geodesic. Last we concretely give the target spaces of the Gauss maps associated with $\mathcal{V}$-submanifolds of the rank one symmetric spaces.

§1. The space of the totally geodesic $\mathcal{V}^{\perp}$-submanifolds

Fix an equivalence class $\mathcal{V}$ in $\mathcal{S}(M,g)$. Denote by $\mathcal{T}_{\mathcal{V}^{\perp}}$ the set of all complete totally geodesic $\mathcal{V}^{\perp}$- submanifolds of M and by $C_{\mathcal{V}}$ the set of the strongly curvature invariant subspaces representing $\mathcal{V}$. We first define a relation on the set $C_{\mathcal{V}}$ in the following: Two subspaces in $C_{\mathcal{V}}$ are *related* to each other if they are normal spaces of a complete totally geodesic $\mathcal{V}^{\perp}$-submanifold. This relation is an equivalence relation since a strongly curvature invariant subspace representing $\mathcal{V}^{\perp}$ determines a unique complete totally geodesic $\mathcal{V}^{\perp}$-submanifold such that the subspace is a tangent space of it ([2]). Denote by $\langle V\rangle$ the equivalence class of V in $C_{\mathcal{V}}$ and by $\mathcal{C}_{\mathcal{V}}$ the set of all the equivalence classes.

Lemma 1.1. *For $S\in\mathcal{T}_{\mathcal{V}^{\perp}}$ the normal spaces $N_pS, p\in S$, of S are related to each other in $C_{\mathcal{V}}$ and the correspondence:*

$$\mathcal{T}_{\mathcal{V}^{\perp}}\ni S\longrightarrow\langle N_pS\rangle\in\mathcal{C}_{\mathcal{V}}$$

is bijective.

Proof. This follows again since a strongly curvature invariant subspace representing $\mathcal{V}^{\perp}$ determines a unique complete totally geodesic $\mathcal{V}^{\perp}$-submanifold such that the subspace is a tangent space of it.

Q.E.D.

Now denote by r the dimension of the subspaces representing $\mathcal{V}$. Let $\Lambda^r(p)$ be the Grassmannian manifold of all the r-dimensional subspaces of T_pM and $\Lambda^r(M)$ the fibre bundle over M with the fibres $\Lambda^r(p), p\in M$. Then, since the isometry group $I(M,g)$ of (M,g) is a Lie transformation group of M, it is also a Lie transformation group of $\Lambda^r(M)$ in the following action: $\varphi\cdot V=\varphi_*(V)$ for $\varphi\in I(M,g)$, $V\in\Lambda^r(M)$. The set $C_{\mathcal{V}}$ is a closed topological subspace of $\Lambda^r(M)$ by (0.1), and it is preserved by this action. Hence the restriction to $C_{\mathcal{V}}$ of this action makes $I(M,g)$ a topological transformation group of $C_{\mathcal{V}}$. Consider the quatient topology on $\mathcal{C}_{\mathcal{V}}$ induced from $C_{\mathcal{V}}$. Then, since the action on $C_{\mathcal{V}}$ preserves the above relation , it also makes $I(M,g)$ a topological transformation group of $\mathcal{C}_{\mathcal{V}}$. Since $I(M,g)$ acts transitively on $C_{\mathcal{V}}$ and $\mathcal{C}_{\mathcal{V}}$, these spaces have unique differentiable structures so that $I(M,g)$ is Lie transformation groups, respectively. Moreover the identity component G of $I(M,g)$ acts transitively on each connected component of $C_{\mathcal{V}}$ (resp. $\mathcal{C}_{\mathcal{V}}$) and all the connected components of $C_{\mathcal{V}}$ (resp. $\mathcal{C}_{\mathcal{V}}$) are quatient manifolds of G diffeomorphic to each other.

Let M^* be a connected component of $\mathcal{C}_{\mathcal{V}}$ and fix a point p_* of M^*. Take a subspace V of T_pM such that $V \in C_{\mathcal{V}}$ and $\langle V \rangle = p_*$. Denote by K, K_* the isotropy subgroups of p, p_* in G, respectively. Denote by s_p the geodesic symmetry at p of (M,g) and by t_p the isometry of (M,g) satisfying that $t_p(p) = p$ and $(t_p)_{*p}x = -x$ or x according as $x \in V$ or $x \in V^{\perp}$. Such t_p uniquely exists from the condition (0.1) and the simple connectedness of M. The isometries induce involutive automorphisms σ, τ of G in the following way: $\sigma(h) = s_p \circ h \circ s_p$, $\tau(h) = t_p \circ h \circ t_p$ for $h \in G$. Then the followings hold ([2] and [5]):

$$(\text{Fix } \sigma)_0 \subset K \subset \text{Fix } \sigma, \qquad \text{and} \qquad (\text{Fix } \tau)_0 \subset K_* \subset \text{Fix } \tau,$$

where Fix $*$ denotes the Lie subgroup of the points fixed by $*$ and (Fix $*)_0$ the identity component of Fix $*$. Hence (G,K) and (G,K_*) are symmetric pairs. Let $\mathfrak{g}$ be the Lie algebra of G and denote by the same notations σ, τ the differentials of σ, τ. Since s_p and t_p commute, the involutive automorphisms σ, τ also commute. Decompose the Lie algebra $\mathfrak{g}$ into the (± 1)-eigenspaces $\mathfrak{g}_{\pm 1}$ of σ, and moreover decompose $\mathfrak{g}_1$ and $\mathfrak{g}_{-1}$ into the (± 1)-eigenspaces $\mathfrak{g}_{1\pm 1}$ and $\mathfrak{g}_{-1\pm 1}$ of τ, respectively. Then the Lie algebras of K, K_* are given by $\mathfrak{g}_1$, $\mathfrak{g}_{11} \oplus \mathfrak{g}_{-11}$ and the following identifications hold:

$$T_pM = \mathfrak{g}_{-1} = \mathfrak{g}_{-11} \oplus \mathfrak{g}_{-1-1}, V = \mathfrak{g}_{-1-1}, V^{\perp} = \mathfrak{g}_{-11},$$

and

$$T_{p*}M^* = \mathfrak{g}_{1-1} \oplus \mathfrak{g}_{-1-1}.$$

These identifications are given by corresponding $X \in \mathfrak{g}$ to the values at p, p_* of vector fields on M, M^* generated by the one parameter subgroup $\exp tX$ of G, respectively.

We define a riemannian metric g_* on M^* as follows. Under the identification $T_pM = \mathfrak{g}_{-1}$ regard the metric g_p on T_pM as an inner product on $\mathfrak{g}_{-1}$. Then the inner product is uniquely extended to a nondegenerate symmetric bilinear form $\langle\ ,\ \rangle$ on $\mathfrak{g}$ such that $\langle \mathfrak{g}_1, \mathfrak{g}_{-1} \rangle = \{0\}$ and that $\text{ad}(X), X \in \mathfrak{g}$, are skew symmetric. Note that $\langle\ ,\ \rangle$ is τ-invariant and so nondegenerate on $\mathfrak{g}_{1-1} \oplus \mathfrak{g}_{-1-1}$. Hence the bi-invariant indefinite metric on G induced by $\langle\ ,\ \rangle$ induces a pseudo-riemannian metric g_* on M^*. This metric is determined independently of the fixed point p of M.

Theorem 1.2 (Naitoh[5]). *The space (M^*, g_*) is a pseudo-riemannian symmetric space. The geodesic symmetry at p_* is induced by the automorphism τ of G. Moreover if (M, g) is compact, the space (M^*, g_*) is a compact riemannian symmetric space.*

§2. Gauss maps for $\mathcal{V}$-submanifolds

Fix an equivalence class $\mathcal{V}$ in $\mathcal{S}(M, g)$ and let S be a $\mathcal{V}$-submanifold of M. Let M^* be the connected component of $\mathcal{C}_{\mathcal{V}}$ which contains the equivalence class $p_* = \langle T_pS \rangle$ for a point p of S. Since S is connected, the space M^* is determined independently of the base point p. On M^* we consider the pseudo-riemannian metric g_* defined in §1. In the following contents we retain the notations in §1.

The *Gauss map* κ is a smooth mapping of S to M^* defined in the following way: $\kappa(p) = \langle T_pS \rangle$ for $p \in S$. We first study the differential κ_* of κ. Fix a point p of S. Let Ω_p be the holonomy algebra at p of (M, g). Since (M, g) is a riemannian symmetric space, it holds that

$$\Omega_p = \{R(x, y) \in \text{End}\,(T_pM); x, y \in T_pM\}_{\mathbb{R}} \tag{2.1}$$

where $\{*\}_{\mathbb{R}}$ denotes the linear subspace of $\text{End}(T_pM)$ spanned by $\{*\}$ over $\mathbb{R}$. Decompose T_pM into the sum of the tangent space T_pS and the normal space N_pS of S and put $E_p^+ = (T_pS)^* \otimes T_pS \oplus (N_pS)^* \otimes N_pS$, $E_p^- = (T_pS)^* \otimes N_pS \oplus (N_pS)^* \otimes T_pS$. Here V^* denotes the dual space of a vector space V. Then they hold that $\text{End}(T_pM) = E_p^+ \oplus E_p^-$ and moreover by the properties (0.1), (2.1) that

$$\Omega_p = \Omega_p^+ \oplus \Omega_p^- \tag{2.2}$$

where $\Omega_p^{\pm} = \Omega_p \cap E_p^{\pm}$. Under the identifications: $T_pS = \mathfrak{g}_{-1-1}$, $N_pS = \mathfrak{g}_{-11}$ the space Ω_p is identified with the adjoint representation $\text{ad}_{\mathfrak{g}_{-1}}(\mathfrak{g}_1)$ of $\mathfrak{g}_1$ on $\mathfrak{g}_{-1}$ ([2]) and the subspaces $\Omega_p^{\pm}$ are identified with the adjoint representations $\text{ad}_{\mathfrak{g}_{-1}}(\mathfrak{g}_{1\pm1})$ of $\mathfrak{g}_{1\pm1}$ on $\mathfrak{g}_{-1}$ since $[\mathfrak{g}_{11}, \mathfrak{g}_{-1\pm1}] \subset \mathfrak{g}_{-1\pm1}$, $[\mathfrak{g}_{1-1}, \mathfrak{g}_{-1\pm1}] \subset \mathfrak{g}_{-1\mp1}$. Moreover $\Omega_p^{\pm}$ are identified with $\mathfrak{g}_{1\pm1}$ since $\text{ad}_{\mathfrak{g}_{-1}}(\mathfrak{g}_1)$ is faithful. Particularly the dimensions of $\Omega_p^{\pm}$ are constant independently of the base point p of S since the isometries s_q, t_q for other points q of S are conjugate to s_p, t_p in G. Put $\Omega = \cup_{p \in S} \Omega_p$ and $\Omega^{\pm} = \cup_{p \in S} \Omega_p^{\pm}$. Then Ω is the vector bundle over S induced by the holonomy bundle of (M, g) and $\Omega^{\pm}$ are vector subbundles of Ω. Now let $\kappa^{-1}TM^*$ be the pull bak of the tangent bundle TM^* by κ. Then it holds that

$$\kappa^{-1}TM^* = \Omega^- \oplus TS. \tag{2.3}$$

This identification is obvious by the following identifications: $T_{p_*}M^* = \mathfrak{g}_{1-1} \oplus \mathfrak{g}_{-1-1}$, $T_pS = \mathfrak{g}_{-1-1}$, and $\Omega_p^- = \mathrm{ad}_{\mathfrak{g}_{-1}}(\mathfrak{g}_{1-1}) = \mathfrak{g}_{1-1}$.

By the virture of (2.3) we regard the differential κ_* of κ as a bundle map of TS to $\Omega^- \oplus TS$. Denote by α the second fundamental form of the submanifold S of M and by B_ξ the shape operator for a normal vector ξ. For $x \in T_pS$ define an endomorphism T_x of T_pM in the following way: $T_x(y) = \alpha(x, y)$ for $y \in T_pS$ and $T_x(\xi) = -B_\xi(x)$ for $\xi \in N_pS$. It obviously follows that $T_x \in E_p^-$ and moreover the followings hold:

Proposition 2.1. *$T_x \in \Omega_p^-$ and*

$$\kappa_{*p}(x) = T_x \ + \ x$$

for $x \in T_pS$.

Proof. Fix a vector x of T_pS and let $\gamma(t)$ be a curve in S such that $\gamma(0) = p$ and $(d\gamma/dt)(0) = x$. Since S is a connected $\mathcal{V}$-submanifold, we can take a curve $u(t)$ in G such that $u(0) = e$, $u(t)(p) = \gamma(t)$, and $u(t)_*(T_pS) = T_{\gamma(t)}S$, where e denotes the identity map in G. Let Y be the Killing vector field on M generated by $u(t)$, i.e., $Y_q = (d/dt)\,|_{t=0}\, u(t)(q)$, $q \in M$. Identify Y with an element of $\mathfrak{g}$ and decompose Y into the sum of Y_{11}, Y_{1-1}, Y_{-1} where $Y_{1\pm1} \in \mathfrak{g}_{1\pm1}$ and $Y_{-1} \in \mathfrak{g}_{-1}$. Put $v(t) = u(t)\cdot\exp(-tY_{11})$. Then, since the one parameter subgroup $\exp(-tY_{11})$ of K satisfies that $(\exp\, -tY_{11})(p) = p$, $(\exp - tY_{11})_*T_pS = T_pS$ for all t, the curve $v(t)$ in G also satisfies that $v(0) = e$, $v(t)(p) = \gamma(t)$, and $v(t)_*(T_pS) = T_{\gamma(t)}S$. Let X be the Killing vector field on M generated by $v(t)$ and decompose X into the sum of X_1, X_{-1} where $X_{\pm1} \in \mathfrak{g}_{\pm1}$. Then it holds that $X_1 \in \mathfrak{g}_{1-1}$ and $X_{-1} \in \mathfrak{g}_{-1-1}$. In fact, it follows since

$$X_q = \frac{d}{dt}\,|_{t=0}\,(u(t)\exp(-tY_{11}))(q) = Y_q - (Y_{11})_q = (Y_{1-1})_q + (Y_{-1})_q$$

for $q \in M$, and

$$X_p = (Y_{-1})_p = x.$$

We first show that $\kappa_{*p}(x) = X$ under the identification: $T_{p_*}M^* = \mathfrak{g}_{1-1} \oplus \mathfrak{g}_{-1-1}$. In fact,regard X as a Killing vector field on M^*. Then it follows that

$$\begin{aligned}\kappa_{*p}(x) &= \frac{d}{dt}\,|_{t=0}\,\kappa(\gamma(t)) = \frac{d}{dt}\,|_{t=0}\,\langle T_{\gamma(t)}S\rangle \\ &= \frac{d}{dt}\,|_{t=0}\,\langle v(t)_*T_pS\rangle = \frac{d}{dt}\,|_{t=0}\,v(t)(p_*) = X_{p_*}.\end{aligned}$$

Hence it holds that $\kappa_{*p}(x) = X$ in $\mathfrak{g}_{1-1} \oplus \mathfrak{g}_{-1-1}$.

We next show that $X_1 = \mathrm{ad}_{\mathfrak{g}_{-1}}(X_1) = T_x$ under the identification: $\mathfrak{g}_{1-1} = \mathrm{ad}_{\mathfrak{g}_{-1}}(\mathfrak{g}_{1-1}) = \Omega_p^-$, while it is obvious that $X_{-1} = x$ under the identification: $\mathfrak{g}_{-1-1} = T_pS$. Denote by D, ∇ the riemannian connections of (M, g), (S, g), respectively. For the Killing vector field X of (M, g) define an endomorphism A_X of T_pM in the following way: $A_X(y) = -D_yX$ for $y \in T_pM$. Then we have the identification: $A_X = -\mathrm{ad}_{\mathfrak{g}_{-1}}(X_1)$ ([3]) since (M, g) is a symmetric space. For a vector y of T_pS define a vector field Y_t tangent to S along γ in the following way: $Y_t = v(t)_*y$ and moreover extend it to a local vector field Y on M around p. Then, since X is a vector field on M generated by $v(t)$, it holds that $[X, Y]_p = 0$ ([3]). Hence it follows that

$$\begin{aligned}\mathrm{ad}_{\mathfrak{g}_{-1}}(X_1)(y) &= -A_X(y) = D_yX = (D_YX)_p \\ &= (D_XY)_p = D_xY = \nabla_xY + \alpha(x, y)\end{aligned}$$

and, since $\mathrm{ad}_{\mathfrak{g}_{-1}}(X_1)y \in N_pS$, it moreover follows that $\mathrm{ad}_{\mathfrak{g}_{-1}}(X_1)(y) = \alpha(x, y)$ and $\nabla_xY = 0$.

Let ξ be a vector of N_pS. Then, since $\mathrm{ad}_{\mathfrak{g}_{-1}}(X_1)(\xi) \in T_pS$, it follows that, for $z \in T_pS$,

$$\begin{aligned}\langle \mathrm{ad}_{\mathfrak{g}_{-1}}(X_1)\xi, z\rangle &= -\langle \xi, \mathrm{ad}_{\mathfrak{g}_{-1}}(X_1)z\rangle \\ &= -g(\xi, \alpha(x, z)) = -g(B_\xi(x), z).\end{aligned}$$

Hence it holds that $\mathrm{ad}_{\mathfrak{g}_{-1}}(X_1)\xi = -B_\xi(x)$. Q.E.D.

Corollary 2.2. *The Gauss map κ is an immersion.*

Denote by ∇^* the Levi-Civita connection of (M^*, g_*). Then ∇^* induces the covariant differentiation ∇^* in the pull back $\kappa^{-1}TM^*$. We study the operation of ∇^* under the identification: $\kappa^{-1}TM^* = \Omega^- \oplus TS$

Proposition 2.3. *For a vector $x \in T_pS$ and a smooth vector field Z on S the covariant derivative ∇_x^*Z is contained in T_pS and it holds that $\nabla_x^*Z = \nabla_xZ$.*

Proof. Fix a vector x of T_pS and let $\gamma(t)$, $v(t)$ be the curves in S, G given in Proposition 2.1, respectively. Moreover for a vector y of T_pS let Y_t be the vector field along γ given in the proposition. Then, in the proof of the proposition, it holds that $\nabla_xY_t = 0$. If it moreover holds that $\nabla_x^*Y_t = 0$, our claim is proved as follows. Let $e_1, \cdots, e_r$ be a basis of T_pS and $(E_1)_t, \cdots, (E_r)_t$ be the base fields along γ constructed from $e_1, \cdots, e_r$ as Y_t is done from y. For a vector field Z on S put $Z_{\gamma(t)} =$

$\sum_{i=1}^r f^i(t)(E_i)_t$. Then it follows that $\nabla_x Z_{\gamma(t)} = \sum_{i=1}^r (df^i/dt)(0)e_i = \nabla_x^* Z_{\gamma(t)}$. Hence it holds that $\nabla_x^* Z = \nabla_x Z \in T_pS$.

We show that $\nabla_x^* Y_t = 0$. Note that the tangent spaces $T_{\gamma(t)}S$ are identified with the subspaces $\mathrm{Ad}(v(t))(\mathfrak{g}_{-1-1})$ in $\mathfrak{g}$ and moreover $\mathfrak{g}$ is identified with the Lie algebra of the Killing vector fields on M^*. Under these identifications let Y_0^* be the Killing vector field on M^* corresponding to the vector y of T_pS. Then the vectors Y_t of $T_{\gamma(t)}S$ correspond to the Killing vector fields $v(t)_*Y_0^*$ on M^*. Hence under the identification (2.1) the vector field Y_t is identified with the TM^*-valued vector field $v(t)_*((Y_0^*)_{p_*})$ along $\kappa \circ \gamma$. Extend this vector field to a local vector field Y^* on M^* around p_*. Next take the element X of $\mathfrak{g}$ defined in Proposition 2.1 and identify it with a Killing vector field X^* on M^*. Then X^* is generated by $v(t)$ and thus it holds that $[X^*, Y^*]_{p_*} = 0$. Let $A^*_{X^*}$ be the endomorphism of $T_{p_*}M^*$ defined as the endomorphism A_X of T_pM. Since $X \in \mathfrak{g}_{1-1} \oplus \mathfrak{g}_{-1-1}$, it holds that $A^*_{X^*} = 0$ ([3]). Then it follows that

$$\begin{aligned}\nabla_x^* Y &= (\nabla^*_{X^*} Y^*)_{p_*} = [X^*, Y^*]_{p_*} + (\nabla^*_{Y^*} X^*)_{p_*} \\ &= -A^*_{X^*}(Y^*_{p_*}) = 0\end{aligned}$$

Q.E.D.

Denote by $D^\perp$ the normal connection of the submanifold S of M. We define a covariant defferentiation D^* in the vector bundle $E^- = \cup_{p\in S} E_p^-$ over S. For a vector x of T_pS and a section K of E^- the covariant derivative D_x^*K in E_p^- is given in the following way: For $y \in T_pS$ and $\xi \in N_pS$ extend them to a tangent local vector field Y on S and a normal local vector field N on S, respectively. Then,

$$(D_x^*K)(y) = D_x^\perp(K(Y)) - K(\nabla_x Y)$$

and

$$(D_x^*K)(\xi) = \nabla_x(K(N)) - K(D_x^\perp N).$$

We here note that D_x^*K is skew symmetric if K is skew symmetric.

Proposition 2.4. *For a vector x of T_pS and a section K of Ω^- the covariant derivatives ∇_x^*K, D_x^*K are contained in Ω_p^- and it holds that $\nabla_x^*K = D_x^*K$.*

Proof. Fix a vector x of T_pS and let $\gamma(t)$, $v(t)$ be the curves in S, G given in Proposition 2.1, respectively . Moreover for $L_0 \in \Omega_p^-$ let L_t be the tensor field along γ given in the following way: $L_t = v(t)^*L_0$. Then the tensors L_t are contained in $\Omega^-(\gamma(t))$ since $v(t)$ are isometries of (M,g) satisfying that $v(t)_*T_pS = T_{\gamma(t)}S$ and $v(t)_*N_pS = N_{\gamma(t)}S$. If it holds that $\nabla^*_x L_t = D^*_x L_t = 0$, our claim can be proved in the same way as Proposition 2.3.

We first show that $\nabla^*_x L_t = 0$. Note that the spaces $\Omega^-(\gamma(t))$ are identified with the subspaces $\mathrm{Ad}(v(t))(\mathfrak{g}_{1-1})$ in $\mathfrak{g}$ and identify the tensors L_t with Killing vector fields L^*_t on M^*. Then, under the identification (2.1), the tensor field L_t is identified with the TM^*-valued vector field $(L^*_t)_{\kappa(\gamma(t))}$ along $\kappa\circ\gamma$ and it holds that $(L^*_t)_{\kappa(\gamma(t))} = v(t)_*((L^*_0)_{p_*})$ for all t. Extend $(L^*_t)_{\kappa(\gamma(t))}$ to a local vector field L^* on M^* around p_*. Then, in the same way as in Proposition 2.3, it follows that

$$\begin{aligned}\nabla^*_x L_t &= (\nabla^*_{X^*}L^*)_{p_*} = [X^*, L^*]_{p_*} + (\nabla^*_{L^*}X^*)_{p_*}\\ &= -A^*_{X^*}((L^*)_{p_*}) = 0\end{aligned}$$

We next show that $D^*_x L_t = 0$. For $y \in T_pS$ put $Y_t = v(t)_*y$. Then, since $\nabla_x Y_t = 0$, it follows that $(D^*_x L_t)(y) = D^\perp_x(L_t(Y_t))$. Note that $L_t(Y_t) = v(t)_*(L_0(y))$ and extend $L_t(Y_t)$ to a local vector field Z on M around p. Then it follows that

$$\begin{aligned}D_x(L_t(Y_t)) &= (D_X Z)_p = [X,Z]_p + (D_Z X)_p\\ &= -A_X(Z_p) = -A_X(L_0(y)) = T_x(L_0(y)) \in T_pS.\end{aligned}$$

(See the proof of Proposition 2.1.) Hence it holds that $(D^*_x L_t)(y) = 0$. Also, since $L_t \in \Omega^-_{\gamma(t)} \subset \Omega_{\gamma(t)}$, the tensors L_t and thus $D^*_x L_t$ are skew symmetric. This, together with the above fact, implies that $(D^*_x L_t)(\xi) = 0$ for $\xi \in N_pS$. Q.E.D.

Now for a smooth mapping f of a riemannian manifold (S,g) to a pseudo-riemannian manifold (M^*, g_*), define a covariant differentiation $\bar{D}f_*$ of the differential f_* in the following way:

$$(\bar{D}f_*)(X,Y) = \nabla^*_X(f_*Y) - f_*(\nabla_X Y)$$

for vector fields X, Y on S. If it holds that $\bar{D}f_* = 0$, the mapping f is called *totally geodesic*. Define a TM^* -valued vector field T_f on S as follows. For $p \in S$,

$$(T_f)_p = (1/\dim S)\sum_{i=1}^{r}(\bar{D}f_*)(e_i, e_i)$$

where $\{e_i\}$ denotes an orthonormal basis of T_pS. If it holds that $T_f = 0$ on S, the mapping f is called *harmonic*. Next a submanifold S of a riemannian manifold (M, g) is called a *parallel submanifold* if it satisfies that

$$(\bar{D}\alpha)(X, Y, Z) = D^{\perp}_X(\alpha(Y, Z)) - \alpha(\nabla_X Y, Z) - \alpha(Y, \nabla_X Z) = 0$$

for vector fields X, Y, Z on S.

Theorem 2.5. *Let $\mathcal{V} \in \mathcal{S}(M, g)$ and let S be a connected $\mathcal{V}$-submanifold of M. Then the followings hold.*

(1) *The submanifold S has the parallel mean curvature vectors if and only if the Gauss map κ is harmonic.*

(2) *The submanifold S is a parallel submanifold if and only if the Gauss map κ is totally geodesic.*

Proof. (1) Define a covariant derivative $\bar{D}H$ of the mean curvature vector field H as follows:

$$(\bar{D}H)(X) = D^{\perp}_X H \quad \text{and} \quad (\bar{D}H)(N) = -{}^t(D^{\perp}H)(N)$$

for a tangent vector field X and a normal vector field N on S, where ${}^t(F)$ denotes the transposed mapping of F. We show that

$$T_\kappa = \bar{D}H.$$

By this our claim (1) is obvious. Fix a point p of S and take an orthonormal local base field $E_1, \cdots, E_r$ on S around p satisfying that $(\nabla_{E_i} E_j)_p = 0$ for all i, j. Then it follows that

$$\begin{aligned}(\dim S)(T_\kappa)_p &= \sum_{i=1}^{r} (\nabla^*_{E_i}(\kappa_*(E_i)))_p \\ &= \sum_{i=1}^{r} (\nabla^*_{E_i}(T_{E_i} + E_i))_p = \sum_{i=1}^{r} (D^*_{E_i} T_{E_i} + \nabla_{E_i} E_i)_p \\ &= \sum_{i=1}^{r} (D^*_{E_i} T_{E_i})_p\end{aligned}$$

by Propositions 2.1, 2.3, and 2.4. Take a vector y of T_pS and extend it to a local vector field Y on S satisfying that $(\nabla_{E_i} Y)_p = 0$ for all i.

Then it follows that

$$\begin{aligned}(D^*_{E_i}T_{E_i})_p(y) &= (D^{\perp}_{E_i}(T_{E_i}(Y)))_p = (D^{\perp}_{E_i}(\alpha(E_i,Y)))_p \\ &= (\bar{D}\alpha)(E_i,E_i,Y)_p + \alpha(\nabla_{E_i}E_i,Y)_p + \alpha(E_i,\nabla_{E_i}Y)_p \\ &= (\bar{D}\alpha)_p(E_i,E_i,Y) = (\bar{D}\alpha)_p(Y,E_i,E_i) \\ &= D^{\perp}_Y(\alpha(E_i,E_i))_p - 2\alpha(\nabla_Y E_i,E_i)_p \\ &= D^{\perp}_y(\alpha(E_i,E_i))\end{aligned}$$

by the Codazzi equation and the condition (0.1). Hence it follows that $\sum_{i=1}^r (D^*_{E_i}T_{E_i})_p(y) = (\dim S)(D^{\perp}_y H)_p$. Since $D^*_{E_i}T_{E_i}$ are skew symmetric, it holds that $T_\kappa = \bar{D}H$.

(2) Define a covariant derivative $\bar{D}B$ of the shape operator B as follows:

$$(\bar{D}B)(X,Y,N) = \nabla_X(B_N(Y)) - B_{D^{\perp}_X N}Y - B_N(\nabla_X Y)$$

for tangent vector fields X, Y and a normal vector field N on S. Then it holds that

$$g(\bar{D}B(X,Y,N),Z) = g(\bar{D}\alpha(X,Y,Z),N) \tag{2.5}$$

for a tangent vector field Z on S. We show that $\bar{D}\kappa_* \in (TS)^* \otimes (TS)^* \otimes \Omega^-$ and the followings hold:

$$(\bar{D}\kappa_*)(X,Y)Z = (\bar{D}\alpha)(X,Y,Z)$$

and

$$(\bar{D}\kappa_*)(X,Y)N = -(\bar{D}B)(X,Y,N).$$

By these our claim(2) is obvious. It first follows that

$$\begin{aligned}(\bar{D}\kappa_*)(X,Y) &= \nabla^*_X(T_Y + Y) - (T_{\nabla_X Y} + \nabla_X Y) \\ &= D^*_X T_Y - T_{\nabla_X Y}\end{aligned}$$

by Propositions 2.1, 2.3, and 2.4. Hence it holds that $\bar{D}\kappa_* \in (TS)^* \otimes (TS)^* \otimes \Omega^-$. It next follows that

$$\begin{aligned}(\bar{D}\kappa_*)(X,Y)Z &= (D^*_X T_Y)(Z) - T_{\nabla_X Y}(Z) \\ &= D^{\perp}_X(\alpha(Y,Z)) - \alpha(Y,\nabla_X Z) - \alpha(\nabla_X Y,Z) \\ &= (\bar{D}\alpha)(X,Y,Z).\end{aligned}$$

Note that $(\bar{D}\kappa_*)(X,Y)$ is skew symmetric. Then by (2.5) it follows that $(\bar{D}\kappa_*)(X,Y)N = -(\bar{D}B)(X,Y,N)$. Q.E.D.

Remark. (a) A complete $\mathcal{V}$-submanifold S of (M, g) is parallel if and only if it is a symmetric submanifold. It has already been proved in [5] that the Gauss map of a symmetric $\mathcal{V}$-submanifold is totally geodesic. The proof is done by a concrete construction of the Gauss image of a geodesic in S. Refer [4], [6] for symmetric submanifolds.

(b) On the "classical" Gauss map for a submanifold of $\mathbb{R}^n$, a theorem of this type has been proved in Vilm [7].

§3. Examples

A *symmetric Lie algebra* $(\mathfrak{g}, \sigma)$ is, by definition, a pair of a semisimple Lie algebra $\mathfrak{g}$ and an involutive automorphism σ of $\mathfrak{g}$ such that the adjoint representation $\mathrm{ad}_{\mathfrak{g}_{-1}}(\mathfrak{g}_1)$ is faithful, where $\mathfrak{g}_{\pm 1}$ denote the (± 1)-eigenspaces of σ. If $\mathfrak{g}$ is of compact type (resp. of noncompact type), the symmetric Lie algebra $(\mathfrak{g}, \sigma)$ is also called *of compact type* (resp. *of noncompact type*). Let $(\mathfrak{g}, \sigma)$ be a symmetric Lie algebra of compact type and take a σ-invariant inner product $\langle\ ,\ \rangle$ on $\mathfrak{g}$ such that the endomorphisms $\mathrm{ad}(X)$, $X \in \mathfrak{g}$, of $\mathfrak{g}$ are skew symmetric. Let G be a compact simply connected Lie group with Lie algebra $\mathfrak{g}$ and K the connected closed subgroup of G with Lie algebra $\mathfrak{g}_1$. Put $M = G/K$ and let g be the riemannian metric on M induced from $\langle\ ,\ \rangle$. Then (M, g) is a compact simply connected riemannian symmetric space. Next put $\hat{\mathfrak{g}} = \mathfrak{g}_1 \oplus \sqrt{-1}\mathfrak{g}_{-1}$ and let $\hat{\sigma}$ be the involutive automorphism of $\hat{\mathfrak{g}}$ induced by σ. Then $(\hat{\mathfrak{g}}, \hat{\sigma})$ is a symmetric Lie algebra of noncompact type. Let $\langle\ \hat{,}\ \rangle$ be the nondegenerate symmetric bilinear form on $\hat{\mathfrak{g}}$ induced by $-\langle\ ,\ \rangle$. Let $\hat{G}$ be a simply connected Lie group with Lie algebra $\hat{\mathfrak{g}}$ and $\hat{K}$ be the connected closed subgroup of $\hat{G}$ with Lie algebra $\mathfrak{g}_1$. Put $\hat{M} = \hat{G}/\hat{K}$ and let $\hat{g}$ be the riemannian metric on $\hat{M}$ induced from $\langle\ \hat{,}\ \rangle$. Then $(\hat{M}, \hat{g})$ is a noncompact simply connected riemannian symmetric space. These spaces (M, g) and $(\hat{M}, \hat{g})$ are called *dual* to each other.

Put $p = K \in M$ and identify $\mathfrak{g}$ with the Lie algebra of the Killing vextor fields of (M, g). Then an isometry φ of (M, g) fixing p induces an automorphism $\varphi_\sharp$ of $\mathfrak{g}$ which commutes with σ and leaves $\langle\ ,\ \rangle$ invariant, in the following way: $\varphi_\sharp(X) = \varphi_*(X)$ for $X \in \mathfrak{g}$. Conversely, such an automorphism of $\mathfrak{g}$ is induced by an isometry of (M, g) in this way. These facts also hold for $(\hat{M}, \hat{g})$. The corresponding notations are denoted by attaching the hat to the notations for (M, g).

Now identify the tangent spaces T_pM, $T_{\hat{p}}\hat{M}$ with the subspaces $\mathfrak{g}_{-1}$, $\sqrt{-1}\mathfrak{g}_{-1}$, respectively. Then the curvature tensor R_p, (resp. $\hat{R}_{\hat{p}}$) is identified as follows: Let x, y, $z \in T_pM$ (resp. $\hat{x}$, $\hat{y}$, $\hat{z} \in T_{\hat{p}}\hat{M}$) and let X, Y, Z (resp. $\hat{X}$, $\hat{Y}$, $\hat{Z}$) be the Killing vector fields corresponding to x, y, z (resp. $\hat{x}$, $\hat{y}$, $\hat{z}$). Then it holds that $R_p(x,y)z = [[Y,X],Z]$ (resp. $\hat{R}_{\hat{p}}(\hat{x},\hat{y})\hat{z} = [[\hat{Y},\hat{X}],\hat{Z}]$). Hence, if a subspace V of T_pM is strongly curvature invariant, the subspace $\sqrt{-1}V$ of $T_{\hat{p}}\hat{M}$ is also strongly curvature invariant. Take an equivalence class $\mathcal{V}$ of $\mathcal{S}(M,g)$ and let V be a subspace in T_pM representing $\mathcal{V}$. Then we define an equivalence class $\hat{\mathcal{V}}$ of $\mathcal{S}(\hat{M},\hat{g})$ by putting $\hat{\mathcal{V}} = [\sqrt{-1}V]$.

Proposition 3.1. *The correspondence:* $\mathcal{S}(M,g) \ni \mathcal{V} \longmapsto \hat{\mathcal{V}} \in \mathcal{S}(\hat{M},\hat{g})$ *is a well-defined bijection.*

Proof. We first show that it is well defined. Let W be another subspace in T_pM representing $\mathcal{V}$. Then there exists an isometry φ of (M,g) such that $\varphi(p) = p$ and $\varphi_*(V) = W$. The isometry φ induces an automorphism $\varphi_\sharp$ of $\mathfrak{g}$. Since $\varphi_\sharp$ commutes with σ and leaves $\langle\ ,\ \rangle$ invariant, it moreover induces an automorphism $\hat{\varphi}_\sharp$ of $\hat{\mathfrak{g}}$ in the following way: $\hat{\varphi}_\sharp(X+\sqrt{-1}Y) = \varphi_\sharp(Y) + \sqrt{-1}\varphi_\sharp(X)$ for $X+\sqrt{-1}Y \in \hat{\mathfrak{g}}$. Then $\hat{\varphi}_\sharp$ commutes with $\hat{\sigma}$ and leaves $\hat{\langle\ ,\ \rangle}$ invariant. Hence $\hat{\varphi}_\sharp$ induces the isometry $\hat{\varphi}$ of $(\hat{M},\hat{g})$ such that $\hat{\varphi}(\hat{p}) = \hat{p}$. It obviously follows that $\hat{\varphi}_*(\sqrt{-1}V) = \sqrt{-1}W$. This implies that $\sqrt{-1}V$ and $\sqrt{-1}W$ are equivalent. Hence the above correspondence is well defined.

The injectivity of the correspondence is proved in the same way as above, and the surjectivity is obvious. Q.E.D.

Now let (M,g) be a compact simply connected riemannian symmetric space and $(\mathfrak{g},\sigma)$ the corresponding symmetric Lie algebra. Let $\mathcal{V}$ be an equivalence class of $\mathcal{S}(M,g)$ and let V be a subspace of T_pM representing $\mathcal{V}$. Let τ be the involutive automorphism of $\mathfrak{g}$ induced by the isometry t_p associated with V, and moreover let $\hat{\tau}$ be the involutive automorphism of $\hat{\mathfrak{g}}$ induced by τ. Then, from the arguements in §1, the target spaces M^*, $\hat{M}^*$ assciated with $\mathcal{V}$, $\hat{\mathcal{V}}$ are locally determined by the symmetric Lie algebras $(\mathfrak{g},\tau)$, $(\hat{\mathfrak{g}},\hat{\tau})$, respectively. We concretely give the symmetric Lie algebras for the case that (M,g) is of rank one. An equivalence class is denoted by the unique complete totally geodesic submanifold which belongs to it, and a symmetric Lie algebra is denoted by the quotient of the Lie algebra by the subalgebra of the points fixed by the involution. Denote by S^n the n-dimensional sphere, by $\mathbb{R}P^n$, $\mathbb{C}P^n$, $\mathbb{Q}P^n$, $\mathbb{C}aP^2$ the n-dimensional real, complex, quaternion projective spaces and the Cayley projective plane, and by $\mathbb{R}H^n$, $\mathbb{C}H^n$, $\mathbb{Q}H^n$, $\mathbb{C}aH^2$ the n-dimensional real, complex, quaternion hyperbolic spaces and the Cayley hyperbolic plane, respectively.

Example 1. Let $(M,g)=S^n$ and $(\hat{M},\hat{g})=\mathbb{R}H^n$. Moreover let $\mathcal{V}$, $\hat{\mathcal{V}}$ be the totally geodesic sphere S^r and the totally geodesic real hyperbolic space $\mathbb{R}H^r$, respectively. Then it holds that

$$(\mathfrak{g},\tau)=\mathfrak{so}(n+1)/\mathfrak{so}(r)\oplus\mathfrak{so}(n+1-r)$$

and

$$(\hat{\mathfrak{g}},\hat{\tau})=\mathfrak{so}(n,1)/\mathfrak{so}(n-r,1)\oplus\mathfrak{so}(r).$$

Example 2. Let $(M,g)=\mathbb{C}P^n$ and $(\hat{M},\hat{g})=\mathbb{C}H^n$.

(1) Let $\mathcal{V}$, $\hat{\mathcal{V}}$ be the totally real totally geodesic submanifolds $\mathbb{R}P^n$, $\mathbb{R}H^n$, respectively. Then it holds that

$$(\mathfrak{g},\tau)=\mathfrak{su}(n+1)/\mathfrak{so}(n+1)$$

and

$$(\hat{\mathfrak{g}},\hat{\tau})=\mathfrak{su}(1,n)/\mathfrak{so}(1,n).$$

(2) Let $\mathcal{V}$, $\hat{\mathcal{V}}$ be the kaehlerian totally geodesic submanifolds $\mathbb{C}P^r$, $\mathbb{C}H^r$, respectively. Then it holds that

$$(\mathfrak{g},\tau)=\mathfrak{su}(n+1)/\mathfrak{s}(\mathfrak{u}(r)\oplus\mathfrak{u}(n+1-r))$$

and

$$(\hat{\mathfrak{g}},\hat{\tau})=\mathfrak{su}(n,1)/\mathfrak{su}(n-r,1)\oplus\mathfrak{su}(r)\oplus\mathbb{T}.$$

Example 3. Let $(M,g)=\mathbb{Q}P^n$ and $(\hat{M},\hat{g})=\mathbb{Q}H^n$.

(1) Let $\mathcal{V}$, $\hat{\mathcal{V}}$ be the quaternionic totally geodesic submanifolds $\mathbb{Q}P^r$, $\mathbb{Q}H^r$, respectively. Then it holds that

$$(\mathfrak{g},\tau)=\mathfrak{sp}(n+1)/\mathfrak{sp}(r)\oplus\mathfrak{sp}(n+1-r)$$

and

$$(\hat{\mathfrak{g}},\hat{\tau})=\mathfrak{sp}(n,1)/\mathfrak{sp}(n-r,1)\oplus\mathfrak{sp}(r).$$

(2) Let $\mathcal{V}$, $\hat{\mathcal{V}}$ be the totally complex totally geodesic submanifolds $\mathbb{C}P^n$, $\mathbb{C}H^n$, respectively. Then it holds that

$$(\mathfrak{g},\tau)=\mathfrak{sp}(n+1)/\mathfrak{u}(n+1)$$

and

$$(\hat{\mathfrak{g}},\hat{\tau})=\mathfrak{sp}(1,n)/\mathfrak{su}(1,n)\oplus\mathbb{T}.$$

Example 4. Let $(M,g)=\mathbb{C}\mathrm{a}P^2$ and $(\hat{M},\hat{g})=\mathbb{C}\mathrm{a}H^2$.

(1) Let $\mathcal{V}$, $\hat{\mathcal{V}}$ be the totally geodesic submanifolds $\mathbb{Q}P^2$, $\mathbb{Q}H^2$, respectively. These imbeddings are induced from the inclusion: $\mathbb{Q} \hookrightarrow \mathbb{C}\mathrm{a}$. Then it holds that

$$(\mathfrak{g}, \tau) = \mathfrak{F}_4/\mathfrak{sp}(3) \oplus \mathfrak{su}(2)$$

and

$$(\hat{\mathfrak{g}}, \hat{\tau}) = \mathfrak{F}_4^2/\mathfrak{sp}(1,2) \oplus \mathfrak{su}(2).$$

(2) Let $\mathcal{V}$, $\hat{\mathcal{V}}$ be the totally geodesic submanifolds S^8, $\mathbb{R}H^8$, respectively. The space S^8 is a line in $\mathbb{C}\mathrm{a}P^2$. Then it holds that

$$(\mathfrak{g}, \tau) = \mathfrak{F}_4/\mathfrak{so}(9)$$

and

$$(\hat{\mathfrak{g}}, \hat{\tau}) = \mathfrak{F}_4^2/\mathfrak{so}(1,8).$$

Remark. (a) On the case of Example 1, if we regard a $\mathcal{V}$-submanifold of S^n as a submanifold in $\mathbb{R}^{n+1}$, our Gauss map is the "classical" Gauss map.

(b) On the case of Example 3 (1), $\mathcal{V}$-submanifolds of M and $\hat{\mathcal{V}}$-submanifolds of $\hat{M}$ are always totally geodesic ([1]).

(c) Refer [5] for the details of these examples and the target spaces M^*, $\hat{M}^*$ in the case that (M, g), $(\hat{M}, \hat{g})$ are other riemannian symmetric spaces.

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Lax Equations Associated with a Least Squares Problem and Compact Lie Algebras

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Abstract.

The gradient flow in a least squares problem on a Lie group takes a Lax form [8]. We associate the Lax equation with homogeneous spaces and symmetric spaces of compact simple Lie groups. The critical points of the Lax equation lie in the Cartan subalgebras of the simple Lie algebras. A reduction from homogeneous spaces to symmetric spaces is described by a 'coalescence' of roots. For the complex Grassmann manifold, it is shown that an initial value problem of the Lax equation can be uniquely solved. Some applications to a least squares fitting problem and a linear programming problem are discussed.

§1. Introduction

In the recent work [8] Brockett studied a critical point problem for a least squares function defined on the space of orthogonal matrices $\mathbf{SO}(n)$ in terms of a gradient flow. He derived the nonlinear ordinary differential equation of Lax type

$$\frac{dL(t)}{dt} = [L(t),\ [N, L(t)]]$$

by projecting the gradient flow on $\mathbf{SO}(n)$ onto the adjoint orbit of $\mathbf{SO}(n)$ via $L(t) = g^{-1}(t)Qg(t)$. Here $g(t) \in \mathbf{SO}(n)$, Q is symmetric and N is also a symmetric matrix with distinct eigenvalues. The origin of such a least squares problem itself goes back to an old result by von Neumann [19]. It is shown in [8] that the Lax equation provides a method for solving the eigenvalue problem of the symmetric matrix Q. The solution $L(t)$ converges as $t \to \infty$ to a diagonal matrix being some permutation of the eigenvalues of Q. This feature of Lax type equation is very similar to

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that of finite nonperiodic Toda equation found by Moser [16] and further analyzed by Deift, Nanda and Tomei [10]. See also the expository paper by Watkins [22]. If both Q and N have distinct eigenvalues, then there are $n!$ critical points on $\mathbf{SO}(n)$ and the least squares problem can solve a combinatorial optimization problem [8]. Thus the above property of the Lax equation turns the optimization problem into a problem in the theory of ODEs.

More recently Bloch [4] asserted that the Lax equation, where $L(t)$ and N are skew Hermitian, could be derived from a Hamiltonian flow on the complex Grassmann manifold $\mathbf{SU}(p+q)/\mathbf{S}(\mathbf{U}(p)\times\mathbf{U}(q))$. This makes crucial use of the (almost) complex structure. Here the Hamiltonian is a least squares fitting function in estimation theory [3], which measures the total perpendicular squares distance of m-points sited in $\mathbb{C}^{p+q}$ onto the q-plane. Hence the least squares fitting problem may be solved by investigating critical points of the Lax equation associated with the complex Grassmann manifold. The usual fitting problem is formulated in $\mathbb{R}^{p+q}$ and is related to the real Grassmann manifold, so that the method in [4] does not apply. However, the observation in [4] allows us to introduce ideas from Lie algebras and symmetric spaces and to carry out a generalization and classification of the Lax equation by setting $L(t)$ to lie on some homogeneous and symmetric spaces.

Bloch, Brockett and Ratiu [5] discussed a generalization of the Lax equation to any semisimple Lie algebra $\mathfrak{g}$. They showed that, by a suitable choice of $L(t)$ and N, the Lax equation is reduced to the generalized finite nonperiodic Toda equations studied by many authors (see, for example, [15]). They considered a decomposition of $\mathfrak{g}$ into the direct sum of the centralizer of N and its vector space complement in $\mathfrak{g}$. It is known [12] that the complement is identified with the tangent space of certain homogeneous space. The diagonal matrix N defines the isotropy subgroup of the corresponding Lie group G, say H, which gives rise to the homogeneous space G/H. Here $L(t) = g^{-1}(t)Qg(t)$ can be viewed as an element of an adjoint orbit through Q of G. Thus it has not been clear how we can regard $L(t)$ as representing a point of the homogeneous space G/H. Moreover, since the Lax equation is the projection of a gradient flow on G into its adjoint orbit, it should be checked that one can reconstruct the gradient flow on G from any solution of the Lax equation. It is also an important problem to propose a method for solving the Lax equation itself.

In this paper we consider another type of decomposition of any simple Lie algebra $\mathfrak{g}$, namely, a decomposition into the centralizer $\mathfrak{k}$ of Q (*not* N) and its complement $\mathfrak{m}$ in $\mathfrak{g}$. This enables us not only to associate the Lax equation with homogeneous spaces G/K but to give an explicit description of reduction of the Lax equation from homogeneous spaces to symmetric spaces. Here K is the isotropy subgroup having the Lie algebra $\mathfrak{k}$ and $L(t)$ can be identified with a point of G/K. Note that (irreducible) symmetric spaces are completely classified in terms of simple Lie algebras [12]. The equivalence between the gradient flow on any simple Lie group G and the Lax equation associated with the homogeneous space G/K is proved in §3. We also see that 'coalescing' of eigenvalues of $\operatorname{ad} Q$ (or, roots of Lie algebras) gives rise to a sequence of reductions in §4. The Lax equations in least squares fitting problem are derived without using the complex structure as Lax equations associated with real and complex Grassmann manifolds. A class of linear programming problems which can be regarded as a generalization of that of Brockett [8] is also discussed. The Lax equation describes an interior flow on a convex polytope. For the complex Grassmann manifolds it is shown in §5 that an initial value problem of the Lax equation can be uniquely solved by a decomposition of the exponential of initial value. This provides a new approach to such linear programming problems in terms of nonlinear ODEs.

§2. Preliminary

First we review some of the basic facts concerning simple Lie algebras, reductive homogeneous spaces and Hermitian symmetric spaces. More details can be found in the book [12, 14]. This section is based upon the works by Bogoyavlensky [7], Fordy and Kulish [11] and this author [17] which discuss a generalization and classification of the Toda equation, the nonlinear Schrödinger equation and the Heisenberg model, respectively.

Let G be a simple Lie group and $\mathfrak{g}$ be its Lie algebra. Let M be a homogeneous space of G, namely, M is a differentiable manifold on which G acts transitively. It is known [12, p.121] that there is a homeomorphism of the coset space G/K onto M for some isotropy subgroup K of G at a point of M. Let $\mathfrak{k}$ be the Lie algebra of K. We consider the decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}, \qquad [\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}, \tag{1}$$

where $\mathfrak{m}$ is the vector space complement of $\mathfrak{k}$ in $\mathfrak{k}$. If $\mathfrak{k}$ and $\mathfrak{m}$ satisfy (1) and

$$[\mathfrak{k}, \mathfrak{m}] \subset \mathfrak{m}, \tag{2}$$

then G/K is called a *reductive homogeneous space.* The flag manifold gives a striking example of the space, as we will see in §3. If $\mathfrak{k}$ and $\mathfrak{m}$

satisfy (1), (2) and

$$[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{k}, \tag{3}$$

then G/K is a *symmetric space.*

Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ which is the maximal abelian subalgebra of diagonalizable elements of $\mathfrak{g}$. In terms of a Weyl basis [12, p.421] we can see that $\mathfrak{g}$ has the following commutation relations

$$\begin{aligned} [H_i, H_j] &= 0, \qquad [H_i, X_\alpha] = \alpha(H_i)X_\alpha, \\ [X_\alpha, X_\beta] &= N_{\alpha,\beta}X_{\alpha+\beta} \quad (\alpha+\beta \in \Delta), \\ &= \sum_{i=1}^{\operatorname{rank}\mathfrak{h}} C_{\alpha,i}H_i \quad (\alpha+\beta = 0), \\ &= 0 \quad (\alpha+\beta \notin \Delta, \alpha+\beta \neq 0) \end{aligned} \tag{4}$$

for any $H_i \in \mathfrak{h}$ and $X_\alpha \in \mathfrak{g} \bmod \mathfrak{h}$, where $N_{\alpha,\beta}$ and $C_{\alpha,i}$ are constants and Δ is a set of nonzero linear functionals $\alpha : \mathfrak{h} \to \mathbb{C}$ called roots. Let Q be an element of $\mathfrak{h}$. In this paper we choose the isotropy K such that its Lie algebra $\mathfrak{k}$ is given by the centralizer $C_\mathfrak{g}(Q)$ of Q in $\mathfrak{g}$,

$$\mathfrak{k} = C_\mathfrak{g}(Q) = \{X \in \mathfrak{g} \mid [X, Q] = 0\}. \tag{5}$$

The existence of such Q is proved in [14, p.261]. Compare $\mathfrak{k}$ in (5) with the subalgebra $C_\mathfrak{g}(N)$ defined in [5]. It is to be noted [12, p.163] that if Q is *regular*, namely, the eigenvalues $\alpha(Q)$ of $\operatorname{ad} Q$ are mutually distinct, then

$$C_\mathfrak{g}(Q) = \mathfrak{h}. \tag{6}$$

From (4) we see $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$. Since $\mathfrak{k} = \mathfrak{h}$ in this case, the corresponding coset space G/K is automatically a reductive homogeneous space. The decomposition (1) is essentially a Cartan decomposition of $\mathfrak{g}$. If Q is not regular, then $\mathfrak{k} = C_\mathfrak{g}(Q) \supset \mathfrak{h}$. This implies that as the eigenvalues $\alpha(Q)$ 'coalesce', $C_\mathfrak{g}(Q)$ becomes larger, and consequently, the homogeneous space G/K becomes smaller. Hence coalescing of $\alpha(Q)$ can give rise to a sequence of reductions from homogeneous spaces to symmetric spaces.

Consider a special but important class of symmetric spaces called (irreducible) *Hermitian symmetric spaces.* In this case the eigenvalues $\alpha(Q)$ have only three distinct values $\{0, \pm\alpha\}$. We can decompose $\mathfrak{g}$ into three

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}^+ \oplus \mathfrak{m}^-. \tag{7}$$

Here if we set $X = X^0 + X^+ + X^-$ with $X^0 \in \mathfrak{k}$ and $X^\pm \in \mathfrak{m}^\pm$ for any $X \in \mathfrak{g}$, then $[Q, X^0] = 0$, $[Q, X^\pm] = \pm\alpha X^\pm$. Namely, the eigenvalues $\alpha(Q)$ take the same value for all $X^\pm \in \mathfrak{m}^\pm$. The second property of the Hermitian symmetric space to be noted here is the existence of the (almost) complex structure [12, p.391]. By multiplying Q by a suitable nonzero constant we have a linear endomorphism $\operatorname{ad} Q : \mathfrak{m} \to \mathfrak{m}$ satisfying

$$(\operatorname{ad} Q)^2 = -1. \tag{8}$$

There are 6-types of Hermitian symmetric spaces [12, p.518]. One of them denoted by **AIII** is the complex Grassmann manifold which will be discussed in §4 and §5.

§3. Lax equations and reductive homogeneous spaces

Let N be a fixed regular element of the Cartan subalgebra $\mathfrak{h}$ of any simple Lie algebra $\mathfrak{g}$. Define the function $f(g)$ on G

$$f(g) = \operatorname{tr}(g^{-1}QgN), \tag{9}$$

where Q is a fixed element of $\mathfrak{h}$ and $g \in G$. Following Brockett [8] we consider the critical point problem for $f(g)$ by investigating the gradient flow on G. It is not hard to derive the gradient flow

$$\frac{dg}{dt} = gNg^{-1}Qg - QgN, \tag{10}$$

where g is a smooth function of $t \in \mathbb{R}$ with value in G. Define a (matrix) group action on G, $\gamma : K \times G \to G$ by $\gamma(k, g) = kg$. Since $f(g)$ is invariant under γ, the gradient flow on G can be projected onto the coset space G/K. We introduce a point on adjoint orbit of G through Q by

$$L(t) = g^{-1}(t)Qg(t) \in \mathfrak{g}. \tag{11}$$

Brockett [8] found that if $g(t)$ satisfy (10), then $L(t)$ holds

$$\frac{dL(t)}{dt} = [L(t),\ [N, L(t)]]. \tag{12}$$

Following [1, p.59] we say that the gradient flow (10) admits the *Lax representation* (12). Noting $[N, L(t)] \in \mathfrak{g}$ we see that when $L(t)$ changes in time, $L(t)$ remains on the adjoint orbit of $\mathfrak{g}$. Hence the invariants of the orbit (for example, eigenvalues of $L(t)$) are first integrals of (10). It is known [5] that (12) is actually the gradient flow on the adjoint orbit (11) endowed with the standard metric

$$ds^2 =< (\mathrm{ad}\, L)^{-1} dL, (\mathrm{ad}\, L)^{-1} dL >,$$

where the bracket is defined by the Killing form. See also [6] for the Grassmannian case. The following proposition shows that we can reconstruct a solution of (10) from any solution of (12) if G/K is reductive.

Proposition 1. *Let G/K be a reductive homogeneous space. If $L(t) = g^{-1}(t)Qg(t)$ satisfies the Lax equation* (12)*, then $kg(t)$ satisfies the gradient flow* (10) *for some $k \in K$.*

Proof. Differentiating $L(t) = g^{-1}(t)Qg(t)$ and using (12) we have

$$[\frac{dg}{dt} g^{-1} - [gNg^{-1}, Q],\ Q] = 0.$$

This implies $\frac{dg}{dt} g^{-1} - [gNg^{-1}, Q] \in \mathfrak{k}$. On the other hand, from $[\mathfrak{k}, \mathfrak{m}] \subset \mathfrak{m}$, we see $[gNg^{-1}, Q] \in \mathfrak{m}$. There is an element k of K such that $\frac{d(kg)}{dt}(kg)^{-1} \in \mathfrak{m}$, and then $\frac{dg'}{dt} g'^{-1} - [g'Ng'^{-1}, Q] \in \mathfrak{m}$, where $g' = kg$. For such k, $g'^{-1}Qg' = L(t)$. Hence we can obtain an element of G from any given $L(t)$ which satisfies the gradient flow (10) on G. Q.E.D.

From the definition, our G/K is automatically a reductive homogeneous space. This fact does not depend on the regularity of Q and enable us to carry out a reduction to symmetric spaces (§4). The matrix N is also permitted to be non-regular in Proposition 1.

It is important to remark the followings. If we consider a decomposition of $\mathfrak{g}$ into the centralizer of N denoted by $C_{\mathfrak{g}}(N)$ and its complement as in [5], then Proposition 1 does not hold unless Q is regular. The Lax equation discussed in [5] is a special case of

$$\frac{dM(t)}{dt} = [M(t),\ [Q, M(t)]] \tag{13}$$

which is derived from (10) by setting

$$M(t) = g(t)Ng^{-1}(t). \tag{14}$$

Here the roles of N and Q are interchanged. From (13) and (14) we have

$$[g^{-1}\frac{dg}{dt} + [g^{-1}Qg, N],\ N] = 0.$$

If both N and Q are regular, then $g^{-1}\frac{dg}{dt} + [g^{-1}Qg, N] \in \mathfrak{k}$. Since G/K is reductive and $N \in \mathfrak{k}$, $[g^{-1}Qg, N] \in \mathfrak{m}$. Note that there is an action $\widetilde{\gamma} : G \times K \to G$, with $\widetilde{\gamma}(g,k) = gk$, which leaves $M(t)$ invariant. We can choose so that $g^{-1}\frac{dg}{dt} \in \mathfrak{m}$ in terms of $\widetilde{\gamma}$. Thus we can derive (10) from (13) providing that both N and Q are regular.

We call (12) the *Lax equation associated with the reductive homogeneous space* G/K. For a while let us suppose that Q is regular as well as N. As was shown in the above (10), (12) and (13) are mutually equivalent. Since $\mathfrak{k}$ and $\mathfrak{m}$ satisfy (1) and (2), the Lax equation (13) is expressed as

$$(15) \qquad \begin{aligned} \frac{dM_{\mathfrak{k}}}{dt} &= [M_{\mathfrak{m}},\ [Q, M_{\mathfrak{m}}]]_{\mathfrak{k}}, \\ \frac{dM_{\mathfrak{m}}}{dt} &= [M_{\mathfrak{k}},\ [Q, M_{\mathfrak{m}}]] + [M_{\mathfrak{m}},\ [Q, M_{\mathfrak{m}}]]_{\mathfrak{m}}, \end{aligned}$$

where $M(t) = M_{\mathfrak{k}} + M_{\mathfrak{m}}$ and the subscript $\mathfrak{k}$ and $\mathfrak{m}$ refer to the components in the vector subspaces $\mathfrak{k}$ and $\mathfrak{m}$, respectively. The Lax equation in [5] is essentially equivalent to (15). This class of Lax equations is visualized in terms of the flag manifold $G/K = \mathbf{SU}(r)/\mathbf{S}(\mathbf{U}(1) \times \cdots \times \mathbf{U}(1))$, where K is the maximal torus of G. We give a simple example, $\mathbf{SU}(3)/\mathbf{S}(\mathbf{U}(1) \times \mathbf{U}(1) \times \mathbf{U}(1))$. Let us set

$$Q = i \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix},$$

$$M_{\mathfrak{k}} = i \begin{pmatrix} b_1 & 0 & 0 \\ 0 & b_2 & 0 \\ 0 & 0 & b_3 \end{pmatrix}, \qquad M_{\mathfrak{m}} = \begin{pmatrix} 0 & a_1 & a_3 \\ -\bar{a}_1 & 0 & a_2 \\ -\bar{a}_3 & -\bar{a}_2 & 0 \end{pmatrix},$$

where the bar denotes the complex conjugate, $\sum_j \lambda_j = 0$, $\lambda_i \neq \lambda_j$, $\sum_j b_j = 0$ and λ_j, b_j are real. Part of the Lax equation (15) is then

$$\begin{aligned} \frac{da_1}{dt} &= -(\lambda_1 - \lambda_2)a_1(b_1 - b_2) - 3i\lambda_3 a_3 \bar{a}_2, \\ \frac{db_1}{dt} &= 2(\lambda_1 - \lambda_2)|a_1|^2 - 2(\lambda_3 - \lambda_1)|a_3|^2. \end{aligned}$$

If we set $a_3 = 0$ and $\lambda_1 - \lambda_2 = 1$, $\lambda_2 - \lambda_3 = 1$, the Lax equation is just the (complex) finite nonperiodic Toda equation discussed in [10, 16].

In [5] it is also proved that $\lim_{t\to\infty} M(t)$ exists and lies in the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. Here the regularity of Q is assumed and then $\mathfrak{h} = \mathfrak{k} = C_{\mathfrak{g}}(Q)$. Here we extend this result to the case where $\operatorname{ad} Q$ is permitted to have repeated eigenvalues, namely, $\mathfrak{h} \subset \mathfrak{k} = C_{\mathfrak{g}}(Q)$.

Proposition 2. *Let* $G = \mathbf{SU}(r)$. *The limits* $\lim_{t\to\infty} L(t)$ *and* $\lim_{t\to\infty} M(t)$ *exist and lie in* $\mathfrak{h}$ *and* $\mathfrak{k}$, *respectively.*

Proof. Since L, $N \in \mathfrak{su}(r)$, $[N, L]^* = -[N, L]$, where $*$ denotes the Hermitian conjugate. Set

$$S(L) = \mathrm{tr}(LN) = f(g). \tag{16}$$

We have

$$\begin{aligned}\frac{dS(L)}{dt} &= \mathrm{tr}([L, [N, L]]N)\\ &= \mathrm{tr}(2LNLN - L^2N^2 - NL^2N)\\ &= \mathrm{tr}([L, N]^2)\\ &= -\mathrm{tr}([L, N][L, N]^*) \leq 0.\end{aligned}$$

Thus $S(L)$ is a monotonically decreasing function of t. Since G is compact, $S(L)$ is bounded from below, and then $\lim_{t\to\infty} S(L)$ exists and $\lim_{t\to\infty} \frac{dS(L)}{dt} = 0$. This implies

$$[\lim_{t\to\infty} L(t),\ N] = 0.$$

Noting that the regularity of N leads to $\mathfrak{h} = C_{\mathfrak{g}}(N)$, we see $\lim_{t\to\infty} L(t) \in \mathfrak{h}$. Differentiating $\mathrm{tr}(MQ) = f(g)$, we can derive

$$[\lim_{t\to\infty} M(t),\ Q] = 0.$$

From this it follows that $\lim_{t\to\infty} M(t) \in \mathfrak{k} = C_{\mathfrak{g}}(Q)$. Q.E.D.

Remark. Critical points of the Lax equation (12) which do not lie in $\mathfrak{h}$ are not stable. Hence the integral curve of (12) generally flows from an initial value toward a matrix lies in $\mathfrak{h}$. A similar property of the Lax equation (13) also can be proved.

As was pointed out in [8, 9] in the case $G = \mathbf{SO}(r)$, when Q is regular, this asymptotic property of solutions of the Lax equations (12) has an application to a simple combinatorial optimization problem. Let us set

$$Q = \mathrm{diag}(q_1, q_2, \cdots, q_r), \qquad N = \mathrm{diag}(n_1, n_2, \cdots, n_r), \tag{17}$$

where $q_i \neq q_j$ and $n_i \neq n_j$. From Proposition 2, $\lim_{t\to\infty} L(t) \in \mathfrak{h}$. Set $\mathrm{diag}(l_1^\infty, l_2^\infty, \cdots, l_r^\infty) = \lim_{t\to\infty} L(t)$. Since $\mathrm{Spec}\, L(t) = \mathrm{Spec}\, Q$ for any $t \in \mathbb{R}$, we see that l_j^∞ takes the form

$$l_j^\infty = q_{\pi(j)},$$

$1 \le j \le r$, for some permutation π of r-words. This implies that the limit of $g(t)$ takes the form $\lim_{t\to\infty} g(t) = D\Pi$, where $D = \mathrm{diag}(d_1, d_2, \cdots, d_r)$ with $|d_j| = 1$ and Π is the permutation matrix associated with π. Thus there is an infinite number of critical points of $f(g)$ on $G = \mathbf{SU}(r)$. We note that $\lim_{t\to\infty} L(t) = \Pi^{-1}Q\Pi$, and there are $r!$ critical points of $S(L)$ on the flag manifold $G/K = \mathbf{SU}(r)/\mathbf{S}(\mathbf{U}(1)\times\cdots\times\mathbf{U}(1))$. Consequently,

$$\lim_{t\to\infty} S(L) = \sum_{j=1}^{r} q_{\pi(j)} n_j \in \mathbb{R}.$$

On exactly one of the critical points on G/K, $S(L)$ takes the maxima (minima) of $\sum_{j=1}^{r} q_{\pi(j)} n_j$. Each local maxima (minima) of $S(L)$ is realized according to the choice of the initial value $L(0)$. Thus the flow $L(t)$ of the Lax equation solves the combinatorial optimization problem.

§4. Reduction to Hermitian symmetric spaces

We discuss a reduction of the Lax equations associated with homogeneous spaces to symmetric spaces by coalescing eigenvalues of $\mathrm{ad}\, Q$. When Q is regular, $\mathfrak{k} = \mathfrak{h}$. As the eigenvalues coalesce, $\mathfrak{k}$ grows larger. If $\mathfrak{k}$ and its complement $\mathfrak{m}$ satisfy $[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}$, $[\mathfrak{k}, \mathfrak{m}] \subset \mathfrak{m}$ and $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{k}$, then G/K is called a symmetric space. In this case the Lax equation (13) reads

$$\text{(18)} \qquad \frac{dM_{\mathfrak{k}}}{dt} = [M_{\mathfrak{m}},\ [Q, M_{\mathfrak{m}}]], \qquad \frac{dM_{\mathfrak{m}}}{dt} = [M_{\mathfrak{k}},\ [Q, M_{\mathfrak{m}}]].$$

We here restrict ourselves to an interesting class of symmetric spaces, the Hermitian symmetric spaces. Let us set $M(t) = M_{\mathfrak{k}} + M_{\mathfrak{m}}^{+} + M_{\mathfrak{m}}^{-}$, where $M_{\mathfrak{m}}^{\pm} \in \mathfrak{m}^{\pm}$. Since $[Q, M_{\mathfrak{m}}^{\pm}] = \pm\alpha(Q)M_{\mathfrak{m}}^{\pm}$, we derive from (18)

$$\text{(19)} \qquad \frac{dM_{\mathfrak{k}}}{dt} = 2\alpha(Q)[M_{\mathfrak{m}}^{-},\ M_{\mathfrak{m}}^{+}], \qquad \frac{dM_{\mathfrak{m}}^{\pm}}{dt} = \pm\alpha(Q)[M_{\mathfrak{k}},\ M_{\mathfrak{m}}^{\pm}].$$

Next we give some examples of the Lax equation (19) associated with the complex projective space $\mathbf{SU}(r)/\mathbf{S}(\mathbf{U}(1)\times\mathbf{U}(r-1))$, type **AIII**, and the Hermitian symmetric space $\mathbf{Sp}(n)/\mathbf{U}(n)$ of type **CI**. When $r = 3$, we may write [14, p.275]

$$Q = i\begin{pmatrix} 2\lambda & 0 & 0 \\ 0 & -\lambda & 0 \\ 0 & 0 & -\lambda \end{pmatrix}, \qquad M_{\mathfrak{k}} = \begin{pmatrix} ib_1 & 0 & 0 \\ 0 & ib_2 & b_4 \\ 0 & -\bar{b}_4 & ib_3 \end{pmatrix},$$

$$M_{\mathfrak{m}}^{+} = \begin{pmatrix} 0 & a_1 & a_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad M_{\mathfrak{m}}^{-} = \begin{pmatrix} 0 & 0 & 0 \\ -\bar{a}_1 & 0 & 0 \\ -\bar{a}_2 & 0 & 0 \end{pmatrix},$$

where $\sum_{j=1}^{3} b_j = 0$. It is easy to see $\alpha(Q) = 3i\lambda$. Part of components of the Lax equation associated with $\mathbf{SU}(3)/\mathbf{S}(\mathbf{U}(1) \times \mathbf{U}(2))$ are:

$$\begin{aligned} \frac{da_1}{dt} &= -3\lambda a_1(b_1 - b_2) + 3i\lambda a_2\bar{b}_4, \\ \frac{db_1}{dt} &= -6\lambda(|a_1|^2 + |a_2|^2), \\ \frac{db_2}{dt} &= 6\lambda|a_1|^2, \qquad \frac{db_4}{dt} = 6\lambda\bar{a}_1a_2. \end{aligned}$$

Suppose we take the Hermitian symmetric space of type **CI**, where $n = 2$ for convenience. In this case we may set [17]

$$Q = i\begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & -\lambda & 0 \\ 0 & 0 & 0 & -\lambda \end{pmatrix},$$

$$M_{\mathfrak{k}} = \begin{pmatrix} 0 & b_1 + ib_2 & 0 & 0 \\ -b_1 + ib_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & b_1 - ib_2 \\ 0 & 0 & -b_1 - ib_2 & 0 \end{pmatrix},$$

$$M_{\mathfrak{m}}^{+} = \begin{pmatrix} 0 & 0 & a_1 & a_2 \\ 0 & 0 & a_2 & a_3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad M_{\mathfrak{m}}^{-} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\bar{a}_1 & -\bar{a}_2 & 0 & 0 \\ -\bar{a}_2 & -\bar{a}_3 & 0 & 0 \end{pmatrix}.$$

Explicit form of the Lax equation associated with $\mathbf{Sp}(2)/\mathbf{U}(2)$ can be easily derived. We here omit it.

We now consider an application of the Lax equations associated with the Grassmann manifold to a least squares fitting problem. Let $y_j = (y_{j1}, y_{j2}, \cdots, y_{jp+q})$, $1 \leq j \leq m$, be m-data in $\mathbb{C}^{p+q}$, where each y_{jk} is measured with observation error ε_{jk}. Suppose that ε_{jk} are independent and follow the normal distribution $N(0,1)$. We set

$$Y = \begin{pmatrix} y_{11} & \cdots & y_{1p+q} \\ \vdots & & \vdots \\ y_{m1} & \cdots & y_{mp+q} \end{pmatrix}. \tag{20}$$

Let P be the orthogonal projection of $\mathbb{C}^{p+q}$ onto a p-plane, namely, $P^2 = P$, $\operatorname{rank} P = p$ and $P^* = P$. The matrix $\widetilde{P}$ defined by $\widetilde{P} = I - P$ is the orthogonal projection of $\mathbb{C}^{p+q}$ onto a q-plane. The total perpendicular squares distance of the m-points y_j onto the q-plane is given by

$$\operatorname{tr}(Y(I-\widetilde{P})(Y(I-\widetilde{P}))^*) = \operatorname{tr}(PY^*Y). \tag{21}$$

Let us consider the total least squares fitting problem of finding the matrix P which minimizes $\operatorname{tr}(PY^*Y)$. Since Y^*Y is Hermitian, there is an element g of $\mathbf{U}(p+q)$ such that $g^{-1}Y^*Yg$ is diagonal. We set $D_Y = g^{-1}Y^*Yg$. It is easy to see $\operatorname{tr}(PY^*Y) = \operatorname{tr}(g^{-1}PgD_Y)$. Moreover, since P is diagonalizable by an element of $\mathbf{U}(p+q)$, we can suppose that P is diagonal. Let us consider the critical point problem for $\operatorname{tr}(PY^*Y)$ via the gradient flow on $\mathbf{U}(p+q)$. Set

$$L_P(t) = g^{-1}Pg, \tag{22}$$

where $g = g(t) \in \mathbf{U}(p+q)$ and $t \in \mathbb{R}$. From Proposition 1, we can prove

Proposition 3. *If D_Y has distinct eigenvalues, namely, the singular values of Y are distinct, then the Lax equation*

$$\frac{dL_P(t)}{dt} = [L_P(t),\ [D_Y, L_P(t)]] \tag{23}$$

describes the gradient flow for the least squares fitting function (21).

Proof. Set $N = iD_Y$. This is an regular element of the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{u}(p+q)$. Note also $iL_P \in \mathfrak{u}(p+q)$. Since ${L_P}^2 = L_P$, $\operatorname{rank} L_P = p$ and ${L_P}^* = L_P$, iL_P can be regarded as an element of an adjoint orbit of $\mathbf{U}(p+q)$ through $iP \in \mathfrak{h}$ and then identified with a point in $\mathbf{U}(p+q)/\mathbf{U}(p) \times \mathbf{U}(q) \approx \mathbf{SU}(p+q)/\mathbf{S}(\mathbf{U}(p) \times \mathbf{U}(q))$, the Grassmann manifold. We set $Q = -iP \in \mathfrak{h}$. From the assumption and Proposition 1, the Lax equation $\frac{dL(t)}{dt} = [L(t),[N,L(t)]]$, where $L(t) = g^{-1}Qg = -iL_P$, is equivalent to the gradient flow for $\operatorname{tr}(PY^*Y)$. Q.E.D.

It follows from Proposition 2 that the critical points of the fitting function $\mathrm{tr}(PY^*Y)$ can be given by those of the flow $L(t)$ on the Grassmann manifold. Consequently, the projection matrix P minimizing $\mathrm{tr}(PY^*Y)$ can be expressed as

$$P_{\min} = \lim_{t\to\infty} L_P(t). \tag{24}$$

The least squares estimate of q-plane is determined by $I - P_{\min}$. This is also the maximum likelihood estimate through the theorem of Gauss and Markov.

There is an identity $[L,[L,[L,N]]] = -[L,N]$ which follows from $P^2 = P$. Substituting $L = g^{-1}Qg$ into this we have

$$[Q,\ [Q,\ [Q,\ gNg^{-1}]]] = -[Q,\ gNg^{-1}]. \tag{25}$$

Let us set $J = \mathrm{ad}\, Q : \mathfrak{m} \to \mathfrak{m}$. Since $[Q, gNg^{-1}] \in \mathfrak{m}$ and $J([Q, gNg^{-1}]) \in \mathfrak{m}$ for any N and g, J defines the (almost) complex structure in the sense of (8). Regarding the fitting function $\mathrm{tr}(PY^*Y) = \mathrm{tr}(LN)$ as a Hamiltonian, Bloch [3] derived the Hamiltonian equation of the Lax form

$$\frac{dL(\tau)}{d\tau} = [L(\tau),\ N]. \tag{26}$$

The Hamiltonian equation has $p+q-1$ independent integrals of motion in involution and can be solved explicitly when $p = 1$. Recently, Bloch [4] showed that the gradient flow for $\mathrm{tr}(LN)$ may be obtained from the Hamiltonian flow (26) by letting a linear transformation $\widetilde{J} = \mathrm{ad}\, L$ act on the Hamiltonian flow. It is easy to check $\widetilde{J}^2([L,N]) = -[L,N]$ and $[L,N] \in \mathfrak{m} = T(G/K)$, where $T(G/K)$ is the tangent space of the complex Grassmann manifold G/K. However, Bloch's mapping $\widetilde{J}$ is not an endomorphism of $\mathfrak{m}$, namely, $\widetilde{J}([L,N]) \in \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$. In Proposition 3, we have derived the Lax equation (23) in the least squares fitting problem without using the complex structure. We should remark here that our derivation naturally fits into the usual situation in least squares estimation in the following sense. A Lax equation similar to (23) can be obtained in the case of real Grassmann manifold, $G/K = \mathbf{SO}(p+q)/\mathbf{S}(\mathbf{O}(p) \times \mathbf{O}(q))$. The resulting Lax equation solves a least squares problem of q-plane fitted to a date in $\mathbb{R}^{p+q}$. Hence the maximum likelihood problem can be turned into a critical point problem of the Lax equation.

Finally in this section we briefly deal with a linear programing problem. Let $G/K = \mathbf{U}(p+q)/\mathbf{U}(p) \times \mathbf{U}(q)$, the Grassmann manifold.

Brockett already pointed out in [8] that when $p = 1$ the Lax equation (12) can solve a linear programming problem. We wish to pursue a somewhat different manner here for any integer p. As in the proof of Proposition 3, we write $f(g) = \mathrm{tr}(g^{-1}QgN) = \mathrm{tr}(L_P D_Y)$, where $N = iD_Y \in \mathfrak{h}$, regular, and $Q = -iP \in \mathfrak{h}$. Here L_P is defined by (22) with $iL_P \in \mathfrak{u}(p+q)$, ${L_P}^2 = L_P$ and ${L_P}^* = L_P$. Set $D_Y = \mathrm{diag}(|\delta_1|^2, |\delta_2|^2, \cdots, |\delta_{p+q}|^2)$. Let $(l_1(t), l_2(t), \cdots, l_{p+q}(t))$ be diagonal elements of $L_P(t)$. Since $\mathrm{tr}(L_P D_Y) = \sum_{j=1}^{p+q} l_j(t)|\delta_j|^2$, the Lax equation (23) is viewed as the gradient flow for $\sum_{j=1}^{p+q} l_j(t)|\delta_j|^2$. It is to be noted from ${L_P}^2 = L_P$, ${L_P}^* = L_P$ and $\mathrm{tr}\, L_P = p$ that the diagonal elements should satisfy the conditions

$$0 \le l_j(t) \le 1, \qquad \sum_{j=1}^{p+q} l_j(t) = p. \tag{27}$$

Such a $(l_1(t), l_2(t), \cdots, l_{p+q}(t))$ may then be regarded as a point in a convex polytope in $\mathbb{R}^{p+q}$. The Lax equation (23) describes an interior flow on it which approaches critical points (vertices of the polytope) as t goes to infinity. This assertion follows from Proposition 2, i.e. $\lim_{t\to\infty} iL_P(t) \in \mathfrak{h}$. The critical point problem for $f(g)$ then corresponds to a linear programming problem. Recently, Bayer and Lagarias [2] considered a dynamical picture of Karmarkar's polynomial time algorithm [13] which solves various linear programming problems as an interior flow on polytopes. They observed that the algorithm is deeply related to the nonlinear integrable system of non-Lax type. It would be most interesting to make clear the link between the Lax equation (23) and the nonlinear system in [2].

§5. Solutions of Lax equations associated with the Grassmann manifold

We come in the position to propose a method for solving the Lax equations associated with the complex Grassmann manifold. Let us express the gradient flow (10) on $G = \mathbf{SU}(p+q)$ as

$$\frac{dg}{dt} \cdot g^{-1} = [gNg^{-1}, Q].$$

Decomposing $\mathfrak{g} = \mathfrak{su}(p+q)$ as in (7) we have

$$\left(\frac{dg}{dt} \cdot g^{-1}\right)^{\pm} = \mp\alpha(Q)(gNg^{-1})^{\pm}, \tag{28}$$

where $X^{\pm}$ denote the $\mathfrak{m}^{\pm}$-part of X, respectively. It is to be remarked from $\overline{\alpha(Q)} = -\alpha(Q)$ that the Hermitian conjugate of the $\mathfrak{m}^+$-part is equivalent to the $\mathfrak{m}^-$-part. Let $g(0) \in \mathbf{SU}(p+q)$ be an initial value for the $\mathfrak{m}^+$-part of (28) and $M(0) = g(0)Ng^{-1}(0) \in \mathfrak{su}(p+q)$. The exponential of matrix $\alpha(Q)tM(0)$ is nonsingular for any $t \in \mathbb{R}$ and always admits the unique decomposition

$$\exp(\alpha(Q)tM(0)) = A^{-1}(t)B(t). \tag{29}$$

Here $A(t) \in \mathbf{SU}(p+q)$, $A(0) = I$ and $B(t)$ is an element of the group of lower triangular matrices with positive diagonal entries such that $B(0) = I$. The decomposition (29) is carried out by a Gram-Schmidt orthogonalization (see, for example, [23]). The following proposition is the key for our construction of solutions of the Lax equation.

Proposition 4. *The initial value problem for the $\mathfrak{m}^+$-part of* (28) *is uniquely solved via the decomposition* (29) *as*

$$g(t) = A(t)g(0). \tag{30}$$

Proof. From (29), $\alpha(Q)AM(0)A^{-1} = -\frac{dA}{dt} \cdot A^{-1} + \frac{dB}{dt} \cdot B^{-1}$. We take the $\mathfrak{m}^+$-part of it and derive

$$(\frac{dA}{dt} \cdot A^{-1})^+ = -\alpha(Q)(AM(0)A^{-1})^+.$$

Setting $A(t) = g(t)g^{-1}(0)$, where $g(0)$ is the initial value, we see that $g(t)$ satisfies the $\mathfrak{m}^+$-part of (28), $(\frac{dg}{dt} \cdot g^{-1})^+ = -\alpha(Q)(gNg^{-1})^+$. Since the gradient flow satisfies the Lipschitz condition for any $t \in \mathbb{R}$, $g(t)$ defined by (30) gives a unique solution of the initial value problem.

Q.E.D.

With a help of Proposition 4, we can easily construct solutions of the Lax equations (12) and (13). Let $A(t)$ be the factor of the unique decomposition (29) and let $L(0) = g^{-1}(0)Qg(0)$ and $M(0) = g(0)Ng^{-1}(0)$ be initial values for the Lax equations (12) and (13) associated with the complex Grassmann manifold. Then

Proposition 5. *$L(t)$ and $M(t)$ defined by*

$$\begin{aligned} L(t) &= g^{-1}(0)A^{-1}(t)QA(t)g(0), \\ M(t) &= A(t)g(0)Ng^{-1}(0)A^{-1}(t) \end{aligned} \tag{31}$$

uniquely solve the initial value problems for (12) *and* (13), *respectively.*

It is well-known that the generalized Toda equations are solved by the QR decomposition of $\exp(tL(0))$ into a product of unitary and upper triangular matrices, where $L(0)$ is a Jacobi matrix [20]. We have shown that the Lax equations (12) and (13) in the specific case are also solved by a similar but slightly different way. From $M(0) = g(0)Ng^{-1}(0)$ we can compute $\exp(\alpha(Q)tM(0))$ and the factor $A(t)$ in (29) by a polynomial time algorithm. This may provide a suggestive picture of a polynomial time calculation process for the linear programming problem in the previous section.

§6. Discussions

We have established a nontrivial generalization and classification of the Lax equation which appears in a least squares problem. A very explicit way of a reduction of the Lax equation from reductive homogeneous spaces to symmetric spaces is given. We also found a method for solving an initial value problem for the Lax equation associated with the complex Grassmann manifold. The Lax equation has some applications to a least squares fitting problem and a linear programming problem. Any algorithm which finds 'optimal' solution must be iterative, and consequently, it describes a dynamical system. It is to be expected that evidence provided by the results will give an impetus to the further design of efficient algorithms for these problems as dynamical processes.

The Lax equation in the least squares problem induces an interior flow on a convex polytope in $\mathbb{R}^{p+q}$. Here let us recall a result by Tomei [21]. He proved that the level manifold of a generalized Toda equation is homeomorphic to a convex polytope in $\mathbb{R}^n$. The Lax equation (15) obviously includes the Toda equation in [21]. However it is not clear how to relate this Toda equation to the Lax equation (23). The level manifold of the original Toda equation itself is $\mathbb{R}^n$ [16]. Recently the author [18] obtained a different generalization of the Toda equation. The resulting level manifold is diffeomorphic to a certain cylinder. It would be interesting to extend the approach developed in this paper to the generalized Toda equation in [18] and clarify the nature of these connections.

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Green Function on Self-Similar Trees

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§1. Introduction

There are typical examples of symmetric homogeneous spaces where the canonical Green functions associated with the Laplacians are explicitly calculated [29]. However, it is usually difficult to calculate them on models without nice symmetry or rich group structure.

In this note we shall study one dimensional models which have self-similar structure instead of symmetric one and shall derive a functional equation via a scaling argument which determines in principle the Green function. An asymptotic expansion of the Green function will also be discussed which gives the decay order of the heat kernel as time goes to infinity.

We are partly motivated by fractal geometry. In fact self-similar trees are typical fractal models [19] [13] [11] and the asymptotic decay order of heat kernels is in general closely related to the so called spectral dimension of fractal models [25]. Note also that the tree structure is omnipresent in the natural world [19] [18].

We hope that our approach also enrich the knowledge on the spectral geometry (or differential geometry) and on the brownian motion on various models.

§2. Self-similar tree

Let X be the self-similar tree network depicted as in the following Figure 1.

Let the length of PP′, PQ and PR be respectively 1, r and s. Here the self-similarity means that the lengths of QS, QT, RU and RV are respectively r^2, rs, rs and s^2 and moreover other branches are defined in the same manner.

First we choose coordinate x such that the point O and P corresponds respectively to $x = 0$ and $x = 1/2$. To simplify the notation, Q

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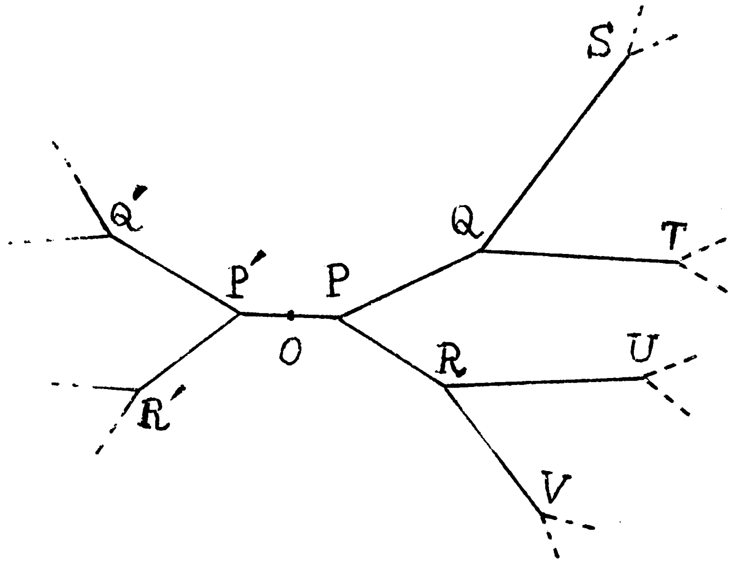

Fig. 1.

and R correspond respectively to $x = 1/2+r$ (on PQ) and $x = 1/2+s$ (on PR) without ambiguity. Then the canonical Green function $G(x, 0, \lambda)$ associated with the canonical Laplacian on X is formally defined for $\lambda \in \mathbf{C} - \mathbf{R}_-$ by

$$(1) \qquad (\lambda - \frac{d^2}{dx^2})G = \delta_0 \text{ and } G \to 0, \text{ as } \mathrm{dist}(x, \mathrm{O}) \to \infty.$$

Lemma. *Set $u(x) = G(x, 0, \lambda)$ modulo constant multiple. Then $u(x)$ can be written as follows.*

For $x \in OP$,

$$(2) \qquad u(x) = e^{\sqrt{\lambda}x} + \theta(\lambda)e^{-\sqrt{\lambda}x},$$

for $x \in PQ$,

$$(3) \qquad u(x) = c(\lambda)\{e^{\sqrt{\lambda}(x-r/2-1/2)} + \theta(r^2\lambda)e^{\sqrt{\lambda}(r/2+1/2-x)}\},$$

and for $x \in PR$,

$$(4) \qquad u(x) = d(\lambda)\{e^{\sqrt{\lambda}(x-s/2-1/2)} + \theta(s^2\lambda)e^{\sqrt{\lambda}(s/2+1/2-x)}\},$$

where c, d and θ can be determined via a functional identity.

Proof. First we give the outline of the proof. The Green function is harmonic on X–{O} which means that $u(x)$ is continuous and should satisfy the conservation law of heat flux at the point P. See [9] for example. These three equations determine the three functions c, d, and θ. Then it suffices to show that $u(x)$ defined above can be extended in such a way that it satisfies the same conditions at other nodal points Q, R, etc.

Next, let us see the details.

a) continuity at $x = 1/2$:

By setting $x = 1/2$ in (2), (3) and (4), we have

$$(5)\quad \begin{aligned} u(x) = e^{\sqrt{\lambda}/2} + \theta(\lambda)e^{-\sqrt{\lambda}/2} &= c(\lambda)\{e^{-r\sqrt{\lambda}/2} + \theta(r^2\lambda)e^{r\sqrt{\lambda}/2}\}, \\ &= d(\lambda)\{e^{-s\sqrt{\lambda}/2} + \theta(s^2\lambda)e^{s\sqrt{\lambda}/2}\}. \end{aligned}$$

b) conservation of heat flux:

At $x = 1/2$ the sum of first derivatives is equal to zero, i.e.,

$$(6)\quad \begin{aligned} e^{\sqrt{\lambda}/2} - \theta(\lambda)e^{-\sqrt{\lambda}/2} = c(\lambda)\{e^{-r\sqrt{\lambda}/2} - \theta(r^2\lambda)e^{r\sqrt{\lambda}/2}\} \\ + d(\lambda)\{e^{-s\sqrt{\lambda}/2} - \theta(s^2\lambda)e^{s\sqrt{\lambda}/2}\}. \end{aligned}$$

In this way c, d and θ are determined by (5) and (6) and we shall get a functional equation for θ.

Now let us proceed to show that if we define u appropriately on QS, QT, RU and RV, then the above conditions a) and b) are also satisfied on the points Q and R. In fact it suffices to define u for $x \in$ QS by

$$(7)\quad u(x) = c(\lambda)c(r^2\lambda)\{e^{\sqrt{\lambda}(x-r^2/2-r-1/2)} + \theta(r^4\lambda)e^{\sqrt{\lambda}(r^2/2+r+1/2-x)}\},$$

and for $x \in$ QT by

$$(8)\quad u(x) = c(\lambda)d(r^2\lambda)\{e^{\sqrt{\lambda}(x-rs/2-r-1/2)} + \theta(r^2s^2\lambda)e^{\sqrt{\lambda}(rs/2+r+1/2-x)}\}.$$

Then note that the conditions at Q are also satisfied for this u since by the definition $c(r^2\lambda)$, $d(r^2\lambda)$ and $\theta(r^2\lambda)$ in (7) and (8) correspond to $c(\lambda)$, $d(\lambda)$ and $\theta(\lambda)$ in (3) and (4) respectively. The same scaling argument can also be applied to other nodal points. Q.E.D.

§3. A functional equation

Let $w \equiv w(\lambda) = e^{\sqrt{\lambda}/2}$. Then (5) gives,

$$c(\lambda) = \frac{w + \theta w^{-1}}{w^{-r} + \theta(r^2\lambda)w^r} \text{ and } d(\lambda) = \frac{w + \theta w^{-1}}{w^{-s} + \theta(s^2\lambda)w^s},$$

and hence (6) implies

$$w^2 - \theta(\lambda) \over w^2 + \theta(\lambda)} = \frac{w^{-2r} - \theta(r^2\lambda)}{w^{-2r} + \theta(r^2\lambda)} + \frac{w^{-2s} - \theta(s^2\lambda)}{w^{-2s} + \theta(s^2\lambda)}. \tag{9}$$

Theorem 1. *Let* $H(\lambda) \equiv 2\sqrt{\lambda}\, G(0,0,\lambda)$ *and* $\phi(\lambda) = \tanh\sqrt{\lambda}/2$. *Then, H satisfies the following functional equation*:

$$\frac{1 - \phi(\lambda)H(\lambda)}{H(\lambda) - \phi(\lambda)} = \frac{1 + \phi(r^2\lambda)H(r^2\lambda)}{H(r^2\lambda) + \phi(r^2\lambda)} + \frac{1 + \phi(s^2\lambda)H(s^2\lambda)}{H(s^2\lambda) + \phi(s^2\lambda)}. \tag{10}$$

Proof. By the definition of G

$$G(0,0,\lambda) = \frac{\theta + 1}{2\sqrt{\lambda}\,(\theta - 1)},$$

which implies $\theta = \frac{H+1}{H-1}$. Therefore from (9) follows the theorem.

Q.E.D.

This may be one of few situations where a functional equation appears naturally in geometry in an explicit manner. Further, this functional equation may itself be interesting. In the next section, however, we shall investigate only the asymptotic behavior of the solution H near $\lambda = 0$, $\lambda \in \mathbf{C} - \mathbf{R}_-$.

§4. Asymptotic expansion I (conjecture)

Let us study how to determine the first terms of the asymptotic expansion of H, as λ tends to zero in $\mathbf{C} - \mathbf{R}_-$. To this end the Tauberian theorem is useful since the heat kernel P_t which is the inverse Laplace transform of G may provide information on G in some cases. We shall state our result as a conjecture since mathematically rigorous proof has not yet been completed.

Conjecture. *For* $g(\lambda) = G(0,0,\lambda)$, *we have the following expansion as* $\lambda \to 0$ *in* $\mathbf{C} - \mathbf{R}_-$:

Case 1. *If* $1/r + 1/s < 1$ *then* $g(\lambda) \sim a_\alpha \lambda^\alpha$, *where* α *satisfies*

$$r^{-2\alpha-1} + s^{-2\alpha-1} = 1 \qquad (-1/2 < \alpha < 0). \tag{11}$$

Case 2. *If* $1/r + 1/s = 1$ *then* $g(\lambda) \sim a_0 \log\lambda$, $\quad (a_0 \neq 0)$.

Case 3. *If* $1/r + 1/s > 1$ *and* $\max(r,s) > 1$, *then there are two subcases. Let* β *be defined by*

$$r^{2\beta-1} + s^{2\beta-1} = \bigl(1/r + 1/s\bigr)^2 \quad (0 < \beta). \tag{12}$$

(i) *Then if* β *is not an integer, i.e.,* $n < \beta < n+1$,

$$g(\lambda) \sim a_0 + a_1\lambda + \cdots + a_{n-1}\lambda^{n-1} + a_\beta\lambda^\beta \quad (a_\beta \neq 0),$$

and (ii) *if* β *is an integer, i.e.,* $\beta = n$,

$$g(\lambda) \sim a_0 + a_1\lambda + \cdots + a_{n-1}\lambda^{n-1} + a_n\lambda^n \log\lambda \quad (a_n \neq 0).$$

where $a_0 = (r+s+rs)/\{4(r+s-rs)\}$.

Let us briefly mention other cases.

If $\max(r,s) \leq 1$ then $g(\lambda)$ may be an analytic function in a neighborhood of $\lambda = 0$. In particular, if $r = s = 1$ this is shown in [8].

The case where $r + s < 1$ is exceptional since then the solution of the functional equation (10) is not unique. In fact as is easily seen, the following function g_1 also satisfies (10):

$$g_1(\lambda) = \frac{a_{-1}}{\lambda} + a_0 + \cdots, \text{ where } a_{-1} = \frac{1-r-s}{1+r+s}. \tag{13}$$

To understand this apparently peculiar situation it suffices to observe that in this case the total length of X is finite ($= \frac{1+r+s}{1-r-s}$). It turns out that g and g_1 correspond respectively to the Dirichlet and Neumann boundary conditions. Note that under both conditions our computation which uses the scaling property of the model X are justified. Further, note that these two boundary conditions make no difference provided $r + s \geq 1$. This phenomenon is explained by means of probability theory. See Proposition in the section 6.

§5. Asymptotic expansion II (computation)

We would like to present the idea of the "proof", although as we have already mentioned, some parts of the following argument have not yet been rigorously verified. The following is actually difficult for us to prove directly from (10).

$$\frac{\log H(\lambda)}{\log\lambda} \longrightarrow c\,, \text{ as } \lambda \to 0,\ \lambda \in \mathbf{C} - \mathbf{R}_-\,, \tag{14}$$

with a nonzero constant c, or equivalently via the Tauberian theorem

$$\text{(15)} \qquad \frac{\log P_t(0,0)}{\log t} \longrightarrow -c - 1/2\,, \text{ as } t \to \infty.$$

In the sequel we shall assume (14).

Remark 1. Since the left hand side of (15) is bounded, we need only to show

$$\frac{\log P_t(0,0)}{\log t} \text{ is a monotone function of } t \text{ for large } t\,.$$

However this kind of property seems to be unknown in general.

"Proof" of Conjecture. Case 1. $1/r + 1/s < 1$. First we know $0 \le c \le 1/2$ since it is known that the Brownian motion on X is recurrent if and only if $1/r + 1/s \le 1$. See [6] [17] [22] for example. Next, as $\phi(\lambda) = \sqrt{\lambda}/2 + \cdots$, (10) implies

$$\text{(16)} \qquad \frac{1}{H(\lambda) - \frac{\sqrt{\lambda}}{2}} = \frac{1}{H(r^2\lambda) + \frac{r\sqrt{\lambda}}{2}} + \frac{1}{H(s^2\lambda) + \frac{s\sqrt{\lambda}}{2}}\,,$$

up to an error $O(\sqrt{\lambda})$ and hence provided $c < 1/2$,

$$\text{(17)} \qquad 1 = \frac{H(\lambda)}{H(r^2\lambda)} + \frac{H(\lambda)}{H(s^2\lambda)}\,,$$

up to the same error.

Consequently, $c = -2\alpha - 1$ with α defined in (11).

Case 2. $1/r + 1/s = 1$. This case corresponds to $c = 1/2$ in previous case. We start from (16). Let us set $h(\lambda) = H(\lambda)/\sqrt{\lambda}$.

Then we can show that $h(\lambda)$ tends slowly to infinity as λ tends to 0 in view of (16) and thus

$$\text{(18)} \qquad \begin{aligned} &\frac{1}{h(\lambda)}\Big(1 + \frac{1}{2h(\lambda)}\Big) \\ =&\frac{1}{rh(r^2\lambda)}\Big(1 - \frac{1}{2h(r^2\lambda)}\Big) + \frac{1}{sh(s^2\lambda)}\Big(1 - \frac{1}{2h(s^2\lambda)}\Big), \end{aligned}$$

up to an error $o(1)$, i.e.,

$$\frac{h(\lambda) - h(r^2\lambda)}{r\,h(r^2\lambda)} + \frac{h(\lambda) - h(s^2\lambda)}{s\,h(s^2\lambda)} = \frac{1}{2h(\lambda)} + \frac{h(\lambda)}{2rh(r^2\lambda)} + \frac{h(\lambda)}{2sh(s^2\lambda)}.$$

Therefore, as $1/r + 1/s = 1$ we may expect that

(19) $$h(\lambda) - h(r^2\lambda) \sim \text{ constant} \neq 0, \text{ as } \lambda \to 0\,,$$

from which follows $h(\lambda) \sim c \log \lambda$, with nonzero c.

Case 3. $1/r + 1/s > 1$ and $\max(r, s) > 1$. In this case we take the Taylor expansion of ϕ up to the order $[\beta]$ and the coefficients a_k, $k = 0, 1, \cdots, [\beta]$ can be computed as far as

(20) $$(1/r + 1/s)^2 \neq r^{2k-1} + s^{2k-1}\,.$$

This fact is easily shown inductively by the same method of comparing terms of the same order. Since the the rest of the proof is analogous to previous cases, it is omitted. Q.E.D.

As a consequence, it turns out that if $r+s \geq 1$ then the solution H is unique. Further, we derive the following from the asymptotic expansion by the Tauberian theorem: *Provided our cojecture is correct.*

(21) $$\begin{aligned} P_t(0,0) &= O(t^{-\alpha-1}) \text{ if } 1/r + 1/s < 1, \\ &= O(t^{-\beta-1}) \text{ if } 1/r + 1/s \geq 1, \end{aligned}$$

where α and β are determined by (11)*and* (12) *respectively. Also note that P_t is rapidly decreasing if* $\max(r, s) \leq 1$.

Remark 2. If $r = s$ the above argument is made rigorous since there exists only one parameter r and the functional equation (10) becomes simpler.

§6. The remaining case : $0 < r < s = \infty$

In the previous sections r and s were finite. Let us treat the case where $s = \infty$. The method is similar and simpler.

Theorem 2. *Denote $e^{-\sqrt{\lambda}}$ by $\rho(\lambda)$. Then for H defined as in Theorem* 1 *, we get*

(22) $$H(r^2\lambda) = \frac{\psi(\lambda)H(\lambda) - \xi(\lambda)}{\eta(\lambda)H(\lambda) - \zeta(\lambda)}\,,$$

where

$$\begin{aligned} \psi &= \rho^{r-1} - \rho^r - 3\rho^{-1} - 1, \\ \xi &= \rho^{r-1} - \rho^r - 3\rho^{-1} + 1, \\ \eta &= \rho^{r-1} - \rho^r + 3\rho^{-1} + 1, \\ \textit{and} \quad \zeta &= \rho^{r-1} + \rho^r + 3\rho^{-1} - 1. \end{aligned}$$

Proof. It suffices to use again the scaling argument due to the self-similarity of X and the details of computation are omitted. Q.E.D.

As before, from Theorem 2. it follows then

Corollary. *As* $\lambda \to 0\,,\ \lambda \in \mathbf{C} - \mathbf{R}_-\,,$

$$(23)\qquad G(0,0,\lambda) = a_0 - \frac{r^2}{(1-r)^3(1+r)}\sqrt{\lambda} + \cdots \quad \textit{if } r < 1,$$
$$= \lambda^{-1/4} \textit{ if } r = 1 \textit{ and } = O(\frac{1}{\sqrt{\lambda}\,\log\,\lambda}) \quad \textit{if } r > 1.$$

Therefore, as $t \to \infty$, $P_t(0,0) = O(t^{-3/2})$, *if* $r < 1$, $= O(t^{-3/4})$, *if* $r = 1$, *and* $= O\left(\frac{1}{\sqrt{t}\,\log\,t}\right)$, *if* $r > 1$.

Remark 3. The case $r = 1$ was already established in [9] and the case $r > 1$ can be easily shown by an alternative probabilistic argument.

Now let us investigate the cases where $r < 1$ and either $s = \infty$ or $1 \leq s < \infty$ from a probabilistic point of view. The following proposition explains why the boundary condition doesn't affect the heat kernel [Green function].

Proposition. *If* $r < 1$ *and* $r + s \geq 1$, *then the Brownian particle never returns to the origin after hitting the boundary.*

Proof. We consider the case $s < \infty$. First we recall that $v(x) \equiv G(x,0,0)/\,G(0,0,0)$ is nothing but the probability that a particle starting from the point at x ever hits the origin (see [3] for example) and this v has already been treated in the preceding sections. In fact, $v(x) = u(x)/\,u(0)$ by the unicity of the Green function, where u is given by (2) and

$$\theta(\lambda) = \frac{2\sqrt{\lambda}G(\lambda) + 1}{2\sqrt{\lambda}G(\lambda) - 1} \sim -1 - 4a_0\sqrt{\lambda},$$

as $\lambda \to 0$. Therefore $c(0) = s/(r+s)$ and $d(0) = r/(r+s)$ in view of the identities in the section 4. Consequently if x is at the middle point of the interval whose length is $r^k s^l$, then

$$v(x) = O(\frac{r^k s^l}{(r+s)^{k+l}}\frac{1+\theta(r^{2k}s^{2l}\lambda)}{1+\theta(\lambda)}) = O((\frac{rs}{r+s})^{k+l}).$$

It follows then that $v(x) \to 0$ as x tends to a boundary point. The case where $s = \infty$ is similar. In this case $c(0) = 1$ but the function θ yields the same conclusion. Q.E.D.

The above phenomenon is a consequence of the dangling effect of H. Kesten [14]. It may be interesting to note that the decay order of the heat kernel does not depend on the particular value of $r < 1$.

§7. Generalizations and questions

Apart from making arguments rigorous in some parts of the section 5, there remains questions and possibility of generalizations.

(1) Trees without strict self-similarity. If trees are not exactly self-similar, our method of explicit computation can not be applied in general. Nevertheless, if the kth branches have lengths not 2^k but $l_k = k^\gamma 2^k$, $-\infty < \gamma < \infty$ for example, small correction will be sufficient for the relevant term in the expansion. In fact, if $r = s = k^\gamma 2^k$ then it is likely that

$$P_t(0,0) = O(\frac{(\log t)^\gamma}{t}). \tag{24}$$

However this question has to be investigated more systematically. Moreover, when the number of branches are not fixed at each nodal point, another difficulty is caused, although self-similar trees with certain periodicity may be treated by the same method as ours.

(2) Models of higher dimension.

(3) On point spectrum. I am indebted to Prof. K. Aomoto for this problem. What can we reduce from (10) on the point spectrum? We have only a partial answer to this question: If λ_0 is a point spectrum the Laplacian, then $G(0,0,\lambda_0) = \infty$, therefore putting $H(\lambda_0) = \infty$ in (10) we get formally

$$-\phi(\lambda_0) = \phi(r^2\lambda_0) + \phi(s^2\lambda_0). \tag{25}$$

Besides this, very few seem to be known on this question. We add only two facts:

It may be interesting to compare our models with the lattice models where the second order differential operator (Laplacian) is replaced by the adjacent second order difference operator when $r = 1$. See [2] [3] [5]

[6] [10] etc. Point spectrums exist in the former model and not in the latter [2].

Another curious thing occurs when $r + s < 1$. It seems that every $\lambda \in \mathbf{R}_-$ is actually a point spectrum for the Neumann problem (i.e., with reflecting boundary condition in terms of probability theory) with respect to the usual line measure on X.

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On a Theorem of Edmonds

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§1. Introduction

For an action of a cyclic group of odd order $m \geqq 3$ on a manifold, the normal bundle of the fixed point set is orientable. It is false for actions of the cyclic group of order 2. Edmonds showed the following

Theorem (Edmonds [E]). *If $\mathbb{Z}_2$ acts smoothly on an n dimensional spin manifold M, preserving its orientation and spin structure, then the fixed point set $F = M^{\mathbb{Z}_2}$ is orientable.*

Bott and Taubes gave another proof in [B-T]. The purpose of this short note is to give an elementary proof of this theorem and consider the spin^c case.

The author is grateful to Professor A. Hattori for showing him private note [H].

§2. Review on Clifford algebras and spin structures

Let V be a vector space with an positive definite inner product. The Clifford algebra $\mathrm{Cl}(V)$ associated to V is defined as the quotient algebra of the tensor algebra over V by the ideal generated by $v \otimes v + |v|^2$ where $v \in V$. $\mathrm{Cl}(V)$ is not an algebra with $\mathbb{Z}$-grading, but there is a filtration as follows:

$$\mathrm{Cl}(V)^k = \textit{linear span of } \{v_1 \cdots v_j \in \mathrm{Cl}(V); v_i \in V, j \leqq k\}.$$

It is easy to see that the associated graded module of filtered module $\mathrm{Cl}(V)$ is the exterior algebra $\Lambda(V)$. $\mathrm{Cl}(V)$ contains the spin group $\mathrm{Spin}(V)$ which is the double covering group of $SO(V)$. More precisely $\mathrm{Spin}(V) = \mathrm{Pin}(V) \cap \mathrm{Cl}(V)_0$, where $\mathrm{Pin}(V)$ is the multiplicative group generated by unit vectors in V, and $\mathrm{Cl}(V)_0$ is the even part of $\mathrm{Cl}(V)$

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with respect to the $\mathbb{Z}_2$-grading. The $SO(V)$ action on V extends to the action on $\mathrm{Cl}(V)$ as algebra automorphisms.

Let M be a Riemannian manifold. $\mathrm{Cl}(M)$ denotes the Clifford algebra bundle on M associated to the tangent bundle TM. A spin structure on an n-dimensional oriented manifold M is a principal $\mathrm{Spin}(n)$-bundle $\widetilde{P}$ on M, which is a double covering of the principal $SO(n)$-bundle P associated to TM. Remark that $\mathrm{Cl}(M)$ contains $\widetilde{P} \times_{\mathrm{Ad}} \mathrm{Spin}(n)$.

§3. Proof

Let τ be an involution on a spin manifold M, preserving orientation and a spin structure $\widetilde{P}$. Then we have $\tau^*\widetilde{P} \cong \widetilde{P}$, which implies that τ can be lifted to $\tilde{\tau} : \widetilde{P} \to \widetilde{P}$. As τ is an involution, $\tilde{\tau}^2$ is a covering transformation of $\widetilde{P} \to P$ and an bundle automorphism of $\widetilde{P} \to M$. Restricted to the fixed point set $F = M^\tau$, $\tilde{\tau}_F : \widetilde{P}|_F \to \widetilde{P}|_F$ is a bundle automorphism acting trivially on the base space F, i.e. a gauge transformation. It is well known that it can be seen as a section of the adjoint bundle. In our case, $\tilde{\tau}_F$ defines a section of $\mathrm{Cl}(M)|_F$, where we choose a $\mathbb{Z}_2$-invariant Riemannian metric on M. Let $TM|_F = TF \oplus N$ be the decomposition into the tangent bundle TF of F and the normal bundle N. Here we recall the following fact.

Fact. *Let $U = V \oplus W$ be a direct sum of vector spaces with positive definite inner products. Only elements in $\mathrm{Cl}(V \oplus W)$ which act on V by -1 and on W by 1 are $\pm e_1 \cdot e_2 \cdot \cdots \cdot e_l$, where $\{e_1, e_2, \cdots, e_l\}$ is an orthonormal basis of V.*

Let $p \in F$. Then τ acts on N_p by -1 and T_pF by 1. Since τ acts on M preserving orientation, the rank of N is even and $\pm v_1 \cdot v_2 \cdot \cdots \cdot v_l$ is an element of $\mathrm{Spin}(N_p)$, where $\{v_1, v_2, \cdots, v_l\}$ is an orthonormal basis of N_p. Thus $\tilde{\tau}_F(p)$ corresponds to $\pm v_1 \cdot v_2 \cdot \cdots \cdot v_l$. As we have seen in §2, the graded algebra bundle associated to the filtered algebra bundle $\mathrm{Cl}(M)$ is the exterior algebra bundle of TM. $\tilde{\tau}_F$ belongs to $\mathrm{Cl}(M)^l$, and corresponds to $\pm v_1 \wedge v_2 \wedge \cdots \wedge v_l$, therefore it determines the orientation of the normal bundle N, which implies the orientablity of the fixed point set F.

§4. Spin^c case

Let $\mathrm{Spin}^c(n) = \mathrm{Spin}(n) \times_{\mathbb{Z}_2} S^1$. A spin^c structure on an n-dimensional oriented manifold is a principal $\mathrm{Spin}^c(n)$-bundle Q such that $Q \times_\rho SO(n)$ is the principal $SO(n)$-bundle associated to TM, and ρ

is the natural homomorphism from $\mathrm{Spin}^c(n)$ to $SO(n)$. We can show the following

Proposition. *Let M be a spin^c manifold and Q a spin^c structure on M. If an smooth involution τ on M can be lifted to Q as a periodic mapping, then the fixed point set F is orientable.*

Proof. As in §2, N denotes the normal bundle of F, and $\tilde{\tau}_F : Q|_F \to Q|_F$ denotes the lifting of τ restricted to F with period s. $\tilde{\tau}_F$ defines a section of $\mathrm{Cl}(M)|_F \otimes \mathbb{C}$, and a section of $\bigwedge(TM)|_F \otimes \mathbb{C}$. For $p \in F$, $\tilde{\tau}_F$ corresponds to $\exp(\frac{2\pi i m}{s}) \cdot v_1 \wedge v_2 \wedge \cdots \wedge v_l$, where $\{v_1, v_2, \cdots, v_l\}$ is an orthonormal vasis of N_p. Consider $\mathrm{Re}(\tilde{\tau}_F)$ or $\mathrm{Im}(\tilde{\tau}_F)$, it defines an orientation of N, which implies the conclusion.

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On the L^2 Cohomology Groups of Isolated Singularities

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Dedicated to Professor Noboru Tanaka on his 60th birthday

Introduction

Let (V, x) be a (complex) n-dimensional isolated singularity. Given a Hermitian metric on $V \backslash \{x\}$, say ds^2, the r-th L^2 cohomology group of V at x is defined as the inductive limit of the L^2 de Rham cohomology groups $H^r_{(2)}(U \backslash \{x\}, ds^2)$, where U runs through the neighbourhoods of x. Recently, L. Saper [10] established a remarkable result that there exist Kähler metrics on $V \backslash \{x\}$, complete near x, for which the r-th L^2 cohomology groups of V at x are zero whenever $r \geq n$. It implies an important fact that the intersection cohomology group of a Kähler variety with isolated singularities carries a canonical Hodge structure. Relying on Saper's result, the author could show that the L^2 cohomology vanishing as above is also true with respect to the restriction of the euclidean metric associated to any holomorphic embedding $(V, x) \hookrightarrow (\mathbf{C}^N, 0)$ (cf. [7]). The purpose of the present article is to complement these works by giving a self-contained version of the latter work. Namely we shall first establish an abstract vanishing theorem as a consequence of a new L^2 estimate with respect to a certain family of metrics and weights which seems to be of interest in itself. Then we shall proceed to apply it to prove a vanishing theorem of Saper type with respect to a certain class of complete Kähler metrics which is actually wider than Saper's ones. Hopefully our method will be available to investigate the L^2 cohomology of spaces with non-isolated singularities. Next we shall give a new proof of our previous result mentioned above. The argument here is essentially the same except that we do not appeal to the existence of a projective variety containing (V, x) and tried to make the argument more transparent. Therefore some part of the proof will be only sketchy.

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§1. Notation and basic facts

We shall first prepare notations and state without proofs several known facts that we use afterwards.

Let (X, ds^2) be a Hermitian manifold of dimension n, and let $C_0(X)$ be the set of compactly supported $\mathbf{C}$-valued C^∞ differential forms on X. We set

$$C_0^r(X) := \{u \in C_0(X); \deg u = r\}$$

and

$$C_0^{p,q}(X) := \{u \in C_0^{p+q}(X); u \text{ is of type } (p,q)\}.$$

Let φ be any real-valued C^∞ function on X. We set

$$(u,v)_\varphi := \int_X e^{-\varphi} u \wedge \overline{*v} \quad \text{for} \quad u, v \in C_0(X),$$

where $*(= *_{ds^2})$ denotes the Hodge's star operator and $\overline{*v}$ the complex conjugate of $*v$. Then $C_0(X)$ is a pre-Hilbert space equipped with the above inner product. We define $L_\varphi(X)(= L_\varphi(X, ds^2))$ to be the completion of $C_0(X)$ with respect to the associated L^2 norm $\| \ \|_\varphi = \sqrt{(\ ,\ )_\varphi}$. We shall refer to φ as the weight of the L^2 norm. For any densely defined closed linear operator, say T, from $L_\varphi(X)$ into itself, we denote its domain, image and kernel by Dom T, Im T and Ker T, respectively. The adjoint of T will be denoted by T_φ^*. As usual φ will not be referred to if $\varphi \equiv 0$. By d we shall denote the exterior derivative, and by $\bar\partial$ (resp.∂) the $(0,1)$-component (resp. $(1,0)$-component) of d. Their maximal closed extensions will be denoted by the same symbol unless there is fear of confusion. By an abuse of language we often identify $\partial\bar\partial\varphi$ with the complex Hessian of φ.

Proposition 0. *Suppose that there exists a C^∞ function $\psi : X \to \mathbf{R}$ such that*

1) $ds^2 = 2\partial\bar\partial\psi$
2) $|\partial\psi|$ *is bounded.*

Then

$$\|u\| \le C(\|\bar\partial u\| + \|\bar\partial^* u\|) \le C(\|du\| + \|d^* u\|)$$

for any $u \in C_0^r(X)$ with $r \neq n$. Here $C = 4 \sup |\partial\psi|$.

For the proof see [8].

We set

$$H^r_{(2)}(= H^r_{(2)}(X, ds^2)) := \operatorname{Ker} d \cap L^r(X) / \operatorname{Im} d \cap L^r(X)$$

$$H^{p,q}_{(2)}(X)(= H^{p,q}_{(2)}(X, ds^2)) := \operatorname{Ker} \bar\partial \cap L^{p,q}(X) / \operatorname{Im} \bar\partial \cap L^{p,q}(X).$$

One can deduce from Proposition 0 the following.

Proposition 1. *Let (X, ds^2) be a complete Kähler manifold equipped with ψ satisfying* 1) *and* 2)*. Then $H^r_{(2)}(X)$ (resp. $H^{p,q}_{(2)}(X)$) is zero whenever $r \neq n$ (resp. $p+q \neq n$). Moreover $H^n_{(2)}(X)$ and $H^{p,n-p}_{(2)}(X)$ $(0 \leq p \leq n)$ are Hausdorff spaces with respect to the quotient topology.*

For the argument needed here, see [1] or [2].

Let V be a reduced irreducible complex space of dimension n which is properly embedded into $\mathbf{C}^N$ so that V contains the origin as the possibly unique singular point. Let $z = (z_1, \cdots, z_N)$ be the coordinate of $\mathbf{C}^N$ and let $\|z\| := (\sum_{i=1}^N |z_i|^2)^{1/2}$. We put $V' = V \backslash \{0\}$ and denote by $\|z\|_{V'}$ the restriction of the function $\|z\|$ to V'. Then $-\partial\bar\partial \log \log (\delta \|z\|_{V'}^{-1})$ defines a complete Kähler metric on $V'_\delta := \{z \in V'; \|z\| < \delta\}$. As a corollary of Proposition 1 we have

Proposition 2.

$$H^r_{(2)}(V'_\delta, -\partial\bar\partial \log \log (\delta \|z\|_{V'}^{-1})) = 0 \quad if \quad r \neq n$$

and

$$H^{p,q}_{(2)}(V'_\delta, -\partial\bar\partial \log \log (\delta \|z\|_{V'}^{-1})) = 0 \quad if \quad p+q \neq n.$$

Moreover
$H^n_{(2)}(V'_\delta, -\partial\bar\partial \log \log (\delta \|z\|_{V'}^{-1}))$ and $H^{p,n-p}_{(2)}(V'_\delta, -\partial\bar\partial \log \log (\delta \|z\|_{V'}^{-1}))$ are Hausdorff spaces.

Proposition 3.

$$\lim_{\delta \to 0} H^r_{(2)}(V'_\delta, \partial\bar\partial(-\log \log \|z\|_{V'}^{-1})) = 0 \quad if \quad r > n$$

and

$$\lim_{\delta \to 0} H^{p,q}_{(2)}(V'_\delta, \partial\bar\partial(-\log \log \|z\|_{V'}^{-1})) = 0 \quad if \quad p+q > n.$$

Furthermore the homomorphism

$$\lim_{\delta \to 0} H^r_{(2)}(V'_\delta, \partial\bar\partial(-\log \log \|z\|_{V'}^{-1})) \to \lim_{\delta \to 0} H^r(V'_\delta)$$

is bijective if $r < n-1$ *and injective if* $r = n-1$, *and the homomorphism*

$$\lim_{\delta\to 0} H^{p,q}_{(2)}(V'_\delta, \partial\bar{\partial}(-\log\log\|z\|^{-1}_{V'})) \to \lim_{\delta\to 0} H^{p,q}(V'_\delta)$$

is bijective if $p+q < n-1$ *and injective if* $p+q = n-1$. *Here* $H^r(\,\cdot\,)$ *and* $H^{p,q}(\,\cdot\,)$ *denote respectively the* r*-th de Rham cohomology group and the Dolbeault cohomology group of type* (p,q).

We put $V_\delta := \{z \in V; \|z\| < \delta\}$ and

$$H^r_{(2)}(V_\delta) := H^r_{(2)}(V'_\delta, \partial\bar{\partial}\|z\|^2_{V'})$$

$$H^{p,q}_{(2)}(V_\delta) := H^{p,q}_{(2)}(V'_\delta, \partial\bar{\partial}\|z\|^2_{V'})$$

by an abuse of notation.

Proposition 4.
(1) $\lim_{\delta\to 0} H^r_{(2)}(V_\delta) = \lim_{\delta\to 0} H^{p,q}_{(2)}(V_\delta) = 0$ *if* $r, p+q > n$.
(2) *The homomorphism*

$$\lim_{\delta\to 0} H^r_{(2)}(V_\delta) \to \lim_{\delta\to 0} H^r(V'_\delta)$$

is bijective if $r < n-1$ *and injective if* $r = n-1$, *and the homomorphism*

$$H^{p,q}_{(2)}(V_\delta) \to H^{p,q}(V'_\delta)$$

is bijective if $p+q < n-1$ *and injective* $p+q = n-1$.

We note that (1) follows from Proposition 3 via a singular perturbation (cf. [5] or [9]), whereas (2) is a consequence of direct application of Andreottei-Vesentini's vanishing theorem (cf. [5, Supplement]).

So far the results have quite straightforward and self-contained proofs. However, to proceed further we must rely on the following deep result.

Theorem (Hironaka [H]). *There exists a complex submanifold* $\tilde{V} \subset \mathbf{C}^N \times \mathbf{P}^{N'}$ *for some* N' *such that the projection* $\mathbf{C}^N \times \mathbf{P}^{N'} \to \mathbf{C}^N$ *induces a proper bimeromorphic morphism from* $\tilde{V}$ *onto* V, *say* π. *Moreover* $(\tilde{V}, \pi)$ *can be chosen so that*

i) $\pi|_{\tilde{V}\setminus\pi^{-1}(0)}$ *is bijective.*

ii) $\pi^{-1}(0)$ *is a divisor whose associated line bundle is isomorphic to the restriction of the pull-back, by the projection* $\mathbf{C}^N \times \mathbf{P}^{N'} \to \mathbf{P}^{N'}$, *of the dual of the hyperplane section bundle.*

iii) *The support of* $\pi^{-1}(0)$ *is a divisor of simple normal crossings.*

Once for all we fix a $(\tilde{V}, \pi)$ satisfying i)~iii). By iii) there exist nonsingular divisors $E_1, \cdots, E_m$ ($E_i \neq E_j$ if $i \neq j$) such that

$$\operatorname{supp}\pi^{-1}(0) = E_1 \cup \cdots \cup E_m.$$

By $(v, w) = (v_1, \cdots, v_k, w_1, \cdots, w_{n-k})$ we denote a coordinate around a k-ple point of $\operatorname{supp}\pi^{-1}(0)$ such that $v_1 \cdot \cdots \cdot v_k = 0$ is a local defining equation of $\operatorname{supp}\pi^{-1}(0)$. By ii) there exist positive integers $p_1, \cdots, p_m$ such that the sheaf $\otimes_{i=1}^m \mathcal{O}(-E_i)^{p_i}$ is very ample. Hence there exists a nonsingular integral $m \times m$ matrix (p_{ij}) with $p_{ij} > 0$ such that

1) $\otimes_{i=1}^m \mathcal{O}(-E_i)^{p_{ij}}$ are ample for all j.

2) Let $1 \leq i_1 < \cdots < i_k \leq m$ $(1 \leq k \leq m)$. Then $\det(p_{i_\alpha i_\beta})_{\alpha,\beta=1}^k \neq 0$ whenever $\bigcap_{\alpha=1}^{k} E_{i_\alpha} \neq \emptyset$.

Therefore we can find C^∞ metrics along the fibers of $\otimes_{i=1}^m \mathcal{O}(-E_i)^{p_{ij}}$, say a_j, whose curvature form is positive. Let $s_i \in \Gamma(\tilde{V}, \mathcal{O}(E_i))$ be so chosen that $E_i = \{y \in \tilde{V}; s_i(y) = 0\}$, and let σ_j be the length of $s_1^{p_{1j}} \cdot \cdots \cdot s_m^{p_{mj}}$ with respect to a_j. Then $-\log \log \sigma_j^{-1}$ is a plurisubharmonic function on a neighbourhood of $\operatorname{supp}\pi^{-1}(0)$, say U. We set

$$d\sigma^2 := -\partial\bar{\partial} \sum_{j=1}^{m} \log \log \sigma_j^{-1} \quad \text{on} \quad U \backslash \operatorname{supp}\pi^{-1}(0).$$

Then $d\sigma^2$ may well be identified via π with a Kähler metric on $V_\delta' := V_\delta \backslash \{0\}$ for sufficiently small δ. We shall refer to $d\sigma^2$ as a Saper metric afterwards. We note that, around any k-ple point of $\operatorname{supp}\pi^{-1}(0)$,

$$(3) \qquad d\sigma^2 \sim \sum_{i=1}^{k} \frac{dv_i d\overline{v_i}}{|v_i|^2 \log^2 |v_i \cdot \cdots \cdot v_k|^{-1}} + \frac{1}{\log |v_1 \cdot \cdots \cdot v_k|^{-1}} \Big(\sum_{i=1}^{k} dv_i d\overline{v_i} + \sum_{j=1}^{n-k} dw_j d\overline{w_j}\Big),$$

where $A \sim B$ means that there exists a $c \in (0, \infty)$ such that $c^{-1}A \leq B \leq cA$.

The following is also an immediate consequence of Proposition 3.

Proposition 5. *For sufficiently small δ and a Saper metric $d\sigma^2$ on V'_δ,*

1) $H^r_{(2)}(V'_\delta, d\sigma^2) = H^{p,q}_{(2)}(V'_\delta, d\sigma^2) = 0$ *if* $r, p+q > n$.

2) *The canonical homomorphisms*

$$H^r_{(2)}(V'_\delta, d\sigma^2) \to H^r(V'_\delta)$$

are bijective if $r < n-1$ and injective if $r = n-1$.

3) *The canonical homomorphisms*

$$H^{p,q}_{(2)}(V'_\delta, d\sigma^2) \to H^{p,q}(V'_\delta)$$

are bijective if $p+q < n-1$ and injective if $p+q = n-1$.

We call a Saper metric $d\sigma^2$ dominating if $d\sigma^2 \gtrsim -\partial\bar{\partial} \log \log \|z\|^{-1}_{V'}$. Here $A \gtrsim B$ means that $cA \geq B$ for some $c \in (0, \infty)$. Existence of a dominating Saper metric is assured also by Hironaka's theorem. Namely, applying Hironaka's desingularization theorem in a more precise form, we can find $(\tilde{V}, \pi)$ so that the maximal ideal of 0 is pulled-back by π to an invertible sheaf (cf. [H]). For such $\tilde{V}$ it is clear that $d\sigma^2 \gtrsim -\partial\bar{\partial} \log \log \|z\|^{-1}_{V'}$.

§2. An abstract L^2 vanishing theorem

In what follows we assume that X admits a C^∞ negative plurisubharmonic function φ such that $-\log(-\varphi)$ is strictly plurisubharmonic, and derive an L^2 estimate for the $\bar{\partial}$-operator with respect to the metrics $d\sigma^2_\varepsilon := 2(-\partial\bar{\partial}\log(-\varphi) + \varepsilon\partial\bar{\partial}\varphi)$ $(\varepsilon \geq 0)$ and weights $-\varepsilon\varphi$.

For simplicity we set

$$L_\varepsilon(X) := L_{-\varepsilon\varphi}(X, d\sigma^2_\varepsilon)$$

$$(u, v)_\varepsilon := \int_X e^{\varepsilon\varphi} u \wedge \overline{*_\varepsilon v},$$

where $*_\varepsilon$ denotes the Hodge's star operator with respect to $d\sigma^2_\varepsilon$, and $\|u\|_\varepsilon := \sqrt{(u,u)_\varepsilon}$.

Note that $L_\varepsilon(X) \supset L_\delta(X)$ if $\varepsilon > \delta$.

The adjoint of an operator T with respect to $(\ ,\ )_\varepsilon$ will be denoted by T^*_ε by an abuse of notation. For simplicity we set $\Lambda_\varepsilon := *^{-1}_\varepsilon e(\sqrt{-1}(\partial\bar{\partial}(-\log(-\varphi) + \varepsilon\varphi)))*_\varepsilon$, where $e(\,\cdot\,)$ stands for the exterior multiplication from the left hand side.

Proposition 6. *If* $p+q<n$,

$$\|u\|_\varepsilon^2 \le 8(\|\bar\partial u\|_\varepsilon^2 + \|\bar\partial_\varepsilon^* u\|_\varepsilon^2)$$

for any $u \in C_0^{p,q}(X)$ *and* $\varepsilon > 0$.

Proof. Since $|\partial \log(-\varphi)|_{d\sigma_\varepsilon^2} \le 1$ we have

$$\begin{aligned}&([\sqrt{-1}e(\partial\bar\partial \log(-\varphi)), \Lambda_\varepsilon]u, u)_\varepsilon \\ \le& \|u\|_\varepsilon(\|\bar\partial u\|_\varepsilon + \|\bar\partial_\varepsilon^* u\|_\varepsilon + \|\partial^* u\|_\varepsilon + \|\partial_\varepsilon u\|_\varepsilon).\end{aligned}$$

Here we put $\partial_\varepsilon := (\partial^*)_\varepsilon^*$. Hence for any $C \ge 1$ and $\sigma > 0$ we have

$$\begin{aligned}(4)\qquad &([\sqrt{-1}e(\partial\bar\partial \log(-\varphi)), \Lambda_\varepsilon]u, u)_\varepsilon \\ \le& 2\sigma\|u\|_\varepsilon^2 + \frac{1}{2}C\sigma^{-1}(\|\bar\partial u\|_\varepsilon^2 + \|\bar\partial_\varepsilon^* u\|_\varepsilon^2 + \|\partial^* u\|_\varepsilon^2 + \|\partial_\varepsilon u\|_\varepsilon^2).\end{aligned}$$

Since

$$\begin{aligned}&\|\partial^* u\|_\varepsilon^2 + \|\partial_\varepsilon u\|_\varepsilon^2 \\ =& \|\bar\partial u\|_\varepsilon^2 + \|\bar\partial_\varepsilon^* u\|_\varepsilon^2 + ([\sqrt{-1}e(\varepsilon\partial\bar\partial\varphi), \Lambda_\varepsilon]u, u)_\varepsilon,\end{aligned}$$

we have

$$\begin{aligned}&([\sqrt{-1}e(\partial\bar\partial \log(-\varphi) - \tfrac{\varepsilon C}{2\sigma}\partial\bar\partial\varphi), \Lambda_\varepsilon]u, u)_\varepsilon - 2\sigma\|u\|_\varphi^2 \\ \le& C\sigma^{-1}(\|\bar\partial u\|_\varepsilon^2 + \|\bar\partial_\varepsilon^* u\|_\varepsilon^2),\end{aligned}$$

so that

$$\begin{aligned}&((1-\frac{C}{2\sigma})[\sqrt{-1}e(\varepsilon\partial\bar\partial\varphi), \Lambda_\varepsilon]u, u)_\varepsilon + (1-2\sigma)\|u\|_\varepsilon^2 \\ \le& C\sigma^{-1}(\|\bar\partial u\|_\varepsilon^2 + \|\bar\partial_\varepsilon^* u\|_\varepsilon^2).\end{aligned}$$

Since $\partial\bar\partial\log(-\varphi) = -\varphi^{-1}\partial\bar\partial\varphi + \varphi^{-2}\partial\varphi\bar\partial\varphi$,

$$([\sqrt{-1}e(\partial\bar\partial\varphi), \Lambda_\varepsilon]u, u)_\varepsilon \le 0$$

if $\deg u < n$. Hence, letting $\sigma = \dfrac{1}{4}$ and $C = 1$ we obtain

$$\|u\|_\varepsilon^2 \le 8(\|\bar\partial u\|_\varepsilon^2 + \|\bar\partial_\varepsilon^* u\|_\varepsilon^2)$$

for all $u \in C_0^{p,q}(X)$ with $p+q<n$. □

Now we can state our vanishing theorem.

Theorem 7. *Let X be a complex manifold of dimension n admitting a negative plurisubharmonic function φ such that $-\partial\bar{\partial}\log(-\varphi)$ is a complet Kähler metric. Take any $f \in L^{p,q}(X, -\partial\bar{\partial}\log(-\varphi))$ with $p+q \leq n$. Then $f \in \operatorname{Im}\bar{\partial}$ if and only if there exist $g_\varepsilon \in L_\varepsilon(X)$ for every $\varepsilon > 0$ such that $\bar{\partial}g_\varepsilon = f$.*

Proof. Since $L_\varepsilon(X) \supset L_0(X)$, 'only if' part is clear. To prove 'if' part, one has only to apply Proposition 6.

□

We note that

$$\partial_\varepsilon u = \partial u + \varepsilon\partial\varphi \wedge u.$$

Hence

$$\|\partial u\|^2_{2\varepsilon} \leq \|\partial_\varepsilon u\|^2_\varepsilon + 4e^{-2}\|u\|^2_\varepsilon,$$

since $\sup e^{\varepsilon\varphi}|\varepsilon\partial\varphi|^2_{d\sigma_0^2} \leq \sup_{t\in(-\infty,0)} e^t \cdot t^2 = 4e^{-2}$.

Therefore we have

$$\|\partial g\|^2_{2\varepsilon} \leq A(\|g\|^2_\varepsilon + \|\bar{\partial}g\|^2_\varepsilon + \|\bar{\partial}^*_\varepsilon g\|^2_\varepsilon) \tag{5}$$

for any $g \in \operatorname{Dom}(\bar{\partial} + \bar{\partial}^*_\varepsilon)$. Here we may choose $A = n \cdot 2^n + 4e^{-2}$. Thus we obtain the following version of Theorem 7.

Theorem 8. *Let X and φ be as above, and take any $f \in L^r(X, -\partial\bar{\partial}\log(-\varphi))$ with $r \leq n$. Then $f \in \operatorname{Im} d$ if and only if there exist $g_\varepsilon \in L^{r-1}_\varepsilon(X)$ for every $\varepsilon > 0$ such that $dg_\varepsilon = f$.*

§3. Application of a topological lemma

Let $(V, 0) \hookrightarrow (\mathbf{C}^N, 0)$ be as before, and let $\rho : W \to V$ be any proper holomorphic map such that $\rho|_{W\setminus\rho^{-1}(0)}$ is bijective and W is nonsingular. We set $W_\delta = \rho^{-1}(V_\delta)$ and $W'_\delta = W_\delta \setminus \rho^{-1}(0)$. The following fact, first pointed out in [4], is crucial for our purpose.

Lemma 9. *The canonical homomorphisms*

$$H^r(W_\delta) \to H^r(\partial W_\delta)$$

are surjective for $r < n$ if $0 < \delta \ll 1$. Here ∂W_δ denotes the boundary of W_δ.

For the proof, see [3] or [6].

Let ds^2 be a Hermitian metric on V'. We put

$$H^r_{(2),0}(W'_\delta, \rho^* ds^2) :=$$
$$\{u \in L^r(W'_\delta, \rho^* ds^2); du = 0 \quad \text{and} \quad \operatorname{supp} u \Subset W_\delta\}$$
$$/\{u \in L^r(W'_\delta, \rho^* ds^2); \exists v \in L^{r-1}(W'_\delta, \rho^* ds^2) \quad \text{such that}$$
$$\operatorname{supp} v \Subset W_\delta \quad \text{and} \quad dv = u\}.$$

Then Lemma 9 implies the following.

Proposition 10. *Let $r < n$. Suppose that the metric ds^2 enjoys a property that $C^r_0(W_\delta) \subset L^r(W'_\delta, \rho^* ds^2)$ for $\delta > 0$. Then the canonical homomorphism*

$$H^{r+1}_{(2),0}(W'_\delta, \rho^* ds^2) \to H^{r+1}_{(2)}(W'_\delta, \rho^* ds^2)$$

is injective for $0 < \delta \ll 1$.

Proof. Let $u \in L^{r+1}(W'_\delta, \rho^* ds^2), \operatorname{supp} u \Subset W_\delta$ and $du = 0$. Assume that there exist a $v \in L^r(W'_\delta, \rho^* ds^2)$ satisfying $dv = u$. If δ is chosen so that $d\|z\|_{V'} \neq 0$ on $\partial V_{\delta'}$ for all $\delta' \in (0, \delta]$, from Lemma 9 there exists a measurable $r-1$ form g on W_δ with $\operatorname{supp} g \cap W_{\delta/2} = \emptyset$ such that g and dg are locally square integrable on W_δ and a locally square integrable d-closed r form w on W_δ, C^∞ on $W_{\delta/2}$, such that $v = w + dg$ outside a compact subset of W_δ. By assumption $v - w - dg \in L^r(W'_\delta, \rho^* ds^2)$. Since $\operatorname{supp}(v - w - dg) \Subset W_\delta$ and $d(v - w - dg) = u$, the assertion was proved. □

Corollary 11. *Under the above situation, suppose moreover that ds^2 is complete and $r = n - 1$. Then the homomorphism*

$$H^n_{(2),0}(W'_\delta, \rho^* ds^2) \to H^n_{(2)}(W'_\delta, \rho^* ds^2) \quad (0 < \delta \ll 1)$$

has a dense image.

Proposition 12. *Let $d\sigma^2$ be a Saper metric on V'_δ associated to a desingularization $\pi : \tilde{V} \to V$, and let $\tilde{V}_\delta := \pi^{-1}(V_\delta)$. Then*

$$C^r_0(\tilde{V}_\delta) \subset L^r(\tilde{V}_\delta \backslash \operatorname{supp} \pi^{-1}(0), d\sigma^2).$$

Proof. Let $u \in C^r_0(\tilde{V}_\delta)$ be any element, and let D be a neighbourhood of a k-ple point of $\operatorname{supp} \pi^{-1}(0)$ with coordinate (v, w) as described before. Since $d\sigma^2$ satisfies (3), we have

$$|u|^2 \lesssim (\log |v_1 \cdot \cdots \cdot v_k|^{-1})^r$$

if $|v| < 1/2$. Let $dV_{(\sigma)}$ be the volume form of $d\sigma^2$. Then (3) implies that

$$dV_{(\sigma)} \sim |v_1 \cdot \cdots \cdot v_k|^{-2}(\log |v_1 \cdot \cdots \cdot v_k|^{-1})^{-n-k}.$$

Therefore, if $r < n$

$$\begin{aligned}
&\int_D |u|^2 dV_{(\sigma)} \\
&\lesssim \int_0^{1/2} \cdots \int_0^{1/2} (t_1 \cdot \cdots \cdot t_k)^{-1} \\
&\qquad \times (\log((t_1 \cdot \cdots \cdot t_k)^{-1})^{-n-k+r} dt_1 \cdot \cdots \cdot dt_k \\
&\leq \int_0^{1/2} \cdots \int_0^{1/2} (t_1 \cdot \cdots \cdot t_k)^{-1} \\
&\qquad \times (\log((t_1 \cdot \cdots \cdot t_k)^{-1})^{-k-1} dt_1 \cdot \cdots \cdot dt_k \\
&\leq (\int_0^{1/2} t^{-1}(\log t^{-1})^{-1-k^{-1}} dt)^k < \infty.
\end{aligned}$$

□

§4. A homotopy operator

Let $\pi : \widetilde{V}_{(\delta)} \to V$ be as before. Once for all we fix C^∞ metrics along the fibers of $\mathcal{O}(E_i)$ and denote by $|s_i|$ the length of the canonical section s_i of $\mathcal{O}(E_i)$. Then we put $s := \min_i |s_i|$ and $\widetilde{V}_{(\delta)} := \{y \in \tilde{V}; s < \sigma\}$. Note that $\widetilde{V}_{(\delta)}$ is a tubular neighbourhood of $\operatorname{supp}\pi^{-1}(0)$ if $0 < \sigma \ll 1$. We may choose δ so that $\partial\widetilde{V}_{(\delta)}$ is piecewise smooth and there exists a piecewise smooth retraction $r_\delta : \widetilde{V}_{(\delta)} \backslash \operatorname{supp}\pi^{-1}(0) \to \partial\widetilde{V}_{(\delta)}$ which is up to a local diffeomorphism of $\widetilde{V}_{(\delta)}$ of the form

$$(v, w) \to \big((\delta + |v_1| - |v_i|)e^{\arg v_1}, \cdots, \delta e^{\arg v_i}, \cdots, (\delta + |v_k| - |v_i|)e^{\arg v_k}, w\big)$$

on $\{y; |v_i(y)| = \min_{1\leq j\leq k} |v_j(y)|\}$. Note that any differential form f on $\widetilde{V}_{(\delta)} \backslash \operatorname{supp}\pi^{-1}(0)$ splits into the sum $ds \wedge f_0 + f_1$, where $f_i = g_i \cdot r_\delta^* h_i$ for some functions g_i and differential forms h_i on $\partial\widetilde{V}_{(\delta)}$ in the piecewise smooth sense. For any $u \in C_0(\widetilde{V}_{(\delta)})$, with a splitting $u = ds \wedge u_0 + u_1$ as above, we put

$$K_\delta u := \int_\delta^s u_0(t, \cdot\,)dt,$$

where t denotes the s-variable. Clearly K_δ is extendable by continuity to a linear operator on the space of locally square integrable forms on $\overline{\widetilde{V}_{(\delta)}}\backslash \mathrm{supp}\pi^{-1}(0)$, which shall be denoted also by K_δ. Note that $d(K_\delta u)$ if $du = 0$ and $\mathrm{supp} u \Subset \widetilde{V}_{(\delta)}$.

§5. L^2 vanishing theorems for isolated singularities

From now on we put $V'_{(\delta)} := \widetilde{V}_{(\delta)} \backslash \mathrm{supp}\pi^{-1}(0)$. Combining Theorem 8 with Proposition 2 and Corollary 11 we obtain the following.

Theorem 13. *Let φ be a C^∞ negative plurisubharmonic function on V'_{δ_0} $(0 < \delta_0 \ll 1)$ such that $ds^2 := 2\partial\bar\partial(-\log(-\varphi))$ is a complete Kähler metric on V'_{δ_0}. Suppose that the following conditions are satisfied.*

(a) $C_0^{n-1}(\widetilde{V}_{(\delta_0)}) \subset L^{n-1}(V'_{\delta_0}, ds^2)$

(b) *K_δ extends to a continuous linear map from $L^n(V'_{\delta_0}, ds^2)$ to $L^{n-1}_{-\varepsilon\varphi}(V'_{\delta_0}, ds^2 + 2\varepsilon\partial\bar\partial\varphi)$ if $\varepsilon > 0$ and $V'_{(\delta)} \Subset \widetilde{V}_{\delta_0}$.*

Then

$$\lim_{\delta\to 0} H^n_{(2)}(V'_\delta, ds^2) = 0.$$

Our next task is to apply Theorem 13 to prove that $\lim\limits_{\delta\to 0} H^n_{(2)}(V'_\delta, d\sigma^2) = 0$ for any Saper metric $d\sigma^2$.

Lemma 14. *Let $d\sigma^2$ be a Saper metric on V'_δ. Then there exists a negative C^∞ plurisubharmonic function φ on V'_δ such that*

(i) $2\partial\bar\partial(-\log(-\varphi)) = d\sigma^2$ *on* $V'_{\delta/2}$.

(ii) $2\partial\bar\partial(-\log(-\varphi))$ *is a complete Kähler metric on* V'_δ.

Proof. Let σ_i be as in §1 and put

$$\varphi_\eta := -\prod_{i=1}^m (-\log\sigma_i)^\eta, \qquad \text{for} \quad \eta \in (0,1).$$

Then

$$\partial\bar\partial\varphi_\eta$$
$$= (-\varphi_\eta)\left\{\sum_{i=1}^m (-\log\sigma_i)^{-\eta}\partial\bar\partial(-(-\log\sigma_i)^\eta) - \eta^2\sum_{i,j}\frac{\partial\log\sigma_i}{\log\sigma_i}\frac{\bar\partial\log\sigma_j}{\log\sigma_j}\right\}.$$

Since $\partial\bar\partial(-(-\log\sigma_i)^\eta) \geq \eta(1-\eta)(-\log\sigma_i)^{\eta-2}\partial\log\sigma_i\bar\partial\log\sigma_i$, we obtain $\partial\bar\partial\varphi_\eta \geq 0$ if $0 < \eta \leq 1/2$. Let λ be a C^∞ convex increasing function such that $\lambda(t) = -\frac{1}{2}\log 2$ on $(-\infty, -\log 2)$ and $\lambda(t) = t$ on $(-\frac{1}{2}\log 2, \infty)$. Then we put

$$\varphi = \varphi_{1/2} + \lambda\big(\log(\delta \|z\|_{V'}^{-1})\big).$$

Clearly φ satisfies (i) and (ii). □

From Proposition 12 it follows immediately that (a) is true for $d\sigma^2$ since so is it for $2\partial\bar{\partial}(-\log(-\varphi))$, where φ is as above. We are going to show that (b) is also true for this choice of φ.

Take any k-ple point $x \in \mathrm{supp}\pi^{-1}(0)$ and a neighbourhood $D \ni x$ with a local coordinate (v, w) around x as before. From the obvious asymptotics of $d\sigma^2$ and $\partial\bar{\partial}\varphi$ around x, the metric $d\sigma_\varepsilon^2 = d\sigma^2 + \varepsilon\partial\bar{\partial}\varphi$ is estimated as

$$d\sigma_\varepsilon^2 \gtrsim \sum_{i=1}^{k} \frac{dv_i d\bar{v}_i}{|v_i|^2(\log s)^2} + \frac{1}{-\log s}\sum_{j=1}^{n-k} dw_j d\bar{w}_j \tag{6}$$

and

$$d\sigma_\varepsilon^2 \lesssim \sum_{i=1}^{k} \frac{dv_i d\bar{v}_i}{|v_i|^2} + \sum_{j=1}^{n-k} dw_j d\bar{w}_j. \tag{7}$$

Let $D_i = \{y \in D; |v_i(y)| = \min_{1\le j\le k} |v_j(y)|\}$. We shall estimate $\|K_{\delta'}u\|_{\varepsilon, D_i}$ ($\delta' \ll \delta$) for each i. Fixing i we set $t_j = |v_j| - |v_i|$ for $j \neq i$. Furthermore we put $\theta_j = \arg v_j$ for $1 \le j \le k$. Then (6) and (7) are rewritten in terms of a (piecewise smooth) local coordinate $(s, t_1, \cdots \overset{i}{\vee} \cdots, t_k, \theta_1, \cdots, \theta_k, w_1, \cdots, w_{n-k})$ as

$$\begin{aligned} \mathrm{Re}\, d\sigma_\varepsilon^2 \gtrsim & \frac{ds^2}{s^2(\log s)^2} + \sum_{j\neq i} \frac{dt_j^2}{(t_j + s)^2(\log s)^2} \\ & + \sum_{i=1}^{k} \frac{d\theta_i^2}{(\log s)^2} + \frac{1}{-\log s}\mathrm{Re}\sum_{j=1}^{n-k} dw_j d\bar{w}_j \end{aligned} \tag{8}$$

and

$$\begin{aligned}\operatorname{Re} d\sigma_\varepsilon^2 \lesssim & \frac{ds^2}{s^2} + \sum_{j\neq i} \frac{dt_j^2}{(t_j+s)^2} + \sum_{i=1}^{k} d\theta_i^2 \\ & + \operatorname{Re} \sum_{j=i}^{n-k} dw_j d\bar{w}_j .\end{aligned} \tag{9}$$

Take any $\delta' > 0$ with $V'_{(\delta')} \Subset \widetilde{V}_\delta$ and let $u = ds \wedge u_0(s,\cdot) + u_1(s,\cdot) \in C_0^n(V'_{(\delta')})$, where u_0 and u_1 are determined as before. Then we put

$$\|u_0\|^2_{(\varepsilon),t} := \int_{\{y;s(y)=t\}} |u_0|^2_\varepsilon dV_{\varepsilon,t} \quad \text{for} \quad t < \sigma',$$

where $dV_{\varepsilon,t}$ denotes the volume form with respect to $d\sigma_\varepsilon^2|\{y; s(y) = t\}$ (in the piecewise smooth sense). Note that $s \lesssim |ds|_\varepsilon \lesssim 1$ and

$$\|u_0\|^2_{(\varepsilon),t} \lesssim (\log t^{-1})^{2n} \|u\|^2_{(0),t}$$

by (8) and (9). Therefore

$$\begin{aligned} & \|K_{\delta'} u\|^2_{\varepsilon,D_i} \\ = & \left\| \int_{\delta'}^{s} u_0(t,\cdot) dt \right\|^2_{\varepsilon,D_i} \\ \lesssim & \int_0^{\delta'} \left(\int_{\delta'}^{s} \|u_0\|^2_{(\varepsilon),s} |ds|_0^{-1} ds \int_{\delta'}^{s} |ds|_0 ds \right) s^{\varepsilon/2} |ds|_\varepsilon^{-1} ds \\ \lesssim & \int_0^{\delta'} \|u\|_0^2 s^{\varepsilon/2-1} (\log s^{-1})^{2n+1} ds \lesssim \|u\|_0^2 \end{aligned}$$

if $\varepsilon > 0$.

Thus we have verified (b) for φ. Consequently we obtain the following.

Theorem 15.

$$\lim_{\delta\to 0} H^n_{(2)}(V'_\delta, d\sigma^2) = 0$$

for any Saper metric $d\sigma^2$.

We now turn our attention to more general metrics. First we prepare a comparison lemma.

Lemma 16. *Let X be a complex manifold, let ds_i^2 $(i = 0, 1)$ be C^∞ Hermitian metrics on X satisfying $ds_0^2 \lesssim ds_1^2$, and let $\Omega \subset X$ be a domain whose boundary $\partial\Omega$ is compact. With respect to the metrics $ds_\varepsilon^2 := \varepsilon ds_0^2 + (1-\varepsilon) ds_1^2$, $\varepsilon \in [0, 1]$, with associated L^2 norms $\| \ \|_\varepsilon$, suppose that ds_i^2 are complete and there exist a compact subset $K \subset \bar{\Omega}$ and a constant C independent of $\varepsilon \in [0, 1]$ such that*

$$\|u\|_{\varepsilon,\Omega} \leq C(\|u\|_{\varepsilon,K} + \|du\|_{\varepsilon,\Omega} + \|d^*_{\varepsilon,\Omega}u\|_{\varepsilon,\Omega}) \tag{10}$$

for any $u \in \mathrm{Dom}(d + d^*_{\varepsilon,\Omega}) \cap L^{r\pm1}(\Omega, ds_\varepsilon^2)$. Here $d^*_{\varepsilon,\Omega}$ denotes the adjoint of d with respect to $\| \ \|_{\varepsilon,\Omega}$ and r is a nonnegative integer. Then $\dim H^{r\pm1}_{(2)}(\Omega, ds_i^2) < \infty$. Moreover

$$\dim H^r_{(2)}(\Omega, ds_0^2) \leq \dim H^r_{(2)}(\Omega, ds_1^2) \tag{11}$$

if

$$\dim H^{r+j}_{(2)}(\Omega, ds_0^2) \leq \dim H^{r+j}_{(2)}(\Omega, ds_\varepsilon^2) \tag{12}$$

hold for $j = \pm1$ and $\varepsilon \in [0, 1]$.

Proof. That $\dim H^{r\pm1}_{(2)}(\Omega, ds_\varepsilon^2) < \infty$ follows from (10) is well known (cf. [2]). Suppose moreover that (12) holds. Then there must exist a constant C' such that

$$\|u\|_{\varepsilon,\Omega} \leq C'(\|du\|_{\varepsilon,\Omega} + \|d^*_{\varepsilon,\Omega}u\|_{\varepsilon,\Omega}) \tag{13}$$

if $u \in \mathrm{Dom}(d + d^*_{\varepsilon,\Omega}) \cap L^{r\pm1}(\Omega, ds_\varepsilon^2) \ominus \mathrm{Ker}(d + d^*_{\varepsilon,\Omega})$. (See [8] for the argument.) (13) shows that $\dim H^r_{(2)}(\Omega, ds_0^2) \leq \dim \mathrm{Ker}(d + d^*_{\varepsilon,\Omega}) = \dim H^r_{(2)}(\Omega, ds_\varepsilon^2)$. □

By Lemma 16, we have the following generalization of Theorem 13.

Proposition 17. *Let φ and V'_{δ_0} be as in Theorem* 13, *and let ψ be a C^∞ plurisubharmonic function on V'_{δ_0} such that*

1) *$\partial\bar{\partial}\psi$ is a complete Kähler metric*
2) *$|\partial\psi|_{\partial\bar{\partial}\psi}$ is bounded*
3) *$\partial\bar{\partial}\psi \lesssim \partial\bar{\partial}(-\log(-\varphi))$.*

Then $\lim_{\delta\to0} H^n_{(2)}(V'_\delta, \partial\bar{\partial}\psi) = 0$.

Proof. We put $ds_0^2 = \partial\bar{\partial}\psi$ and $ds_1^2 = \partial\bar{\partial}(-\log(-\varphi))$. Then we can apply Lemma 16 in virtue of Proposition 1. □

Thus the existence of dominating Saper metrics implies the following.

Corollary 18. $\lim_{\delta \to 0} H^n_{(2)}(V'_\delta, \partial\bar\partial(-\log\log \|z\|^{-1}_{V'})) = 0.$

Finally we shall prove the L^2 cohomology vanishing with respect to $\partial\bar\partial\|z\|^2_{V'}$. For that purpose we prepare another lemma.

Lemma 19. *Let Ω and ds^2_ε be as in Lemma* 16 *except that ds^2_0 is not necessarily complete and instead of* (10) *we assume the estimate*

$$\|\eta_\varepsilon u\|_{\varepsilon,\Omega} \le C(\|u\|_{\varepsilon,K} + \|du\|_{\varepsilon,\Omega} + \|d^*_{\varepsilon,\Omega} u\|_{\varepsilon,\Omega}) \tag{14}$$

for any $\varepsilon \in (0,1]$ and $u \in \mathrm{Dom}(d + d^*_{\varepsilon,\Omega}) \cap L^{r\pm1}(\Omega, ds^2_\varepsilon)$. Here η_ε are continuous functions on Ω with values in $(1,\infty)$ such that
(15) *$\eta_\varepsilon \to \eta_0$ uniformly on compact subsets of Ω.*
(16) *There exists a sequence of C^∞ functions $\{\chi_\mu\}_{\mu=1}^\infty$ on $\bar\Omega$ satisfying*
 i) $|d\chi_\mu|_{ds^2_0} \le \eta_0$
 ii) $\mathrm{supp}\chi_\mu$ *is compact and* $\bigcup_{\mu=1}^\infty \mathrm{supp}\chi_\mu = \bar\Omega$
 iii) $0 \le \chi_\mu \le 1$ *and* $\chi_\mu \equiv 1$ *on* $\mathrm{supp}\chi_{\mu-1}$.
Assume moreover that

$$\dim H^{r\pm1}_{(2)}(\Omega, ds^2_0) \le \dim H^{r\pm1}_{(2)}(\Omega, ds^2_1). \tag{17}$$

Then $\dim H^r_{(2)}(\Omega, ds^2_1) \le \dim H^r_{(2)}(\Omega, ds^2_1)$.

Proof. To be precise, let $d_{\max}$ and $d_{\min}$ denote respectively the maximal and the minimal closed extensions of d on $L(\Omega, ds^2_0)$. By (16,i) we have

$$\mathrm{Dom}\, d_{\max} \cap \{u \in L(\Omega, ds^2_0); \|\eta_0 u\|_{0,\Omega} < \infty\} \subset \mathrm{Dom}\, d_{\min}.$$

Similarly $u \in \mathrm{Dom}\, d^*_{\max}$ if $\|\eta_0 u\|_{0,\Omega} < \infty$ and $\chi_\mu u \in \mathrm{Dom}\, d^*_{\max}$ for all μ. Suppose that $\dim H^r_{(2)}(\Omega, ds^2_0) > \dim H^r_{(2)}(\Omega, ds^2_1)$. Then there must exist a finite dimensional subspace $W \subset L^r(\Omega, ds^2_0) \cap \mathrm{Ker}\, d_{\max}$ consisting of 0 and non-$d_{\max}$-exact forms, and a sequence $f_\mu \in W$ ($\mu = 1, 2, \cdots$) such that $\|f_\mu\|_0 = 1$ and $\chi_\mu f_\mu \perp \mathrm{Ker}\,(d + d^*_{1/\mu,\Omega})$ in $L^r(\Omega, ds^2_{1/\mu})$. Therefore, by (14) and (17) there must exist a constant $C', g_\mu \in L^{r-1}(\Omega, ds^2_{1/\mu})$ and $h_\mu \in L^{r+1}(\Omega, ds^2_{1/\mu})$ such that

$$\begin{cases} \chi_\mu f_\mu = dg_\mu + d^*_{1/\mu,\Omega} h_\mu \\ \|\varphi_{1/\mu} g_\mu\|_{1/\mu,\Omega} \le C' \\ \|\varphi_{1/\mu} h_\mu\|_{1/\mu,\Omega} \le C'. \end{cases}$$

Choosing weakly convergent subsequences of f_μ, g_μ and h_μ we thus obtain $f \in W$ $g \in \operatorname{Dom} d_{\min}$ and $h \in \operatorname{Dom} d^*_{\max}$ such that $f = d_{\min} g + d^*_{\max} h$. Since $f \in \operatorname{Ker} d_{\max}, d^*_{\max} h = 0$. Therefore $f = 0$. On the other hand $f \neq 0$ since $\|f_\mu\|_{1/\mu} = 1$ and W is finite dimensional. This is a contradiction.

□

Combining Corollary 18 and Lemma 19 we obtain the following.

Theorem 20.

$$\lim_{\delta \to 0} H^n_{(2)}(V'_\delta, \partial\bar{\partial}\|z\|^2_{V'}) = 0.$$

Proof. Put $ds_0^2 = \partial\bar{\partial}\|z\|^2_{V'}, ds_1^2 = \partial\bar{\partial}(-\log\log\|z\|^{-1}_{V'})$ and let η_ε be the smallest eigenvalue of $\partial\bar{\partial}(-\log\log\|z\|^{-1}_{V'})$ with respect to $(1-\varepsilon)ds_0^2 + \varepsilon ds_1^2$. Since the other eigenvalues of $\partial\bar{\partial}(-\log\log\|z\|^{-1}_{V'})$ are equal to each other, we have the estimate (14) for $r = n$ (cf. [8]). (16) follows from the fact that $(t \log t)^{-1}$ is non-integrable on $(0, 1/2)$. (15) is trivial. (17) is a consequence of Corollary 18 together with Proposition 3 and Proposition 4. □

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A Geometric Construction of Laguerre-Forsyth's Canonical Forms of Linear Ordinary Differential Equations

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§0. Introduction

The purpose of this paper is to reformulate the fundamental results of E.J. Wilczynski's book [5] by applying E. Cartan's method of moving frames.

Let E be a vector bundle over a 1-dimensional manifold M. By taking a local coordinate system t in M and a moving frame $\{e_1, \cdots, e_r\}$ of E, we express each cross section s of E in the form:

$$s = \sum_{\alpha=1}^{r} y_\alpha e_\alpha, \tag{0.1}$$

where $y_1, \cdots, y_r$ are functions on M. Let $\mathcal{D}$ be a system of homogeneous linear ordinary differential equations on E of order n given in the form:

$$\left(\frac{d}{dt}\right)^n y_\alpha + \sum_{k=1}^{n}\sum_{\beta=1}^{r} a^{(k)}_{\alpha\beta}(t)\left(\frac{d}{dt}\right)^{n-k} y_\beta = 0, \quad \alpha = 1, \cdots, r. \tag{0.2}$$

Corresponding to (0.2), we define $r \times r$ matrices $A^{(1)}(t), \cdots, A^{(n)}(t)$ by

$$A^{(k)}(t) = (a^{(k)}_{\alpha\beta}(t)), \quad k = 1, \cdots, n. \tag{0.3}$$

For another local coordinate system in M and another moving frame, we also express $\mathcal{D}$ in a similar way as (0.2) and give $r \times r$ matrices as (0.3).

In modern terminology, Wilczynski showed the following facts:

(A) There exists a pair $(t, \{e_\alpha\})$ satisfying the following condition

$$A^{(1)}(t) = 0 \quad \text{and} \quad \mathrm{Tr}\ A^{(2)}(t) = 0. \tag{L.F}$$

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(B) If both $(t, \{e_\alpha\})$ and $(t', \{e'_\alpha\})$ satisfy the condition (L.F), then there exists $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R})$ and $C = (c_{\alpha\beta}) \in GL(r, \mathbb{R})$ such that

$$t' = \frac{at+b}{ct+d},$$
$$e'_\beta = (ct+d)^{n-1} \sum_{\alpha=1}^{r} c_{\alpha\beta} e_\alpha, \quad \beta = 1, \cdots, r.$$

(C) Let $(t, \{e_\alpha\})$ be a pair satisfying the condition (L.F). For each integer k, $2 \leqq k \leqq n$, let θ_k be the cross section of the vector bundle $\overset{k}{\otimes} T(M)^* \otimes \mathrm{End}(E)$ which corresponds to the $r \times r$ matrix valued k covariant tensor field

$$\sum_{j=0}^{k-2} (-1)^j \frac{(2k-j-2)!(n-k+j)!}{j!(k-j-1)!} \left(\frac{d}{dt}\right)^j A^{(k-j)}(t)(dt)^k$$

under the trivialization with respect to $\{e_\alpha\}$. Then the definition of θ_k does not depend on the choice of $(t, \{e_\alpha\})$, and hence θ_k is an invariant of $\mathcal{D}$. Moreover $\theta_2, \cdots, \theta_n$ form a fundamental system of invariants of $\mathcal{D}$.

He studied mainly the case $r = 1$ and gave

$$\frac{k!(k-2)!}{n!(2k-3)!}\theta_k, \quad k = 3, \cdots, n,$$

as a fundamental system of invariants. (In this case, the invariant θ_2 automatically vanishes.) On the other hand, for the case $r \geqq 2$, he did not give a fundamental system of invariants. Following him, we will call the system (0.2) with the condition (L.F) a Laguerre-Forsyth's canonical form of $\mathcal{D}$.

Let us proceed to the description of the contents of this paper. Let $\{s_1, \cdots, s_{nr}\}$ be a fixed family of linearly independent solutions of $\mathcal{D}$. We define a map κ of M into the Grassmann manifold $\mathrm{Gr}(\mathbb{R}^{nr}, r)$ as follows: By using a pair $(t, \{e_\alpha\})$, we define an $nr \times r$ matrix $Y_1(x) = (y_{\alpha\beta}(x))$, $x \in M$, by

$$s_\alpha(x) = \sum_{\beta=1}^{r} y_{\alpha\beta}(x) e_{\beta x}, \quad \alpha = 1, \cdots, nr, \tag{0.4}$$

and an $nr \times nr$ matrix $Y(x)$, $x \in M$, by

$$
(0.5) \qquad \begin{aligned} Y(x) &= (Y_1(x), \cdots, Y_n(x)), \\ Y_k(x) &= \frac{1}{(k-1)!}\left(\frac{d}{dt}\right)^{k-1} Y_1(x), \quad k = 2, \cdots, n. \end{aligned}
$$

Then we define $\kappa(x)$ to be the point of $\mathrm{Gr}(\mathbb{R}^{nr}, r)$ corresponding to the r-dimensional subspace of $\mathbb{R}^{nr}$ spanned by the column vectors of $Y_1(x)$.

The general linear group $GL(nr, \mathbb{R})$ acts transitively on $\mathrm{Gr}(\mathbb{R}^{nr}, r)$. Let K be the isotropy subgroup of $GL(nr, \mathbb{R})$ at an origin of $\mathrm{Gr}(\mathbb{R}^{nr}, r)$. Then we regard $GL(nr, \mathbb{R})$ as a principal K-bundle over $\mathrm{Gr}(\mathbb{R}^{nr}, r)$ with projection π_0. Let P be the principal K-bundle over M defined by

$$
P = \{(x, Z) \in M \times GL(nr, \mathbb{R}) | \kappa(x) = \pi_0(Z)\},
$$

and let ω be the $gl(nr, \mathbb{R})$ valued 1-form on P induced by the Maurer-Cartan form of $GL(nr, \mathbb{R})$.

The main result is to show the unique existence of a normal reduction Q of P. Here the normal reduction Q is defined by the condition that the restriction of ω to Q is $\mathfrak{h} + \mathfrak{m}$ valued, where $\mathfrak{h}$ (resp. $\mathfrak{m}$) is the subalgebra (resp. the subspace) of $gl(nr, \mathbb{R})$ defined in §2. The restriction of ω to Q is decomposed into the two components $\chi_{\mathfrak{h}}$ and $\chi_{\mathfrak{m}}$. The 1-form $\chi_{\mathfrak{h}}$ is a flat Cartan connection in Q and the 1-form $\chi_{\mathfrak{m}}$ induces differential invariants in Q corresponding to $\theta_1, \cdots, \theta_n$. Using the absolute parallelism induced by the Cartan connection $\chi_{\mathfrak{h}}$, we give a vector field on Q whose integral curve corresponds to a Laguerre-Forsyth's canonical form of $\mathcal{D}$. We apply Cartan's reduction method developed in [1] to the construction of the normal reduction Q.

In Appendix, we construct Q as a reduction of the frame bundle $\mathcal{F}(J^{n-1}(E))$ of $J^{n-1}(E)$, where $J^{n-1}(E)$ is the $(n-1)$-th jet bundle of E. The connection form of the affine connection of $J^{n-1}(E)$ which is associated with $\mathcal{D}$ takes the place of the 1-form ω, in this case.

Preliminary remarks

1. Throughout this paper, we always assume the differentiability of class C^∞, though the argument goes through in complex analytic category with suitable modifications.

2. As we are mainly concerned with local properties of linear ordinary differential equations, base manifolds will be assumed to be simply connected unless otherwise stated.

3. We will frequently write any $nr \times nr$ matrix A in the form:

$$A = (A_1 \cdots A_n) \quad \text{or} \quad A = \begin{pmatrix} A_{11} & \dots & A_{1n} \\ \vdots & & \vdots \\ A_{n1} & \dots & A_{nn} \end{pmatrix},$$

where A_i, $1 \leqq i \leqq n$ and A_{ij}, $1 \leqq i,\ j \leqq n$ are $nr \times r$ matrices and $r \times r$ matrices respectively. We sometimes simply write $A = (A_i)$ or $A = (A_{ij})$.

4. As to Lie groups and principal bundles, we use the standard notations and terminology as in [2]. Especially let K be a Lie group and P a principal K-bundle over a base manifold M. For $A \in K$, R_A denotes the right translation induced by A. Let $\mathfrak{k}$ be the Lie algebra of K. For $X \in \mathfrak{k}$, X^* denotes the vertical vector field on P induced by the 1-parameter group of right translations $\{R_{\exp tX}\}$. The vector field X^* is called the fundamental vector field corresponding to X.

5. Cartan connections. Let H/H_0 be a homogeneous space of a Lie group H over its closed subgroup H_0. Let $\mathfrak{h}$ and $\mathfrak{h}_0$ be the Lie algebras of H and H_0 respectively. Let Q be a principal H_0-bundle over a manifold M, where dim $M =$ dim H/H_0, and let ω be a $\mathfrak{h}$ valued 1-form on Q. Then we say that ω is a Cartan connection on Q of type H/H_0 if the following conditions are satisfied:

(C.1) $\omega(v) \neq 0$ for every non-zero tangent vector v of Q.

(C.2) $R_{A*}\omega = \mathrm{Ad}(A^{-1})\omega, \quad A \in H_0$.

(C.3) $\omega(X^*) = X$ for every $X \in \mathfrak{h}_0$.

Let Ω be the 2-form on Q defined by

$$\Omega = d\omega + \frac{1}{2}[\omega \wedge \omega].$$

The 2-form Ω is called the curvature form of the Cartan connection ω. If $\Omega = 0$, then the Cartan connection ω is said to be flat.

§1. Characteristic maps

1.1. Characteristic maps

Let E be a vector bundle over a 1-dimensional manifold M. As noted in Preliminary remarks, we assume that M is simply connected. Let $\mathcal{D}$ be a system of homogeneous linear ordinary differential equations on E as considered in Introduction.

Let Sol($\mathcal{D}$) denote the space of all solutions of $\mathcal{D}$. As is well known, Sol($\mathcal{D}$) is an nr-dimensional vector space. For a moment, let us fix a basis $\{s_1, \cdots, s_{nr}\}$ of Sol($\mathcal{D}$).

Taking a local coordinate system t in M and a moving frame $\{e_1, \cdots, e_r\}$ of E, we define, for each $x \in M$, the matrices $Y_1(x), \cdots, Y_n(x)$ and $Y(x)$ by (0.4) and (0.5). It is well known that $Y(x)$ is non-singular, and in particular the r column vectors of $Y_1(x)$ are linearly independent.

Taking another coordinate system t' in M and another moving frame $\{e'_1, \cdots, e'_r\}$ of E, we define matrices $Y'_1(x), \cdots, Y'_n(x)$ and $Y'(x)$ in the same way. We write $e'_1, \cdots, e'_n$ in the form:

$$e'_{\beta x} = \sum_{\alpha=1}^{r} c_{\alpha\beta}(x) e_{\alpha x}, \quad x \in M,$$

where $C(x) = (c_{\alpha\beta}(x)) \in GL(r, \mathbb{R})$. The proof of the next lemma is straightforward.

Lemma 1.1. (1) $Y'_1(x) = Y_1(x)^t C(x)^{-1}$, $x \in M$.
(2) *For every* k, $2 \leqq k \leqq n$, $Y'_k(x)$ *can be written in the form:*

$$Y'_k(x) = \left(\frac{dt}{dt'}\right)^{k-1} Y_k(x)^t C(x)^{-1} + \sum_{j=1}^{k-1} Y_j(x) D_{jk}(x),$$

where $D_{1k}(x), \cdots, D_{k-1\ k}(x)$ *are* $r \times r$ *matrices.*

Let $\mathrm{Gr}(\mathbb{R}^{nr}, r)$ be the Grassmann manifold consisting of all r-dimensional subspaces of $\mathbb{R}^{nr}$. For each $x \in M$, let $\kappa(x)$ $(\in \mathrm{Gr}(\mathbb{R}^{nr}, r))$ be the subspace of $\mathbb{R}^{nr}$ spanned by the r column vectors of $Y_1(x)$. By (1) of Lemma 1.1, the definition of $\kappa(x)$ does not depend on the choice of $(t, \{e_\alpha\})$. It is not difficult to see that the assignment $x \to \kappa(x)$ gives an immersion of M into $\mathrm{Gr}(\mathbb{R}^{nr}, r)$. We call the map κ the characteristic map of $\mathcal{D}$ (corresponding to the basis $\{s_\alpha\}$ of $\mathrm{Sol}(\mathcal{D})$).

1.2. The induced principal bundles P

Consider the general linear group $GL(nr, \mathbb{R})$ acting on $\mathbb{R}^{nr}$ on the left. The group $GL(nr, \mathbb{R})$ acts on the Grassmann manifold $\mathrm{Gr}(\mathbb{R}^{nr}, r)$ in a natural way. Let o be the point of $\mathrm{Gr}(\mathbb{R}^{nr}, r)$ corresponding to the subspace spanned by the first r vectors of the natural basis of $\mathbb{R}^{nr}$. The isotropy subgroup K of $GL(nr, \mathbb{R})$ at o is given by:

$$K = \{A \in GL(nr, \mathbb{R}) | A_{21} = \cdots = A_{n1} = 0\}.$$

We denote by π_0 the natural projection of $GL(nr, \mathbb{R})$ onto $\mathrm{Gr}(\mathbb{R}^{nr}, r)$ under the identification $\mathrm{Gr}(\mathbb{R}^{nr}, r) \simeq GL(nr, \mathbb{R})/K$.

Using the characteristic map κ, we define a submanifold P of the direct product $M \times GL(nr, \mathbb{R})$ by

$$P = \{(x, Z) \in M \times GL(nr, \mathbb{R}) | \kappa(x) = \pi_0(Z)\}.$$

We denote by π_M (resp. π_G) the projection of P onto M (resp. the projection of P onto $GL(nr, \mathbb{R})$). Clearly P is a principal K-bundle over M with the projection π_M.

The $gl(nr, \mathbb{R})$-valued 1-form ω on P. Let ω be the pull back of the Maurer-Cartan form of $GL(nr, \mathbb{R})$ by the projection π_G. The 1-form ω possesses the following properties:

(ω.1) $\omega(v) \neq 0$ for every non-zero tangent vector v of P.

(ω.2) $R_{A*}\omega = \mathrm{Ad}(A^{-1})\omega$ for all $A \in K$.

(ω.3) $\omega(X^*) = X$ for all $X \in \mathfrak{k}$, where $\mathfrak{k}$ stands for the Lie algebra of K.

The standard cross sections. We fix a local coordinate system t in M and a moving frame $\{e_\alpha\}$ of E. We define $Y(x) \in GL(nr, \mathbb{R})$, $x \in M$, by (0.5). Let σ be the map of M into $M \times GL(nr, \mathbb{R})$ defined by

$$\sigma(x) = (x, Y(x)), \quad x \in M.$$

Obviously σ is a cross section of P, that is, $\sigma(x) \in P$ for all $x \in M$. The cross section σ will be called the standard cross section of P (corresponding to $(t, \{e_\alpha\})$).

1.3. The bundle homomorphism $\varepsilon : P \to \mathcal{F}(E)$

Let $\mathcal{F}(E)$ be the frame bundle of E. Let $\varepsilon : K \to GL(r, \mathbb{R})$ be the homomorphism defined by

$$\varepsilon(A) = {}^tA_{11}^{-1}, \quad A \in K.$$

Here, we will define a natural bundle homomorphism ε of P onto $\mathcal{F}(E)$ corresponding to the group homomorphism $\varepsilon : K \to GL(r, \mathbb{R})$.

For each $p \in P$, we write p in the form:

$$p = (x, Z), \quad x \in M, \quad Z \in GL(nr, \mathbb{R}).$$

We further write Z in the form: $Z = (Z_1, \cdots, Z_n)$.

Lemma 1.2. *There exists a unique basis $\{\varepsilon_1(p), \cdots, \varepsilon_r(p)\}$ of E_x such that*

$$s_\alpha(x) = \sum_{\beta=1}^{r} z_{\alpha\beta}\varepsilon_\beta(p), \quad \alpha = 1, \cdots, nr,$$

where $Z_1 = (z_{\alpha\beta})$.

Proof. We take a local coordinate system t in M and a moving frame $\{e_\alpha\}$ of E. Let $Y(x)$ be the $nr \times nr$-matrix defined by (0.5). Since $\pi_0(Z_1) = \pi_0(Y_1(x))$, we can write Z_1 in the form:

$$Z_1 = Y_1(x)C, \quad \text{where} \quad C \in GL(r, \mathbb{R}).$$

Let $\{\varepsilon_1(p), \cdots, \varepsilon_r(p)\}$ be the basis of E_x defined by

$$\varepsilon_\beta(p) = \sum_{\alpha=1}^{r} c'_{\alpha\beta} e_{\alpha x}, \quad 1 \leqq \beta \leqq r$$

where ${}^tC^{-1} = (c'_{\alpha\beta})$. It is easy to see that $\{\varepsilon_\alpha(p)\}$ is the desired basis of E_x. The uniqueness is obvious. Q.E.D.

Let $\varepsilon : P \to \mathcal{F}(E)$ be the map defined by

$$\varepsilon(p) = \{\varepsilon_\alpha(p)\}.$$

It is easily checked that $\varepsilon : P \to \mathcal{F}(E)$ is a bundle homomorphism corresponding to $\varepsilon : K \to GL(r, \mathbb{R})$.

Remark. Let σ be the standard cross section of P corresponding to $(t, \{e_\alpha\})$. Then,

$$\varepsilon(\sigma(x)) = \{e_{\alpha x}\} \quad \text{for all} \quad x \in M. \tag{1.1}$$

1.4. Expressions in local coordinate systems

We fix a local coordinate system t in M and a moving frame $\{e_\alpha\}$ of E. Here, we will give a local expression of the 1-form ω by using $(t, \{e_\alpha\})$. We express $\mathcal{D}$ as (0.2).

Proposition 1.3. *Let σ be the standard cross section of P corresponding to $(t, \{e_\alpha\})$. For each $x \in M$, we write the $nr \times nr$ matrix $(\sigma^*\omega)(d/dt)_x$ in the form:*

$$(\sigma^*\omega)\left(\frac{d}{dt}\right)_x = (X_{ij}(x)),$$

where $X_{ij}(x)$ are $r \times r$ matrices. Then,

$$X_{k+1\ k}(x) = kI, \ k = 1, \cdots, n-1. \tag{1.2}$$

$$X_{n-k+1\ n}(x) = -\frac{(n-k)!}{(n-1)!}\ {}^tA^{(k)}(t(x)), \ k = 1, \cdots, n. \tag{1.3}$$

$$X_{ij}(x) = 0 \text{ for the remaining pairs of indices } (i, j). \tag{1.4}$$

Proof. We define $Y(x) = (Y_k(x)) \in GL(nr,\mathbb{R})$, $x \in M$, by (0.5). Then we have the following equalities:

$$\frac{dY_k}{dt}(x) = kY_{k+1}(x), \quad k = 1,\cdots,n-1,$$
$$\frac{dY_n}{dt}(x) = -\sum_{k=1}^{n}\frac{(n-k)!}{(n-1)!}Y_{n-k+1}(x)\ {}^tA^{(k)}(t(x)).$$

On the other hand, since $Y(x)^{-1}dY/dt(x) = X(x)$, we have

$$\frac{dY_k}{dt}(x) = \sum_{j=1}^{n} Y_j(x)X_{jk}(x), \quad k = 1,\cdots,n.$$

The assertion follows from these equalities. Q.E.D.

Remark. It is easy to verify that if a cross section σ of P satisfies (1.1), (1.2) and (1.4), then σ is the standard cross section corresponding to $(t,\{e_\alpha\})$.

1.5. The compatibility relative to the choice of the basis of Sol($\mathcal{D}$)

Let $\{s'_1,\cdots,s'_{nr}\}$ be a basis of Sol($\mathcal{D}$) such that

$$s'_\alpha = \sum_{\beta=1}^{nr} a_{\alpha\beta}s_\beta, \quad \alpha = 1,\cdots,nr,$$

where $A = (a_{\alpha\beta}) \in GL(nr,\mathbb{R})$. With respect to $\{s'_\alpha\}$, we define κ', P', ω' and ε' as before. We define a transformation Φ of $M \times GL(nr,\mathbb{R})$ by

$$\Phi(x,Z) = (x,AZ), \quad x \in M \quad \text{and} \quad Z \in GL(nr,\mathbb{R}).$$

The next proposition is obvious.

Proposition 1.4. (1) $\kappa' = A^\circ\kappa$.

(2) *Φ maps P onto P' and induces a bundle isomorphism Φ_P of P onto P' such that $\Phi_P * \omega' = \omega$ and $\varepsilon' \circ \Phi_P = \varepsilon$.*

In view of Proposition 1.4, the subsequent argument goes through without reference to the choice of $\{s_\alpha\}$.

§2. A reduction theorem for the principal bundle P

2.1. Notations

Let $S^{n-1}(\mathbb{R}^2)$ be the symmetric tensor product of $\mathbb{R}^2$ of degree $n-1$ and $S^{n-1}(\mathbb{R}^2) \otimes \mathbb{R}^r$ be the tensor product of $S^{n-1}(\mathbb{R}^2)$ and $\mathbb{R}^r$. Let $\{v_1, v_2\}$ be the natural basis of $\mathbb{R}^2$ and $\{u_1, \cdots, u_r\}$ be the natural basis of $\mathbb{R}^r$. Taking $\{\binom{n-1}{0}v_1^{n-1} \otimes u_1, \cdots, \binom{n-1}{0}v_1^{n-1} \otimes u_r, \binom{n-1}{1}v_1^{n-2} \otimes v_2 \otimes u_1, \cdots, \binom{n-1}{1}v_1^{n-2} \otimes v_2 \otimes u_r, \cdots, \binom{n-1}{n-1}v_2^{n-1} \otimes u_1, \cdots, \binom{n-1}{n-1}v_2^{n-1} \otimes u_r\}$ as a basis of $S^{n-1}(\mathbb{R}^2) \otimes \mathbb{R}^r$, we identify $S^{n-1}(\mathbb{R}^2) \otimes \mathbb{R}^r$ with $\mathbb{R}^{nr}$.

The homomorphism $\rho : SL(2, \mathbb{R}) \to GL(nr, \mathbb{R})$. Let ρ_1 be the representation of $SL(2, \mathbb{R})$ on $S^{n-1}(\mathbb{R}^2)$ defined by

$$\rho_1(A)w_1 \cdots w_r = Aw_1 \cdots Aw_r, \quad A \in SL(2, \mathbb{R}), \quad w_1, \cdots, w_r \in \mathbb{R}^2,$$

and let ρ_2 be the trivial representation of $SL(2, \mathbb{R})$, i.e.,

$$\rho_2(A)w = w, \quad A \in SL(2, \mathbb{R}), \quad w \in \mathbb{R}^r.$$

Then we define ρ to be the homomorphism corresponding to the representation $\rho_1 \otimes \rho_2$ of $SL(2, \mathbb{R})$ on $S^{n-1}(\mathbb{R}^2) \otimes \mathbb{R}^r$ under the identification of $S^{n-1}(\mathbb{R}^2) \otimes \mathbb{R}^r$ with $\mathbb{R}^{nr}$. Let ρ_* denote the homomorphism of $sl(2, \mathbb{R})$ onto $gl(nr, \mathbb{R})$ induced by ρ. Let $\{s_{-1}, s_0, s_1\}$ be the basis of $sl(2, \mathbb{R})$ defined by

$$s_{-1} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad s_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad s_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Then ρ_* maps s_{-1}, s_0 and s_1 respectively to

$$U_{-1} = \begin{pmatrix} 0 & & & & & & \\ I & 0 & & & & 0 & \\ & 2I & 0 & & & & \\ & & \ddots & \ddots & & & \\ & & & & 0 & & \\ & 0 & & & (n-2)I & 0 & \\ & & & & & (n-1)I & 0 \end{pmatrix},$$

$$U_0 = \begin{pmatrix} (n-1)I & & & & \\ & (n-3)I & & & 0 \\ & & \ddots & & \\ & 0 & & -(n-3)I & \\ & & & & -(n-1)I \end{pmatrix},$$

$$U_1 = \begin{pmatrix} 0 & (n-1)I & & & & \\ & 0 & (n-2)I & 0 & & \\ & & 0 & \ddots & & \\ & & & 2I & & \\ & 0 & & 0 & I & \\ & & & & 0 \end{pmatrix},$$

where I stands for the $r \times r$ unit matrix.

The homomorphism $\lambda : GL(r, \mathbb{R}) \to GL(nr, \mathbb{R})$. Let λ be the homomorphism of $GL(r, \mathbb{R})$ onto $GL(nr, \mathbb{R})$ defined by

$$\lambda(C) = \begin{pmatrix} {}^tC^{-1} & & & \\ & {}^tC^{-1} & & 0 \\ 0 & & \ddots & \\ & & & {}^tC^{-1} \end{pmatrix}, \quad C \in GL(r, \mathbb{R}).$$

The subgroups H and H_0 of $GL(nr, \mathbb{R})$. Let H be the subgroup of $GL(nr, \mathbb{R})$ defined by

$$H = \rho(SL(2, \mathbb{R})) \cdot \lambda(GL(r, \mathbb{R})),$$

and let H_0 be the subgroup of K defined by

$$H_0 = H \cap K.$$

Let $SL(2, \mathbb{R})_0$ be the set of all upper triangular matrices of $SL(2, \mathbb{R})$. Then it should be noted that

$$H_0 = \rho(SL(2, \mathbb{R})_0) \cdot \lambda(GL(r, \mathbb{R})),$$

and hence

$$H/H_0 \simeq SL(2, \mathbb{R})/SL(2, \mathbb{R})_0 \simeq P(\mathbb{R}^2),$$

where $P(\mathbb{R}^2)$ is the real projective line. We denote by $\mathfrak{h}$ and $\mathfrak{h}_0$ the Lie algebras of H and H_0 respectively.

The subspace $\mathfrak{m}$ of $gl(nr, \mathbb{R})$. Let $\mathfrak{m}$ be the subspace of $gl(nr, \mathbb{R})$ consisting of all elements $X = (X_{ij})$ of $gl(nr, \mathbb{R})$ satisfying

$$\begin{aligned} & X_{ij} = 0 \quad \text{for} \quad j \neq n, \\ & X_{nn} = 0, \\ & \operatorname{Tr} X_{n-1\, n} = 0. \end{aligned}$$

One should note that $\mathfrak{m}$ is $\operatorname{Ad}(H_0)$ invariant.

2.2. The normal reduction of P to H_0

Let P be the principal K-bundle over M defined as in 1.2. A reduction Q of P to H_0 is said to be normal if the restriction of ω to Q takes values in $\mathfrak{h} + \mathfrak{m}$.

Proposition 2.1. *Let Q be a normal reduction of P to H_0, and χ be the restriction of ω to Q. Let $\chi_{\mathfrak{h}}$ and $\chi_{\mathfrak{m}}$ be the $\mathfrak{h}$-component of χ and the $\mathfrak{m}$-component of χ respectively. Then,*

(1) $\chi_{\mathfrak{h}}$ *is a flat Cartan connection of type H/H_0 in Q.*

(2) $\chi_{\mathfrak{m}}$ *is a tensorial form, that is, the following equalities are satisfied:*

$$R_{A*}\chi_{\mathfrak{m}} = \mathrm{Ad}(A^{-1})\chi_{\mathfrak{m}} \qquad \text{for all} \quad A \in H_0.$$
$$\chi_{\mathfrak{m}}(X^*) = 0 \qquad \text{for all} \quad X \in \mathfrak{h}_0.$$

Proof. Since both $\mathfrak{h}$ and $\mathfrak{m}$ are $\mathrm{Ad}(H_0)$ invariant, we have

$$R_{A*}\chi_{\mathfrak{h}} = \mathrm{Ad}(A^{-1})\chi_{\mathfrak{h}} \quad \text{and} \quad R_{A*}\chi_{\mathfrak{m}} = \mathrm{Ad}(A^{-1})\chi_{\mathfrak{m}}$$

for any $A \in H_0$. By definition, we have

$$\chi_{\mathfrak{h}}(X^*) = X \quad \text{and} \quad \chi_{\mathfrak{m}}(X^*) = 0 \quad \text{for any} \quad X \in \mathfrak{h}_0.$$

Clearly we have $\chi_{\mathfrak{h}}(v) \neq 0$ for every non-zero tangent vector v of Q. Therefore $\chi_{\mathfrak{h}}$ is a Cartan connection of type H/H_0 in Q. Since M is 1-dimensional, $\chi_{\mathfrak{h}}$ is flat. Q.E.D.

We are now in a position to state the main theorem.

Theorem 2.2. *There exists a unique normal reduction of P to H_0.*

This theorem will be proved in the next section. The uniqueness of the normal reduction yields the following

Proposition 2.3. *Let Q be the normal reduction of P. Let σ be a cross section of P and $\sigma^*\omega$ be the pull back of ω by σ. If $\sigma^*\omega$ takes values in $\mathfrak{h} + \mathfrak{m}$, then $\sigma(x) \in Q$ for all $x \in M$. In other words, σ is a cross section of Q.*

Proof. Let Q' be the reduction of P to H_0 which contains the subset $\sigma(M)$ of P. We want to show that the reduction Q' is normal. Let χ' be the restriction of ω to Q'. We first show that $\chi'(v) \in \mathfrak{h} + \mathfrak{m}$, for any $v \in T(Q)_{\sigma(x)}$, $x \in M$. It is sufficient to consider the following two cases:

Case 1. v is vertical. In this case v can be written in the form $v = X^*$, where $X \in \mathfrak{h}_0$. Hence we have

$$\chi'(v) = X \in \mathfrak{h}_0 \subset \mathfrak{h} + \mathfrak{m}.$$

Case 2. $v = \sigma_*(d/dt)$, where t is a local coordinate system in M. We have

$$\chi'(v) = (\sigma^*\omega)\left(\frac{d}{dt}\right) \in \mathfrak{h} + \mathfrak{m}.$$

We can write any point q of Q' in the form:

$$q = \sigma(x)A, \quad x \in M, \quad A \in H_0.$$

From the equality $R_{A*}\chi'_q = \mathrm{Ad}(A^{-1})\chi'_{\sigma(x)}$ and the fact that $\mathfrak{h} + \mathfrak{m}$ is $\mathrm{Ad}(H_0)$ invariant, it follows that χ'_q takes values in $\mathfrak{h} + \mathfrak{m}$. Therefore Q' is a normal reduction. Q.E.D.

§3. Proof of Theorem 2.2

In this section, we will prove Theorem 2.2.

3.1. Algebraic preliminaries

For each integer k, $-n+1 \leqq k \leqq n-1$, let $\mathfrak{g}_k$ (resp. $\mathfrak{g}^{(k)}$) be the subspace of $gl(nr, \mathbb{R})$ consisting of all elements $X = (X_{ij})$ such that

$$X_{ij} = 0 \quad \text{for} \quad j \neq i + k$$
$$(\text{resp. } X_{ij} = 0 \quad \text{for} \quad j < j + k).$$

It is easy to see that $gl(nr, \mathbb{R})$ becomes a graded Lie algebra with the direct sum decomposition $gl(nr, \mathbb{R}) = \sum_k \mathfrak{g}_k$, that is, $[\mathfrak{g}_j, \mathfrak{g}_k] \subset \mathfrak{g}_{j+k}$, for any $-n+1 \leqq j$, $k \leqq n-1$. One should note that, for any $k \geqq 0$, $\mathfrak{g}^{(k)}$ becomes a graded subalgebra of $gl(nr, \mathbb{R})$ with the direct sum decomposition $\mathfrak{g}^{(k)} = \sum_{j \geq k} \mathfrak{g}_j$.

For $k = -1$, 0, 1, let ℓ_k be the subspace of $sl(2, \mathbb{R})$ defined by $\ell_k = \mathbb{R}s_k$. It is also easy to see that $sl(2, \mathbb{R})$ becomes a graded Lie algebra with the direct sum decomposition $sl(2, \mathbb{R}) = \sum_k \ell_k$, and that $\rho_*(\ell_k) \subset \mathfrak{g}_k$ for $k = -1$, 0, 1.

The operators ∂_k. For each integer k, $-n+2 \leqq k \leqq n$, we put $C^{k,0} = \mathfrak{g}_{k-1}$ and $C^{k,1} = \mathfrak{g}_{k-1} \otimes \ell^*_{-1}$. On the analogy of the so-called Spencer complex, we define an operator $\partial_k : C^{k,0} \to C^{k-1,1}$ by

$$(\partial_k X)(s) = [\rho_*(s), X], \quad X \in C^{k,0} \quad \text{and} \quad s \in \ell_{-1}.$$

A simple calculation shows the following

Lemma 3.1. (1) Ker $\partial_k = 0$, $2 \leqq k \leqq n$.

(2) $C^{k,1} = \mathrm{Im}\ \partial_{k+1} + (\mathfrak{h} + \mathfrak{m})_{k-1} \otimes \ell^*_{-1}$, *for any* k, *where* $(\mathfrak{h} + \mathfrak{m})_{k-1} = \mathfrak{g}_{k-1} \cap (\mathfrak{h} + \mathfrak{m})$.

The subgroup $H^{(0)}$ of K. Let G_0 and $G^{(0)}$ be the subgroups of $GL(nr, \mathbb{R})$ defined respectively by

$$G_0 = \{A = (A_{ij}) |\ A_{ij} = 0 \quad \text{for} \quad i \neq j\},$$
$$G^{(0)} = \{A = (A_{ij}) |\ A_{ij} = 0 \quad \text{for} \quad i > j\}.$$

Clearly, $G_0 \subset G^{(0)} \subset K$. It is easy to see that the Lie algebras of G_0 and $G^{(0)}$ agree with $\mathfrak{g}_0$ and $\mathfrak{g}^{(0)}$ respectively. The next lemmas are obvious.

Lemma 3.2. (1) $\mathfrak{g}_k$ *is* $\mathrm{Ad}(G_0)$*-invariant for every* k.

(2) $\mathfrak{g}^{(k)}$ *is* $\mathrm{Ad}(G^{(0)})$*-invariant for every* $k \geqq 0$.

Lemma 3.3. (1) *Every element* A *of* $G^{(0)}$ *can be written uniquely in the form:*

$$A = A_0 \exp X_1 \cdots \exp X_{n-1},$$

where $A_0 \in G_0$ *and* $X_k \in \mathfrak{g}_k$. *Moreover the assignment* $A \to A_0$ *gives a homomorphism of* $G^{(0)}$ *onto* G_0.

(2) *Every element* A *of* H_0 *can be written uniquely in the form:*

$$A = A_0 \exp X_1, \quad A_0 \in G_0 \cap H, \quad X_1 \in \mathfrak{g}_1 \cap \mathfrak{h}.$$

Let $H^{(0)}$ *be the set of all elements* A *of* $G^{(0)}$ *of the form:*

$$A = A_0 \exp X_1 \cdots \exp X_{n-1}, \quad A_0 \in G_0 \cap H, \quad X_k \in \mathfrak{g}_k. \tag{3.1}$$

Clearly $H_0 \subset H^{(0)} \subset K$. *By (1) of Lemma* 3.3, $H^{(0)}$ *is a subgroup of* $G^{(0)}$ *whose Lie algebra* $\mathfrak{h}^{(0)}$ *coincides with the subspace* $\mathfrak{h}_0 + \mathfrak{g}^{(1)}$ *of* $gl(nr, \mathbb{R})$.

Lemma 3.4. (1) $\mathbb{R}U_{-1} + \mathfrak{g}^{(0)}$ *is* $\mathrm{Ad}(H^{(0)})$*-invariant.*

(2) $\mathfrak{m}$ *is* $\mathrm{Ad}(H^{(0)})$*-invariant.*

Proof. We will prove only (1). The proof of (2) is much easier. Let A be any element of $H^{(0)}$. By definition A can be written in the form

$$A = A_0 \exp X_1 \cdots \exp X_{n-1}, \quad A_0 \in G_0 \cap H, \quad X_k \in \mathfrak{g}_k.$$

Since $A \in G^{(0)}$, we have $\mathrm{Ad}(A)\mathfrak{g}^{(0)} = \mathfrak{g}^{(0)}$. Hence to prove the assertion, it suffices to show that $\mathrm{Ad}(A)U_{-1} \in \mathbb{R}U_{-1} + \mathfrak{g}^{(0)}$. Clearly we have, for any $k \geqq 1$,

$$\mathrm{Ad}(\exp X_k)(U_{-1}) \equiv U_{-1} \pmod{\mathfrak{g}^{(0)}},$$

and hence

$$\mathrm{Ad}(A)(U_{-1}) \equiv \mathrm{Ad}(A_0)(U_{-1}) \pmod{\mathfrak{g}^{(0)}}.$$

Since $A_0 \in H$, we have

$$\mathrm{Ad}(A_0)(U_{-1}) \in \mathfrak{h} \subset \mathbb{R}U_{-1} + \mathfrak{g}^{(0)}.$$

Therefore we have the assertion. Q.E.D.

The subgroups $H^{(1)}, \cdots, H^{(n-1)}$ of K. Let $H^{(1)}, \cdots, H^{(n-1)}$ be the subgroups of K defined inductively by

$$H^{(k)} = \{A \in H^{(k-1)} \mid \mathrm{Ad}(A) \text{ preserves } \mathbb{R}U_{-1} + \mathfrak{h}^{(k-1)} + \mathfrak{m}\},$$

where $\mathfrak{h}^{(k-1)}$ stands for the Lie algebra of $H^{(k-1)}$.

Lemma 3.5. (1) *For any integer $k \geqq 1$, $H^{(k)}$ consists of the elements A of $G^{(0)}$ of the form:*

$$A = A_0 \exp X_1 \exp X_{k+1} \cdots \exp X_{n-1},$$

where $A_0 \in G_0 \cap H$, $X_1 \in \mathfrak{g}_1 \cap \mathfrak{h}$ and $X_j \in \mathfrak{g}_j$ for $j \geqq k+1$. In particular $H^{(n-1)} = H_0$.

(2) *For every $A \in H^{(k-1)} \backslash H^{(k)}$,*

$$\mathrm{Ad}(A)(\mathbb{R}U_{-1} + \mathfrak{h}^{(k-1)} + \mathfrak{m}) \cap (\mathbb{R}U_{-1} + \mathfrak{h}^{(k-1)} + \mathfrak{m}) = \mathfrak{h}^{(k-1)} + \mathfrak{m}.$$

Proof. (1) We first remark that

$$H^{(k)} = \{A \in H^{(k-1)} \mid \mathrm{Ad}(A)U_{-1} \in \mathbb{R}U_{-1} + \mathfrak{h}^{(k-1)} + \mathfrak{m}\}.$$

In fact $\mathfrak{h}^{(k-1)}$ is $\mathrm{Ad}(H^{(k-1)})$-invariant, and by Lemma 3.4, $\mathfrak{m}$ is also $\mathrm{Ad}(H^{(k-1)})$-invariant.

The proof is by induction on k. We first consider the case where $k = 1$. By definition, any element A of $H^{(0)}$ can be written as (3.1). Since $\mathrm{Ad}(A_0)U_{-1} \in \mathfrak{h} \subset \mathbb{R}U_{-1} + \mathfrak{h}^{(0)}$, we have $A_0 \in H^{(1)}$. Hence $A \in H^{(1)}$, if and only if $A_0^{-1}A \in H^{(1)}$. In a similar way as in the proof of Lemma 3.4, we can show that

$$\mathrm{Ad}(A_0^{-1}A)(U_{-1}) \equiv [X_1, U_{-1}] \pmod{\mathbb{R}U_{-1} + \mathfrak{h}^{(0)}}.$$

Therefore $A \in H^{(1)}$ if and only if $[X_1, U_{-1}] \in \mathfrak{g}_0 \cap \mathfrak{h}$. A simple calculation shows that this is equivalent to $X_1 \in \mathfrak{g}_1 \cap \mathfrak{h}$.

Assume that the assertion is true for $k-1$. Then, any element A of $H^{(k-1)}$ can be written in the form: $A = A_0 \exp X_1 \exp X_k \cdots \exp X_{n-1}$, where $A_0 \in G_0 \cap H$, $X_1 \in \mathfrak{g}_1 \cap \mathfrak{h}$ and $X_j \in \mathfrak{g}_j$ for $j \geqq k$. Hence $\mathfrak{h}^{(k-1)} = \mathfrak{h}_0 + \mathfrak{g}^{(k)}$. As above, we can show that $A_0 \exp X_1 \in H^{(k)}$. Accordingly, without loss of generality, we may assume that A is of the form

$$A = \exp X_k \cdots \exp X_{n-1}, \quad X_j \in \mathfrak{g}_j \quad j = k, \cdots, n-1.$$

Then we have

$$\mathrm{Ad}(A)(U_{-1}) \equiv [X_k, U_{-1}] \pmod{\mathbb{R}U_{-1} + \mathfrak{h}^{(k-1)} + \mathfrak{m}}.$$

Therefore $A \in H^{(k)}$ if and only if

$$[X_k, U_{-1}] \in \mathfrak{g}_{k-1} \cap (\mathfrak{h} + \mathfrak{m}).$$

This is equivalent to $X_k = 0$. (2) follows from the above arguments.

Q.E.D.

3.2. The normal reduction of P to $H^{(0)}$

A reduction $Q^{(0)}$ of P to $H^{(0)}$ is said to be normal if the restriction of ω to $Q^{(0)}$ takes values in $\mathbb{R}U_{-1} + \mathfrak{g}^{(0)}$.

Proposition 3.6. *There exists a unique normal reduction of P to $H^{(0)}$.*

Proof. We need the next lemma, which follows immediately from the definition of the group $H^{(0)}$.

Lemma 3.7. *Let A, $A' \in GL(nr, \mathbb{R})$. The following conditions are mutually equivalent:*

(i) *There exists $B \in H^{(0)}$ such that $A' = AB$.*

(ii) *There exist $a \in \mathbb{R}\backslash\{0\}$, $C \in GL(r, \mathbb{R})$ and a family of $r \times r$ matrices D_{jk}, $j < k$, such that*

$$A'_k = a^{k-1} A_k C + \sum_{j=1}^{k-1} A_j D_{jk} \quad k = 1, \cdots, n.$$

We first show the existence. We take a local coordinate system t in M and a moving frame $\{e_\alpha\}$ of E. Let σ be the standard cross section of P corresponding to $(t, \{e_\alpha\})$. Let $Q^{(0)}$ be the unique reduction of P which contains the submanifold $\sigma(M)$. By Lemmas 1.1 and 3.7, it follows that the definition of $Q^{(0)}$ does not depend on the choice of $(t, \{e_\alpha\})$.

We want to show that $Q^{(0)}$ is a normal reduction of P to $H^{(0)}$. Let $\omega^{(0)}$ be the restriction of ω to $Q^{(0)}$. We assume that, for any $x \in M$, $\omega^{(0)}$ takes values in $\mathbb{R}U_{-1} + \mathfrak{g}^{(0)}$ at $\sigma(x)$. It suffices to consider the following two cases:

Case 1. v is vertical. In this case, v can be written in the form $v = X^*$, where $X \in \mathfrak{h}^{(0)}$. Hence we have

$$\omega^{(0)}(v) = X \in \mathfrak{h}^{(0)} \subset \mathbb{R}U_{-1} + \mathfrak{g}^{(0)}.$$

Case 2. $v = \sigma_*(d/dt)$. The assertion is a consequence of Proposition 1.3.

For a general point q of $Q^{(0)}$, we can write q in the form

$$q = \sigma(x)A, \quad x \in M \quad \text{and} \quad A \in H^{(0)}.$$

Now the assertion follows from Lemma 3.4 and the equality

$$R_{A*}\omega^{(0)} = \mathrm{Ad}(A^{-1})\omega^{(0)}.$$

We next show the uniqueness. Let $Q^{(0)'}$, be a normal reduction of P to $H^{(0)}$. We must show that $Q^{(0)'}$, coincides with $Q^{(0)}$. Let σ' be any cross section of $Q^{(0)'}$. We write σ' in the form

$$\sigma'(x) = (x, Y'(x)), \quad x \in M, \quad Y'(x) \in GL(nr, \mathbb{R}).$$

By assumption, $\sigma'^*\omega$ takes values in $\mathbb{R}U_{-1} + \mathfrak{g}^{(0)}$. We can take a coordinate system t in M in such a way that $\mathbb{R}U_{-1}$-component of the function $(\sigma'^*\omega)(d/dt)$ is equal to U_{-1}. Then we have, for every $x \in M$,

$$\begin{aligned}
&Y'(x)^{-1}\frac{dY'}{dt}(x) \\
&= (\sigma'^*\omega)\left(\frac{d}{dt}\right)(x) \\
&= \begin{pmatrix}
X_{11}(x) & \cdot & \cdot & \cdot & \cdot\ \cdot & X_{1n}(x) \\
I & X_{22}(x) & & & & \cdot \\
 & 2I & \ddots & & & \cdot \\
 & \ddots & & & & \cdot \\
0 & & (n-2)I & X_{n-1\ n-1}(x) & & \cdot \\
 & & & (n-1)I & & X_{nn}(x)
\end{pmatrix},
\end{aligned}$$

where $X_{jk}(x)$, $j \leqq k$, are $r \times r$ matrices. From this we obtain the following equalities

$$(3.2) \qquad Y'_{k+1}(x) = \frac{1}{k}\frac{dY'}{dt}k(x) - \frac{1}{k}\sum_{j\leq k} Y'_j(x)X_{jk}(x), \quad k = 1, \cdots, n.$$

We take a moving frame $\{e_\alpha\}$ of E in such a way that $e_\alpha(x) + \varepsilon_\alpha(\sigma'(x))$, $\alpha = 1, \cdots, r$, for all $x \in M$, where $\varepsilon = \{\varepsilon_\alpha\}$ denotes the bundle homomorphism of P to $\mathcal{F}(E)$ defined in the paragraph 1.3. We write the standard cross section σ in the form

$$\sigma(x) = (x, Y(x)), \quad x \in M, \quad Y(x) \in GL(nr, \mathbb{R}).$$

Then, for every $x \in M$, we have the following equalities

$$(3.3) \qquad Y_{k+1}(x) = \frac{1}{k}\frac{d}{dt}Y_k(x), \quad x \in M, \quad k = 1, \cdots, n-1.$$

Furthermore we have

$$(3.4) \qquad Y'_1(x) = Y_1(x) \quad x \in M.$$

Combining (3.2), (3.3) and (3.4), we can inductively show that there exists a family of $r \times r$ matrices $D_{jk}(x)$, $j < k$, such that

$$Y'_k(x) = Y_k(x) + \sum_{j=1}^{k-1} Y_j(x)D_{jk}(x),$$

for $k = 1, \cdots, n$. From Lemma 3.7, we see that there exists $B(x) \in H^{(0)}$ such that $\sigma'(x) = \sigma(x)B(x)$. This means that $\sigma'(x) \in Q^{(0)}$. Q.E.D.

3.3. The normal reduction of P to $H^{(k)}$

For each integer $k \geqq 1$, we say that a reduction $Q^{(k)}$ of P to $H^{(k)}$ is normal if the restriction of ω to $Q^{(k)}$ takes values in $\mathbb{R}U_{-1} + \mathfrak{h}^{(k-1)} + \mathfrak{m}$. Since $H^{(n-1)} = H_0$ and $\mathbb{R}U_{-1} + \mathfrak{h}^{(n-2)} + \mathfrak{m} = \mathfrak{h} + \mathfrak{m}$, a normal reduction of P to $H^{(n-1)}$ is nothing but a normal reduction of P to H_0 defined in §2. Hence Theorem 2.2 follows from the following

Proposition 3.8. *For every $k \geqq 1$, there exists a unique normal reduction of P to $H^{(k)}$.*

Proof. The proof is by induction on k. Let us consider the case where $k = 1$. We first show the existence. Let $Q^{(0)}$ be the normal reduction of P to $H^{(0)}$ and $\omega^{(0)}$ be the restriction of ω. We define $Q^{(1)}$ to be the set of all points q of $Q^{(0)}$ such that $\omega_q^{(0)}$ takes values in $\mathbb{R}U_{-1} + \mathfrak{h}^{(0)} + \mathfrak{m}$.

Now we want to show that, for every $x \in M$, the fiber $Q_x^{(1)}$ of $Q^{(1)}$ over x is non-empty. For this purpose, we fix an arbitrary point q of $Q_x^{(0)}$ and an arbitrary subspace V_q of $T(Q^{(0)})_q$ such that the projection $\pi_{M*} : V_q \to T(M)_x$ is an isomorphism. For each j, $-n+1 \leqq j \leqq n-1$, let $\omega_j^{(0)}$ denote the $\mathfrak{g}_j$-component of $\omega^{(0)}$. Since $Q^{(0)}$ is a normal reduction, $(\omega_{-1}^{(0)})_q$ takes values in $\mathbb{R}U_{-1}$ $(= \rho_*(\ell_{-1}))$ and gives an isomorphism of V_q onto $\mathbb{R}U_{-1}$. Hence, for every $s \in \ell_{-1}$, there exists a unique element $v(s)$ of V_q such that $(\omega_{-1}^{(0)})(v(s)) = s$.

This being prepared, for each j, $-n+2 \leqq j \leqq n$, we define $u_j \in C^{j,1}$ $(= \mathfrak{g}_{j-1} \otimes \ell_{-1}^*)$ by

$$u_j(s) = (\omega_{j-1}^{(0)})(v(s)), \quad s \in \ell_{-1}. \tag{3.5}$$

Since $Q^{(0)}$ is a normal reduction, we have

$$\begin{aligned} u_j &= 0 \quad \text{for} \quad j \leqq -1, \\ u_0 &= \rho_*. \end{aligned}$$

Taking any $X \in \mathfrak{g}_1$, we put $A = \exp X$ and $q' = qA$. For each j, $-n+2 \leqq j \leqq n$, we also define $u'_j \in C^{j,1}$ by

$$u'_j(s) = (\omega_{j-1}^{(0)})(R_{A*}v(s)), \quad s \in \ell_{-1}. \tag{3.6}$$

Since $R_{A*}\omega_j^{(0)} = \omega_j^{(0)}$ for every $j \leqq -1$ and $R_{A*}\omega_0^{(0)} = \omega_0^{(0)} - [X, \omega_{-1}^{(0)}]$, we have

$$\begin{aligned} u'_j &= u_j = 0 \quad \text{for all} \quad j \leqq -1 \\ u'_0 &= u_0 = \rho_* \\ u'_{-1} &= u_1 + \partial_2 X. \end{aligned} \tag{3.7}$$

By Lemma 3.1, we can choose X in such a way that

$$u'_1 \in (\mathfrak{h} + \mathfrak{m})_0 \otimes \ell_{-1}^*. \tag{3.8}$$

Then, by (3.7) and (3.8), we have

$$\omega^{(0)}(R_{A*}v(s)) \in \mathbb{R}U_{-1} + \mathfrak{h}^{(0)} + \mathfrak{m}$$

for all $s \in \ell_{-1}$. On the other hand, we have

$$\omega^{(0)}(v) \in \mathfrak{h}^{(0)}$$

for any vertical tangent vector v of $Q^{(0)}$ at q'. Therefore we see that $\omega^{(0)}_{q'}$ takes values in $\mathbb{R}U_{-1} + \mathfrak{h}^{(0)} + \mathfrak{m}$ and hence $q' \in Q^{(1)}_x$.

From (2) of Lemma 3.5 and the definitions of $H^{(1)}$ and $Q^{(1)}$, it is clear that $Q^{(1)}$ is a normal reduction of P to $H^{(1)}$.

Next we show the uniqueness. Let $Q^{(1)'}$ be a normal reduction of P to $H^{(1)}$. Let $Q^{(0)'}$ be the reduction of P to $H^{(0)}$ which contains $Q^{(1)'}$. It is not difficult to see that $Q^{(0)'}$ is a normal reduction of P to $H^{(0)}$. Hence, $Q^{(1)'} \subset Q^{(0)'} = Q^{(0)}$. Take any point q' of $Q^{(1)'}$. From the assumption that $Q^{(1)'}$ is normal, we easily see that $\omega^{(0)}_{q'}$ takes values in $\mathbb{R}U_{-1} + \mathfrak{h}^{(0)} + \mathfrak{m}$. Hence we have $q' \in Q^{(1)}$. We have thus proved the case $k = 1$.

Assume that the assertion is true for $k-1$. Let $Q^{(k-1)}$ be the normal reduction of P to $H^{(k-1)}$ and $\omega^{(k-1)}$ be the restriction of ω to $Q^{(k-1)}$. In a similar way as in the case $k = 1$, let $Q^{(k)}$ be the set of all points q of $Q^{(k-1)}$ such that $\omega^{(k-1)}_q$ takes values in $\mathbb{R}U_{-1} + \mathfrak{h}^{(k-1)} + \mathfrak{m}$. We claim that, for every $x \in M$, the fiber $Q^{(k)}_x$ of $Q^{(k)}$ at x is non-empty. We fix an arbitrary point q of $Q^{(k-1)}_x$ and an arbitrary subspace V_q of $T(Q^{(k-1)})_q$ such that $\pi_{M*} : V_q \to T(M)_x$ is an isomorphism. Taking any $X \in \mathfrak{g}_k$, we define u_j, $u'_j \in C^{j,1}$ respectively as (3.5) and (3.6). Then we have

$$(3.9) \qquad \begin{aligned} & u'_j = u_j = 0 && \text{for} \quad j \leqq -1, \\ & u'_0 = u_0 = \rho_*, && \\ & u'_j = u_j \in (\mathfrak{h} + \mathfrak{m})_{j-1} \otimes \ell^*_{-1} && \text{for} \quad 1 \leqq j \leqq k-1, \\ & u'_k = u_k + \partial_{k+1} X. && \end{aligned}$$

By Lemma 3.1, we can take $X \in \mathfrak{g}_k$ in such a way that

$$(3.10) \qquad u'_k \in (\mathfrak{h} + \mathfrak{m})_{k-1} \otimes \ell^*_{-1}.$$

Then, from (3.9) and (3.10), we see that $q \exp X \in Q^{(k)}$, proving the assertion. It is clear that $Q^{(k)}$ is a normal reduction of P to $H^{(k)}$.

The uniqueness can be shown in quite similar manner as in the case $k = 1$. Q.E.D.

§4. Canonical forms and differential invariants

Let Q be the normal reduction of P to H_0. As before, let χ be the restriction of ω to Q and $\chi_{\mathfrak{h}}$ (resp. $\chi_{\mathfrak{m}}$) be the $\mathfrak{h}$-component of χ (resp. the $\mathfrak{m}$-component of χ).

4.1. The vector field U_{-1}^* and the functions $A^{(1)}, \cdots, A^{(n)}$

Since $\chi_{\mathfrak{h}}$ is a Cartan connection in Q, $\chi_{\mathfrak{h}}$ gives a linear isomorphism of $T(Q)_q$ onto $\mathfrak{h}$ at each point q of Q. For any $X \in \mathfrak{h}$, let X^* be the vector field on Q defined by

$$\chi_{\mathfrak{h}}(X^*) = X. \tag{4.1}$$

For $X \in \mathfrak{h}_0$, the above notation X^* is compatible with the standard notation of the fundamental vector field corresponding to X (see Preliminary remarks).

Lemma 4.1. (1) *For any* $T = \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \in SL(2,\mathbb{R})_0$ *and any* $C \in GL(r,\mathbb{R})$, *the following equalities are satisfied:*

$$R_{\rho(T)*}(U_{-1}^*) = a^2 U_{-1}^* - abU_0^* - b^2 U_1^*,$$
$$R_{\lambda(C)*}(U_{-1}^*) = U_{-1}^*.$$

(2) $[U_0^*, U_{-1}^*] = -2U_{-1}^*$ *and* $[U_1^*, U_{-1}^*] = U_0^*$.

Proof. These assertions follow immediately from the facts that

$$R_{A*}\chi_{\mathfrak{h}} = \mathrm{Ad}(A^{-1})\chi_{\mathfrak{h}} \quad \text{and} \quad d\chi_{\mathfrak{h}} + \frac{1}{2}[\chi_{\mathfrak{h}} \wedge \chi_{\mathfrak{h}}] = 0,$$

where $A \in H_0$. Q.E.D.

We write the $nr \times nr$ matrix $\chi_{\mathfrak{m}}(U_{-1}^*)_q$, $q \in Q$, in the form:

$$\chi_{\mathfrak{m}}(U_{-1}^*)q = (X_{ij}(q)),$$

where $X_{ij}(q)$ are $r \times r$ matrices. Since $\chi_{\mathfrak{m}}(U_{-1}^*)_q \in \mathfrak{m}$, we have

$$X_{ij}(q) = 0 \quad \text{for} \quad j \neq n,$$
$$X_{nn}(q) = 0,$$
$$\mathrm{Tr}\ X_{n-1\ n}(q) = 0.$$

For each integer k, $2 \leqq k \leqq n$, let $A^{(k)}(q) = (a^{(k)}_{\alpha\beta}(q))$, $q \in Q$, be the $r \times r$-matrices defined by the equation:

$$A^{(k)}(q) = -\frac{(n-1)!}{(n-k)!}\, {}^t X_{n-k+1\ n}(q).$$

Note that $\mathrm{Tr} A^{(2)}(q) = 0$.

Lemma 4.2. (1) *For any* $T = \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \in SL(2,\mathbb{R})_0$ *and any* $C \in GL(r,\mathbb{R})$, *the following equalities are satisfied:*

$$A^{(k)}(q\rho(T)) = \sum_{j=0}^{k-2} j! \binom{k-1}{j} \binom{n-k+j}{j} a^{-2k+j}(-b)^j A^{(k-j)}(q),$$

$$A^{(k)}(q\lambda(C)) = C^{-1}A(q)C.$$

(2) $U_0^* A^{(k)} = -2kA^{(k)}$,

$$U_1^* A^{(k)} = -(k-1)(n-k+1)A^{(k-1)}.$$

Proof. (1) We will show only the first equality. The second equality is also proved in a similar manner. The proof is based on (2) of Proposition 2.1 and on (1) of Lemma 4.1. We have

$$\begin{aligned}(4.2)\qquad \chi_{\mathfrak{m}}\ (U^*_{-1})_{q\rho(T)} &= a^{-2}\chi_{\mathfrak{m}}\ (R_{\rho(T)*}(U^*_{-1})_q) \\ &\quad + a^{-1}b\chi_{\mathfrak{m}}\ (U^*_0)_{q\rho(T)} + a^{-2}b^2\chi_{\mathfrak{m}}\ (U^*_1)_{g\rho(T)} \\ &= a^{-2}\,\mathrm{Ad}(\rho(T)^{-1})\chi_{\mathfrak{m}}\ (U^*_{-1})_q.\end{aligned}$$

We write $\rho(T)$ and $\rho(T)^{-1}$ in the form $\rho(T) = (T_{ij})$ and $\rho(T)^{-1} = (T'_{ij})$, where T_{ij}, T'_{ij}, $1 \leqq i, j \leqq n$ are $r \times r$ matrices. Then it is easy to see that

$$\begin{aligned}(4.3)\qquad T_{ij} &= T'_{ij} = 0 \quad \text{for} \quad i > j, \qquad \text{for} \quad i \leqq j, \\ T_{ij} &= \binom{n-i}{j-i} a^{n+1-i-j} b^{j-i} I \qquad \text{for} \quad i \leqq j, \\ T'_{ij} &= \binom{n-i}{j-i} a^{-n-1+i+j}(-b)^{j-i} I \quad \text{for} \quad i \leqq j.\end{aligned}$$

From (4.2) and (4.3) it follows that

$$X_{n-k+1\ n}(q\rho(T)) = a^{-2}\sum_{j=0}^{k-1} T'_{n-k+1\ n-k+j+1} X_{n-k+j+1\ n}(q) T_{nn}$$
$$= \sum_{j=0}^{k-1} \binom{k-1}{j} a^{-2k+j}(-b)^j X_{n-k+j+1\ n}.$$

The assertion follows from this equality.

(2) Substituting $\exp ts_0$ for T in the first equality of (1), we have

$$A^{(k)}(q\ \exp tU_0) = \exp(-2kt)A^{(k)}(q).$$

By differentiating both sides of this equality with respect to the parameter t, we obtain the first equality of (2). The second equality is proved quite similarly. Q.E.D.

4.2. Laguerre-Forsyth's canonical forms

Here, we will show that there exists one to one correspondence between the Laguerre-Forsyth's canonical forms of $\mathcal{D}$ and the integral curves of the vector field U^*_{-1}.

Let $\gamma(t)$ be an integral curve of U^*_{-1}. We write $\gamma(t)$ in the form

$$\gamma(t) = (x(t), Y(t)), \quad x(t) \in M, \quad Y(t) \in GL(nr, \mathbb{R}).$$

Clearly the map $t \to x(t)$ is an immersion and hence the parameter t can be regarded as a local coordinate system in M. Let σ be the cross section of Q defined by the assignment $\sigma : x(t) \to \gamma(t)$. Then we have

$$(\sigma^*\omega)(\frac{d}{dt})_{x(t)} = \chi(U^*_{-1})_{\gamma(t)} = U_{-1} + \chi_{\mathfrak{m}}\ (U^*_{-1})_{\gamma(t)}.$$

Hence equations (1.2) and (1.4) are satisfied. From the remark following Proposition 1.3, we see that σ is the standard cross section of P with respect to $(t, \{e_\alpha\})$, where $\{e_\alpha\}$ is defined by

$$e_\alpha(x) = \varepsilon_\alpha(\sigma(x)), \quad x \in M, \quad \alpha = 1, \cdots, r. \tag{4.4}$$

By Proposition 1.3, $\mathcal{D}$ is written in the form

$$\left(\frac{d}{dt}\right)^n y_\alpha + \sum_{k=1}^{n}\sum_{\beta=1}^{r} a^{(k)}_{\alpha\beta}(\gamma(t)) \left(\frac{d}{dt}\right)^{n-k} y_\beta = 0, \quad \alpha = 1, \cdots, r. \tag{4.5}$$

This is nothing but one of the Laguerre Forsyth's canonical forms of $\mathcal{D}$.

Theorem 4.3. *For any integral curve $\gamma(t)$ of U^*_{-1}, (4.5) gives a Laguerre-Forsyth's canonical form of $\mathcal{D}$. Conversely every Laguerre-Forsyth's canonical form can be thus obtained.*

Proof. It is sufficient to show the converse. We take any Laguerre-Forsyth's canonical form of $\mathcal{D}$. Let t be the corresponding coordinate system in M and $\{e_\alpha\}$ be the corresponding moving frame of E, and let σ be the standard cross section of P corresponding to $(t, \{e_\alpha\})$. By Proposition 1.3, the 1-form $\sigma^*\omega$ takes values in $\mathfrak{h}+\mathfrak{m}$, and by Proposition 2.3, σ is a cross section of Q. Let $\gamma(t)$ be the curve in Q satisfying $\gamma(t(x)) = \sigma(x)$. Clearly $\gamma(t)$ is an integral curve of U^*_{-1}. It is obvious that the Laguerre-Forsyth's canonical form corresponds to this integral curve $\gamma(t)$. Q.E.D.

4.3. Transformations between canonical forms

Here, we will interpret the fact (B) in terms of the relation between the integral curves of U^*_{-1}. Let $\gamma(t)$ and $\gamma'(t)$ be integral curves of U^*_{-1}. Regarding t and t' as local coordinate systems in M, we write t' in the form: $t' = t'(t)$.

Theorem 4.4. *There exist $S = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R})$ and $C = (c_{\alpha\beta}) \in GL(r,\mathbb{R})$ such that the following equalities are satisfied.*

$$\begin{aligned} t'(t) &= \frac{at+b}{ct+d}, \\ \gamma'(t'(t)) &= \gamma(t)\rho(T(t))\lambda(C), \end{aligned} \tag{4.6}$$

where $T(t)$ is the element of $SL(2,\mathbb{R})_0$ defined by

$$T(t) = \begin{pmatrix} (ct+d)^{-1} & -c \\ 0 & ct+d \end{pmatrix}.$$

Proof. For any $S = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R})$ and any $C \in GL(r,\mathbb{R})$, we define a curve $\delta(s)$ by the right-hand side of (4.6), i.e.,

$$\begin{aligned} s = s(t) &= \frac{at+b}{ct+d}, \\ \delta(s(t)) &= \gamma(t)\rho(T(t))\lambda(C). \end{aligned}$$

We claim that $\delta(s)$ is an integral curve of U^*_{-1}. For this purpose, we write $\gamma(t)$ and $\delta(s)$ in the form:

$$\begin{aligned} \gamma(t) &= (x(t), Y(t)), \quad x(t) \in M, \quad Y(t) \in GL(r, \mathbb{R}), \\ \delta(s) &= (z(s), W(s)), \quad z(s) \in M, \quad W(s) \in GL(r, \mathbb{R}). \end{aligned}$$

We have $z(s(t)) = x(t)$ and $W(s(t)) = Y(t)\rho(T(t))\lambda(C)$. Hence, it follows that

$$\begin{aligned} \chi_{\mathfrak{h}}\left(\delta_*\left(\frac{d}{ds}\right)\right) &= [W(s)^{-1}\frac{dW}{ds}(s)]_{\mathfrak{h}} \\ &= \left(\frac{dt}{ds}\right)\lambda(C)^{-1}\rho(T(t))^{-1}[Y(t)^{-1}\frac{dY}{dt}(t)]_{\mathfrak{h}}\ \rho(T(t))\lambda(C) \\ &+ \left(\frac{dt}{ds}\right)\lambda(C)^{-1}\rho(T(t))^{-1}\frac{d}{dt}\rho(T(t))\lambda(C), \end{aligned}$$

where $[Y(t)^{-1}dY/dt(t)]_{\mathfrak{h}}$ and $[W(s)^{-1}dW/ds(s)]_{\mathfrak{h}}$ denote the $\mathfrak{h}$ components of $Y(t)^{-1}dY/dt(t)$ and $W(s)^{-1}dW/ds(s)$ respectively. Using the equality $[Y(t)^{-1}dY/dt(t)]_{\mathfrak{h}} = U_{-1} = \rho_*(s_{-1})$ and the fact that ρ is a homomorphism of $SL(2, \mathbb{R})$ onto $GL(nr, \mathbb{R})$, we have

$$\begin{aligned} &\rho(T(t)^{-1})[Y(t)^{-1}\frac{dY}{dt}(t)]_{\mathfrak{h}}\ \rho(T(t)) = \rho_*(\mathrm{Ad}(T(t)^{-1}s_{-1})), \\ &\rho(T(t)^{-1})\frac{d}{dt}\rho(T(t)) = \rho_*(T(t)^{-1}\frac{dT}{dt}(t)). \end{aligned}$$

A direct calculation shows that

$$\mathrm{Ad}(T(t)^{-1})s_{-1} + T(t)^{-1}\frac{dT}{dt}(t) = \frac{1}{(ct+d)^2}s_{-1}.$$

From these equalities and $dt/ds = (ct+d)^2$, we conclude that

$$\chi_{\mathfrak{h}}\left(\delta_*\left(\frac{d}{ds}\right)\right) = U_{-1}.$$

Therefore $\delta(s)$ is an integral curve of U^*_{-1}.

We can choose $T \in SL(2, \mathbb{R})$ and $C \in GL(r, \mathbb{R})$ in such a way that $\gamma'(t_0) = \delta(t_0)$ for some $t_0 \in \mathbb{R}$. Since both $\gamma'(t')$ and $\delta(s)$ are integral curves of U^*_{-1}, $\gamma'(t')$ coincides with $\delta(s)$, i.e., $t' = s$ and $\gamma'(t') = \delta(s)$.

Q.E.D.

Corollary 4.5. *Under the same notations as in Theorem 4.4, the following equalities are satisfied:*

$$\varepsilon_\beta(\gamma'(t')) = (ct+d)^{n-1}\sum_{\alpha=1}^{r} c_{\alpha\beta}\varepsilon_\alpha(\gamma(t)), \quad \beta = 1,\cdots,r,$$

$$A^{(k)}(\gamma'(t')) = \sum_{j=0}^{k-2} j!\binom{k-1}{j}\binom{n-k+j}{j}\cdot (ct+d)^{2k-j}c^j C^{-1}A^{(k-j)}(\gamma(t))C.$$

Proof. The first assertion follows from the fact that

$$\varepsilon\circ\rho(T(t)) = (ct+d)^{n-1}I \quad \text{and} \quad \varepsilon\circ\lambda(C) = C.$$

The second assertion follows from (1) of Lemma 4.2. Q.E.D.

4.4. Differential invariants $\Omega_2,\cdots,\Omega_n$

We say that an $r\times r$ matrix valued function $\Omega = (\Omega_{\alpha\beta})$ on Q is said to be an invariant of weight k if it satisfies the following two conditions:

(I.1)
$$\Omega(q\rho(T)) = a^{-2k}\Omega(q) \quad \text{for all} \quad T = \begin{pmatrix} a & b \\ 0 & a^{-1}\end{pmatrix} \in SL(2,\mathbb{R})_0 \quad \text{and all } q\in Q.$$

(I.2)
$$\Omega(q\lambda(C)) = C^{-1}\Omega(q)C \quad \text{for all} \quad C\in GL(r,\mathbb{R}) \quad \text{and all } q\in Q.$$

We may replace condition (I.1) by the following condition:

(I.1)′
$$U_0^*\Omega = -2k\Omega \quad \text{and} \quad U_1^*\Omega = 0.$$

Let χ_{-1} denote the $\mathbb{R}U_{-1}$ component of χ with respect to the direct sum decomposition $\mathfrak{h} = \mathbb{R}U_{-1} + \mathfrak{h}_0$. The next lemma is obvious.

Lemma 4.6. (1) $\chi_{-1}(v) = 0$ *for any vertical tangent vector v of Q.*

(2) $R_{\rho(T)*}\chi_{-1} = a^2\chi_{-1}$ for all $T = \begin{pmatrix} a & b \\ 0 & a^{-1}\end{pmatrix} \in SL(2,\mathbb{R})_0$, $R^*_{\lambda(C)}\chi_{-1} = \chi_{-1}$ for all $C\in GL(r,\mathbb{R})$.

For any invariant $\Omega = (\Omega_{\alpha\beta})$ of weight k on Q, we define an $r \times r$ matrix valued covariant tensor field $\Omega' = (\Omega'_{\alpha\beta})$ of degree k on Q by

$$\Omega'_q = \overbrace{\chi_{-1} \otimes \cdots \otimes \chi_{-1}}^{k \text{ times}} \otimes \Omega(q).$$

Proposition 4.7. (1) $\Omega'(v_1, \cdots, v_k) = 0$ *whenever at least one of the tangent vectors* v_i *of* Q *is vertical.*

(2) $R_{\rho(T)*}\Omega' = \Omega'$ *for any* $T \in SL(2, \mathbb{R})_0$.
$R_{\lambda(C)*}\Omega' = C^{-1}\Omega' C$ *for any* $C \in GL(r, \mathbb{R})$.

(3) *There exists a unique cross section* θ *of the vector bundle* $\overset{k}{\otimes} T(M)^* \otimes \mathrm{End}(E)$ *such that, for every* $q \in Q$ *and every* β, $1 \leqq \beta \leqq r$,

$$\theta_q(\pi_* v_1, \cdots, \pi_* v_k)\varepsilon_\beta(q) = \sum_{\alpha=1}^{r} \Omega'_{\alpha\beta}(v_1, \cdots, v_k)\varepsilon_\alpha(q),$$

where $v_1, \cdots, v_k \in T(Q)_q$ *and* π *denotes the projection of* Q *onto* M.

Proof. (1) and (2) follow from the definition of the invariants and Lemma 4.6.

(3) For each $x \in M$, we take any $q \in Q$ such that $\pi(q) = x$. From (1), we see that there exists a unique element θ'_q of $\overset{k}{\otimes} T(M)^*_x \otimes \mathrm{End}(E_x)$ such that

$$\theta'_q(\pi_* v_1, \cdots, \pi_* v_k)\varepsilon_\beta(q) = \sum_{\alpha=1}^{r} \Omega'_{\alpha\beta}(v_1, \cdots, v_k)\varepsilon_\alpha(q),$$

for all $v_1, \cdots, v_k \in T(Q)_q$. We put $\theta_x = \theta'_q$. To see that the definition of θ_x does not depend on the choice of q, it suffices to remark the following equalities:

$$\varepsilon_\alpha(q\rho(T)) = a^{-n+1}\varepsilon_\alpha(q), \ 1 \leqq \alpha \leqq r, \text{for all} \quad T \in SL(2, \mathbb{R})_0$$

$$\varepsilon_\beta(q\lambda(C)) = \sum_{\alpha=1}^{r} c_{\alpha\beta}\varepsilon_\beta(q), \ 1 \leqq \beta \leqq r, \text{for all} \quad C \in GL(r, \mathbb{R}).$$

Q.E.D.

For each integer k, $2 \leqq k \leqq n$, let θ_k be the $r \times r$-matrix-valued function on Q defined by

$$\theta_k(x) = \sum_{j=0}^{k-2} (-1)^j \frac{(2k-j-2)!(n-k+j)!}{j!(k-j-1)!} (U^*_{-1})^j A^{(k-j)}(x), \quad x \in M.$$

Remark. In the case where $r = 1$, the invariant Ω_2 automatically vanishes.

Theorem 4.8. *Ω_k is an invariant of weight k.*

Proof. For the sake of simplicity, we put

$$A^{(k,j)} = (U_{-1}^*)^j A^{(k)}.$$

The assertion is an immediate consequence of the following

Lemma 4.9. (1) $U_0^* A^{(k,j)} = -2(k+j)A^{(k,j)}$.

(2) $U_1^* A^{(k,j)} = -(2k+j-1)j\ A^{(k,j-1)} - (k-1)(n-k+1)A^{(k-1,j)}$.

(3) $A^{(k,j)}(q\lambda(C)) = C^{-1}A^{(k,j)}(q)C$ *for all* $q \in Q$ *and* $C \in GL(r,\mathbb{R})$.

Proof. By (2) of Lemma 4.1 and by (2) of Lemma 4.2, we have

$$\begin{aligned} U_0^* A^{(k,j)} &= U_0^* (U_{-1}^*)^j A^{(k)} \\ &= \sum_{i=1}^{j} (U_{-1}^*)^{j-i} [U_0^*, U_{-1}^*](U_{-1}^*)^{i-1} A^{(k)} \\ &\qquad + (U_{-1}^*)^j U_0^* A^{(k)} \\ &= -2j(U_{-1}^*)^j A^{(k)} - 2k(U_{-1}^*)^j A^{(k)} \\ &= -2(k+j)A^{(k,j)}, \end{aligned}$$

proving (1). Similarly we have

$$\begin{aligned} U_1^* A^{(k,j)} &= U_1^* (U_{-1}^*)^j A^{(k)} \\ &= \sum_{i=1}^{j} (U_{-1}^*)^{j-i} [U_1^*, U_{-1}^*](U_{-1}^*)^{i-1} A^{(k)} \\ &\qquad + (U_{-1}^*)^j U_1^* A^{(k)} \\ &= \sum_{i=1}^{j} (U_{-1}^*)^{j-i} U_0^* A^{(k,i-1)} \\ &\qquad - (k-1)(n-k+1)A^{(k-1,j)}. \end{aligned}$$

From (1), it follows that

$$\begin{aligned} U_1^* A^{(k,j)} &= -\sum_{i=1}^{j} 2(k+i-1)A^{(k,j-1)} \\ &\quad - (k-1)(n-k+1)A^{(k-1,j)} \\ &= -(2k+j-1)j\ A^{(k,j-1)} - (k-1)(n-k+1)A^{(k-1,j)}, \end{aligned}$$

proving (2). (3) follows directly from (1) of Lemma 4.1 and (1) of Lemma 4.2. Q.E.D.

Let θ_k be the cross section of $\overset{k}{\otimes} T(M)^* \otimes \mathrm{End}(E)$ corresponding to Ω_k. Let $\gamma(t)$ be an integral curve of U_{-1}^*. As in 4.2, we take t as a local coordinate system in M and give the moving frame $\{e_\alpha\}$ of E defined by (4.4). In terms of $(t, \{e_\alpha\})$, θ_k is expressed by the $r \times r$-matrix-valued k-covariant tensor field

$$\sum_{j=0}^{k-2} (-1)^j \frac{(2k-j-2)!(n-k+j)!}{j!(k-j-1)!} \left(\frac{d}{dt}\right)^j A^{(k-j)}(\gamma(t))(dt)^k.$$

Note that $A^{(j)}(\gamma(t))$, $j = 2, \cdots, n$, are coefficients of (4.5).

§5. Normal reductions and isomorphisms

In this section, we will show the compatibility of the normal reductions of Theorem 2.2 with isomorphisms of linear ordinary differential equations.

5.1. Isomorphisms of linear ordinary differential equations

Let E' be a vector bundle over a 1-dimensional manifold M' of the same rank as E. As in §1, we consider a system $\mathcal{D}'$ of linear ordinary differential equations on E'.

Let ϕ be a bundle isomorphism of E onto E'. For each cross section s' of E', we define a cross section $\phi^* s'$ of E by

$$(\phi^* s')(x) = \phi_x^{-1}(s'(\phi'(x)),$$

where $x \in M$ and ϕ' denotes the diffeomorphism of M onto M' induced by ϕ.

We say that ϕ is an isomorphism of $\mathcal{D}$ onto $\mathcal{D}'$ if $\mathcal{D}'$ corresponds to $\mathcal{D}$ under ϕ. In this case, we have $\phi^* s' \in \mathrm{Sol}(\mathcal{D}')$ for any $s' \in \mathrm{Sol}(\mathcal{D}')$.

5.2 Normal reductions and isomorphisms

We fix a basis $\{s'_1, \cdots, s'_{nr}\}$ of $\mathrm{Sol}(\mathcal{D}')$. For any isomorphism ϕ of $\mathcal{D}$ onto $\mathcal{D}'$, we define an $nr \times nr$-matrix $A(\phi) = (a_{\alpha\beta}(\phi))$ as follows: As we have remarked above, $\phi^* s'_\alpha \in \mathrm{Sol}(\mathcal{D})$ for every α. Hence $\phi^* s'_\alpha$ can be written as a linear combination of $s_1, \cdots, s_{nr}$. We define $A(\phi)$ by

$$\phi^* s'_\alpha = \sum_{\beta=1}^{r} a_{\alpha\beta}(\phi) s_\beta, \quad \alpha = 1, \cdots, nr.$$

Using the matrix $A(\phi)$, we define a bundle isomorphism Φ of $M \times GL(nr, \mathbb{R})$ onto $M' \times GL(nr, \mathbb{R})$ as follows

$$\Phi(x, Z) = (\phi'(x), A(\phi)Z),$$

where $x \in M$ and $Z \in GL(nr, \mathbb{R})$.

Associated with $\mathcal{D}'$, we define a principal K-bundle P' over M' and a $gl(nr, \mathbb{R})$-valued 1-form ω' on P' as before. Let Q' be the normal reduction of P' to H_0, and let χ' be the restriction of ω' to Q'.

Theorem 5.1. (1) *For every isomorphism ϕ of $\mathcal{D}$ onto $\mathcal{D}'$, Φ maps Q onto Q' and induces a bundle isomorphism Φ_Q of Q onto Q' satisfying $\Phi_Q^* \chi' = \chi$.*

(2) *For every bundle isomorphism Ψ of Q onto Q' satisfying $\Psi^* \omega' = \omega$, there exists a unique isomorphism ϕ of $\mathcal{D}$ onto $\mathcal{D}'$ such that $\Phi_Q = \Psi$.*

Proof. (1): We first show that Φ maps P onto P'. We choose a moving frame $\{e_\alpha\}$ of E and a moving frame $\{e'_\alpha\}$ of E' in such a way that $\phi^* e'_\alpha = e_\alpha$, $\alpha = 1, \cdots, n$, and then define Y_1 and Y'_1 by (0.4). It is easy to see that

$$Y'_1(\phi'(x)) = A(\phi) Y_1(x).$$

From this we easily see that Φ maps P onto P' and induces a bundle isomorphism Φ_P of P onto P'. Moreover we have $\Phi_P^* = \omega$. From the uniqueness of the normal reduction of P to H_0, we conclude that Φ_P maps Q onto Q', that is, Φ maps Q onto Q'. Hence Φ induces a bundle isomorphism Φ_Q of Q onto Q'. It is clear that $\Phi_Q^* \chi' = \chi$.

(2): We first remark that there exists a unique bundle isomorphism ϕ of E onto E' which makes the following diagram commutative:

$$\begin{array}{ccc} Q & \xrightarrow{\ \Psi\ } & Q' \\ \varepsilon \downarrow & & \varepsilon' \downarrow \\ \mathcal{F}(E) & \xrightarrow{\ \phi_{\mathcal{F}}\ } & \mathcal{F}(E') \end{array}$$

where $\phi_{\mathcal{F}}$ is the natural bundle isomorphism of $\mathcal{F}(E)$ onto $\mathcal{F}(E')$ induced by ϕ.

It suffices to show that ϕ is an isomorphism of $\mathcal{D}$ onto $\mathcal{D}'$. For this purpose, we take an integral curve $\gamma(t)$ of the vector field U_{-1}^* on Q. Let $\gamma'(t)$ be the curve in Q' defined by $\gamma'(t) = \Psi(\gamma(t))$. From the equality $\Psi^*\chi' = \chi$, we see that $\gamma'(t)$ is an integral curve of $U_{-1}^{*\prime}$, where U_{-1}^* is the vector field on Q' defined in the same way as U_{-1}^*. It is obvious that the canonical form of $\mathcal{D}'$ corresponding to $\gamma'(t)$ coincides with that of $\mathcal{D}$ corresponding to $\gamma(t)$. Q.E.D.

§6. Appendix

6.1. Jet bundles

Let E be a vector bundle over a 1-dimensional manifold M. We denote by $\Gamma(E)$ the space of all cross sections of E on M. In this section, we will not necessarily assume that M is simply connected, unless otherwise stated. For each $k \geqq 0$, let $J^k(E)$ be the k-th jet bundle of E, and for any $s \in \Gamma(E)$ and any $x \in M$, $j_x^k(s)$ denotes the k-th jet of s at x. For each integer k, $0 \leqq k \leqq n-1$, π_k^{n-1} denotes the natural projection of $J^{n-1}(E)$ onto $J^k(E)$.

We put $F^k(E) = \operatorname{Ker} \pi_K^{n-1}$ for $0 \leqq k \leqq n-1$, and $F^{-1}(E) = J^{n-1}(E)$. Then we have a natural filtration

$$0 = F^{n-1}(E) \subset F^{n-2}(E) \subset \cdots \subset F^0 \subset F^{-1}(E) = J^{n-1}(E).$$

Moreover there is a natural bundle isomorphism of the quotient bundle $F^{k-1}(E)/F^k(E)$ onto the vector bundle $\overset{k}{\otimes} T(M)^* \otimes W$. This is defined as follows: We take a local coordinate system t in M and a moving frame $\{e_\alpha\}$ of E. Fixing a point x of M, to each $s \in \Gamma(E)$ with $j_x^{k-1}(s) = 0$, we assign

$$(6.1) \qquad \frac{1}{k!}\sum_{\alpha=1}^{r}(\frac{d}{dt})^k y_\alpha(x)(dt)^k \otimes e_{\alpha x} \quad (\in \overset{k}{\otimes} T(M)_x^* \otimes E_x),$$

where y_α, $1 \leqq \alpha \leqq r$, are functions defined by (0.1). It is easy to see that the definition of (6.1) does not depend on the choice of $(t, \{e_\alpha\})$. It is also easy to see that (6.1) is equal to zero for every $s \in \Gamma(E)$ with $j_x^k(s) = 0$. Thus the assignment of s to the expression (6.1) induces the isomorphism $F^{k-1}(E)/F^k(E) \simeq \overset{k}{\otimes} T(M)^* \otimes E$.

6.2. The typical fiber of $J^{n-1}(E)$

Let V be an r-dimensional vector space with a fixed bases $\{v_1, \cdots, v_r\}$ and W a 1-dimensional vector space with a fixed bases $\{w\}$. Let F be the vecotr space defined by

$$F = V + W \otimes V + \cdots + \overset{n-1}{\otimes} W \otimes V \quad \text{(direct sum)}$$

Taking $\{v_1, \cdots, v_r, w \otimes v_1, \cdots, w \otimes v_r, \cdots, w^{n-1} \otimes v_1, \cdots, w^{n-1} \otimes v_r\}$ as a basis of F, we identify F with $\mathbb{R}^{nr}$. We put

$$F^k = \sum_{j>k} \overset{j}{\otimes} W \otimes V.$$

Then we have a natural filtration:

$$0 = F^{n-1} \subset F^{n-2} \subset \cdots \subset F^0 \subset F^{-1} = F$$

Note that F^{k-1}/F^k is isomorphic to $\overset{k}{\otimes} W \otimes V$.

6.3. The principal $H^{(0)}$-bundle $R^{(0)}$

Let $\mathcal{F}(J^{n-1}(E))$ be the frame bundle of $J^{n-1}(E)$. For every $x \in M$, we regard $\mathcal{F}(J^{n-1}(E))_x$ as the set of all linear isomorphisms of F onto $J^{n-1}(E)_x$.

Let $R_x^{(0)}$ be the set of all elements p of $\mathcal{F}(J^{n-1}(E))_x$ satisfying the following two conditions.

(R.1) $\qquad p(F^k) = F^k(E) \quad \text{for every} \quad 0 \leqq k \leqq n-1.$

By the first condition (R.1), every element p of $R_x^{(0)}$ induces an isomorphism p^k of F^{k-1}/F^k onto $F^{k-1}(E)_x/F^k(E)_x$ for each k, $0 \leqq k \leqq n-1$. Furthermore the induced isomorphism p^0 can be regarded as an isomorphism of V onto E_x. The second condition is
(R.2) There exists a linear isomorphism A of W onto $T(M)_x^*$ which makes the following diagram commutative:

$$\begin{array}{ccc} F^{k-1}/F^k & \xrightarrow{p^k} & F^{k-1}(E)_x/F^k(E)_x \\ \downarrow & & \downarrow \\ \overset{k}{\otimes} W \otimes V & \xrightarrow{A^k \otimes p^0} & \overset{k}{\otimes} T(M)_x^* \otimes E_x \end{array}$$

where $A^k : \overset{k}{\otimes} W \to \overset{k}{\otimes} T(M)_x^*$ denotes the isomorphism defined by

$$A^k(w_1 \otimes \cdots \otimes w_k) = Aw_1 \otimes \cdots \otimes Aw_k, \quad w_1, \cdots, w_k \in W.$$

We put $R^{(0)} = \bigcup_{x\in M} R_x^{(0)}$. It is not difficult to see that $R^{(0)}$ is a principal $H^{(0)}$-bundle over M under the identification of F with $\mathbb{R}^{nr}$.

Let t be a local coordinate system in M and $\{e_\alpha\}$ a moving frame of E. For each $x \in M$, let σ_x be the linear isomorphism of F onto $J^{n-1}(E)_x$ defined by

$$\sum_{k=0}^{n-1} \sum_{\alpha=1}^{r} \frac{1}{k!}(\frac{d}{dt})^k y_\alpha(x) w^k \otimes v_\alpha \to j_x^{n-1}(s),$$

where $s = \sum y_\alpha e_\alpha \in \Gamma(E)$. It can be easily verified that $\sigma_x \in R_x^{(0)}$. Thus we have a cross section σ of $R^{(0)}$, which will be called the standard cross section of $R^{(0)}$ corresponding $(t, \{e_\alpha\})$.

6.4. The associated connection of $J^{n-1}(E)$

As before, let $\mathcal{D}$ be a system of linear ordinary differential equations on E of order n. Let us consider the system of first-order differential equations associated with $\mathcal{D}$. This can be regarded as a system of first-order equations on the $(n-1)$-th jet bundle $J^{n-1}(E)$ of E as follows; Let σ be the standard cross section of $R^{(0)}$ corresponding to a pair $(t, \{e_\alpha\})$. For every $\zeta \in \Gamma(J^{n-1}(E))$, we define a family of functions $f_{k\alpha}$, $1 \leqq k \leqq n$, $1 \leqq \alpha \leqq r$ on M by

$$\sigma_x^{-1}\zeta(x) = \sum_{k=1}^{n} f_{k\alpha}(x) w^{k-1} \otimes v_\alpha, \quad x \in M.$$

Then the system of first-order equations is given by

$$\frac{df_{k\alpha}}{dt} - k f_{k+1\ \alpha} = 0, \quad 1 \leqq k \leqq n-1,$$
$$\frac{df_{n\alpha}}{dt} + \sum_{k=1}^{n} \sum_{\beta=1}^{r} \frac{(n-k)!}{(n-1)!}\, a_{\alpha\beta}^{(k)} f_{k\beta} = 0.$$

Now we consider the affine connection of $J^{n-1}(E)$ associated with this system. Let ω be its connection form in $\mathcal{F}(J^{n-1}(E))$ and let $\omega^{(0)}$ be the restriction of ω to $R^{(0)}$. It is easy to see that a similar assertion as in Proposition 1.3 holds for the pair $(R^{(0)},\ {}^t\omega^{(0)})$. Therefore we have the following

Proposition 6.1. *${}^t\omega^{(0)}$ takes values in $\mathbb{R}U_{-1} + \mathfrak{g}^{(0)}$.*

6.5. A reduction theorem for $R^{(0)}$

A reduction R of $R^{(0)}$ to H_0 is said to be normal if the restriction of ${}^t\omega^{(0)}$ to R takes values in $\mathfrak{h}+\mathfrak{m}$. In view of Proposition 6.1, it is obvious that the proof of Proposition 3.8 is still valid, if we replace $Q^{(0)}$ by $R^{(0)}$. Therefore we obtain the following

Theorem 6.2. *There exists a unique normal reduction of $R^{(0)}$ to H_0.*

Remark. Assume that M is simply connected. We fix a basis $\{s_1, \cdots, s_{nr}\}$ of $\mathrm{Sol}(\mathcal{D})$. Then the system $\{j^{n-1}(s_1), \cdots, j^{n-1}(s_{nr})\}$ gives a trivialization: $J^{n-1}(E) \simeq M \times \mathbb{R}^{nr}$. Hence we can naturally identify $\mathcal{F}(J^{n-1}(E))$ with $M \times GL(nr, \mathbb{R})$. It is obvious that, under this identification, the correspondence $(x, y) \in R^{(0)} \to (x, {}^tY^{-1}) \in P$ maps the normal reduction of $R^{(0)}$ to that of P constructed in §3.

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Editor's Note. The present paper was written by the author before he died by an accident in 1989, when he was an associate professor of Department of Mathematics, Kitami Institute of Technology, Kitami 090, Japan. Correspondence should be sent to Professor K. Yamaguchi, Department of Mathematics, Faculty of Science, Hokkaido University, Sapporo 060, Japan.

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The Length Function of Geodesic Parallel Circles

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Dedicated to Professor T. Otsuki on his 75th birthday

§0. Introduction

The isoperimetric inequalities for a simply closed curve C on a Riemannian plane Π (i.e., a complete Riemannian manifold homeomorphic to $\mathbf{R}^2$) was first investigated by Fiala in [1] and later by Hartman in [2]. These inequalities were generalized by the first named author in [3],[4] for a simply closed curve on a finitely connected complete open surface and by both authors in [5] for a simply closed curve on an infinitely connected complete open surface. Here a noncompact complete and open Riemannian 2-manifold M is called *finitely connected* if it is homeomorphic to a compact 2-manifold without boundary from which finitely many points are removed, and otherwise M is called *infinitely connected.* Fiala and Hartman investigated certain properties of geodesic parallel circles $S(t) := \{x \in \Pi \; ; \; d(x, C) = t\}$, $t \geq 0$ around C of a Riemannian plane Π in order to prove the isoperimetric inequalities, where d denotes the Riemannian distance function. Fiala proved in [1] that if a Riemannian plane Π and a simple closed curve C on Π are *analytic*, then $S(t)$ is a finite union of piecewise smooth simple closed curves except for t in a discrete subset of $[0, \infty)$ and its length $L(t)$ is *continuous* on $[0, \infty)$. If Π and C are *not analytic but smooth*, then $L(t)$ is *not always continuous* as pointed out by Hartman in [2]. What is worse is that $S(t)$ does not always admit its length. Under the assumption of low differentiability of Π and C, Hartman proved that $S(t)$ is a finite union of piecewise smooth simple closed curves except for t in a closed subset of Lebesgue measure zero in $[0, \infty)$. This result was recently extended by the authors [5] to an arbitrary given simply closed curve C in an arbitrary given complete, connected, oriented and noncompact Riemannian 2-manifold M.

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The normal exponential map along C induces a local chart and a function $L(t)$ for all $t \geq 0$ is well defined with the aid of this local chart. As mentioned above, $L(t)$ for all $t \geq 0$ defines the length of $S(t)$ whenever $S(t)$ is a finite union of piecewise smooth simple closed curves. However we do not know the geometric meaning of $L(t)$ for the other t-values. Hartman introduced a certain monotone function $J\colon [0,\infty) \to \mathbf{R}$ by using this local chart and proved in Theorem 6.2 ; [2] that the following function

$$(*) \qquad H(t) := J(t) + L(t)$$

is absolutely continuous on every compact interval of $[0,\infty)$.

The purpose of the present article is to extend the absolute continuity of H as defined in $(*)$ for an arbitrary given simple closed curve C in an arbitrary given connected, complete, noncompact and oriented Riemannian 2-manifold M. The cut locus and focal locus to C are essential in our discussion. In §1 we introduce the notations concerning with the cut points and focal points to C as used in [2],[5]. Under our situation $M \setminus C$ has at most two components. The type of cut locus and focal locus changes as the number of components of $M \setminus C$. In §2 we deal with the simpler case where $M \setminus C$ has two components and prove the absolute continuity of $(*)$ in this case (see Theorem 2.2). We also need to modify the definition of $J(t)$ in the case where $M \setminus C$ is connected. In §3 we prove the absolute continuity of $(*)$ in the case where $M \setminus C$ is connected (see Theorem 3.2).

§1. Preliminaries

From now on let M be a connected, oriented, complete and noncompact Riemannian 2-manifold and C a smooth simply closed curve on M. Since our discussion proceeds in the same manner as developed by Hartman , we shall employ the same terminologies as used in [2],[5]. Let L_0 be the length of C. A point on C is expressed as $z_0(s)$ with respect to the arclength parameter $s \in [0, L_0]$. $z_0(s)$ and other functions of s will be considered periodic of period of L_0 for convenience. Let g be the Riemannian metric on M and N a unit normal field along C with $N_0 = N_{L_0}$. A map $z : \mathbf{R}\times[0, L_0] \to M$ is defined by

$$z(t,s) := \exp_{z_0(s)} tN_s$$

where $\exp_p$ is the exponential map of M at p. If $|t|$ is sufficiently small, then z gives a coordinate system (t,s) and $g\left(\dfrac{\partial z}{\partial t}, \dfrac{\partial z}{\partial t}\right) = 1$ holds around

C and $g\left(\frac{\partial z}{\partial t},\frac{\partial z}{\partial s}\right)=0$ follows from Gauss Lemma. For every $s\in[0,L_0]$ let $\gamma_s\colon R\to M$ be a geodesic with $\gamma_s(t)=z(t,s)$ and $e_s(t)$ a unit parallel vector field along γ_s with $e_s(0)=\frac{\partial z}{\partial s}(0,s)$. For each s let $Y_s(t)$ denote the Jacobi field along γ_s with $Y_s(0)=e_s(0)$, $g(Y_s(t),\gamma_s'(t))=0$. By setting $f(t,s)=g(Y_s(t),e_s(t))$, we have $f(0,s)=1$, $f_t(0,s)=\kappa(s)$ and $g\left(\frac{\partial z}{\partial s},\frac{\partial z}{\partial s}\right)=f^2(t,s)$, where $\kappa(s)$ is the geodesic curvature of C at $z_0(s)$ and $f_t=\frac{\partial f}{\partial t}$. Since Y_s is a Jacobi field we have $f_{tt}(t,s)+G(z(t,s))f(t,s)=0$, where $f_{tt}=\frac{\partial^2 f}{\partial t^2}$.

Let $P(s)$ (respectively $N(s)$) denote the least positive (respectively the largest negative) t with $f(s,t)=0$, or $P(s)=+\infty$ (respectively $N(s)=-\infty$) if there is no such zero. If $P(s_0)<+\infty$ (respectively $N(s_0)>-\infty$), then P (respectively N) is smooth around s_o and $z(P(s_0),s_0)$, (respectively $z(N(s_0),s_0)$ is called the first positive (respectively negative) focal point to C along γ_{s_0}.

A unit speed geodesic $\sigma\colon[0,\ell]\to M$ is called a *C-segment* iff $\sigma(0)\in C$ and $d(\sigma(t)$,C$)=t$ holds for all $t\in[0,\ell]$. Every C-segment is a subarc of some γ_s . Let $\rho(s):=\sup\{t>0\ ;\ d(\gamma_s(t),C)=t\}$ and $\nu(s):=\inf\{t<0\ ;\ d(\gamma_s(t),C)=-t\}$. $\rho(s)$ (respectively $\nu(s)$) is the cut point distance to C along $\gamma_s|[0,\infty)$ (respectively $\gamma_s|(-\infty,0]$). $z(\rho(s),s)$ is called a cut point to C along γ_s and $\gamma_s|[0,\rho(s)]$ is a maximal C-segment contained in $\gamma_s|[0,\infty)$. A cut point is a first focal point of a C-segment or the intersection of at least two distinct C-segments.

A cut point at C is called *normal* if it is the endpoint of exactly two distinct C-segments and is not a first focal point along either of them. A cut point to C which is not normal is called *anormal.* An anormal cut point $z(\rho(s),s)$ (or $z(\nu(s),s)$) is called *totally nondegenerate* iff $z(\rho(s),s)$ (or $z(\nu(s),s)$) is not a first focal point to C along any C-segment ending at $z(\rho(s),s)$ (or $z(\nu(s),s)$). An anormal cut point is called *degenerate* iff it is not totally nondegenerate. A number $t>0$ is called *anormal* iff there exists a value $s\in\rho^{-1}(t)$ (or $s\in\nu^{-1}(-t)$) such that $z(t,s)$ (or $z(-t,s)$) is anormal. It $t>0$ is not anormal, then t is called *normal.* Also $t>0$ is called *exceptional* iff it is either anormal or normal but there exists an s such that $\rho(s)=t$ (or $\nu(s)=-t$) and $\rho'=0$ (or $\nu'=0$) at s. A positive number t is by definition *non-exceptional* iff it is not exceptional.

§2. The case where C bounds a domain

Throughout this section let $M \setminus C$ have two components and M_1 the component containing $\{z(\rho(s), s)\ ;\ \rho(s) < \infty\}$. Note that the sets $\{z(\rho(s), s)\ ;\ \rho(s) < \infty\}$ and $\{z(\nu(s), s)\ ;\ \nu(s) > -\infty\}$ have no common point. We only restrict to consider M_1 , since the same discussion holds for $M \setminus M_1$.

We begin with the discussion of degenerate cut points that was not discussed in [2]. It seems to the authors that the lack of degenerate cut points in [2] would cause unclearness in the proof of Theorem 6.2 in [2]. The following Lemma 2.1 is useful to prove our results.

Lemma 2.1. *The set* $F = \{s \in [0, L_0]\ ;\ \rho(s) < P(s)$, *but* $z(\rho(s), s) \in M_1$ *is a degenerate cut point along some* C*-segment*$\}$ *is of Lebesgue measure zero.*

Proof. It suffices for the proof to show that for any $s \in F$ there exists a positive δ such that $F \cap (s - \delta, s + \delta)$ is of Lebesgue measure zero. Let $s_0 \in F$ and set $p = z(\rho(s_0), s_0)$. Choose a small positive ϵ such that B_ϵ is an open normal convex ϵ-ball around p. For each $s \in [0, L_0]$ with $z(\rho(s), s) = p$ let s' denote the common point of ∂B_ϵ and $\gamma_s([0, \rho(s_0)])$. The circle ∂B_ϵ is naturally oriented. Define the oriented open subarc from s'_1 to s'_2 of ∂B_ϵ by $(s'_1,\ s'_2)$. For each $s \in [0, L_0] \setminus \{s_0\}$ with $z(\rho(s), s) = p$ let $D(s'_0, s')$ (respectively $D(s', s'_0)$) be the disk domain bounded by three arcs $\gamma_{s_0}|[\rho(s_0) - \epsilon, \rho(s_0)]$, $\gamma_s|[\rho(s_0) - \epsilon, \rho(s_0)]$ and (s'_0, s') (respectively $D(s', s'_0)$). Since $\rho(s_0) < P(s_0)$, there exist $s_+, s_- \in [0, L_0]$ such that $z(\rho(s_+), s_+) = z(\rho(s_-), s_-) = p$ and such that $D_+ := D(s'_0, s'_+)$ and $D_- := D(s'_-, s'_0)$ are disjoint and they do not contain any C-segment passing through p. Let (π, NC, M) be the normal bundle over C with projection π, total space NC and base space M. Since p is not a focal point to C along γ_{s_0},there exist a neighborhood V of $\rho(s_0) \cdot \dot\gamma_{s_0}(0)$ in NC and a neighborhood U of p in M such that the restriction $\exp_V$ of the normal exponential map to V is a diffeomorphism of V onto U. Since p is a degenerate cut point, there is a C-segment ending at p along which p is the first focal point to C. Suppose $P(s_+) = \rho(s_+)$. Choose a positive number ϵ_1 such that U contains $z(\rho(s), s)$ and $z(P(s), s)$ for all $s \in [s_+ - \epsilon_1, s_+ + \epsilon_1]$. From construction of D_+ we can choose a positive number $\delta_1 < \epsilon_1$ such that if $z(\rho(s_1), s_1) = z(\rho(s), s)$ for $s_1 \in [0, L_0]$, $s \in (s_0, s_0 + \delta_1)$, then $s = s_1$ or $s_1 \in (s_+ - \epsilon_1, s_+)$. Let $v : (s_+ - \epsilon_1, s_+) \to (s_1, s_0 + \delta_1)$ be defined as

$$v(s) := z_0^{-1} \circ \pi \circ (\exp_V^{-1})(z(\rho(s), s))$$

If $s \in (s_+ - \epsilon, s_+)$ satisfies $P'(s) = 0$ and $P(s) = \rho(s)$, then $v'(s) = 0$, and hence s is a critical point of v. Let $K \subset (s_+ - \epsilon_1, s_+)$ be the set of all critical points of v. If $s \in (s_0, s_0 + \delta_1)$ is an element of F, then there exists an $s_1 \in [0, L_0]$ such that $z(\rho(s), s) = z(\rho(s_1), s_1)$, $P(s_1) = \rho(s_1)$. It follows from the choice of δ_1 and Proposition 2.1 in [5] that $P'(s_1) = 0$ and $s_1 \in (s_+ - \epsilon_1, s_+)$. Therefore we find an $s_1 \in K$ such that $z(\rho(s), s) = z(\rho(s_1), s_1) = z(P(s_1), s_1)$. This fact means that $(s_0, s_0 + \delta_1) \cap F$ is contained entirely in $v(K)$. The Sard Theorem implies that $v(K)$ is of Lebesgue measure zero. If $\rho(s_+) < P(s_+)$, then there exists a positive number δ such that $(s_0, s_0 + \delta) \cap F = \emptyset$. Summing up these discussion we observe that there exists a positive number δ_1 such that $(s_0, s_0 + \delta_1) \cap F$ is of measure zero.

An analogous discussion applies to D_- to prove that $(s_0 - \delta_1', s_0) \cap F$ is of measure zero for some positive number δ_1'. This completes the proof of Lemma 2.1.

Let $D := \{(t, s)\ ;\ 0 \le t < \rho(s), 0 \le s \le L_0\}$ and $\chi(t, s)$ the characteristic function of D such that $\chi(s, t) = 1$ or 0 according as $(t, s) \in D$ or not. For any $t \ge 0$ set

$$L(t) := \int_0^{L_0} \chi(t, s) f(t, s)\, ds$$

This $L(t)$ is the length of $S(t) = \{x \in M_1 | d(x, C) = t\}$ if t is a non-exceptional value. We define for $t \ge 0$ the set $Q(t)$ as follows.

$$Q(t) := \{s \in \rho^{-1}(t)\ ;\ z(s, t) \text{ is normal and } \rho'(s) = 0\}.$$

$Q(t)$ has the property that elements in it are pairwise disjoint, and hence it is of Lebesgue measure zero except for an at most countable set of $[0, \infty)$. We define for $t \ge 0$ the function

$$J(t) := \sum_{0 \le u \le t} \int_{Q(u)} f(u, s)\, ds.$$

Note that L and also J is discontinuous at $t = t_0$ iff the Lebesgue measure of $Q(t_0)$ is positive.

In order to prove Theorem 2.2 we shall need some basic tools from measure theory which is referred to [6]. Let h be a continuous function of bounded variation defined on a closed interval $[a, b]$. Then the function h defines a *Lebesgue-Stieltjes measure* Λ_h such that $\Lambda_h((x, y])$ for each subinterval $(x, y]$ of $[a, b]$ equals the total variation of h on $[x, y]$. It is known that any Borel set B in $[a, b]$ is Λ_h-measurable. For each Lebesgue measurable set $S \subset \mathbf{R}$, $|S|$ denotes its Lebesgue measure.

Theorem 2.2. *The function $H(t) = L(t) + J(t)$ is absolutely continuous on any compact subinterval of* $[0, \infty)$.

Proof. Let $[a, b]$ be a compact subinterval of $[0, \infty)$. In order to prove the theorem we shall show that for any positive ϵ there exists a positive $\eta = \eta(\epsilon, a, b)$ such that if $\delta_1, \delta_2, \dots, \delta_k$ are non-overlapping subintervals of $[a, b]$, then

$$\sum_{i=1}^{k} |\delta_i H| < (L_0 + 2)\epsilon \text{ whenever } \sum_{i=1}^{k} |\delta_i| < \eta \tag{2.1}$$

where $\delta_i H = H(\tau) - H(\sigma)$, $|\delta_i| = \tau - \sigma$ if $\delta_i = (\sigma, \tau]$. Let $\epsilon > 0$ be fixed. It follows from Proposition 3.1 in [5] that the set $T_b := \{s \in [0, L_0] \ ; \ \rho(s) \leq b, z(\rho(s), s) \text{ is a totally nondegenerate anormal point}\}$ is finite. Let $c = c(b)$ be a constant satisfying

$$|f(t, s)| \leq c, \ |f_t(t, s)| \leq c, \ (t, s) \in [0, b] \times [0, L_0]$$

By Lemma 2.1 the set F^ϵ defined by

$$F^\epsilon = \{s \in [0, L_0] \ ; \ \rho(s) \leq b, \ s \in F, \ f(\rho(s), s) \geq \epsilon/2\}$$

is compact and of Lebesgue measure zero. Here there exists a set V^ϵ with $|V^\epsilon| < \epsilon/c$ consisting of a finite number of open subintervals of $[0, L_0]$ such that $V^\epsilon \supset T_b \cap F^\epsilon$. Let Q^ϵ be the set

$$Q^\epsilon := \{s \in [0, L_0] \ ; \ \rho(s) \leq b, f(\rho(s), s) \leq \epsilon/2\}.$$

Since Q^ϵ is compact, Q^ϵ can be covered by a set S^ϵ consisting of a finite number of open subintervals of $[0, L_0]$ on which $f(\rho(s), s) < 3\epsilon/4$. Then the set $R^\epsilon = [0, L_0] - (S^\epsilon \cup V^\epsilon)$ consists of a finite number of closed subintervals $I_1, \dots, I_p$ of $[0, L_0]$. It follows from construction of R^ϵ and from Proposition 2.2 in [5] that ρ is smooth at each point $s \in R^\epsilon$ if $\rho(s) \leq b$. Hence the function $\rho_b :=\mathbf{Max}\ \{\rho, b\}$ is Lipschitz continuous on each closed intervals $I_j, j = 1, \dots, p$. In particular the restriction ρ_j of ρ_b to I_j is of bounded variation. If Λ_j denotes the Lebesgue-Stieltjes measure defined by ρ_j, then we observe from Corollary 3.1 in [2] that

$$\sum_{j=1}^{k} \Lambda_j(\rho_j^{-1}(\delta_i)) = \int_\sigma^\tau n(r)\, dr \tag{2.2}$$

where $n(r)$ is the Lebesgue summable function defined by the number of the elements of the set $\{s \in R^\epsilon \ ; \ \rho(s) = r\}$. Let $O(i)$ be an open set containing $R(i) = \cup_{\sigma<t\leq\tau}Q(t)$ such that $|O(i) - R(i)| < |\delta_i|$. Setting $S(i) = \rho^{-1}(\delta_i)$, we define

$$\begin{aligned} S_1 &= (S(i) - R(i)) \cap O(i) \\ S_2 &= (S(i) - R(i)) \cap [\{s \,;\, f(\rho(s), s) < \epsilon\} \cup V^\epsilon] \\ S_3 &= (S(i) - R(i)) - (S_1 \cup S_2). \end{aligned}$$

Making use of the inequality (6.20) in [2], we obtain

$$\begin{aligned} |\delta_i H| &\leq \sum_{j=1}^{3} \int_{S_j} f(\rho(s), s)\, ds + 2cL_0|\delta_i| \\ &\leq c|\delta_i| + \epsilon|S(i)| + c|V^\epsilon \cap S(i)| + c|S_3| + 2cL_0|\delta_i| \end{aligned} \tag{2.3}$$

Since $S_3 \subset R^\epsilon$ and $S_3 \cap O(i) = \emptyset$, ρ is smooth at each point of S_3 and $|\rho'| \geq c_1$ on S_3 holds for some positive constant $c_1 = c_1(\epsilon, a, b)$. From the property of the Lebesgue-Stieltjes measure Λ_j we obtain

$$\sum_{j=1}^{p} \Lambda_j(I_j \cap S_3) \geq c_1 \sum_{j=1}^{p} |I_j \cap S_3| = c_1|R^\epsilon \cap S_3| = c_1|S_3|.$$

From (2.2) and the above inequality, we get

$$|S_3| \leq c_1^{-1} \sum_{j=1}^{p} \Lambda_j(I_j \cap S_3) \leq c_1^{-1} \sum_{j=1}^{p} \Lambda_j(I_j \cap \rho^{-1}(\delta_i)) = c_1^{-1} \int_\sigma^\tau n(r)\, dr. \tag{2.4}$$

From inequalities (2.3) and (2.4) we have

$$\sum_{i=1}^{k} |\delta_i H| \leq c(1 + 2L_0) \sum_{i=1}^{k} |\delta_i| + (L_0 + 1)\epsilon + cc_1^{-1} \sum_{i=1}^{k} \int_{\delta_i} n(r)\, dr. \tag{2.5}$$

The inequality (2.5) implies that we can find a positive $\eta = \eta(\epsilon, a, b)$ satisfying (2.1). Note that the function $n(r)$ is Lebesgue summable.

§3. The case where C bounds no domain

We deal with the case where a closed curve C does not bound any domain of M. Our situation means that there exists a cut point $p \in M$

to C such that $p = z(\rho(s_1), s_1) = z(\nu(s_2), s_2)$ for some $s_1, s_2 \in [0, L_0]$. Three types of cut points to C appear. A cut point p to C is by definition of ρ-type (respectively ν-type) iff all C-segments ending at p are tangent to N (respectively to $-N$) at their starting points. A cut point p to C is of *mixed type* iff $p = z(\rho(s_1), s_1) = z(\nu(s_2), s_2)$ for some $s_1, s_2 \in [0, L_0]$. For a mixed type cut point to C the normality, anormality, degeneracy and all other properties are well defined by the same manner as before. These properties are defined for t-value where $S(t)$ contains a mixed type cut point having the corresponding properties. Let F_+, F_- be the sets

$$F_+ := \{s \in [0, L_0] \ ; \ \rho(s) < P(s), \quad \text{but } z(\rho(s), s) \text{ is a degenerate cut point}\}$$

$$F_- := \{s \in [0, L_0] \ ; \ \nu(s) > Q(s), \quad \text{but } z(\nu(s), s) \text{ is a degenerate cut point}\}.$$

Since the proof of Lemma 2.1 is done by a local discussion in a small convex ball around a cut point, we obtain the following lemma by a similar discussion.

Lemma 3.1. *The set $F := F_+ \cup F_-$ is of Lebesgue measure zero.*

Let $D_+ := \{(t, s) \ ; \ 0 \le t < \rho(s), s \in [0, L_0]\}$ and $D_- := \{(t, s) \ ; \ \nu(s) < t \le 0, s \in [0, L_0]\}$. We then define two functions L_+ and L_- on $[0, \infty)$ by

$$L_+(t) := \int_0^{L_0} \chi_+(t, s) f(t, s) \, ds$$
$$L_-(t) := \int_0^{L_0} \chi_-(t, s) f(-t, s) \, ds$$

where $\chi_+(t, s)$ and $\chi_-(t, s)$ are the characteristic functions of D_+ and D_- respectively. If $t > 0$ is non-exceptional, then the function

$$L(t) := L_+(t) + L_-(t)$$

is nothing but the length of $S(t) = \{x \in M \ ; \ d(x, C) = t\}$.

Note that if $t_0 > 0$ is a normal exceptional value, then $S(t_0)$ consists of a set of piecewise smooth curves. However the length of $S(t_0)$ is not necessarily equal to $L(t_0)$ but equal to

$$L(t_0) + \frac{1}{2}\{\int_{Q_+(t_0)} f(t_0, s)\, ds + \int_{Q_-(t_0)} f(-t_0, s)\, ds\}.$$

Here we set

$$Q_+(t) := \{s \in \rho^{-1}(t)\ ;\ z(t, s) \text{ is normal and } \rho'(s) = 0\},$$
$$Q_-(t) := \{s \in \nu^{-1}(-t)\ ;\ z(-t, s) \text{ is normal and } \nu'(s) = 0\}.$$

In order to define $J(t)$ in this case we need to set

$$J_+(t) := \sum_{0 \le u \le t} \int_{Q_+(t)} f(u, s)\, ds,$$
$$J_-(t) := \sum_{0 \le u \le t} \int_{Q_-(t)} f(-u, s)\, ds.$$

We then define $J(t)$ as follows.

$$J(t) := J_+(t) + J_-(t).$$

By a similar discussion as in the proof of Theorem 2.2 we obtain the following

Theorem 3.2. *The function $H(t) = L(t) + J(t)$ is absolutely continuous on any compact subinterval of $[0, \infty)$.*

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Diameter and Area Estimates for S^2 and P^2 with Nonnegatively Curved Metrics

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§0. Introduction

We consider the quantity

$$F(M) := \frac{\mathrm{Vol}(M)}{\mathrm{Diam}(M)^n}$$

for any closed Riemannian n-manifold M, which is a homothety invariant, where Vol and Diam denote the volume and the diameter respectively. If the Ricci curvature of M is nonnegative everywhere, Bishop's volume comparison theorem implies that $F(M) < \pi$. A.D. Alexandrov conjectured in [A, p.417] (see also [BZ, p.42]) that for any nonnegatively curved metric g on the 2-sphere S^2,

$$F(S^2, g) \leq \frac{\pi}{2},$$

and the equality holds only if g is homothetic to the metric of the double of the Euclidean unit disk $\bar{B}(1) := \{x \in \mathbf{R}^2 \mid d(x, o) \leq 1\}$, which is a singular metric of nonnegative Toponogov curvature. Note that Alexandrov deals a class of surfaces containing such a singular space, namely surfaces of bounded curvature in the sense of [AZ]. The volume and the diameter of any such singular surface can be approximated by those of Riemannian 2-manifolds, and thus it suffices to consider only regular metrics.

Alexandrov's conjecture has not been proved as of now. Concerning this, there are two known results as follows.

Theorem (Sakai, [S]). *For any nonnegatively curved Riemannian metric g on the 2-sphere S^2,*

$$F(S^2, g) < 0.985\pi.$$

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Theorem (Grove-Petersen, [GP1, Theorem B]). *For any integer $n \geq 2$ there exists an $\epsilon(n) > 0$ such that any compact Riemannian n-manifold M with nonnegative sectional curvature satisfies*

$$F(M) < V(n) - \epsilon(n),$$

where $V(n)$ is the volume of the n-dimensional Euclidean unit ball.

In the present paper, we try to extend the above estimates in the 2-dimensional case, i.e. the estimates for the 2-sphere S^2 and the real projective 2-space P^2 with nonnegatively curved metrics, and the 2-torus T^2 and the Klein bottle K^2 with flat metrics. We easily observe that, when $M = (T^2, g)$ or (K^2, g) for a flat metric g, then $F(M) \leq 2$, where the equality holds only if g is the canonical flat metric. Sakai's proof cannot be extended to the case of P^2. On the other hand, although Grove-Petersen's theorem is more general, their proof gives no calculable constant. Accordingly, we develop a proving method independent of the topology and have the following finer estimates.

Main Theorem. (1) *For any nonnegatively curved Riemannian metric g on the 2-sphere S^2, we have*

$$F(S^2, g) \leq \left(\frac{5}{2}\sqrt{10} - 7\right)\pi < 0.906\pi.$$

(2) *For any nonnegatively curved Riemannian metric g on the real projective 2-space P^2, we have*

$$F(P^2, g) \leq \frac{7\sqrt{7} - 10}{9}\pi < 0.947\pi.$$

Different from the case of S^2, the maximum of $F(P^2, g)$ for all nonnegatively curved metrics g on P^2 seems to be $F(P^2, g_c) = 8/\pi > 0.810\pi$, where g_c is the canonical metric on P^2, namely the metric of constant curvature 1.

§1. Preliminaries

Let M be a (not necessarily closed) complete Riemannian 2-manifold without boundary and p a fixed point in M. Consider the metric balls $B(p, r) := \{\, x \in M \mid d(p, x) < r \,\}$ and the metric spheres $S(p, r) := \{\, x \in M \mid d(p, x) = r \,\}$ centered at p for radii $r > 0$, where d denotes the distance function of M induced from the metric. Following Hartman [H] we define the notion of an exceptional radius as follows (actually, he called it an exceptional t-value).

Definition [H]. A radius $r > 0$ is said to be *exceptional* if and only if there exists a cut point q in $S(p,r)$ from p satisfying one of the following three conditions.

(1) q is a first conjugate point of p along some minimal geodesic segment joining p and q.

(2) There exist more than two distinct minimal geodesic segments joining p and q.

(3) There exist exactly two geodesic segments joining p and q, and moreover the angle between these segments at q is equal to π.

A radius is said to be *nonexceptional* if and only if it is not exceptional.

Note that if M is compact, $S(p,r)$ for any sufficiently large radius $r > 0$ is empty and hence any such r is nonexceptional. Hartman has proved in [H] that the set of all exceptional radii is a closed and measure zero subset of $\mathbf{R}$ and that $S(p,r)$ for each nonexceptional $r > 0$ consists of finitely many simple closed curves of class C^∞ except the cut points in $S(p,r)$ from p, the number of which is finite. For any nonexceptional $r > 0$ we denote by $q_{r,1}, \dots, q_{r,n(r)}$ $(0 \le n(r) < +\infty)$ the cut points in $S(p,r)$ from p. Then $S(p,r) - \{q_{r,1}, \dots, q_{r,n(r)}\}$ consists of $n(r)$ disjoint smooth open arcs $\alpha_{r,1}, \dots, \alpha_{r,n(r)}$. Define a continuous function $\rho\colon M \to \mathbf{R} \cup \{+\infty\}$ by

$$\rho(x) := \sup_{y \in M} d(x,y) \qquad \text{for } x \in M.$$

Clearly, $\rho(x) = +\infty$ if and only if M is open. Denote by $F_{r,i}$ the set of interior points of the minimal segments joining p and all points in $\alpha_{r,i}$ for any nonexceptional $0 < r < \rho(p)$ and any $1 \le i \le n(r)$. Then, $F_{r,i}$ is the open disk bounded by the triangle whose sides are $\alpha_{r,i}$ and two minimal segments joining p and the endpoints of $\alpha_{r,i}$ provided $n(r) \ge 1$. Denote by $\kappa_{r,i}(u)$ the integral of the geodesic curvature of the arc $S(p,u) \cap F_{r,i}$ with respect to $B(p,u)$ for any nonexceptional u and r with $0 < u < r < \rho(p)$ and for any $i = 1, \dots, n(r)$. Now we will prove

$$(*) \qquad \operatorname{Vol}(F_{r,i}) = \int_0^r \int_0^t \kappa_{r,i}(u)\, du\, dt.$$

Indeed, considering the geodesic polar coordinates (θ, t) on $F_{r,i}$ (θ is the angle at p and t is the distance from p), the volume of $F_{r,i}$ is expressed as

$$\operatorname{Vol}(F_{r,i}) = \int_0^r \int_0^{\Theta_{r,i}} \left\| \frac{\partial}{\partial \theta} \right\| d\theta\, dt,$$

where $\Theta_{r,i}$ is the inner angle of $F_{r,i}$ at p. Moreover, since the geodesic curvature of $S(p,t) \cap F_{r,i}$ with respect to $B(p,t)$ is equal to

$$\left\| \frac{\partial}{\partial \theta} \right\|^{-2} \left\langle \nabla_{\frac{\partial}{\partial \theta}} \frac{\partial}{\partial \theta}, -\frac{\partial}{\partial t} \right\rangle = \frac{\partial}{\partial t} \left\| \frac{\partial}{\partial \theta} \right\|,$$

we have

$$\int_0^{\Theta_{r,i}} \left\| \frac{\partial}{\partial \theta} \right\| d\theta = \int_0^t \kappa_{r,i}(u)\, du.$$

This proves (*).

In particular, if $\bar{B}(p,r) := B(p,r) \cup S(p,r)$ contains no cut points from p, we have

$$\mathrm{Vol}(B(p,r)) = \int_0^r \int_0^t \kappa(B(p,u))\, du\, dt,$$

where $\kappa(D)$ denotes the sum of the integral of the geodesic curvature of the boundary ∂D of D with respect to D and of the exterior angles at all vertices of D (we remark that $B(p,r)$ has no vertices in this case). Fiala [F] and Hartman [H] have extended this to the case where $B(p,r)$ may contain cut points from p as follows.

Lemma [F], [H]. *For any $0 < r \leq \rho(p)$ we have*

$$(**) \qquad \mathrm{Vol}(B(p,r)) = \int_0^r \int_0^t [\kappa(B(p,u)) - h_p(u)]\, du\, dt$$

where h_p is the nonnegative function defined by

$$h_p(u) := \sum_{i=1}^{n(u)} \left(2 \tan \frac{\varphi_{u,i}}{2} - \varphi_{u,i} \right)$$

and where $\varphi_{u,i}$ for each nonexceptional $0 < u < \rho(p)$ denotes the angle at $q_{u,i}$ between the two minimal segments joining p and $q_{u,i}$.

Note that Fiala and Hartman deal only with the case where $M = (\mathbf{R}^2, g)$ (Fiala [F] proved (**) for manifolds with real analytic metrics and Hartman [H] later extended this to the case of manifolds with C^2-metrics). However, we observe that their discussions are independent of the topology of M (see [ST]).

§2. Some partial estimates

Assume that M is a nonnegatively curved Riemannian 2-manifold diffeomorphic to either S^2 or P^2 the diameter of which is normalized as $\mathrm{Diam}(M) = 1$. Every curve in M is assumed to have arclength parameter and is often identified with its image. For a while, let p be any fixed point in M.

First we state a basic topological lemma.

Lemma 1. *Let $0 < r < \rho(p)$ be any nonexceptional radius. Then the Euler characteristic $\chi(B(p,r))$ of $B(p,r)$ satisfies*

$$\chi(B(p,r)) \leq 1,$$

and the equality holds if and only if $B(p,r)$ is a disk.

Note that $B(p,r)$ for a nonexceptional $r > 0$ is a disk if and only if it is contractible.

Proof. Since $B(p,r)$ is not closed, the 2-dimensional homology $H_2(B(p,r), \mathbf{Z})$ vanishes, and the first Betti number $b_1(B(p,r))$ is equal to zero if and only if $B(p,r)$ is contractible, namely a disk. Moreover we have

$$\chi(B(p,r)) = 1 - b_1(B(p,r)).$$

This completes the proof. Q.E.D.

Remark. It follows from Lemma 1 and the Gauss-Bonnet theorem that

$$\kappa(B(p,r)) = 2\pi\chi(B(p,r)) - c(B(p,r)) \leq 2\pi$$

for any nonexceptional $0 \leq r < \rho(p)$, where $c(D)$ denotes the total curvature of D, namely the integral $\int_D K\,dv$ of Gaussian curvature K over D with respect to the volume element dv of M.

Applying (**) to $B(q, \inf\rho)$ for a point q in M with $\rho(q) = \inf\rho$ and using the above remark, the following consequence is directly proved.

Proposition 2. *We have*

$$\mathrm{Vol}(M) \leq \pi \cdot (\inf\rho)^2.$$

Note that this is also obtained from Bishop's volume comparison theorem.

The following two lemmas are needed to prove Propositions 5 and 6.

Lemma 3. *Let $0 < R < \rho(p)$ and $a \geq 0$ be any given constants. If $\kappa(B(p,r)) \leq a$ for every nonexceptional r with $R < r < \rho(p)$, then*

$$\mathrm{Vol}(M) \leq \frac{a}{2} + (2\pi - a)\left(R - \frac{R^2}{2}\right).$$

Proof. By (**) and $\rho(p) \leq 1$ we have

$$\begin{aligned}\mathrm{Vol}(M) &\leq \int_0^{\rho(p)} \int_0^t \kappa(B(p,u))\, du\, dt \\ &\leq \int_0^R \int_0^t 2\pi\, du\, dt + \int_R^{\rho(p)} \left(\int_0^R 2\pi\, du + \int_R^t a\, du\right) dt \\ &\leq \frac{a}{2} + (2\pi - a)\left(R - \frac{R^2}{2}\right).\end{aligned}$$

Q.E.D.

Lemma 4. *If $\bar{B}(p,r)$ for a number $0 < r < \rho(p)$ is not contractible, then there exists a geodesic loop with base point p which is entirely contained in $\bar{B}(p,r)$.*

Proof. Take a continuous loop γ: $[0,l] \to \bar{B}(p,r)$ with base point p such that

$$L(\gamma) = \inf\{\, L(c) \mid c \text{ is a loop with base point } p \text{ which is not homotopic to the point } p \text{ in } \bar{B}(p,r)\,\}.$$

If γ does not intersect $S(p,r)$, it is a geodesic loop. Thus we consider the case where γ intersect $S(p,r)$. Then $l = L(\gamma) \geq 2r$. Let us first prove the following

Claim. *γ forms a geodesic biangle consisting of two geodesics with length r.*

It suffices to show that $2r = l$. Now suppose that $2r < l$. For a minimal segment σ of M joining p and a point $\gamma(t)$ with $r < t < l - r$, one of the two closed curves $\gamma([0,t]) \cup \sigma$ and $\gamma([t,l]) \cup \sigma$ is not homotopic to the point p in $\bar{B}(p,r)$. Denoting this by γ_1 we have

$$L(\gamma_1) < L(\gamma)$$

because of $L(\sigma) \leq r$. This contradicts the definition of γ and completes the proof of the claim.

We will prove that γ does not break at $\gamma(r)$. Suppose the contrary. For each $0 \le t \le r$ we take a minimal segment σ_t joining $\gamma(r-t)$ and $\gamma(r+t)$ and set $\gamma_t := \gamma([0, r-t]) \cup \sigma_t \cup \gamma([r+t, 2r])$. Since γ breaks we have

$$L(\gamma_t) < L(\gamma) = 2r \qquad \text{and hence} \qquad \gamma_t \subset B(p, r)$$

for any $0 < t \le r$. Moreover, there is a small $\epsilon > 0$ such that $[0, \epsilon] \times [0, 1] \ni (t, s) \longmapsto \gamma_t(sL(\gamma_t))$ is a smooth variation entirely contained in $\bar{B}(p, r)$, which is a homotopy with $\gamma_0 = \gamma$ in particular. This contradicts the definition of γ. Q.E.D.

Proposition 5. *Let $0 < R < \rho(p)$. If there exists a number $0 < r_0 \le R$ such that $\bar{B}(p, r_0)$ is not contractible, then*

$$\mathrm{Vol}(M) \le \frac{\pi}{2}(1 + 2R - R^2).$$

Proof. Take any fixed nonexceptional r with $R < r < \rho(p)$. If $B(p, r)$ is not a disk, Lemma 1 implies $\chi(B(p, r)) \le 0$ and hence

$$\kappa(B(p, r)) \le 0$$

by the Gauss-Bonnet theorem. In the case where $B(p, r)$ is a disk, Lemma 4 implies that $\bar{B}(p, r_0)$ contains a geodesic loop, which bounds a disk in $B(p, r)$ whose total curvature greater than π, because of the Gauss-Bonnet theorem. Therefore we have $c(B(p, r)) > \pi$ and hence by Lemma 1

$$\kappa(B(p, r)) = 2\pi\chi(B(p, r)) - c(B(p, r)) < \pi.$$

As a result, in either case we have $\kappa(B(p, r)) < \pi$ for any nonexceptional r with $R < r < \rho(p)$. Applying Lemma 3 under $a := \pi$, the proof is completed. Q.E.D.

Proposition 6. *Let $0 < R < \rho(p)$. Then we have*

$$\mathrm{Vol}(M) \le \pi - \frac{1}{2}(1 - R)^2 \min\{\, c(B(p, R)), 2\pi \,\}.$$

Proof. It follows from the Gauss-Bonnet theorem and Lemma 1 that $\kappa(B(p, r)) \le 2\pi - c(B(p, r))$ for all nonexceptional r with $R < r < \rho(p)$. Since the function $t \mapsto c(B(p, t))$ is monotone nondecreasing, we have

$$\kappa(B(p, r)) \le 2\pi - c(B(p, R))$$

for all nonexceptional r with $R < r < \rho(p)$. Setting

$$a := \max\{\, 2\pi - c(B(p,R)), 0\,\},$$

Lemma 3 completes the proof. Q.E.D.

§3. Proof of Main Theorem

Lemma 7. *Let $0 < R < \rho(p)$. If $B(p,r)$ for every $0 < r \leq R$ is contractible, then*

$$\mathrm{Vol}(B(p,R)) \geq \frac{1}{2}R^2(2\pi - c(B(p,R))).$$

Proof. In the case where R is exceptional, the above inequality for every nonexceptional R' with $0 < R' < R$ yields the conclusion since the set of nonexceptional radii is dense in $[0,+\infty)$. Thus we may consider only the case where R is nonexceptional. Under the notations as in section 1, it follows from the Gauss-Bonnet theorem that $\kappa_{R,i}(t) = \Theta_{R,i} - c(F_{R,i} \cap B(p,t)) \geq \Theta_{R,i} - c(F_{R,i})$ for all nonexceptional $0 < t \leq R$. This and $(*)$ imply

$$\mathrm{Vol}(F_{R,i}) \geq \int_0^R \int_0^r (\Theta_{R,i} - c(F_{R,i}))\, dt\, dr$$

and hence, by setting $F_R := \bigcup_{i=1}^{n(R)} F_{R,i}$ and $\Theta_R := \sum_{i=1}^{n(R)} \Theta_{R,i}$,

$$\mathrm{Vol}(B(p,R)) \geq \mathrm{Vol}(F_R) \geq \int_0^R \int_0^r (\Theta_R - c(F_R))\, dt\, dr.$$

On the other hand, since $B(p,R) - F_R$ is the union of $n(R)$ disks bounded by geodesic biangles, the Gauss-Bonnet theorem shows that

$$c(B(p,R) - F_R) > 2\pi - \Theta_R.$$

Thus we have

$$\begin{aligned}\mathrm{Vol}(B(p,R)) &\geq \int_0^R \int_0^r (2\pi - c(B(p,R)))\, dt\, dr \\ &= \frac{1}{2}R^2(2\pi - c(B(p,R))).\end{aligned}$$

Q.E.D.

Lemma 8. *For a given constant $R > 0$ we have*

$$\mathrm{Vol}(M) \geq \frac{c(M) \inf_{p \in M} \mathrm{Vol}(B(p,R))}{\sup_{p \in M} c(B(p,R))}.$$

Recall that

$$c(M) = \begin{cases} 4\pi & \text{if } M \cong S^2 \\ 2\pi & \text{if } M \cong P^2. \end{cases}$$

Proof. It suffices to show that

$$\int_M c(B(p,R))\, dp = \int_M K(p) \,\mathrm{Vol}(B(p,R))\, dp,$$

where dp is the volume element with respect to a variable p of M. Define the function $\varphi\colon M \times M \to \mathbf{R}$ by

$$\varphi(p,q) := \begin{cases} 1 & \text{if } d(p,q) < R \\ 0 & \text{if } d(p,q) \geq R \end{cases} \qquad \text{for all } p, q \in M.$$

By Fubini's theorem we have

$$\begin{aligned} \int_M c(B(p,R))\, dp &= \int_M \int_M \varphi(p,q)\, K(q)\, dq\, dp \\ &= \int_M K(q) \int_M \varphi(p,q)\, dp\, dq \\ &= \int_M K(q) \,\mathrm{Vol}(B(q,R))\, dq. \end{aligned}$$

Q.E.D.

Proof of Main Theorem. Let us define a constant $0 < R < 1$ by

$$R := \frac{4 - \sqrt{4 + 3c(M)/2\pi}}{4 - c(M)/2\pi} = \begin{cases} 2 - \sqrt{10}/2 & \text{if } M \cong S^2 \\ (4 - \sqrt{7})/3 & \text{if } M \cong P^2. \end{cases}$$

In the case where $\inf \rho \leq R$, Proposition 2 implies

$$\mathrm{Vol}(M) \leq \pi R^2 < \begin{cases} 0.176\pi & \text{if } M \cong S^2 \\ 0.204\pi & \text{if } M \cong P^2, \end{cases}$$

which concludes Main Theorem in particular. Thus assume that $\inf \rho > R$. If there is a point p in M such that $c(B(p,R)) \geq 2\pi$, then by

Proposition 6 we have

$$\mathrm{Vol}(M) \leq \pi \cdot (2R - R^2) < \begin{cases} 0.663\pi & \text{if } M \cong S^2 \\ 0.700\pi & \text{if } M \cong P^2. \end{cases}$$

If there are a point p in M and a radius $0 < r_0 \leq R$ such that $\bar{B}(p, r_0)$ is not contractible, then Proposition 5 implies

$$\mathrm{Vol}(M) \leq \frac{\pi}{2}(1 + 2R - R^2) < \begin{cases} 0.832\pi & \text{if } M \cong S^2 \\ 0.850\pi & \text{if } M \cong P^2. \end{cases}$$

Therefore, it suffices to consider the case where $c(B(p, R)) < 2\pi$ and $\bar{B}(p, r)$ is contractible for all points p in M and all $0 < r \leq R$. Now, setting

$$c := \sup_{p \in M} c(B(p, R)),$$

we have $0 < c \leq 2\pi$. Lemmas 7 and 8 show

$$\mathrm{Vol}(M) \geq \frac{R^2 c(M)(2\pi - c)}{2c}.$$

On the other hand, we have by Proposition 6

$$(\#) \qquad \mathrm{Vol}(M) \leq \pi - \frac{1}{2}(1 - R)^2 c.$$

Combining these two formulas, we have the quadratic inequality:

$$(1 - R)^2 c^2 - (2\pi + R^2 c(M))c + 2\pi R^2 c(M) \leq 0,$$

which gives the estimate of c:

$$c \geq \frac{2\pi + R^2 c(M) - \sqrt{b}}{2(1 - R)^2},$$

where b is the constant defined by

$$b := (2\pi + R^2 c(M))^2 - 8\pi R^2 (1 - R^2) c(M).$$

By this and $(\#)$ we obtain

$$\mathrm{Vol}(M) \leq \frac{\pi}{2} - \frac{1}{4}(R^2 c(M) - \sqrt{b}).$$

This completes the proof of Main Theorem. Q.E.D.

Note that R is determined as the last estimate is finest.

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On the Poles of Riemannian Manifolds of Nonnegative Curvature

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Dedicated to Professor Masahisa Adachi on his 60th birthday

Abstract.

The diameter of the set of poles on Riemannian manifolds of nonnegative sectional curvature is estimated by a constant defined by Maeda. We study the constant for elliptic paraboloids and show that our estimate is sharp.

§1. Introduction

Let M be a noncompact complete Riemannian manifold. In [2] M. Maeda defined a constant $d_o(M)$ which describes how M expands at infinity. For a point p of M let $S_t(p) = \{q \in M \ ; \ d(p,q) = t\}$ denote the metric sphere centered at p with radius $t \geq 0$ and $D_t(p)$ the diameter $\operatorname{diam} S_t(p)$ of $S_t(p)$. He defined

$$d_o(M) = \limsup_{t \to \infty} \frac{D_t(p)^2}{t}$$

and showed that d_0 does not depend on the choice of p and the distance between two poles does not exceed $d_o(M)$ if M is of nonnegative sectional curvature, where a point q of M is said to be a pole if the exponential mapping $\exp_q : T_qM \to M$ is a diffeomorphism. In this paper we shall improve his estimate as follows:

Theorem 1.1. *Let M be a noncompact and complete Riemannian manifold of nonnegative sectional curvature. Then the distance between two poles does not exceed $d_o(M)/8$.*

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The distance of two poles of an elliptic paraboloid defined by

$$x_0^2/a_0 + x_1^2/a_1 = 2x_2$$

with $0 < a_0 < a_1$ goes towards $d_0/8$ as $a_0 \to 0$. Hence our estimate is sharp.

We note that elliptic paraboloids are Liouville surfaces. So, by deforming elliptic paraboloids through Liouville surfaces, we can construct various surfaces of nonnegative curvature with two poles and $d_o < \infty$.

On the other hand, M. Tanaka [4] studied the poles on surfaces of revolution and showed that the center of revolution is the only pole if and only if d_o is finite. Hence we conjecture

Conjecture 1.2. *If the constant $d_o(M)$ is finite for a Riemannian manifold M of nonnegative sectional curvature, then the number of poles of M is finite or at most two.*

In §2 we shall give a proof of Theorem 1.1. In §3 we shall study the behavior of geodesics on elliptic paraboloids using the elliptic coordinates to show that two umbilic points are the poles. In §4 we shall give the exact value of d_0 for an elliptic paraboloid and show that our estimate is sharp.

§2. The proof of Theorem 1.1

In this section let M denote a Riemannian manifold of nonnegative sectional curvature and all geodesics of M are assumed to be parametrized by arc length.

Lemma 2.1. *Let $\gamma : [0, \infty) \to M$ be a ray emanating from p, i.e., $\gamma|[0,t]$ is minimizing for any $t > 0$. Let $\alpha : [0, s] \to M$ be a geodesic from $\gamma(t_o)$ to q and θ the angle $-\dot\gamma(t_o)$ and $\dot\alpha(0)$ make. Then*

$$t_o - s\cos\theta \leq d(p,q).$$

Proof. First we assume that α is a minimizing geodesic. Toponogov's comparison theorem for a triangle $\triangle\gamma(t_0)\gamma(t)q$ with $t > t_0$ implies

$$\begin{aligned} d(q,\gamma(t))^2 &\leq d(q,\gamma(t_0))^2 + d(\gamma(t),\gamma(t_0))^2 \\ &\qquad - 2d(q,\gamma(t_0))d(\gamma(t),\gamma(t_0))\cos(\pi-\theta) \\ &= s^2 + (t-t_0)^2 + 2s(t-t_0)\cos\theta \\ &= ((t-t_0) + s\cos\theta)^2 + s^2(1-\cos^2\theta). \end{aligned}$$

Hence we get

$$d(q,\gamma(t))-((t-t_0)+s\cos\theta)\leq\frac{s^2(1-\cos^2\theta)}{d(q,\gamma(t))+((t-t_0)+s\cos\theta)}=O(1/t).$$

If $d(p,q) < t_0 - s\cos\theta$, then there is a positive constant ϵ such that $d(p,q) < t_0 - s\cos\theta - \epsilon$. Hence we get

$$\begin{aligned} t = d(p,\gamma(t)) &\leq d(p,q) + d(q,\gamma(t)) \\ &< (t_0 - s\cos\theta - \epsilon) + (t - t_0 + s\cos\theta) + O(1/t) \\ &= t - \epsilon + O(1/t) < t \end{aligned}$$

for large t, which contradicts the assumption that γ is a ray.

If α is not minimizing, then we divide α into minimizing arcs $\alpha|[s_{i-1}, s_i]$ $(i = 1\ldots k)$ with $0 = s_0 < s_1 < \ldots < s_k = s$. We consider a polygon $\bar\gamma(t)\bar\alpha(s_0)\ldots\bar\alpha(s_k)$ in the two-dimensional Euclidean space which corresponds to $\gamma(t)\alpha(s_0)\ldots\alpha(s_k)$ with

$$\begin{aligned} d(\bar\gamma(t),\bar\alpha(s_i)) &= d(\gamma(t),\alpha(s_i)) \qquad (i=0\ldots k) \\ d(\bar\alpha(s_{i-1}),\bar\alpha(s_i)) &= d(\alpha(s_{i-1}),\alpha(s_i))\ (i=1\ldots k). \end{aligned}$$

Then Toponogov's comparison theorem implies that the polygon is convex. Therefore we easily get

$$\begin{aligned} &\lim_{t\to\infty}\{t - t_0 + s\cos\theta - d(\alpha(s),\gamma(t))\} \\ &\quad = \lim_{t\to\infty}\{t - t_0 + s\cos\theta - d(\bar\alpha(s),\bar\gamma(t))\} \geq 0. \end{aligned}$$

Hence the assertion is clear.

Let p_1 and p_2 be poles of M and $a = d(p_1,p_2)$. Let $\gamma_1,\gamma_2 : [0,\infty) \to M$ be two rays with $\gamma_1(0) = p_1$, $\gamma_1(a) = p_2$, $\gamma_2(0) = p_2$ and $\gamma_2(a) = p_1$. Let $q_1 = \gamma_1(t)$, $q_2 = \gamma_2(t+a)$ and $v = d(q_1,q_2)$. Then $v \leq D_{p_1}(t)$ because $d(p_1,q_1) = d(p_1,q_2) = t$. Let q be the middle point of a minimizing geodesic between q_1 and q_2. Let $\theta_i = \angle p_1q_iq$ $(i = 1,2)$. Then Toponogov's comparison theorem for a triangle $\triangle p_1q_2q$ implies

(2.1) $$d(p_1,q)^2 \leq t^2 + v^2/4 - tv\cos\theta_2.$$

And from Lemma 2.1, we get

(2.2) $$t - (v/2)\cos\theta_1 \leq d(p_1,q)$$

$$(t+a) - v\cos\theta_2 \leq d(p_2,q_1) = t - a.$$

Hence we have

$$2a \leq v \cos \theta_2.$$

Since $\limsup v^2/t \leq d_o < \infty$, we may assume the left side of (2.2) is positive. Therefore (2.1) combined with (2.2) yields

$$(t - (v/2)\cos \theta_1)^2 \leq t^2 + v^2/4 - tv \cos \theta_2,$$

which is reduced to

$$v^2 \cos^2 \theta_1 - 4tv \cos \theta_1 + 4tv \cos \theta_2 - v^2 \leq 0. \tag{2.3}$$

Toponogov's comparison theorem for a triangle $\Delta p_2 q_1 q$ gives

$$d(p_2, q)^2 \leq (t-a)^2 + v^2/4 - (t-a)v \cos \theta_1. \tag{2.4}$$

And from Lemma 2.1 we get

$$(t+a) - (v/2)\cos \theta_2 \leq d(p_2, q). \tag{2.5}$$

Hence (2.4) combined with (2.5) yields

$$((t+a) - (v/2)\cos \theta_2)^2 \leq (t-a)^2 + v^2/4 - (t-a)v \cos \theta_1,$$

which is reduced to

$$v^2 \cos^2 \theta_2 - 4(t+a)v \cos \theta_2 + 4(t-a)v \cos \theta_1 + 16at - v^2 \leq 0.$$

Deleting $v^2 \cos^2 \theta_2$, we get

$$\frac{4(t-a)v \cos \theta_1 + 16at - v^2}{4(t+a)} \leq v \cos \theta_2.$$

We substitute this inequality to (2.3). Then (2.3) becomes

$$v^2 \cos^2 \theta_1 - \frac{8atv}{t+a} \cos \theta_1 + \frac{16at^2 - 2tv^2 - av^2}{t+a} \leq 0.$$

Deleting $v^2 \cos^2 \theta_1$, we get

$$2t - \frac{(2t+a)v^2}{8at} \leq v \cos \theta_1. \tag{2.6}$$

Applying Toponogov's comparison theorem to a triangle $\Delta p_2 q_1 q_2$, we get

$$(t+a)^2 \leq (t-a)^2 + v^2 - 2(t-a)v \cos \theta_1.$$
$$4at \leq v^2 - 2(t-a)v \cos \theta_1.$$

Substituting (2.6) to this inequality, we get

$$4at \le v^2 - 2(t-a)\left(2t - \frac{(2t+a)v^2}{8at}\right)$$
$$4at + 4t(t-a) \le v^2\left(1 + \frac{(t-a)(2t+a)}{4at}\right).$$

Dividing both sides by t^2 and letting $t \to \infty$, we get

$$4 \le \frac{d_o}{2a},$$

i.e.,

$$a \le \frac{d_o}{8}.$$

§3. Geodesics on elliptic paraboloids

H. von Mangoldt studied the behavior of geodesics of hyperboloids in [3] and stated that his method could be applied to show that two umbilic points of an elliptic paraboloid are the only poles. In this section we study the behavior of geodesics of elliptic paraboloids and prove his assertion. Our argument mainly relies on [1, §3.5]. Let us consider an elliptic paraboloid

$$M = \{(x_0, x_1, x_2) \in \mathbf{R}^3 \ ; \ x_0^2/a_0 + x_1^2/a_1 = 2x_2\}$$

with $0 < a_0 < a_1$.

We introduce the elliptic coordinates $(u_1, u_2) \in]a_0, a_1[\times]a_1, \infty[$:

$$x_0^2 = \frac{a_0(a_0-u_1)(a_0-u_2)}{a_1-a_0}$$
$$x_1^2 = \frac{a_1(a_1-u_1)(a_1-u_2)}{a_0-a_1}$$
$$x_2 = \frac{u_1+u_2-a_0-a_1}{2}.$$

Note that $u_1 = u_2 = a_1$ corresponds to the umbilic points

$$(\pm\sqrt{a_0}\sqrt{a_1-a_0}, 0, \frac{a_1-a_0}{2})$$

and the distance between two umbilic points of M is equal to

$$\sqrt{a_1-a_0}\sqrt{a_1} + a_0 \log\left|\sqrt{\frac{a_1-a_0}{a_0}} + \sqrt{\frac{a_1}{a_0}}\right|.$$

The first fundamental form is expressed in the elliptic coordinates as follows:

$$ds^2 = (-u_1 + u_2)(U_1 du_1^2 + U_2 du_2^2),$$

where

$$U_i = \frac{(-1)^i u_i}{f(u_i)} \ ; \ f(u_i) = 4(a_0 - u_i)(a_1 - u_i).$$

For a real number γ, $a_0 < \gamma < a_1$ or $a_1 < \gamma$, we consider a coordinate change

$$\begin{aligned} du_1' &= \sqrt{-u_1 + \gamma}\sqrt{U_1}\, du_1 \pm \sqrt{u_2 - \gamma}\sqrt{U_2}\, du_2 \\ du_2' &= \frac{\sqrt{U_1}}{\sqrt{-u_1 + \gamma}}\, du_1 \mp \frac{\sqrt{U_2}}{\sqrt{u_2 - \gamma}}\, du_2. \end{aligned} \tag{3.1}$$

Then

$$ds^2 = du_1'^2 + (-u_1 + \gamma)(u_2 - \gamma) du_2'^2.$$

From this expression of the first fundamental form, we see that u_1'-parameter curves are geodesics. Hence we get

Theorem 3.1 ([1, Theorem 3.5.5]). *In the elliptic coordinates geodesics of M are characterized by*

$$\frac{\sqrt{U_1}}{\sqrt{-u_1 + \gamma}}\dot{u}_1 \mp \frac{\sqrt{U_2}}{\sqrt{u_2 - \gamma}}\dot{u}_2 = 0,$$

together with the condition $E(u, \dot{u}) = \mathrm{const}$, where $E = ds^2/2$ is the energy function. Here γ is a constant with value in $]a_0, a_1[$ or $]a_1, \infty[$.

The constant γ is called the parameter of the geodesic.

Corollary 3.2 (cf. [1, Corollary 3.5.6]). *Denote by $(T_1M)'$ the open and dense subset of the unit tangent bundle T_1M formed by those unit tangent vectors which are tangent to a geodesic with parameter γ, $\gamma \in]a_0, a_1[$ or $\gamma \in]a_1, \infty[$. Define $F : (T_1M)' \to \mathbf{R}$ in elliptic tangent coordinates $(u, \dot{u})$ by*

$$F(u, \dot{u}) = (-u_1 + u_2)(u_2 U_1 \dot{u}_1^2 + u_1 U_2 \dot{u}_2^2).$$

Then F is a first integral of the geodesic flow on T_1M. And if $u(t) = (u_1(t), u_2(t))$ is a geodesic parametrized by arc length with parameter γ, then $F(u(t), \dot{u}(t)) = \gamma$.

If we denote by $\mu(X)$ the angle between $X \in (T_1M)'$ and the u_1-parameter line through $\tau_M X$, then we may also write

$$F(X) = u_1(\tau_M X)\sin^2\mu(X) + u_2(\tau_M X)\cos^2\mu(X),$$

where $\tau_M : T_1M \to M$ denotes the canonical projection.

We now go to the co-geodesic flow ϕ_t on the cotangent bundle T^*M. The cotangent coordinates (u, v) are related to the tangent coordinates $(u, \dot{u})$ by

$$\dot{u}_i = g^{ij}(u)v_j = \frac{v_i}{(-u_1 + u_2)U_i}, \; i = 1, 2.$$

The functions E, F correspond to the following functions on $(T^*M)'$:

$$E^*(u, v) = \frac{1}{2(-u_1 + u_2)} \left(\frac{1}{U_1} v_1^2 + \frac{1}{U_2} v_2^2 \right)$$
$$F^*(u, v) = \frac{1}{(-u_1 + u_2)} \left(\frac{u_2}{U_1} v_1^2 + \frac{u_1}{U_2} v_2^2 \right).$$

Theorem 3.3 (cf. [1, Theorem 3.5.7]). *For $\gamma \in]a_0, a_1[$ or $\gamma \in]a_1, \infty[$ the ϕ_t-invariant set $\{F^* = \gamma\}$ in the total unit cotangent space T_1^*M consists of two embedded 2-dimensional cylinders which we denote by $T_\gamma^\pm$.*

We distinguish the cases $\gamma \in]a_1, \infty[$ or $\gamma \in]a_0, a_1[$ as type I *and* II*, respectively.*

The flow lines on the cylinder $T_\gamma^\pm$ of type I *correspond, under the projection $\tau_M^* : T_1^*M \to M$, to geodesics which monotonously wind x_2-axis, while descending to tangent to a u_1-parameter line $\{u_2 = \gamma\}$ then ascending to $x_2 = \infty$. The cylinder of Type* I *corresponds, under τ_M^*, to $\{(u_1, u_2) \; ; \; a_0 \le u_1 \le a_1, \gamma \le u_2\}$.*

The flow lines on the cylinders of type II *correspond, under τ_M^*, to geodesics which oscillate between the two u_2-parameter lines $\{u_1 = \gamma\}$. The cylinder of type* II *corresponds, under τ_M^*, to $\{(u_1, u_2) \; ; \; a_0 \le u_1 \le \gamma, a_1 \le u_2\}$.*

As γ goes towards a_0, the cylinders $T_\gamma^\pm$ become degenerate, i.e., we get two embedded curves given by the unit tangent vectors to the curve $M \cap \{x_1 = 0\}$.

Proof. Let

$$u_1 = a_0 \cos^2 \psi_1 + a_1 \sin^2 \psi_1$$
$$u_2 = a_1 + \psi_2^2 \tag{3.2}$$

with $(\psi_1, \psi_2) \in \mathbf{R}/2\pi \times \mathbf{R}$. Then equations $2E^* = 1$ and $F^* = \gamma$ yield

$$v_1^2 = U_1(\gamma - u_1) \; ; \; v_2^2 = U_2(u_2 - \gamma).$$

For the cotangent coordinates (Ψ_1, Ψ_2) corresponding to (ψ_1, ψ_2) we get

$$\Psi_1 = v_1 \frac{\partial u_1}{\partial \psi_1} = 2(a_1 - a_0) v_1 \sin\psi_1 \cos\psi_1$$
$$\Psi_2 = v_2 \frac{\partial u_2}{\partial \psi_2} = 2 v_2 \psi_2.$$

Hence

$$\text{(3.3)} \qquad \begin{aligned} \Psi_1^2 &= (\gamma - u_1)u_1 \\ \Psi_2^2 &= (u_2 - \gamma)u_2/(u_2 - a_0). \end{aligned}$$

With $u_i = u_i(\psi_i)$ as in (3.2), we get $\Psi_i = \Psi_i(\psi_i)$.

Consider now type I, i.e., $a_1 < \gamma$. Then $\Psi_2 = \Psi_2(\psi_2)$ describes a simple non-closed curve in the (ψ_2, Ψ_2)-plane. $\Psi_1 = \Psi_1(\psi_1)$ yields two non-closed curves in the (ψ_1, Ψ_1)-plane, one with $\Psi_1 > 0$, the other with $\Psi_1 < 0$, since $\Psi_1(\psi_1)$ is always $\neq 0$. However, in T_1^*M, $\Psi_1 = \Psi_1(\psi_1)$, $\psi_1 \in S^1$, describes two closed curves, since the (u, v) are periodic in ψ_1. Thus, $T_1^*M \cap \{F^* = \gamma\}$ consists of two embedded cylinders.

The discussion of type II, i.e., $a_0 < \gamma < a_1$, is similar.

Let $P(t, \gamma) = (-t)(\gamma - t)(a_0 - t)(a_1 - t)$. For $\gamma \in]a_0, a_1[$ define $\omega_2 = (\omega_{12}, \omega_{22})$ with

$$\omega_{12} = 4 \int_{a_0}^{\gamma} \frac{-t(\gamma - t)}{\sqrt{P(t,\gamma)}}\, dt \ ; \quad \omega_{22} = 4 \int_{a_0}^{\gamma} \frac{-t}{\sqrt{P(t,\gamma)}}\, dt.$$

For $\gamma \in]a_1, \infty[$ define $\omega_2 = (\omega_{12}, \omega_{22})$ with

$$\omega_{12} = 4 \int_{a_0}^{a_1} \frac{-t(\gamma - t)}{\sqrt{P(t,\gamma)}}\, dt \ ; \quad \omega_{22} = 4 \int_{a_0}^{a_1} \frac{-t}{\sqrt{P(t,\gamma)}}\, dt.$$

In each case, put $-\omega_{21} : \omega_{22} = \omega(\gamma)$.

Theorem 3.4 (cf. [1, Theorem 3.5.10]). *The geodesic flow on each of the invariant cylinders $T_\gamma^\pm$ in appropriate coordinates, is equivalent to the linear flow of slope $\omega(\gamma)$ on the flat cylinder.*

Proof. Let $\gamma \in]a_0, a_1[$. The differentials du_1', du_2' in (3.1) determine functions $u_1'(u_1, u_2), u_2'(u_1, u_2)$ on $T_\gamma^\pm$, i.e.,

$$u_1' = \int_{a_0}^{u_1} \sqrt{-u_1 + \gamma}\sqrt{U_1}\, du_1 \pm \textstyle\int_{a_1}^{u_2} \sqrt{u_2 - \gamma}\sqrt{U_2}\, du_2 \ ;$$
$$u_2' = \int_{a_0}^{u_1} \frac{\sqrt{U_1}}{\sqrt{-u_1 + \gamma}}\, du_1 \mp \int_{a_1}^{u_2} \frac{\sqrt{U_2}}{\sqrt{u_2 - \gamma}}\, du_2.$$

Denote by T_ω the flat cylinder $\mathbf{R}^2/\mathbf{Z}\omega_2$. Then the functions $u' = u'(u)$ give a transformation from $T_\gamma^{\pm}$ to T_ω. The geodesic lines go into the u_1'-parameter lines.

The case $\gamma \in]a_1, \infty[$ is treated in exactly the same manner.

Theorem 3.5 (cf. [1, Theorem 3.5.16]). *The flow-invariant set* $\{F^* = a_1\} \cap T_1^*M$ *is formed by those flow lines which, when projected into* M, *yield the geodesics which pass through the umbilic points. And the umbilic points are the only poles of* M.

Proof. Solve equations

$$E^* = 1/2 \; ; \; F^* = a_1 \tag{3.4}$$

at a point $p \in M$ which does not lie on the x_0x_2-plane. Since (3.4) is equivalent to

$$\Psi_1^2 = (a_1 - u_1)u_1 \; ; \; \Psi_2^2 = (u_2 - a_1)u_2/(u_2 - a_0), \tag{3.5}$$

we see that there are four solutions of the equation in T_1^*M. On the other hand there are at least four geodesics between p and the umbilic points even if we take the directions of geodesics in consideration. If $\gamma \neq a_1$, the equations (3.3) and (3.5) have no common solutions. Hence each solution of (3.5) corresponds to a geodesic between p and an umbilic point and there is only one geodesic between p and each umbilic point. Therefore umbilic points are poles. From Theorem 3.3 we easily see any geodesic half-lines with $F^* \neq a_1$ are not rays.

§4. The constant d_o for an elliptic paraboloid

In this section we give the exact value of the constant d_o for a paraboloid M in $\mathbf{R}^3$ defined by an equation

$$x_0^2/a_0 + x_1^2/a_1 = 2x_2$$

with $0 < a_0 < a_1$ in §3.

Let $M(t) = \{(x_0, x_1, x_2) \in M \; ; \; x_2 = t\}$ and let $p = (0,0,0), q_0(t) = (\sqrt{2a_0t}, 0, t)$ and $q_1(t) = (0, \sqrt{2a_1t}, t)$. Let $\ell_0(t)$ (resp. $\ell_1(t)$) denote the distance between p and $q_0(t)$ (resp. $q_1(t)$) along $M \cap \{x_1 = 0\}$ (resp. $\{x_0 = 0\}$). Then

$$\ell_i(t) = \sqrt{t^2 + \frac{a_i t}{2}} + \frac{a_i}{2} \log \left| \sqrt{\frac{2t}{a_i}} + \sqrt{\frac{2t}{a_i} + 1} \right| \quad (i = 1, 2).$$

And

$$d(p, M(t)) = \ell_0(t).$$

Let $\ell_0(t) = \ell_1(t')$. Then the metric circle $S_{\ell_0(t)}(p)$ is located between two planes $\{x_2 = t\}$ and $\{x_2 = t'\}$ and

$$\left|\operatorname{diam} S_{\ell_0(t)}(p) - \operatorname{diam} M(t)\right| \leq 2(\ell_1(t) - \ell_1(t')). \tag{4.1}$$

Lemma 4.1. $\displaystyle\lim_{t\to\infty} \frac{2 \operatorname{diam} M(t)}{\operatorname{length} M(t)} = 1.$

Proof. Let c be a minimizing geodesic of M from $q_0(t)$ to $-q_0(t)$. Let $t_2 = \min_c x_2$ and $t_1 = \sqrt{2a_1t_2}$. Let

$$C_1 = \{(x_0, x_1, x_2) \in M \ ; \ x_1 = t_1 \text{ and } x_2 \leq t\}$$
$$C_2 = \{(x_0, x_1, x_2) \in M \ ; \ x_2 = t_2 \text{ and } x_1 \geq 0\}$$

Since c satisfies $x_1 \circ c \leq t_1$ and $x_2 \circ c \geq t_2$ (cf. §3),

$$\operatorname{length}(c) \geq \operatorname{length} C_i \ (i = 1, 2).$$

We note

$$\begin{aligned}
\operatorname{length} M(t) &= \sqrt{2t} \int_0^{2\pi} \sqrt{a_0 \sin^2\theta + a_1 \cos^2\theta}\, d\theta, \\
\operatorname{length} C_1 &= \sqrt{t^2 + \frac{a_1 t}{2}} + \frac{a_1}{2} \log\left|\sqrt{\frac{2t}{a_1}} + \sqrt{\frac{2t}{a_1} + 1}\right| \\
&\quad - \sqrt{t_2^2 + \frac{a_1 t_2}{2}} - \frac{a_1}{2} \log\left|\sqrt{\frac{2t_2}{a_1}} + \sqrt{\frac{2t_2}{a_1} + 1}\right| \\
\operatorname{length} C_2 &= \sqrt{\frac{t_2}{2}} \int_0^{2\pi} \sqrt{a_0 \sin^2\theta + a_1 \cos^2\theta}\, d\theta.
\end{aligned}$$

If $\limsup t_2/t = 1$, then

$$\begin{aligned}
1 \geq \limsup_{t\to\infty} \frac{2 \operatorname{diam} M(t)}{\operatorname{length} M(t)} &= \limsup_{t\to\infty} \frac{2 \operatorname{length}(c)}{\operatorname{length} M(t)} \\
&\geq \limsup_{t\to\infty} \frac{2 \operatorname{length} C_2}{\operatorname{length} M(t)} = 1.
\end{aligned}$$

Suppose $\limsup t_2/t < 1$. Then

$$\operatorname{length} C_1 \sim \text{const.} t \gg \text{const.} \sqrt{t} \sim \frac{1}{2} \operatorname{length} M(t).$$

as $t \to \infty$. Hence

$$\operatorname{diam} M(t) \geq \operatorname{length}(c) \geq \operatorname{length} C_1 > \frac{1}{2} \operatorname{length} M(t)$$

for large t, which contradicts $\operatorname{diam} M(t) \leq \frac{1}{2} \operatorname{length} M(t)$.

Since $\ell_1(t) - \ell_1(t') \sim \text{const.} \log t$ and $\operatorname{diam} M(t) \sim \text{const.}\sqrt{t}$ as $t \to \infty$, the inequality (4.1) combined with Lemma 4.1 yields

Lemma 4.2. $\lim_{t\to\infty} \frac{\operatorname{diam} S_{\ell_0(t)}(p)}{\operatorname{diam} M(t)} = 1.$

From Lemma 4.2 we easily get

Proposition 4.3. $d_o(M) = \frac{1}{2}\left(\int_0^{2\pi} \sqrt{a_0 \sin^2\theta + a_1 \cos^2\theta\, d\theta}\right)^2.$

Hence the distance between two umbilic points goes towards

$$a_1 = \lim_{a_0 \to 0} d_0/8$$

as $a_0 \to 0$, so the estimate in Theorem 1.1 is sharp.

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Bubbling of Minimizing Sequences for Prescribed Scalar Curvature Problem

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§1. Introduction

Let (M, g) be a compact Riemannian manifold of dimension n (≥ 3) and K be a smooth function on M. In this paper we consider the problem of finding a metric conformal to g having the scalar curvature K. Any conformal metric to g can be written $\tilde{g} = u^{2/(n-2)}g$ where u is a positive smooth function on M. From the transformation law for the scalar curvature, this problem is equivalent to solve the nonlinear partial differential equation

$$
(1.1) \qquad \begin{gathered} L_g u := -\kappa \Delta_g u + Ru = Ku^{N-1}\,, \qquad u > 0 \qquad \text{in } M\,, \\ \kappa = \frac{4(n-1)}{n-2}\,, \qquad N = \frac{2n}{n-2}\,, \end{gathered}
$$

where Δ_g denotes the negative definite Laplacian and R is the scalar curvature of g. The linear elliptic operator L_g is called the conformal Laplacian of (M, g). In the case K is a constant the problem was first studied by Yamabe [26]. For general K the problem was presented by Kazdan-Warner [16], [17]. Since their pioneer work, the problem has drawn attentions of both geometers and analysts (for example, see [3], [11], [14]).

As proved in [15], the problem can be reduced to the case where scalar curvature R is everywhere either positive, zero or negative. Here we consider only the case that R is positive everywhere. In this case, we easily see that a necessary condition for the solvability of (1.1) is that K is positive somewhere. For such function K, the problem has the variational formulation. We consider the functional

$$
E(u) = \int_M (\kappa |\nabla u|^2 + Ru^2)\, dV\,,
$$

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on the constraint set $\mathcal{C}_K = \{ u \in H^1(M) \mid F_K(u) = 1 \}$ where

$$F_K(u) = \int_M K|u|^N dV ,$$

and dV is the volume element of (M, g). Here $H^1(M)$ is the Sobolev space of L^2 functions whose first derivatives are in $L^2(M)$. The condition that K is positive somewhere guarantees that $\mathcal{C}_K$ is not empty. We set

$$\lambda_K = \lambda_K(M, g) = \inf\{ E(u) \mid u \in \mathcal{C}_K \} .$$

From the Sobolev inequality we see that λ_K is a positive constant. As stated in [11], [14], if a function u of $\mathcal{C}_K$ achieves the infimum λ_K, then $u/\lambda_K^{(n-2)/4}$ is a smooth solution of (1.1).

Take a minimizing sequence $\{u_j\}$ of $\mathcal{C}_K$ for E, that is, $E(u_j)$ tends to λ_K as $j \to \infty$. We may assume that each u_j is non-negative almost everywhere. In fact, if $\{u_j\}$ is a minimizing sequence, then so is $\{|u_j|\}$. Since $\mathcal{C}_K$ is closed in $H^1(M)$, the infimum is achieved if $\{u_j\}$ is compact in $H^1(M)$. Aubin [2] showed that any minimizing sequence is compact in $H^1(M)$ if the strict inequality

$$\lambda_K < \Lambda/(\max K)^{2/N}$$
$$\text{where} \quad \Lambda = \lambda_{K=1}(S^n, g_0) = n(n-1)\,\mathrm{vol}\,(S^n)^{2/n} ,$$

holds (also, see [7], [14]). However, the non-existence results of Kazdan-Warner [17] and Bourguignon-Ezin [4] for the equation (1.1) lead to the fact that no minimizing sequence is compact in $H^1(M)$.

The purpose of this paper is to describe how a minimizing sequence behaves if its compactness in $H^1(M)$ fails. In section 2 we prove the following result.

Theorem A. *Let $\{u_j\} \subset \mathcal{C}_K$ be a minimizing sequence for E with $u_j \geq 0$ almost everywhere. If $\{u_j\}$ is not compact in $H^1(M)$, then there exist*

(i) *a subsequence $\{k\} \subset \{j\}$,*
(ii) *a point $a \in M$,*
(iii) *a sequence $\{r_k\}$ of $\mathbb{R}_+$ with $r_k \to 0$ as $k \to \infty$, and*
(iv) *a sequence $\{a_k\}$ of M with $a_k \to a$ as $k \to \infty$,*

satisfying the following conditions :

(1) *u_k converges to 0 in $H^1_{\mathrm{loc}}(M\backslash\{a\})$.*
(2) *The measure $u_k^N\, dV$ converges to $K(a)^{-1}\delta_a$ weakly in the sense of measures on M where δ_a denotes Dirac measure.*

(3) *The renormalized sequence* $\tilde{u}_k(x) = r_k^{(n-2)/2} u_k(\exp_{a_k}(r_k x))$ *converges to the function*

$$v(x) = \left(\frac{2^n}{\mathrm{vol}(S^n)K(a)^2}\right)^{(n-2)/2n}\left(\frac{\rho}{\rho^2 + |x-b|^2}\right)^{(n-2)/2},$$

in $H^1_{\mathrm{loc}}(\mathbb{R}^n)$ *for some* $\rho > 0$, $b \in \mathbb{R}^n$. *Here,* $\exp_{a_k}$ *denotes the exponential map of* M *at* a_k.

(4) $\lambda_K = \Lambda/(\max K)^{(n-2)/n}$.

(5) *The point* a *attains the maximum of* K.

A similar phenomenon to Theorem A has been observed in various nonlinear problems and called *bubbling* or *concentration* (for example, see [5], [9], [20], [22], [23]). P. L. Lions obtained the same results of (2) and (5) in Theorem A by using his theory of concentration-compactness principle [19]. Our proof differs from his. We only use the notion of the concentration function introduced in [19]. the statement (1), (3) and (4) of our result give a more precise description of the behavior of minimizing sequences. In the case $K \equiv 1$ the results corresponding to the statements (1)–(4) are proved in [24]. We note that the above mentioned result of Aubin can also be derived from Theorem A (4).

In section 3 we consider the case $(M, g) = (S^n, g_0)$ where g_0 denotes the standard metric of the sphere. We prove the following result.

Theorem B. *Let* $\{u_j\} \subset \mathcal{C}_K$ *be a minimizing sequence for* E *with* $u_j \geq 0$ *almost everywheree. If* $\{u_j\}$ *is not relatively compact in* $H^1(S^n)$, *then there exist*

(i) *a subsequence* $\{k\} \subset \{j\}$,

(ii) *a sequence* $\{\psi_k\}$ *of conformal transformations on* S^n,

(iii) *a conformal transformation* ψ *on* S^n,

such that

(1) *the renormalized sequence* $\{\tilde{u}_k\}$ *defined by*
$\tilde{u}_k^{N-2} g_0 = \psi_k^*(u_k^{N-2} g_0)$
converges to a positive smooth function u_0 *in* $H^1(S^n)$, *and*

(2) *the function* u_0 *is determined by the equality*
$u_0^{N-2} g_0 = (\max K)^{-2/n} \psi^*(g_0)$.

This theorem states that on the sphere we are able to take the globally defined renormalized sequence by using conformal transformations. In the case $K \equiv 1$, Lee-Parker [18] proved an analogous result for the special minimizing sequence of approximate solutions for (1.1).

Finally, in Section 4 we state some results related to Theorems A and B.

§2. Proof of Theorem A

We first recall several known facts about the minimizing problem. Take a minimizing sequence $\{u_j\}$ of $\mathcal{C}_K$ for E with $u_j \geq 0$ almost everywhere. Since the quantity $\sqrt{E(\,\cdot\,)}$ is equivalent to the Sobolev norm $\|\cdot\|_{H^1}$, we see that $\{u_j\}$ is bounded in $H^1(M)$. Therefore, $\{u_j\}$ is compact in the weak topology of $H^1(M)$. Using the Rellich compactness theorem, we may assume

$$u_j \longrightarrow u \qquad \begin{aligned} &\text{weakly in } H^1(M), \\ &\text{strongly in } L^2(M), \\ &\text{almost everywhere on } M, \end{aligned}$$

for some $u \in H^1(M)$ with $u \geq 0$ a.e. From the general theory of calculus of variation we obtain

$$L_g u_j - \lambda_K K(x) u_j^{N-1} \longrightarrow 0 \qquad \text{in } H^{-1}(M) = (H^1(M))^*, \tag{2.1}$$

(for example, see [10], [13]).

Proposition 2.1. *If the weak limit $u \not\equiv 0$, then*

(1) *u belongs to $\mathcal{C}_K$ and achieves the infimum λ_K.*
(2) *$\{u_j\}$ converges to u in the strong topology of $H^1(M)$.*

Proof. Passing to the limit in (2.1), we get

$$L_g u = \lambda_K K(x) u^{N-1} \in H^{-1}(M). \tag{2.2}$$

From the regularity result of Brezis-Kato [6] and Trudinger [26] we see that u is a smooth function. Multiplying the both side of (2.2) by u and integrating over M, we have $E(u) = \lambda_K F_K(u)$. By the assumption that $u \not\equiv 0$, we get $F_K(u) > 0$. From the definition of λ_K we have

$$\lambda_K \leq E(u/F_K(u)^{1/N}) = E(u)/F_K(u)^{2/N} = \lambda_K F_K(u)^{2/n}.$$

This shows $F_K(u) \geq 1$. On the other hand, since E is weakly lower semi-continuous, we have

$$\lambda_K F_K(u) = E(u) \leq \liminf_{j\to\infty} E(u_j) = \lambda_K.$$

Then, we obtain $F_K(u) = 1$ and $E(u) = \lambda_K$. Since u_j converges to u weakly in $H^1(M)$, we get

$$E(u_j) = \kappa \int_M |\nabla(u_j - u)|^2 dV + E(u) = \lambda_K + o(1)\,.$$

Hence, we have

$$\int_M |\nabla(u_j - u)|^2 dV = o(1)\,.$$

The proof is completed.

The next theorem plays a crucial role in the proof of our main results.

Theorem 2.2. (Local convergence theorem). *Let Ω be a domain in M. Suppose that sequences $\{u_j\} \subset H^1(\Omega)$ and $\{\lambda_j\} \subset \mathbb{R}$ satisfy*

(1) $u_j \longrightarrow u$ *weakly in* $H^1(\Omega)$,
(2) $\lambda_j \longrightarrow \lambda > 0$,
(3) $L_g u_j - \lambda_j K(x)|u_j|^{N-2}u_j \longrightarrow 0$ *in* $H^{-1}(\Omega) = (H_0^1(\Omega))^*$.

If each u_j satisfies

$$(2.3)\qquad \int_\Omega K^+|u_j|^N dV \le \epsilon \qquad \textit{for some } \epsilon < \Big(\frac{\Lambda}{\lambda}\Big)^{n/2}(\max K)^{-(n-2)/2},$$

where $K^+(x) = \max\{K(x),\, 0\}$, then $u_j \longrightarrow u$ in $H^1_{\mathrm{loc}}(\Omega)$.

This theorem was proved in [24] in case K is a constant. For general K the proof in [24] also works with a slight modification.

Remark 2.3. We may replace the condition (2.3) by

$$(2.4)\qquad \int_\Omega |u_j|^N dV \le \epsilon/(\max K)\,.$$

We now give a proof of Theorem A. From Proposition 2.1 we know the weak limit u is identically zero if $\{u_j\}$ is not compact in $H^1(M)$.

Proof of statement (1). We take ϵ as in Theorem 2.2. We define the set $\mathcal{S}$ as

$$(2.5)\qquad \mathcal{S} = \bigcap_{r>0}\Big\{\, x \in M \;\Big|\; \liminf_{j\to\infty} \int_{B(x,r)} K^+|u_j|^N dV \ge \epsilon \,\Big\}\,,$$

where $B(x,r)$ is the open geodesic ball with center x and radius r. As proved in [23], $\mathcal{S}$ is a compact subset of M. We first show that $K(x) > 0$ for any $x \in \mathcal{S}$.

Take any point y in M with $K(y) \leq 0$. Since $K^+(x)$ is Lipschitz continuous, we have

$$\begin{aligned}\int_{B(y,r)} & K^+ u_j^N dV \\ & \leq K^+(y) \int_{B(y,r)} u_j^N dV + \max_{B(y,r)} |K^+(\cdot) - K^+(y)| \int_{B(y,r)} u_j^N dV \\ & \leq O(r).\end{aligned}$$

This leads to $y \notin \mathcal{S}$.

We next show that a subsequence $\{u_k\}$ of $\{u_j\}$ converges to 0 in $H^1_{\text{loc}}(M \setminus \mathcal{S})$. If y in $M \setminus \mathcal{S}$, then there exist $r > 0$ and infinitely many j such that the inequality

$$\int_{B(y,r)} K^+ u_j^N \, dV \leq \epsilon$$

holds. By Theorem 2.2 we show that such u_j converges to 0 strongly in $H^1(B(y, r/2))$. By a diagonal subsequence argument, a subsequence $\{u_k\}$ of $\{u_j\}$ converges to 0 strongly in $H^1(\Omega)$ for each $\Omega \Subset M \backslash \mathcal{S}$.

We finally show that $\mathcal{S}$ consists of a single point. Note that we may take $\epsilon = 1 - \delta$ for any sufficiently small $\delta > 0$. For $r > 0$, we take a maximal family $\{ B(x_1, r), \cdots, B(x_I, r) \}$ of $I = I(r)$ disjoint geodesic balls of radius r with center $x_i \in \mathcal{S}$. By maximality $\mathcal{S}$ is covered by $B(x_1, 2r), \cdots, B(x_I, 2r)$. Since each x_i lies in $\mathcal{S}$, for any $\delta > 0$

$$\int_{B(x_i,r)} K^+ u_k^N dV \geq (1-\delta)^2,$$

holds if k is sufficiently large. Summing up these, we get

$$\begin{aligned} I & \leq \frac{1}{(1-\delta)^2} \sum_{i=1}^{I} \int_{B(x_i,r)} K^+ u_k^N dV \leq \frac{1}{(1-\delta)^2} \int_M K^+ u_k^N dV \\ & \leq \frac{1}{(1-\delta)^2} \Big(1 + \int_M K^- u_k^N dV \Big) \\ & \leq \frac{1}{(1-\delta)^2} \Big(1 + \|K\|_\infty \int_{M^-} u_k^N dV \Big), \end{aligned}$$

where $M^- = \{ x \in M \mid K \leq 0 \}$. Since M^- is a compact set in $M \backslash \mathcal{S}$, u_k converges to 0 in the strong H^1 topology on some neighborhood of

M^-. Thus, by taking k sufficiently large, we have

$$I \leq \frac{1+\delta}{(1-\delta)^2}\,.$$

This shows $\mathcal{H}^0(\mathcal{S}) \leq (1+\delta)/(1-\delta)^2$ where $\mathcal{H}^0$ denotes the 0-dimensional Hausdorff measure on M. If we choose δ small enough, we have $\mathcal{H}^0(\mathcal{S}) < 2$. Since the 0-dimensional Hausdorff measure coincides with the counting measure, either $\mathcal{S} = \{a\}$ for some $a \in M$ or $\mathcal{S}$ is empty. If $\mathcal{S}$ is empty, $\{u_k\}$ converges strongly in $H^1(M)$ because of Theorem 2.2 and the compactness of M. Thus, we obtain the desired result.

Proof of statement (2). For each k, we define the Radon measure μ_k on M by

$$\mu_k(A) = \int_A u_k^N\,dV \qquad \text{for } A \subset M.$$

Since $\{u_k\}$ is bounded in $L^N(M)$, the total variation of μ_k is uniformly bounded. Then, taking a subsequence if necessary, μ_k converges to some Radon measure μ weakly. Since $\{u_k\}$ converges to 0 in $H^1_{\text{loc}}(M\backslash\{a\})$, the support of the measure μ is contained in $\{a\}$. Thus, we have $\mu = \alpha\delta_a$ for some $\alpha \geq 0$. Since each u_k lies in $\mathcal{C}_K$, we have

$$1 = \lim_{k\to\infty}\int_M K u_k^N dV = \lim_{k\to\infty}\int_M K\,d\mu_k = K(a)\alpha\,.$$

The proof is completed.

Proof of statement (3)–(5). We take a normal coordinate neighborhood W of a and a normal coordinate system x of M centered at a. Through this coordinate W can be regarded as a neighborhood of the origin 0 in $\mathbb{R}^n$. So we note that the metric g satisfies $g_{\alpha\beta} = \delta_{\alpha\beta} + O(|x|^2)$ in the x-coordinate. Let $B(x,r)$ be the open ball with center x and radius r and let $B(r) = B(0,r)$. We choose $R > 0$ small enough. As in [19] and [24], we introduce the concentration function

$$Q_j(t) = \sup_{y\in B(R)}\int_{B(y,t)} u_j^N\,dx \qquad \text{for } 0 \leq t \leq R.$$

Each function Q_j is continuous and non-decreasing in t, and $Q_j(0) = 0$. We fix an arbitrary small $\delta > 0$. By the definition of the point a,

$$Q_j(R) \geq \int_{B(R)} u_j^N\,dx \geq (1-\delta)/(\max K)$$

holds for sufficiently large j. By continuity of Q_j, there exist $0 < r_j < R$ and $a_j \in \overline{B(R)}$ such that

$$Q_j(r_j) = \int_{B(a_j, r_j)} u_j^N \, dx = \epsilon(1 - 2\delta)/(\max K)\,.$$

Then we easily see that

$$r_j \longrightarrow 0 \qquad \text{and} \qquad a_j \longrightarrow 0 \qquad \text{as} \quad j \longrightarrow \infty.$$

We set $U(j) = B(a_j/r_j, 2R/r_j) \subset \mathbb{R}^n$ and

$$\tilde{u}_j(x) = r_j^{(n-2)/2} u_j(a_j + r_j x)\,.$$

Since a_j lies in $B(R/2)$ for sufficiently large j, we have $B(R/r_j) \subset U(j)$ which leads to $U(j) \longrightarrow \mathbb{R}^n$ as $j \to \infty$. We fix any bounded domain Ω of $\mathbb{R}^n$. Then we have

$$\int_\Omega |\nabla \tilde{u}_j|^2 dx \le (1 + C_1 R^2) \int_M |\nabla u_j|^2 dV \le C_2 < \infty\,,$$
$$\int_\Omega \tilde{u}_j^N dx \le (1 + C_1 R^2) \int_M u_j^N dV \le C_3 < \infty\,.$$

From (2.1) we have

$$\kappa \Delta_j \tilde{u}_j - R(a_j + r_j \cdot) r_j^2 \tilde{u}_j + \lambda_j K(a_j + r_j \cdot) \tilde{u}_j^{N-1} \longrightarrow 0 \qquad \text{in} \quad H^{-1}(\Omega),$$

where Δ_j is the Laplacian with respect to the metric $g_j = g(a_j + r_j)$. Since g is the standard Euclidean metric up to second order, we have

$$\text{(2.6)} \qquad \kappa \Delta \tilde{u}_j - \lambda_K K(a) \tilde{u}_j^{N-1} \longrightarrow 0 \qquad \text{in} \quad H^{-1}(\Omega)\,.$$

Using the diagonal subsequence argument, we can take a subsequence $\{k\} \subset \{j\}$ so that for each domain $\Omega \Subset \mathbb{R}^n$,

$$\tilde{u}_k \longrightarrow v \qquad \text{weakly in } H^1(\Omega),$$
$$\tilde{u}_k(x) \longrightarrow v(x) \qquad \text{almost everywhere on } \mathbb{R}^n,$$

for some $v \in H^1_{\text{loc}}(\mathbb{R}^n)$ with $v \ge 0$ almost everywhere. Passing to the limit in (2.2), we know that v is a weak solution of

$$\text{(2.7)} \qquad \kappa \Delta v + \lambda_K K(a) v^{N-1} = 0\,.$$

By the regularity theorem in [6], [26] and the maximum principle, v is either a positive smooth function or identically zero.

We prove that $\{\tilde{u}_k\}$ converges in $H^1_{\mathrm{loc}}(\mathbb{R}^n)$. Fix any $z \in \mathbb{R}^n$. By the definition of a_k, r_k, we have

$$\int_{B(z,1)} \tilde{u}_k^N \, dx \le \int_{B(1)} \tilde{u}_k^N \, dx = (1-2\delta)/(\max K) < (1-\delta)/(\max K).$$

By Theorem 2.2 and Remark 2.3, $\tilde{u}_k$ converges to v strongly in $H^1(B(z,1/2))$. Also, we obtain $v \not\equiv 0$, that is, v is positive everywhere.

From the result of Gidas-Ni-Nirenberg [12], all positive solutions of (2.7) are completely determined. Hence, we have

$$v(x) = (\frac{4n(n-1)}{\lambda_K K(a)})^{(n-2)/4} (\frac{\rho}{\rho^2 + |x-b|^2})^{(n-2)/2},$$

for some $\rho > 0$ and $b \in \mathbb{R}^n$. By the result of Talenti [25] on the Sobolev inequality such v satisfies the equality

$$(\int_{\mathbb{R}^n} v^N dx)^{2/N} = \frac{\kappa}{\Lambda} \int_{\mathbb{R}^n} |\nabla v|^2 \, dx.$$

Then we have

$$\int_{\mathbb{R}^n} v^N \, dx = (\frac{\Lambda}{\lambda_K K(a)})^{n/2}.$$

From the result of Aubin [2], we have the upper estimate of λ_K as

$$\lambda_K \le \Lambda/(\max K)^{2/N}.$$

Thus we have

$$\begin{aligned} 1 &\le (\frac{\Lambda}{\lambda_K (\max K)^{2/N}})^{n/2} \le (\frac{\Lambda}{\lambda_K K(a)^{2/N}})^{n/2} = K(a) \int_{\mathbb{R}^n} v^N dx, \\ &\le \liminf_{k\to\infty} \int_{\mathbb{R}^n} K(\exp_{a_k}(r_k x)) \tilde{u}_k(x)^N \, dx, \\ &\le \liminf_{k\to\infty} \int_M K|u_k|^N dV + C_1 R^2 \le 1 + C_1 R^2. \end{aligned}$$

Hence we obtain

$$\lambda_K = \Lambda/(\max K)^{2/N}, \qquad K(a) = \max K.$$

The proof is completed.

§3. Proof of Theorem B

Since $\{u_j\}$ is not compact in $H^1(M)$, Theorem A implies that the bubbling phenomenon occurs at a point a of S^n. We may assume that a is the south pole. Let $P = (0, \cdots, 0, 1)$ be the north pole and $\pi : S^n \backslash \{P\} \longrightarrow \mathbb{R}^n$ be the stereographic projection. We take the local coordinate system defined by π. Using the similar argument to the proof of (3) in Theorem A, we can choose

(a) a subsequence $\{k\} \subset \{j\}$,
(b) a sequence $\{r_k\}$ of $\mathbb{R}_+$ with $r_k \to 0$ as $k \to \infty$, and
(c) a sequence $\{a_k\}$ of $\mathbb{R}^n$ with $a_k \to a$ as $k \to \infty$,

so that the sequence $\{r_k^{(n-2)/2} u_k(\pi^{-1}(r_k \cdot + a_k))\}$ converges in $H^1_{\mathrm{loc}}(\mathbb{R}^n)$.

We define the mapping $\psi_k : S^n \longrightarrow S^n$ by $\psi_k(x) = \pi^{-1}(r_k \pi(x) + a_k)$. Then we easily see that each ψ_k is a conformal transformation of S^n. We set the renormalized sequence $\{\tilde{u}_k\}$ by the relation $\tilde{u}_k^{N-2} g_0 = \psi_k^*(u_k^{N-2} g_0)$. We easily obtain

$$\tilde{u}_k(x) = r_k^{(n-2)/2} u_k(\pi^{-1}(r_k \pi(x) + a_k)) .$$

Thus, we get

$$\tilde{u}_k \longrightarrow u_0 \qquad \begin{array}{l} \text{weakly in } H^1(S^n) , \\ \text{strongly in } H^1_{\mathrm{loc}}(S^n \backslash \{P\}) , \\ \text{almost everywhere on } S^n , \end{array}$$

for some $u_0 \in H^1(S^n)$ with $u_0 \geq 0$, $u_0 \not\equiv 0$.

We show the statement (2). Using the same argument as the proof of Theorem A (3), we see that the sequence $\{\tilde{u}_k\}$ satisfies

$$L_g \tilde{u}_k - \lambda_K (\max K) \tilde{u}_k^{N-1} \longrightarrow 0 \qquad \text{in} \quad H^{-1}_{\mathrm{loc}}(S^n \backslash \{P\}) . \tag{3.1}$$

Passing to the limit in (3.1), we have

$$L_g u_0 + \lambda_K (\max K) u_0^{N-1} = 0 \in H^{-1}_{\mathrm{loc}}(S^n \backslash \{P\}) .$$

By the regularity theorem in [6], [26] and the maximum principle, u_0 is a positive smooth function on $S^n \backslash \{P\}$ satisfying

$$L_g u_0 = \lambda_K (\max K) u_0^{N-1} , \tag{3.2}$$

in $S^n \backslash \{P\}$. The result of Caffarelli-Gidas-Spruck [8] implies that u_0 can be extended to a positive function defined on the whole of S^n and satisfies (3.2) on S^n. This means that the conformal metric $u_0^{N-2} g_0$ on S^n has constant scalar curvature. From the result of Obata [21], we can take a conformal transformation ψ so that the statement (2) holds.

Finally we prove that $\{\tilde{u}_k\}$ converges in $H^1(S^n)$. The result of Obata [21] leads to

$$\Lambda = \inf\{\, E(u)/\|u\|_N^2 \;\big|\; u \in H^1(S^n),\ u \not\equiv 0 \,\} = E(u_0)/\|u_0\|_N^2 \,.$$

Multiplying (3.2) by u_0 and integrating over S^n, we have

$$E(u_0) = \lambda_K (\max K) \|u_0\|_N^N \,.$$

Noting that the relation $\lambda_K (\max K)^{2/N} = \Lambda$, we obtain

$$\|u_0\|_N = (\max K)^{-1/N}\,, \qquad E(u_0) = \lambda_K \,.$$

Since the functional E is conformally invariant, we have

$$E(\tilde{u}_k) = E(u_k) = \lambda_K + o(1)\,.$$

Thus, we get

$$\int_{S^n} |\nabla(\tilde{u}_k - u_0)|^2 dV = \frac{1}{\kappa}\big(E(\tilde{u}_k) - E(u_0) \big) = o(1)\,.$$

The proof is completed.

§4. Some remarks

We first remark that the bubbling phenomenon in Theorem A may occur at each point where K achieves the maximum.

Proposition 4.1. *Suppose (M, g) and K satisfy the equality $\lambda_K = \Lambda/(\max K)^{2/N}$. For each $p \in M$ with $K(p) = \max K$, there exists a minimizing sequence $\{u_j\} \subset C_K$ satisfying*

(1) *each u_j is a non-negative smooth function on S^n,*
(2) *$u_j^N\, dV \longrightarrow (\max K)^{-1} \delta_p$ weakly in the sense of measures on M.*

Proof. We take a radial cutoff function $\eta \in C_0^\infty(\mathbb{R}^n)$ satisfying

$$\eta(x) = \begin{cases} 1 & \text{if } |x| \le 1, \\ 0 & \text{if } |x| \ge 2, \end{cases}$$
$$0 \le \eta \le 1, \qquad |\nabla \eta| = |\partial \eta / \partial r| \le 2.$$

Take a normal coordinate of M centered at p. For small $\epsilon > 0$ and $\rho > 0$ we define

$$u_{\epsilon,\rho}(x) = \eta(\frac{x}{\epsilon})(\frac{\rho}{\rho^2 + |x|^2})^{(n-2)/2}.$$

If we choose ϵ small enough, we have $F_K(u_{\epsilon,\rho}) > 0$ for any $\lambda > 0$. The calculation in [1], [14] gives

$$\lambda_K \leq E(u_{\epsilon,\rho})/F_K(u_{\epsilon,\rho})^{2/N} \leq \Lambda(1 + C\epsilon)(1 + C\rho)/(\max K)^{2/N}.$$

Taking sequences $\epsilon_j \to 0$ and $\rho_j \to 0$ as $j \to \infty$, we obtain the sequence $u_j(x) = u_{\epsilon_j,\rho_j}(x)/F_K(u_{\epsilon_j,\rho_j})^{1/N}$ having the desired properties.

We next consider the case that (M, g) is the sphere S^n with the standard metric g_0. Consider the case that K is a constant. We remark that the functional F_K is conformally invariant if K is a constant. This implies that renormalized sequence $\{\tilde{u}_k\}$ in Theorem B is also a minimizing sequence of E. Thus we obtain the following theorem as a corollary of Theorem B.

Theorem 4.2. *If $(M, g) = (S^n, g_0)$ and K is a constant, then every minimizing sequence of $\mathcal{C}_K$ for E can be renormalized to converge in the strong topology of $H^1(S^n)$ by conformal transformations.*

On the other hand, if K is not a constant, then the following nonexistence result was proved.

Theorem 4.3 (Kazdan [15]). *If $(M, g) = (S^n, g_0)$ and K is not a constant, then the infimum λ_K is never achieved.*

Proof. Suppose that there exists a function $u \in \mathcal{C}_K$ with $E(u) = \lambda_K$. We may assume that u is a positive smooth function on S^n. From the definition of Λ, we have

$$\lambda_K = \frac{E(u)}{F_K(u)^{2/N}} \geq \frac{E(u)}{(\max K)^{2/N}\|u\|_N^2} \geq \frac{\Lambda}{(\max K)^{2/N}} = \lambda_K\,.$$

This leads to

$$F_K(u) = \int_{S^n} K|u|^N dV = \max K \int_{S^n} |u|^N dV\,.$$

Since u is positive everywhere, K is a constant. the proof is completed.

Thus we obtain the following.

Corollary 4.4. *If K is not a constant, then no minimizing sequence is compact in $H^1(S^n)$.*

Also, we observe that the renormalized sequence in Theorem B is not a minimizing one.

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Geometry of Laplace-Beltrami Operator on a Complete Riemannian Manifold

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§0. Introduction

This is a survey paper on recent developments of analytic and geometric aspects of the Laplace-Beltrami operator on a complete Riemannian manifold. Systematic treatments from a Riemannian geometric viewpoint have been already appreared in Berger, Gauduchon & Mazet ['71], Kotake, Maeda, Ozawa & Urakawa ['81], Bérard & Berger ['83], Bérard ['86], Chavel ['84], Gilkey ['84] and Sunada ['88]. But they are mainly concerned with compact case, except Chavel ['84]. In this paper, we shall focus on recent developments of spectral geometry of a *noncompact* complete Riemannian manifold. It seems that the materials may be divided into three parts:

(1) the distribution of the (essential) spectrum of the Laplacian,

(2) the heat kernel of a complete Riemannian manifold, and

(3) harmonic functions, and Green functions on such a manifold.

More precisely,

(1) in §3, we treat mainly results on estimates of the bottom of the (essential) spectrum of the Laplacian of a noncompact complete Riemannian manifold.

(2) In §4, following Ito ['88], Dodziuk ['83], we construct the (minimal) heat kernel of a noncompact complete Riemannian manifold, and show results on uniqueness and estimates of such heat kernel, under certain curvature conditions.

(3) In §5, we will treat positive harmonic functions, the Martin boundary, and Liouville type theorems for harmonic functions on complete manifolds.

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Contents

§1. Preliminaries

1.1. The Laplace-Beltrami operator

In this section, we prepare some basic materials about spectral theory of selfadjoint operators, and the Laplace-Beltrami operator on a noncompact complete Riemannian manifold.

All Riemannian manifolds we consider in this paper will be C^∞ connected noncompact complete Riemannian manifolds without boundary (unless otherwise stated).

Let (M, g) be a complete Riemannian manifold without boundary. Define the Laplace-Beltrami operator (we call it the Laplacian briefly hereafter) Δ_g acting on the space $C^\infty(M)$ of all C^∞ real valued functions on M by

$$
\begin{aligned}
(1.1) \qquad \Delta_g f &= \delta d f = -\text{ div grad } f \\
&= -\frac{1}{\sqrt{g}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x^i} \left(\sqrt{g} g^{ij} \frac{\partial f}{\partial x^j} \right) \\
&= -\sum_{i,j=1}^{n} g^{ij} \left(\frac{\partial^2 f}{\partial x^i \partial x^j} - \sum_{k=1}^{n} \Gamma_{ij}^k \frac{\partial f}{\partial x^k} \right) \\
&= -\sum_{i=1}^{n} \{ e_i (e_i f) - (\nabla_{e_i} e_i) f \},
\end{aligned}
$$

where $\sqrt{g} = \sqrt{\det(g_{ij})}$, $(g^{ij}) = (g_{ij})^{-1}$ (the inverse matrix), $g_{ij} = g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})$, and Γ_{ij}^k is Christoffel's symbol of g for a local coordinate $(x^1, \ldots, x^n)$, and $\{e_1, \ldots, e_n\}$ is a locally defined orthonormal frame field on M. Moreover, we denote by $\mathcal{X}(M)$ the space of all C^∞ vector fields on M, and define the divergence $\operatorname{div}(X)$ of $X \in \mathcal{X}(M)$ by

$$\operatorname{div}(X) = \frac{1}{\sqrt{g}} \sum_{i=1}^{n} \frac{\partial}{\partial x^i} \left(\sqrt{g}\, X^i \right),$$

where $X = \sum_i X^i \frac{\partial}{\partial x^i}$. The gradient vector field X = grad $f \in \mathcal{X}(M)$ is defined by $g(Y, X) = df(Y) = Yf$, $Y \in \mathcal{X}(M)$, i.e.,

$$\operatorname{grad}(f) = \sum_{i=1}^{n} e_i(f)\, e_i = \sum_{i,j=1}^{n} g^{ij} \frac{\partial f}{\partial x^i} \frac{\partial}{\partial x^j}.$$

We denote by $L^2(M)$ the space of all square integrable real valued functions on M. We define the inner product $(\ ,\ )$ on $L^2(M)$ by

$$(f_1, f_2) = \int_M f_1(x) f_2(x)\, v_g, \qquad f_1, f_2 \in L^2(M),$$

and put $\| f \| = \sqrt{(f, f)}$, $f \in L^2(M)$. We also define the global inner product $(\ ,\ )$ for tensor fields α, β by

$$(\alpha, \beta) = \int_M \langle \alpha, \beta \rangle\ v_g,$$

where $\langle\ ,\ \rangle$ denotes the pointwise inner product on $\otimes T_x M \otimes T_x^* M$, $x \in M$, and put $\| \alpha \| = \sqrt{(\alpha, \alpha)}$. Put

$$\begin{aligned} C_o^\infty(M) &= \{ f \in C^\infty(M);\ \operatorname{supp}(f), \quad \text{compact} \}, \\ A_o^1(M) &= \left\{ \omega \in A^1(M);\ \operatorname{supp}(\omega), \quad \text{compact} \right\}, \\ \mathcal{X}_o(M) &= \{ X \in \mathcal{X}(M);\ \operatorname{supp}(X), \quad \text{compact} \}. \end{aligned}$$

Here $A^1(M)$ is the space of all smooth 1 forms on M. Then the following is well-known:

Proposition 1.2. *For all $f, f_1, f_2 \in C_o^\infty(M), \omega \in A_o^1(M)$, and $X \in \mathcal{X}_o(M)$, we get*

(i) $$(f, \operatorname{div}(X)) = -(\operatorname{grad} f, X), \quad (df, \omega) = (f, \delta\omega),$$

(ii) $$(\Delta_g f_1, f_2) = (\operatorname{grad} f_1, \operatorname{grad} f_2) = (f_1, \Delta_g f_2),$$

(iii) $$\int_M \operatorname{div}(X)\, v_g = 0,$$

(iv) $$(\Delta_g f, f) \geq 0,$$

where v_g *is the canonical measure of* (M, g) *given locally by*

$$v_g = \sqrt{\det(g_{ij})}\, dx^1 \cdots dx^n.$$

Corollary 1.3. *The Laplacian* $\Delta_g : C_o^\infty(M) \longrightarrow C_o^\infty(M)$ *can be extended to a symmetric,* i.e., *formally selfadjoint operator of* $L^2(M)$ *into itself (see the definition below Theorem* 1.5).

Definition 1.4. We define the following spaces which are called *domains* of the differential operators div, d, δ, and Δ by

$$\begin{aligned} D(\mathrm{div}) &= \{\text{measurable vector field } X \text{ on } M;\, \|X\| < \infty,\, \|\mathrm{div}(X)\| < \infty\}, \\ D(d) &= \{\text{measurable function } f \text{ on } M;\, \|f\| < \infty,\, \|df\| < \infty\}, \\ D(\delta) &= \{\text{measurable 1 form } \alpha \text{ on } M;\, \|\alpha\| < \infty,\, \|\delta\alpha\| < \infty\}, \\ D(\Delta) &= \{\text{measurable function } f \text{ on} M;\, f \in D(d), df \in D(\delta)\} \\ &= \{\text{measurable function } f \text{ on } M;\, \|f\| < \infty, \|df\| < \infty, \|\delta df\| < \infty\}. \end{aligned}$$

Then we have:

Theorem 1.5 (Gaffney ['51] $\sim$ ['55]). *Let* (M, g) *be a complete Riemannian manifold. Then:* (i) *if* $X \in D(\mathrm{div})$, *and* $|X|$ *and* $\mathrm{div}(X)$ *are integrable, then*

$$\int_M \mathrm{div}(X)\, v_g = 0.$$

(ii) *If* $f \in D(d)$, $\omega \in D(\delta)$, *and* $X \in D(\mathrm{div})$, *then*

$$(df, \omega) = (f, \delta\omega),$$
$$(f, \mathrm{div}(X)) = -(\mathrm{grad}\ f, X).$$

(iii) (*symmetry*) *For* $f_1, f_2 \in D(\delta)$,

$$(\Delta f_1, f_2) = (f_1, \Delta f_2).$$

(iv) (*positivity*) *For* $f \in D(\Delta)$,

$$(\Delta f, f) \geq 0.$$

(v) *The closure to* $L^2(M)$ *of* $\Delta : C_o^\infty(M) \longrightarrow C_o^\infty(M)$ *is selfadjoint.*

In general, an operator $A : D \subset H \longrightarrow H$ of a Hilbert space H which is defined on a dense subset D is said to be *symmetric* (formally selfadjoint)

$$(Au, v) = (u, Av), \; u, v \in D.$$

A symmetric operator A; $D \subset H \longrightarrow H$ is said to be *selfadjoint* if

$$(Au, v) = (u, v^*) \quad \forall u \in D \Longrightarrow v \in D \quad \& \quad v^* = Av.$$

Then it is well-known that:

Theorem 1.6 (Spectral Resolution). *Let $A : D \subset H \longrightarrow H$ be selfadjoint. Then:*

(1) *A has the following resolution:*

$$A = \int_{-\infty}^{\infty} \lambda \, dE(\lambda),$$

where $\{E(\lambda)\,; \lambda \in \mathbb{R}\}$ is a one parameter family of projections of H satisfying the following conditions (i), (ii), (iii):

(i) $$\lambda < \mu \Longrightarrow E(\lambda) \le E(\mu),$$

(ii) $$E(\infty) = I \ (\textit{identity operator}), E(-\infty) = 0 \ (\textit{null operator}),$$

(iii) $$E(\lambda + 0) = E(\lambda).$$

(2) *The spectrum of A is contained in the set of real numbers:* $\mathrm{Spect}(A) \subset \mathbb{R}$.

Here let us recall the notions of resolvent, (essential-)spectrum, eigenvalues of a selfadjoint operator.

Definition 1.7. (i) The *resolvent* $\mathrm{Resolv}(A)$ of a selfadjoint operator A is the set of $\lambda \in \mathbb{C}$ satisfyng that $\mathrm{Ker}(A - \lambda I) = \{0\}, \mathrm{Range}(A - \lambda I) \subset H$ is dense, and $(A - \lambda I)^{-1}$ is a bounded operator. The *spectrum* of A, $\mathrm{Spect}(A)$, is by definition $\mathbb{C} \setminus \mathrm{Resolv}(A)$.

(ii) $\lambda \in C\,\mathrm{Spect}(A)$ (the *continuous spectrum*) if $\mathrm{Range}(A - \lambda I) \subset H$ is dense, but $(A - \lambda I)^{-1}$ is not a bounded operator.

(iii) A real number $\lambda \in \mathbb{R}$ is an *eigenvalue* of A if there exists a nonzero $u \in D(A)$ such that $Au = \lambda u$. $\mathrm{Ker}(A - \lambda I)$ is called the eigenspace, and $\dim \mathrm{Ker}(A - \lambda I)$ is called the multiplicity. Let $\mathrm{Spect}_o(A)$ be the set of all the eigenvalues which are isolated in $\mathrm{Spect}(A)$ and have finite multiplicities, and call $\mathrm{Ess\ Spect}(A) = \mathrm{closure}(\mathrm{Spect}(A) \setminus \mathrm{Spect}_o(A))$ the *essential spectrum*.

It is known that:

$$\lambda \in \operatorname{Spect}(A) \iff \exists 0 \neq f_n \in D(A)\ ; \quad \|Af_n - \lambda f_n\| \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty,$$

$$\lambda \in \operatorname{Ess\,Spect}(A) \iff \exists \{f_n\}_{n=1}^{\infty} \subset D(A) \text{ (noncompact set) } ; \quad \|Af_n - \lambda f_n\| \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty,$$

and

$$\lambda \in \operatorname{Ess\,Spect}(A) \iff \text{either } \lambda \in C\operatorname{Spect}(A) \text{ or the eigenvalue with infinite multiplicity.}$$

1.2. Discreteness of spectrum

We mainly deal with the following three types of the eigenvalue problems:

(1) (*Boundary Value Problem*) Let (M, g) be a complete Riemannian manifold without boundary, $\Omega \subset M$ a relatively compact domain in M.

(1-i) (*Dirichlet Eigenvalue Problem*):

$$\begin{cases} \Delta u = \lambda u & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

(1-ii) (*Neumann Eigenvalue Problem*):

$$\begin{cases} \Delta u = \lambda u & \text{in} \quad \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{a.e.} \quad \partial\Omega. \end{cases}$$

(2) (*Free Boundary Problem*) For a smooth function V on M,

$$(\Delta + V)u = \lambda u \quad \text{on} \quad M.$$

Then it is well-known that:

Theorem 1.8. *Let (M, g) be a complete Riemannian manifold, $\Omega \subset M$ a relatively compact domain. Then:*

(1) (1-i) *The spectrum of the Dirichlet eigenvalue problem has a discrete spectrum of eigenvalues with finite multiplicities.*

(1-ii) *If* Ω *satisfies, furthermore, the segment property* (cf. Agmon ['65, p.13], Reed & Simon ['78, p.256]), i.e., Ω *is a finite union of coordinate neighborhoods in* $M, (U_i, \phi_i)$, $\phi_i(U_i) \subset \mathbb{R}^n, n = \dim(M)$, *with the property that there exists* $y_i \in U_i$ *such that* $\phi_i(x) + t\,\phi_i(y_i) \in \phi_i(U_i)$ $(0 < \forall t < 1, \forall x \in \partial\Omega \cap U_i)$. *Then the Neumann problem has a discrete spectrum of eigenvalues with finite multiplicities.*

(2) *Let* V *be a smooth function on* M *satisfying the following exhaustion condition:*

$$\{x \in M \ ; \ V(x) \leq C\} \quad \textit{is compact, for all} \quad C > 0.$$

Then the free boundary problem for $\Delta + V$ *has a discrete spectrum of eigenvalues with finite multiplicities.*

Remark. (1) If M is compact, then $\Delta + V$ has a discrete spectrum for any smooth function V. (2) If Ω has a piecewise smooth boundary, then it satisfies the segment property.

Outline of Proof. To prove (1), we set $\mathcal{M}(\Omega)$ to be the set of all real valued measurable functions on Ω, and

$$L^2(\Omega) = \{u \in \mathcal{M}(\Omega) \ ; \ \int_\Omega |u(x)|^2 \, v_g \ < \infty\}.$$

The inner product $(\ ,\ )$ on $L^2(\Omega)$ is given by $(u, v) = \int_\Omega u(x)\, v(x)\, v_g$, $u, v \in L^2(\Omega)$. We also set the Sobolev space

$$H^1(\Omega) = \left\{u \in L^2(\Omega) \ ; \ |du| \in L^2(\Omega)\right\},$$

and define the inner product $(\ ,\ )_1$, and the norm $||\ \ ||_1$ by

$$(u, v)_1 = \int_\Omega u\, v\, v_g + \int_\Omega \langle du, dv \rangle v_g, \quad u, v \in H^1(\Omega),$$

$$||u||_1 = \sqrt{(u, v)_1}\,.$$

Let $\overset{\circ}{H}{}^1(\Omega)$ be the closure of $C_o^\infty(\Omega)$ in $H^1(\Omega)$, i.e.,

$$\overset{\circ}{H}{}^1(\Omega) = \left\{u \in H^1(\Omega) \ ; \ \exists u_n \in C_o^\infty(\Omega), ||u_n - u||_1 \longrightarrow 0\,(n \longrightarrow \infty)\right\}.$$

Lemma 1.9 (Green). *For* $u, v \in C^\infty(\bar{\Omega})$,

$$\int_\Omega u\, \Delta v\, v_g - \int_\Omega \langle du, dv \rangle v_g = \int_{\partial\Omega} u \frac{\partial v}{\partial \nu}\, d\sigma\,,$$

where $\dfrac{\partial v}{\partial \nu}$ *is the derivative of* v *with respect to the inward unit normal at* $\partial\Omega$*, and* $d\sigma$ *is the area element of* $\partial\Omega$*. In particular, we get*

$$\int_{\Omega} \{(\Delta u)\, v - u\,(\Delta v)\}\ v_g = \int_{\partial\Omega} \left\{ u\,\frac{\partial v}{\partial \nu} - \frac{\partial u}{\partial \nu}\, v \right\} d\sigma.$$

(*Dirichlet Problem*). By Green's theorem (cf. Lemma 1.9), the operator $\Delta : C_o^\infty(\Omega) \longrightarrow C_o^\infty(\Omega)$ is symmetric. If we define

$$D(\Delta_D) = \overset{\circ}{H}{}^1(\Omega) \cap \left\{ u \in L^2(\Omega)\ ;\ \Delta u \in L^2(\Omega) \right\},$$

then Δ can be extended to a selfadjoint operator

$$\Delta_D : D(\Delta_D) \longrightarrow L^2(\Omega),$$

and $(\Delta_D\, u, u) \geq 0$, $\forall u \in D(\Delta_D)$. Each element in $\overset{\circ}{H}{}^1(\Omega)$ can be regarded as the one in $H^1(M)$ by defining to be zero outside Ω.

Lemma 1.10 (Rellich). *If* S *is a bounded subset of* $H^1(M)$*, then* $\{ u\,|_\Omega\ ;\ u \in S\}$ *is relatively compact in* $L^2(\Omega)$*.*

Lemma 1.11. *If* S *is a bounded subset of* $L^2(\Omega)$*, then* $(\Delta_D + I)^{-1}(S) \subset \overset{\circ}{H}{}^1(\Omega)$ *is bounded.*

In fact, if $u = (\Delta_D + I)^{-1} f, f \in S$, then $u \in D(\Delta_D)$ and

$$\| \, u \, \|_1^2 = (\Delta u, u) + (u, u) = (f, u) \leq \| \, f \, \| \, \| \, u \, \| \leq \| \, f \, \| \, \| \, u \, \|_1 \,.$$

We get $\| \, u \, \|_1 \leq \| \, f \, \|$.

Therefore $(\Delta_D + I)^{-1} : L^2(\Omega) \longrightarrow D(\Delta_D) \subset L^2(\Omega)$ is a compact operator. In fact, if $S \subset L^2(\Omega)$ is bounded, then $(\Delta_D + I)^{-1}(S) \subset \overset{\circ}{H}{}^1(\Omega)$ is also bounded by Lemma 1.11. Then it is relatively compact in $L^2(\Omega)$ by Lemma 1.10.

Hence $\mathrm{Spect}(\Delta_D)$ is a discerete set of eigenvalues with finite multiplicities.

(*Neumann Problem*). For $u, v \in C^\infty(\bar{\Omega})$, satisfying $\dfrac{\partial u}{\partial \nu} = 0$, $\dfrac{\partial v}{\partial \nu} = 0$ on $\partial\Omega$, we get by Green's theorem (cf. Lemma 1.9),

$$(\Delta u, v) = (u, \Delta v), \qquad (\Delta u, u) = \int_{\Omega} \| \, \mathrm{grad}\ u \, \|^2 \ v_g \geq 0.$$

We now set

$$D(\Delta_N) = \{u \in L^2(\Omega) \ ; \ \Delta u \in L^2(\Omega) \text{ and } u \text{ satisfies } (N)\},$$

where $u \in H^1(\Omega)$ is said to satisfy the condition (N) if

$$(\Delta u, v) - (\text{grad } u, \text{grad } v) = 0 \quad \forall v \in H^1(\Omega).$$

Note that, due to Green's theorem, the left hand side coincides with $\int_{\partial\Omega} \frac{\partial u}{\partial \nu} v \, d\sigma$ for smooth functions u, v.

Then Δ can be extended to a selfadjoint operator

$$\Delta_N \quad : \quad D(\Delta_N) \longrightarrow L^2(\Omega)$$

which satisfies $(\Delta_N u, u) \geq 0, \quad \forall u \in D(\Delta_N)$, and $D(\Delta_N)$ is dense in $H^1(\Omega)$. Moreover, if Ω satisfies the *segment property*, then S is relatively compact in $L^2(\Omega)$ for all bounded $S \subset H^1(\Omega)$. Also, if $S \subset L^2(\Omega)$ is bounded, then $(\Delta_N + I)^{-1} \subset H^1(\Omega)$ is bounded. Therefore the operator

$$(\Delta_N + I)^{-1} \quad : \quad L^2(\Omega) \longrightarrow D(\Delta_N) \subset L^2(\Omega)$$

is a compact operator. Hence $\text{Spect}(\Delta_N)$ is a discrete set of eigenvalues with finite multiplicities.

(*Free Boundary Problem*). We assume that V is a function on M with the property that every $\{x \in M \ ; \ V(x) \leq C\}$ is compact. Then

$$\gamma = \min_{x \in M} V(x) < \infty.$$

So we put $H_o = \Delta + V : C_o^\infty(M) \longrightarrow C_o^\infty(M)$, which satisfies

$$(H_o u, u) \geq \gamma\,(u, u), \qquad (H_o u, v) = (u, H_o v), \quad u, v \in C_o^\infty(M).$$

Therefore H_o can be uniquely extended to a selfadjoint operator

$$H : \quad D(H) \longrightarrow L^2(M),$$

where

$$D(H) = H^1(M) \cap \{u \in L^2(M) \ ; \ \Delta u \in L^2(M)\}, \quad \text{and}$$
$$H^1(M) = \{u \in L^2(M) \ ; \ |du| \in L^2(M)\}.$$

Lemma 1.12. *For* $u, v \in D(H)$,

$$\sqrt{V + |\gamma|}\, u \in L^2(M),$$
$$\big((H + |\gamma|)\, u, v\big) = (\text{grad } u, \text{grad } v) + \big(\sqrt{V + |\gamma|}\, u, \sqrt{V + |\gamma|}\, v\big).$$

In particular,

$$\big((H+|\gamma|)\,u,u\big) = \|\ \mathrm{grad}\ u\ \|^2 + \|\ \sqrt{V+|\gamma|}\,u\ \|^2\ .$$

Proof. If $u \in D(H)$, there exists $u_n \in C_o^\infty(M)$ such that $u_n \longrightarrow u$ and $(\Delta+V)\,u_n \longrightarrow Hu$ as $n \longrightarrow \infty$. Since $u_n \in C_o^\infty(M)$,

$$\big((\Delta+V+|\gamma|)\,u_n\,,u_n\big) = \|\ \mathrm{grad}\ u_n\ \|^2 + \|\ \sqrt{V+|\gamma|}\,u_n\ \|^2,$$

where the left hand side converges to $\big((H+|\gamma|)\,u,u\big)$. Therefore $\{u_n\}_{n=1}^\infty$ is a Cauchy sequence in $H^1(M)$, and $u_n \longrightarrow u$ in $H^1(M)$. This u satisfies

$$\sqrt{V+\gamma} \in L^2(M), \quad \text{and}$$
$$\big((H+|\gamma|\,u),u\big) = \|\ \mathrm{grad}\ u\ \|^2 + \|\ \sqrt{V+|\gamma|}\,u\ \|^2\ .$$

The rest of the statement can be proved in a similar way. □

Lemma 1.13. *Assume that S is a subset of $L^2(M)$ which satisfies $\|\ f\ \| \leq C$ for all $f \in S$. Then, for all $u \in (H+|\gamma|+1)^{-1}(S)$,*

$$\|\ u\ \|_1 \leq C \qquad \textit{and} \qquad \|\ \sqrt{V+|\gamma|}\,u\ \| \leq C.$$

Proof. Since $|\gamma|+1 \in \mathrm{Resolv}(H)$, $\mathrm{Range}(H+|\gamma|+1) = L^2(M)$, and hence $u = (H+|\gamma|+1)^{-1}\,f \in D(H)$. Then $u \in H^1(M)$ and

$$\big((H+|\gamma|)\,u,u\big) = \|\ \mathrm{grad}\ u\ \|^2 + \|\ \sqrt{V+|\gamma|}\,u\ \|^2\ .$$

Then we get

$$\begin{aligned}\|\ u\ \|_1^2 &= \|\ \mathrm{grad}\ u\ \|^2 + \|\ u\ \|^2\\ &= \big((H+|\gamma|)\,u,u\big) - \|\ \sqrt{V+|\gamma|}\ \|^2 + \|\ u\ \|^2\\ &\leq \big((H+|\gamma|)\,u,u\big) + \|\ u\ \|^2\\ &= (f,u) \leq \|\ f\ \|\ \|\ u\ \|_1,\end{aligned}$$

we get $\|\ u\ \|_1 \leq \|\ f\ \| \leq C$. We get also the second inequality in a similar way. □

Lemma 1.14. (1) *We put $u_n = (H+|\gamma|+1)^{-1}\,f_n$ for any sequence $\{f_n\}_{n=1}^\infty$ in S. Then there exists a subsequence $\{u_k\}$ such that for every*

relatively compact domain $\Omega \subset M$, $\{u_k|_\Omega\}$ *is convergent in* $L^2(\Omega)$.
(2) *The sequence* $\{u_k\}$ *is convergent strongly in* $L^2(M)$.

Proof. (1) Take a sequence $0 < R_1 < R_2 < \cdots < R_j \longrightarrow \infty$, and a point $x_o \in M$. Put

$$K_j = \{x \in M \; ; \; d(x, x_o) \geq R_j\}.$$

By Rellich's theorem (cf. Lemma 1.10), there exist a subsequence $\{u_{1,k}\}$ of $\{u_n\}$ which is convergent strongly in $L^2(K_1)$, and a subsequence $\{u_{2,k}\}$ of $\{u_{1,k}\}$ which is convergent strongly in $L^2(K_2), \ldots$, and inductively subsequences $\{u_{j,k}\}$ which is strongly convergent in $L^2(K_j)$ for each j. Then, putting $u_k = u_{k,k}$, we get the desired subsequence $\{u_k\}$. (2) In the case M is compact, taking $\Omega = M$, (1) implies (2). When M is noncompact, due to the assumption of V, for all $N > 0$, there exists $R(N) > 0$ such that

$$d(x, x_o) \geq R(N) \quad \Longrightarrow \quad V(x) \geq N.$$

Then we get

$$\begin{aligned} \int_{d(x,x_o) \geq R(N)} |u_k|^2 v_g &= \int_{d(x,x_o)\geq R(N)} |\sqrt{V + |\gamma|}\, u_k|^2 \, (V + |\gamma|)^{-1} \, v_g \\ &\leq (N + |\gamma|)^{-1} \int_{d(x,x_o)\geq R(N)} |\sqrt{V + |\gamma|}\, u_k|^2 v_g \\ &\leq C/N, \end{aligned}$$

by Lemma 1.13. Then, for every $\epsilon > 0$, there exist $N > 0$ and $R(N) > 0$ such that for all k,

$$\int_{d(x,x_o)\geq R(N)} |u_k|^2 \, v_g \leq \epsilon/3.$$

By (1) in Lemma 1.14, there exists $k_o = k_o(\epsilon) > 0$ such that

$$\|u_\ell - u_k\|^2_{B_{R(N)}} \leq \epsilon/3, \qquad \forall k, \ell \geq k_o(\epsilon),$$

where $B_{R(N)} = \{x \in M \; ; \; d(x, x_o) \leq R(N)\}$. Therefore for all $k, \ell \geq k_o(\epsilon)$,

$$\begin{aligned} \|u_\ell - u_k\|^2 &= \|u_\ell - u_k\|^2_{B_{R(N)}} + \int_{d(x,x_o)\geq R(N)} |u_\ell - u_k|^2 \, v_g \\ &\leq \epsilon/3 + \int_{d(x,x_o)\geq R(N)} |u_\ell|^2 \, v_g + \int_{d(x,x_o)\geq R(N)} |u_k|^2 \, v_g \\ &\leq \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon, \end{aligned}$$

whence $\{u_k\}$ is a Cauchy sequence. $\square$

Therefore the operator $(H + |\gamma| + 1)^{-1} : L^2(M) \longrightarrow L^2(M)$ is compact. Thus the spectrum of H is discrete, i.e., $\mathrm{Spect}(\Delta + V)$ consists of only eigenvalues with finite multiplicities.

Example 1.15. (*Harmonic Oscillator*) (1) On the standard line $(\mathbb{R}, g_o)$, the eigenvalue problem

$$-\frac{d^2}{dx^2}\, u + x^2\, u = \lambda\, u, \quad u \in L^2(\mathbb{R}) \cap C^\infty(\mathbb{R}),$$

has the following spectrum, for $m = 0, 1, 2, \cdots$,

$$\begin{cases} \text{eigenvalue} & : \quad \lambda_m = 2m + 1, (\text{multiplicity } 1), \\ \text{eigenfunction} & : \quad \varphi_m(x) = C_m\, H_m(x) \exp(-\dfrac{x^2}{2}), \end{cases}$$

where $H_m(x)$ is the Hermite polynomial and $C_m = \sqrt{\pi}\, 2^m\, m!$.

(2) On the standard Euclidean space $(\mathbb{R}^n, g_o)$, the eigenvalue problem:

$$(\Delta + |x|^2)\, u = \lambda\, u, \quad u \in L^2(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n),$$

has also the following spectrum: for $\mathbf{m} = (m_1, \cdots, m_n) \in \mathbb{Z}_+ \times \cdots \times \mathbb{Z}_+$, where $\mathbb{Z}_+ = \{0, 1, 2, \cdots\}$,

$$\begin{cases} \text{eigenvalue} & : \quad \lambda_{\mathbf{m}} = 2(m_1 + \cdots + m_n) + n, \\ \text{eigenfunction} & : \quad \varphi_{\mathbf{m}}(x) = C_{\mathbf{m}}\, H_{m_1}(x_1) \cdots H_{m_n}(x_n) \exp(-\dfrac{|x|^2}{2}), \end{cases}$$

for $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$, where $C_{\mathbf{m}} = C_{m_1} \cdots C_{m_n}$.

§2. Asymptotic distribution of discrete spectrum

2.1. Mini-Max Principle

In this section, we consider the following three eigenvalue problems: the Dirichlet problem, the Neumann problem, and the free boundary problem, which have the discrete spectra of the eigenvalues with finite multiplicities as in section 1.2.

Definition 2.1. In each eigenvalue problem, we count the eigenvalues with their multiplicities:

$$\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots \longrightarrow \infty.$$

Remark 2.2. In case of a compact manifold M, the eigenvalue $\lambda_1 = 0$ corresponds to the constant functions.

We will characterize the k-th eigenvalue λ_k of each problem by Mini-Max Principle:

Let A be a selfadjoint operator defined on a dense subspace $D(A)$ of the Hilbert space $L^2(A)$, namely, in each eigenvalue problem, we take A, $D(A)$, $L^2(A)$ as follows:

Case (1-i) (*Dirichlet Eigenvalue Problem*)

$$A = \Delta_D,$$
$$D(A) = D(\Delta_D) = \overset{\circ}{H}{}^1(\Omega) \cap \left\{u \in L^2(\Omega); \Delta u \in L^2(\Omega)\right\},$$

and

$$L^2(A) = L^2(\Omega).$$

Case (1-ii) (*Neumann Eigenvlaue Problem*)

$$A = \Delta_N,$$
$$D(A) = D(\Delta_N) = \left\{u \in L^2(\Omega); \Delta u \in L^2(\Omega) \text{ and } u \text{ staitfies } (N)\right\},$$

and

$$L^2(A) = L^2(\Omega).$$

Case (2) (*Free Boundary Problem*)

$$A = H, \text{ the selfadjoint extension of } H_o = \Delta + V,$$
$$D(A) = D(H) = H^1(M) \cap \left\{u \in L^2(M); \Delta u \in L^2(M)\right\},$$

and

$$L^2(A) = L^2(M).$$

We define the Rayleigh-Ritz quotient as follows: for $0 \neq f \in D(A)$,

$$R(f) = \int_\Omega |df|^2\, v_g \Big/ \int_\Omega f^2\, v_g, \text{ or } \quad \int_M |df|^2\, v_g \Big/ \int_M f^2\, v_g,$$

where $\int_\Omega f^2\, v_g \neq 0$ for Dirichlet and Neumann boundary problems or $\int_M f^2\, v_g \neq 0$ for free boundary problem, respectively. Then the k-th eigenvalue λ_k is obtained by the following Mini-Max Principle:

Theorem 2.2. *The k-th eigenvalue of each eigenvalue problems is given by*

$$\lambda_k = \sup \Lambda(L_{k-1}),$$

where L_{k-1} runs through all $(k-1)$-dimensional subspaces of $D(A)$, and $\Lambda(L_{k-1})$ is defined by

$$\Lambda(L_k) = \inf \{ R(f)\,;\, D(A) \ni f \neq 0, f \text{ orthogonal to } L_{k-1} \}.$$

Here the orthogonality means that with respect to the inner product

$$(f_1, f_2) = \int_\Omega f_1(x) f_2(x)\, v_g \quad or \quad \int_M f_1(x) f_2(x)\, v_g.$$

Theorem 2.3. *The k-th eigenvalue of each eigenvalue problem is given also by*

$$\lambda_k = \inf \tilde{\Lambda}(L_k),$$

where L_k runs through all k-dimensional subspaces of $D(A)$, and $\tilde{\Lambda}(L_k)$ is

$$\tilde{\Lambda}(L_k) = \sup \{ R(f)\,;\, D(A) \ni f \neq 0 \}.$$

For proofs and applications, see Bérard ['86] or Bando & Urakawa ['83].

2.2. Asymptotic distributions (I)

See also Protter ['87] for a survey of this topic.

Theorem 2.4 (Minakshisundaram-Pleijel's expansion). *Let Ω be a relatively compact domain in a complete Riemannian manifold (M, g). We assume the segment property of Ω for the Neumann problem. Then the zeta function*

$$Z(t) = \sum_{i=1}^{\infty} e^{-\lambda_i t}, \; t > 0$$

has the following asymptotic expansion:
(1) (*Boundary value problems*)

$$Z(t) \sim (4\pi t)^{-\frac{n}{2}} \left\{ a_0 + a_{\frac{1}{2}}\, t^{\frac{1}{2}} + a_1\, t + \cdots \right\} \quad as \quad t \longrightarrow 0,$$

where the coefficients a_i are given by:

$$a_0 = \mathrm{Vol}(\Omega), a_{\frac{1}{2}} = \mp 4^{-1}\sqrt{4\pi}\ \mathrm{Vol}_{n-1}(\partial\Omega),$$

$$a_1 = 6^{-1}\left\{\int_\Omega \{\kappa_g + 6V\}\, v_g \ - \ 2\int_{\partial\Omega} J\, d\sigma\right\}, etc\cdots.$$

Here κ_g is the scalar curvature of (M, g), J the mean curvature of $\partial\Omega$ in (M, g), $d\sigma$ the $(n-1)$-dimensional area element of $\partial\Omega$ and the sign $-$ (resp. $+$) above corresponds to the Dirichlet (resp. Neumann) problem.
(2) (*Free boundary problem*) *If M is compact, then, for $\Delta + V$,*

$$Z(t) \ \sim \ (4\pi t)^{-\frac{n}{2}}\, \{a_0 + a_1\, t\, + \cdots\} \ \ as \ \ t \longrightarrow 0,$$

where the coefficients a_i are given as:

$$a_0 = \mathrm{Vol}(M, g),\ a_1 = 3^{-1}\int_M \{\kappa_g + 3V\}\, v_g,\ \ etc.\cdots.$$

For proofs, see McKean & Singer ['67], Branson & Gilkey ['90] for the boundary value problems of a relatively compact domain of a complete Riemannian manifold (M, g), and see also Minakshisundaram & Pleijel ['49], Berger ['68], Sakai ['71], Gilkey ['75-1], ['75-2] for the free boundary problem of $\Delta + V$ on a compact Riemannian manifold (M, g).

Corollary 2.5 (Weyl's formula). *Let Ω be a relatively compact domain in a complete Riemannian manifold (M, g). We assume the segment property of Ω for the Neumann problem. Let*

$$N(\lambda) = \sharp\{k;\ \lambda_k\ \leq\ \lambda\}$$

be the (counting) number of the eigenvalues less than or equal to a positive real number λ. For the boundary problems or the free boundary problem of a compact Riemannian manifold (M, g), the asymptotic behavior of $N(\lambda)$ is given by:

$$N(\lambda) \sim \begin{cases} C_n\ \mathrm{Vol}(\Omega)\, \lambda^{\frac{n}{2}}, & (\textit{boundary value problems}), \\ C_n\ \mathrm{Vol}(M, g)\, \lambda^{\frac{n}{2}}, & (\textit{free boundary probelm}),\ as\ \lambda \longrightarrow \infty, \end{cases}$$

where

$$C_n = (2\sqrt{\pi})^{-n}\, \Gamma(\tfrac{n}{2} + 1)^{-1} = (2\pi)^{-n}\ \mathrm{Vol}(B_1),$$

B_1 being the unit ball in $\mathbb{R}^n$.

Remark 2.6. Moreover, the following best possible estimates of the remainder term of $N(\lambda)$ hold for the Dirichlet problem (1-i) of any smooth bounded domain Ω in the standard Euclidean space $(\mathbb{R}^n, g_0)$, and for the free boundary problem (2) of $\Delta + V$ of a compact Riemannian manifold (M, g):

$$\text{(1-i)} \qquad N(\lambda) = (2\pi)^{-n}\ \mathrm{Vol}(B_1)\ \mathrm{Vol}(\Omega)\, \lambda^{\frac{n}{2}} + O(\lambda^{\frac{n-1}{2}}),$$

$$\text{(2)} \qquad N(\lambda) = (2\pi)^{-n}\ \mathrm{Vol}(B_1)\ \mathrm{Vol}(M)\, \lambda^{\frac{n}{2}} + O(\lambda^{\frac{n-1}{2}}).$$

For proofs, see Seeley ['78], Pham The Lai ['81] for (1-i), and Avakumovic ['56], Hörmander ['68] for (2).

For more precise asymptotic behavior of $N(\lambda)$ of the boundary problems, we have:

Theorem 2.7 (Weyl's conjectrure). *Let (M, g) be a complete Riemannian manifold, and Ω a relatively compact domain in M with smooth boundary $\partial\Omega$. We assume the following geodesic concave condition for the boundary $\partial\Omega$: the set of periodic points of the geodesic billiard, i.e., the union of the geodesic segments of (M, g) lying on the inside of Ω and reflecting 'normally' at the boundary $\partial\Omega$, has measure zero. Then the asymptotic behaviors of $N(\lambda)$ are given by:*

$$N(\lambda) = C_n\ \mathrm{Vol}(\Omega)\, \lambda^{\frac{n}{2}} \mp \frac{1}{4} C_{n-1}\ \mathrm{Vol}_{n-1}(\partial\Omega)\, \lambda^{\frac{n-1}{2}} + o(\lambda^{\frac{n-1}{2}}), \quad \text{as } \lambda \longrightarrow \infty,$$

where the sign $-$ (resp. $+$) of the right hand side corresponds to the Dirichlet (resp. Neumann) problem, and the constants C_n, C_{n-1} are those given in Corollary 2.5.

For proofs, see Ivrii ['80], Melrose ['80].

Remark 2.8. Bérard ['83] gave examples of domains Ω in S^2, for which $N(\lambda)$ has no asymptotic behavior such as

$$N(\lambda) = C\ \mathrm{Vol}(\Omega)\, \lambda^{\frac{n}{2}} \mp C'\ \mathrm{Vol}(\partial\Omega)\, \lambda^{\frac{n-1}{2}} + o(\lambda^{\frac{n-1}{2}}), \quad \text{as } \lambda \longrightarrow \infty,$$

for some constants C, C'.

(*Polya's conjecture*). Due to Corollary 2.5, the asymptotic behavior of k-th eigenvalues of the eigenvalue problems for a domain Ω satisfies

$$\lambda_k \sim \left((2\pi)^{-n}\,\mathrm{Vol}(B_1)\,\mathrm{Vol}(\Omega)\right)^{-\frac{2}{n}} k^{\frac{2}{n}} \quad \text{as } k \longrightarrow \infty.$$

Furthemore, Polya ['61] and Kellner ['66] conjectured the following inequalities: Let λ_k^D, (resp. λ_k^N) be the k-th eigenvalue of the Dirichlet, (resp. Neumann) boundary problems for a bounded domain Ω in $\mathbb{R}^n$. Then

$$\lambda_k^N \le \left((2\pi)^{-n}\,\mathrm{Vol}(B_1)\,\mathrm{Vol}(\Omega)\right)^{-\frac{2}{n}} k^{\frac{2}{n}} \le \lambda_k^D, \quad \text{for all } k = 1, 2, \cdots.$$

They showed these inequalities for a tiling bounded domain Ω, i.e., an infinite number of non-overlapping domains which are congruent to Ω, cover $\mathbb{R}^n$ except a measure zero set.

Li & Yau ['83] showed:

$$\frac{n}{n+2}\left((2\pi)^{-n}\,\mathrm{Vol}(B_1)\,\mathrm{Vol}(\Omega)\right)^{-\frac{2}{n}} k^{\frac{2}{n}} \le \lambda_k^D, \text{ for all } k = 1, 2, \cdots,$$

and Urakawa ['84] showed:

$$\delta(\Omega)^{\frac{2}{n}}\left((2\pi)^{-n}\,\mathrm{Vol}(B_1)\,\mathrm{Vol}(\Omega)\right)^{-\frac{2}{n}} k^{\frac{2}{n}} \le \lambda_k^D, \text{ for all } k = 1, 2, \cdots.$$

Here the constant $\delta(\Omega)$ is the packing density of Ω, and $\delta(\Omega) = 1$ if Ω is a tiling domain.

2.3. Asymptotic distribution (II)

In this section, we are concerned with the free boundary problem of $\Delta + V$ on a noncompact complete Riemannian manifold with V satisfying the exhausion condition in Theorem 1.8.

We conjecture that, if a noncompact complete Riemannian manifold (M, g) has non-negative Ricci curvature Ric_M, and a function V on M satisfies the exhausion condition:

$$\{x \in M; V(x) \le C\} \quad \text{is compact in } M, \text{ for all } C > 0,$$

then the counting number $N(\lambda) = \sharp\{\lambda_n; \lambda_n \le \lambda\}$ would be asymptotically

$$N(\lambda) \sim C_n \int_M (\lambda - V(x))_+{}^{\frac{n}{2}}\, v_g, \quad \text{as} \quad \lambda \longrightarrow \infty,$$

where

$$C_n = (2\sqrt{\pi})^{-n}\,\Gamma(\frac{n}{2}+1)^{-1} = (2\pi)^{-n}\,\mathrm{Vol}(B_1),$$
$$f(x)_+ = \max(f(x), 0).$$

In fact, it is believed to hold that:

"Theorem" 2.9. *Let V be a continuous function on $\mathbb{R}^n$ which satisfies the above exhausion condition and $V(x) \geq 1, \quad \forall x \in M$. Let $N(\lambda)$ be the counting function of the free boundary problem for $\Delta + V$ on $L^2(\mathbb{R}^n)$. Then*

$$N(\lambda) \sim C_n \int_{\mathbb{R}^n} (\lambda - V(x))_+{}^{\frac{n}{2}}\, dx$$
$$= (2\pi)^{-n} \operatorname{Vol}\left(\{(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n\,;\, |\xi|^2 + V(x) < \lambda\}\right).$$

We follow the argument in Rosenbljum ['74] and show the precise statement of his theorem. See also Fefferman ['83], Tachizawa ['90] for these topics.

Let $Q = Q_d$ be a cube of edge d in $\mathbb{R}^n$. For each positive number a, let us consider the Dirichlet, Neumann eigenvalue problems for $\Delta + a$ on Q, and let us denote the counting numbers of the eigenvalue problems by $N_D(\lambda, a, Q)$, $N_N(\lambda, a, Q)$, respectively. Then due to Mini-Max Principle (cf. Theorems 2.2, 2.3), we get:

Lemma 2.10. *For all $0 < \epsilon < 1$, there exist positive constants $C_1(\epsilon)$ and $C_2(\epsilon)$ such that*

$$\text{(2.11)} \quad N_D(\lambda, a, Q) \geq (1-\epsilon)^{\frac{n}{2}}\, C_n \operatorname{Vol}(Q)\, \left((\lambda - a) - C_1(\epsilon)\, d^{-2}\right)_+{}^{\frac{n}{2}},$$

$$\text{(2.12)} \quad N_N(\lambda, a, Q) \leq (1+\epsilon)^{\frac{n}{2}}\, C_n \operatorname{Vol}(Q)\, \left((\lambda - a) + C_2(\epsilon)\, d^{-2}\right)_+{}^{\frac{n}{2}},$$

for all $\lambda > 0$.

Let Ξ be an arbitrary lattice of $\mathbb{R}^n$ defined by cubes of edge 1, and assume that V satisfies the following conditions:

(I) There exist a decreasing function ν on the interval $[1, \infty)$ satisfying $\nu(t) \longrightarrow 0$ as $t \longrightarrow \infty$, and $0 \leq \alpha \leq \dfrac{1}{2}$ such that

$$|V(x) - V(y)| \leq |x-y|^{2\alpha}\, V(x)^{1+\alpha}\, \nu(V(x)), \quad \text{for all} \quad x, y \in Q' \backslash \partial Q',$$

for any cube Q' of the lattice Ξ.

(II) Letting $\sigma(\lambda, V) = \operatorname{Vol}(\{x \in \mathbb{R}^n; V(x) < \lambda\})$, there exists a constant $C_3 > 0$ such that

$$\sigma(2\lambda, V) \leq C_3\, \sigma(\lambda, V), \quad \text{for large} \quad \lambda \gg 1.$$

We say $V \in W_\alpha(\Xi)$ if V satisfies the conditions (I), (II).

Theorem 2.13. *Assume that $V \in W_\alpha(\Xi)$ for some Ξ and $0 < \alpha < 1$. Then we get:*

$$N(\lambda) \sim \Phi(\lambda, V),$$
$$\Phi(\lambda, V) = C_n \int_{\mathbb{R}^n} (\lambda - V(x))_+{}^{\frac{n}{2}}\, dx.$$

Proof. For arbitrarily fixed $\epsilon, \epsilon_1 > 0$, we choose $\epsilon_2 > 0$ in such a way that

$$C_1(\epsilon_1)\,\epsilon_2 < \epsilon, \quad C_2(\epsilon_1)\,\epsilon_2 < \epsilon \text{ and } \quad \lambda\,\epsilon_2 > 1.$$

We also choose a positive integer k such that

$$\epsilon_2\,\lambda \leq 4\,k^2 < 4\,\epsilon_2\,\lambda.$$

We divide all unit cubes of the lattice Ξ into cubes Q of edge $d = \frac{1}{k}$. Then by Mini-Max Principle (Theorems 2.2, 2.3), we get:

$$\sum_Q N_D(\lambda, V_Q^+, Q) \leq N(\lambda) \leq \sum_Q N_N(\lambda, V_Q^-, Q),$$

where Q in the sums run through all the above unit cubes of Ξ, and

$$V_Q^+ = \operatorname*{ess\,sup}_{x \in Q} V(x), \quad V_Q^- = \operatorname*{ess\,inf}_{x \in Q} V(x).$$

By Lemma 2.10, the right hand side of the above inequality is smaller than or equal to

$$(1+\epsilon_1)^{\frac{n}{2}}\, C_n \sum_Q \mathrm{Vol}(Q)\, \left(\lambda - V_Q^- + C_2(\epsilon_1) d^{-2}\right)_+^{\frac{n}{2}}$$
$$\leq (1+\epsilon_1)^{\frac{n}{2}}\, C_n \sum_Q \mathrm{Vol}(Q)\, \left(\lambda(1+\epsilon) - V_Q^-\right)_+^{\frac{n}{2}},$$

since $C_2(\epsilon_1)\,d^{-2} = C_2(\epsilon_1)\,k^2 < C_2(\epsilon_1)\,\epsilon_2\,\lambda < \epsilon\,\lambda$, by the choice of d, and ϵ_2. Here the cubes Q in the above sums run through, indeed, a finite number of the ones satisfying $\lambda(1+\epsilon) > V_Q^-$, divided from the unit cubes of the lattice Ξ. For $t > 0$, we denote by $\sum_{\mathrm{I}}$, the sum running over the Q's satisfying $V_Q^- < t$, and by $\sum_{\mathrm{II}}$, the sum running over the Q's satisfying $t \leq V_Q^- < \lambda(1+\epsilon)$.

Then we get

$$\sum_{\mathrm{I}} \mathrm{Vol}(Q) \left(\lambda(1+\epsilon) - V_Q^-\right)_+^{\frac{n}{2}} \leq (\lambda(1+\epsilon))^{\frac{n}{2}} \, \sigma(t,V),$$

$$\begin{aligned}
&\sum_{\mathrm{II}} \mathrm{Vol}(Q) \left(\lambda(1+\epsilon) - V_Q^-\right)_+^{\frac{n}{2}} \\
&\quad \leq \sum_{\mathrm{II}} \int_Q \left(\lambda(1+\epsilon) - V(x) + |V(x) - V_Q^-|\right)_+^{\frac{n}{2}} dx \\
&\quad \leq \sum_{\mathrm{II}} \int_Q \left(\lambda(1+\epsilon) - V(x) + d^{2\alpha}\, (V_Q^-)^{1+\alpha}\, \nu(t)\right)_+^{\frac{n}{2}} dx,
\end{aligned}$$

since $V_Q^- \geq V(x) - |V(x) - V_Q^-|$ in the second inequality, and ν is decreasing in $t \leq V_Q^-$ in the last inequality.

Here we take a large t in such a way

$$2^{2\alpha}\, {\epsilon_2}^{-\alpha}\, (1+\epsilon)^{1+\epsilon}\, \nu(t) \leq \epsilon.$$

Then

$$\begin{aligned}
d^{2\alpha}\, (V_Q^-)^{1+\alpha}\, \nu(t) &< d^{2\alpha}\, (\lambda(1+\epsilon))^{1+\alpha}\, \nu(t) \\
&= k^{-2\alpha}\, \lambda^{1+\alpha}\, (1+\epsilon)^{1+\alpha}\, \nu(t) \\
&\leq \left(2^2\, {\epsilon_2}^{-1}\, \lambda^{-1}\right)^{\alpha}\, \lambda^{1+\alpha}\, (1+\epsilon)^{1+\alpha}\, \nu(t) < \epsilon\, \lambda.
\end{aligned}$$

Therefore the right hand side of the last inequality is smaller than or equal to

$$\sum_{\mathrm{II}} \int_Q (\lambda(1+2\epsilon) - V(x))_+^{\frac{n}{2}}\, dx \leq \int_{\mathbb{R}^n} (\lambda(1+2\epsilon) - V(x))_+^{\frac{n}{2}}\, dx.$$

Hence we have

$$\begin{aligned}
N(\lambda) &\leq (1+\epsilon_1)^{\frac{n}{2}} \\
&\quad \times C_n \left\{ \int_{\mathbb{R}^n} (\lambda(1+2\epsilon) - V(x))_+^{\frac{n}{2}}\, dx + \lambda^{\frac{n}{2}}\, (1+\epsilon)^{\frac{n}{2}}\, \sigma(t,V) \right\}.
\end{aligned}$$

Thus we obtain

$$\begin{aligned}
&\limsup_{\lambda \longrightarrow \infty} \frac{N(\lambda)}{C_n\, \Phi(\lambda, V)} \\
&\quad \leq \limsup_{\lambda \longrightarrow \infty} \frac{(1+\epsilon_1)^{\frac{n}{2}} \int_{\mathbb{R}^n} (\lambda(1+2\epsilon) - V(x))_+^{\frac{n}{2}}\, dx}{\Phi(\lambda, V)},
\end{aligned}$$

since $\Phi(\lambda, V) = o(\lambda^{\frac{n}{2}})$ by definition of $\Phi(\lambda, V)$. Thus, letting $\epsilon \longrightarrow 0$ and then $\epsilon_1 \longrightarrow 0$, we obtain

$$\limsup_{\lambda \longrightarrow \infty} \frac{N(\lambda)}{C_n \, \Phi(\lambda, V)} \leq 1.$$

In a similar manner, we also have

$$\liminf_{\lambda \longrightarrow \infty} \frac{N(\lambda)}{C_n \, \Phi(\lambda, V)} \geq 1.$$

In consequence, we obtain Theorem 2.13. □

§3. The bottom of the (essential-)spectrum

3.1. Definitions of analytic and geometric quantities

In this section, we discuss the bottom of the spectrum of the Laplacian Δ of a *noncompact* complete Riemannian manifold (M, g). Namely, the following problems are considered:

(1) When are the Ess Spect(Δ), and the point-spectrum nonempty ?
(2) How to estimate the infima $\lambda_o(\Delta)$, $\lambda_o^{\text{ess}}(\Delta)$ of Spect(Δ), and Ess Spect(Δ) ?
(3) Compare such quantities to the other geometric ones.

Definition 3.1. For a noncompact complete Riemannian manifold (M, g), we define the following quantities:

(1) The bottom of the spectrum of the Laplacian Δ is

$$\begin{aligned} \lambda_1 &= \lambda_1(M, g) = \inf \ (\text{Spect}(\Delta)) \\ &= \inf \left\{ \frac{\|df\|^2}{\|f\|^2}; 0 \neq f \in C_o^\infty(M) \right\}. \end{aligned}$$

(2) The bottom of the essential spectrum of the Laplacian Δ is

$$\begin{aligned} \lambda_1^{\text{ess}} &= \lambda_1^{\text{ess}}(M, g) = \inf \ (\text{Ess Spect}(\Delta)) \\ &= \sup \{\lambda_1(M \backslash K) \,;\, K \subset M, \ \text{compact}\}, \end{aligned}$$

where $\lambda_1(M \setminus K)$ is the bottom of the spectrum for $M \setminus K$ in (1).

(3) The exponential growth of volume of (M, g) is

$$\mu = \mu(M, g) = \limsup_{r \to \infty} \frac{1}{r} \log V(r),$$

where $V(r) = \mathrm{Vol}(B(r))$ is the volume of the geodesic ball of radius r of some point $p \in M$. Note that the definition of μ does not depend on the choice of the point p.

(4) The Cheeger's constant of (M,g) is

$$h = h(M,g) = \inf\left\{\frac{\mathrm{Vol}_{n-1}(\partial D)}{\mathrm{Vol}(D)}\,;\, D \subset M,\ \text{compact subdomain}\right\}.$$

(5) The isoperimetric growth of (M,g) is

$$\bar{h} = \bar{h}(M,g) = \limsup_{r\to\infty} \frac{S(r)}{V(r)},$$

where $V(r) = \mathrm{Vol}(B(r)), S(r) = \mathrm{Vol}_{n-1}(\partial B(r))$.

3.2. (Essential-)spectrum

In this section, we show results on the existence of the essential spectrum of the Laplacian Δ. In the next section, we will show results which compare the above quantities.

Theorem 3.2 (Donnelly['81-1]). *Let (M,g) be an n-dimensional noncompact complete Riemannian manifold with the Ricci curvature* $\mathrm{Ric}_M \geq -(n-1)c$, $c \geq 0$. *Then the essential spectrum appears* i.e.,

$$\text{Ess Spect}(\Delta) \cap \left[0, \frac{(n-1)^2}{4}c\right] \neq \emptyset.$$

Moreover, we know:

Theorem 3.3 (Donnelly['81-1]). *Let (M,g) be an n-dimensional simply connected complete Riemannian manifold with nonpositive sectional curvature. Let*

$$\phi(r) = \sup\{\,|K(x,\Pi)+c|\,;\, d(x,p) \geq r, \Pi \subset T_xM,\ \textit{plane},\ x \in M\},$$

where $p \in M$ is an arbitrarily fixed point and $c \geq 0$ is a constant, and $K(x,\Pi)$ is the sectional curvature of a plane Π. Assume that

$$\lim_{r\to\infty} \phi(r) = 0.$$

Then we get

$$\text{Ess Spect}(\Delta) = [\frac{(n-1)^2}{4}c, \infty).$$

Let (M, g) be a simply connected n-dimensional complete Riemannian manifold with nonpositive curvature. Fix $p \in M$. Let $\gamma(\omega, r)$ be a geodesic emanating from p, parametrized with distance r from p, and with unit direction ω at p, and let $K(\omega, r, \theta)$ be the curvature of the plane obtained by parallel translation along $\gamma(\omega, r)$ of (ω, θ) plane at p. We denote by $\|F\|$, the supremum of $F(\omega, r, \theta)$ where (ω, θ) run through $S^{n-1} \times S^{n-1}$, and by D the covariant derivative of the standard unit sphere (S^{n-1}, can). Then we have:

Theorem 3.4 (Donnely ['81-2], see also Pinsky ['78], ['81]). *Let (M, g) be as above. Suppose that the sectional curvature of (M, g) satisfies the following decay condition along geodesics emanating from a fixed point p:*

$$\text{i)} \quad \int_0^\infty r\,\|K+1\|\,dr < d_1,$$

$$\text{ii)} \quad \int_0^\infty \|D_\omega K\|\,e^{2r}\,dr < d_2,$$

$$\text{iii)} \quad \int_0^\infty \|{D_\omega}^2\,K\|\,e^{2r}\,dr < d_3\,, \quad \textit{and}$$

$$\text{iv)} \quad \lim_{r\to\infty} r\,\|K+1\| = 0,$$

for some positive constants d_1, d_2, d_3. Then:

(1) *Δ has no eigenvalue in $\left(\dfrac{(n-1)^2}{4}, \infty\right)$.*

(2) *Moreover, if the sectional curvature of (M, g) is bounded above by -1, then*

$$\mathrm{Spect}(\Delta) = C\ \mathrm{Spect}(\Delta) = [\frac{(n-1)^2}{4}, \infty).$$

Examples 3.5. As particular cases, we consider homogeneous spaces. Let G be a semi-simple Lie group, K be a maximal compact subgroup, and $\mathfrak{g}, \mathfrak{k}$ their Lie algebras. Let B be the Killing form of $\mathfrak{g}$, and define the positive definite inner product $\langle\,,\,\rangle$ on $\mathfrak{g}$ defined by

$$\langle X, Y\rangle = -B(X, Y), \quad X, Y \in \mathfrak{k} \quad ; \quad \langle X, Y\rangle = B(X, Y), \quad X, Y \in \mathfrak{p},$$

where $\mathfrak{p}$ is the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to B. Let g_o be a G-invariant Riemannian metric on the symmetric space G/K corresponding to the inner product $\langle\,,\,\rangle$ on $\mathfrak{p}$, and g be the left invariant Riemannian metric on G corresponding to the inner product to $\langle\,,\,\rangle$ on $\mathfrak{g}$. Then:

(1) the spectrum of the Laplacian of the Euclidean space $(\mathbb{R}^n, \mathrm{can})$ is

$$\mathrm{Spect}(\Delta_{\mathrm{can}}) = C\ \mathrm{Spect}(\Delta_{\mathrm{can}}) = [0, \infty).$$

(2) The spectrum $\mathrm{Spect}(\Delta_{g_o})$ of the Laplacian of $(G/K, g_o)$ satisfies that

$$\mathrm{Spect}(\Delta_{g_o}) = C\ \mathrm{Spect}(\Delta_{g_o}) = [\,|\rho|^2, \infty),$$

where $|\rho|^2 = \langle \rho, \rho \rangle$ and ρ is half of the sum of all positive restricted root system of $(\mathfrak{g}, \mathfrak{k})$ (cf. Donnely ['79], Urakawa ['80]). And there is an interesting example, i.e.,

(3) if $G = SL(2, \mathbb{R})$, then the spectrum and the set of all eigenvalues of the Laplacian of (G, g) are given as follows (cf. Kobayashi, Ono & Sunada ['89]):

$$\mathrm{Spect}(\Delta_g) = [\frac{1}{8}, \infty), \qquad \text{and}$$

$$\begin{aligned} \text{the set of all eigenvalues of } \Delta_g &= \{\frac{1}{8}(n^2 + 4nm + 2m^2 + 1)\,; \\ &\qquad n = 1, 2, 3, \cdots, m = 1, 3, 5, \cdots\} \\ &= \left\{1, \frac{15}{8}, 3, 4, \cdots\right\}. \end{aligned}$$

On the other hand, in the following cases the essential spectrum does not appear:

Theorem 3.6 (cf. Donnelly & Li ['79]). *Let (M, g) be a noncompact complete Riemannian manifold. We denote by $K(x, \Pi)$, the sectional curvature of a plane Π in the tangent space $T_x(M), x \in M$, and fix $p \in M$, define*

$$\bar{K}(r) = \sup\{K(x, \Pi);\ d(x, p) \geq r, \Pi \subset T_x(M),\ \textit{plane},\ x \in M\}.$$

Assume that

$$\bar{K}(r) \longrightarrow -\infty, \qquad \textit{as } r \longrightarrow \infty.$$

(1) *If (M, g) is simply connected and has negative curvature, then*

$$\mathrm{Spect}(\Delta) = \mathrm{Spect}_o(\Delta), \ \textit{i.e.,}\ \mathrm{Ess\ Spect}(\Delta) = \emptyset.$$

(2) *If* $\dim(M) = 2$ *and the fundamental group $\pi_1(M)$ is finitely generated, then we have the same conclusion as* (1).

3.3. Estimates of the bottom of the spectrum

In this section, we show results comparing several quantities defined in section 3.1.

Theorem 3.7 (McKean ['70]). *Let (M, g) be a complete simply connected Riemannian manifold whose sectional curvature K satisfies $K \leq -k^2 < 0$. Then the bottom of the spectrum, λ_1, satisfies:*

$$\lambda_1(M, g) \geq \frac{k^2}{4}.$$

Remark 3.8. The sectional curvature condition of Theorem 3.7 can be relaxed to some Ricci curvature one by Setti ['91].

Theorem 3.9 (Pinsky ['81]). *Let (M, g) be a simply connected complete Riemannian manifold with nonpositive sectional curvature K. Fix $p \in M$. Let $\psi(r)$ denote*

$$\sup\{|K(\gamma(r), \Pi) + c| \ ; \ \gamma(r) \text{ a geodesic emanating } p \text{ with tangent } \omega \in T_p(M), \|\omega\| = 1, \Pi \subset T_{\gamma(r)}(M)\},$$

where $c > 0$ is a constant. Then we have:

(i) *If either $\int_1^\infty \psi(r)\, dr < \infty$, or $K \equiv -c < 0$ outside a compact subset, then* $0 < \lambda_1(M, g) \leq \dfrac{(n-1)^2}{4}\, c$.

(ii) *If $\int_1^\infty \psi(r)\, dr < \infty$, and $K \leq -c < 0$ everywhere M, then* $\lambda_1(M, g) = \dfrac{(n-1)^2}{4}\, c$.

Moreover we get:

Theorem 3.10 (Osserman ['79]). (i) *Let (M, g) be a complete n-dimensional Riemannian manifold with a pole p,* i.e., *the exponential mapping* $\exp : T_p(M) \longrightarrow M$ *is an onto diffeomorphism. Assume that there exist constants C and $r_o > 0$ such that*

$$S'' \leq c V'', \quad \forall r \geq r_o.$$

Then we get: $\lambda_o(M, g) \leq \dfrac{1}{4} c^2$.

(ii) *Assume that* $\dim(M) = 2$ *and (M, g) has nonpositive curvature K. If $-c \leq K \leq -d < 0$ for some positive constants c, d, then* $\lambda_1(M, g) \leq \dfrac{1}{4}\dfrac{c}{d}$.

Theorem 3.11 (Brooks ['81]). *Let (M, g) be a noncompact complete Riemannian manifold. Assume that* $\mathrm{Vol}(M, g) = \infty$. *Then we* *have:*

$$\frac{1}{4}h^2 \le \lambda_o^{\mathrm{ess}} \le \frac{1}{4}\mu^2.$$

Theorem 3.12 (Urakawa ['89]). (i) *Let (M,g) be a simply connected complete Riemannian manifold without focal point (not necessarily, nonpositive curvature). Then we have:*

$$\frac{1}{4}m^2 \le \frac{1}{4}h^2 \le \lambda_1 \le \frac{1}{4}\mu^2 \le \frac{1}{4}\bar{h}^2,$$

where m is the infimum of the mean curvature of $\partial B(r), 0 < r < \infty$, and $B(r)$ is the geodesic ball of radius r of some fixed point.

(ii) *In particular, let (M,g) be a simply connected Riemannian symmetric space G/K of noncompact type whose metric comes from the Killing form of the Lie algebra $\mathfrak{g}$ of G. Then we have:*

$$\lambda_1 = \frac{1}{4}\mu^2 = \frac{1}{4}\bar{h}^2 = \|\rho\|^2; \quad m = \inf\{2\rho(H); H \in \mathfrak{a}^+, \|H\| = 1\},$$

where $\mathfrak{a}^+$ is a positive restricted Weyl chamber. Moreover, if M is rank one, i.e., $\dim(\mathfrak{a}) = 1$, *then*

$$\frac{1}{4}m^2 = \frac{1}{4}h^2 = \lambda_1 = \frac{1}{4}\mu^2 = \frac{1}{4}\bar{h}^2 = \|\rho\|^2.$$

There is the following striking result about the bottom of the spectrum for the Laplacian:

Theorem 3.13 (Brooks ['81-1]). *Let (M,g) be a compact Riemannian manifold and $(\tilde{M},\tilde{g})$ the universal covering Riemannian manifold. Then:*

$\lambda_1(\tilde{M},\tilde{g})=0 \Longleftrightarrow$ *the fundamental group $\pi_1(M)$ is an amenable group.*

Furthermore, Sunada ['89] clarifies the above Brooks' theorem as follows: Let $(X,\tilde{g}) \longrightarrow (M,g)$ be a normal Riemannian covering of a compact Riemannian manifold with covering transformation group G. For $\rho; G \longrightarrow U(V)$, a unitary representation of G, let E_ρ be a flat vector bundle over M associated to ρ, and Δ_ρ the Laplacian acting on the vector bundle E_ρ. Define

$$\lambda_1(\rho) = \inf(\mathrm{Spect}(\Delta_\rho)); \quad \delta(\rho,\mathbf{1}) = \inf_{v\in V, \|v\|=1} \sup_{\sigma\in A} \|\rho(\sigma)\,v - v\|,$$

where A is a finite set of generators of G. Then Theorem 3.13 follows from the following theorem:

Theorem 3.14 (cf. Sunada ['89]). (i) *There exist positive constants C_1, C_2 such that for all unitary representation ρ of G,*

$$C_1\,\delta(\rho,\mathbf{1})^2 \le \lambda_1(\rho) \le C_2\,\delta(\rho,\mathbf{1})^2.$$

(ii) *If ρ is the regular representation of G, then $\lambda_1(\rho) = \lambda_1(X,\tilde{g})$, and*

$$\delta(\rho,\mathbf{1}) = 0 \iff G \text{ is amenable.}$$

Ono ['88] showed:

Theorem 3.15. *Let (M,g) be a compact Riemannian spin manifold, and $(\tilde{M},\tilde{g})$ be its universal Riemannian covering. Assume that the A-roof genus of M does not vanish. Then we get:*

$$\lambda_1(\tilde{M},\tilde{g}) \le \frac{1}{4}\left(-\min_{x\in M}\kappa(x)\right),$$

where κ is the scalar curvature of (M,g).

§4. Heat kernel of a complete Riemannian manifold

4.1. Construction of heat kernel

In this section, we construct the heat kernel of a Riemannian manifold. This was done by Ito ['79], Dodziuk ['83], Strichartz ['83], and Yau ['78]. There are two ways to construct the heat kernel. The one is to apply an abstract semigroup theory of $e^{t\Delta}$ on L^2 space of a Riemannian manifold (M,g) which are due to Yau ['78] and Strichartz ['83]. The other is a more or less constructive way due to Ito ['79] and Dodziuk ['83], which takes the following steps:

(1) taking an exhausion sequence of relatively compact domains D_i of M,

(2) define

$$p(x,y,t) = \lim_{i\to\infty} p_i(x,y,t),$$

where $p_i(x,y,t)$ is the Dirichlet heat kernels of D_i.

(3) And show that $p(x,y,t)$ is the heat kernel on (M,g).

In the following, we show the latter way more precisely, following Dodziuk ['83].

Definition 4.1. Let (M,g) be an arbitrary Riemannian manifold, $T > 0$, and u_o a continuous function on M. Then a continuous function $u\,;\, M\times(0,T) \longrightarrow \mathbb{R}$ is said to be a *solution of the Cauchy problem* of the heat equation on $M \times [0,T)$ with the initial data u_o, if $u(x,t)$ is C^2 in x and C^1 in t, and satisfies

$$\begin{cases} \Delta_x u + \dfrac{\partial u}{\partial t} = 0 & \text{on} \quad M \times [0,T), \\ u(x,0) = u_o(x), & x \in M. \end{cases}$$

Definition 4.2. A continuous function $p(x,y,t)$ on $M \times M \times (0,\infty)$ is a *fundamental solution of the heat equation*, i.e., *heat kernel* on M if, for all bounded continuous function u_o on M,

$$u(x,t) = \begin{cases} \displaystyle\int_M p(x,y,t)\, u_o(y)\, v_g(y)\,, \; t > 0, \\ u_o(x), \qquad t = 0, \end{cases}$$

is a solution of the Cauchy problem of the initial data u_o.

In order to construct the heat kernel on M, we consider the heat kernel $p_D(x,y,t)$ on a relatively compact domain D in M with C^∞ boundary with Dirichlet condition. Then it satisfies that:

Proposition 4.3. *The function p_D is C^∞ on $\bar{D} \times \bar{D} \times (0,\infty)$, and $p_D(x,y,t) = 0$ if x or $y \in \partial D$. Moreover,*

(1) $p_D(x,y,t) > 0, \quad p_D(x,y,t) = p_D(y,x,t),\ x,y \in D,\ t > 0.$

(2) $(\Delta_x + \frac{\partial}{\partial t})\, p_D \equiv 0.$

(3) $\displaystyle\int_D p_D(x,z,t) p_D(z,y,s) v_g(z) = p_D(x,y,t+s),\ s,t > 0, x,y \in \bar{D}.$

(4) *For all relatively compact smooth domain $D \subset M$, there exists a C^∞ function Φ on $D \times D$ such that $\Phi(x,x) \equiv 1, x \in D$, and*

$$p_D(x,y,t) - (4\pi t)^{-\frac{n}{2}} \exp\left(-\frac{d^2(x,y)}{4t}\right) \Phi(x,y)$$
$$= O\left(t^{-\frac{n}{2}+1} \exp\left(-\frac{d^2(x,y)}{4t}\right)\right), x,y \in D, \quad as \quad t \longrightarrow 0,$$

where the convergence in the right hand side is uniform on any compact subset of $D \times D$.

(5) *For relatively compact domains $D_1, D_2 \subset M$, let p_1, p_2 be the corresponding heat kernels. Then for all $x,y \in D_1 \cap D_2$, and for all $N > 0$,*

$$p_1(x,y,t) - p_2(x,y,t) = O(t^N) \quad as \quad t \longrightarrow 0,$$

where the estimate is uniform if x, y run on a compact subset of $D_1 \cap D_2$.

(6) *For all $x \in D, t > 0$,*

$$\int_D p_D(x, y, t)\, v_g(y) \; < \; 1.$$

(7) *For $D_1 \subset D_2$, we get*

$$p_{D_1}(x, y, t) \; \leq \; p_{D_2}(x, y, t), \quad x, y \in D_1, t \; > \; 0.$$

We omit its proof, but only note here that we need the following strong maximum principle to get (1), (6), and (7) in Proposition 4.3:

Lemma 4.4. (*strong maximum principle*) *Let $D \subset M$ be a relatively compact domain, u a bounded continuous function on $D \times [0, T]$ which is C^2 on $D \times (0, T)$ and satisfies*

$$\left(\Delta + \frac{\partial}{\partial t} \right) u \; \leq \; 0, \qquad on\ D \times (0, T).$$

Suppose that there exists $(x_o, t_o) \in D \times (0, T]$ such that

$$u(x_o, t_o) \; = \; \max_{(x,t) \in D \times [0,T]} u(x, t).$$

Then we get

$$u(x, t) \; = \; u(x_o, t_o) \quad for\ all \quad x \in D, t \; \leq \; t_o.$$

For a proof, see Nirenberg ['53, p.171].

The candidate of a heat kernel on M can be constructed as follows: We take an exhausion

$D_1 \subset D_2 \subset \cdots$; D_i are relatively compact domains with smooth ∂D_i.

I.e.,

$$\bar{D}_i \; \subset \; D_{i+1}, \qquad \bigcup_{i=1}^{\infty} D_i \; = \; M.$$

Definition 4.5. We define

$$p(x, y, t) \; = \; \lim_{i \to \infty} p_i(x, y, t),$$

where p_i is the heat kernel of D_i with the Dirichlet condition. Note that the limit allows infinity, but exists because of (7) in Proposition 4.3. Moreover we obtain:

Theorem 4.6. *The function $p(x,y,t)$ is C^∞ and a fundamental solution in the sense of Definition* 4.2. *Moreover,*

(1) $p(x,y,t) > 0, \quad p(x,y,t) = p(y,x,t), \quad t > 0,\, x,y \in M.$

$$\left(\Delta_x + \frac{\partial}{\partial t}\right) p \equiv 0. \tag{2}$$

$$\int_M p(x,z,t)\, p(z,y,s)\, v_g(z) = p(x,y,t+s),\ t,s > 0,\ x,y \in M. \tag{3}$$

(4) *$p(x,y,t)$ does not depend on the choice of an exhausion in its definition, and satisfies that*

$$p(x,y,t) = \sup_{D \subset M} p_D(x,y,t), \quad t > 0, x,y \in M,$$

where $D \subset M$ run over all relatively compact domains in M.

(5) *$p(x,y,t)$ is the smallest positive fundamental solution,* i.e., *for any $q(x,y,t)$ positive fundamental solution,*

$$p(x,y,t) \le q(x,y,t).$$

Outline of Proof. We only show the covergence of p_i to p. Fix $y \in M$. Let us consider $u_i(x,t) = p_i(x,y,t)$, and show that $\{u_i\}_{i=1}^\infty$ converges uniformly to a C^∞ solution of the heat equation on $D \times [t_1, t_2]$ for a relatively compact domain $D \subset M$, and $0 < t_1 < t_2$. For this, we need:

Lemma 4.7. *Let (N,h) be a Riemannian manifold, $a,b \in \mathbb{R}$ with $0 < a < b < \infty$. Let $\{u_i\}_{i=1}^\infty$ be a nondecreasing sequence of solutions of the heat equations on $N \times (a,b)$. Assume that*

$$\int_N |u_i(x,t)|\, v_h(x) \le C,$$

where C is a constant independent on $i, t \in (a,b)$. Then $u = \lim_{i\to\infty} u_i$ is a smooth solution of the heat equation, and the convergence of u_i to u is uniform with respect to the C^∞ topology on a relatively compact domains, and the derivatives of all orders converge.

In fact, let $D \subset N$ be a relatively compact smooth domain, $a < t_1 < t_2 < b$. Choose a function $h \in C_o^\infty(D)$, with $h \equiv 1$ on an open subset $V \subset D$. If $v(x,t)$ is a solution of the heat equation, then, for $x \in V, t \in (t_1, t_2)$, by Green's formula and Duhamel's principle, we get

$$\begin{aligned} v(x,t) &= \int_D v(y,t_1)\, h(y)\, p_D(x,y,t-t_1)\, v_h(y) \\ &+ \int_{t_1}^t ds \int_D v(y,t)\, \Delta h(y)\, p_D(x,y,t-s)\, v_h(y) \\ &+ 2 \int_{t_1}^t ds \int_D v(y,s)\, \langle \nabla h(y), \nabla_y\, p_D(x,y,t-s)\rangle\, v_h(y). \end{aligned}$$

Since $\Delta h \equiv 0, \nabla h \equiv 0$ in a neighborhood of x, arbitrary large order derivatives of $v(x,t)$ with respect to x are estimated by the terms of the L^1-norm of v, and the same is true for all derivatives by means of $\dfrac{\partial u}{\partial t} = \Delta u$. Applying this to $\{u_i\}$, $\{u_i\}, \{\nabla u_i\}$ are locally bounded, by the assumption, and then $u = \lim_{i \to \infty} u_i$ is finite and continuous. By Dini's theorem the convergence is uniform on a compact subset. Repeating this to the differentials of $\{u_i\}$, u is C^∞ and satisfies the heat equation. □

(Proof of Theorem continued) For fixed $y \in M$, dut to Proposition 4.3 (6), the function

$$u_i(x,t) = p_i(x,y,t), x \in M,\ t > 0$$

satisfies the conditions of Lemma 4.7, and then the limit $p(x,y,t)$ satisfies the heat equation in the variable (x,t). Moreover, $p(x,y,t)$ is a C^∞ function on $M \times M \times (0,\infty)$: In fact, we consider the heat equation on $(M \times M) \times (0,\infty)$:

$$(*) \qquad \left(\Delta_x + \Delta_y + 2\frac{\partial}{\partial t}\right) v(x,y,t) = 0.$$

Fix a relatively compact domain $D \subset M$. Then, for large $i \gg 1$, $p_i(x,y,t)$ satisfies the equation $(*)$ on $D \times D \times (0,\infty)$, and by (6) of Proposition 4.3,

$$\int_{D \times D} p_i(x,y,t)\, v_g(x)\, v_g(y) \le Vol(D).$$

Thus by Lemma 4.7, $p(x,y,t) = \lim_{l \to \infty} p_i(x,y,t)$ satisfies a C^∞ solution of $(*)$.

Now we show the function $p(x,y,t)$ is a fundamental solution of the heat equation in the sense of Definition 4.2: (1) For a bounded continuous function u_o on M,

$$u(x,t) = \begin{cases} \int_M p(x,y,t)\, u_o(y)\, v_g(y),\ t > 0, \\ u_o(x), \qquad t = 0, \end{cases}$$

is bounded and continuous.

In fact, we first show, for any open subset $U \subset M$ and $x \in U$,

$$\text{(a)} \qquad \lim_{t \downarrow 0} \int_U p(x,y,t) v_g(y) = 1.$$

Because note that, by (4) of Proposition 4.3,

$$\text{(b)} \qquad \lim_{t \downarrow 0} \int_D p_D(x,y,t) v_g(y) = 1, \quad x \in D.$$

Then by (b), (6) of Proposition 4.3, and positivity of p, we get

$$\begin{aligned} 1 &\geq \liminf_{t \downarrow 0} \int_M p(x,y,t)\, v_g(y) \geq \liminf_{t \downarrow 0} \int_U p_D(x,y,t)\, v_g(y) \\ &\geq \lim_{t \downarrow 0} \int_D p_D(x,y,t)\, v_g(y) = 1, \quad x \in D, \end{aligned}$$

for a relatively compact smooth domain $D \subset U$. We get (a). Moreover, by (6) of Proposition 4.3, we get

$$\text{(c)} \qquad \int_M p(x,y,t)\, v_g(y) \leq 1, \quad x \in M.$$

By (a), (c) and positivity of p, we obtain (1).

(2) The function $u(x,t)$ is a solution of the heat equation. In fact, we may assume $u_o \geq 0$. Then the function

$$u(x,t) = \lim_{i \to \infty} \int_M p_i(x,y,t)\, u_o(y)\, v_g(y)$$

is a limit of nondecreasing sequence of solutions of the heat equation, and the L^1-norm of each function is bounded above by a constant independent on i. Therefore by Lemma 4.7, $u(x,t)$ satisfies the heat equation. Thus $p(x,y,t)$ is the heat kernel. The properties (4), (5) of Theorem 4.6 follow from the maximum principle. □

4.2. Uniqueness of solution of heat equation

In this section, we show uniqueness results on the heat equation. Namely, let (M, g) be a complete Riemannian manifold. For a continuous function $f(x)$ on M, let us consider the Cauchy problem:

$$\begin{cases} \left(\Delta_x + \dfrac{\partial}{\partial t}\right) u = 0, & \text{on } M \times (0, \infty), \\ u(x, 0) = f(x), \; x \in M, \end{cases} \tag{4.8}$$

where $u(x, t)$ is a continuous function on $M \times [0, \infty)$, and C^2 in x, and C^1 in t. Then we get:

Theorem 4.9 (cf. Dodziuk ['83]). *Let (M, g) be a complete Riemannian manifold with the Ricci curvature satisfying $\mathrm{Ric}_M \geq -C$, $C > 0$. Then a bounded solution of* (4.8) *is determined uniquely by the initial data f.*

For a proof, see Dodziuk ['83] or Chavel ['84].

Theorem 4.10 (cf. Donnelly ['83]). *Under the same assumption of Theorem* 4.9, *a non-negative solution of* (4.8) *is uniquely determined by the initial data.*

Theorem 4.11 (cf. Li ['84]). *Let (M, g) be a noncompact complete Riemannian manifold whose Ricci curvature satisfies*

$$\mathrm{Ric}_M(x) \geq -C\left(1 + r(x)^2\right), \quad \forall x \in M,$$

for some positive constant C, where $r(x) = d(x, p)$, $x \in M$, for some fixed point p. Then

(1) *any L^1-solution of* (4.8) *is uniquely determined by the initial data in $L^1(M)$.*

(2) $1 < p < \infty$. *Then any L^p-solution of* (4.8) *is uniquely determined by the initial data in $L^p(M)$.*

(3) (cf. Li & Yau ['86]) *any solution of* (4.8) *which is bounded below, is uniquely determined by the initial data.*

Theorem 4.12 (cf. Li & Karp ['91]). *Let (M, g) be a complete Riemannian manifold satisfying that there exist a point $p \in M$ and a constant C such that, either*

$$\mathrm{Vol}(B_r(p)) \leq \exp(C r^2), \quad \forall r, \tag{1}$$

where $B_r(p)$ is the geodesic ball centered p with radius r, or

$$\text{Ric}_M(x) \geq -C\left(1 + r^2(x)\right), \quad \forall x \in M, \tag{2}$$

where $r(x) = d(x,p)$, $x \in M$. Then any bounded solution of (4.8) *is determined by the initial data.*

Theorem 4.13 (cf. Nagasawa ['91]). *Let (M,g) be a complete Riemannian manifold, and $u(x,t)$ a continuous solution of* (4.8), *and assume that there exist $p \in M$ and $C > 0$ such that*

$$\int_{B_{r+1}(p)\setminus B_r(p)} |u(x,t)|^2\, v_g(x) \leq \exp\left(C(1+r^2)\right), \quad \forall r > 0.$$

Then $u(x,t) \equiv 0$, $\forall t > 0$, if $u(x,0) = f(x) \equiv 0$. In particular, let

$$K_p(r) = \inf_{x \in B_r(p)} \text{Ric}_M(x), \text{ and } K_p^+(r) = \max\{K_p(r), 0\}$$

$$p \in M,\ r > 0.$$

Assume that there exist $p \in M$ and $C > 0$ such that

$$K_p^+ \leq C(1+r^2), \forall r.$$

Then any nonnegative continuous solution of (4.8) *is uniquely determined by the initial data.*

4.3. Estimates of the heat kernel

In this section, we show results on the asymptotic behavior, and upper and lower estimates of the heat kernel of a complete Riemannian manifold.

We first show the following asymptotic behavior of the heat kernel $p(x,y,t)$ of a complete Riemannian manifold (M,g), as t tends to zero.

Theorem 4.14 (cf. Cheng, Li, & Yau ['81, p.1040]). *Let (M,g) be an arbitrary complete Riemannian manifold, $p(x,y,t)$ be the heat kernel. Then we get:*

$$\lim_{t \downarrow 0} -4t \log p(x,y,t) = d^2(x,y), \quad \forall\, x,y \in M.$$

Remark 4.15. The above theorem was obtained by Varadhan ['67] when $(\mathbb{R}^n, g)$, where g satisfies the uniform Hölder condition and the uniform ellipticity condition. One can also see a proof of the above theorem in Chavel ['84, p.201], when (M,g) is a complete Riemannian manifold with Ricci curvature bounded from below.

On the other hand, the asymptotic behaviors of the heat kernel $p(x,y,t)$, as t tends to $+\infty$, are given as follows:

Theorem 4.16 (cf. Li ['86]). *Let (M,g) be an n-dimensional complete Riemannian manifold, $p(x,y,t)$ be the heat kernel of (M,g). Then*

$$(1) \qquad \lim_{t\to\infty} \frac{\log p(x,y,t)}{t} = -\lambda_1(M,g), \quad \forall\, x,y \in M.$$

(2) *Assume that (M,g) has nonnegative Ricci curvature $\mathrm{Ric}_M \geq 0$, and there exist a point p and a positive constant θ such that*

$$\liminf_{r\to\infty} \frac{\mathrm{Vol}(B_r(p))}{r^n} = \theta.$$

Then we have:

$$\lim_{t\to\infty} \mathrm{Vol}(B_{\sqrt{t}}(p))\, p(x,y,t) = \mathrm{Vol}_n(B_1)(4\pi)^{-\frac{n}{2}},$$

where B_1 is the unit ball in $\mathbb{R}^n$.

The lower and upper bounds of the heat kernel $p(x,y,t)$ are given as follows:

(Lower bounds) We first prepare some terminologies: For any Riemannian manifold (M,g), and a fixed point p, let $m(r,\theta)$ be the mean curvature function at point (r,θ), of $\partial B_r(p)$ with $\partial B_r(p) \cap C$ deleted. Here $\partial B_r(p)$ is the distance sphere centered with p, radius r, and C is the cut locus of p. Moreover, we call a Riemannian manifold $\mathcal{M}$ to be an *open model* if the following conditions hold:

(1) For some point $z \in \mathcal{M}$ and $0 < R \leq \infty$, $\mathcal{M} = B_R(z)$ and the exponential map $\exp_z$; $B_R(0) \longrightarrow B_R(z)$ is a diffeomorphism.
(2) For all $r < R$, the mean curvature of the distance sphere $\partial B_r(z)$ is constant on $\partial B_r(z)$, denoted by $m(r)$.

Then we get by definition:

Proposition 4.17. *Let $\mathcal{M}$ be an open model. Then its heat kernel $p(\tilde{x},\tilde{y},t) = p(d(\tilde{x},\tilde{y}),t), \tilde{x},\tilde{y} \in \mathcal{M}$, depends only on $r = d(\tilde{x},\tilde{y})$, and t.*

Then the heat kernel $p(x,y,t)$ can be estimated as follows:

Theorem 4.18 (cf. Cheeger & Yau ['81]). *Let (M, g) be a complete Riemannian manifold, $\mathcal{M}$ an open model. Assume that*

$$m(r, \theta) \leq m(r), \quad \forall\, 0 < r \leq R.$$

Then we have:

$$p(d(x, y), t) \leq p(x, y, t), \quad \forall\, x, y \in M,\, t > 0,$$

and the equality holds if and only if (M, g) is isometric to $\mathcal{M}$ and $m(r, \theta) = m(r), \forall\, r$.

Moreover, it is known that:

Theorem 4.19 (cf. Li & Yau ['86]). *Let (M, g) be a complete Riemannian manifold with nonnegative Ricci curvature* $\mathrm{Ric}_M \geq 0$. *Then for all $\epsilon > 0$, there exists a constant $C(\epsilon)$ such that*

$$p(x, y, t) \geq C(\epsilon)^{-1}\, \mathrm{Vol}\left(B_{\sqrt{t}}(x)\right)^{-1} \exp\left\{\frac{-d(x, y)^2}{(4+\epsilon)t}\right\},$$

$$p(x, y, t) \geq C(\epsilon)^{-1}\, \mathrm{Vol}\left(B_{\sqrt{t}}(x)\right)^{-\frac{1}{2}}\, \mathrm{Vol}\left(B_{\sqrt{t}}(y)\right)^{-\frac{1}{2}} \exp\left\{\frac{-d(x, y)^2}{(4+\epsilon)t}\right\},$$

where the constant $C(\epsilon)$ tends to $+\infty$ as $\epsilon \longrightarrow 0$.

(Upper bounds) In general, we obtain the following estimates:

Theorem 4.20 (cf. Cheng, Li & Yau ['81, p.1037]). *Let (M, g) be a complete Riemannian manifold. Then, for all $\beta > 1$, $T > 0$, and $x \in M$, there exists a constant $C = C(\beta, T, x)$ such that*

$$\int_{M \setminus B_R(x)} p(x, y, t)^2 v_g(y) \leq C\, t^{-\frac{n}{2}} \exp\left\{\frac{-R^2}{2\beta t}\right\}, \qquad \forall\, t \in [0, T],\, \forall\, R > 0,$$

where the constant C tends to $+\infty$ as $\beta \longrightarrow 0$.

In particular, we obtain:

Theorem 4.21 (cf. Cheng, Li & Yau ['81, p.1046]). *Let (M, g) be a complete Riemannian manifold with bounded curvature,* i.e., *whose sectional curvature is bounded. Then for all $\alpha > 4$, $T > 0$ and $x \in M$, there exists a constant $C' = C'(\alpha, T, x)$ such that*

$$p(x, y, t) \leq C'\, t^{-\frac{n}{2}} \exp\left\{-\frac{d(x, y)^2}{\alpha t}\right\}, \quad \forall\, t \in [0, T],\, \forall\, y \in M.$$

Theorem 4.22 (cf. Varopoulos ['84]). *Assume that (M, g) satisfies the same conditions of Theorem 4.21 and the injectivity radius is bounded below by a positive constant. Then the heat kernel satisfies that, for all $0 < \epsilon < 0.1$, there exist C_1, $C_2 > 0$ such that*

$$\sup_{x,y\in M} p(x,y,t) \le \min\left\{C_1\, t^{-\frac{1}{2}+\epsilon},\, C_2\, t^{-\frac{1}{2}}\, (\log t)^{1+\epsilon}\right\}, \quad \forall t > 1.$$

Theorem 4.23 (cf. Li & Yau ['86 p.175]). *Let (M, g) be a complete Riemannian manifold with nonnegative Ricci curvature : $\mathrm{Ric}_M \ge 0$. Then, for $\forall\, 0 < \epsilon < 1$, there exists a constant $C(\epsilon)$ such that*

$$p(x,y,t) \le C(\epsilon)\, \mathrm{Vol}\left(B_{\sqrt{t}}(x)\right)^{-1} \exp\left\{-\frac{d^2(x,y)}{(4+\epsilon)t}\right\},$$
$$\forall x, y \in M,\ \forall t > 0,$$

where the constant $C(\epsilon)$ tends to $+\infty$ as $\epsilon \longrightarrow 0$.

Theorem 4.24 (cf. Davies ['87]). *Let (M, g) be a Riemannian manifold whose heat kernel $p(x,y,t)$ satisfies*

$$p(x,y,t) \le a\, t^{-\frac{n}{2}}, \quad \forall x, y \in M,\ t > 0,$$

for some positive constant a. Then, for all $\delta > 0$, there exists a constant $C(\delta)$ such that

$$p(x,y,t) \le C(\delta)\, t^{-\frac{n}{2}} \exp\left\{-\frac{d(x,y)^2}{4(1+\delta)t}\right\}, \quad \forall x, y \in M, \forall t > 0.$$

Remark 4.25. The assumption of the heat kernel $p(x,y,t)$ in Theorem 4.24 is equivalent to the following:

$$\|f\|_{\frac{2n}{n-2}}{}^2 \le a\,(\Delta f, f), \quad \forall\, 0 \le f \in C_o^\infty(M),$$

which is satisfied, if the Ricci curvature of (M, g) is bounded below: $\mathrm{Ric}_M \ge -c$, $c > 0$, and the injectivity radius is bounded below by a positive constant.

Remark 4.26. Recently the following Lichnerowicz conjecture is solved negatively by E. Damek and F. Ricci ['91]: A noncompact complete Riemannian manifold whose heat kernel $p(x,y,t)$ depends only on the distance $r(x,y)$ and t, is the Euclidean space or a symmetric space of rank one.

§5. Harmonic functions

5.1. Green functions

In this section, we are concerned with Green function on a relatively compact domain $\Omega \subset M$ of a complete Riemannian manifold (M, g).

Definition 5.1. Let $\Omega_D = \{(x,x) \in \Omega \times \Omega; x \in \Omega\}$. Then a function $G_\Omega; \bar{\Omega} \times \bar{\Omega} \backslash \Omega_D \longrightarrow \mathbb{R}$ is said to be a *Green function* of Ω if

(1) it is C^2 function on $\Omega \times \Omega \setminus \Omega_D$,
(2) $\Delta_y G_\Omega = 0, \quad \forall x, y \in \Omega, x \neq y$,
(3) $G_\Omega(x,y) = 0, \quad x \in \Omega, y \in \Gamma = \partial\Omega$,
(4) G_Ω can be written in a neighborhood of Ω_D by $G_\Omega(x,y) = \psi(x,y) + h(x,y)$, where $h \in C^o(\bar{\Omega} \times \bar{\Omega}) \cap C^2(\Omega \times \Omega)$, and

$$\psi(x,y) = \begin{cases} \dfrac{1}{C_{n-1}} \dfrac{d(x,y)^{2-n}}{n-2}, & n > 2, \\ \dfrac{1}{2\pi} \left(-\log d(x,y)\right), & n = 2, \end{cases}$$

$d(x,y)$, x, $y \in M$ being the geodesic distance in (M, g), and $C_{n-1} = \dfrac{2\,\pi^{n/2}}{\Gamma(n/2)}$, the $(n-1)$-volume of the unit sphere in $\mathbb{R}^n$.

For the existence of such a function G_Ω, see John ['82], for example.

Definition 5.2 (cf. Aomoto ['66]). Let $M_D = \{(x,x) \in M \times M; x \in M\}$ for a Riemannian manifold (M, g). A C^2 function $G; M \times M \backslash M_D \longrightarrow \mathbb{R}$ is called a *Green function* of (M, g) if the following hold:

(1) $\Delta_y G = 0, \quad \forall x, y \in M, x \neq y$,
(2) G can be written in a neighborhood of M_D by $G(x,y) = \psi(x) + h(x,y)$, where $h \in C^2(M \times M)$, and $\psi(x,y)$ satisfies the same properties as (4) in Definition 5.1.
(3) For all $y \in M$, there exists $\delta > 0$ such that $G(x,y)$ is a bounded function in x on $M_\delta = \{x \in M\,;\, d(x,y) > \delta\}$.

Definition 5.3. A function f on (M, g) is said to be *superharmonic* if the following hold:

(1) $-\infty < f(x) \leq \infty$, and f does not vanish identically on M.
(2) f is lower semi continuous on M.
(3) Let $\Omega \subset M$ be a relatively compact smooth domain. If a function w which is continuous on $\bar{\Omega}$, and harmonic on Ω, satisfies that $w(x) \leq f(x), \quad \forall x \in \partial\Omega$, then $w(x) \leq f(x), \quad \forall x \in \Omega$.

Note that a C^2 function f on (M,g) is superharmonic if and only if $\Delta f \geq 0$ everywhere on M. Here notice that our Laplacian is $\Delta = \delta d$. Then it is known that:

Theorem 5.4 (cf. Ito ['64-1], ['64-2]). *Let (M,g) be a Riemannian manifold, $p(x,y,t)$ be the heat kernel defined in §4. Define*

$$G(x,y) = \int_0^\infty p(x,y,t)\,dt,\ x,\ y \in M.$$

Then $G(x,y)$ gives a Green function of (M,g) if and only if there exists a nonconstant positive superharmonic function on (M,g).

Definition 5.5. A Riemannian manifold (M,g) is said to be *hyperbolic* if it has a nonconstant positive superharmonic function, *parabolic* otherwise. For these examples, see section 5.3.

5.2. The Martin boundary

In this section, we introduce the notion of the Martin boundary. To do this, we first prepare the Harnack inequality, the Harnack principle, and the maximum principle:

Theorem 5.6 (Harnack inequality) (cf. Moser ['61]). *Let $\Omega \subset M$ be a relatively compact domain in a complete Riemannian manifold (M,g). Let $\Omega' \Subset \Omega$ be a domain whose closure is contained in Ω. Let u be a positive harmonic function on Ω. Then we get:*

$$\sup_{x\in\Omega'} u(x) \leq C \inf_{x\in\Omega'} u(x),$$

where C is a positive constant which depends only on Ω, Ω', and the curvature of (M,g).

Theorem 5.7 (Harnack principle). *Let Ω be a relatively compact domain in a complete Riemannian manifold (M,g). Let $\{u_n\}_{n=1}^\infty$ be a sequence of harmonic functions on Ω. Assume that there exists a positive constant K such that $|u_n| \leq K$, $n = 1,2,\cdots$. Then $\{u_n\}_{n=1}^\infty$ is a normal family,* i.e., *there exists a subsequence which is convergent to a harmonic function on Ω and the convergence is uniform on each compact subset of Ω.*

For a proof, see Tsuji ['59], Kishi ['74], Doob ['83].

Theorem 5.8 (Maximum principle). *Let $\Omega \subset M$ be a relatively compact domain in a complete Riemannian manifold (M,g). Assume that*

$$\Delta u = 0 \quad \text{on } \Omega, \quad \text{and } u \leq 0 \text{ on } \partial\Omega.$$

Then we get $u \leq 0$ *on* Ω.

For a proof, see Protter & Weinberger ['84].

Assume that (M, g) is hyperbolic, i.e., it has a nonconstant positive superharmonic function. Let $o \in M$ be a fixed point. For x, $y \in M$, let

$$K_y(x) = K(y,x) = \begin{cases} G(y,x)/G(y,o), & y \neq o, \\ 0, & y = o, \, x \neq o, \\ 1, & x = y = o, \end{cases}$$

then the function K satisfies the following:

(1) For each fixed $y \in M$, K_y is a nonnegative harmonic function in $x, x \neq y$,
(2) $K_y(o) = 1$, and
(3) for each fixed $x \in M$, $K(y, x)$ is a continuous function in $y, y \neq x$.

Assume that $\{y_n\}_{n=1}^{\infty}$ is a sequence in M which has no accumulation point in M. By the Harnack principle (cf. Theorem 5.7), a sequence $\{K_{y_n}|_{\Omega}\}_{n=1}^{\infty}$ has a subsequence which converges to a harmonic function on Ω, for every relatively compact domain $\Omega \subset M$. Take an exhausion $\Omega_1 \subset \Omega_2 \subset \cdots \subset M, \cup_i \Omega_i = M$, and use the diagonal method as in the proof of Lemma 1.14 to get a subsequence $K_{y_{n_k}}$ of K_{y_n} which converges to a harmonic function on M, say K.

Definition 5.9. A sequence $\{y_n\}$ in M is said to be *fundamental* if K_{y_n} converges to a harmonic function K on M.

By the above argument, we get:

Lemma 5.10. *Assume that a complete Riemannian manifold* (M, g) *is hyperbolic. Then any sequence in* M *which has no accumulation point, has a fundamental subsequence.*

Definition 5.11. Let (M, g) be a hyperbolic Riemannian manifold. Then two fundamental sequences in (M, g) are *equivalent* if the corresponding limit harmonic functions in Lemma 5.10 coincides each other. The *Martin boundary* or *ideal boundary* $\mathcal{M}$ of (M, g) is the equivalence classes of all fundamental sequences of (M, g).

Note that, for $[Y] \in \mathcal{M}$,

$$K_Y(x) = \lim_{i \to \infty} K_{y_i}(x), \; x \in M, \tag{5.12}$$

where $\{y_i\}$ is a fundamental sequence associated to $[Y] \in \mathcal{M}$, and K_Y is a positive harmonic function satisfying $K_Y(o) = 1$. Therefore each $[Y] \in \mathcal{M}$ corresponds to a unique positive harmonic function K_Y on M with $K_Y(o) = 1$.

Definition 5.13. Put $\tilde{M} = M \cup \mathcal{M}$, and define the following metric ρ on $\tilde{M}$:

$$\rho(Y, Y') = \int_{B_1(o)} \frac{|K_Y(x) - K_{Y'}(x)|}{1 + |K_Y(x) - K_{Y'}(x)|} v_g(x),\ Y,\, Y' \in \tilde{M},$$
$$\left(\text{or} \quad \sup_{x \in B_1(o)} |K_Y(x) - K_{Y'}(x)| \right),$$

where $B_1(o)$ is the geodesic ball centered at o with radius 1 in (M, g).

Proposition 5.14 (cf. Martin ['41]). *This ρ is actually a complete metric on $\tilde{M}$, and $(\tilde{M}, \rho)$ is compact, M is open in $\tilde{M}$, and $\mathcal{M}$ is the boundary of $\tilde{M}$. The relative topology of M with respect to ρ coincides with the original topology of M. Moreover, for each $x \in M$, the mapping $Y \longmapsto K_Y(x)$ is continuous on $\tilde{M} \setminus \{x\}$ with respect to ρ.*

Then Martin showed

Theorem 5.15 (Representation theorem) (cf. Martin ['41]). *For each nonnegative harmonic function u on (M, g), there exists a Borel measure μ on $\mathcal{M}$ such that*

$$u(x) = \int_{\mathcal{M}} K_Y(x)\, d\mu(Y),\ x \in M. \tag{5.16}$$

Conversely, for any Borel measure μ on $\mathcal{M}$, (5.16) *gives a nonnegative harmonic function on (M, g), and $\mu(\mathcal{M}) = u(o)$.*

Definition 5.17. A positive harmonic function u on (M, g) is *minimal* if any positive harmonic function v with $v(x) \leq u(x)$, $\forall x \in M$ is a constant multiple of u.

Note that, if u is a positive minimal harmonic function, then there exists a positive constant C such that $u = C K_Y$ for some $Y \in \mathcal{M}$. Then we define:

Definition 5.18. Put $\mathcal{M}_1 = \{Y \in \mathcal{M};\ K_Y \text{ minimal}\}$, and $\mathcal{M}_o = \mathcal{M} \setminus \mathcal{M}_1$. A Borel measure μ on $\mathcal{M}$ is *canonical* if $\mu(\mathcal{M}_o) = 0$.

Theorem 5.19 (Canonical representation theorem) (cf. Martin ['41]). *For any nonnegative harmonic function u on (M,g), there exsists a unique canonical Borel measure μ on $\mathcal{M}$ such that*

$$u(x) = \int_{\mathcal{M}} K_Y(x)\, d\mu(Y),\ x \in M.$$

Moreover, Brelot ['56] showed:

Theorem 5.20 (Solvability of Dirichlet problem). *Let ν be a canonical Borel measure on $\mathcal{M}$ and f a continuous function on $\mathcal{M}$. Define a function $\mathcal{P}_f$ on M by*

$$\mathcal{P}_f(x) = \int_{\mathcal{M}} f(Y)\, K_Y(x)\, d\nu(Y),\ x \in M.$$

Then $\mathcal{P}_f$ is a harmonic function on (M,g), and satisfies that

$$\lim_{x \to Y'} \mathcal{P}_f(x) = f(Y'),\ Y' \in \mathcal{M}.$$

The function $\mathcal{P}_f$ is called a *Poisson integral* on the Martin boundary. For a proof, see Brelot ['56], Doob ['83, p.207, p.101], and Ito ['88].

5.3. Examples of hyperbolic Riemannian manifolds

In this section, we give examples of complete hyperbolic Riemannian manifolds (M,g) and realize their Martin boundaries.

Let G be a real semisimple Lie group with finite center, K a maximal compact subgroup, and $M = G/K$ a symmetric space of noncompact type as in Example 3.5. Let g_o be the Riemannian metric on M induced from the Killing form of the Lie algebra $\mathfrak{g}$ of G, and $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$, the Cartan decomposition of $\mathfrak{g}$. Then:

Theorem 5.21 (cf. Furstenberg ['63]). *Let (M,g_o) be as above. Then it is hyperbolic and its Martin boundary $\mathcal{M}$ coincides with the homogeneous space $K/Z_K(A) = G/B$. Here $Z_K(A)$ is the centralizer of A in K, A is the analytic subgroup of G corresponding to a maximal abelian subalgebra $\mathfrak{a}$ of $\mathfrak{g}$ contained in $\mathfrak{p}$ and B is a Borel subgroup of G. Moreover, the Poison integral coincides with the integral*

$$\mathcal{P}_f(x) = \int_K f(xk)\, dk, x \in G,$$

for a continuous function f on K which satisfies $f(km) = f(k), k \in K$, $m \in Z_K(A)$, and dk is the Haar measure on K.

For more interesting results about the Poisson integrals on symmetric spaces, see Korányi ['69], Helgason ['70-'76], Kashiwara, Kowata, Minemura, Okamoto, Oshima and Tanaka ['78].

In case of a general (not necessary homogeneous) Riemannian manifold, we get:

Theorem 5.22 (Aomoto ['66]). *Let (M,g) be an n-dimensional simply connected complete Riemannian manifold. Assume that the sectional curvature K of (M,g) is nonpositive in case of $n \geq 3$, and $K \leq -C$, $C > 0$ in case of $n = 2$. Then (M,g) is hyperbolic,* i.e., *it has nonconstant positive superharmonic function.*

Moreover, there are the following criteria telling which (M,g) is hyperbolic or parabolic, due to Kasue ['82]:

Let (M,g) be a complete noncompact connected Riemannian manifold. For $x \in M$, let $\sigma : [0,\infty) \longrightarrow M$, a geodesic emanating x with unit speed, and define functions R_x and $f_x : [0,\infty) \longrightarrow \mathbb{R}$, such that

$$R_x(t) \leq \frac{1}{n-1} \operatorname{Ric}_M(\dot{\sigma}(t)), \forall t \in [0,\infty),$$

where Ric_M is the Ricci curvature of (M,g), and

$$f_x{}'' + R_x f_x = 0, f_x(0) = 0 \text{ and } f_x{}'(0) = 1.$$

Then we get:

Theorem 5.23 (Kasue ['82]). *Assume that (M,g) has a positive Green function $G(x,y)$. Then the following holds:*

$$G(x,y) \geq \frac{1}{\omega_{n-1}} \int_{d(x,y)}^{\infty} f_x(t)^{1-n}\, dt,$$

where ω_{n-1} is the $(n-1)$-dimensional volume of the unit sphere of $\mathbb{R}^n$. In particular, if $\int^{\infty} f_x(t)^{1-n}\, dt = \infty$, then (M,g) has no positive Green function, therefore parabolic, i.e., *(M,g) does not have any nonconstant positive superharmonic function.*

On the contrary, let us define functions $\bar{K}_x$, $\bar{F}_x$; $[0,\infty) \longrightarrow \mathbb{R}$, by

$$\bar{K}_x(t) \geq K(\Pi), \forall\, \dot{\sigma}(t) \in \Pi \subset T_{\sigma(t)}M, \text{ plane},$$

where $K(\Pi)$ is the sectional curvature of the plane Π, and

$$\bar{F}_x'' + \bar{K}_x \bar{F}_x = 0, \ \bar{F}_x(0) = 0, \ \text{and} \ \bar{F}_x'(0) = 1.$$

Then we get:

Theorem 5.24.
(1) (Kasue ['82]) *Let $i(x)$ be the injectivity radius of (M, g) at $x \in M$. Assume that $i(x) = \infty$, and $\int^{\infty} \bar{F}_x(t)^{1-n} dt < \infty$ for all $x \in M$. Then (M, g) admit a positive Green function G which satisfies*

$$G(x, y) \leq \frac{1}{\omega_{n-1}} \int_{d(x,y)}^{\infty} \bar{F}_x(t)^{1-n} \, dt, \ \forall x \neq y \in M,$$

where ω_{n-1} is as in Theorem 5.23.
(2) (cf. Li & Tam ['87-1]) *Let (M, g) be a Riemannian manifold whose sectional curvature is nonnegative outside some compact subset. Assume that (M, g) has at least one large end (see section* 5.5 *for definition). Then (M, g) is hyperbolic.*

Example 5.25. Let $M = \mathbb{R} \times_f F$ be the warped product where F is an n-dimensional compact Riemannian manifold and f is a positive C^{∞} function on $\mathbb{R}$. Then:

(1) If $\displaystyle\int_{-\infty}^{\infty} f(t)^{-n} \, dt < \infty$, then M is hyperbolic and admit a non-constant harmonic function with finite Dirichlet integral.

(2) If $\displaystyle\int_0^{\infty} f(t)^{-n} \, dt = \infty$ and $\displaystyle\int_{-\infty}^{0} f(t)^{-n} \, dt = \infty$. Then M is parabolic.

Example 5.26. Let (M, g) be a complete Riemannian manifold with nonnegative Ricci curvature. Then a theorem of Cheeger and Gromoll says that $(M, g) = (N, h) \times (\mathbb{R}^k, g_o)$, the Riemannian product, and g_o is the standard metric on $\mathbb{R}^n$. Moreover, the following hold:

(1) If $k \geq 3$, then (M, g) is hyperbolic.
(2) If $k \leq 2$ and N is compact, then (M, g) is parabolic.

For a proof of these examples, see also Kasue ['82].

On the other hand, Lyons & Sullivin ['84] studied a Riemannian manifold (M, g) which admits a positive Green function. Note that a Riemannian manifold (M, g) has the property (1) : (M, g) admits a positive Green function, is equivalent to (2) : (M, g) admits a nonconstant bounded subharmonic function, and also equivalent to (3) : the Brownian motion on (M, g) is transient.

They constructed (M, g) which does not admit a nonconstant positive harmonic function, but have a positive Green function, as a Corollary of the following theorems: Let $\Gamma \subset \mathrm{Isom}(M)$ be a discrete subgroup of the isometry group of (M, g), whose quotient space $N = M/\Gamma$ is smooth.

Definition 5.27. M is an *Abelian* (resp. *nilpotent, solvable, ω-nilpotent*) *cover* of N , if Γ is Abelian (resp. nilpotent, solvable, Γ is a infinite union of normal subgroups Z_i of Γ with Z_{i+1} contained in the center of Γ/Z_i).

Then they obtained:

Theorem 5.28.

(1) *Let (N, h) be a compact Riemannian manifold. Then any Riemannian nilpotent covering space of (N, h) admits no nonconstant positive harmonic function.*

(2) *Let (M, g) be a Riemannian ω-nilpotent cover of (N, h). Assume that (N, h) admits no positive Green function. Then a bounded harmonic function on (M, g) is always constant.*

(3) *Let (M, g) be a Riemannian non-amenable cover of (N, h). Then (M, g) admits a nonconstant bounded harmonic function.*

(4) *Let (M, g) be a Riemannian Abelian cover of (N, h). Then (M, g) admits a positive Green function if and only if the rank of Γ is bigger than or equal to* 3.

Corollary 5.29. *The universal Riemannian cover of a compact negatively curved manifold admits a nonconstant bounded harmonic function.*

They also extended a theorem of Kelvin, Nevanlinna, & Royden:

Theorem 5.30. *Let (M, g) be a complete Riemannian manifold. Then it admits a positive Green function if and only if there exists a vector field V on M such that*

$$\int_M |\operatorname{div} V|^2 v_g < \infty, \int_M |V|^2 v_g < \infty, \text{ and } \int_M \operatorname{div} V \, v_g \neq 0.$$

Corollary 5.31. *Let (X, g), (Y, h) be complete Riemannian manifolds. Assume that they are quasi-isometric. Then the one admits a positive Green function if and only if the other does so.*

(*Open Problem*). Under the assumption of Corollary 5.31, does the property that the one admits a bounded harmonic function if and only if the other does so, hold ?

5.4. The Martin boundary and the ideal boundary

We first, in this section, introduce the ideal boundary of a Riemannian manifold of nonpotive curvature following Eberlein & O'Neill ['73], and show results of Anderson ['83], Sullivan ['83], and Anderson & Schoen ['85]. In this section, we assume (M, g) is a complete Riemannian manifold of nonpositive curvature.

Definition 5.32. A geodesic ray $\gamma\,;\,[0,\infty) \longrightarrow M$ is a geodesic of (M,g) parametrized with arc length, and each of whose segment is minimal between its endpoints. Two geodesic rays γ_1, γ_2 are said to be *asymptotic* if $\sup_{0\leq t<\infty} d(\gamma_1(t),\gamma_2(t)) < \infty$. Let $S(\infty)$ be the set of all asymptotic classes of geodesic rays, which is called the *ideal boundary* or *geometric boundary.* For a geodesic ray γ, we denote by $\gamma(\infty)$ the asymptotic classes containing γ.

Note that for each $p \in M$ and $x \in S(\infty)$, there exists a unique geodesic ray γ_{px} such that $\gamma_{px}(0) = p$, and $\gamma_{px}(\infty) = x$.

Let $\bar{M} = M \cup S(\infty)$, and introduce the topology, called the *cone topology*, which is compatible to that of M, and with respect to which $S(\infty)$ is homeomorphic to the $(n-1)$-dimensional unit sphere S^{n-1}: Let $p \in M$, a, $b \in \bar{M}$, $p \neq a$, b. The *angle subtended* by a, b at $p \in M$, denoted by $\sphericalangle_p(a,b)$, is the angle $\sphericalangle({\gamma_{pa}}'(o), {\gamma_{pb}}'(0))$ between the geodesics γ_{pa}, γ_{pb} at p. Then for $\pi > \epsilon > 0$, and $v \in S(p)$, the unit sphere in the tangent space T_pM, let us define the *cone* of vertex p, axis v and angle ϵ by

$$C(v,\epsilon) = \left\{b \in \bar{M}\,;\, \sphericalangle_p(\gamma_v(\infty), b) < \epsilon\right\}.$$

We can define the topology on $\bar{M}$ (called the *cone topology*) in such a way that, for each point $x \in S(\infty)$, a collection of the set

$$\{C(v,\epsilon)\,;\, x \in C(v,\epsilon), v \in S(p),\, p \in M,\, \pi > \epsilon > 0\}$$

is a neighborhood system of x in $\bar{M}$.

Propsition 5.33 (Eberlein & O'Neill ['73, p.54]). *Let* $B(p) = \{v \in T_pM;\, \|v\| < 1\}$, *and* $S(p) = \{v \in T_pM;\, \|v\| = 1\}$ *for* $p \in M$. *Let* $f : [0,1] \longrightarrow [0,\infty]$ *be a homeomorphism. Then* $\varphi\,;\, B(p) \ni v \longmapsto \exp\left(f(\|v\|)\, v\right) \in M$ *gives a homeomorphism of* $S(p)$ *onto* $S(\infty)$.

Theorem 5.34 (cf. Anderson ['83], Sullivan [83]). *Assume that (M, g) be a complete simply connected Riemannian manifold whose sectional curvature K satisfies $-\infty < -b^2 \leq K \leq -a^2 < 0$, for some positive constants a, b. Then, for every continuous function φ on $S(\infty)$, there exists a unique function $u \in C^\infty(M) \cap C^o(\bar{M})$ such that*

$$\begin{cases} \Delta u = 0, \\ u \,|\, _{S(\infty)} = \varphi. \end{cases}$$

Furthermore, Anderson & Schoen ['85] showed that $\mathcal{M} = S(\infty)$ under the same assumption of Theorem 5.34. Namely,

Theorem 5.35. *Under the same assumption of (M, g) in Theorem* 5.34 *there exists a homeomorphism Φ of $\mathcal{M}$ onto $S(\infty)$. Therefore, if we put, for a fixed point $o \in M$,*

$$K(x, Q) = \lim_{y \to Q} \frac{G(y, x)}{G(y, o)}, \; Q \in S(\infty), \; x \in M,$$

each positive harmonic function u on (M, g) can be uniquely expressed by a finite positive Borel measure on $S(\infty)$ such that

$$u(x) = \int_{S(\infty)} K(x, Q) \, d\mu(Q),$$

and has nontangential limit at a.e. *$Q \in S(\infty)$,* i.e., *for every nontangential domian Ω at Q, $\lim_{\Omega \ni x \to Q} u(x)$ exists, and the limit is the absolutely continuous part of the Borel measure μ on $S(\infty)$ corresponding to u.*

Here, a domain $\Omega \subset M$ is a *nontangential* domain at $Q \in S(\infty)$ if $\Omega \cap S(\infty) = \{Q\}$, and there exists a neighborhood V of Q which is contained in a nontangential cone at Q. The *nontangential cone* at Q is, by definition, $T_c = \{x \in M \,;\, \rho(x, \gamma) < c\}$ for a positive constant c, where $\gamma : [0, \infty) \longrightarrow M$ is a geodesic ray in (M, g) with $\gamma(0) = o, \gamma(\infty) = Q$, and ρ is the metric of Definition 5.13 on $\bar{M}$ under the identification $S(\infty)$ and $\mathcal{M}$.

Remark 5.36. (1) Sasaki ['84] showed that if (M, g) is a simply connected complete Riemannian manifold whose curvature is negative and asymptotically constant curvature $-c^2$, $c > 0$, then $S(\infty)$ is homeomorphic to the Martin boundary $\mathcal{M}$ and $\mathcal{M} = \mathcal{M}_1$ (cf. Definition 5.17).

(2) Ancona ['87] extends Theorem 5.19 to a general elliptic operator (see also Ito ['64-2]).

(3) Arai ['87], ['89] studied Fatou type theorems of the boundary behavior of harmonic functions, and BMO on negatively curved manifolds.

5.5. Liouville type theorems for harmonic functions

One of the first remarkable results on existence of harmonic functions on a complete Riemannian manifold is the following theorem:

Theorem 5.37 (cf. Yau ['75]). *Let (M, g) be a complete Riemannian manifold with nonnegative Ricci curvature:* $\mathrm{Ric}_M \geq 0$.

(1) *Then any positive harmonic function on (M, g) must be a constant.*

(2) *Moreover assume that (M, g) has a point $p \in M$ whose cut locus is empty. Then any harmonic function f on (M, g) satisfying*

$$\inf_{x \in M} (f(x) + \gamma(x)^s) > -\infty, \quad \text{for some } 0 \leq s < 1,$$

must be a constant. Here $\gamma(x) = d(x, p), x \in M$.

Theorem 5.38 (cf. Yau ['76]). *Let (M, g) be a complete Riemannian manifold. Let $f \in C^\infty(M)$ satisfy $f \Delta f \leq 0$, where $\Delta = \delta d$. If $\int_M f^p \, v_g < \infty$, for some $p > 1$, then f is a constant.*

He also showed in the paper that

(1) There is no nonconstant holomorphic L^p functions on a complete Kähler manifold for some $p > 1$.

(2) Any L^2 harmonic 1 form on a complete Riemannian manifold with nonnegative Ricci curvature is parallel.

(3) As their applications, any noncompact complete Riemannian manifold (M, g) with nonnegative Ricci curvature has infinite volume $\mathrm{Vol}(M, g) = \infty$.

In the case of L^1 harmonic functions, the following is known:

Theorem 5.39 (cf. Li ['84]). *Let (M, g) be a noncompact complete Riemannian manifold whose Ricci curvature satisfies that*

$$\mathrm{Ric}_M(x) \geq -C\left(1 + \gamma(x)^2\right), \forall\, x \in M,$$

where $\gamma(x) = d(x, p)$, $x \in M$ for some $p \in M$. Then any L^1 subharmonic function must be a constant.

The condition of nonnegativity of Ricci curvature can be relaxed as follows:

Theorem 5.40 (Li ['85]). *Let (M, g) be an n-dimensional complete Riemannian manifold.*

(1) *Assume that there exist $C > 0$, $\alpha > 0$ such that the Ricci curvature satisfies*

$$\mathrm{Ric}_M(x) \geq -C\left(1+\gamma(x)^2\right)\left\{\log\left(1+\gamma(x)^2\right)\right\}^{-\alpha}, \forall x \in M.$$

Then any L^1 nonnegative subharmonic function is constant.

(2) *Assume that there exist positive constants $C \leq \delta(n)$ depending only on n such that $\mathrm{Ric}_M(x) \geq -C\,\gamma(x)^{-2}$, $x \in M$. Then any L^p nonnegative subharmonic function is constant for all $0 < p < 1$.*

(3) *Assume that either (M, g) is simply connected and has nonpositive sectional curvature, or (M, g) satisfies $\mathrm{Ric}_M \geq -c$, for some $c > 0$, and $\mathrm{Vol}(B_1(x)) \geq d > 0$, for all $x \in M$. Then each nonnegative L^p subharmonic function is constant for all $0 < p < 1$.*

Kanai ['85] introduced the notions of rough isometry, rough isometric.

Definition 5.41. For two metric spaces $(X, d_X), (Y, d_Y)$, a map (not necessarily continuous) $\phi;\, X \longrightarrow Y$ is said to be *rough isometric* if

(1) image of ϕ is full in Y, i.e., the ϵ-ball of the image of ϕ coincides with Y, and

(2) there exists constsnts $a \geq 1$ and $b \geq 0$ such that

$$\begin{aligned} a^{-1}\, d_X(x_1, x_2) - b &\leq d_Y(\phi(x_1), \phi(x_2)) \\ &\leq a\, d_X(x_1, x_2) + b, \forall\, x_1,\, x_2 \in X. \end{aligned}$$

Then he showed the following:

Theorem 5.42 (Kanai ['85]). (1) *Let (M, g) be an n-dimensional complete Riemannian manifold whose Ricci curvature satisfies $\mathrm{Ric}_M \geq -c$, for some constant $c > 0$. Assume that (M, g) is rough isometric to the standard Euclidean space $(\mathbb{R}^m, g_o)$ with $m \geq n$. Then any positive harmonic function on (M, g) is constant.*

(2) *Let (M, g), (N, h) be complete Riemannian manifolds whose Ricci curvatures are bounded below. Assume that these Riemannian manifolds are rough isometric. Then (M, g) is parabolic if and only*

if (N, h) is parabolic. Here let us recall (M, g) is parabolic if all positive superharmonic function on (M, g) is constant.

He also introduced (cf. Kanai ['85]) the notion of a *parabolic net* of a Riemannian manifold and showed its relation to parabolicity of Riemannian manifold:

Definition 5.43. A countable set P of points of a Riemannian manifold (M, g) is called to be a *net* if there corresponds to $\{N_p\}_{p \in P}$ such that

(1) for all $p \in P$, N_p is a finite subset of P, and
(2) for all p, $q \in P$, $p \in N_q \iff q \in N_p$.

A sequence $\mathbf{P} = \{p_o, \ldots, p_s\}$ of a net P is said to be a *path* if $p_k \in N_{p_{k-1}}$ for all $k = 1, \ldots, s$. The net P is *connected* if each two point can be joined by a path. We define the Laplacian Δ_P acting on functions on P by

$$(\Delta_P f)(p) = -\frac{1}{\# N_p} \sum_{q \in N_p} f(q) + f(p), p \in P,$$

for a function f on P. Then a function f on P is said to be *superharmonic* if $\Delta_P f \geq 0$. A net P is said to be *parabolic* if each positive superharmonic function on P is constant. A subset P of M is *ϵ-separated* if $d(p, q) \geq \epsilon$, for all p, $q \in P$, $p \neq q$. If we take a maximal ϵ-separated subset P in M, it has a net structure, in fact, for each $p \in P$, we may set $N_p = \{q \in P\,;\, 0 < d(p, q) \leq 3\epsilon\}$. We call this P a ϵ-net in (M, g). Then

Theorem 5.44 (Kanai ['85]). *Let (M, g) be a complete Riemannian manifold whose Ricci curvature is bounded below. Then (M, g) is parabolic if and only if for $\forall \epsilon > 0$, any ϵ-net P is parabolic.*

Furthermore, Kanai ['85] defined the notion of Green function on a net P: For each $k = 0, 1, 2, \ldots$, define inductively $\pi_k\,;\, P \times P \longrightarrow \mathbb{R}$ by

$$\pi_o(p, q) = \begin{cases} 1, & p = q, \\ 0, & p \neq q, \end{cases}$$

$$\pi_{k+1}(p, q) = \sum_{r \in P} \pi_k(p, r)\, \pi(r, q),$$

where

$$\pi(p,q) = \begin{cases} \dfrac{1}{\#N_p}, & q \in N_p, \\ 0, & q \notin N_p. \end{cases}$$

Then the *Green function* G_P on a net P is defined by

$$G_P(p,q) = \sum_{k=0}^{\infty} \pi_k(p,q), \quad p,\, q \in P,$$

if the sum is convergent. Then

Theorem 5.45 (Kanai ['85]). *A net P of a complete Riemannian manifold (M,g) is hyperbolic if and only if*

$$G_P(p,q) < \infty, \forall p \neq q \in P.$$

See Gaveau & Okada ['91] for de Rham-Hodge theory and the heat kernels on graphs, and see also Dodziuk ['81], Bérard ['90] about the vanishing theorems of L^2 harmonic sections of a vector bundle.

5.6. Miscellaneous topics of harmonic functions

In this section, we treat with the problem which a complete Riemannian manifold admits a nonconstant bounded harmonic function.

We consider, in this section, a Riemannian manifold (M,g) whose sectional curvature K satisfies $K \geq 0$ outside some compact subset, following Li & Tam ['87-1], ['87-2], and Li ['90]. For such one (M,g), an end E is said to be *large* if, for a fixed point $p \in M$, we put $V_E(t) = \mathrm{Vol}(E \cap B_t(p))$, it holds that $\displaystyle\int_1^{\infty} \frac{t}{V_E(t)}\, dt < \infty$, and we call it *small* otherwise. Then:

Theorem 5.46 (Li & Tam ['87-2]). *Let (M,g) be as above.*

(1) *Let E be a large end. Then there exists a unique positive harmonic function f on (M,g) such that $\lim_{E\ni x\to\infty} f(x) = 1$, and $\lim_{D\ni x\to\infty} f(x) = 0$ for other large end D (if exists).*

(2) *(M,g) admits at least one large end and one small end, say E. Then there exists a unique (up to a positive constant multiple), positive harmonic function g such that $\lim_{E\ni x\to\infty} g(x) = \infty$, $\lim_{D\ni x\to\infty} g(x) = 0$, for any large end D, and g is bounded on the other small end if any.*

(3) *If* (M, g) *has only small ends, then it is parabolic.*

Theorem 5.47 (Li & Tam ['87-2]). *Let* (M, g) *be as above. We denote by* $\mathcal{H}_\infty$ *the space of all bounded harmonic functions on* (M, g). *Then:*

(1) *if* (M, g) *has only small ends. Then* $\dim \mathcal{H}_\infty = 1$, i.e., *any bounded harmonic function is constant.*

(2) *If* (M, g) *has large ends, say* $\{E_i\,;\, i = 1, \ldots, k\}$. *Then* $\dim \mathcal{H}_\infty = k$. *Here we can take as a basis of* $\mathcal{H}_\infty$, *the unique positive harmonic function as in* (1) *of Theorem* 5.46, $f_i, i = 1, \ldots, k$ *on* (M, g) *satisfying* $\lim_{E_i \ni x \to \infty} f_i(x) = 1$, *and* $\lim_{E_j \ni x \to \infty} f_i(x) = 0\,(\forall j \neq i)$.

Theorem 5.48 (Li & Tam ['87-2]). *Let* (M, g) *be as above. We denote by* $\mathcal{H}_+$, *the positive cone of positive harmonic functions. Then:*

(1) *if* (M, g) *has only small ends, then* $\mathcal{H}_+ = \{constant\ functions\}$.

(2) *If* (M, g) *has only* k *large ends, then* $\mathcal{H}_+ \subset \mathcal{H}_\infty$, *and any positive harmonic function is a nonnegative linear combination of* $\{f_i\,;\, i = 1, \ldots, k\}$ *in* (2) *of Theorem* 5.47.

(3) *If* (M, g) *has* k *small ends and* s *large ends, then any positive harmonic function is a nonnegative linear combination of* $\{f_i\,; i = 1, \ldots, k\}$ *as in* (2) *of Theorem* 5.47 *and* $\{g_j\,;\, j = 1, \ldots, s\}$ *positive harmonic functions as in* (2) *of Theorem* 5.46 *corresponding to* s *small ends.*

Next we consider a noncompact complete Kähler manifold (M, g) whose sectional curvature is nonnegative outside some compact subset. Then one gets:

Theorem 5.49 (Li ['90]). *Let* (M, g) *be as above. Assume that* (M, g) *has* $k(\geq 2)$ *large ends* $\{E_i\,;\, i = 1, \ldots, k\}$. *Then there exists a unique bounded harmonic function* h *on* (M, g) *satisfying* $\lim_{E_1 \ni x \to \infty} h(x) = 1$, *and* $\lim_{E_i \ni x \to \infty} h(x) = 0$ *for* $i \neq 1$. *At infinity of each small end,* h *is asymptotically a constant in the interval* $(0, 1)$, *and has a finite Dirichlet integral over* M.

Corollary 5.50 (Li ['90]). *Let* (M, g) *as above. Assume that* (M, g) *has at least* 2 *ends. Then its all ends are small, and there exists a compact subset* $D \subset M$ *such that* $M \setminus D$ *is isometrically product of a compact Kähler manifold with nonnegative sectional curvature and nonnegatively curved Riemann surface with boundary.*

5.7. Open problems

Finally we gather open problems about the Laplacian on a complete

Riemannian manifold:

(1) The first main problem is to determine the spectrum Spect($\Delta + V$) of a complete Riemannian manifold (M, g), The bottom of the (essential) spectrum for a noncompact Riemannian manifold is particularly interesting.

(1-1) The index (i.e., the number of negative eigenvalues of the second variation operator of the volume) of a complete minimal submanifold has been studied by many people. Then the essential spectrum, and the distribution of discrete spectrum of minimal submanifolds with infinite index must be studied next.

(1-2) Show the counting number $N(\lambda) = \#\{\lambda_n\,;\ \lambda_n \leq \lambda\}$ for $\Delta + V$ of (M, g) behaves asymptotically

$$N(\lambda) \sim C_n \int_M (\lambda - V(x))_+{}^{\frac{n}{2}} v_g, \quad \text{as} \quad \lambda \longrightarrow \infty,$$

under certain Ricci curvature condition of (M, g) and the exhausion one of N (cf. section 2.3).

(2) (due to T. Nagasawa) Extend to a complete Riemannian manifold, the following Täcklind's theorem on the Euclidean space for uniqueness of solution of the heat equation: For a positive measurable function on the interval $(0, \infty)$, the only solution of

$$\Delta u + \frac{\partial u}{\partial t} = 0 \quad \text{on} \quad \mathbb{R}^n \times (0, \infty),$$

satisfying $u(x, 0) \equiv 0$, and

$$|u(x,t)| \leq \exp\{|x| h(|x|)\},$$

is $u \equiv 0$ if and only if

$$\int_1^\infty \frac{dr}{h(r)} = \infty.$$

See Nagasawa ['91], for more detail.

(3) Extend theories (existence and Liouville type theorems, etc.) about harmonic functions on a complete Riemannian manifold to ones about harmonic maps between complete Riemannian manifolds. See for example Akutagawa ['89], ['90], Li & Tam ['91-2].

(4) For two quasi- or rough isometric Riemannian manifolds, does the property that the one admits a bounded harmonic function if and only if the other does so, hold ? See section 5.3.

(5) Study the Martin boundary of a Riemannian manifold of non positive curvature outside a compact set (cf. Sasaki ['84], Freire ['91]). Recently a remarkable progress on a study of the Martin boundary of a strictly pseudo-convex domain has been made by H. Arai ['91].

(6) Construct two isospectral bounded domains in the Euclidean space with *smooth* boundaries which are not isometric each other. Examples of isospectral plane domains with piecewise smooth boundaries like tangrams have been constructed by Gordon, Webb & Wolpert ['91].

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Super Lie Groups

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In recent years the theory of super Lie groups has been studied by many authors in different formulations. See [1] for general references. We have developed the theory of super manifolds in previous notes [2] and [3]. With the same formulation used in the latter, here we shall consider super Lie groups and prove some fundamental existence theorems.

§1. Preliminary

In this note we shall basically follow the arguments and notations in [3]. However, we shall make some change in notations so that our arguments will be more coherent with the theory of ordinary Lie groups.

Let M be a super manifold and $\mathcal{O}_z$ the set of all germs of super smooth functions at a point $z \in M$. A super tangent vector at $z \in M$ was defined in [3]. But in this note we define a super tangent vector as follows.

A mapping v of $\mathcal{O}_z$ into Λ whose image of $f \in \mathcal{O}_z$ is written by $v \cdot f \in \Lambda$ is called a *super tangent vector* at $z \in M$ if v satisfies the following conditions: for $f, g \in \mathcal{O}_z$ and $a \in \Lambda$,

$$1) \quad v \cdot (f + g) = v \cdot f + v \cdot g,$$
$$2) \quad v \cdot (fa) = (v \cdot f)a,$$
$$3) \quad v \cdot (fg) = (v \cdot f)g(z) + (-1)^{fg}(v \cdot g)f(z),$$

where f, g in $(-1)^{fg}$ denote their parities of f, g. Then the set of all super tangent vectors at $z \in M$ forms a super vector space called the *super tangent space* at $z \in M$, denoted by $T_z(M)$. This change is not at all essential. Actually, this $T_z(M)$ can be identified with the old $T_z(M)$ in [3] in a natural way. See [1] for the details of super linear algebra. When (z^i) is a local coordinate around $z \in M$, $\{(\frac{\vec{\partial}}{\partial z^i})_z\}$ forms a base

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of the super vector space $T_z(M)$. Then *super vector fields* on a domain in M are defined as usual. The *bracket* $[X,Y]$ of super vector fields X,Y is defined as follows:

$$[X,Y]\cdot f = X\cdot(Y\cdot f) - (-1)^{XY}Y\cdot(X\cdot f),$$

for a super smooth function f on a domain M. On the underlying non-super manifold of a super manifold M, we have defined a non-super tangent space $\mathcal{T}_z(M)$ at a point $z\in M$, which can be identified with the even part $T_z(M)_{[0]}$ of the super tangent space $T_z(M)$ as [3]. Thus each even super vector field X on a super manifold M can be regarded as a vector field $\tilde{X}$ on the underlying non-super manifold of M. Since both operate on functions on the left hand side, we have the following:

$$\widetilde{[X,Y]} = [\tilde{X},\tilde{Y}],$$

for even super vector fields X,Y on M. This formula is different from the previous one in [3].

The almost super structures on a super manifold M have been defined in [3]. Even after our change, the almost super structures are defined as before, since they are defined in terms of multiplications of even super numbers. That is, for an even super tangent vector $v\in T_z(M)_{[0]}$ and $K\in\Gamma_{[0]}$, *the almost super structure* J^K is defined by

$$J^K(\tilde{v}) = \widetilde{(\zeta^K v)}.$$

Then the almost super structure J^K is a linear endomorphism of the non-super tangent space $\mathcal{T}_z(M)$ of M.

After our change of the signature as above, all the previous theorems obtained in [3] hold without any changes.

§2. Super Lie groups and super Lie algebras

A *super Lie algebra* $\mathfrak{g}$ is a super vector space provided with a bracket operation $[\,,\,]$ which satisfies the following: for $u,v,w\in\mathfrak{g}$ and $a,b\in\Lambda$,

(1) $[au,vb] = a[u,v]b,$

(2) $[u+v,w] = [u,w]+[v,w],$

(3) $[u,v+w] = [u,v]+[u,w],$

(4) $[u,v]+(-1)^{uv}[v,u] = 0,$

(5) $[u,[v,w]]+(-1)^{u(v+w)}[v,[w,u]]+(-1)^{w(u+v)}[w,[u,v]] = 0.$

The formula (5) is called *the super Jacobi identity*, which is equivalent to the following (5′).

$$(5') \qquad [u,[v,w]] = [[u,v],w] + (-1)^{uv}[v,[u,w]].$$

The even part of $\mathfrak{g}$ will be denoted by $\mathfrak{g}_{[0]}$. When $\mathfrak{g}$ is $(m|n)$ dimensional as a super vector space, the space $\mathfrak{g}_{[0]}$ can be regarded as an $(m|n)$-dimensional super manifold $\mathbf{R}^{(m|n)}$ in a natural way.

A group G is called a *super Lie group* if G satisfies the following conditions.

(1) G is also a super manifold and
(2) the multiplication in G is super smooth. That is, the mapping $G \times G \ni (a,b) \longrightarrow ab^{-1} \in G$ is super smooth.

When G is a super Lie group, the N-th skeleton G_N $(N \geq 0)$ of the super Lie group G is an ordinary Lie group, whose Lie algebra will be denoted by $\mathfrak{g}_N$. For $N = 0$, the 0-th skeleton G_0 is usually called the *body* of a super Lie group G and is denoted by G_B whose Lie algebra will be denoted by $\mathfrak{g}_B$.

A left-invariant super vector field on a super Lie group is super smooth. We denote by $\mathfrak{g}$ the set of all *left-invariant* super vectors on G. When the super Lie group G is $(m|n)$-dimensional as a super manifold, the super tangent space $T_e(M)$ at the identity $e \in G$ is an $(m|n)$-dimensional super vector space and hence, so is $\mathfrak{g}$. Moreover $\mathfrak{g}$ is a super Lie algebra in a natural way and is called *the super Lie algebra* of the super Lie group G. The set of all even elements of $\mathfrak{g}$ is denoted by $\mathfrak{g}_{[0]}$, which is an infinite-dimensional Lie algebra over $\mathbf{R}$, and is $\mathbf{R}$-linearly isomorphic with the tangent space $\mathcal{T}_e(G)$ of the underlying non-super manifold of G. Then the above argument implies that the Lie algebra of the projective limit of the family $\{\mathfrak{g}_N\}$ is canonically isomorphic with the Lie algebra $\mathfrak{g}_{[0]}$ as a Lie algebra over $\mathbf{R}$. Thus from this point, we shall identify these Lie algebras as follows: $\mathfrak{g}_{[0]} = \varprojlim \mathfrak{g}_N$.

For $N \geq 1$, we denote by p^N_{N-1} the projection of G_N onto G_{N-1} ($\mathfrak{g}_N$ onto $\mathfrak{g}_{N-1}$) and moreover by A_N ($\mathfrak{a}_N$) the kernel of $p^N_{N-1} : G_N \longrightarrow G_{N-1}$ ($p^N_{N-1} : \mathfrak{g}_N \longrightarrow \mathfrak{g}_{N-1}$, respectively). Then the Lie algebra $\mathfrak{a}_N$ is abelian and A_N is homeomorphic to a Euclidean space. We obtain the following exact sequences of Lie algebras and Lie groups.

$$\{0\} \longrightarrow \mathfrak{a}_N \longrightarrow \mathfrak{g}_N \longrightarrow \mathfrak{g}_{N-1} \longrightarrow \{0\},$$

$$\{e\} \longrightarrow A_N \longrightarrow G_N \longrightarrow G_{N-1} \longrightarrow \{e\},$$

where $N \geq 1$ and the 1st exact sequence is splitting.

§3. Fundamental Existence Theorems

We shall prove two fundamental theorems on super Lie groups.

Theorem 1. *Let G be a super Lie group with the super Lie algebra $\mathfrak{g}$ and $\mathfrak{h}$ a super Lie subalgebra of $\mathfrak{g}$. Then there exists uniquely a connected super Lie subgroup H of G whose super Lie algebra is $\mathfrak{h}$. Furthermore if the body H_B of H is a closed subgroup of G_B, then H itself is a closed super Lie subgroup of the super Lie group G.*

Proof. The super Lie subalgebra $\mathfrak{h}$ defines a super differential system D on G. That is, $D_a = (L_a)_{*e}(\mathfrak{h}) \subset T_a(G)$ for $a \in G$ where L_a denotes the left-translation by $a \in G$ and $\mathfrak{h}$ is regarded as a subspace of $T_e(G)$. Then D is an involutive super differential system on G. By Frobenius' Theorem on a super manifold obtained in [3], there exists uniquely a maximum connected integral super submanifold H of the super differential system D through the identity $e \in G$. In fact this super submanifold H is a unique connected super Lie subgroup with the super Lie algebra $\mathfrak{h}$. The second part of the theorem can be proved inductively.

The following lemma gives a sufficient condition for a projective limit of a projective family of Lie groups to be a super Lie group.

Lemma. *Let $\mathfrak{g}$ be a finite dimensional super Lie algebra and $\{G_N\}_{N\geq 0}$ a projective family of connected Lie groups such that the Lie algebra of G_N is the N-th skeleton $\mathfrak{g}_N$ of the even part $\mathfrak{g}_{[0]}$ of $\mathfrak{g}$. If each kernel A_N of the projection p^N_{N-1} of G_N onto G_{N-1} is homeomorphic to a Euclidean space, then the projective limit G of the family $\{G_N\}_{N\geq 0}$ is a super Lie group with the super Lie algebra $\mathfrak{g}$.*

Proof. Let V_B an open set in $\mathfrak{g}_B = \mathfrak{g}_0$ which is diffeomorphic with $U_B = \exp(V_B) \subset G_B = G_0$ through the exponential mapping exp of $\mathfrak{g}_B$ into G_B. Let $U = (p_B)^{-1}(U_B)$ be a domain of G, whose N-th skeleton is denoted by U_N ($N \geq 0$). Similarly we define $V = (p_B)^{-1}(V_B) \subset \mathfrak{g}_{[0]}$ and $V_N = p_N(V) \subset \mathfrak{g}_N$. Then by induction on $N \geq 0$, we can prove that $\exp : \mathfrak{g}_N \longrightarrow G_N$ is a diffeomorphism of V_N onto U_N for each $N \geq 0$ since the kernel A_N of $p^N_{N-1} : G_N \longrightarrow G_{N-1}$ is abelian and homeomorphic to a Euclidean space. Thus $\exp : \mathfrak{g}_{[0]} \longrightarrow G$ defines a diffeomorphism of $V \subset \mathfrak{g}$ onto $U \subset G$. Through this diffeomorphism, we introduce a super manifold structure on $U \subset G$ regarding the domain $V \subset \mathfrak{g}_{[0]}$ a super manifold. We shall prove that for any $a \in U$ the left-translation L_a is a super diffeomorphism around the identity $e \in G$. It is sufficient to show that $(L_a)_* \circ J^K = J^K \circ (L_a)_*$ on $\mathcal{T}_e(G) = \mathfrak{g}_{[0]}$

for each almost super structure J^K $(K \in \Gamma_{[0]})$. Let $a = \exp(X) \in U$ $(X \in V \subset \mathfrak{g}_{[0]} = T_e(G))$. Then we have the following.

$$(\exp)_{*X} = (L_a)_{*e} \circ (\frac{1 - e^{-ad(X)}}{ad(X)}) \quad \text{on} \quad \mathcal{T}_X(\mathfrak{g}_{[0]}) = \mathfrak{g}_{[0]}$$

where $\mathfrak{g}_{[0]}$ is regarded as a non-super regular manifold. Since we introduce a super structure on $U \subset G$ through the exponential mapping and $ad(X) \circ J^K = J^K \circ ad(X)$ on $\mathfrak{g}_{[0]}$, the above formula shows that $(L_a)_{*e} \circ J^K = J^K \circ (L_a)_{*e}$ on $\mathcal{T}_e(G)$. Thus it follows that $(L_a)_{*g} \circ J^K = J^K \circ (L_a)_{*g}$ on $\mathcal{T}_g(G)$ if $g \in U$ is sufficiently close to $e \in G$. And hence L_a is super smooth around the identity e. Now we take $\{(L_a \circ \exp)^{-1}, L_a(U)\}_{a \in G}$ as an atlas defining the super structure on G. The above shows that this is a well-defined super structure on G. Let $a = \exp(X) \in G$. Then $ad(a) = e^{ad(X)}$ on $\mathfrak{g}_{[0]}$ which is commutative with J^K $(K \in \Gamma_{[0]})$. This implies that the right-translation R_a is also super smooth. Therefore the mapping $G \times G \ni (a, b) \longmapsto ab \in G$ is super smooth. Let ψ be the mapping of G onto G which maps $a \in G$ to $a^{-1} \in G$. Then on $\mathcal{T}_e(G)$, we have $(\psi)_{*e} = -id$ and hence $\psi_{*e} \circ J^K = J^K \circ \psi_{*e}$. On the other hand we have $\psi_{*g} = (R_{g^{-1}})_{*e} \circ \psi_{*e} \circ (L_{g^{-1}})_{*g}$ on $\mathcal{T}_g(G)$. Therefore ψ is also super smooth and hence G is a super Lie group with the super Lie algebra $\mathfrak{g}$.

Theorem2. *Let $\mathfrak{g}$ be a finite dimensional super Lie algebra. Then there exists a super Lie group whose super Lie algebra is $\mathfrak{g}$.*

Proof. For $N \geq 0$, let G_N be the connected and simply connected Lie group whose Lie algebra is the N-th skeleton $\mathfrak{g}_N$ of $\mathfrak{g}_{[0]}$. Then the projection p^N_{N-1} of $\mathfrak{g}_N$ onto $\mathfrak{g}_{N-1}$ induces the projection p^N_{N-1} of G_N onto G_{N-1} for each $N \geq 1$. Since the kernel of $p^N_{N-1} : \mathfrak{g}_N \longrightarrow \mathfrak{g}_{N-1}$ is an abelian Lie algebra, the kernel of $p^N_{N-1} : G_N \longrightarrow G_{N-1}$ is an abelian Lie group, denoted by A_N. Then we have the exact sequence of the Lie groups:

$$\{e\} \longrightarrow A_N \longrightarrow G_N \longrightarrow G_{N-1} \longrightarrow \{e\}.$$

Therefore we have the following long exact sequence of homotopy groups:

$$\begin{aligned} \cdots \longrightarrow &\pi_2(G_{N-1}) \longrightarrow \\ \longrightarrow \pi_1(A_N) \longrightarrow \pi_1(G_N) \longrightarrow &\pi_1(G_{N-1}) \longrightarrow \\ \longrightarrow \pi_0(A_N) \longrightarrow \pi_0(G_N) \longrightarrow &\pi_0(G_{N-1}) \longrightarrow \{0\}. \end{aligned}$$

By the assumption, both G_N and G_{N-1} are connected and simply connected and then we have $\pi_1(G_N) = \pi_1(G_{N-1}) = \{e\}$ and $\pi_0(G_N) = \{e\}$.

On the other hand, it is well known that $\pi_2(G) = \{e\}$ for any Lie group G. Thus the kernel A_N is abelian and homeomorphic to a Euclidean space. Therefore the family $\{G_N\}_{N\geq 0}$ satisfies the conditions in the above lemma, and hence the projective limit G of the family $\{G_N\}_{N\geq 0}$ is a super Lie group with the super Lie algebra $\mathfrak{g}$.

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Differential Systems Associated with Simple Graded Lie Algebras

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Dedicated to Professor Noboru Tanaka on his sixtieth birthday

§0. Introduction

This is a survey paper on differential systems associated with simple graded Lie algebras. By a differential system (M, D), we mean a pfaffian system D (or a distribution in Chevalley's sense) on a manifold M, that is, D is a subbundle of the tangent bundle $T(M)$ of M. Our primary subject will be the Lie algebra (sheaf) $\mathcal{A}(M, D)$ of all infinitesimal automorphisms of (M, D).

Let $\mathfrak{g}$ be a simple Lie algebra over the field $\mathbb{R}$ of real numbers. A gradation $\{\mathfrak{g}_p\}_{p\in\mathbb{Z}}$ of $\mathfrak{g}$ is a direct decomposition $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ such that

$$[\mathfrak{g}_p, \mathfrak{g}_q] \subset \mathfrak{g}_{p+q} \qquad \text{for } p,\ q \in \mathbb{Z}.$$

Let $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ be a simple graded Lie algebra over $\mathbb{R}$ satisfying $\mathfrak{g}_p = [\mathfrak{g}_{p+1}, \mathfrak{g}_{-1}]$ for $p < -1$. We denote by G the adjoint group of $\mathfrak{g}$ and let G' be the normalizer of $\mathfrak{g}' = \bigoplus_{p\geqq 0} \mathfrak{g}_p$ in G;

$$G' = \{\, \sigma \in G \mid \sigma(\mathfrak{g}') = \mathfrak{g}' \,\}.$$

We consider the homogeneous space $M_\mathfrak{g} = G/G'$, which is a real or complex manifold (R-space) depending on whether the complexification $\mathbb{C}\mathfrak{g}$ of $\mathfrak{g}$ is simple or $\mathfrak{g}$ is complex simple (see Proposition 3.3 in §3.2 and §4.1). By identifying $\mathfrak{g}$ with the Lie algebra of left invariant vector fields on G, the G'-invariant subspace $\mathfrak{f}^{-1} = \mathfrak{g}_{-1} \oplus \mathfrak{g}'$ induces a G-invariant differential system $D_\mathfrak{g}$ on $M_\mathfrak{g}$, which is a holomorphic differential system when $\mathfrak{g}$ is complex simple. $(M_\mathfrak{g}, D_\mathfrak{g})$ is called the standard differential system of type $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ (§4.1).

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The main purpose of this article is to give an overview of the basic materials both on the geometry of differential systems and on the structure of simple graded Lie algebras over $K = \mathbb{R}$ or $\mathbb{C}$, which culminates to show the following (Corollary 5.4):

The Lie algebra $\mathcal{A}(M_{\mathfrak{g}}, D_{\mathfrak{g}})$ (or more precisely, each stalk $\mathcal{A}_x(M_{\mathfrak{g}}, D_{\mathfrak{g}})$ of the Lie algebra sheaf $\mathcal{A}(M_{\mathfrak{g}}, D_{\mathfrak{g}})$) is isomorphic with $\mathfrak{g}$, except when $(M_{\mathfrak{g}}, D_{\mathfrak{g}})$ is locally isomorphic with the canonical (or contact) system on a real or complex jet space.

For the precise statement, see Corollary 5.4 in §5.2.

Historically E. Cartan, in the course of the classification of simple Lie algebras over $\mathbb{C}$, indicated some of simple Lie algebras of exceptional type as the Lie algebras of the invariance groups of certain pfaffian systems ([C1], [C2]), (thus exihibiting the existence of simple Lie algebras of these types). These discoveries seem to be forgotten during the course of the modern development of the structure theory of semisimple Lie algebras or of the Lie group theory (cf. Introduction of [He]).

On the other hand, after E. Cartan, the equivalence problems of differential sytems, or more generally of geometric structures subordinate to differential systems were investigated and developed by N. Tanaka in [T1], [T2], [T3] and [T4]. Utilizing his theory and the structure theory of simple Lie algebras over $\mathbb{R}$ and $\mathbb{C}$, we shall show the above result, which also reestablishes Cartan's discoveries cited above (see examples in §1.3 and §5.3).

Now let us proceed to the description of the contents of this paper. In §§1 and 2, we shall review the Tanaka theory of regular differential systems. He introduced the graded algebras $\mathfrak{m}(x) = \bigoplus_{p<0} \mathfrak{g}_p(x)$ of a regular differential system (M, D) at each $x \in M$ as the first invariant for the equivalence of differential systems, which are nilpotent graded Lie algebras satisfying $\mathfrak{g}_{-1}(x) = D(x)$ and $\mathfrak{g}_p(x) = [\mathfrak{g}_{p+1}(x), \mathfrak{g}_{-1}(x)]$ for $p < -1$ (see §1.2 for the definition). Let $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ be a fundamental graded algebra, that is, a nilpotent graded Lie algebra satisfying $\mathfrak{g}_p = [\mathfrak{g}_{p+1}, \mathfrak{g}_{-1}]$ for $p < -1$. Then (M, D) is called of type $\mathfrak{m}$, if $\mathfrak{m}(x)$ is isomorphic with $\mathfrak{m}$ at each $x \in M$. Moreover, given a fundamental graded algebra $\mathfrak{m}$, we can construct a model differential system $(M(\mathfrak{m}), D_{\mathfrak{m}})$ of type $\mathfrak{m}$ group theoretically, which is called the standard differential system of type $\mathfrak{m}$ (§1.2). Here we note that, when $\mathfrak{m}$ is the negative part of a simple graded Lie algebra, $(M(\mathfrak{m}), D_{\mathfrak{m}})$ is naturally identified with an open dense submanifold of $(M_{\mathfrak{g}}, D_{\mathfrak{g}})$ (see §4.1).

For a fundamental graded algebra $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$, Tanaka [T2] introduced the notion of the algebraic prolongation $\mathfrak{g}(\mathfrak{m})$ of $\mathfrak{m}$; $\mathfrak{g}(\mathfrak{m}) = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p(\mathfrak{m})$ is a graded Lie algebra satisfying the following conditions:

(1) $\mathfrak{g}_p(\mathfrak{m}) = \mathfrak{g}_p$ for $p < 0$.

(2) For $k \geqq 0$, if $X \in \mathfrak{g}_k(\mathfrak{m})$ and $[X, \mathfrak{m}] = \{0\}$, then $X = 0$.

(3) $\mathfrak{g}(\mathfrak{m})$ is maximum among graded algebras satisfying conditions (1) and (2) above.

Moreover, among the graded Lie algebra $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ satisfying conditions (1) and (2) above, $\mathfrak{g}(\mathfrak{m})$ is characterized by the vanishing of the first cohomology groups $H^{p,1}(\mathfrak{m}, \mathfrak{g})$ for $p \geqq 0$. Here $H^q(\mathfrak{m}, \mathfrak{g}) = \bigoplus_{p\in\mathbb{Z}} H^{p,q}(\mathfrak{m}, \mathfrak{g})$ is the Lie algebra cohomology associated with the representation $\mathrm{ad}\colon \mathfrak{m} \to \mathfrak{gl}(\mathfrak{g})$, which is called the generalized Spencer cohomology of the graded Lie algebra $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ (§2.3). The prolongation $\mathfrak{g}(\mathfrak{m})$ plays a fundamental role in the equivalence problems of regular differential systems of type $\mathfrak{m}$. Especially $\mathfrak{g}(\mathfrak{m})$ describes the structure of the Lie algebra $\mathcal{A}(M(\mathfrak{m}), D_{\mathfrak{m}})$ of all infinitesimal automorphisms of the standard differential system $(M(\mathfrak{m}), D_{\mathfrak{m}})$ of type $\mathfrak{m}$. In particular $\mathcal{A}(M(\mathfrak{m}), D_{\mathfrak{m}})$ is isomorphic with $\mathfrak{g}(\mathfrak{m})$ when $\mathfrak{g}(\mathfrak{m})$ is finite dimensional. We shall review these facts in §§1 and 2 following [T2] and also discuss the Hilbert-Cartan equation as an example (§1.3).

With these preparations, we shall be concerned with the following question: When does $\mathfrak{g}(\mathfrak{m})$ become finite dimensional and simple? The answer to this question (Theorems 5.2 and 5.3) gives us the result stated above. In order to answer this question, we first classify, for a simple Lie algebra $\mathfrak{g}$ over $K = \mathbb{R}$ or $\mathbb{C}$, the gradations $\{\mathfrak{g}_p\}_{p\in\mathbb{Z}}$ of $\mathfrak{g}$ satisfying $\mathfrak{g}_p = [\mathfrak{g}_{p+1}, \mathfrak{g}_{-1}]$ for $p < -1$, which turns out to be equivalent to the classification of parabolic subalgebras $\mathfrak{g}' = \bigoplus_{p\geqq 0} \mathfrak{g}_p$ of $\mathfrak{g}$. This allows us to describe the gradation $\{\mathfrak{g}_p\}_{p\in\mathbb{Z}}$ of $\mathfrak{g}$ in terms of the root (or restricted root) space decomposition of $\mathfrak{g}$ (cf. [K-A]) and to apply the method of Kostant [K] to compute $H^{p,1}(\mathfrak{m}, \mathfrak{g})$ for $p \geqq 0$, which is carried out in §5.2. Namely, for a complex simple Lie algebra $\mathfrak{g}$, let us fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and a simple root system $\Delta = \{\alpha_1, \dots, \alpha_\ell\}$ of the root system Φ relative to $\mathfrak{h}$. Take any non-empty subset Δ_1 of Δ and put

$$\Phi_k^+ = \{\, \alpha = \sum_{i=1}^{\ell} n_i(\alpha)\, \alpha_i \in \Phi^+ \mid \sum_{\alpha_i \in \Delta_1} n_i(\alpha) = k \,\} \quad \text{for } k \geqq 0.$$

Then we obtain a gradation $\{\mathfrak{g}_p\}_{p\in\mathbb{Z}}$ of $\mathfrak{g}$ satisfying $\mathfrak{g}_p = [\mathfrak{g}_{p+1}, \mathfrak{g}_{-1}]$ for

$p < -1$ by putting

$$\mathfrak{g}_0 = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi_0^+} (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}),$$

$$\mathfrak{g}_k = \bigoplus_{\alpha \in \Phi_k^+} \mathfrak{g}_\alpha, \quad \mathfrak{g}_{-k} = \bigoplus_{\alpha \in \Phi_k^+} \mathfrak{g}_{-\alpha} \qquad (k > 0),$$

where $\mathfrak{g}_\alpha$ denotes the root space corresponding to $\alpha \in \Phi$. We denote the simple graded Lie algebra $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ obtained from Δ_1 in this manner by (X_ℓ, Δ_1), when $\mathfrak{g}$ is a simple Lie algebra of type X_ℓ. Here X_ℓ stands for the Dynkin diagram of $\mathfrak{g}$ representing Δ and Δ_1 is a subset of vertices of X_ℓ. Then every complex graded Lie algebra $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ satisfying $\mathfrak{g}_p = [\mathfrak{g}_{p+1}, \mathfrak{g}_{-1}]$ for $p < -1$ is conjugate to (X_ℓ, Δ_1) for some $\Delta_1 \subset \Delta$ (Theorem 3.12). In the real case, we can utilize the Satake diagram to describe the gradation of $\mathfrak{g}$ (see §3.4).

Now we can state one of the main results of this paper

Theorem 5.2′. *Let $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ be a simple graded Lie algebra over $\mathbb{C}$ such that $\mathfrak{g}_p = [\mathfrak{g}_{p+1}, \mathfrak{g}_{-1}]$ for $p < -1$. Then $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ is the prolongation of $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ except for the following three cases.*

(1) $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$.

(2) $\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$ *is a complex contact gradation, that is,* $\dim \mathfrak{g}_{-2} = 1$.

(3) $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ *is isomorphic with* $(A_\ell, \{\alpha_1, \alpha_i\})$ $(1 < i < \ell)$ *or* $(C_\ell, \{\alpha_1, \alpha_\ell\})$.

We shall obtain also the real version of this theorem (Theorem 5.3).

In §4, we shall discuss the standard differential system $(M_\mathfrak{g}, D_\mathfrak{g})$ of type $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$. First we shall consider the contact gradation of $\mathfrak{g}$ and show that every complex simple Lie algebra other than $\mathfrak{sl}(2, \mathbb{C})$ admits a unique complex contact gradation up to conjugacy. We discuss the unified description of the standard contact manifolds $(J_\mathfrak{g}, C_\mathfrak{g})$ associated with this contact gradation via the adjoint representation of $\mathrm{Int}(\mathfrak{g})$, which were originally found by Boothby [Bo] as compact simply connected homogeneous complex contact manifolds. Moreover we shall reproduce the explicit matrix description, due to Takeuchi [Tk1], of the root space decompositions of simple Lie algebras over $\mathbb{C}$ of the classical type, which gives us explicit pictures of $M_\mathfrak{g}$ in these cases. With the aid of this description, we shall discuss those standard differential systems $(M_{\mathfrak{g}}, D_{\mathfrak{g}})$ which are isomorphic with the canonical systems on Grassmann bundles (geometric jet spaces). These are obviously exceptions for the assertion of our main result stated at the beginning of this introduction. More precisely we shall show that the standard differential system $(M_{\mathfrak{g}}, D_{\mathfrak{g}})$ of types $(A_\ell, \{\alpha_1, \alpha_{i+1}\})$ and $(C_\ell, \{\alpha_1, \alpha_\ell\})$ are isomorphic with the canonical system $(J(\mathbb{C}P^\ell, i), C)$ on the Grassmann bundle $J(\mathbb{C}P^\ell, i)$ over the complex projective space $\mathbb{C}P^\ell$, consisting of i-dimensional contact elements to $\mathbb{C}P^\ell$, and the canonical system $(L(\mathbb{C}P^{2\ell-1}), E)$ on the Lagrange-Grassmann bundle $L(\mathbb{C}P^{2\ell-1})$ over the odd dimensional (contact) projective space $\mathbb{C}P^{2\ell-1}$ respectively.

In §5, we shall first review the harmonic theory of Kostant [K] for the Lie algebra cohomology and apply his method to compute $H^{p,1}(\mathfrak{m}, \mathfrak{g})$ and $H^{p,2}(\mathfrak{m}, \mathfrak{g})$ for $p \geqq 0$, which gives us the main results (Theorems 5.2, 5.3 and Corollary 5.4) of this article. Here we include the computation of $H^{p,2}(\mathfrak{m}, \mathfrak{g})$ for $p \geqq 0$, which is important to know the fundamental invariants of the normal Cartan connection, constructed by Tanaka [T4], for the geometric structures associated with a simple graded Lie algebra $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ such that $\mathfrak{g}$ is the prolongation of $(\mathfrak{m}, \mathfrak{g}_0)$. Especially, by these computations combined with Theorem of Tanaka [T4], we can find many examples of regular differential systems (M, D) of type $\mathfrak{m}$ with no local invariants, whose Lie algebra $\mathcal{A}(M, D)$ of all infinitesimal automorphisms are finite dimensional and simple. Finally in §5.4, we shall discuss the reducible primitive actions of finite dimensional Lie groups, following [Go], [K-N, I and II] and [Gu], and characterize the standard differential systems $(M_{\mathfrak{g}}, D_{\mathfrak{g}})$ of type $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ whose isotropy subalgebra $\mathfrak{g}'$ are maximal parabolic, as homogeneous differential systems which have nonlinear reducible primitive actions of Lie groups (cf. [O1], [Go]).

The main results of this paper (Theorems 5.2, 5.3 and Corollary 5.4) were obtained by the author around 1985 (unpublished) by a different method based on the finite dimensionality criterion of the prolongation $\mathfrak{g}(\mathfrak{m})$ (Corollary 2 to Theorem 11.1 of [T2]) due to Tanaka. The present cohomological method with the powerful theorem of Kostant has the advantage to produce the result for the second cohomology $H^{p,2}(\mathfrak{m}, \mathfrak{g})$ at the same time.

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§1. Symbol algebras of (M, D)

1.1. Regular differential systems

By a differential system (M, D), we mean a subbundle D of the tangent bundle $T(M)$ of a manifold M of dimension n. Locally D is defined by 1-forms $\omega_1, \dots, \omega_{n-r}$ such that $\omega_1 \wedge \cdots \wedge \omega_{n-r} \neq 0$ at each point, where r is the rank of D;

$$D = \{\, \omega_1 = \cdots = \omega_{n-r} = 0 \,\}.$$

For two differential systems (M, D) and $(\hat{M}, \hat{D})$, a diffeomorphism ϕ of M onto $\hat{M}$ is called an isomorphism of (M, D) onto $(\hat{M}, \hat{D})$ if the differential map ϕ_* of ϕ sends D onto $\hat{D}$. Our subject will be the Lie algebra $\mathcal{A}(M, D)$ of infinitesimal automorphisms of (M, D). For a vector field X on M, X belongs to $\mathcal{A}(M, D)$ if and only if

$$L_X \omega_i \equiv 0 \pmod{\omega_1, \dots, \omega_{n-r}} \qquad \text{for } i = 1, \dots, n - r,$$

or equivalently, if and only if

$$[X, \mathcal{D}] \subset \mathcal{D},$$

where $\mathcal{D} = \Gamma(D)$ denotes the space of sections of D.

By the Frobenius theorem, we know that D is completely integrable if and only if

$$d\omega_i \equiv 0 \pmod{\omega_1, \dots, \omega_{n-r}} \qquad \text{for } i = 1, \dots, n - r,$$

or equivalently, if and only if

$$[\mathcal{D}, \mathcal{D}] \subset \mathcal{D}.$$

When D is completely integrable, it is easily seen that $\mathcal{A}(M, D)$ is infinite dimensional.

Thus, for a non-integrable differential system D, we are led to consider the derived system ∂D of D, which is defined, in terms of sections, by

$$\partial \mathcal{D} = \mathcal{D} + [\mathcal{D}, \mathcal{D}].$$

In general ∂D is obtained as a subsheaf of the tangent sheaf of M (for the precise argument, see [T2] or [Y1]). Moreover higher derived systems $\partial^k D$ are usually defined successively by

$$\partial^k D = \partial(\partial^{k-1} D),$$

where we put $\partial^0 D = D$ for convention.

On the other hand we define the k-th weak derived system $\partial^{(k)} D$ of D inductively by

$$\partial^{(k)}\mathcal{D} = \partial^{(k-1)}\mathcal{D} + [\mathcal{D}, \partial^{(k-1)}\mathcal{D}],$$

where $\partial^{(0)} D = D$ and $\partial^{(k)}\mathcal{D}$ denotes the space of sections of $\partial^{(k)} D$.

A differential system (M, D) is called regular, if $D^{-(k+1)} = \partial^{(k)} D$ are subbundles of $T(M)$ for every integer $k \geqq 1$. For a regular differential system (M, D), we have ([T2, Proposition 1.1])

$$\begin{cases} (1) & \textit{There exists a unique integer } \mu > 0 \textit{ such that, for all } k \geqq \mu, \\ & \qquad D^{-k} = \cdots = D^{-\mu} \supsetneqq D^{-\mu+1} \supsetneqq \cdots \supsetneqq D^{-2} \supsetneqq D^{-1} = D, \\ (2) & [\mathcal{D}^p, \mathcal{D}^q] \subset \mathcal{D}^{p+q} \qquad \text{for all } p,\ q < 0. \end{cases}$$

where $\mathcal{D}^p$ denotes the space of sections of D^p. (2) can be checked easily by induction on q.

Thus $D^{-\mu}$ is the smallest completely integrable differential system, which contains $D = D^{-1}$.

1.2. Graded algebras associated with (M, D)

Let (M, D) be a regular differential system such that $T(M) = D^{-\mu}$. As a first invariant for non-integrable differential systems, we now define the *graded algebra* $\mathfrak{m}(x)$ *associated with a differential system* (M, D) at $x \in M$, which was introduced by N. Tanaka [T2].

We put $\mathfrak{g}_{-1}(x) = D^{-1}(x)$, $\mathfrak{g}_p(x) = D^p(x)/D^{p+1}(x)$ $(p < -1)$ and

$$\mathfrak{m}(x) = \bigoplus_{p=-1}^{-\mu} \mathfrak{g}_p(x).$$

Let ϖ_p be the projection of $D^p(x)$ onto $\mathfrak{g}_p(x)$. Then, for $X \in \mathfrak{g}_p(x)$ and $Y \in \mathfrak{g}_q(x)$, the bracket product $[X, Y] \in \mathfrak{g}_{p+q}(x)$ is defined by

$$[X, Y] = \varpi_{p+q}([\tilde{X}, \tilde{Y}]_x),$$

where $\tilde{X}$ and $\tilde{Y}$ are any element of $\mathcal{D}^p$ and $\mathcal{D}^q$ respectively such that $\varpi_p(\tilde{X}_x) = X$ and $\varpi_q(\tilde{Y}_x) = Y$. From

$$[f\tilde{X}, g\tilde{Y}] = f \cdot g[\tilde{X}, \tilde{Y}] + f(\tilde{X}g)\tilde{Y} - g(\tilde{Y}f)\tilde{X},$$

for vector fields $\tilde{X}$, $\tilde{Y}$ and functions f, g on M, it follows immediately that $[X, Y] \in \mathfrak{g}_{p+q}(x)$ is well-defined for $X \in \mathfrak{g}_p(x)$ and $Y \in \mathfrak{g}_q(x)$ (cf. [T2, Lemma 1.1]).

Endowed with this bracket operation, by (2) above, $\mathfrak{m}(x)$ becomes a nilpotent graded Lie algebra such that $\dim \mathfrak{m}(x) = \dim M$ and satisfies

$$\mathfrak{g}_p(x) = [\mathfrak{g}_{p+1}(x), \mathfrak{g}_{-1}(x)] \qquad \text{for } p < -1.$$

We call $\mathfrak{m}(x)$ the *symbol algebra of* (M, D) at $x \in M$ for short.

Furthermore, let $\mathfrak{m}$ be a fundamental graded Lie algebra of μ-th kind, that is,

$$\mathfrak{m} = \bigoplus_{p=-1}^{-\mu} \mathfrak{g}_p$$

is a nilpotent graded Lie algebra such that

$$\mathfrak{g}_p = [\mathfrak{g}_{p+1}, \mathfrak{g}_{-1}] \qquad \text{for } p < -1.$$

Then (M, D) is called of type $\mathfrak{m}$ if the symbol algebra $\mathfrak{m}(x)$ is isomorphic with $\mathfrak{m}$ at each $x \in M$.

Conversely, given a fundamental graded Lie algebra $\mathfrak{m}$, we can construct a model differential system of type $\mathfrak{m}$ as follows: Let $M(\mathfrak{m})$ be the simply connected Lie group with Lie algebra $\mathfrak{m}$. Identifying $\mathfrak{m}$ with the Lie algebra of left invariant vector fields on $M(\mathfrak{m})$, $\mathfrak{g}_{-1}$ defines a left invariant subbundle $D_{\mathfrak{m}}$ of $T(M(\mathfrak{m}))$. By definition of symbol algebras, it is easy to see that $(M(\mathfrak{m}), D_{\mathfrak{m}})$ is a regular differential system of type $\mathfrak{m}$. $(M(\mathfrak{m}), D_{\mathfrak{m}})$ is called the standard differential system of type $\mathfrak{m}$.

1.3. The Hilbert-Cartan equation

As a good illustration of our previous discussion, we shall now calculate the symbol algebras of a differential system (R, D), which is associated with the following underdetermined ordinary differential equation studied by Hilbert [H] and Cartan [C3]:

$$\text{(H.C)} \qquad \frac{dv}{dx} = \left(\frac{d^2u}{dx^2}\right)^2$$

As usual, we consider a hypersurface R', defined by (H.C), in the space J^2 of 2-jets for 2-unknown and 1-independent variables with coordinate sytem $(x, u, v, u', v', u'', v'')$;

$$R' = \{ v' = (u'')^2 \}.$$

Our differential system (R', D') is obtained by restricting to R' the canonical (or contact) system on J^2;

$$D' = \{ \omega_1' = \omega_2' = \omega_3' = \omega_4' = 0 \},$$

where

$$\begin{cases} \omega_1' = dv - (u'')^2 dx, \\ \omega_2' = du - u' dx, \\ \omega_3' = du' - u'' dx, \\ \omega_4' = d(u'')^2 - v'' dx = 2u'' du'' - v'' dx. \end{cases}$$

For the regularity condition, we shall work on the domain $R = \{ u'' \neq 0 \}$ in R' and take (x, u, v, p, r, t) as a coordinate system on R, where $p = u'$, $r = u''$ and $t = \frac{1}{2}(u'')^{-1} v''$. Then (R, D) is given on this coordinate system by

$$D = \{ \omega_1 = \omega_2 = \omega_3 = \omega_4 = 0 \},$$

where $\omega_1 = dv - r^2 dx$, $\omega_2 = du - p\,dx$, $\omega_3 = dp - r\,dx$ and $\omega_4 = dr - t\,dx$.

First we calculate

$$\begin{cases} d\omega_1 = 2r\,dx \wedge dr = 2r\,dx \wedge \omega_4, \\ d\omega_2 = dx \wedge dp = dx \wedge \omega_3, \\ d\omega_3 = dx \wedge dr = dx \wedge \omega_4, \\ d\omega_4 = dx \wedge dt. \end{cases} \tag{1.1}$$

To locate the derived system ∂D, we look at the equalities (1.1) modulo the ideal spanned by 1-forms ω_1, ω_2, ω_3 and ω_4:

$$\begin{cases} d\omega_1 \equiv d\omega_2 \equiv d\omega_3 \equiv 0, \\ d\omega_4 \equiv dx \wedge dt. \end{cases} \quad (\text{mod } \omega_1, \omega_2, \omega_3, \omega_4)$$

Then, since $d\omega_i(X, Y) = -\omega_i([X, Y])$ for X, $Y \in \mathcal{D}$ ($i = 1, 2, 3$), it follows that

$$D^{-2} = \partial D = \{ \omega_1 = \omega_2 = \omega_3 = 0 \}.$$

To locate $\partial^2 D$, we proceed to look at $d\omega_1$, $d\omega_2$ and $d\omega_3$ modulo 1-forms ω_1, ω_2 and ω_3. Putting $\widetilde{\omega}_1 = \omega_1 - 2r\,\omega_3$, we have

$$d\widetilde{\omega}_1 = 2\omega_3 \wedge dr = 2\omega_3 \wedge \omega_4 + 2t\,\omega_3 \wedge dx.$$

Hence we get

$$D^{-2} = \{\,\widetilde{\omega}_1 = \omega_2 = \omega_3 = 0\,\},$$

and

$$\begin{cases} d\widetilde{\omega}_1 \equiv d\omega_2 \equiv 0, \\ d\omega_3 \equiv dx \wedge \omega_4. \end{cases} \quad (\mathrm{mod}\ \widetilde{\omega}_1, \omega_2, \omega_3)$$

This implies that $\partial^2 D = \partial(D^{-2})$ is defined by $\widetilde{\omega}_1$ and ω_2. However, in this case, from $\mathrm{rank}\, D^{-2} = \mathrm{rank}\, D^{-1} + 1$, we have

$$\mathcal{D}^{-2} + [\mathcal{D}^{-1}, \mathcal{D}^{-2}] = \mathcal{D}^{-2} + [\mathcal{D}^{-2}, \mathcal{D}^{-2}].$$

Namely we have $D^{-3} = \partial^2 D$. To proceed, we put $2\bar{\omega}_1 = \widetilde{\omega}_1 + 2t\,\omega_2$. Then we have

$$D^{-3} = \{\,\bar{\omega}_1 = \omega_2 = 0\,\},$$

and

$$\text{(1.2)} \qquad \begin{cases} d\bar{\omega}_1 = \omega_3 \wedge \omega_4 + \omega_6 \wedge \omega_2, \\ d\omega_2 = \omega_5 \wedge \omega_3, \\ d\omega_3 = \omega_5 \wedge \omega_4, \\ d\omega_4 = \omega_5 \wedge \omega_6. \end{cases}$$

where $\omega_5 = dx$ and $\omega_6 = dt$. From $d\bar{\omega}_1 \equiv \omega_3 \wedge \omega_4$, $d\omega_2 \equiv \omega_5 \wedge \omega_3$ $(\mathrm{mod}\ \bar{\omega}_1, \omega_2)$, we obtain

$$T(R) = \partial^3 D.$$

On the other hand, to locate D^{-4}, we should ignore the contributions of elements in $[\mathcal{D}^{-3}, \mathcal{D}^{-3}]$, which are not contained in $[\mathcal{D}^{-1}, \mathcal{D}^{-3}]$. Thus we must look at $d\bar{\omega}_1$ and $d\omega_2$ modulo $\bar{\omega}_1$, ω_2 and $\omega_3 \wedge \omega_4$:

$$\begin{cases} d\bar{\omega}_1 \equiv 0, \\ d\omega_2 \equiv \omega_5 \wedge \omega_3. \end{cases} \quad (\mathrm{mod}\ \bar{\omega}_1, \omega_2, \omega_3 \wedge \omega_4)$$

This implies that

$$D^{-4} = \{\,\bar{\omega}_1 = 0\,\}.$$

Furthermore, we have

$$d\bar{\omega}_1 \equiv \omega_3 \wedge \omega_4 + \omega_6 \wedge \omega_2 \quad (\mathrm{mod}\ \bar{\omega}_1, \omega_2 \wedge \omega_3, \omega_2 \wedge \omega_4)$$

Hence we get

$$T(R) = D^{-5}.$$

Thus we see that (R, D) is a regular differential system of type $\mathfrak{m}_6$, where

$$\mathfrak{m}_6 = \bigoplus_{p=-1}^{-5} \mathfrak{g}_p,$$

is the fundamental graded algebra of 5-th kind, whose Maurer-Cartan equation is given by (1.2). Namely $\mathfrak{m}_6$ is a 6-dimensional nilpotent graded Lie algebra, which is described as follows: There exists a basis $\{e_1, \dots, e_6\}$ of $\mathfrak{m}_6$ such that each $\mathfrak{g}_p$ is spanned by the following vectors

$$\mathfrak{g}_{-5} = \langle e_1 \rangle, \qquad \mathfrak{g}_{-4} = \langle e_2 \rangle, \qquad \mathfrak{g}_{-3} = \langle e_3 \rangle,$$
$$\mathfrak{g}_{-2} = \langle e_4 \rangle, \qquad \mathfrak{g}_{-1} = \langle e_5, e_6 \rangle,$$

and that the bracket product is given by

$$[e_6, e_5] = e_4, \qquad [e_4, e_5] = e_3, \qquad [e_3, e_5] = e_2,$$
$$[e_2, e_6] = [e_4, e_3] = e_1, \qquad [e_i, e_j] = 0 \quad \text{otherwise.}$$

A notable fact for (R, D) is that we obtain the strict equalities (1.2) instead of mod equalities for defining 1-forms $\bar{\omega}_1$, ω_2, ω_3, ω_4 of D, that is, (R, D) is isomorphic with the standard differential system of type $\mathfrak{m}_6$. Because of this fact, we shall see later in §5.2 that the Lie algebra $\mathcal{A}(R, D)$ of infinitesimal automorphisms of (R, D) is isomorphic with the 14-dimensional simple Lie algebra G_2 (cf. [C3], [A-K-O]). In fact we shall encounter $\mathfrak{m}_6$ in §3.4 in connection with the root space decomposition of G_2.

Another example of a historical interest is the following differential system (X, E) on $X = \mathbb{R}^5$, which was found by E. Cartan [C2];

$$E = \{\, \omega_1 = \omega_2 = \omega_3 = 0 \,\},$$

where

$$\begin{cases} \omega_1 = dx_1 + (x_3 + \frac{1}{2} x_4 x_5)\, dx_4, \\ \omega_2 = dx_2 + (x_3 - \frac{1}{2} x_4 x_5)\, dx_5, \\ \omega_3 = dx_3 + \frac{1}{2}(x_4\, dx_5 - x_5\, dx_4), \end{cases}$$

and $(x_1, x_2, x_3, x_4, x_5)$ is a coordinate system of $X = \mathbb{R}^5$. We have

$$\begin{cases} d\omega_1 = \omega_3 \wedge \omega_4, \\ d\omega_2 = \omega_3 \wedge \omega_5, \\ d\omega_3 = \omega_4 \wedge \omega_5, \end{cases} \tag{1.3}$$

where $\omega_4 = dx_4$ and $\omega_5 = dx_5$. In this case we may calculate symbol algebras of (X, E) as follows. We take a dual basis $\{X_1, \ldots, X_5\}$ of vector fields on X to a basis of 1-forms $\{\omega_1, \ldots, \omega_5\}$ given above;

$$X_1 = \frac{\partial}{\partial x_1}, \quad X_2 = \frac{\partial}{\partial x_2}, \quad X_3 = \frac{\partial}{\partial x_3},$$
$$X_4 = \frac{\partial}{\partial x_4} + \frac{1}{2}x_5\frac{\partial}{\partial x_3} - (x_3 + \frac{1}{2}x_4x_5)\frac{\partial}{\partial x_1},$$
$$X_5 = \frac{\partial}{\partial x_5} - \frac{1}{2}x_4\frac{\partial}{\partial x_3} - (x_3 - \frac{1}{2}x_4x_5)\frac{\partial}{\partial x_2}.$$

Then we calculate, or from (1.3),

$$[X_5, X_4] = X_3, \quad [X_5, X_3] = X_2, \quad [X_4, X_3] = X_1,$$

and $[X_i, X_j] = 0$ otherwise. This implies that $E^{-2} = \{\, \omega_1 = \omega_2 = 0\,\}$, $E^{-3} = T(X)$ and that (X, E) is isomorphic with the standard differential system of type $\mathfrak{m}_5$, where

$$\mathfrak{m}_5 = \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$$

is the fundamental graded algebra of third kind, whose Maurer-Cartan equation is given by (1.3). Here we note that the Lie algebra structure of $\mathfrak{m}_5$ is uniquely determined by the requirement that $\mathfrak{m}$ is fundamental, $\dim \mathfrak{g}_{-3} = \dim \mathfrak{g}_{-1} = 2$ and $\dim \mathfrak{g}_{-2} = 1$ (cf. [C2], [T2]). In fact $\mathfrak{m}_5$ is the universal fundamental graded algebra of third kind with $\dim \mathfrak{g}_{-1} = 2$ (see [T2, §3]). We shall encounter $\mathfrak{m}_5$ in §3.4 in connection with the root space decomposition of G_2.

§2. Algebraic prolongation of $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$

2.1. Review of the prolongation of G-structure

We first review the notion of the algebraic prolongation of the usual G-structure theory (cf. [St], [K2]). Let G be a Lie subgroup of $GL(V)$, where V is a real vector space of dimension n. A G-structure on a manifold M of dimension n is, by definition, a G-reduction P_G of the frame bundle $F(M)$ of M. Let $\mathfrak{g}$ be the Lie algebra of G. As is well-known (cf. [St], [K2]), the notion of the (algebraic) prolongation of $\mathfrak{g}$ originates from the calculation of infinitesimal automorphisms of the flat G-structure. A basis $\{e_1, \ldots, e_n\}$ of V gives a global trivialization of the frame bundle $F(V)$ of V. Then the flat G-structure on V is given as the G-subbundle $P_G^o = V \times G$ of $F(V) = V \times GL(V)$.

To seek infinitesimal automorphisms of P_G^o, we may proceed as follows: Take a linear coordinate system $(x_1, \ldots, x_n)$ given by the above basis of V. Owing to the global trivialization of $F(V)$, every vector field X on V is identified with a V-valued function f_X on V by putting

$$f_X = \xi(X) = (\xi_1, \cdots, \xi_n),$$

where $\xi = (dx_1, \ldots, dx_n)$ is a V-valued 1-form on V and

$$X = \sum_{i=1}^{n} \xi_i \frac{\partial}{\partial x_i}.$$

By utilizing the V-valued 1-form ξ, the derivatives of f_X can be expressed as the coefficient matrix f_X^0 of $d\,f_X$ with respect to ξ, that is, the $\mathfrak{gl}(V) = V \otimes V^*$-valued function f_X^0 on V is defined by

$$v(f_X) = d\,f_X(v) = f_X^0(\xi(v)) = f_X^0(v) \qquad \text{for } v \in V \cong T_x(V),$$

Here we regard v as a tangent vector at $x \in V$ on the left side of the equalities and as a vector in V on the right side. We shall write this equality, in short, as

$$d\,f_X^{-1} = [f_X^0, \xi],$$

where $f_X^{-1} = f_X$. The second derivatives of f_X can be obtained as the coefficient matrix f_X^1 of $d\,f_X^0$ with respect to ξ, that is, the $\mathfrak{gl}(V) \otimes V^*$-valued function f_X^1 on V is defined by

$$d\,f_X^0 = [f_X^1, \xi].$$

Here, by the compatibility condition for second derivatives (or by the chain rule), f_X^1 actually takes values in $V \otimes S^2(V^*) \subset \mathfrak{gl}(V) \otimes V^* = V \otimes V^* \otimes V^*$. Inductively the $(k+1)$-th derivatives of f_X can be expressed as the coefficient matrix f_X^k of $d\,f_X^{k-1}$ with respect to ξ, that is, the $V \otimes S^{k+1}(V^*)$-valued function f_X^k on V is defined by

$$d\,f_X^{k-1} = [f_X^k, \xi],$$

where $S^{k+1}(V^*)$ denotes the $(k+1)$-th symmetric power of V^*.

Now, for a vector field X on V, let $\tilde{X}$ be the lift of X to $F(V)$, that is, $\tilde{X}$ is a vector field on $F(V)$ generated by the differential flow $(\phi_t)_*$ of the (local) flow ϕ_t of X. Then X is an infinitesimal automorphism of the flat G-structure P_G^o if and only if $\tilde{X}$ is tangent to P_G^o. This is equivalent to the condition that f_X^0 is a $\mathfrak{g}$-valued function on V. Thus, for higher order derivatives, we see that f_X^k takes values in

$$\mathfrak{g}^{(k)} = \mathfrak{g} \otimes \otimes^k V^* \cap V \otimes S^{k+1}(V^*).$$

Here $\mathfrak{g}^{(k)}$ is called the k-th prolongation of $\mathfrak{g}$. Especially the $(k+1)$-th coefficient of the Taylor expansion of f_X takes values in $\mathfrak{g}^{(k)}$ at the origin of V. Conversely, for an element $a \in \mathfrak{g}^{(k)}$, there exists a unique polynomial (of homogeneous degree $k+1$) vector field X such that X is an infinitesimal automorphism of P_G^o and that the coefficient of the Taylor expansion of f_X at the origin coincides with $a \in \mathfrak{g}^{(k)}$.

In this way the structure of the Lie algebra of infinitesimal automorphisms of P_G^o can be expressed by the graded Lie algebra;

$$\bigoplus_{p=-1}^{\infty} \mathfrak{g}^{(p)},$$

where $\mathfrak{g}^{(-1)} = V$, $\mathfrak{g}^{(0)} = \mathfrak{g}$, and the bracket operation is defined accordingly. Here we note that $\mathfrak{g}^{(-1)} = V$ corresponds to constant coefficient vector fields. For the details, we refer the reader to [K2] or [St].

2.2. Infinitesimal automorphisms of $(M(\mathfrak{m}), D_{\mathfrak{m}})$

Let $\mathfrak{m}$ be a fundamental graded Lie algebra of μ-th kind. In the same spirit as in the previous section, we are going to seek infinitesimal automorphisms of our model (flat) differential system of type $\mathfrak{m}$, that is, the standard differential system $(M(\mathfrak{m}), D_{\mathfrak{m}})$ of type $\mathfrak{m}$.

Let ξ be the Maurer-Cartan form on $M(\mathfrak{m})$, that is, ξ is a $\mathfrak{m}$-valued 1-form on $M(\mathfrak{m})$ such that

$$\xi(X_x) = X \qquad \text{for } X \in \mathfrak{m} \text{ and } x \in M(\mathfrak{m}),$$

where $\mathfrak{m}$ is identified with the Lie algebra of left invariant vector fields on $M(\mathfrak{m})$. Then, for $p < 0$, $D_{\mathfrak{m}}^p = \partial^{(-p-1)} D_{\mathfrak{m}}$ is given by

$$D_{\mathfrak{m}}^p = \{\, \xi^{-\mu} = \cdots = \xi^{p-1} = 0 \,\} = \{\, \xi^s = 0 \quad (s < p) \,\},$$

where ξ^p is the $\mathfrak{g}_p$-component of ξ. Namely we have a global trivialization of $F(M(\mathfrak{m}))$ by a basis of $\mathfrak{m}$. Thus every vector field X on $M(\mathfrak{m})$ is identified with a $\mathfrak{m}$-valued function f_X by putting

$$f_X(x) = \xi(X_x) \qquad \text{at } x \in M(\mathfrak{m}).$$

In particular f_X is a constant function if and only if X is left invariant. For two vector fields X, Y on $M(\mathfrak{m})$, we have

$$f_{[X,Y]}(x) = [f_X(x), f_Y(x)] + X_x(f_Y) - Y_x(f_X), \tag{2.1}$$

at $x \in M(\mathfrak{m})$. Here the bracket product on the right side is that of $\mathfrak{m}$. Moreover, according to the decomposition of $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$, f_X is written as a sum

$$f_X = \sum_{p<0} f_X^p,$$

where f_X^p is a $\mathfrak{g}_p$-valued function on $M(\mathfrak{m})$.

Now recall that a vector field X is an infinitesimal automorphism of $(M(\mathfrak{m}), D_{\mathfrak{m}})$ if and only if

$$[X, \mathcal{D}_{\mathfrak{m}}] \subset \mathcal{D}_{\mathfrak{m}},$$

where $\mathcal{D}_{\mathfrak{m}}$ is the space of sections of $D_{\mathfrak{m}}$. Thus $X \in \mathcal{A}(M(\mathfrak{m}), D_{\mathfrak{m}})$ if and only if

$$f^p_{[X,Y]} = 0 \qquad \text{for } p < -1 \text{ and } Y \in \mathcal{D}_{\mathfrak{m}}.$$

By (2.1), this condition is equivalent to the following equalities;

$$Y(f_X^p) = [f_X^{p+1}, f_Y^{-1}] \qquad \text{for } p < -1 \text{ and } Y \in \mathcal{D}_{\mathfrak{m}},$$

or equivalently

$$d f_X^p \equiv [f_X^{p+1}, \xi^{-1}] \quad (\text{mod } \xi^s \quad (s < -1)) \quad \text{for } p < -1. \tag{2.2}$$

The equalities (2.2) express the condition for a vector field X to be an infinitesimal automorphism of $(M(\mathfrak{m}), D_{\mathfrak{m}})$ in terms of f_X. However, from the generating condition of $\mathfrak{m}$: $\mathfrak{g}_p = [\mathfrak{g}_{p+1}, \mathfrak{g}_{-1}]$ for $p < -1$, $X \in \mathcal{A}(M(\mathfrak{m}), D_{\mathfrak{m}})$ satisfies additional equalities as follows: First we calculate

$$\begin{aligned} Y(Z(f_X^p)) &= Y([f_X^{p+1}, f_Z^{-1}]) = [Y(f_X^{p+1}), f_Z^{-1}] + [f_X^{p+1}, Y(f_Z^{-1})] \\ &= [[f_X^{p+2}, f_Y^{-1}], f_Z^{-1}] + [f_X^{p+1}, Y(f_Z^{-1})], \end{aligned}$$

for vector fields Y, $Z \in \mathcal{D}_{\mathfrak{m}}$ and $p < -2$. Then, by (2.1), we get

$$[Y, Z](f_X^p) = Y(Z(f_X^p)) - Z(Y(f_X^p)) = [f_X^{p+2}, f^{-2}_{[Y,Z]}],$$

for $p < -2$. From $[\mathfrak{g}_{-1}, \mathfrak{g}_{-1}] = \mathfrak{g}_{-2}$, this implies

$$W(f_X^p) = [f_X^{p+2}, f_W^{-2}] = [f_X^{p+2}, W],$$

for $p < -2$ and $W \in \mathfrak{g}_{-2}$, since $f_W = W$ is a constant function for $W \in \mathfrak{g}_{-2}$. Proceeding by induction on r, we see that, for a fixed $p < 0$, the same calculation as above yields

$$Y(f_X^p) = [f_X^{p-r}, Y],$$

for $r > p$ and $Y \in \mathfrak{g}_r$. Summarizing, we obtain

$$(2.3) \qquad d\,f_X^p \equiv \sum_{r=-1}^{p+1} [f_X^{p-r}, \xi^r] \pmod{\xi^s \quad (s < p+1)} \quad \text{for } p < -1.$$

Starting from (2.3), we are going to seek all the (higher) derivatives of f_X. In order to do so, we first introduce a $\bigoplus_{p<0} \mathfrak{g}_p \otimes \mathfrak{g}_p^*$-valued function f_X^0 by

$$(f_X^0(x))(Y) = Y_x(f_X^p) \qquad \text{for } Y \in \mathfrak{g}_p \text{ and } x \in M(\mathfrak{m}).$$

Here we regard Y as a vector field on $M(\mathfrak{m})$ on the right side of the equality and as a vector in $\mathfrak{g}_p$ on the left side. We write this equality in short as

$$Y(f_X^p) = [f_X^0, Y] \qquad \text{for } Y \in \mathfrak{g}_p.$$

Equivalently we can say that f_X^0 is defined by the following equalities;

$$d\,f_X^p \equiv \sum_{r=-1}^{p} [f_X^{p-r}, \xi^r] \pmod{\xi^s \quad (s < p)} \quad \text{for } p < 0.$$

Namely we have strengthened the mod equalities (2.3) and add $d\,f_X^{-1} \equiv [f_X^0, \xi^{-1}] \pmod{\xi^s \quad (s < -1)}$. From these equalities, it follows a compatibility condition for f_X^0: For $Y \in \mathfrak{g}_r$ and $Z \in \mathfrak{g}_s$ (r, $s < 0$), we calculate as above and get

$$(2.4) \qquad [Y, Z](f_X^\ell) = [[f_X^k, Y], Z] - [[f_X^k, Z], Y],$$

where $k = \ell - (r+s)$. This equality is valid as far as f_X^ℓ and f_X^k are defined. When $\ell = r + s$, by definition of f_X^0, we obtain

$$[f_X^0, [Y, Z]] = [[f_X^0, Y], Z] - [[f_X^0, Z], Y].$$

This implies that f_X^0 takes values in

$$(\text{p.0}) \quad \mathfrak{g}_0(\mathfrak{m}) = \{\, u \in \bigoplus_{p<0} \mathfrak{g}_p \otimes \mathfrak{g}_p^* \mid u([Y, Z]) = [u(Y), Z] + [Y, u(Z)] \,\}.$$

Here we note that $\mathfrak{g}_0(\mathfrak{m})$ is the Lie algebra of all (gradation preserving) derivations of the graded Lie algebra $\mathfrak{m}$.

Now we continue this procedure and introduce a $\mathfrak{g}_k(\mathfrak{m})$-valued function f_X^k for positive integer k inductively as follows: Assume that $\mathfrak{g}_\ell = \mathfrak{g}_\ell(\mathfrak{m})$ and f_X^ℓ are defined for $\ell < k$ such that

$$d\,f_X^\ell \equiv \sum_{r=-1}^{\ell-k+1} [f_X^{\ell-k}, \xi^r] \quad (\mathrm{mod}\ \xi^s \quad (s < \ell - k + 1)) \quad \text{for } \ell < k-1.$$

Here we understand that $\xi^r = 0$ for $r < -\mu$. We introduce a $\bigoplus_{p<0} \mathfrak{g}_{p+k} \otimes \mathfrak{g}_p^*$-valued function f_X^k by

$$(f_X^k(x))(Y) = Y_x(f_X^{p+k}) \qquad \text{for } Y \in \mathfrak{g}_p \text{ and } x \in M(\mathfrak{m}),$$

or equivalently by the following equalities;

$$\text{(2.5)} \qquad d\,f_X^\ell \equiv \sum_{r=-1}^{\ell-k} [f_X^{\ell-r}, \xi^r] \quad (\mathrm{mod}\ \xi^s \quad (s < \ell - k)) \quad \text{for } \ell < k.$$

Here we write $f_X^k(Y) = [f_X^k, Y]$ in short. Then, by definition of f_X^k and (2.4), we have

$$[f_X^k, [Y, Z]] = [[f_X^k, Y], Z] - [[f_X^k, Z], Y],$$

for $Y \in \mathfrak{g}_r$, $Z \in \mathfrak{g}_s$ $(r,\ s < 0)$. Namely f_X^k takes values in

$$\text{(p.k)} \quad \mathfrak{g}_k(\mathfrak{m}) = \{\, u \in \bigoplus_{p<0} \mathfrak{g}_{p+k} \otimes \mathfrak{g}_p^* \mid u([Y, Z]) = [u(Y), Z] - [u(Z), Y] \,\}.$$

This finishes our inductive definition of f_X^k and $\mathfrak{g}_k(\mathfrak{m})$.

One should note that, for a fixed ℓ, (2.5) becomes a strict equality when k increases sufficiently large. Thus, for a family $\{f_X^k\}_{k \geqq -\mu}$ of functions on $M(\mathfrak{m})$, we obtain ([T2, Lemma 6.2])

$$\text{(2.6)} \qquad d\,f_X^k = \sum_{r=-1}^{-\mu} [f_X^{k-r}, \xi^r].$$

In this way we get the whole information of all higher derivatives of f_X.

2.3. Algebraic prolongation of $\mathfrak{m}$

Motivated by the above discussion, we now give the definition of the *algebraic prolongation* $\mathfrak{g}(\mathfrak{m})$ of the fundamental graded Lie algebra $\mathfrak{m}$, which was introduced by N. Tanaka [T2].

Let $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ be a fundamental graded Lie algebra of μ-th kind defined over a field K. Here K denotes the field of real numbers $\mathbb{R}$ or that of complex numbers $\mathbb{C}$. We put

$$\mathfrak{g}(\mathfrak{m}) = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p(\mathfrak{m}),$$

where $\mathfrak{g}_p(\mathfrak{m}) = \mathfrak{g}_p$ for $p < 0$ and $\mathfrak{g}_k(\mathfrak{m})$ is defined inductively by (p.k) for $k \geqq 0$. Thus, as a vector space over K, $\mathfrak{g}_k(\mathfrak{m})$ is a linear subspace of $\mathrm{End}(\mathfrak{m}, \mathfrak{m}^k) = \mathfrak{m}^k \otimes \mathfrak{m}^*$, where $\mathfrak{m}^k = \mathfrak{m} \oplus \mathfrak{g}_0(\mathfrak{m}) \oplus \cdots \oplus \mathfrak{g}_{k-1}(\mathfrak{m})$. The bracket operation of $\mathfrak{g}(\mathfrak{m})$ is given as follows: First, since $\mathfrak{g}_0(\mathfrak{m})$ is the Lie algebra of all (gradation preserving) derivations of graded Lie algebra $\mathfrak{m}$, we see that $\bigoplus_{p\leqq 0} \mathfrak{g}_p(\mathfrak{m})$ becomes a graded Lie algebra by putting

$$[u, X] = -[X, u] = u(X) \qquad \text{for } u \in \mathfrak{g}_0(\mathfrak{m}) \text{ and } X \in \mathfrak{m}.$$

Similarly, for $u \in \mathfrak{g}_k(\mathfrak{m}) \subset \mathfrak{m}^k \otimes \mathfrak{m}^*$ $(k > 0)$ and $X \in \mathfrak{m}$, we put $[u, X] = -[X, u] = u(X)$ (this justifies our use of $[\,,\,]$ in the previous paragraph). Now, for $u \in \mathfrak{g}_k(\mathfrak{m})$ and $v \in \mathfrak{g}_\ell(\mathfrak{m})$ (k, $\ell \geqq 0$), by induction on the integer $k + \ell \geqq 0$, we define $[u, v] \in \mathfrak{m}^{k+\ell} \otimes \mathfrak{m}^*$ by

$$[u, v](X) = [[u, X], v] + [u, [v, X]] \qquad \text{for } X \in \mathfrak{m}.$$

Here we note that, as the first case $k = \ell = 0$, this definition begins with that of the bracket product in $\mathfrak{g}_0(\mathfrak{m})$. It follows easily that $[u, v] \in \mathfrak{g}_{k+\ell}(\mathfrak{m})$. With this bracket product, $\mathfrak{g}(\mathfrak{m})$ becomes a graded Lie algebra. In fact the Jacobi identity

$$[[u, v], w] + [[v, w], u] + [[w, u], v] = 0,$$

for $u \in \mathfrak{g}_p(\mathfrak{m})$, $v \in \mathfrak{g}_q(\mathfrak{m})$ and $w \in \mathfrak{g}_r(\mathfrak{m})$, follows by definition when one of p, q or r is negative, and can be shown by induction on the integer $p + q + r \geqq 0$, when all of p, q and r are non-negative.

Let $\mathfrak{g}_0$ be a subalgebra of $\mathfrak{g}_0(\mathfrak{m})$. We define a sbspace $\mathfrak{g}_k$ of $\mathfrak{g}_k(\mathfrak{m})$ for $k \geqq 1$ inductively by

$$\mathfrak{g}_k = \{\, u \in \mathfrak{g}_k(\mathfrak{m}) \mid [u, \mathfrak{g}_{-1}] \subset \mathfrak{g}_{k-1} \,\}.$$

Then, putting

$$\mathfrak{g}(\mathfrak{m}, \mathfrak{g}_0) = \mathfrak{m} \oplus \bigoplus_{k\geqq 0} \mathfrak{g}_k,$$

we see, with the generating condition of $\mathfrak{m}$, that $\mathfrak{g}(\mathfrak{m}, \mathfrak{g}_0)$ is a graded subalgebra of $\mathfrak{g}(\mathfrak{m})$. $\mathfrak{g}(\mathfrak{m}, \mathfrak{g}_0)$ is called the prolongation of $(\mathfrak{m}, \mathfrak{g}_0)$.

Remark 2.1. The notion of the prolongation of $\mathfrak{m}$ or $(\mathfrak{m}, \mathfrak{g}_0)$ plays quite an important role in the equivalence problems for the geometric structures subordinate to regular differential systems of type $\mathfrak{m}$, e.g., CR-structures, pseudo-product structures or Lie contact structures (cf. [T3], [T5], [S-Y]). We could not touch upon the more important geometric aspect of the prolongation theory of these structures. On these subjects, we refer the reader to foundational papers [T2], [T3], [T4] of N. Tanaka, although we shall discuss some consequences of our results related to [T4] in §5.3.

Now, going back to the discussion in 2.2, we shall see how $\mathfrak{g}(\mathfrak{m})$ describes the structure of $\mathcal{A}(M(\mathfrak{m}), D_{\mathfrak{m}})$, following the argument in §6 of [T2] rather closely. First let us fix a point $x \in M(\mathfrak{m})$. Then $\{f_X^\ell(x)\}_{\ell \geqq -\mu}$ has all the information of higher derivatives of f_X at x. Conversely, given an element a of $\mathfrak{g}(\mathfrak{m})$, we can construct an infinitesimal automorphism whose "Taylor expansion" at x coincides with a. Namely we have ([T2, Lemma 6.3]):

Let $a = \sum_{p \leqq k} a^p$ be any element of $\mathfrak{g}(\mathfrak{m})$, where $a^p \in \mathfrak{g}_p(\mathfrak{m})$. Then there is a unique $X \in \mathcal{A}(M(\mathfrak{m}), D_{\mathfrak{m}})$ such that

$$\begin{cases} f_X^p(x) = a^p & \text{for } p \leqq k, \\ f_X^\ell \equiv 0 & \text{for } \ell > k. \end{cases}$$

By (2.6), in order to construct such X, we need to solve the following differential equations for $\mathfrak{g}_\ell(\mathfrak{m})$-valued functions $u^\ell = f_X^\ell$ $(-\mu \leqq \ell \leqq k)$;

$$du^\ell = \sum_{\ell < s \leqq k} [u^s, \xi^{\ell - s}] \qquad \text{for } \ell = -\mu, \dots, k,$$

under the condition $u^\ell(x) = a^\ell \in \mathfrak{g}_\ell(\mathfrak{m})$ (here we understand that $\xi^r = 0$ for $r < -\mu$ as before). However this can be accomplished by the Frobenius theorem. In fact, on $M(\mathfrak{m}) \times \mathfrak{m}^{k+1}$, we consider a differential system E defined by

$$\alpha^\ell = du^\ell - \sum_{\ell < s \leqq k} [u^s, \xi^{\ell - s}] \qquad \text{for } \ell = -\mu, \dots, k,$$

where u^ℓ is the linear coordinate on $\mathfrak{g}_\ell(\mathfrak{m})$. Then it follows

$$d\alpha^\ell + \sum_{\ell < s \leqq k} [\alpha^s \wedge \xi^{\ell - s}] = 0.$$

Namely E is completely integrable. Thus, since $M(\mathfrak{m})$ is simply connected, the graph of $(f_X^\ell)_{-\mu\leqq\ell\leqq k}$ is obtained as a leaf of E passing through $(x,a)\in M(\mathfrak{m})\times\mathfrak{m}^{k+1}$. One should note here that, when $a\in\mathfrak{m}$, we actually obtain a right invariant vector field X on $M(\mathfrak{m})$.

Thus, by fixing a point of $M(\mathfrak{m})$, we obtain a linear isomorphism of $\mathfrak{g}(\mathfrak{m})$ into $\mathcal{A}(M(\mathfrak{m}),D_\mathfrak{m})$. For the correspondence of bracket operation, we have ([T2, Lemma 6.4]): For X, $Y\in\mathcal{A}(M(\mathfrak{m}),D_\mathfrak{m})$,

$$f_{[X,Y]}^\ell=-\sum_{r+s=\ell}[f_X^r,f_Y^s]\qquad\text{for }\ell\geqq-\mu. \tag{2.7}$$

In fact, (2.7) follows easily from (2.1) and (2.6) when $\ell<0$. Thus, putting $g^\ell=-\sum_{r+s=\ell}[f_X^r,f_Y^s]$, we have $g^p=f_{[X,Y]}^p$ for $p<0$. Moreover, by (2.6), we calculate

$$\begin{aligned}dg^\ell&=-\sum_{r+s=\ell}\{[d\,f_X^r,f_Y^s]+[f_X^r,d\,f_Y^s]\}\\&=-\sum_{s+t+u=\ell}[[f_X^t,\xi^u],f_Y^s]-\sum_{p+q+r=\ell}[f_X^r,[f_Y^p,\xi^q]]\\&=\sum_{r<0}[g^{\ell-r},\xi^r]\qquad\text{for }\ell\geqq-\mu.\end{aligned}$$

Then, by the definition (2.5) of $f_{[X,Y]}^\ell$ for $\ell\geqq0$, we conclude $g^\ell=f_{[X,Y]}^\ell$ for $\ell\geqq-\mu$.

In this way the structure of the Lie algebra $\mathcal{A}(M(\mathfrak{m}),D_\mathfrak{m})$ can be described by $\mathfrak{g}(\mathfrak{m})$. Especially $\mathcal{A}(M(\mathfrak{m}),D_\mathfrak{m})$ is isomorphic with $\mathfrak{g}(\mathfrak{m})$, when $\mathfrak{g}(\mathfrak{m})$ is finite dimensional. In the subsequent sections, we shall be concerned with the following question: When does $\mathfrak{g}(\mathfrak{m})$ or $\mathfrak{g}(\mathfrak{m},\mathfrak{g}_0)$ become finite dimensional and simple?

Remark 2.2. (1) In infinite dimensional case, the completion $\overline{\mathfrak{g}}(\mathfrak{m})$ of $\mathfrak{g}(\mathfrak{m})$ gives the formal algebra of the transitive Lie algebra sheaf $\mathcal{A}$ of infinitesimal automorphisms of $(M(\mathfrak{m}),D_\mathfrak{m})$. On this subject, we refer the reader to the further discussion in §6 of [T2].

(2) We remark here that the discussions in §§1 and 2 are valid also in the complex analytic category. Thus, for a fundamental graded Lie algebra $\mathfrak{m}=\bigoplus_{p<0}\mathfrak{g}_p$ over $\mathbb{C}$, the standard differential system $(M(\mathfrak{m}),D_\mathfrak{m})$ of type $\mathfrak{m}$ is a holomorphic differential system on a complex Lie group $M(\mathfrak{m})$. Furthermore the prolongation $\mathfrak{g}(\mathfrak{m})$ of $\mathfrak{m}$ over $\mathbb{C}$ describes the stalk of the Lie algebra sheaf $\mathcal{A}$ of holomorphic infinitesimal automorphisms of $(M(\mathfrak{m}),D_\mathfrak{m})$.

2.4. Generarized Spencer cohomology

We now give some remarks on the algebraic prolongation $\mathfrak{g}(\mathfrak{m})$ of $\mathfrak{m}$. First $\mathfrak{g}(\mathfrak{m})$ is characterized as the graded Lie algebra which satisfies the following conditions:

(1) $\mathfrak{g}_p(\mathfrak{m}) = \mathfrak{g}_p$ for $p < 0$, where $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$.
(2) For $k \geqq 0$, if $X \in \mathfrak{g}_k(\mathfrak{m})$ and $[X, \mathfrak{m}] = \{0\}$, then $X = 0$.
(3) $\mathfrak{g}(\mathfrak{m})$ is maximum among graded algebras satisfying conditions (1) and (2) above.

More precisely, let $\mathfrak{h} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{h}_p$ be any graded algebra satisfying (1) and (2). Then $\mathfrak{h}$ is imbedded in $\mathfrak{g}(\mathfrak{m})$ as a graded subalgebra.

In fact (1) and (2) are obvious. The imbedding ι of $\mathfrak{h}$ into $\mathfrak{g}(\mathfrak{m})$ is obtained as follows: Since $\bigoplus_{p\leqq 0} \mathfrak{h}_p = \mathfrak{m} \oplus \mathfrak{h}_0$ is a graded subalgebra, we get a homomorphism ι_0 of $\mathfrak{h}_0$ into $\mathfrak{g}_0(\mathfrak{m})$, which is injective by condition (2) above. Then, by definition (p.k) of $\mathfrak{g}_k(\mathfrak{m})$, we obtain a linear map ι_k of $\mathfrak{h}_k$ into $\mathfrak{g}_k(\mathfrak{m})$ by induction on $k \geqq 1$, which is also injective by (2). ι is obviously a homomorphism.

In the presence of the generating condition of $\mathfrak{m}$, the condition (2) above is equivalent to the following condition:

$$\text{For } k \geqq 0, \text{ if } X \in \mathfrak{g}_k(\mathfrak{m}) \text{ and } [X, \mathfrak{g}_{-1}] = \{0\}, \text{ then } X = 0.$$

From this, it follows that $\mathfrak{g}_{k+1}(\mathfrak{m}) = \{0\}$ if $\mathfrak{g}_k(\mathfrak{m}) = \{0\}$, that is, $\mathfrak{g}_\ell(\mathfrak{m}) = \{0\}$ for $\ell \geqq k$ if $\mathfrak{g}_k(\mathfrak{m}) = \{0\}$. Hence $\mathfrak{g}(\mathfrak{m})$ becomes finite dimensional if and only if $\mathfrak{g}_k(\mathfrak{m}) = \{0\}$ for some $k \geqq 0$.

Now we shall turn to another characterization of $\mathfrak{g}(\mathfrak{m})$. First, recall that the prolongation $\mathfrak{g}^{(k)}$ of a linear Lie algebra $\mathfrak{g} \subset \mathfrak{gl}(V)$ is defined also by the following exact sequence;

$$0 \to \mathfrak{g}^{(k+1)} \to C^{k+1,1} = \mathfrak{g}^{(k)} \otimes V^* \xrightarrow{\partial} C^{k,2} = \mathfrak{g}^{(k-1)} \otimes \wedge^2 V^*,$$

where the coboundary operator $\partial\colon C^{k+1,1} \to C^{k,2}$ is given by

$$(\partial p)(X, Y) = [p(X), Y] - [p(Y), X].$$

In the same way, we can define $\mathfrak{g}_k(\mathfrak{m})$ as follows. First we decompose $\bigwedge^2 \mathfrak{m}^* = \bigoplus_{j<-1} \wedge_j^2 \mathfrak{m}^*$ according to the gradation $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$, where

$$\wedge_j^2 \mathfrak{m}^* = \bigoplus_{p+q=j} \mathfrak{g}_p^* \wedge \mathfrak{g}_q^*.$$

Putting $C^{k,1} = \bigoplus_{p<0} \mathfrak{g}_{p+k} \otimes \mathfrak{g}_p^*$ and $C^{k-1,2} = \bigoplus_{j<-1} \mathfrak{g}_{j+k} \otimes \wedge_j^2 \mathfrak{m}^*$, we can define $\mathfrak{g}_k = \mathfrak{g}_k(\mathfrak{m})$ for $k \geqq 0$ inductively by the following exact sequence;

$$0 \to \mathfrak{g}_k \to C^{k,1} \xrightarrow{\partial} C^{k-1,2},$$

where the coboundary operator $\partial\colon C^{k,1} \to C^{k-1,2}$ is given by

$$(\partial p)(X,Y) = [X,p(Y)] - [Y,p(X)] - p([X,Y]).$$

We shall utilize this characterization in the following situation. Let $\mathfrak{h} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{h}_p$ be a graded Lie algebra such that $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{h}_p$ is a fundamental graded algebra of μ-th kind. To check whether $\mathfrak{h}$ is the prolongation of $\mathfrak{m}$ or $(\mathfrak{m}, \mathfrak{h}_0)$, we consider the Lie algebra cohomology $H^q(\mathfrak{m}, \mathfrak{h})$ associated with the representation ad: $\mathfrak{m} \to \mathfrak{gl}(\mathfrak{h})$. Namely, putting $C(\mathfrak{m},\mathfrak{h}) = \bigoplus C^q(\mathfrak{m},\mathfrak{h})$, $C^q(\mathfrak{m},\mathfrak{h}) = \mathfrak{h} \otimes \bigwedge^q \mathfrak{m}^*$, we have the coboundary operator $\partial\colon C^q \to C^{q+1}$;

$$\begin{aligned}(\partial p)(X_1,\ldots,X_{q+1}) = &\sum_i (-1)^{i+1}[X_i, p(X_1,\ldots,\check{X}_i,\ldots,X_{q+1})] \\ &+ \sum_{i<j}(-1)^{i+j} p([X_i,X_j],X_1,\ldots,\check{X}_i,\ldots,\check{X}_j,\ldots,X_{q+1}),\end{aligned}$$

for $p \in C^q(\mathfrak{m},\mathfrak{h})$ and $X_1, \ldots, X_{q+1} \in \mathfrak{m}$. $H^q(\mathfrak{m},\mathfrak{h})$ is the cohomology group of this cochain complex $(C(\mathfrak{m},\mathfrak{h}),\partial)$. According to the gradation of $\mathfrak{h}$, this complex has a bigradation given as follows ([T4, §1]): First $\bigwedge^q \mathfrak{m}^*$ has the decomposition $\bigwedge^q \mathfrak{m}^* = \bigoplus_{j\leqq -q} \wedge_j^q \mathfrak{m}^*$, where

$$\wedge_j^q \mathfrak{m}^* = \bigoplus_{i_1+\cdots+i_q=j} \mathfrak{h}_{i_1}^* \wedge \cdots \wedge \mathfrak{h}_{i_q}^*.$$

Then the bigradation of $C(\mathfrak{m},\mathfrak{h})$ is introduced by

$$C^{p,q}(\mathfrak{m},\mathfrak{h}) = \bigoplus_{j\leqq -q} \mathfrak{h}_{j+p+q-1} \otimes \wedge_j^q \mathfrak{m}^*.$$

Here we note that

$$C^{p,0} = \mathfrak{h}_{p-1},\; C^{p,1} = \bigoplus_{j<0} \mathfrak{h}_{j+p} \otimes \mathfrak{h}_j^*,\; C^{p,2} = \bigoplus_{j<-1} \mathfrak{h}_{j+p+1} \otimes \wedge_j^2 \mathfrak{m}^*$$

and ∂ sends $C^{p,q}$ into $C^{p-1,q+1}$. With this bigradation,

$$H^q(\mathfrak{m},\mathfrak{h}) = \bigoplus_p H^{p,q}(\mathfrak{m},\mathfrak{h})$$

is called the *generalized Spencer cohomology group* of the graded Lie algebra $\mathfrak{h}$.

Utilizing this cohomology group, we have (cf. [T4, Lemma 1.14])

Lemma 2.1. *Let $\mathfrak{h} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{h}_p$ be a graded Lie algebra such that $\mathfrak{h}_p = [\mathfrak{h}_{p+1}, \mathfrak{h}_{-1}]$ for $p < -1$. Then $\mathfrak{h}$ is the prolongation of $\mathfrak{m}$ (resp. of $(\mathfrak{m}, \mathfrak{h}_0)$) if and only if the following two conditions hold:*

(1) *For $k \geqq 0$, if $X \in \mathfrak{h}_k$ and $[X, \mathfrak{m}] = \{0\}$, then $X = 0$.*

(2) *$H^{p,1}(\mathfrak{m}, \mathfrak{h}) = \{0\}$ for $p \geqq 0$ (resp. $p \geqq 1$).*

With this criterion in mind, in order to answer the question posed at the end of 2.3, we proceed as follows: First, for a (finite dimensional) simple Lie algebra $\mathfrak{g}$, we shall classify, in §3, the gradations $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ of $\mathfrak{g}$ such that $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ is fundamental. Then we calculate $H^{p,1}(\mathfrak{m}, \mathfrak{g})$ by the method of Kostant [K] in §5.2.

§3. Simple graded Lie algebras

3.1. Semisimple graded Lie algebras

We begin with generalities of semisimple graded Lie algebras (cf. [Hu], [K-N], [T4]). Let $\mathfrak{g}$ be a (finite dimensional) semisimple Lie algebra over $\mathbb{R}$. A gradation of $\mathfrak{g}$ is a direct decomposition $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ such that

$$[\mathfrak{g}_p, \mathfrak{g}_q] \in \mathfrak{g}_{p+q} \qquad \text{for } p,\ q \in \mathbb{Z}.$$

As is well-known, there exists a unique element $E \in \mathfrak{g}_0$ such that

$$\mathfrak{g}_p = \{ X \in \mathfrak{g} \mid [E, X] = pX \} \qquad \text{for } p \in \mathbb{Z}.$$

In fact, for a graded Lie algebra $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$, we have a derivation D of $\mathfrak{g}$ given by $D(X) = pX$ for $X \in \mathfrak{g}_p$. Then, since $\mathfrak{g}$ is semisimple, there exists a unique $E \in \mathfrak{g}$ such that $D = \mathrm{ad}(E)$. Obviously we have $E \in \mathfrak{g}_0$. In particular $\mathfrak{g}_0 \neq \{0\}$. E is called the *characteristic element* of the semisimple graded Lie algebra $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$.

Moreover we get easily ([K2, p. 131, Proposition 4.1], [T4, Lemma 1.2])

Lemma 3.1.

(1) *$B(\mathfrak{g}_p, \mathfrak{g}_q) = 0$ if $p + q \neq 0$.*

(2) *The restriction of the Killing form B to $\mathfrak{g}_p \times \mathfrak{g}_{-p}$ is non-degenerate if $\mathfrak{g}_p \neq \{0\}$.*

Namely gradations of a semisimple Lie algebra $\mathfrak{g}$ are always symmetric, that is, $\mathfrak{g}_p \neq \{0\}$ if and only if $\mathfrak{g}_{-p} \neq \{0\}$ and the Killing form B gives a duality between $\mathfrak{g}_p$ and $\mathfrak{g}_{-p}$. The largest integer μ such that

$\mathfrak{g}_\mu \neq \{0\}$ is called the *depth* of $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$. Furthermore we see that $\mathfrak{g}$ is non-compact if the gradation is not trivial, that is, if $\mathfrak{g}_p \neq \{0\}$ for some $p \neq 0$.

Now we consider the decomposition of $\mathfrak{g}$ into simple ideals;

$$\mathfrak{g} = \bigoplus_s \mathfrak{g}^s.$$

Then the characteristic element E decomposes as $E = \sum_s E^s$. For $X \in \mathfrak{g}_p$, we have $X = \sum_s X^s$. Thus, from $pX = [E, X] = \sum_s [E^s, X^s]$, we get $[E^s, X^s] = pX^s$. Namely E^s defines a gradation $\mathfrak{g}^s = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p^s$ of $\mathfrak{g}^s$, where $\mathfrak{g}_p^s = \mathfrak{g}^s \cap \mathfrak{g}_p$ and

$$\mathfrak{g}_p = \bigoplus_s \mathfrak{g}_p^s.$$

Therefore $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ is a direct sum of simple graded Lie algebras $\mathfrak{g}^s = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p^s$.

A graded Lie algebra $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ is called effective if $\mathfrak{g}' = \bigoplus_{p\geqq 0} \mathfrak{g}_p$ contains no ideals of $\mathfrak{g}$. Then, by the above argument, we see that $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ is effective if and only if none of simple ideals $\mathfrak{g}^s = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p^s$ has a trivial gradation.

Some conditions on the gradation forces $\mathfrak{g}$ to be a simple graded Lie algebra. Among these, we quote here the following two conditions: A gradation $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ is called a *contact gradation* if $\mathfrak{g}$ is effective and $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ satisfies

(1) $\mathfrak{m} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$ such that $\dim \mathfrak{g}_{-2} = 1$.

(2) The bracket operation $[\,,\,] \colon \mathfrak{g}_{-1} \times \mathfrak{g}_{-1} \to \mathfrak{g}_{-2}$ is nondegenerate.

In fact it follows from (1) that there exists a unique ideal $\mathfrak{g}^{s_o}$ such that $\mathfrak{g}_{-2} = \mathfrak{g}_{-2}^{s_o}$ and that $\mathfrak{g}^s = \mathfrak{g}_{-1}^s \oplus \mathfrak{g}_0^s \oplus \mathfrak{g}_1^s$ for $s \neq s_o$. Then condition (2) forces $\mathfrak{g}_{-1}^s = \{0\}$ for $s \neq s_o$. Thus the effectiveness of $\mathfrak{g}$ implies $\mathfrak{g} = \mathfrak{g}^{s_o}$. We shall see later in §4 that each simple Lie algebra over $\mathbb{C}$ has a unique complex contact gradation up to conjugacy.

A gradation $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$, such that $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ is fundamental, is called *primitive* if $\mathfrak{g}$ is effective and $\mathrm{ad}\colon \mathfrak{g}_0 \to \mathfrak{gl}(\mathfrak{g}_{-1})$ is irreducible. It follows easily that $\mathfrak{g}$ is simple if it is primitive. More generally we shall discuss primitive actions of finite dimensional Lie groups in §5.4.

For simple graded Lie algebras, we prepare (cf. [T4, Lemmas 1.3, 1.6])

Lemma 3.2. *Let $\mathfrak{g} = \bigoplus_{p=-\mu}^{\mu} \mathfrak{g}_p$ be a simple graded Lie algebra over $\mathbb{R}$ of depth μ such that $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ is fundamental.*

Then, for every $p > -\mu$,

(1) *If $X \in \mathfrak{g}_p$ and $[X, \mathfrak{g}_{-1}] = \{0\}$, then $X = 0$.*
(2) $\mathfrak{g}_p = [\mathfrak{g}_{p-1}, \mathfrak{g}_1]$.

In particular the centralizer $Z_{\mathfrak{g}}(\mathfrak{m})$ of $\mathfrak{m}$ in $\mathfrak{g}$ coincides with $\mathfrak{g}_{-\mu}$.

Proof. From the generating condition of $\mathfrak{m}$: $\mathfrak{g}_p = [\mathfrak{g}_{p+1}, \mathfrak{g}_{-1}]$ for $p < -1$, it follows $\mathfrak{g}_k = [\mathfrak{g}_{k-1}, \mathfrak{g}_1]$ for $k > 1$ (for this fact, see 3.3). Then we see that a linear subspace $\mathfrak{a}$ of $\mathfrak{g}$ is an ideal of $\mathfrak{g}$ if $\mathfrak{a}$ is $\mathrm{ad}(\mathfrak{g}_i)$-invariant for $i = -1$, 0, 1.

Now let us fix an integer q ($-\mu \leqq q \leqq \mu$) and put

$$\mathfrak{a}^q(q) = \{ X \in \mathfrak{g}_q \mid [X, \mathfrak{g}_{-1}] = 0 \}.$$

We define a linear subspace $\mathfrak{a}^q = \bigoplus_{p=q}^{\mu} \mathfrak{a}^q(p)$ of $\mathfrak{g}$ inductively by

$$\mathfrak{a}^q(p+1) = [\mathfrak{a}^q(p), \mathfrak{g}_1] \subset \mathfrak{g}_{p+1}.$$

By the Jacobi identity, we see that $\mathfrak{a}^q(q)$ is $\mathrm{ad}(\mathfrak{g}_0)$-invariant. Moreover one can check that $\mathfrak{a}^q(p)$ is $\mathrm{ad}(\mathfrak{g}_0)$-invariant and $[\mathfrak{a}^q(p+1), \mathfrak{g}_{-1}] \subset \mathfrak{a}^q(p)$, by induction on $p \geqq q$. Thus $\mathfrak{a}^q$ is an ideal of $\mathfrak{g}$. When $q > -\mu$, $\mathfrak{a}^q$ is a proper ideal of $\mathfrak{g}$. Hence, by the simplicity of $\mathfrak{g}$, $\mathfrak{a}^q = \{0\}$, which proves (1). When $q = -\mu$, we have $\mathfrak{a}^{-\mu}(-\mu) = \mathfrak{g}_{-\mu}$. Hence $\mathfrak{a}^{-\mu} = \mathfrak{g}$, which implies (2).

This lemma shows, in particular, that condition (1) of Lemma 2.1 in §2.4 is always satisfied by a simple graded Lie algebra $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ such that $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ is fundamental. In other words, $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ is a graded subalgebra of the prolongation $\mathfrak{g}(\mathfrak{m})$ of $\mathfrak{m}$.

3.2. Complexification of $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$

Let $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ be a simple graded Lie algebra over $\mathbb{R}$. Let $\mathbb{C}\mathfrak{g} = \mathbb{C} \otimes_{\mathbb{R}} \mathfrak{g}$ be the complexification of $\mathfrak{g}$. Then $\mathbb{C}\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathbb{C}\mathfrak{g}_p$ becomes a semisimple graded Lie algebra over $\mathbb{C}$. First we recall the following fact (cf. [He, p. 443, Proposition 1.5]).

The simple Lie algebras over $\mathbb{R}$ fall into two disjoint classes:

A. *The simple Lie algebras over $\mathbb{C}$, considered as real Lie algebras.*
B. *The real forms of simple Lie algebras over $\mathbb{C}$.*

More precisely, a real simple Lie algebra $\mathfrak{g}$ belongs to class A if $\mathbb{C}\mathfrak{g}$ is not simple and there exists a complex structure J on $\mathfrak{g}$ such that $(\mathfrak{g}, J)$ is a simple Lie algebra over $\mathbb{C}$. In this case we have

$$\mathbb{C}\mathfrak{g} = \mathfrak{g}^{1,0} \oplus \mathfrak{g}^{0,1},$$

where $\mathfrak{g}^{1,0} = \{ X - \sqrt{-1}JX \mid X \in \mathfrak{g} \}$ and $\mathfrak{g}^{0,1} = \{ X + \sqrt{-1}JX \mid X \in \mathfrak{g} \}$ are simple ideals of $\mathbb{C}\mathfrak{g}$, which are isomorphic with $(\mathfrak{g}, J)$.

When a simple graded Lie algebra $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ belongs to class A, we note that, since $\mathrm{ad}(E)\cdot J = J\cdot\mathrm{ad}(E)$ for the characteristic element E, $\mathfrak{g}_p = \{ X \in \mathfrak{g} \mid [E, X] = pX \}$ is a complex subspace of $(\mathfrak{g}, J)$. Namely, for a real simple Lie algebra $\mathfrak{g}$ of class A, any gradation of $\mathfrak{g}$ as a real Lie algebra is in fact a gradation as a complex Lie algebra. Thus we obtain

Proposition 3.3. *The simple graded Lie algebras* $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ *over* $\mathbb{R}$ *fall into two disjoint classes*:

A. *The simple graded Lie algebras over* $\mathbb{C}$, *considered as real graded Lie algebras.*
B. *The real forms of simple Lie algebras over* $\mathbb{C}$ *so that* $\mathbb{C}\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ *become simple graded Lie algebras over* $\mathbb{C}$.

Now we give some remarks on the generalized Spencer cohomology of $\mathbb{C}\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathbb{C}\mathfrak{g}_p$. We denote by $H^q_{\mathbb{C}}(\mathbb{C}\mathfrak{m}, \mathbb{C}\mathfrak{g})$ the complex cohomology group associated with the complex representation $\mathrm{ad}\colon \mathbb{C}\mathfrak{m} \to \mathfrak{gl}(\mathbb{C}\mathfrak{g})$. Namely we consider $C^q_{\mathbb{C}}(\mathbb{C}\mathfrak{m}, \mathbb{C}\mathfrak{g}) = \mathbb{C}\mathfrak{g} \otimes_{\mathbb{C}} \wedge^q \mathbb{C}\mathfrak{m}^*$, which is naturally identified with the complexification $\mathbb{C}C^q(\mathfrak{m}, \mathfrak{g}) = \mathbb{C} \otimes_{\mathbb{R}} C^q(\mathfrak{m}, \mathfrak{g})$ of $C^q(\mathfrak{m}, \mathfrak{g}) = \mathfrak{g} \otimes_{\mathbb{R}} \wedge^q \mathfrak{m}^*$. Under this identification, the coboundary operator $\partial\colon C^q_{\mathbb{C}} \to C^{q+1}_{\mathbb{C}}$ is a real operator. Hence $H^q_{\mathbb{C}}(\mathbb{C}\mathfrak{m}, \mathbb{C}\mathfrak{g})$ is naturally identified with the complexification $\mathbb{C}H^q(\mathfrak{m}, \mathfrak{g})$ of $H^q(\mathfrak{m}, \mathfrak{g})$. The bigradation is also preserved under this identification. Thus, by Lemma 3.2 and Lemma 2.1 in §2.4, we have

Lemma 3.4. *Let* $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ *be a simple graded Lie algebra over* $\mathbb{R}$ *such that* $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ *is fundamental.*

Then $\mathfrak{g}$ *is the prolongation of* $\mathfrak{m}$ (*resp. of* $(\mathfrak{m}, \mathfrak{g}_0)$) *if and only if* $H^{p,1}_{\mathbb{C}}(\mathbb{C}\mathfrak{m}, \mathbb{C}\mathfrak{g}) = \{0\}$ *for* $p \geqq 0$ (*resp.* $p \geqq 1$).

Let $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ be of class A, that is, a simple graded Lie algebra over $\mathbb{C}$. In this case we have two cohomology groups $H^q_{\mathbb{C}}(\mathfrak{m}, \mathfrak{g})$ and $H^q_{\mathbb{R}}(\mathfrak{m}, \mathfrak{g})$ associated with $\mathrm{ad}\colon \mathfrak{m} \to \mathfrak{gl}_{\mathbb{C}}(\mathfrak{g}) \subset \mathfrak{gl}_{\mathbb{R}}(\mathfrak{g})$. Namely $H^q_{\mathbb{C}}$ is obtained from the cochain complex $(C_{\mathbb{C}}(\mathfrak{m}, \mathfrak{g}), \partial)$, $C_{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{C}} \wedge\mathfrak{m}^*$, whereas $H^q_{\mathbb{R}}$ is obtained from the cochain complex $(C_{\mathbb{R}}(\mathfrak{m}, \mathfrak{g}), \partial)$, $C_{\mathbb{R}} = \mathfrak{g} \otimes_{\mathbb{R}} \wedge\mathfrak{m}^*$.

From the complex structure J on $\mathfrak{g}$, $C_{\mathbb{R}}$ inherits a complex structure $J \otimes_{\mathbb{R}} \mathrm{id}$ such that ∂ is complex linear. Hence $H^q_{\mathbb{R}}(\mathfrak{m}, \mathfrak{g})$ is a complex vector space. Then we have

Lemma 3.5. *Let $\mathfrak{g} = \bigoplus_{p=-\mu}^{\mu} \mathfrak{g}_p$ be a simple graded Lie algebra over $\mathbb{C}$ of depth μ such that $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ is fundamental. Then*

(1) *$H^{p,1}_{\mathbb{C}}(\mathfrak{m}, \mathfrak{g})$ and $H^{p,1}_{\mathbb{R}}(\mathfrak{m}, \mathfrak{g})$ are isomorphic for $p > 0$.*

(2) *$H^{0,1}_{\mathbb{C}}(\mathfrak{m}, \mathfrak{g})$ and $H^{0,1}_{\mathbb{R}}(\mathfrak{m}, \mathfrak{g})$ are isomorphic when $\mu > 1$.*

Proof. Since $\mathbb{C}H^1_{\mathbb{R}}(\mathfrak{m}, \mathfrak{g})$ is isomorphic with $H^1_{\mathbb{C}}(\mathbb{C}\mathfrak{m}, \mathbb{C}\mathfrak{g})$, we first calculate $H^1_{\mathbb{C}}(\mathbb{C}\mathfrak{m}, \mathbb{C}\mathfrak{g})$. Utilizing the decomposition $\mathbb{C}\mathfrak{g} = \mathfrak{g}^{1,0} \oplus \mathfrak{g}^{0,1}$, we have

$$C_{\mathbb{C}}(\mathbb{C}\mathfrak{m}, \mathbb{C}\mathfrak{g}) = \mathfrak{g}^{1,0} \otimes \wedge \mathbb{C}\mathfrak{m}^* \oplus \mathfrak{g}^{0,1} \otimes \wedge \mathbb{C}\mathfrak{m}^*.$$

This is the eigenspace decomposition of the complex structure $J \otimes_{\mathbb{R}} \mathrm{id}$. Obviously ∂ preserves this decomposition. Thus to calculate $H^1_{\mathbb{R}}(\mathfrak{m}, \mathfrak{g})$, we need only to calculate the cohomology of $(\bar{C}, \partial)$, where $\bar{C} = \mathfrak{g}^{1,0} \otimes \wedge \mathbb{C}\mathfrak{m}^*$. Moreover we have the decomposition of $\wedge \mathbb{C}\mathfrak{m}^*$;

$$\wedge^q \mathbb{C}\mathfrak{m}^* = \bigoplus_{r+s=q} \wedge^{r,s} \mathbb{C}\mathfrak{m}^*,$$

which is induced from $\mathbb{C}\mathfrak{m} = \mathfrak{m}^{1,0} \oplus \mathfrak{m}^{0,1}$. Thus we have $\bar{C}^0 = \mathfrak{g}^{1,0}$, $\bar{C}^1 = \mathfrak{g}^{1,0} \otimes (\wedge^{1,0}\mathfrak{m}^* \oplus \wedge^{0,1}\mathfrak{m}^*)$ and $\bar{C}^2 = \mathfrak{g}^{1,0} \otimes (\wedge^{2,0}\mathfrak{m}^* \oplus \wedge^{1,1}\mathfrak{m}^* \oplus \wedge^{0,2}\mathfrak{m}^*)$. Then, from $[\mathfrak{g}^{1,0}, \mathfrak{g}^{0,1}] = \{0\}$, we get

$$\begin{aligned} \partial\, \mathfrak{g}^{1,0} &\subset \mathfrak{g}^{1,0} \otimes \wedge^{1,0}\mathfrak{m}^*, \\ \partial\, (\mathfrak{g}^{1,0} \otimes \wedge^{1,0}\mathfrak{m}^*) &\subset \mathfrak{g}^{1,0} \otimes \wedge^{2,0}\mathfrak{m}^*, \\ \partial\, (\mathfrak{g}^{1,0} \otimes \wedge^{0,1}\mathfrak{m}^*) &\subset \mathfrak{g}^{1,0} \otimes (\wedge^{1,1}\mathfrak{m}^* \oplus \wedge^{0,2}\mathfrak{m}^*). \end{aligned}$$

Here we note that $\mathfrak{g}^{1,0} \otimes \wedge^{1,0}\mathfrak{m}^*$ (resp. $\mathfrak{g}^{1,0} \otimes \wedge^{0,1}\mathfrak{m}^*$) is naturally identified with the space of complex linear (resp. conjugate linear) mappings of $(\mathfrak{m}, J)$ into $(\mathfrak{g}, J)$. Hence $H^1_{\mathbb{R}}(\mathfrak{m}, \mathfrak{g})$ is isomorphic with

$$H^1_{\mathbb{C}}(\mathfrak{m}, \mathfrak{g}) \oplus \bar{Z},$$

where $\bar{Z} = \{p \colon \mathfrak{m} \to \mathfrak{g};\ \text{conjugate linear} \mid \partial p = 0\}$. For a conjugate linear map $p \colon \mathfrak{m} \to \mathfrak{g}$, we calculate

$$\begin{aligned} (\partial p)(JX, Y) &= [JX, p(Y)] - [Y, p(JX)] - p([JX, Y]) \\ &= J\{[X, p(Y)] + [Y, p(X)] + p([X, Y])\}, \end{aligned}$$

for X, $Y \in \mathfrak{m}$. Hence $\partial p = 0$ if and only if $[X, p(Y)] = 0$ and $p([X,Y]) = 0$ for X, $Y \in \mathfrak{m}$. Then, by Lemma 3.2, we get

$$\bar{Z} = \{\, p\colon \mathfrak{m} \to \mathfrak{g};\ \text{conjugate linear} \mid p(\mathfrak{g}_{-1}) \subset \mathfrak{g}_{-\mu} \text{ and } p(\mathfrak{g}_q) = \{0\} \text{ for } q < -1 \,\}.$$

Therefore we obtain

$$\bar{Z} \subset \mathfrak{g}_{-\mu} \otimes \mathfrak{g}_{-1}^{*} \subset C_{\mathbb{R}}^{-\mu+1,1}(\mathfrak{m}, \mathfrak{g}),$$

which completes the proof.

Thus, if $H_{\mathbb{C}}^{p,1}(\mathfrak{m}, \mathfrak{g}) = \{0\}$ for $p \geqq 0$ ($\mu > 1$), a simple graded Lie algebra $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ over $\mathbb{C}$, such that $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ is fundamental, is the prolongation of $\mathfrak{m}$ as a graded Lie algebra over $\mathbb{R}$ as well as over $\mathbb{C}$. In this case, the standard differential system $(M(\mathfrak{m}), D_{\mathfrak{m}})$ of type $\mathfrak{m}$ is a holomorphic differential system on a complex Lie group $M(\mathfrak{m})$. Then Lemma 3.5 implies that, if $H_{\mathbb{C}}^{p,1}(\mathfrak{m}, \mathfrak{g}) = \{0\}$ for $p \geqq 0$, every real infinitesimal automorphism of $(M(\mathfrak{m}), D_{\mathfrak{m}})$ is necessarily holomorphic.

In view of the discussion in this paragraph, we shall be mainly concerned with simple graded Lie algebras over $\mathbb{C}$ in the susequent discussion.

3.3. Gradation and the root space decomposition

Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbb{C}$. We shall describe the gradation of $\mathfrak{g}$ in terms of the root space decomposition of $\mathfrak{g}$. Our standard reference in this section are [Hu] and [He].

Let E be the characteristic element of $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$. Since $\operatorname{ad}(E)$ is a semisimple endomorphism of $\mathfrak{g}$, we can take a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ such that $E \in \mathfrak{h}$. Let Φ be the set of roots of $\mathfrak{g}$ relative to $\mathfrak{h}$. Then we have the root space decomposition of $\mathfrak{g}$;

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha,$$

where $\mathfrak{g}_\alpha = \{\, X \in \mathfrak{g} \mid [H, X] = \alpha(H)\, X \text{ for all } H \in \mathfrak{h} \,\}$ is the root space for $\alpha \in \Phi$. It follows from $E \in \mathfrak{h}$ that

$$\mathfrak{g}_0 = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi_0} \mathfrak{g}_\alpha,$$

$$\mathfrak{g}_p = \bigoplus_{\alpha \in \Phi_p} \mathfrak{g}_\alpha \quad (p \neq 0),$$

where $\Phi_p = \{\alpha \in \Phi \mid \alpha(E) = p\}$. Moreover, since $\alpha(E) \in \mathbb{Z}$ for $\alpha \in \Phi$, E belongs to the real part $\mathfrak{h}_{\mathbb{R}} = \{X \in \mathfrak{h} \mid \alpha(X) \in \mathbb{R}$ for $\alpha \in \Phi\}$ of $\mathfrak{h}$. Let $\mathfrak{h}^\sharp = \langle\Phi\rangle_{\mathbb{R}}$ be the real linear subspace of $\mathfrak{h}^*$ spanned by all roots of $\mathfrak{g}$. Identifying $\mathfrak{h}^*$ with $\mathfrak{h}$ by the Killing form B of $\mathfrak{g}$, we know that $\mathfrak{h}^\sharp$ corresponds to $\mathfrak{h}_{\mathbb{R}}$ and that the Killing form B gives a positive definite inner product $(\ ,\ )$ on $\mathfrak{h}_{\mathbb{R}}$. Then, by fixing a Weyl chamber D of $\mathfrak{h}_{\mathbb{R}}$ such that its closure $\bar{D}$ contains E, we can choose a simple root system $\Delta = \{\alpha_1, \ldots, \alpha_\ell\}$ of Φ such that $\alpha(E) \geqq 0$ for all $\alpha \in \Delta$. Then E determines a partition $\Phi^+ = \cup_{k\geqq 0}\Phi_k^+$ of the set Φ^+ of positive roots by $\Phi_k^+ = \{\alpha \in \Phi^+ \mid \alpha(E) = k\}$ such that

$$(3.1)\qquad \begin{aligned} \mathfrak{g}_0 &= \mathfrak{h} \oplus \bigoplus_{\alpha\in\Phi_0^+} (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}),\\ \mathfrak{g}_k &= \bigoplus_{\alpha\in\Phi_k^+} \mathfrak{g}_\alpha, \quad \mathfrak{g}_{-k} = \bigoplus_{\alpha\in\Phi_k^+} \mathfrak{g}_{-\alpha} \qquad (k > 0). \end{aligned}$$

This explains the symmetry of gradations of semisimple graded Lie algebras. Here we note that $\Phi_0 = \{\alpha \in \Phi \mid \alpha(E) = 0\}$ forms a subsystem of the root system Φ with a simple root system $\Delta_0 = \{\alpha \in \Delta \mid \alpha(E) = 0\}$.

Conversely let us fix a Cartan subalgebra $\mathfrak{h}$ of a semisimple Lie algebra $\mathfrak{g}$ and choose a simple root system $\Delta = \{\alpha_1, \ldots, \alpha_\ell\}$ of the root system Φ of $\mathfrak{g}$ relative to $\mathfrak{h}$. Then, given a ℓ-tuple $(a_1, \ldots, a_\ell)$ of non-negative integers, we see that an element $E \in \mathfrak{h}_{\mathbb{R}}$, which is defined by $\alpha_i(E) = a_i$, gives a gradation $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ of $\mathfrak{g}$ such that (3.1) holds.

With the above choice of $\mathfrak{h}$ and Δ, putting $\mathfrak{g}' = \bigoplus_{p\geqq 0} \mathfrak{g}_p$, we have

$$\mathfrak{g}' = \bigoplus_{\alpha\in\Phi_0^+} \mathfrak{g}_{-\alpha} \oplus \mathfrak{h} \oplus \bigoplus_{\alpha\in\Phi^+} \mathfrak{g}_\alpha = \bigoplus_{\alpha\in\Phi_0^+} \mathfrak{g}_{-\alpha} \oplus \mathfrak{B}(\Delta),$$

where $\mathfrak{B}(\Delta) = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Phi^+} \mathfrak{g}_\alpha$ is a standard Borel subalgebra of $\mathfrak{g}$ relative to $\mathfrak{h}$ ([Hu, Chapter IV]). Hence $\mathfrak{g}'$ is a parabolic subalgebra of $\mathfrak{g}$. In fact $\mathfrak{g}' = \mathfrak{P}(\Delta_0)$ is the standard parabolic subalgebra corresponding to Δ_0. For the subalgebra $\mathfrak{g}_0$, we have

Proposition 3.6. *$\mathfrak{g}_0$ is a reductive Lie algebra such that*

(1) *Dimension of the center $Z(\mathfrak{g}_0)$ of $\mathfrak{g}_0$ is equal to the number of simple roots in $\Delta \setminus \Delta_0$.*
(2) *$[\mathfrak{g}_0, \mathfrak{g}_0]$ is a semisimple Lie algebra with the root system Φ_0 and is a Levi subalgebra of $\mathfrak{g}'$.*

Proof. Let $\mathfrak{h}_0 = \langle \Delta_0 \rangle_{\mathbb{C}}$ be the linear subspace of $\mathfrak{h}^*$ spanned by elements of Δ_0. Identifying $\mathfrak{h}^*$ with $\mathfrak{h}$ via the Killing form duality, we have an orthogonal decomposition of $\mathfrak{h}$;

$$\mathfrak{h} = \mathfrak{h}_0 \oplus \mathfrak{h}_0^{\perp},$$

which in fact arises from an orthogonal decomposition in $\mathfrak{h}_{\mathbb{R}}$. Then we have $[\mathfrak{h}_0^{\perp}, \mathfrak{g}_0] = \{0\}$ and

$$[\mathfrak{g}_0, \mathfrak{g}_0] = \mathfrak{h}_0 \oplus \bigoplus_{\alpha \in \Phi_0^+} (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}).$$

Thus, by Serre's Theorem ([Hu, Theorem 18.3]), $[\mathfrak{g}_0, \mathfrak{g}_0]$ is a semisimple Lie algebra with a simple root system Δ_0. Hence we have $\mathfrak{h}_0^{\perp} = Z(\mathfrak{g}_0)$.

Remark 3.7. Let $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ be a semisimple graded Lie algebra over $\mathbb{C}$ such that $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ is fundamental. $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ is called primitive if $\mathfrak{g}$ is effective and $\mathrm{ad}\colon \mathfrak{g}_0 \to \mathfrak{gl}(\mathfrak{g}_{-1})$ is irreducible. If $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ is primitive, then $\mathfrak{g}$ is simple and it follows from Schur's Lemma that $\dim Z(\mathfrak{g}_0) = 1$. Then, by Proposition 3.6, $\mathfrak{g}'$ is a maximal parabolic subalgebra. In fact $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ is primitive if and only if $\mathfrak{g}$ is simple and $\mathfrak{g}'$ is a maximal parabolic subalgebra of $\mathfrak{g}$ (cf. the proof of Lemma 3.8. See also §5.4).

Let $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ be a semisimple graded Lie algebra over $\mathbb{R}$. In the real case, we should start with a Cartan decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

of $\mathfrak{g}$ such that $E \in \mathfrak{p}$ (cf. [M]). In fact such a Cartan decomposition can be found by Theorem 7.1 of [He, p. 182]. We first take a (complex) Cartan subalgebra $\widehat{\mathfrak{h}}$ of $\mathbb{C}\mathfrak{g}$ such that $E \in \widehat{\mathfrak{h}}$. Moreover we take a compact real form $\mathfrak{u}$ of $\mathbb{C}\mathfrak{g}$ by choosing a Weyl basis of $\mathfrak{g} = \widehat{\mathfrak{h}} \oplus \bigoplus_{\alpha \in \widehat{\Phi}} \mathfrak{g}_\alpha$. Then we have $E \in \widehat{\mathfrak{h}}_{\mathbb{R}} \subset \sqrt{-1}\mathfrak{u}$. Let σ and τ denote the conjugations of $\mathbb{C}\mathfrak{g}$ with respect to $\mathfrak{g}$ and $\mathfrak{u}$ respectively. Putting $N = \sigma \cdot \tau$, we have $N(E) = -E$. Hence $P(E) = E$ for $P = N^2$. By Theorem 7.1 of [He], a Cartan decomposition of $\mathfrak{g}$ is obtained by putting

$$\mathfrak{k} = \mathfrak{g} \cap \varphi(\mathfrak{u}),$$
$$\mathfrak{p} = \mathfrak{g} \cap \varphi(\sqrt{-1}\mathfrak{u}),$$

where $\varphi = P^{\frac{1}{4}}$. Then, from $\varphi(E) = E$, we see that $E \in \mathfrak{p}$. Here we note that, from $\tau_o(E) = -E$, the conjugation τ_o with respect to $\varphi(\mathfrak{u})$ reverses the gradation of $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$, that is, $\tau_o(\mathfrak{g}_p) = \mathfrak{g}_{-p}$.

Let us take a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$ such that $E \in \mathfrak{a}$. Moreover let $\mathfrak{h}$ be a maximal abelian subalgebra of $\mathfrak{g}$ containing $\mathfrak{a}$. Then $\mathbb{C}\mathfrak{h}$ is a Cartan subalgebra of $\mathbb{C}\mathfrak{g}$ such that $\mathfrak{a} = (\mathbb{C}\mathfrak{h})_{\mathbb{R}} \cap \mathfrak{g}$ ([He, p. 259, Lemma 3.2]). Hence the root space decomposition $\mathbb{C}\mathfrak{g} = \mathbb{C}\mathfrak{h} \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha$ of $\mathbb{C}\mathfrak{g}$ or more directly the simultaneous diagonalization of $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{a})$ induces the restricted root space decomposition of $\mathfrak{g}$;

$$\mathfrak{g} = Z(\mathfrak{a}) \oplus \bigoplus_{\lambda\in\Sigma} \mathfrak{g}_\lambda,$$

where $Z(\mathfrak{a})$ is the centralizer of $\mathfrak{a}$ in $\mathfrak{g}$ and Σ is the set of restricted roots of $\mathfrak{g}$ relative to $\mathfrak{a}$ ([He, p. 263]). A restricted root $\lambda \in \mathfrak{a}^*$ is a non-zero linear form on $\mathfrak{a}$ obtained as the restriction of some root $\alpha \in \Phi \subset (\mathbb{C}\mathfrak{h})^*$ to the subspace $\mathfrak{a}$ of $(\mathbb{C}\mathfrak{h})_{\mathbb{R}}$. Σ forms a root system in $\mathfrak{a}^*$, which in general is not reduced ([He, Chapter VII]). Thus, by fixing a Weyl chamber D of $\mathfrak{a}$ such that $E \in \bar{D}$, we have a simple root system $\widehat{\Delta} = \{\lambda_1, \ldots, \lambda_p\}$ of Σ such that $\lambda_i(E) \geqq 0$ for $\lambda_i \in \widehat{\Delta}$. Then the gradation of $\mathfrak{g}$ can be described as

$$\mathfrak{g}_0 = Z(\mathfrak{a}) \oplus \bigoplus_{\lambda\in\Sigma_0^+} (\mathfrak{g}_\lambda \oplus \mathfrak{g}_{-\lambda}),$$

$$\mathfrak{g}_k = \bigoplus_{\lambda\in\Sigma_k^+} \mathfrak{g}_\lambda, \quad \mathfrak{g}_{-k} = \bigoplus_{\lambda\in\Sigma_k^+} \mathfrak{g}_{-\lambda} \qquad (k > 0),$$

where $\Sigma_k^+ = \{\, \lambda \in \Sigma^+ \mid \lambda(E) = k \,\}$. For the details, we refer the reader to [K-A].

3.4. Generating condition of $\mathfrak{m}$

Let $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ be a simple graded Lie algebra over $\mathbb{C}$. As in the previous paragraph, let us fix a Cartan subalgebra $\mathfrak{h}$ and a simple root system Δ such that $E \in \mathfrak{h}$ and $\alpha(E) \geqq 0$ for any $\alpha \in \Delta$. Then, for the generating condition of $\mathfrak{m}$, we have (cf. [K-A, Lemma 2.3])

Lemma 3.8. *$\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ satisfies $\mathfrak{g}_p = [\mathfrak{g}_{p+1}, \mathfrak{g}_{-1}]$ for $p < -1$, if and only if $\alpha(E) = 0$ or 1 for any $\alpha \in \Delta$.*

Proof. We have $\mathfrak{g}_{-k} = \bigoplus_{\alpha\in\Phi_k^+} \mathfrak{g}_{-\alpha}$ for $k > 0$, where $\Phi_k^+ = \{\alpha \in \Phi^+ \mid \alpha(E) = k\}$ and $\Phi^+ = \cup_{k\geqq 0} \Phi_k^+$. Then it follows that $\mathfrak{g}_{-(k+1)} = [\mathfrak{g}_{-k}, \mathfrak{g}_{-1}]$ if and only if each $\alpha \in \Phi_{k+1}^+$ can be written as a sum $\alpha = \beta + \gamma$ of some $\beta \in \Phi_k^+$ and $\gamma \in \Phi_1^+$. Hence $\mathfrak{m}$ satisfies the generating condition if and only if each $\alpha \in \Phi_k^+$ can be written as a sum of k elements of Φ_1^+.

Therefore, if $\mathfrak{m}$ satisfies the generating condition, every simple root must belong to Φ_1^+ or Φ_0^+.

Conversely assume that $\alpha(E) = 0$ or 1 for any $\alpha \in \Delta$. We start with the following property of roots (cf. [Hu, p. 50, Lemma A]):

If $\beta \in \Phi$ is positive but not simple, then $\beta - \alpha \in \Phi^+$ for some $\alpha \in \Delta$.

Hence each $\beta \in \Phi^+$ can be written as $\beta = \alpha_1 + \cdots + \alpha_k$ $(\alpha_i \in \Delta)$ such that $\alpha_1 + \cdots + \alpha_i \in \Phi^+$ for $i = 1, 2, \ldots, k$. This implies a root vector of $\mathfrak{g}_\beta$ can be written as $[x_{\alpha_k}, [\cdots, [x_{\alpha_2}, x_{\alpha_1}]\cdots]]$, where x_{α_i} is a root vector of $\mathfrak{g}_{\alpha_i}$ $(\alpha_i \in \Delta)$. By our assumption, x_α belong to $\mathfrak{g}_0$ or $\mathfrak{g}_1$ for any $\alpha \in \Delta$. Therefore it follows that

(1) $\mathfrak{m} \oplus \mathfrak{g}_0 = \bigoplus_{p \leqq 0} \mathfrak{g}_p$ is generated by $\mathfrak{g}_{-1}$ and $\mathfrak{g}_0$,

(2) $\widehat{\mathfrak{m}} \oplus \mathfrak{g}_0 = \bigoplus_{p \geqq 0} \mathfrak{g}_p$ is generated by $\mathfrak{g}_1$ and $\mathfrak{g}_0$.

Moreover $\mathfrak{m} = \bigoplus_{p=-1}^{-\mu} \mathfrak{g}_p$ for some $\mu > 0$ such that $\mathfrak{g}_p \neq \{0\}$ for $p = -1, -2, \ldots, -\mu$.

Now starting from $\mathfrak{a}_\mu = \mathfrak{g}_\mu$, we define a subspace $\mathfrak{a}_p$ of $\mathfrak{g}_p$ for $p < \mu$ inductively by $\mathfrak{a}_p = [\mathfrak{a}_{p+1}, \mathfrak{g}_{-1}]$ and put

$$\mathfrak{a} = \bigoplus_{p=-\mu}^{\mu} \mathfrak{a}_p.$$

Then, as in the proof of Lemma 3.2, we can check that $\mathfrak{a}_p$ is $\mathrm{ad}(\mathfrak{g}_0)$-invariant and satisfies $[\mathfrak{a}_p, \mathfrak{g}_1] \subset \mathfrak{a}_{p+1}$ by (backward) induction on p. $\mathfrak{a}$ is $\mathrm{ad}(\mathfrak{g}_{-1})$-invariant by definition. Since $\mathfrak{g}$ is generated by $\mathfrak{g}_{-1}$, $\mathfrak{g}_0$ and $\mathfrak{g}_1$, we conclude $\mathfrak{a}$ is a non-trivial ideal of $\mathfrak{g}$. Then the simplicity of $\mathfrak{g}$ forces $\mathfrak{a} = \mathfrak{g}$. Especially $\mathfrak{g}_p = [\mathfrak{g}_{p+1}, \mathfrak{g}_{-1}]$ for $p < -1$.

Now let $\mathfrak{g}$ be a simple Lie algeba over $\mathbb{C}$. Let us fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and a simple root system $\Delta = \{\alpha_1, \ldots, \alpha_\ell\}$ of Φ. Take any non-empty subset Δ_1 of Δ and put

$$\Phi_k^+ = \{\, \alpha = \sum_{i=1}^{\ell} n_i(\alpha)\, \alpha_i \in \Phi^+ \mid \sum_{\alpha_i \in \Delta_1} n_i(\alpha) = k \,\} \quad \text{for } k \geqq 0.$$

Then, by Lemma 3.8, we can construct a (non-trivial) gradation of $\mathfrak{g}$

satisfying the generating condition for $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ by putting

$$\mathfrak{g}_0 = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi_0^+} (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}),$$

$$\mathfrak{g}_k = \bigoplus_{\alpha \in \Phi_k^+} \mathfrak{g}_\alpha, \quad \mathfrak{g}_{-k} = \bigoplus_{\alpha \in \Phi_k^+} \mathfrak{g}_{-\alpha} \qquad (k > 0),$$

or equivalently by defining the characteristic element $E \in \mathfrak{h}$ by

$$\alpha_i(E) = \begin{cases} 1 & \text{if } \alpha_i \in \Delta_1, \\ 0 & \text{if } \alpha_i \in \Delta_0 = \Delta \setminus \Delta_1. \end{cases}$$

We denote the simple graded Lie algebra $\mathfrak{g} = \bigoplus_{p=-\mu}^{\mu} \mathfrak{g}_p$ obtained from Δ_1 in this manner by (X_ℓ, Δ_1), when $\mathfrak{g}$ is a simple Lie algebra of type X_ℓ. Namely X_ℓ stands for the Dynkin diagram of $\mathfrak{g}$ representing Δ and Δ_1 is a subset of vertices of X_ℓ.

In this case the depth μ of (X_ℓ, Δ_1) can be computed by means of the heighest root θ of Φ. In fact we have $\theta \in \Phi_\mu^+$, because θ is the unique maximal root relative to the partial order $\succ$ of Φ, where $\alpha \succ \beta$ means that $\alpha - \beta$ is a sum of positive roots or $\alpha = \beta$ (cf. [Hu, Lemma 10.4.A]). Thus μ is given by

$$\mu = \sum_{\alpha_i \in \Delta_1} n_i(\theta),$$

where $\theta = \sum_{i=1}^{\ell} n_i(\theta)\, \alpha_i$.

As an illustration, let us examine the case of G_2. The Dynkin diagram of G_2 is given by

$$\underset{\alpha_1}{\circledcirc} \Lleftarrow \underset{\alpha_2}{\circledcirc}$$

and the set Φ^+ of positive roots consists of six elements (cf. [Bu]):

$$\Phi^+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2, 3\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2\}.$$

Here $\theta = 3\alpha_1 + 2\alpha_2$ and we have three choices for $\Delta_1 \subset \Delta = \{\alpha_1, \alpha_2\}$. Namely $\Delta_1 = \{\alpha_1\}$, $\{\alpha_2\}$ or $\{\alpha_1, \alpha_2\}$. Then the structure of each (G_2, Δ_1) is described as follows.

(1) $(G_2, \{\alpha_1\})$. We have $\mu = 3$ and Φ^+ decomposes as follows;

$$\Phi_3^+ = \{3\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2\}, \qquad \Phi_2^+ = \{2\alpha_1 + \alpha_2\},$$
$$\Phi_1^+ = \{\alpha_1, \alpha_1 + \alpha_2\}, \qquad \Phi_0^+ = \{\alpha_2\}.$$

Thus $\dim \mathfrak{g}_{-3} = \dim \mathfrak{g}_{-1} = 2$, $\dim \mathfrak{g}_{-2} = 1$ and $\dim \mathfrak{g}_0 = 4$. Hence $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ is isomorphic with $\mathbb{C}\mathfrak{m}_5$ in §1.3.

(2) $(G_2, \{\alpha_2\})$. We have $\mu = 2$ and Φ^+ decomposes as follows;

$$\Phi_2^+ = \{3\alpha_1 + 2\alpha_2\}, \qquad \Phi_0^+ = \{\alpha_1\},$$
$$\Phi_1^+ = \{\alpha_2, \alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2, 3\alpha_1 + \alpha_2\}.$$

Thus $\dim \mathfrak{g}_{-2} = 1$ and $\dim \mathfrak{g}_{-1} = \dim \mathfrak{g}_0 = 4$. Hence this is a contact gradation (cf. §4.2).

(3) $(G_2, \{\alpha_1, \alpha_2\})$. We have $\mu = 5$ and Φ^+ decomposes as follows;

$$\Phi_5^+ = \{3\alpha_1 + 2\alpha_2\}, \qquad \Phi_4^+ = \{3\alpha_1 + \alpha_2\}, \qquad \Phi_3^+ = \{2\alpha_1 + \alpha_2\},$$
$$\Phi_2^+ = \{\alpha_1 + \alpha_2\}, \qquad \Phi_1^+ = \{\alpha_1, \alpha_2\}, \qquad \Phi_0^+ = \emptyset.$$

Namely $(G_2, \{\alpha_1, \alpha_2\})$ is a gradation according to the height of roots and $\mathfrak{g}' = \bigoplus_{p \geqq 0} \mathfrak{g}_p$ is a Borel subalgebra. In this case, by utilizing a Chevalley basis of $\mathfrak{g}$ (cf. [Hu, p. 147]), one can check that $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ is isomorphic with $\mathbb{C}\mathfrak{m}_6$ in §1.3 (cf. example (3) in §5.3).

We shall see in §5.2 that G_2 is the prolongation of $(\mathfrak{m}, \mathfrak{g}_0)$ in case (2), and is the prolongation of $\mathfrak{m}$ in case (1) and (3).

Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{R}$ such that $\mathbb{C}\mathfrak{g}$ is simple. In the real case, we can utilize the Satake diagram S_ℓ of $\mathfrak{g}$ to describe gradations of $\mathfrak{g}$.

Let us fix a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$ and a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ containing $\mathfrak{a}$. $\mathbb{C}\mathfrak{h}$ is a Cartan subalgebra of $\mathbb{C}\mathfrak{g}$ such that $\mathfrak{a} = (\mathbb{C}\mathfrak{h})_{\mathbb{R}} \cap \mathfrak{g}$. Let Φ be the root system of $\mathbb{C}\mathfrak{g}$ relative to $\mathbb{C}\mathfrak{h}$ and Σ be the restricted root system of $\mathfrak{g}$ relative to $\mathfrak{a}$. Each $\lambda \in \Sigma$ is obtained by restricting some $\alpha \in \Phi$ to $\mathfrak{a} \subset (\mathbb{C}\mathfrak{h})_{\mathbb{R}}$.

Let σ denote the conjugation of $\mathbb{C}\mathfrak{g}$ with respect to $\mathfrak{g}$. Let us take a σ-fundamental system $\Delta = \{\alpha_1, \ldots, \alpha_\ell\}$ of Φ ([Sa]). Namely Δ is a simple root system of Φ satisfying the following property:

$$\text{If } \alpha \in \Phi^+ \text{ and } \alpha|_{\mathfrak{a}} \neq 0, \text{ then } \alpha^\sigma \in \Phi^+,$$

where $\alpha^\sigma \in \Phi$ is defined by $\alpha^\sigma(H) = \overline{\alpha(\sigma(H))}$ for $H \in \mathbb{C}\mathfrak{h}$. Put $\Delta^\bullet = \{\, \alpha \in \Delta \mid \alpha|_{\mathfrak{a}} = 0 \,\}$ and $\Delta^\circ = \Delta \setminus \Delta^\bullet$. Then there exists a permutation ν of order 2 of Δ° such that

$$\beta^\sigma = \nu(\beta) + \sum_{\alpha_i \in \Delta^\bullet} m_i\, \alpha_i,$$

for $\beta \in \Delta^\circ$ ([Sa, Lemma 1]). The Satake diagram S_ℓ of $\mathfrak{g}$ is constructed from the Dynkin diagram X_ℓ of $\mathbb{C}\mathfrak{g}$ representing Δ, firstly by marking simple roots of $\Delta^\bullet$ by black vertices and secondly by connecting two white vertices α_i and α_j of Δ° by an arrow when $\alpha_i|_{\mathfrak{a}} = \alpha_j|_{\mathfrak{a}}$, that is, when $\alpha_i = \nu(\alpha_j)$. A non-compact real form $\mathfrak{g}$ of a simple Lie algebra over $\mathbb{C}$ is determined by its Satake diagram S_ℓ. For an explicit construction of real form $\mathfrak{g}$ from its Satake diagram S_ℓ in terms of root vectors of $\mathbb{C}\mathfrak{g}$, we refer the reader to §4 of [Tk1]. Thus, from a σ-fundamental system Δ of Φ, we obtain a simple root system $\widehat{\Delta} = \{\lambda_1, \ldots, \lambda_p\}$ of Σ, by restricting $\alpha_i \in \Delta$ to $\mathfrak{a}$.

Now take any non-empty subset $\widehat{\Delta}_1$ of $\widehat{\Delta}$ and define $E \in \mathfrak{a}$ by

$$\lambda_i(E) = \begin{cases} 1 & \text{if } \lambda_i \in \widehat{\Delta}_1, \\ 0 & \text{if } \lambda_i \in \widehat{\Delta}_0 = \widehat{\Delta} \setminus \widehat{\Delta}_1. \end{cases}$$

Here we note that $\alpha(E) = 0$ or 1 for any $\alpha \in \Delta$ and that $\Delta_1 = \{\, \alpha \in \Delta \mid \alpha|_{\mathfrak{a}} \in \widehat{\Delta}_1 \,\}$ is a subset of the Satake diagram S_ℓ of $\mathfrak{g}$ which consists of white vertices and is stable under $\nu\colon \Delta^\circ \to \Delta^\circ$, that is, Δ_1 is a ν-invariant subset of Δ°. Then, by Lemma 3.8, E defines a gradation of $\mathfrak{g}$ such that $\mathbb{C}\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathbb{C}\mathfrak{g}_p$ satisfies the generating condition for $\mathbb{C}\mathfrak{m} = \bigoplus_{p<0} \mathbb{C}\mathfrak{g}_p$. Hence $\mathfrak{g} = \bigoplus_{p=-\mu}^{\mu} \mathfrak{g}_p$ is a simple graded Lie algebra over $\mathbb{R}$ such that $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ is fundamental. Moreover $\Delta_0 = \Delta \setminus \Delta_1$ is a σ-subsystem containing $\Delta^\bullet$, which corresponds to the parabolic subalgebra $\mathfrak{g}' = \bigoplus_{p\geqq 0} \mathfrak{g}_p$. We denote the simple graded Lie algebra over $\mathbb{R}$ obtained in this manner by (S_ℓ, Δ_1). In this case, the depth μ of (S_ℓ, Δ_1) can be computed by means of the highest root θ of the σ-fundamental system Δ;

$$\mu = \sum_{\alpha_i \in \Delta_1} n_i(\theta),$$

where $\theta = \sum_{i=1}^{\ell} n_i(\theta)\, \alpha_i$.

3.5. Conjugacy of simple graded Lie algebras

Let $\mathfrak{g}$ be a simple Lie algebra over $K = \mathbb{R}$ or $\mathbb{C}$. We denote by $\mathrm{Aut}(\mathfrak{g})$ the group of Lie algebra isomorphisms of $\mathfrak{g}$ over K, and by $\mathrm{Int}(\mathfrak{g})$ the adjoint group of $\mathfrak{g}$. $\mathrm{Int}(\mathfrak{g})$ coincides with the identity component of $\mathrm{Aut}(\mathfrak{g})$. We shall consider the conjugacy problems for gradations of $\mathfrak{g}$ satisfying the generating condition for $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ under the group $\mathrm{Aut}(\mathfrak{g})$ or $\mathrm{Int}(\mathfrak{g})$. Two gradations $\{\mathfrak{g}_p\}_{p\in\mathbb{Z}}$ and $\{\widehat{\mathfrak{g}}_p\}_{p\in\mathbb{Z}}$ are called conjugate under G if there exists $\varphi \in G$ such that $\varphi(\mathfrak{g}_p) = \widehat{\mathfrak{g}}_p$ for all $p \in \mathbb{Z}$, where $G = \mathrm{Aut}(\mathfrak{g})$ or $\mathrm{Int}(\mathfrak{g})$.

Let $\mathfrak{g} = \bigoplus_{p=-\mu}^{\mu} \mathfrak{g}_p$ be a simple graded Lie algebra over K such that $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ is fundamental. First we consider the filtration $\{\mathfrak{f}^p\}_{p\in\mathbb{Z}}$ of $\mathfrak{g}$ defined by

$$\mathfrak{f}^p = \bigoplus_{q \geqq p} \mathfrak{g}_q \qquad \text{for } p \in \mathbb{Z}.$$

Then $[\mathfrak{f}^p, \mathfrak{f}^q] \subset \mathfrak{f}^{p+q}$ for p, $q \in \mathbb{Z}$ and we have $\mathfrak{f}^p = \mathfrak{g}$ for $p \leqq -\mu$, $\mathfrak{f}^k = \{0\}$ for $k > \mu$ and $\mathfrak{f}^0 = \mathfrak{g}'$. Recall, by the argument in 3.3, that $\mathfrak{g}' = \mathfrak{P}$ is a parabolic subalgebra of $\mathfrak{g}$ (when $K = \mathbb{R}$, a subalgebra $\mathfrak{P}$ of $\mathfrak{g}$ is called parabolic if $\mathbb{C}\mathfrak{P}$ is parabolic in $\mathbb{C}\mathfrak{g}$). Furthermore, by Lemma 3.1, Lemma 3.2 and the generating condition of $\mathfrak{m}$, we have

Lemma 3.10. *The filtration $\{\mathfrak{f}^p\}_{p\in\mathbb{Z}}$ of $\mathfrak{g}$ is determined solely by $\mathfrak{P} = \mathfrak{f}^0$ and given as follows.*

(1) $\mathfrak{f}^1 = \{ X \in \mathfrak{P} \mid B(X, \mathfrak{P}) = 0 \}$ *and is the nilradical of* $\mathfrak{P}$.
(2) $\mathfrak{f}^k = C^k\mathfrak{f}^1 = [\mathfrak{f}^1, C^{k-1}\mathfrak{f}^1]$ *for* $k \geqq 2$, *where* $\mathfrak{f}^1 = C^1\mathfrak{f}^1$ *by convention.*
(3) $\mathfrak{f}^{-1} = \{ X \in \mathfrak{g} \mid [X, \mathfrak{f}^1] \subset \mathfrak{f}^0 \}$.
(4) $\mathfrak{f}^{-k} = C^k\mathfrak{f}^{-1} = [\mathfrak{f}^{-1}, C^{k-1}\mathfrak{f}^{-1}]$ *for* $k \geqq 2$, *where* $\mathfrak{f}^{-1} = C^1\mathfrak{f}^{-1}$ *by convention.*

The last statment in (1) can be obtained by describing the gradation in terms of the root space decomposition of $\mathbb{C}\mathfrak{g}$ as in 3.3.

By Lemma 3.10, we note that, for a simple graded Lie algebra $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ such that $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ is fundamental, the gradation is recovered from the parabolic subalgebra $\mathfrak{P} = \mathfrak{g}'$ firstly by forming the filtration $\{\mathfrak{f}^p\}_{p\in\mathbb{Z}}$ given by Lemma 3.10 and secondly by passing to the associated graded Lie algebra of $\{\mathfrak{f}^p\}_{p\in\mathbb{Z}}$. This observation leads us to the following.

Proposition 3.11. *Let $\{\mathfrak{g}_p\}_{p\in\mathbb{Z}}$ and $\{\widehat{\mathfrak{g}}_p\}_{p\in\mathbb{Z}}$ be two gradations of a simple Lie algebra $\mathfrak{g}$ over $K = \mathbb{R}$ or $\mathbb{C}$. Then $\{\mathfrak{g}_p\}_{p\in\mathbb{Z}}$ and $\{\widehat{\mathfrak{g}}\}_{p\in\mathbb{Z}}$ are conjugate under* $\mathrm{Aut}(\mathfrak{g})$ *(resp.* $\mathrm{Int}(\mathfrak{g})$*) if and only if $\mathfrak{P} = \bigoplus_{p\geqq 0} \mathfrak{g}_p$ and $\widehat{\mathfrak{P}} = \bigoplus_{p\geqq 0} \widehat{\mathfrak{g}}_p$ are conjugate under* $\mathrm{Aut}(\mathfrak{g})$ *(resp.* $\mathrm{Int}(\mathfrak{g})$*).*

Proof. Only if part is trivial. Let φ be an automorphism of $\mathfrak{g}$ such that $\varphi(\mathfrak{P}) = \widehat{\mathfrak{P}}$. Then, by Lemma 3.10, φ is an isomorphism as a filtered Lie algebra, that is, $\varphi(\mathfrak{f}^p) = \widehat{\mathfrak{f}}^p$ for all $p \in \mathbb{Z}$. Let $\widehat{\varpi}_p$ be the projection of $\widehat{\mathfrak{f}}^p$ onto $\widehat{\mathfrak{g}}_p$ corresponding to the decomposition $\widehat{\mathfrak{f}}^p = \bigoplus_{q\geqq p} \widehat{\mathfrak{g}}_q$. φ induces a graded map $\widehat{\varphi}$ of $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ onto $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \widehat{\mathfrak{g}}_p$ by

$$\widehat{\varphi}(X) = (\widehat{\varpi}_p \cdot \varphi)(X) \qquad \text{for } X \in \mathfrak{g}_p.$$

It is easy to see that $\widehat{\varphi}$ is a graded Lie algebra isomorphism of $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ onto $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \widehat{\mathfrak{g}}_p$. This finishes the proof for $\mathrm{Aut}(\mathfrak{g})$. Furthermore put $\psi = \widehat{\varphi}^{-1} \cdot \varphi$. Then ψ is a filtration preserving automorphism of $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$. Hence, by Lemma 1.7 of [T4], ψ can be written uniquely in the form

$$\psi = \varphi_0 \cdot \exp X_1 \cdots \exp X_\mu,$$

where $\varphi_0 \in G_0$, $X_k \in \mathfrak{g}_k$ and G_0 is the subgroup of $\mathrm{Aut}(\mathfrak{g})$ consisting of all gradation preserving automorphisms of $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$. Thus we obtain

$$\varphi = \widehat{\varphi}_0 \cdot \exp X_1 \cdots \exp X_\mu,$$

where $\widehat{\varphi}_0 = \widehat{\varphi} \cdot \varphi_0$ is a graded Lie algebra isomorphism of $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ onto $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \widehat{\mathfrak{g}}_p$, which completes the proof for $\mathrm{Int}(\mathfrak{g})$.

Thus the conjugacy of gradations of a simple Lie algebra $\mathfrak{g}$ over $K = \mathbb{R}$ or $\mathbb{C}$ satisfying the generating condition for $\mathfrak{m}$ is reduced to that of parabolic subalgebras of $\mathfrak{g}$.

The classification of parabolic subalgebras of a simple Lie algebra over $\mathbb{C}$ is achieved by the conjugacy of Borel subalgebras of $\mathfrak{g}$ (cf. [Hu, Chapter IV]): Every parabolic subalgebra in $\mathfrak{g}$ is conjugate to a standard parabolic subalgebra $\mathfrak{P}(\Delta_0)$, where Δ_0 is a subset of Δ. Moreover the conjugacy class of parabolic subalgebras under $\mathrm{Aut}(\mathfrak{g})$ is one to one correspondent to the equivalence class of (X_ℓ, Δ_0) under the diagram automorphisms of X_ℓ, where X_ℓ stands for the Dynkin diagram of $\mathfrak{g}$ and Δ_0 is any subset of X_ℓ. Similarly, in the real case, we have ([M, p. 431, Theorem 3.1]); the conjugacy class of parabolic subalgebras under $\mathrm{Aut}(\mathfrak{g})$ is one to one correspondent to the equivalence of (S_ℓ, Δ_0) under the diagram automorphisms of S_ℓ, where S_ℓ stands for the Satake diagram of $\mathfrak{g}$ and Δ_0 is any σ-subsystem containing $\Delta^\bullet$. For the details, we refer the reader to [M].

Summarizing we obtain (cf. [K-A, Theorem 2.7]. For the notation see 3.4.)

Theorem 3.12. *Let $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ be a simple graded Lie algebra over $K = \mathbb{R}$ or $\mathbb{C}$ such that $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ satisfies $\mathfrak{g}_p = [\mathfrak{g}_{p+1}, \mathfrak{g}_{-1}]$ for $p < -1$.*

(1) *The complex case. Let X_ℓ be the Dynkin diagram of $\mathfrak{g}$. Then $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ is isomorphic with a graded Lie algebra (X_ℓ, Δ_1) for some $\Delta_1 \subset \Delta$. Moreover (X_ℓ, Δ_1) and (X_ℓ, Δ_1') are isomorphic if and only if there exists a diagram automorphism ϕ of X_ℓ such that $\phi(\Delta_1) = \Delta_1'$.*

(2) *The real case. Let* S_ℓ *be the Satake diagram of* $\mathfrak{g}$. *Then* $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ *is isomorphic with a graded Lie algebra* (S_ℓ, Δ_1) *for some* ν-*invariant subset* Δ_1 *of* Δ°. *Moreover* (S_ℓ, Δ_1) *and* (S_ℓ, Δ_1') *are isomorphic if and only if there exists a diagram automorphism* ϕ *of* S_ℓ *such that* $\phi(\Delta_1) = \Delta_1'$.

§4. Standard differential system $(M_\mathfrak{g}, D_\mathfrak{g})$ of type $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$

4.1. Standard differential system $(M_\mathfrak{g}, D_\mathfrak{g})$

First we shall give general remarks on the model space associated with a simple graded Lie algebra.

Let $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ be a simple graded Lie algebra over $K = \mathbb{R}$ or $\mathbb{C}$ such that $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ is fundamental. We denote by $\mathrm{Int}(\mathfrak{g})$ the adjoint group of $\mathfrak{g}$. Let G_0 be the automorphism group of the graded Lie algebra $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$, that is, the subgroup of $\mathrm{Aut}(\mathfrak{g})$ consisting of elements which preserves the gradation. Then the Lie algebra of G_0 coincides with $\mathfrak{g}_0$ ([T3, Lemma 2.4]). Moreover let G' be the automorphism group of the filtered Lie algebra $\mathfrak{g} = \mathfrak{f}^{-\mu}$ (cf. Lemma 3.10). The Lie algebra of G' is $\mathfrak{g}' = \bigoplus_{p\geqq 0} \mathfrak{g}_p$.

Now we define an open subgroup G of $\mathrm{Aut}(\mathfrak{g})$ by

$$G = \mathrm{Int}(\mathfrak{g}) \cdot G' = \mathrm{Int}(\mathfrak{g}) \cdot G_0.$$

We consider the homogeneous space $M_\mathfrak{g} = G/G'$. $M_\mathfrak{g}$ is connected and compact (this is a consequence of the Iwasawa decomposition of G), on which G acts effectively. $M_\mathfrak{g} = G/G'$ is the model space for the normal Cartan connection of type $\mathfrak{g}$ constructed by N. Tanaka [T4]. Furthermore, when $\mu > 1$, by identifying $\mathfrak{g}$ with the Lie algebra of left invariant vector fields on G, $\mathfrak{f}^{-1}$ defines a left invariant subbundle of $T(G)$, which is also preserved by the right action of G' on G. Hence $\mathfrak{f}^{-1}$ induces a G-invariant differential system $D_\mathfrak{g}$ on $M_\mathfrak{g}$.

Here we remark that, when $\mathfrak{g}$ is a real simple Lie algebra of class A in Proposition 3.3, that is, when $\mathfrak{g}$ is a complex simple Lie algebra regarded as a real simple Lie algebra, the identity component $\mathrm{Int}(\mathfrak{g})$ of G is a complex Lie group. Hence $M_\mathfrak{g} = \mathrm{Int}(\mathfrak{g})/G' \cap \mathrm{Int}(\mathfrak{g})$ is a complex manifold such that $D_\mathfrak{g}$ is a holomorphic differential system on $M_\mathfrak{g}$. However G does not act on $M_\mathfrak{g}$ as a group of holomorphic transformations, although $\mathrm{Int}(\mathfrak{g})$ does. Namely the Lie group G changes depending on whether we regard $\mathfrak{g}$ as a real Lie algebra or as a complex Lie algebra, whereas $M_\mathfrak{g}$ remains the same. In fact the group of all automorphisms of $(M_\mathfrak{g}, D_\mathfrak{g})$, which coincides with G by Theorem 2.7 of [T4], under the assumption that $\mathfrak{g}$ is the prolongation of $\mathfrak{m}$, differs depending on whether we regard $(M_{\mathfrak{g}}, D_{\mathfrak{g}})$ as a real or a holomorphic differential differential system, whereas the Lie algebra $\mathcal{A}(M_{\mathfrak{g}}, D_{\mathfrak{g}})$ of all infinitesimal automorphisms remains the same (cf. Remark at the end of §3.2).

Thus $(M_{\mathfrak{g}}, D_{\mathfrak{g}})$ may be called the *standard differential system* of type $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$. In fact, let us fix a reference point o of $M_{\mathfrak{g}}$. Let $\widehat{M}$ be the analytic subgroup of G with Lie algebra $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$. Then, since $\widehat{M} \subset G \subset GL(\mathfrak{g})$, the unipotent linear subgroup $\widehat{M}$ is simply connected. Moreover, since $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{g}'$, $\widehat{M}$ has an open orbit through o. (This orbit is in fact diffeomorphic with $\widehat{M}$. This follows from the generalized Bruhat decomposition [Tk1, Theorem 8].) Thus the restriction of the projection $G \to M_{\mathfrak{g}} = G/G'$ gives a (local) diffeomorphism p of $\widehat{M} = M(\mathfrak{m})$ into $M_{\mathfrak{g}}$ such that $p(id_{\widehat{M}}) = o$. By the definition of $D_{\mathfrak{g}}$, we see that p is a (local) isomorphism of $(M(\mathfrak{m}), D_{\mathfrak{m}})$ into $(M_{\mathfrak{g}}, D_{\mathfrak{g}})$. We shall see in §5.2 that, in many cases, G coincides with the group $\mathrm{Aut}(M_{\mathfrak{g}}, D_{\mathfrak{g}})$ of all automorphisms of $(M_{\mathfrak{g}}, D_{\mathfrak{g}})$.

By the previous argument in §3, we know that $M_{\mathfrak{g}}$ is in fact a R-space, that is, $M_{\mathfrak{g}} = G/G'$ is a quotient space of a simple algebraic group G by a parabolic subgroup G' (cf. [Tt1], [Tk1]). Especially, when $\mathfrak{g}$ is complex simple, we know that $M_{\mathfrak{g}}$ is a compact simply connected projective algebraic manifold (cf. [Wa], [Se], [Tt1], [Tk1]). Hence, in this case, starting from any connected complex Lie group $\widetilde{G}$ with Lie algebra $\mathfrak{g}$, we can construct $M_{\mathfrak{g}}$ as $\widetilde{G}/\widetilde{G}'$, where $\widetilde{G}'$ is the analytic subgroup of $\widetilde{G}$ with Lie algebra $\mathfrak{g}'$.

Now let $\widehat{G}$ be the simply connected Lie group with Lie algebra $\mathfrak{g}$ and (ρ, V) be an irreducible representation of $\widehat{G}$ with the highest weight Λ, which is strongly associated to Φ_0 in the sense of Borel-Weil [Se]. Namely Λ is a dominant weight of $\mathfrak{g}$ such that $(\Lambda, \alpha) = 0$ for $\alpha \in \Delta_0$ and $(\Lambda, \alpha) > 0$ for $\alpha \in \Delta_1$. Then we obtain a $\widehat{G}$-equivariant projective imbedding of $M_{\mathfrak{g}}$ by taking a $\widehat{G}$-orbit passing through $[v_\Lambda]$ in the projective space $P(V)$ consisting of all lines in V, where v_Λ is a maximal vector in V of the heighest weight Λ. For the discussion of the real case, we refer the reader to [Tk1].

In the following, we shall give an example of this construction and also discuss explicit examples of $(M_{\mathfrak{g}}, D_{\mathfrak{g}})$ for simple Lie algebras of the classical type. Our emphasis will be on the differential system $D_{\mathfrak{g}}$.

Remark 4.0. In the complex case, since $M_{\mathfrak{g}}$ is a compact complex manifold, the group $\mathrm{Aut}(M_{\mathfrak{g}})$ of all holomorphic transformations of $M_{\mathfrak{g}}$ is a Lie transformation group acting on $M_{\mathfrak{g}}$. It is known ([On]) that

$\mathrm{Int}(\mathfrak{g})$ coincides with the identity component of $\mathrm{Aut}(M_{\mathfrak{g}})$ except when $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ is isomorphic with $(C_\ell, \{\alpha_1\})$ ($\ell \geqq 2$), $(B_\ell, \{\alpha_\ell\})$ ($\ell \geqq 3$) or $(G_2, \{\alpha_2\})$. In these exceptions, $M_{\mathfrak{g}}$ is biholomorphic with $P^{2\ell-1}(\mathbb{C})$, $SO(2\ell+2)/U(\ell+1)$ or $Q^5(\mathbb{C})$ (complex quadric) and the Lie algebra of $\mathrm{Aut}(M_{\mathfrak{g}})$ is of type $A_{2\ell}$, $D_{\ell+1}$ or B_3 respectively. These facts are pointed out to us by the referee (see also Remark 4.3 (1)).

4.2. Contact gradation

For each simple Lie algebera over $\mathbb{C}$, we shall show the existence of a complex contact gradation which is unique up to conjugacy (cf. [Bo], [Wo], [Ch], [Tk2]).

Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$. First assume that $\mathfrak{g}$ has a contact gradation, that is, $\mathfrak{g}$ admits a gradation of depth 2 such that $\mathfrak{g}_{-1} \neq \{0\}$ and $\dim \mathfrak{g}_{-2} = 1$;

$$\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2.$$

By Lemma 3.2 (1), the bracket operation $[\,,\,]\colon \mathfrak{g}_{-1} \times \mathfrak{g}_{-1} \to \mathfrak{g}_{-2}$ is non-degenerate. Let us fix a Cartan subalgebra $\mathfrak{h}$ and a simple root system $\Delta = \{\alpha_1, \dots, \alpha_\ell\}$ such that $E \in \mathfrak{h}$ and $\alpha(E) \geqq 0$ for $\alpha \in \Delta$. We have a partition of positive roots Φ^+;

$$\Phi^+ = \Phi_2^+ \cup \Phi_1^+ \cup \Phi_0^+.$$

Then, since $\dim \mathfrak{g}_{-2} = 1$, we have $\Phi_2^+ = \{\theta\}$, where θ is the highest root. Moreover, from the non-degeneracy of $[\,,\,]\colon \mathfrak{g}_{-1} \times \mathfrak{g}_{-1} \to \mathfrak{g}_{-2}$, we see that, for each $\alpha \in \Phi_1^+$, there exists $\beta \in \Phi_1^+$ such that $\alpha + \beta = \theta$. Hence $\Phi_1^+ = \{\, \alpha \in \Phi^+ \mid \theta - \alpha \text{ is a root}\,\}$. Since $\Phi^+ = \Phi_2^+ \cup \Phi_1^+ \cup \Phi_0^+$ is a partition, we get $\Phi_0^+ = \{\, \alpha \in \Phi^+ \setminus \{\theta\} \mid \theta - \alpha \text{ is not a root}\,\}$. On the other hand, since θ is a long root and $\theta + \alpha$ is not a root for any $\alpha \in \Phi^+$, we have (cf. [Hu, 9.4])

$$\langle \alpha, \theta \rangle = 0 \quad \text{or} \quad 1 \qquad \text{for any } \alpha \in \Phi^+ \setminus \{\theta\},$$

where $\langle \alpha, \theta \rangle = \dfrac{2(\alpha, \theta)}{(\theta, \theta)}$ is a Cartan integer. Moreover, by considering the α-string through θ, we see that $\langle \theta, \alpha \rangle = 0$ if and only if $\theta - \alpha$ is not a root. Therefore we obtain

$$\Phi_k^+ = \{\, \alpha \in \Phi^+ \mid \langle \alpha, \theta \rangle = k \,\} \qquad \text{for } k = 0,\ 1,\ 2.$$

This implies that the characteristic element E of $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ is given by $h_\theta = \dfrac{2\,t_\theta}{(\theta, \theta)} \in \mathfrak{h}$, where $t_\theta \in \mathfrak{h}$ is defined by $B(t_\theta, H) = \theta(H)$ for

$H \in \mathfrak{h}$. Conversely the above argument shows that $h_\theta \in \mathfrak{h}$ indeed defines a contact gradation of $\mathfrak{g}$.

Thus a contact graded Lie algebra $\mathfrak{g} = \bigoplus_{p=-2}^{2} \mathfrak{g}_p$ is isomorphic with (X_ℓ, Δ_θ), where $\Delta_\theta = \{ \alpha \in \Delta \mid \langle \alpha, \theta \rangle = 1 \}$. Here we note that, since depth of (X_ℓ, Δ_θ) is two, Δ_θ should consist of two elements $\{\alpha_i, \alpha_j\}$ of Δ satisfying $n_i(\theta) = n_j(\theta) = 1$ or consist of a single element α_i of Δ satisfying $n_i(\theta) = 2$. In fact the information of Δ_θ is expressed in the *extended Dynkin diagram* of $\mathfrak{g}$ and the former case can occur only when $\mathfrak{g}$ is of type A_ℓ ($\ell \geqq 2$). Thus Δ_θ is the subset of Δ consisting of simple roots which are connected to $-\theta$ in the extended Dynkin diagram of $\mathfrak{g}$.

Summarizing, we obtain (cf. [Wo, Theorem 4.2], [Ch], [Tk2, §1])

Theorem 4.1. *Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$ such that* rank $\mathfrak{g} \geqq 2$. *Then $\mathfrak{g}$ admits a unique complex contact gradation up to conjugacy. This gradation is isomorphic with (X_ℓ, Δ_θ), where $\Delta_\theta = \{ \alpha \in \Delta \mid \langle \alpha, \theta \rangle = 1 \}$ and θ is the highest root. Furthermore the characteristic element of (X_ℓ, Δ_θ) is given by $E = h_\theta \in \mathfrak{h}$.*

In the next page we show the extended Dynkin diagrams with the coefficient of the highest root.

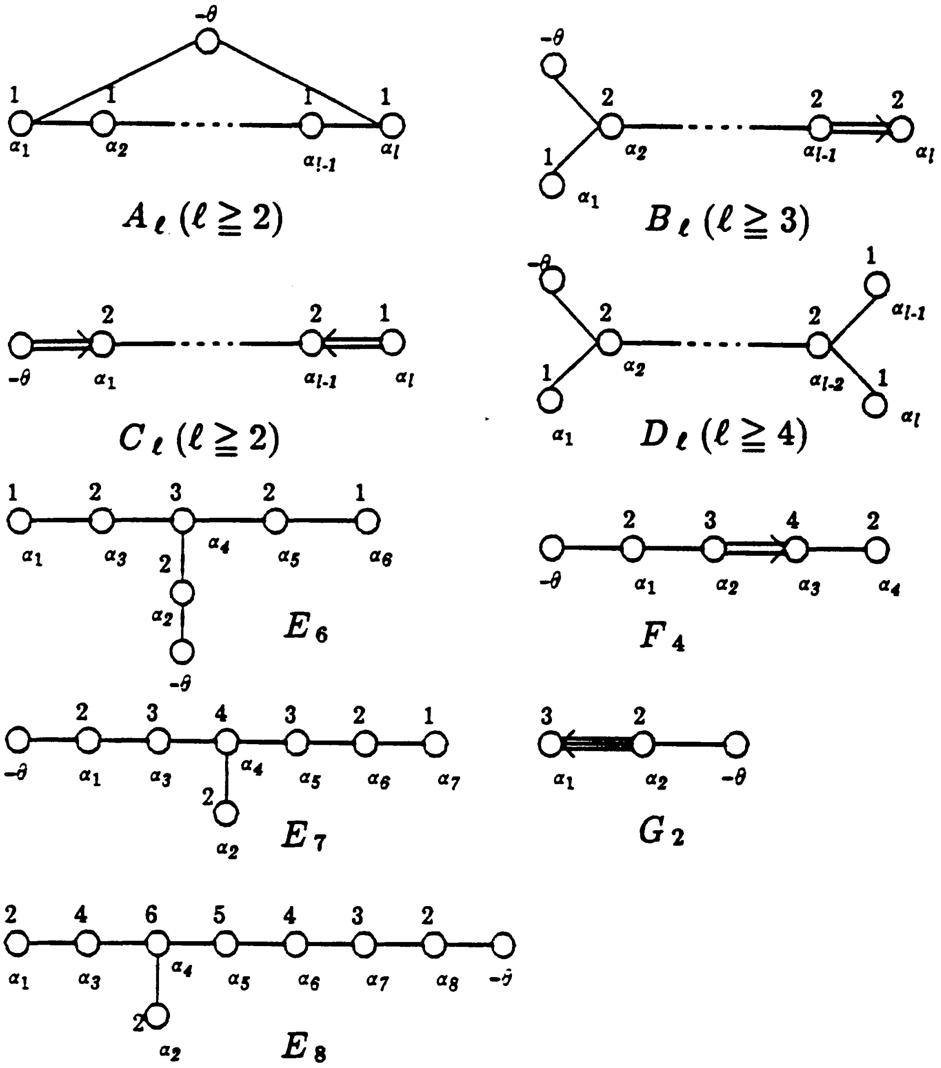
Remark 4.2. Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{R}$ such that $\mathbb{C}\mathfrak{g}$ is simple and rank $\mathbb{C}\mathfrak{g} \geqq 2$. By Theorem 4.1, to seek a real contact gradation of $\mathfrak{g}$, we need only to check whether Δ_θ is a ν-invariant subset of Δ^o in its Satake diagram or whether the lowest root $-\theta$ is not connected to any black vertex in the extended Satake diagram of $\mathfrak{g}$ ([Tk2, §3]). In this way, we obtain (cf. [Ch, Theorem 3])

*A real simple Lie algebra $\mathfrak{g}$ of class B admits a unique real contact gradation (S_ℓ, Δ_θ) up to conjugacy except for the cases when S_ℓ is of type*AI ($\ell = 1$), AII, BII, CII, DII, EIV *or* FII *in the list of table VI in* [He, Chapter X, p. 532]. *In the latter cases, they do not admit a contact gradation.*

For the details, we refer the reader to [Ch] and [Tk2, §1].

4.3. Standard contact manifolds

We shall discuss the standard differential sytem $(M_\mathfrak{g}, D_\mathfrak{g})$ associated with a contact gradation of a simple Lie algebra $\mathfrak{g}$ over $\mathbb{C}$ as an illustration of the method, mentioned in 4.1, of constructing the model space via a certain representation. Here we note that, by Theorem 4.1, the heighest root θ is a dominant weight strongly associated to $\Delta \setminus \Delta_\theta$, and θ is the heighest weight of the adjoint representation ad: $\mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$. In



The extended Dynkin diagrams

fact the standard differential system $(M_{\mathfrak{g}}, D_{\mathfrak{g}})$ of type (X_ℓ, Δ_θ) can be constructed via the adjoint representation as follows.

Let $\mathfrak{g}$ be a simple Lie algebra over $K = \mathbb{R}$ or $\mathbb{C}$ and let $\mathfrak{g} = \bigoplus_{p=-2}^{2} \mathfrak{g}_p$ be a contact gradation over K. Let $G = \mathrm{Int}(\mathfrak{g})$ be the adjoint group of $\mathfrak{g}$. Let us fix a non-zero vector $X_o \in \mathfrak{g}_2$. First we consider the adjoint orbit S of G passing through X_o. Since the adjoint representation and coadjoint representation of G are equivalent via the Killing form duality, it is well-known (cf. [A]) that S has a symplectic structure (over K). The symplectic structure on S is given as follows: Let ω_o be the covector corresponding to X_o, that is, $\omega_o \in \mathfrak{g}^*$ is defined by $\omega_o(X) = B(X_o, X)$ for $X \in \mathfrak{g}$. Then we have

$$\mathfrak{f}^{-1} = \bigoplus_{p \geqq -1} \mathfrak{g}_p = \{ X \in \mathfrak{g} \mid \omega_o(X) = 0 \}. \tag{4.1}$$

Let $\bar{G}$ be the isotropy subgroup of G at $X_o \in \mathfrak{g}$;

$$\bar{G} = \{ g \in G \mid \mathrm{Ad}(g)(X_o) = X_o \} = \{ g \in G \mid \mathrm{Ad}^*(g)(\omega_o) = \omega_o \}.$$

Then the Lie algebra $\bar{\mathfrak{g}}$ of $\bar{G}$ is given by

$$\bar{\mathfrak{g}} = \{ X \in \mathfrak{g} \mid [X, X_o] = 0 \} = Z_{\mathfrak{g}}(\mathfrak{g}_2).$$

On the other hand we see from the root space description of the contact gradation in 4.1 that $Z_{\mathfrak{g}}(\mathfrak{g}_2)$ is an ideal of $\mathfrak{g}' = \bigoplus_{p \geqq 0} \mathfrak{g}_p$ of codimension 1 such that $\mathfrak{g}' = \langle E \rangle_K \oplus Z_{\mathfrak{g}}(\mathfrak{g}_2)$ and that $\mathfrak{g}'$ is the normalizer $N_{\mathfrak{g}}(\mathfrak{g}_2)$ of $\mathfrak{g}_2$ in $\mathfrak{g}$. In particular $\bar{\mathfrak{g}} \subset \mathfrak{f}^{-1} = \{ X \in \mathfrak{g} \mid \omega_o(X) = 0 \}$. Then, for a left invariant 1-form ω_o on G, we have $R_g^* \omega_o = \mathrm{Ad}^*(g^{-1}) \omega_o = \omega_o$ for $g \in \bar{G}$ and $\omega_o(X) = 0$ for $X \in \bar{\mathfrak{g}}$, which implies that ω_o is projectable to $S = G/\bar{G}$. Namely there exists a G-invariant 1-form α on S such that $\pi^* \alpha = \omega_o$, where $\pi \colon G \to S = G/\bar{G}$ is the projection. Moreover, since $\iota(X) d\omega_o = L_X \omega_o$ for a left invariant vector field $X \in \mathfrak{g}$, we see from (4.2) that $X \in \bar{\mathfrak{g}}$ if and only if $\iota(X) d\omega_o = 0$. Therefore $d\alpha$ is a symplectic form on S. (For an arbitrary coadjoint orbit S_ω passing through $\omega \in \mathfrak{g}^*$, only $d\omega$ is projectable to S_ω.)

Now let us take a G-orbit $J_{\mathfrak{g}}$ passing through $[X_o] = \mathfrak{g}_2$ in the projective space $P(\mathfrak{g})$ over K. Let G' be the isotropy subgroup of G at $[X_o] \in P(\mathfrak{g})$:

$$\begin{aligned} G' &= \{ g \in G \mid \mathrm{Ad}(g)(X_o) = \rho(g) \cdot X_o \} \\ &= \{ g \in G \mid \mathrm{Ad}^*(g^{-1})(\omega_o) = \rho(g) \cdot \omega_o \}, \end{aligned}$$

where $\rho \colon G' \to K^\times$ defines a 1-dimensional representation of G'. From the existence of the characteristic element E, we see that ρ is not trivial. Hence we get $\mathrm{Ker}\, \rho = \bar{G}$, $G'/\bar{G}$ is isomorphic with K^* and the Lie algebra of G' coincides with $\mathfrak{g}' = N_{\mathfrak{g}}(\mathfrak{g}_2)$, where $K^* = \mathbb{C}^*$ when $K = \mathbb{C}$ and $K^* = \mathbb{R}^+$ or $\mathbb{R}^\times$ when $K = \mathbb{R}$ (see Remark 4.3 below). In particular S is stable under the K^*(scalar)-action of the ambient vector space $\mathfrak{g}$. Let p be the projection of S onto $J_{\mathfrak{g}}$, which is the restriction of the projection $p \colon \mathfrak{g} \setminus \{0\} \to P(\mathfrak{g})$. Then $(S, J_{\mathfrak{g}}, p)$ is a principal K^*-bundle over $J_{\mathfrak{g}}$. From $R_g^* \omega_o = \rho(g) \cdot \omega_o$ for $g \in G'$ and $\omega_o(X) = 0$ for $X \in \mathfrak{g}'$, we have $R_a^* \alpha = a \cdot \alpha$ for $a \in K^*$ and $\mathrm{Ker}\, p_* \subset \mathrm{Ker}\, \alpha = \{ X \in T(S) \mid \alpha(X) = 0 \}$, where $\operatorname{Ker} p_*$ is the vertical subbundle of $T(S)$ of the projection $p\colon S \to J_{\mathfrak{g}}$. Hence a G-invariant 1-form α on S defines a G-invariant differential system $C_{\mathfrak{g}}$ on $J_{\mathfrak{g}}$ of codimension 1 by

$$C_{\mathfrak{g}}(u) = p_*(\operatorname{Ker} \alpha(x)) \qquad \text{at each } u = p(x) \in J_{\mathfrak{g}}.$$

From (4.1), we see that $(J_{\mathfrak{g}}, C_{\mathfrak{g}})$ is a standard differential system of type $\mathfrak{g} = \bigoplus_{p=-2}^{2} \mathfrak{g}_p$. $(J_{\mathfrak{g}}, C_{\mathfrak{g}})$ is called the standard contact manifold of type $\mathfrak{g}$.

Furthermore we have an imbedding γ of S into $T^*(J_{\mathfrak{g}})$, which commutes with K^*-actions of S and $T^*(J_{\mathfrak{g}})$. In fact, since $\operatorname{Ker} p_* \subset \operatorname{Ker} \alpha$, for each $x \in S$, α determines a covector $\gamma(x) \in T_u^*(J_{\mathfrak{g}})$ at $u = p(x)$ such that $\gamma(x)(p_*(X)) = \alpha(X)$ for $X \in T_x(S)$. Then, via γ, $(S, d\alpha)$ is identified with the symplectification of $(J_{\mathfrak{g}}, C_{\mathfrak{g}})$ when $K = \mathbb{C}$ and with a connected component of the symplectification $(J_{\mathfrak{g}}, C_{\mathfrak{g}})$ when $K = \mathbb{R}$ (cf. [A], [K1]).

Standard contact manifolds associated with simple Lie algebras over $\mathbb{C}$ were first found by Boothby [Bo] as compact simply connected homogeneous complex contact manifolds. The above construction was also given in [Wo]. The advantage of this construction is a clarification of the contact structure on $M_{\mathfrak{g}}$ in a unified manner. We shall give a more explicit picture of $(J_{\mathfrak{g}}, C_{\mathfrak{g}})$ for the classical type in 4.5.

Remark 4.3. (1) In the complex case, it is known ([Wo]) that $\operatorname{Int}(\mathfrak{g})$ coincides with the identity component of the group $\operatorname{Aut}(J_{\mathfrak{g}}, C_{\mathfrak{g}})$ of all holomorphic contact transformations of $(J_{\mathfrak{g}}, C_{\mathfrak{g}})$.

(2) In the real case, $G'/\bar{G}$ is not necessarily connected. In fact $G'/\bar{G}$ is connected if and only if $(J_{\mathfrak{g}}, C_{\mathfrak{g}})$ admits a global contact form, or equivalently, if and only if the symplectification of $(J_{\mathfrak{g}}, C_{\mathfrak{g}})$ has two connected components. For example, $G'/\bar{G}$ is connected when $\mathfrak{g} = \mathfrak{su}(r+1, \ell - r)$ $(0 \leqq r \leqq [\frac{n-1}{2}])$ and is not connected when $\mathfrak{g} = \mathfrak{sl}(\ell + 1, \mathbb{R})$ or $\mathfrak{sp}(\ell, \mathbb{R})$.

4.4. Gradation and matrices

Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$ of the classical type. We shall describe gradations of $\mathfrak{g}$ in terms of matrices. Here we reproduce the matrices description of the root space decomposition of $\mathfrak{g}$ from §7 of [Tk1] (cf. [K-A], [V, Chapter 4.4]), which gives us explicit pictures of $M_{\mathfrak{g}}$.

(1) A_ℓ type $(\ell \geqq 1)$. $\mathfrak{g} = \mathfrak{sl}(\ell+1, \mathbb{C})$. We take a Cartan subalgebra $\mathfrak{h}$ consisting of all diagonal elements of $\mathfrak{sl}(\ell + 1, \mathbb{C})$, whose member we denote by $\operatorname{diag}(a_1, \ldots, a_{\ell+1})$. Let $\lambda_1, \ldots, \lambda_{\ell+1}$ be the linear form on

$\mathfrak{h}$ defined by λ_i: $\mathrm{diag}(a_1, \dots, a_{\ell+1}) \mapsto a_i$. We write E_{ij} $(1 \leqq i, j \leqq \ell + 1)$ for the matrix whose (i, j)-component is 1 and all of whose other components are 0. Then we have

$$[H, E_{ij}] = (\lambda_i - \lambda_j)(H)\, E_{ij} \qquad \text{for } H \in \mathfrak{h}.$$

Hence $\Phi = \{\lambda_i - \lambda_j \in \mathfrak{h}^* \ (1 \leqq i, j \leqq \ell + 1,\ i \neq j)\}$ and E_{ij} spans the root subspace for $\lambda_i - \lambda_j \in \Phi$. Let us choose a simple root system $\Delta = \{\alpha_1, \dots, \alpha_\ell\}$ by putting

$$\alpha_i = \lambda_i - \lambda_{i+1}.$$

We have $\lambda_i - \lambda_j = \alpha_i + \cdots + \alpha_{j-1}$ when $i < j$. Hence $\theta = \alpha_1 + \cdots + \alpha_\ell$. Then we see that the gradation of $(A_\ell, \{\alpha_i\})$ is given by $\mathfrak{sl}(\ell + 1, \mathbb{C}) = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$;

$$\mathfrak{g}_{-1} = \left\{ \begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix} \mid C \in M(j, i) \right\}, \ \mathfrak{g}_1 = \left\{ \begin{pmatrix} 0 & D \\ 0 & 0 \end{pmatrix} \mid D \in M(i, j) \right\},$$

$$\mathfrak{g}_0 = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \mid A \in M(i, i),\ B \in M(j, j) \text{ and } \operatorname{tr} A + \operatorname{tr} B = 0 \right\},$$

where $j = \ell - i + 1$ and $M(p, q)$ denotes the set of $p \times q$ matrices. This decomposition can be described schematically by the following diagram;

	i	j
i	0	1
j	-1	0

where the vertical (resp. horizontal) line stands for the i-th vertical (resp. horizontal) intermediate line of a matrix in $\mathfrak{sl}(\ell + 1, \mathbb{C})$. Then, for example, the diagram of $(A_\ell, \{\alpha_i, \alpha_j\})$ $(i < j)$ is obtained by superposing the diagrams of $(A_\ell, \{\alpha_i\})$ and $(A_\ell, \{\alpha_j\})$;

0	1
-1	0

0	1
-1	0

$\Rightarrow$

0	1	2
-1	0	1
-2	-1	0

In general the diagram of $(A_\ell, \{\alpha_{i_1}, \dots, \alpha_{i_k}\})$ is obtained by superposing the k diagrams of $(A_\ell, \{\alpha_{i_1}\})$, ... , $(A_\ell, \{\alpha_{i_k}\})$. Namely the gradation of $(A_\ell, \{\alpha_{i_1}, \dots, \alpha_{i_k}\})$ is obtained by subdividing matrices by both vertical and horizontal k lines. Here i-th intermediate line corresponds to the simple root α_i.

By this description of gradations, we see that the model space $M_{\mathfrak{g}}$ of $(A_\ell, \{\alpha_i\})$ is the complex Grassmann manifold $Gr(i, V)$ consisting of all i-dimensional subspaces of $V = \mathbb{C}^{\ell+1}$. Furthermore the model space $M_{\mathfrak{g}}$ of $(A_\ell, \{\alpha_{i_1}, \dots, \alpha_{i_k}\})$ $(1 \leqq i_1 < \cdots < i_k \leqq \ell)$ is the flag manifold $F(i_1, \dots, i_k; V)$ consisting of all flags $\{V_1 \subset \cdots \subset V_k\}$ in V such that $\dim V_j = i_j$ for $j = 1, \dots, k$ (cf. [Tt1]).

(2) C_ℓ type $(\ell \geqq 2)$. Let $(V, \langle\, , \,\rangle)$ be a symplectic vector space over $\mathbb{C}$ of dimension 2ℓ, that is, $\langle\, , \,\rangle$ is a non-degenerate skew symmetric bilinear form on V. Then $\mathfrak{g} = \mathfrak{sp}(V)$. Let us take a symlectic basis $\{e_1, \dots, e_\ell, f_1, \dots, f_\ell\}$ of V such that $\langle e_i, e_j\rangle = \langle f_i, f_j\rangle = 0$ and $\langle f_i, e_{\ell+1-j}\rangle = \delta_{ij}$ for $i, j = 1, \dots, \ell$. Thus we have a matrix representation

$$\mathfrak{g} = \{\, X \in \mathfrak{gl}(2\ell, \mathbb{C}) \mid {}^tXJ + JX = 0\,\}, \quad \text{where } J = \begin{pmatrix} 0 & K \\ -K & 0 \end{pmatrix},$$

and K is the $\ell \times \ell$ matrix whose (i,j)-component is $\delta_{i,\ell+1-j}$. We put $A' = KAK$ for $A \in \mathfrak{gl}(\ell, \mathbb{C})$. Namely A' is the "transposed" matrix of A with respect to the anti-diagonal line. Each $X \in \mathfrak{g}$ is expressed as a matrix of the following form;

$$X = \begin{pmatrix} A & B \\ C & -A' \end{pmatrix},$$

where A, B, C are $\ell \times \ell$ matrices such that B and C satisfy $B = B'$ and $C = C'$. Namely both B and C are symmetric with respect to the anti-diagonal line. Thus we see that X is determined by its upper anti-diagonal part. In the following we write $X = (A, B, C)$ in short.

We take a Cartan subalgebra $\mathfrak{h}$ consisting of all diagonal elements of the form $H = (\mathrm{diag}(a_1, \dots, a_\ell), 0, 0)$. Let $\lambda_1, \dots, \lambda_\ell$ be the linear form on $\mathfrak{h}$ defined by $\lambda_i \colon H \mapsto a_i$. We put $F_{ij} = E_{ij} + E'_{ij}$, where $E'_{ij} = E_{\ell+1-j, \ell+1-i}$. Then we have

$$\begin{aligned} [H, (E_{ij}, 0, 0)] &= (\lambda_i - \lambda_j)(H)(E_{ij}, 0, 0), \\ [H, (0, F_{ij}, 0)] &= (\lambda_i + \lambda_{\ell+1-j})(H)(0, F_{ij}, 0), \\ [H, (0, 0, F_{ij})] &= -(\lambda_{\ell+1-i} + \lambda_j)(H)(0, 0, F_{ij}). \end{aligned}$$

Hence $\Phi = \{\lambda_i - \lambda_j \ (i \neq j),\ \pm(\lambda_i + \lambda_j)\ (1 \leqq i \leqq j \leqq \ell)\}$ and $(E_{ij}, 0, 0)$, $(0, F_{i,\ell+1-j}, 0)$, $(0, 0, F_{\ell+1-i,j})$ are root vectors for $\lambda_i - \lambda_j$,

$\lambda_i+\lambda_j$, $-(\lambda_i+\lambda_j)\in\Phi$ respectively. Let us choose a simple root system $\Delta=\{\alpha_1,\dots,\alpha_\ell\}$ by putting

$$\begin{cases}\alpha_i=\lambda_i-\lambda_{i+1} & \text{for } i=1,\,\dots\,,\,\ell-1,\\ \alpha_\ell=2\,\lambda_\ell. & \end{cases}$$

We have

$$\begin{cases}\lambda_i-\lambda_j=\alpha_i+\cdots+\alpha_{j-1} \quad (1\leqq i<j\leqq\ell),\\ \lambda_i+\lambda_j=(\alpha_i+\cdots+\alpha_{\ell-1})+(\alpha_j+\cdots+\alpha_\ell) \quad (1\leqq i\leqq j\leqq\ell).\end{cases}$$

Hence $\theta=2\,\alpha_1+\cdots+2\,\alpha_{\ell-1}+\alpha_\ell$. Then we see that the gradation of $(C_\ell,\{\alpha_i\})$ is given by the following diagram;

	i		i
i	0	1	2
	−1	0	1
i	−2	−1	0

$(1\leqq i<\ell)$

0	1
−1	0

$(i=\ell)$

Then the diagram of $(C_\ell,\{\alpha_{i_1},\dots,\alpha_{i_k}\})$ is obtained by superposing the k diagrams of $(C_\ell,\{\alpha_{i_1}\})$, ... , $(C_\ell,\{\alpha_{i_k}\})$. Here two intermediate lines (i-th and $(2\ell-i)$-th lines) correspond to the simple root $\{\alpha_i\}$ for $i=1$, ... , $\ell-1$ and the center line corresponds to $\{\alpha_\ell\}$.

By this description of gradation, we see that the model space $M_{\mathfrak{g}}$ of $(C_\ell,\{\alpha_i\})$ is the Grassmann manifold $Sp\text{-}Gr(i,V)$ consisting of all i-dimensional isotropic subspaces of $(V,\langle\,,\,\rangle)$. Furthermore the model space $M_{\mathfrak{g}}$ of $(C_\ell,\{\alpha_{i_1},\dots,\alpha_{i_k}\})$ $(1\leqq i_1<\cdots<i_k\leqq\ell)$ is the flag manifold $Sp\text{-}F(i_1,\dots,i_k;V)$ consisting of all flags $\{V_1\subset\cdots\subset V_k\}$ in V such that V_j is an i_j dimensional isotropic subspace of $(V,\langle\,,\,\rangle)$ (cf. [Tt1]).

(3) B_ℓ $(\ell\geqq 3)$, D_ℓ $(\ell\geqq 4)$ type. Let $(V,(\,|\,))$ be an inner product space over $\mathbb{C}$ of dimension 2ℓ or $2\ell+1$, that is, $(\,|\,)$ is a non-degenerate symmetric bilinear form on V. Then $\mathfrak{g}=\mathfrak{o}(V)$. Let us take a basis $\{e_1,\dots,e_\ell,e_{\ell+1},f_1,\dots,f_\ell\}$ of V such that $(e_i|e_j)=(e_{\ell+1}|e_i)=(e_{\ell+1}|f_i)=(f_i|f_j)=0$, $(e_{\ell+1}|e_{\ell+1})=1$ and $(e_i|f_{\ell+1-j})=\delta_{ij}$ for i, $j=1$, ... , ℓ. Here we neglect $e_{\ell+1}$, when $\dim V=2\ell$. Then we have a matrices representation

$$\mathfrak{g}=\{\,X\in\mathfrak{gl}(n,\mathbb{C})\mid {}^tXS+SX=0\,\},\quad \text{where } S=\begin{pmatrix}0&0&K\\0&1&0\\K&0&0\end{pmatrix}$$

and $n = 2\ell$ or $2\ell + 1$. Each $X \in \mathfrak{g}$ is expressed as a matrix of the form

$$X = \begin{pmatrix} A & a & B \\ \xi & 0 & -a' \\ C & -\xi' & -A' \end{pmatrix}$$

where A, B, C are $\ell \times \ell$ matrices such that $B = -B'$, $C = -C'$ and a, ξ are column and row ℓ-vector respectively such that a' and ξ' are given by $a' = (a_\ell, \dots, a_1)$, $\xi' = {}^t(\xi_\ell, \dots, \xi_1)$ for $a = {}^t(a_1, \dots, a_\ell)$, $\xi = (\xi_1, \dots, \xi_\ell)$ respectively. Here the center column and the center row of X should be deleted when $\dim V = 2\ell$. Both B and C are skew symmetric with respect to the anti-diagonal line. In particular all the anti-diagonal components $x_{i,n+1-i}$ of X are 0. Thus X is determined by its upper anti-diagonal part. We write $X = (A, B, C, a, \xi)$, in short.

We take a Cartan subalgebra $\mathfrak{h}$ consisting of all diagonal elements of the form $H = (\operatorname{diag}(a_1, \dots, a_\ell), 0, 0, 0, 0)$. Let $\lambda_1, \dots, \lambda_\ell$ be the linear form on $\mathfrak{h}$ defined by $\lambda_i \colon H \mapsto a_i$. We put $G_{ij} = E_{ij} - E'_{ij}$ and $E_i = (\delta_{1i}, \dots, \delta_{\ell i}) \in \mathbb{C}^\ell$. Then we have

$$\begin{aligned}
[H, (E_{ij}, 0, 0, 0, 0)] &= (\lambda_i - \lambda_j)(H)(E_{ij}, 0, 0, 0, 0), \\
[H, (0, G_{ij}, 0, 0, 0)] &= (\lambda_i + \lambda_{\ell+1-j})(H)(0, G_{ij}, 0, 0, 0), \\
[H, (0, 0, G_{ij}, 0, 0)] &= -(\lambda_{\ell+1-i} + \lambda_j)(H)(0, 0, G_{ij}, 0, 0), \\
[H, (0, 0, 0, E_i, 0)] &= \lambda_i(H)(0, 0, 0, E_i, 0), \\
[H, (0, 0, 0, 0, E_i)] &= -\lambda_i(H)(0, 0, 0, 0, E_i).
\end{aligned}$$

Hence we have

$$\Phi = \begin{cases} \{\lambda_i - \lambda_j \ (i \neq j),\ \pm(\lambda_i + \lambda_j)\ (1 \leqq i < j \leqq \ell)\} & \text{if } n = 2\ell, \\ \{\pm\lambda_i \ (1 \leqq i \leqq \ell),\ \lambda_i - \lambda_j \ (i \neq j), & \\ \qquad\qquad \pm(\lambda_i + \lambda_j)\ (1 \leqq i < j \leqq \ell)\} & \text{if } n = 2\ell + 1. \end{cases}$$

$(E_{ij}, 0, 0, 0, 0)$, $(0, G_{i,\ell+1-j}, 0, 0, 0)$, $(0, 0, G_{\ell+1-i,j}, 0, 0)$, $(0, 0, 0, E_i, 0)$ and $(0, 0, 0, 0, E_i)$ are root vectors for $\lambda_i - \lambda_j$, $\lambda_i + \lambda_j$, $-(\lambda_i + \lambda_j)$, λ_i and $-\lambda_i \in \Phi$ respectively. Let us choose a simple root system $\Delta = \{\alpha_1, \dots, \alpha_\ell\}$ by putting

(i) B_ℓ type $\quad \begin{cases} \alpha_i = \lambda_i - \lambda_{i+1} & \text{for } i = 1, \dots, \ell - 1, \\ \alpha_\ell = \lambda_\ell. \end{cases}$

(ii) D_ℓ type $\quad \begin{cases} \alpha_i = \lambda_i - \lambda_{i+1} & \text{for } i = 1, \dots, \ell - 1, \\ \alpha_\ell = \lambda_{\ell-1} + \lambda_\ell. \end{cases}$

Then we have

(i) B_ℓ type

$$\begin{cases} \lambda_i - \lambda_j = \alpha_i + \cdots + \alpha_{j-1} & (1 \leqq i < j \leqq \ell), \\ \lambda_i = \alpha_i + \cdots + \alpha_\ell & (1 \leqq i \leqq \ell), \\ \lambda_i + \lambda_j = \alpha_i + \cdots + \alpha_{j-1} + 2\,\alpha_j + \cdots + 2\,\alpha_\ell & (1 \leqq i < j \leqq \ell). \end{cases}$$

Hence $\theta = \alpha_1 + 2\,\alpha_2 + \cdots + 2\,\alpha_\ell$.

(ii) D_ℓ type

$$\begin{cases} \lambda_i - \lambda_j = \alpha_i + \cdots + \alpha_{j-1} & (1 \leqq i < j \leqq \ell), \\ \lambda_i + \lambda_\ell = \alpha_i + \cdots + \alpha_{\ell-2} + \alpha_\ell & (1 \leqq i \leqq \ell - 2), \\ \lambda_{\ell-1} + \lambda_\ell = \alpha_\ell & \\ \lambda_i + \lambda_{\ell-1} = \alpha_i + \cdots + \alpha_{\ell-1} + \alpha_\ell & (1 \leqq i \leqq \ell - 2), \\ \lambda_i + \lambda_j = \alpha_i + \cdots + \alpha_{j-1} + 2\,\alpha_j + \cdots + 2\,\alpha_{\ell-2} + \alpha_{\ell-1} + \alpha_\ell & \\ & (1 \leqq i < j \leqq \ell - 2). \end{cases}$$

Hence $\theta = \alpha_1 + 2\,\alpha_2 + \cdots + 2\,\alpha_{\ell-2} + \alpha_{\ell-1} + \alpha_\ell$.

Then we see that the gradation of $(B_\ell, \{\alpha_i\})$ is given by the following diagram;

1	$n-2$	1
0	1	$*$
-1	0	1
$*$	-1	0

$(i = 1)$

i	$n-2i$	i
0	1	2
-1	0	1
-2	-1	0

$(1 < i \leqq \ell)$

The gradation of $(D_\ell, \{\alpha_i\})$ is given by the same diagram as above for $i = 1, \ldots, \ell - 2$ and the above diagram with $i = \ell - 1$ is that of $(D_\ell, \{\alpha_{\ell-1}, \alpha_\ell\})$. Moreover the diagrams of $(D_\ell, \{\alpha_{\ell-1}\})$ and $(D_\ell, \{\alpha_\ell\})$ are given as follows

$\ell - 1$	0	1	0	1
1	-1	0	$*$	0
1	0	$*$	0	1
$\ell - 1$	-1	0	-1	0

$(i = \ell - 1)$

ℓ	0	1
ℓ	-1	0

$(i = \ell)$

Clearly, by interchanging e_ℓ and f_1, matrices representations of $(D_\ell, \{\alpha_{\ell-1}\})$ and $(D_\ell, \{\alpha_\ell\})$ transforms each other, i.e., $(D_\ell, \{\alpha_{\ell-1}\})$ and $(D_\ell, \{\alpha_\ell\})$ are conjugate. The other gradations of B_ℓ or D_ℓ type can be obtained by the principle of superposition as in the previous cases. Here two intermediate lines (i-th and $(n-i)$-th lines) correspond to the simple root $\{\alpha_i\}$ for $i = 1, \dots, \ell$ in case of type B_ℓ and for $i = 1, \dots, \ell - 2$ in case of type D_ℓ. Moreover in case of type D_ℓ, $(\ell-1)$-th and $(\ell+1)$-th intermediate lines correspond to the pair $\{\alpha_{\ell-1}, \alpha_\ell\}$ and the center line corresponds to $\{\alpha_\ell\}$.

By this description of gradations, we see that the Grassmann manifold $O\text{-}Gr(i, V)$ consisting of all i-dimensional isotropic subspaces of $(V, (\,|\,))$ is the model space $M_\mathfrak{g}$ of $(B_\ell, \{\alpha_i\})$ or $(D_\ell, \{\alpha_i\})$ according as $\dim V = 2\ell + 1$ or 2ℓ, except for the case when $i = \ell - 1$ and $\dim V = 2\ell$. In the latter case $O\text{-}Gr(\ell-1, V)$ is the model space $M_\mathfrak{g}$ of $(D_\ell, \{\alpha_{\ell-1}, \alpha_\ell\})$, where $\dim V = 2\ell$. Thus, for D_ℓ type, we make a following convention for a subset Δ_1 of Δ: If $\alpha_{\ell-1} \in \Delta_1$ and $\alpha_\ell \notin \Delta_1$, we replace $\alpha_{\ell-1}$ by α_ℓ (the conjugacy class of (D_ℓ, Δ_1) does not change by this replacement), and if both $\alpha_{\ell-1}$ and $\alpha_\ell \in \Delta_1$, we write $\alpha^*_{\ell-1} = \{\alpha_{\ell-1}, \alpha_\ell\}$. Under this convention, we see that the model space $M_\mathfrak{g}$ of $(B_\ell, \{\alpha_{i_1}, \dots, \alpha_{i_k}\})$ or $(D_\ell, \{\alpha_{i_1}, \dots, \alpha_{i_k}\})$ $(1 \leqq i_1 < \cdots < i_k \leqq \ell)$ is the flag manifold $O\text{-}F(i_1, \dots, i_k; V)$ consisting of all flags $\{V_1 \subset \cdots \subset V_k\}$ in V such that V_j is an i_j-dimensional isotropic subspace of $(V, (\,|\,))$, according as $\dim V = 2\ell + 1$ or 2ℓ (cf. [Tt1]).

4.5. Canonical systems on Grassmann bundles

First we recall the notion of canonical systems on Grassmann bundles ([Y1], [Y2]). Let M be a (real or complex) manifold of dimension $m + n$. We consider the Grassmann bundle $J(M, n)$ over M consisting of all n-dimensional contact elements to M;

$$J(M, n) = \bigcup_{x \in M} J_x(M, n),$$

where $J_x(M, n) = Gr(n, T_x(M))$ is the Grassmann manifold of all n-dimensional subspaces of the tangent space $T_x(M)$ to M at x. Let π be the projection of $J(M, n)$ onto M. Each element $u \in J(M, n)$ is a linear subspace of $T_x(M)$ of codimension m, where $x = \pi(u)$. Hence we have a differential system C of codimension m on $J(M, n)$ by putting

$$C(u) = \pi_*^{-1}(u) \subset T_u(J(M, n)) \qquad \text{at } u \in J(M, n).$$

C is called the canonical system on $J(M, n)$. $J(M, n)$ is the (geometrical) 1-jet space for n-dimensional submanifolds in M and C is the contact system on this jet space (cf. [Y2, §1]). In fact let us fix $u_o \in J(M,n)$ and take an inhomogeneous Grassmann coordinate system $(x^1,\dots,x^n,y^1,\dots,y^m,p_i^\alpha)$ $(1 \leqq i \leqq n,\ 1 \leqq \alpha \leqq m)$ of $J(M,n)$ in a neighborhood U of u_o, that is, $(x^1,\dots,x^n,y^1,\dots,y^m)$ is a pull back of a coordinate system on M around $x_o = \pi(u_o)$ such that $dx^1 \wedge \cdots \wedge dx^n \mid_u \neq 0$ for $u \in U$ and $p_i^\alpha(u)$ is defined by $dy^\alpha|_u = \sum_{i=1}^n p_i^\alpha(u) dx^i|_u$. Then the canonical system C is given in this coordinate system by

$$C = \{\, \varpi^1 = \cdots = \varpi^m = 0 \,\},$$

where $\varpi^\alpha = dy^\alpha - \sum_{i=1}^n p_i^\alpha dx^i$ $(1 \leqq \alpha \leqq m)$.

Furthermore, starting from a contact manifold (J,C) of dimension $2n+1$, which can be regarded locally as a space of 1-jets for one unknown function by Darboux's theorem, we can construct a geometric second order jet space $(L(J),E)$ as follows. We consider the Lagrange-Grassmann bundle $L(J)$ over J consisting of all n-dimensional integral elements of (J,C);

$$L(J) = \bigcup_{u \in J} L_u(J),$$

where $L_u(J) = \text{Sp-Gr}(n, C(u))$ is the Grassmann manifold of all lagrangian (or legendrian) subspaces of the symplectic vector space $(C(u), d\varpi)$. Here ϖ is a local contact form on J. Let π be the projection of $L(J)$ onto J. Then the canonical system E on $L(J)$ is defined by

$$E(v) = \pi_*^{-1}(v) \subset T_v(L(J)) \qquad \text{at } v \in L(J).$$

Starting from a canonical coordinate system $(x^1,\dots,x^n,z,p_1,\dots,p_n)$ of (J,C), we can introduce a coordinate system (x^i,z,p_j,p_{ij}) $(1 \leqq i \leqq j \leqq n)$ of $L(J)$ such that $p_{ij} = p_{ji}$ and E is defined by

$$E = \{\, \varpi = \varpi_1 = \cdots = \varpi_n = 0 \,\},$$

where $\varpi = dz - \sum_{i=1}^n p_i dx^i$ and $\varpi_i = dp_i - \sum_{j=1}^n p_{ij} dx^j$ $(1 \leqq i \leqq n)$. For the details, we refer the reader to [Y1].

These canonical systems appear among our standard differential systems in the following cases.

(1) $(A_\ell, \{\alpha_1, \alpha_{i+1}\})$ $(1 \leqq i < \ell)$. Let V be a complex vector space of dimension $\ell + 1$. By the discussion in 4.4, we know that the model space $M_\mathfrak{g}$ of $(A_\ell, \{\alpha_1, \alpha_{i+1}\})$ is given by

$$M_\mathfrak{g} = \{\, ([v], W) \in P(V) \times Gr(i+1, V) \mid [v] \subset W \,\}.$$

Let p be the projection of $M_{\mathfrak{g}}$ onto $P(V)$. Each fibre of $p\colon M_{\mathfrak{g}} \to P(V)$ is a Grassmann manifold $Gr(i, V/[v])$. At each $x = [v] \in P(V)$, we can naturally identify $T_x(P(V))$ with the quotient space $V/[v]$. With this identification, we have a fibre-preserving diffeomorphism φ of $M_{\mathfrak{g}}$ onto $J(P(V), i)$ defined by

$$\varphi(u) = W/[v] \subset V/[v] \cong T_x(P(V)) \qquad \text{for } u = ([v], W) \in M_{\mathfrak{g}}.$$

Moreover let us fix a basis $\{e_0, \dots, e_\ell\}$ of V and put $x_0 = [e_0]$ and $u_0 = ([e_0], W_0)$, where $W_0 = \langle e_0, \dots, e_i \rangle$. Let π^1 and π^2 denote the projection of $G = SL(V)$ onto $P(V)$ and $M_{\mathfrak{g}}$ defined by $\pi^1(g) = g(x_0)$ and $\pi^2(g) = g(u_0)$ for $g \in G$ respectively. Then, from the matrices description of $(A_\ell, \{\alpha_1, \alpha_{i+1}\})$ in 4.4, we see that

$$(\pi^1_*)^{-1}(W_0/x_0) = \mathfrak{f}^{-1}.$$

Hence it follows from $p \cdot \pi^2 = \pi^1$ that $(M_{\mathfrak{g}}, D_{\mathfrak{g}})$ is isomorphic with the canonical differential system $(J(P(V), i), C)$ via φ. Especially $(J(P(V), \ell - 1), C)$ is the standard contact manifold of type A_ℓ, which is also naturally identified with the projective cotangent bundle $PT^*(P(V))$ over $P(V)$ with its contact structure induced from the symplectic structure on $T^*(P(V))$ (cf. [Bo], [A]). Here we note that the above argument is valid also for the normal real form $\mathfrak{sl}(\ell + 1, \mathbb{R})$ of $\mathfrak{sl}(\ell + 1, \mathbb{C})$.

(2) $(C_\ell, \{\alpha_1, \alpha_\ell\})$. Let us start with the contact gradation $(C_\ell, \{\alpha_1\})$. From 4.4, we see that the model space of $(C_\ell, \{\alpha_1\})$ is the projective space $P(V)$, where $(V, \langle\ ,\ \rangle)$ is a symplectic vector space over $\mathbb{C}$ of dimension 2ℓ. Let us take a symplectic basis $\{e_1, \dots, e_\ell, f_1, \dots, f_\ell\}$ as in 4.4 and let π^1 denote the projection of $G = Sp(V)$ onto $P(V)$ given by $\pi^1(g) = g([e_1])$ for $g \in G$. Then, under the identification $V/[e_1] \cong T_{x_0}(P(V))$, $x_0 = [e_1]$, we see from the matrices description in 4.4 that $(\pi^1_*)^{-1}([e_1]^\perp/[e_1]) = \mathfrak{f}^{-1}$, where $[e_1]^\perp = \{\, v \in V \mid \langle v, e_1 \rangle = 0 \,\}$. Thus we see that the contact structure C on $P(V)$ is given by (cf. [K1])

$$C(x) = \operatorname{Ker}\alpha/[v] \subset V/[v] \cong T_x(P(V)) \quad \text{at each } x = [v] \in P(V),$$

where α is the linear symplectic form defined on V by $\alpha_v(w) = \langle v, w \rangle$ for v, $w \in V$.

The model space of $(C_\ell, \{\alpha_1, \alpha_\ell\})$ is given by

$$M_{\mathfrak{g}} = \{\, ([v], L) \in P(V) \times \text{Sp-Gr}(\ell, V) \mid [v] \subset L \,\}.$$

Let p be the projection of $M_{\mathfrak{g}}$ onto $P(V)$. We have a fibre-preserving diffeomorphism φ of $M_{\mathfrak{g}}$ onto the Lagrange-Grassmann bundle $L(P(V))$

over $P(V)$ defined by

$$\varphi(u) = L/[v] \subset \operatorname{Ker}\alpha/[v] \cong C(x) \quad \text{for } u = ([v], L) \text{ and } x = [v].$$

Let π^2 denote the projection of G onto $M_{\mathfrak{g}}$ given by $\pi^2(g) = g(u_o)$ for $g \in G$, where $u_o = ([e_1], L_o)$ and $L_o = \langle e_1, \dots, e_\ell \rangle$. Then we have

$$(\pi^1_*)^{-1}(L_o/[e_1]) = \mathfrak{f}^{-1}$$

from the following diagram for $(C_\ell, \{\alpha_1, \alpha_\ell\})$;

1	0	1	2	3
$\ell - 1$	−1	0	1	2
$\ell - 1$	−2	−1	0	1
1	−3	−2	−1	0

Hence it follows from $p \cdot \pi^2 = \pi^1$, that $(M_{\mathfrak{g}}, D_{\mathfrak{g}})$ is isomorphic with the canonical differential system $(L(P(V)), E)$ via φ. We here note that the above argument is valid also for the normal real form $\mathfrak{sp}(\ell, \mathbb{R})$ of $\mathfrak{sp}(\ell, \mathbb{C})$.

Finally we shall add another construction of standard contact manifolds of type B_ℓ or D_ℓ and those of their real forms. Let $(V, (\,|\,))$ be an inner product space over $K = \mathbb{R}$ or $\mathbb{C}$, that is, $(\,|\,)$ is a non-degenerate symmetric bilinear form over K on V. In the real case, we assume that $(V, (\,|\,))$ is indefinite and admits 2-dimensional isotropic subspaces.

Let $W = V \oplus V$ be the direct sum of two copies of V. The inner product $(\,|\,)$ induces a skew symmetric bilinear form $\langle\, , \rangle$ on W by

$$\langle (x_1, y_1), (x_2, y_2) \rangle = (x_1|y_2) - (x_2|y_1).$$

Then $(W, \langle\, , \rangle)$ is a symplectic vector space. Let ω be the 1-form on W defined by $\omega = (x|dy) - (y|dx)$, where (x, y) is the linear coordinate system of W. Put $\alpha = \frac{1}{2}\omega$. Thus $(W, d\alpha)$ is a symplectic manifold.

$GL(2, K)$ acts on W on the right as follows;

$$(x, y)\sigma = (ax + cy, bx + dy) \qquad \text{for } \sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, K).$$

We have $R^*_\sigma \alpha = (\det \sigma)\, \alpha$ for $\sigma \in GL(2, K)$. Hence $SL(2, K)$ acts on $(W, d\alpha)$ as a group of symplectic transformations. Let $\{X, H, Y\}$ be the basis of $\mathfrak{sl}(2, K)$ given by

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Then $SL(2,K)$-action on W induces hamiltonian vector fields

$$X^* = \sum_{i=1}^{n} x_i \frac{\partial}{\partial y_i}, \quad H^* = \sum_{i=1}^{n} (x_i \frac{\partial}{\partial x_i} - y_i \frac{\partial}{\partial y_i}), \quad Y^* = \sum_{i=1}^{n} y_i \frac{\partial}{\partial x_i},$$

with hamiltonians $\alpha(X^*) = \frac{1}{2}(x|x)$, $\alpha(H^*) = -(x|y)$, $\alpha(Y^*) = -\frac{1}{2}(y|y)$ respectively. Thus we have a momentum mapping $f\colon W \to \mathfrak{sl}(2,K)^*$ given by

$$f(w)(Z) = \alpha_w(Z^*) \qquad \text{for } w \in W \text{ and } Z \in \mathfrak{sl}(2,K).$$

Then we have

$$f^{-1}(0) = \{\, (x,y) \in W \mid (x|x) = (x|y) = (y|y) = 0 \,\}.$$

By our assumption on $(\,|\,)$, $f^{-1}(0)$ is a non-empty variety in W. Let F be the regular part of $f^{-1}(0)$;

$$F = \{\, (x,y) \in W \mid (x|x) = (x|y) = (y|y) = 0, x \wedge y \neq 0 \,\}.$$

$GL(2,K)$ acts freely on F on the right. Moreover the orthogonal group $O(V)$ of $(V, (\,|\,))$ acts on F in the obvious way. As is well-known (cf. [A, Appendix 5]), the reduced phase space $S = F/SL(2,K)$ is a symplectic manifold over K. In fact, since α is $SL(2,K)$-invariant, the restriction $\theta = \alpha|_F$ of α to F projects to $S = F/SL(2,K)$ so that $d\theta$ is a symplectic form on S. Furthermore the quotient space $J = F/GL(2,K)$ is naturally identified with $O\text{-}Gr(2,V)$. Thus F is the total space of the universal 2-frame bundle over $J = O\text{-}Gr(2,V)$ and (S,J,p) is a principal $K^\times$-bundle over J, where $p\colon S \to J$ denotes the natural projection. Then as in 4.3, the contact structure C on J is defined by $\theta = 0$ so that $(S, d\theta)$ is the symplectification of (J,C). From the equivalence of the adjoint representation and the exterior representation on $\wedge^2 V$ for $O(V)$, it follows that $(S, d\theta)$ is isomorphic with the adjoint orbit constructed in 4.3, which implies that (J,C) is isomorphic with the standard contact manifold of type B_ℓ, D_ℓ or one of their real forms.

§5. Infinitesimal automorphisms of $(M_\mathfrak{g}, D_\mathfrak{g})$

5.1. Review of harmonic theory (Kostant's Theorem)

We here review the harmonic theory of Kostant [K] for the Lie algebra cohomology, which enables us to compute the generalized Spencer cohomology groups $H^q(\mathfrak{m}, \mathfrak{g})$ (cf. [O2]).

Let $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ be a simple graded Lie algebra over $\mathbb{C}$ such that $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ is fundamental. Let us fix a Cartan subalgebra $\mathfrak{h}$ containing E. In accordance with Kostant's paper [K], let us fix a simple root system $\Delta = \{\alpha_1, \ldots, \alpha_\ell\}$ such that $\alpha(E) = 0$ or -1 for $\alpha \in \Delta$ throughout this section. Thus, by putting $\widehat{\Phi}^+ = \Phi^+ \setminus \Phi_0^+$, $\mathfrak{m}$ is a direct sum of positive root subspaces:

$$\mathfrak{m} = \bigoplus_{\alpha\in\widehat{\Phi}^+} \mathfrak{g}_\alpha.$$

Let us take a compact real form $\mathfrak{u}$ of $\mathfrak{g}$ by choosing a Weyl basis of the root space decomposition relative to $\mathfrak{h}$ (cf. [He, p. 421]). Let τ denote the conjugation of $\mathfrak{g}$ with respect to $\mathfrak{u}$. Then $E \in \mathfrak{h}_\mathbb{R} \subset \sqrt{-1}\mathfrak{u}$ and we have a hermitian inner product $\{\ ,\ \}$ of $\mathfrak{g}$, which is given by

$$\{X, Y\} = -B(X, \tau(Y)) \qquad \text{for } X,\, Y \in \mathfrak{g}.$$

By our choice of $\mathfrak{u}$, we have $\tau(\mathfrak{g}_\alpha) = \mathfrak{g}_{-\alpha}$ for $\alpha \in \Phi$. For a linear subspace $\mathfrak{a}$ of $\mathfrak{g}$, we put $\widehat{\mathfrak{a}} = \tau(\mathfrak{a})$ and $\mathfrak{a}^o = \{X \in \mathfrak{g} \mid B(X, \mathfrak{a}) = 0\}$. Then the orthogonal complement $\mathfrak{a}^\perp$ of $\mathfrak{a}$ with respect to $\{\ ,\ \}$ coincides with $\widehat{\mathfrak{a}}^o$. By definition of $\{\ ,\ \}$, it follows that the Killing form B gives a non-degenerate pairing of $\mathfrak{a}$ and $\widehat{\mathfrak{a}}$ (cf. Lemma 3.1). Especially we have

$$\widehat{\mathfrak{m}} = \bigoplus_{p>0} \mathfrak{g}_p, \qquad \mathfrak{h} = \mathfrak{B}(\Delta) \cap \widehat{\mathfrak{B}(\Delta)},$$

where $\mathfrak{B}(\Delta) = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Phi^+} \mathfrak{g}_\alpha$ is a standard Borel subalgebra relative to $\mathfrak{h}$. Moreover, starting from a parabolic subalgebra $\mathfrak{P}_- = \bigoplus_{p\leqq 0} \mathfrak{g}_p$ containing $\mathfrak{B}(\Delta)$, we have

$$\mathfrak{m} = \mathfrak{P}_-^o, \qquad \mathfrak{g}_0 = \mathfrak{P}_- \cap \widehat{\mathfrak{P}}_-,$$

and the orthogonal decomposition of $\mathfrak{g}$;

$$\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{g}_0 \oplus \widehat{\mathfrak{m}}.$$

Thus $\mathfrak{m}$ is a nilpotent Lie summand of $\mathfrak{g}$ in the sense of §5 in [K] and the argument in [K] is thoroughly applicable to our situation.

We shall summarize the argument in [K] in the following. Let $(C(\mathfrak{m}, \mathfrak{g}), \partial)$ be the cochain complex (the generalized Spencer complex) associated with the representation ad: $\mathfrak{m} \to \mathfrak{gl}(\mathfrak{g})$. Namely $C(\mathfrak{m}, \mathfrak{g}) = \mathfrak{g} \otimes \wedge\mathfrak{m}^*$ and $\partial\colon C^q(\mathfrak{m}, \mathfrak{g}) \to C^{q+1}(\mathfrak{m}, \mathfrak{g})$ is given as in §2.4. The hermitian inner product $\{\ ,\ \}$ of $\mathfrak{g}$ induces the hermitian inner product of

$C(\mathfrak{m},\mathfrak{g}) = \mathfrak{g} \otimes \wedge\mathfrak{m}^*$ and $\mathfrak{g} \otimes \wedge\mathfrak{g}$ in a natural manner. Let $\{e_1,\dots,e_{n_1}\}$ be a (orthnormal) basis of $\mathfrak{m}$ and let $\{e_1^*,\dots,e_{n_1}^*\}$ be the dual basis of $\widehat{\mathfrak{m}}$ under the Killing form duality. Then the adjoint operator $\partial^*\colon C^{q+1}(\mathfrak{m},\mathfrak{g}) \to C^q(\mathfrak{m},\mathfrak{g})$ of ∂ with respect to this inner product is given by the following formula ([T4, Lemma 1.10], [K, Lemma 4.2]);

$$(\partial^* p)(X_1,\dots,X_q) = \sum_j [e_j^*, p(e_j, X_1,\dots,X_q)] + \tfrac{1}{2}\sum_{i,j}(-1)^{i+1} p([e_j^*, X_i]_-, e_j, X_1,\dots,\check{X}_i,\dots,X_q),$$

for $p \in C^{q+1}(\mathfrak{m},\mathfrak{g})$ and $X_1, \dots, X_q \in \mathfrak{m}$, where $[e_j^*, X_i]_-$ denotes the $\mathfrak{m}$-component of $[e_j^*, X_i]$ with respect to the decomposition $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{g}'$. Here we note that ∂^* sends $C^{p,q+1}(\mathfrak{m},\mathfrak{g})$ into $C^{p+1,q}(\mathfrak{m},\mathfrak{g})$ and $\partial^* p$ does not depend on the choice of the basis $\{e_1,\dots,e_{n_1}\}$, hence, nor on the choice of the compact real form $\mathfrak{u}$.

Now, in order to describe the harmonic space $\mathcal{H} = \operatorname{Ker}\square$ for the Laplacian $\square = \partial\partial^* + \partial^*\partial$, we shall utilize the natural representation of $\mathfrak{g}_0$ on the cochain space $C(\mathfrak{m},\mathfrak{g}) = \mathfrak{g} \otimes \wedge\mathfrak{m}^*$. In fact $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ is $\operatorname{ad}(\mathfrak{g}_0)$-invariant. Hence, from the $\mathfrak{g}_0$-module $\mathfrak{m}$, we have the $\mathfrak{g}_0$-module $\mathfrak{m}^*$ contragradient to $\mathfrak{m}$. Let

$$\rho\colon \mathfrak{g}_0 \to \mathfrak{gl}(C(\mathfrak{m},\mathfrak{g}))$$

be the representation of $\mathfrak{g}_0$ on $C(\mathfrak{m},\mathfrak{g})$ formed by taking the tensor product of $\operatorname{ad}\colon \mathfrak{g}_0 \to \mathfrak{gl}(\mathfrak{g})$ and the exterior representation of $\mathfrak{g}_0$ on $\wedge\mathfrak{m}^*$. ρ is a completely reducible representation of the reductive Lie algebra $\mathfrak{g}_0$ (cf. Proposition 3.6). Here we note that $\mathfrak{h}$ is also a Cartan subalgebra of $\mathfrak{g}_0$. Let $\widehat{G}$ be the simply connected Lie group with Lie algebra $\mathfrak{g}$ and let $\widehat{G}_0$ be the analytic subgroup of $\widehat{G}$ with Lie algebra $\mathfrak{g}_0$. Then an irreducible representation of $\mathfrak{g}_0$, which is induced from a representation of $\widehat{G}_0$, is described as a standard cyclic module with heighest weight $\xi \in D_0$ ([K, §5.5], [Hu, Chapter VI]). Here ξ is a dominant integral weight in

$$D_0 = \{\, \mu \in \Lambda \mid \langle \mu, \alpha \rangle \geqq 0 \quad \text{for each } \alpha \in \Phi_0^+ \,\},$$

where $\Lambda = \{\, \mu \in \mathfrak{h}^\sharp \mid \langle \mu, \alpha \rangle \in \mathbb{Z} \text{ for each } \alpha \in \Phi \,\}$. Moreover, under the identification of $\mathfrak{g}\otimes\wedge\mathfrak{m}^*$ with $\mathfrak{g}\otimes\wedge\widehat{\mathfrak{m}}$ via the Killing form duality between $\mathfrak{m}$ and $\widehat{\mathfrak{m}}$, the representation ρ is equivalent to the subrepresentation $\widehat{\rho} = \widehat{\operatorname{ad}}|_{\mathfrak{g}\otimes\wedge\widehat{\mathfrak{m}}}$ on $\mathfrak{g} \otimes \wedge\widehat{\mathfrak{m}}$ of the tensor representation $\widehat{\operatorname{ad}}$ of $\mathfrak{g}_0$ on $\mathfrak{g} \otimes \wedge\mathfrak{g}$ induced from $\operatorname{ad}\colon \mathfrak{g}_0 \to \mathfrak{gl}(\mathfrak{g})$. Hence the weight space decomposition of $\mathfrak{g}_0$-module $\mathfrak{g} \otimes \wedge\widehat{\mathfrak{m}}$ is provided by the root space decomposition of $\mathfrak{g}$. More precisely, let $\Lambda^{\hat\rho}$ be the set of weights of $\widehat{\rho}$. Then we have

$$\Lambda^{\hat\rho} = \{\, \xi = \alpha - \langle A \rangle \in \Lambda \mid \alpha \in \Phi \cup \{0\}, \quad A \subset \widehat{\Phi}^+ \,\},$$

where $\langle A \rangle = \sum_{\alpha \in A} \alpha$.

By Lemma 3.1, we know that the restriction of the Killing form B to $\mathfrak{g}_0$ is non-degenerate. Let $C_\rho \in \mathfrak{gl}(C(\mathfrak{m}, \mathfrak{g}))$ be the Casimir operator corresponding to the restriction of B to $\mathfrak{g}_0$, that is,

$$C_\rho = \sum_{i=1}^{n_0} \rho(X_i) \cdot \rho(Y_i),$$

where $\{X_1, \ldots, X_{n_0}\}$ and $\{Y_1, \ldots, Y_{n_0}\}$ are basis of $\mathfrak{g}_0$ such that $B(X_i, Y_i) = \delta_{ij}$. We put

$$\delta = \tfrac{1}{2} \sum_{\alpha \in \Phi^+} \alpha, \quad \delta_1 = \tfrac{1}{2} \sum_{\alpha \in \widehat{\Phi}^+} \alpha \quad \text{and} \quad F = t_{\delta_1} \in \mathfrak{h},$$

where t_{δ_1} is defined by $B(t_{\delta_1}, H) = \delta_1(H)$ for $H \in \mathfrak{h}$. Let σ_α denote the reflection in $\mathfrak{h}^\sharp = \langle \Phi \rangle_{\mathbb{R}}$ corresponding to $\alpha \in \Phi$, that is,

$$\sigma_\alpha(\beta) = \beta - \langle \beta, \alpha \rangle\, \alpha \qquad \text{for } \beta \in \mathfrak{h}^\sharp.$$

From $\sigma_\alpha(\delta_1) = \delta_1$ for $\alpha \in \Phi_0$, we have $(\delta_1, \alpha) = 0$ for $\alpha \in \Phi_0$, which implies that $F \in Z(\mathfrak{g}_0)$ by Proposition 3.6. Then we have the following expression of the Laplacian $\square$ on $C(\mathfrak{m}, \mathfrak{g})$ ([K, Theorem 5.7]);

$$\square = \tfrac{1}{2}(|\delta + \theta|^2 - |\delta|^2) \cdot id - (\rho(F) + \tfrac{1}{2} C_\rho), \tag{5.1}$$

where $|\alpha|$ denotes the length of $\alpha \in \mathfrak{h}^\sharp$ and θ is the highest root. This expression of $\square$ can be obtained by expressing the operators ∂ and ∂^* in terms of elementary operations in $\mathfrak{g} \otimes \wedge\mathfrak{g}$ under the identification of $\mathfrak{g} \otimes \wedge\mathfrak{m}^*$ with $\mathfrak{g} \otimes \wedge\widehat{\mathfrak{m}} \subset \mathfrak{g} \otimes \wedge\mathfrak{g}$. For the details, we refer the reader to the discussion in §§3 and 4 of [K].

The important fact on the representation $\rho \colon \mathfrak{g}_0 \to \mathfrak{gl}(C(\mathfrak{m}, \mathfrak{g}))$ is that each $\rho(Z) \in \mathfrak{gl}(C(\mathfrak{m}, \mathfrak{g}))$, $Z \in \mathfrak{g}_0$, commutes with both operators ∂ and ∂^*, which can be easily checked by utilizing the above expression of ∂ and ∂^* ([K, §5], [T4, Lemma 1.11]). Thus the orthogonal decomposition of $C(\mathfrak{m}, \mathfrak{g})$;

$$C(\mathfrak{m}, \mathfrak{g}) = \operatorname{Im} \partial \oplus \operatorname{Im} \partial^* \oplus \mathcal{H}$$

is stable under $\rho(Z)$ for all $Z \in \mathfrak{g}_0$ and $C(\mathfrak{m},\mathfrak{g})$ has the isotypic decomposition as a $\mathfrak{g}_0$-module;

$$C(\mathfrak{m},\mathfrak{g}) = \bigoplus_{\xi \in D_0} C^{\xi},$$

where C^{ξ} is the isotypic component of $C(\mathfrak{m},\mathfrak{g})$ with highest weight $\xi \in D_0$. Namely C^{ξ} is the sum of irreducible components in $C(\mathfrak{m},\mathfrak{g})$ with highest weight ξ. Then, by (5.1) and the Schur's Lemma, the Laplacian $\square$ reduces to a scalar on each isotypic component C^{ξ} and this scalar is given by ([K, Theorem 5.7])

$$\tfrac{1}{2}(|\delta+\theta|^2 - |\delta+\xi|^2).$$

Hence $\mathcal{H}$ consists of isotypic components C^{ξ} of $C(\mathfrak{m},\mathfrak{g})$ such that $|\delta+\theta| = |\delta+\xi|$.

Thus, to describe the harmonic space $\mathcal{H}$, we need to find $\xi \in \Lambda^{\hat{\rho}}$ such that $|\delta+\theta| = |\delta+\xi|$. This is accomplished by the Weyl group W of the root system Φ as follows. For an element $\sigma \in W$, we put $\Phi^- = -\Phi^+$, $\Phi_\sigma = \sigma(\Phi^-) \cap \Phi^+$ and define the subset W^0 of W by putting

$$W^0 = \{\, \sigma \in W \mid \Phi_\sigma \subset \widehat{\Phi}^+ \,\}.$$

Put $\xi_\sigma = \sigma(\delta+\theta) - \delta$ for $\sigma \in W^0$. Then, from $\sigma(\delta) = \delta - \langle \Phi_\sigma \rangle$, we obtain $\xi_\sigma = \sigma(\theta) - \langle \Phi_\sigma \rangle \in \Lambda^{\hat{\rho}}$ and $|\delta+\theta| = |\delta+\xi_\sigma|$. Since $\delta+\theta$ is a strongly dominant weight, the mapping $\sigma \mapsto \xi_\sigma$ of W^0 into $\Lambda^{\hat{\rho}}$ is one to one. In fact ([K, Lemma 5.12], [Cr]), this mapping gives a bijection of W^0 onto the set of highest weights in $\Lambda^{\hat{\rho}}$ appearing in the isotypic decomposition of $\mathcal{H}$ and $\dim V_{\xi_\sigma} = 1$, where V_{ξ_σ} is the weight space of weight ξ_σ in $\mathfrak{g} \otimes \wedge\widehat{\mathfrak{m}} \cong C(\mathfrak{m},\mathfrak{g})$. Furthermore we put

$$W(q) = \{\, \sigma \in W \mid n(\sigma) = q \,\} \quad \text{and} \quad W^0(q) = W^0 \cap W(q),$$

where $n(\sigma)$ is the number of roots in Φ_σ. For an element $\sigma \in W^0(q)$, we put $\hat{x}_{\Phi_\sigma} = x_{-\beta_1} \wedge \cdots \wedge x_{-\beta_q}$, where $\Phi_\sigma = \{\beta_1,\ldots,\beta_q\} \subset \widehat{\Phi}^+$ and $x_{-\beta_i}$ is a root vector for the root $-\beta_i \in \widehat{\Phi}^- = -\widehat{\Phi}^+$. Then we have

$$V_{\xi_\sigma} = \langle x_{\sigma(\theta)} \otimes \hat{x}_{\Phi_\sigma} \rangle_{\mathbb{C}} \subset \mathfrak{g} \otimes \wedge^q \widehat{\mathfrak{m}},$$

which implies $C^{\xi_\sigma} \subset \mathfrak{g} \otimes \wedge^q \mathfrak{m}^*$ for $\sigma \in W^0(q)$. We denote by x_{Φ_σ} the element in $\wedge^q \mathfrak{m}^*$ which corresponds to $\hat{x}_{\Phi_\sigma}$ under the identification of $\wedge\mathfrak{m}^*$ with $\wedge\widehat{\mathfrak{m}}$ via the Killing form duality.

Summarizing we can state

Theorem (Kostant). *Let* $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ *be a simple graded Lie algebra over* $\mathbb{C}$ *such that* $\mathfrak{g}_p = [\mathfrak{g}_{p+1}, \mathfrak{g}_{-1}]$ *for* $p < -1$. *Then the irreducible decomposition of the harmonic space* $\mathcal{H} = \operatorname{Ker} \Box$ *of the generalized Spencer complex, as a* $\mathfrak{g}_0$*-module, is given by*

$$\mathcal{H} = \bigoplus_{\sigma \in W^0} \mathcal{H}^\sigma,$$

where $\mathcal{H}^\sigma$ *is the irreducible* $\mathfrak{g}_0$*-module with highest weight* $\xi_\sigma = \sigma(\theta + \delta) - \delta$ *generated by the highest weight vector* $x_{\sigma(\theta)} \otimes x_{\Phi_\sigma} \in \mathfrak{g} \otimes \wedge \mathfrak{m}^*$. *Moreover degree-wise, for any non-negative integer* q,

$$\mathcal{H}^q = \bigoplus_{\sigma \in W^0(q)} \mathcal{H}^\sigma.$$

Utilizing this theorem, we shall compute $H^{p,1}(\mathfrak{m}, \mathfrak{g})$ and $H^{p,2}(\mathfrak{m}, \mathfrak{g})$ for $p \geqq 0$ in the following paragraph.

5.2. Theorem on infinitesimal automorphisms of $(M_\mathfrak{g}, D_\mathfrak{g})$

First we shall compute $H^{p,1}(\mathfrak{m}, \mathfrak{g})$ for $p \geqq 0$ by virtue of Kostant's theorem. In order to apply the theorem to our computation, we note here that each $\rho(Z) \in \mathfrak{gl}(C(\mathfrak{m}, \mathfrak{g}))$, $Z \in \mathfrak{g}_0$, preserves the bigradation of $C(\mathfrak{m}, \mathfrak{g})$ given by $C^{p,q}(\mathfrak{m}, \mathfrak{g}) = \bigoplus_{j \leqq -q} \mathfrak{g}_{j+p+q-1} \otimes \wedge^q_j \mathfrak{m}^*$. Hence each irreducible component $\mathcal{H}^\sigma$ of the $\mathfrak{g}_0$-module $\mathcal{H}$ is a subspace of some $C^{p,q}(\mathfrak{m}, \mathfrak{g})$. Recall that $\mathfrak{g}_p$ is a direct sum of the root subspaces $\mathfrak{g}_\beta$ satisfying $p = \beta(E)$ and that $\mathfrak{g}_p^*$ is identified with $\mathfrak{g}_{-p}$ by the Killing form. Then, for the generator $x_{\sigma(\theta)} \otimes x_{\Phi_\sigma}$ of $\mathcal{H}^\sigma$, we have $x_{\sigma(\theta)} \in \mathfrak{g}_{\sigma(\theta)(E)}$ and $x_{\Phi_\sigma} \in \wedge^q_j \mathfrak{m}^*$, where $q = n(\sigma)$ and $j = \sum_{i=1}^q \beta_i(E)$ for $\Phi_\sigma = \{\beta_1, \ldots, \beta_q\}$. Hence we have

$$\mathcal{H}^\sigma \subset C^{p,q}(\mathfrak{m}, \mathfrak{g}),$$

where $q = n(\sigma)$ and p can be computed from the following equality;

$$\sigma(\theta)(E) = \sum_{i=1}^{q} \beta_i(E) + p + q - 1, \tag{5.2}$$

One important consequence of Kostant's theorem is that $H^q(\mathfrak{m}, \mathfrak{g})$ never vanishes for $q = 1$. Thus our task is to find $\Delta_1 \subset \Delta$ and $\sigma \in W^0(1)$ so that $\mathcal{H}^\sigma \subset C^{p,1}(\mathfrak{m}, \mathfrak{g})$ for some $p \geqq 0$.

In the following we denote by $\sigma_i = \sigma_{\alpha_i}$ the reflection in $\mathfrak{h}^\sharp$ corresponding to the simple root $\alpha_i \in \Delta$. Then $W(1) = \{ \sigma_i \in W \mid i = 1,$

$\dots, \ell\}$ and $\Phi_{\sigma_i} = \{\alpha_i\}$ (cf. [Hu, Lemma 10.3.A]). Thus we have

$$W^0(1) = \{\, \sigma_i \in W \mid \alpha_i \in \Delta_1 \,\}.$$

Recall that the depth μ of $\mathfrak{g} = \bigoplus_{p=-\mu}^{\mu} \mathfrak{g}_p$ is given by $\mu = \sum_{\alpha_i \in \Delta_1} n_i(\theta)$, where $\theta = \sum_{i=1}^{\ell} n_i(\theta)\, \alpha_i$ (see §3.4). Then, by our choice of the simple root system Δ in 5.1, (5.2) reduces to

$$-\mu + \langle \theta, \alpha_i \rangle = p_i - 1 \qquad \text{for } \sigma_i \in W^0(1).$$

Hence we obtain

$$\mathcal{H}^{\sigma_i} \subset C^{p_i,1}(\mathfrak{m}, \mathfrak{g}) \qquad \text{for } \sigma_i \in W^0(1),$$

where $p_i = \langle \theta, \alpha_i \rangle - \mu + 1$.

On the other hand, from the extended Dynkin diagram in §4.2, we know that $\langle \theta, \alpha_i \rangle = 0$, 1 or 2, which implies that $p_i \geqq 0$ occurs only when $\mu \leqq 3$. More precisely $\langle \theta, \alpha_i \rangle = 2$ if and only if $\mathfrak{g}$ is of type C_ℓ or A_1 and $\alpha_i = \alpha_1$, and $\langle \theta, \alpha_i \rangle = 1$ if and only if $\mathfrak{g}$ is not of type C_ℓ nor A_1 and $\alpha_i \in \Delta_\theta$ (see 4.2). Especially if $\Delta_1 \cap \Delta_\theta = \emptyset$, we have $p_i = 1 - \mu$ for each $\sigma_i \in W^0(1)$. Hence $p_i \geqq 0$ occurs if and only if $\mu = 1$. Namely $\Delta_1 = \{\alpha_{i_o}\}$ such that $n_{i_o}(\theta) = 1$. In this case (cf. [O2]) we have $W^0(1) = \{\sigma_{i_o}\}$ and $p_{i_o} = 0$, that is,

$$\mathcal{H}^1 = \mathcal{H}^{\sigma_{i_o}} \subset C^{0,1}(\mathfrak{m}, \mathfrak{g}).$$

Now assume that $\Delta_1 \cap \Delta_\theta \neq \emptyset$. If $\mathfrak{g}$ is of type C_ℓ, we have $p_i = 3 - \mu$ and $\alpha_1 \in \Delta_1$. Then $p_i \geqq 0$ occurs only when $\mu = 2$ or 3, which forces $\Delta_1 = \{\alpha_1\}$ or $\{\alpha_1, \alpha_\ell\}$. In these cases we have

(1) $\Delta_1 = \{\alpha_1\}$ $\qquad \mathcal{H}^1 = \mathcal{H}^{\sigma_1} \subset C^{1,1}(\mathfrak{m}, \mathfrak{g})$,

(2) $\Delta_1 = \{\alpha_1, \alpha_\ell\}$ $\qquad \mathcal{H}^1 = \mathcal{H}^{\sigma_1} \oplus \mathcal{H}^{\sigma_\ell} \subset C^{0,1}(\mathfrak{m}, \mathfrak{g}) \oplus C^{-2,1}(\mathfrak{m}, \mathfrak{g})$

In the other cases we have $p_i = 2 - \mu$. Moreover, except for type A_ℓ, we have $\Delta_\theta \subset \Delta_1$ and $\mu \geqq 2$. Hence, in these cases, $p_i \geqq 0$ occurs only when $\Delta_1 = \Delta_\theta$. Namely $p_i \geqq 0$ occurs only if $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ is a contact gradation. In these cases we have

$$\mathcal{H}^1 = \mathcal{H}^{\sigma_{i_o}} \subset C^{0,1}(\mathfrak{m}, \mathfrak{g}),$$

where $\Delta_\theta = \{\alpha_{i_o}\}$. Finally, if $\mathfrak{g}$ is of type A_ℓ, we may assume $\alpha_1 \in \Delta_1$, up to conjugacy. Then $p_i \geqq 0$ occurs only when $\mu = 1$ or 2, which forces

$\Delta_1 = \{\alpha_1\}$, $\{\alpha_1, \alpha_j\}$ $(1 < j < \ell)$ or $\{\alpha_1, \alpha_\ell\}$. In these cases we have

$$\begin{aligned}
&(1) \quad \Delta_1 = \{\alpha_1\} && \mathcal{H}^1 = \mathcal{H}^{\sigma_1} \subset C^{1,1}(\mathfrak{m}, \mathfrak{g}) \ (\ell \geqq 2),\\
& && \mathcal{H}^1 = \mathcal{H}^{\sigma_1} \subset C^{2,1}(\mathfrak{m}, \mathfrak{g}) \ (\ell = 1).\\
&(2) \quad \Delta_1 = \{\alpha_1, \alpha_j\} && \mathcal{H}^1 = \mathcal{H}^{\sigma_1} \oplus \mathcal{H}^{\sigma_j} \subset C^{0,1}(\mathfrak{m}, \mathfrak{g}) \oplus C^{-1,1}(\mathfrak{m}, \mathfrak{g}).\\
&(3) \quad \Delta_1 = \{\alpha_1, \alpha_\ell\} && \mathcal{H}^1 = \mathcal{H}^{\sigma_1} \oplus \mathcal{H}^{\sigma_\ell} \subset C^{0,1}(\mathfrak{m}, \mathfrak{g}).
\end{aligned}$$

Summarizing we have (here we follow [Bu] for the numbering of simple roots)

Proposition 5.1. *Let $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ be a simple graded Lie algebra over $\mathbb{C}$ such that $\mathfrak{g}_p = [\mathfrak{g}_{p+1}, \mathfrak{g}_{-1}]$ for $p < -1$. Then $H^{p,1}(\mathfrak{m}, \mathfrak{g}) \neq \{0\}$ for some $p \geqq 0$ occurs only in the following cases.*

(1) *$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is of depth 1 (cf. [O2]), that is, it is isomorphic with $(A_\ell, \{\alpha_i\})$ $(1 \leqq i \leqq [\frac{\ell+1}{2}])$, $(B_\ell, \{\alpha_1\})$, $(C_\ell, \{\alpha_\ell\})$, $(D_\ell, \{\alpha_1\})$, $(D_\ell, \{\alpha_\ell\})$, $(E_6, \{\alpha_1\})$ or $(E_7, \{\alpha_7\})$. In these cases*

$$\begin{aligned}
&\text{(i)} \quad (A_\ell, \{\alpha_1\}) && H^{2,1}(\mathfrak{m}, \mathfrak{g}) \cong \mathcal{H}^{\sigma_1} \ (\ell = 1),\\
& && H^{1,1}(\mathfrak{m}, \mathfrak{g}) \cong \mathcal{H}^{\sigma_1} \ (\ell \geqq 2),\\
&\text{(ii)} \quad \textit{otherwise} && H^{0,1}(\mathfrak{m}, \mathfrak{g}) \cong \mathcal{H}^{\sigma_{i_o}}.
\end{aligned}$$

(2) *$\mathfrak{g} = \bigoplus_{p=-2}^{2} \mathfrak{g}_p$ is a contact gradation, that is, it is isomorphic with $(A_\ell, \{\alpha_1, \alpha_\ell\})$, $(B_\ell, \{\alpha_2\})$, $(C_\ell, \{\alpha_1\})$, $(D_\ell, \{\alpha_2\})$, $(E_6, \{\alpha_2\})$, $(E_7, \{\alpha_1\})$, $(E_8, \{\alpha_8\})$, $(F_4, \{\alpha_1\})$ or $(G_2, \{\alpha_2\})$. In these cases*

$$\begin{aligned}
&\text{(i)} \quad (A_\ell, \{\alpha_1, \alpha_\ell\}) && H^{0,1}(\mathfrak{m}, \mathfrak{g}) \cong \mathcal{H}^{\sigma_1} \oplus \mathcal{H}^{\sigma_\ell},\\
&\text{(ii)} \quad (C_\ell, \{\alpha_1\}) && H^{1,1}(\mathfrak{m}, \mathfrak{g}) \cong \mathcal{H}^{\sigma_1},\\
&\text{(iii)} \quad \textit{otherwise} && H^{0,1}(\mathfrak{m}, \mathfrak{g}) \cong \mathcal{H}^{\sigma_{i_o}}.
\end{aligned}$$

(3) *$\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ is isomorphic with $(A_\ell, \{\alpha_1, \alpha_i\})$ $(1 < i < \ell)$ or $(C_\ell, \{\alpha_1, \alpha_\ell\})$. In these cases*

$$H^{0,1}(\mathfrak{m}, \mathfrak{g}) \cong \mathcal{H}^{\sigma_1}.$$

Combined with Lemma 2.1 in §2.4, we obtain

Theorem 5.2. *Let $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ be a simple graded Lie algebra over $\mathbb{C}$ such that $\mathfrak{g}_p = [\mathfrak{g}_{p+1}, \mathfrak{g}_{-1}]$ for $p < -1$. Then $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ is the prolongation (over $\mathbb{C}$) of $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ except for the following three cases.*

(1) $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ *is of depth* 1.

(2) $\mathfrak{g} = \bigoplus_{p=-2}^{2} \mathfrak{g}_p$ *is a (complex) contact gradation.*

(3) $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ *is isomorphic with* $(A_\ell, \{\alpha_1, \alpha_i\})$ $(1 < i < \ell)$ *or* $(C_\ell, \{\alpha_1, \alpha_\ell\})$.

Furthermore $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ is the prolongation of $(\mathfrak{m}, \mathfrak{g}_0)$ except when $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ is isomorphic with $(A_\ell, \{\alpha_1\})$ or $(C_\ell, \{\alpha_1\})$.

Here $(A_\ell, \{\alpha_1\})$ is the graded Lie algebra $\mathfrak{g} = V \oplus \mathfrak{gl}(V) \oplus V^*$ of depth 1 associated with the (complex) projective structure (cf. [K2, Chapter IV]) and $(C_\ell, \{\alpha_1\})$ is known as the projective contact algebra (cf. [T2, p. 29]).

Now, by Lemmas 3.4, 3.5 and their proof, we have the real version of Theorem 5.2, which answers the question posed in §2.3. Here we note that, in the Satake diagram of type A (resp. C), $\Delta_1 = \{\alpha_1\}$ or $\{\alpha_1, \alpha_i\}$ $(1 < i < \ell)$ (resp. $\Delta_1 = \{\alpha_1\}$ or $\{\alpha_1, \alpha_\ell\}$) is ν-invariant subset of Δ^0 only for the normal real form AI (resp. CI).

Theorem 5.3. *Let $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ be a simple graded Lie algebra over $\mathbb{R}$ such that $\mathfrak{g}_p = [\mathfrak{g}_{p+1}, \mathfrak{g}_{-1}]$ for $p < -1$. Then $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ is the prolongation of $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ except for the following three cases.*

(1) $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ *is of depth* 1.

(2) $\mathfrak{g} = \bigoplus_{p=-2}^{2} \mathfrak{g}_p$ *is a real or complex contact gradation.*

(3) $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ *is isomorphic with* $(A_\ell, \{\alpha_1, \alpha_i\})$, $(C_\ell, \{\alpha_1, \alpha_\ell\})$ *or their normal real forms* $(AI, \{\alpha_1, \alpha_i\})$, $(CI, \{\alpha_1, \alpha_\ell\})$ $(1 < i < \ell)$.

Furthermore $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ is the prolongation of $(\mathfrak{m}, \mathfrak{g}_0)$ except when $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ is isomorphic with $(A_\ell, \{\alpha_1\})$, $(C_\ell, \{\alpha_1\})$ or their normal real forms $(AI, \{\alpha_1\})$, $(CI, \{\alpha_1\})$.

Real simple graded Lie algebra of depth 1 were classified by Kobayashi and Nagano [K-N]. In this case $M_{\mathfrak{g}}$ is a symmetric R-space (cf. [K-N], [Tk1]). The exceptional cases (2) and (3) are already discussed in §4. In these cases $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ is a symbol algebra of canonical systems on real or complex jet spaces.

Let $\mathfrak{g} = \bigoplus_{p=-\mu}^{\mu} \mathfrak{g}_p$ be a simple graded Lie algebra over $K = \mathbb{R}$ or $\mathbb{C}$ with $\mu > 1$. Let $\mathcal{A}(M_\mathfrak{g}, D_\mathfrak{g})$ denote the Lie algebra sheaf of all infinitesimal automorphisms (in the real or complex analytic category) of the standard differential system $(M_\mathfrak{g}, D_\mathfrak{g})$ of type $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$. We denote by $\mathcal{A}_x(M_\mathfrak{g}, D_\mathfrak{g})$ the stalk of $\mathcal{A}(M_\mathfrak{g}, D_\mathfrak{g})$ at $x \in M_\mathfrak{g}$. Then we have

Corollary 5.4. *Let $\mathfrak{g} = \bigoplus_{p=-\mu}^{\mu} \mathfrak{g}_p$ be a simple graded Lie algebra over $K = \mathbb{R}$ or $\mathbb{C}$ with $\mu > 1$. Then the following holds either in the real or complex analytic category.*

$\mathcal{A}_x(M_\mathfrak{g}, D_\mathfrak{g})$ is isomorphic with $\mathfrak{g}$ at each $x \in M_\mathfrak{g}$ except when $(M_\mathfrak{g}, D_\mathfrak{g})$ is locally isomorphic with a canonical system on a real or complex jet space. The latter case occurs if and only if $(M_\mathfrak{g}, D_\mathfrak{g})$ is one of the standard contact manifolds $(J_\mathfrak{g}, C_\mathfrak{g})$ over K, the canonical system $(J(P(V), i), C)$ $(1 \leqq i < \ell - 1)$ on the Grassmann bundle over the ℓ-dimensional projective space $P(V)$ over K or the canonical system $(L(P(V)), E)$ on the Lagrange-Grassmann bundle over the odd dimensional (contact) projective space $P(V)$ over K, where $K = \mathbb{C}$ in the complex category and $K = \mathbb{R}$ or $\mathbb{C}$ in the real category.

5.3. Calculation of $H^{p,2}(\mathfrak{m}, \mathfrak{g})$

First we shall compute $H^{p,2}(\mathfrak{m}, \mathfrak{g})$ for $p \geqq 0$, which is important to know the fundamental invariants of the normal Cartan connection for the geometric structures subordinate to regular differential system of type $\mathfrak{m}$ (cf. [T4, §2]).

For simple reflections $\sigma_i = \sigma_{\alpha_i}$, $\alpha_i \in \Delta$, we put $\sigma_{ij} = \sigma_i \cdot \sigma_j$ for $i \neq j$. Then we see that $\sigma_{ij} = \sigma_{ji}$ if and only if $\langle \alpha_i, \alpha_j \rangle = 0$ and that

$$\Phi_{\sigma_{ij}} = \{\alpha_i, \alpha_j - \langle \alpha_j, \alpha_i \rangle \alpha_i\}.$$

Thus $W^0(2)$ consists of $\sigma_{ij} \in W(2)$ such that one of the following holds:

(a) Both α_i and α_j belong to Δ_1.

(b) $\alpha_i \in \Delta_1$ and $\alpha_j \in \Delta_0$ such that $\langle \alpha_i, \alpha_j \rangle \neq 0$.

Then, by (5.2), we have

$$\mathcal{H}^{\sigma_{ij}} \subset C^{p_{ij},2}(\mathfrak{m}, \mathfrak{g}),$$

where

$$p_{ij} = \begin{cases} 1 - \mu + \langle \theta, \alpha_i \rangle + \langle \theta, \alpha_j \rangle - (\langle \theta, \alpha_j \rangle + 1) \langle \alpha_j, \alpha_i \rangle & \text{in case (a),} \\ -\mu + \langle \theta, \alpha_i \rangle - (\langle \theta, \alpha_j \rangle + 1) \langle \alpha_j, \alpha_i \rangle & \text{in case (b).} \end{cases}$$

First assume that $\langle \alpha_j, \alpha_i \rangle = 0$ for $\sigma_{ij} \in W^0(2)$. Then we have $\sigma_{ij} = \sigma_{ji}$, $\{\alpha_i, \alpha_j\} \subset \Delta_1$ and $p_{ij} = 1 - \mu + \langle \theta, \alpha_i \rangle + \langle \theta, \alpha_j \rangle$. Especially we have $\mu \geqq n_i(\theta) + n_j(\theta) \geqq 2$. Hence $p_{ij} < 0$ if $\Delta_1 \cap \Delta_\theta = \emptyset$, that is, if $\langle\theta, \alpha_i\rangle = \langle\theta, \alpha_j\rangle = 0$. If $\Delta_1 \cap \Delta_\theta \neq \emptyset$, from the diagram in §4.2, we know that $\mu \geqq 3$ except for A_ℓ-type. Thus $p_{ij} \geqq 0$ occurs only when $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ is isomorphic with $(C_\ell, \{\alpha_1, \alpha_\ell\})$ $(\ell \geqq 3)$, $(A_\ell, \{\alpha_1, \alpha_j\})$ $(2 < j < \ell)$, $(A_\ell, \{\alpha_1, \alpha_\ell\})$ or $(A_\ell, \{\alpha_1, \alpha_j, \alpha_\ell\})$ $(1 < j \leqq [\frac{\ell}{2}])$. In fact we have $p_{1\ell} = 0$, $p_{1j} = 0$ $(2 < j < \ell)$, $p_{1\ell} = 1$ and $p_{1\ell} = 0$ $(1 < j \leqq [\frac{\ell}{2}])$ in each case.

Secondly assume that $\langle\alpha_j, \alpha_i\rangle \neq 0$ and $\{\alpha_i, \alpha_j\} \cap \Delta_\theta = \emptyset$ for $\sigma_{ij} \in W^0(2)$. Then we have

$$p_{ij} = \begin{cases} 1 - \mu - \langle\alpha_j, \alpha_i\rangle & \text{in case (a),} \\ -\mu - \langle\alpha_j, \alpha_i\rangle & \text{in case (b).} \end{cases}$$

Moreover $\langle\alpha_j, \alpha_i\rangle = -2$ or -1 and $\langle\alpha_j, \alpha_i\rangle = -2$ occurs only for $\langle\alpha_{\ell-1}, \alpha_\ell\rangle$ in type B_ℓ $(\ell \geqq 4)$, $\langle\alpha_\ell, \alpha_{\ell-1}\rangle$ in type C_ℓ $(\ell \geqq 3)$ and $\langle\alpha_2, \alpha_3\rangle$ in type F_4. Thus, if $\langle\alpha_j, \alpha_i\rangle = -2$, we see, from the diagram in §4.2, that $p_{ij} \geqq 0$ occurs only when $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ is isomorphic with $(C_\ell, \{\alpha_{\ell-1}, \alpha_\ell\})$ in case (a) and isomorphic with $(B_\ell, \{\alpha_\ell\})$ or $(C_\ell, \{\alpha_{\ell-1}\})$ in case (b). In fact we have $p_{\ell-1\,\ell} = 0$, $p_{\ell\,\ell-1} = 0$ and $p_{\ell-1\,\ell} = 0$ in each case. If $\langle\alpha_j, \alpha_i\rangle = -1$, $p_{ij} \geqq 0$ occurs only when $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ is isomorphic with $(A_\ell, \{\alpha_i, \alpha_{i+1}\})$ $(1 < i \leqq [\frac{\ell}{2}])$ in case (a) and isomorphic with $(A_\ell, \{\alpha_2\})$, $(A_\ell, \{\alpha_i\})$ $(2 < i \leqq [\frac{\ell+1}{2}])$, $(C_\ell, \{\alpha_\ell\})$ $(\ell \geqq 3)$, $(D_\ell, \{\alpha_\ell\})$ $(\ell \geqq 5)$, $(E_6, \{\alpha_1\})$ or $(E_7, \{\alpha_7\})$ in case (b). In fact we have $p_{i\,i+1} = p_{i+1\,i} = 0$ $(1 < i \leqq [\frac{\ell}{2}])$, $p_{23} = 0$, $p_{i\,i-1} = p_{i\,i+1} = 0$ $(2 < i \leqq [\frac{\ell+1}{2}])$, $p_{\ell\,\ell-1} = 0$, $p_{\ell\,\ell-2} = 0$, $p_{13} = 0$ and $p_{76} = 0$ in each case.

Thirdly assume that $\langle\alpha_j, \alpha_i\rangle \neq 0$ and $\{\alpha_i, \alpha_j\} \cap \Delta_\theta \neq \emptyset$ for $\sigma_{ij} \in W^0(2)$. Then, from the diagram in §4.2, $\{\alpha_i, \alpha_j\}$ equals to $\{\alpha_1, \alpha_2\}$ or $\{\alpha_{\ell-1}, \alpha_\ell\}$ in type A_ℓ, $\{\alpha_1, \alpha_2\}$ or $\{\alpha_2, \alpha_3\}$ in type B_ℓ, $\{\alpha_1, \alpha_2\}$ in type C_ℓ, $\{\alpha_1, \alpha_2\}$ or $\{\alpha_2, \alpha_3\}$ in type D_ℓ, $\{\alpha_2, \alpha_4\}$ in type E_6, $\{\alpha_1, \alpha_3\}$ in type E_7, $\{\alpha_7, \alpha_8\}$ in type E_8, $\{\alpha_1, \alpha_2\}$ in type F_4 or $\{\alpha_1, \alpha_2\}$ in type G_2. Now assume further $\langle\alpha_j, \alpha_i\rangle = -1$ and $\operatorname{rank}\mathfrak{g} \geqq 3$. (In fact $\langle\alpha_j, \alpha_i\rangle < -1$ occurs only for $\langle\alpha_2, \alpha_1\rangle$ in type C_2 $(\cong B_2)$, $\langle\alpha_2, \alpha_1\rangle$ in type G_2 and $\langle\alpha_2, \alpha_3\rangle$ in type B_3.) Then $\{\alpha_i, \alpha_j\} \cap \Delta_\theta$ consists of a single element.

In case (a), we have $\{\alpha_i, \alpha_j\} \subset \Delta_1$ and

$$p_{ij} = 2 - \mu + \langle\theta, \alpha_i\rangle + 2\,\langle\theta, \alpha_j\rangle.$$

More precisely, if $\mathfrak{g}$ is of type C_ℓ, $p_{ij} = 4 - \mu$ or $6 - \mu$ according to $\alpha_i \in \Delta_\theta$ or $\alpha_j \in \Delta_\theta$. In other cases $p_{ij} = 3 - \mu$ or $4 - \mu$ according to $\alpha_i \in \Delta_\theta$ or $\alpha_j \in \Delta_\theta$. For the exceptional types, from the diagram in §4.2, we observe that $\mu \geqq n_i(\theta) + n_j(\theta) = 5$. Hence

$p_{ij} < 0$ if $\mathfrak{g}$ is of type E_6, E_7, E_8 or F_4. For the classical types, $p_{ij} \geqq 0$ occurs only when $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ is isomorphic with $(A_\ell, \{\alpha_1, \alpha_2\})$, $(A_\ell, \{\alpha_1, \alpha_2, \alpha_k\})$ $(2 < k \leqq \ell)$, $(A_\ell, \{\alpha_1, \alpha_2, \alpha_k, \alpha_m\})$ $(2 < k < m \leqq \ell)$, $(B_\ell, \{\alpha_1, \alpha_2\})$, $(B_\ell, \{\alpha_2, \alpha_3\})$, $(C_\ell, \{\alpha_1, \alpha_2\})$, $(C_\ell, \{\alpha_1, \alpha_2, \alpha_k\})$ $(2 < k \leqq \ell)$, $(D_\ell, \{\alpha_1, \alpha_2\})$, $(D_\ell, \{\alpha_1, \alpha_2, \alpha_\ell\})$ or $(D_\ell, \{\alpha_2, \alpha_3\})$.

In case (b), we have $p_{ij} = 3-\mu$ if $\mathfrak{g}$ is of type C_ℓ and $p_{ij} = 2-\mu$ otherwise. Hence $p_{ij} \geqq 0$ occurs only when $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ is isomorphic with $(A_\ell, \{\alpha_1\})$, $(A_\ell, \{\alpha_1, \alpha_j\})$ $(2 < j < \ell)$, $(A_\ell, \{\alpha_2\})$, $(A_\ell, \{\alpha_2, \alpha_j\})$ $(2 < j \leqq \ell)$, $(B_\ell, \{\alpha_1\})$, $(B_\ell, \{\alpha_3\})$, $(C_\ell, \{\alpha_1, \alpha_\ell\})$, $(C_\ell, \{\alpha_2\})$, $(C_\ell, \{\alpha_2, \alpha_\ell\})$, $(D_\ell, \{\alpha_1\})$, $(D_\ell, \{\alpha_1, \alpha_\ell\})$, $(D_\ell, \{\alpha_3\})$ or contact gradations of each type.

We leave it to the reader to check the remaining cases, that is, the cases $\mathfrak{g}$ is of type A_2, $B_2 = C_2$, G_2 or B_3.

Summarizing we obtain

Proposition 5.5. *Let (X_ℓ, Δ_1) be a simple graded Lie algebra over $\mathbb{C}$ described in §3.4. Then the following are the list of (X_ℓ, Δ_1) and p_{ij} such that $p_{ij} \geqq 0$ holds for the irreducible component $\mathcal{H}^{\sigma_{ij}} \subset C^{p_{ij},2}(\mathfrak{m}, \mathfrak{g})$ of the harmonic space $\mathcal{H}^2 \cong H^2(\mathfrak{m}, \mathfrak{g})$ corresponding to $\sigma_{ij} \in W^0(2)$ in Kostant's theorem.*

(I) *A_ℓ-type* $(\ell \geqq 2)$.

(1) $\{\alpha_1\}$	$p_{12} = 2$ $(\ell = 2)$,
	$p_{12} = 1$ $(\ell \geqq 3)$.
(2) $\{\alpha_2\}$	$p_{21} = 1$, $p_{23} = 0$.
(3) $\{\alpha_i\}$	$p_{i\,i-1} = p_{i\,i+1} = 0$ $(2 < i \leqq [\frac{\ell+1}{2}])$.
(4) $\{\alpha_1, \alpha_2\}$	$p_{12} = p_{21} = 3$ $(\ell = 2)$,
	$p_{12} = 1$, $p_{21} = 2$ $(\ell \geqq 3)$.
(5) $\{\alpha_1, \alpha_i\}$	$p_{12} = p_{1i} = 0$ $(2 < i < \ell - 1)$.
(6) $\{\alpha_1, \alpha_{\ell-1}\}$	$p_{12} = p_{1\,\ell-1} = p_{\ell-1\,\ell} = 0$ $(\ell \geqq 4)$.
(7) $\{\alpha_1, \alpha_\ell\}$	$p_{12} = p_{\ell\,\ell-1} = 0$, $p_{1\ell} = 1$ $(\ell \geqq 3)$.
(8) $\{\alpha_2, \alpha_3\}$	$p_{21} = p_{23} = p_{32} = p_{34} = 0$ $(\ell = 4)$,
	$p_{21} = p_{23} = p_{32} = 0$ $(\ell \geqq 5)$.
(9) $\{\alpha_2, \alpha_i\}$	$p_{21} = 0$ $(3 < i < \ell - 1)$.
(10) $\{\alpha_2, \alpha_{\ell-1}\}$	$p_{21} = p_{\ell-1\,\ell} = 0$ $(\ell \geqq 5)$.
(11) $\{\alpha_i, \alpha_{i+1}\}$	$p_{i\,i+1} = p_{i+1\,i} = 0$ $(2 < i \leqq [\frac{\ell}{2}])$.
(12) $\{\alpha_1, \alpha_2, \alpha_\ell\}$	$p_{13} = p_{12} = p_{32} = 0$, $p_{21} = p_{23} = 1$ $(\ell = 3)$,
	$p_{1\ell} = p_{12} = 0$, $p_{21} = 1$ $(\ell \geqq 4)$.
(13) $\{\alpha_1, \alpha_i, \alpha_\ell\}$	$p_{1\ell} = 0$ $(2 < i \leqq [\frac{\ell}{2}])$.
(14) $\{\alpha_1, \alpha_2, \alpha_i, \alpha_j\}$	$p_{21} = 0$ $(2 < i < j \leqq \ell)$.

(15) $\{\alpha_1, \alpha_2, \alpha_{\ell-1}, \alpha_\ell\}$ $p_{21} = p_{\ell-1\,\ell} = 0$.

(II) *B_ℓ-type* ($\ell \geqq 3$).

(1)	$\{\alpha_1\}$	$\mu = 1$	$p_{12} = 1$.
(2)	$\{\alpha_2\}$	$\mu = 2$	$p_{21} = p_{23} = 0$.
(3)	$\{\alpha_3\}$	$\mu = 2$	$p_{32} = 2$ $(\ell = 3)$, $p_{32} = 0$ $(\ell \geqq 4)$.
(4)	$\{\alpha_\ell\}$	$\mu = 2$	$p_{\ell\,\ell-1} = 0$ $(\ell \geqq 4)$.
(5)	$\{\alpha_1, \alpha_2\}$	$\mu = 3$	$p_{21} = 0$, $p_{12} = 1$.
(6)	$\{\alpha_1, \alpha_3\}$	$\mu = 3$	$p_{32} = 1$ $(\ell = 3)$.
(7)	$\{\alpha_2, \alpha_3\}$	$\mu = 3$	$p_{32} = 2$ $(\ell = 3)$, $p_{32} = 0$ $(\ell \geqq 4)$.
(8)	$\{\alpha_1, \alpha_2, \alpha_3\}$	$\mu = 5$	$p_{32} = 1$ $(\ell = 3)$.

(III) *C_ℓ-type* ($\ell \geqq 2$).

(1)	$\{\alpha_\ell\}$	$\mu = 1$	$p_{21} = 2$ $(\ell = 2)$, $p_{\ell\,\ell-1} = 0$ $(\ell \geqq 3)$.
(2)	$\{\alpha_1\}$	$\mu = 2$	$p_{12} = 2$ $(\ell = 2)$, $p_{12} = 1$ $(\ell \geqq 3)$.
(3)	$\{\alpha_2\}$	$\mu = 2$	$p_{21} = 2$ $(\ell = 2)$, $p_{21} = 1$ $(\ell \geqq 4)$, $p_{21} = 1$, $p_{23} = 0$ $(\ell = 3)$.
(4)	$\{\alpha_{\ell-1}\}$	$\mu = 2$	$p_{\ell-1\,\ell} = 0$ $(\ell \geqq 4)$.
(5)	$\{\alpha_1, \alpha_\ell\}$	$\mu = 3$	$p_{12} = 2$, $p_{21} = 3$ $(\ell = 2)$, $p_{1\ell} = p_{12} = 0$ $(\ell \geqq 3)$.
(6)	$\{\alpha_2, \alpha_\ell\}$	$\mu = 3$	$p_{21} = p_{23} = 0$ $(\ell = 3)$, $p_{21} = 0$ $(\ell \geqq 4)$.
(7)	$\{\alpha_{\ell-1}, \alpha_\ell\}$	$\mu = 3$	$p_{\ell-1\,\ell} = 0$ $(\ell \geqq 4)$.
(8)	$\{\alpha_1, \alpha_2\}$	$\mu = 4$	$p_{12} = 0$, $p_{21} = 2$ $(\ell \geqq 3)$.
(9)	$\{\alpha_1, \alpha_2, \alpha_\ell\}$	$\mu = 5$	$p_{21} = 1$.
(10)	$\{\alpha_1, \alpha_2, \alpha_i\}$	$\mu = 6$	$p_{21} = 0$ $(2 < i < \ell)$.

(IV) *D_ℓ-type* ($\ell \geqq 4$).

(1)	$\{\alpha_1\}$	$\mu = 1$	$p_{12} = 1$.
(2)	$\{\alpha_\ell\}$	$\mu = 1$	$p_{\ell\,\ell-2} = 0$ $(\ell \geqq 5)$.
(3)	$\{\alpha_2\}$	$\mu = 2$	$p_{21} = p_{23} = 0$.
(4)	$\{\alpha_3\}$	$\mu = 2$	$p_{32} = 0$ $(\ell \geqq 5)$.
(5)	$\{\alpha_1, \alpha_\ell\}$	$\mu = 2$	$p_{12} = 0$.
(6)	$\{\alpha_1, \alpha_2\}$	$\mu = 3$	$p_{12} = 1$, $p_{21} = 0$.
(7)	$\{\alpha_1, \alpha_2, \alpha_\ell\}$	$\mu = 4$	$p_{12} = 0$.
(8)	$\{\alpha_2, \alpha_3\}$	$\mu = 4$	$p_{32} = 0$ $(\ell \geqq 5)$.

(V) *Exceptional types.*

(1) $(E_6,\{\alpha_1\})$, $(E_7,\{\alpha_7\})$ $\mu = 1$ $p_{ij} = 0$, *where* $\{\alpha_i\} = \Delta_1$ *and* $\langle \alpha_i, \alpha_j \rangle \neq 0$.

(2) $(E_6,\{\alpha_2\})$, $(E_7,\{\alpha_1\})$, $(E_8,\{\alpha_8\})$, $(F_4,\{\alpha_1\})$ *and* $(G_2,\{\alpha_2\})$.
Contact gradations: $\mu = 2$ $p_{ij} = 0$, *where* $\{\alpha_i\} = \Delta_\theta$ *and* $\langle \alpha_i, \alpha_j \rangle \neq 0$.

(3) $(G_2,\{\alpha_1\})$ $\mu = 3$ $p_{12} = 3$.

(4) $(G_2,\{\alpha_1,\alpha_2\})$ $\mu = 5$ $p_{12} = 3$.

Now we shall give some remarks on regular differential systems of type $\mathfrak{m}$.

Let $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ be a simple graded Lie algebra over $\mathbb{R}$ such that $\mathfrak{m}$ is fundamental. Let M be a manifold with a $G_0^\sharp$-structure of type $\mathfrak{m}$ in the sense of [T4] (for the precise definition, see §2 of [T4]). In [T4], under the assumption that $\mathfrak{g}$ is the prolongation of $(\mathfrak{m}, \mathfrak{g}_0)$, N. Tanaka constructed a normal Cartan connection (P,ω) of type $\mathfrak{g}$ over M, which settles the equivalence problem for the $G_0^\sharp$-structure of type $\mathfrak{m}$ in the following sense: Let M and $\widehat{M}$ be two manifolds with $G_0^\sharp$-structures of type $\mathfrak{m}$. Let (P,ω) and $(\widehat{P},\widehat{\omega})$ be the normal connections of type $\mathfrak{g}$ over M and $\widehat{M}$ respectively. Then a diffeomorphism φ of M onto $\widehat{M}$ preserving the $G_0^\sharp$-structures lifts uniquely to an isomorphism $\varphi^\sharp$ of (P,ω) onto $(\widehat{P},\widehat{\omega})$ and vice versa ([T4, Theorem 2.7]).

Here we note that, if $\mathfrak{g}$ is the prolongation of $\mathfrak{m}$, a $G_0^\sharp$-structure on M is nothing but a regular differential system of type $\mathfrak{m}$ (see [T4, §2.2]). Moreover let K be the curvature of the normal connection (P,ω), which can be regarded as a $C^2(\mathfrak{m},\mathfrak{g})$-valued function on P ([T4, Lemma 2.2]). Then, by the normality condition for K: $K^p = 0$ for $p < 0$ and $\partial^* K^p = 0$ for $p \geqq 0$, where K^p is the $C^{p,2}(\mathfrak{m},\mathfrak{g})$-component of K, and the Bianchi identity, it is further shown ([T4, Theorem 2.9]) that the harmonic part $H(K)$ of K, with respect to the orthogonal decomposition $C^2(\mathfrak{m},\mathfrak{g}) = \operatorname{Im}\partial \oplus \operatorname{Im}\partial^* \oplus \mathcal{H}$, gives a fundamental system of invariants of the connection (P,ω). Namely K vanishes if and only if $H(K)$ vanishes. Hence, as a corollary to Theorems 2.7 and 2.9 of [T4], we have

Let $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ *be a simple graded Lie algebra over* $\mathbb{R}$ *such that* $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ *is fundamental. Assume that* $\mathfrak{g}$ *is the prolongation of* $\mathfrak{m}$ *and* $H^{p,2}(\mathfrak{m},\mathfrak{g}) = \{0\}$ *for* $p \geqq 0$. *Then every regular differential system* (M,D) *of type* $\mathfrak{m}$ *is locally isomorphic with the standard differential system* $(M(\mathfrak{m}), D_{\mathfrak{m}})$ *of type* $\mathfrak{m}$.

Thus, by Proposition 5.5, we can find many examples of regular differential systems (M, D) of type $\mathfrak{m}$ with no local invariants, whose Lie algebra $\mathcal{A}(M, D)$ of all infinitesimal automorphisms are isomorphic with simple Lie algebras over $\mathbb{R}$.

We shall give below some examples of fundamental graded algebras $\mathfrak{m} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$ of the second kind whose prolongation $\mathfrak{g}(\mathfrak{m})$ become finite dimensional and simple. Namely we shall describe the structure of $\mathfrak{m} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$ of several simple graded Lie algebras (X_ℓ, Δ_1) over $\mathbb{C}$ and their normal real forms such that $H^{p,1}(\mathfrak{m}, \mathfrak{g})$ vanishes for $p \geqq 0$ and $\mu = 2$. In the following we shall discuss in the complex analytic or the real C^∞ category depending on whether we treat complex simple graded Lie algebras (X_ℓ, Δ_1) or their normal real forms.

(1) $(B_\ell, \{\alpha_\ell\})$ $(\ell \geqq 3)$. First we have (see §4.4)

$$\Phi_2^+ = \{\alpha_{ij} = \alpha_i + \cdots + \alpha_{j-1} + 2\,\alpha_j + \cdots + 2\,\alpha_\ell \quad (1 \leqq i < j \leqq \ell)\},$$
$$\Phi_1^+ = \{\beta_i = \alpha_i + \cdots + \alpha_\ell \quad (1 \leqq i \leqq \ell)\}.$$

Each $\alpha_{ij} \in \Phi_2^+$ is uniquely written as a sum $\alpha_{ij} = \beta_i + \beta_j$ of roots in Φ_1^+. We have $\dim \mathfrak{g}_{-1} = \ell$ and $\dim \mathfrak{g}_{-2} = \frac{1}{2}\ell(\ell-1)$. Hence the structure of $\mathfrak{m} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$ is described by

$$\mathfrak{g}_{-2} = \wedge^2 V,$$

where we put $\mathfrak{g}_{-1} = V$. Namely $\mathfrak{m}$ is the universal fundamental graded algebra of second kind such that $\dim \mathfrak{g}_{-1} = \ell \geqq 3$. In this case, it is easy to see that $\mathfrak{g}_0$ is naturally identified with $\mathfrak{gl}(\mathfrak{g}_{-1})$ (see also the matrix representation of $(B_\ell, \{\alpha_\ell\})$ in §4.4). This example was first found by Tanaka [T1, p. 245]. The standard differential system $(M(\mathfrak{m}), D_\mathfrak{m})$ of type $\mathfrak{m}$ is given as follows: Let $(x_1, \ldots, x_\ell, x_{ij})$ $(1 \leqq i < j \leqq \ell)$ be a coordinate system of $M(\mathfrak{m}) = K^{\frac{1}{2}\ell(\ell+1)}$. Then $D_\mathfrak{m}$ is defined by the following $\frac{1}{2}\ell(\ell-1)$ forms

$$\varpi_{ij} = dx_{ij} - \tfrac{1}{2}(x_i\, dx_j - x_j\, dx_i) \quad (1 \leqq i < j \leqq \ell).$$

$\mathcal{A}(M(\mathfrak{m}), D_\mathfrak{m})$ is isomorphic with $\mathfrak{o}(2\ell+1, \mathbb{C})$ or $\mathfrak{o}(\ell+1, \ell)$ depending on $K = \mathbb{C}$ or $\mathbb{R}$.

(2) $(D_\ell, \{\alpha_{\ell-1}, \alpha_\ell\})$ $(\ell \geqq 4)$. First we have

$$\begin{aligned}\Phi_2^+ = \{&\alpha_{i\,\ell-1} = \alpha_i + \cdots + \alpha_{\ell-1} + \alpha_\ell \quad (1 \leqq i \leqq \ell-2),\\ &\alpha_{ij} = \alpha_i + \cdots + \alpha_{j-1} + 2\,\alpha_j + \cdots + 2\,\alpha_{\ell-2} + \alpha_{\ell-1} + \alpha_\ell\\ &\qquad\qquad (1 \leqq i < j \leqq \ell-2)\},\\ \Phi_1^+ = \{&\beta_i = \alpha_i + \cdots + \alpha_{\ell-1}\ (1 \leqq i \leqq \ell-1), \quad \gamma_{\ell-1} = \alpha_\ell,\\ &\gamma_i = \alpha_i + \cdots + \alpha_{\ell-2} + \alpha_\ell\ (1 \leqq i \leqq \ell-2)\}.\end{aligned}$$

Each $\alpha_{ij} \in \Phi_2^+$ $(1 \leqq i < j \leqq \ell-1)$ is written as a sum

$$\alpha_{ij} = \beta_i + \gamma_j = \beta_j + \gamma_i$$

of roots in Φ_1^+ in two ways. We have $\dim \mathfrak{g}_{-1} = 2(\ell-1)$ and $\dim \mathfrak{g}_{-2} = \frac{1}{2}(\ell-1)(\ell-2)$. By the explicit matrix representation of $(D_\ell, \{\alpha_{\ell-1}, \alpha_\ell\})$ in §4.4, we can describe the structure of $\mathfrak{m} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$ as follows: There exist basis $\{X_1, \ldots, X_{\ell-1}, Y_1, \ldots, Y_{\ell-1}\}$ of $\mathfrak{g}_{-1}$ and $\{Z_{ij}$ $(1 \leqq i < j \leqq \ell-1)\}$ of $\mathfrak{g}_{-2}$ such that

$$\begin{gathered} Z_{ij} = [X_i, Y_j] = [Y_i, X_j] \quad (1 \leqq i < j \leqq \ell-1),\\ [X_i, X_j] = [Y_i, Y_j] = 0.\end{gathered}$$

Thus the standard differential system $(M(\mathfrak{m}), D_\mathfrak{m})$ of type $\mathfrak{m}$ is given as follows: Let $(x_1, \ldots, x_{\ell-1}, y_1, \ldots, y_{\ell-1}, z_{ij})$ $(1 \leqq i < j \leqq \ell-1)$ be a coordinate system of $M(\mathfrak{m}) = K^{\frac{1}{2}(\ell-1)(\ell+2)}$. Then $D_\mathfrak{m}$ is defined by the following $\frac{1}{2}(\ell-1)(\ell-2)$ forms

$$\varpi_{ij} = dz_{ij} - (x_i\, dy_j + y_i\, dx_j) \quad (1 \leqq i < j \leqq \ell-1).$$

$\mathcal{A}(M(\mathfrak{m}), D_\mathfrak{m})$ is isomorphic with $\mathfrak{o}(2\ell, \mathbb{C})$ or $\mathfrak{o}(\ell, \ell)$ depending on $K = \mathbb{C}$ or $\mathbb{R}$.

Furthermore, by Proposition 5.5 (IV), we see that $H^{p,2}(\mathfrak{m}, \mathfrak{g})$ vanishes for $p \geqq 0$ when $\ell \geqq 5$. Hence, in this case $(\ell \geqq 5)$, every regular differential system (M, D) of type $\mathfrak{m}$ is locally isomorphic with $(M(\mathfrak{m}), D_\mathfrak{m})$ given above. Namely assume that (M, D) is a differential system which has local defining 1-forms ϖ_{ij};

$$D = \{\, \varpi_{ij} = 0 \quad (1 \leqq i < j \leqq \ell-1)\,\},$$

satisfying the following structure equation, for $1 \leqq i < j \leqq \ell-1$

$$d\varpi_{ij} \equiv \omega_i \wedge \varpi_j + \varpi_i \wedge \omega_j \quad (\mathrm{mod}\ \varpi_{rs} \quad (1 \leqq r < s \leqq \ell-1)),$$

where $\{\varpi_{ij}\ (1 \leqq i < j \leqq \ell-1), \omega_1, \dots, \omega_{\ell-1}, \varpi_1, \dots, \varpi_{\ell-1}\}$ is a local (free) basis of 1-forms on M. Then there exists a local coordinate system $(x_1, \dots, x_{\ell-1}, y_1, \dots, y_{\ell-1}, z_{ij})$ $(1 \leqq i < j \leqq \ell-1)$ of M such that

$$D = \{\, dz_{ij} - (x_i\, dy_j + y_i\, dx_j) = 0 \quad (1 \leqq i < j \leqq \ell-1)\,\}.$$

(3) $(F_4, \{\alpha_4\})$. Here we shall show that the standard differential system $(M(\mathfrak{m}), D_{\mathfrak{m}})$ of type $\mathfrak{m}$ in this case has a following description, which was discovered by E. Cartan [C1]: Let $(z, x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4, x_{ij})$ $(1 \leqq i < j \leqq 4)$ be a coordinate system on $M_F = K^{15}$. Let D_F be a differential system on M_F defined by the following 7 forms;

$$\begin{cases} \varpi = dz - y_1\, dx_1 - y_2\, dx_2 - y_3\, dx_3 - y_4\, dx_4, \\ \varpi_{ij} = dx_{ij} - (x_i\, dx_j - x_j\, dx_i + y_h\, dy_k - y_k\, dy_h) \quad (1 \leqq i < j \leqq 4) \end{cases}$$

where (h, k) is determined by the requirement that (i, j, h, k) is an even permutation of $(1, 2, 3, 4)$. By taking the dual vector fields Z, X_{ij}, X_i, Y_j of the basis $\{\varpi, \varpi_{ij}, dx_i, dy_j\}$ of 1-forms on M_F, we have

$$(*) \qquad \begin{cases} Z = [Y_i, X_i] \quad (i = 1,\ 2,\ 3,\ 4), \\ 2\, X_{ij} = [X_i, X_j] = [Y_h, Y_k] \quad (1 \leqq i < j \leqq 4), \end{cases}$$

where (i, j, h, k) is an even permutation of $(1, 2, 3, 4)$. Namely (M_F, D_F) is the standard differential system of type $\mathfrak{m}_F$. Here $\mathfrak{m}_F = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$ is the fundamental graded algebra of the second kind such that there exist bases $\{Z, X_{12}, X_{13}, X_{14}, X_{23}, X_{24}, X_{34}\}$ of $\mathfrak{g}_{-2}$ and $\{X_1, X_2, X_3, X_4, Y_1, Y_2, Y_3, Y_4\}$ of $\mathfrak{g}_{-1}$ satisfying $(*)$ above. Thus our aim here is to show that $\mathfrak{m}_F$ is isomorphic with the negative part $\mathfrak{m} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$ of the simple graded Lie algebra $(F_4, \{\alpha_4\})$ or its normal real form.

For $(F_4, \{\alpha_4\})$, we have (cf. [Bu, p. 272, Planche VIII])

$$\begin{aligned} \Phi_2^+ = \{&\alpha_{14} = 0122,\ \alpha_{13} = 1122,\ \alpha_{12} = 1222, \\ &\alpha = 1232,\ \alpha_{34} = 1242,\ \alpha_{24} = 1342,\ \alpha_{23} = 2342\}, \\ \Phi_1^+ = \{&\beta_1 = 0001,\ \gamma_2 = 0011,\ \gamma_3 = 0111,\ \beta_4 = 0121, \\ &\gamma_1 = 1231,\ \beta_2 = 1221,\ \beta_3 = 1121,\ \gamma_4 = 1111\}. \end{aligned}$$

where $a_1\, a_2\, a_3\, a_4$ stands for the coefficients of the positive root with respect to the simple roots α_1, α_2, α_3 and α_4. Each root in Φ_2^+ is written as a sum of roots in Φ_1^+ as follows.

$$\begin{cases} \alpha = \beta_i + \gamma_i \quad (i = 1,\ 2,\ 3,\ 4), \\ \alpha_{ij} = \beta_i + \beta_j = \gamma_h + \gamma_k \quad (1 \leqq i < j \leqq 4), \end{cases}$$

where $\{i, j, h, k\} = \{1, 2, 3, 4\}$.

Let us take a Chevalley basis $\{x_\alpha \ (\alpha \in \Phi); h_i \ (1 \leqq i \leqq 4)\}$ of F_4 and put $y_\beta = x_{-\beta}$ for $\beta \in \Phi^+$ (cf. [Hu, Chapter VII]). We consider the structure of the negative part $\mathfrak{m}$ of $(F_4, \{\alpha_4\})$ in terms of $\{y_\beta\}_{\beta \in \Phi_1^+ \cup \Phi_2^+}$. Here we note that $\alpha \in \Phi_2^+$ and all roots in Φ_1^+ are short roots in Φ, whereas the other roots in Φ_2^+ are long roots in Φ (see [Bu, Planche VIII]). Moreover, in the root system $\widehat{\Phi}$ of type A_2 or $C_2 = B_2$, we observe that, if $\alpha + \beta \in \widehat{\Phi}$, the α-string through β starts from $\beta - \alpha$ when α, β are short and $\alpha + \beta$ is a long root, and starts from β otherwise. (See [Hu, p. 44].) These observations readily show that $\mathfrak{m}$ satisfies $(*)$ above up to signs of the structure constants. However the question of signs is a subtle point of the Chevalley basis (cf. [Tt2]). We are obliged to check the question of signs as follows: First let us choose signs of $y_i = y_{\alpha_i}$ $(i = 1, 2, 3, 4)$ corresponding to the simple roots α_1, α_2, α_3, α_4 by fixing the root vectors $y_i \in \mathfrak{g}_{-\alpha_i}$. We fix the signs of y_β for $\beta \in \Phi_1^+ \cup \Phi_2^+$ by the following;

$$\begin{aligned}
y_{\beta_1} &= y_4, & y_{\gamma_2} &= [y_3, y_{\beta_1}], & y_{\gamma_3} &= [y_2, y_{\gamma_2}], \\
y_{\gamma_4} &= [y_1, y_{\gamma_3}], & y_{\beta_3} &= [y_3, y_{\gamma_4}], & y_{\beta_2} &= [y_2, y_{\beta_3}], \\
y_{\beta_4} &= [y_3, y_{\gamma_3}], & y_{\gamma_1} &= [y_3, y_{\beta_2}], & y_\alpha &= [y_4, y_{\gamma_1}], \\
2\, y_{\alpha_{14}} &= [y_4, y_{\beta_4}], & 2\, y_{\alpha_{13}} &= [y_4, y_{\beta_3}], & 2\, y_{\alpha_{12}} &= [y_4, y_{\beta_2}], \\
2\, y_{\alpha_{34}} &= [y_3, y_\alpha], & y_{\alpha_{24}} &= [y_2, y_{\alpha_{34}}], & y_{\alpha_{23}} &= [y_1, y_{\alpha_{24}}].
\end{aligned}$$

Then, by the repeated application of Jacobi identity, one can check that, by putting

$$\begin{aligned}
X_i &= y_{\beta_i}, & Y_i &= (-1)^i y_{\gamma_i} \quad (i = 1, 2, 3, 4), \\
Z &= y_\alpha, & Z_{ij} &= y_{\alpha_{ij}} \quad (1 \leqq i < j \leqq 4),
\end{aligned}$$

$\{Z, Z_{ij}, X_i, Y_j\}$ satisfies $(*)$ above, that is, $\mathfrak{m}$ is isomorphic with $\mathfrak{m}_F$.

Finally we remark that, by Proposition 5.5 (V) and Tanaka's Theorem [T4], every regular differential system (M, D) of type $\mathfrak{m}_F$ is locally isomorphic with (M_F, D_F).

5.4. Reducible primitive actions

We shall characterize the standard differential system $(M_\mathfrak{g}, D_\mathfrak{g})$ of type $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$, whose isotropy subalgeras $\mathfrak{g}' = \bigoplus_{p \geqq 0} \mathfrak{g}_p$ are maximal parabolic, as homogeneous differential systems which have nonlinear reducible primitive actions of Lie groups (cf. [O1], [Go]).

We shall consider reducible primitive actions of finite dimensional Lie groups, following the arguments in [Go], [K-N, I and II] and [Gu]. We shall discuss in either real or complex category.

Let L be a connected Lie group acting transitively and effectively on a manifold M. Let L' be the isotropy subgroup of L at a point o of M so that $M = L/L'$. We denote by $\mathcal{L}$ and $\mathcal{L}'$ the Lie algebras of L and L' respectively. Let $\gamma\colon L' \to GL(T_o(M))$ be the linear isotropy representation of L' given by

$$\begin{array}{ccc} \mathcal{L} & \xrightarrow{\mathrm{Ad}(g)} & \mathcal{L} \\ \pi_* \downarrow & & \downarrow \pi_* \\ T_o(M) & \xrightarrow{\gamma(g)} & T_o(M) \end{array}$$

for $g \in L'$, where $\pi\colon L \to M$ is the projection defined by $\pi(g) = g(o)$. Then a $\gamma(L')$-invariant subspace D_o of $T_o(M)$ corresponds to an $\mathrm{Ad}(L')$-invariant subspace $\mathcal{L}^{-1}$ of $\mathcal{L}$ containing $\mathcal{L}'$, which further corresponds to a L-invariant differential system D on M such that $D(o) = D_o$. We say that L acts primitively on M if L leaves invariant no completely integrable differential systems on M (cf. [Go, Definition 1.3]). From the above diagram, it follows that L acts primitively on M if and only if $\mathcal{L}'$ is a maximally $\mathrm{Ad}(L')$-invariant subalgebra of $\mathcal{L}$. Namely (cf. [Go, Theorem 2.1])

If $\mathfrak{h}$ *is a subalgebra of* $\mathcal{L}$ *satisfying* $\mathfrak{h} \supset \mathcal{L}'$ *and* $\mathrm{Ad}(L')(\mathfrak{h}) = \mathfrak{h}$, *then either* $\mathfrak{h} = \mathcal{L}$ *or* $\mathfrak{h} = \mathcal{L}'$.

Here we note that $\mathcal{L}'$ is self-normalizing in $\mathcal{L}$. In fact the normalizer $N(\mathcal{L}')$ of $\mathcal{L}'$ in $\mathcal{L}$ is obviously preserved by $\mathrm{Ad}(L')$. Hence we have $N(\mathcal{L}') = \mathcal{L}'$ or $\mathcal{L}$. However $N(\mathcal{L}') = \mathcal{L}$ implies $\mathcal{L}'$ is an ideal of $\mathcal{L}$, which contradicts to the assumption that L acts effectively on M. Thus $N(\mathcal{L}') = \mathcal{L}'$.

Now we consider the following situation: Assume that L acts primitively on M and the linear isotropy representation $\gamma\colon L' \to GL(T_o(M))$ is reducible. Namely L acts primitively on M and leaves invariant a differential system D on M (which is, of course, non-integrable). Let us take D to be minimal, that is, $D_o = D(o)$ is a $\gamma(L')$-irreducible subspace of $T_o(M)$. $\mathcal{L}$ is naturally identified with the Lie algebra of vector fields on M induced by the L-action. We introduce a filtration $\{\mathcal{L}^p\}_{p\in\mathbb{Z}}$ of $\mathcal{L}$ induced from the L-invariant differential system D as follows ([T2, §6], [We], [Gu, §7], [Go, §4]), which will be the main tool in our argument.

Put $\mathcal{L}^{-1} = \pi_*^{-1}(D_o)$ or equivalently

$$\mathcal{L}^{-1} = \{ X \in \mathcal{L} \mid X_o \in D(o) \},$$

under the above identification. Starting from $(\mathcal{L}, \mathcal{L}^{-1}, \mathcal{L}')$, we first define $\mathcal{L}^p$ for $p < -1$ inductively by

$$\mathcal{L}^p = \mathcal{L}^{p+1} + [\mathcal{L}^{p+1}, \mathcal{L}^{-1}].$$

We put $\mathcal{L}^0 = \mathcal{L}'$ and define $\mathcal{L}^k$ for $k > 0$ inductively by

$$\mathcal{L}^k = \{ X \in \mathcal{L}^{k-1} \mid [X, \mathcal{L}^{-1}] \subset \mathcal{L}^{k-1} \}.$$

Here we note that, since $\mathcal{L}^0$ is self-normalizing, $\mathcal{L}^1$ is properly contained in $\mathcal{L}^0$. Obviously $\mathcal{L}^p$ is $\mathrm{Ad}(L')$-invariant for all $p \in \mathbb{Z}$. It is easy to check that $\{\mathcal{L}^p\}_{p\in\mathbb{Z}}$ satisfies

$$[\mathcal{L}^p, \mathcal{L}^q] \subset \mathcal{L}^{p+q} \qquad \text{for all } p,\, q \in \mathbb{Z}.$$

Since $\mathcal{L}$ is finite dimensional, there exist integers $\mu > 1$ and $\nu \geqq 0$ such that

$$\mathcal{L}^p = \mathcal{L}^{-\mu} \supsetneqq \mathcal{L}^{-\mu+1} \quad \text{for } p \leqq -\mu, \quad \mathcal{L}^\nu \supsetneqq \mathcal{L}^{\nu+1} = \mathcal{L}^k \quad \text{for } k \geqq \nu + 1.$$

Then $\mathcal{L}^{-\mu}$ is a $\mathrm{Ad}(L')$-invariant subalgebra of $\mathcal{L}$ properly containing $\mathcal{L}^0$ and $\mathcal{L}^{\nu+1}$ is an ideal of $\mathcal{L}$ properly contained in $\mathcal{L}^0$. Hence, by our assumption that L acts primitively and effectively on M, we obtain $\mathcal{L} = \mathcal{L}^{-\mu}$ and $\mathcal{L}^{\nu+1} = \{0\}$. Thus $\mathcal{L} = \{\mathcal{L}^p\}_{p\in\mathbb{Z}}$ becomes a (transitive) filtered Lie algebra. This filtration $\{\mathcal{L}^p\}_{p\in\mathbb{Z}}$ is called the Weisfeiler filtration of $(\mathcal{L}, \mathcal{L}^0)$ in §7 of [Gu] and §4 of [Go].

We now consider the associated graded Lie algebra $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ of $\mathcal{L} = \{\mathcal{L}^p\}_{p\in\mathbb{Z}}$. Namely we put $\mathfrak{g}_p = \mathcal{L}^p/\mathcal{L}^{p+1}$ for $p \in \mathbb{Z}$ and put

$$\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p.$$

Let ϖ_p be the projection of $\mathcal{L}^p$ onto $\mathfrak{g}_p = \mathcal{L}^p/\mathcal{L}^{p+1}$. Then, for $X \in \mathfrak{g}_p$ and $Y \in \mathfrak{g}_q$, the bracket product $[X, Y] \in \mathfrak{g}_{p+q}$ is defined by

$$[X, Y] = \varpi_{p+q}([\tilde{X}, \tilde{Y}]),$$

where $\tilde{X} \in \mathcal{L}^p$ and $\tilde{Y} \in \mathcal{L}^q$ are any element such that $\varpi_p(\tilde{X}) = X$ and $\varpi_q(\tilde{Y}) = Y$ (cf. §1.2). For each $g \in L'$, the graded map $\hat{\varphi}_g$ of $\mathrm{Ad}(g)$ is a graded Lie algebra automorphism of $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ (cf. Proposition

3.11). Thus we have a representation $\beta\colon L' \to \mathrm{Aut}_g(\mathfrak{g})$ by $\beta(g) = \hat{\varphi}_g$, where $\mathrm{Aut}_g(\mathfrak{g})$ is the group of all graded Lie algebra automorphisms of $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$. $G_0' = \beta(L')$ is a Lie subgroup of $\mathrm{Aut}_g(\mathfrak{g})$ with Lie algebra isomorphic with $\mathfrak{g}_0 = \mathcal{L}^0/\mathcal{L}^1$.

Then, by our choice of $\mathcal{L}^{-1}$ and the construction of $\{\mathcal{L}^p\}_{p\in\mathbb{Z}}$, we have

$$(5.3) \quad \begin{cases} \text{(i)} & \mathfrak{g}_p = [\mathfrak{g}_{p+1}, \mathfrak{g}_{-1}] \text{ for } p < -1, \\ \text{(ii)} & \text{For } k \geqq 0, \text{ if } X \in \mathfrak{g}_k \text{ and } [X, \mathfrak{g}_{-1}] = \{0\}, \text{ then } X = 0, \\ \text{(iii)} & G_0' \text{ acts irreducibly on } \mathfrak{g}_{-1}. \end{cases}$$

Here we note that, from the structure equation of L, it follows that $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ gives the symbol algebra of (M, D) (cf. [T2, §6]). Any subalgebra $\mathfrak{a}$ of $\mathcal{L}$ becomes a filtered subalgebra of $\mathcal{L} = \{\mathcal{L}^p\}_{p\in\mathbb{Z}}$ with the filtration $\{\mathfrak{a}^p\}_{p\in\mathbb{Z}}$ given by $\mathfrak{a}^p = \mathfrak{a} \cap \mathcal{L}^p$ for $p \in \mathbb{Z}$. Its associated graded Lie algrebra $\widehat{\mathfrak{a}} = \bigoplus_{p\in\mathbb{Z}} \widehat{\mathfrak{a}}_p$ is a graded subalgebra of $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ satisfying $\dim \mathfrak{a} = \dim \widehat{\mathfrak{a}}$. Especially $\widehat{\mathcal{L}^0} = \bigoplus_{p\geqq 0} \mathfrak{g}_p$. Moreover $\widehat{\mathfrak{a}}$ is an ideal of $\mathfrak{g}$ if $\mathfrak{a}$ is an ideal of $\mathcal{L}$. With these preparation, we have ([K-N, I, p. 878, Lemmas 1 and 2])

Lemma 5.6. *$\mathcal{L}$ is simple.*

Proof. Let $\mathfrak{c}$ be an $\mathrm{Ad}(L')$-invariant ideal of $\mathcal{L}$. Since $\mathcal{L}' = \mathcal{L}^0$ is a maximally $\mathrm{Ad}(L')$-invariant subalgebra and contains no ideal of $\mathcal{L}$, we have $\mathcal{L} = \mathfrak{c} + \mathcal{L}^0$. Then we have $\mathfrak{g} = \widehat{\mathfrak{c}} + \mathfrak{g}'$, where $\mathfrak{g}' = \bigoplus_{p\geqq 0} \mathfrak{g}_p$. Hence $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p \subset \widehat{\mathfrak{c}}$. Here we note that $\widehat{\mathfrak{c}}$ is abelian if $\mathfrak{c}$ is so. On the other hand, by our assumption; $\mu > 1$, $\mathfrak{m}$ is not abelian. Hence $\mathcal{L}$ has no abelian ideals, which is $\mathrm{Ad}(L')$-invariant. However if the radical $\mathfrak{r}$ of $\mathcal{L}$ is non-trivial, the last ideal in the derived series of $\mathfrak{r}$ is a non-trivial abelian ideal, which is obviously invariant by $\mathrm{Ad}(L')$. Therefore $\mathcal{L}$ is semisimple. Then, since $\mathrm{Ad}(L')$ is a subgroup of the adjoint group $\mathrm{Int}(\mathcal{L}) = \mathrm{Ad}(L)$, each simple ideal of $\mathcal{L}$ is $\mathrm{Ad}(L')$-invariant. For two simple ideals of $\mathfrak{c}_1$ and $\mathfrak{c}_2$ of $\mathcal{L}$, we have $\widehat{\mathfrak{c}}_1 \cap \widehat{\mathfrak{c}}_2 \supset \mathfrak{m}$. Thus $[\widehat{\mathfrak{c}}_1, \widehat{\mathfrak{c}}_2] \neq \{0\}$, which implies $\mathfrak{c}_1 = \mathfrak{c}_2$. Therefore $\mathcal{L}$ is simple.

Remark 5.7. When the linear isotropy representation $\gamma\colon L' \to GL(T_o(M))$ is irreducible, the nonlinearity of the action; $\mathrm{Ker}\,\gamma$ is non-discrete, is necessary to conclude that $\mathcal{L}$ is simple (see [K-N, I, Lemma 2]). In fact when $\mathcal{L}$ is not simple, the structure of the pair $(\mathcal{L}, \mathcal{L}')$ is determined by Morosov and Golubitsky (see [Go, Proposition 2.3]). Especially $\mathcal{L}/\mathcal{L}'$ is $\mathcal{L}'$-irreducible in this case. Lemma 5.6 follows also from this fact.

The structure of $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ is determined by the following Lemma due to Weisfeiler and Golubitsky ([We], [Go, Theorem 4.3]).

Lemma 5.8. *Let $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ be a graded Lie algebra over $K = \mathbb{R}$ or $\mathbb{C}$ satisfying conditions in* (5.3). *Then*

(1) *If $\mathfrak{g}_1 \neq \{0\}$, $\mathfrak{g}$ is semisimple.*
(2) *If $\mathfrak{g}_1 = \{0\}$, $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{g}_0$, that is, $\mathfrak{g}_k = \{0\}$ for $k \geqq 1$, and $\mathfrak{g}_0$ is reductive.*

Proof. We reproduce the proof from Lemma 8.1 of [Gu] and Lemma 4.2 of [Go]. Let δ_o be the derivation of $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ defined by $\delta_o(X) = pX$ for $X \in \mathfrak{g}_p$. We consider the radical $\mathfrak{r}$ of $\mathfrak{g}$. $\mathfrak{r}$ is preserved by any Lie algebra automorphism of $\mathfrak{g}$. Hence $\mathfrak{r}$ is invariant by G'_0 and by δ_o as well. Thus $\mathfrak{r}$ is a graded ideal of $\mathfrak{g}$, that is, $\mathfrak{r} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{r}_p$, where $\mathfrak{r}_p = \mathfrak{r} \cap \mathfrak{g}_p$. Then $\mathfrak{r}_{-1}$ is a G'_0-invariant subspace of $\mathfrak{g}_{-1}$. Hence, by (iii) of (5.3), we have two cases to distinguish; (1) $\mathfrak{r}_{-1} = \{0\}$ or (2) $\mathfrak{r}_{-1} = \mathfrak{g}_{-1}$.

In case (1), by (ii) of (5.3), we get $\mathfrak{r}_k = \{0\}$ for $k \geqq 0$ by induction on $k \geqq 0$. Let $\mathfrak{g} = \mathfrak{r} \oplus \mathfrak{s}$ be a Levi decomposition of $\mathfrak{g}$. With respect to the filtration $\{\mathfrak{f}^p\}_{p\in\mathbb{Z}}$, $\mathfrak{f}^p = \bigoplus_{j\geqq p} \mathfrak{g}_j$, of $\mathfrak{g}$, we take the associated graded Lie algebras of both sides of $\mathfrak{g} = \mathfrak{r} \oplus \mathfrak{s}$. Then, since $\mathfrak{r}$ is graded, we get $\mathfrak{g} = \mathfrak{r} + \widehat{\mathfrak{s}}$. Hence, from $\mathfrak{r}_k = \{0\}$ for $k \geqq -1$, $\widehat{\mathfrak{s}} \supset \mathfrak{g}_{-1} \oplus \mathfrak{f}^0$. Thus, by (i) of (5.3), we obtain $\widehat{\mathfrak{s}} = \mathfrak{g}$. From $\dim \mathfrak{s} = \dim \widehat{\mathfrak{s}}$, it follows that $\mathfrak{g} = \mathfrak{s}$ and $\mathfrak{r} = \{0\}$. Hence $\mathfrak{g}$ is semisimple in this case. In particular $\mathfrak{g}_1 \neq \{0\}$.

In case (2), $\mathfrak{r}$ is a G'_0-invariant graded ideal of $\mathfrak{g}$ containing $\mathfrak{g}_{-1}$. First we shall show that $\mathfrak{g}_1 = \{0\}$ in this case, which implies $\mathfrak{g}_k = \{0\}$ for $k > 1$ by (ii) of (5.3) and $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{g}_0$. Assume the contrary; $\mathfrak{g}_1 \neq \{0\}$. Then we claim

If $\mathfrak{c}$ is a G'_0-invariant graded ideal of $\mathfrak{g}$ containing $\mathfrak{g}_{-1}$, then $[\mathfrak{c}, \mathfrak{c}]$ is also a G'_0-invariant graded ideal of $\mathfrak{g}$ containing $\mathfrak{g}_{-1}$.

In fact, obviously, $[\mathfrak{c}, \mathfrak{c}]$ is a G'_0-invariant graded ideal of $\mathfrak{g}$. By (ii) of (5.3), $[\mathfrak{g}_{-1}, \mathfrak{g}_1] \neq \{0\}$ if $\mathfrak{g}_1 \neq \{0\}$. Since $\mathfrak{c}$ is an ideal satisfying $\mathfrak{c}_{-1} = \mathfrak{g}_{-1}$, we get $\mathfrak{c}_0 \neq \{0\}$. Then, again by (ii) of (5.3), $[\mathfrak{c}_{-1}, \mathfrak{c}_0] \neq \{0\}$. Since $\mathfrak{c}$ is G'_0-invariant, we obtain $[\mathfrak{c}_{-1}, \mathfrak{c}_0] = \mathfrak{g}_{-1}$ by (iii) of (5.3). The above claim implies that $\mathfrak{c}$ cannot be solvable. Therefore $\mathfrak{g}_1 = \{0\}$ in case (2).

Finally we shall show that $\mathfrak{g}_0$ is reductive following Lemma 4.2 of [Go]. We consider the representation ad: $\mathfrak{g}_0 \to \mathfrak{gl}(\mathfrak{g}_{-1})$. Let us take a nonzero $ad(\mathfrak{g}_0)$-irreducible subspace V of $\mathfrak{g}_{-1}$. For a graded Lie algebra automorphism $\varphi \in G'_0$, $\varphi(V)$ is also $\mathrm{ad}(\mathfrak{g}_0)$-irreducible and is isomorphic with V as a $\mathfrak{g}_0$-module. Put $W = \sum_{\varphi\in G'_0} \varphi(V)$. Then W is a non-trivial G'_0-invariant subspace of $\mathfrak{g}_{-1}$. Hence, by (iii) of (5.3), we get $\mathfrak{g}_{-1} = W$.

Thus $\mathfrak{g}_{-1}$ can be written as a direct sum of ad($\mathfrak{g}_0$)-irreducible subspaces. Hence ad: $\mathfrak{g}_0 \to \mathfrak{gl}(\mathfrak{g}_{-1})$ is completely reducible and also faithful by (ii) of (5.3), which shows that $\mathfrak{g}_0$ is reductive (cf. [V, Theorem 3.16.3]).

Next we recall the following Lemma ([K-N, IV, Theorem 4.1], [Gu, Proposition 7.2]), which enables us to determine the structure of the filtered Lie algebra $\mathcal{L} = \{\mathcal{L}^p\}_{p\in\mathbb{Z}}$ in case (1) of Lemma 5.8.

Lemma 5.9. *Let $\mathcal{L} = \{\mathcal{L}^p\}_{p\in\mathbb{Z}}$ be a filtered Lie algebra over $K = \mathbb{R}$ or $\mathbb{C}$, whose associated graded Lie algebra $\mathfrak{g} = \bigoplus_{p=-\mu}^{\nu} \mathfrak{g}_p$ satisfies conditions* (i) *and* (ii) *of* (5.3). *Then if $\mathfrak{g}_0$ contains an element E such that*

$$[E, X] = -X \qquad \textit{for } X \in \mathfrak{g}_{-1},$$

then $\mathcal{L}$ is isomorphic with $\mathfrak{g}$ as a filtered Lie algebra, where the filtration $\{\mathfrak{f}^p\}_{p\in\mathbb{Z}}$ of $\mathfrak{g}$ is given by $\mathfrak{f}^p = \bigoplus_{j\geqq p} \mathfrak{g}_j$ for $p \in \mathbb{Z}$.

Proof. First we note that, for all $p \in \mathbb{Z}$,

$$[E, X] = pX \qquad \text{for } X \in \mathfrak{g}_p. \tag{5.4}$$

In fact, for $p < 0$, this follows from the generating condition (i) of (5.3). For $p \geqq 0$, we have

$$[Y, [E, X]] = [Y, X] + [E, [Y, X]] \qquad \text{for } Y \in \mathfrak{g}_{-1} \text{ and } X \in \mathfrak{g}_p.$$

Then, for $X \in \mathfrak{g}_0$, we get $[Y, [E, X]] = 0$ for all $Y \in \mathfrak{g}_{-1}$. Hence, by (ii) of (5.3), we get $E \in Z(\mathfrak{g}_0)$. Thus, for $p \geqq 0$, (5.4) follows from (ii) of (5.3) by induction on $p \geqq 0$.

Let us take an element $\widehat{E}$ of $\mathcal{L}^0$ such that $\varpi_0(\widehat{E}) = E$. Then, by (5.4), we see that the eigenvalues of ad($\widehat{E}$) are $-\mu, \ldots, \nu$ and $\mathcal{L}^p$ is the direct sum of the primary components $\mathcal{L}_j = \mathrm{Ker}(\mathrm{ad}(\widehat{E}) - j \cdot \mathrm{id})^{n_j}$ of ad($\widehat{E}$) for the eigenvalues $j = p, p+1, \ldots, \nu$. Moreover $[\mathcal{L}_p, \mathcal{L}_q] \subset \mathcal{L}_{p+q}$ (cf. [Hu, §15.1]). Namely the primary decomposition $\mathcal{L} = \bigoplus_{p=-\mu}^{\nu} \mathcal{L}_p$ with respect to ad($\widehat{E}$) gives a gradation of $\mathcal{L}$ such that $\mathcal{L}^p = \bigoplus_{j=p}^{\nu} \mathcal{L}_j$. By definition of the associated graded Lie algebra, it follows that $\mathcal{L} = \bigoplus_{p\in\mathbb{Z}} \mathcal{L}_p$ is isomorphic with $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ as a graded Lie algebra.

Now we have

Theorem 5.10. *Let L be a connected real* (*or complex*) *Lie group acting transitively and effectively on a real* (*or complex*) *manifold M and L' be the isotropy subgroup of L at a point o of M so that $M = L/L'$. Let $\mathcal{L}$ and $\mathcal{L}'$ be the Lie algebras of L and L' respectively. Assume that*

L acts primitively on M and leaves invariant a differential system D on M. Then L is simple. Moreover let us take D to be minimal and introduce a filtration $\{\mathcal{L}^p\}_{p\in\mathbb{Z}}$ of $\mathcal{L}$ induced from D. Assume further $\mathcal{L}^1 \neq \{0\}$. Then the following holds in either real or complex category.

(1) *$\mathcal{L}'$ is a maximal parabolic subalgebra of $\mathcal{L}$.*

(2) *$\mathcal{L} = \{\mathcal{L}^p\}_{p\in\mathbb{Z}}$ is naturally isomorphic with the associated graded Lie algebra $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ as a filtered Lie algebra. In particular $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ is a simple graded Lie algebra such that $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ is fundamental, and the filtration $\{\mathcal{L}^p\}_{p\in\mathbb{Z}}$ of $\mathcal{L}$ is the one uniquely determined by $\mathcal{L}'$ as in* Lemma 3.10.

(3) *M is a covering space over $M_{\mathfrak{g}}$ such that D is the lift of $D_{\mathfrak{g}}$, where $(M_{\mathfrak{g}}, D_{\mathfrak{g}})$ is the standard differential system of type $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$. Especially (M, D) is isomorphic with $(M_{\mathfrak{g}}, D_{\mathfrak{g}})$ always in the complex category and when $\mathcal{L}$ is complex simple in the real category.*

(4) *Except when (M, D) is locally isomorphic with a real or complex standard contact manifold, $\mathcal{A}_x(M, D)$ is isomorphic with $\mathcal{L}$ at each $x \in M$, where $\mathcal{A}_x(M, D)$ denotes the stalk at x of the Lie algebra sheaf $\mathcal{A}(M, D)$ of all infinitesimal automorphisms of (M, D).*

Proof. By Lemma 5.6, $\mathcal{L}$ is simple over $K = \mathbb{R}$ or $\mathbb{C}$, depending on whether we work in the real or complex category. Put $G = \mathrm{Int}(\mathcal{L})$ and let G' be the normalizer of $\mathcal{L}'$ in G:

$$G' = \{\, g \in G \mid \mathrm{Ad}(g)(\mathcal{L}') = \mathcal{L}' \,\}.$$

Since $\mathcal{L}'$ is self-normalizing, G' is the largest Lie subgroup of G with Lie algebra $\mathcal{L}'$. Then, for the adjoint representation $\mathrm{Ad}\colon L \to GL(\mathcal{L})$ (in the category we are working), we have $\mathrm{Ad}(L) = G$ and $\mathrm{Ad}\colon L \to G$ is a covering homomorphism such that $\mathrm{Ad}(L') \subset G'$. Put $\widehat{L}' = \mathrm{Ad}^{-1}(G')$. Then $\widehat{L}'$ is a closed subgroup of L containing L' such that $L/\widehat{L}'$ is diffeomorphic with G/G'. Thus we see that the projection $p\colon M = L/L' \to G/G'$, defined by the following commutative diagram, is a covering map;

$$\begin{array}{ccc} L & \xrightarrow{\mathrm{Ad}} & G \\ \downarrow & & \downarrow \\ M & \xrightarrow{p} & G/G' \end{array} \tag{5.5}$$

Now assume that $\mathcal{L}^1 \neq \{0\}$. Then the assertion (2) follows from Lemmas 5.8 and 5.9 and $\mathcal{L}'$ is a parabolic subalgebra of $\mathcal{L}$. The last statement of (2) is a consequence of (i) of (5.3) and Lemma 3.10. Then, by the construction of the standard differential system $(M_{\mathfrak{g}}, D_{\mathfrak{g}})$ of type $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ (see §4.1) and (5.5), we see that $G/G' = M_{\mathfrak{g}}$ and $D = p_*^{-1}(D_{\mathfrak{g}})$, which shows the first assertion in (3).

Next let us show the second assertion in (3) and the assertion (1). First we treat the case when $\mathcal{L}$ is a simple Lie algebra over $\mathbb{C}$. In this case, G and G' are complex Lie groups. It is well-known (cf. [Wa], [Tt1], [Tk1]) that the complex R-space $M_{\mathfrak{g}} = G/G'$ is simply connected, which implies the second assertion in (3) and that G' is connected. Hence $\mathrm{Ad}(L') = G'$ in this case. Then, by (iii) of (5.3), $\mathfrak{g}_{-1}$ is $\mathrm{ad}(\mathfrak{g}_0)$-irreducible, which implies $\mathcal{L}'$ is maximal parabolic (see Remark 3.7). Moreover, from the assumption that L acts effectively on $M = L/L'$, it is easy to see that $\mathrm{Ad}\colon L \to G$ is an isomorphism such that $\mathrm{Ad}(L') = G'$ (in the category we are working) in this case.

Now we treat the case when $\mathcal{L}$ is a simple Lie algebra over $\mathbb{R}$ such that $\mathbb{C}\mathcal{L}$ is complex simple. We put $\mathbb{C}G = \mathrm{Int}(\mathbb{C}\mathcal{L})$ and

$$\mathbb{C}G' = \{\, g \in \mathbb{C}G \mid \mathrm{Ad}(g)(\mathbb{C}\mathcal{L}') = \mathbb{C}\mathcal{L}' \,\}.$$

Then G is identified with the identity component of the closed real Lie subgroup of $\mathbb{C}G$ consisting of all elements of $\mathbb{C}G$ which commutes with the conjugation with respect to the real form $\mathcal{L}$ of $\mathbb{C}\mathcal{L}$ (cf. [He, Chapter III, Lemma 6.2]). We have $G' = G \cap \mathbb{C}G'$ and $\mathbb{C}G'$ is connected. If there exists a proper subalgebra $\mathfrak{h}$ of $\mathcal{L}$ containing $\mathcal{L}'$ properly, $\mathbb{C}\mathfrak{h}$ is $\mathbb{C}G'$-invariant by the connectivity of $\mathbb{C}G'$. Hence $\mathfrak{h}$ is G'-invariant and also $\mathrm{Ad}(L')$-invariant from $\mathrm{Ad}(L') \subset G'$, which contradicts the assumption that L acts primitively on $M = L/L'$. Therefore $\mathcal{L}'$ is maximal parabolic, which completes the proof of (1).

Finally, observing that $\mathfrak{g}' = \bigoplus_{p\geqq 0} \mathfrak{g}_p$ is not maximal parabolic in case (3) of Theorem 5.3, the assertion (4) follows from (3) and Corollary 5.4.

Remark 5.11. (1) Since the Lie algebera of $\mathrm{Ker}\,\gamma$ coincides with $\mathcal{L}^{\mu}$ in case (1) and vanishes in case (2) of Lemma 5.8, the condition $\mathcal{L}^1 \neq \{0\}$ is equivalent to the nonlinearity of the action: $\mathrm{Ker}\,\gamma$ is nondiscrete. The finite dimensional nonlinear primitive Lie algebras $(\mathcal{L}, \mathcal{L}')$ were first classified by Ochiai [O1], where a primitive subalgebra $\mathcal{L}'$ of $\mathcal{L}$ is, by definition, a maximal subalgebra of $\mathcal{L}$. In the present article, we follow the definition given in [Go] for the primitive action of a connected Lie group L. Fixing a Lie algebra pair $(\mathcal{L}, \mathcal{L}')$, where $\mathcal{L}$ is the Lie algebra of L, this notion of primitivity depends on the choice of L', although if $\mathcal{L}'$ is maximal, L acts primitively on L/L' for any choice of L'. In fact Golubitsky [Go] has shown many examples of $(\mathcal{L}, \mathcal{L}')$ such that $\mathcal{L}'$ is nonmaximal and $L = \mathrm{Int}(\mathcal{L})$ acts primitively on L/L', where L' is the normalizer of $\mathcal{L}'$ in L. Moreover he has shown that this phenomenon (nonmaximality of $\mathcal{L}'$) occurs only when $\mathcal{L}$ is simple and $\mathcal{L}'$ is reductive. For the details, we refer the reader to the original paper [Go].

(2) The nonlinearity of the action: $\mathcal{L}^1 \neq \{0\}$ is necessary in Theorem 5.10 as the following example shows (cf. [D]): We consider the simple Lie algebra $\mathcal{L}$ of type G_2. Let us fix a Cartan subalgebra $\mathfrak{h}$ and simple root system $\Delta = \{\alpha_1, \alpha_2\}$ as in §3.4. Let $\mathcal{L}(\alpha_1)$ be the subalgebra of $\mathcal{L}$ generated by the root vectors for the roots α_2, $-\theta$, $-\alpha_2$ and θ, where $\theta = 3\,\alpha_1 + 2\,\alpha_2$ is the highest root. Then we have

$$\mathcal{L}(\alpha_1) = \mathfrak{g}_{\theta-\alpha_2} \oplus \mathfrak{g}_{\theta} \oplus \mathfrak{g}_{-\alpha_2} \oplus \mathfrak{h} \oplus \mathfrak{g}_{\alpha_2} \oplus \mathfrak{g}_{-\theta} \oplus \mathfrak{g}_{\alpha_2-\theta}.$$

$\mathcal{L}(\alpha_1)$ is a maximal simple subalgebra of type A_2. This is an example of the construction of regular semisimple subalgebras due to Dynkin [D]. Moreover we have an $\mathrm{ad}(\mathcal{L}(\alpha_1))$-irreducible decomposition of $\mathcal{L}$;

$$\mathcal{L} = V_1 \oplus V_2 \oplus \mathcal{L}(\alpha_1),$$

where

$$\begin{cases} V_1 = \mathfrak{g}_{\alpha_1} \oplus \mathfrak{g}_{\alpha_1+\alpha_2} \oplus \mathfrak{g}_{-(2\alpha_1+\alpha_2)}, \\ V_2 = \mathfrak{g}_{-\alpha_1} \oplus \mathfrak{g}_{-(\alpha_1+\alpha_2)} \oplus \mathfrak{g}_{2\alpha_1+\alpha_2}. \end{cases}$$

In fact we have $[\mathcal{L}(\alpha_1), V_i] = V_i$ $(i = 1,\ 2)$, $[V_1, V_1] = V_2$, $[V_2, V_2] = V_1$ and $[V_1, V_2] = \mathcal{L}(\alpha_1)$. Put $L = \mathrm{Int}(\mathcal{L})$ and let L' be the analytic subgroup of L with Lie algebra $\mathcal{L}(\alpha_1)$. Then, since $\mathcal{L}(\alpha_1)$ is a maximal subalgebra, L acts primitively and effectively on L/L' such that the linear isotropy representation is reducible. Since V_1 and V_2 are isomorphic as an $\mathcal{L}(\alpha_1)$-module, there are many minimal $\mathrm{Ad}(L')$-invariant subspaces $\mathcal{L}^{-1}$ containing $\mathcal{L}(\alpha_1)$. However, for any choice of $\mathcal{L}^{-1}$, we see that the associated graded Lie algebra $\mathfrak{g}$ has a following description;

$$\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0,$$

such that $\mathfrak{g}_{-2} = \wedge^2 V$ and $\mathfrak{g}_0 = \mathfrak{sl}(V)$ by putting $V = \mathfrak{g}_{-1}$. Namely $\mathfrak{m} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$ is isomorphic with the universal fundamental graded algebra of second kind with $\dim \mathfrak{g}_{-1} = 3$ (cf. [T2, §3]) and $\mathfrak{g}_0 = \mathfrak{sl}(\mathfrak{g}_{-1}) \subset \mathfrak{gl}(\mathfrak{g}_{-1})$, where $\mathfrak{gl}(\mathfrak{g}_{-1})$ is naturally identified with the Lie algebra of all gradation preserving derivations of $\mathfrak{m}$.

Finally we note that, if we take L' to be the normalizer of $\mathcal{L}(\alpha_1)$ in L, L acts primitively and effectively on L/L' such that the linear isotropy representation is irreducible.

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Tits Metric and Visibility Axiom

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§1. Introduction

An Hadamard manifold H or H^n, i.e., a complete connected simply-connected n-dimensional Riemannian manifold with non-positive sectional curvature is called *a visibility manifold* if the angles at a fixed point subtended by geodesics going far away are arbitrarily small enough no matter how long they are. This condition given by P. Eberlein and B. O'Neill [3] plays basic roles in the study of Hadamard manifolds. They also defined the concept of points at infinity, $H(\infty)$, and it is known that H is a visibility manifold if and only if any different two points at infinity $x_1, x_2 \in H(\infty)$ can be joined by a geodesic of H. This property is called *the axiom* 1. The next two theorems determining this condition are classical:

Theorem 1 ([2],[3]). *If the sectional curvature of H is bounded above by a negative constant, then H is a visibility manifold.*

Theorem 2 ([1]). *In the case of H^n being a surface H^2, it is a visibility surface if and only if for every sector S of H^2, the total curvature of S,*

$$\iint_S K\, dv = -\infty$$

holds, where K is the Gaussian curvature and a sector S is a piece of surface which is cut off by two different rays starting a common point.

Theorem 1 is proved in [2] Lemma 9-10, and also in [3] Proposition 5-9 with an extended form using the idea of curvature order. These proofs in any cases depend essentially on the so-called Gauss-Bonnet theorem on surfaces. Similarly, using the Gauss-Bonnet theorem, we can prove easily Theorem 2 (cf. [1] page 57). Paying attention to the polar coordinate expression around a point in Theorem 2, K. Uesu [5]

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succeeded in generalizing Theorem 2 to higher dimensional case which is stated in terms of the growth rate of the length of Jacobi vector field, and proved directly the relation with the visibility axiom by estimating the angular length:

Theorem 3 ([5]). *H^n is a visibility manifold if and only if there exists a point p of H^n such that for every Lipschitz curve $c\colon [0.1] \to S(p)$ with non-zero length,*

$$\lim_{r\to\infty} \int_0^1 \|Y_t\|'(r)\,dt = \infty$$

holds, where $S(p)$ is the unit tangent sphere at p with natural metric and Y_t is the Jacobi vector field along the ray $[0,\infty) \ni r \mapsto \exp_p rc(t) \in H^n$ such that $Y_t(0) = 0$, $Y_t'(0)$ (= the covariant derivative of Y_t at $r = 0$) $= \dot{c}(t)$ with natural identification.

The next special case is useful.

Theorem 4. *Assume that there exists a point p of H^n such that*

$$\lim_{r\to\infty} \|Y_{v,w}\|'(r) = \infty$$

holds for any orthonormal vectors $\{v, w\}$ of H_p, where $Y_{v,w}$ is the Jacobi vector field along the ray $[0,\infty) \ni r \mapsto \exp_p rv \in H^n$ such that $Y_{v,w}(0) = 0$, $Y_{v,w}'(0) = w$. Then H^n is a visibility manifold.

On the other hand, in [1] lecture 1, W. Ballmann, M. Gromov and V. Schroeder investigated under the transparent idea the fundamental properties of Hadamard manifolds and derived the importance of notion of Tits metric $Td(x_1, x_2)$ in $H(\infty)$. We note in particular that the following view points of their arguments are essential. (1) They discussed elementarily, based only on the convexity of distance function which means $K \leq 0$, the law of cosine in the constant-negatively curved manifold and on the Rauch and Toponogov comparison theorems. (2) As a consequence, they showed that

$$Td(x_1, x_2) = \lim_{r\to\infty} \frac{1}{r}\, d_r(\,\gamma_{x_1}(r), \gamma_{x_2}(r)\,)$$

holds for any $x_1, x_2 \in H(\infty)$, where d_r is the interior metric of the distance sphere $S_r(p)$ of radius r at p and two γ_{x_i} are the rays directed towards x_i with a common starting point p, and that H is a visibility manifold if and only if $Td(x_1, x_2) = \infty$ for all distinct $x_1, x_2 \in H(\infty)$

as well as other equivalent properties. At a glance we see that Theorem 3 is similar to (2) and that Theorem 2 is proved without Gauss-Bonnet theorem as a special case of Theorem 3. As a matter of fact the converse of Theorem 4 is not generally true (cf. example 5-10 of [3]). Theorem 4 is proved directly and more simply than Theorem 3 is, and we may say that the proof of Theorem 4 gives the essential part of one of Theorem 3. Moreover, if H satisfies a sort of "*symmetry*" with respect to directions, (such a manifold is studied in detail and called *model* in [4],) the visibility axiom is determined completely by Theorem 4, namely, we have the following:

Theorem 5. *Let H be an Hadamard manifold with the following condition: there exist a point $p \in H$ and a continuous function $k:[0,\infty) \to [0,\infty)$ such that for every ray $\gamma:[0,\infty) \to H$ starting at $p = \gamma(0)$, $t \geq 0$ and for every section σ containing $\dot\gamma(t)$, the sectional curvature $K_\sigma = -k(t)$ holds. Then H is a visibility manifold if and only if*

$$\lim_{r\to\infty} \|Y_{v,w}\|'(r) = \infty$$

holds for any orthonormal vectors $\{v,w\}$ of H_p.

In this paper we prove these theorems systematically from the point of view of Tits metric, i.e., of the above (1) and (2) without employing Gauss-Bonnet theorem.

§2. Notations and preliminaries

In the following, let H be a complete simply-connected n-dimensional Riemannian manifold with sectional curvature $K \leq 0$ which is called *an Hadamard manifold.* H is diffeomorphic to $\mathbf{R}^n$ and any geodesics of H are minimal. We assume geodesics are always parametrized by arc-lengths if not stated otherwise. A geodesic $\gamma:[0,\infty) \to H$ ($\mathbf{R} \to H$) is called *a ray* (*line* , respectively) and two rays γ_1, γ_2 are said to be *asymptotic* if $\lim_{r\to\infty} d(\gamma_1(r),\gamma_2(r)) < \infty$, or equivalently, if the function $r \mapsto d(\gamma_1(r),\gamma_2(r))$ is monotone non-increasing on $[0,\infty)$, where $d(p_1,p_2)$ is the distance between p_1 and p_2 of H. This is an equivalent relation and the equivalent class of γ is called *a point at infinity* and denoted by $\gamma(\infty)$. The set of all $\gamma(\infty)$ of rays γ is called *the ideal boundary of H* and denoted by $H(\infty)$. For every $p \in H$ and $q \in H$ ($H(\infty)$) there exists a unique geodesic (ray, resp.) γ_{pq} from p to q. For any q_1, $q_2 \in H \cup H(\infty)$ different from $p \in H$, the angle $\angle(\dot\gamma_{pq_1}(0), \dot\gamma_{pq_2}(0))$ is called *the angle subtended by q_1, q_2 at p* and denoted by $\angle_p(q_1,q_2)$.

An Hadamard manifold H is said to satisfy *the visibility axiom* or simply to be *a visibility manifold* if for a point $p \in H$ and any $\varepsilon > 0$ there exists $r = r(p,\varepsilon) > 0$ such that for every geodesic $\gamma: [a,b] \to H$ satisfying $d(p,\gamma) \geq r$, $\angle_p(\gamma(a),\gamma(b)) \leq \varepsilon$ holds. It must be conscious that the choice of $p \in H$ in this definition may be arbitrarily fixed and moreover this property is equivalent to *the axiom* 1 , that is to say, for any distinct $x_1, x_2 \in H(\infty)$ there exists a line γ in H such that $\gamma(-\infty) := \gamma_-(\infty) = x_1$ and $\gamma(\infty) = x_2$, γ_- being the line with $\gamma_-(r) := \gamma(-r)$. For a point $p \in H$ and $r > 0$, the distance sphere centered at p with radius r, $S_r(p) := \{ q \in H \mid d(p,q) = r \}$ is a compact hypersurface of H. Let d_r be the distance function of $S_r(p)$ naturally induced by the metric. For each $x_1, x_2 \in H(\infty)$ the function $(0,\infty) \ni r \mapsto \frac{1}{r} d_r(\gamma_{px_1}(r), \gamma_{px_2}(r))$ is monotone non-decreasing and we call $Td(x_1,x_2) := \lim_{r\to\infty} \frac{1}{r} d_r(\gamma_{px_1}(r), \gamma_{px_2}(r)) \in \mathbf{R} \cup \{\infty\}$ *the Tits distance.* It must be conscious too that this definition does not depend on the choosed point p and that H is a visibility manifold if and only if $Td(x_1,x_2) = \infty$ holds for every distinct $x_1, x_2 \in H(\infty)$ or equivalently $Td(x_1,x_2) \geq a$, $a > 0$ being a constant. (cf. [1], [3])

For every $v \in S(p) := \{ v \in H_p \mid \|v\| = 1 \}$, $\gamma_v(r) := \exp_p rv$ is the ray of initial vector $\dot\gamma_v(0) = v$ where $r \geq 0$. For any $w \in H_p$ orthogonal to v, let $Y_{v,w}$ be the Jacobi vector field along γ_v such that $Y_{v,w}(0) = 0$, $Y'_{v,w}(0) = w$ which is expressed by

$$Y_{v,w}(r) = \exp_{p*} r\mathbf{I}_{rv} w$$

for any $r \geq 0$ where for each $u \in H_p$, $\mathbf{I}_u : H_p \to (H_p)_u$ is the natural isomorphism defined by $\mathbf{I}_u w := \dot c_{u,w}(0)$, $c_{u,w}(t) := u + tw$ for any $t \in \mathbf{R}$ and $w \in H_p$. According to $K \leq 0$, $\|Y_{v,w}\|'' \geq 0$ holds, namely, the function $\|Y_{v,w}\|$ is convex and $\|Y_{v,w}\|'$ is monotone non-decreasing on $[0,\infty)$. In particular, if H^n is of constant curvature $K = -c^2$, $c > 0$, $Y_{v,w}$ is expressed by $Y_{v,w}(r) = \frac{1}{c}\sinh(cr) \cdot X(r)$ where X is the parallel vector field along γ_v with $X(0) = w$.

For any $p \in H$ we set $G_p := \{ \sigma \mid \sigma$ is a 2-dimensional vector subspace of $H_p \}$. The well-known Rauch comparison theorem means the following: Let v, $w \in {H^n}_p$ be orthonormal and we take another triple $\{\tilde v, \tilde w, \tilde H^{\tilde n}\}$. Denoting the corresponding terms by $\sim$, we assume $n \leq \tilde n$ and $K_\sigma \leq K_{\tilde\sigma}$ for any $r \geq 0$, $\dot\gamma_v(r) \in \sigma \in G_{\gamma_v(r)}$ and $\dot{\tilde\gamma}_{\tilde v}(r) \in \tilde\sigma \in G_{\tilde\gamma_{\tilde v}(r)}$. Then it follows that

$$\|Y_{v,w}\| \geq \|\tilde Y_{\tilde v,\tilde w}\|, \quad \frac{\|Y_{v,w}\|'}{\|Y_{v,w}\|} \geq \frac{\|\tilde Y_{\tilde v,\tilde w}\|'}{\|\tilde Y_{\tilde v,\tilde w}\|}$$

on $(0,\infty)$ and

$$\frac{\|Y_{v,w}\|(r_1)}{\|Y_{v,w}\|(r_2)} \le \frac{\|\tilde{Y}_{\tilde{v},\tilde{w}}\|(r_1)}{\|\tilde{Y}_{\tilde{v},\tilde{w}}\|(r_2)}$$

for all $r_2 > r_1 \ge 0$. (cf. [7])

Given three distinct points $p_i \in H$ and geodesics $\gamma_i\colon [0, l_i] \to H$ ($i = 0,1,2$) such that $\gamma_i(l_i) = \gamma_{i+1}(0) = p_{i+2}$ (mod 3), the triple (p_0, p_1, p_2) or $(\gamma_0, \gamma_1, \gamma_2)$ is said to form *a geodesic triangle.* For each $i = 0,1,2$, $\theta_i := \pi - \angle(\dot\gamma_{i+1}(l_{i+1}), \dot\gamma_{i+2}(0))$ (mod 3) is called *the angle at* p_i. In the Hadamard manifold $H^n(-c^2)$ of constant negative curvature $K = -c^2$, $c > 0$, the law of cosine,

$$\cosh(cl_0) = \cosh(cl_1)\cdot\cosh(cl_2) - \sinh(cl_1)\cdot\sinh(cl_2)\cdot\cos\theta_0$$

holds for every geodesic triangle $(\gamma_0, \gamma_1, \gamma_2)$, and if $l_1 = l_2$, then we get

$$\sinh\frac{cl_0}{2} = \sinh(cl_1)\cdot\sin\frac{\theta_0}{2}.$$

§3. Proofs of Theorems 4, 1 and 5

Proof of Theorem 4. We show that

$$Td(x_1, x_2) = \lim_{r\to\infty}\frac{1}{r}d_r(\gamma_{px_1}(r), \gamma_{px_2}(r)) = \infty$$

holds for any $x_1, x_2 \in H(\infty)$, $x_1 \ne x_2$. Let $A := \{\,(v,w) \in H_p \times H_p \mid \|v\| = \|w\| = 1,\ \langle v,w\rangle = 0\,\}$. By the assumption, for any $(v,w) \in A$ and $M > 0$ there exists a $r(v,w) > 0$ such that $r \ge r(v,w)$ implies $\|Y_{v,w}\|'(r) > M$. Since $\|Y_{v,w}\|'(r)$ is continuous as to $(v,w) \in A$ and monotone non-decreasing relative to $r \in [0,\infty)$, there exists a neighbourhood $U = U(v,w)$ of (v,w) in A such that for any $(v',w') \in U$ and $r \ge r(v,w)$,

$$\|Y_{v',w'}\|'(r) \ge \|Y_{v',w'}\|'(\,r(v,w)\,) > M$$

holds. There exists a finite covering $\bigcup_{i=1}^{k} U(v_i, w_i) \supset A$ because A is compact. So we take $r_0 := \max\{r(v_i,w_i) \mid i = 1,\dots,k\} > 0$. Hence for any $(v,w) \in A$ and $r \ge r_0$, we have

$$\|Y_{v,w}\|(r) \ge \|Y_{v,w}\|(r) - \|Y_{v,w}\|(r_0) \ge (r-r_0)\|Y_{v,w}\|'(r_0) \ge (r-r_0)M.$$

In every distance sphere $S_r(p)$ we take a minimal geodesic $c_r\colon [0,1] \to S_r(p)$ from $\gamma_{px_1}(r)$ to $\gamma_{px_2}(r)$ which is expressed by $c_r(t) = \exp_p r\tilde{c}_r(t)$

for all $t \in [0,1]$ where $\tilde{c}_r : [0,1] \to S(p) \subset H_p$ is a differentiable curve. Accordingly we have

$$d_r(\gamma_{px_1}(r), \gamma_{px_2}(r)) = L(c_r) = \int_0^1 \|\dot{c}_r\|(t)\, dt$$

and

$$\begin{aligned} \dot{c}_r(t) &= \exp_{p*}(r\tilde{c}_r)'(t) \\ &= \exp_{p*} r\mathbf{I}_{r\tilde{c}_r(t)} \cdot {\mathbf{I}_{\tilde{c}_r(t)}}^{-1} \cdot \dot{\tilde{c}}_r(t) \\ &= Y_{\tilde{c}_r(t), w_r(t)}(r) \\ &= \|\dot{\tilde{c}}_r\|(t) \cdot Y_{\tilde{c}_r(t), w_r(t)/\|w_r(t)\|}(r) \end{aligned}$$

where $w_r(t) := {\mathbf{I}_{\tilde{c}_r(t)}}^{-1} \cdot \dot{\tilde{c}}_r(t) \in H_p$. Therefore, for any $r > 2r_0$ we have

$$\begin{aligned} \frac{1}{r} d_r(\gamma_{px_1}(r), \gamma_{px_2}(r)) &\geq \frac{1}{r} \int_0^1 \|\dot{\tilde{c}}_r\|(t) \cdot (r - r_0) M\, dt \\ &\geq M\left(1 - \frac{r_0}{r}\right) L(\tilde{c}_r) \\ &\geq \frac{1}{2} M \angle_p(x_1, x_2) \end{aligned}$$

and get $Td(x_1, x_2) = \infty$.

Proof of Theorem 1. We prove Theorem 1 using Theorem 4. We assume $K \leq -c^2$ for a positive constant c and have only to show $\lim_{r\to\infty} \|Y_{v.w}\|'(r) = \infty$ for an arbitrary orthonormal pair $\{v, w\}$ of TH. We take a geodesic $\tilde{\gamma}$ and a Jacobi vector field $\tilde{Y}$ along $\tilde{\gamma}$ in the Hadamard manifold $H^n(-c^2)$ of constant negative curvature $-c^2$ such that $\langle \tilde{Y}, \dot{\tilde{\gamma}} \rangle = 0$, $\tilde{Y}(0) = 0$ and $\|\tilde{Y}'(0)\| = 1$. Then we have the expression $\tilde{Y}(r) = \frac{1}{c} \sinh(cr) \cdot \tilde{X}(r)$ where $\tilde{X}$ is the parallel vector field along $\tilde{\gamma}$ with $\tilde{X}(0) = \tilde{Y}'(0)$. Hence, applying Rauch comparison theorem, we get

$$\|Y_{v,w}\|'(r) \geq \|\tilde{Y}\|'(r) \frac{\|Y_{v,w}\|(r)}{\|\tilde{Y}\|(r)} \geq \|\tilde{Y}\|'(r) = \cosh(cr)$$

for any $r > 0$, and consequently

$$\lim_{r\to\infty} \|Y_{v,w}\|'(r) = \infty.$$

Remark (1). It is also possible to prove Theorem 1 directly by the law of cosine without using Theorem 4 such as following: We take a fixed point $p \in H$ and any different $x_1, x_2 \in H(\infty)$. Using Rauch-Alexandrov comparison theorem (cf. [6]) and the law of cosine, we have

$$d(\gamma_{px_1}(r), \gamma_{px_2}(r)) \geq \frac{2}{c}\sinh^{-1}\left(\sin\frac{1}{2}\angle_p(x_1,x_2)\cdot\sinh(cr)\right)$$

hence

$$\begin{aligned} Td(x_1,x_2) &= \lim_{r\to\infty}\frac{1}{r}d_r(\gamma_{px_1}(r),\gamma_{px_2}(r)) \\ &\geq \lim_{r\to\infty}\frac{1}{r}d(\gamma_{px_1}(r),\gamma_{px_2}(r)) \\ &\geq \lim_{r\to\infty}\frac{2}{cr}\sinh^{-1}\left(\sin\frac{1}{2}\angle_p(x_1,x_2)\cdot\sinh(cr)\right) \\ &= \lim_{r\to\infty}\frac{2}{c}\frac{\sin\frac{1}{2}\angle_p(x_1,x_2)\cdot c\cdot\cosh(cr)}{\left(1+\left(\sin\frac{1}{2}\angle_p(x_1,x_2)\cdot\sinh(cr)\right)^2\right)^{\frac{1}{2}}} \\ &= 2. \end{aligned}$$

This implies that H is a visibility manifold.

Remark (2). In general, an Hadamard manifold with smaller curvature than one of a visibility manifold satisfies the visibility axiom too. That is to say more precisely, the following assertion is obvious by Rauch comparison theorem.

Let H^n, $\tilde{H}^{\tilde{n}}$ be two Hadamard manifolds, $p \in H^n$, $\tilde{p} \in \tilde{H}^{\tilde{n}}$, $n \leq \tilde{n}$ and $\iota\colon {H^n}_p \longrightarrow {\tilde{H}^{\tilde{n}}}_{\tilde{p}}$ an isometric isomorphism. We assume that $K_\sigma \leq K_{\tilde{\sigma}}$ holds for every ray $\gamma\colon [0,\infty) \longrightarrow H$ starting at p, $r \geq 0$ and every σ, $\tilde{\sigma}$ such that $\dot{\gamma}(r) \in \sigma \in G_{\gamma(r)}$, $\dot{\tilde{\gamma}}(r) \in \tilde{\sigma} \in G_{\tilde{\gamma}(r)}$, where $\tilde{\gamma}\colon [0,\infty) \longrightarrow \tilde{H}^{\tilde{n}}$ is the ray starting at $\tilde{p}$ with $\dot{\tilde{\gamma}}(0) = \iota\dot{\gamma}(0)$. Then if $\tilde{H}$ is a visibility manifold, H is so too.

Proof of Theorem 5. We prove the converse of Theorem 4, that is, $\lim_{r\to\infty} \|Y_{v,w}\|'(r) = \infty$ for any two orthonormal vectors $\{v, w\}$ of H_p under the visibility condition. We take the curve $c\colon [0, \pi/2] \to S(p)$ and the variation $V: [0, \infty) \times [0, \pi/2] \to H$ by $c(t) := v \cos t + w \sin t$, $V(r, t) := \exp_p rc(t)$ for every $r \in [0, \infty)$ and $t \in [0, \pi/2]$. Then $V_* \frac{\partial}{\partial t}|_{(r,t)} = Y_{c(t),w(t)}(r)$ holds where $w(t) := -v \sin t + w \cos t = \mathbf{I}_{c(t)}{}^{-1} \dot{c}(t)$. Since clearly $Y_{v,w}(0) = 0 = Y_{c(t),w(t)}(0)$ and $\|Y'_{v,w}\|(0) = 1 = \|Y'_{c(t),w(t)}\|(0)$ are satisfied, we can apply Rauch comparison theorem to $Y_{v,w} = Y_{c(0),w(0)}$ and $Y_{c(t),w(t)}$ owing to the assumption on curvatures and get $\|Y_{v,w}\|(r) = \|Y_{c(t),w(t)}\|(r)$ for every $r \geq 0$ and $t \in [0, \pi/2]$. So we have

$$\|Y_{v,w}\|'(r) \geq \frac{1}{r}\|Y_{v,w}\|(r) = \frac{2}{\pi r}\int_0^{\frac{\pi}{2}} \|Y_{c(t),w(t)}\|(r)\,dt = \frac{2}{\pi r}L(\,V(r,\cdot)\,)$$

$$\geq \frac{2}{\pi r}d_r(\,V(r,0),\,V(r,\frac{\pi}{2})\,) = \frac{2}{\pi r}d_r(\gamma_v(r), \gamma_w(r)),$$

and $\lim_{r\to\infty} \|Y_{v,w}\|'(r) = \infty$ because $Td(\gamma_v(\infty), \gamma_w(\infty)) = \infty$.

§4. Proofs of Theorems 3 and 2

Proof of Theorem 3. We assume H is a visibility manifold and $p \in H$. For any Lipschitz curve $c\colon [0,1] \to S(p)$ with $L(c) \neq 0$ we take $0 \leq t_1 < t_2 \leq 1$ such that $c(t_1) \neq c(t_2)$. Since $\|Y_t\|'(r) \geq \frac{1}{r}\|Y_t\|(r)$ holds for every $r > 0$ and for almost all $t \in [0,1]$ because of $K \leq 0$, we have

$$\int_0^1 \|Y_t\|'(r)\,dt \geq \int_{t_1}^{t_2} \frac{1}{r}\|Y_t\|(r)\,dt$$

$$= \frac{1}{r}L(\,\exp_p rc(\cdot)|_{[t_1,t_2]}\,)$$

$$\geq \frac{1}{r}d_r(\gamma_{c(t_1)}(r), \gamma_{c(t_2)}(r))$$

and

$$\lim_{r\to\infty}\int_0^1 \|Y_t\|'(r)\,dt \geq Td(\gamma_{c(t_1)}(\infty), \gamma_{c(t_2)}(\infty))$$

$$= \infty.$$

Conversely, assume there exist two different x_1, $x_2 \in H(\infty)$ such that $Td(x_1, x_2) = \lim_{r\to\infty} \frac{1}{r} d_r(\gamma_{px_1}(r), \gamma_{px_2}(r)) < \infty$. We take a divergent monotone-increasing sequence (r_k) and a family of minimal curves $c_k\colon [0,1] \to S_{r_k}(p)$ from $\gamma_{px_1}(r_k)$ to $\gamma_{px_2}(r_k)$ in $S_{r_k}(p)$, where we parametrize so that each $\tilde{c}_k := \frac{1}{r_k} \exp_p{}^{-1} c_k : [0,1] \to S(p)$ is proportional to arc-length. Then we have

$$\dot{c}_k(t) = \exp_{p*} r_k \mathbf{I}_{r_k \tilde{c}_k(t)} \cdot \mathbf{I}_{\tilde{c}_k(t)}{}^{-1} \cdot \dot{\tilde{c}}_k(t) = Y_{\tilde{c}_k(t), w_k(t)}(r_k)$$

for any $t \in [0,1]$ where $w_k(t) := \mathbf{I}_{\tilde{c}_k(t)}{}^{-1} \cdot \dot{\tilde{c}}_k(t)$, and so

$$\|\dot{c}_k(t)\| \geq r_k \|w_k(t)\| = r_k \|\dot{\tilde{c}}_k(t)\|$$

by Rauch theorem, hence

$$\begin{aligned} L(\tilde{c}_k) = \int_0^1 \|\dot{\tilde{c}}_k\|(t)\, dt \leq \frac{1}{r_k} L(c_k) = \frac{1}{r_k} d_{r_k}(\gamma_{px_1}(r_k), \gamma_{px_2}(r_k)) \\ \leq Td(x_1, x_2) < \infty. \end{aligned}$$

Applying the Ascoli-Arzelà theorem we get a convergent subsequence of $(\tilde{c}_k)$ and also denote it by $(\tilde{c}_k)$ for simplicity. The limit curve $\tilde{c}_0 := \lim_{k\to\infty} \tilde{c}_k : [0,1] \to S(p)$ is a Lipschitz curve with $L(\tilde{c}_0) \neq 0$ as $x_1 \neq x_2$, whose convergence is uniform. Since $\lim_{r\to\infty} \int_0^1 \|Y_t\|'(r)\, dt = \infty$ holds by the assumption with $Y_t(r) := Y_{\tilde{c}_0(t), w(t)}(r)$ and $w(t) := \mathbf{I}_{\tilde{c}_0(t)}{}^{-1} \cdot \dot{\tilde{c}}_0(t)$, for a constant $M = 6Td(x_1, x_2) > 0$ there exists $r_0 > 0$ such that $r \geq r_0$ implies $\int_0^1 \|Y_t\|'(r)\, dt > M$.

Set $c_{k,r}(t) := \exp_p r\tilde{c}_k(t)$ for every $k \in \mathbf{N} \cup \{0\}$, $r \geq 0$ and $t \in [0,1]$, so it follows $c_k = c_{k,r_k}$ for every $k \in \mathbf{N}$. We choose $k_0 \in \mathbf{N}$ such that $r_{k_0} > 2r_0$. Since $L(c_{0,r}) \leq \liminf_{k\to\infty} L(c_{k,r})$ holds for each fixed $r \geq 0$, for $\varepsilon := \frac{1}{6} M r_{k_0} > 0$ there exists $k_1 = k_1(r_{k_0}) > k_0$ such that $k > k_1$ implies $L(c_{0,r_{k_0}}) - \varepsilon < L(c_{k,r_{k_0}})$. Using comparisom theorem, we have

$$\begin{aligned} L(c_{k,r_k}) &= \int_0^1 \|Y_{\tilde{c}_k(t), w_k(t)}\|(r_k)\, dt \\ &\geq \int_0^1 \|Y_{\tilde{c}_k(t), w_k(t)}\|(r_{k_0}) \frac{r_k}{r_{k_0}}\, dt \\ &= \frac{r_k}{r_{k_0}} L(c_{k,r_{k_0}}) \end{aligned}$$

for any $k > k_1$ and get finally

$$\begin{aligned} Td(x_1, x_2) &\geq \frac{1}{r_k} d_{r_k}(\gamma_{px_1}(r_k), \gamma_{px_2}(r_k)) = \frac{1}{r_k} L(c_k) = \frac{1}{r_k} L(c_{k,r_k}) \\ &\geq \frac{1}{r_{k_0}} L(c_{k,r_{k_0}}) \\ &> \frac{1}{r_{k_0}} (L(c_{0,r_{k_0}}) - \varepsilon) = \frac{1}{r_{k_0}} \left(\int_0^1 \|Y_t\|(r_{k_0})\, dt - \varepsilon \right) \\ &\geq \frac{1}{r_{k_0}} \left(\int_0^1 \left((r_{k_0} - r_0)\|Y_t\|'(r_0) + \|Y_t\|(r_0) \right) dt - \varepsilon \right) \\ &> \frac{1}{r_{k_0}} \left((r_{k_0} - r_0) M - \varepsilon \right) \\ &> \frac{1}{2} M - \frac{1}{6} M = \frac{1}{3} M \\ &= 2Td(x_1, x_2) > 0 \end{aligned}$$

which contradicts.

Proof of Theorem 2. Let $H^2 = (\mathbf{R}^2,\, ds^2 = dr^2 + f(r,\theta)^2 d\theta^2)$ be a geodesic polar coordinate around p, that is, the differentiable function $f : [0,\infty) \times S^1 \to [0,\infty)$ be assumed to satisfy $f(0,\theta) = 0$, $f_r(0,\theta) = 1$ and $f_{rr}(r,\theta) \geq 0$. Then for every sector $S = \{ (r,\theta) \mid r \geq 0,\ a \leq \theta \leq b \}$ $(0 \leq a < b \leq 2\pi)$ with vertex p, we have

$$\begin{aligned} \iint_S K\, dv &= \lim_{r\to\infty} \iint_{S_r(p)\cap S} K\, dv \\ &= \lim_{r\to\infty} \int_a^b d\theta \int_0^r \frac{f_{rr}}{-f} \cdot f|_{(t,\theta)}\, dt \\ &= -\lim_{r\to\infty} \int_a^b f_r(r,\theta)\, d\theta + (b-a). \end{aligned}$$

We denote by γ_θ the ray from p with a direction θ and by X_θ the parallel vector field along γ_θ with $\|X_\theta\| = 1$ and $\langle X_\theta, \dot\gamma_\theta \rangle = 0$, then we have $Y_{\dot\gamma_\theta(0), X_\theta(0)}(r) = f(r,\theta) \cdot X_\theta(r)$ for all $r \geq 0$ and $\theta \in S^1$, hence $f_r(r,\theta) = \|Y_{\dot\gamma_\theta(0), X_\theta(0)}\|'(r)$. This fact and Theorem 3 imply Theorem 2.

Remark. If H is a surface H^2 in Theorem 5, namely, if the function $f(r,\theta)$ with $f(0,\theta) = 0$, $f_r(0,\theta) = 1$ and $f_{rr}(r,\theta) \geq 0$ which is adopted in the above proof depends only on r, then H^2 is a visibility surface if and only if $\lim_{r\to\infty} f'(r) = \infty$, and $K < 0$ is equivalent to $f'' > 0$ on $(0, \infty)$ and $\lim_{r\downarrow 0} \dfrac{f''(r)}{f(r)} \in (0, \infty)$. Therefore we gain easily an example of Hadamard surface with $K < 0$ which does not satisfy the visibility axiom. For example, for given $c > 0$ and $\varepsilon_2 > \varepsilon_1 > 0$ we are able to construct a C^∞-function $f(r)$ so as to satisfy $f(r) = \frac{1}{c}\sinh(cr)$ on $[0, \varepsilon_1]$, $f(r) = M_1 + M_2 \int_{\varepsilon_2}^{r} \tan^{-1} t\, dt$ on $[\varepsilon_2, \infty)$ and $f'' > 0$ on $[\varepsilon_1, \varepsilon_2]$ by choosing M_1, $M_2 > 0$ so large enough.

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