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Analysis on Anticommuting Self-Adjoint Operators

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Dedicated to Professor ShigeToshi Kuroda on the occasion of his 60th birthday

I. Introduction

Two *bounded* linear operators A and B in a Hilbert space $\mathcal{H}$ are said to anticommute if $AB + BA = 0$. However, if A and B are *unbounded*, then this definition of anticommutativity does not work, because $AB + BA$ may not make sense on any vector in $\mathcal{H}$.

A proper notion of anticommutativity of (*unbounded*) self-adjoint operators was given by Vasilescu [23]. Samoilenko [21] and Pedersen [20] gave several equivalent characterizations of the anticommutativity and discussed some aspects of anticommuting self-adjoint operators.

Following [20], we say that *two self-adjoint operators A and B in a Hilbert space anticommute if*

$$e^{itA}B \subset Be^{-itA}$$

for all $t \in \mathbb{R}$. We remark that this definition is symmetric in A and B [20] and gives an extension of the notion of anticommutativity of bounded operators mentioned above.

Families of anticommuting self-adjoint operators are not only interesting in its own right (in particular, from representation theoretical points of view), but also may be important in applications (e.g., analysis of operators of Dirac's type [3, 5-8, 13, 16] and supersymmetric quantum theory [1, 2, 4, 9, 15, 17, 18]).

In [10, 11] the present author has developed analysis on anticommuting self-adjoint operators; The paper [10] is concerned with algebraic properties of the partial isometries assoicated with anticommuting self-adjoint operators and analysis of the sum of two anticommuting

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self-adjoint operators, while the paper [11] gives a characterization of the anticommutativity of self-adjoint operators in connection with Clifford algebra and discusses some consequences of it, one of which can be applied to the self-adjointness problem of some classes of operators of Dirac's type in both finite and infinite dimensions. In this paper we summarize the main results obtained in [10, 11].

II. Product of two anticommuting self-adjoint operators

In this section we describe some results on a product of two anticommuting self-adjoint operators. We denote by $D(A)$ the domain of the operator A.

For the reader's convenience, we first summarize as a lemma some known facts on anticommuting self-adjoint operators.

Let A be a self-adjoint operator in a Hilbert space with the spectral family $\{E_A(\lambda)|\lambda \in \mathbb{R}\}$. Then the polar decomposition of A is given by

$$A = U_A|A|$$

with

$$U_A = 1 - E_A(0) - E_A(-0),$$

see, e.g., [19, p.358]. We call U_A the partial isometry associated with the self-adjoint operator A.

Lemma 2.1 [20, 23]. *Let A and B be anticommuting self-adjoint operators in a Hibert space. Then the following* (i)–(vii) *hold:*

(i) $U_B A \subset -AU_B$ *and* $U_A B \subset -BU_A$.
(ii) $U_B|A| \subset |A|U_B$ *and* $U_A|B| \subset |B|U_A$.
(iii) $|A|$ *and* $|B|$ *commute.*
(iv) $U_A U_B = -U_B U_A$.
(v) A *and* $|B|$ *commute and* B *and* $|A|$ *commute.*
(vi) $D(A) \cap D(B) \cap D(AB) = D(A) \cap D(B) \cap D(BA)$ *and*

$$(AB + BA)f = 0, \quad f \in D(A) \cap D(B) \cap D(AB).$$

(vii) $A + B$ *is self-adjoint.*

For two anticommuting self-adjoint operators A and B in a Hilbert space, we consider the product

$$C_0(A, B) = iAB$$

with $D(C_0(A,B)) = D(A) \cap D(B) \cap D(AB)$. It follows from Lemma 2.1(vi) that

$$D(C_0(A,B)) = D(C_0(B,A)),$$

and

$$[C_0(A,B) + C_0(B,A)]f = 0, \quad f \in D(C_0(A,B)).$$

In particular, $C_0(A,B)$ is symmetric.

Theorem 2.2 [10]. *Let A and B be anticommuting self-adjoint operators in a Hilbert space. Then*

(i) *$C_0(A,B)$ is essentially self-adjoint.*
(ii) *Let $C(A,B)$ be the closure of $C_0(A,B)$. Then*

$$C(A,B) = -C(B,A).$$

(iii) *The operator $C(A,B)$ is essentially self-adjoint on every core for $A^2 + B^2$.*

Remark. We can find a dense domain $\mathcal{D}$ on which $C_0(A,B)^k$ is essentially self-adjoint for all $k \in \mathbb{N}$ [10, Theorem 2.3].

By Lemma 2.1 (vi) we have

$$AC_0(A,B) + C_0(A,B)A = 0, \quad BC_0(A,B) + C_0(A,B)B = 0,$$

on a suitable domain, respectively. Hence $C(A,B)$ may have a chance to anticommute with A and B. In fact, the following theorem holds.

Theorem 2.3 [10]. *The operator $C(A,B)$ anticommutes with A, B, and $A+B$.*

III. Algebraic properties of the partial isometries associated with anticommuting self-adjoint operators

Theorem 2.3 shows that, given two anticommuting self-adjoint operators A and B in a Hilbert space, we have a triple $\{A, B, C(A,B)\}$ of mutually anticommuting self-adjoint operators. It is interesting to investigate structures of this triple. We do it by analyzing the algebraic structure of the partial isometries of U_A, U_B, and $U_{C(A,B)}$. Thus our first task is to compute products of these partial isometries. A key tool for this purpose is the following formula for the partial isometry associated with a self-adjoint operator.

Lemma 3.1. *Let A be a self-adjoint operator. Then*

$$U_A = s - \lim_{\epsilon \to +0} A(A^2 + \epsilon)^{-1/2}.$$

Proof. This can be proven by the functional calculus for self-adjoint operators. For the details, see [10]. Q.E.D.

Remark. Let A be a self-adjoint operator and P_A be the orthogonal projection onto $(\mathrm{Ker}\ A)^{\perp}$. Then:

$$U_A = \mathrm{sgn}(A) P_A,$$

where $\mathrm{sgn}(\lambda) = \lambda/|\lambda|, \lambda \in \mathbb{R} \setminus \{0\}$.

We also note the following fact.

Lemma 3.2 [10]. *Let A and B be anticommuting self-adjoint operators in a Hibert space. Then:*

(i) *P_A and P_B commute.*
(ii) *P_A and U_B commute, and P_B and U_A commute.*

Using Lemmas 3.1, 3.2 and some technical facts, we can obtain the following results .

Theorem 3.3 [10]. *Let A and B be anticommuting self-adjoint operators in a Hilbert space. Then:*

$$\begin{aligned} U_A U_B &= -iU_{C(A,B)}, \\ U_{C(A,B)} U_A &= -iP_A U_B = -iU_B P_A, \\ U_{C(A,B)} U_B &= iP_B U_A = iU_A P_B. \end{aligned}$$

In the rest of this section, we assume that A *and* B *are anticommuting self-adjoint operators in a Hilbert space* $\mathcal{H}$. To rewrite the formulas given in Theorem 3.3 as commutation relations, we introduce

$$\begin{aligned} &X_1 = i\frac{U_A}{2},\ X_2 = i\frac{U_B}{2},\ X_3 = i\frac{U_{C(A,B)}}{2}, \\ &Y_1 = 1,\ Y_2 = P_B,\ Y_3 = P_A,\ Y_4 = P_A P_B. \end{aligned}$$

For bounded linear operators X, Y on $\mathcal{H}$, we define

$$[X, Y] = XY - YX.$$

Theorem 3.4 [10]. *The following commutation relations hold:*

$$[X_j, X_k] = \sum_{\ell=1}^{3} \epsilon_{jk\ell} X_\ell Y_j, \quad j,k = 1,2,3,$$
$$[X_j, Y_m] = [Y_m, Y_n] = 0, \quad j = 1,2,3, \; m,n = 1,2,3,4,$$

where $\epsilon_{jk\ell}$ *is the Levi-Civita symbol with* $\epsilon_{123} = 1$.

Proof (*Outline*). This follows from Theorem 3.3, Lemma 3.2, Lemma 2.1 (iv), and the fact that $P_A^2 = P_A$. Q.E.D.

The vector space of all bounded linear operators on $\mathcal{H}$ is a Lie algebra with the Lie bracket $[\,\cdot\,,\,\cdot\,]$. We denote it by $\mathfrak{L}(\mathcal{H})$. Theorem 3.4 implies the following result.

Theorem 3.5 [10]. *Let* $\mathfrak{M} \subset \mathfrak{L}(\mathcal{H})$ *be the subspace spanned by* $X_k Y_m, k = 1,2,3, m = 1,\ 2,3,4$. *Then* $\mathfrak{M}$ *is a Lie subalgebra of* $\mathfrak{L}(\mathcal{H})$.

As is well-known, the Lie algebra $\mathfrak{su}\ (2,\mathbb{C})$ of the special unitary group $SU(2)$ is the set of 2×2 complex skew-Hermitian matrices of trace zero and has a basis $\{e_j\}_{j=1}^3$ which satisfy the commutation relations

$$[e_j, e_k] = \sum_{\ell=1}^{3} \epsilon_{jk\ell} e_\ell, \quad j,k = 1,2,3.$$

We define a linear map $\varrho : \mathfrak{su}\ (2,\mathbb{C}) \to \mathfrak{L}(\mathcal{H})$ by

$$\varrho(\sum_{j=1}^{3} \alpha_j e_j) = \sum_{j=1}^{3} \alpha_j X_j, \quad \alpha_j \in \mathbb{C}, j = 1,2,3.$$

Theorem 3.6 [10]. *Suppose that* A *and* B *are injective. Then* ϱ *is an isomorphism between* $\mathfrak{su}\ (2,\mathbb{C})$ *and* $\mathfrak{M}$.

Proof. We need only to note that, in the present case, $P_A = P_B = 1$. Q.E.D.

In the case where A and B are not necessarily injective, we can proceed as follows. Let

$$\mathcal{H}_0 = (\text{Ker } A + \text{Ker } B)^\perp$$

and define the operators A_0 and B_0 acting in $\mathcal{H}_0$ by

$$A_0 f = Af, \quad f \in D(A_0),$$
$$B_0 f = Bf, \quad f \in D(B_0)$$

with

$$D(A_0) = D(A) \cap \mathcal{H}_0, \quad D(B_0) = D(B) \cap \mathcal{H}_0.$$

It has been proven in [20] that A_0 and B_0 are injective, self-adjoint, and anticommute. We define the operators

$$X_1^{(0)} = i\frac{U_{A_0}}{2}, \quad X_2^{(0)} = i\frac{U_{B_0}}{2}, \quad X_3^{(0)} = i\frac{U_{C(A_0,B_0)}}{2}.$$

and the map $\varrho_0 : \mathfrak{su}\ (2,\mathbb{C}) \to \mathfrak{L}(\mathcal{H}_0)$ by

$$\varrho_0(\sum_{j=1}^{3} \alpha_j e_j) = \sum_{j=1}^{3} \alpha_j X_j^{(0)}, \quad \alpha_j \in \mathbb{C}, j = 1,2,3.$$

Applying Theorem 3.6 with A and B replaced by A_0 and B_0, respectively, we have the following result.

Theorem 3.7 [10]. *The map ϱ_0 is an isomorphism between $\mathfrak{su}\ (2,\mathbb{C})$ and the Lie algebra $\mathfrak{M}_0$ generated by $X_j^{(0)}, j = 1,2,3$.*

Theorem 3.7 implies that ϱ_0 is a faithful representation of $\mathfrak{su}\ (2,\mathbb{C})$ on the Hilbert space $\mathcal{H}_0$. If $\mathcal{H}_0$ is infinite dimensional, then ϱ_0 gives an infinite dimensional representation of $\mathfrak{su}\ (2,\mathbb{C})$. The structure of the representation ϱ_0 may be interesting. We have the following theorem.

Theorem 3.8 [10]. *Let $\mathcal{H}$ be separable and $\mathcal{H}_0$ be infinite dimensional. Then there exists a sequence $\{\mathcal{M}_n\}_{n=1}^{\infty}$ of subspaces in $\mathcal{H}_0$ with the following properties:*

(i) *For each m and n with $m \neq n$, $\mathcal{M}_m$ and $\mathcal{M}_n$ are orthogonal.*
(ii) $\mathcal{H}_0 = \oplus_{n=1}^{\infty} \mathcal{M}_n$.
(iii) *For all $n \in \mathbb{N}$, $\dim \mathcal{M}_n = 2$ and $\mathcal{M}_n$ is left invariant by $\{X_j^{(0)}\}_{j=1}^{3}$.*

In particular, the representation ϱ_0 is completely reducible with the heighest weight of each irreducible component being $1/2$.

In concluding this section, we give a remark on a relevance of anticommuting self-adjoint operators to Clifford algebra theory. The Clifford algebra $\mathfrak{A}_n$ associated with the n-dimensional Euclidean space $\mathbb{R}^n$ is the algebra generated by elements $\gamma_j, j = 1, \cdots, n$, and identity 1 satisfying

$$\gamma_j\gamma_k + \gamma_k\gamma_j = 2\delta_{jk}, \quad j, k = 1, \cdots, n. \tag{3.1}$$

Let A and B be anticommuting self-adjoint operators in the Hilbert space $\mathcal{H}$ and define

$$\Gamma_1 = U_{A_0}, \quad \Gamma_2 = U_{B_0}, \quad \Gamma_3 = U_{C(A_0,B_0)}.$$

Then the operators $\Gamma_j, j = 1, 2, 3$, are self-adjoint on $\mathcal{H}_0$. Moreover we have

$$\Gamma_j\Gamma_k + \Gamma_k\Gamma_j = 2\delta_{jk}, \quad j, k = 1, 2, 3,$$

and Γ_j leaves $\mathcal{M}_n$ invariant. Let $\Gamma_j^{(n)}$ be the restriction of Γ_j to $\mathcal{M}_n$, so that we have

$$\Gamma_j = \bigoplus_{n=1}^{\infty} \Gamma_j^{(n)}.$$

Let $\mathfrak{C}_n$ be the algebra generated by $\Gamma_j^{(n)}, j = 1, 2, 3$. Then we have the following result.

Theorem 3.9 [10]. *For each $n = 1, 2, \cdots$, the algebra $\mathfrak{C}_n$ is the spin representation of $\mathfrak{A}_3$.*

IV. The sum of two anticommuting self-adjoint operators

Let A and B be anticommuting self-adjoint operators in the Hilbert space $\mathcal{H}$. As we have seen in Lemma 2.1 (vii), $A+B$ is self-adjoint. This section concerns more detailed properties of the operator $A + B$.

4.1. The case where B is injective

In this case, the partial isometry U_B is unitary with the spectrum $\sigma(U_B) = \{\pm 1\}$, so that we have the orthogonal decomposition

$$\mathcal{H} = \mathcal{H}_+ \bigoplus \mathcal{H}_- = \left\{ \begin{pmatrix} f \\ g \end{pmatrix} | f \in \mathcal{H}_+, g \in \mathcal{H}_- \right\} \tag{4.1}$$

with

$$\mathcal{H}_\pm = \mathrm{Ker}\ (U_B \mp 1).$$

Theorem 4.1 [10]. *Let A and B be anticommuting self-adjoint operators in $\mathcal{H}$ and B be injective. Then A, B, and P_A have the following matrix representations with respect to (w.r.t.) the decomposition* (4.1)*:*

$$A = \begin{pmatrix} 0 & a^* M_- \\ a M_+ & 0 \end{pmatrix}, \quad B = \begin{pmatrix} B_+ & 0 \\ 0 & -B_- \end{pmatrix}, \qquad (4.2)$$
$$P_A = \begin{pmatrix} a^* a & 0 \\ 0 & a a^* \end{pmatrix},$$

where a is a partial isometry from $\mathcal{H}_+$ to $\mathcal{H}_-$, B_+ (resp. B_-) and M_+ (resp. M_-) are commuting nonnegative self-adjoint operators in $\mathcal{H}_+$ (resp. $\mathcal{H}_-$), and $aB_+ \subset B_- a$.

This theorem is a generalization of [20, Corollary 3.3] which gives matrix representations of A and B similar to (4.2) in the case where both of A and B are injective.

We consider the diagonalization of $A + B$ w.r.t. the decomposition (4.1). By the commutativity of $|A|$ and $|B|$ [Lemma 2.1(iii)], we can define, via the functional calculus,

$$\Lambda = \mathrm{Arctan}(|A||B|^{-1}),$$

which is bounded and self-adjoint. Since $-iX_3$ and Λ are commuting bounded self-adjoint operators, $-iX_3\Lambda$ is bounded and self-adjoint. Hence the operator

$$V = e^{X_3 \Lambda}$$

is unitary. It turns out that V implements the diagonalization of $A + B$ w.r.t. the decomposition (4.1):

Theorem 4.2 [10]. *Let A and B be anticommuting self-adjoint operators and B be injective. Then*

$$\begin{aligned} V(A+B)V^{-1} &= U_B(A^2 + B^2)^{1/2} \\ &= \begin{pmatrix} (L_A^* L_A + B_+^2)^{1/2} & 0 \\ 0 & -(L_A L_A^* + B_-^2)^{1/2} \end{pmatrix}, \end{aligned} \qquad (4.3)$$

where

$$L_A = aM_+.$$

Remark. Formula (4.3) can be regarded as an abstract and non-perturbative version of the so-called *Tani-Foldy-Wouthuysen transformation* of the usual Dirac operator in three space dimensions (e.g., [14]).

Theorem 4.2 can be proven by using the following lemma.

Lemma 4.3 [10].

$$VX_1V^{-1} = (1 - P_A)X_1 + P_A(X_1 \cos\Lambda + X_2 \sin\Lambda),$$
$$VX_2V^{-1} = (1 - P_A)X_2 + P_A(-X_1 \sin\Lambda + X_2 \cos\Lambda).$$

4.2. The case where B is not injective

In this case, we note the following fact.

Lemma 4.4 [10]. *The operator P_B commutes with A, V, U_B, and $(A^2 + B^2)^{1/2}$.*

Lemma 4.4 implies that A, B, V, U_B, and $(A^2+B^2)^{1/2}$ can be reduced to $(\text{Ker } B)^{\perp}$ in which B is injective. Thus we can apply the preceeding result in Section 4.1 to obtain the following theorem.

Theorem 4.5 [10]. *Let A and B be anticommuting self-adjoint operators. Then* (4.3) *holds on* $(\text{Ker } B)^{\perp}$.

Remark. In the case of abstract Dirac operators, results similar to Theorems 4.2 and 4.5 have been obtained in [22].

V. Characterization of anticommutativity of self-adjoint operators in connection with Clifford algebra

In Section III we have seen that two anticommuting self-adjoint operators are related to the Clifford algebra $\mathfrak{A}_3$. This fact suggests that it may be more natural to characterize anticommutativity of self-adjoint operators in connection with Clifford algebra. In fact, such a characterization is possible as we shall present below.

Let $\mathcal{H}$ be a Hilbert space. We say that $\{\gamma_j\}_{j=1}^n$ *is a self-adjoint representation of the Clifford algebra $\mathfrak{A}_n$ on $\mathcal{H}$ if each γ_j is a bounded self-adjoint operator on $\mathcal{H}$ satisfying* (3.1).

The first of the main results in this section is the following.

Theorem 5.1 [11]. *Let A and B be self-adjoint operators in a Hilbert space $\mathcal{H}$. Suppose that there exists a self-adjoint representation $\{\gamma_1, \gamma_2\}$ of $\mathfrak{A}_2$ on $\mathcal{H}$ such that each γ_j commutes with A and B. Then A and B anticommute if and only if*

$$e^{is\gamma_1 A}e^{it\gamma_2 B} = e^{it\gamma_2 B}e^{is\gamma_1 A}$$

for all $s, t \in \mathbb{R}$.

Remark. If γ_1 commutes with A, then $\gamma_1 A$ is self-adjoint with $\gamma_1 A = A\gamma_1$. The same holds for the pair $\{\gamma_2, B\}$. Hence $\exp(is\gamma_1 A)$ and $\exp(it\gamma_2 B)$ can be defined via the functional calculus.

Theorem 5.1 has some interesting consequences. We fix a self-adjoint representation $\{\gamma_1, \gamma_2\}$ of $\mathfrak{A}_2$ on a Hilbert space $\mathcal{K}$. We denote by $\mathcal{K} \otimes \mathcal{H}$ the tensor product of $\mathcal{K}$ and $\mathcal{H}$.

Theorem 5.2 [11]. *Let A and B be self-adjoint operators in a Hilbert space $\mathcal{H}$. Then A and B anticommute if and only if $\gamma_1 \otimes A$ and $\gamma_2 \otimes B$ commute in the Hilbert space $\mathcal{K} \otimes \mathcal{H}$.*

Remark. A simple example of $\mathcal{K}$ and $\{\gamma_1, \gamma_2\}$ is given by

$$\mathcal{K} = \mathbb{C}^2,$$
$$\gamma_1 = \sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_2 = \sigma_2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

The matrices σ_1 and σ_2 are the first two of the so-called *Pauli matrices*.

We have a "dual" version of Theorem 5.2:

Theorem 5.3 [11]. *Let A and B be self-adjoint operators in a Hilbert space $\mathcal{H}$. Then A and B commute if and only if $\gamma_1 \otimes A$ and $\gamma_2 \otimes B$ anticommute in the Hilbert space $\mathcal{K} \otimes \mathcal{H}$.*

Remark. In the case where $\mathcal{K} = \mathbb{C}^2$ and $\gamma_j = \sigma_j, j = 1, 2$, the necessary condition in Theorem 5.3 has been proven in [13] by a method different from that in [11].

Theorem 5.3 can be applied to the self-adjointness problem of operators of Dirac's type. We first recall a basic result due to Vasilescu [23]:

Lemma 5.4 [23]. *Let $\{A_j\}_{j=1}^n$ be a family of mutually anticommuting self-adjoint operators in a Hilbert space ($n < \infty$). Then $\sum_{j=1}^n A_j$ is self-adjoint and*

$$\left(\sum_{j=1}^n A_j\right)^2 = \sum_{j=1}^n A_j^2.$$

Using this lemma and Theorem 5.3, we can prove the following fact:

Theorem 5.5 [11]. *Let $\{A_j\}_{j=1}^n$ be a family of mutually commuting self-adjoint operators in a Hilbert space $\mathcal{H}$ $(n < \infty)$. Let $\{\gamma_j\}_{j=1}^n$ be a self-adjoint representation of $\mathfrak{A}_n$ on a Hilbert space $\mathcal{K}$. Then the operator*

$$\not{D} := \sum_{j=1}^{n} \gamma_j \otimes A_j$$

is self-adjoint in $\mathcal{K} \otimes \mathcal{H}$ and

$$\not{D}^2 = \sum_{j=1}^{n} I \otimes A_j^2.$$

We next consider a countable family $\{A_n\}_{n=1}^\infty$ of self-adjoint operators. We can define the operator

$$A := \sum_{n=1}^{\infty} A_n$$

by the relation

$$D(A) = \left\{ f \in \bigcap_{n=1}^{\infty} D(A_n) \mid \mathrm{w} - \lim_{N\to\infty} \sum_{n=1}^{N} A_n f \text{ exists} \right\},$$

$$Af = \mathrm{w} - \lim_{N\to\infty} \sum_{n=1}^{N} A_n f, \quad f \in D(A).$$

The following lemma is an extension of Lemma 5.4.

Lemma 5.6 [23]. *Let $\{A_n\}_{n=1}^\infty$ be a family of mutually anticommuting self-adjoint operators in a Hilbert space $\mathcal{H}$ such that $D(\sum_{n=1}^\infty A_n)$ is dense in $\mathcal{H}$. Then $\sum_{n=1}^\infty A_n$ is self-adjoint and*

$$\left(\sum_{n=1}^{\infty} A_n \right)^2 = \sum_{n=1}^{\infty} A_n^2.$$

Using Lemma 5.6, we can obtain an extension of Theorem 5.5:

Theorem 5.7 [11]. *Let $\{A_n\}_{n=1}^\infty$ be a family of mutually commuting self-adjoint operators in a Hilbert space $\mathcal{H}$. Let $\{\gamma_n\}_{n=1}^\infty$ be a* *self-adjoint representation of* $\mathfrak{A}_\infty$ *on a Hilbert space* $\mathcal{K}$. *Suppose that* $D(\sum_{n=1}^\infty \gamma_n \otimes A_n)$ *is dense in* $\mathcal{K} \otimes \mathcal{H}$. *Then the operator*

$$\not{D}_\infty := \sum_{n=1}^\infty \gamma_n \otimes A_n$$

is self-adjoint in $\mathcal{K} \otimes \mathcal{H}$ *and*

$$\not{D}_\infty^2 = \sum_{n=1}^\infty I \otimes A_n^2.$$

The operator $\not{D}$ (resp. $\not{D}_\infty$) in Theorem 5.5 (resp. Theorem 5.7) gives a class of operators of Dirac's type in an abstract form. Hence Theorems 5.5 and 5.7 solve the self-adjointness problem for such Dirac operators. Examples to which Theorems 5.5 and 5.7 are applicable include: (i) the Dirac-Weyl operator with a strongly singular gauge potential [13] (cf. also [12]); (ii) classes of operators of Dirac's type in an abstract Boson-Fermion Fock space (infinite dimensional Dirac operators) [3, 4, 7, 9, 11].

VI. Anticommuting self-adjoint operators and supersymmetric quantum theory

As a final topic in this paper, we discuss a connection of the theory of anticommuting self-adjoint operators with supersymmetric quantum theory (SSQT).

We first give an abstract definition of SSQT (e.g., [1, 2, 4, 17, 25]). Let $N \geq 1$ be an integer. A SSQT with N-supersymmetry is defined to be a quadruple $\{\mathcal{H}, \{Q_n\}_{n=1}^N, H, N_F\}$ consisting of a Hilbert space $\mathcal{H}$, a set of self-adjoint operators $\{Q_n\}_{n=1}^N$ ("supercharges"), self-adjoint operators H ("supersymmetric Hamiltonian") and N_F ("Fermion number operator") acting in $\mathcal{H}$, which satisfies the following conditions:

(S.1) $N_F^2 = I$ (identity on $\mathcal{H}$) and $N_F \neq \pm I$.

(S.2) $H = Q_n^2, \quad n = 1, \cdots, N$.

(S.3) For each $n = 1, \cdots, N$, N_F leaves $D(Q_n)$ invariant and

$$N_F Q_n + Q_n N_F = 0 \quad \text{on } D(Q_n),\ n = 1, \cdots, N.$$

(S.4) For all $n, m = 1, \cdots, N$, with $n \neq m$,

$$(Q_n\psi, Q_m\phi) + (Q_m\psi, Q_n\phi) = 0, \quad \psi, \phi \in D(Q_n) \cap D(Q_m),$$

where $(\cdot, \cdot)$ is the inner product of $\mathcal{H}$.

Note that (S.3) means that N_F and Q_n anticommute in a "naive" sense, while (S.4) shows that Q_n and Q_m $(n \neq m)$ anticommute in the sense of quadratic form on $D(Q_n) \cap D(Q_m)$. It is natural to ask if they anticommute in the proper sense given in the Introduction.

The following fact is known.

Lemma 6.1 [23]. *Let T be a bounded self-adjoint operator and Q be a self-adjoint operator in a Hilbert space. Suppose that T leaves $D(Q)$ invariant and*

$$TQ + QT = 0 \quad \text{on } D(Q).$$

Then T and Q anticommute.

Applying Lemma 6.1 to $T = N_F$ and $Q = Q_n$, we have the following result.

Proposition 6.2. *In any SSQT $\{\mathcal{H}, \{Q_n\}_{n=1}^N, H, N_F\}$, each Q_n and N_F anticommute.*

As for (S.4), we can apply the following theorem.

Theorem 6.3. *Let Q_1 and Q_2 be self-adjoint operators in a Hilbert space $\mathcal{H}$ such that*

$$Q_1^2 = Q_2^2$$

and

$$(Q_1\psi, Q_2\phi) + (Q_2\psi, Q_1\phi) = 0, \quad \psi, \phi \in D(Q_1) \cap D(Q_2). \tag{6.1}$$

Then Q_1 and Q_2 anticommute.

Proof. We have $L \equiv |Q_1| = |Q_2|$. Hence $D(Q_1) = D(Q_2) = D(L)$ and the polar decompositions of Q_1 and Q_2 are given by

$$Q_1 = U_1 L, \quad Q_2 = U_2 L,$$

where $U_j = U_{Q_j}$. Putting these formulas into (6.1), we have

$$(U_1\widetilde{\psi}, U_2\widetilde{\phi}) + (U_2\widetilde{\psi}, U_1\widetilde{\phi}) = 0 \tag{6.2}$$

with $\widetilde{\psi} = L\psi, \widetilde{\phi} = L\phi, \quad \psi, \phi \in D(L)$.

We first consider the case where L is injective and hence so is Q_j $(j = 1, 2)$. Then U_1 and U_2 are unitary, self-adjoint and Ran L is dense in $\mathcal{H}$. Hence (6.2) implies that

$$U_j U_k + U_k U_j = 2\delta_{jk}, \quad j, k = 1, 2.$$

Let $\mathcal{D} = \cup_{n=1}^{\infty} \text{Ran } E_L([0, n])$. Then $\mathcal{D}$ is dense in $\mathcal{H}$. Since U_j commutes with L, U_1 and U_2 leave $\mathcal{D}$ invariant and hence so do Q_1 and Q_2. It is easy to see that $\mathcal{D}$ is a set of entire analytic vectors for each Q_j and $Q_1 Q_2 + Q_2 Q_1 = 0$ on $\mathcal{D}$. Hence we can apply [20, Proposition 5.2] to conclude that Q_1 and Q_2 anticommute.

In the case where L is not injective, Q_1 and Q_2 are reduced to $\mathcal{H}_0 \equiv (\text{Ker } L)^{\perp} = (\text{Ker } Q_1)^{\perp} = (\text{Ker } Q_2)^{\perp}$. We can apply the preceeding result to $\widetilde{Q}_j \equiv Q_j \restriction \mathcal{H}_0$ to conclude that $\widetilde{Q}_1$ and $\widetilde{Q}_2$ anticommute. This implies the anticommutativity of Q_1 and Q_2 in $\mathcal{H}$. Q.E.D.

Theorem 6.3 gives the following result.

Proposition 6.4. *In any SSQT $\{\mathcal{H}, \{Q_n\}_{n=1}^{N}, H, N_F\}$, Q_n and Q_m $(n, m = 1, \cdots, N, n \neq m)$ anticommute.*

Remark. The SSQT considered above is a non-relativistic one. In relativistic cases, condition (S.2) have to be replaced by a more complicated one (e.g., [9, 24]).

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Asymptotics for the Painlevé II Equation: Announcement of Result

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Submitted in honor of Professor S.T. Kuroda, from whom we have learned so much

1. Introduction

In this paper we study the asymptotics of a class of solutions of the (homogeneous) Painlevé II (PII) equation

(1.1) $$u_{xx} = 2u^2 + xu\ , \qquad x \in \mathbb{R}\ ,$$

as $x \to \pm\infty$. Following the work of Flaschka and Newell [FN] and Jimbo, Miwa and Ueno [JMU] the PII equation can be solved by means of a Riemann-Hilbert (RH) factorization problem as follows ([FA]; see also [IN]). Let $\Sigma^{(1)}$ denote the oriented contour consisting of six rays, $\Sigma^{(1)} = \bigcup_{k=1}^{6} \left\{ \Sigma_j^{(k)} = e^{i(k-1)\pi/3}\mathbb{R}_+ \right\}$,

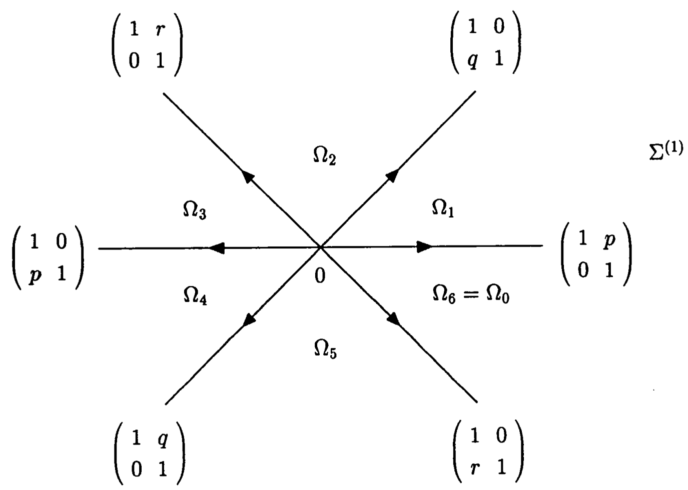

Fig. 1.2

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with associated jump matrix $v^{(1)}\colon \Sigma^{(1)} \to M_2(\mathbb{C})$, $v^{(1)} \restriction \Sigma_1^{(1)} = \begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix}$, etc., where p, q and r are complex numbers satisfying the relation

$$p + q + r + pqr = 0 \ . \tag{1.3}$$

For $x \in \mathbb{R}$ and $z \in \Sigma^{(1)}$, set

$$\left.\begin{aligned} v_x^{(1)}(z) &= e^{-i(\frac{4z^3}{3}+xz)\sigma_3}\, v^{(1)}(z)\, e^{i(\frac{4}{3}z^3+xz)\sigma_3} \\ &\equiv e^{-i(\frac{4z^3}{3}+xz)ad\,\sigma_3}\, v^{(1)}(z) \end{aligned}\right\} \tag{1.4}$$

where σ_3 is the Pauli matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

Now let $m^{(1)}(z) = m^{(1)}(x,z)$ be a (2×2 matrix valued) holomorphic function defined on $\mathbb{C} \setminus \Sigma^{(1)}$ solving the RH problem

$$\left.\begin{aligned} &m_+^{(1)}(z) = m_-^{(1)}(z)\, v_x^{(1)}(z)\ , \quad 0 \neq z \in \Sigma^{(1)}\ , \\ &m^{(1)}(z) \to I \quad \text{as} \quad z \to \infty\ , \end{aligned}\right\} \tag{1.5}$$

where $m_+^{(1)}(z)$ (resp. $m_-^{(1)}(z)$) denotes as usual the boundary value of $m^{(1)}(z)$ from the left (resp. right) side of the oriented contour $\Sigma^{(1)}$. (Thus for $z \in \mathbb{R}_+$ in particular, we have $m_\pm^{(1)}(z) = \lim_{\epsilon\downarrow 0} m^{(1)}(z \pm i\epsilon)$, etc.). Then

$$u(x) \equiv 2i(m_1^{(1)}(x))_{12} = -2i(m_1^{(1)}(x))_{21}\ , \tag{1.6}$$

solves PII, where

$$m^{(1)}(z) = m^{(1)}(z;x) = I + \frac{m_1^{(1)}(x)}{z} + O(\frac{1}{z^2}) \tag{1.7}$$

as $z \to \infty$.

For general p, q, r and x, $p+q+r+pqr = 0$, $x \in \mathbb{R}$, the RH problem (1.5) may fail to have a solution. However, every (local) solution of the Cauchy problem for (1.1) can be obtained from the RH problem for suitable p, q and r by the above prescription. Indeed (see e.g. [FA], [IN]) there is an inejctive map (the *Direct Transform*)

$$\begin{aligned} (1.8)\quad &(u(0), u'(0)) \mapsto \\ &\quad (p_0 = p(u(0),u'(0)),\ q_0 = q(u(0),u'(0)),\ r_0 = r(u(0),u'(0))) \\ &\qquad \in \{(p,q,r)\colon p+q+r+pqr = 0\} \end{aligned}$$

with the property that the RH problem (1.5) with $p = p_0$, $q = q_0$ and $r = r_0$ has a solution for all x in a neighborhood of zero, and if $u(x; p_0, q_0, r_0)$

is the solution of PII obtained from (1.6), then $u(x;p_0,q_0,r_0)$ is the unique (local) solution of PII with the given initial data $(u(0),u'(0))$. Moreover, allowing complex values for x in (1.4), $u(x;p_0,q_0,r_0)$ gives a meromorphic continuation of the solution to the entire complex plane.

We are interested in particular in solutions of (1.1) that exist for all x in $\mathbb{R}$. A sufficient condition (see [FZ]) for the RH problem (1.5) to have a solution for all $x \in \mathbb{R}$, is that

$$|q-\bar{p}| < 2 \quad \text{and} \quad r \in \mathbb{R}\,. \tag{1.8}$$

Real solutions of PII correspond to RH data with the symmetry

$$p = -\bar{q}\,, \qquad r \in i\mathbb{R}\,, \tag{1.9}$$

and pure imaginary solutions correspond to data with

$$p = \bar{q}\,, \qquad r \in \mathbb{R} \tag{1.10}$$

(cf. [FA], [IN]). From (1.3), (1.8), (1.9) and (1.10), we see that for any real q,

$$-1 < q < 1\,, \qquad p = -q\,, \qquad r = 0 \tag{1.11}$$

formula (1.6) leads to a global, real solution of PII, while for any $q \in \mathbb{C}$

$$q\,, \quad p = \bar{q}\,, \quad r = -\left[(q+\bar{q})/(1+|q|^2)\right] \in \mathbb{R} \tag{1.12}$$

formula (1.6) leads to a global, purely imaginary solution of PII. Furthermore (see below) a special argument shows that for

$$q = \pm 1\,, \qquad p = \mp 1\,, \qquad r = 0 \tag{1.13}$$

(1.6) also leads to a global, real solution of (1.1).

We will study the asymptotic behavior of the solutions of PII in these three cases (1.1), (1.2) and (1.3). The results are as follows.

Theorem 1.14 (global real solutions). *For*

$$-1 < q < 1\,, \quad p = -q \quad \textit{and} \quad r = 0\,,$$

(1.15)

$$u(x) = \frac{\sqrt{2\nu}}{(-x)^{1/4}} \cos\left(\frac{2}{3}(-x)^{3/2} - \frac{3}{2}\nu\log(-x) + \phi\right) + O\left(\frac{\log(-x)}{(-x)^{5/4}}\right)$$

$$\textit{as} \quad x \to -\infty\,,$$

where

$$\nu = \nu(q) = \frac{-1}{2\pi} \log(1 - q^2) \tag{1.16}$$

$$\phi = -3\nu \log 2 + \log \Gamma(i\nu) + \frac{\pi}{2} \operatorname{sgn} q - \frac{\pi}{4} \tag{1.17}$$

(*here* Γ *denotes the Gamma function*) *and*

$$u(x) = \frac{q}{2\sqrt{\pi} x^{1/4}} e^{-(2/3)x^{3/2}} (1 + O(1)) \qquad \text{as} \quad x \to +\infty \ . \tag{1.18}$$

Theorem 1.19 (global purely imaginary solutions). *For*

$$q \in \mathbb{C} \ , \quad p = \bar{q} \quad \text{and} \quad r = -[(q + \bar{q})/(1 + |q|^2)]$$

(1.20)

$$u(x) = \frac{i(-2\nu)^{1/2}}{(-x)^{1/4}} \sin\left(\frac{2}{3}(-x)^{3/2} - \frac{3}{2}\nu \log(-x) + \phi\right) + O\left(\frac{\log(-x)}{(-x)^{5/4}}\right)$$
$$\text{as} \quad x \to -\infty \ ,$$

where

$$\nu = \frac{-1}{2\pi} \log(1 + |q|^2) \tag{1.21}$$

and

$$\phi = -3\nu \log 2 + \frac{\pi}{4} + \arg \Gamma(i\nu) - \arg q \ . \tag{1.22}$$

For $\operatorname{Re} q \neq 0$ (*equivalently* $r \neq 0$)
(1.23)

$$u(x) = \sigma i \sqrt{\frac{x}{2}} - \frac{\sigma i \sqrt{\nu}}{(2x)^{1/2}} \cos\left(\frac{2\sqrt{2}}{3} x^{3/2} - \frac{3}{2}\nu \log x + \phi\right) + O\left(\frac{1}{x^{(1/2)-\epsilon}}\right)$$
$$\text{as} \quad x \to +\infty$$

where ϵ *is any positive number and*

$$\nu = \frac{1}{\pi} \log \frac{1 + |q|^2}{2|\operatorname{Re} q|} \tag{1.24}$$

$$\phi = \frac{\pi}{4} - \frac{7}{2}\nu \log 2 + \arg \Gamma(i\nu) + \arg(1 - q^2) \ , \tag{1.25}$$

$$\sigma = \operatorname{sgn}(\operatorname{Re} q) \ , \tag{1.26}$$

and for Re $q = 0$ (*equivalently* $r = 0$)

$$u(x) = \frac{1}{2\sqrt{\pi}x^{1/4}} e^{-(2/3)x^{3/2}} (1 + o(1)) \qquad as \qquad x \to +\infty \ . \tag{1.27}$$

Theorem 1.28 (global real solutions: singular case). *For*

$$q = \pm 1 \ , \quad p = \mp 1 \quad \text{and} \quad r = 0 \ ,$$

$$u(x) = \pm [\big(-\frac{x}{2}\big)^{1/2} - \frac{1}{2^{7/2}}(-x)^{-5/2} + O\Big((-x)^{-11/2}\Big)] \qquad as \quad x \to -\infty \tag{1.29}$$

and

$$u(x) = \pm \frac{1}{2\sqrt{\pi}x^{1/4}} e^{-(2/3)x^{3/2}} (1 + o(1)) \qquad as \qquad x \to +\infty \ . \tag{1.30}$$

Theorem 1.14 is due to Ablowitz and Segur (see [SA1], [SA2]). A rigorous justification of the beautiful heuristic calculations in [SA1], [SA2] is given in [HM] and [CM], at least up to the phase shift (1.17), using a Gelfand-Levitan type equation derived earlier by Ablowitz and Segur in [AS] Theorem 1.28, at least to leading order in x, is due to Hastings and Mcleod and appears in [HM]. A Gelfand-Levitan approach is only possible in the special case of Theorems 1.14 and 1.28 when $r = 0$ and the contour $\Sigma^{(1)}$ for the RH problem can be reduced to a single line. Theorem 1.19 is due to Its and Kapaev [IK]. In the case $r \neq 0$, the contour does not reduce to a line and the RH problem must be solved directly as a RH problem on a nontrivial contour with self-interactions. The authors in [IK] use the so called "isomonodromy method" which they have developed, together with Novokshenov and others, in a wonderful series of papers over the last eight or nine years. An exposition of the method, together with a discussion of the many results that have been obtained, can be found in [IN]. The method is a descendent of the original method of Zakharov and Manikov [ZM] which they derived in analyzing the long-time behavior of the nonlinear Schrödinger equation. Another derivation of Theorem 1.14, using the isomonodromy method, was given by Suleimanov [S]. We note, however, that certain technical difficulties in [IK] and [S] remain, and a completely rigorous justification of the isomonodromy method poses a deep and very interesting challenge.

The above results solve the so-called "connection problem" for global solutions of PII. For example, if we observe the asymptotics of a purely imaginary solution of P II as $x \to -\infty$, then ν and ϕ in (1.20) are known and hence q can be computed from (1.21) and (1.22), and so $\nu(q)$, $\phi(q)$ and $\sigma(q)$ can be determined from (1.24), (1.25) and (1.26), which yields the asymptotics as $x \to +\infty$ through (1.23). Conversely if we observe the asymptotics of a purely imaginary solution as $x \to +\infty$, then ν, ϕ and σ in (1.23) are known. But then $r = \dfrac{-2\,\mathrm{Re}\, q}{1+|q|^2}$ is known from (1.24) and (1.26). This relation can be rewritten as $|r^{-1}+q|^2 = r^{-2}-1$. On the other hand $\arg(1+rq) = \arg((1-q^2)/1+|q|^2) = \arg(1-q^2)$, and hence $\arg(r^{-1}+q)$ can be determined from (1.25). Thus $r^{-1}+q$ and, hence q, is known. Substitution in (1.21) and (1.22) then yields the asymptotics as $x \to -\infty$ through (1.20). Thus if we know the behavior of the solution as $x \to +\infty$ (resp. $x \to -\infty$), we can "connect" the solution to its asymptotics as $x \to -\infty$ (resp. $x \to +\infty$).

The six Painlevé equations PI-PVI were introduced by Painlevé and Gambier at the beginning of this century on purely mathematical grounds, but recently they have appeared in a wide range of physical applications, including self-similar solutions of the Korteweg deVries equation, correlation functions for the transverse Ising chain in the infinite temperature limit, nonperturbative 2D quantum gravity, amongst many others. A comprehensive survey of recent results and applications of Painlevé equations can be found in [FI]. It is increasingly recognized that the Painlevé equations play a role in modern mathematical physics analogous to the role played by the classical special functions of the last century. Many of the applications of classical special function theory rest on the fact that the asymptotics and the associated connection problem for the special functions can be solved explicitly. Theorems 1.14, 1.19 and 1.28 should be viewed as providing the analogous information for P II.

We now make some additional remarks about the asymptotic formulae in the above theorems. Note that if $q = \pm 1$, $p = \pm 1$ and $r = \mp 1$ in (1.24)–(1.26) then $\nu = 0$ and the lower order oscillatory term in (1.23) is absent. This case plays a special role in our analysis of PII and provides a model problem by means of which the general case of Theorem 1.19 can be analyzed. Moreover, in this case, the solution has full fractional expansion as $x \to +\infty$ identical in form to (1.29).

It is interesting to consider Theorems 1.14 and 1.28 from the following point of view. Observe that for large positive x, PII reduces to the Airy equation. *Question* (see [HM]): for real q does there exist a real global solution of P II that is asymptotic to $qA_i(x)$ as $x \to +\infty$? (Here

$A_i(x)$ is the standard Airy function.) Theorems 1.14 and 1.28 show that this is so for $|q| \leq 1$ (the asymptotics for $A_i(x)$ can be found, for example, in [AbSt]). However, if $|q| > 1$ and the solution $u(x) \sim q\, A_i(x)$ as $x \to +\infty$, then $u(x)$ must blow up for some x. This result is due to Hastings and McLeod (see [HM]). In fact, as shown by Kapaev and Novokshenov, $u(x)$ blows up at an infinite number of points $x_n \to -\infty$ (see [KN], [IN]).

An analysis of the asymptotics of solutions $u(x)$ of P II as $x \to \infty$ along a ray in the complex plane has been given by Boutroux [B]. Further interesting developments can be found in Novokshenov [N] and Kapaev [K], who use the isomonodromy method.

Recently the authors have introduced a new and general nonlinear steepest descent-type method for analyzing the asymptotics of oscillatory RH problems [DZ1]. The method has been used to derive rigorously the long-time asymptotics for the modified Korteweg de Vries (MKdV) equation [DZ1], for the nonlinear Schrödinger (NLS) equation [DIZ], and for the doubly infinite, compactly perturbed Toda lattice [Kam]. The method has also been used to announce the derivation of the collisionless shock region of Ablowitz-Segur for the Korteweg de Vries (KdV) equation [DZ3], and to obtain the long-time asymptotics for the autocorrelation function of the transverse Ising chain at the critical magnetic field [DZ2].

As indicated above, many of the results in Theorems 1.14, 1.19 and 1.28 have not yet been justified rigorously. Moreover, the methods of the authors, and in particular the isomonodromy method, require an a priori ansatz for the form of the solution. The purpose of this paper is to derive Theorems 1.14, 1.19 and 1.28 rigorously and directly with error bounds using the steepest descent method of [DZ1]. Our approach is algorithmic and requires no ansatz for the asymptotic form of the solution. The method proceeds by deforming contours, and in the simplest cases, we are left with the localized RH problem near the points of stationary phase. These localized RH problems can then be solved explicitly in terms of classical special functions. This is the case for MKdV in the similarity region and also for the asymptotics on (1.15) and (1.20). This is not the case, however, for the asymptotics in (1.23) and (1.29): here the RH problem localizes on a line segment rather than at the stationary phase points. A similar situation arises in the analysis of the collisionless shock region in KdV (see [DZ3]). This is a new and essentially nonlinear feature of the steepest descent method, and its resolution occupies the main part of the work.

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Eigenvalue Properties of Schrödinger Operators

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Abstract.

In Evans-Lewis [5] and Evans-Lewis-Saitō [6], [7], [8], [9] we have been discussing conditions for the finiteness and for the infiniteness of bound states of Schrödinger-type operators using geometric methods. Here the ideas and results obtained so far are summarized and presented in an expository manner. These bound states correspond to eigenvalues below the essential spectrum of the operator. After basic results are presented, Schrödinger operators of atomic type will be discussed to show how these basic results can be applied to various types of N-body Schrödinger operators.

Introduction

In [5], [6], [7], [8] and [9] we have been considering criteria for the bound states of Schrödinger-type operators

$$(0.1)\qquad P = -\sum_{j,k=1}^{n} \partial_j a^{jk}(x)\partial_k + q(x) \qquad x \in \mathbf{R}^n,\ \partial_j = \frac{\partial}{\partial x_j}$$

to be finite or infinite (see Assumption 1.1 for the properties satisfied by the coefficients $a^{jk}(x)$ and $q(x)$). These bound states correspond to eigenvalues below the essential spectrum of the operator. The goal of this paper is twofold:

(1) *In* §1 *the basic results for the operator* (0.1) *will be presented in a more self-contained and unified way, which we hope makes these basic results easier to be understood. Our arguments are based on the geometric method using the Agmon spectral function which was introduced in Agmon* [1]. *We are going to show that our arguments become smoother and more streamlined by restricting the operator P using only smooth cut-off functions. This was introduced in* [9]. *Here we have an opportunity to modify our way of deriving the basic results obtained in* [5] *and*

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[6]. *Other important ingredients are the results of Glazman* [10, *Chapter* 1] *on counting the eigenvalues of an abstract selfadjoint operator in a given interval. Since the proofs of some of his theorems in* [10] *are too succinct, we are proving these theorems in a more self-contained way so that our main theorems will be understood more easily.*

(2) *In* §2 *we shall discuss the Schrödinger operator of atomic type*

$$P = P_N = \sum_{i=1}^{N} \left(-\frac{1}{2m_i}\Delta_i + v_{0i}(x^i) \right) + \sum_{1 \le i < j \le N} v_{ij}(x^i - x^j), \tag{0.2}$$

where

$$x^i = (x_1^i, x_2^i, \cdots, x_\nu^i) \in \mathbf{R}^\nu, \tag{0.3}$$

(*see Assumption* 2.1). *We chose the operator* (0.2) *as an example to give an idea how the general results obtained in* §1 *can be applied to various types of* N*-body Schrödinger operators since it is easier to be treated without being bothered by technical troubles. We are going to compare our results to the celebrated results by Zhislin* ([22], [23], [24], [25]), Yafaev ([20], [21]) *and others for the atomic Hamiltonian given by*

$$P = P(N, Z) = \sum_{i=1}^{N} (-\frac{1}{2m}\Delta_i - \frac{Z}{|x^i|}) + \sum_{1 \le i < j \le N} \frac{1}{|x^i - x^j|}. \tag{0.4}$$

We are also giving an another proof for the finiteness of the bound states of the operator (0.2) *with "short-range" potentials* v_{jk}, $0 \le j < k \le N$, *i.e.*,

$$v_{jk} \in L_2^{\nu/2}(\mathbf{R}^\nu) \qquad (0 \le j < k \le N). \tag{0.5}$$

The results given in §1 can be applied to other types of N-body Schrödinger operators. In [8] we discussed N-body Schrödinger operators with their center of mass removed. Then the operator becomes unitarily equivalent to the operator in §1, and hence we can develop essentially the same theory as in §1 and §2. Thus we are able to treat molecular Hamiltonians. We also discussed molecular Hamiltonians with symmetry restrictions in [9]. The N-body Schrödinger operator with its center of mass removed is considered in the L_2 space whose elements are square integrable functions over

$$X = \{x \in \mathbf{R}^{\nu N} : m_1 x^1 + m_2 x^2 + \cdots + m_N x^N = 0\} \tag{0.6}$$

satisfying specified symmetry conditions. Again we found that we can construct a parallel theory to those in §1 and §2. For these details see [8] and [9].

While we try to make this work self-contained, we refer to our works [5], [6], [7] and [8] when we use the exactly same propositions given in the above papers. Some technical lemmas and theorems are proved in the Appendices.

§1. The bound states of Schrödinger-type operators

Consider the Schrödinger-type operator

$$P = -\sum_{j,k=1}^{n} \partial_j a^{jk}(x)\partial_k + q(x) \qquad x \in \mathbf{R}^n,\ \partial_j = \frac{\partial}{\partial x_j}. \tag{1.1}$$

Assumption 1.1. The coefficients a^{jk} and q of the operator P is assumed to satisfy the following (i) $\sim$ (iii):

(i) Each a^{jk} is a bounded, continuous, real-valued function on $\mathbf{R}^n$.

(ii) The matrix $A(x) = \big(a^{jk}(x)\big)$ is uniformly positive definite on $\mathbf{R}^n$, i.e., there exists a constant $c_0 > 0$ such that

$$\sum_{j,k=1}^{n} a^{jk}(x)\xi_j\overline{\xi_k} \geq c_0 \sum_{j=1}^{n} |\xi|^2 \tag{1.2}$$

for all $x \in \mathbf{R}^n$ and $(\xi_1, \xi_2, \cdots, \xi_n) \in \mathbf{C}^n$.

(iii) $q \in L_1(\mathbf{R}^n)_{\mathrm{loc}}$.

We start with the following definition.

Definition 1.2. (i) Let η be a nonnegative, bounded C^∞ function on $\mathbf{R}^n$. Let the sesquilinear form on $C_0^\infty(\mathbf{R}^n) \times C_0^\infty(\mathbf{R}^n)$ be defined by

$$\rho_\eta[\phi, \varphi] = \int_{\mathbf{R}^n} \{< \nabla(\eta\phi), \nabla(\eta\varphi) >_A + q\eta^2\phi\overline{\varphi}\}\, dx, \tag{1.3}$$

where

$$< \xi, \zeta >_A = \sum_{j,k=1}^{n} a^{jk}\xi_j\overline{\zeta_k} \qquad (\xi = (\xi_1, \xi_2, \cdots, \xi_n), \zeta = (\zeta_1, \zeta_2, \cdots, \zeta_n) \in \mathbf{C}^n). \tag{1.4}$$

We set $\rho_\eta[\phi] := \rho_\eta[\phi, \phi]$. For $\eta \equiv 1$ we denote $\rho_1[\ ,\ ]$ simply by $\rho[\ ,\ ]$.

(ii) Define the Hilbert space $L_{2,\eta}(\mathbf{R}^n)$ by

$$L_{2,\eta}(\mathbf{R}^n) = L_2(\mathbf{R}^n, \eta^2 dx). \tag{1.5}$$

The inner product and norm of $L_{2,\eta}(\mathbf{R}^n)$ are denoted by $(\ ,\ )_\eta$ and $\|\ ,\ \|_\eta$, respectively. For $\eta \equiv 1$ we simply write $L_2(\mathbf{R}^n)$, $(\ ,\ )$, and $\|\ \|$.

The following assumption guarantees that ρ_η is closable on $L_{2,\eta}(\mathbf{R}^n)$.

Assumption 1.3. For every $\epsilon \in (0,1)$ there is a $C(\epsilon) > 0$ such that

$$\int_{\mathbf{R}^n} q_-|\phi|^2 dx \le \epsilon \int_{\mathbf{R}^n} |\nabla\phi|^2 dx + C(\epsilon) \int_{\mathbf{R}^n} |\phi|^2 dx, \qquad \phi \in C_0^\infty(\mathbf{R}^n), \tag{1.6}$$

where $q_-(x) = \max(-q(x), 0)$.

It is known (Schechter [16, Theorem 7.3, p.138]) that (1.6) holds if q_- belongs to the Kato class, i.e.,

$$\lim_{r\to 0} \int_{|x-y|<r} g(x,y)|q(y)|dy = 0, \tag{1.7}$$

where

$$g(x,y) = \begin{cases} |x-y|^{2-n} & \text{if } n \ge 3, \\ |\ln|x-y|| & \text{if } n = 2, \text{ and} \\ 1 & \text{if } n = 1. \end{cases} \tag{1.8}$$

Remark 1.4. Assumptions 1.1 and 1.3 are slightly more strict than those given in [6], [7], [8], [9] although usual N-body Schrödinger operators satisfy our assumptions. Since we assume that the matrix $A(x)$ is uniformly positive, the condition on q_- seems to be easier to check (cf. the condition $\mathcal{H}(1)$ in [6, p.383]).

Proposition 1.5. *Let Assumptions* 1.1 *and* 1.3 *be satisfied. Let* ρ_η *be as in Definition* 1.2. *Then* ρ_η *is densely defined, symmetric, bounded below, and closable in* $L_{2,\eta}(\mathbf{R}^n)$.

Proof. (1) Since it is easy to see that ρ_η is densely defined, symmetric, and bounded below, we are going to give the proof that ρ_η is closable. Let $\{\phi_j\}$ be a sequence in $C_0^\infty(\mathbf{R}^n)$ such that

$$\begin{cases} \|\phi_j\|_\eta \to 0 & (j \to \infty), \\ \rho_\eta[\phi_j - \phi_k] \to 0 & (j, k \to \infty). \end{cases} \tag{1.9}$$

We have only to prove that $\rho_\eta[\phi_j] \to 0$ as $j \to \infty$.

(2) It follows from Assumption 1.1, (ii) and Assumption 1.3 that there exists a positive constant C_1 such that

$$(1.10)\quad \|\phi\|_{\rho_\eta}^2 \equiv \rho_\eta[\phi] + C_1\|\phi\|_\eta^2 \geq \int_{\mathbf{R}^n} \{\frac{c_0}{2}|\nabla(\eta\phi)|^2 + (q_+ + 1)\eta^2|\phi|^2\}\, dx$$

for $\phi \in C_0^\infty(\mathbf{R}^n)$, where $q_+(x) = \max\{q(x), 0\}$, and hence $\{\eta\phi_j\}$ is a Cauchy sequence in both $H^1(\mathbf{R}^n)$ and $L_2(\mathbf{R}^n, q_+dx)$. Further, since $\eta\phi_j \to 0$ in $L_2(\mathbf{R}^n)$ as $j \to \infty$, it follows that

$$(1.11)\qquad \mathrm{s} - \lim_{j\to\infty} \eta\phi_j = 0 \qquad (j \to \infty)$$

in both $H^1(\mathbf{R}^n)$ and $L_2(\mathbf{R}^n, q_+dx)$ which implies that $\rho_\eta[\phi_j] \to 0$.

Q.E.D.

Definition 1.6. Let ρ_η be as above. Denote the closure of ρ_η by $\tilde{\rho}_\eta$. Let H_η be the selfadjoint operator in $L_{2,\eta}(\mathbf{R}^n)$ associated with $\tilde{\rho}_\eta$ (see, e.g., Kato [13, Chapter VI]). For $\eta \equiv 1$ H_1 will be denoted simply by H. Define $\Sigma(H_\eta)$ by

$$(1.12)\qquad \Sigma(H_\eta) = \inf \sigma_e(H_\eta),$$

where $\sigma_e(H_\eta)$ is the essential spectrum of H_η.

Now we are in a position to introduce the Agmon spectral function.

Definition 1.7. Let S^{n-1} be the unit sphere. For any set $U \subset S^{n-1}$ and for positive numbers R and δ define

$$\begin{aligned} U_\delta &:= \{\omega \in S^{n-1} : \mathrm{dist}(\omega : U) < \delta\}; \\ \Gamma(U_\delta, R) &:= \{x \in \mathbf{R}^n : x = t\omega \text{ for } \omega \in U_\delta \text{ and } t > R\} \\ K(U_\delta, R; P) &:= \inf\{\rho[\varphi] :\ \varphi \in C_0^\infty(\Gamma(U_\delta, R)), \|\varphi\| = 1\}; \\ (1.13)\qquad K(U : P) &:= \lim_{\delta\downarrow 0} \lim_{R\to\infty} K(U_\delta, R; P); \end{aligned}$$

and

$$\mathcal{M} := \{\omega \in S^{n-1} : K(\omega : P) = \inf_{\omega\in S^{n-1}} K(\omega : P)\},$$

where we write $K(\omega : P)$ instead of $K(\{\omega\} : P)$, and the set $\mathcal{M} \subset S^{n-1}$ is called the minimizing set.

The following properties of the Agmon spectral function are important.

Proposition 1.8. *Suppose that Assumptions* 1.1 *and* 1.3 *hold. Let* H *be as in Definition* 1.6.

(i) *Then* $K(\omega : P)$ *is a lower semicontinuous function on* S^{n-1} *and we have*

$$\Sigma(H) := \min_{\omega \in S^{n-1}} K(\omega : P), \tag{1.14}$$

and the minimizing set $\mathcal{M}$ *is a compact set in* S^{n-1} *with*

$$\mathcal{M} = \{\omega \in S^{n-1} : K(\omega : P) = \min_{\omega \in S^{n-1}} K(\omega : P)\}. \tag{1.15}$$

(ii) *For any* $U \subset S^{n-1}$,

$$K(U : P) = K(\overline{U} : P) = \inf_{\omega \in \overline{U}} K(\omega : P) \tag{1.16}$$

The first part of the above proposition is due to Agmon [1, Lemma 2.7, p.38]. For the proof of (ii) see [6, Lemma 5, p.380].

Let us give a necessary condition for the bound states to be finite.

Theorem 1.9 ([6, Theorem 8]). *Let Assumptions* 1.1 *and* 1.3 *hold. Let* H *be the selfadjoint operator associated with the closure* $\tilde{p}$ *of* p *in* $L^2(\mathbf{R}^n)$. *A necessary condition for the finiteness of the number of eigenvalues of* H *below* $\Sigma(H)$ *is that for some* $\delta_0 > 0$ *and some* $R_0 > 0$

$$K(\mathcal{M}_\delta, R; P) = K(M : P) = \Sigma(H) \qquad \text{for all } \delta \geq \delta_0 \text{ and } R \geq R_0. \tag{1.17}$$

Before proving the theorem we mention a simple fact on a linear space.

Lemma 1.10. *Let* Y *be a vector space over* $\mathbf{C}$. *Let* Y_1 *and* Y_2 *be linear subspaces of* Y *such that* $\dim Y_2 < \infty$ *and* Y *is the direct sum of* Y_1 *and* Y_2 (*i.e.,* $Y_1 \cap Y_2 = \{0\}$, *and* $Y = Y_1 + Y_2$). *Let* Y_0 *be another linear subspace of* Y *such that* $\dim Y_0 > \dim Y_2$. *Then we have* $\dim(Y_0 \cap Y_1) \geq 1$.

Proof. Let $\dim Y_2 = m$ and let $\phi_1, \phi_2, \cdots, \phi_m$ be a base of Y_2. Let $\{f_j\}$, $j = 1, 2, \cdots, m+1$, be a set of $m+1$ independent vectors in Y_0. Since Y is the direct sum of Y_1 and Y_2, there exist $u_j \in Y_1$,

$j = 1, 2, \cdots, m+1$ and $a_{jk} \in \mathbf{C}$, $j = 1, 2, \cdots, m+1$, $k = 1, 2, \cdots, m$ such that

$$f_j = u_j + \sum_{k=1}^{m} a_{jk}\phi_k \qquad (j = 1, 2, \cdots, m+1). \tag{1.18}$$

Note that the system of linear equations

$$\sum_{j=1}^{m+1} c_j a_{jk} = 0 \qquad k = 1, 2, \cdots, m \tag{1.19}$$

has a nontrivial solution $(c_1, c_2, \cdots, c_{m+1})$. Then we have

$$f_0 := \sum_{j=1}^{m+1} c_j f_j = \sum_{j=1}^{m+1} c_j u_j \tag{1.20}$$

is nontrivial and belongs to $Y_0 \cap Y_1$, which completes the proof. Q.E.D.

Proof of Theorem 1.9. We are going to prove that the number of eigenvalues of H below $\Sigma(H)$ is infinite if

$$K(\mathcal{M}_\delta, R; P) < K(M : P) = \Sigma(H) \qquad (\delta > 0 \text{ and } R > 0). \tag{1.21}$$

The proof is divided into several steps.

(1) It follows from (1.21) that for each $j = 1, 2, \cdots$ there exist a positive number R_j and $\phi_j \in C_0^\infty(\Gamma(\mathcal{M}_{(1/j)}, R_j))$ such that

$$\begin{cases} \text{(a) } R_1 < R_2 < \cdots < R_j < \cdots \to \infty, \\ \text{(b) } \|\phi_j\|^2 = 1 \qquad (j = 1, 2, \cdots), \\ \text{(c) } \operatorname{supp}(\phi_j) \cap \operatorname{supp}(\phi_k) = \emptyset \qquad (j \neq k), \\ \text{(d) } \rho[\phi_j] < \Sigma(H) \qquad (j = 1, 2, \cdots). \end{cases} \tag{1.22}$$

Let X_0 be the linear subspace spanned by $\{\phi_j\}_{j=1}^\infty$. Note that it follows from (b) and (c) of (1.22) that

$$\rho[f] < \Sigma(H)\|f\|^2 \tag{1.23}$$

for any $f \in X_0$.

(2) Let s be a positive number such that

$$\rho[\phi] + s\|\phi\|^2 \geq 0 \qquad (\phi \in C_0^\infty(\mathbf{R}^n)). \tag{1.24}$$

Define the sesquilinear form $\rho^{(s)}$ on $C_0^\infty(\mathbf{R}^n) \times C_0^\infty(\mathbf{R}^n)$ by

$$\rho^{(s)}[\phi, \varphi] = \rho[\phi, \varphi] + s(\phi, \varphi) \qquad (\phi, \varphi \in C_0^\infty(\mathbf{R}^n)). \tag{1.25}$$

Since the potential $q^{(s)} = q(x) + s$ satisfies Assumptions 1.1 and 1.3, $\rho^{(s)}$ is closable with its closure $\tilde{\rho}^{(s)}$. Let $H^{(s)}$ be the nonnegative selfadjoint operator determined through $\tilde{\rho}^{(s)}$. Obviously we have $D(\tilde{\rho}^{(s)}) = D(\tilde{\rho})$. It follows from the uniqueness of the selfadjoint operator determined by a symmetric closed sesquilinear form (Kato [13, Chapter VI, Theorem 2.1 and Corollary 2.4, pp.322–323]) that $H^{(s)} = H + sI$, where I is the identity operator on $L_2(\mathbf{R}^n)$. Let $E^{(s)}(\cdot)$ be the spectral measure associated with $H^{(s)}$. Applying the second representation theorem (Kato [13, Chapter VI, Theorem 2.23, p.331]) to the nonnegative closed sesquilinear form $\tilde{\rho}^{(s)}$, we see that

$$\tilde{\rho}^{(s)}[f] = \int_{\mathbf{R}} \lambda \, d\|E^{(s)}(\lambda)f\|^2 \qquad (f \in D(\tilde{\rho}^{(s)}) = D(\tilde{\rho})). \tag{1.26}$$

Therefore we have

$$\begin{cases} E^{(s)}((-\infty, \lambda)) = E((-\infty, \lambda - s)) & (\lambda \in \mathbf{R}) \\ \tilde{\rho}[f] = \displaystyle\int_{\mathbf{R}} \lambda \, d\|E(\lambda)f\|^2 & (f \in D(\tilde{\rho})), \end{cases} \tag{1.27}$$

where $E(\cdot)$ is the spectral measure associated with H.

(3) Suppose that $\dim E(-\infty, \Sigma(H)) = m < \infty$. Then, setting

$$\begin{cases} Y_1 = E([\Sigma(H), \infty))L_2(\mathbf{R}^n), \\ Y_2 = E((-\infty, \Sigma(H))L_2(\mathbf{R}^n), \\ Y_0 = X_0 \end{cases} \tag{1.28}$$

in Lemma 1.10, we see that there exists a nonzero $f_0 \in L_2(\mathbf{R}^n)$ which belongs to both X_0 and $E([\Sigma(H), \infty))L_2(\mathbf{R}^n)$. Therefore, f_0 satisfies (1.23) with f replaced by f_0, and it follows from the second relation of (1.27) that

$$\begin{aligned} \rho[f_0] &= \int_{\mathbf{R}} \lambda \, d\|E(\lambda)f_0\|^2 \\ &= \int_{\Sigma(H)}^{\infty} \lambda \, d\|E(\lambda)f_0\|^2 \\ &\geq \Sigma(H)\|f_0\|^2. \end{aligned} \tag{1.29}$$

These two inequalities contradict each other, which completes the proof.

Q.E.D.

In order to give a sufficient condition for the bound states of H to be finite we are going to start with

Proposition 1.11 (cf. [5, Theorem 15], [6, Theorem 10]). *Suppose that Assumptions* 1.1 *and* 1.3 *hold. Let η be a nonnegative, bounded C^∞ function on $\mathbf{R}^n$. Let H_η be the selfadjoint operator given by Definition* 1.6. *For any $R > 0$ define*

$$(1.30) \qquad \begin{cases} K_R = K_R(H_\eta) = \inf\{\rho_\eta[\phi] : \phi \in C_0^\infty(E_R),\ \|\phi\|_\eta = 1\}, \\ K_\infty = K_\infty(H_\eta) = \lim_{R\to\infty} K_R, \end{cases}$$

where

$$(1.31) \qquad E_R = \{x \in \mathbf{R}^n : |x| > R\}.$$

Then, setting $K_\infty = \lim_{R\to\infty} K_R$, we have

$$(1.32) \qquad K_\infty = \Sigma(H_\eta).$$

Since the idea of the proof is essentially the same as the proof of [5], Theorem 10 or [6], Theorem 15, we are going to give the proof in Appendix.

The following corollary will be used later.

Corollary 1.12. *Let η be a nonnegative, bounded C^∞ function on $\mathbf{R}^n$ such that all the first derivatives $\partial_j\eta$, $j = 1, 2, \cdots, n$ are also bounded on $\mathbf{R}^n$. Let ρ_η and $\rho = \rho_1$ be as in Definition* 1.2.

(i) *Then we have $D(\tilde\rho) \subset D(\tilde\rho_\eta)$, i.e., for $u \in D(\tilde\rho)$ and for any sequence $\{\phi_j\} \subset C_0^\infty(\mathbf{R}^n)$ such that $\phi_j \to u$ in $D(\tilde\rho)$, we have*

$$(1.33) \qquad \begin{cases} s - \lim_{j\to\infty} \phi_j = u \qquad \text{in } D(\tilde\rho_\eta), \\ \lim_{j\to\infty} \rho_\eta[\phi_j] = \tilde\rho_\eta[u]. \end{cases}$$

(ii) *On the other hand, for $u \in \tilde\rho_\eta$ we have $\eta u \in D(\tilde\rho)$*

Proof. Since it follows from (1.10) that $\{\phi_j\}$ is a Cauchy sequence both in $H^1(\mathbf{R}^n)$ and $L_2(\mathbf{R}^n, q_+dx)$. Then it is easy to see that $\{\phi_j\}$ is also a Cauchy sequence in the norm $\|\ \|_{\rho_\eta}$. The second part of the corollary follows directly from the fact that $\rho_\eta[\phi] = \rho[\eta\phi]$ for any $\phi \in C_0^\infty(\mathbf{R}^n)$. Q.E.D.

Assumption 1.13. Let $\mathcal{M}$ be the minimizing set associated with the operator P give in Definition 1.7. We assumed that $\mathcal{M}$ is a *proper subset* of the unit sphere S^{n-1}.

Assumption 1.13 is introduced to exclude a phenomenon known as the Efimov effect in the case of N-body Schrödinger operators. For more detailed discussion and the references, see [6, p.381–382].

Lemma 1.14. *Let Assumption* 1.13 *be satisfied. Let δ be a sufficiently small positive number, and let R be a positive number. Then there exist $\alpha = \alpha_{\delta,R}, \beta = \beta_{\delta,R} \in C_0^\infty(\mathbf{R}^n)$ satisfying*

(i) *$\alpha(x), \beta(x) \in [0,1]$ and $\alpha(x)^2 + \beta(x)^2 \equiv 1$ for all $x \in \mathbf{R}^n$;*

(ii) *$\mathrm{supp}(\alpha) \subset \Gamma(\mathcal{M}_\delta; R/2)$, with $\alpha \equiv 1$ in $\Gamma(\mathcal{M}_{\frac{\delta}{2}}; R)$;*

(iii) *$\mathrm{supp}(\beta) \subset X \setminus \Gamma(\mathcal{M}_{\frac{\delta}{2}}; R)$;*

(iv) *α and β are homogeneous of degree 0 in $\mathbf{R}^n \setminus B(R)$; and*

(v) *given $\epsilon > 0$ there exists $C(\epsilon) > 0$ such that*

$$|\nabla\alpha(x)|^2 + |\nabla\beta(x)|^2 \le \left(\epsilon\alpha(x)^2 + C_\epsilon\beta(x)^2\right)\chi_\Delta/|x|^2 \qquad (x \in \mathbf{R}^n),$$

where χ_Δ is the characteristic function of the set $\Delta := \Gamma(\mathcal{M}_\delta; R/2) \setminus \Gamma(\mathcal{M}_{\frac{\delta}{2}}; R)$, and $\mathcal{M}$ is the minimizing set.

For the proof see [6, Lemmas 9, 10, and Definition 11]. See also [9, Lemma 3.1]. We can take β as w in (i) of Definition 11 of [6].

Proposition 1.15. *Let Assumptions* 1.1, 1.3 *and* 1.13 *be satisfied. Let $\beta = \beta_{\delta,R}$, $\delta, R > 0$, be as in Lemma* 1.14. *Let ρ_β and H_β be as in Definitions* 1.2 *and* 1.6 *with η replaced by β, respectively. Then we have*

$$\Sigma(H_\beta) > \Sigma(H). \tag{1.34}$$

Proof. Set $\mathcal{N}(\delta) = S^{n-1} \setminus \mathcal{M}_\delta$, and let γ be a (sufficiently small) positive number. Set

$$\mathcal{N}(\delta)_\gamma = \{\omega \in S^{n-1} : \mathrm{dist}(\omega : \mathcal{N}(\delta)) < \gamma\}. \tag{1.35}$$

Let $T > R$. Then it follows that

$$K_T(H_\beta) \ge K(\mathcal{N}(\delta)_\gamma, T : P), \tag{1.36}$$

where $K_T(H_\beta)$ is as in Proposition 1.11, $K(\mathcal{N}(\delta)_\gamma, T : P)$ is as in (1.13), and we should note that $\rho_\beta[\phi] = \rho[\beta\phi]$, $\|\phi\|_\beta = \|\beta\phi\|$, and the cone

$\Gamma(\mathcal{N}(\delta)_\gamma, T)$ contains $\Gamma(\mathcal{N}(\delta), T)$. Letting $T \to \infty$ first and letting $\gamma \to 0$ next, we obtain

$$K_\infty(H_\beta) \geq K(\mathcal{N}(\delta) : P), \tag{1.37}$$

which implies by Proposition 1.11 that

$$\Sigma(H_\beta) \geq K(\mathcal{N}(\delta) : P). \tag{1.38}$$

Since $\operatorname{dist}(\mathcal{N}(\delta), \mathcal{M}) > 0$, Proposition 1.8 can be applied to get

$$\Sigma(H_\beta) \geq K(\mathcal{N}(\delta) : P) > K(\mathcal{M} : P) = \Sigma(H), \tag{1.39}$$

which completes the proof Q.E.D.

Theorem 1.17, which is one of our main results in this section, is the application of an abstract result by Glazman [10] to the operator H. Here we are going to give his result as follows:

Proposition 1.16 (Glazman, [10, p.13–15]). *Let A be a selfadjoint operator defined in a Hilbert space $\mathcal{H}$. Let λ_0 be a fixed real number. Let $E(\cdot)$ be the spectral measure associated with A. Then the dimension of $E((-\infty, \lambda_0))\mathcal{H}$ is finite if and only if there exists a linear subspaces F and G of $\mathcal{H}$ such that $\dim G < \infty$, $\mathcal{H}$ is the direct sum of F and G, and*

$$(Af - \lambda_0 f, f) \geq 0 \qquad (f \in F \cap D(A)), \tag{1.40}$$

where $(\ ,\ )$ denotes the inner product of $\mathcal{H}$, and $D(A)$ denotes the domain of A. Then the number of eigenvalues λ of A such that $\lambda < \lambda_0$ does not exceed the dimension of G.

Since the proof is given rather implicitly in Glazman [10], we shall give a proof in Appendix.

Let $\epsilon > 0$. In order to give a sufficient condition for the finiteness of the bound states of H, we are going to introduce an operator P_ϵ defined by

$$\begin{cases} P_\epsilon = -\displaystyle\sum_{j,k=1}^{n} \partial_j a^{jk}(x)\partial_k + q_\epsilon(x), \\ q_\epsilon(x) = q(x) - \dfrac{\epsilon}{|x|^2}\chi_\Delta, \end{cases} \tag{1.41}$$

where χ_Δ is as in Lemma 1.14. Since the behavior of q_ϵ at infinity is the same as q, we have

$$\Sigma(H_\epsilon) = K(M : P_\epsilon) = K(M : P) = \Sigma(H). \tag{1.42}$$

Theorem 1.17 ([7, Theorem 13]). *Let Assumptions* 1.1, 1.3, *and* 1.13 *hold. Suppose that there exist* $\delta_0 > 0$, $\epsilon > 0$, *and* $R_0 > 0$ *such that*

$$K(M_\delta, R; P_\epsilon) = \Sigma(H) \qquad \text{for all } \delta \le \delta_0, \text{ and } R \ge R_0. \tag{1.43}$$

Then H has no more than a finite number of eigenvalues in $(-\infty, \Sigma(H))$.

Proof. (1) Let $\alpha = \alpha_{\delta_0,R_0}$, $\beta = \beta_{\delta_0,R_0}$ be as in Lemma 1.14. Let $\phi \in C_0^\infty(\mathbf{R}^n)$. Then, using the IMS localization formula (Ismagilov [12], Morgan [14], Morgan and Simon [15]), and (v) of Lemma 1.14, we have

$$\begin{aligned} \rho[\phi] &= \int_{\mathbf{R}^n} \{|\nabla(\alpha\phi)|_A^2 + q|\alpha\phi|^2 - (|\alpha|_A^2 + |\beta|_A^2)|\phi|^2\}\, dx + \rho_\beta[\phi] \\ &\ge \int_{\mathbf{R}^n} \{|\nabla(\alpha\phi)|_A^2 + q_\epsilon|\alpha\phi|^2\}\, dx + \rho_\beta[\phi] - \int_{\mathbf{R}^n} \frac{C_\epsilon}{|x|^2}\chi_\Delta|\beta\phi|^2\, dx, \end{aligned} \tag{1.44}$$

where C_ϵ is a positive constant depending only on ϵ and χ_Δ is as in Lemma 1.14 with R and δ replaced by R_0 and δ_0. Then (1.44) is combined with (1.43) to give

$$\rho[\phi] \ge \Sigma(H)\|\alpha\phi\|^2 + \rho_\beta[\phi] - \int_{\mathbf{R}^n} \frac{C_\epsilon}{|x|^2}\chi_\Delta|\beta\phi|^2\, dx \qquad (\phi \in C_0^\infty(\mathbf{R}^n)). \tag{1.45}$$

(2) Define the linear form ρ'_β on $C_0^\infty(\mathbf{R}^n) \times C_0^\infty(\mathbf{R}^n)$ by

$$\rho'_\beta[\phi, \varphi] = \rho_\beta[\phi, \varphi] - \int_{\mathbf{R}^n} \frac{C_\epsilon}{|x|^2}\chi_\Delta\beta\phi\beta\overline{\varphi}\, dx. \tag{1.46}$$

Then, since the potential

$$q'(x) = q(x) - \frac{C_\epsilon}{|x|^2}\chi_\Delta(x) \tag{1.47}$$

satisfies Assumptions 1.1 and 1.3, the linear form ρ'_β is closable with its closure $\tilde{\rho}'_\beta$. Let H'_β be the selfadjoint operator in $L_{2,\beta}(\mathbf{R}^n)$ determined through $\tilde{\rho}'_\beta$. Thus, using the denseness of $C_0^\infty(\mathbf{R}^n)$ in $D(\tilde{\rho})$ and Corollary 1.12, we obtain from (1.45)

$$\tilde{\rho}[u] \ge \Sigma(H)\|\alpha u\|^2 + \tilde{\rho}'_\beta[u] \qquad (u \in D(\tilde{\rho})). \tag{1.48}$$

(3) By noting that $q'(x) - q(x) \to 0$ uniformly as $|x| \to \infty$, it follows from Propositions 1.11 and 1.15 that

$$\Sigma(H'_\beta) = \Sigma(H_\beta) > \Sigma(H). \tag{1.49}$$

Therefore, the spectrum of H'_β in $(-\infty, \Sigma(H))$ is only a finite number of eigenvalues with finite multiplicity. Let $\varphi_1, \varphi_2, \cdots, \varphi_m$ be the eigenfunctions corresponding to these eigenvalues. Set

$$F = \{u \in L_2(\mathbf{R}^n) : (u, \beta^2\varphi_j) = 0,\ j = 1, 2, \cdots, m\}. \tag{1.50}$$

Then $F^\perp$ is the linear m-dimensional subspace spanned by φ_j, $j = 1, 2, \cdots, m$. Let $u \in D(\tilde{\rho}) \cap F$. Then it follows from the second representation theorem of the closed symmetric linear form (e.g., Kato [13, Chapter IV, Theorem 2.23]) that

$$\begin{aligned} \tilde{\rho}'_\beta[u] &= \int_{\mathbf{R}} \lambda \, d\|E'_\beta(\lambda)u\|_\beta^2 \\ &= \int_{\Sigma(H)}^{\infty} \lambda \, d\|E'_\beta(\lambda)u\|_\beta^2 \\ &\geq \Sigma(H)\|\beta u\|^2, \end{aligned} \tag{1.51}$$

where $E'_\beta(\cdot)$ is the spectral measure associated with H'_β. This, together with (1.48), gives

$$\tilde{\rho}[u] \geq \Sigma(H)\{\|\alpha u\|^2 + \|\beta u\|^2\} = \Sigma(H)\|u\|^2 \tag{1.52}$$

for any $u \in D(\tilde{\rho}) \cap F$, and hence we have

$$(Hu - \Sigma(H)u, u) \geq 0 \qquad (u \in D(H) \cap F). \tag{1.53}$$

Thus, the condition (1.40) in Proposition 1.16 was verified, which completes the proof. Q.E.D.

Corollary 1.18. *The number of eigenvalues of H below $\Sigma(H)$ is less than the number of eigenvalues of H'_β below $\Sigma(H)$ for H'_β given above.*

Remark 1.19. Notice the gap between the conditions (1.17) in Theorem 1.9 and (1.43) in Theorem 1.17. We are led to the following question:

Under Assumptions 1.1, 1.3 *and* 1.13, *are there conditions which can be imposed upon M that will insure that* (1.17) *is a necessary and sufficient condition for the finiteness of $\sigma(H) \cap (-\infty, \Sigma(H))$?*

When stronger conditions are imposed on q, then (1.17) (with S^{n-1} substituted for M_δ and the location of M left unspecified) is known to be a sufficient condition for the finiteness of $\sigma(H) \cap (-\infty, \Sigma(H))$, see Simon [18, pp.517–518], and the related "open question" on p.518 of that article. However, these stronger conditions do not include N-body systems for $N \geq 3$.

Recently Donig [3] answered the open question in the affirmative. While the conditions imposed on his potential is slightly more strict than ours, Coulomb potentials satisfy his condition.

§2. Schrödinger operators of atomic type

In this section we consider the $(N+1)$-body Schrödinger operator of atomic-type

$$P = P_N = \sum_{i=1}^{N} \left(-\frac{1}{2m_i}\Delta_i + v_{0i}(x^i) \right) + \sum_{1 \leq i < j \leq N} v_{ij}(x^i - x^j), \tag{2.1}$$

in $\mathbf{R}^{\nu N}$, where $N \geq 3$,

$$\begin{cases} x^i = (x_1^i, x_2^i, \cdots, x_\nu^i) \in \mathbf{R}^\nu & (i = 1, 2, \cdots, N), \\ m_i > 0 \quad (i = 1, 2, \cdots, N), \\ x = (x^1, x^2, \cdots, x^N) \in \mathbf{R}^{\nu N}, \end{cases} \tag{2.2}$$

and Δ_i is the Laplacian in $\mathbf{R}^\nu$ with respect to the variables $x^i = (x_1^i, x_2^i, \cdots, x_\nu^i)$ with $\nu \geq 3$. The atomic Hamiltonian is given by

$$P = P(N, Z) = \sum_{i=1}^{N} \left(-\frac{1}{2m}\Delta_i - \frac{Z}{|x^i|}\right) + \sum_{1 \leq i < j \leq N} \frac{1}{|x^i - x^j|}, \tag{2.3}$$

where N, ν are as above, and m and Z are positive numbers corresponding to the mass and charge of the nucleus, respectively.

The sesquilinear form ρ associated with the operator (2.1) is given by

$$\begin{aligned} \rho[\phi, \varphi] = &\sum_{i=1}^{N} \frac{1}{2m_i} \int_{\mathbf{R}^{\nu N}} \nabla^i \phi(x) \cdot \overline{\nabla^i \varphi}\, dx \\ &+ \sum_{i=1}^{N} \int_{\mathbf{R}^{\nu N}} v_{0i}(x^i) \phi(x) \overline{\varphi(x)}\, dx \\ &+ \sum_{1 \leq i < j \leq N} \int_{\mathbf{R}^{\nu N}} v_{ij}(x^i - x^j) \phi(x) \overline{\varphi(x)}\, dx \end{aligned} \tag{2.4}$$

for $\phi, \varphi \in C_0^\infty(\mathbf{R}^{\nu N})$, where

$$\nabla^i = (\frac{\partial}{\partial x_1^i}, \frac{\partial}{\partial x_2^i}, \cdots, \frac{\partial}{\partial x_\nu^i}) \tag{2.5}$$

for $i = 1, 2, \cdots, N$. For the potentials v_{ij}, we assume the following

Assumption 2.1. For $0 \leq i < j \leq N$, v_{ij} is a real-valued function satisfying

(i) $v_{ij} \in L^1_{\mathrm{loc}}(\mathbf{R}^\nu)$,
(ii) $\lim_{|y| \to \infty} v_{ij}(y) = 0$, and
(iii) $(v_{ij})_- \in M(\mathbf{R}^\nu)$.

Then, setting

$$V_{ij}(x) = \begin{cases} v_{0j}(x^j) & (i = 0, j = 1, 2, \cdots, N), \\ v_{ij}(x^i - x^j) & (1 \leq i < j \leq N), \end{cases} \tag{2.6}$$

where $x = (x^1, x^2, \cdots, x^N) \in \mathbf{R}^{\nu N}$ as in (2.2), and

$$q(x) = \sum_{j=1}^{N} V_{0j}(x) + \sum_{1 \leq i < j \leq N} V_{ij}(x), \tag{2.7}$$

we easily see that $q(x)$ satisfies Assumptions 1.1 and 1.3 (see Agmon [1, Lemma 4.7] for the proof that $q_- \in M(\mathbf{R}^{\nu N})$. Thus, the corresponding sesquilinear form ρ (or, more exactly, the closure $\tilde{\rho}$ of ρ) determines a selfadjoint operator in $L_2(\mathbf{R}^{\nu N})$. Henceforth, the selfadjoint realization will be denoted by P again.

We are now introducing the subsystems of the operator P.

Definition 2.2 (Subsystems of P).

Let $S^{\nu N-1}$ be the unit sphere of $\mathbf{R}^{\nu N}$. For $\omega \in S^{\nu N-1}$ define the subsystem P_ω of P by

$$P_\omega = -\sum_{j=1}^{N} \frac{1}{2m_j} \Delta_j + \sum_{\omega^i = 0} v_{0i}(x^i) + \sum_{\omega^i = \omega^j} v_{ij}(x^i - x^j), \tag{2.8}$$

where $\omega = (\omega^1, \omega^2, \cdots, \omega^N)$ and $\sum_{\omega^i=0}$ [or $\sum_{\omega^i=\omega^j}$] means summation over those indices i for which $\omega^i = 0$ [or those pair of indices (i, j), $1 \leq i < j \leq N$, for which $\omega^i = \omega^j$]. The selfadjoint realization of P_ω in $L^2(\mathbf{R}^{\nu N})$ will continue to be denoted by P_ω.

The following fact given by Agmon [1, Lemma 4.8, p.66] will play an important role:

Proposition 2.3 ($K(\omega)$ and subsystems (Agmon [1, Lemma 4.8])). *Let P be as in* (2.2) *and satisfy Assumption* 2.1. *Let P_ω be the subsystem of P defined above. Then, for any $\omega \in S^{\nu N-1}$*

$$K(\omega; P) = K(\omega; P_\omega) = \Sigma(P_\omega) = \Lambda(P_\omega), \tag{2.9}$$

where $\Lambda(A)$ and $\Sigma(A)$ denote the infimum of the spectrum and essential spectrum of A, respectively.

Let $\mathcal{M}$ be the minimizing set for the Schrödinger operator P of atomic type (see Definition 1.7).

Definition 2.4 (Sets $\mathcal{M}_i$ and subsystems P_i). For $i = 1, 2, \cdots, N$, define

(2.10)
$$\mathcal{M}_i = \{\omega = (\omega^1, \omega^2, \cdots, \omega^N) \ : \omega^j = \delta_{ij}\eta \text{ for } \eta \in S^{\nu-1}, j = 1, 2, \cdots, N\},$$

where $\delta_{ii} = 1$ and $\delta_{ij} = 0$ for $j \neq i$. The set $\mathcal{M}_i$ is a closed subset of $S^{\nu N-1}$. Let P_ω be given by (2.8). Since for any $\omega \in \mathcal{M}_i$ the subsystem P_ω has the same form, we set $P_\omega = P_i$ for $\omega \in \mathcal{M}_i$, i.e.,

$$P_i = -\sum_{j=1}^{N} \frac{1}{2m_j}\Delta_j + \sum_{j\neq i} v_{oj}(x^j) + \sum_{\substack{1\le j<k\le N \\ j\neq i \text{ and } k\neq i}} v_{jk}(x^j - x^k). \tag{2.11}$$

The subsystem P_i is the subsystem of $(N-1)$ electrons $x^1, \cdots, x^{i-1}, x^{i+1}, \cdots, x^N$.

In this section we assume that the lower bound $\Sigma(P)$ of the essential spectrum of P is determined only by subsystems of $N-1$ electrons.

Assumption 2.5. Let P be the atomic-type Hamiltonian (2.1). Let $\mathcal{M}$ be the minimizing set of P. Assume that

$$\mathcal{M} \subset \bigcup_{i=1}^{N} \mathcal{M}_i. \tag{2.12}$$

Assumption 2.5 implies that the minimizing set $\mathcal{M}$ is not only a closed set of $S^{\nu N-1}$, but also a proper subset of $S^{\nu N-1}$. Thus, this assumption implies Assumption 1.12 for our operator P.

Definition 2.6 (Operators P_i' and L_i). Let P be as above and for each $i = 1, 2, \cdots, N$ define

$$(2.13)\qquad P_i' = -\sum_{j\neq i} \frac{1}{2m_j}\Delta_j + \sum_{j\neq i} v_{0j}(x^j) + \sum_{\substack{1\leq j<k\leq N\\ j\neq i \text{ and } k\neq i}} v_{jk}(x^j - x^k),$$

The selfadjoint realization of P_i' in $L_2(\mathbf{R}^{\nu(N-1)})$ is also denoted by P_i'. We also set

$$(2.14)\qquad L_i = P - P_i' = -\frac{1}{2m_i}\Delta_i + v_{0i}(x^i) + \sum_{\substack{1\leq j<k\leq N\\ j=i \text{ or } k=i}} v_{jk}(x^j - x^k).$$

Now we are in a position to give a criterion for the finiteness of the bound states of the atomic-type Hamiltonian P.

Theorem 2.7 (Finiteness of bound states ([7, Theorem 3.4)).
Let P be given by (2.1) *and let Assumptions* 2.1 *and* 2.5 *be satisfied. Let P_i' and L_i be as above. Suppose there exist positive numbers δ_0, R_0, and ϵ such that*

$$(2.15)\qquad (L_i\phi, \phi)_{L^2(\mathbf{R}^{\nu N})} \geq \int_{\mathbf{R}^{\nu N}} \frac{\epsilon}{|x|^2}|\phi|^2\, dx$$

for each $i = 1, 2, \cdots, N$ such that $\mathcal{M}_i \subset \mathcal{M}$ and for every $\phi \in C_0^\infty(\Gamma((\mathcal{M}_i)_{\delta_0}; R_0))$. Then P has at most a finite number of bound states.

For the proof see the proof of Theorem 3.4 in [7].

Let us next discuss the infiniteness of the bound states. It follows from Assumption 2.5 that there exist some $i \subset \{1, 2, \cdots, N\}$ such that $\mathcal{M}_i \subset \mathcal{M}$. In view of Theorem 1.9 we are looking for a condition which guarantees the existence of a sequence of functions $\{F_n\}$ such that

$$(2.16)\qquad F_n \in C_0^\infty(\Gamma(\mathcal{M}_i)_{\delta_n}; R_n))$$

with $\delta_n \downarrow 0$ and $R_n \uparrow \infty$ as $n \to \infty$, and

$$(2.17)\qquad \rho[F_n] = (PF_n, F_n)_{L^2(\mathbf{R}^{\nu N})} < \Sigma(P) \qquad (n = 1, 2, \cdots, N),$$

which gives the inequality (1.21) immediately. Write $x \in \mathbf{R}^{\nu N}$ as

$$(2.18)\qquad x = (x^i, x') \qquad (x' = (x^1, \cdots, x^{i-1}, x^{i+1}, \cdots, x^N)).$$

We are going to find F_n with the form

$$F_n(x^i, x') = \theta_n(x^i)\phi_n(x') \qquad (n = N_0, N_0 + 1, \cdots), \tag{2.19}$$

where N_0 is a positive integer determined later. As for θ_n, we have the following

Proposition 2.8 [7, Proposition 4.4]. *Let $q > 1$. Then there exists a sequence $\{\theta_n\} = \{\theta_{n,q}\}$ of functions on $\mathbf{R}^\nu$ such that, for $n = 1, 2, \cdots$,*
1) $\theta_n \in C_0^\infty(\mathbf{R}^\nu)$,
2) $\|\theta_n\|_{L^2(\mathbf{R}^\nu)} = 1$,
3) $\operatorname{supp}\theta_n \subset \{x^i \in \mathbf{R}^\nu : n^q \leq |x^i| \leq 5n^q\}$,
4) *there exists a constant $C_2 = C_2(q)$, independent of $n = 1, 2, \cdots$, satisfying*

$$0 \leq (-\frac{1}{2m_i}\Delta_i\theta_n, \theta_n)_{L^2(\mathbf{R}^\nu)} \leq \frac{C_2}{n^{2q}}.$$

The construction of θ_n, $n = 1, 2, \cdots$, is easy and direct. See the proof of Proposition 4.4 of [7].

In order to discuss the construction of $\phi_n(x')$, we need the next

Assumption 2.9. The potentials v_{ij}, $0 \leq i < j \leq N$, satisfy

$$v_{ij} \in M_{\mathrm{loc}}(\mathbf{R}^\nu). \tag{2.20}$$

Let i be as above. Then it follows from the HVZ theorem (see [11], [19], [22]) combined with Assumption 2.5 that $\Sigma(P) = \Lambda(P_i') < 0$ and $\Sigma(P_i') > \Lambda(P_i')$, and hence $\Lambda(P_i')$ is the lowest eigenvalue (ground state) of P_i' with the eigenfunction $\Phi_i(x')$. In fact, suppose that $\Sigma(P_i') = \Lambda(P_i')$. Then we see from the HVZ theorem that there should exist a subsystem P_i'' of P_i' such that

$$\Lambda(P_i'') = \Sigma(P_i') = \Lambda(P_i') = \Sigma(P). \tag{2.21}$$

This contradicts Assumption 2.5 since the lower bound $\Lambda(P_i'')$ of the subsystem P_i'', which is different from any P_j', $j = 1, 2, \cdots, N$ coincides with $\Sigma(P)$. It follows from [1, Theorem 5.9] that the eigenfunction $\Phi_i(x')$ decays exponentially. Similarly, using Assumption 2.9, too, we can prove that any first derivatives of $\Phi_i(x')$ decay exponentially ([7, Proposition 4.2]). Now we shall prove that $\phi_n(x')$ in (2.19) can be constructed by truncating Φ_i using a smooth function, and then approximating with functions in $C_0^\infty(\mathbf{R}^{\nu(N-1)})$.

Proposition 2.10. *Let Assumptions* 2.1, 2.5, *and* 2.9 *hold. Then, for some positive integer* N_0 *and each integer* $n \geq N_0$, *there exists* $\phi_n \in C_0^\infty(\mathbf{R}^{\nu(N-1)})$ *satisfying*

$$\begin{cases} \|\phi_n\|_{L^2(\mathbf{R}^{\nu(N-1)})} = 1, \\ \operatorname{supp}\phi_n \subset \{x' \in \mathbf{R}^{\nu(N-1)} \ : |x'| \leq 2n\}, \\ (P_i'\phi_n, \phi_n)_{L^2(\mathbf{R}^{\nu(N-1)})} \leq \Sigma(P) + C_1 \dfrac{e^{-nc_0}}{n} \end{cases} \tag{2.22}$$

with positive constants c_0 *and* C_1.

For an integer $1 \leq i \leq N$ set

$$I_i(x) = v_{0i}(x^i) + \sum_{\substack{1\leq j<k\leq N \\ j=i \text{ or } k=i}} v_{jk}(x^j - x^k). \tag{2.23}$$

We have $P = P_i + I_i$.

Theorem 2.11 (Infiniteness of bound states, [7, Theorem 4.7]). *Let Assumptions* 2.1, 2.5 *and* 2.9 *be satisfied. Suppose that there exists an integer* $1 \leq i \leq N$, $\mathcal{M}_i \subset \mathcal{M}$, *positive numbers* δ_0, R_0, c_*, *and* $s \in (0, 2)$ *such that*

$$I_i(x) \leq -c_*|x^i|^{-s} \qquad (x \in \Gamma((\mathcal{M}_i)_{\delta_0}; R_0)). \tag{2.24}$$

Then P *has infinitely many bound states.*

For the proof see the proof of [7, Theorem 4.7] and [7, Proposition 4.5]. We have only to show that the sequence $\{F_n\}$ above satisfies (2.17).

The following theorem on the finiteness and infiniteness of the bound states for the atomic Hamiltonian is well-known: Zhislin ([22], [23], [24], [25]), Yafaev ([20], [21]), and others.

Theorem 2.12 (Zhislin ([22], [23], [24], [25]), Yafaev ([20], [21]), and others). *Let* $N, \nu \geq 3$ *be integers. Suppose that Assumption* 3.2 *is satisfied for* $P = P(N, Z)$ *given by* (2.2).

(i) *Suppose that*

$$Z \leq N - 1. \tag{2.25}$$

Then $P = P(N, Z)$ *has at most a finite number of bound states.*

(ii) *Suppose that*

$$Z > N - 1. \tag{2.26}$$

Then $P = P(N, Z)$ has infinitely many bound states.

Using Theorems 2.7 and 2.11 we can give a proof of the above celebrated theorem except the case $Z = N - 1$. Let $Z < N - 1$. Since it is easy to see that, for $x \in \Gamma((\mathcal{M}_i)_\delta; R)$ with $0 < 2\delta < 1$, we have

$$\begin{cases} |x^i| > (1-\delta)|x|, \\ |x^i - x^j| < (1+2\delta)|x|, \end{cases} \tag{2.27}$$

it follows that, for $\phi \in C_0^\infty(\Gamma((\mathcal{M}_i)_\delta; R))$,

$$\begin{aligned} (L_i\phi, \phi)_{L^2(\mathbf{R}^{\nu N})} &\geq \int_{\mathbf{R}^{\nu N}} [-\frac{Z}{|x^i|} + \sum_{j \neq i} |x^i - x^j|]|\phi|^2\, dx \\ &\geq \int_{\mathbf{R}^{\nu N}} [\frac{N-1}{(1+2\delta)|x|} - \frac{Z}{(1-\delta)|x|}]|\phi|^2\, dx \\ &\geq \int_{\mathbf{R}^{\nu N}} \frac{\epsilon}{|x|^2}|\phi|^2\, dx \end{aligned} \tag{2.28}$$

if $\delta > 0$ is sufficiently small and $R > 1$, where

$$\epsilon = \frac{N-1}{1+2\delta} - \frac{Z}{1-\delta} > 0. \tag{2.29}$$

Thus we see that the condition (2.15) in Theorem 2.7 is satisfied for every $i = 1, 2, \cdots, N$. In the case that $Z > N - 1$, see the proof of Theorem 4.8 of [7].

Concerning Assumption 2.5, [7] gave a proof of the following theorem (Theorem 5.2):

Theorem 2.13. *Let $N, \nu \geq 3$ be integers and $P = P(N, Z)$ be as in* (2.2). *Suppose that*

$$Z > N - 2. \tag{2.30}$$

Then the operator $P = P(N, Z)$ satisfies Assumption 2.5, *i.e., the lower bound of P is determined only by subsystems of $N - 1$ electrons.*

Finally consider the case where the potentials are "short-range", i.e., $v_{ij} \in L_{\nu/2}(\mathbf{R}^\nu)$. It is known that the bound states are finite in this case (Sigal [17]). We are going to give another simple proof for the slightly more general version.

Theorem 2.14. *Let Assumptions* 2.1 *and* 2.5 *hold. Suppose that*

$$(v_{jk})_-(\cdot) \in L_{\nu/2}(\mathbf{R}^\nu) \qquad (0 \le j < k \le N,\ j = i \ \ or\ k = i) \tag{2.31}$$

for any i such that $\mathcal{M}_i \subset \mathcal{M}$, where $(v_{jk})_-$ is the negative part of v_{jk}. Then the operator P given by (2.1) *has at most finite bound states.*

Proof. Let $\epsilon > 0$. Let δ_0 be a positive number such that $1 - 2\delta_0 > 0$. Then there exists $R_0 > 0$ satisfying

(2.32)
$$\Big[\int_{|y|>cR_0} \{(v_{jk})_-(y)\}^{\nu/2}\,dy\Big]^{2/\nu} < \epsilon \qquad \le j < k \le N,\ j = i \text{ or } k = i),$$

where $c = 1 - 2\delta_0$. Let $\phi \in C_0^\infty(\Gamma((\mathcal{M}_i)_{\delta_0}; R_0))$. Since we have

$$x \in \Gamma((\mathcal{M}_i)_{\delta_0}; R_0) \implies \begin{cases} |x^i| > (1-\delta_0)R_0, \\ |x^i - x^j| > (1 - 2\delta_0)R_0, \end{cases} \tag{2.33}$$

it follows from the Hölder inequality that

(2.34)
$$\begin{cases} \displaystyle\int_{\mathbf{R}^\nu} (v_{0i})_-(x^i)|\phi|^2\,dx^i \\ \displaystyle\qquad \le \Big[\int_{|y|>cR_0} \{(v_{0i})_-\}^{\nu/2}\,dx^i\Big]^{2/\nu}\Big[\int_{\mathbf{R}^\nu} |\phi|^{2\nu/(\nu-2)}\,dx^i\Big]^{(\nu-2)/\nu}, \\ \displaystyle\int_{\mathbf{R}^\nu} (v_{jk})_-(x^j - x^k)|\phi|^2\,dx^i \\ \displaystyle\qquad \le \Big[\int_{|y|>cR_0} \{(v_{jk})_-\}^{\nu/2}\,dx^i\Big]^{2/\nu}\Big[\int_{\mathbf{R}^\nu} |\phi|^{2\nu/(\nu-2)}\,dx^i\Big]^{(\nu-2)/\nu}, \end{cases}$$

where $1 \le j < k \le N$, $j = i$ or $k = i$. It follows from a Sobolev-type inequality (e.g., [4, Theorem III.3.6]) that

$$\Big[\int_{\mathbf{R}^\nu} |\phi|^{2\nu/(\nu-2)}\,dx^i\Big]^{(\nu-2)/\nu} \le \gamma \int_{\mathbf{R}^\nu} |\nabla^i \phi|^2\,dx^i, \tag{2.35}$$

γ being a positive constant depending only on ν. Then we obtain from

(2.34) and (2.35) that

(2.36)
$$\int_{\mathbf{R}^\nu} [(v_{0i})_-(x^i) + \sum_{\substack{1\le j<k\le N \\ j=i \text{ or } k=i}} (v_{jk})_-(x^j - x^k)] |\nabla^i \phi|^2 \, dx^i \le (2N-1)\epsilon \int_{\mathbf{R}^\nu} |\nabla^i \phi|^2 \, dx^i$$

for any $\phi \in C_0^\infty(\Gamma((\mathcal{M}_i)_{\delta_0}; R_0))$. The inequality (2.36) is combined with the Hardy inequality

$$\int_{\mathbf{R}^\nu} \frac{|\phi|^2}{|x|^2} \, dx^i \le \int_{\mathbf{R}^\nu} \frac{|\phi|^2}{|x^i|^2} \, dx^i \le \frac{4}{(\nu-2)^2} \int_{\mathbf{R}^\nu} |\nabla^i \phi|^2 \, dx^i, \tag{2.37}$$

where $\phi \in C_0^\infty(\mathbf{R}^{\nu N})$, to give

(2.38)
$$(L_i\phi, \phi)_{L_2(\mathbf{R}^{\nu N})} - \int_{\mathbf{R}^{\nu N}} \frac{\epsilon|\phi|^2}{|x|^2} \, dx \ge \int_{\mathbf{R}^{\nu N}} [1 - (2N-1)\gamma\epsilon - \frac{4\epsilon}{(\nu-2)^2}] |\nabla^i \phi|^2 \, dx$$

for any $\phi \in C_0^\infty(\Gamma((\mathcal{M}_i)_{\delta_0}; R_0))$. Therefore, choosing $\epsilon > 0$ sufficiently small, we see that the right-hand side of (2.37) is nonnegative for $\phi \in C_0^\infty(\Gamma((\mathcal{M}_i)_{\delta_0}; R_0))$. Thus the condition (2.15) is satisfied, which complete the proof. Q.E.D.

Appendices

A.1 The infimum of the essential spectrum of H_η

Proof of Proposition 1.11.

(1) Let $\Lambda \in \sigma_e(H_\eta)$ with a singular sequence $\{u_j\}$, i.e.,

$$\begin{cases} \text{(a) } u_j \in D(H_\eta) \qquad (j = 1, 2, \cdot), \\ \text{(b) } \|u_j\|_\eta = 1 \qquad (j = 1, 2, \cdot), \\ \text{(c) w} - \lim\limits_{j\to\infty} u_j = 0 \qquad in L_{2,\eta}(\mathbf{R}^n), \\ \text{(d) s} - \lim\limits_{j\to\infty} (H_\eta - \lambda) u_j = 0 \qquad in L_{2,\eta}(\mathbf{R}^n). \end{cases} \tag{A.1.1}$$

Introduce an inner product $(\ ,\ )_{\rho_\eta}$ and norm $\|\ \|_{\rho_\eta}$ in $C_0^\infty(\mathbf{R}^n)$ by

$$\text{(A.1.2)} \qquad \begin{cases} (\phi,\varphi)_{\rho_\eta} = \rho_\eta[\phi,\varphi] + C_1(\phi,\varphi)_\eta, \\ \|\phi\|_{\rho_\eta} = [(\phi,\phi)_{\rho_\eta}]^{1/2}, \end{cases}$$

where the positive constant C_1 is as in (1.10). Note that we obtain from (1.10)

$$\text{(A.1.3)} \qquad \begin{cases} \displaystyle\int_{\mathbf{R}^n} |\nabla(\eta\phi)|^2\,dx \le \frac{2}{c_0}\|\phi\|^2_{\rho_\eta}, \\ \|\phi\|_\eta \le \|\phi\|_{\rho_\eta} \end{cases}$$

for $\phi \in C_0^\infty(\mathbf{R}^n)$. Then $C_0^\infty(\mathbf{R}^n)$ becomes a pre-Hilbert space with the inner product $(\ ,\ )_{\rho_\eta}$ and norm $\|\ \|_{\rho_\eta}$, and the domain $D(\tilde\rho_\eta)$ of the closed linear form $\tilde\rho_\eta$ is the completion of $C_0^\infty(\mathbf{R}^n)$ by $\|\ \|_{\rho_\eta}$. The inner product and norm of $D(\tilde\rho_\eta)$ will be denoted again by $(\ ,\ )_{\rho_\eta}$ and norm $\|\ \|_{\rho_\eta}$. We have

$$\text{(A.1.4)} \qquad \begin{cases} (u,v)_{\rho_\eta} = \tilde\rho_\eta[u,v] + C_1(u,v)_\eta, \\ \|u\|_{\rho_\eta} \ge \|\phi\|^2_\eta \end{cases}$$

for $u, v \in D(\tilde\rho_\eta)$.

(2) Since $C_0^\infty(\mathbf{R}^n)$ is dense in the Hilbert space $D(\tilde\rho_\eta)$, there exists a sequence $\{\phi_j\} \subset C_0^\infty(\mathbf{R}^n)$ such that

$$\text{(A.1.5)} \qquad \|u_j - \phi_j\|_{\rho_\eta} \to 0 \qquad (j \to \infty).$$

Then it follows that

$$\text{(A.1.6)} \qquad \begin{cases} \text{(a) } \|\phi_j\|_\eta \to 1 \qquad (j\to\infty), \\ \text{(b) } \mathrm{w} - \displaystyle\lim_{j\to\infty} \phi_j = 0 \qquad \text{in } D(\tilde\rho_\eta), \\ \text{(c) } \rho_\eta[\phi_j] \to \lambda \qquad (j \to \infty). \end{cases}$$

In fact, (a) follows directly from (b) of (A.1.1) and (A.1.3). As for (b), we have for any $v \in D(\tilde\rho_\eta)$

(A.1.7)

$$\begin{aligned} (\phi_j, v)_{\rho_\eta} &= (\phi_j - u_j, v)_{\rho_\eta} + (u_j, v)_{\rho_\eta} \\ &= (\phi_j - u_j, v)_{\rho_\eta} + ((H_\eta - \lambda)u_j, v)_\eta + (\lambda + C_1)(u_j, v) \\ &\to 0 \end{aligned}$$

as $j \to \infty$, where we have used (c), (d) of (A.1.1), and we should note that

$$\tilde{\rho}_\eta[u_j, v] = (H_\eta u_j, v)_\eta \tag{A.1.8}$$

(see, e.g., Kato [13, Theorem VI.2.1, p.322]). Finally, since we have

$$\begin{aligned}\rho_\eta[\phi_j] &= \|\phi_j\|^2_{\rho_\eta} - C_1\|\phi_j\|^2_\eta \\ &= \|u_j\|^2_{\rho_\eta} - C_1\|u_j\|^2_\eta + \gamma_j \\ &= \tilde{\rho}_\eta[u_j] + \gamma_j \\ &= ((H_\eta - \lambda)u_j, u_j)_\eta + \lambda\|u_j\|^2_\eta + \gamma_j \\ &= \lambda + ((H_\eta - \lambda)u_j, u_j)_\eta + \gamma_j,\end{aligned} \tag{A.1.9}$$

where $\gamma_j \to 0$ and $((H_\eta - \lambda)u_j, u_j)_\eta$ converge to 0 as $j \to \infty$, we obtain (d).

(3) Let $\alpha(x)$ be a C^∞ function on $\mathbf{R}^n$ such that

$$\alpha(x) = \begin{cases} 0 & x \in B_R = \{x \in \mathbf{R}^n : |x| \le R\}, \\ 1 & x \in E_{R+1}, \end{cases} \tag{A.1.10}$$

$0 \le \alpha \le 1$, and $|\nabla\alpha|$ is bounded on $\mathbf{R}^n$. Set $|\xi|_A = [< \xi, \xi >_A]^{1/2}$ for $\xi \in \mathbf{C}^n$. Then it follows from the identity

$$|\nabla(\alpha\eta\phi)|^2_A = \alpha^2|\nabla(\eta\phi)|^2_A + |\nabla\alpha|^2_A|\eta\phi|^2 + 2\alpha\eta\Re\{\overline{\phi} < \nabla(\eta\phi), \nabla\alpha >_A\}, \tag{A.1.11}$$

where $\phi \in C_0^\infty(\mathbf{R}^n)$, that

$$|\nabla(\alpha\eta\phi)|^2_A \le (1+\delta)|\nabla(\eta\phi)|^2_A + C_\delta\chi_{R,R+1}|\eta\phi|^2 \tag{A.1.12}$$

for $\phi \in C_0^\infty(\mathbf{R}^n)$, where δ is an arbitrary positive number, $\chi_{R,R+1}$ is the characteristic function of $\{x \in \mathbf{R}^n : R < |x| \le R+1\}$, and

$$C_\delta = (1 + \delta^{-1}) \max_{x \in \mathbf{R}^n} |\nabla\alpha|^2_A. \tag{A.1.13}$$

Further we have

$$\begin{aligned} q(x)\alpha^2|\eta\phi|^2 &= q_+\alpha^2|\eta\phi|^2 - q_-\alpha^2|\eta\phi|^2 \\ &= \alpha^2 q_+|\eta\phi|^2 - q_-|\eta\phi|^2 + (1-\alpha^2)q_-|\eta\phi|^2 \\ &\le q|\eta\phi|^2 + (1-\alpha^2)q_-|\eta\phi|^2. \end{aligned} \tag{A.1.14}$$

Therefore, it follows that

(A.1.15)

$$\rho_\eta[\alpha\phi] \le (1+\delta)\rho_\eta[\phi] + C_\delta \int_{B_{R+1}} |\eta\phi|^2\,dx + \int_{\mathbf{R}^n} (1-\alpha^2)q_-|\eta\phi|^2\,dx.$$

Here it follows from Assumption 1.3 and (A.1.3) that

(A.1.16)

$$\begin{aligned}
&\int_{\mathbf{R}^n} (1-\alpha^2)q_-|\eta\phi|^2\,dx \\
&\quad \le \Big\{\int_{\mathbf{R}^n} q_-|\eta\phi|^2\,dx\Big\}^{1/2}\Big\{\int_{\mathbf{R}^n} q_-|(1-\alpha^2)\eta\phi|^2\,dx\Big\}^{1/2} \\
&\quad \le \Big\{\int_{\mathbf{R}^n} |\nabla(\eta\phi)|^2\,dx + C(1)\|\phi\|_\eta^2\Big\}^{1/2} \\
&\qquad \Big\{\delta^2\int_{\mathbf{R}^n} |\nabla((1-\alpha^2)\eta\phi)|^2\,dx + C(\delta^2)\|(1-\alpha^2)\phi\|_\eta^2\Big\}^{1/2} \\
&\quad \le C_2\|\phi\|_{\rho_\eta}\Big\{\delta\|\phi\|_{\rho_\eta} + C'(\delta)\Big[\int_{B_{R+1}} |\eta\phi|^2\,dx\Big]^{1/2}\Big\},
\end{aligned}$$

where $C(1)$ and $C(\delta^2)$ are as in (1.6) with ϵ replaced by 1 and δ^2, respectively, C_2 is a positive constant independent of δ, and $C'(\delta)$ is a positive constant which may depend on δ. Thus, combining (A.1.14) with (A.1.15), substituting $\phi = \phi_j$, and taking note of the definition of K_R, we obtain with another constants C_2' and $C''(\delta)$

(A.1.17)

$$\begin{aligned}
&K_R\|\alpha\phi_j\|_\eta^2 \\
&\quad \le \rho_\eta[\alpha\phi_j] \\
&\quad \le (1+\delta)\rho_\eta[\phi_j] + C_2'\|\phi_j\|_{\rho_\eta}\Big\{\delta\|\phi_j\|_{\rho_\eta} + C''(\delta)\Big[\int_{B_{R+1}} |\eta\phi_j|^2\,dx\Big]^{1/2}\Big\}.
\end{aligned}$$

(4) Using (A.1.6) and the Rellich theorem, we see that, for any $0 < R < \infty$,

$$\int_{B_R} |\eta\phi_j|^2\,dx \to 0 \tag{A.1.18}$$

as $j \to \infty$, where we should note that (c) of (A.1.1) and (A.1.5) imply that

$$\mathrm{w}-\lim_{j\to\infty} \phi_j = 0 \qquad \text{in } L_{2,\eta}(\mathbf{R}^n). \tag{A.1.19}$$

From (A.1.18) we see that

$$\begin{aligned}
\text{(A.1.20)} \qquad & \lim_{j\to\infty} \|\alpha\phi_j\|_\eta^2 \\
&= \lim_{j\to\infty} \{ \int_{\mathbf{R}^n} |\phi_j \eta|^2 \, dx + \int_{\mathbf{R}^n} (1-\alpha^2)|\phi_j\eta|^2 \, dx \} \\
&= \lim_{j\to\infty} \|\phi_j\|_\eta^2 \\
&= 1.
\end{aligned}$$

Thus, by letting $j \to \infty$ in (A.1.17) and using (c) of (A.1.6), (A.1.18), and (A.1.20), it follows that

$$\text{(A.1.21)} \qquad K_\infty \le (1+\delta)\lambda + \delta C_2' C_3$$

with $C_3 = \sup_j \|\phi_j\|_{\rho_\eta}$. Since δ is arbitrary, we have proved that $K_\infty \le \lambda$ for any $\lambda \in \sigma_e(H_\eta)$, i.e., $K_\infty \le \Sigma(H_\eta)$.

(5) Let $\mu < \Sigma(H_\eta)$. Then in $(-\infty, \mu]$ the spectrum $\sigma(H_\eta)$ of H_η consists of a finite number (M say) of eigenvalues λ_k, $k = 1, 2, \cdots, M$, repeated according to multiplicity, with corresponding eigenfunctions $\varphi_k \in D(H_\eta) \subset D(\tilde{\rho}_\eta)$. Let $E_\eta(\cdot)$ be the spectral measure associated with H_η. Then note that we have

$$\begin{aligned}
\text{(A.1.22)} \qquad \tilde{\rho}_\eta[\phi] &= (H_\eta \phi, \phi)_\eta \\
&= \sum_{k=1}^{M} \lambda_k |(\phi, \varphi_k)_\eta|^2 + \int_\mu^\infty \lambda \, d(E_\eta(\lambda)\phi, \phi)_\eta \\
&\ge \sum_{k=1}^{M} \lambda_k |(\phi, \varphi_k)_\eta|^2 + \mu \|E_\eta((\mu, \infty))\phi\|_\eta^2 \\
&= \sum_{k=1}^{M} (\lambda_k - \mu) |(\phi, \varphi_k)_\eta|^2 + \mu \|\phi\|_\eta^2
\end{aligned}$$

for $\phi \in D(H_\eta)$. Further, since $D(H_\eta)$ is dense in $D(\tilde{\rho}_\eta)$, the inequality (A.1.22) holds for any $\phi \in D(\tilde{\rho}_\eta)$. Now choose $\{\phi_j\} \subset C_0^\infty(\mathbf{R}^n)$ such that

$$\text{(A.1.23)} \qquad \begin{cases} \text{(a)} \ \lim_{j\to\infty} \rho_\eta[\phi_j] = R_\infty, \\ \text{(b)} \ \|\phi_j\|_\eta = 1 \qquad (j = 1, 2, \cdots), \\ \text{(c)} \ \operatorname{supp}\phi_j \cap \operatorname{supp}\phi_\ell = \emptyset \qquad (j, \ell = 1, 2, \cdots, \ j \ne \ell). \end{cases}$$

Let $\phi = \phi_j$ and make $j \to \infty$ in (A.1.22). Then it follows that

$$\text{(A.1.24)} \qquad K_\infty \ge \mu,$$

where we should note that ϕ_j converges to 0 weakly in $L_{2,\eta}(\mathbf{R}^n)$ as $j \to \infty$. Since $\mu < \Sigma(H_\eta)$ is arbitrary, we obtain $K_\infty \geq \Sigma(H_\eta)$, which completes the proof. Q.E.D.

A.2 Proof of Glazman's theorem

Proof of Proposition 1.16.

(1) Suppose that the dimension of $E((-\infty, \lambda_0))\mathcal{H}$ is finite. Then set

$$\text{(A.2.1)} \qquad \begin{cases} F = E([\lambda_0, \infty))\mathcal{H}, \\ G = E((-\infty, \lambda_0))\mathcal{H}. \end{cases}$$

Then the dimension of G is finite, and $\mathcal{H}$ is the direct sum of F and G. Further, for $f \in D(A) \cap F$ we have

$$\text{(A.2.2)} \qquad \begin{aligned} (Af, f) &= \int_{\lambda_0}^{\infty} \lambda\, d\|E(\lambda)f\|^2 \\ &\geq \lambda_0 \|E([\lambda_0, \infty))f\|^2 \\ &= \lambda_0 (f, f), \end{aligned}$$

where $\| \quad \|$ denotes the norm of $\mathcal{H}$, and we have used the relation $\|E([\lambda_0, \infty))f\| = \|f\|$ for $f \in F$. This implies that (1.40) is satisfied.

(2) Suppose that there exists subspaces F and G of $\mathcal{M}$ satisfying the conditions in Proposition 1.16. Set $m = \dim G$ and suppose that

$$\text{(A.2.3)} \qquad \dim E((-\infty, \lambda_0))\mathcal{H} \geq m + 1.$$

Then it follows from Lemma A.1.1 that

$$\text{(A.2.4)} \qquad E((-\infty, \lambda_0))\mathcal{H} \cap F \neq \emptyset.$$

In fact we can assume that there exists a nonzero element f_0 such that

$$\text{(A.2.5)} \qquad f_0 \in E((-\infty, \lambda_0 - \mu))\mathcal{H} \cap F \cap D(A)$$

with $\mu > 0$ because we can choose the $m+1$ independent elements $f_1, f_2, \cdots, f_{m+1}$ in $E((-\infty, \lambda_0)\mathcal{H}$ so that all f_j belong to $E((-\infty, \lambda_0 - \mu)\mathcal{H} \cap D(A)$, which is possible in either case where the spectrum of A in $(-\infty, \lambda_0)$ contains the essential spectrum or it consists only the discrete spectrum. Thus it follows that

$$\text{(A.2.6)} \qquad \begin{aligned} (Af_0, f_0) &= \int_{-\infty}^{\lambda_0 - \mu} \lambda\, d\|E(\lambda)f_0\|^2 \\ &< \lambda_0 \|E((-\infty, \lambda_0 - \mu))f_0\|^2 \\ &= \lambda_0 (f_0, f_0). \end{aligned}$$

This contradicts (1.40). Therefore, we have shown that

(A.2.7) $$\dim E((-\infty, \lambda_0))\mathcal{H} \leq m,$$

which completes the proof. Q.E.D.

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Stationary Phase Method with Estimate of Remainder Term over a Space of Large Dimension

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Abstract.

Let $r_d(\nu)$ denote the remainder term of the stationary phase method over R^d. Then an estimate of $\nu^{d/2+1}r_d(\nu)$, as $d \to \infty$, is given under certain assumptions, which are tolerable for application to Feynman path integrals.

§1. Stationary phase method

Stationary phase method is a method to evaluate asymptotically, as $\nu \to \infty$, oscillatory integrals over R^d of the following form:

$$I(S, a, \nu) = \int_{R^d} e^{-i\nu S(x)} a(x) dx,$$

where $S(x)$ is a real valued C^∞ function called the phase function, $a(x)$ is a C^∞ function called the amplitude and ν is a large positive parameter. In the simplest case that $a(x) \in C_0^\infty(R^d)$ and that $S(x)$ has only one critical point x^*, where $\operatorname{Hess} S(x^*)$ is non-degenerate, it gives

$$I(S, a, \nu) = \left(\frac{2\pi}{i\nu}\right)^{d/2} [\det\{\operatorname{Hess} S(x^*)\}]^{-1/2} (e^{-i\nu S(x^*)} a(x^*) + r_d(\nu))$$

and an estimate of the remainder term

$$r_d(\nu) = O(\nu^{-d/2-1}).$$

If support of $a(x)$ is not compact, we have to require some additional assumption that control the behaviour of $a(x)$ at the infinity. For instance (cf. [1]), the same conclusion holds if we assume the following

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Hypothesis (H.0). (i) $\sup_x \mid \partial_x^\alpha S(x) \mid < \infty$ *for any multi-index* α *with* $\mid \alpha \mid \geq 2$. (ii) *There exists a constant* $\delta > 0$ *such that* $\mid \det \mathrm{Hess}\, S(x) \mid \geq \delta$. (iii) *For any multi-index* α, $\sup_x \mid \partial_x^\alpha a(x) \mid < \infty$.

Since the stationary phase method is closely related to the mathematical theory of Feynman path integrals (cf. [3], [4], [5] and [6]), we wish to investigate the following

Question. *Can one control* $\nu^{d/2+1} r_d(\nu)$ *as* $d \to \infty$ *?*

We give a positive answer to this question. Detailed discussions can be found in [2]. Applications are discussed in [4], [5] and [6].

§2. Statement of results

We shall treat the following oscillatory integral over $L-1$ dimensional space:

$$I(\{t_j\}, S, a, \nu)(x_L, x_0)$$
$$= \prod_{j=1}^{L} \Big(\frac{\nu i}{2\pi t_j}\Big)^{1/2} \int_{R^{L-1}} e^{-i\nu S(x_L,\ldots,x_0)} a(x_L, \ldots, x_0) \prod_{j=1}^{L-1} dx_j,$$

with large positive parameter ν and small positive parameters $\{t_j\}$. Our hypothesis for the phase function is

Hypothesis (H.1). $S(x_L, \ldots, x_0)$ *is of the form*

$$S(x_L, \ldots, x_0) = \sum_{j=1}^{L} S_j(t_j, x_j, x_{j-1}),$$

where

$$S_j(t_j, x_j, x_{j-1}) = \frac{\mid x_j - x_{j-1} \mid^2}{2t_j} + t_j \omega_j(t_j, x_j, x_{j-1}).$$

For any $m \geq 2$ *there exists a positive constant* κ_m *such that*

$$\sup_{x_j, x_{j-1}} \mid \partial_{x_j}^\alpha \partial_{x_{j-1}}^\beta \omega_j(t_j, x_j, x_{j-1}) \mid \leq \kappa_m$$

if $2 \leq \alpha + \beta \leq m$.

We will give two examples of phase functions satisfying hypothesis (H.1).

Example 1. Let $L(\xi, x) = \frac{1}{2}\xi^2 - V(x)$, $(\xi, x) \in R^2$, be a Lagrangian with a potential $V(x)$. Assume that the potential $V(x)$ is a real-valued C^∞-function satisfying estimates:

$$\sup_x \mid V^{(k)}(x) \mid < \infty \quad \text{for any} \quad k \geq 2.$$

Then for a small $T > 0$, there exists a unique classical orbit $\gamma^{cl}(t)$ such that $\gamma^{cl}(0) = y, \gamma^{cl}(T) = x$. Let

$$S^{cl}(T, x, y) = \int_0^T L(\dot{\gamma}^{cl}(t), \gamma^{cl}(t))dt$$

be the classical action. Then $S^{cl}(T, x, y)$ is of the form

$$S^{cl}(T, x, y) = \frac{\mid x - y \mid^2}{2T} + T\phi^{cl}(T, x, y)$$

and for any $m \geq 2$ there exists a constant C_m such that

$$\sup_x \mid \partial_x^\alpha \partial_y^\beta \phi^{cl}(T, x, y) \mid \leq C_m$$

if $2 \leq \alpha + \beta \leq m$. Therefore, $S(x_L, \ldots, x_0) = \sum_{j=1}^L S(t_j, x_j, x_{j-1})$ satisfies the hypothesis (H.1).

Example 2. Let $L(\xi, x)$ be the same lagrangian. Let $\gamma^{ln}(t)$ be the straight line connecting $(0, y)$ and (T, x) in the time-space, i.e.,

$$\gamma^{ln}(t) = \frac{t}{T}x + \frac{T - t}{T}y.$$

Let

$$S^{ln}(T, x, y) = \int_0^T L(\dot{\gamma}^{ln}(t), \gamma^{ln}(t))dt.$$

Then function $S^{ln}(T, x, y)$ is of the form

$$S^{ln}(T, x, y) = \frac{\mid x - y \mid^2}{2T} + T\phi^{ln}(T, x, y)$$

and for any $m \geq 2$ there exists a positive constant C_m such that

$$\sup_x \mid \partial_x^\alpha \partial_y^\beta \phi^{ln}(T, x, y) \mid \leq C_m$$

if $2 \leq \alpha + \beta \leq m$. Therefore, $S^{ln}(x_L, \dots, x_0) = \sum_{j=1}^{L} S^{ln}(t_j, x_j, x_{j-1})$ satisfies the hypothesis (H.1).

Under hypothesis (H.1) the critical point of the function $(x_{L-1}, \dots, x_1) \to S(x_L, x_{L-1}, \dots, x_1, x_0)$ is unique if $T_L = \sum_{j=1}^{L} t_j$ is small. We denote it by $(x_{L-1}^*, \dots, x_1^*)$. We abbreviate $S(x_L, x_{L-1}^*, \dots, x_1^*, x_0)$ as $S(\overline{x_L, x_0})$. We can write the Hessian of S at the critical poit as $H + W$, where

$$H = \begin{pmatrix} \frac{1}{t_1} + \frac{1}{t_2} & -\frac{1}{t_2} & 0 & 0 & \dots \\ -\frac{1}{t_2} & \frac{1}{t_2} + \frac{1}{t_3} & -\frac{1}{t_3} & 0 & \dots \\ 0 & -\frac{1}{t_3} & \frac{1}{t_3} + \frac{1}{t_4} & -\frac{1}{t_4} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

and

$$W = \begin{pmatrix} t_1\partial_{x_1}^2\omega_1 + t_2\partial_{x_1}^2\omega_2 & t_2\partial_{x_1}\partial_{x_2}\omega_2 & 0 & \dots \\ t_2\partial_{x_1}\partial_{x_2}\omega_2 & t_2\partial_{x_2}^2\omega_2 + t_3\partial_{x_2}^2\omega_3 & t_3\partial_{x_2}\partial_{x_3}\omega_3 & \dots \\ 0 & t_3\partial_{x_2}\partial_{x_3}\omega_3 & t_3\partial_{x_3}^2\omega_3 + t_4\partial_{x_3}^2\omega_4 & \dots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix}.$$

It is clear that

$$\det H = \frac{T_L}{t_1 t_2 \dots t_L} \neq 0.$$

We can state our first result.

Theorem 1. *Under the hypothesis* (H.1) *there exists a positive constant δ_1 independent of L such that if $T_L = t_1 + \dots + t_L \leq \delta_1$ then*

$$\begin{aligned} &I(\{t_j\}, S, 1, \nu)(x_L, x_0) \\ &= \left(\frac{\nu i}{2\pi T_L}\right)^{1/2} e^{-i\nu S(\overline{x_L, x_0})} \left[\det(I + H^{-1}W)\right]^{-1/2} (1 + r(\nu, x_L, x_0)), \end{aligned}$$

where the remainder term $r(\nu, x_L, x_0)$ satisfies the estimate: For any $K \geq 0$ there exists positive constants C_K such that if $|\alpha_0|, |\alpha_L| \leq K$

$$|\partial_{x_0}^{\alpha_0}\partial_{x_L}^{\alpha_L} r(\nu, x_L, x_0)| \leq C_K T_L^3 \nu^{-1}.$$

Remark. δ_1 and C_K are independent of L as far as T_L is bounded. Therefore, we can control $r(\nu, x_L, x_0)$ even when L tends to ∞.

In order to state the result for general integral with amplitude $a(x)$, we require a little more preparations. Let $1 \leq k \leq l \leq L$. Then the critical point of the function $(x_{l-1}, \ldots, x_{k+1}) \to \sum_{j=k+1}^{l} S_j(t_j, x_j, x_{j-1})$ is unique if $t_{k+1} + \ldots + t_l$ is small. Let $(x_{l-1}^*, \ldots, x_{k+1}^*)$ denote the critical point, which is a function of x_l and x_k. We abbreviate $a(x_L, \ldots, x_l, x_{l-1}^*, \ldots, x_{k+1}^*, x_k, \ldots, x_0)$ to $a(x_L, \ldots, x_{l+1}, \overline{x_l, x_k}, x_{k-1}, \ldots, x_0)$.

Our hypothesis concerning the amplitude function is the following:

Hypothesis (H.2). *For any integer $K \geq 0$ there exists a positive constant A_K with the following properties:* (i) *If $\mid \alpha_j \mid \leq K$ for $j = 0, 1, \ldots, L$, then*

$$\mid \prod_{j=0}^{L} \partial_{x_j}^{\alpha_j} a(x_L, \ldots, x_0) \mid \leq A_K.$$

(ii) *For any sequence of positive integers $\{j_1, \ldots, j_s\}$ satisfying*

$$0 = j_0 < j_1 - 1 < j_1 < j_2 - 1 < \ldots < j_s - 1 < j_s < L$$

we have

$$\mid \partial_{x0}^{\alpha_0} \partial_{xL}^{\alpha_L} \prod_{k=1}^{s} \partial_{x_{j_k-1}}^{\alpha_{j_k-1}} \partial_{x_{j_k}}^{\alpha_{j_k}} a(\overline{x_L, x_{j_s}}, \overline{x_{j_s-1}, x_{j_{s-1}}}, \ldots, \overline{x_{j_1-1}, x_{j_0}}) \mid \leq A_K,$$

as far as $\mid \alpha_j \mid \leq K$ for $j = 0, j_1 - 1, j_1, \ldots, j_s - 1, j_s, L$.

Before stating our second theorem, we give an example of amplitude functions satisfying hypothesis (H.2).

Example. Let $b_j(x_j, x_{j-1})$, $j = 1, \ldots, L$, be functions bounded together with their derivatives of all order, i.e., for any positive integer K there exists C_K such that

$$\sup_x \mid \partial_{x_j}^{\alpha_j} \partial_{x_{j-1}}^{\alpha_{j-1}} b_j(x_j, x_{j-1}) \mid \leq C_K \qquad 0 \leq \alpha_j, \alpha_{j-1} \leq K.$$

Then $a(x_L, \ldots, x_0) = e^{(\sum_{j=1}^{L} t_j b_j(x_j, x_{j-1}))}$ satisfies hypothesis (H.2) above.

Now we can state our main

Theorem 2. *Under the hypotheses* (H.1) *and* (H.2) *there exists a positive constant δ_1 such that if $0 < T_L \leq \delta_1$*

$$I(\{t_j\}, S, a, \nu)(x_L, x_0)$$
$$= \left(\frac{\nu i}{2\pi T_L} \right)^{1/2} e^{-i\nu S(\overline{x_L, x_0})} \left[\det(I + H^{-1} W) \right]^{-1/2}$$
$$\times \left(a(\overline{x_L, x_0}) + r(\nu, x_L, x_0) \right),$$

where $r(\nu, x_L, x_0)$ *satisfies the estimate: For any* $K \geq 0$ *there exists positive constants* C_K *and* $M(K)$ *such that if* $|\alpha_0|, |\alpha_L| \leq K$ *we have*

$$|\partial_{x_0}^{\alpha_0}\partial_{x_L}^{\alpha_L} r(\nu, x_L, x_0)| \leq C_K T_L \nu^{-1} A_{M(K)}.$$

Remark. δ_1 , C_K and $M(K)$ are independent of L as far as T_L is bounded. Therefore, we can control $r(\nu, x_L, x_0)$ even when L tends to ∞.

§3. Sketch of the proof

We begin with our key lemma, which is valid under hypothesis (H.3) weaker than (H.2) and is interesting in its own sake.

Hypothesis (H.3). *For any integer* $K \geq 0$ *there exists a positive constant* A_K *such that if* $|\alpha_j| \leq K$ *for* $j = 0, 1, \dots, L$,

$$|\prod_{j=0}^{L} \partial_{x_j}^{\alpha_j} a(x_L, \dots, x_0)| \leq A_K.$$

We can state

Key Lemma. *Under the hypotheses* (H.1) *and* (H.3) *there exists a positive constant* δ_0 *such that if* $T_L \leq \delta_0$ *we have*

$$I(\{t_j\}, S, a, \nu)(x_L, x_0)$$
$$= \left(\frac{\nu i}{2\pi T_L}\right)^{1/2} e^{-i\nu S(\overline{x_L, x_0})} \left[\det(I + H^{-1}W)\right]^{-1/2} b(\nu, x_L, x_0),$$

where $b(\nu, x_L, x_0)$ *satisfies the estimate: For any* $K \geq 0$ *there exists positive constants* $C_1(K)$ *and* $M(K)$ *such that if* $|\alpha_0|, |\alpha_L| \leq K$ *we have*

$$|\partial_{x_0}^{\alpha_0}\partial_{x_L}^{\alpha_L} b(\nu, x_L, x_0)| \leq C_1(K)^L A_{M(K)}.$$

Remark. $C(K)$ and $M(K)$ are independent of $\{t_j\}, L, (x_L, x_0)$ and ν as long as $T_L \leq \delta_0$.

Above Lemma can be proved by modifying the proof of Theorem 6.8 in Chapt. 10 of Kumano-go [7].

Omitting the proof of lemma we proceed to the proof of Theorem 2. To make notations simpler we denote $\frac{\nu i}{2\pi}$ by E. With this notation we can write

$$I(\{t_j\}, S, a, \nu)(x_L, x_0)$$
$$= \prod_{j=1}^{L} \left(\frac{E}{t_j}\right)^{1/2} \int_{R^{L-1}} e^{-i\nu S(x_L,\ldots,x_0)} a(x_L, \ldots, x_0) \prod_{j=1}^{L-1} dx_j.$$

We perform integration over x_1-space. Using stationary phase method, we have

$$\prod_{j=1}^{2} \left(\frac{E}{t_j}\right)^{1/2} \int_R e^{-i\nu\{S_2(t_2,x_2,x_1)+S_1(t_1,x_1,x_0)\}} a(x_L, \ldots, x_2, x_1, x_0) dx_1$$
$$= \left(\frac{E}{T(2,1)}\right)^{1/2} e^{-i\nu S^*_{21}(x_2,x_0)} \left(P_1 a(x_L, \ldots, x_2, x_0) + R_1 a(x_L, \ldots, x_2, x_0)\right).$$

Here $T(2,1) = t_2 + t_1$, $S^*_{21}(x_2, x_0)$ denotes the critical value of $S_2(t_2, x_2, x_1) + S_1(t_1, x_1, x_0)$ with respect to the variable x_1, $P_1 a$ is the main part and $R_1 a$ is the remainder term of the stationary phase method.

Remark. (A) Clearly, we have

$$P_1(a)(x_L, \ldots, x_2, x_0) = a(x_L, x_{L-1}, \ldots, \overline{x_2, x_0}) D(S_1 + S_2; x_2, x_0)^{-1/2}$$

here

$$D(S_1+S_2; x_2, x_0) = 1 + \frac{t_1 t_2}{t_1 + t_2} \left(t_2 \partial^2_{x_1} \omega_2(t_2, x_2, x^*_1) + t_1 \partial^2_{x_1} \omega_1(t_1, x^*_1, x_0)\right).$$

(B) The remainder term $R_1 a$ is a very complicated function with respect to x_2 but is simple with respect to the variable $(x_L, \ldots, x_3, x_0)$. In fact, we have $\partial_{x_j}(R_1 a) = R_1 \partial_{x_j} a$ for $j = 0$ and $3 \le j \le L$. And $R_1 a$ is small in the following sense: For any integer $K \ge 0$ there exists a constant C_K such that

$$| \, \partial^{\alpha_0}_{x_0} \partial^{\alpha_2}_{x_2} \ldots \partial^{\alpha_L}_{x_L} R_1 a(x_L, \ldots, x_2, x_0) \, |$$
$$\le C_K \nu^{-1} \frac{t_1 t_2}{t_1 + t_2} \max \sup_{x_1} | \, \partial^{\alpha_0}_{x_0} \partial^{\beta_1}_{x_1} \partial^{\beta_2}_{x_2} \partial^{\alpha_3}_{x_3} \ldots \partial^{\alpha_L}_{x_L} a(x_L, \ldots, x_2, x_1, x_0) \, | \, .$$

Here max is taken with respect to β_1, β_2 for $\beta_1 \le \alpha_2 + 4, \beta_2 \le \alpha_2$.

Next we integrate the term P_1a over x_2-space and apply the stationary phase method. We obtain

$$\left(\frac{E}{t_3}\right)^{1/2}\left(\frac{E}{T(2,1)}\right)^{1/2}$$
$$\int_R e^{-i\nu\{S_3(t_3,x_3,x_2)+S_{21}^*(x_2,x_0)\}}P_1a(x_L,\dots,x_2,x_0)dx_2$$
$$=\left(\frac{E}{T(3,1)}\right)^{1/2}e^{-i\nu S_{31}^*(x_3,x_0)}$$
$$(P_2P_1a(x_L,\dots,x_3,x_0)+R_2P_1a(x_L,\dots,x_3,x_0)).$$

Here $S_{31}^*(x_3,x_0)$ denotes the critical value of the function $x_2\to S_3(t_3,x_3,x_2)+S_{21}^*(x_2,x_0)$, P_2P_1a is the main term and R_2P_1a is the remainder. Since P_2P_1a is a simple function of x_3, we integrate it over x_3-space and apply the stationary phase method. The main term includes $P_3P_2P_1a$ and the remainder includes $R_3P_2P_1a$.

Repeating this procedure $L-1$ times, we obtain

$$A_0(x_L,x_0)=\left(\frac{E}{T(L,1)}\right)^{1/2}e^{-i\nu S_{L1}^*(x_L,x_0)}P_{L-1}\dots P_1a(x_L,x_0),$$

which is nothing but the main term of Theorem 2.

Now we must treat the remainder term. Since R_1a is a complicated function of x_2, we skip integration over x_2 space and perform integration over x_3-space. Then we obtain

$$\left(\frac{E}{t_4}\right)^{1/2}\left(\frac{E}{t_3}\right)^{1/2}\left(\frac{E}{T(2,1)}\right)^{1/2}$$
$$\int_R e^{-i\nu\{S_4(t_4,x_4,x_3)+S_3(t_3,x_3,x_2)+S_{21}^*(x_2,x_0)\}}R_1a(x_L,\dots,x_4,x_3,x_2,x_0)dx_3$$
$$=\left(\frac{E}{T(4,3)}\right)^{1/2}\left(\frac{E}{T(2,1)}\right)^{1/2}e^{-i\nu\{S_{43}^*(x_4,x_2)+S_{21}^*(x_2,x_0)\}}$$
$$(P_3R_1a(x_L,\dots,x_4,x_2,x_0)+R_3R_1a(x_L,\dots,x_4,x_2,x_0))\,.$$

Here $S_{43}^*(x_4,x_2)$ denotes the critical value of the function $x_3\to S_4(t_4,x_4,x_3)+S_3(t_3,x_3,x_2)$, P_3R_1a denotes the main term and R_3R_1a is the remainder. P_3R_1a is a simple function of the variable x_4 but R_3R_1a is not. We integrate P_3R_1a over x_4-space but we skip integration of R_3R_1a over x_4-space.

Similarly, we skip integration of R_2P_1a over x_3-space and integrate it over x_4-space. We obtain

$$\left(\frac{E}{t_5}\right)^{1/2}\left(\frac{E}{t_4}\right)^{1/2}\left(\frac{E}{T(3,1)}\right)^{1/2}$$

$$\int_R e^{-i\nu\{S_5(t_5,x_5,x_4)+S_4(t_4,x_4,x_3)+S^*_{31}(x_3,x_0)}R_2P_1a(x_L,\ldots,x_4,x_3,x_0)dx_4$$

$$=\left(\frac{E}{T(5,4)}\right)^{1/2}\left(\frac{E}{T(3,1)}\right)^{1/2}e^{-i\nu\{S^*_{54}(x_5,x_3)+S^*_{31}(x_3,x_0)\}}$$

$$(P_4R_2P_1a(x_L,\ldots,x_5,x_3,x_0)+R_4R_2P_1a(x_L,\ldots,x_5,x_3,x_0))\,.$$

We continue this process. *The rule is that we apply the stationary phase method when we integrate over x_k-space and if R_k appears then we skip integration over x_{k+1}-space.* We finally obtain the following expression:

$$I(\{t_j\},S,a,\nu)(x_L,x_0)=A_0(x_L,x_0)+\sum^{*}A_{j_sj_{s-1}\ldots j_1}(x_L,x_0),$$

where $\sum^*$ denotes summation with respect to indices $(j_s,\ldots,j_1)$ satisfying

$$1<j_1<j_2-1<j_2<j_3-1<\ldots<j_s-1<j_s,$$

and each term is an oscillatory integral

$$A_{j_1j_2\ldots j_s}(x_L,x_0)$$

$$=\prod_{m=1}^{s}\left(\frac{E}{T(j_m,j_m-1)}\right)^{1/2}$$

$$\int_{R^s}e^{-i\nu S_{j_s\ldots j_1}(x_L,x_{j_s},\ldots,x_{j_1},x_0)}b_{j_s\ldots j_1}(x_L,x_{j_s},\ldots,x_{j_1},x_0)\prod_{m=1}^{s}dx_{j_m},$$

whose phase function is

$$S_{j_s\ldots j_1}(x_L,x_{j_s},\ldots,x_{j_1},x_0)$$

$$=S^*_{Lj_s}(x_L,x_{j_s})+S^*_{j_sj_{s-1}}(x_{j_s},x_{j_{s-1}})+\ldots+S^*_{j_10}(x_{j_1},x_0)$$

and the amplitude is

$$b_{j_s\ldots j_1}(x_L,x_{j_s},\ldots,x_{j_1},x_0)=Q_{L-1}Q_{L-2}\ldots Q_1a(x_L,x_{j_s},\ldots,x_{j_1},x_0),$$

with

$$Q_j = \begin{cases} Id, & \text{for } j = j_s, j_{s-1}, \dots, j_1, \\ R_j, & \text{for } j = j_s - 1, j_{s-1} - 1, \dots, j_1 - 1, \\ P_j, & \text{otherwise.} \end{cases}$$

Furthermore, we can prove that $b_{j_s \dots j_1}(x_L, x_{j_s}, \dots, x_{j_1}, x_0)$ satisfies hypothesis (H.3).

Proposition. *For any integer $K \geq 0$ there exist positive constants $C_2(K)$ and integer $m(K)$ such that*

$$\begin{aligned} &| \partial_{x_L}^{\alpha_L} \partial_{x_{j_s}}^{\alpha_{j_s}} \dots \partial_{x_{j_1}}^{\alpha_{j_1}} \partial_{x_0}^{\alpha_0} b_{j_s \dots j_1}(x_L, x_{j_s}, \dots, x_{j_1}, x_0) | \\ \leq &C_2(K)^s A_{m(K)} \prod_{k=1}^{s} \nu^{-1} t_{j_k}. \end{aligned}$$

Now we apply our key lemma to $A_{j_s j_{s-1} \dots j_1}(x_L, x_0)$ and use the proposition above. Then we obtain

$$A_{j_s j_{s-1} \dots j_1}(x_L, x_0) = \left(\frac{E}{T_{L,1)}} \right)^{1/2} e^{-i\nu S(\overline{x_L, x_0})} a_{j_s j_{s-1} \dots j_1}(x_L, x_0),$$

where the function $a_{j_s j_{s-1} \dots j_1}(x_L, x_0)$ satisfies the following estimates: For any integer $K \geq 0$ we have

$$| \partial_{x_L}^{\alpha_L} \partial_{x_0}^{\alpha_0} a_{j_s j_{s-1} \dots j_1}(x_L, x_0) | \leq C_1(K)^s C_2(M(K))^s A_{m(M(K))} \prod_{k=1}^{s} \nu^{-1} t_{j_k}.$$

This implies that the remainder term $r(\nu, x_L, x_0)$ can be written as

$$r(\nu, x_L, x_0) = \sum^{*} a_{j_s j_{s-1} \dots j_1}(x_L, x_0).$$

If $\alpha_0, \alpha_L \leq K$ we have

$$\begin{aligned} | \partial_{x_L}^{\alpha_L} \partial_{x_0}^{\alpha_0} r(\nu, x_L, x_0) | &\leq \sum^{*} | \partial_{x_L}^{\alpha_L} \partial_{x_0}^{\alpha_0} a_{j_s j_{s-1} \dots j_1}(x_L, x_0) | \\ &\leq \sum^{*} C_3(K)^s A_{m(M(K))} \prod_{k=1}^{s} \nu^{-1} t_{j_k} \\ &\leq A_{m(M(K))} \left(\prod_{j=1}^{L} (1 + C_3(K) \nu^{-1} t_j) - 1 \right), \end{aligned}$$

where we abbreviated $C_1(K)C_2(M(K))$ as $C_3(K)$. This proves Theorem 2.

Theorem 1 can be proved similarly.

More detailed dicussions are given by [2].

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Commutator Algebra and Resolvent Estimates

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§1. Introduction

In studying the detailed properties of Schrödinger operators, the method of micro-localization seems to be indispensable. For the many-body problem, this point of view was introduced by Enss [3], Mourre [11] and then by Sigal-Soffer [13] to investigate the propagation properties of the unitary group. These sorts of estimates not only lead us to a deep understanding of the space-time behavior of the solution to the Schrödinger equation, but also give us many applications. The aim of this paper is to prove a certain variation of these kinds of estimates for the resolvent of the N-body Schrödinger operator.

We consider a system of N-particles moving in $\mathbf{R}^\nu$ with mass m_i and position $x^i \in \mathbf{R}^\nu (1 \leq i \leq N)$. Let $\mathcal{X}$ be defined by

$$\mathcal{X} = \{(x^1, \cdots, x^N); \sum_{i=1}^N m_i x^i = 0\},$$

and consider the Schrödinger operator

$$H = H_0 + \sum_{i<j} V_{ij},$$

where $-H_0$ is the Laplace-Beltrami operator on $\mathcal{X}$ equipped with the Riemannian metric induced from $ds^2 = 2\sum_{i=1}^N m_i (dx^i)^2$ on $\mathbf{R}^{N\nu}$. Each pair potential $V_{ij} = V_{ij}(x^i - x^j)$ is assumed to be a real-valued C^∞-function on $\mathbf{R}^\nu$ and satisfies for some constant $\rho > 0$

$$(1.1) \qquad |\partial_y^m V_{ij}(y)| \leq C_m <y>^{-m-\rho},$$

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for any $m = 0, 1, 2, \cdots$, where ∂_y^m denotes an arbitrary derivative of order m and $<y> = (1 + |y|^2)^{1/2}$. Let $R(z) = (H - z)^{-1}$. Let Λ be the set of thresholds of H. For $\lambda \in \sigma_{\rm ess}(H) \cap \sigma_p(H)^c \cap \Lambda^c$, we define

$$a(\lambda) = \inf\{\lambda - \mu; \mu \in \Lambda, \mu < \lambda\}. \tag{1.2}$$

Note that $a(\lambda) = \lambda$ if $\lambda > 0$, which follows from the absence of positive eigenvalues of Schrödinger operators (see [5]). We consider a pseudo-differential operator (Ps.D.Op.) P with symbol $p(x, \xi)$ belonging to the following class. For a positive integer k and $a \in \mathbf{R}$, let $\mathcal{R}^k(a)$ be the set of C^∞-functions $p(x, \xi)$ having the following estimates:

$$|\partial_x^m \partial_\xi^n p(x, \xi)| \leq C <x>^{-m} <\xi>^{-k}, \quad 0 \leq m, n \leq k, \tag{1.3}$$

and also satisfying

$$\sup_{x,\xi} \frac{x \cdot \xi}{<x>} < a \quad \text{on} \quad \operatorname{supp} p(x, \xi). \tag{1.4}$$

A typical example of the element of $\mathcal{R}^k(a)$ is given as follows. We take $\rho(t) \in C^\infty(\mathbf{R})$ such that $\rho(t) = 1$ if $t < a - 2\epsilon$, $\rho(t) = 0$ if $t > a - \epsilon$, ϵ being a small positive constant. Then

$$\rho(\frac{x \cdot \xi}{<x>}) <\xi>^{-2k} \in \mathcal{R}^k(a).$$

For a Ps.D.Op. P, $P \in \mathcal{R}^k(a)$ means that the symbol of P belongs to $\mathcal{R}^k(a)$. As is well-known, for a sufficiently large k, $P \in \mathcal{R}^k(a)$ is L^2-bounded. Let $\mathbf{B}$ denote the totality of bounded operators on $L^2(\mathcal{X})$. The main result of this paper is the following

Theorem 1.1. *For any $s > -1/2$ and $t > 1$, there exists $k = k(s) > 0$ such that*

$$<x>^s PR(\lambda + i0) <x>^{-s-t} \in \mathbf{B}$$

for any $P \in \mathcal{R}^k(\sqrt{a(\lambda)})$.

Although the above theorem is formulated by Ps.D.Op.'s, the main part of the proof consists in the calculus of commutators in an algebra consisting of functions of several operators, which is one of the interesting features of the many-body problem. This commutator calculus has its origin in the work of Mourre [11], was developed by Sigal-Soffer [13], [14] with great success and is now considered as a basic tool for the many-body problem.

One of the authors proved a slightly weaker theorem in [15] and we should note that the above Theorem 1.1 is implicitly suggested in [16], where the commutator of H and

$$\tilde{A} = \frac{1}{2i}(x \cdot \nabla_x + \nabla_x \cdot x) - C{<}x{>}$$

was used. In this paper, we shall explain a method which treats directly the resolvent. Our idea is very close to those of Sigal-Soffer [13] and Derezinski [2]. One of the applications of the above theorem is the study of the detailed structure of the S-matrix ([9], [10], [16]). Other applications will be given elsewhere.

Finally, we remark that throughout the paper we neglect the domain question and treat freely the product of unbounded operators. This is justified by defining them by quadratic forms on $\mathcal{S} \times \mathcal{S}$, where $\mathcal{S}$ is the space of rapidly decreasing functions.

§2. Commutator Algebra

For two operators P and A, we introduce their multiple commutators by

$$\begin{aligned} &\mathrm{ad}_0(P, A) = P, \\ &\mathrm{ad}_n(P, A) = [\mathrm{ad}_{n-1}(P, A), A], \quad n \geq 1. \end{aligned}$$

The fundamental formulas to calculate the commutators are as follows:

$$\begin{aligned} (\mathrm{ad}_n(P, A))^* &= (-1)^n \,\mathrm{ad}_n(P^*, A^*), \\ \mathrm{ad}_n(PQ, A) &= \sum_{k=0}^{n} \binom{n}{k} \mathrm{ad}_{n-k}(P, A)\, \mathrm{ad}_k(Q, A), \\ [P, A^n] &= \sum_{k=1}^{n} c_{n,k}\, \mathrm{ad}_k(P, A) A^{n-k}, \end{aligned}$$

$c_{n,k}$ being constants.

We choose the coordinates $x = (x_1, \cdots, x_{(N-1)\nu})$ on $\mathcal{X}$ such that

$$H_0 = -\sum_{i=1}^{(N-1)\nu} (\partial/\partial x_i)^2.$$

As in [2] and [13], an important role is played by the self-adjoint operator B defined by

$$B = \frac{1}{2i}\left(\frac{x}{{<}x{>}} \cdot \nabla_x + \nabla_x \cdot \frac{x}{{<}x{>}}\right). \tag{2.1}$$

We first consider the commutation relations between H, B and $X = <x>$. Let L_0 be the differential operator defined by

$$L_0 = \sum_{i=1}^{(N-1)\nu} x_i \frac{\partial}{\partial x_i}.$$

Let $\mathcal{V}$ be the set of C^∞-functions v on $\mathcal{X}$ such that $L_0^n v$ is bounded on $\mathcal{X}$ for any $n \geq 0$. This set $\mathcal{V}$ forms an algebra and is independent of the choice of the Jacobi coordinates.

Example. If $v \in C^\infty(\mathbf{R}^\nu)$ satisfies $|\partial_y^m v(y)| \leq C_m < y >^{-m}$, $\forall m \geq 0$, then $v(x^i - x^j) \in \mathcal{V}$. In particular, each two-body potential $V_{ij}(x^i - x^j)$ belongs to $\mathcal{V}$.

Let $\mathcal{V}_m = X^m \mathcal{V}$. Let $\mathcal{P}_{k,m}$ be the set of differential operators of order k with coefficients $\in \mathcal{V}_m$. $\mathcal{V}_m$ is invariant by the action of L_0, which implies that, if $L \in \mathcal{P}_{k,m}$, $[L, B] \in \mathcal{P}_{k,m-1}$. We have, therefore,

Lemma 2.1. *For $n \geq 1$, we have*
(1) $\mathrm{ad}_n(X, B) \in \mathcal{P}_{0,1-n}$.
(2) $\mathrm{ad}_n(H, B) \in \mathcal{P}_{2,-n}$.
(3) $\mathrm{ad}_n(B, H) \in \mathcal{P}_{n+1,-1}$.

These commutation relations suggest us to introduce the following

Definition 2.2. $P \in \mathcal{O}p^m(X) \quad (m \in \mathbf{R}) \Longleftrightarrow$
$X^\alpha \,\mathrm{ad}_n(P, B) X^\beta \in \mathbf{B}$, for any $\alpha, \beta \in \mathbf{R}$ and $n \geq 0$ such that $\alpha + \beta = n - m$.

The analogy of the class $\mathcal{O}p^m(X)$ to that of Ps.D.Op.'s is apparent when one thinks of Beals' characterization of the standard class of Ps.D.Op.'s ([1]). The basic properties of $\mathcal{O}p^m(X)$ are summarized in the following lemma whose proof follows easily from the definition.

Lemma 2.3. (1) $P \in \mathcal{O}p^m(X) \Longleftrightarrow$ *There exists* $P_0 \in \mathcal{O}p^0(X)$ *such that* $P = X^m P_0$.
(2) $P \in \mathcal{O}p^m(X) \Longrightarrow [P, B] \in \mathcal{O}p^{m-1}(X)$.
(3) $P \in \mathcal{O}p^m(X) \Longrightarrow X^k P X^l \in \mathcal{O}p^{m+k+l}(X), \quad \forall k, l \in \mathbf{R}$.
(4) $P \in \mathcal{O}p^m(X) \Longrightarrow P^* \in \mathcal{O}p^m(X)$.
(5) $P \in \mathcal{O}p^m(X)$, $Q \in \mathcal{O}p^n(X) \Longrightarrow PQ \in \mathcal{O}p^{m+n}(X)$.

Therefore, $\cup_m \mathcal{O}p^m(X)$ forms an algebra which is our basic tool in this paper.

The basic subject of this section is to calculate the commutators of functions of operators. For $m \in \mathbf{R}$, let $\mathcal{F}^m$ be the set of C^∞-functions on $\mathbf{R}$ such that

$$|f^{(k)}(x)| \le C_k(1+|x|)^{m-k}, \quad \forall k \ge 0.$$

Then for $f \in \mathcal{F}^m$ ($m \in \mathbf{R}$), there exists $F(z) \in C^\infty(\mathbf{C})$, called an almost analytic extension of f, having the following properties:

$$F(x) = f(x), \quad x \in \mathbf{R},$$

$$|\overline{\partial_z} F(z)| \le C_N < z >^{m-1-N} |\mathrm{Im} z|^N, \quad \forall N \ge 0,$$

$$\mathrm{supp} F(z) \subset \{|\mathrm{Im} z| \le \epsilon(1+|\mathrm{Re} z|)\}, \quad 0 < \epsilon \ll 1.$$

Furthermore, $\partial_x^k F(z)$ is an almost analytic extension of $f^{(k)}(x)$ (see [6]). Let $f \in \mathcal{F}^{-\epsilon}$ ($\epsilon > 0$) and F be its almost analytic extension. Then for any self-adjoint operator A we have

$$f(A) = \frac{1}{2\pi i}\int_{\mathbf{C}} \overline{\partial_z} F(z)(z-A)^{-1} dz \wedge d\bar{z} \tag{2.2}$$

(see [8]). One can also prove the following formula of the asymptotic expansion of the commutator: If $f \in \mathcal{F}^m$ ($m \in \mathbf{R}$) and A is self-adjoint, we have

$$[P, f(A)] = \sum_{n=1}^{N-1} (-1)^{n-1}/n! \ \mathrm{ad}_n(P,A) f^{(n)}(A) + R_N, \tag{2.3}$$

$$R_N = \frac{1}{2\pi i}\int_{\mathbf{C}} \overline{\partial_z} F(z)(A-z)^{-1} \,\mathrm{ad}_N(P,A)(A-z)^{-N} dz \wedge d\bar{z}. \tag{2.4}$$

R_N is bounded if there exists k such that $m+k < N$ and $\mathrm{ad}_N(P,A)(A+i)^{-k} \in \mathbf{B}$. This commutator expansion formula turns out to be a powerful tool of analysis (see also [6], [7]).

An important example of the element of $\mathcal{O}p^m(X)$ is given by

Lemma 2.4. *$f(H)$, $f(B) \in \mathcal{O}p^0(X)$ if $f \in \mathcal{F}^{-\epsilon}$, $\epsilon > 0$.*

Proof. By (2.2), we have

$$\mathrm{ad}_n(f(H), B) = \frac{1}{2\pi i}\int_{\mathbf{C}} \overline{\partial_z} F(z) \,\mathrm{ad}_n((z-H)^{-1}, B) dz \wedge d\bar{z}.$$

For $\alpha,\ \beta \in \mathbf{R}$ such that $\alpha + \beta = n$, one can show

$$\|X^{\alpha}\,\mathrm{ad}_n((H-z)^{-1}, B)X^{\beta}\| \leq C <z>^{\gamma}\,|\mathrm{Im}z|^{-\gamma-1},$$

with $\gamma = \gamma(\alpha,\beta) > 0$. The above mentioned properties of the almost analytic extensions then prove that $f(H) \in \mathcal{O}p^0(X)$. To prove the lemma for $f(B)$, we have only to note that

$$\|X^n(B-z)^{-1}X^{-n}\| \leq C_n|\mathrm{Im}z|^{-n-1}, \quad \forall n \geq 0. \qquad \square$$

It is convenient to introduce the following notation : Let $P_n \in \mathcal{O}p^{k(n)}(X)$, $k(1) > k(2) > \cdots \to -\infty$. Then an operator P is said to have the asymptotic expansion $\sum_{n\geq 1} P_n$, written as $P \sim \sum_{n\geq 1} P_n$, if and only if

$$P - \sum_{n=1}^{N-1} P_n \in \mathcal{O}p^{k(N)}(X), \quad \forall N \geq 2.$$

Using (2.3), one can show the following

Lemma 2.5. *Let $P \in \mathcal{O}p^m(X)$, $f \in \mathcal{F}^n$, $m, n \in \mathbf{R}$. Then*

$$[P, f(B)] \sim \sum_{k\geq 1} P_k f^{(k)}(B), \quad P_k \in \mathcal{O}p^{m-k}(X).$$

By the same methods as above, we can also show

Lemma 2.6. *Let $\varphi \in C_0^\infty(\mathbf{R})$ and $f \in \mathcal{F}^m, m \in \mathbf{R}$. Then we have:*

(1) $\qquad \mathrm{ad}_n(\varphi(H), X) \in \mathcal{O}p^0(X), \quad n \geq 0.$

(2) $\qquad [\varphi(H), f(X)] \sim \sum_{n\geq 1}(-1)^{n-1}/n!\ \mathrm{ad}_n(\varphi(H), X)f^{(n)}(X).$

Lemma 2.7. *Let $f \in \mathcal{F}^m, g \in \mathcal{F}^n, m, n \in \mathbf{R}$. Then we have:*

(1) $\qquad \mathrm{ad}_k(g(X), B) \in \mathcal{O}p^{n-k}(X), \quad k \geq 0.$

(2) $\qquad [g(X), f(B)] \sim \sum_{k\geq 1}(-1)^{k-1}/k!\ \mathrm{ad}_k(g(X), B)f^{(k)}(B).$

§3. Resolvent Estimates (1)

We fix $\lambda \in \sigma_{\text{ess}}(H) \cap \sigma_p(H)^c \cap \Lambda^c$ and let $C_0(\lambda) = a(\lambda) - \epsilon$ for small $\epsilon > 0$. Let $\varphi \in C_0^\infty(\mathbf{R})$ be such that $\varphi(t) = 1$ if $|t - \lambda| < \delta$, $\varphi(t) = 0$ if $|t - \lambda| > 2\delta$. Our starting point is the following Mourre type estimate which holds for small $\delta > 0$ ([4]):

$$\varphi(H)i[H, A]\varphi(H) \geq 2C_0(\lambda)\varphi(H)^2, \tag{3.1}$$

where

$$A = \frac{1}{2i}(x \cdot \nabla_x + \nabla_x \cdot x).$$

We now introduce

Definition 3.1. $f \in \mathcal{F}^m_-(\lambda)$, $m \in \mathbf{R} \iff f \in \mathcal{F}^m$, supp $f \subset (-\infty, \sqrt{a(\lambda)})$.

For a small $\epsilon_0 > 0$, we take $F_0(t) \in \mathcal{F}^0_-(\lambda)$ such that

$$\begin{cases} F_0(t) = 0 & \text{if} \quad t > \sqrt{C_0(\lambda) - \epsilon_0}, \\ F_0(t) = 1 & \text{if} \quad t < \sqrt{C_0(\lambda) - 2\epsilon_0}, \\ F_0(t) \geq 0, & \sqrt{F_0(t)} \in \mathcal{F}^0_-(\lambda), \\ F_0'(t) \leq 0, & \sqrt{-F_0'(t)} \in \mathcal{F}^0_-(\lambda). \end{cases}$$

For $0 < \epsilon_1 < \epsilon_0$, let $C_1(\lambda) = \sqrt{C_0(\lambda) - \epsilon_1}$ and define

$$F_m(t) = (C_1(\lambda) - t)^m F_0(t),$$

$$\widetilde{F_{2m+1}}(t) = (C_1(\lambda) - t)F_m(t)^2.$$

In the following arguments, $(*)$ denotes an operator having the asymptotic expansion:

$$\sum_{n \geq 2} P_n f_n(B), \quad P_n \in \mathcal{O}p^{2m+1-n}(X),$$

$$f_n \in \mathcal{F}^{2m+1-n}_-(\lambda), \quad \text{supp } f_n \subset \text{supp } F_0.$$

The crucial step is the following lemma.

Lemma 3.2. *Let $m > -1/2$. With $F_m(t)$ and $\varphi(t)$ introduced above, we define $P_m = X^m F_m(B)\varphi(H)$. Then there exists a constant $C_0 > 0$ such that*

$$- \mathit{Re}\, \varphi(H)i[H, X^{2m+1}\widetilde{F_{2m+1}}(B)]\varphi(H) \geq C_0 P_m^* P_m + (*).$$

Proof. To calculate the commutator $i[H, X^{2m+1}\widetilde{F_{2m+1}}(B)]$ in the category of the algebra explained in §2, we make the following device. Let $\varphi_1(t) \in C_0^\infty(\mathbf{R})$ be such that $\varphi_1(t) = 1$ on supp φ, and put $\psi(t) = t\varphi_1(t)$. Then

$$\begin{aligned}
&\varphi(H)i[H, X^{2m+1}\widetilde{F_{2m+1}}(B)]\varphi(H)\\
&= \varphi(H)i[\psi(H), X^{2m+1}\widetilde{F_{2m+1}}(B)]\varphi(H)\\
&= \varphi(H)i[\psi(H), X^{2m+1}]\widetilde{F_{2m+1}}(B)\varphi(H)\\
&\quad + \varphi(H)X^{2m+1}i[\psi(H), \widetilde{F_{2m+1}}(B)]\varphi(H).
\end{aligned}$$

We first show that

$$\begin{aligned}
&- \operatorname{Re}\varphi(H)X^{2m+1}i[\psi(H), \widetilde{F_{2m+1}}(B)]\varphi(H)\\
&\geq (2m+1)P_m^*(2C_0(\lambda) - 2B^2 - \epsilon_2)P_m + (*),
\end{aligned} \tag{3.2}$$

ϵ_2 being a sufficiently small positive constant. In fact, we have

$$\frac{d}{dt}\widetilde{F_{2m+1}}(t) = -(2m+1)F_m(t)^2 - G(t),$$

where

$$G(t) = -2(C_1(\lambda) - t)^{2m+1}F_0(t)F_0'(t).$$

Then using (2.3), we see that the left-hand side of (3.2) is written as

$$\begin{aligned}
&(2m+1)\operatorname{Re}\varphi(H)X^{2m+1}i[\psi(H), B]F_m(B)^2\varphi(H)\\
&+ \operatorname{Re}\varphi(H)X^{2m+1}i[\psi(H), B]G(B)\varphi(H) + (*).
\end{aligned}$$

Taking note of the relation,

$$\begin{aligned}
&\varphi(H)X^{1/2}i[H, B]X^{1/2}\varphi(H)\\
&= \varphi(H)(i[H, A] - 2B^2 + K)\varphi(H),
\end{aligned}$$

K being a compact operator, we have

$$\begin{aligned}
&\operatorname{Re}\varphi(H)X^{2m+1}i[\psi(H), B]G(B)\varphi(H)\\
&= X^m\sqrt{G(B)}\varphi(H)X^{1/2}i[H, B]X^{1/2}\varphi(H)\sqrt{G(B)}X^m + (*)\\
&\geq X^m\sqrt{G(B)}\varphi(H)(2C_0(\lambda) - 2B^2 + K)\varphi(H)\sqrt{G(B)}X^m + (*)\\
&\geq (*),
\end{aligned}$$

where we have used Lemmas 2.5, 2.6 and 2.7 in the first line, (3.1) in the second line and the fact that $-2t^2 \geq -2(C_0(\lambda) - \epsilon_0)$ on supp $G(t)$ in the third line. We can then see that the left-hand side of (3.2) is estimated from below by

$$
\begin{aligned}
&(2m+1)F_m(B)X^m\varphi(H)X^{1/2}i[H,B]X^{1/2}\varphi(H)X^mF_m(B)+(*)\\
&\geq (2m+1)P_m^*(2C_0(\lambda)-2B^2-\epsilon_2)P_m+(*).
\end{aligned}
$$

We next show that

$$
\begin{aligned}
&-\operatorname{Re}\varphi(H)i[\psi(H),X^{2m+1}]\widetilde{F_{2m+1}}(B)\varphi(H)\\
&\geq (2m+1)P_m^*(2B^2-2C_1(\lambda)^2)P_m+(*).
\end{aligned}
\tag{3.3}
$$

In fact, the left-hand side of (3.3) is written as

$$
\begin{aligned}
&-\operatorname{Re}\varphi(H)i[H,X^{2m+1}]\widetilde{F_{2m+1}}(B)\varphi(H)+(*)\\
&=-\operatorname{Re}2(2m+1)\varphi(H)X^{2m}B(C_1(\lambda)-B)F_m(B)^2\varphi(H)+(*).
\end{aligned}
$$

Since $t \leq C_1(\lambda)$ on supp $F_m(t)$, we have

$$
-B(C_1(\lambda)-B)F_m(B)^2 \geq (B^2-C_1(\lambda)^2)F_m(B)^2,
$$

which proves (3.3).

The lemma now follows from (3.2) and (3.3). □

Let $F_m(t)$ be as above. We call $X^mF_m(B)$ the operator of canonical type.

Lemma 3.3. *Let $m \in \mathbf{R}$, $P \in \mathcal{O}p^{2m}(X)$ and $f \in \mathcal{F}_-^{2m}(\lambda)$. Take $n > m$. Then for any $N \geq 1$, there exist the operators of canonical type $X^{n-k/2}F_{n-k/2}(B)$ $(k=1,\cdots,N-1)$, $P_N \in \mathcal{O}p^{2n-N}(X)$ and a constant $C > 0$ such that*

$$
Re\,Pf(B) \leq C\sum_{k=0}^{N-1}F_{n-k/2}(B)X^{2n-k}F_{n-k/2}(B)+P_N.
$$

Proof. By enlarging the support of $F_n(t)$ suitably, we see that $\psi(t)=f(t)F_n(t)^{-2} \in \mathcal{F}_-^{-\epsilon}(\lambda)$, $\epsilon > 0$. Then we have

$$
\begin{aligned}
Pf(B)&=P\psi(B)F_n(B)^2\\
&=F_n(B)P\psi(B)F_n(B)+[P\psi(B),F_n(B)]F_n(B).
\end{aligned}
$$

One can then see that

$$
\operatorname{Re}F_n(B)P\psi(B)F_n(B)=F_n(B)X^nP_0X^nF_n(B),
$$

where $P_0 = P_0^* \in \mathcal{O}p^0(X)$. Therefore, for a suitable constant $C > 0$,

$$\mathrm{Re}\ F_n(B)P\psi(B)F_n(B) \le CF_n(B)X^{2n}F_n(B),$$

$X^nF_n(B)$ being the operator of canonical type. Since $[P\psi(B), F_n(B)]$ has an asymptotic expansion:

$$[P\psi(B), F_n(B)] \sim \sum_{k\ge 1} P_kF_n^{(k)}(B),\ \ P_k \in \mathcal{O}p^{2m-k}(X),$$

we repeat the above procedure to conclude the lemma. □

The main purpose of this section is the following

Theorem 3.4. *Let $m > -1/2, t > 1$ and $F \in \mathcal{F}_-^m(\lambda)$. Then we have*

$$X^mF(B)\varphi(H)R(\lambda + i0)X^{-m-t} \in \mathbf{B}.$$

Proof. We take $\psi \in C_0^\infty(\mathbf{R})$ such that $\psi = 1$ on supp φ. Let $u = \psi(H)R(\lambda + i\epsilon)f, \epsilon > 0$. By Lemma 3.3, we have only to consider the case where $X^mF(B)$ is the operator of canonical type $X^mF_m(B)$.

We introduce a notation here: $Q \in \mathcal{O}p_-^m(\lambda; X)$ if and only if $Q = Pf(B)$ for some $P \in \mathcal{O}p^m(X)$ and $f \in \mathcal{F}_-^m(\lambda)$.

By Lemma 3.2, we have

$$\begin{aligned} C_0\|X^mF_m(B)\varphi(H)u\|^2 \le & - \mathrm{Re}\ (i[H,Q]\varphi(H)u, \varphi(H)u) \\ & + \mathrm{Re} \sum_{n=2}^{N-1} (Q_nu, u) + (Q_Nu, u), \end{aligned} \tag{3.4}$$

where $Q = X^{2m+1}\widetilde{F_{2m+1}}(B)$, $Q_n \in \mathcal{O}p_-^{2m+1-n}(\lambda; X)$ and $Q_N \in \mathcal{O}p^{2m+1-N}(X)$. Note that

$$\begin{aligned} & - \mathrm{Re}\ (i[H,Q]\varphi(H)u, \varphi(H)u) \\ = & \ \mathrm{Im}\ \{(Q\varphi(H)f, \varphi(H)u) - (Q\varphi(H)u, \varphi(H)f)\} \\ & - \ 2\epsilon\ \mathrm{Re}\ (Q\varphi(H)u, \varphi(H)u). \end{aligned}$$

Let $\delta = t - 1$. Since Q is written as

$$Q = \sum_{i=0}^{N-1} \widetilde{P}_i^{\,*} P_i + Q_N,$$

where $P_i \in \mathcal{O}p_-^{m-i-\delta}(\lambda; X)$, $\widetilde{P}_i \in \mathcal{O}p_-^{m+t}(\lambda; X)$ and $Q_N \in \mathcal{O}p^{2m+1-N}(X)$, we have

$$|(Q\varphi(H)u, \varphi(H)f)| \le \sum_{i=0}^{N-1} \|P_i\varphi(H)u\|^2 + C\|X^{m+t}f\|^2.$$

Here and in the sequel C denotes a constant independent of $\epsilon > 0$. $|(Q\varphi(H)f, \varphi(H)u)|$ is estimated from above in the same way. Since Q can be written as

$$\begin{aligned} Q &= \sqrt{\widetilde{F_{2m+1}}(B)}X^{2m+1}\sqrt{\widetilde{F_{2m+1}}(B)} \\ &+ [\sqrt{\widetilde{F_{2m+1}}(B)}, [\sqrt{\widetilde{F_{2m+1}}(B)}, X^{2m+1}]], \end{aligned}$$

one can show that

$$-\text{ Re } \varphi(H)Q\varphi(H) \le \sum_{i\ge 0} P_i^* P_i + Q_N,$$

with a finite number of $P_i \in \mathcal{O}p_-^{m-1/2-i}(\lambda; X)$, and $Q_N \in \mathcal{O}p^{-N}(X)$. Therefore

$$-\text{ Re } (Q\varphi(H)u, \varphi(H)u) \le \sum_{i\ge 0} \|P_i u\|^2 + C\|X^{m+t}f\|^2.$$

Re $(Q_n u, u)$ in (3.4) is estimated from above similarly. We then arrive at

$$\|X^m F_m(B)\varphi(H)u\|^2 \le \sum_{i\ge 0} \|P_i u\|^2 + C\|X^{m+t}f\|^2, \tag{3.5}$$

with a finite number of $P_i \in \mathcal{O}p_-^{m-\delta}(\lambda; X)$. In view of Lemma 3.3, one can use (3.5) with m replaced by $m - \delta$ to estimate $\|P_i u\|^2$. We repeat this procedure and finally obtain

$$\|X^m F_m(B)\varphi(H)u\|^2 \le C(\|X^{-s}u\|^2 + \|X^{m+t}f\|^2),$$

with $s > 1/2$. The limiting absorption principle then implies the theorem (see [12]). □

§4. Resolvent Estimates (2)

In this section, we shall give the proof of Theorem 1.1 which consists in translating Theorem 3.4 in terms of Ps.D.Op.'s. Let $\varphi(H)$ be as in the previous section. Then by Lemma 2.4,

$$X^m(1-\varphi(H))R(\lambda+i0)X^{-m} \in \mathbf{B}, \quad \forall m \in \mathbf{R}.$$

Therefore to prove Theorem 1.1, we have only to consider $\varphi(H)R(\lambda+i0)$. For a small $\epsilon_0 > 0$, we define $C(\lambda) = \sqrt{a(\lambda)-\epsilon} + 3\epsilon_0$ so that $C(\lambda) < \sqrt{a(\lambda)}$. We take $F_-(t) \in \mathcal{F}^0$ such that $F_-(t) = 1$ if $t < C(\lambda) - \epsilon_0$, $F_-(t) = 0$ if $t > C(\lambda)$. Let $F_+(t) = 1 - F_-(t)$. Throughout this section, we shall use the Weyl calculus of Ps.D.Op.'s.

Let $P \in \mathcal{R}^k(\sqrt{a(\lambda)})$. Then for $s > -1/2$ one can take k large enough so that $X^sP <B>^{-s} X^{-s} \in \mathbf{B}$. Therefore by Theorem 3.4,

$$X^sPF_-(B)\varphi(H)R(\lambda+i0)X^{-s-t}$$

$$= X^sP <B>^{-s} X^{-s} \cdot X^s <B>^s F_-(B)\varphi(H)R(\lambda+i0)X^{-s-t} \in \mathbf{B}$$

for $s > -1/2$ and $t > 1$.

The proof of Theorem 1.1 is thus completed if we show the following assertion : For any $s > 0$, there exists $k = k(s) > 0$ such that

$$X^sPF_+(B)X \in \mathbf{B},\ \forall P \in \mathcal{R}^k(\sqrt{a(\lambda)}). \tag{4.1}$$

Applying Lemma 2.7 to $[X, F_+(B)]$, we see that (4.1) follows from the following assertion: For any $s > 0$, there exists $k = k(s) > 0$ such that

$$X^sPF_+(B) \in \mathbf{B},\ \forall P \in \mathcal{R}^k(\sqrt{a(\lambda)}). \tag{4.2}$$

Suppose (4.2) is proved for some $s \geq 0$. Let $C_1(\lambda) = \sqrt{a(\lambda)-\epsilon} + \epsilon_0$. Then by taking ϵ and ϵ_0 small enough we have

$$\frac{x \cdot \xi}{< x >} \leq C_1(\lambda) - \epsilon_0$$

on supp $p(x,\xi)$ and $t \geq C_1(\lambda) + \epsilon_0$ on supp $F_+(t)$. Let $B_1 = B - C_1(\lambda)$ and consider

$$P(t) = e^{-tB_1}F_+(B)P^*X^{2s+1}PF_+(B)e^{-tB_1}, \quad t \geq 0.$$

Let $b_1(x,\xi)$ be the symbol of B_1. Namely,

$$b_1(x,\xi) = \frac{x \cdot \xi}{<x>} - C_1(\lambda).$$

Then on supp $p(x,\xi)$, $b_1(x,\xi) < -\epsilon_0$. Let P_0 be the Ps.D.Op. with symbol

$$p_0(x,\xi) = (-b_1(x,\xi))^{1/2}p(x,\xi).$$

As is easily seen $P_0 \in \mathcal{R}^{k-1}(\sqrt{a(\lambda)})$. We now take k large enough and apply the standard symbolic calculus to obtain

$$\begin{aligned}2P_0^* X^{2s+1} P_0 = &- B_1 P^* X^{2s+1} P - P^* X^{2s+1} P B_1 \\ &+ \mathrm{Re} \sum_i^{finite} \tilde{P}_i^* X^{2s} P_i + Q,\end{aligned}$$

where $P_i, \tilde{P}_i \in \mathcal{R}^l(\sqrt{a(\lambda)})$, $l = l(k,s)$ satisfies $l(k,s) \to \infty$ as $k \to \infty$, and the symbol of Q is rapidly decreasing in x. We have, therefore,

$$\begin{aligned}&B_1 P^* X^{2s+1} P + P^* X^{2s+1} P B_1 \\ &\le \mathrm{Re} \sum_i \tilde{P}_i^* X^{2s} P_i + Q.\end{aligned}$$

Hence by the induction hypothesis

$$\begin{aligned}-\frac{d}{dt} P(t) &\le e^{-tB_1} F_+(B) (\mathrm{Re} \sum_i \tilde{P}_i^* X^{2s} P_i + Q) F_+(B) e^{-tB_1} \\ &\le C e^{-t\epsilon_0},\end{aligned}$$

with some constant $C > 0$, if k is chosen large enough. Since

$$F_+(B) P^* X^{2s+1} P F_+(B) = P(0) = - \int_0^\infty \frac{d}{dt} P(t) dt,$$

one can see that $X^{s+1/2} P F_+(B) \in \mathbf{B}$, which completes the proof of Theorem 1.1.

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Scattering Theory in the Energy Space for a Class of Nonlinear Wave Equations

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Dedicated to Professor ShigeToshi Kuroda on his sixtieth birthday

§1. Introduction

The purpose of this talk is to present a survey of the theory of scattering for a class of nonlinear wave equations of the form

$$\Box\varphi \equiv \partial_t^2\varphi - \Delta\varphi = -f(\varphi) \tag{1.1}$$

in a space of initial data and asymptotic states as large as the energy space associated with that equation. The exposition will follow the treatment given in [12]. Here φ is a complex valued function defined in space time $\mathbb{R}^{n+1}$, Δ is the Laplace operator in $\mathbb{R}^n$, and f is a nonlinear suitably regular complex valued function satisfying polynomial bounds at zero and at infinity. A large amount of work has been devoted to the theory of scattering for the equation (1.1) and for several other equations, and we shall devote most of this introduction to a partial review of nonlinear scattering in order to put the subsequent treatment of (1.1) into perspective.

The general setting is the following. One considers a semilinear equation

$$\partial_t u = Lu + F(u) \tag{1.2}$$

where L is a linear antiselfadjoint operator in some Hilbert space $\mathcal{H}$, and generates a one parameter unitary group $U(t) = \exp(tL)$ in $\mathcal{H}$. One is interested in situations where the global Cauchy problem for (1.2) is well understood in some space X (which may or may not coincide with $\mathcal{H}$). In particular any initial data $u_0 \in X$ should generate a unique global X valued solution of (1.2) with $u(0) = u_0$ and with suitable regularity

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in time. One is then interested in studying the asymptotic behaviour in time of the solutions of (1.2) by comparison with the solutions of the linear equation

$$\partial_t u = Lu \tag{1.3}$$

hereafter referred to as the free equation. That study gives rise to the following two questions.
(1) Given $u_\pm \in X$, does there exist a (unique) solution u of the equation (1.2) that behaves at $t \to \pm\infty$ as the solution $U(\cdot)u_\pm$ of the free equation (1.3) generated by $u_\pm$, for instance in the sense that

$$\|u(t) - U(t)u_\pm; X\| \longrightarrow 0 \qquad \text{as } t \to \pm\infty \tag{1.4}$$

or

$$\|U(-t)u(t) - u_\pm; X\| \longrightarrow 0 \qquad \text{as } t \to \pm\infty. \tag{1.4$'$}$$

If that is the case, one defines the wave operators $\Omega_\pm$ as the maps $u_\pm \to u(0)$ thereby obtained. This first question is referred to as that of the existence of the wave operators. Actually, one may be interested in comparing solutions of (1.2) and (1.3) in a sense different from and in fact stronger than (1.4) (1.4$'$). For instance one may require that

$$\|u - U(\cdot)u_\pm; \mathcal{X}([T, \pm\infty))\| \longrightarrow 0 \qquad \text{as } T \to \pm\infty \tag{1.5}$$

where $\mathcal{X}(I)$ is a space of X valued functions defined in a time interval I with prescribed behaviour in time. Such a convergence is in fact needed in order to develop a consistent theory of scattering.

The second question is somehow the converse of the first one.
(2) Given a generic X valued solution of (1.2) generated by initial data $u(0) = u_0 \in X$, does there exist $u_\pm \in X$ such that u behaves asymptotically as $U(\cdot)u_\pm$ as $t \to \pm\infty$ in the same sense as above. If that is the case for all $u_0 \in X$, one says that asymptotic completeness (AC) holds in X. Note that this notion of asymptotic completeness is very restrictive, since the only asymptotic evolution which is used is the free evolution. In the linear quantum mechanical many body problem, this would correspond to the case where asymptotic completeness is achieved by the completely free channel, a situation typical of purely repulsive interactions.

A general method to prove the existence of the wave operators, and the one to be used in all the examples to follow, consists in solving the Cauchy problem for (1.2) with infinite initial time. In fact the Cauchy problem for (1.2) with initial data u_0 at time t_0 is equivalent to the integral equation

$$u(t) = U(t-t_0)u_0 + \int_{t_0}^{t} d\tau U(t-\tau)F(u(\tau)). \tag{1.6}$$

The solution u expected to behave as $U(\cdot)u_\pm$ at $t \to \pm\infty$ should then be obtained by taking $u_0 = U(t_0)u_\pm$ and letting $t_0 \to \pm\infty$. Restricting one's attention to positive times for definiteness, one obtains the equation

$$u(t) = U(t)u_+ - \int_{t}^{\infty} d\tau U(t-\tau)F(u(\tau)) \tag{1.7}$$

to be solved for u for given u_+. One can then try to solve (1.7) by a contraction method in a time interval $[T, \infty)$ for T sufficiently large, and then continue the solution u thereby obtained to all times by using the known results on the Cauchy problem at finite times. The contraction step requires the use of a space $\mathcal{X}([T, \infty))$ of X valued functions of time with a suitable time decay, in order to control the integral in (1.7). That time decay has to be satisfied by the solutions $U(\cdot)u_+$ of the free equation. As a standard by product of the previous method, one obtains a proof of the existence of global solutions and of asymptotic completeness for small data. The method also requires that $F(u)$ exhibit a suitable decay in time for u in the relevant space $\mathcal{X}([T, \infty))$. This in turns requires that the function F tend to zero sufficiently fast when u tends to zero. In the case where F satisfies power bounds in u as $u \to 0$, that condition reduces to lower bounds on the associated exponents.

Asymptotic completeness for general data, once the previous results are available, reduces to proving that generic solutions of (1.2) with initial data in X exhibit the time decay that is used in the definition of the space $\mathcal{X}(\cdot)$ used to solve the Cauchy problem at infinity. The question of AC therefore reduces to the derivation of a priori estimates and depends in a specific way on the invariances and conservation laws of the equation at hand. As should be clear from a previous remark, it always requires a repulsivity condition on the interaction term F.

We now review briefly some of the available results for the most studied equations, namely the nonlinear Schrödinger (NLS) equation

$$i\partial_t \varphi = -(1/2)\Delta\varphi + f(\varphi), \tag{1.8}$$

the nonlinear wave (NLW) equation (1.1), and the nonlinear Klein Gordon (NLKG) equation

$$\Box\varphi + m^2\varphi + f(\varphi) = 0 \tag{1.9}$$

which differs from (1.1) by the presence of a mass term $m^2\varphi$. For clarity we restrict our attention ot the case where the nonlinear interaction term is a single power

$$f(\varphi) = \lambda|\varphi|^{p-1}\varphi. \tag{1.10}$$

For those three equations, the global Cauchy problem is well understood in the energy space X_0, to be defined below, for $\lambda \geq 0$ and $1 \leq p < p_* \equiv 1 + 4/(n-2)$ in space dimension $n \geq 2$.

For the NLS equation, one takes $u = \varphi$ and $F(u) = -if(\varphi)$, the free evolution group is $U(t) = \exp(i(t/2)\Delta)$, the conserved energy is

$$E(\varphi) = (1/2)\|\nabla\varphi\|_2^2 + \int dx V(\varphi) \tag{1.11}$$

where $\|\cdot\|_r$ denotes the norm in $L^r \equiv L^r(\mathbb{R}^n)$ and

$$V(\varphi) = 2\lambda(p+1)^{-1}|\varphi|^{p+1} \tag{1.12}$$

in the special case (1.10). Furthermore the L^2 norm of φ is also conserved, and the energy space is the standard Sobolev space $X_0 = H^1$.

For the NLW and NLKG equations, one takes $u = (\varphi, \partial_t\varphi)$ and $F(u) = (0, -f(\varphi))$. The solution of the free equation generated by the initial data $u_0 = (\varphi_0, \psi_0)$ at time $t = 0$ is

$$\varphi^{(0)}(t) = \dot{K}(t)\varphi_0 + K(t)\psi_0 \tag{1.13}$$

where $K(t) = \omega^{-1}\sin\omega t$, $\dot{K}(t) = \cos\omega t$, $\omega = \sqrt{-\Delta}$ for NLW ($\omega = \sqrt{-\Delta + m^2}$ for NLKG), so that the free evolution group is

$$U(t) = \begin{pmatrix} \dot{K}(t) & K(t) \\ -\omega^2 K(t) & \dot{K}(t) \end{pmatrix}. \tag{1.14}$$

The energy is

$$E(\varphi, \psi) = \|\psi\|_2^2 + \|\nabla\varphi\|_2^2(+m^2\|\varphi\|_2^2) + \int dx V(\varphi) \tag{1.15}$$

for NLW(NLKG), and is conserved in the sense that $E(\varphi, \partial_t\varphi) = \text{Const.}$ for solutions of the equation. The energy space is $X_0 = (\dot{H}^1 \cap L^{p+1}) \oplus L^2$ for NLW and $X_0 = H^1 \oplus L^2$ for NLKG, where $\dot{H}^1$ is the homogeneous Sobolev space associated with H^1.

We now summarize the main results available on the existence of the wave operators for the NLS, NLW and NLKG equations with power nonlinearity (1.10). For the NLS equation [5, 6, 7, 9, 17, 25, 42, 45, 46], the wave operators are known to exist in the energy space $X_0 = H^1$ for $4/n < p - 1 < 4/(n-2)$ [7]. In the smaller space $X = \Sigma$ defined by

$$\Sigma = H^1 \cap \mathcal{F}(H^1) \equiv \{\varphi \colon \varphi \in H^1 \text{ and } x\varphi \in L^2\} \tag{1.16}$$

the wave operators are known to exist for $4/(n+2) < p - 1 < 4/(n-2)$ [5]. Finally for $0 < p - 1 \leq 2/n$, the wave operators do no exist even in the L^2-sense, namely (1.4) with $X = L^2$ implies $u_\pm = 0$ and $u = 0$ [41].

There is a huge literature on the theory of scattering and related problems (including global existence for small data) for the NLW equation [10–12, 14–16, 18, 19, 22–24, 28, 29, 33–36, 38–42]. For that equation, the wave operators are known to exist in the space

$$X = X_0 \cap \{(\varphi, \psi) \colon (x \otimes \nabla)\varphi \in L^2, x\psi \in L^2\} \tag{1.17}$$

for $p_1(n) < p < p_*$, where $p_1(n)$ is the larger root of the equation [28, 29]

$$n(n-1)p^2 - (n^2 + 3n - 2)p + 2 = 0. \tag{1.18}$$

That lower bound on p is not expected to be optimal however. One expects the same result to hold (possibly in a smaller space) for $p_0(n) < p < p_*$, where $p_0(n)$ is the larger root of the equation

$$(n-1)p(p-1) = 2(p+1). \tag{1.19}$$

That result is proved only in dimensions $n = 2$ and 3 and on special sets of regular asymptotic states [15, 18, 36]. For $p \leq p_0(n)$, the wave operators are expected not to exist, in view of the existing finite time blow up results for small solutions [14, 18, 19, 39]. In the energy space X_0, the wave operators exist under assumptions on f which barely fail to include (1.10) with $p = p_*$, the reason being that the lower limit on p required for the existence of the wave operators turns out to be $p > p_*$ in that case and conflicts with the condition $p < p_*$ required to solve the global Cauchy problem at finite times [12]. It is one of the purposes of this talk to present that theory.

For the NLKG equation [3, 4, 9, 31], the wave operators are expected to exist for $4/n < p-1 < 4/(n-2)$ in the energy space and for $p_0(n+1) < p < p_*$ in a suitably smaller space, but the available treatments of the problem in the literature do not seem to be optimal.

We next summarize the main results available on the question of asymptotic completeness (AC) for the same equations. As mentioned above, the proof of AC requires a repulsivity condition, namely $\lambda \geq 0$ in the case (1.10), and reduces basically to the proof of a priori estimates for generic solutions of the equations. There are essentially two methods available. The first method applies to the NLS and NLW equations (but not to the NLKG equation) and exploits the approximate pseudoconformal invariance of the NLS equation and conformal invariance of the NLW equation. For the NLS equation, it yields AC in the space $X = \Sigma$ defined by (1.16) for $p_0(n+1) \leq p < p_*$ [5, 17, 45, 46]. For the NLW equation [10, 33, 40] it yields AC in the space X defined by (1.17) in a rather simple way for $4/(n-1) < p-1 < 4/(n-2)$. There are some results for lower values of p and not too high dimension ($2 \leq n \leq 4$), but they are much harder to derive and probably not optimal [10, 11]. The second method of proof of AC is based on the Morawetz inequality [30], itself a variant of the approximate dilation invariance of the equation at hand. That method has been applied first to the NLKG equation [3, 4, 9, 31] and to the NLS equation [7, 9, 25]. It is especially well suited to the proof of AC in the energy space X_0 and allows for such a proof both for the NLS and NLKG equation for $4/n < p-1 < 4/(n-2)$. Remarkably enough, that method also applies to the NLW equation in the energy space, in spite of the weakness of the time decay available in that case [12], and yields AC under conditions on f that again barely fail to include the power interaction (1.10) with $p = p_*$. It is the second purpose of this talk to present the basic steps of that method and its application to the NLW equation.

The treatment of the theory of scattering for the NLW equation to be given below is interesting for several reasons. First, it allows for a test of the power of the methods in a case where on the one hand only weak time decay is available, but where on the other hand the space time homogeneity of the free equation somewhat alleviates the algebraic complications. This situation is to be contrasted with the better behaved but more complicated case of the NLKG equation. Second, it requires the study of the NLW equation in the critical case $p = p_*$, thereby leading to a number of results of direct relevance to the Cauchy problem in that case, for which there has been a strong interest recently.

The problem of existence of the wave operators will be treated in Section 2 below, and that of asymptotic completeness in Section 3. The exposition follows closely [12] to which we refer for a more detailed treatment and in particular for all the proofs.

§2. Existence of the wave operators

In this section we shall prove the existence of the wave operators for the NLW equation (1.1) by following closely the method sketched in the introduction, namely we shall solve the associated equation (1.7) by a contraction method for large times and continue the solution thereby obtained to all times by using the known results on the Cauchy problem at finite times. An essential role in proof will be played by the space time integrability properties (STIP) associated with the free wave equation $\Box\varphi = 0$, by which we mean properties of the following type. Consider a linear evolution equation

$$(2.1) \equiv (1.3) \qquad \partial_t u = Lu$$

where L is a linear antiselfadjoint operator in some Hilbert space, typicaly $\mathcal{H} = L^2$, and let again $U(t) = \exp(tL)$. Then any initial data $u_0 \in L^2$ generates a solution of (2.1)

$$(2.2) \qquad U(\cdot)u_0 \in (\mathcal{C} \cap L^\infty)(\mathbb{R}, L^2)$$

where by $\mathcal{C}(I, X)$ (resp. $L^q(I, X)$) we mean the space of continuous (resp. L^q) functions of time from some interval I to a Banach space X. Now if one is willing to give up some regularity in time, it may happen that one gains some regularity in space, namely that

$$(2.3) \qquad U(\cdot)u_0 \in L^q(\mathbb{R}, X)$$

for some q, $2 \leq q < \infty$, where X may be L^r for some $r > 2$, or a Sobolev space W_r^ρ for some $r \geq 2$ and some $\rho \in \mathbb{R}$, preferably $\rho \geq 0$, or some more general space. Such properties exist for a large class of dispersive equations and have a long history [7, 13, 20, 21, 26, 27, 35, 37, 43, 44, 47]. A recent and hopefully didactic account appears in [13]. Since the wave equation is somewhat complicated in that respect, we shall first explain the basic facts on the simplest example of the Schrödinger equation $i\partial_t\varphi = -(1/2)\Delta\varphi$. In that case the unitary group $U(t)$ can be represented by the operator of convolution in space

$$(2.4) \qquad U(t) = \exp[i(t/2)\Delta] = (2\pi i t)^{-n/2} \exp[ix^2/(2t)] *_x$$

so that by the Young inequality, for any $f \in L^1$,

$$(2.5) \qquad \|U(t)f\|_\infty \leq (2\pi|t|)^{-n/2} \|f\|_1$$

and by interpolation with unitarity in L^2,

$$(2.6) \qquad \|U(t)f\|_r \leq (2\pi|t|)^{-\delta(r)} \|f\|_{\bar{r}}$$

for all $f \in L^{\bar{r}}$, where $2 \leq r \leq \infty$, r and $\bar{r}$ denote pairs of Hölder conjugate exponents, namely $1/r + 1/\bar{r} = 1$, and here and in what follows

$\delta(r) = n(1/2-1/r)$. Let now f be a function of space time and introduce the operator

$$U *_t f \equiv \int d\tau U(t-\tau) f(\tau). \tag{2.7}$$

From (2.6) and from the Hardy-Littlewood-Sobolev inequality in time, it follows that for $0 \le \delta(r) = 2/q < 1$

$$\| U *_t f; L_t^q(\mathbb{R}, L_x^r) \| \le C \| f; L_t^{\bar{q}}(\mathbb{R}, L_x^{\bar{r}}) \| \tag{2.8}$$

where the subscripts t and x serve as reminders of the variable of interest. At this point, an elementary and by now well known duality argument (see Lemma 2.1 in [13]) yields the following two results. First, the following inequalities also hold

$$\| U *_t f; L_t^{q_1}(\mathbb{R}, L_x^{r_1}) \| \le C \| f; L_t^{\bar{q}_2}(\mathbb{R}, L_x^{\bar{r}_2}) \| \tag{2.9}$$

where $0 \le \delta(r_i) = 2/q_i < 1$, $i = 1, 2$, the main point and difference with (2.8) being that now the pairs of exponents (q, r) in the left hand side and in the right hand side are completely decoupled. Second, for any $u_0 \in L^2$ and $0 \le \delta(r) = 2/q < 1$, the following estimate also holds

$$\| U(\cdot) u_0; L_t^q(\mathbb{R}, L_x^r) \| \le C \| u_0 \|_2. \tag{2.10}$$

Estimates of the type (2.9), (2.10) are especially convenient to study the Cauchy problem for semilinear equations of the type (1.2) in the form of the integral equation (1.6). In fact, one can use the estimates of the type (2.10) to control the free solution and the estimates of the type (2.9) to control the integral in the right hand side of (1.6).

We now turn to the case of the wave equation $\square\varphi = 0$. We recall that the solution with initial data $u_0 = (\varphi_0, \psi_0)$ at time zero is given by

$$\varphi^{(0)}(t) = \dot{K}(t)\varphi_0 + K(t)\psi_0 \tag{2.11 $\equiv$ 1.13}$$

where $K(t) = \omega^{-1} \sin \omega t$, $\dot{K}(t) = \cos \omega t$ and $\omega = \sqrt{-\Delta}$. The STIP of the wave equation are best expressed in terms of homogeneous Besov spaces $\dot{B}_r^\rho \equiv \dot{B}_{r,2}^\rho(\mathbb{R}^n)$. Those spaces are to be thought of as technically more adequate substitutes for the homogeneous Sobolev spaces $\dot{W}_r^\rho$ (spaces of distributions with derivatives of order exactly ρ in L^r). In order to avoid technicalities, we refrain from giving an explicit definition. We refer for that and for a summary of basic properties to the appendix of [8] or [12], and for a more extensive treatment to [1], Chap. 6.

The basic estimate which replaces (2.6) in the case of the wave equation is the following [2, 32].

Lemma 2.1. *The following estimates hold for all r, $2 \le r \le \infty$*

$$\| \exp(i\omega t) f; \dot{B}_r^{-\beta(r)} \| \le C|t|^{-\gamma(r)} \| f; \dot{B}_{\bar{r}}^{\beta(r)} \| \tag{2.12}$$

where the loss of derivatives and the time decay exponents are given by $\beta(r) = \frac{n+1}{2}(1/2 - 1/r)$ *and* $\gamma(r) = (n-1)(1/2 - 1/r)$.

By exactly the same arguments as in the Schrödinger case, one obtains the following analogue of (2.9).

Lemma 2.2. *The following estimates hold*

$$\|K *_t f; L^{q_1}(\mathbb{R}, \dot{B}_{r_1}^{1-\beta(r_1)})\| \le C \|f; L^{\bar{q}_2}(\mathbb{R}, \dot{B}_{\bar{r}_2}^{\beta(r_2)})\| \tag{2.13}$$

for $0 \le \gamma(r_i) = 2/q_i < 1$, $i = 1, 2$.

We define the energy space for the wave equation as the space

$$X_0 = (\dot{H}^1 \cap L^{2^*}) \oplus L^2 \tag{2.14}$$

where $2^* = 2n/(n-2)$ and we restrict our attention from now on to space dimension $n \ge 3$. Finite energy initial data, namely initial data $(\varphi_0, \psi_0) \in X_0$ generate solutions of the free wave equation through (2.11). In the same way as for the Schrödinger equation, one obtains the following STIP for those solutions, in the form of inequalities similar to (2.10).

Lemma 2.3. *Let $(\varphi_0, \psi_0) \in X_0$ and define $\varphi^{(0)}$ by (2.11). Then*

$$\|\varphi^{(0)}; L^q(\mathbb{R}, \dot{B}_r^\rho)\| \le C \|(\varphi_0, \psi_0); X_0\| \tag{2.15}$$

for all triples (ρ, r, q) which are admissible in the sense that

$$0 \le \delta(r) \le n/2 \qquad (\textit{equivalently: } 2 \le r \le \infty) \tag{2.16}$$

$$0 \le 1/q = \rho + \delta(r) - 1 \equiv \sigma < 1/2 \tag{2.17}$$

$$\rho + \beta(r) \le 1 \qquad (\textit{equivalently: } 2\sigma \le \gamma(r)). \tag{2.18}$$

The STIP of Lemma 2.3 are best visualized in the (σ - ρ) plane, where σ is the variable defined by the second equality in (2.17). The variable σ characterizes the homogeneity of the relevant norms in the space variable. In particular Sobolev inequalities allow to control a given

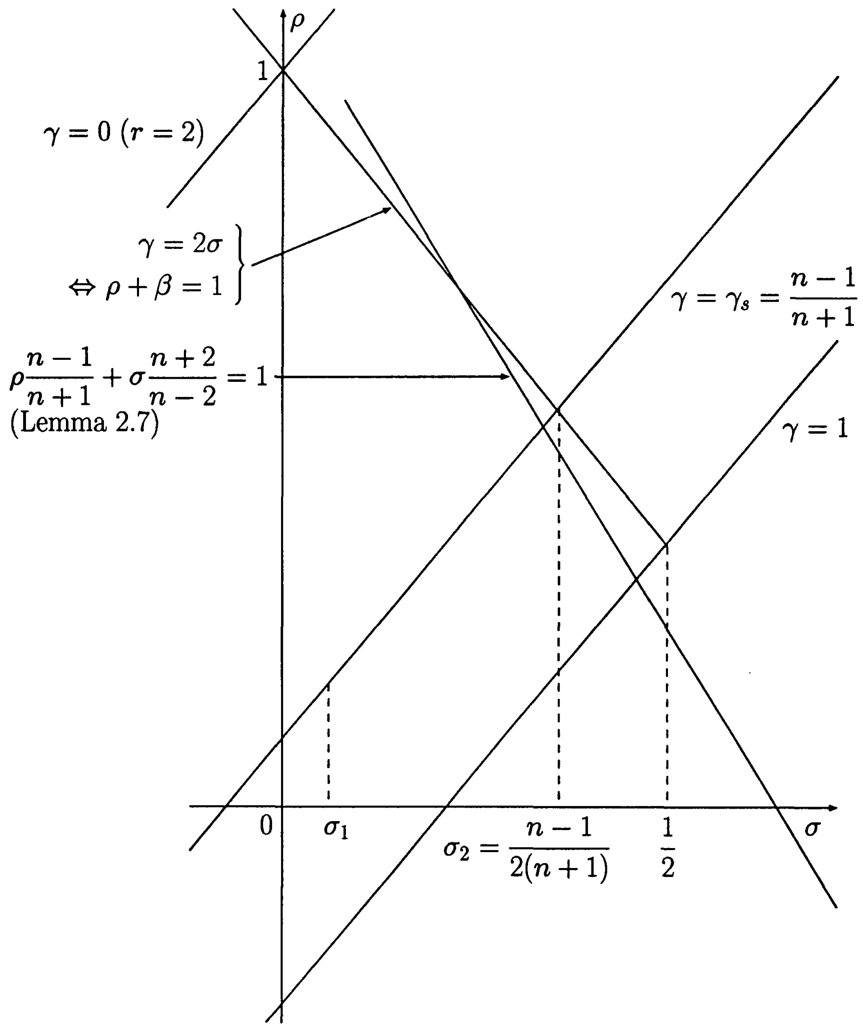

Fig. 1. STIP of $\square\varphi = 0$

$\dot{B}^\rho_r$ norm in terms of other such norms with the same σ and higher values of ρ. The admissible region (2.16)–(2.18) is represented in Figure 1. For instance the point $(\sigma = 0, \rho = 0)$ corresponds to $L^\infty(\mathbb{R}, L^{2^*})$, the point $\sigma = 0$, $\rho = 1$ to $L^\infty(\mathbb{R}, \dot{H}^1)$, etc. Following Lemma 2.3, it is natural to introduce the following spaces of pairs of functions. For any interval $I \subset \mathbb{R}$, we define

$$(2.19)\quad \mathcal{Y}_0(I) = \{(\varphi, \psi) \colon \varphi \in L^\infty(I, L^{2^*}) \cap L^q(I, \dot{B}^\rho_r) \text{ and } \psi \in L^q(I, \dot{B}^{\rho-1}_r) \text{ for all admissible } (\rho, r, q)\}.$$

Lemma 2.3 says in particular that initial data $(\varphi_0, \psi_0) \in X_0$ generate solutions $\varphi^{(0)}$ such that $(\varphi^{(0)}, \partial_t\varphi^{(0)}) \in \mathcal{Y}_0(\mathbb{R})$.

Although we shall never need to assume faster space decay on φ than is contained in the condition $\varphi \in L^{2^*}$, it is worthwhile to remark that such a decay is preserved in time for functions in $\mathcal{Y}_0(\cdot)$ in the following sense (see Proposition 2.1 in [12]).

Lemma 2.4. *Let $(\varphi, \partial_t\varphi) \in \mathcal{Y}_0(I)$ for some interval I and let $\varphi(s) \in L^k$ for some $s \in I$ and for some k, $2 \le k \le 2^*$. Then $\varphi(t) \in L^k$ for all $t \in I$, $\varphi \in \mathcal{C}(I, L^k)$ and φ satisfies the estimate*

$$\|\varphi(t)\|_k \le C(1 + |t|)^{1-\delta(k)} \tag{2.20}$$

for all $t \in I$, where C depends on the norms of $\varphi(s)$ in L^k and of $(\varphi, \partial_t\varphi)$ in $\mathcal{Y}_0(I)$ but not otherwise on I.

We now turn to the study of finite energy solutions of the equation (1.1). We assume from now on that f satisfies the following assumption:

(H1) $f \in \mathcal{C}^1(\mathbb{C}, \mathbb{C})$ and for some p, $1 < p < \infty$,

$$|f'(z_1) - f'(z_2)| \le C \begin{cases} |z_1 - z_2| \max |z_i|^{p-2} & \text{if } p \ge 2 \\ |z_1 - z_2|^{p-1} & \text{if } p \le 2 \end{cases} \tag{2.21}$$

for all $z_1, z_2 \in \mathbb{C}$, where f' stands for any of $\partial f/\partial z, \partial f/\partial \bar{z}$.

Of special interest will be the case where $p = p_*$.

We recall that the NLW equation (1.1) can be recast in the form (1.2) with $u = (\varphi, \partial_t\varphi)$ and $U(\cdot)$ given by (1.14), so that the integral equation (1.6) reduces in that case to

$$\varphi(t) = \dot{K}(t - t_0)\varphi_0 + K(t - t_0)\psi_0 - \int_{t_0}^{t} d\tau K(t - \tau) f(\varphi(\tau)) \tag{2.22}$$

and to a second equation for $\partial_t\varphi$ which is nothing but the time derivative of (2.22) and which we shall therefore omit. Similarly the equation (1.7) which leads to the definition of the wave operators reduces to

$$\varphi(t) = \dot{K}(t)\varphi_+ + K(t)\psi_+ + \int_t^\infty d\tau K(t - \tau) f(\varphi(\tau)) \tag{2.23}$$

and to the time derivative thereof, which we again omit. All subsequent results in this section are derived from the equations (2.22), (2.23) by estimating the free solution and the integral in the right hand sides by Lemmas 2.3 and 2.2 respectively. That requires in addition estimates for the nonlinear interaction $f(\varphi)$ in the integrand. Besov spaces are especially convenient for that purpose since they allow for Leibniz type estimates of the following form (see Lemma 2.3 in [12] for more general results).

Lemma 2.5. *Let f satisfy* (H1) *for some $p \geq 2$. Let $1 \leq r, s, k \leq \infty$, $1/s = 1/r + 1/k$, and $0 < \rho < \min(1, n/r)$. Then*

$$\|f(\varphi); \dot{B}^\rho_s\| \leq C\|\varphi; \dot{B}^\rho_r\| \; \| \, |\varphi|^{p-2}\|_k. \tag{2.24}$$

We first give some preliminary results on finite energy solutions of the equation (1.1). As a preliminary to the proof of asymptotic completeness in the next section, one can easily show that solutions of (1.1) in $\mathcal{Y}_0$ have asymptotic states (see Proposition 2.3 in [12]).

Lemma 2.6. *Let f satisfy* (H1) *with $p = p_*$. Let φ be a solution of* (1.1) *such that $u = (\varphi, \partial_t \varphi) \in \mathcal{Y}_0(I)$ for some interval $I \subset \mathbb{R}$. Then $u \in \mathcal{C}(I, X_0)$. Furthermore if I is infinite, say $I = [T, \infty)$ then*

$$\exists \operatorname*{s-lim}_{t \to \infty} U(-t)u(t) = u_+ \qquad \textit{in } X_0 \tag{2.25}$$

The next result says basically that "some" of the STIP of solutions of (1.1) included in the definition of $\mathcal{Y}_0$ imply all such STIP.

Lemma 2.7. *Let f satisfy* (H1) *with $p = p_*$, let I be an interval of $\mathbb{R}$, and let φ be a solution of* (1.1) *with $\varphi \in L^q(I, \dot{B}^\rho_r)$ for one admissible triple (ρ, r, q) such that*

$$\rho(n-1)/(n+1) + \sigma(n+2)/(n-2) \geq 1. \tag{2.26}$$

Then $(\varphi, \partial_t \varphi) \in \mathcal{Y}_0(I)$.

The region defined by (2.26) in the (σ - ρ) plane is the upper right corner of the admissible region indicated on Figure 1.

We are now in a position to attack the local resolution of the equation (1.1) in the form of the integral equations (2.22) or (2.23). As mentioned earlier, we first solve that problem locally in time by a contraction method, actually by a partial contraction method whereby all the norms defining the relevant space are reproduced by the right hand side of (2.22), (2.23), but only part of them are contracted on bounded sets of that space. One could use for that purpose the space $\mathcal{Y}_0$ defined by (2.19), but it is technically more convenient to use intermediate spaces of functions φ satisfying only part of the STIP and to rely on

Lemma 2.7 to prove that the solutions thereby obtained belong to $\mathcal{Y}_0$. A convenient choice of intermediate spaces is

$$\mathcal{X}_0(I) = \bigcap_{i=1,2} L^{q_i}(I, \dot{B}^{\rho_i}_{r_s}) \tag{2.27}$$

where $r_s = 2(n+1)/(n-1)$ and (ρ_i, r_s, q_i) are two admissible triples satisfying

(2.28)

$$0 < \sigma_1 \le \min\left(\frac{n-2}{2(n+1)}, \frac{n+2}{(n+1)(n-2)}\right) < \sigma_2 = \frac{1}{2}\gamma(r_s) = \frac{n-1}{2(n+1)}.$$

The value $r = r_s$ corresponds to $\beta = 1/2$, namely to the case where there is neither gain nor loss of derivatives in (2.13). The point (ρ_2, r_s, q_2) lies on the upper boundary of the admissible region and satisfies (2.26), so that Lemma 2.7 will be applicable to solutions in $\mathcal{X}_0(I)$ (see Figure 1).

We can now state the basic local existence result (see Proposition 3.1 in [12]).

Proposition 2.1. *Let f satisfy* (H1) *with $p = p_*$.*
(1) *Let $(\varphi_0, \psi_0) \in X_0$. Then there exists $T > 0$ such that the equation* (2.22) *has a unique solution $\varphi \in \mathcal{X}_0(I)$, where $I = [t_0 - T, t_0 + T]$. Furthermore $(\varphi, \partial_t\varphi) \in \mathcal{Y}_0(I)$.*
(2) *Let $(\varphi_+, \psi_+) \in X_0$. Then there exists $T > 0$ such that the equation* (2.23) *has a unique solution $\varphi \in \mathcal{X}_0(I)$, where $I = [T, \infty)$. Furthermore $(\varphi, \partial_t\varphi) \in \mathcal{Y}_0(I)$. In particular φ satisfies* (2.25).
(3) *Let $(\varphi_0, \psi_0) \in X_0$ (resp. $(\varphi_+, \psi_+) \in X_0$) be small in X_0 norm. Then there exists a unique solution $\varphi \in \mathcal{X}_0(\mathbb{R})$ of the equation* (2.22) *(resp.* (2.23)*). Furthermore $(\varphi, \partial_t\varphi) \in \mathcal{Y}_0(\mathbb{R})$, φ satisfies* (2.25) *and its analogue as $t \to -\infty$.*

As remarked before the contraction method yields as a by product the existence of global solutions for small data (part (3) of Proposition 2.1.) and asymptotic completeness for small data.

Corollary 2.1. *Let f satisfy* (H1) *with $p = p_*$. Then the wave operators $\Omega_\pm$ exist as bijections of X_0 locally in a neighborhood of zero.*

The second step in the construction of the wave operators consists in extending the solutions obtained in Proposition 2.1 part (2) to all times. For that purpose we need to solve the global Cauchy problem at finite times. According to standard methods, this requires a priori estimates of solutions of (1.1) in the energy space, which in turn follow from energy conservation. We assume

(H2) (gauge invariance) There exists a function $V \in \mathcal{C}^1(\mathbb{C}, \mathbb{R})$ with $V(0) = 0$ such that $f(z) = \partial V/\partial \bar{z}$ and $V(z) = V(|z|) \geq -a^2|z|^2$ for all $z \in \mathbb{C}$.

The energy is then defined by (1.15) (with $m = 0$) and one can prove energy conservation in the following sense (see Proposition 3.6 in [12]).

Lemma 2.8. *Let f satisfy* (H1) *with $p = p_*$ and* (H2). *Let I be an interval of $\mathbb{R}$, let $t_0 \in I$ and $(\varphi_0, \psi_0) \in X_0$ and let φ be a solution of* (2.22) *such that $(\varphi, \partial_t\varphi) \in \mathcal{Y}_0(I)$. Then for all s and t in I*

$$E(\varphi(t), \partial_t\varphi(t)) = E(\varphi(s), \partial_t\varphi(s)). \tag{2.29}$$

The question arises at this point whether one can solve the Cauchy problem globally in time for the NLW equation in the energy space for the critical value $p = p_*$ of the exponent in f. There has been recently a strong interest for that problem, for which Proposition 3.1, especially part (1), and Lemma 2.8 are directly relevant. The answer is most probably yes but the existing proofs are restricted either to finite energy *radial* solutions, or to smooth solutions in space dimension $n \leq 7$. In order to proceed safely, we therefore assume in addition that f satisfies the assumption (H1) both for $p = p_*$ and for some $p < p_*$. It is at this point that single power nonlinearities (1.10) barely escape from the present theory. One can then derive the final result on the existence of the wave operators.

Proposition 2.2. *Let f satisfy* (H1) *both for $p = p_*$ and for $p = p_2 < p_*$, and* (H2).
(1) *Let $(\varphi_+, \psi_+) \in X_0$. Then the equation* (2.23) *has a unique solution φ such that $(\varphi, \partial_t\varphi) \in \mathcal{Y}_0([T, \infty))$ for all $T \in \mathbb{R}$. That solution satisfies* (2.29) *for all s and t in $\mathbb{R}$ and satisfies* (2.25). *In particular the wave operator $\Omega_+ \colon (\varphi_+, \varphi_+) \to (\varphi(0), \partial_t\varphi(0))$ is well defined from X_0 to X_0. Similar results hold for negative times.*
(2) *Let in addition $V \geq 0$ (i.e. $a = 0$). Then $\Omega_\pm$ and $\Omega_\pm^{-1}$ are bounded operators in X_0 norm.*

Part (2) of Proposition 2.2 follows from the simple remark that for all $t \in \mathbb{R}$

$$\|u(t); X_0\|^2 \leq E \leq \|u(t); X_0\|^2 + C\|u(t); X_0\|^{2^*} \tag{2.30}$$

where E is the energy, the first inequality follows from the positivity of V and the second one from a Sobolev inequality. The same double inequality holds for $U(-t)u(t)$ instead of $u(t)$ because $U(\cdot)$ is isometric in X_0, and for u_+ by Lemma 2.6, so that $\|u(0); X_0\| \leq m(\|u_+; X_0\|)$ and $\|u_+; X_0\| \leq m(\|u(0); X_0\|)$ with $m(s) = (s^2 + Cs^{2^*})^{1/2}$.

§3. Asymptotic completeness

In this section, we sketch the proof of asymptotic completeness for the NLW equation (1.1) in the energy space by using the method originally devised in [31] for the NLKG equation, in the version given in [7, 9, 12]. It follows from Propositions 3.1 and 3.2 that the proof of AC reduces to proving that generic finite energy solutions, namely solutions of (2.22) with initial data in X_0, belong to $\mathcal{Y}_0(\mathbb{R})$. By Lemma 2.7, it suffices to prove that such solutions belong to $L^q(\mathbb{R}, \dot{B}^\rho_r)$ for one admissible triple (ρ, r, q) satisfying (2.26). The proof then reduces to a priori estimates on those solutions. We continue to restrict our attention to space dimension $n \geq 3$ to begin with. However the proof will require at some point the existence of one norm of the solutions with integrable decay in time, namely $\gamma(r) > 1$, and will therefore only apply in space dimension $n \geq 4$, since $\gamma(\infty) = 1$ for $n = 3$.

The essence of the proof consists in squeezing the given solution between two conflicting estimates which force it to decay. The first of those estimates is the Morawetz inequality [30]. For f satisfying (H2), we introduce the auxiliary potential

$$W_1(z) = \bar{z}f(z) - V(z). \tag{3.1}$$

For f a single power (1.10), W_1 reduces to $W_1(z) = (\lambda/2)(p-1)|z|^{p+1}$. We introduce also the functions $g(x) = (x^2 + a^2)^{-1/2}$ and $g_1(x) = \nabla \cdot (xg)$. One checks easily that $(n-1)g \leq g_1 \leq ng$ and that $\Delta g_1 \leq 0$ for $n \geq 3$. We can now state the Morawetz inequality (see Lemma 4.3 in [12]).

Lemma 3.1. *Let f satisfy* (H1) *with $p = p_*$ and* (H2), *let I be an interval, $t_0 \in I$, $(\varphi_0, \psi_0) \in X_0$ and φ a solution of* (2.22) *with $(\varphi, \partial_t\varphi) \in \mathcal{Y}_0(I)$. Then for all s and t in I, $s \leq t$,*

$$\int_s^t d\tau \int dx g_1(x) W_1(\varphi(\tau, x)) \leq - \mathrm{Re}\langle \partial_t \varphi, (xg \cdot \nabla + \nabla \cdot xg)\varphi \rangle \Big|_t^s . \tag{3.2}$$

The Morawetz inequality is a modified version of dilation invariance. In fact the operator $x \cdot \nabla + \nabla \cdot x$ is the generator of space dilations. It fails to be defined in the energy space because of the factor x, and the function g serves to compensate for that defect. Let $A = xg \cdot \nabla + \nabla \cdot xg$.

The formal proof of (3.2) consists in computing the time derivative

$$\begin{aligned} -\partial_t \operatorname{Re}\langle \partial_t \varphi, A\varphi \rangle &= -\operatorname{Re}\langle \partial_t^2 \varphi, A\varphi\rangle \\ &= -\operatorname{Re}\langle \Delta\varphi, A\varphi\rangle + \operatorname{Re}\langle f(\varphi), A\varphi\rangle \end{aligned}$$

by using the antisymmetry of A and the equation (1.1). By elementary computations, the term with Δ is easily seen to be non negative, while

$$\operatorname{Re}\langle f(\varphi), A\varphi\rangle = \int g_1 W_1(\varphi) dx.$$

The regularity of φ provided by $\mathcal{Y}_0$ is sufficient to convert the formal proof into an actual proof. For positive V the right hand side of (3.2) is bounded by $2E$ uniformly in s, t and a. For $I = \mathbb{R}$ and $W_1 \geq 0$, and after taking the harmless limit $a \downarrow 0$, one obtains from (3.2)

$$\int dt dx |x|^{-1} W_1(\varphi(t,x)) \leq 2E/(n-1). \tag{3.3}$$

The meaning of (3.3) is best understood by seing what it forbids: it forbids in particular that φ be a localized solution travelling at finite speed. In fact, if $\varphi(t,x) = h(x - vt)$ and if f is a single power (1.10), then the left hand side of (3.3) becomes approximately for large t

$$C \int^{\infty} t^{-1} dt \; \|h\|_{p+1}^{p+1} = \infty.$$

This fact suggests that φ must either spread out in space, or recede to infinity with unbounded velocity. The second possibility is however forbidden by the second basic estimate, namely the finiteness of the propagation speed coming from the hyperbolicity of the equation. That estimate is best expressed for the present purposes in terms of the propagation of local energy. For any $\Lambda \subset \mathbb{R}^n$, we denote the complement of Λ by $\Lambda' = \mathbb{R}^n \backslash \Lambda$ and we define the energy in Λ by

$$E(\varphi, \psi; \Lambda) = \int_\Lambda dx (|\psi|^2 + |\nabla\varphi|^2 + V(\varphi)). \tag{3.4}$$

We shall also need the balls in $\mathbb{R}^n$

$$B(x_0, R) = \{x \in \mathbb{R}^n \colon |x - x_0| < R\}.$$

We can now state the local energy propagation as follows (see Lemma 4.2 in [12]).

Lemma 3.2. *Let f satisfy* (H1) *with $p = p_*$ and* (H2) *with $V \geq 0$. Let $I \subset \mathbb{R}$ with $0 \in I$, let $(\varphi_0, \psi_0) \in X_0$ and let φ be a solution of* (2.22) *with $(\varphi, \partial_t\varphi) \in \mathcal{Y}_0(I)$. Then for all $x_0 \in \mathbb{R}^n, R > 0$ and $t \in I$*

$$E(\varphi(t), \partial_t\varphi(t); B(x_0, R - |t|)) \leq E(\varphi(0), \partial_t\varphi(0); B(x_0, R)) \tag{3.5}$$

$$E(\varphi(t), \partial_t\varphi(t); B'(x_0, R + |t|)) \leq E(\varphi(0), \partial_t\varphi(0); B'(x_0, R)). \tag{3.6}$$

The formal proof of Lemma 3.2 consists in noting that the energy momentum vector

$$\begin{aligned} \theta_0 &= |\partial_t\varphi|^2 + |\nabla\varphi|^2 + V(\varphi) \\ \theta &= -2\,\mathrm{Re}\,\partial_t\bar{\varphi}\nabla\varphi \end{aligned}$$

is time like and applying the Green formula to the truncated cones $0 \leq |\tau| \leq |t|$, $|x| \leq R \pm |\tau|$. Again the regularity provided by $\mathcal{Y}_0$ is sufficient to convert the formal proof into an actual proof. Lemma 3.2 will be used in the form of the following easily derived corollary (see Lemma 4.6 in [12]).

Corollary 3.1. *Under the same assumptions as in Lemma* 3.2 *with $I = \mathbb{R}$, for any $\eta > 0$*

$$\|\varphi(t); L^{2^*}(B'(0, (1+\eta)|t|)\| \longrightarrow 0 \qquad \textit{when } |t| \to \infty. \tag{3.7}$$

There remains the hard task of combining the estimates (3.3) and (3.7) to derive a priori estimates for the norm of φ in $L^q(\mathbb{R}, \dot{B}^\rho_r)$ for a suitable admissible triple (ρ, r, q). We choose such a triple with $\gamma(r) = 1 + \varepsilon$ and $\sigma = 1/2 - \varepsilon$ for some small $\varepsilon > 0$ (such a triple satisfies (2.26) for ε small enough). It is as this point that we have to restrict our attention to space dimension $n \geq 4$. For φ a solution of (2.22) with $t_0 = 0$, $(\varphi_0, \psi_0) \in X_0$ and $(\varphi, \partial_t\varphi) \in \mathcal{Y}_{0,\mathrm{loc}}(\mathbb{R})$ we define

$$k_0(t) = \|\dot{K}(t)\varphi_0 + K(t)\psi_0; \dot{B}^\rho_r\| \tag{3.8}$$

$$k(t) = \|\varphi(t); \dot{B}^\rho_r\|. \tag{3.9}$$

One of the main technical steps of the proof consists in deriving a set of integral inequalities for k by applying the estimates (2.12), (2.13), (2.15) to the integral equation (2.22). Some of these inequalities will require that f satisfy (H1) both for some $p_2 < p_*$ and for some $p_1 > p_*$. One can then prove:

Lemma 3.3. *Let $n \geq 4$ and let f satisfy* (H1) *both for $p = p_2 < p_*$ and for $p = p_1 > p_*$. Let (ρ, r, q) be an admissible triple with $\gamma(r) = 1+\varepsilon$ and $\sigma = 1/2 - \varepsilon$ for some small $\varepsilon > 0$ and let φ be a solution of* (2.22) *with $t_0 = 0$, $(\varphi_0, \psi_0) \in X_0$ and $(\varphi, \partial_t\varphi) \in \mathcal{Y}_{0,\mathrm{loc}}(\mathbb{R})$. Then for some $\eta > 0$ depending only on ε, p_1, p_2 and some $M(E)$ depending only on the energy E, φ satisfies the inequalities*

$$k(t) \leq k_0(t) + M(E) \int_0^t d\tau \min |t-\tau|^{-(1\pm\eta)} \min k(\tau)^{1\pm\eta}. \tag{3.10}$$

It is easy to see by homogeneity that $\varepsilon = 0$ and $p = p_*$ would yield (3.10) with $\eta = 0$. The combination of signs $(--)$ in (3.10) yields an information on the local regularity of k and requires only the assumption (H1) with $p_2 < p_*$ while the combination $(++)$ yields the (most important) information at infinity in time and requires $p_1 > p_*$. The crossed terms with $(+-)$ and $(-+)$ would hold with only $p = p_*$.

The next step in the proof consists in combining the estimates (3.3), (3.7) with the inequalities (3.10) in the case $(\pm, -)$ to show that $k(t)$ is small in suitable large intervals. That step requires the assumption (H1) only with $p = p_*$ and $p = p_2 < p_*$. In addition it requires the following additional repulsivity condition in order to exploit (3.3).

(H3) For some $c > 0$, for $p_2 < p_* < p_1$ and for all $z \in \mathbb{C}$

$$W_1(z) \geq c \min(|z|^{p_1+1}, |z|^{p_2+1}). \tag{3.11}$$

We introduce the auxiliary norms

$$\|\varphi; \ell^\infty(L^q(I, \dot{B}^\rho_r))\| \equiv \|k(t); \ell^\infty(L^q(I))\| = \sup_{t:\ [t,t+1]\subset I} \|k; L^q([t, t+1])\|.$$

One can then prove (see Lemma 4.5 in [12]):

Lemma 3.4. *Let $n \geq 4$, let f satisfy* (H1) *both for $p = p_*$ and for $p = p_2 < p_*$,* (H2) *with $V \geq 0$ and* (H3). *Let φ and k be as in Lemma 3.3. Then for any $\varepsilon_1 > 0$ and for any $\ell > 0$, there exists $a > 0$ such that*

$$\|k; \ell^\infty(L^q([a, a+\ell]))\| \leq \varepsilon_1. \tag{3.12}$$

The proof of Lemma 3.4 consists in estimating the integral in (2.22) with $t_0 = 0$ by splitting the integration region for large t in four subregions, for some small θ_1 and some large $\theta_2 < t$:

(1) In the region $t - \theta_1 \leq \tau \leq t$, one uses the estimate (3.10) with signs

$(--)$,
(2) In the region $0 \le \tau \le t - \theta_2$, one uses the estimate (3.10) with signs $(+-)$, thereby obtaining two estimates sublinear in k with small coefficients. In the intermediate region $t - \theta_2 \le \tau \le t - \theta_1$, one essentially splits the x integration in two subregions:
(3) For $t - \theta_2 \le \tau \le t - \theta_1$ and $|x| \le 2\tau$, one uses a modified version of the estimate (3.10) with signs $(\pm-)$ and a consequence of the estimate (3.3).
(4) For $t - \theta_2 \le \tau \le t - \theta_1$ and $|x| \ge 2\tau$, one uses a modified version of the estimate (3.10) with signs $(\pm-)$ and the estimate (3.7).
Lemma 3.4 means essentially that k tends to be small in suitably located, but arbitrarily large intervals. Using that information, which is a weak form of the fact that k tends to zero at infinity, and the superlinear part of Lemma 3.3, namely the inequalities (3.10) with signs $(\pm+)$, one then proves that $k(t)$ exhibits the same time decay at infinity as $k^{(0)}$, namely belongs to $L^q(\mathbb{R})$. That last step is an elementary abstract argument based on (3.10) and (3.12) and is otherwise independent of any additional property of the equation.

Combining all previous steps yields the final result (see Proposition 4.2 in [12]).

Proposition 3.1. *Let $n \ge 4$, let f satisfy* (H1) *both for $p = p_2 < p_*$ and for $p = p_1 > p_*$,* (H2) *with $V \ge 0$ and* (H3) *(namely* (3.11)*). Let $(\varphi_0, \psi_0) \in X_0$ and let φ be a solution of* (2.22) *with $(\varphi, \partial_t\varphi) \in \mathcal{Y}_{0,\mathrm{loc}}(\mathbb{R})$. Then $(\varphi, \partial_t\varphi) \in \mathcal{Y}_0(\mathbb{R})$. In particular asymptotic completeness holds in X_0.*

We finally comment on a problem left open by the preceding proof. Although any finite energy initial data $(\varphi_0, \psi_0) \in X_0$ generates a solution in $\mathcal{Y}_0(\mathbb{R})$, no estimate is obtained for the norm of $(\varphi, \partial_t\varphi)$ in $\mathcal{Y}_0(\mathbb{R})$ in terms of the energy of the solution. This is due to the fact that the proof starts from some time translation invariant information and ends up with information of the same type, in the form of time integrals, by going at some intermediate stages through estimates which are pointwise in time. It would be interesting to know whether the map $(\varphi_0, \psi_0) \to (\varphi, \partial_t\varphi)$ is bounded from X_0 to $\mathcal{Y}_0(\mathbb{R})$. This would probably require a simplified time translation invariant version of the preceding proof.

Another challenging question would be to extend the present results — if true — to the purely critical case where f is assumed to satisfy (H1) for $p = p_*$ only.

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The Relativistic Boltzmann Equation Near Equilibrium

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§1. A remark on spectral theory

The Boltzmann equation, linearized around the equilibrium, has the form

$$\frac{\partial f}{\partial t} + A f + K f = 0.$$

We want to deduce the exponential decay of $f(t)$ as $t \to \infty$. The operator $A + K$ is neither symmetric nor skew-symmetric. Nor is K compact. However, it enjoys the following properties:

(i) Spec $(A + K) \subset \{\text{Re } \lambda \geq 0\}$

(ii) $A + K$ has no point spectrum on Re $\lambda = 0$.

(iii) Spec $(A) \subset \{\text{Re } \lambda \geq \alpha_0\}$ for some $\alpha_0 > 0$.

(iv) K is A -smoothing.

Property (iv) means, roughly, that the operator

$$e^{-t_1 A} K e^{-t_2 A} K \ldots e^{-t_\ell A} K$$

is compact for all $t_1 > 0, \ldots, t_\ell > 0$.

Theorem [Vidav, Shizuta]. *The spectrum of $A + K$ in the strip $\{0 \leq \text{Re } \lambda < \alpha_0\}$ is discrete, and*

$$\| e^{-t(A+K)} \| \leq e^{-\alpha_1 t}$$

for some $\alpha_1 > 0$.

This is a generalization of Weyl's classical theorem on the perturbation of spectra. We will see at the end of the lecture how this theorem proves the stability of the equilibrium of the relativistic Boltzmann equation.

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§2. The relativistic Boltzmann equation

Consider a gas with particle density $F(t, x, v)$ where $t =$ time, $x =$ position, and $v =$ momentum. The particles interact only through collision. Thus

$$v_0\, \partial_t F + v \cdot \nabla_x F = \text{scattering term}.$$

If the particles are treated relativistically, then the momentum v is any vector in $\mathbf{R}^3$ and the velocity $\hat{v}$ satisfies $|\hat{v}| < c$. They are related by

$$v_0^2 - |v|^2 = m^2 c^2, \quad \hat{v} = c\frac{v}{v_0}.$$

The mass of a particle is m and the energy is cv_0. Henceforth we set $c = m = 1$ and rewrite the equation as

$$\text{(RB)} \qquad \partial_t F + \hat{v} \cdot \nabla_x F = Q(F)$$

with the scattering term

$$Q(F)(v) = \int_{\mathbf{R}^3} \int_{S^2} V_M \sigma [F(u')F(v') - F(u)F(v)]\, d\Omega\, du.$$

Here u and v are interpreted as the momenta of a pair of incoming particles, and u' and v' as the scattered ones. Thus the term $F(u')F(v')$ represents the gain and $F(u)F(v)$ the loss. Conservation of momentum and energy is expressed by

$$u + v = u' + v', \quad u_0 + v_0 = u_0' + v_0',$$

where $v_0 = \sqrt{1 + |v|^2}$, $u_0' = \sqrt{1 + |u_0'|^2}$, etc. (This is in contrast to the classical non-relativistic case where $v_0 = \text{const} \cdot |v|^2$). The scattering kernel is the product of two quantities. The Møller velocity V_M is given by

$$V_M^2 = |\hat{v} - \hat{u}|^2 - |\hat{v} \times \hat{u}|^2.$$

The scattering cross-section $\sigma = \sigma(g, \Theta)$ is a function of the generalized momentum difference g and the generalized scattering angle Θ. Notice that, for a given incoming momentum v, the three vectors u, u' and v' are constrained by the four scalar conservation laws given above. The integration in the scattering term runs over the five remaining variables.

A solution of (RB) has the conserved quantities

$$\int\!\!\int F\, dv\, dx, \int\!\!\int v\, F\, dv\, dx, \int\!\!\int v_0\, F\, dv\, dx,$$

the mass, momentum and energy, respectively. Furthermore, the entropy increases:

$$\frac{d}{dt} \int\!\!\int F \log F\, dv\, dx \leq 0.$$

(The last integral is the negative entropy.) The equilibrium of greatest entropy comes from minimizing the negative entropy subject to fixed mass, momentum and energy. It is

$$\mu(v) = e^{a+b\cdot v-c\sqrt{1+|v|^2}},$$

the maxwellian distribution. Our goal is to prove the asymptotic stability of $\mu(v)$.

In the classical case, $\mu(v)$ is a gaussian. After the introduction of the equation by Boltzmann in 1872, it was not until Carleman in 1933 that the stability was proved for the case of space-independent solutions. Grad in a series of papers around 1963 proved the stability for a finite time for general solutions. Finally in 1974 Ukai proved the asymptotic stability, and hence the global existence of solutions near equilibrium, in the case of spatial periodicity. Then Nishida and Imai and Ukai in 1976 solved the problem without a periodicity assumption. Many others have made substantial contributions to the classical theory in the last 15 years. Here we announce the resolution of the relativistic problem with spatial periodicity.

Main Theorem. *Assume that the scattering cross-section σ satisfies $k_1 g(1+g)^{-1} \leq \sigma(g,\Theta) \leq k_2$ for some constants $k_1, k_2 > 0$. Let the initial distribution F^0 satisfy*

(i) $F^0(x,v) \geq 0$

(ii) F^0 *is continuous*

(iii) F^0 *is periodic in* x

(iv) $\int\int (a+b\cdot v - c\sqrt{1+|v|^2})\,[F^0(x,v)-\mu(v)]\,dv\,dx = 0$ *for all* a, b, c.

(v) $\left|F^0(x,v)-\mu(v)\right| \leq \varepsilon\sqrt{\mu(v)}\,(1+|v|)^{-\gamma-3/2}$

for some $\gamma > 0$ and for sufficiently small ε. Then there exists a global, continuous, x-periodic solution of (RB) *with $F(0,x,v) = F^0(x,v)$, and there exist $\delta > 0$ and $c_1 > 0$ such that*

$$|F(t,x,v)-\mu(v)| \leq c_1 \varepsilon e^{-\delta t}\sqrt{\mu(v)}$$

for $0 \leq t < \infty$.

This theorem is also true with C^k and H^k norms for arbitrarily large k. Hence there exist arbitrarily smooth solutions. It is also true under more general conditions on σ.

§3. Sketch of the proof of stability

We may normalize $\mu(v) = \exp(-\sqrt{1+|v|^2})$. Next we write the perturbation as $F - \mu = \sqrt{\mu} f$, so that f satisfies

$$\partial_t f + A f + K f = \tilde{Q}(f),$$

where

$A = \hat{v} \cdot \nabla_x + \alpha(v)$,

$K =$ a linear integral operator in v,

$\tilde{Q} =$ a quadratic term.

We wish to solve this equation globally with small initial data. To do this, we choose a space Y on which $\tilde{Q}$ is bounded:

$$\| \tilde{Q} f \|_Y \leq c \| F \|_Y^2$$

and a similar Lipschitz property for $\tilde{Q}f - \tilde{Q}g$, together with decay of the linearized problem:

$$\int_0^\infty \| e^{-t(A+K)} \|_{\mathcal{L}(Y,Y)} dt < \infty.$$

It is a standard fact that these two properties imply the asymptotic stability.

To prove the Main Theorem, we choose the space Y of continuous functions $f(x,v)$, periodic in x, which satisfy

$$\int\int (a + b \cdot v + c\sqrt{1+|v|^2}) \sqrt{\mu} f \, dv \, dx = 0$$

for all a, b, c, such that the norm

$$\| f \|_Y = \sup_{x,v} (1+|v|)^{\gamma+3/2} |f(x,v)|$$

is finite. We omit the proof of boundedness of $\tilde{Q}$ on this space in order to concentrate on the linearized problem.

The linearized entropy identity is

$$\begin{aligned} &\langle Af + Kf, f \rangle \\ &= \int\int\int\int V_M \sigma \mu(u) \mu(v) \left[\frac{f(v')}{\sqrt{\mu(v')}} + \frac{f(u')}{\sqrt{\mu(u')}} - \frac{f(\mu)}{\sqrt{\mu(u)}} - \frac{f(v)}{\sqrt{\mu(v)}} \right]^2 \\ &\qquad \times du \, d\Omega \, dv \, dx. \end{aligned}$$

This expression is manifestly non-negative, and in fact is positive for all $f \neq 0$ in Y because of the orthogonality conditions. Thus properties (i) and (ii) from the beginning of this lecture are satisfied. Furthermore, $A = \hat{v} \cdot \nabla_x + \alpha(v)$ where

$$\alpha(v) = \frac{1}{2} \int \int V_M \sigma(g, \Theta) \mu(u) du \, d\Omega$$

is bounded above and below: $0 < \alpha_0 \leq \alpha(v) \leq \alpha_2 < \infty$. Hence $\langle Af, f \rangle \geq \alpha_0 \| f \|_{L^2}^2$.

In the classical case $\alpha(v)$ is like a constant times $|v|$, which means that the dissipation is large for large $|v|$. In the mid-1970's Shizuta showed how the concept of an A-smoothing operator can be applied to the classical Boltzmann equation. In fact, Grad showed in the 1960's that

$$Kf(t, x, v) = \int k(u, v) f(t, x, u) du$$

where $k(u, v) \leq c_1 |u - v|^{-1} \exp(-c_2 |u - v|^2)$, in the case of the hard sphere.

In the relativistic case the exponent is much weaker. Nevertheless we can improve the denominator to obtain

$$|k(u, v)| \leq c_1 \frac{e^{-c_2|u-v|}}{|u - v| + |u \times v|}.$$

This estimate implies that

$$\sup_v \int \left(|k| + |k|^2 \right) du < \infty$$

and, for all $\beta \geq 0$,

$$\int (1 + |u|)^\beta \, |k| du \leq c \, (1 + |v|)^{-\beta - 1}.$$

Following Shizuta, we approximate the kernel as a sum

$$k(u, v) \sim \sum p_j(u) q_j(v)$$

with nice functions p_j and q_j. Therefore the A-smoothing property of K would follow from the compactness of the operator

$$e^{-t_1 A} QP e^{-t_2 A} QP e^{-t_3 A} QP \cdots e^{-t_\ell A} QP$$

where Q is multiplication by $q_j(v)$ and P is integration with $p_j(u)$. In this string of operators it suffices to prove the boundedness of the various factors and the compactness of one of the factors. In fact, one string of three factors is

$$Pe^{-tA}Qf(x) = \int e^{-tA}[q(v)f(x)]p(v)dv$$
$$= \int e^{-t\alpha(v)}q(v)f(x-t\hat{v})p(v)dv.$$

We apply ∂_x to both sides of this identity. Inside the integral, ∂_x is converted to $t^{-1}\partial_{\hat{v}}$. A change of variables from v to $\hat{v}$ thus leads to the identity

$$\partial_x[Pe^{-tA}Qf] = \frac{1}{t}\int \frac{\partial}{\partial\hat{v}}(\text{a kernel})\cdot f\cdot dv.$$

Thus we gain regularity in x and therefore $Pe^{-tA}Q$ is compact. For details, see [3].

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On Spectral Theory for Schrödinger Operators with Magnetic Potentials

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Abstract.

In this survey, we want to analyze the effect of the presence of a magnetic potential on the spectrum of the Schrödinger operator with magnetic field. We consider three connected problems:
— study of the bottom of the spectrum
— study of the bottom of the essential spectrum
— study of the decay of the eigenfunctions.
We think this survey is complementary to other presentations of the subject in [12], [20] and [49].

§1. Qualitative Theory

Let $V \in C^\infty(\mathbb{R}^n)$ be an electrical potential s.t.

$$V \geq C \qquad \text{for some constant } C, \tag{1.1}$$

and let $A = (A_1, \dots, A_n)$ be a magnetic potential in $C^\infty(\mathbb{R}^n, \mathbb{R}^n)$. We denote by

$$\omega_A = \sum_j A_j \, dx_j \tag{1.2}$$

the corresponding 1-form and by

$$\sigma_B = d\sigma_A = \sum_{j<k} b_{jk} \, dx_j \wedge dx_k \tag{1.3}$$

the corresponding magnetic 2-form.

The Schrödinger operator with magnetic field is usually defined by

$$P_{A,V}(h) = \sum_{1 \leq j \leq n} (hD_{x_j} - A_j)^2 + V \tag{1.4}$$

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and we shall denote by $P^{\Omega}_{A,V}$ the Dirichlet realization in a connected open set Ω with bounded regular boundary (cf. [57]). If the operator is with compact resolvent, for example (see also the results in Section 2) if

$$V \text{ tends to } \infty, \text{ as } |x| \to \infty \tag{1.5}$$

we know by the Kato's inequality that (cf. [12])

$$\lambda^{\Omega}_{0,A,V}(h) \geq \lambda^{\Omega}_{0,0,V}(h) \tag{1.6}$$

where $\lambda^{\Omega}_{0,A,V}(h)$ is the first eigenvalue of $P^{\Omega}_{A,V}$.

In the case when $P^{\Omega}_{A,V}$ is not with compact resolvent, one easily get

$$\inf \mathrm{Sp}\, P^{\Omega}_{A,V} \geq \inf \mathrm{Sp}\, P^{\Omega}_{0,V} \tag{1.7}$$

observing that it is true (cf. (1.6)) when V is replaced by $V_\varepsilon = V + \varepsilon|x|^2$ and that

$$\inf \mathrm{Sp}\, P^{\Omega}_{A,V_\varepsilon} \to \inf \mathrm{Sp}\, P^{\Omega}_{A,V} \qquad \text{as } \varepsilon \to 0 \ (\varepsilon > 0). \tag{1.8}$$

Finally let us observe that due to the characterization of the essential spectrum by Persson [54] (see also Agmon [1]) we have also for the essential spectrum

$$\inf \mathrm{EssSp}\, P^{\Omega}_{A,V} \geq \inf \mathrm{EssSp}\, P^{\Omega}_{0,V}. \tag{1.9}$$

We are now interested to the cases where we have equality. Let us first recall the following result due essentially to Lavine-O'Caroll [41], (see also [21]).

Proposition 1.1. *Let $h > 0$ be fixed and Ω as above; let us assume that we have the assumptions* (1.1)–(1.5); *then the following properties are equivalent:*

(i) $\lambda^{\Omega}_{0,A,V}(h) = \lambda^{\Omega}_{0,0,V}(h)$

(ii) $P^{\Omega}_{A,V}$ *and* $P^{\Omega}_{0,V}$ *are unitary equivalent.*

(iii) (a) $\sigma_B = 0$ *in* Ω *and*

(b) *for all closed path in* Ω, $(2\pi h)^{-1} \int_\gamma \omega_A \in \mathbb{Z}$.

Sketch of the proof. If u_0 is the first eigenfunction of $P^{\Omega}_{0,V}(h)$ attached to the eigenvalue $\lambda^{\Omega}_0(h)$, (we know that u_0 does not vanish in Ω and we can then assume that $u_0 > 0$ in Ω and $\|u_0\| = 1$) we have the following identity

$$\begin{aligned} &\|(h\nabla - iA - h(\nabla u_0/u_0))\phi\|^2 \\ &\qquad = \langle (P^{\Omega}_A(h) - \lambda^{\Omega}_0)\phi \mid \phi\rangle \qquad \forall \phi \in C^\infty_0(\Omega) \end{aligned} \tag{1.10}$$

The first consequence is of course that we get another proof of (1.6). Let us briefly sketch the proof of (i) $\Rightarrow$ (iii) (which is the non trivial part of the statements). From (1.10) we deduce, using a minimizing sequence tending in L^2 to a normalized eigenfunction of $P^{\Omega}_{A,V}(h)$ u_A corresponding to $\lambda_A = \lambda_0$

$$(1.11) \qquad (h\nabla - iA - h(\nabla u_0/u_0))u_A = 0 \qquad \text{in } \mathcal{D}'(\Omega).$$

We rewrite (1.11) on the form

$$(1.12) \qquad (h\nabla - iA)\varphi_A = 0 \qquad \text{in } \mathcal{D}'(\Omega), \text{ with } \varphi_A = u_A/u_0.$$

It is easy to prove that

$$(1.13) \qquad \varphi_A \neq 0 \qquad \text{in } \Omega.$$

By differentiation we get $\varphi_A d\omega_A = 0$ and finally $d\omega_A = 0$. In the case when Ω is simply connected we get the existence of θ such that $\omega_A = d\theta$ and we have immediately

$$\int_\gamma \omega_A = \int_\gamma d\theta = 0.$$

In the general case, we use (1.12) which can be written locally

$$(1.14) \qquad hd(\operatorname{Log}\varphi_A) = i\,\omega_A.$$

Hence $|\varphi_A|$ is locally constant (and then constant by connectedness) and because φ_A is univalued, we get $(\text{iii})_\text{b}$. (iii) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (i) are much easier.

Remark 1.2. The same result can be obtained under the weaker assumption (replacing (1.5)).

(1.15)
The bottom of the spectrum of $P^{\Omega}_{0,V}$ is an isolated eigenvalue $\lambda^{\Omega}_{0,0,V}$,

with (i) replaced by the apparently different

$$(\text{i})' \qquad \inf \operatorname{Sp} P^{\Omega}_{A,V} = \lambda^{\Omega}_{0,0,V}.$$

We observe indeed that (1.15) implies

$$(1.16) \qquad \inf \operatorname{EssSp} P^{\Omega}_{0,V} > \lambda^{\Omega}_{0,0,V}.$$

Using (1.16) and (1.9), we get that if (i)$'$ is satisfied then there is at least one eigenvalue $\lambda^{\Omega}_{0,A,V}$ and the proof goes after in the same way.

§2. More on the essential spectrum

In this section, we present essentially the results of Helffer-Mohamed ([22], [23]) with more recent improvements due to Iwatsuka [34], Mohamed-Nourrigat [47], Meftah [44] .... We consider an electric potential of the form

$$(2.1) \quad V(x) = \sum_{j=1}^{p} V_j(x)^2 + V_0(x) \qquad \text{where } V_j \in C^\infty(\mathbb{R}^n),\ V_0(x) \geq 0$$

and a C^∞ magnetic potential $\omega_A = \sum_j A_j\, dx_j$. Because V is semibounded we know that $P_{A,V}$ admits a unique selfadjoint realization on $L^2(\mathbb{R}^n)$ (cf. Schechter [58], Avron-Herbst-Simon [3] or Reed-Simon [57]). Moreover $C_0^\infty(\mathbb{R}^n)$ is dense in $D(P_{A,V})$. In Avron-Herbst-Simon [3], Dufresnoy [13], Helffer-Mohamed [22], sufficient conditions were given which imply compact resolvent. These sufficient conditions are not far to be necessary (cf. Dufresnoy [13] and Iwatsuka [34], and also Remark 5 in Mohamed [45]). We shall give here two extensions of the basic result given in [22]. It is probably possible to establish a unique statement containing the two results. For the sufficient conditions we recall that it is sufficient to prove (cf. Avron-Herbst-Simon [3] or Iwatsuka [34]) the following inequality

$$(2.2) \qquad \forall u \in C_0^\infty(\mathbb{R}^n), \qquad \|\phi u\|^2 \leq C(\langle P_{A,V} u, u\rangle + \|u\|^2)$$

where ϕ is a continuous function tending to $+\infty$ as $|x|$ tends to ∞. For all $r \in \mathbb{Z}$, we introduce

$$(2.3) \qquad m_r(x) = 1 + |V_0(x)| + \sum_{j=1}^{p} \sum_{|\alpha|=0}^{r} |\partial_x^\alpha V_j(x)| + \sum_{i,j=1}^{n} \sum_{|\alpha|=0}^{r-1} |\partial_x^\alpha b_{ij}|.$$

The following theorem is due to Meftah [44] and is an improvement of [22] (see also Mohamed-Raikov [49] or Simon [61]):

Theorem 2.1. *Let us assume that* (2.1) *is satisfied and that there exists* $r \in \mathbb{N}$, $0 \leq \delta < 1$ *and* $c_1 > 0$ *such that*

$$(2.4) \ |\operatorname{grad} V_0| + \sum_{j=1}^{p} \sum_{|\alpha|=r+1} |\partial_x^\alpha V_j(x)| + \sum_{i,j=1}^{n} \sum_{|\alpha|=r} |\partial_x^\alpha b_{ij}| \leq c_1 m_r(x)^{1+\delta},$$

then there exists a constant c_2 *s.t.*

$$(2.5) \qquad \|(m_r(x))^k u\|^2 \leq c_2(\langle P_{A,V} u \mid u\rangle + \|u\|^2) \qquad \forall u \in C_0^\infty(\mathbb{R}^n)$$

where $k = (1 - \delta(2^{r+1} - 3))/2^r$.

Corollary 2.2. *If we assume in addition that*

$$m_r(x) \to +\infty \qquad \textit{as } |x| \to \infty \tag{2.6}$$

and

$$\delta < 1/(2^{r+1} - 3) \tag{2.7}$$

then $P_{A,V}$ *is with compact resolvent.*

Remark 2.3. The case $\delta = 0$ corresponds to the result given in [22]. As $r = 1$, $V_j = 0$, $n = 2$, Corollary 2.2 says that, if $|b_{12}(x)| \to \infty$ as $|x| \to \infty$ and if there exists $C > 0$ and $\delta < 1$ s.t. $|\nabla b_{12}| \leq C(|b_{12}|^{1+\delta}+1)$, then $P_{A,V}$ is with compact resolvent. The counterexamples given by Iwatsuka [34] and Dufresnoy [13] correspond to the case where $|\nabla b_{12}|$ is of the order of $|b_{12}|^2$.

The proof is an adaptation of the proof given in [22] (cf. also Helffer [20] or Mohamed-Raikov [49] for a presentation) and is based on ideas coming from a proof given by J.J. Kohn [37] for the hypoellipticity of Hörmander's operators.

Remark 2.4. As observed in Mohamed-Nourrigat [47], the choice of V of the form (2.1) is not necessary. We refer also to Guibourg [16] for other proofs in this direction or to the surveys of Mohamed-Raikov [49] and Nourrigat [51]. Other generalizations are given in Iftimie [32].

Remark 2.5. Necessity. Under the assumption (2.4), Corollary 2.2 gives in fact a necessary and sufficient condition for compactness of the resolvent. Indeed if there exists a sequence of points in $\mathbb{R}^n y_k$ such that $|y_k|$ tends to ∞ and s.t. $m_r(y_k)$ is bounded, then (taking possibly a subsequence) $m_r(x)$ remains bounded in a union of disjoints balls $B(y_k, C)$ and using the proof (see p.102–103 in Helffer-Mohamed [22]) characterizing the essential spectrum we get the existence of some essential spectrum. Let us also observe that an assumption like (2.4) permits the control of the variation of $m_r(x)$ in suitable balls and the comparison of the above statements with the statements of Iwatsuka [34].

In order to characterize the essential spectrum of $P_{A,V}$ in the case when $m_r(x)$ does not tend to ∞ we introduce stronger assumptions in place of (2.4). Let us first consider a slowly varying function ϕ on $\mathbb{R}^n$ that satisfies for some τ, $c > 0$ the conditions

$$\forall x \in \mathbb{R}^n, \qquad \phi(x) \geq 1 \tag{2.8a}$$

$$|x-y| < \tau\phi(x) \Longrightarrow c^{-1}\phi(y) \leq \phi(x) \leq c\,\phi(y) \tag{2.8}$$

and

$$\phi(x) \to +\infty \qquad \text{as } |x| \to \infty \tag{2.9}$$

and let us assume now that our potentials have a polynomial behavior in the following sense

$$\begin{aligned} |\operatorname{grad} V_0| + \sum_{j=1}^{p} \sum_{|\alpha|=r+1} |\partial_x^\alpha V_j(x)| \\ + \sum_{i,j=1}^{n} \sum_{r\leq|\alpha|\leq(r+2)} \phi^{(|\alpha|-r)} |\partial_x^\alpha b_{ij}| \leq C\,\phi(x)^{-1}. \end{aligned} \tag{2.10}$$

We then introduce the following "limit set" at ∞.

Definition 2.6. $\mathcal{B}_\infty$ is described as the set of the

$$z = (v_0, (v_j^\alpha)_{|\alpha|\leq r, j=1,\ldots,p}, (B_{ij}^\alpha)_{|\alpha|\leq(r-1), 1\leq i\leq j\leq n})$$

s.t. there exists a sequence y_ν ($\nu \in \mathbb{N}$) with the following properties:

$$\text{(a)} \qquad |y_\nu| \to \infty \qquad \text{as } |\nu| \to \infty \tag{2.11a}$$

$$\text{(b)} \qquad \partial_x^\alpha V_j(y_\nu) \to (v_j^\alpha) \qquad \text{as } |\nu| \to \infty \tag{2.11b}$$

$$\text{(c)} \qquad \partial_x^\alpha b_{ij}(y_\nu) \to (B_{ij}^\alpha) \qquad \text{as } |\nu| \to \infty \tag{2.11c}$$

We now associate to each $z \in \mathcal{B}_\infty$
— an electric potential:

$$V_z(x) = v_0 + \sum_{1\leq j\leq p} (\sum_{|\alpha|\leq r} x^\alpha v_j^\alpha/\alpha!)^2, \tag{2.12a}$$

— a magnetic potential:

$$(A_z(x))_i = \sum_{1\leq j\leq n} (\sum_{|\alpha|\leq(r-1)} x^\alpha B_{ij}^\alpha x_j/(\alpha!\cdot(2+|\alpha|)), \tag{2.12b}$$

and the corresponding Schrödinger operator P_{A_z,V_z}.

We then introduce the following subset of $\mathbb{R}$

$$S_\infty = \bigcup_{z\in\mathcal{B}_\infty} \operatorname{Sp}(P_{A_z,V_z}). \tag{2.13}$$

The theorem of [22] gives the link between the union of the spectra of these "limit Schrödinger operators" and the essential spectrum of $P_{A,V}$. This is quite natural if you remember the statement of Persson's Theorem (see [54] or Agmon [1])

$$\inf \mathrm{EssSp}(P_{A,V}) = \sup_{K \in \mathcal{K}} \inf \mathrm{Sp}(P_{A,V}^{\mathbb{R}^n \setminus K}) \tag{2.14}$$

where $\mathcal{K}$ is the family of the compacts in $\mathbb{R}^n$ or the second version $\inf \mathrm{EssSp}(P_{A,V}) = \lim_{R \to +\infty} \inf \mathrm{Sp}(P_{A,V}^{\mathbb{R}^n \setminus B(0,R)})$

Theorem 2.7 (cf. Helffer-Mohamed [22]). *Under assumption* (2.10)*, we have*

$$\mathrm{EssSp}(P_{A,V}) = \overline{S_\infty}. \tag{2.15}$$

Actually we shall give in Section 6 a sketch of the unpublished result of Helffer-Mohamed [23] saying that

Theorem 2.8.

$$S_\infty \textit{ is closed in } \mathbb{R}. \tag{2.16}$$

With this theorem we can effectively give a reasonable answer to the question of the equality

$$\inf \mathrm{EssSp}(P_{A,V}) = \inf \mathrm{EssSp}(P_{0,V}).$$

But first we can understand from a new point of view the inequality (1.9). For this, we compare $\mathcal{B}_\infty(A, V)$ and $\mathcal{B}_\infty(0, V)$. We observe first of all that

$$\mathcal{B}_\infty(A, V) \subset \mathcal{B}_\infty(0, V). \tag{2.17}$$

If we use what we know for the spectrum (cf. (1.7)), we get from (2.16) the existence of $z \in \mathcal{B}_\infty(A, V)$ s.t.

$$\inf S_\infty(A, V) = \inf \mathrm{Sp}\, P_{A_z, V_z} \geq \inf \mathrm{Sp}\, P_{0, V_z}$$

Then (2.17) implies

$$\inf S_\infty(0, V) \leq \inf \mathrm{Sp}\, P_{0, V_z} \leq \inf S_\infty(A, V). \tag{2.18}$$

because $z \in \mathcal{B}_\infty(A, V) \subset \mathcal{B}_\infty(0, V)$. In order to simplify we just discuss the case where $V = 0$ and we get

Proposition 2.9. *Under assumptions* (2.1), (2.10) *with* $V = 0$. *Then* $\inf \mathrm{EssSp}\, P_{A,0} = \inf \mathrm{EssSp}\, P_{0,0}$ *if and only if there exists a sequence* y_ν *s.t.* $|y_\nu|$ *tends to* ∞ *and* $|\partial_x^\alpha b_{ij}(y_\nu)| \to 0$ *as* $|\nu| \to \infty$ *for* $|\alpha| \leq (r-1)$ *and* $1 \leq i < j \leq n$.

§3. Semi-classical results

3.1. The Schrödinger case

In [21], we gave an estimate as h tends to 0 of $\lambda^\Omega_{0,A,V}(h) - \lambda^\Omega_{0,0,V}(h)$ when condition (iii) is not satisfied. Under suitable assumptions on V (V has a unique non degenerate minimum in Ω[1] at a point x_0, $V(x_0) = 0$ and V creates a sufficiently strong barrier around $\partial\Omega$), we prove that a magnetic potential (with 0 corresponding σ_B in Ω) creates a splitting of the type

$$
\begin{aligned}
(3.1)\quad & \lambda^\Omega_{0,A,V}(h) - \lambda^\Omega_{0,0,V}(h) \\
& = h^{1/2}\exp(-S_1/h)(a(h)(1-\cos(\int_\gamma \omega_A/h)) + O(\exp(-\varepsilon)/h)))
\end{aligned}
$$

where

- $a(h)$ is a symbol (independent of A) which is (under suitable generic assumptions) elliptic,
- ε is strictly positive,
- S_1 is the minimal length of a closed path starting of x_0 and not homotop to the trivial path in Ω.

Here the length is measured according to the Agmon metric $V \cdot dx^2$. The sentence "*creates a sufficiently strong barrier*" means mathematically that

$$S_1 < 2S_0$$

where S_0 is the Agmon distance of x_0 to $\mathbb{R}^2 \setminus \Omega$.

The proof is based on a comparison of $\lambda_A(h)$ with a problem (independent of A) on the covering of Ω. Another important fact in the proof is the decay of the eigenfunctions which is controlled by Agmon estimates (cf. Agmon [1], Helffer-Sjöstrand [28] and Section 5). As a consequence of these estimates we get also by perturbation

$$
\begin{aligned}
(3.2)\quad & \lambda_{0,A,V}(h) - \lambda_{0,0,V}(h) \\
& = h^{1/2}\exp(-S_1/h)a(h)((1-\cos(\int_\gamma \omega_A/h)) + O(\exp(-\varepsilon)/h)))
\end{aligned}
$$

[1]We have assumed to simplify that Ω was the complementary of a disc in $\mathbb{R}^2$

where $\lambda_{0,A,V}$ is now attached to the problem in $\mathbb{R}^n$.

3.2. The direct effect

When $2S_0 < S_1$, it is explained in Helffer [21] how to produce under suitable assumptions a direct effect of the magnetic field whose order is effectively $\exp(-2S_0/h)$.

3.3. The paramagnetic inequality

As a first application we obtain (following [21]) a new version of the counterexample (given by Avron-Simon [7]) to a conjecture on the existence of a paramagnetic inequality due to Hogreve-Schrader-Seiler [30] and we think that this gives also some interesting information in the discussion around the existence of the Bohm-Aharonov effect (cf. [2], [54], [8], and the references in this paper). We treat the case of dimension 2 but the arguments are more general in nature. Let us consider the Dirac operator in $\mathbb{R}^2$ with a magnetic field

$$D(A)(h) = \sum_{j=1}^{2} \sigma_j (hD_{x_j} - A_j), \tag{3.3}$$

where the σ_j are the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
$$\sigma_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

which is a selfadjoint operator on $L^2(\mathbb{R}^2) \otimes \mathbb{C}^2$.

Then the Pauli operator is classically defined as the square of the Dirac operator

$$P(A)(h) = (D(A)(h))^2 = \sum_{j=1}^{2} (hD_{x_j} - A_j)^2 \cdot \mathrm{Id} + h \begin{pmatrix} B & 0 \\ 0 & -B \end{pmatrix} \tag{3.4}$$

with $B(x) = (\partial_{x_2} A_1 - \partial_{x_1} A_2)$. If V satisfies (1.1), we are interested in the validity of the paramagnetic inequality

$$\inf \mathrm{Sp}((D(A)(h))^2 + V \cdot I) \leq \inf \mathrm{Sp}(-h^2 \Delta + V). \tag{3.5}$$

If $\lambda_{0,0,V}(h)$ denotes the first eigenvalue of $(-h^2\Delta + V)$ and if we denote by $\lambda^{\pm}_{A,V}$ the first eigenvalues of $((D(A)(h))^2 + V \cdot I)$, the question is to know if the following inequality is true:

$$\inf_{\pm} \lambda^{\pm}_{A,V}(h) \leq \lambda_{0,0,V}(h) \tag{3.6}$$

Let us recall that in Section 1 we have mentioned the opposite inequality:

$$\lambda_{0,A,V}(h) \geq \lambda_{0,0,V}(h) \tag{3.7}$$

It is then an easy corollary of (3.2) that, under the same assumptions, (3.6) is false for a convenient choice of A and h small enough. We observe indeed that according to the decay properties of the corresponding eigenfunctions, we have

$$\lambda^{\pm}_{A,V}(h) - \lambda^{0}_{A,V}(h) = O(\exp(-2(S_0 - \varepsilon)/h)), \qquad \forall \varepsilon > 0$$

which is a smaller effect that the effect due to the flux (this was the argument we use to go from (3.1) to (3.2)).

3.4. The Dirac operator in dimension 3

We consider the Dirac operator with magnetic potential A

$$(\sum_{j=1}^{3} \alpha_j(hD_{x_j} - A_j) + \beta + V) \tag{3.8}$$

in $L^2(\mathbb{R}^3;\mathbb{C}^4)$, where $(\alpha_j)_{j=1,2,3}$ and $\beta = \alpha_4$ are the Dirac matrices, $(A_j)_{j=1,2,3}$ is a magnetic vector potential and V a scalar potential. Let us assume that:

$$\limsup_{|x|\to\infty} V(x) < 1$$

which implies that the spectrum is discrete in the neighborhood of 0. We assume also that Ω is the complementary of an infinite cylinder $\mathcal{C}$ in the x_3 direction and that $B = 0$ in Ω. We assume that V creates a sufficiently strong barrier around $\mathcal{C}$ and that V has a unique non-degenerate extremum in Ω at some point say $x_0 = (0,1,0)$ and that $V(x_0) = 1$. Finally we assume generical assumptions on V (unique "non degenerate" minimal path around $\mathcal{C}$ starting from x_0). In the case where A is zero we know from X.P. Wang [64] that due to the Kramers theorem all the eigenspaces appear with even dimension (see also [53]). Near 0 the "first" eigenvalue $\lambda_0(h)$ is determined modulo $O(h^2)$ by some quadratic approximation and separated from the rest of the spectrum by (h/C) (cf. [64]). Moreover the multiplicity is exactly 2. The argument fails as the magnetic field is introduced and the purpose of the work of B. Parisse [53] was to study the effect of the magnetic field on the splitting which by perturbation arguments will in this context be "exponentially small". B. Parisse proves the following theorem:

Theorem 3.1. *If $\lambda_0(h)$ is the double eigenvalue of $D_0(h)$, then for h small enough, the operator $D_A(h)$ admits two eigenvalues $\lambda_A^{\pm}(h)$ satisfying to*

$$(3.9) \quad \lambda_A^{\pm}(h) = \lambda_0(h) + h^{1/2}\exp(-S_1/h)(\Re(c(h)\cdot[\exp(\pm i\phi/h)-1]) + O(\exp(-\varepsilon)/h)))$$

where $\varepsilon > 0$, $c(h) = a(h) + i\,b(h)$ is a complex elliptic symbol, $\phi = \int_\gamma \omega_A$, $\varepsilon > 0$ and S_1 is the minimal length of a closed path starting of x_0 and not homotop to the trivial path in Ω. Here the length is measured according to the Agmon metric $(1 - V^2)_+ dx^2$.

Modulo some technicalities due to the fact that we now deal with systems, the scheme of the proof is the same as for Schrödinger. It is more delicate to prove that $c(h) \neq 0$ and this a consequence of the WKB constructions.

Let us remark that as a consequence of (3.9) we get the following formula for the splitting

$$(3.10) \quad \lambda_A^{+}(h) - \lambda_A^{-}(h) = -2h^{1/2}\exp(-S_1/h)\cdot(b(h)\sin(\phi/h) + O(\exp(-\varepsilon)/h)))$$

It would be very interesting to prove that generically $b(h)$ is elliptic or that under additional symmetries $b(h)$ is exponentially small. As suggested by B. Parisse it would also be interesting to look to the non relativistic limit where we will find a problem similar to the case treated in Subsection 3.3.

§4. The case of systems

(after Hebbar, Kuwabara, Manabe, Shigekawa ...)

4.1. Introduction

The idea to look at systems is very natural and physically motivated (see for example T.T. Wu and C.N. Yang [65]). But O. Hebbar found more recently that R. Kuwabara treats the case with $V = 0$ in 1982 [40]. As we shall see, the case $V \neq 0$ is not essentially more difficult. Anyway the result of Hebbar [18] is a little more general that the result of [40] *also* in the case $V = 0$.

4.2. The results of Kuwabara revisited

Let (M, g) be a compact n-dimensional C^∞ manifold without boundary and E a be a complex vector bundle over M with rank r. We assume that E has a C^∞ Hermitian structure $\langle\cdot\rangle$. Let us denote by $A^0(M,E) = C^\infty(E)$ the set of the C^∞ sections of E. More generally we denote by $A^p(M)$ the set of the C^∞ p-forms on M and by $A^p(M,E)$ the set of E-valued C^∞ p-forms on M. Let $\tilde{d}\colon A^0(M,E) \to A^1(M,E)$ be a linear connection on E compatible with the Hermitian structure. There is also a natural extension of $\tilde{d} = \tilde{d}_0$ on the p-forms given by

$$\tilde{d}_p(s \otimes t) = (\tilde{d}_0 s) \otimes t + s \otimes dt \tag{4.1}$$

for all $s \in A^0(M,E)$ and $t \in A^p(M)$. There is a natural inner product on $A^p(M,E)$ and we can then define the L^2 p-forms with a natural Hilbertian structure. The Laplace operator on the p-forms is then defined by:

$$L^{(p)} = \tilde{d}_p^* \tilde{d}_p + \tilde{d}_{p-1}.\tilde{d}_{p-1}^* \tag{4.2}$$

We shall concentrate on: $L = L^0$ and will write sometimes $L(E,\tilde{d})$ to mention the dependence with respect to the fiber bundle and the connection. Of course L is an elliptic operator (of order 2) with compact resolvent and admits as spectrum an increasing sequence of eigenvalues $\lambda_j(E,\tilde{d})$ tending to $+\infty$ and because the Laplacian is positive, we have of course $\lambda_0(E,\tilde{d}) \geq 0$. If $E = M \times \mathbb{C}$, and if we take the trivial connection d, we get the usual spectrum of the Laplace-Beltrami operator $\lambda_j(M)$ with $\lambda_0(M) = 0$. The problem we want to address is now: *Under which conditions on E and $\tilde{d}$ do we have $\lambda_0(M) = \lambda_0(E,\tilde{d})$, or more generally $\lambda_0(M) = \lambda_j(E,\tilde{d})$ for $j = 0,\dots,k-1$ for some k.* Let us remark that if a section s satisfies $Ls = 0$ (we shall say that s is harmonic) then it satisfies $\tilde{d}s = 0$ (that is s is a parallel section). Kuwabara proves the following proposition (Proposition 3.1 in [40]):

Proposition 4.1. (i) *If L has a zero eigenvalue with multiplicity k ($k \leq r$) then*

$$E = E' \oplus T_k \qquad (\textit{Whitney sum}), \tag{4.3}$$

where T_k is a trivial bundle of rank k.
(ii) *If L has zero eigenvalue with multiplicity r, then E is a trivial bundle and the curvature Ω of the connection vanishes.*

The proof is a direct consequence of the fact that an orthonormal system of k independent eigenfunctions u_k gives actually a system of k independent sections giving a natural orthogonal basis for a trivial subbundle of E. The second point is as in the study of the scalar Bohm-Aharonov effect.

The second result given in [40] is the following:

Proposition 4.2. *If L has zero eigenvalue, then* $\mathrm{Sp}(M,g) \subset \mathrm{Sp}(M,g,E,\tilde{d})$. *Moreover, if L has zero eigenvalue with multiplicity r, then* $\mathrm{Sp}(M,g,E,\tilde{d}) = r \cdot \mathrm{Sp}(M,g)$ *where* $r \cdot \mathrm{Sp}(M,g) = \mathrm{Sp}(M,g) \cup \cdots \cup \mathrm{Sp}(M,g)$ *(r times).*

Proof. Since $0 \in \mathrm{Sp}(M,g,E,\tilde{d})$, there is a non zero f in $C^\infty(E)$ s.t.

$$\tilde{d}f = 0. \tag{4.4}$$

We have already seen that it does not vanish anywhere. Suppose $\lambda \in \mathrm{Sp}(M,g)$ and let ϕ be a non zero eigenvector

$$-\Delta\phi = \lambda\phi. \tag{4.5}$$

Then, using elementary computations, (4.4) and (4.5), we get that $s = \phi f$ is an eigenvector for L. The other part is also easy.

Actually, O. Hebbar will deduce these results from the following:

Lemma 4.3 (see [18]). *If L has a zero eigenvalue with multiplicity k ($k \leq r$) then the connection split according to the decomposition:*

$$E = T_k^\perp \oplus T_k \qquad (\textit{orthogonal decomposition})$$
$$\tilde{d} = \tilde{d}_1 \oplus \tilde{d}_2$$

As a consequence we have a direct decomposition of the Laplacian

$$L(M,g,E,\tilde{d}) = L(M,g,T_k^\perp,\tilde{d}_1) \oplus L(M,g,T_k,\tilde{d}_2)$$

with

$$\mathrm{Sp}(M,g,E,\tilde{d}) = \mathrm{Sp}(M,g,E,\tilde{d}_1) \cup \mathrm{Sp}(M,g,E,\tilde{d}_2)$$

and moreover $L(M,g,T_k,\tilde{d}_2)$ has zero eigenvalue with multiplicity k. Then we get the following improvement of Proposition:

Proposition 4.4. *If L has a zero eigenvalue with multiplicity k ($k \leq r$) then*

(i) $E = T_k^\perp \oplus T_k$ *(Whitney sum),* $\tilde{d} = \tilde{d}_1 \oplus \tilde{d}_2$

(ii) *$L(M,g,T_k,\tilde{d}_2)$ has zero eigenvalue with multiplicity k*

(iii) *T_k is a trivial bundle and the curvature of $\tilde{d}_2$ vanishes*

(iv) $\mathrm{Sp}(L(M,g,E,\tilde{d}) \supset \mathrm{Sp}\, L(M,g,T_k,\tilde{d}_2) = k\,\mathrm{Sp}(M,g)$.

To go further, we have to analyze more precisely and introduce the notion of gauge transformations. Recall that a gauge transformation on a vector bundle E with the Hermitian structure is a diffeomorphism $\Phi\colon E \to E$ which maps each fiber E_x isometrically and linearly onto itself. For a linear connection $\tilde{d}$ on E, we get a new connection $\Phi^*\tilde{d} = \Phi^{-1}\tilde{d}\Phi$. Two connections $\tilde{d}$ and $\tilde{d}'$ on E are called gauge equivalent to each other (and we write $\tilde{d} \sim \tilde{d}'$) if there exists a gauge transformation such that: $\tilde{d}' = \Phi^*\tilde{d}$. Of course, we have in this case

$$\mathrm{Sp}(L(M, g, E, \tilde{d}) = \mathrm{Sp}(L(M, g, E, \tilde{d}').$$

The problem we are looking at is to give now a good characterization of two gauge equivalent connections. Kuwabara [40] gives the following criterion:

Proposition 4.5. *Let E be a line-bundle on M then $\tilde{d} \sim \tilde{d}'$ if and only if the corresponding connection 1-forms ω and ω' satisfy $(\omega - \omega')/2\pi$* [2] *is an integral 1-form.*

This was already observed in Section 1. For a general fiber bundle, there is a similar criterion using the notion of matrix of holonomy attached to a connection and a closed path γ. Using the theorem that a connection with 0 curvature is locally gauge-equivalent to 0, it is natural to attach to each curve γ a class of equivalence of unitary matrices in $U(\mathbb{C}^r)$: $U_\gamma = I$. We have then the following criterion (cf. for example [18] but it is probably well known in Topology):

Proposition 4.6. *Let E be a trivial hermitian fiber bundle on M and let d_0 be the connection associated to the 1-form 0; then $\tilde{d} \sim d_0$ if and only if the corresponding connection 1-form ω*

(a) *ω has 0-curvature*

and

(b) *$U_\gamma = I$ for any closed path γ.*

As a conclusion of this subsection, we get following Hebbar [18] the following extension of the results in [40]:

[2] which is a global 1-form on M

Theorem 4.7. *Let E be a Hermitian bundle over (M, g), and $\tilde{d}$ a linear connection on E which is compatible with the Hermitian structure. Then the following properties are equivalent:*

(i) *L has a zero eigenvalue with at least multiplicity k $(k \leq r)$*

(ii) *$E = T_k^{\perp} \oplus T_k$ (Whitney sum), $\tilde{d} = \tilde{d}_1 \oplus \tilde{d}_2$ with[3] $\tilde{d}_2 \sim \tilde{d}_0$ where d_0 denotes the canonical connection on the trivial bundle T_k whose 1-form is 0.*

(iii) $\mathrm{Sp}(L(M, g, E, \tilde{d}) \supset k\,\mathrm{Sp}(M, g)$

4.3. Extension to the Bochner-Laplace-Schrödinger equation

Here we explain the results of [18]. More precisely we shall explain how to deduce the results with non zero V from the corresponding results with $V = 0$. But note that it is possible because we are on a compact manifold. For other cases (boundary problems) we must of course take the problem directly (as Hebbar did). The theorem obtained by Hebbar [18], generalizing results of ([21], [40], [59], [43]), is the following (we limit ourselves to the case when M is compact):

Theorem 4.8. *Let E be a Hermitian bundle over (M, g), and $\tilde{d}$ a linear connection on E which is compatible with the Hermitian structure. Let V be a C^∞ potential on M. Let $\lambda_0(M, g, V)$ be the first eigenvalue of the Laplace-Beltrami-Schrödinger operator on M: $-\Delta + V$. Then the following properties are equivalent:*

(i) *$L + V$ has $\lambda_0(M, g, V)$ with at least multiplicity k $(k \leq r)$.*

(ii) *$E = T_k^{\perp} \oplus T_k$ (Whitney sum), $\tilde{d} = \tilde{d}_1 \oplus \tilde{d}_2$ with (cf. preceding Footnote) $\tilde{d}_2 \sim d_0$ where d_0 denotes the canonical connection on the trivial bundle T_k whose 1-form is 0.*

(iii) $\mathrm{Sp}(L(M, g, V, E, \tilde{d})) \supset k\,\mathrm{Sp}(M, V, g)$

Remark 4.9. In particular, if $k = r$, we get the equivalent of the theorem given in Section 1.

Corollary 4.10. *Let E be a Hermitian bundle over (M, g) with rank r; then the following properties are equivalent:*

(i) *$L + V$ has $\lambda_0(M, g, V)$ as an eigenvalue with multiplicity r.*

(ii) *E is a trivial bundle and $\tilde{d} \sim d_0$ where d_0 denotes the canonical connection on E whose 1-form is 0.*

---

[3]and Proposition 4.6 gives a good criterion to verify the property

(iii) *$L(M, g, V, E, \tilde{d})$ is gauge equivalent to $(-\Delta + V)$. Id defined on the trivial fiber bundle $M \times \mathbb{C}^r$.*

The equivalence of (i) and (iii) was proved in [43].

Sketch of the proof (following partially [18]*).* We extend the Lavine-O'Caroll formula to this case. For $s \in C^\infty(E)$, we have the identity

$$\|\tilde{d}s - (du_0/u_0) \otimes s\|^2 = \langle L(M, g, V, E, \tilde{d})s \mid s\rangle - \lambda_0(M, g, V)\|s\|^2 \tag{4.6}$$

where we take the L^2-canonical scalar products. From this, we get that an eigenfunction s_j of $L(M, g, V, E, \tilde{d})$ with eigenvalue $\lambda_0(M, g, V)$ has the property that (s_j/u_0) is parallel for $\tilde{d}$. This was the essential point to get all the statements in Subsection 4.2.

Remark 4.11. It is possible to quantify this result by semi-classical methods in the spirit of the results of Section 3. The problem is studied by Hebbar in [18].

§5. Some decay results for the eigenfunctions

5.1. Decay at ∞

We want to present in this subsection some results on the decay at ∞ (or locally as the Planck constant tends to 0) of the eigenfunctions of $P_{A,V}$. For the first result, we consider the simpler case where A and V are polynomials with

$$V \geq 0. \tag{5.1}$$

As in Helffer-Nourrigat [24] and also Feffermann [12] we introduce

$$M(x) = \sum_\alpha |\partial^\alpha V(x)|^{1/(|\alpha|+2)} + \sum_{\alpha,j,k} |\partial^\alpha b_{j,k}(x)|^{1/(|\alpha|+2)}. \tag{5.2}$$

In this simpler case, the compactness criterion given in Corollary 2.2 was obtained in [24], where it is also proved that, if $M(x)$ tends to ∞, every solution in $\mathcal{S}'(\mathbb{R}^n)$ of $H\psi = \lambda\psi$, $\lambda > 0$ is actually in $\mathcal{S}(\mathbb{R}^n)$. In the case when $V(x)$ itself tends to ∞, the decay of the eigenfunction ψ is associated with the Agmon metric $(V - \lambda)_+ dx^2$. Of course it is not necessary to assume that we have compact resolvent and it is for example sufficient to assume that λ satisfies

$$\lambda < \inf \operatorname{EssSp} P_{A,V} \tag{5.3}$$

in order to get some decay like $\exp((\lambda - \inf \operatorname{EssSp} P_{A,V})|x|)$. We refer to [1] and references therein for a discussion. But let us come back to the case when $M(x)$ tends to ∞. The heuristic idea is that the role played by V is replaced by $M(x)^2$. We shall loose a little in precision because one has to remember that $M(x)$ is only defined up to some multiplicative constant. For all λ we introduce the "well"

$$U(\lambda) = \{x \in \mathbb{R}^n, M(x)^2 \leq \lambda\} \tag{5.4}$$

and denote by $d_\lambda(x) = d(x, U(\lambda))$ the distance of x to $U(\lambda)$ for the modified Agmon's metric $ds^2 = M(x)^2 dx^2$. The principal result obtained in [25] is the following:

Theorem 5.1. *There exist constants $C > 0$ and $\varepsilon > 0$, depending only on the dimension n of the space and on the largest degree r of the polynoms A_j and $V \geq 0$, s.t. for any solution $\psi \in \mathcal{S}(\mathbb{R}^n)$ of $P_{A,V}\psi = \lambda\psi, \lambda > 0$, the following inequality is satisfied*

$$|\psi(x)| \leq C\lambda^{n/4} \exp(-\varepsilon d_{C\lambda}(x)) \|\psi\|_{L^2}, \qquad \textit{for all } x \in \mathbb{R}^n. \tag{5.5}$$

Remark 5.2. As we have implicitly seen in Section 3 (and as it appears clearly in [26] or in [60], [62]), the Agmon's type estimates have a natural transcription in the semi-classsical context and play a basic role in the estimate of the tunneling effect. The estimates are then local but asymptotic for h tending to 0. A semi-classical version of this theorem was obtained by Brummelhuis [10] (see also [25] Section 6).

Example 5.3. $n = 2$; $A_1(x_1, x_2) = x_1^2 x_2$, $A_2(x_1, x_2) = -x_2^2 x_1$; $V = 0$. We have in this case: $b_{1,2}(x_1, x_2) = x_1^2 + x_2^2$ and $M(x_1, x_2)^2 \approx (1 + x_1^2 + x_2^2)$.

Remark 5.4. The polynomial character is only assumed for simplification. One can certainly extend the results under assumptions of the type given in (2.11) (see Guibourg [16] for results in this direction).

Some words on the proof. The L^2 estimates in (5.4) follows closely the Agmon's proof replacing V by M^2. In order to get the L^∞ estimates, a global Sobolev's theorem is used in [25] whose proof is based on the proof of maximal estimates in adapted Sobolev spaces appearing in [24]. The proof is then a consequence of the nilpotent Lie groups techniques which will be presented very shortly in Section 6 (See the book [24] or the surveys of Helffer [19] or Nourrigat [51]).

5.2. Semiclassical aspects for the decay

As it was already mentioned in the context of the study of the decay of the eigenfunctions at ∞, we can also study the decay in the semiclassical context and the first result proved in Helffer-Sjöstrand [28] is that if

$$P_{A,V}(h)u_h = \lambda(h)u_h \tag{5.5}$$

with $\lambda(h) \to E$ and $\|u_h\|_{L^2} = 1$ then we have on every compact K and for every $\varepsilon > 0$

$$|u_h(x)| \leq C_{\varepsilon,K} \exp(\varepsilon/h) \exp(-d_{(V-E)_+}(x, U_E)/h) \tag{5.6}$$

where U_E is the well: $V \leq E$ and $d_{(V-E)_+}(x, y)$ is the Agmon distance attached to the potential $(V - E)_+$. As we observed in Subsection 5.1 and as one can easily compute for examples of the type

$$-\sum_j (h\partial_{x_j} - i\sum_k b_{jk}x_k)^2 + |x|^2,$$

this estimate is not at all optimal. It can be useful (at least to understand heuristically the problem) to look for WKB constructions in the case where V has a unique non-degenerate minimum at 0 and is analytic in a neighborhood of 0. We assume here that

$$\inf V = 0.$$

It is proved in [28] that for t small enough it is possible to construct a WKB solution for $P_{tA,V}(h)$ of the form

$$h^{-n/4}a(t, x, h) \exp(-\phi(t, x, h)/h) \tag{5.7}$$

where $\phi(t, x, h)$ is a solution in a neighborhood of 0 of the eikonal equation

$$(\nabla_x \phi - itA)^2 = V \tag{5.8}$$

Admitting that this WKB approximation gives effectively an approximation of one eigenfunction (and this is proved for t small enough in [28]), then $\Re\,\phi$ gives the control of the decay with respect to t. We admit the existence of $\phi(t, x)$ (also proved in [28]) and taking the real part and the imaginary part of (5.8) we get

$$|\nabla(\Re\,\phi_t)|^2 = V + |\nabla(\Im\,\phi_t) - tA|^2 \tag{5.9}$$

and

$$\nabla(\Re\,\phi_t)\cdot(\nabla(\Im\,\phi_t)-tA)=0 \tag{5.10}$$

and we can take

$$\Re\,\phi_t\geq 0. \tag{5.11}$$

Equations (5.9) and (5.11) permit to say that in a neighborhood of 0, $\Re\,\phi_t$ is the Agmon distance to 0 for the potential: $(V+|\nabla\Im\,\phi_t-tA|^2)$. This gives the general inequality

$$\Re\,\phi_t\geq\Re\,\phi_0 \qquad \text{in a neighborhood of } 0. \tag{5.12}$$

Then we observe that

$$\nabla(\Re\,\phi_t+\phi_0)\nabla(\Re\,\phi_t-\phi_0)=|\nabla\Im\,\phi_t-tA|^2 \tag{5.13}$$

and

$$(\Re\,\phi_t-\phi_0)(0)=0. \tag{5.14}$$

Then we get in a suitable (but independent of t with $|t|\leq t_0$) neighborhood of 0 that $(\Re\,\phi_t-\phi_0)(x)=0$ implies that: $\nabla(\Im\,\phi_t)-tA=0$ along the integral curve of the vector field $\nabla(\Re\,\phi_t+\phi_0)$ joining x and 0. In particular if $(\Re\,\phi_t-\phi_0)(x_j)=0$ in an open set on some sphere around 0 then we get by analyticity that $\nabla(\Im\,\phi_t)-tA=0$ in a neighborhood of 0 which gives that A is locally exact.

§6. Nilpotent Lie group techniques

In this section we shall give the proof of Theorem 2.8. We assume that the reader is somewhat familiar with the theory of nilpotent Lie groups (see [15]) and we emphasize that all these techniques were developped first for the study of hypoellipticity. For $n,p,s\in\mathbb{N}$, let us introduce the "maximal" universal Lie Algebra $\mathcal{G}^{(n,p,s)}$ with the following properties

$$\mathcal{G}^{(n,p,s)} \text{ is graded of rank of nilpotency } s, \tag{6.1}$$

i.e.

$$\mathcal{G}^{(n,p,s)}=\mathcal{G}_1\oplus\mathcal{G}_2\oplus\ldots\oplus\mathcal{G}_s$$

and

$$\begin{aligned} &[\mathcal{G}_i, \mathcal{G}_j] \subset \mathcal{G}_{i+j}, && \text{if } (i+j) \leq s, \\ &[\mathcal{G}_i, \mathcal{G}_j] = 0, && \text{if } (i+j) \geq (s+1) \end{aligned}$$

$$\mathcal{G}_1 = \mathcal{G}_1' \oplus \mathcal{G}_1'', \qquad \text{with } \dim \mathcal{G}_1' = n,\ \dim \mathcal{G}_1'' = p \tag{6.2}$$

$$\mathcal{G}_1 \text{ generates } \mathcal{G}. \tag{6.3}$$

$$[\mathcal{G}_1'' \oplus \mathcal{G}^2, \mathcal{G}_1'' \oplus \mathcal{G}^2] = 0 \tag{6.4}$$

where

$$\mathcal{G}^2 = \mathcal{G}_2 \oplus \cdots \oplus \mathcal{G}_s \tag{6.5}$$

and

$\mathcal{G}^{(n,p,s)}$ is of maximal dimension with the above properties.

The algebra $\mathcal{G}^{(n,p,s)}$ has the following universal property: Let $(Y_j')_j$ be a basis of $\mathcal{G}_1'$, $(Y_k'')_k$ a basis of $\mathcal{G}_1''$; then there exists a partial homomorphism of rank s, λ, s.t.:

$$\lambda(Y_j') = X_j' \qquad \lambda(Y_k'') = X_k'', \tag{6.6}$$

where

$$X_j' = \partial_{x_j} - iA_j(x) \qquad \text{for } j = 1, \ldots, n;\ X_k'' = iV_k(x)$$

for $k = 1, \ldots, p$ (with $s = r + 1$). We refer to R. Goodman [15] for this property or to [24] where this type of Lie Algebras is studied in Chapter XI. We observe (cf. Chapter XI of [24])

$$P_{A_z, V_z} = \Pi_{\ell_z, \mathcal{H}}(-\Delta), \tag{6.7}$$

where

$$\mathcal{H} = \mathcal{G}_1'' \oplus \mathcal{G}^2, \tag{6.8}$$

ℓ_z is the element of $\mathcal{G}^*$ (dual of $\mathcal{G} = \mathcal{G}^{(n,p,s)}$) associated to $z \in \mathcal{B}_\infty$ by the relations

$$\ell_z/\mathcal{G}_1' = 0, \tag{6.9}$$

$$\ell_z((adY')^\alpha Y_k'') = v_k^\alpha, \qquad \text{for } |\alpha| \le s-1, \tag{6.10}$$

$$\ell_z((adY')^\alpha [Y_i', Y_j']) = B_{ij}^\alpha, \qquad \text{for } |\alpha| \le (s-2), \tag{6.11}$$

$$\Delta := \sum_j Y_j'^2 + \sum_k Y_k''^2 \tag{6.12}$$

and $\Pi_{\ell,\mathcal{H}}$ is the induced representation associated to ℓ and to a subalgebra $\mathcal{H}$ satisfying

$$\ell([\mathcal{H},\mathcal{H}]) = 0.$$

Let us introduce

$$\Lambda(\ell,\mathcal{H}) = G\cdot(\ell+\mathcal{H}^\perp) \qquad \text{in } \mathcal{G}^*.$$

In a first step we use the techniques of [24] in order to prove:

Proposition 6.1.

$$\sigma(P_{A_z,V_z}) = \bigcup_{\rho\in\overline{\Lambda}} \sigma(\Pi_\rho(-\Delta)) \tag{6.13}$$

The map $\rho \longrightarrow \Pi_\rho$ is the classical Kirillov's map from $\mathcal{G}^*$ onto $\hat{G}$ (the set of equivalence classes of irreducible representations of the simply connected Lie group associated to $\mathcal{G}$, $G := \exp\mathcal{G}$) and G acts on $\mathcal{G}^*$ by the coadjoint map.

Proof of Proposition 6.1. Let us first observe that the different operators appearing in formula (6.13) P_{A_z,V_z} and $\Pi_\rho(-\Delta)$ are essentially selfadjoint starting from respectively $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}_{\Pi_\rho}$, the space of C^∞-vectors of the representation. Proposition 2.21 of Chapter II in [24] gives immediately the following equivalences for $\lambda\in\mathbb{R}$ and $C>0$

$$\|(P_{A_z,V_z}-\lambda)u\| \ge C^{-1}\|u\|, \qquad \forall u\in C_0^\infty(\mathbb{R}^n) \tag{6.14}$$

$$\|(\Pi_\rho(-\Delta-\lambda)f\|_{H^0_{\Pi_\rho}} \ge C^{-1}\|f\|_{H^0_{\Pi_\rho}}, \qquad \forall f\in\mathcal{S}_{(\Pi_\rho)} \text{ and } \forall\rho\in\overline{\Lambda(\ell_z,\mathcal{H})} \tag{6.15}$$

where $H^0_{\Pi_\rho}$ is the space of the representation Π_ρ.

We shall write $(6.14)_{\lambda,C}$ (resp. $(6.15)_{\lambda,C}$) in order to say that the inequality (6.14) (resp. (6.15)) is satisfied for specific constants (λ, C).

This equivalence between (6.14) and (6.15) implies immediately the property

$$\sigma(P_{A_z,V_z}) = \overline{\mathcal{C}} \tag{6.16}$$

with

$$\mathcal{C} = \left(\bigcup_{\rho \in \overline{(\Lambda(\ell_z,\mathcal{H})}} \sigma(\Pi_\rho(-\Delta)) \right),$$

and the way to go from (6.16) to the stronger (6.13) is of the same type as the object of Theorem 2.8.

Proof of (6.13). Let us assume that for some $\lambda \in \mathbb{R}$, we have the following property

$$\forall \rho \in \Gamma_z = \overline{\Lambda(\ell_z,\mathcal{H})},\ \exists C_\rho > 0$$

s.t. $(6.15)_{\lambda,c}$ is satisfied with $C = C_\rho$.

We wish to show $(6.15)_{\lambda,c}$ with C independent of $\rho \in \Gamma_z$. This problem is quite analogous to the problems solved in [24]. The only new point is that Γ_z is closed and invariant by G but not stable by dilation. We refer to [19] which is more adapted to our problem. A first important remark coming from the hypoellipticity of Δ in G is the existence of a constant $D > 0$ s.t.

$$\|u\|^2_{H^2_{\pi_{\ell_z,\mathcal{H}}}} \leq D(\|P_{A_z,V_z}u\|^2 + \|u\|^2), \qquad \forall u \in \mathcal{S}(\mathbb{R}^n), \tag{6.17}$$

and (cf. Proposition 2.2.1, Chapter II in [24]),

$$\|f\|^2_{H^2_{\pi_\rho}} \leq D(\|\Pi_\rho(-\Delta)f\|^2_{H^0_{\pi_\rho}} + \|f\|^2_{H^0_{\pi_\rho}}), \qquad \forall f \in \mathcal{S}_{\Pi_\rho}. \tag{6.18}$$

where H^m_π (for $m \in \mathbb{N}$ and π a representation) is the space of the $u \in H^0_\pi$ s.t. $\pi(Y)^\alpha u \in H^0_\pi$, for $|\alpha| \leq m$, with the natural Hilbertian norm. (6.18) shows that the problem to prove $(6.15)_{\lambda,C}$ with C independent of ρ is equivalent to the apparently stronger result (but more stable):

Property $\mathbf{P_1}$. Let us assume that, for all $\rho \in \Gamma_z$, there exists $C_\rho > 0$ s.t.

(6.19)
$$(\|(\Pi_\rho)(-\Delta) - \lambda)f\|^2_{H^0_{\pi_\rho}} \leq C^{-1}\|f\|^2_{H^0_{\pi_\rho}}, \qquad \forall f \in \mathcal{S}_{\Pi_\rho} \text{ with } C = C_\rho,$$

then there exists $C > 0$ s.t. $(6.19)_C$ is satisfied for all $\rho \in \Gamma_z$.

On the same way, Proposition 6.1 results of the following stronger property:

Property $\mathbf{P_2}$. Let us assume that, for all $z \in \mathcal{B}_\infty$ and for all $\rho \in \Gamma_z$ there exists $C_{\rho,z} > 0$ s.t. $(6.19)_C$ is satisfied with $C = C_{\rho,z}$, then there exists $C > 0$ s.t. $(6.19)_C$ is satisfied for all $z \in \mathcal{B}_\infty$ and $\rho \in \Gamma_z$.

Here we introduce as a new subset of $\mathcal{G}^*$

$$\Gamma = \bigcup_{z \in \mathcal{B}_\infty} \Gamma_z \tag{6.20}$$

whose properties are given in the following:

Proposition 6.2. *$\ell \in \Gamma$ if and only if there exists $z \in \mathcal{B}_\infty$ s.t.: $\ell/(\mathcal{G}^2 \oplus \mathcal{G}_1'') = \ell_z/(\mathcal{G}^2 \oplus \mathcal{G}_1'')$ where ℓ_z is defined in* (6.9–6.11). *Moreover Γ is closed in $\mathcal{G}^*$ and stable by the action of G.*

Proof of Proposition 6.2. We can define Γ on the following way which is quite similar to Definition 2.4 in chapter I of [24]

$$\ell \in \Gamma \Longleftrightarrow \exists((x_q, \xi_q)_{q \in \mathbb{N}} \quad \text{s.t. } |x_q| + |\xi_q| \to \infty \text{ as } q \to \infty \text{ and } \ell = \lim_{q \to \infty} \lambda^*_{x_q} \xi_q$$

where λ is the partial homomorphism of rank s introduced in (6.6): $(\lambda^*_{x,\xi})(Z) := i^{-1}\sigma(\lambda(Z))(x, \xi)$,

$$\forall Z \in \mathcal{G}$$

(If X is a vector field, $\sigma(X)$ is by definition the symbol of the corresponding differential operator). The proof that Γ is closed is the same as in [24] (Corollary 2.4, Section 2, Chapter IV). We observe that if

$$\ell = \lim_{q \to \infty} \lambda^*_{x_q} \xi_q$$

then, for $(y, \eta) \in \mathbb{R}^{2n}$,

$$\ell_{y,\eta} = \lim_{q \to \infty} \lambda^*_{x_q}(\xi_q + \eta)$$

is well defined in Γ and that we have

$$\ell_{(y,\eta)}/(\mathcal{G}_1'' \oplus \mathcal{G}^2) = (\exp(y \cdot Y')) \cdot \ell/(\mathcal{G}_1'' \oplus \mathcal{G}^2).$$

As (y, η) varies in $\mathbb{R}^{2n}$, we verify that $\ell_{(y,\eta)}$ describes the orbit of ℓ, which proves the stability of Γ by the action of G.

Proof of P_2 (the proof of P_1 is similar). Let us assume that for each $\rho \in \Gamma$, we have $(6.19)_{C_\rho}$ with $C_\rho > 0$. In order to come back to a more homogeneous situation, we introduce a new Lie algebra $\hat{\mathcal{G}}$

$$\hat{\mathcal{G}} = \mathcal{G} \oplus \mathbb{R} \cdot Z, \tag{6.21}$$

where the law (and the graduation) for $\hat{\mathcal{G}}$ is deduced from $\mathcal{G}$'s law by imposing

$$\hat{\mathcal{G}}_1 = \mathcal{G}_1 \oplus \mathbb{R} \cdot Z \tag{6.22}$$

and

$$[\mathcal{G}, Z] = \{0\}. \tag{6.23}$$

Let us now introduce $\mathcal{P}_\lambda \in \mathcal{U}_2(\hat{\mathcal{G}})$ ($\mathcal{U}(\hat{\mathcal{G}})$ is the enveloping algebra of $\hat{\mathcal{G}}$ and $\mathcal{U}_2(\hat{\mathcal{G}})$ is the subspace of the 2-homogeneous elements for the natural dilation)

$$\mathcal{P}_\lambda = -\Delta + \lambda \cdot Z^2. \tag{6.24}$$

We associate to Γ the set $\hat{\Gamma}$ defined by

$$\hat{\Gamma} = \{\hat{\rho} \in \hat{\mathcal{G}}^*;\ \hat{\rho} = (\rho, \zeta),\ \rho \in \Gamma \text{ and } \zeta = 1\}. \tag{6.25}$$

It is clear that $\hat{\Gamma}$ is closed in $\hat{\mathcal{G}}^*$, $\hat{G}$-stable and that there is a natural identification

$$\Pi_\rho(-\Delta) - \lambda = \Pi_{\hat{\rho}}(\mathcal{P}_\lambda). \tag{6.26}$$

Consequently, we have

$$\forall \hat{\rho} \in \hat{\Gamma},\ \exists C_{\hat{\rho}} = C_\rho > 0$$

s.t.

$$\|\Pi_{\hat{\rho}}(\mathcal{P}_\lambda) f\|^2_{H^0_{\pi_{\hat{\rho}}}} \geq C^{-1} \|f\|^2_{H^0_{\pi_{\hat{\rho}}}}, \qquad \forall f \in \mathcal{S}_{\Pi_{\hat{\rho}}} \tag{6.27}$$

with

$$C = C_{\hat{\rho}}.$$

Unfortunately, we can not directly apply the statements of [24] but the proof of Theorem 4.7 as sketched in [19] can be adapted in our context by modifying the assumptions on the following way:

Theorem 6.3. *Let $\mathcal{G}$ be a graded Lie algebra of rank s and Γ a closed G-stable subset in $\mathcal{G}^*$. Let us assume moreover that:* $[\mathcal{G}^2, \mathcal{G}^2] = 0$ *and that $\mathcal{G}_1$ generates $\mathcal{G}$. Let $\mathcal{P} \in \mathcal{U}_m(\mathcal{G})$ and let us assume that*

$$\text{(H1)} \qquad \forall \rho \in \Gamma,\ \exists C_\rho > 0 \ s.t.\ \|f\|^2_{H^m_{\pi_\rho}} \leq C_\rho \|\Pi_\rho(\mathcal{P})f\|^2_{H^0_{\pi_\rho}},\ \forall f \in \mathcal{S}_{\Pi_\rho}$$

(H2)

$$\exists \hat{C} > 0 s.t\, \forall \rho \in \Gamma,\ \|f\|^2_{H^m_{\pi_\rho}} \leq \hat{C}[\|\Pi_\rho(\mathcal{P})f\|^2_{H^0_{\pi_\rho}} + \|f\|^2_{H^0_{\pi_\rho}}],\ \forall f \in \mathcal{S}_{\Pi_\rho}$$

$$\text{(H3)} \qquad \inf_{(g,\rho) \in G \times \Gamma} |g \cdot \rho| \geq (1/2).$$

Then there exists $C > 0$ s.t. for all $\rho \in \Gamma$ we have:

$$\text{(6.28)} \qquad \|f\|^2_{H^m_{\pi_\rho}} \leq C \|\Pi_\rho(\mathcal{P})f\|^2_{H^0_{\pi_\rho}} \qquad \forall f \in \mathcal{S}_{\Pi_\rho}$$

It is easy to see, using (6.18) and (6.19), Proposition 6.2 and the property $|g \cdot \hat{\rho}| \geq |\zeta| = 1$, that all the assumptions of Theorem 6.3 are satisfied with $\mathcal{G} = \hat{\mathcal{G}}$, $\Gamma = \hat{\Gamma}$ and $\mathcal{P} = \mathcal{P}_\lambda \in \mathcal{U}_2(\hat{\mathcal{G}})$. (6.28) will give Property (P_2).

Indications on the proof of Theorem 6.3. We follow closely the sketch given in [19] p.228 (proof of Theorem 4.7). Let us mention that J. Nourrigat [50] has improved this theorem, but it is sufficient to use the above theorem in our context. If we compare with Theorem 4.7 in [19], we do not make a proof by induction nor a restriction to $|\ell_s| = 1$. Assumption (H2) replaces (4.21) and (H3) replaces (4.22) in [19]. Modulo these modifications the proof is the same (in this article $s = r$). We introduce for $j = 1, \ldots, s$ and $(\ell_1, \ldots, \ell_s)$ the set

$$\Gamma^j(\ell_1, \ldots, \ell_s) = \{\hat{\ell} \in \Gamma,\ \exists g \in G \text{ with } g \cdot \hat{\ell}/\mathcal{G}^j = (\ell_j, \ldots, \ell_s)\}$$

where $\mathcal{G}^j = \mathcal{G}_j \oplus \cdots \oplus \mathcal{G}_s$. Note that ($\Gamma^{s+1} = \Gamma$) and that $\Gamma^j(\ell_j, \ldots, \ell_s)$ is just the orbit of $\ell \in \Gamma$ if $\ell \in \Gamma$ and $\emptyset$ if $\ell \notin \Gamma$.

Lemma 6.4. *Let us assume* (H2), (H3) *and the following property: For all $(\ell_j, \ldots, \ell_s) \in \mathcal{G}_j^* \times \cdots \times \mathcal{G}_s^*$, $\exists C(\ell_j, \ldots, \ell_s)$ s.t. $\forall \tilde{\ell} \in \Gamma^j(\ell_j, \ldots, \ell_s)$,*

$$(\mathrm{Q}_j) \qquad \|f\|^2_{H^m_{\pi_{\tilde{\ell}}}} \leq C(\ell_j, \ldots, \ell_s) \|\Pi_{\tilde{\ell}}(\mathcal{P})f\|^2_{H^0_{\pi_{\tilde{\ell}}}} \qquad \forall f \in \mathcal{S}_{\Pi_{\tilde{\ell}}},$$

then we have Property $(\mathrm{Q}_{(j+1)})$.

Note now that $(\mathrm{Q}_1) = (\mathrm{H1})$ and that $(\mathrm{Q}_{(s+1)})$ is the conclusion of the theorem. According to the remarks before the lemma, the proof of Lemma 6.4 is almost identical to the proof of Lemma 4.10 in [19] by observing that the assumptions of Theorem 4.9 in [19] are satisfied ($|\ell_s| = 1$ is no more true but (H3) replaces this assumption). This ends the proof of Theorem 6.3.

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H^1-Blow up Solutions for Peker-Choquard Type Schrödinger Equations

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§1. Introduction and the main results

In this paper, we study the H^1-solution for the following nonlinear Schrödinger equation

$$\text{(1-1)} \qquad \begin{cases} i\partial_t u = -\Delta_x u - (r^{-\gamma} * |u|^2)u \\ u(0,x) = u_0(x) \in H^1(\mathbf{R}^N) \end{cases},$$

where $r = |x|$ and $2 \leq \gamma < 4$, $\gamma \leq N-1$, and show a sufficient condition of 'H^1-blowing up'. Here we say that u is an H^1-local solution of (1-1) when for some $T > 0$, $u \in C([0,T); H^1)$ and satisfies next integral equation

$$\text{(1-2)} \qquad u(t) = U(t)u_0 - i\int_0^t U(t-s)\{(r^{-\gamma} * |u^2|)u\}(s)ds,$$

where $U(t) = \exp(it\Delta_x)$ is the evolution operator for the free Schrödinger equation. Above type nonlinear Schrödinger equation is appeared in some approximations of many body problems, so-called Hartree approximation. As for detailed arguments of this approximation, see e.g. [5], [6] and [7].

Before stating the main results, we define several notations. For $p \in [1,\infty]$ and $k \in \overline{\mathbf{N}}$, we define Sobolev space

$$W^{k,p} \equiv \{f \in \mathcal{S}' : \|f\|_{W^{k,p}} \equiv \sum_{|\alpha|\leq k} \|\partial_x^\alpha f\|_p < \infty\},$$

where $\|\cdot\|_p$ is usual L^p-norm. $H^k \equiv W^{k,2}$ and $H^{-k} \equiv (H^k)^*$. For an interval I and a Banach space X, $C^k(I;X)$ is the space of X-valued C^k-functions on I, $k = 0,1,2...$ and $L^p(I;X)$ is the space of L^p-functions. We say $u \in L^p_{\text{loc}}(I;X)$ if $u \in L^p(J;X)$ for any compact $J \subset I$.

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For the existence of H^1-local solution of (1-1) and (1-2), we have obtained following theorem. (e.g. [2],[3])

Theorem 0. *Let $2 \leq \gamma \leq 4$, $\gamma < N$ and $u_0 \in H^1$. Then, there exist $T^* > 0$ and $u \in C([0,T^*); H^1)$, which satisfies* (1-2), *and has following properties* (1) $\sim$ (4).

(1) *u is unique solution of* (1-2) *in $L^\theta_{\mathrm{loc}}(0,T^*; W^{1,p})$, where $1/p = 1/2 - (\gamma-2)/4N$ and $\theta = 8/(\gamma-2)$.*

(2) *u satisfies following conservation laws.*

$$\text{(1-3)} \quad \|u(t)\|_2 = \|u_0\|_2,$$

$$\text{(1-4)} \quad E(u(t)) \equiv \|\nabla_x u(t)\|_2^2 - 1/2(|u(t)|^2, r^{-\gamma} * |u(t)|^2) = E(u_0),$$

for $t \in [0,T^*)$. Here $(\cdot,\cdot)$ is L^2-dual coupling.

(3) *If $2 \leq \gamma < 4$ and $T^* < \infty$, then $\|\nabla_x u(t)\|_2 \to \infty$ as $t \to T^*$.*

(4) *u satisfies* (1-1) *in H^{-1} sense.*

Remark. (1) If u satisfies $\|u(t)\|_2 \to \infty$ as $t \to T^*$ for some $T^* < \infty$, we say u blows up at blow up time T^*.

(2) The assumption $2 \leq \gamma$ is not essential. Since the space in which u is unique becomes simple, we state this assumption. On the other hand, the assumption $4 \geq \gamma$ is essential for the existence of H^1-local solution.

On the blow up of H^1-solutions, $2 \leq \gamma$ is a necessary condition, i.e. when $0 \leq \gamma < 2$, the H^1-solution with any initial data $u_0 \in H^1$ is global. On the other hand, it is well-known that when $2 \leq \gamma$, $u_0 \in H^1 \cap L^2(\mathbf{R}^N; |x|^2 dx)$ and $E(u_0) < 0$, the H^1-solution of (1-1) blows up in finite time (e.g. [1]). K. Kurata and T. Ogawa ([4]) dealt with more complicated potential $-(r^{-\gamma_1} * |u|^2)u - (r^{-\gamma_2} * |u|^2)u$, and showed there exists a blow up solution under the assumption $\gamma_1 < 2 < \gamma_2 < 4$ and $\gamma_2 < N-1$. Recently, in the local nonlinear case, i.e. $-|u|^{p-1}u$ instead of $-(r^{-\gamma} * |u|^2)u$, T. Ogawa and Y. Tsutsumi ([8]) showed that for any radially symmetric H^1-initial data u_0, the H^1-solution of corresponding equation blows up in finite time. We shall prove that we can use their methods in the non-local nonlinear case in this paper. Our main result is following.

Theorem 1. *Let $2 \leq \gamma < 4$ and $\gamma + 1 \leq N$. Suppose that u_0 be radially symmetric in $H^1(\mathbf{R}^N)$ and $E(u_0) < 0$. Then the H^1-solution u blows up in finite time.*

Remark. (1) Since u_0 is unique in $L^\theta_{\mathrm{loc}}(0,T^*; W^{1,p})$ and the equation is symmetric by spatial rotation, u is also radially symmetric.

(2) Since $E(K\phi) = K^2\|\nabla_x\phi\|_2^2 - K^4/2 \cdot (|\phi|^2, r^{-\gamma} * |\phi|^2)$ for any $\phi \in H^1$ and $K > 0$, $E(u_0) < 0$ is attained by some $u_0 \in H^1$. This observation shows the assumption $E(u_0) < 0$ means 'u_0 is not small'.

§2. General lemmas

In this chapter, we state two well-known lemmas which hold in H^1. The first one is so-called Gagliardo-Nirenberg's inequality.

Lemma 2-1. *Let $u \in H^1(\mathbf{R}^N)$ and $N \geq 3$. Then, there exists a constant C such that*

$$\|u\|_p \leq C\|\nabla_x u\|_2^a \|u\|_2^{1-a}, \tag{2-1}$$

where $1/p = 1/2 - a/N$.

The second one holds on radially symmetric functions.

Lemma 2-2 (Strauss[9]). *Let u be a radially symmetric function in $H^1(\mathbf{R}^N)$. Then, there exists a constant C such that for any $R > 0$ and $p \in [2, \infty]$,*

$$\|u\|_{L^p(R<|x|)} \leq CR^{-(1/2-1/p)(N-1)}\|u\|_{L^2(R<|x|)}^{1/2+1/p}\|\nabla_x u\|_{L^2(R<|x|)}^{1/2-1/p}. \tag{2-2}$$

§3. Proof of Theorem 1

Choose $\phi \in W^{3,\infty}([0,\infty))$ such that

$$\phi(r) = \begin{cases} r & \text{for } \ 0 \leq r \leq 1, \\ r - (r-1)^3 & \text{for } \ 1 \leq r \leq 1 + \sqrt{3}/3, \\ \text{smooth and } \phi' \leq 0 & \text{for } \ 1 + \sqrt{3}/3 \leq r \leq 2, \\ 0 & \text{for } \ 2 \leq r, \end{cases} \tag{3-1}$$

and put

$$\phi_m(r) = m \cdot \phi(r/m),$$
$$\psi_m(x) = x/|x| \cdot \phi_m(|x|).$$

Remark that if we put $\Phi(r) = \int_0^r \phi_m(s)ds$, $\Phi \in L^\infty(\mathbf{R}^N)$ and $\nabla_x\Phi = \psi_m$. We also obtain next lemma.

Lemma 3-1. *Let u be the H^1-solution of* (1-1). *Then,*

(3-2)

$$\begin{aligned}
&\Im\int u_0\,\psi_m\cdot\nabla_x\overline{u_0}\,dx-\Im\int u(t)\psi_m\cdot\nabla_x\overline{u(t)}dx\\
&=\int_0^t[2\Re\sum_{j,k}\int\partial_j(\psi_m)_k\,\partial_j u(\tau)\partial_k\overline{u(\tau)}dx\\
&\quad-1/2\int\Delta_x(\nabla_x\cdot\psi_m)\cdot|u(\tau)|^2,dx+\gamma E(u_0)-\gamma\|\nabla_x u(\tau)\|_2^2\\
&\quad+\gamma/2\int\int_{|x|\vee|y|\ge m}a(x,y)|x-y|^{-\gamma-2}|u(\tau,x)|^2|u(\tau,y)|^2dxdy]d\tau\\
&\qquad\text{for all}\quad t\in[0,T^*),
\end{aligned}$$

where $\Im$ and $\Re$ mean imaginary and real parts respectively, $(\psi_m)_k$ is k^{th} component of ψ_m and

$$a(x,y)=|x-y|^2-(\psi_m(x)-\psi_m(y))\cdot(x-y). \tag{3-3}$$

Now, remarking that u is radially symmetric, we have

(3-4)

$$\begin{aligned}
&2\Re\sum_{j,k}\int\partial_j(\psi_m)_k\partial_j u\,\partial_k\overline{u}\,dx\\
&\qquad=2\int_{|x|\le m}|\nabla_x u|^2dx+2\int_{m\le|x|\le 2m}\phi'_m|\nabla_x u|^2dx.
\end{aligned}$$

And, simple calculation shows that there exists a constant C such that

$$|\Delta_x(\nabla_x\cdot\psi_m(x))|\quad\begin{cases}\le Cm^{-2} & \text{for}\quad m\le|x|\le 2m,\\ =0 & \text{for otherwise.}\end{cases} \tag{3-5}$$

The next lemma is the key estimate to obtain our result.

Lemma 3-2. *Let $0<\alpha<1$ and $m\gg 1$. For $|x|\vee|y|\ge m$ and $|x-y|\le m^\alpha$, there exists a constant C, which is independent of x,y and m, such that*

$$a(x,y)\le C(b(|x|)+b(|y|))|x-y|^2. \tag{3-6}$$

Here

$$b(r)=\begin{cases}0 & \textit{for}\quad r\le m,\\ 1-\phi'_m(r) & \textit{for}\quad m\le r\le 2m,\\ 1 & \textit{for}\quad 2m\le r.\end{cases} \tag{3-7}$$

Using this lemma, we obtain

(3-8)

$$\int\int_{|x|\vee|y|\geq m,|x-y|\leq m^\alpha} a(x,y)|x-y|^{-\gamma-2}|u(x)|^2|u(y)|^2dxdy$$

$$\leq C\int\int_{|x|\vee|y|\geq m,|x-y|\leq m^\alpha} (b(|x|)+b(|y|))|x-y|^{-\gamma}|u(x)|^2|u(y)|^2dxdy$$

$$\leq 2C\int_{|x|\geq m} b(r)|u(x)|^2(\{\chi(\{r\leq m^\alpha\})\cdot r^{-\gamma}\}*|u|^2)(x)dx$$

$$\leq 2C\|b^{1/2}(r)u(x)\|^2_{L^\infty(|x|\geq m)}\|\chi(\{r\leq m^\alpha\})\cdot r^{-\gamma}\|_1\cdot\|u_0\|_2^2$$

(by Hölder's and Young's inequalities)

$$\leq Cm^{-(N-1)}\|\nabla_x\{b^{1/2}(r)u(x)\}\|^2_{L^2(|x|\geq m)}\cdot m^{\alpha(N-\gamma)}\|u_0\|_2^2$$

(by Lemma 2-2)

$$\leq Cm^{\alpha(N-\gamma)-(N-1)}\|u_0\|_2^2\int b(r)|\nabla_x u(x)|^2dx + Cm^{\alpha(N-\gamma)-(N-1)}\|u_0\|_2^4.$$

Here we used L^2-conservation law (1-3) and defined

$$\chi(A)(x)=\begin{cases}1 & x\in A,\\ 0 & x\notin A.\end{cases}$$

On the other hand, since $|(\psi_m(x)-\psi_m(y))\cdot(x-y)|\leq\|\psi_m'\|_\infty|x-y|^2$, we get

$$\int\int_{|x|\vee|y|\geq m,|x-y|\geq m^\alpha} a(x,y)|x-y|^{-\gamma-2}|u(x)|^2|u(y)|^2dxdy$$

(3-9) $$\leq C\int\int_{|x|\vee|y|\geq m,|x-y|\geq m^\alpha} |x-y|^{-\gamma}|u(x)|^2|u(y)|^2dxdy$$

$$\leq Cm^{-\gamma\alpha}\|u_0\|_2^4.$$

After all, by (3-2),(3-4),(3-8) and (3-9), we have

$$
\begin{aligned}
&\Im \int u_0 \psi_m \cdot \nabla_x \overline{u_0} dx - \Im \int u(t) \psi_m \cdot \nabla_x \overline{u(t)} dx \\
&\le \int_0^t [\gamma E(u_0) - (\gamma - 2)\|\nabla_x u(\tau)\|_2^2 \\
&\qquad - 2 \int b(r) |\nabla_x u(\tau)|^2 dx - C m^{-2} \|u_0\|_2^2 \\
&\qquad + C(m^{-\gamma\alpha} + m^{\alpha(N-\gamma)-(N-1)}) \|u_0\|_2^4 \\
&\qquad + C m^{\alpha(N-\gamma)-(N-1)} \|u_0\|_2^2 \int b(r) |\nabla_x u(\tau)|^2 dx] d\tau.
\end{aligned} \tag{3-10}
$$

Thus, if we take sufficiently large m such that

$$
\gamma E(u_0) + C(m^{-\gamma\alpha} + m^{\alpha(N-\gamma)-(N-1)}) \|u_0\|_2^4 \equiv -\eta < 0,
$$

and

$$
C m^{\alpha(N-\gamma)-(N-1)} \|u_0\|_2^2 - 2 \le 0,
$$

we obtain

$$
\Im \int u_0 \, \psi_m \cdot \nabla_x \overline{u_0} \, dx - \Im \int u(t) \, \psi_m \cdot \nabla_x \overline{u(t)} dx \ge \eta t. \tag{3-11}
$$

Since

$$
d/dt(\int \Psi |u(t)|^2 dx) = -2\Im \int u(t) \, \psi_m \cdot \nabla_x \overline{u(t)} dx,
$$

integrating the both hands of (3-12), we deduce that

$$
\begin{aligned}
\int \Psi |u(t)|^2 dx &\le -\eta t^2 - 2t \, \Im \int u_0 \, \psi_m \cdot \nabla_x \overline{u_0} \, dx \\
&\quad + \int \Psi |u_0|^2 dx \quad \text{for all } t \in [0, T^*).
\end{aligned} \tag{3-12}
$$

Now, we assume u is a global solution. Then, (3-12) is satisfied for any $t < \infty$ and the r.h.s. of (3-12) is negative for sufficiently large t. This is contradiction since the l.h.s. of (3-12) is non-negative. Thus, u is not global solution and $T < \infty$. Using Theorem 0.(3), we obtain $\|\nabla_x u(t)\|_2 \to \infty$ as $t \to T^*$. This means our desired result.

§4. The proofs of lemmas

Proof of Lemma 3-1. We first assume $u_0 \in H^2$. Under this assumption, the solution u belongs to $C([0, T^*); H^2) \cap C^1([0, T^*); L^2)$ and satisfies (1-1) in L^2-sense (see e.g. [2]). Note that the maximum existence time T^* is the same as that of the H^1-solution. We take the real part of L^2-inner product between (1-1) and $\psi_m \cdot \nabla_x u$. Here, using equality (1-1) and integrating by parts, we have

$$
\begin{aligned}
&2\Re(i\partial_t u\,,\, \psi_m \cdot \nabla_x u) \\
&= i\int \partial_t u\, \psi_m \cdot \nabla_x \overline{u}\, dx - i\int \psi_m \cdot \nabla_x u\, \partial_t \overline{u}\, dx \\
&= i\, d/dt \int u\, \psi_m \cdot \nabla_x \overline{u}\, dx + \int \nabla_x \cdot \psi_m |u|^2 (r^{-\gamma} * |u|^2) dx \\
&\quad - \int \nabla_x \cdot \psi_m |\nabla_x u|^2 dx + 1/2 \int \Delta_x (\nabla_x \cdot \psi_m) |u|^2 dx,
\end{aligned}
\tag{4-1}
$$

$$
\begin{aligned}
&2\Re(-\Delta_x u\,,\, \psi_m \cdot \nabla_x u) \\
&\quad = 2\Re \sum_{j,k} \int \partial_j (\psi_m)_k\, \partial_j u\, \partial_k \overline{u}\, dx - \int \nabla_x \cdot \psi_m |\nabla_x u|^2 dx,
\end{aligned}
\tag{4-2}
$$

and

$$
\begin{aligned}
&2\Re(u(r^{-\gamma} * |u|^2)\,,\, \psi_m \cdot \nabla_x \overline{u}) \\
&= \int (\nabla_x \cdot \psi_m) |u|^2 (r^{-\gamma} * |u|^2) dx + \int |u|^2 \psi_m \cdot \nabla_x (r^{-\gamma} * |u|^2) dx.
\end{aligned}
\tag{4-3}
$$

Here, since

$$
\begin{aligned}
&1/2 \int |u(x)|^2 \psi_m(x) \cdot \nabla_x \Big(\int |x-y|^{-\gamma} |u(y)|^2 dy\Big) dx \\
&= 1/2 \int |u(x)|^2 \Big\{\nabla_x \Big(\int \psi_m(x) |x-y|^{-\gamma} |u(y)|^2 dy\Big) \\
&\quad - (\nabla_x \psi_m)(x) \cdot \int |x-y|^{-\gamma} |u(y)|^2 dy\Big\} dx \\
&= 1/2 \int |u(x)|^2 \nabla_x \cdot \Big[\int \{(\psi_m(x) - \psi_m(y)) |x-y|^{-\gamma} |u(y)|^2 \\
&\quad + |x-y|^{-\gamma} \psi_m(y) |u(y)|^2\} dy\Big] dx \\
&\quad - 1/2 \int (\nabla_x \cdot \psi_m)(x) |u(x)|^2 \Big(\int |x-y|^{-\gamma} |u(y)|^2 dy\Big) dx \\
&= -1/2 \int \nabla_x |u(x)|^2 \cdot \Big(\int \psi_m(y) |x-y|^{-\gamma} |u(y)|^2 dy\Big) dx
\end{aligned}
$$

$$
\begin{aligned}
&+1/2\int |u(x)|^2[\int \nabla_x\cdot\{(\psi_m(x)-\psi_m(y))|x-y|^{-\gamma}\}|u(y)|^2dy]dx\\
&-1/2\int(\nabla_x\cdot\psi_m)(x)|u(x)|^2(\int |x-y|^{-\gamma}|u(y)|^2dy)dx\\
=&-1/2\int |u(y)|^2\psi_m(y)\cdot(\int \nabla_x|u(x)|^2|x-y|^{-\gamma}dx)dy\\
&+1/2\int |u(x)|^2\{\int(\psi_m(x)-\psi_m(y))\cdot(\nabla r^{-\gamma})(x-y)|u(y)|^2dy\}dx\\
=&-1/2\int |u(x)|^2\psi_m(x)\cdot\nabla_x(r^{-\gamma}*|u|^2)(x)dx\\
&-\gamma/2\int |u(x)|^2\{\int(\psi_m(x)-\psi_m(y))\cdot(x-y)\,|x-y|^{-\gamma-2}|u(y)^2dy\}dx,
\end{aligned}
$$

the second term of r.h.s. of (4-3) is equal to

$$
-\gamma/2\int\int |u(x)|^2(a(x,y)|x-y|^{-\gamma-2}-|x-y|^{-\gamma})|u(y)|^2\,dy\,dx.
$$

Thus, by (4-1)$\sim$(4-3), we get

$$
\begin{aligned}
&i\,d/dt\int u\,\psi_m\cdot\nabla_x\overline{u}\,dx+1/2\int \Delta_x(\nabla_x\cdot\psi_m)|u|^2dx\\
=&2\Re\sum_{j,k}\int \partial_j(\psi_m)_k\,\partial_j u\,\partial_k\overline{u}\,dx\\
&+\gamma/2\int\int |u(x)|^2a(x,y)|x-y|^{-\gamma-2}|u(y)|^2dy\,dx\\
&-\gamma/2\int |u|^2(r^{-\gamma}*|u|^2)dx.
\end{aligned}
$$

Taking real part of b.h.s. and using the definition of energy (1-4), we obtain

$$
\begin{aligned}
&-d/dt\,\Im\int u\,\psi_m\cdot\nabla_x\overline{u}\,dx\\
=&2\Re\sum_{j,k}\int \partial_j(\psi_m)_k\,\partial_j u\,\partial_k\overline{u}\,dx-1/2\int \Delta_x(\nabla_x\cdot\psi_m)|u|^2dx\\
&+\gamma E(u_0)-\gamma\|\nabla_x u\|_2^2\\
&+\gamma/2\int\int |u(x)|^2a(x,y)|x-y|^{-\gamma-2}|u(y)|^2dy\,dx.
\end{aligned}
\tag{4-4}
$$

Thus, integrating (4-4) over $[0,T^*)$ by t, we obtain (3-3).

For the case of $u_0 \in H^1$, we take $\{u_{0,l}\} \subset H^2$ such that $u_{0,l} \to u_0$ in H^1 as $l \to \infty$. For each $u_{0,l}$, we can construct strong solutions $u_l(t)$ of (1-1) in a certain common time interval $[0,T]$, and $\{u_l(t)\}$ converges to the H^1-solution $u(t)$ in H^1 uniformly. (See [2].) Thus, we obtain (3-2) on $[0,T]$. Since T is depend only on $\|u_0\|_{H^1}$, we can repeat this procedure, and we obtain (3-2) as long as $u(t)$ exists. Q.E.D.

Proof of Lemma 3-2. It suffices to consider on x, y 2-dimensional plain, then let $x = (r\cos\theta, r\sin\theta)$ and $y = (\rho, 0)$. By taking m sufficiently large and using renormalization, we can assume $m = 1$ and $\theta \ll 1$. For the case of $1 \le r, \rho \le 1 + \sqrt{3}/3$, we calculate

$$\begin{aligned}
&|x-y|^2 - (\phi(x) - \phi(y)) \cdot (x-y) \\
=&(r-\rho)\{(r-\phi(r)) - (\rho - \phi(\rho))\} \\
&\quad + (1-\cos\theta)\{r(\rho - \phi(\rho)) + \rho(r - \phi(r))\} \\
=&(r-\rho)\{(r-1)^3 - (\rho-1)^3\} + (1-\cos\theta)\{r(\rho-1)^3 + \rho(r-1)^3\} \\
=&(r-\rho)^2\{(r-1)^2 + (r-1)(\rho-1) + (\rho-1)^2\} \\
&\quad + (1-\cos\theta)\{r(\rho-1)^3 + \rho(r-1)^3\}.
\end{aligned}$$

Since $b(r) = 3(r-1)^2$ on $1 \le r \le 1 + \sqrt{3}/3$, it suffices to show that there exists a constant C, independent of r and ρ, such that

$$\begin{aligned}
&(r-\rho)^2\{(r-1)^2 + (r-1)(\rho-1) + (\rho-1)^2\} \\
&\quad + (1-\cos\theta)\{r(\rho-1)^2 + \rho(r-1)^2\} \\
\le& C[(r-\rho)^2\{(r-1)^2 + (\rho-1)^2\} \\
&\quad + 2(1-\cos\theta)r\rho\{(r-1)^2 + (\rho-1)^2\}].
\end{aligned}$$

This is possible obviously since $1 \le r, \rho$. For the case of $r \wedge \rho < 1$, the similar calculation shows the statement, and we omit the details.

Q.E.D.

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On Scattering by Two Degenerate Convex Bodies

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§1. Introduction

Let n be an odd integer ≥ 3, and let $\mathcal{O}$ be a bounded open set in $\mathbb{R}^n$ such that

$$\Omega = \mathbb{R}^n - \overline{\mathcal{O}} \quad \text{is connected.} \tag{1.1}$$

We assume that

$$\Gamma = \partial\mathcal{O} \quad \text{is smooth.}$$

Denote by $\mathcal{S}(z)$ the scattering matrix for $\mathcal{O}$. The scattering matrix $\mathcal{S}(z)$ is an $\mathcal{L}(L^2(S^{n-1}))$-valued holomorphic function defined in $\{z \in \mathbb{C};\ \operatorname{Re} z < 0\}$, where we denote by $\mathcal{L}(E)$ the space of all the bounded operators from E into itself. As a fundamental property of the scattering matrix, it is shown in Lax-Phillips [7]:

Theorem 5.1 of Chapter V. *The scattering matrix $\mathcal{S}(z)$ is holomorphic on the real axis and meromorphic in the whole plane, having a pole at exactly those points z for which there is a nontrivial z-outgoing local solution of*

$$\begin{cases} (\triangle + z^2)\, u = 0 & \text{in } \Omega \\ \quad u = 0 & \text{on } \ \Gamma. \end{cases}$$

In the study of scattering by obstacles, the problem to know relationships between the geometry of obstacles and the distribution of poles of scattering matrices is one of the most interesting and important problems. It is conjectured that the more rays of geometric optics are trapped by $\mathcal{O}$ the more solutions of the wave equation are trapped by $\mathcal{O}$, and that the more solutions of the wave equation are trapped, the nearer to the real axis it appears the poles of the scattering matrix.

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Concerning this problem, Melrose [9] proved that, if $\mathcal{O}$ is nontrapping in the sense of geometric optics, for any $a > 0$ the logarithmic domain

$\{z;\ \mathrm{Im}\, z \leq a \log(|z| + 1)\}$ has at most a finite number of poles of $\mathcal{S}(z)$.

For trapping obstacles, Bardos-Guillot-Ralston [1] considered the following example:

$$\mathcal{O} = \mathcal{O}_1 \cup \mathcal{O}_2$$

where

$$\mathcal{O}_l, \quad l = 1, 2 \text{ are strictly convex and } \overline{\mathcal{O}_1} \cap \overline{\mathcal{O}_1} = \phi.$$

They showed that, for any $\varepsilon > 0$, the logarithmic domain

$$\{z;\ \mathrm{Im}\, z \leq \varepsilon \log(|z| + 1)\} \quad \text{has an infinite number of poles of } \mathcal{S}(z).$$

Next Ikawa [3] considered the same example and showed the following result: Set $d = \text{distance}(\mathcal{O}_1, \mathcal{O}_2)$, and let A_l, $l = 1, 2$, be the point on $\Gamma_l = \partial\mathcal{O}_l$ such that

$$\text{distance}(\mathcal{O}_1, \mathcal{O}_2) = |A_1 - A_2|.$$

Then, there is a positive constant c_0 determined by d and the geometry of Γ_l near A_l $(l = 1, 2)$ such that, in the strip $\{z;\ 0 < \mathrm{Im}\, z < \frac{2}{3}c_0\}$ the poles of $\mathcal{S}(z)$ distribute asymptotically at the points $\frac{\pi}{d}j + \sqrt{-1}c_0$, $j = 0, \pm 1, \pm 2, \cdots$.

After that, Gérard [2] proved that, for any $a > 0$, the poles of $\mathcal{S}(z)$ in the strip $\{z;\ 0 < \mathrm{Im}\, z < a\}$ distribute asymptotically on the points

$$\frac{\pi}{d}j + \sqrt{-1}c_m, \quad j = 0, \pm 1, \pm 2, \cdots, \quad m = 0, 2, \cdots, m_0$$

where

$$0 < c_0 \leq c_1 \leq c_2 \leq \cdots \leq c_{m_0} < a.$$

The constants c_m, $m \geq 1$ are also determined by d and the geometry of Γ_l near A_l, $l = 1, 2$.

The formula which gives c_m indicates that, when all the principal curvatures of Γ_l at A_l, $l = 1, 2$, become small, the constants c_m become also small, and when all the principal curvatures vanish at A_l $(l = 1, 2)$, all the c_m determined by the formula are equal to 0.

This fact indicates us that, if all the principal curvatures vanish at A_l, $\mathcal{S}(z)$ may have a sequence of of poles converging to the real axis.

But the methods used in [3] and [2] are no more valid in the case where all the principal curvatures vanish. We considered in [4] an example of $\mathcal{O} = \mathcal{O}_1 \cup \mathcal{O}_2$ in $\mathbb{R}^3$ such that the principal curvatures of Γ_l vanish only at A_l of finite order, and showed that there exist an infinite number of poles in a domain $\{z;\ \mathrm{Im}\, z \leq |\mathrm{Re}\, z|^{-\gamma}\}$ for some positive constant γ. The proof of this result is based on the trace formula due to Bardos-Guillot-Ralston [1]. On the other hand, as to the position of poles near to the real axis, by taking account of the results of [3] and [2], it seems very likely that the poles of $\mathcal{S}(z)$ in the domain $\{z;\ \mathrm{Im}\, z \leq |\mathrm{Re}\, z|^{-\gamma}\}$ exist only near the points $\frac{\pi}{d}j$, $j = \pm 1, \pm 2, \cdots$. But it seems very difficult to get more information on the distribution of poles by the means of the trace formula.

In this paper we shall consider an example of obstacle in $\mathbb{R}^2$ consisting of two convex bodies, whose curvature vanishes of finite order at A_l. Precisely, let $\mathcal{O}_1$ be a bounded open set in $\mathbb{R}^2$ with smooth boundary Γ_1 such that

(1) $\mathcal{O}_1 \subset \{x = (x_1, x_2) \in \mathbb{R}^2; x_2 < 0\}$,
(2) $A_1 = (0, 0) \in \Gamma_1$,
(3) Γ_1 is represented near A_1 as

$$x_2 = -{x_1}^{2m}$$

where m is a positive integer≥ 2,
(4) the curvature of Γ_1 does not vanish on $\Gamma_1 - \{A_1\}$.

Let $\mathcal{O}_2$ be a bounded open set in $\mathbb{R}^2$ with smooth boundary Γ_2 such that

(1) $\mathcal{O}_2 \subset \{x = (x_1, x_2) \in \mathbb{R}^2; x_2 > d\}$ where d is a positive constant,
(2) $A_2 = (0, d) \in \Gamma_2$,
(3) Γ_2 is represented near A_2 as

$$x_2 = d + {x_1}^{2m},$$

(4) the curvature of Γ_2 does not vanish on $\Gamma_2 - \{A_2\}$.

We set

$$\mathcal{O} = \mathcal{O}_1 \cup \mathcal{O}_2, \quad \Gamma = \Gamma_1 \cup \Gamma_2$$

and

$$\Omega = \mathbb{R}^2 - \overline{\mathcal{O}}.$$

Consider the following boundary value problem with parameter $\mu \in \mathbb{C}$

$$(1.2) \qquad \begin{cases} (\triangle + \mu^2)u(x) = 0 & \text{in } \Omega \\ u(x) = g(x) & \text{in } \Gamma \end{cases}$$

for $g(x) \in C^\infty(\Gamma)$. For $\operatorname{Im}\mu < 0$, (1.1) has a unique solution in $L^2(\Omega)$. Denote the solution $u(x)$ as

$$u(x) = (U(\mu)g)(x).$$

Then by using the regularity theorem for elliptic operators, $U(\mu)$ can be regarded as a continuous operator from $C^\infty(\Gamma)$ into $C^\infty(\overline{\Omega})$ for each μ such that $\operatorname{Im}\mu < 0$. Thus, $U(\mu)$ becomes an $\mathcal{L}(C^\infty(\Gamma), C^\infty(\overline{\Omega}))$-valued holomorphic function in $\{\mu; \operatorname{Im}\mu < 0\}$, where $\mathcal{L}(E, F)$ denotes the set of all the continuous operators from E into F.

We would like to consider the analytic continuation of $U(\mu)$ into $\{\mu; \operatorname{Im}\mu \geq 0\}$. The result that I will show is the following theorem:

Theorem 1. *Assume that*

$$(1.3) \qquad m \geq 4,$$

and set

$$\alpha = \frac{1}{m-1}.$$

Then, for any ε_1, $\varepsilon_2 > 0$, there exists a positive constant $C_{\varepsilon_1 . \varepsilon_2}$ such that $U(\mu)$ can be continued analytically into

$$(1.4) \qquad \begin{aligned} &\{\mu;\ \operatorname{Im}\mu \leq |\operatorname{Re}\mu|^{-(1+2\alpha)^{-1}-\varepsilon_1},\ |\operatorname{Re}\mu| \geq C_{\varepsilon_1,\varepsilon_2}\} \\ &\quad - \cup_{r=-\infty}^{\infty}\{\mu;\ \operatorname{Im}\mu \geq 0 \text{ and } |\frac{\pi}{d}r - \operatorname{Re}\mu| < \varepsilon_2\}. \end{aligned}$$

Recall that the poles of $\mathcal{S}(z)$ coincide with those of $U(\mu)$. Therefore, even though Theorem 1 is of the analytic continuation of $U(\mu)$ for an obstacle in $\mathbb{R}^2$, it gives us a partial answer to the above question.

§2. Geometric optics near the periodic ray a_1a_2

In order to consider the solution of the reduced wave equation (1.2) for high frequency, that is, for $|\operatorname{Re}\mu|$ large, the geometric optics in Ω plays an important role. Especially, it is essential to know the asymptotic behavior of rays trapped by $\mathcal{O}$, which are the ones approaching to the periodic rays when $t \to \infty$. In our case, the periodic ray in Ω is only the one going and returning between A_1 and A_2. Thus we consider the behavior of rays in the domain $\Omega(\delta)$ $(\delta > 0)$ surrounded by the following four curves

$$x_1 = \delta, \quad x_1 = -\delta, \quad x_2 = -{x_1}^{2m}, \quad x_2 = d + {x_1}^{2m}$$

and set

$$S_l(\delta) = \overline{\Omega(\delta)} \cap \Gamma_l, \quad l = 1, 2.$$

From now on, in this section we shall denote the point in $\mathbb{R}^2$ as $Q = (x, y),\ x, y \in \mathbb{R}$.

Let

$$Q = (x, -x^{2m}) \in S_1(\delta) \ \text{ and } \ \Xi = (\xi, \sqrt{1 - \xi^2}) \in S^1,$$

and denote by $X(Q, \Xi)$ the ray starting from Q in the direction Ξ, that is,

$$X(Q, \Xi) = \{Q + s\Xi; s \geq 0\}.$$

Denote by Q' and Ξ' the first fitting point of $X(Q, \Xi)$ at Γ_2 and the direction of the reflected ray respectively. Setting $Q' = (x', d + {x'}^{2m})$, we have

$$\Xi' = \Xi - 2(\Xi,\ N(Q'))N(Q')$$

where $N(Q')$ denotes the unit outer (with respect to $\mathcal{O}_2$) normal of Γ_2, that is,

$$N(Q') = (1 + (2m{x'}^{2m-1})^2)^{-1/2}\,(2m{x'}^{2m-1}, -1).$$

Set $\Xi' = (\xi', -\sqrt{1 - {\xi'}^2})$. Then we have a mapping

$$T : (x, \xi) \to (x', \xi').$$

It is obvious that, when the both x and ξ tend to zero, x' and ξ' also tend to zero. As an approximation of the mapping T we shall consider the following mapping $\tilde{T}$, which maps (x, ξ) to $\tilde{T}(x, \xi) = (x', \xi')$ given by

$$(2.1) \qquad \begin{cases} x' = x + \xi \\ \xi' = \xi + 4m{x'}^{2m-1} = \xi + 4m(x + \xi)^{2m-1}. \end{cases}$$

Let $f(s)$ be a smooth function defined for s near to 0, and let $\{m_j\}_{j=0}^{\infty}$ be an increasing sequence such that $m_j \to \infty$ as $j \to \infty$. We say that $f(s)$ has an asymptotic expansion for $s \to 0$

$$f(s) \sim a_0\, s^{m_0} + a_1\, s^{m_1} + \cdots + a_j\, s^{m_j} + \cdots$$

when, for any $M > 0$, there are j_0 and C_M such that

$$|f(s) - \sum_{j=0}^{j_0} a_j\, s^{m_j}| \le C_M\, |s|^M.$$

Lemma 2.1. *Suppose that $m \ge 2$. Then, there is a one parameter family of a pair of functions $g(s)$ and $h(s)$ defined for small s having asymptitic expansions*

$$(2.2)\qquad \begin{cases} g(s) \sim a_0\, s^{\alpha} + a_1\, s^{\alpha+1} + \cdots + a_j\, s^{\alpha+j} + \cdots \\ h(s) \sim b_0\, s^{\alpha+1} + b_1\, s^{\alpha+2} + \cdots + b_j\, s^{\alpha+j+1} + \cdots \end{cases}$$

and satisfying

$$(2.3)\qquad \tilde{T}(g(s), h(s)) - (g(\frac{s}{s+1}), h(\frac{s}{s+1})) \sim 0.$$

In the asymptotic expansion (2.2), a_1 is a free parameter, $a_0 = \pm(\alpha/2)^{\alpha}$ and $b_0 = \mp\alpha a_0$ are independent of a_1, and b_1 is given by

$$b_1 = a_0\, \frac{\alpha(\alpha+1)}{2} - (\alpha+1)a_1,$$

and a_j and b_j $(j \ge 2)$ depend on a_1.

Proof. We look for formal series

$$(2.4)\qquad \begin{cases} g(s) \sim a_0 s^{\gamma} + a_1 s^{\gamma+1} + \cdots \\ h(s) \sim b_0 s^{\beta} + b_1 s^{\beta+1} + \cdots \end{cases}$$

as they satisfy (2.3), which can be written as

$$(2.5)\qquad g(\frac{s}{1+s}) - g(s) \sim h(s),$$

$$(2.6)\qquad h(\frac{s}{1+s}) - h(s) \sim 4m(g(s) + h(s))^{2m-1}.$$

We choose a_j, b_j $(j = 0, 1, \cdots)$ so that (2.5) and (2.6) hold. Note that for $p \in \mathbb{R}$ we have

$$\left(\frac{s}{s+1}\right)^p \sim s^p - ps^{p+1} + \frac{1}{2}p(p+1)s^{p+2} - \cdots.$$

Substitute (2.4) and the above expansion into (2.5) and (2.6). Equating the leading terms of the both sides of (2.5) we have

$$-a_0\gamma s^{\gamma+1} = b_0 s^{\beta},$$

which implies that $\beta = \gamma + 1$ and $b_0 = -\gamma a_0$. Substituting the just obtained relations into (2.6) and equating the leading terms of the both sides of (2.6) we have

$$-\beta b_0 s^{\beta+1} = 4m\left(a_0 s^{\gamma}\right)^{2m-1}.$$

Therefore it follows that

$$\begin{aligned}
&\gamma(2m-1) = \beta + 1 = \gamma + 2,\\
&4m{a_0}^{2m-1} = -b_0\beta = a_0\gamma\beta = a_0\gamma(\gamma+1).
\end{aligned}$$

Thus we have

$$\begin{aligned}
\gamma &= (m-1)^{-1} = \alpha,\\
4m{a_0}^{2m-2} &= \frac{1}{m-1}\frac{m}{m-1} = m\alpha^2.
\end{aligned}$$

Now, substitute these γ, β, a_0 and b_0 and equate the second terms of the both sides of (2.5). The we have

$$\left\{a_0\frac{\alpha(\alpha+1)}{2} - a_1(\alpha+1)\right\} s^{\alpha+2} = b_1 s^{\alpha+2}.$$

Choose arbitrary a_1 and take b_1 as

$$b_1 = a_0\frac{\alpha(\alpha+1)}{2} - a_1(\alpha+1).$$

Then the second term of the left hand side of (2.6) is

$$\left\{b_0\frac{\beta(\beta+1)}{2} - b_1(\beta+1)\right\} s^{\beta+2}.$$

On the other hand, the second term of the right hand side of (2.6) is

$$4m(2m-1){a_0}^{2m-1}(a_1+b_0)s^{(2m-2)\alpha+\alpha+1}.$$

Evidently it holds that $(2m-2)\alpha + \alpha + 1 = \beta + 2$, and we can check easily by a direct calculus that

$$b_0\frac{\beta(\beta+1)}{2} - b_1(\beta+1) = 4m(2m-1){a_0}^{2m-1}(a_1+b_0).$$

For $j \geq 2$, the $(j+1)$-th term of the left hand side of (2.5) is

$$\{-(\alpha+j)a_j + \text{ linear combination of } a_0, a_1, \cdots, a_{j-1}\}\, s^{\alpha+j+1}.$$

Thus, b_j should satisfy

$$-(\alpha+j)a_j + \text{ linear combination of } a_0,\ a_1, \cdots, a_{j-1} = b_j.$$

Similarly, $(j+1)$-th term of the left hand side of (2.6) is

$$\{-(\beta+j)b_j + \text{ linear combination of } b_0,\ b_1, \cdots, b_{j-1}\}\, s^{\beta+j+1}.$$

The $(j+1)$-th term of the right hand side of (2.6) is

$$4m\left\{(2m-1){a_0}^{2m-2}a_j + \text{terms determined by } b_0,\ b_1, \cdots, b_{j-1} \text{ and } a_0,\ a_1, \cdots, a_{j-1},\ a_j\right\} s^{\beta+j+1}.$$

Now consider a linear equation in unknown $(a_j,\, b_j)$:

$$\begin{cases} (\alpha+j)a_j + b_j = F_j \\ (\alpha+1)(\alpha+2)a_j + (\beta+j)b_j = G_j. \end{cases}$$

Since $(\alpha+j)(\beta+j) - (\alpha+1)(\alpha+2) \neq 0$ for all $j \geq 2$, the above equation has a unique solution for any given $(F_j,\, G_j)$. Thus for $j \geq 2$, we can choose the coefficients a_j, b_j successively in such a way that the asymptotic expansions of the both sides of (2.3) are equal. Q.E.D.

Lemma 2.1 gives us an asymptotic behavior of broken rays in Ω. Choose j_0 and set

$$g^{(j_0)} = a_0\, s^{\alpha} + a_1\, s^{\alpha+1} + \cdots + a_{j_0}\, s^{\alpha+j_0},$$
$$h^{(j_0)} = b_0\, s^{\alpha+1} + b_1\, s^{\alpha+2} + \cdots + b_{j_0}\, s^{\alpha+1+j_0}$$

and

$$x_n^0 = g^{(j_0)}(n^{-1}), \quad \xi_n^0 = h^{(j_0)}(n^{-1}).$$

Since $\dfrac{s}{1+s} = (n+1)^{-1}$ for $s = n^{-1}$, we have

$$x_{n+1}^0 = g^{(j_0)}\Big(\frac{n^{-1}}{1+n^{-1}}\Big), \quad \xi_{n+1}^0 = h^{(j_0)}\Big(\frac{n^{-1}}{1+n^{-1}}\Big).$$

Then, for any M fixed, if we choose j_0 sufficiently large, we have the following estimate

$$|\tilde{T}(x_n^0, \xi_n^0) - (x_{n+1}^0, \xi_{n+1}^0)| \leq C_M\, n^{-M} \quad \text{for all } n.$$

This gives us an approximate behavior of broken ray in Ω which converges to the periodic ray $a_1 a_2$.

Let us denote as

$$\tilde{T}(x_n^0 + s, \xi_n^0 + t) = (\tilde{x}_{n+1}^0 + s', \tilde{\xi}_{n+1}^0 + t'),$$

where we set $(\tilde{x}_{n+1}^0, \tilde{\xi}_{n+1}^0) = \tilde{T}(x_n^0, \xi_n^0)$. Define mapping $\tilde{T}_n$ by

$$\tilde{T}_n : (s,t) \to (s',t'),$$

which maps a neighborhood of $(0,0) \in \mathbb{R}^2$ in to a neighborhood of $(0,0) \in \mathbb{R}^2$. Then we have

$$\begin{aligned} A_n &= \left.\frac{\partial(s',t')}{\partial(s,t)}\right|_{s=t=0} \\ &= \begin{pmatrix} 1 & 1 \\ 2m(2m-1)\left(x_{n+1}^0\right)^{2m-2} & 1 + 2m(2m-1)\left(x_{n+1}^0\right)^{2m-2} \end{pmatrix}. \end{aligned}$$

Substituting the expansion of x_n^0, we have

$$A_n \sim \begin{pmatrix} 1 & 1 \\ d_0 n^{-2} + d_1 n^{-3} + \cdots & 1 + d_0 n^{-2} + d_1 n^{-3} + \cdots \end{pmatrix},$$

where $d_0 = (\alpha+1)(\alpha+2)$. Set

$$S_j^n(s,t) = \tilde{T}_n \circ \tilde{T}_{n-1} \circ \cdots \circ \tilde{T}_j(s,t) = (X_j^n(s,t), \Xi_j^n(s,t)),$$

and

$$D_j^n(s,t) = \frac{\partial S_j^n(s,t)}{\partial(s,t)} = \begin{pmatrix} g_{j,11}^n(s,t) & g_{j,12}^n(s,t) \\ g_{j,21}^n(s,t) & g_{j,22}^n(s,t) \end{pmatrix}.$$

Evidently we have

$$D_1^n(0,0) = A_n \circ A_{n-1} \circ \cdots \circ A_1.$$

Lemma 2.2. *Suppose that $m \geq 4$. Then, we have an asymptotic expansion of $D_j^n(0,0)$ in $n^{-\alpha}$ of the form*

$$D_j^n(0,0)\, D_j \sim \begin{pmatrix} n^{\alpha+2} + a_{11,1} n^{\alpha+1} + \cdots & n^{-\alpha-1} + a_{12,1} n^{-\alpha-2} + \cdots \\ (\alpha+2) n^{\alpha+1} + a_{21,1} n^{\alpha} + \cdots & -(\alpha+1) n^{-\alpha-2} + a_{22,1} n^{-\alpha-3} + \cdots \end{pmatrix},$$

where D_j is a nonsingular 2×2-matrix.

Proof. In this proof, we write $D_j^n(0,0)$ as D^n for the simplicity. Suppose that D^n has an asymptotic expansion of the form

$$D^n \sim \left(g_{ij}(n^{-1})\right)_{i,j=1,2} = G(n^{-1})$$

where $g_{ij}(s)$ are functions with asymptotic expansion for $s \to 0$

$$g_{ij}(s) \sim a_{ij,0}s^{\gamma_{ij}} + a_{ij,1}s^{\gamma_{ij}+\alpha} + a_{ij,2}s^{\gamma_{ij}+2\alpha} + \cdots .$$

Since

$$\begin{aligned} D^{n+1} &= A_{n+1}D^n \\ &= \left\{E + \begin{pmatrix} 0 & 1 \\ d_0n^{-2} + d_1n^{-3} + \cdots & d_0n^{-2} + d_1n^{-3} + \cdots \end{pmatrix}\right\} D^n, \end{aligned}$$

we have

$$D^{n+1} - D^n = \begin{pmatrix} 0 & 1 \\ d_0n^{-2} + d_1n^{-3} + \cdots & d_0n^{-2} + d_1n^{-3} + \cdots \end{pmatrix} D^n.$$

Thus, it suffices to look for 2×2-valued function $G(s)$ satisfying

$$G(\frac{s}{1+s}) - G(s) = \begin{pmatrix} 0 & 1 \\ d_0s^2 + d_1s^3 + \cdots & d_0s^2 + d_1s^3 + \cdots \end{pmatrix} G(s).$$

By the same argument as of Lemma 2.1 we get an aymptotic expansion of $G(s)$ for $s \to 0$, and $D^n = G(n^{-1})$ satisfies the required properties. Here we use essentially the assumption $m \geq 4$ for the purpose of the possibility of successive determination of the coefficients of $G(s)$.

Lemma 2.3. *For any multi-index γ we have*

$$\left|\partial_{s,t}^{\gamma} X_1^n(s,t)\big|_{s=t=0}\right| \leq C^{|\gamma|}\, n^{-\alpha}(n^{2+2\alpha})^{|\gamma|},$$
$$\left|\partial_{s,t}^{\gamma} \Xi_1^n(s,t)\big|_{s=t=0}\right| \leq C^{|\gamma|}\, n^{-\alpha-1}(n^{2+2\alpha})^{|\gamma|},$$

where $C > 0$ is a constant independent of γ.

Take other functions

$$\tilde{g}(s) \sim \tilde{a}_0s^{\alpha} + \tilde{a}_1s^{\alpha+1} + \cdots,$$
$$\tilde{h}(s) \sim \tilde{b}_0s^{\alpha} + \tilde{b}_1s^{\alpha+1} + \cdots$$

with the properties of Lemma 2.1 of the type (2.3), that is,

$$\tilde{T}(\tilde{g}(s), \tilde{h}(s)) - (\tilde{g}(\frac{s}{1+s}), \tilde{h}(\frac{s}{1+s})) \sim 0.$$

Set

$$\begin{aligned} y_n^0 &= \tilde{a}_0 n^{-\alpha} + \tilde{a}_1 n^{-(\alpha+1)} + \cdots \\ \eta_n^0 &= \tilde{b}_0 n^{-(\alpha+1)} + \tilde{b}_1 n^{-(\alpha+2)} + \cdots \end{aligned}$$

and define $\tilde{S}_n$ by

$$\tilde{S}_n(s,t) = (s', t'),$$

where (s,t) and (s',t') are combined by the relation $\tilde{T}(y_n^0 + s, \eta_n^0 + t) = (\tilde{y}_{n+1}^0 + s', \tilde{\eta}_{n+1}^0 + t')$. We set similarly

$$\tilde{S}_j^n = \tilde{S}_n \circ \tilde{S}_{n-1} \circ \cdots \circ \tilde{S}_j.$$

Now by using Lemmas 2.2 and 2.3 we have

Proposition 2.4. *Let $j_{0,l}$ $(l = 1, 2)$ be fixed. Then there are functions $k_l(s)$, $(l = 1, 2)$ with asymptotic expansion for $s \to 0$*

$$k_l(s) \sim c_{l,0} s^{\alpha+2} + c_{l,1} s^{\alpha+3} + \cdots$$

satisfying

$$S_{j_{0,1}}^n(0, k_1(\frac{1}{n})) \sim \tilde{S}_{j_{0,2}}^n(0, k_2(\frac{1}{n})) \quad \text{for } n \to \infty.$$

§3. Construction of asymptotic solutions

From now on we shall use again the notation $x = (x_1, x_2)$ to denote a point of $\mathbb{R}^2$. Let us construct an asymptotic solution of (1.2) for an oscillatory data. Since the curvature of the boundary Γ_l is positive except at A_l, the behavior of asymptotic solutions going out from $\Omega(\delta)$ is same as in the case that the bodies are strictly convex. Therefore it is essential to consider asymptotic solutions in $\Omega(\delta)$ for oscillatory data given on $S_1(\delta)$. Let $\omega \in S^1 = \{\omega \in \mathbb{R}^2; |\omega| = 1\}$, and let $f(x) \in C_0^\infty(S_1(\delta))$, and set

$$g(x, \mu) = e^{-i\mu x \cdot \omega} f(x). \tag{3.1}$$

We shall use a standard method for construction, but as remarked in Section 2 it is crucial to know the behavior of the phase functions when the number of reflections increases to the infinity.

With aid of Proposition 2.4 we have the following

Proposition 3.1. *Let ω be an element of S^1 near $(0,1)$, and set*

$$\varphi_1(x) = x \cdot \omega.$$

For any positive integer N, there is a sequence of real valued smooth functions defined in a neighborhood of $\Omega(\delta)$ with the following expansions in $n^{-\alpha}$:

$$\frac{\partial \varphi_n}{\partial x_2}(x) = b_0(x)n^{-1-\alpha} + b_1(x)n^{-1-2\alpha} + \cdots + b_M(x)n^{-1-(M+1)\alpha},$$

$$\varphi_{2n}(x) = c_0(x) + 2nd + c_1(x)n^{-1-2\alpha} + \cdots + c_M(x)n^{-1-(M+1)\alpha},$$

$$\varphi_{2n+1}(x) = \tilde{c}_0(x) + (2n+1)d + \tilde{c}_1(x)n^{-1-2\alpha} + \tilde{c}_2(x)n^{-1-3\alpha} + \cdots + \tilde{c}_M(x)n^{-1-(M+1)\alpha},$$

where M is a positive integer and $b_j(x)$, $c_j(x)$, $\tilde{c}_j(x)$, $j = 1, 2, \cdots, M$, are smooth functions.

Moreover, $\varphi_j(x)$, $j = 1, 2, \cdots$, satisfy the eikonal equation

$$|\nabla \varphi_j(x)| = 1 \quad \text{in } \Omega(\delta)$$

and the difference $\varphi_{j+1} - \varphi_j$ on the boundary satisfies

$$(\varphi_{2n} - \varphi_{2n-1})(x) = e_0(x) + e_{N-1}(x)n^{-1-N\alpha} + e_N(x)n^{-1-(N+1)\alpha} + \cdots + e_M(x)n^{-1-(M+1)\alpha} \quad \text{for all } x \in S_1(\delta),$$

$$(\varphi_{2n+1} - \varphi_{2n})(x) = \tilde{e}_0(x) + \tilde{e}_{N-1}(x)n^{-1-N\alpha} + \tilde{e}_N(x)n^{-1-(N+1)\alpha} + \cdots + \tilde{e}_M(x)n^{-1-(M+1)\alpha} \quad \text{for all } x \in S_2(\delta),$$

where $e_0(x)$ and $\tilde{e}_0(x)$ satisfy the following estimate

$$|e_0(x)|, \ |\tilde{e}_0(x)| \leq C_N |x_1|^N.$$

Now we construct a sequence of asymptotic solutions by using the sequence $\{\varphi_j\}_{j=1}^{\infty}$ of phase functions in the above proposition. First let

$\mu = k + i\sigma$ with $\sigma < 0$ and set

$$u_j(x,\mu) = \exp(-i\mu\varphi_j(x))\, v_j(x,\mu),$$
$$v_j(x,\mu) = \sum_{p=0}^{P} v_{jp}(x)(i\mu)^{-p},$$

and we shall construct v_{jp} successively by the following procedure:

Set
$$T_j = 2\nabla\varphi_j \cdot \nabla + \triangle\varphi_j.$$

Let $v_{00}(x)$ be solution of

$$\begin{cases} T_0\, v_{00} = 0 & \text{in } \Omega(\delta), \\ v_{00}(x) = f(x) & \text{on } S_1(\delta) \end{cases}$$

and $v_{0p}(x),\ p = 1, 2, \cdots, P$ be the successive solutions of

$$\begin{cases} T_0\, v_{0p} = -\triangle\, v_{0,p-1} & \text{in } \Omega(\delta), \\ v_{0p}(x) = 0 & \text{on } S_1(\delta). \end{cases}$$

Let $j \geq 1$ and suppose that $v_{j-1,p}(x)$ are defined. Define v_{jp} as the solutions of

$$\begin{cases} T_j\, v_{jp} = \triangle\, v_{j,p-1} & \text{in } \Omega(\delta), \\ v_{jp}(x) = v_{j-1,p} & \text{on } S_{\epsilon(j)}(\delta) \end{cases}$$

where we take $v_{j,-1} \equiv 0$ and

$$\epsilon(j) = \begin{cases} 1 & \text{for } j \text{ even}, \\ 2 & \text{for } j \text{ odd}. \end{cases}$$

About the asymptotic behavior of v_{np} for $n \to \infty$, we have the following lemma which is a direct consequence of the properties of $\varphi_n(x)$ in Proposition 3.1.

Lemma 3.2. *For each p fixed, we get the following asymptotic expansion of $v_{np}(x)$ in $n^{-\alpha}$:*

$$v_{2n,p}(x) \sim w_{p0}(x)n^p + w_{p1}(x)n^{p-\alpha} + w_{p2}(x)n^{p-2\alpha} + \cdots + w_{pK}(x)n^{p-K\alpha},$$

and

$$v_{2n+1,p}(x) \sim \tilde{w}_{p0}(x)n^p + \tilde{w}_{p1}(x)n^{p-\alpha} + \tilde{w}_{p2}(x)n^{p-2\alpha} + \cdots + \tilde{w}_{pK}(x)n^{p-K\alpha},$$

where $w_{pj}(x)$ and $\tilde{w}_{pj}(x)$ are smooth.

Now define $u(x,\mu)$ for $\operatorname{Im}\mu=\sigma<0$ by

$$u(x,\mu)=\sum_{n=0}^{\infty}(-1)^n u_n(x,\mu). \tag{3.2}$$

It is evident that $u(x,\mu)$ converges absolutely, and we see from the construction of u_j that the following relations hold:

$$(\triangle+\mu^2)\,u(x,\mu)=(i\mu)^{-P}\sum_{n=0}^{\infty}\exp(-i\mu\varphi_n(x))\,\triangle v_{nP}(x), \tag{3.3}$$

$$\begin{aligned}&[u(x,\mu)-\exp(-i\mu\varphi_0(x))\,f(x)]_{S_1(\delta)}\\ &=\sum_{n=1}^{\infty}\{\exp(-i\mu\varphi_{2n}(x))-\exp(-i\mu\varphi_{2n-1}(x))\}\,v_{2n}(x,\mu)\end{aligned} \tag{3.4}$$

and

(3.5)

$$u(x,\mu)|_{S_2(\delta)}=\sum_{n=0}^{\infty}\{\exp(-i\mu\varphi_{2n+1}(x))-\exp(-i\mu\varphi_{2n}(x))\}\,v_{2n+1}(x,\mu).$$

Let η and ε_0 be an arbtrary positive constant. With the aid of Lemma 3.2 we have from (3.3)

$$|(\triangle+\mu^2)\,u(x,\mu)|\le C_{N,\eta,\varepsilon_0}\,|\mu|^{-P} \qquad \text{for all } \operatorname{Im}\mu\le-\varepsilon_0,\quad x\in\Omega(\delta). \tag{3.6}$$

Similarly we have from (3.4)

$$|u(x,\mu)-g(x,\mu)|\le C_{N,\eta,\varepsilon_0}\,|\mu|^{-\eta N} \qquad \text{for all } x\in S_1(|\mu|^{-\eta}) \text{ and } \operatorname{Im}\mu\le-\varepsilon_0 \tag{3.7}$$

and from (3.5)

$$|u(x,\mu)|\le C_{N,\eta,\varepsilon_0}\,|\mu|^{-\eta N} \qquad \text{for all } x\in S_2(|\mu|^{-\eta}) \text{ and } \operatorname{Im}\mu\le-\varepsilon_0. \tag{3.8}$$

Now, note that for any broken ray starting from a point in $\Omega(\delta)$ and for any $a>0$ it holds the either of the following two cases:

(i) the broken ray fits $S_1(a)$ within $[a^{-2(m-1)}]$-times reflections.

(ii) the broken ray goes out from $\Omega(\delta)$ within $[a^{-2(m-1)}]$-times reflections.

Then, by using the techniques in Ikawa [4] and that of Vainberg [10] jointly, we can easily construct $\tilde{u}(x,\mu)$ by an explicit procedure from $u(x,\mu)$ satisfying the following estimates, which show that $u(x,\mu)$ an good approximate solution to (1.2) for an oscillatry data $g(x,\mu)$ defined (3.1):

For any $N > 0$ and $\varepsilon_0 > 0$ we have for all $\operatorname{Im}\mu \leq -\varepsilon_0$

(i) $\quad \tilde{u}(\cdot,\mu)$ is $C^\infty(\overline{\Omega})$-valued holomorphic function,

(ii) $\quad (\triangle + \mu^2)\tilde{u}(x,\mu) = 0 \quad$ in Ω,

(iii) $\quad |\tilde{u}(x,\mu) - g| \leq C_{N,\varepsilon_0}|\mu|^{-N} \quad$ for all $x \in \Gamma_1$,

(iv) $\quad |\tilde{u}(x,\mu)| \leq C_{N,\varepsilon_0}|\mu|^{-N} \quad$ for all $x \in \Gamma_2$.

When we want to extend the above results beyond the real axis, there is a difficulty that the convergence of $u_n(x,\mu)$ is not exponential with respect to $n \to \infty$. But the summation (3.2) is of a similar form to the zeta functions. Thus, we shall use the technique of analytic continuation of the zeta functions. We shall consider in the next section the analytic continuation and estimates of the zeta function so that we may use it for the analytic extension of $u(x,\mu)$ beyond the real axis.

§4. Analytic continuation of the zeta function and its generalization

In order to consider the analytic continuation of $u(x,\mu)$ defined by (3.2), we express $u(x,\mu)$ as a sum of zeta functions.

Even though the analytic continuation of the zeta function is well known (see for example Veech [11]), we shall give a proof because the one used here is modified a little and we need estimates of the dependency of the functions on parameters.

In this section, several notations will be used in different meanings from the ones in the previous sections, except α.

Let m be a positive integer and let z and s be complex numbers. For $|z| < 1$ we define the function $F(z,s:m)$ by

$$F(z,s:m) = \sum_{n \geq m} z^n n^{-s}. \tag{4.1}$$

Obviously, the right hand side of (4.1) converges absolutely for $|z| < 1$, which implies that the function $F(z,s:m)$ is holomorphic in $\{z; |z| < 1\}$ for any $s \in \mathbb{C}$.

We consider the analytic continuation of F. First assume $\operatorname{Re} s > 0$, and set

$$I(z, s : m) = \int_0^\infty \frac{z^m e^{-mx} x^{s-1}}{1 - ze^{-x}} dx.$$

We see that, for each $\operatorname{Re} s > 0$, $I(z, s : m)$ is holomorphic in $z \in D = \mathbb{C} - [1, \infty)$. As it is well known, $F(z, s : m)$ has the following integral representation:

$$F(z, s : m) = \frac{1}{\Gamma(s)} I(z, s : m) \qquad \text{for } |z| < 1. \tag{4.2}$$

On the other hand, the definition (4.1) gives us

$$z \frac{\partial F}{\partial z}(z, s : m) = F(z, s - 1 : m) \qquad \text{for all } \ |z| < 1.$$

Let a be a positive integer. Then we have for $\operatorname{Re} s > 0$ and $|z| < 1$ the expression

$$F(z, s - a : m) = \frac{1}{\Gamma(s)} \left(z \frac{\partial}{\partial z} \right)^a I(z, s : m). \tag{4.3}$$

By means of the above integral representation we shall show the following lemma:

Lemma 4.1. *For any $s \in \mathbb{C}$ and m positive integer, $F(z, s : m)$ as a function in z variable can be continued holomorphically into the domain $D = \mathbb{C} - [1, \infty)$. Moreover, we have the following estimate:*

$$|F(z, s : m)| \leq C_{K,a} \frac{\Gamma(\operatorname{Re} s + a)}{|\Gamma(s + a)|} m^{-\operatorname{Re} s} |z|^m (1 + |z|)^a \tag{4.4}$$
$$\text{for all } \operatorname{Re} s > -a \text{ and } z \in K,$$

where K is an arbitrary compact subset of D, a is an arbitrary positive integer and $C_{K,a}$ is a constant independent of m.

Proof. By using the fact that $I(z, s : m)$ is holomorphic in $z \in D$ for any $\operatorname{Re} s > 0$, the expression (4.3) proves Lemma 4.1 except the estimate (4.4). It is easy to show by the induction that

$$\left(z \frac{\partial}{\partial z} \right)^a \frac{z^m}{1 - ze^{-x}} = \frac{m^a z^m}{(1 - ze^{-x})^{a+1}} \{1 + c_{a,1}(m) ze^{-x} +$$
$$+ c_{a,2}(m)(ze^{-x})^2 + \cdots + c_{a,a}(m)(ze^{-x})^a\},$$

where the coefficients $c_{a,l}(m)$, $l = 1, 2, \dots, a$ are polynomials of m^{-1} of order less than a, and they satisfy

$$|c_{a,l}(m)| \le C_a \qquad \text{for all } m.$$

Thus, if we set

$$\max_{\substack{x \ge 0 \\ z \in K}} |1 - ze^{-x}| = c_K,$$

we have for all $\operatorname{Re} s > 0$

$$\begin{aligned}
&\left| \left(z \frac{\partial}{\partial z} \right)^a I(z, s : m) \right| \\
\le & m^a |z|^m (c_K)^{-(a+1)} C_a (1 + |z|)^a \int_0^\infty e^{-mx} |x^{s-1}| \, dx \\
\le & m^a |z|^m (c_K)^{-(a+1)} C_a (1 + |z|)^a m^{-\operatorname{Re} s} \Gamma(\operatorname{Re} s).
\end{aligned}$$

Substituting this estimate into (4.3) we get immediately for all $\operatorname{Re} s > 0$

$$|F(z, s - a : m)| \le (c_K)^{-(a+1)} C_a \frac{\Gamma(\operatorname{Re} s)}{|\Gamma(s)|} m^{a - \operatorname{Re} s} |z|^m (1 + |z|)^a.$$

Denoting $s - a$ in the above inequality by s anew, we get (4.4). Q.E.D.

In order to consider the analytic continuation of $u(x, \mu)$ beyond the real axis, we have to consider the analytic continuation of the following function originally defined for $\operatorname{Im} \mu < 0$:

$$\begin{aligned}
\text{(4.5)} \qquad R_\beta(\mu : q) = \sum_{n \ge |k|^\beta} \exp\big(& -i\mu(n + c_0 n^{-1-2\alpha} \\
& + c_1 n^{-1-3\alpha} + \cdots + c_M n^{-1-(M+2)\alpha})\big) \, n^q.
\end{aligned}$$

Let us set

$$D_{r,\beta,\varepsilon} = \{\mu = ik + \sigma; 2r\pi + \varepsilon \le |k| \le 2(r+1)\pi - \varepsilon, \ \sigma \le r^{-\beta}\}.$$

For $\sigma < 0$, as remarked in the above, the right hand side converges absolutely. Now consider the holomorphic extension of $R_\beta(\mu : q)$ into $\sigma > 0$.

Lemma 4.2. *Let $\beta > (1 + 2\alpha)^{-1}$ and let $\varepsilon > 0$. For any positive integer r, $R_\beta(\mu : q)$ can be prolonged analytically into $D_{r,\beta,\varepsilon}$. Moreover, we have the following estimates:*

$$\text{(4.6)} \qquad |R_\beta(\mu : q)| \le C_{\beta,\varepsilon} \, r^{q\beta} \qquad \textit{for all } \mu \in D_{r,\beta,\varepsilon}$$

and

$$\left|R_\beta(\mu:q)-F(e^{-i\mu},-q:[r^\beta])\right| \le C_{\beta,\varepsilon}\, c_1\, r^{q\beta-\gamma} \quad \text{for all } \mu\in D_{r,\beta,\varepsilon},$$

where $\gamma=(1+\alpha)\beta-1>0$.

Proof. First suppose that $c_j=0$ for all $j\ge 1$. For each $n\ge 0$ we have

$$\exp\bigl(-i\mu(n+c_0n^{-1-2\alpha})\bigr)\,n^q = z^n\sum_{l=0}^{\infty}\frac{(-i\mu)^l}{l!}\,c_0{}^l\,n^{-(1+2\alpha)l}\,n^q, \tag{4.8}$$

where we set $z=\exp(-i\mu)$. Suppose that $\mu\in D_{r,\beta,\varepsilon}$ $(r>0)$ and set $m=[r^\beta]$. Note that

$$\sum_{n\ge m} z^n\,n^{-(1+2\alpha)l}\,n^q = F(z,(1+2\alpha)l-q:m).$$

Let $|z|<1$ and take the summation in $n\ge m$ of the both sides of (4.8). Since the both summations converges absolutely we have a relation

$$R_\beta(\mu:q)=\sum_{l=0}^{\infty}\frac{(-i\mu)^l}{l!}\,c_0{}^l\,F(z,(1+2\alpha)l-q:m),$$

which implies

$$R_\beta(\mu:q)-F(e^{-i\mu},-q:m) = \sum_{l=1}^{\infty}\frac{(-i\mu)^l}{l!}\,c_0{}^l\,F(e^{-i\mu},(1+2\alpha)l-q:m). \tag{4.9}$$

We see easily that $\{z=\exp(-i\mu);\ \mu\in D_{r,\beta,\varepsilon}\}$ is contained in a compact subset K of $D=\mathbb{C}-[1,\infty)$ for all r. Then by Lemma 4.1, each term of the right hand side of (4.9) can be extended holomorphically into $D_{r,\beta,\varepsilon}$. Therefore, if we show that the right hand side of (4.9) converges absolutely in $D_{r,\beta,\varepsilon}$, it follows that $R_\beta(\mu:q)$ can be extended analytically into $D_{r,\beta,\varepsilon}$.

Thus, by applying the previous lemma we have for all $\mu\in D_{r,\beta,\varepsilon}$

$$\left|\frac{(-i\mu)^l}{l!}\,c_0{}^l\,F(z,(1+2\alpha)l-q,m)\right| \le C_{K,q}|z|^m(1+|z|)^q\frac{|k|^l}{l!}\,|c_0|^l\,m^{-(1+2\alpha)l+q}.$$

Here we applied Lemma 4.1 by taking $s = (1+2\alpha)l - q$, and used the fact that $\Gamma(\operatorname{Re} s + a) = |\Gamma(s+a)|$. Note that

$$|z^m| = |e^{-ikm+m\sigma}| = e^{m\sigma} \le Ce^{r^{-\beta}\cdot r^{\beta}} = C,$$
$$m^{-(1+2\alpha)l}\,|k|^l \le (|k|^{-(1+2\alpha)\beta+1})^l = C|k|^{-\gamma l}$$

where we set $\gamma = (1+2\alpha)\beta - 1 > 0$, and

$$|z| \le \exp(r^{-\beta}) \le C \quad \text{for all} \quad z = e^{-i\mu},\ \mu \in D_{r,\beta,\varepsilon}.$$

Then we have

$$\begin{aligned}&|R_\beta(\mu) - F(e^{-i\mu}, -q : m)|\\ \le& C_{K,q}\,|k|^{q\beta} \sum_{l=1}^{\infty} \frac{1}{l!}(c_0 k^{-\gamma})^l \le C_{K,q}\, c_0 |k|^{q\beta-\gamma}.\end{aligned}$$

Thus the desired properties of $R_\beta(\mu : q)$ are proved for the special case.

Next consider the general case, that is, the case that c_j, $(j \ge 1)$ are not necessarily zero. We introduce some notations. Set

$$l = (l_0, l_1, \cdots, l_M) \in \{0, 1, \cdots\}^{M+1},$$
$$c = (c_0, c_1, \cdots, c_M) \quad \text{and} \quad A = (0, 1, \cdots, M),$$

and denote as

$$|l| = l_0 + l_1 + \cdots + l_M, \quad A\cdot l = l_1 + 2l_2 + \cdots + M\,l_M,$$
$$c^l = \prod_{j=0}^{M} c_j^{l_j}, \quad l! = \prod_{j=0}^{M} l_j!.$$

Now we have the following expansion

$$\begin{aligned}&\exp\Big(-\mu(n + c_0 n^{-1-2\alpha} + \cdots + c_M n^{-1-(2+M)\alpha})\Big)\, n^q\\ &\quad = z^n \sum \frac{(-i\mu)^{|l|}}{l!}\, c^l\, n^{-1-(2|l|+A\cdot l)\alpha}\, n^q\end{aligned}$$

Thus, by replacing the expansion (4.8) by the above one, we can achieve the same argument as the special case, and get Lemma 4.2. Q.E.D.

§5. Proof of Theorem 1

First we shall show that the function $u(x,\mu)$ defined by (3.2) can be extended analytically into the domain

$$D_{\beta,\varepsilon} = \cup_{|r|\geq C_{\beta,\varepsilon}} D_{r,\beta,\varepsilon}$$

where $C_{\beta,\varepsilon}$ is a positive integer depending on β and ε. Secondly, we shall show that $u(x,\mu)$ is a good approximation of the solution of (1.2) for all $\mu \in D_{\beta,\varepsilon}$

Let $\mu \in D_{r,\beta,\varepsilon}$ and set $m_r = |r|^{\beta}$. We express the function $u(x,\mu)$ defined by (3.2) as follows:

$$\begin{aligned}(5.1)\qquad u(x,\mu) =& \sum_{n=0}^{m_r} u_{2n}(x,\mu) + \sum_{n>m_r} u_{2n}(x,\mu) \\ & - \sum_{n=0}^{m_r} u_{2n+1}(x,\mu) - \sum_{n>m_r} u_{2n+1}(x,\mu) \\ =& u_{\mathrm{e}}^{(1)}(x,\mu) + u_{\mathrm{e}}^{(2)}(x,\mu) - u_{\mathrm{o}}^{(1)}(x,\mu) - u_{\mathrm{o}}^{(2)}(x,\mu).\end{aligned}$$

Recall Proposition 3.1 and Lemma 3.2. Then we have

$$\begin{aligned}u_{\mathrm{e}}^{(2)}(x,\mu) =& \sum_{p=0}^{P} (-i\mu)^{-p} \sum_{n\geq m_r} \exp(-i\mu(c_0(x) + 2nd + c_1(x)n^{-1-2\alpha} \\ & + \cdots + c_M(x)n^{-1-(M+1)\alpha}))\,\{w_{p0}(x)n^p + \cdots + w_{pK}n^{p-K\alpha}\}.\end{aligned}$$

Thus, for each $x \in \Omega(\delta)$ fixed, $u_{\mathrm{e}}^{(2)}(x,\mu)$ can be expressed by a summation of following terms:

$$(-i\mu)^{-p} R_\beta(\mu : p - j\alpha),\ p = 0,\cdots,P,\ j = 0,\cdots,K,$$

from which we see that $u_{\mathrm{e}}^{(2)}(x,\mu)$ can be extended analytically into $D_{r,\beta,\varepsilon}$ beyond the real axis. Moreover, applying the estimate in Lemma 4.2 we have

$$(5.2)\qquad |u_{\mathrm{e}}^{(2)}(x,\mu)| \leq C_{N,\beta,\varepsilon} \sum_{p=0}^{P} |\mu|^{-p}\,|r|^{\beta p} \leq C'_{N,\beta,\varepsilon}$$

$$\text{for all } \mu \in D_{r,\beta,\varepsilon}$$

(Recall that the constants P, K, M are determined by N through Proposition 3.1). Similarly we see that $u_{\mathrm{o}}^{(2)}(x,\mu)$ also can be extended analytically into $D_{r,\beta,\varepsilon}$ and

$$(5.3)\qquad |u_{\mathrm{o}}^{(2)}(x,\mu)| \le C_{N,\beta,\varepsilon} \sum_{p=0}^{P} |\mu|^{-p}\,|r|^{\beta p} \le C'_{N,\beta,\varepsilon} \qquad \text{for all } \mu \in D_{r,\beta,\varepsilon}$$

holds.

Consider $u_{\mathrm{e}}^{(1)}(x,\mu)$. Since it is a finite sum of entire functions, it is also an entire function. But it is important to get an estimate for $\mu \in D_{r,\beta,\varepsilon}$. For all $n \le m_r$ we have

$$\operatorname{Re}\varphi_{2n} \le 2n\sigma \le m_r\,\sigma \le s_0,$$

where s_0 is independent of r. Therefore we have for all $\mu \in D_{r,\beta,\varepsilon}$

$$|u_{2n}(x,\mu)| \le e^{s_0} \sum_{p=0}^{P} m_r^p\,|\mu|^{-p} \le C_\beta,$$

from which it follows that

$$(5.4)\qquad |u_{\mathrm{e}}^{(1)}(x,\mu)| \le C_\beta\, m_r.$$

Evidently the same estimate holds for $u_{\mathrm{o}}^{(1)}(x,\mu)$. Thus we have the following

Lemma 5.1. *The function $u(x,\mu)$ defined by* (3.1) *can be extended analytically into $D_{\beta,\varepsilon}$ and the following estimate holds:*

$$(5.5)\qquad |u(x,\mu)| \le C_{N,\beta,\varepsilon}\,|\mu|^\beta \quad \textit{for all } x \in \Omega(\delta),\ \mu \in D_{\beta,\varepsilon}.$$

Next consider $(\triangle + \mu^2)u(x,\mu)$. By applying the above argument to the expression (3.3), we get easily

$$(5.6)\qquad |(\triangle + \mu^2)u(x,\mu)| \le C_{N,\beta,\varepsilon}\,|\mu|^{-P}\,|r|^{\beta P} \qquad \text{for all } x \in \Omega(\delta),\quad \mu \in D_{r,\beta,\varepsilon}.$$

For the estimate of boundary value, we can use the same argument as above.

An in Section 3, by using the techniques in [4] and that of [10] jointly, we can easily construct by an explicit procedure from $u(x,\mu)$ a function $\tilde{u}(x,\mu)$ with the following properties:

Proposition 5.2. *Let $N > 0$, $\varepsilon_0 > 0$ and $\beta > (1+2\alpha)^{-1}$ be fixed. For the oscillatory data $g(x,\mu)$ of the form (3.1) we can construct a function $\tilde{u}(x,\mu)$, which is $C^\infty(\overline{\Omega})$-valued holomorphic function in $D_{\beta,\varepsilon}$, satisfying for all $\mu \in D_{\beta,\varepsilon}$*

(i) $\quad (\triangle + \mu^2)\tilde{u}(x,\mu) = 0 \quad$ *in* Ω,

(ii) $\quad |\tilde{u}(x,\mu) - g| \le C_{N,\beta,\varepsilon_0} |\mu|^{-N} \quad$ *for all* $x \in \Gamma_1$,

(iii) $\quad |\tilde{u}(x,\mu)| \le C_{N,\beta,\varepsilon_0} |\mu|^{-N} \quad$ *for all* $x \in \Gamma_2$.

Theorem 1 in Introduction can be derived from the above proposition by a standard arugument.

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Blowing-up Behavior for Solutions of Nonlinear Elliptic Equations

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Dedicated to Prof. S.T. Kuroda on his 60th birthday

Abstract.

We consider the following nonlinear elliptic equations with real parameter λ:

$$(P_\lambda) \qquad \Delta u + f(u,\lambda) = 0, \quad u > 0 \text{ in } \Omega; \quad u = 0 \quad \text{on } \partial\Omega,$$

where Ω is a smooth bounded domain in R^n $(n \geq 2)$ and $f \geq 0$ satisfies an inequality:

$$f(u,\lambda) \leq c_1 + c_2 u^p$$

$$(\ c_1, c_2 > 0,\ p > 1 \text{ constants}).$$

We suppose the existence of a family of solutions $\{(u_s, \lambda_s)\}_{0<s\leq 1}$ of (P_λ) with the following properties: $(u_s, \lambda_s) \in C(\overline{\Omega}) \times R$ is continuous in s, λ_s is bounded, and $\max u_s \to \infty$ $(s \downarrow 0)$.
We investigate the asymptotic behavior of solutions near blowing-up points.

§1. Introduction

In this paper we consider the following nonlinear elliptic equations with real parameter λ:

$$(P_\lambda) \qquad \begin{cases} \Delta u + f(u,\lambda) = 0, \quad u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Here Ω is a smooth bounded domain in R^n $(n \geq 2)$ and a smooth function f satisfies the following inequality:

$$0 \leq f(u,\lambda) \leq c_1 + c_2 u^p \qquad (u \geq 0)$$

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where $c_1, c_2 > 0$, and $p > 1$ are constants. Recently many works have been done in the case that (P_λ) is Yamabe type problem, i.e., when $n \geq 3$ and f has (nearly) critical Sobolev exponents such as

$$\text{(i)} \qquad f = u^{\frac{n+2}{n-2}} + \lambda u,$$

$$\text{(ii)} \qquad f = u^{\frac{n+2}{n-2} - \lambda} \qquad (\lambda > 0).$$

See, e.g., [1, 3, 4, 6, 8, 10, and references therein]. We recall the results on the asymptotic behavior of solutions of (P_λ) when f is (i) or (ii). There are two types of results. The first one is on the behavior of solutions when Ω is a ball with center 0. In this case it is known that a family of solutions $\{(u_s, \lambda_s)\}_{s \in (0,1]}$ $\big(\subset C^2(\Omega) \times R\big)$ exists with the following properties:

(A1) (u_s, λ_s) $\big(\subset C(\overline{\Omega}) \times R\big)$ is continuous in s;

(A2) λ_s is bounded;

(A3) $\max u_s \to \infty$;

(A4) $u_s(0) \to \infty, \quad u_s(x) \to 0 \quad (x \in \Omega, x \neq 0)$ as $s \downarrow 0$.

(We call such a point as $x = 0$ a blowing-up point.) For more detailed behavior see [1, 3, 4, 10].

The second one is on the behavior of solutions of (P_λ) which satisfy a minimizing sequence property for the (Sobolev) inequality:

$$\frac{\int_\Omega | \nabla u_s |^2 \, dx}{\| u_s \|_{p+1}^2} \to S_n \qquad \text{as} \quad s \downarrow 0,$$

where $p = \frac{n+2}{n-2}$ or $p = \frac{n+2}{n-2} - \lambda$ respectively, and S_n is the best Sobolev constant in R^n. Under appropriate assumptions it is proved that a blowing-up point is unique and that (A3) and similar behavior to (A4) hold ([3, 4, 6, 8, 10]).

We would like to investigate the asymptotic behavior in a neighborhood of a blowing-up point for more general domains and for more general functions.

Throughout the paper we assume that there exists a family of solutions $\{(u_s, \lambda_s)\}_{0<s\leq 1}$ of (P_λ) with the properties (A1)–(A3).

Before proceeding to state our result, we give the definition of blowing up points. From our assumption it follows that there exist a family of points $\{x_j\}\,(\subset \Omega)$, a point $x_0 \in \overline{\Omega}$, $s_j \in (0, 1]$, and λ_0 such that $x_j \to x_0, \lambda_{s_j} \to \lambda_0, u_{s_j}(x_j) \to \infty$ as $j \uparrow \infty$. We call (x_0, λ_0) or simply x_0 a *blowing-up point* with respect to $\{(u_{s_j}, \lambda_{s_j})\}_{j=1}^\infty$.

Our result is

Theorem. *Under above hypotheses the following statement holds. For each blowing-up point $x_0 \in \Omega$ there exists $r_0 > 0$ such that for each fixed r $(0 < r \leq r_0)$ there exists s $(0 < s < 1)$ such that*

$$k_1 r^{-2/(p-1)} \leq u_s(x) \leq k_2 r^{-2/(p-1)}$$

$$(| x - x_0 | \leq r).$$

Here $k_1, k_2 > 0$ are constants depending only on Ω,c_1,c_2, and p.

As a direct consequence of Theorem we have

Corollary. *Let $n \geq 3$ and let $p = \frac{n+2}{n-2}$. Then for each blowing-up point $x_0 \in \Omega$ there exists $r_0 > 0$ such that for each fixed r $(0 < r \leq r_0)$ there exists s $(0 < s < 1)$ such that*

$$\int_{|x-x_0|\leq r} u_s(x)^{\frac{2n}{n-2}}\, dx \geq k_3.$$

Here $k_3 > 0$ is a constant depending only on Ω,c_1,c_2, and p.

In Section 2 we give the proof of Theorem in the case $n = 2$. In Section 4 we sketch the proof of it in the case $n \geq 3$.

§2. Proof of Theorem $(n = 2)$

In this section we prove Theorem in the case $n = 2$. For the proof of it we need the following two lemmas.

Lemma 1. *Let $n = 2$. Suppose that a family of functions $\{v_s\}_{0<s\leq 1} \subset C^2(\Omega) \cap C(\overline{\Omega})$ satisfies the following hypotheses :*
(i) *v_s satisfies the following differential inequality*

$$\Delta v_s + k e^{v_s} \geq 0 \qquad in \quad \Omega$$

where $k > 0$ is a constant.
(ii) *$v_s \in \left(C(\overline{\Omega})\right)$ is continuous in s .*
Let $r > 0$ be such that $B(x_0, r) \equiv \{x :| x - x_0 |\leq r\} \subset \Omega$, and

$$\int_{B(x_0,r)} e^{v_1(x)} dx < \frac{4\pi}{k}.$$

Assume that for some $0 < s_1 < 1$ the following inequality holds for all $s_1 \leq s \leq 1$,

$$[e^{v_s}]_r \equiv \left\{ \frac{1}{2\pi} \int_0^{2\pi} e^{v_s(x_0+x(\theta))/2}\, d\theta \right\}^2$$
$$< \frac{2}{kr^2}, \qquad x(\theta) \equiv r(\cos\theta, \sin\theta).$$

Then for all $s_1 \le s \le 1$

(i) $$e^{v_s(x_0)} < 4[e^{v_s}]_r,$$

(ii) $$\int_{B(x_0,r)} e^{v_s(x)} dx < \frac{4\pi}{k}.$$

Lemma 2. *Let* $n = 2$. *Let* $x_0 \in \Omega$ *be a blowing-up point. Let* r *be such that* $B(x_0, r) \subset \Omega$. *Suppose that for some* $0 < s_1 < 1$ *a solution* u_s *of* (P_λ) *with* $\lambda = \lambda_s$ *satisfies*

$$u_s(x) < \left(\frac{2}{k}\right)^{1/(p-1)} \mid x - x_0 \mid^{-2/(p-1)} \qquad (x \in B(x_0, r))$$

for all $s \in [s_1, 1]$. *Then* $v_s \equiv (p-1)\ln u_s$ *satisfies a differential inequality:*

$$\Delta v_s + ke^{v_s} \ge 0 \qquad (x \in B(x_0, r))$$

for all $s \in [s_1, 1]$, *where* k *is a constant independent of* x_0,r,s_1.

For the proof of Lemma 1 see [7; Proposition] or [2]. In Section 3 we prove Lemma 2.

Proof of Theorem. We set $v_s \equiv (p-1)\ln u_s$. Let $k > 0$ be a constant as in Lemma 2. Let r_0 be so small that $B(x_0, r_0) \subset \Omega$,

(1) $$\int_{B(x_0,r_0)} e^{v_1(x)} dx < \frac{4\pi}{k},$$
$$e^{v_1(x)} < \frac{2}{k} \mid x - x_0 \mid^{-2} \qquad (x \in B(x_0, r_0)).$$

Let $0 < r \le r_0$ be fixed. Suppose that for some $s_1 > 0$, v_s satisfies

(2) $$e^{v_s(x)} \equiv u_s^{p-1}(x) < \frac{2}{k} \mid x - x_0 \mid^{-2} \qquad (x \in B(x_0, r))$$

for all $s \in [s_1, 1]$. Then by lemma 2, v_s satisfies

(3) $$\Delta v_s + ke^{v_s} \ge 0 \qquad (x \in B(x_0, r)).$$

Let $x_s \in B(x_0, r)$ be a maximal point of u_s in $B(x_0, r)$:

$$u_s(x_s) = \max_{B(x_0,r)} u_s(x).$$

Then by (2)

$$e^{v_s(x_s)} < \frac{2}{k} \mid x_s - x_0 \mid^{-2}.$$

We consider u_s a solution of the following linear elliptic equation

$$\Delta u_s + c_s(x) u_s = 0 \, ; \quad c_s(x) \equiv \frac{f(u_s(x), \lambda_s)}{u_s(x)}.$$

Since x_0 is a blowing-up point, we may assume that $u_s(x) \geq 1$ for $x \in B(x_0, r)$. Then $c_s(x)$ satisfies

$$c_s(x) \leq c_1 + c_2 u_s^{p-1}(x) \leq c_1 + \frac{2c_2}{k} \mid x_s - x_0 \mid^{-2}.$$

Hence by Harnack's theorem there is a constant c' such that

$$u_s(x_s) \leq c' \min u_s(x) \tag{4}$$

for all x with $\mid x - x_0 \mid \leq \mid x_s - x_0 \mid$. Here c' depends only on p,c_1,c_2. On the other hand, since (1),(2), and (3) hold, we have by Lemma 1

$$u_s^{p-1}(x_0) \equiv e^{v_s(x_0)} < 4[e^{v_s(x)}]_r \qquad s \in [s_1, 1].$$

Hence, by (2), (4)

$$\begin{aligned} u_s(x_s) &\leq c' u_s(x_0) \\ &\leq 2^{3/(p-1)} c' k^{-1/(p-1)} r^{-2/(p-1)}. \end{aligned}$$

Applying Harnack's theorem again we get an inequality:

$$u_s(x_s) \leq c \min_{B(x_0,r)} u_s(x). \tag{5}$$

Here c is a constant depending only on p,c_1,c_2. Since x_0 is a blowing-up point, this implies that (2) does not hold for all $s \in (0, 1]$.
Set

$$s_2 \equiv \inf\{s' : (2) \text{ holds for all} s \in [s', 1]\}.$$

Then $s_2 > 0$, and (2) does not hold for $s = s_2$, i.e., there exists $x' \in B(x_0, r)$ such that

$$u_{s_2}^{p-1}(x') \equiv e^{v_{s_2}(x')} = \frac{2}{k} \mid x' - x_0 \mid^{-2} \quad .$$

On the other hand, by Harnack's inequality (5) we have

$$c^{-1}u_{s_2}(x) \leq u_{s_2}(x') \leq cu_{s_2}(x) \qquad (x \in B(x_0, r))\,.$$

Hence we have

$$\begin{aligned}\frac{2}{k}r^{-2} &\leq u_{s_2}^{p-1}(x') \leq \max_{B(x_0,r)} u_{s_2}^{p-1}(x) \\ &\leq c^{p-1} \min_{B(x_0,r)} u_{s_2}^{p-1}(x) \leq \frac{2c^{p-1}}{k} r^{-2}.\end{aligned}$$

Thus we obtain

$$k_1 r^{-2/(p-1)} \leq u_{s_2}(x) \leq k_2 r^{-2/(p-1)},$$

$$k_1 \equiv c^{-1}\left(\frac{2}{k}\right)^{\frac{1}{p-1}}, \qquad k_2 \equiv c\left(\frac{2}{k}\right)^{\frac{1}{p-1}}.$$

Q.E.D.

§3. Proof of Lemma 2

Proof of Lemma 2. Since u_s is a solution of (P_λ) with $\lambda = \lambda_s$, $v_s(x)$ satisfies

$$\Delta v_s + \frac{1}{p-1} \mid \nabla v_s \mid^2 + (p-1)\frac{f(u_s, \lambda_s)}{u_s} = 0.$$

On the other hand, by our assumption on f

$$\frac{f(u, \lambda)}{u} \leq c_1 + c_2 u^{p-1} \qquad (u \geq 1).$$

Hence we get a differential inequality

$$\Delta v_s + \frac{1}{p-1} \mid \nabla v_s \mid^2 + (p-1)c_3 e^{v_s} \geq 0$$

$$(c_3 = c_1 + c_2).$$

Therefore if we can estimate the term $\mid \nabla v_s \mid^2$ by e^{v_s} , i.e.,

$$\mid \nabla v_s \mid^2 \leq c_4' e^{v_s} \quad \text{or} \quad \mid \nabla u_s \mid^2 \leq c_4 u_s^{(p+1)}\,,$$

then we get a differential inequality

$$\Delta v_s + k' e_s^v \geq 0$$

$$(k' = (p-1)(c_3 + c_4)).$$

In the following we estimate the term $|\nabla u_s|^2$ by u_s^{p+1}.
Set

$$M_s \equiv \max_{B(x_0,r)} u_s(x), \quad m_s \equiv \min_{B(x_0,r)} u_s(x),$$

and choose $K_1 > \frac{M_1}{m_1}$. Then by the continuity of u_s $\left(\subset C(\overline{\Omega})\right)$ in s, we have for some $s_2 > 0$

$$M_s \leq K_1 u_s(x) \qquad (x \in B(x_0, r)) \tag{6}$$

for $s_2 \leq s \leq 1$. On the other hand, by Sperb's lemma [9; Lemma 5.1]

$$P_s(x) \equiv \frac{|\nabla u_s(x)|^2}{2} + \int_0^{u_s(x)} f(t, \lambda_s)\, dt \qquad (x \in B(x_0, r))$$

attains its maximum at the point where $\nabla u_s = 0$ or on $\partial B(x_0, r)$. Since x_0 is a blowing-up point, we may assume that P_s attains its maximum where $\nabla u_s(x) = 0$. Hence we have

$$|\nabla u_s|^2 \leq 2\left(c_1 + \frac{c_2}{p+1}\right) M_s^{p+1} \qquad (x \in B(x_0, r)) \tag{7}$$

for $s_2 \leq s \leq 1$. By (6) and (7)

$$\frac{|\nabla u_s|^2}{u_s^2} \leq 2K_1^{p+1}\left(c_1 + \frac{c_2}{p+1}\right) u_s^{p-1}.$$

Therefore we get a differential inequality

$$\Delta v_s + K_2 e^{v_s} \geq 0 \quad ; \qquad K_2 \equiv \left(2K_1^{p+1}\left(c_1 + \frac{c_2}{p+1}\right) + c_3\right)(p-1).$$

We may assume that

$$K_2 \geq 1, \qquad K_2 > k,$$

where k is the constant determined by (11) which is independent of x_0,r,s_2. Note that K_2 depends on x_0,r,s_2 . In the following we improve the above differential inequality and obtain:

$$\Delta v_s + k e^{v_s} \geq 0 \qquad (x \in B(x_0, r)).$$

If necessary, by choosing $r > 0$ sufficiently small we may assume that

$$e^{v_1(x)} < \frac{2}{K_2} r^{-2} \qquad (|x - x_0| \leq r), \tag{8}$$

$$K_2 \int_{B(x_0,r)} e^{v_1(x)}\, dx < 4\pi.$$

By the continuity of v_s in s, it follows that for some $s' > 0$, (8) holds for all $s' \le s \le 1$. Hence by Lemma 1 we have

$$e^{v_s(x_0)} \le 4[e^{v_s(x)}]_r < 8r^{-2} \tag{9}$$

for all $s' \le s \le 1$. On the other hand, by Harnack's theorem there exists a constant c' such that

$$\max_{|x-x_0|\le r} u_s(x) \le c' u_s(x_0).$$

Hence by (9) we have

$$u_s(x_s) \le 2^{3/(p-1)} c' r^{-2/(p-1)}.$$

Applying Harnack's theorem again we get

$$u_s(x_s) \le c u_s(x) \qquad (x \in B(x_0, r))$$

for all $s' \le s \le 1$. Here c is a constant depending only on p,c_1,c_2. Then repeating the above arguments we get a differential inequality

$$\Delta v_s + k e^{v_s} > 0 \qquad (x \in B(x_0, r)), \tag{10}$$

$$k \equiv \left(2c^{p+1}\left(c_1 + \frac{c_2}{p+1}\right) + c_3\right)(p-1). \tag{11}$$

Since $k < K_2$, from the continuity of $u_s(x)$ in s it follows that there exists s'' such that for all $s'' \le s \le 1$

$$u_s(x)^{p-1} \equiv e^{v_s(x)} < \frac{2}{k} r^{-2} \qquad x \in B(x_0, r), \tag{12}$$

$$\int_{B(x_0,r)} e^{v_s(x)} dx < \frac{4\pi}{k}. \tag{13}$$

Set

$$s^* \equiv \inf\{s'' : (10),(12) \text{ hold for} s'' \le s \le 1\}$$

Suppose that $s_1 < s^*$. Then repeating the above argument we conclude that a differential inequality (10) holds for all $s \in [s^*, 1]$. This contradicts the definition of s^*. Thus we have $s^* = s_1$. Q.E.D.

§4. Proof of Theorem $n \geq 3$

In this section we sketch the proof of Theorem when $n \geq 3$. We may assume that $0 \notin \Omega$ and introduce spherical coordinates:

$$x = r\omega \quad \left(r = \mid x \mid, \quad \omega \in S^{n-1}\right).$$

Let $x_0 \in \Omega$ be a blowing-up point. Let $r_0 > 0$ be such that $B(x_0, r_0) \subset \Omega$.
Suppose that

$$u_s(x) \leq \mid x - x_0 \mid^{-2/(p-1)} \qquad (x \in B(x_0, r_0)).$$

Then we have

$$\mid u_s \mid_{C^2(B(x_0,r_0))} \leq c'(c_1 + c_2 M_s^p), \qquad M_s \equiv \max_{B(x_0,r_0)} u_s(x).$$

On the other hand, by Sperb's lemma [9; Lemma 5.2] we get

$$\mid \nabla u_s \mid^2 \leq 2\left(c_1 M_s + \frac{c_2}{p+1} M_s^{p+1}\right).$$

Hence $v_s \equiv (p-1)\ln u_s$ satisfies a differential inequality

$$(v_s)_{rr} + \frac{(v_s)_r}{r} + c\frac{M_s^{p+1}}{u_s^2} \geq 0,$$

where c is a constant depending only on Ω,c_1,c_2, and p. We consider $v_s(r\omega)$ a function $w_{s,\omega}(y)$ defined in R^2 near $\mid y \mid = \mid x_0 \mid$:

$$w_{s,\omega}(y) \equiv v_s(r\omega), \qquad \mid y \mid = r, \quad y \in R^2.$$

Now we have a two-parameter family of functions $\{w_{s.\omega}\}_{s,\omega}$. Repeating similar arguments as in Sections 2 and 3 we can conclude the assertion in Theorem. Q.E.D.

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Mapping Properties of Functions of Schrödinger Operators between L^p-Spaces and Besov Spaces

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Abstract.

Sufficient conditions are given for the boundedness of $f(H)$, $H = -\triangle + V$, in $L^p(\mathbf{R}^d)$, $1 \le p \le \infty$. Optimal results with respect to the decay of f are obtained for L^p-boundedness of $e^{-itH}f(H)$ and the nearly-optimal norm-estimate $\left\|e^{-itH}f(H)\right\|_{\mathcal{B}(L^p)} \le C(1+|t|)^\gamma$, $t \in \mathbf{R}$, $\gamma > d|1/2 - 1/p|$ is proved. Results are also obtained on the mapping properties of e^{-itH} between certain Besov spaces.

§1. Introduction

In this paper we consider mapping properties of functions $f(H)$ of a Schrödinger operator $H = -\triangle + V$ between L^p-spaces. Let f be a bounded Borel function on $\mathbf{R}$. Then $f(H)$ is defined using the functional calculus and is a bounded operator on $L^2(\mathbf{R}^d)$. For $1 \le p < \infty$ the operator $f(H)$ is densely defined on $L^p(\mathbf{R}^d)$ and one may ask whether it can be extended to a bounded operator on $L^p(\mathbf{R}^d)$. Results for $p = \infty$ are obtained from those for $p = 1$ via duality. If $H = H_0 = -\triangle$, then $f(H_0)$ is a Fourier multiplier, and conditions for L^p-boundedness are well-known. One of the goals of this paper is to extend to $f(H)$ results from the theory of Fourier multipliers.

The results in this paper extend and complement the results obtained in [JN]. The main new ingredient here is a scaling result. We also

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obtain several results on mapping properties between Besov spaces. To state the results, we need some definitions. Our main assumption on the potential V is the following:

Assumption A. V is real-valued function on $\mathbf{R}^d$, and it is decomposed as $V(x) = V_+(x) - V_-(x)$ such that $V_\pm \geq 0$, $V_+ \in K_d^{\mathrm{loc}}$ and $V_- \in K_d$, where K_d is the Kato class of potentials.

For the sake of completeness, we recall the definitions of K_d and K_d^{loc} (cf. [S, Section A2] for details, discussion and examples):

Definition 1.1. $V \in K_d$, if:

$$\text{For} \quad d \geq 3, \quad \lim_{r\to 0} \sup_{x\in\mathbf{R}^d} \int_{|x-y|\leq r} \frac{|V(y)|}{|x-y|^{d-2}} dy = 0;$$

$$\text{For} \quad d = 2, \quad \lim_{r\to 0} \sup_{x\in\mathbf{R}^d} \int_{|x-y|\leq r} \log\{|x-y|^{-1}\}|V(y)|dy = 0;$$

$$\text{For} \quad d = 1, \quad \sup_{x\in\mathbf{R}^d} \int_{|x-y|\leq 1} |V(y)|dy < \infty.$$

$V \in K_d^{\mathrm{loc}}$ if $\chi_{\{|x|<R\}}(x)V(x) \in K_d$ for any $R > 0$, where χ_Ω denotes the characteristic function of Ω.

Let V satisfy Assumption A. Then $H = -\triangle + V$ is defined on $L^2(\mathbf{R}^d)$ using the quadratic form technique, see [S] for the details.

We consider functions in the following symbol class, which may be denoted by $S^\alpha = S(\langle\lambda\rangle^\alpha, d\lambda^2/\langle\lambda\rangle^2)$ in the notation of Hörmander's $S(m,g)$-class of pseudodifferential operators. Here $\langle\lambda\rangle = (1+\lambda^2)^{1/2}$ as usual.

Definition 1.2. Let $\alpha \in \mathbf{R}$. $f \in S^\alpha$ if and only if $f \in C^\infty(\mathbf{R})$ and for any $k \geq 0$,

$$\left|\partial_\lambda^k f(\lambda)\right| \leq C_k \langle\lambda\rangle^{\alpha-k}, \qquad \lambda \in \mathbf{R}.$$

We now describe our main results and the contents of the paper. In §2 we prove three main theorems. The following result is a variant of one of the results in [JN].

Theorem 1.3. *Let $\varepsilon > 0$. If $f \in S^{-\varepsilon}$, then $f(H)$ is extended to a bounded operator in $L^p(\mathbf{R}^d)$, $1 \leq p \leq \infty$.*

The results in [JN] on the t-dependence of the norm of $e^{-itH}f(H)$ are extended in the following result:

Theorem 1.4. *Let* $1 \leq p \leq \infty$ *and let* $\beta > d\left|\frac{1}{2} - \frac{1}{p}\right|$, $\gamma > d\left|\frac{1}{2} - \frac{1}{p}\right|$. *If* $f \in S^{-\beta}$, *then* $e^{-itH}f(H)$ *is bounded in* $L^p(\mathbf{R}^d)$ *and*

$$\left\|e^{-itH}f(H)\right\| \leq C\,\langle t\rangle^{\gamma}, \qquad t \in \mathbf{R}.$$

This result is optimal with respect to the decay of f in the sense that for $H = H_0 = -\triangle$ the L^p-boundedness of $e^{-itH_0}(H_0+1)^{-\gamma}$ implies $\gamma \geq d|\frac{1}{2} - \frac{1}{p}|$, see [Sj]. For results with optimal t-estimates, see [JN] and the comments in §2.

We prove the following resolvent estimate:

Theorem 1.5. *Let* $1 \leq p \leq \infty$ *and let* $\beta = d\left|\dfrac{1}{2} - \dfrac{1}{p}\right|$. *Then there exists* $C > 0$ *such that*

$$\left\|(H-z)^{-1}\right\|_{\mathcal{B}(L^p)} \leq C\frac{\langle z\rangle^{\beta}}{|\operatorname{Im} z|^{\beta+1}}, \qquad z \notin \mathbf{R}.$$

This estimate was proved by Pang [P] with $\beta = d$. Computing the L^1-norm of the explicit integral kernel of the free resolvent one finds that this estimate holds with $\beta = (d-1)/2$ $(p=1)$. Thus we have no reason to believe that our estimate is optimal.

An alternate method for obtaining L^p-boundedness of $f(H)$ can be based on resolvent estimates as Theorem 1.5 and the representation formula (cf. [HS])

$$f(H) = \frac{1}{2\pi i}\int_{\mathbf{C}} (\partial_{\bar z}\tilde f(z))(H-z)^{-1}dz\,d\bar z,$$

where $\tilde f$ is an almost analytic continuation of f. We discuss this approach and give some results in §3 and in the Appendix.

In §§4–5 we obtain results on mapping properties of e^{-itH} between Besov spaces. We first introduce a class of generalized Besov spaces and then show that under certain regularity assumptions on V these spaces can be identified with ordinary Besov spaces. Generalized Besov spaces have previously been considered in [Pe] in a different context. For one particular case this approach was also used in [JP]. The advantage of using the Besov spaces is that one obtains results for e^{-itH} directly, avoiding the localization $f(H)$. The main result is stated as Theorem 5.2.

§2. Scaling and L^p-estimates

In this section we show that estimates in [JN] are uniform with respect to the scaling: $H \to \theta H$, $0 < \theta \leq 1$, and apply it to improve L^p-estimates for $f(H)$ and $e^{-itH}f(H)$. Throughout this section, we suppose V satisfies Assumption A and assume $\sigma(H) \subseteq [0, \infty)$ without loss of generality.

Theorem 2.1. *Let $1 \leq p \leq \infty$, $\beta > d\left|\frac{1}{2} - \frac{1}{p}\right|$, and let $g \in C_0^\infty(\mathbf{R})$. Then there exists $C > 0$ such that*

$$\left\|g(\theta H)e^{-it\theta H}\right\|_{\mathcal{B}(L^p)} \leq C\langle t\rangle^\beta, \qquad t \in \mathbf{R}, \tag{2.1}$$

uniformly in $0 < \theta \leq 1$. In addition, the estimate is uniform with respect to g, if g runs in a bounded set G in C_0^∞, i.e., if there is $R > 0$ such that $\operatorname{supp} g \subset [-R, R]$ *and $|\partial_\lambda^\alpha g| \leq C_\alpha$ for any α and any $g \in G$.*

Proof. The scaling operator $U_p(\theta)$ on $L^p(\mathbf{R}^d)$ is given by

$$U_p(\theta)\varphi(x) = \theta^{d/p}\varphi(\theta x), \qquad 0 < \theta \leq 1, \quad x \in \mathbf{R}^d,$$

and $U_p(\theta)$ is an isometry in $L^p(\mathbf{R}^d)$. Then we have

$$\theta H = U_p(\sqrt{\theta})^{-1}(H_\theta)U_p(\sqrt{\theta})$$

where $H_\theta = H_0 + V_\theta$ and $V_\theta(x) = \theta V(\sqrt{\theta}x)$. In particular, this holds for $p = 2$, and by the functional calculus we learn

$$f(\theta H) = U_p(\sqrt{\theta})^{-1}f(H_\theta)U_p(\sqrt{\theta}),$$

in $L^2(\mathbf{R}^d)$, which in turn holds in any $L^p(\mathbf{R}^d)$ by a density argument. Thus it suffices to show

$$\left\|g(H_\theta)e^{-itH_\theta}\right\|_{\mathcal{B}(L^p)} \leq C\langle t\rangle^\beta, \qquad t \in \mathbf{R} \tag{2.2}$$

uniformly in $0 < \theta \leq 1$.

The idea of the proof is now to check all the computations in [JN] in order to conclude that the proof of (2.2) with $\theta = 1$ can be carried out with constants uniform in $0 < \theta \leq 1$. It seems that two points in the argument require some comments. We discuss only these two points and omit other details.

At first, the proof of [JN, Theorem 2.1] uses the Gaussian kernel estimate for e^{-tH}. We note that if T on $L^p(\mathbf{R}^d)$ has an integral kernel

$T(x,y)$, then the scaled operator $T(\theta) = U_p(\sqrt{\theta})TU_p(\sqrt{\theta})^{-1}$ has the integral kernel given by $\theta^{d/2}T(\sqrt{\theta}x, \sqrt{\theta}y)$. Thus

$$e^{-tH_\theta} = U_p(\sqrt{\theta})e^{-t\theta H}U_p(\sqrt{\theta})^{-1}$$

has the integral kernel

$$e^{-tH_\theta}(x,y) = \theta^{d/2}\left(e^{-t\theta H}\right)(\sqrt{\theta}x, \sqrt{\theta}y). \tag{2.3}$$

On the other hand, under Assumption A, the integral kernel of e^{-tH} satisfies the bound

$$\left|e^{-tH}(x,y)\right| \le C_\varepsilon t^{-d/2}e^{Lt}\exp\left(-\frac{|x-y|^2}{4(1+\varepsilon)t}\right), \qquad t>0, \quad x,y \in \mathbf{R}^d,$$

for some $L > 0$ and any $\varepsilon > 0$ (see, e.g., [S, Theorem B.6.7] or [D]). Hence $e^{-t\theta H}$ satisfies

$$\left|e^{-t\theta H}(x,y)\right| \le C_\varepsilon \theta^{-d/2}t^{-d/2}e^{L\theta t}\exp\left(-\frac{|x-y|^2}{4(1+\varepsilon)\theta t}\right). \tag{2.4}$$

Combining (2.3) and (2.4), we derive

$$\begin{aligned}\left|e^{-tH_\theta}(x,y)\right| &\le C_\varepsilon t^{-d/2}e^{L\theta t}\exp\left(-\frac{|x-y|^2}{4(1+\varepsilon)t}\right)\\ &\le C_\varepsilon t^{-d/2}e^{Lt}\exp\left(-\frac{|x-y|^2}{4(1+\varepsilon)t}\right),\end{aligned}$$

which is uniform in $0 < \theta \le 1$.

The second part is concerned with the commutator estimates in [JN, §3], where we need to have estimates for the operator norms on $L^2(\mathbf{R}^d)$ for $\|(H_\theta + M)^{-1/2}\|$ and $\|\partial_x(H_\theta + M)^{-1/2}\|$ on $L^2(\mathbf{R}^d)$ ($M > 0$ is a sufficiently large constant). The former one is clear because it is bounded by $M^{-1/2}$. The latter follows once more from the scaling argument:

$$\begin{aligned}\left\|\partial_x(H_\theta + M)^{-1/2}\right\| &= \left\|\partial_x U_2(\sqrt{\theta})(\theta H + M)^{-1/2}U_2(\sqrt{\theta})^{-1}\right\|\\ &= \left\|\left(U_2(\sqrt{\theta})^{-1}\partial_x U_2(\sqrt{\theta})\right)(\theta H + M)^{-1/2}\right\|\\ &= \left\|\partial_x(H + \theta^{-1}M)^{-1/2}\right\| \le \left\|\partial_x(H + M)^{-1/2}\right\|.\end{aligned}$$

Remark 2.2. Under additional assumptions, e.g., if $d \le 3$, we know that (2.1) holds with $\theta = 1$, $\beta = d\,|1/2 - 1/p|$ (see [JN, Theorems 1.4, 5.2]). In these cases, the estimate also holds with $\beta = d|1/2 - 1/p|$ uniformly in $0 < \theta \le 1$. The modifications needed are essentially the same as above, so we omit the details.

Proof of Theorem 1.3. Without loss of generality, we may suppose $\operatorname{supp} f \subset [-1, \infty)$. We choose $\varphi \in C_0^\infty(1/2, 2)$ so that

$$\sum_{n=-\infty}^{\infty} \varphi(2^n \lambda) = 1, \qquad \lambda > 0.$$

We let

$$\varphi_k(\lambda) = \varphi(2^{-k}\lambda), \qquad \lambda \in \mathbf{R},\ k = 1, 2, \dots,$$

and let $\varphi_0(\lambda) \in C_0^\infty(\mathbf{R})$ such that

$$\varphi_0(\lambda) + \sum_{k=1}^{\infty} \varphi_k(\lambda) = 1, \qquad \lambda \ge -1.$$

We decompose f using $\{\varphi_k(\lambda)\}$ as follows:

$$f(\lambda) = \sum_{k=0}^{\infty} f(\lambda)\varphi_k(\lambda) = \sum_{k=0}^{\infty} f_k(2^{-k}\lambda),$$

where $f_k(\mu) = \varphi(\mu) f(2^k \mu)$ for $k \ge 1$. Then it is easy to see that $\operatorname{supp} f_k \subset (1/2, 2)$ for $k \ge 1$, and

$$\left|\partial_\mu^\alpha f_k(\mu)\right| \le C_\alpha 2^{k\alpha} \left\langle 2^k \mu \right\rangle^{-\varepsilon-\alpha} \le C_\alpha 2^{-\varepsilon k}, \qquad \mu \in \mathbf{R}, \quad k \ge 0.$$

Hence $\left\{2^{\varepsilon k} f_k(\mu)\right\}_{k=0}^{\infty}$ is a bounded set in $C_0^\infty(\mathbf{R})$. By Theorem 2.1, we learn

$$\left\|f_k(2^{-k}H)\right\|_{\mathcal{B}(L^p)} \le C\ 2^{-\varepsilon k}, \qquad k \ge 0. \tag{2.5}$$

Thus we conclude

$$\|f(H)\|_{\mathcal{B}(L^p)} \le \sum_{k=0}^{\infty} \left\|f_k(2^{-k}H)\right\|_{\mathcal{B}(L^p)} \le C \sum_{k=0}^{\infty} 2^{-\varepsilon k} < \infty.$$

Proof of Theorem 1.4. Let $f \in S^{-\beta}$ with $\beta > d|1/2 - 1/p|$ and fix γ so that $d\,|1/2 - 1/p| < \gamma < \beta$. Let φ_k and f_k be chosen as in the proof of Theorem 1.3. Then by the above argument and Theorem 2.1, we learn

$$\left\|e^{-it\theta H} f_k(\theta H)\right\|_{\mathcal{B}(L^p)} \le C 2^{-\beta k} \langle t\rangle^{\gamma}, \quad t \in \mathbf{R},\ k \ge 0,\ 0 < \theta \le 1.$$

Setting $\theta = 2^{-k}$, $t = 2^k s$, we have

$$\left\|e^{-isH} f_k(2^{-k} H)\right\|_{\mathcal{B}(L^p)} \le C\ 2^{-\beta k} \left\langle 2^k s\right\rangle^{\gamma} \le C\ 2^{-(\beta-\gamma)k} \langle s\rangle^{\gamma}.$$

Summing over k we obtain

$$\begin{aligned}\left\|e^{-isH} f(H)\right\|_{\mathcal{B}(L^p)} &\le \sum_{k=0}^{\infty} \left\|e^{-isH} f_k(2^{-k} H)\right\|_{\mathcal{B}(L^p)} \\ &\le C\,\langle s\rangle^{\gamma} \sum_{k=0}^{\infty} 2^{-(\beta-\gamma)k} \le C\,\langle s\rangle^{\gamma}.\end{aligned}$$

Lemma 2.3. *Let $m > d/2$ be an integer. Then there exists $C > 0$ such that for $z \in \{z \in \mathbf{C} \setminus \mathbf{R} \mid |z| \le 2\}$,*

$$\left\|(H - z)^{-1}(H + 1)^{-m}\right\|_{\mathcal{B}(L^1)} \le C|\operatorname{Im} z|^{-1-d/2}. \tag{2.6}$$

Moreover, the estimate holds uniformly in $\theta \in (0, 1]$, if we replace H by θH.

Proof. The idea is to mimic the proof of [JN, Theorems 1.1, 1.3], so we give only a sketch. For the notation and the details, we refer to [JN].

By commutator computations as in the proof of [JN, Lemma 3.2], we have

$$\sup_{n \in \mathbf{Z}^d} \left\|\langle \cdot - n\rangle^{l} (H - z)^{-1} \langle \cdot - n\rangle^{-l}\right\| \le C_l |\operatorname{Im} z|^{-l-1}$$

for $z \in \{z \in \mathbf{R} \setminus \mathbf{R} \mid |z| \le 2\}$ with any $l \in \mathbf{N}$. This implies

$$\begin{aligned}\left|\!\left|\!\left|(H - z)^{-1}\right|\!\right|\!\right|_l \equiv \left\|(H - z)^{-1}\right\| + \sup_{n \in \mathbf{Z}^d} \left\|\langle \cdot - n\rangle^{l} (H - z)^{-1} \chi_{C(n)}\right\| \\ \le C|\operatorname{Im} z|^{-l-1}.\end{aligned}$$

We let $l > d/2$ and apply [JN, Theorem 2.4] to obtain

$$\begin{aligned}\left\|(H-z)^{-1}\right\|_{\mathcal{B}(l^1(L^2))} &\le C\,\left|\!\left|\!\left|(H-z)^{-1}\right|\!\right|\!\right|_l^{d/2l}\,\left\|(H-z)^{-1}\right\|_{\mathcal{B}(L^2)}^{1-d/2l}\\ &\le C\,|\mathrm{Im}\,z|^{-(l+1)d/2l}\,|\mathrm{Im}\,z|^{-(1-d/2l)}\\ &= C\,|\mathrm{Im}\,z|^{-1-d/2}\,.\end{aligned}$$

On the other hand, $(H+1)^{-m}$ is bounded from $L^1(\mathbf{R}^d)$ to $l^1(L^2)$ ([JN, Theorem 2.1]), hence

$$\begin{aligned}&\left\|(H-z)^{-1}(H+1)^{-m}\right\|_{\mathcal{B}(L^1)}\\ &\qquad\le \left\|(H-z)^{-1}\right\|_{\mathcal{B}(l^1(L^2),L^1)}\,\left\|(H+1)^{-m}\right\|_{\mathcal{B}(L^1,l^1(L^2))}\\ &\qquad\le C\,\left\|(H-z)^{-1}\right\|_{\mathcal{B}(l^1(L^2))}\le C\,|\mathrm{Im}\,z|^{-1-d/2}\,.\end{aligned}$$

The proof of the last statement is analogous to the proof of Theorem 2.1, so we omit the details.

Proof of Theorem 1.5. It suffices to consider the case $p=1$. Other cases follow by complex interpolation. We let $\beta = d/2$ and let $m > d/2$ be an integer. We first consider the case $|z| \le 2$. We write $z = x+iy$, and suppose $0 < y < 2$. By iterations of the first resolvent equation (recall that we assume $\sigma(H) \subseteq [0,\infty)$), we have

$$(H-z)^{-1} = \sum_{k=1}^{m}(z+1)^{k-1}(H+1)^{-k} + (z+1)^m(H-z)^{-1}(H+1)^{-m}. \tag{2.7}$$

The first term is uniformly bounded, and we estimate the second term by Lemma 2.3 to obtain

$$\left\|(H-z)^{-1}\right\|_{\mathcal{B}(L^1)} \le C|\mathrm{Im}\,z|^{-\beta-1}, \qquad |z|\le 2. \tag{2.8}$$

Now we use the scaling argument again. By the last statement in Lemma 2.3 we may replace H by θH in (2.8) :

$$\left\|(\theta H-z)^{-1}\right\|_{\mathcal{B}(L^1)} \le C|\mathrm{Im}\,z|^{-\beta-1}, \qquad |z|\le 2,\ 0<\theta\le 1.$$

For $|z|>1$, we let $z = |z|\cdot\hat{z}$, $|\hat{z}|=1$, and let $\theta = |z|^{-1}$. Then we obtain

$$\begin{aligned}\left\|(H-z)^{-1}\right\|_{\mathcal{B}(L^1)} &= \left\|(|z|(|z|^{-1}H-\hat{z}))^{-1}\right\|_{\mathcal{B}(L^1)}\\ &= |z|^{-1}\,\left\|(\theta H-\hat{z})^{-1}\right\|_{\mathcal{B}(L^1)}\\ &\le C|z|^{-1}\,|\mathrm{Im}\,\hat{z}|^{-\beta-1} = C|z|^{\beta}|\mathrm{Im}\,z|^{-\beta-1}.\end{aligned}$$

This completes the proof.

Remark 2.4. We could have used Theorem 1.4 instead of Lemma 2.3 to estimate the second term in the right hand side of (2.7). This gives, however, a slightly weaker result, namely, the estimate with $\beta > d|1/2 - 1/p|$.

§3. The almost analytic continuation and L^p-boundedness

In this section we discuss an alternative approach to the proof of the L^p-boundedness of functions of Schrödinger operators. The idea is to combine the almost analytic continuation method with resolvent estimates.

We introduce the following definition concerning the almost analytic continuation. A construction is discussed in the Appendix, and it is used in the proof of Theorem 3.3.

Definition 3.1. Let $f \in S^\alpha$ for some $\alpha \in \mathbf{R}$. A function $\tilde{f}$ on $\mathbf{C}$ is called an almost analytic continuation of f, if it satisfies

(1) $\tilde{f}$ is a smooth function on $\mathbf{C}$ and $\tilde{f}(x) = f(x)$ for $x \in \mathbf{R}$.
(2) For any $N \geq 0$,

$$\left|\partial_{\bar{z}}\tilde{f}(z)\right| \leq C_N \langle z\rangle^{\alpha-1-N} |\mathrm{Im}\, z|^N, \qquad z \in \mathbf{C}, \tag{3.1}$$

where $\partial_{\bar{z}}\tilde{f}(x+iy) = (\partial_x + i\partial_y)\tilde{f}(x+iy)$.

If $f \in S^{-\varepsilon}$, $\varepsilon > 0$, and if A is a selfadjoint operator in a Hilbert space, then it is known that $f(A)$ can be represented by the almost analytic continuation of f and the resolvent of A:

$$f(A) = \frac{1}{2\pi i}\int_{\mathbf{C}} \left(\partial_{\bar{z}}\tilde{f}(z)\right)(A-z)^{-1} dz d\bar{z} \tag{3.2}$$

(see [HS] and [G, Appendix]).

In order to apply this formula to Schrödinger operators on $L^p(\mathbf{R}^d)$, we need *a priori* estimates for the resolvent. Since the discussion of this section is methodological in its nature, we start from the following hypothesis, which includes the result of Theorem 1.5 as a special case.

Hypothesis (RE(β)). Let H be a Schrödinger operator on an $L^p(\mathbf{R}^d)$-space. We say that H satisfies RE(β), if

$$\left\|(H-z)^{-1}\right\|_{\mathcal{B}(L^p)} \leq C\frac{\langle z\rangle^\beta}{|\mathrm{Im}\, z|^{\beta+1}}, \qquad z \in \mathbf{C}\setminus\mathbf{R}.$$

Theorem 3.2. *Suppose H satisfies Hypothesis RE(β) on $L^p(\mathbf{R}^d)$ with $\beta \geq 0$, and suppose $f \in S^{-\varepsilon}$ with $\varepsilon > 0$. Then $f(H)$ is extended to a bounded operator in $L^p(\mathbf{R}^d)$.*

Proof. We take $N > \beta$, construct the almost analytic continuation, and then apply (3.1) and (3.2) to obtain

$$\begin{aligned}\|f(H)\|_{\mathcal{B}(L^p)} &\leq \frac{1}{2\pi}\int_{\mathbf{C}} C\frac{\langle z\rangle^{\beta}}{|\mathrm{Im}\, z|^{\beta+1}}\langle z\rangle^{-\varepsilon-1-N}|\mathrm{Im}\, z|^{N}dzd\bar{z}\\ &\leq C\int_{\mathbf{C}}\frac{\langle z\rangle^{-1-\varepsilon-(N-\beta)}}{|\mathrm{Im}\, z|^{1-(N-\beta)}}dzd\bar{z} < \infty.\end{aligned}$$

Theorem 1.3 follows easily from Theorem 3.2 and Theorem 1.5 (the proof of which is independent of Theorem 1.3) or a result by Pang [P]. We can also prove an analogue of Theorem 1.4 using the same idea. To simplify the argument, we consider only the case $f \in C_0^\infty(\mathbf{R})$.

Theorem 3.3. *Suppose H satisfies RE(β) on $L^p(\mathbf{R}^d)$ with $\beta \geq 0$ and let $f \in C_0^\infty(\mathbf{R})$. Then for any $\gamma > \beta + 1$,*

$$\left\|e^{-itH}f(H)\right\|_{\mathcal{B}(L^p)} \leq C\langle t\rangle^{\gamma}, \qquad t \in \mathbf{R}. \tag{3.3}$$

Proof. Let $f_t(x) = e^{-itx}f(x)$. Then it is easy to see that for any $s > 0$, $\|f_t\|_{H^s} \leq C_s\langle t\rangle^s$, $t \in \mathbf{R}$. Hence, by Lemma A.2 we learn

$$\int_{\mathbf{C}}|\mathrm{Im}\, z|^{-\gamma+\varepsilon}\left|\partial_{\bar{z}}\tilde{f}_t(z)\right|dzd\bar{z} \leq C_\varepsilon\langle t\rangle^{\gamma}, \qquad \varepsilon > 0,\ t \in \mathbf{R},$$

where $\tilde{f}_t(z)$ is the almost analytic continuation of f_t as constructed in the Appendix. Now letting $\varepsilon = \gamma - \beta - 1 > 0$, we obtain from RE($\beta$)

$$\begin{aligned}\left\|e^{-itH}f(H)\right\|_{\mathcal{B}(L^p)} &\leq \frac{1}{2\pi}\int\left|\partial_{\bar{z}}\tilde{f}_t(z)\right|\left\|(H-z)^{-1}\right\|_{\mathcal{B}(L^p)}dzd\bar{z}\\ &\leq C\int\frac{\langle z\rangle^{\beta}}{|\mathrm{Im}\, z|^{\beta+1}}\left|\partial_{\bar{z}}\tilde{f}_t(z)\right|dzd\bar{z}\\ &\leq C\int|\mathrm{Im}\, z|^{-\gamma+\varepsilon}\left|\partial_{\bar{z}}\tilde{f}_t(z)\right|dzd\bar{z}\\ &\leq C\langle t\rangle^{\gamma},\end{aligned}$$

since $\tilde{f}_t$ is compactly supported.

Combining this result with Theorem 1.5, we obtain (3.3) with $\gamma > 1 + d/2$ for $p = 1$. Thus this direct approach does not give the optimal result. Even if we had RE$((d-1)/2)$ (free case), we would only get (3.3) with $\gamma > (d+1)/2$. We have lost at least order $O(\langle t \rangle^{1/2})$ in this procedure. There is another possibility, however. In the proof of Theorem 2.1 (or Theorem 1 in [JN]), the representation

$$f(H) = \int_{-\infty}^{\infty} e^{-isR} g(s) ds, \qquad R = (H+M)^{-1},$$

is used to obtain estimates for $\|\!| f(H) |\!\|_\beta$ and $\left|\!\left\| e^{-itH} f(H) \right|\!\right\|_\beta$ (see the proof of Lemma 2.3 for the definition of $\|\!|\cdot|\!\|_\beta$). An alternative is to use the representation (3.2) instead, and then we obtain optimal estimates for the t-dependence.

Remark 3.4. The almost analytic continuation technique was introduced by L. Hörmander in a series of lectures on Fourier integral operators held in 1969, see also [H1] and [H2, Chapter 3]. It was used extensively by A. Melin and J. Sjöstrand in their work on Fourier integral operators with complex phase functions. The representation formula (3.2) first appeared in [HS], and has recently been used extensively in the study of many-body Schrödinger operators.

Remark 3.5. An axiomatic approach to the functional calculus based on (3.2) and RE(β) has been given by Davies [D2].

§4. Generalized Besov spaces

Throughout this section we consider a fixed selfadjoint operator H on the Hilbert space $L^2(\mathbf{R}^d)$. Our goal is to associate with H a family of spaces in such a manner that this family becomes the usual Besov spaces for $H = -\triangle$. We define the spaces for an arbitrary selfadjoint operator on $L^2(\mathbf{R}^d)$, under certain assumptions on this operator, which are verified for $H = -\triangle + V$ by Theorem 2.1.

Assumption 4.1. For any $\varphi \in C_0^\infty(\mathbf{R})$ let $\varphi(H)$ denote the bounded operator on $L^2(\mathbf{R}^d)$ obtained via the functional calculus. Assume that $\varphi(H)$ extends to a bounded operator on $L^p(\mathbf{R}^d)$, $1 \le p \le \infty$.

Remark. As mentioned in §1 the operator $\varphi(H)$ on $L^\infty(\mathbf{R}^d)$ is obtained as the adjoint of the corresponding operator on $L^1(\mathbf{R}^d)$, hence is uniquely determined.

Assumption 4.2. Let H satisfy Assumption 4.1. Let $\varphi \in C_0^\infty(\mathbf{R})$. Assume that for any p, $1 \le p \le \infty$, $\|\varphi(\theta H)\|_{\mathcal{B}(L^p)} \le c$ for all $\theta \in [0, 1]$, with c independent of θ.

If V satisfies Assumption A, then $H = -\Delta + V$ satisfies Assumption 4.2 by Theorem 2.1 (with $t = 0$). Fix $\varphi \in C_0^\infty(\mathbf{R})$ with $\operatorname{supp}(\varphi) \subseteq \{\lambda \,|\, 1/4 \le |\lambda| \le 4\}$ and

$$\sum_{j=-\infty}^{\infty} \varphi(4^{-j}\lambda) = 1, \qquad \lambda \ne 0.$$

Define

$$\psi_0(\lambda) = 1 - \sum_{j=1}^{\infty} \varphi(4^{-j}\lambda), \qquad \lambda \in \mathbf{R},$$

and

$$\psi_j(\lambda) = \varphi(4^{-j}\lambda), \qquad j = 1, 2, \ldots, \quad \lambda \in \mathbf{R}.$$

Definition 4.3. Let H satisfy Assumption 4.2. Let p, q, s satisfy $1 \le p \le \infty$, $1 \le q < \infty$, and $s \ge 0$. For $v \in L^p(\mathbf{R}^d)$ define

$$\|v\|_{\boldsymbol{B}_p^{s,q}(H)} = \left(\sum_{j=0}^{\infty} \left(2^{sj} \left\| \psi_j(H) v \right\|_p \right)^q \right)^{1/q} \tag{4.1}$$

For $q = \infty$ the definition is modified in the obvious way. The generalized Besov space is defined by

$$\boldsymbol{B}_p^{s,q}(H) = \{ v \in L^p(\mathbf{R}^d) \,|\, \|v\|_{\boldsymbol{B}_p^{s,q}(H)} < \infty \}.$$

Lemma 4.4. *Let H satisfy Assumption* 4.1. *Let $u \in L^p(\mathbf{R}^d)$. Then*

$$\|u\|_p \le \sum_{j=0}^{\infty} \|\psi_j(H) u\|_p \,,$$

where the sum may equal $+\infty$.

Proof. Let $1 \le p < \infty$. If $u \in L^p(\mathbf{R}^d) \cap L^2(\mathbf{R}^d)$, then we have $u = \sum_{j=0}^{\infty} \psi_j(H) u$, and the assertion is clear. It follows for general

$u \in L^p(\mathbf{R}^d)$ by the density argument. The case $p = \infty$ follows by the duality argument.

Proposition 4.5. *For $s > 0$, $1 \leq p, q \leq \infty$ and $s = 0$, $q = 1$, $1 \leq p \leq \infty$, the space $\boldsymbol{B}_p^{s,q}(H)$ is a Banach space with the norm given by* (4.1). *It is a subspace of $L^p(\mathbf{R}^d)$.*

Proof. It is easy to see that (4.1) defines a norm on $\boldsymbol{B}_p^{s,q}(H)$. Let $(v^k)_{k\in\mathbf{N}}$ be a Cauchy sequence in $\boldsymbol{B}_p^{s,q}(H)$. Consider first the case $q = 1$. Then by Lemma 4.4 and $s \geq 0$

$$\|u\|_p \leq \sum_{j=0}^{\infty} \|\psi_j(H)u\|_p \leq \sum_{j=0}^{\infty} 2^{sj} \|\psi_j(H)u\|_p = \|u\|_{\boldsymbol{B}_p^{s,1}(H)}$$

Let $q > 1$. Let q' denote the exponent conjugate to q. Then $q' < \infty$ and for $s > 0$ we have

$$\|u\|_p \leq \sum_{j=0}^{\infty} \|\psi_j(H)u\|_p \leq \left(\sum_{j=o}^{\infty} 2^{-sjq'}\right)^{1/q'} \|u\|_{\boldsymbol{B}_p^{s,q}(H)} \cdot$$

In either case we conclude that $\boldsymbol{B}_p^{s,q}(H)$ is a subspace of $L^p(\mathbf{R}^d)$ and that the given sequence $(v^k)_{k\in\mathbf{N}}$ is a Cauchy sequence in $L^p(\mathbf{R}^d)$, hence convergent in L^p to a limit $v \in L^p(\mathbf{R}^d)$. Define

$$\begin{aligned} \xi_j^k &= 2^{sj} \left\|\psi_j(H)v^k\right\|_p \\ \xi_j &= 2^{sj} \|\psi_j(H)v\|_p \,. \end{aligned}$$

Then $\xi_j^k \to \xi_j$ as $k \to \infty$ for each $j = 0, 1, 2, \ldots$. Furthermore, since $\left\|v^k\right\|_{\boldsymbol{B}_p^{s,q}(H)} \leq c$ for all k, we conclude that $(\xi_j)_{j\in\mathbf{N}} \in \ell^q(\mathbf{N})$.

We have now proved $v \in \boldsymbol{B}_p^{s,q}(H)$. It remains to prove convergence of the sequence $\boldsymbol{\xi}^k = (\xi_j^k)_{j\in\mathbf{N}}$ to $\boldsymbol{\xi} = (\xi_j)_{j\in\mathbf{N}}$ in $\ell^q(\mathbf{N})$. Since $(\boldsymbol{\xi}^k)_{k\in\mathbf{N}}$ is a Cauchy sequence in $\ell^q(\mathbf{N})$ and the components converge, this result is straightforward to prove. Details are omitted.

Now we prove a mapping property of e^{-itH} between abstract Besov spaces associated with H.

Theorem 4.6. *Let V satisfy Assumption* A *and let $H = -\triangle + V$. Assume $s \geq 0$, $1 \leq p$, $q \leq \infty$, and $\beta > d|\frac{1}{2} - \frac{1}{p}|$. Then*

$$e^{-itH} \in \mathcal{B}(\boldsymbol{B}_p^{s+2\beta,q}(H), \boldsymbol{B}_p^{s,q}(H)) \tag{4.2}$$

with norm bounded by $c\langle t\rangle^{\beta}$.

Remark 4.7. Note that the above result holds with $\beta = d\left|\frac{1}{2} - \frac{1}{p}\right|$ under restrictions on d (e.g. $d \le 3$) or under additional assumptions on V, see [JN, §5].

Proof. Fix $\chi \in C_0^\infty(\mathbf{R})$ such that $\varphi(\lambda) = \chi(\lambda)\varphi(\lambda)$ for all $\lambda \in \mathbf{R}$. For $j \ge 1$ and $u \in L^p(\mathbf{R}^d)$ we have from Theorem 2.1

$$\begin{aligned} 2^{sj}\left\|\varphi(4^{-j}H)e^{-itH}u\right\|_p &= 2^{sj}\left\|\chi(4^{-j}H)e^{-i(4^jt)4^{-j}H}\varphi(4^{-j}H)u\right\|_p \\ &\le c\,2^{j(s+2\beta)}\langle t\rangle^{\beta}\left\|\varphi(4^{-j}H)u\right\|_p. \end{aligned}$$

The estimate for $j = 0$ follows from Theorem 2.1. The result now follows from the definition of the norm (4.1) and the covering argument.

We note the following results, which are useful in the next section.

Proposition 4.8. *Let* V *satisfy Assumption* A. *Assume* $1 \le p, q, q_1 \le \infty$ *and* $s \ge s_1 > 0$. *If either* $s > s_1$ *or* $s = s_1$ *and* $q \le q_1$, *then* $\boldsymbol{B}_p^{s,q}(H)$ *is continuously embedded in* $\boldsymbol{B}_p^{s_1,q_1}(H)$.

Proof. The argument in the proof of [BTW, Theorem 2.2.1] carries over unchanged to our generalized Besov spaces.

Lemma 4.9. *Let* V *satisfy Assumption* A *and let* $s \ge 0$, $1 \le p, q \le \infty$. *Let* $M \in \mathbf{R}$. *Then* $\boldsymbol{B}_p^{s,q}(H + M) = \boldsymbol{B}_p^{s,q}(H)$ *with equivalent norms.*

Proof. A simple covering argument, which is omitted.

§5. Identification with ordinary Besov spaces

We have chosen the definition of $\boldsymbol{B}_p^{s,q}(H)$ in such a manner that for $H = -\triangle$ this space is identical with the usual Besov space, which we here denote $B_p^{s,q}$. In applications it is of interest to know conditions on V which imply $\boldsymbol{B}_p^{s,q}(H) = B_p^{s,q}$ (with equivalent norms).

Our result on this question is based on the real interpolation method and interpolation spaces defined via semigroups. We refer to [BL] for the results needed. We recall a few results from [BL, Section 6.7]. Let

$G(t)$, $t > 0$, be a strongly continuous bounded semigroup on a Banach space $\mathcal{X}$ with infinitesimal generator Λ. For $u \in \mathcal{X}$ define

$$\omega(t,u) = \sup_{s<t} \|G(s)u - u\|_{\mathcal{X}} .$$

The real interpolation method constructs a family of Banach spaces between the domain $\mathcal{D}(\Lambda)$ of Λ (with the graph norm) and $\mathcal{X}$, denoted $(\mathcal{X}, \mathcal{D}(\Lambda))_{\theta,q}$, $0 < \theta < 1$, $1 \le q \le \infty$. In [BL, Theorem 6.7.3] it is shown that the norm $\|u\|_{(\mathcal{X},\mathcal{D}(\Lambda))_{\theta,q}}$ is equivalent to the norm given by

$$\|u\|_{\mathcal{X}} + \left(\int_0^\infty t^{-\theta q - 1} \omega(t,u)^q dt \right)^{1/q} . \tag{5.1}$$

The usual L^p-type Sobolev space of order $m \in \mathbf{N}$ is denoted $W_p^m(\mathbf{R}^d)$.

Assumption $B(p,m)$**.** Let $1 \le p \le \infty$ and let $m \in \mathbf{N}$. Let V satisfy Assumption A, and let $H = -\Delta + V$. Assume there exists $M \ge 0$ such that $(H+M)^{-m}$ is a bounded map from $L^p(\mathbf{R}^d)$ to $W_p^m(\mathbf{R}^d)$ with a bounded inverse.

Theorem 5.1. *Let V satisfy Assumption $B(p,m)$ for some $m \in \mathbf{N}$ and $1 \le p \le \infty$. Then for $1 \le q \le \infty$, $0 < s < 2m$, $B_p^{s,q}(H) = B_p^{s,q}$ (with equivalent norms).*

Proof. Let V satisfy Assumption $B(p,m)$. We first show that $-(H+M)^m - L$ generates a strongly continuous bounded semigroup with $M, L > 0$ and the domain of the generator is $W_p^m(\mathbf{R}^d)$. Without loss of generality we may assume $M = 0$ and $H > 1$. Then by Theorem 1.3, $U(t) = e^{-tH^m}$ is bounded in $L^p(\mathbf{R}^d)$. Moreover, by Theorem 2.1 $U(t) = e^{-(t^{1/m}H)^m}$ is uniformly bounded with respect to $t \in (0,1]$. Hence there is $L \in \mathbf{R}$ such that $\|U(t)\|_{\mathcal{B}(L^p)} \le Ce^{Lt}$ for any $t > 0$. Thus $-(H^m + L)$ generates a bounded C_0 semigroup. The strong continuity follows from the fact that it is strongly continuous in $L^2(\mathbf{R}^d)$. The expression of the resolvent by the semigroup:

$$(\Lambda + K)^{-1} = -\int_0^\infty e^{-Kt} U(t) dt, \qquad K > L,$$

where Λ is the generator of $U(t)$, implies $(H^m + K)^{-1} = (\Lambda + K)^{-1}$, and hence the domain of Λ is $W_p^m(\mathbf{R}^d)$. We assume $L = 0$ in the sequel in order to simplify the notation.

Now we let $\Lambda = -H^m$ and let $G(t)$, $t > 0$, denote the semigroup generated by Λ. Let $\mathcal{D} = \mathcal{D}(\Lambda) = W_p^m(\mathbf{R}^d)$. Note that the usual Sobolev norm and the graph norm of Λ are equivalent norms on $\mathcal{D}$, as can be seen using the closed graph theorem.

Fix q, $1 \leq q < \infty$ (the case $q = \infty$ requires obvious modifications in the arguments below) and s, $0 < s < 2m$. Define $\theta = \dfrac{s}{2m}$. It follows from the real interpolation method (see [BL]) that

$$B_p^{s,q} = (L^p, \mathcal{D})_{\theta,q}.$$

Thus to prove the theorem it suffices to prove

$$\boldsymbol{B}_p^{s,q}(H) = (L^p, \mathcal{D})_{\theta,q}$$

with equivalent norms. We follow essentially the arguments in [BL, p. 160–1]. Let φ, ψ_j denote the functions from §4 used in our definition of the generalized Besov spaces.

Assume first $u \in (L^p, \mathcal{D})_{\theta,q}$. Let $\Phi(\lambda) = \varphi(\lambda)(\exp(-\lambda^m) - 1)^{-1}$, $\lambda \in \mathbf{R}$. Note $\Phi \in C_0^\infty(\mathbf{R})$. Using Theorem 2.1 we find $\left\|\Phi(4^{-j}H)\right\|_{\mathcal{B}(L^p)} \leq c$ for $j = 0, 1, 2, \ldots$. Therefore

$$\|\psi_j(H)u\|_p = \left\|\Phi(4^{-j}H)(G(4^{-mj}) - 1)u\right\|_p \leq c\,\omega(4^{-mj}, u).$$

Using (4.1) we conclude

$$\|u\|_{\boldsymbol{B}_p^{s,q}(H)} \leq c\left(\|u\|_p + \left(\sum_{j=0}^{\infty}(2^{sj}\omega(2^{-2mj}, u))^q\right)^{1/q}\right).$$

Since $\omega(t, u)$ is an increasing function of t and we have

$$\int_{2^{-2mj}}^{2^{-2m(j-1)}} t^{-\theta q - 1}\,dt = c\,2^{2m\theta jq} = c\,2^{sjq},$$

we get

$$\begin{aligned}
\sum_{j=0}^{\infty} 2^{sjq}\omega(2^{-2mj}, u)^q &= c\sum_{j=0}^{\infty}\int_{2^{-2mj}}^{2^{-2m(j-1)}} t^{-\theta q-1}\omega(2^{-2mj}, u)^q\,dt \\
&\leq c\sum_{j=0}^{\infty}\int_{2^{-2mj}}^{2^{-2m(j-1)}} t^{-\theta q-1}\omega(t, u)^q\,dt \\
&\leq c\int_0^{\infty} t^{-\theta q-1}\omega(t, u)^q\,dt.
\end{aligned}$$

Using (5.1) we conclude

$$\|u\|_{B_p^{s,q}(H)} \le c\,\|u\|_{(L^p,\mathcal{D})_{\theta,q}} \tag{5.2}$$

which proves the first half of the theorem. To prove the second half, assume $u \in B_p^{s,q}(H)$. Theorem 2.1 implies

$$\|\Lambda\psi_j(H)u\|_p \le c\,4^{mj}\,\|\psi_j(H)u\|_p, \qquad j = 0,1,2,\dots.$$

Using

$$\|G(s)u-u\|_p \le \int_0^s \|G(\tau)\Lambda u\|_p\,d\tau$$

and

$$\|G(s)u-u\|_p \le 2\,\|u\|_p$$

we get (see also Lemma 4.4)

$$\omega(t,u) \le c\sum_{j=0}^{\infty}\min\{1,t\,4^{mj}\}\,\|\psi_j(H)u\|_p\,.$$

We estimate the integral term in (5.1). The integral is split as

$$\int_0^{\infty}\cdots\,dt = \int_1^{\infty}\cdots\,dt + \sum_{k=0}^{\infty}\int_{4^{-m(k+1)}}^{4^{-mk}}\cdots\,dt. \tag{5.3}$$

We introduce the notation $\alpha_j = \|\psi_j(H)u\|_p$. For $t \in (4^{-m(k+1)}, 4^{-mk})$ we have $\min\{1,t\,4^{mj}\} = 1$, if $j \ge k+1$, and $\min\{1,t\,4^{mj}\} = t\,4^{mj}$, if $j \le k$. This result is inserted in the sum in (5.3) to get

$$\begin{aligned}
&\sum_{k=0}^{\infty}\int_{4^{-m(k+1)}}^{4^{-mk}} t^{-\theta q-1}\left(\sum_{j=0}^{k} t\,4^{mj}\alpha_j + \sum_{j=k+1}^{\infty}\alpha_j\right)^q dt\\
&\qquad\le c\sum_{k=0}^{\infty}\left[\sum_{j=0}^{k}4^{mj-(1-\theta)mk}\alpha_j\right]^q + c\sum_{k=0}^{\infty}\left[\sum_{j=k+1}^{\infty}4^{\theta mk}\alpha_j\right]^q\\
&= c\sum_{k=0}^{\infty}\left[\sum_{j=0}^{k}4^{m(1-\theta)(j-k)}(4^{\theta jm}\alpha_j)\right]^q + c\sum_{k=0}^{\infty}\left[\sum_{j=k+1}^{\infty}4^{\theta m(k-j)}(4^{\theta jm}\alpha_j)\right]^q.
\end{aligned}$$

Since $u \in B_p^{s,q}(H)$, $(4^{\theta jm}\alpha_j)_{j\in\mathbf{N}} \in \ell^q(\mathbf{N})$, and in both cases above we have convolution by a sequence in ℓ^1, so we use Young's inequality to conclude

$$\left(\int_0^1 t^{-\theta q-1}\omega(t,u)^q dt\right)^{1/q} \leq c\left(\sum_{j=0}^{\infty}\left(4^{\theta mj}\left\|\psi_j(H)u\right\|_p\right)^q\right)^{1/q}.$$

The other term in (5.3) is estimated using Hölder's inequality:

$$\left(\int_1^\infty t^{-\theta q-1}\omega(t,u)^q dt\right)^{1/q} \leq \left(\int_1^\infty t^{-\theta q-1}\left(\sum_{j=0}^{\infty}\alpha_j\right)^q dt\right)^{1/q}$$

$$= c\sum_{j=0}^{\infty}\alpha_j \leq c\left(\sum_{j=0}^{\infty}\left(4^{\theta mj}\alpha_j\right)^q\right)^{1/q}.$$

Combining these estimates we get

$$\|u\|_{(L^p,\mathcal{D})_{\theta,q}} \leq c\,\|u\|_{\boldsymbol{B}_p^{s,q}(H)} \tag{5.4}$$

which proves the second half of the theorem.

Theorem 5.1 combined with Theorem 4.6 implies the following mapping property of e^{-itH} between (usual) Besov spaces.

Theorem 5.2. *Let V satisfy Assumptions* A *and $B(p,m)$, and let $H=-\Delta+V$. Assume $1\leq p,\ q\leq\infty$, $\beta>d|\frac{1}{2}-\frac{1}{p}|$, $\gamma>d|\frac{1}{2}-\frac{1}{p}|$, and $0\leq s<2(m-\beta)$. Then*

$$e^{-itH}\in\mathcal{B}(B_p^{s+2\beta,q},B_p^{s,q}) \tag{5.5}$$

with norm bounded by $c\,\langle t\rangle^\gamma$.

Concerning the Assumption $B(p,m)$ we note that for $m=1$ we can use standard perturbation results to show that if V is bounded relative to the Laplacian on $L^p(\mathbf{R}^d)$ with relative bound less than one, then the condition is satisfied. Several sufficient conditions for this to hold can be found in [Sc]. For $m>1$ some regularity is needed. If $V\in C^\infty(\mathbf{R}^d)$ with all derivatives bounded, then Assumption $B(p,m)$ holds for all $m\geq 1$ and all p, $1\leq p\leq\infty$.

Remark 5.3. Note that the proof of Theorem 5.2 also yields

$$e^{-itH}\in\mathcal{B}(B_p^{2\beta,q},L^p(\mathbf{R}^d)) \tag{5.6}$$

under the same assumptions. In this form the result is a direct generalization of the results on the free Schrödinger equation in [BTW].

Remark 5.4. In the proof of Theorem 5.1 we have shown that $-(H+M)^m - L$ generates a bounded C_0 semigroup. This result has also been obtained by Davies [D2] in an abstract framework, cf. Remark 3.5.

Appendix. A construction of an almost analytic continuation

In this appendix we propose a construction of an almost analytic continuation, and discuss its properties. We start by constructing an almost analytic continuation of $f \in C_0^\infty(-2,2)$.

We fix $\chi \in C_0^\infty(\mathbf{R})$ such that $0 \le \chi(x) \le 1$,

$$\chi(x) = \begin{cases} 1, & \text{if } |x| \le 1, \\ 0, & \text{if } |x| \ge 2, \end{cases}$$

and let $\rho(x) = \int_0^x \chi(y)dy$. For $f \in C_0^\infty(-2,2)$, we define $\tilde{f}(z)$, $z \in \mathbf{C}$ by

$$\tilde{f}(x+iy) = (2\pi)^{-1/2}\chi(x/2)\chi(y)\int_{-\infty}^{\infty} e^{-\rho(y\xi)}e^{ix\xi}\hat{f}(\xi)d\xi, \tag{A.1}$$

where $\hat{f}(\xi)$ denotes the Fourier transform of $f(x)$.

Lemma A.1. *$\tilde{f}(z)$ is an almost analytic continuation of $f(x)$.*

Proof. It is easy to see that $\tilde{f}(z) \in C_0^\infty(\mathbf{C})$ because $\hat{f} \in \mathcal{S}$, and $e^{-\rho(y\xi)}$ is a smooth bounded function. It is also easy to see that $\tilde{f}(x) = f(x)$ for $x \in \mathbf{R}$ since $\rho(0) = 0$. It remains to show (3.1). By direct computation we have

$$\begin{aligned}
\left(\partial_{\bar{z}}\tilde{f}\right)(x+iy) &= (\partial_x + i\partial_y)\tilde{f}(x+iy) \\
&= (2\pi)^{-1/2}\chi(x/2)\chi(y)\int i\xi(1-\rho'(y\xi))e^{-\rho(y\xi)}e^{ix\xi}\hat{f}(\xi)d\xi \\
&\quad + (2\pi)^{-1/2}2^{-1}\chi'(x/2)\chi(y)\int e^{-\rho(y\xi)}e^{ix\xi}\hat{f}(\xi)d\xi \\
&\quad + i(2\pi)^{-1/2}\chi(x/2)\chi'(y)\int e^{-\rho(y\xi)}e^{ix\xi}\hat{f}(\xi)d\xi \\
&= \mathrm{I} + \mathrm{II} + \mathrm{III}.
\end{aligned} \tag{A.2}$$

To estimate the first term, we note that $\rho'(y\xi) = \chi(y\xi) = 1$ if $|y\xi| \le 1$, hence

$$|1-\rho'(y\xi)| = |1-\chi(y\xi)| \le \chi_{\{|y\xi|\ge 1\}}(y,\xi)$$

where χ_Ω denotes the characteristic function of Ω. Then we have

$$|\mathrm{I}| \le C \int |\xi| \chi_{\{|y|\,|\xi| \ge 1\}}(y,\xi) |\hat{f}(\xi)| d\xi$$

$$\le C \int |\xi|^{N+1} |y|^N |\hat{f}(\xi)| d\xi \le C|y|^N \tag{A.3}$$

since $\hat{f} \in \mathcal{S}$. To estimate the second term, we note that $f(x) \equiv 0$ on $\operatorname{supp} \chi'(x/2)$. Hence

$$0 = \chi'\Big(\frac{x}{2}\Big) \sum_{k=0}^{N} \frac{(iy)^k f^{(k)}(x)}{k!} = (2\pi)^{-1/2} \chi'\Big(\frac{x}{2}\Big) \int \sum_{k=0}^{N} \frac{(-y\xi)^k}{k!} e^{ix\xi} \hat{f}(\xi) d\xi.$$

We subtract this from (II) to obtain

$$|\mathrm{II}| \le C \int \left| \left(e^{-\rho(y\xi)} - \sum_{k=0}^{N} \frac{(-y\xi)^k}{k!} \right) \right| |\hat{f}(\xi)| d\xi$$

$$\le C \int |y|^{N+1} |\xi|^{N+1} |\hat{f}(\xi)| d\xi$$

$$\le C|y|^{N+1}. \tag{A.4}$$

The estimate for (III) is easy since it is supported away from the real axis.

Once an almost analytic extension is constructed for a C_0^∞-function, it is then standard procedure to extend it to $f \in S^\alpha$. We include the construction for the sake of completeness. Let $\varphi \in C_0^\infty(1/2, 2)$ as in the proof of Theorem 1.3, and let $\varphi_j(x) \in C_0^\infty(\mathbf{R})$ defined by

$$\varphi_{\pm k}(x) = \varphi(\pm 2^{-k} x), \quad k = 1, 2, \dots, \; x \in \mathbf{R},$$

$$\varphi_0(x) = 1 - \sum_{k \ne 0} \varphi_k(x), \quad x \in \mathbf{R}.$$

We decompose $f \in S^\alpha$ as

$$f(x) = \sum_{j=-\infty}^{\infty} f(x)\varphi_j(x) = \sum_{j=-\infty}^{\infty} f_j(2^{-|j|}x),$$

where $f_k(y) = \varphi(\operatorname{sign}(k)y) f(2^{|k|}y)$ for $k \ne 0$ and $f_0(y) = f(y)\varphi_0(y)$. Now we can apply the above construction to each $f_j(x)$ to obtain $\tilde{f}_j(z)$. Note that we can modify the construction such that $\tilde{f}_j(z)$ is supported in $\{z \mid \operatorname{Re} z \in [1/4,\ 4], |\operatorname{Im} z| \le 2\}$ for $j > 0$ and in $\{z \mid \operatorname{Re} z \in [-4,\ -1/4],$ $|\operatorname{Im} z| \le 2\}$ for $j < 0$. Then $\tilde{f}(z) = \sum_{j=-\infty}^{\infty} \tilde{f}_j(2^{-j}z)$ defines an almost analytic continuation of f. Further details are omitted.

Compared with the other known constructions of an almost analytic continuation, our method seems to have the advantage of being straightforward, namely, we do not use asymptotic sums. On the other hand, we need no differentiability of f to define $\tilde{f}(z)$, and the proof of Lemma A.1 shows that (3.1) with $N = a \in \mathbf{R}_+$ follows from $f \in H_0^s$, $s > a + 3/2$. In fact, it is known that $f \in C_0^{1+a}(\mathbf{R})$ is sufficient to construct $\tilde{f}(z)$ satisfying (3.1) with $N = a$ (E. B. Davies, private communication, see also [D2]). Our construction may be not as precise as Davies', but the next lemma is sufficient for our application in §3.

Lemma A.2. *Let $R > 0$ be fixed, and let $f \in H_0^s([-R, R])$ with $s \ge 1$. Then for any $\varepsilon > 0$ there is $C = C(R, \varepsilon)$ such that*

$$\int_{\mathbf{C}} |\operatorname{Im} z|^{-s+\varepsilon} \left|\partial_{\bar{z}} \tilde{f}(z)\right| dz d\bar{z} \le C \|f\|_{H_0^s}. \tag{A.5}$$

Proof. It suffices to consider the case $R = 1$, and we may assume $\tilde{f}(z)$ is defined by (A.1). As in the proof of Lemma A.1, we decompose $\partial_{\bar{z}} \tilde{f}$ as $\partial_{\bar{z}} \tilde{f} = \mathrm{I} + \mathrm{II} + \mathrm{III}$. We start by estimating (I). As in the computation to derive (A.3), for each y we have

$$\begin{aligned}\left(\int \left|\xi(1 - \rho'(y\xi))\hat{f}(\xi)\right|^2 d\xi\right)^{1/2} &\le |y|^{s-1} \left(\int |\xi|^{2s} |\hat{f}(\xi)|^2\right)^{1/2} \\ &\le C|y|^{s-1} \|f\|_{H_0^s}.\end{aligned}$$

Hence by Plancherel's theorem, we have

$$\begin{aligned}\int_{\mathbf{C}} |\operatorname{Im} z|^{-s+\varepsilon} |\mathrm{I}(z)| dz d\bar{z} &= \int_{|x|,|y| \le 2} |y|^{-s+\varepsilon} |\mathrm{I}(x + iy)| dx dy \\ &\le C \int_{|y| \le 2} \left(\int |\mathrm{I}(x + iy)|^2 dx\right)^{1/2} |y|^{-s+\varepsilon} dy \\ &\le C \int_{|y| \le 2} \left(|y|^{s-1} \|f\|_{H_0^s}\right) |y|^{-s+\varepsilon} dy \\ &= C \left(\int_{|y| \le 2} |y|^{-1+\varepsilon} dy\right) \|f\|_{H_0^s} = C \|f\|_{H_0^s}.\end{aligned}$$

On the other hand, (A.4) implies

$$|\,\mathrm{II}(x+iy)| \le C|y|^{s-1}\,\|f\|_{H_0^s},$$

and the estimate for (II) follows from this. The estimate for (III) is easy and we omit it.

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Absolute Continuity of the Essential Spectrum for some Linearized MHD Operator

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§1. Introduction

The magnetohydrodynamic (MHD) motion of plasma is described by the system of equations which consist of the compressible Euler equation and the reduced Maxwell equation with the mutual interaction terms given by the Lorenz force and Ohm's low. Related to the plasma confinement experiment, the study of the behavior of plasma motion around the equilibrium is very important. The MHD motion in the vicinity of the equilibrium is described by the following linearized MHD equation:

$$\begin{aligned} (1.1) \qquad \rho_0 \frac{\partial^2 \xi}{\partial t^2} &= -K\xi \equiv \operatorname{grad}\{\gamma P_0 (\operatorname{div} \xi) + (\operatorname{grad} P_0) \cdot \xi\} \\ &\quad + B_0 \times \operatorname{rot}(\operatorname{rot}(B_0 \times \xi)) - (\operatorname{rot} B_0) \times \operatorname{rot}(B_0 \times \xi), \end{aligned}$$

for the Lagrangian displacement vector field $\xi : \Omega \subset R^3 \to R^3$. Here, the equilibrium quantities ρ_0 (=density), P_0 (=pressure), B_0 (=magnetic field), are given bounded smooth functions which satisfy the equilibrium condition:

$$(1.2) \qquad \begin{aligned} &\operatorname{grad} P_0 = j_0 \times B_0, \quad \operatorname{div} B_0 = 0, \\ &\text{with } j_0 = \operatorname{rot} B_0 \ (\text{=electric current density}), \\ &P_0 \geq c_P > 0, \quad \rho_0 \geq c_\rho > 0 : \text{arbitrary}. \end{aligned}$$

We assume in (1.1) that the specific heat ratio γ is a positive constant. We impose a slip condition: $\xi \cdot n = 0$ on the boundary $\partial\Omega$ where n is the unit normal on the boundary.

In this paper, we shall study some spectral properties of the operator $\rho_0^{-1} K$ in a Hilbert space $L^2(\Omega; \rho_0 dr)^3$. In particular, we shall prove the absolute continuity of the essential spectrum and the discreteness of the embedded eigenvalues in the continuum under some assumptions on the

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shape of the region Ω and the symmetry of the equilibrium. We assume hereafter that Ω is a flat torus in R^3:

$$\Omega = \{(x, y, z) : x, y, z \in S \equiv R/2\pi Z\} = S^3.$$

We consider the equilibrium where the quantities B_0 and P_0 depend only on one variable x. Then, these one-dimensional equilibrium quantities are given as:

$$\begin{aligned} B_0 &= (0, b(x)\sin\phi(x), b(x)\cos\phi(x)) \\ P_0 &= c - \frac{1}{2}b(x)^2, \end{aligned}$$

where $b(x)$ and $\phi(x)$ are arbitrary smooth functions with the property:

$$b(0) = b(2\pi),\ \ \phi(0) = \phi(2\pi) \mod 2\pi,$$

and c is a sufficiently large positive constant. Due to the symmetry of the coefficients, we can decompose ξ into (m,n) Fourier modes:

$$e^{imy+inz}\xi(x),\ \ m, n : \text{integers}$$

and the force operator ${\rho_0}^{-1}K$ is realized in the decomposed space as a selfadjoint operator with a form (see Kako [1]):

$$K = \begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \tag{1.3}$$

where A, B, B^* and C are differential/multiplication operators given as

$$\begin{aligned} A &= -\frac{d}{dx}(b^2 + \gamma P_0)\frac{d}{dx} + b^2{n_\phi}^2 \\ B &= \left(-i\frac{d}{dx}(b^2 + \gamma P_0)m_\phi, -i\frac{d}{dx}\gamma P_0 n_\phi\right) \\ B^* &= \begin{pmatrix} -i(b^2 + \gamma P_0)m_\phi \frac{d}{dx} \\ \\ -i\gamma P_0 n_\phi \frac{d}{dx} \end{pmatrix} \\ C &= \begin{pmatrix} m_\phi^2(b^2 + \gamma P_0) + b^2 n_\phi^2 & m_\phi n_\phi \gamma P_0 \\ \\ m_\phi n_\phi \gamma P_0 & n_\phi^2 \gamma P_0 \end{pmatrix} \end{aligned} \tag{1.4}$$

with

$$n_\phi = n(\cos\phi) + m(\sin\phi),\ \ m_\phi = m(\cos\phi) - n(\sin\phi). \tag{1.5}$$

We can construct a selfadjoint operator ${\rho_0}^{-1}K$ in $L^2(S;\rho_0 dx)^3$ with the resolvent expression:

$$(1.6)\quad (\rho_0^{-1}K+\lambda)^{-1} = \begin{pmatrix} E_\lambda^{-1} & -E_\lambda^{-1}BC_\lambda^{-1} \\ -C_\lambda^{-1}B^*E_\lambda^{-1} & C_\lambda^{-1}+C_\lambda^{-1}B^*E_\lambda^{-1}BC_\lambda^{-1} \end{pmatrix}\rho_0,$$

where $E_\lambda = A_\lambda - BC_\lambda^{-1}B^*$ with $A_\lambda = A+\lambda\rho_0$ and $C_\lambda = C+\lambda\rho_0$ (see Kako [1]).

§2. The essential spectrum of $\rho_0^{-1}K$

From the resolvent expression (1.6), we can extract some spectral properties of the operator $\rho_0^{-1}K$ such as the range of the essential spectrum.

Theorem 2.1 (Kako [1]). *The operator $\rho_0^{-1}K$ has a natural self-adjoint realization in the Hilbert space $L^2(S;\rho_0 dx)^3$, and the essential spectrum of $\rho_0^{-1}K$ consists of σ_A and σ_S with*

$$\sigma_A = \{\lambda : \lambda = \omega_A(x),\ 0 \le x \le 2\pi\}$$

and

$$(2.1)\qquad \sigma_S = \{\lambda : \lambda = \omega_S(x),\ 0 \le x \le 2\pi\},$$

where

$$\omega_A \equiv b^2{n_\phi}^2/\rho_0\ (Alfv\acute{e}n\ frequency)$$

and

$$(2.2)\qquad \omega_S \equiv \omega_A\gamma P_0/(b^2+\gamma P_0)\ (slow\ magnetosonic\ frequency).$$

The proof of this theorem is based on the following expression of the resolvent:

$$(2.3)\qquad (\rho_0^{-1}K+\lambda_0)^{-1} = \begin{pmatrix} 0 & -GF_{\lambda_0}^{-1} \\ -F_{\lambda_0}^{-1}G^* & F_{\lambda_0}^{-1} \end{pmatrix}\rho_0 + R_1.$$

with $G = A_{\lambda_0}^{-1}B$ and $G^* = B^*A_{\lambda_0}^{-1}$. Where the remainder R_1 is a trace class operator in $L^2(S;\rho_0 dx)^3$ and G is a Hilbert-Schmidt class operator from $L^2(S;\rho_0 dx)^2$ to $L^2(S;\rho_0 dx)$, and F_{λ_0} is a multiplication operator:

$$(2.4)\qquad F_{\lambda_0} = \rho_0\begin{pmatrix} \omega_A(x)+\lambda_0 & 0 \\ 0 & \omega_S(x)+\lambda_0 \end{pmatrix}.$$

Introducing unitary operators U and U^* in $L^2(S;\rho_0 dx)^3$ as

$$(2.5)\qquad U=\rho_0^{-1}\exp\begin{pmatrix}0 & G\\ -G^* & 0\end{pmatrix} \text{ and } U^*=\rho_0^{-1}\exp\begin{pmatrix}0 & -G\\ G^* & 0\end{pmatrix},$$

we have

$$(2.6)\qquad U(\rho_0^{-1}K+\lambda_0)^{-1}U^*=\begin{pmatrix}0 & 0\\ 0 & F_{\lambda_0}^{-1}\end{pmatrix}\rho_0+R_2,$$

where R_2 is a trace class operator which maps $L^2(S;\rho_0 dx)^3$ to the Sobolev space of order two: $H^2(S;\rho_0 dx)^3$. Applying the trace class perturbation theory (see Kato [2]), we can prove that there exists an absolutely continuous spectrum which consists of the union of the ranges of functions $\omega_A(x)$ and $\omega_S(x)$ (see Kako [1]).

§3. Application of Mourre's estimate

We shall apply Mourre's commutator estimate to the present problem and prove the discreteness of embedded eigenvalues in the continuum as well as the absolute continuity of the continuous spectrum in the complement of eigenvalues under the following assumption.

Assumption. The functions ω_A and ω_S are smooth and a number of critical points $x_A^c(k), k=1,2,...,M$ and $x_S^c(l), l=1,2,...,N$:

$$\omega_A'(x_A^c(k))=\omega_S'(x_S^c(l))=0$$

are finite.

We define functions $H_A(x)$ and $H_S(x)$ as

$$H_A(x)=(\omega_A(x)+\lambda_0)^{-1} \text{ and } H_S(x)=(\omega_S(x)+\lambda_0)^{-1}.$$

Let T be an unitary operator from $L^2(S;\rho_0 dx)^3$ to $L^2(S)^3$:

$$(3.1)\qquad T: L^2(S;\rho_0 dx)^3 \ni f \mapsto \rho_0^{1/2} f \in L^2(S)^3.$$

Then the operator $T\rho_0^{-1}KT^{-1}=\rho_0^{-1/2}K\rho_0^{-1/2}$ is unitarily equivalent to $\rho_0^{-1}K$. We denote this selfadjoint operator in $L^2(S)^3$ by K'. We introduce a conjugate operator H to K' as

$$(3.2)\qquad H\equiv\begin{pmatrix}0 & 0 & 0\\ 0 & H_A'(x)\frac{d}{dx}+\frac{d}{dx}H_A'(x) & 0\\ 0 & 0 & H_S'(x)\frac{d}{dx}+\frac{d}{dx}H_S'(x)\end{pmatrix}.$$

Proposition 1. *Under the assumption of the smoothness of ω_A and ω_S, the operator iH with domain:*

$$\begin{aligned}\mathcal{D}(iH) = \{f : f =&(f_1, f_2, f_3)^t, f_k \in L^2(S), k = 1,2,3,\\ &H'_A(x)\frac{d}{dx}f_2, H'_S(x)\frac{d}{dx}f_3 \in L^2(S)\}\end{aligned}$$

is skew selfadjoint in $L^2(S)^3$

Proof. Let $a(x)$ be a real valued continuously differentiable function. We claim that an operator A defined as

$$\begin{aligned}\mathcal{D}(A) =&\{f : f \in L^2(S), a(x)\frac{d}{dx}f \in L^2(S)\}\\ Af =&i(a(x)\frac{d}{dx}f + \frac{d}{dx}a(x)f)\end{aligned}$$

is selfadjoint. In fact, for $f, g \in L^2(S)$ with the property that $a(x)\frac{d}{dx}g$, $a(x)\frac{d}{dx}f \in L^2(S)$, we have

$$\int_S \frac{d}{dx}(a(x)g(x)\overline{f(x)})dx = 0. \tag{3.3}$$

Using this identity, we can prove that A is closed and symmetric. The denseness of the range of $A \pm i$ can be shown in the standard way.

Q.E.D.

Let $E(\cdot)$ and $E_0(\cdot)$ be spectral resolutions of $D \equiv (K' + \lambda_0)^{-1}$ and

$$D_0 \equiv \begin{pmatrix} 0 & 0 \\ 0 & F_{\lambda_0}^{-1} \end{pmatrix} \rho_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & H_A(x) & 0 \\ 0 & 0 & H_S(x) \end{pmatrix}.$$

Then the commutator $[H, D_0] \equiv HD_0 - D_0H$ between H and D_0 can be calculated as

$$HD_0 - D_0H = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2H'_A(x)^2 & 0 \\ 0 & 0 & 2H'_S(x)^2 \end{pmatrix}. \tag{3.4}$$

This operator is nonnegative. Using Proposition 1 and this expression of the commutator $[H, D_0]$, we can prove the following lemma.

Lemma 2. *Let $\Delta \subset R$ be such that*

$$(H_A^{-1}(\Delta) \cup H_S^{-1}(\Delta)) \cap \{x_A^c(k)\}_{k=1}^N \cap \{x_S^c(l)\}_{l=1}^M = \emptyset$$

and also let the intersection between Δ and the point spectrum of $(K' + \lambda_0)^{-1}$ be empty. Then we have the following Mourre type estimate:

$$E(\Delta)[H, D]E(\Delta) \geq \alpha E(\Delta) + Q,\ \alpha > 0, \tag{3.5}$$

where Q is a compact operator.

Proof. Since $D - D_0$ is compact, $E(\Delta) - E_0(\Delta)$ is also compact. Furthermore, $(D - D_0)H$ is a compact operator in $L^2(S)^3$, since the difference $D - D_0$ is bounded from $L^2(S)^3$ to $H^2(S)^3$. Hence we have that the operator $E(\Delta)[H, D]E(\Delta) - E_0(\Delta)[H, D_0]E_0(\Delta)$ is compact. Using the non-negativity of the commutator $[H, D_0]$ and the assumption for the interval Δ, we have the estimate (3.5). Q.E.D.

From this lemma, applying the results of Mourre (see [4 , Theorem 4.7 and Theorem 4.9] and [3]), we have the follwing theorem.

Theorem 3. *Let Δ be as in Lemma 2. Then the operator $(K' + \lambda_0)^{-1}$ restricted to the subspace $E(\Delta)L^2(S)^3$ is absolutely continuous except for some discrete set. The absolutely continuous part is unitarily equivalent to a part of the multiplication operator $F_{\lambda_0}^{-1}\rho_0$.*

From this theorem, we can have the corresponding results for the absolute continuity of the continuous spectrum of $\rho_0^{-1}K$ and the unitary equivalence between the absolutely continuous part of the operator $\rho_0^{-1}K$ and the multiplication opetator $\rho_0^{-1}F_0$.

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Singularities of Solutions to System of Wave Equations with Different Speed

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Dedicated to the sixtieth anniversary of Professor ShigeToshi Kuroda

§1. Introduction and results

We consider the following system of wave equations

$$
\begin{cases} \Box_{c_1} u = f(u, v) \\ \Box_{c_2} v = g(u, v) \end{cases} \tag{1.1}
$$

where $\Box_c = (1/c^2)\partial^2/\partial t^2 - \sum_{j=1}^n \partial^2/\partial x_j^2$ and c_1 and c_2 are positive constants. We assume that $f(\cdot, \cdot)$ and $g(\cdot, \cdot)$ are in C^∞. In what follows, we shall study the singularities of the solutions to (1.1) when the solutions are 'conormal distributions' to some hyperplanes. Before the statement of main theorems, we define conormal distributions.

Definition (Conormal distributions). *Let $\Omega \subset \mathbb{R}^n$ be a domain. Let L be a C^∞-manifold in Ω. We call that u is in $H^s(L, \infty)$ in Ω if*

$$M_1 \circ M_2 \circ \cdots \circ M_l u \in H^s_{loc}(\Omega) \quad \textit{for } l = 0, 1, 2, \dots ,$$

where each M_j is a C^∞ vector field which is tangent to L.

We can define the space of conormal distributions not only for a C^∞-manifold but also for a union of two hypersurfaces which intersect each other transversally.

Now we shall state the main results. Let $\omega \in S^{n-1}$ and $L_{ij} = \{(t, x) \in \mathbb{R}^n; c_i t + (-1)^j \omega \cdot x = 0\}$ for i, $j = 1$, 2.

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Theorem 1. *Let Ω be a neighborhood of the origin of $\mathbb{R}^{n+1}$, $i = 1$ or 2 and $j = 1$ or 2. Suppose that u, v are in $H^s_{\mathrm{loc}}(\Omega)$ for $s > (n+1)/2$, u and v are solutions to* (1.1) *and*

$$u,\, v \in\ H^s(L_{ij}, \infty) \quad \textit{in } \Omega \cap \{t < 0\},$$

then

$$u,\, v \in\ H^s(L_{ij}, \infty) \quad \textit{in } K$$

where K is the domain of dependence with respect to $\Omega \cap \{t < 0\}$.

Theorem 2. *Let Ω be a neighborhood of the origin of $\mathbb{R}^{n+1}$ and i, i', j, $j' \in \mathbb{N}$ with $i + i' = 3$, $j + j' = 3$. Suppose that $0 < c_1 < c_2$, u, v are in $H^s_{\mathrm{loc}}(\Omega)$ for $s > (n+1)/2$, u and v are solutions to* (1.1) *and*

$$u,\, v \in\ H^s(L_{ij} \cup L_{i'j}, \infty) \quad \textit{in } \Omega \cap \{t < 0\},$$

then

$$u,\, v \in\ H^s(L_{ij} \cup L_{i'j} \cup L_{ij'} \cup L_{i'j'}, \infty) \quad \textit{in } K$$

where K is the domain of dependence with respect to $\Omega \cap \{t < 0\}$.

Theorem 3. *Let Ω be a neighborhood of the origin of $\mathbb{R}^{n+1}$ and i, i', j, $j' \in \mathbb{N}$ with $i + i' = 3$, $j + j' = 3$. Suppose that $0 < c_1 < c_2$, u, v are in $H^s_{\mathrm{loc}}(\Omega)$ for $s > (n+1)/2$, u and v are solutions to* (1.1) *and*

$$u,\, v \in\ H^s(L_{ij} \cup L_{ij'}, \infty) \quad \textit{in } \Omega \cap \{t < 0\},$$

then

$$u,\, v \in\ H^s(L_{ij} \cup L_{i'j} \cup L_{ij'} \cup L_{i'j'}, \infty) \quad \textit{in } K$$

where K is the domain of dependence with respect to $\Omega \cap \{t < 0\}$.

J.M. Bony has obtained the same result for scalar strictly hyperbolic equations in [3]. So our results are not full of originalities. But the author believes that our proofs are new and simple.

§2. Proof of Theorem 1

We set $M = t\partial_t + x \cdot \partial_x$ and $M_k = \omega_k \partial_t + c_i \partial_{x_k}$ for $k = 1, \ldots, n$. It is easy to prove the following proposition.

Proposition 1. *$M_1, \dots, M_n$ are linearly independent on $\mathbb{R}^{n+1}$ and M, $M_1, \dots, M_n$ are linearly independent on $\mathbb{R}^{n+1}\backslash L_{ij}$.*

Proof of Theorem 1.

$$\begin{aligned}\square(Mu) =&[\square, M]u + Mf(u,v)\\ =&2\square u + Mf(u,v)\\ =&2f(u,v) + Mf(u,v).\end{aligned} \tag{2.1}$$

Similarly we have

$$\square(Mu) = 2g(u,v) + Mf(u,v). \tag{2.2}$$

Since u and v are in $H^s_{\text{loc}}(\Omega)$, we have that $2f(u,v) + Mf(u,v)$ and $2g(u,v)+Mf(u,v)$ are in $H^{s-1}_{\text{loc}}(\Omega)$ and Mu, Mv are in $H^s_{\text{loc}}(\Omega\cap\{t<0\})$. Using the energy estimate for $\square_{c_1}$ and $\square_{c_2}$, we consequently have that Mu, $Mv \in H^s_{\text{loc}}(K)$. Repeating this argument, we have

$$M^l u,\ M^l v \in H^s_{\text{loc}}(K). \tag{2.3}$$

It is easy to see that

$$M^l_k u,\ M^l_k v \in H^s_{\text{loc}}(K) \quad \text{for } \forall k,\ \forall l \in \mathbb{N}. \tag{2.4}$$

(2.3) and (2.4) yield Theorem 1.

§3. Proof of Theorem 2 and Theorem 3

Proof of Theorem 2. We put $M_a = t\partial_t + (x-a)\cdot\partial_x$ for $a \in \mathbb{R}^n$. Using the same argument as in the proof of Theorem 1, we have

$$M^l_a u,\ M^l_a v \in H^s_{\text{loc}}(K) \quad \text{for } \forall a \text{ with } a\cdot\omega = 0 \text{ and } \forall l \in \mathbb{N}. \tag{3.1}$$

We divide $K\backslash\bigcup^2_{i,j=1} L_{ij}$ into the following three parts,

$$\begin{aligned}K_1 &= \{(t,x)\in K; c_1t - \omega\cdot x > 0,\ c_1t + \omega\cdot x > 0\}\\ K_2 &= \{(t,x)\in K; c_1t - \omega\cdot x < 0,\ c_2t - \omega\cdot x > 0\}\cup\\ &\quad\ \{(t,x)\in K; c_2t + \omega\cdot x > 0,\ c_1t + \omega\cdot x < 0\}\\ K_3 &= \{(t,x)\in K; c_2t - \omega\cdot x < 0 \text{ or } c_2t + \omega\cdot x < 0\}.\end{aligned}$$

We prove first that u, $v \in C^\infty$ in K_1. Let (t_0, x_0) be any point in K_1. Let $(t_0, x_0, \tau_0, \xi_0)$ be any point in $T^*_{(t_0,x_0)}\backslash 0$. We use the same argument as in the proof of the main theorem of M. Beals [1]. If M_a is elliptic at $(t_0, x_0, \tau_0, \xi_0)$ for some $a \in \mathbb{R}^n$, then from (3.1) we have

$$u,\ v \in H^{s+1} \quad \text{at} \quad (t_0, x_0, \tau_0, \xi_0). \tag{3.2}$$

When M_a is not elliptic at $(t_0, x_0, \tau_0, \xi_0)$ for all $a \in \mathbb{R}^n$, $\Box_{c_1}$ and $\Box_{c_2}$ are elliptic at $(t_0, x_0, \tau_0, \xi_0)$. In fact, we can choose $a_0 \in \mathbb{R}^n$ with $a_0 \cdot \omega = 0$ such that $c_1^2 t_0^2 - |x_0 - a_0|^2 > 0$. Then we have

$$\begin{aligned} c_1 t_0 \left(\frac{1}{c_1} |\tau_0| - |\xi_0| \right) &< t_0 |\tau_0| - |\xi_0| |x_0 - a_0| \\ &= |\xi_0 \cdot (x_0 - a_0)| - |\xi_0| |x_0 - a_0| \\ &\leq 0 . \end{aligned}$$

The same argument works for $\Box_{c_2}$. Hence

$$u,\ v \in H^{s+1} \quad \text{at} \quad (t_0, x_0, \tau_0, \xi_0). \tag{3.3}$$

From (3.2) and (3.3), we have

$$u, v \in H^{s+1} \quad \text{at } (t_0, x_0).$$

Repeating this argument, we have

$$u, v \in C^\infty \quad \text{at } (t_0, x_0). \tag{3.5}$$

Next we prove that u, v is in C^∞ on K_2. Let (t_0, x_0) be any point in K_2. Let $(t_0, x_0, \tau_0, \xi_0)$ be any point in $T^*_{(t_0, x_0)} \backslash 0$. When M_a is elliptic at $(t_0, x_0, \tau_0, \xi_0)$ for some $a \in \mathbb{R}^n$, then from (3.1) we have

$$u,\ v \in H^{s+1} \quad \text{at} \quad (t_0, x_0, \tau_0, \xi_0). \tag{3.5}$$

When M_a is not elliptic at $(t_0, x_0, \tau_0, \xi_0)$ for all $a \in \mathbb{R}^n$, the same method as in the first step proves that $\Box_{c_2}$ is elliptic at $(t_0, x_0, \tau_0, \xi_0)$. So it suffices to show that $\Box_{c_1}$ is elliptic at $(t_0, x_0, \tau_0, \xi_0)$. Since $\tau_0 t_0 + (x_0 - a) \cdot \xi_0 = 0$ for all $a \in \mathbb{R}^n$ with $a \cdot \omega = 0$, $\tau_0 t_0 + x_0 \cdot \xi_0 = a \cdot \xi_0 = 0$. Then $a \cdot \xi_0 = 0$ for all $a \in \mathbb{R}^n$ with $a \cdot \omega = 0$. Hence ξ is parallel to ω. We decompose $x_0 = x_0^{(1)} + x_0^{(2)}$ such that $x_0^{(1)}$ is parallel to ω and $x_0^{(2)}$ is perpendicular to ω. We put $a_0 = x_0^{(2)}$. Hence $x_0 - a_0 = x_0^{(1)}$ is parallel to ω. Since $c_1^2 |t_0|^2 < |x_0 - a|^2$ for all $a \in \mathbb{R}^n$, we have

$$\begin{aligned} c_1 t_0 \left(\frac{1}{c_1} |\tau_0| - |\xi_0| \right) &> t_0 |\tau_0| - |\xi_0| |x_0 - a_0| \\ &= t_0 |\tau_0| - |\xi_0 \cdot (x_0 - a_0)| \\ &= 0 \quad (\text{since } t_0 \tau_0 - \xi_0 \cdot (x_0 - a_0) = 0). \end{aligned}$$

Consequently we have

(3.6) $$u,\ v \in H^{s+2} \quad \text{at} \quad (t_0, x_0, \tau_0, \xi_0).$$

From (3.5) and (3.6), we have

$$u, v \in H^{s+1} \quad \text{at } (t_0, x_0).$$

Repeating this argument, we have

(3.7) $$u, v \in C^{\infty} \quad \text{at } (t_0, x_0).$$

The same argument for u in the second step yields that

(3.8) $$u, v \in C^{\infty} \quad \text{in } K_3.$$

(3.1), (3.4), (3.7) and (3,8) imply Theorem 2.

We can prove Theorem 3 by the same argument as in the proof of Theorem 2.

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An $L^{q,r}$-Theory for Nonlinear Schrödinger Equations

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§1. Introduction

Consider the nonlinear Schrödinger equation:

$$\text{(NLS)} \qquad \partial_t u = i(\Delta u - F(u)), \quad t \in \mathbb{R}, \quad x \in \mathbb{R}^m,$$

where $F(u) = F \circ u$ is, for example, a Nemyckii operator defined by a function $F : \mathbb{C} \to \mathbb{C}$. There is an extensive literature on this problem, but it seems that all existing work assumes that either the initial value $\phi = u(0) = u(0, \cdot)$ or the limit $\phi_\pm = \lim_{t \to \pm\infty} e^{-it\Delta} u(t)$ is in L^2. The present paper is an attempt to solve (NLS) with the data in a larger class of functions.

As in most of the work on (NLS), we convert (NLS) into integral equations such as

$$\text{(INT)} \qquad u = \Phi u \equiv u_0 - iGF(u), \quad \text{or} \quad u = \Phi_\pm u \equiv u_\pm - iG_\pm F(u).$$

Here u_0 or $u_\pm$ is a *free wave* (solution of the free Schrödinger equation $\partial_t u = i\Delta u$), and G or $G_\pm$ is an integral operator defined by

$$(1.1) \qquad \begin{aligned} Gf(t) &= \int_0^t U(t-s) f(s)\, ds, \\ G_\pm f(t) &= \int_{\pm\infty}^t U(t-s) f(s)\, ds, \quad U(t) = e^{it\Delta}. \end{aligned}$$

The free term u_0 in (INT) is usually related to the initial value $u(0) = \phi$ by

$$(1.2) \qquad u_0 = \Gamma\phi, \quad \Gamma\phi(t) = U(t)\phi,$$

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but it is often convenient to take any *free wave* without regard to the initial value. The dual operator to Γ is formally given by

$$\Gamma^* f = \int_{-\infty}^{\infty} U(-s) f(s)\, ds. \tag{1.3}$$

We note that

$$G_1 \equiv G_- - G_+ = \Gamma\Gamma^*. \tag{1.4}$$

To deal with the different operators G, $G_\pm$ and G_1 simultaneously, it is convenient to consider operators of the general form

$$G_a f(t) = \int_{-\infty}^{\infty} a(t,s) U(t-s) f(s)\, ds, \tag{1.5}$$

where a is a measurable function such that $|a(t,s)| \leqslant 1$ (cf. Yajima [14]).

Our first task is to study the continuity properties of the operators Γ and G_a between wider classes of spaces than hitherto considered. Set $L^p = L^p(\mathbb{R}^m)$, $L^{q,r} = L^r(L^q) = L^r(\mathbb{R}; L^q)$. The following results are well known (see e.g. [7]). Γ is bounded on L^2 to $L^{q,r}$ if

$$1/q + 2/mr = 1/2, \quad 1/2 - 1/m < 1/q \leqslant 1/2. \tag{1.6}$$

G_a is bounded on $L^{s,t}$ to $L^{q,r}$ if either

$$1/q + 2/mr = 1/2 \quad \text{and} \quad 1/s + 2/mt = 1/2 + 2/m, \tag{1.7}$$

or

$$1/q + 1/s = 1 \quad \text{and} \quad 1/t - 1/r = 1 - (m/2)(1/s - 1/q), \tag{1.8}$$

with the parameters restricted by

$$1/2 - 1/m < 1/q \leqslant 1/2 \leqslant 1/s < 1/2 + 1/m \tag{1.9}$$

in either case. (Note that these results do not depend on a. This is obvious since they were deduced from the Sobolev inequalities using only absolute value estimates for the Green function of $U(t)$.)

We shall extend these results to wider ranges of the parameters.

Geometric notation. In order to describe various estimates in concise form, we find it convenient to use the geometric notation introduced in [7]. Slightly deviating from [7], we denote by $\square$ the closed unit square in $\mathbb{R}^2$, defined by $0 \leqslant x, y \leqslant 1$. Then we set $L(P) = L^{q,r}$ if

$P = (1/q, 1/r) \in \square$, and write $1/q = x(P)$, $1/r = y(P)$; $y(P)$ is sometimes called the *height* of P. The norm in $L(P)$ is denoted by $|||\ : L(P)|||$ or, more briefly, by $|||\ : P|||$. (If $y(P) = 0$, it is often convenient to replace $L(P) = L^{q,\infty}$ by $BC(L^q)$, where BC is the class of bounded and continuous functions. For simplicity, we do not use this modification in the present paper.)

The segment connecting $P, Q \in \square$ is denoted by $[PQ]$, $[PQ[$, $]PQ]$, or $]PQ[$, according as it is closed, open, etc. Sometimes we regard each $P \in \square$ also as a 2-vector (with origin $O = (0,0)$), so that $P + Q$ and kP $(k > 0)$ make sense as long as they are in $\square$.

The convenience of such notations will be seen from the following rules (see [7]).

(1.10a) $\quad L(P)^* = L(P') \quad \text{if} \quad P + P' = (1,1), \quad y(P) > 0,$

(1.10b)

$$|||fg : P + Q||| \leqslant |||f : P||| \, |||g : Q|||, \quad |||f^k : kP||| = |||f : P|||^k, \quad k > 0,$$

(1.10c) $\quad L(P) \cap L(Q) \subset L(R) \subset L(P) + L(Q) \quad \text{for} \quad R \in [PQ].$

We introduce some special points in $\square$:

$$B = (1/2, 0), \quad C = (1/2 - 1/m, 1/2) \quad (C = (0, 1/4) \text{ if } m = 1),$$
$$E = (1/2 - 1/m, 1), \quad F = (1/2 - 1/m, 0)$$
$$(E = (0, 1/2),\ F = (0,0) \text{ if } m = 1),$$
$$B' = (1/2, 1), \quad C' = (1/2 + 1/m, 1/2) \quad (C' = (1, 3/4) \text{ if } m = 1),$$
$$E' = (1/2 + 1/m, 0), \quad F' = (1/2 + 1/m, 1)$$
$$(E' = (1, 1/2),\ F' = (1,1) \text{ if } m = 1).$$

We further introduce the triangles $T = \triangle(BEF)$ and $T' = \triangle(B'E'F')$; these are assumed to be open except that B and B' are included. Note that $[BC[\in T$, $[B'C'[\in T'$.

With these notations, the known results (1.6)–(1.9) can be stated as follows.

(i) Γ is bounded on L^2 to $L(P)$ for any $P \in [BC[$.

(ii) G_a is bounded on $L(\bar{P})$ to $L(P)$ if either

(iia) $P \in [BC[$ and $\bar{P} \in [B'C'[$, or

(iib) $P \in T$ and $\bar{P} \in T'$ with

$$x(P) + x(\bar{P}) = 1, \quad x(\bar{P}) + 2y(\bar{P})/m - x(P) - 2y(P)/m = 2/m.$$

§2. The operator G_a

In this section we generalize the estimates (ii) for G_a given in Section 1, using the geometric notation throughout. It is convenient to introduce the linear functional

$$\pi(P) = x + 2y/m \quad \text{for} \quad P = (x, y) \in \square. \tag{2.1}$$

Theorem 2.1. *G_a is bounded on $L(\bar{P})$ to $L(P)$ if $P \in T$, $\bar{P} \in T'$ with $\pi(\bar{P}) - \pi(P) = 2/m$.*

Remark. Theorem 2.1 can be improved by admitting certain points P on $[BF[$ and $\bar{P}$ on $[B'F'[$. The improvement requires deeper results, and will be given in next section.

Theorem 2.1 may be expressed in still another way. The set of $P \in \mathbb{R}^2$ with $\pi(P) = \text{const}$ is a straight line with slope $-m/2$; such a line [or a segment on it] will be called a π-line [or π-segment]. $[BC[$ and $[B'C'[$ are π-segments. T is composed of a one-parameter family of π-segments l (such as $[BC[$), and likewise T' by a family of segments $\bar{l}$ of π-segments (such as $[B'C'[$). The constant value of $\pi(P)$ for $P \in l$ will be denoted by $\pi(l)$, and similarly for $\bar{l}$. The possible values of $\pi(l)$ range over $(1/2 - 1/m, 1/2 + 1/m)$ ($(0, 1)$ if $m = 1$), and those of $\pi(\bar{l})$ over $(1/2 + 1/m, 1/2 + 3/m)$ ($(2, 3)$ if $m = 1$); these intervals do not overlap. l will be said to be *conjugate* to $\bar{l}$, and vice versa, if $\pi(\bar{l}) - \pi(l) = 2/m$. For each l, there is a conjugate $\bar{l}$, and vice versa. In particular, $[BC[$ and $[B'C'[$ are conjugate. It is easy to see that a conjugate pair l, $\bar{l}$ have equal length, while the upper end of l and the lower end of $\bar{l}$ have equal height.

Theorem 2.1 is equivalent to saying that given any conjugate pair l, $\bar{l}$, G_a is bounded on $L(\bar{P})$ to $L(P)$ for any $P \in l$ and any $\bar{P} \in \bar{l}$.

It is obvious how Theorem 2.1 generalizes the known results (iia) and (iib) (see Section 1). In (iia), P and $\bar{P}$ were restricted on a particular conjugate pair $[BC[$, $[B'C'[$. In (iib), P may be on any l and $\bar{P}$ on any $\bar{l}$ if l, $\bar{l}$ are conjugate, but they had to correspond to each other one to one due to the condition $x(P) + x(\bar{P}) = 1$. Theorem 2.1 unites these two cases by eliminating the restrictions.

Theorem 2.1 will be proved by interpolating between these special cases using the following lemma.

Interpolation Lemma. *Assume that none of P, $\bar{P}$, Q, $\bar{Q}$ has height zero. If a linear operator maps $L(\bar{P})$ into $L(P)$ and $L(\bar{Q})$ into*

$L(Q)$ (continuously), then it maps $L((1-\theta)\bar{P}+\theta\bar{Q})$ into $L((1-\theta)P+\theta Q)$, where $0<\theta<1$.

This lemma follows directly from Bergh-Löfström [1;Theorem 5.1.2], which shows that $(L(P), L(Q))_{[\theta]} = L((1-\theta)P+\theta Q)$ with equal norm.

To prove Theorem 2.1, we may assume that $y(P), y(\bar{P}) > 0$, since the only case to the contrary is $P = B$, $\bar{P} \in [B'C'[$, for which the result is known by (iia). We begin the proof by invoking the map $\bar{P} \to P$ involved in (iib); it is defined by $x(P) + x(\bar{P}) = 1$ and $\pi(\bar{P}) - \pi(P) = 2/m$, and can be extended to an *affine* map Λ of $\mathrm{cl}(T')$ onto $\mathrm{cl}(T)$ (cl denotes the closure). Λ sends B' into B, E' into F, and F' into E. The known special case (iib) shows that G_a is bounded on $L(\bar{P})$ to $L(P)$ if $P = \Lambda(\bar{P})$, *provided that $P \in T$, $\bar{P} \in T'$.*

Now take any pair $P \in T$, $\bar{P} \in T'$ with $\pi(\bar{P}) - \pi(P) = 2/m$. We have to show that G_a maps $L(\bar{P})$ to $L(P)$. First take the case that $\bar{P}$ is above $[B'C'[$, which implies that P is above $[BC[$. Take a point $\bar{Q} \in T'$ sufficiently close to F' that the prolongation of $[\bar{Q}\bar{P}]$ meets $[B'C'[$, say at $\bar{R}$. Let Q be the image of $\bar{Q}$ under Λ, so that Q is close to E. Prolong $[QP]$ until it meets $[BC[$, say at R (this is possible if Q is sufficiently close to E, which is guaranteed if $\bar{Q}$ is close enough to F').

G_a maps $L(\bar{Q})$ to $L(Q)$ by (iib), because $Q = \Lambda(\bar{Q})$. G_a maps $L(\bar{R})$ into $L(R)$ by (iia), because $R \in [BC[$ and $\bar{R} \in [B'C'[$. According to Interpolation Lemma, therefore, the theorem will follow if we show that P divides $[QR]$ at the same ratio as $\bar{P}$ does $[\bar{Q}\bar{R}]$.

This is a simple geometric problem. Indeed, let θ be such that $\bar{P} = (1-\theta)\bar{Q}+\theta\bar{R}$. Since π is linear, we have $\pi(\bar{P}) = (1-\theta)\pi(\bar{Q})+\theta\pi(\bar{R})$. On the other hand, $\pi(\bar{R}) = \pi(R) + 2/m$, $\pi(\bar{Q}) = \pi(Q) + 2/m$, and $\pi(\bar{P}) = \pi(P) + 2/m$, by conjugacy. Hence $\pi(P) = \pi((1-\theta)Q + \theta R)$. But π is injective on $[QR]$, which has slope different from $-m/2$. It follows that $P = (1-\theta)Q + \theta R$, as required.

The case that $\bar{P}$ is below $[B'C'[$ follows from this by duality, or one may repeat the above arguments with $\bar{Q}$ close to E'. This completes the proof of Theorem 2.1.

§3. The operators Γ and Γ^*

According to the known result (i) (see Section 1), Γ is bounded on L^2 to $L(P)$ if $P \in [BC[$. In this section, we generalize this result to some other domain spaces, and deduce corresponding results for the dual operator Γ^*. We begin by noting that certain $L(P)$'s are never realized by Γ.

Lemma 3.1. *If $P \in \Box$, $P \neq B$, is on or to the right of $[BE]$ (i.e. $x(P) + y(P)/m \geqslant 1/2$), there is no nontrivial $\phi \in \mathcal{S}'$ such that $\Gamma\phi \in L(P)$. (Note that $[BE]$ has slope $-m$, twice the slope of π-lines.)*

This is an immediate consequence of the following lemma (due to Strauss [10] for $q \geqslant 2$), which limits the decay rate of a free wave.

Decay Lemma. *For any nontrivial $\phi \in \mathcal{S}'$ and $1 \leqslant q \leqslant \infty$, one has*

$$\|U(t)\phi\|_q \geqslant K\langle t\rangle^{m(1/q-1/2)}, \quad t \in \mathbb{R}, \quad \langle t\rangle = (1+t^2)^{1/2},$$

where $K > 0$ is a constant depending on ϕ. (Set $\|\psi\|_q = +\infty$ if $\psi \notin L^q$.)

Proof. Let $u = \Gamma\phi$, $v = \Gamma\psi$, with $0 \neq \phi \in \mathcal{S}'$, $\psi \in \mathcal{S}$. Then $\langle u(t), v(t)\rangle = \langle \phi, \psi\rangle \equiv K$, hence $|K| \leqslant \|u(t)\|_q \|v(t)\|_{q'}$. If we choose a special function $\psi(x) = \exp[-(x-a)^2/4s]$, $s > 0$, a direct computation gives $\|v(t)\|_{q'} = c\langle t\rangle^{m(1/q'-1/2)}$. Hence $\|u(t)\|_q \geqslant c|K|\langle t\rangle^{m(1/q-1/2)}$. This proves the required result if we can show that $K \neq 0$ for some choice of a and s. But $K = 0$ for all a and s would imply that $e^{-s\Delta}\phi = 0$ for $s > 0$, as is seen from Green's formula. On passing to the limit $s \to 0$, this gives $\phi = 0$, a contradiction.

We now prove that Γ maps certain L^p's into certain $L(P)$'s. To this end we introduce further special points

$$D = ((m-2)/2(m-1), m/2(m-1)) \in [BE[,$$
$$(D = E = (0, 1/2) \ \text{ if } \ m = 1),$$
$$D' = (m/2(m-1), (m-2)/2(m-1)) \in [B'E'[,$$
$$(D' = E' = (1, 1/2) \ \text{ if } \ m = 1).$$

(Note that O, C, D are colinear.) We set $\hat{T} = \triangle(BCD) \subset T$, which is supposed to include the side $]CD[$ (except for $m = 2$) but no other boundary points. Similarly we define $\hat{T}' = \triangle(B'C'D') \subset T'$.

Theorem 3.2. *Let $1/2 < 1/p < m/2(m-1)$ ($1/2 < 1/p \leqslant 1$ if $m = 1$). Then Γ is bounded on L^p to $L(P)$ for any $P \in \hat{T}$ with $\pi(P) = 1/p$. Γ^* is bounded on $L(\bar{P})$ to $L^{p'}$ for any $\bar{P} \in \hat{T}'$ with $\pi(\bar{P}) = 1/p' + 2/m$.*

Corollary 3.3. *If $(2m+2)/(m+2) < p \leqslant 2$, Γ is bounded on L^p to $L^q(\mathbb{R} \times \mathbb{R}^m)$ for $q = (m+2)p/m$.*

Remark. Corollary 3.3 generalizes the well known result of Strichartz [12]. The restriction on p comes from the fact that the line

$\pi(x,y) = 1/p$ must meet the diagonal $x = y$ *inside* $\hat{T}$. The lower limit of the possible values of q is $2 + 2/m$, and corresponds to the maximal decay.

Proof of Theorem 3.2. The following is an adaptation of a method used by Giga [4] for the heat operator $e^{-t\Delta}$. First fix q such that

$$(3.1) \qquad 1/2 - 1/m < 1/q < 1/2 \quad (0 \leqslant 1/q < 1/2 \ \text{ if } \ m = 1).$$

Let $Q \in [BC[$ with $x(Q) = 1/q$, so that $\pi(Q) = 1/2$. The special case (i) (Section 1) shows that Γ maps L^2 (continuously) into $L(Q)$. On the other hand, $\phi \in L^{q'}$ implies that $\|U(t)\phi\|_q \leqslant c|t|^{-m(1/2-1/q)}\|\phi\|_{q'}$. Thus Γ maps $L^{q'}$ into $L_*(R)$, where $R = (1/q, m(1/2 - 1/q)) \in [BE[$, hence $\pi(R) = 1/q'$, and where L_* denotes the *weak* L-space with respect to the time variable. Since Q and R are on the same vertical line $x = 1/q$, it follows from Marcinkiewitz's interpolation theorem that if

$$(3.2) \qquad 1/2 < 1/p < 1/q',$$

then Γ maps L^p into $L(P)$ with

$$(3.3) \qquad x(P) = 1/q \quad \text{and} \quad \pi(P) = 1/p,$$

provided that

$$(3.4) \qquad y(P) \leqslant 1/p.$$

We now change the viewpoint and vary q, with $p < 2$ fixed. Then (3.3) shows that P moves on a π-segment with $x(P) = 1/q$, restricted by $1/2 - 1/m < x(P) < 1/p'$, due to (3.1) and (3.2). This proves the theorem for $m \leqslant 2$, since (3.4) is automatically satisfied. If $m \geqslant 3$, (3.4) introduces a new restriction; combined with (3.3), it requires that $y(P) \leqslant \pi(P) = x(P) + 2y(P)/m$, hence $x(P)/y(P) \geqslant (m-2)/m$. This means that P must be below the ray extending $[OD[$. Thus P must belong to $\hat{T}$. Summing up, we have proved Theorem 3.2.

If $p > 2$, Theorem 3.2 is not true. However, there is an analogous result with L^p replaced by a certain subspace. As is well known, the Fourier transform $\mathcal{F}$ on $\mathbb{R}^m$ maps $L^{p'}$ into L^p. We shall denote its image by $\tilde{L}^p$, and make it into a normed space with the norm $\|\phi\|_p^\sim = \|\mathcal{F}^{-1}\phi\|_{p'}$. Obviously $\tilde{L}^p$ is a Banach space, isometrically isomorphic with $L^{p'}$.

Theorem 3.4. *Let $2 \leqslant p \leqslant \infty$. The map Γ is bounded on $\tilde{L}^p$ to $L(P)$ if P is in the triangle $\triangle(OBC)$ with $\pi(P) = 1/p$. The triangle is assumed to exclude $]OC[$ but otherwise closed.*

Corollary 3.5. *If $2 \leqslant p \leqslant \infty$, Γ is bounded on $\tilde{L}^p$ to $L^q(\mathbb{R} \times \mathbb{R}^m)$ for $q = (m+2)p/m$.*

Proof of Theorem 3.4. In view of the definition of $\tilde{L}^p$, Theorem 3.4 is equivalent to saying that $\Gamma \circ \mathcal{F}$ maps $L^{p'}$ into $L(P)$ if P is as stated in the theorem. This is true for $p' = 2 = p$ by (i). Moreover, $\Gamma \circ \mathcal{F}$ maps L^1 into $BC(L^\infty)$. Indeed, $\psi \in L^1$ implies $U(t)\mathcal{F}\psi = \mathcal{F}\omega(t)$, where $\omega(t)(\xi) = \exp(-it\xi^2)\psi(\xi)$, so that $\omega \in BC(L^1)$, hence $\Gamma\mathcal{F}\psi = \mathcal{F}\omega \in BC(L^\infty)$. The assertion then follows by another application of the interpolation theorem [1;Theorem 5.1.2] to the pair $BC(L^\infty) \subset L(O)$ and $L(P)$, with P varying on $[BC[$.

Unfortunately, the range of the P's in Theorems 3.2, 3.4 does not cover the basic triangle T. But this does not mean that the region left out cannot be realized. In fact it is easy to see that $\Gamma\phi \in L(P)$ for all $P \in \Box$ to the left of $[BE[$, if ϕ is a sufficiently nice function. Actually we are not so much interested in P outside the triangle $T = \triangle(BEF)$. Thus the following theorem gives a convenient criterion; here Σ denotes the Ginibre-Velo class $H^1 \cap L^2_1$, where L^2_1 is the weighted L^2-space $\langle x \rangle^{-1} L^2$, $\langle x \rangle = (1 + |x|^2)^{1/2}$.

Theorem 3.6. *For any $P \in T \cup [BF[$, Γ is bounded on Σ to $L(P)$. For any $\bar{P} \in T' \cup [B'F'[$, Γ^* is bounded on $L(\bar{P})$ to Σ^*.*

Proof. $\phi \in \Sigma$ implies that $\phi \in L^{q'}$ for $1/2 \leqslant 1/q' < 1/2 + 1/m$ and that $\phi \in H^1$. Hence $\|U(t)\phi\|_q \leqslant K\langle t \rangle^{-m(1/2-1/q)}$ (maximal decay) for $1/2 - 1/m < q \leqslant 1/2$, which implies that $\Gamma\phi \in L^{q,r}$ for $0 \leqslant 1/r < m(1/2 - 1/q)$. Thus $\Gamma\phi \in L(P)$ for any $P \in T \cup [BF[$. The second part of the theorem follows by duality.

Finally we prove the promised improvement of Theorem 2.1. For this we need another set of special points. Let

$$H = ((m-2)/2(m-1), 0), \quad H' = (m/2(m-1), 1)$$
$$(H = (0,0),\ H' = (1,1) \text{ if } m = 1).$$

Theorem 2.1 (improved). *Let $P \in T \cup [BH[$ and $\bar{P} \in T' \cup [B'H'[$ with $\pi(\bar{P}) - \pi(P) = 2/m$. Then G_a is bounded on $L(\bar{P})$ to $L(P)$. (H and H' are introduced to avoid empty statement.)*

Proof. It suffices to consider the case $P \in [BH[$ or $\bar{P} \in [B'H'[$. In the first case, let $P = (1/q, 0) \in [BH[$ and set $g = G_a f$, $f \in L(\bar{P})$. Then

$$
\begin{aligned}
g(t) &= \int a(t,s)U(t-s)f(s)\,ds \\
&= \int a(t,s+t)U(-s)f(s+t)\,ds = \Gamma^*(a_t f_t),
\end{aligned}
\tag{3.5}
$$

where

$$
a_t(s) = a(t, s+t), \quad f_t(s) = f(s+t). \tag{3.6}
$$

But Γ^* is bounded on $L(\bar{P})$ to L^q by Theorem 3.2, since $\pi(\bar{P}) - 1/q = 2/m$. Hence $\|g(t)\|_q \leqslant c|||a_t f_t : \bar{P}||| \leqslant c|||f_t : \bar{P}||| = c|||f : \bar{P}|||$. This shows that G_a is bounded on $L(\bar{P})$ to $L^{q,\infty} = L(P)$. The case $\bar{P} \in [B'H'[$ then follows by duality.

§4. Further estimates

1. *Free waves.* By a free wave in general we mean a solution $u \in \mathcal{S}'(\mathbb{R} \times \mathbb{R}^m)$ of the free Schrödinger equation $\partial_t u - i\Delta u = 0$. Such u may be identified with a function $u \in C^\infty(\mathbb{R}; \mathcal{S}')$, where $\mathcal{S}' = \mathcal{S}'(\mathbb{R}^m)$ (see Schwartz [8]). Equivalently, we may write $u = \Gamma\phi$, where $\phi = u(0) \in \mathcal{S}'$. In fact $\{U(t)\}$ forms a C^∞-group on $\mathcal{S}'$. Thus $\Gamma\phi$ is a general form of the free wave if we allow all $\phi \in \mathcal{S}'$. It is also well known that $U(t)$ forms a strongly continuous group on Σ (for Σ see Section 3). Since Σ is a Hilbert space, it follows by duality that $U(t)$ also forms a strongly continuous group on Σ^*. However, these groups are not uniformly bounded.

2. *Free waves in $L(P)$.* We denote by $\underline{L}(P)$ the set of free waves belonging to $L(P)$. It is easy to see that $\underline{L}(P)$ is a closed linear manifold in $L(P)$. Lemma 3.1 shows that $\underline{L}(P) = \{0\}$ if P is on or to the right of $[BE]$; otherwise $\underline{L}(P)$ is a rather large space, as is seen from Theorem 3.6.

Lemma 4.1. *Let $P \in T$. If $u \in \underline{L}(P)$, then $u \in \dot{C}(\mathbb{R}; \Sigma^*)$. ($\dot{C}$ denotes the class of continuous functions that tend to zero as $t \to \pm\infty$.)*

Proof. $u \in \underline{L}(P)$ implies that $u(s) \in L^q$ for almost all s, where $1/q = x(P)$. But $L^q \subset \Sigma^*$, since $\Sigma \subset L^{q'}$ by $1/2 - 1/m < 1/q \leqslant 1/2$. Since $u(t) = U(t-s)u(s)$, it follows that $u \in C(\mathbb{R}; \Sigma^*)$.

To analyze the behavior of $u(t)$ for large t, let $\psi \in \Sigma$ and $v(t) = U(t)\psi \in \Sigma$. We shall estimate $\langle u(t), \psi\rangle$.

$$
|\langle u(t), \psi\rangle| = |\langle u(t+s), v(s)\rangle| \leqslant \|u(t+s)\|_q \|v(s)\|_{q'}.
$$

But $\|\omega\|_{q'} \leqslant \|\langle x\rangle\omega\|_2 \|\langle x\rangle^{-1}\|_\sigma$ for any $\omega \in L^{q'}$, where $1/\sigma = 1/q' - 1/2 = 1/2 - 1/q < 1/m$ (see above) so that $\|\langle x\rangle^{-1}\|_\sigma = c < \infty$. Thus

$$\begin{aligned}\|v(s)\|_{q'} \leqslant c\|\langle x\rangle v(s)\|_2 = c\|\langle x\rangle U(s)\psi\|_2 = c\|U(s)\langle x + 2is\partial\rangle\psi\|_2 \\ = c\|\langle x + 2is\partial\rangle\psi\|_2 \leqslant c\langle s\rangle\|\psi\|_\Sigma.\end{aligned}$$

(Here we have used the operator calculus involving $x\cdot$ and $U(s)$ (see e.g. Ginibre-Velo [5]).) Thus we obtain

$$|\langle u(t), \psi\rangle| \leqslant c\langle s\rangle\|u(t+s)\|_q\|\psi\|_\Sigma.$$

We integrate this inequality in s, after multiplying with a weight function $\kappa(s) \geqslant 0$ with L^1-norm one, with a bounded support including $s = 0$. Since $\|u(\cdot)\|_q$ has finite L^r-norm $|||u : P|||$, where $1/r = y(P)$, it follows that $|\langle u(t), \psi\rangle| \leqslant c|||\kappa u_t : P|||\|\psi\|_\Sigma$, where $u_t(s) = u(t+s)$. Since this is true for any $\psi \in \Sigma$, we conclude that

$$u(t) \in \Sigma^* \quad \text{with} \quad \|u(t)\|_{\Sigma^*} \leqslant c|||\kappa u_t : P|||.$$

Since $|||u : P|||$ is finite, the right member tends to zero as $t \to \pm\infty$ if $y(P) > 0$.

This argument does not work if $y(P) = 0$. But $y(P) = 0$ occurs only if $P = B$, in which case $u(t) \in L^2$ for almost all t, hence $u \in L(Q)$ for every $Q \in [BC[$ by (i) (Section 1). Choosing any such Q with $y(Q) > 0$, we see that the required result holds also for $P = B$.

Remark. Given $u \in \underline{L}(P)$ with $P \in T$, how can one characterize $\phi = u(0)$, or $u(t)$ in general? Unfortunately we have no answer to this question, beyond the fact that $u(t) \in \Sigma^*$.

3. *The range of G_a.* In Section 2 we proved that G_a is bounded on $L(\bar{P})$ to $L(P)$ for certain P and $\bar{P}$. Since G_a is an integral operator, it is expected that the functions produced by G_a are continuous in some sense or other, unless the function a is ill-behaved.

Lemma 4.2. *Suppose that a has the property that for each $t \in \mathbb{R}$, $t_n \to t$ implies $a(t_n, s + t_n) \to a(t, s+t)$ for almost every $s \in \mathbb{R}$. (This condition is satisfied for $G_a = G$, $G_\pm$.) If $f \in L(\bar{P})$ with $\bar{P} \in T'$, then $G_a f \in \dot{C}(\mathbb{R}; \Sigma^*)$.*

Proof. Let $g = G_a f$ where $f \in L(\bar{P})$, $\bar{P} \in T'$. Then we have the relations (3.5–6). Since Γ^* maps $L(\bar{P})$ continuously into Σ^* (see Theorem 3.6), we have $g(t) \in \Sigma^*$, with $\|g(t)\|_{\Sigma^*} \leqslant c|||f : \bar{P}|||$.

Next we prove that $g(t) \in \Sigma^*$ is continuous in t. To this end we compute

$$g(\tau) - g(t) = \Gamma^*(a_\tau f_\tau - a_t f_t) = \Gamma^*[a_\tau(f_\tau - f_t) + (a_\tau - a_t) f_t].$$

It suffices to show that the expression in [ ] tends to zero in $L(\bar{P})$ as $\tau \to t$ along any sequence t_n. This is true of $a_\tau(f_\tau - f_t)$, since translation is continuous on $L(\bar{P})$. The same is true of $(a_\tau - a_t)f_t$ by dominated convergence, since by hypothesis $a(t_n, s + t_n) \to a(t, s+t)$ as $t_n \to t$, for almost all s. This proves the continuity of $g(t)$.

It remains to show that $g(t) \to 0$ in Σ^* as $t \to \pm\infty$. To this end we take any $\epsilon > 0$ and write $f = f' + f''$, where f' is supported on $(-\infty, \tau)$ and f'' on (τ, ∞), with τ sufficiently large that $|||f'' : \bar{P}||| < \epsilon$. Set $g' = G_a f'$, $g'' = G_a f''$. It follows from the preceding results that both $g'(t)$ and $g''(t)$ are continuous and bounded in Σ^*, with $\|g''(t)\|_{\Sigma^*} \leqslant c\epsilon$. On the other hand $g'(t)$ coincides with a free wave for $t > \tau$. Thus Lemma 4.1 shows that $g'(t)$ tends in Σ^* to zero as $t \to \infty$. Since ϵ may be arbitrarily small, we have shown that $g(t) \to 0$ as $t \to \infty$. Similarly we can prove the same result for $t \to -\infty$.

Lemma 4.3. *Suppose that for each $t \in \mathbb{R}$, $t_n \to t$ implies $a(t_n, s) \to a(t,s)$ for almost all s. (This condition is met for $G_a = G$, $G_\pm$.) Let $h(t) = U(-t)(G_a f)(t)$, where $f \in L(\bar{P})$ with $\bar{P} \in T'$. Then $h \in BC(\mathbb{R}; \Sigma^*)$. If, in particular, $G_a = G_+$ $[G_-]$, then $h(t) \to 0$ in Σ^* as $t \to \infty$ $[-\infty]$.*

Proof. We have

$$h(t) = \int_{-\infty}^{\infty} a(t,s) U(-s) f(s)\, ds = \Gamma^* q_t, \quad q_t(s) = a(t,s) f(s).$$

Since $|||q_t : \bar{P}||| \leqslant |||f : \bar{P}|||$, the result follows as in the proof of Lemma 4.2, except that h need not tend to zero as $t \to \pm\infty$. (In fact h is constant if $a \equiv 1$.)

If $G_a = G_+$, then $a(t,s) = 0$ for $s < t$, so that $q_t \to 0$ in $L(\bar{P})$ as $t \to \infty$. Hence $h(t) \to 0$ in Σ^* as $t \to \infty$. G_- can be handled in the same way.

§5. A miniature scattering theory for NLS

In this section we shall construct a scattering theory for small solutions of (NLS), assuming, for simplicity, that

(5.1) $\quad |F'(\zeta)| \leqslant M' |\zeta|^{k-1}, \quad F(0) = 0,$ where $k > 1$ is a constant.

This implies that $|F(\zeta)| \leqslant M|\zeta|^k$ with some M; we may set $M' = M$.

Our solution u will belong to $L(P)$, where $P \in T$ is a *k-point*, by which we mean that P and kP form a conjugate pair (see Section 2). Obviously $y(P) > 0$ for a k-point P.

If P is a k-point, then $kP \in T'$ and $(k-1)\pi(P) = \pi(kP) - \pi(P) = 2/m$, hence

$$\pi(P) = 2/(k-1)m. \tag{5.2}$$

Thus $\pi(P)$ is determined by k only and decreases with increasing k. Moreover, since $P \in T$ implies $1/2 - 1/m < \pi(P) < 1/2 + 1/m$, it follows from (5.2) that $1 + 4/(m+2) < k < 1 + 4/(m-2)$. But this is not sufficient; we have

Lemma 5.1. *In order that there exist a k-point, it is necessary and sufficient that*

$$[m + 2 + (m^2 + 12m + 4)^{1/2}]/2m < k < 1 + 4/(m-2). \tag{5.3}$$

The right member should read ∞ *if* $m \leqslant 2$.

Remark. Lemma 5.1 wil be proved below. (5.3) is a familiar condition that recurs in various situations for NLS, see e.g. [2, 3, 11, 13]. It is of some interest that it occurs here as a simple geometric condition. Under condition (5.3), a typical k-point is given by

$$P = (1/(k+1), 1/(k-1) - m/2(k+1)). \tag{5.4}$$

Of course any points sufficiently close to P on the π-line through P are k-points.

In what follows we have to do with free waves that are *asymptotic* to solutions u of (NLS). In general we say that two functions $u, v \in C(\mathbb{R}; \mathcal{S}')$ are asymptotic to each other at ∞, and write "$u \sim v$ at ∞", if $U(-t)(u(t) - v(t)) \to 0$ as $t \to \infty$. Similarly we define "$u \sim v$ at $-\infty$". Obviously the relation $u \sim v$ is invariant under simultaneous translation of u, v in t. We also note that given u, there is at most one *free wave* v such that $u \sim v$ at ∞, and similarly at $-\infty$. This follows from the fact that $U(-t)v(t) = v(0)$ for a free wave v.

Theorem 5.2. *Let P be a k-point, and $u \in L(P)$ a solution of* (NLS). *Then there are unique free waves $u_\pm \in \underline{L}(P)$ that are asymptotic to u at $\pm\infty$. The maps $u \mapsto u_\pm$ are continuous and injective from $L(P)$ to $\underline{L}(P)$, and in fact uniformly continuous on bounded sets in $L(P)$.*

Proof. Uniqueness of $u_\pm$ is obvious from the remark above. We shall construct u_+ (u_- can be similarly handled). Set $w = -iG_+F(u) \in$

$L(P)$, which exists because $F(u) \in L(kP)$ (by (1.10b)) and P, kP are conjugate. Then $(\partial_t - i\Delta)w = -iF(u)$. Since $(\partial_t - i\Delta)u = -iF(u)$, we have $(\partial_t - i\Delta)(u-w) = 0$, so that $u_+ \equiv u - w \in \underline{L}(P)$, and we can write $u = u_+ - iG_+F(u)$. That $u \sim u_+$ at ∞ follows from Lemma 4.3. The map $u \mapsto u_+$ is uniformly continuous on bounded sets since $u \mapsto F(u)$ from $L(P)$ to $L(kP)$ and $F(u) \mapsto w = G_+F(u)$ from $L(kP)$ to $L(P)$ have the same property (see Theorem 2.1).

The proof that $u \mapsto u_+$ is injective is more complicated. Suppose that there is another solution $v \in L(P)$ of (NLS). Then we have as above $v = v_+ - iG_+F(v)$, where $v_+ \in \underline{L}(P)$ and $v \sim v_+$ at ∞. we claim that if $v_+ = u_+$ then $v = u$. Indeed $v_+ = u_+$ implies

$$u - v = -iG_+(F(u) - F(v)) \tag{5.5}$$

on subtraction. We divide $(-\infty, \infty)$ into a finite number of subintervals $I_0 = (-\infty, T_1)$, $I_1 = (T_1, T_2)$, ... , $I_n = (T_n, \infty)$, and set $u_j = \chi_j u$, $v_j = \chi_j v$, where χ_j is the characteristic function of I_j. Since $|||u : P|||$ and $|||v : P|||$ are finite, for any $\epsilon > 0$ we can choose n and the I_j so that $|||u_j : P|||^{k-1} + |||v_j : P|||^{k-1} \leqslant \epsilon$.

Let us compute $u_j - v_j$ by multiplying (5.5) with χ_j. Since G_+ is of Volterra type, with integration on (t, ∞), there is no contribution from the parts u_i, v_i with $i \leqslant j$. Since G_+ is bounded on $L(kP)$ to $L(P)$ and since

$$|F(u_i) - F(v_i)| \leqslant cM|u_i - v_i|(|u_i|^{k-1} + |v_i|^{k-1}),$$

we obtain (cf. [7] for this computation)

$$\begin{aligned} |||u_j - v_j : P||| &\leqslant c\sum_{i=j}^{n} |||F(u_i) - F(v_i) : kP||| \\ &\leqslant cM\sum_{i=j}^{n} |||u_i - v_i : P|||(|||u_i : P|||^{k-1} + |||v_i : P|||^{k-1}) \\ &\leqslant cM\epsilon\sum_{i=j}^{n} |||u_j - v_j : P|||. \end{aligned} \tag{5.6}$$

Now assume that ϵ is chosen so small that $cM\epsilon < 1$. If we set $j = n$ in (5.6), we obtain $|||u_n - v_n : P||| \leqslant cM\epsilon|||u_n - v_n : P|||$, hence $u_n = v_n$. On setting $j = n - 1$, then, we have $|||u_{n-1} - v_{n-1} : P||| \leqslant cM\epsilon|||u_{n-1} - v_{n-1} : P|||$, hence $u_{n-1} = v_{n-1}$. Proceding in the same way, we obtain $u_j = v_j$ for $j = 0, 1, \ldots, n$, hence $u = v$.

We now construct a scattering theory for small solutions in $L(P)$.

Theorem 5.3. *Let P be a k-point. Then there exist balls $B_\pm$ in $\underline{L}(P)$ and a ball B in $L(P)$, with center O and positive radii, with the following properties.*

(a) *If $u_- \in B_-$,* (NLS) *has a unique global solution $u \in B$ such that $u \sim u_-$ at $-\infty$.*

(b) *There is a unique free wave $u_+ \in \underline{L}(P)$ such that $u \sim u_+$ at ∞.*

(c) *The scattering operator $S : u_+ = Su_-$ is well defined and is continuous and injective on B_- to $\underline{L}(P)$.*

(d) *The range of S covers B_+.*

(e) *All u and $u_\pm$ belong to $\dot{C}(\mathbb{R}; \Sigma^*)$.*

Remark. Our scattering operator S acts on space-time functions, and differs from the conventional ones, which act on space functions. Our viewpoint is in conformity with the idea of Segal (see e.g. [9]).

Proof. To construct the solution u, we solve the integral equation $u = \Phi_-(u) \equiv u_- - iG_-F(u)$ by a routine method (such as was used in [6,7]; see Section 1 for $G_\pm$). Indeed, given $v \in L(P)$, we have $F(v) \in L(kP)$, with $|||F(v) : kP||| \leqslant M|||v : P|||^k$. Since P and kP are conjugate, we obtain $|||\Phi_-(v) : P||| \leqslant |||u_- : P||| + cM|||v : P|||^k$ by Theorem 2.1. It follows that Φ_- sends a certain ball B of $L(P)$ into itself if $|||u_- : P|||$ is sufficiently small. An analogous estimate using the Lipschitz continuity of F shows that Φ_- is a contraction on B. Thus Φ_- has a unique fixed point u in B, which is a (weak) solution of (NLS). Lemma 4.3 then shows that $u \sim u_-$ at $-\infty$.

Since we are using the contraction theorem, the uniqueness of u in B is obvious. Moreover, the continuity of the map $u_- \mapsto u$ follows easily.

The existence of u_+, hence of S too, follows from Theorem 5.2. Since the map $u \mapsto u_+$ is injective and uniformly continuous on bounded sets, the same is true of S. Property (e) follows from Lemmas 4.1–2.

Finally we note that the role of u_- and u_+ may be reversed to construct the inverse operator $S^{-1} : u_- = S^{-1}u_+$ for sufficiently small $u_+ \in \underline{L}(P)$. Since $|||u_- : P||| \leqslant \text{const}|||u_+ : P|||$ for sufficiently small $|||u_+ : P|||$ (due to the uniform continuity proved above), we have $S^{-1}B_+ \subset B_-$ if B_+ is sufficiently small. This shows that the range of S covers B_+.

Proof of Lemma 5.1. We recall some properties of the generic conjugate pair l, $\bar{l}$. l and $\bar{l}$ are parallel and have the same length; the upper end Q of l is on the vertical side $]EF[$ of T, the lower end $\bar{Q}$ of $\bar{l}$ is on the vertical side $]E'F'[$ of T', and $Q, \bar{Q}$ have the same height, which we denote by h. Let R denote the lower end of l, and $\bar{R}$ the upper end of $\bar{l}$.

Obviously a k-point $P \in l$ exists *with some* $k > 1$ if and only if there is a ray OX from the origin O that meets both l and $\bar{l}$; in this case

$k = \pi(\bar{l})/\pi(l)$, since l and $\bar{l}$ are parallel, so that k does not depend on the exact position of the ray.

If $h \leqslant 1/2$ so that l is on or below $[BC[$, R is on the bottom side $[BF[$ of T. Thus the ray $O\bar{P}$ meets l if $\bar{P} \in \bar{l}$ is sufficiently low, hence k-points exist on l for some k. If we let $h \to 0$, so that l shrinks to the point $F = (1/2 - 1/m, 0)$, and $\bar{l}$ to $E' = (1/2 + 1/m, 0)$, the ratio $k = \pi(\bar{l})/\pi(l)$ approaches $(1/2 + 1/m)/(1/2 - 1/m) = (m+2)/(m-2)$. If $h = 1/2$, then $l = [BC[$, $\bar{l} = [B'C'[$, and $k = 1 + 4/m$.

The case that l is above $[BC[$ is more complicated. In this case R is on the hypotenuse BE of T and $\bar{R}$ is on the upper side $[B'F'[$ of T'. If h is not too large, the ray OR is still below the ray $O\bar{R}$, so that there is a ray OX that meets both l and $\bar{l}$. If h is increased, this ceases to be the case eventually. The critical value of h can be determined by the condition that the two rays OR and $O\bar{R}$ coincide. An elementary algebra gives the value of h, then of k, which turns out to be the value on the left side of (5.3). Since k decreases with increasing h, we have proved the lamma.

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Helmholtz-Type Equation on Non-compact Two-Dimensional Riemannian Manifolds

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§1. Introduction

We shall consider the existence, or rather non-existence of square integrable solutions of the equation $-\triangle f = \lambda f$ on a non-compact Riemannian manifold which is homeomorphic to $\mathbf{R}^n$ minus a ball, where $\triangle$ is the Laplace-Beltrami operator and λ is an arbitrary positive constant. The source of this problem is the study of the non-existence of positive eigenvalues of the Schrödinger operator $-\triangle + q$ in a region of $\mathbf{R}^n$, and the method used there was found to be applicable to problems of the above type.

There may be several ways of physical interpretation of the equation $-\triangle f = \lambda f$ on manifolds. But probably the most essential one is as follows: Let a Riemannian manifold $\mathcal{M}$ represent a non-Euclidean space which is filled up with a medium whose displacement on some quantity, e.g. pressure, electric field etc., obeys Hooke's law isotropically and homogeneously in each small portion of the medium. We suppose further that the displacement is transferred entirely to the neighboring portions without influence of the curvature. (This situation occurs, for example, if $\mathcal{M}$ is a surface and the medium is distributed on and moving along $\mathcal{M}$ without friction or obstruction.) Then, the displacement D should enjoy the "wave equation" $D_{tt} = \triangle D$ (by taking an appropriate scale), therefore $-\triangle f = \lambda f$ describes the standing wave $D = e^{i\sqrt{\lambda}t} f(x)$.

We notice that the *total energy* $\displaystyle\int_{\mathcal{M}} (|D_t|^2 + |\nabla D|^2) d\mathcal{M}$ is finite if and only if $\displaystyle\int_{\mathcal{M}} |f|^2 d\mathcal{M}$ is finite. Therefore, what we are asking is the conditions for $\mathcal{M}$ not to admit a standing wave of finite energy.

Before describing the general statement, let us see examples of $\mathcal{M}$ which have L^2-solutions.

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Examples. (a) $\mathcal{M}$ is the semi-infinite cylinder whose metric $ds^2 = dr^2 + \rho_0^2 d\theta^2$, $r \in (r_0, \infty)$, $\theta \in \mathbf{S}^1$, where ρ_0 is a positive constant. Since $\triangle = \partial^2/\partial r^2 + \rho_0^{-2}\partial^2/\partial\theta^2$, the function $f = e^{-ar+2\pi ni\theta}$ is an L^2-solution for $\lambda = -a^2 + 4\pi^2 n^2/\rho_0^2$ if the constant a and the integer n satisfy $0 < a < 2\pi n/\rho_0$. (b) Let $ds^2 = dr^2 + e^{2ar}d\theta^2$, $r \in (r_0, \infty)$, $\theta \in \mathbf{S}^1$. If a and b are constants such that $0 < b < a < 2b$, then $f = e^{-br}$ is a solution for $\lambda = b(a-b)$ which is square integrable since $d\mathcal{M} = e^{ar}drd\theta$.

The above examples suggest that, in so far as 2-dimensional *rotationally symmetric* manifolds are concerned, the following theorem is in some sense a satisfactory one.

Theorem 1 ([2],[4]). *If $\mathcal{M}$ is a two-dimensional manifold whose metric has the form*

$$ds^2 = dr^2 + \rho(r)^2 d\theta^2, \quad r \in (r_0, \infty),\ \theta \in \mathbf{S}^1$$

where $\rho(r)$ is a positive absolutely continuous nondecreasing function of r which enjoys (i) $\rho(r) \to \infty$ $(r \to \infty)$ *and* (ii) $\displaystyle\int_{r_0}^{\infty} \frac{dr}{\rho(r)} = \infty$, *then for any constant $\lambda > 0$ and any nontrivial locally square integrable solution of $-\triangle f = \lambda f$, there exist constants $C > 0$ and $r_1 \geq r_0$ such that*

$$\int_{r_0<r<R} |f|^2 d\mathcal{M} \geq C \int_{r_0}^{R} \frac{dr}{\rho(r)} \quad (R \geq r_1)$$

holds where $d\mathcal{M} = \rho(r)drd\theta$. (Therefore $f \notin L^2(\mathcal{M})$ unless $f \equiv 0$.)

The previous example (a) does not satisfy (i), while (b) breaks (ii).

Corollary. *Let $\mathcal{M}$ be a surface of revolution in $\mathbf{R}^3$ obtained by rotating the graph of an arbitrary absolutely continuous function $z = z(\rho)$, $\rho_0 < \rho < \infty$, around the z-axis. Then $\mathcal{M}$ has the same property with respect to the natural metric. In particular, any non-vanishing solution of $-\triangle f = \lambda f$, $\lambda > 0$ can not be square integrable.*

As to the higher dimensional cases, we have the following theorem.

Theorem 2 ([3],[4]). *Let $\mathcal{M} = \{(r, \omega) \mid r_0 < r < \infty,\ \omega \in \mathbf{S}^{n-1}\}$ $(n \geq 2)$ with the metric $ds^2 = dr^2 + \rho(r)^2 d\tilde{s}^2$ where ρ is a positive function and $d\tilde{s}$ is the line element of the $(n-1)$-sphere $\mathbf{S}^{n-1}$. Suppose that*

(i) $\rho \in C^2(r_0, \infty)$, $\rho'(r) > 0$ *and* $\rho(r) \to \infty$ $(r \to \infty)$.
(ii) $\rho'(r)/\rho(r) \to 0$ $(r \to \infty)$.

(iii) $\rho''(r)/\rho'(r) \to 0$ $(r \to \infty)$.

(iv) *There exists a number* $\alpha > 0$ *such that*

$$\int_{r_0}^{\infty} \frac{dr}{\rho(r)^{\alpha}} = \infty.$$

Then for any $\lambda > 0$ *and any non-zero solution of* $-\triangle f = \lambda f$ *and for an arbitrary* $\varepsilon > 0$, *we can take* $C > 0$ *and* $r_1 \geq r_0$ *so that*

$$\int_{r_0<r<R} |f|^2 d\mathcal{M} \geq C \int_{r_0}^{R} \frac{dr}{\rho(r)^{\varepsilon}} \quad (R \geq r_1).$$

We see that Theorem 2 assumes weaker growth of $\rho(r)$ than Theorem 1. Moreover, the obtained estimate is better. But it requires higher smoothness of ρ and restricts the magnitude of ρ'' in return.

§2. Not symmetric manifolds

T. Tayoshi's work [6] treated the case in which the metric itself is not rotationally symmetric but approaches such one asymptotically. His theorem is a generalization of Theorem 2 above, though not completely. Here we want to have an extension of Theorem 1.

Let $\mathcal{M}$ be a two-dimensional manifold whose metric has the form

$$ds^2 = a(r,\theta)dr^2 + 2b(r,\theta)\rho(r)drd\theta + c(r,\theta)\rho(r)^2 d\theta^2,$$

where a, b, c and ρ are real-valued functions. To describe the conditions altogether, let us begin with definitions.

Definition 1. (i) $t(r) = \exp(-\int_{r_0}^{r} \frac{ds}{\rho(s)})$.

(ii) For each number $m > 0$, the quantity $h(r;m)$ is the one that satisfies

$$\int_{r}^{r+h(r;m)} \frac{ds}{\rho(s)} = mt(r).$$

(iii) $$\varphi(r;m) = \operatorname*{ess\,inf}_{r \leq s \leq r+h(r;m)} \rho(s)^2 \rho'(s).$$

Assumption on ρ.

(i) $\rho(r)$ is positive, nondecreasing and absolutely continuous with $\rho'(r) > 0$ a.e.

(ii) $\rho(r) \to \infty$ $(n \to \infty)$.

(iii) $\displaystyle\int_{r_0}^{\infty} \frac{dr}{\rho(r)} = \infty.$

(iv) $t(r)/\rho'(r) \to 0 \quad (r \to \infty).$

(v) $\displaystyle\int_{r_0}^{\infty} \frac{\varphi(r;m)}{\rho(r+h(r;m))} dr = \infty.$

Remark. If $\rho(r)t(r)$ is bounded and $\rho'(r) \leq 1$, and moreover $\rho(r)^2\rho'(r)$ is nondecreasing or nonincreasing, then the condition (v) is fulfilled.

Definition 2. $g = \sqrt{ac - b^2}, \quad A = a/g, \quad B = b/g, \quad C = c/g.$

Definition 3. A function $f(t, \theta)$ is said to satisfy *the condition of Definition* 3 if it enjoys the inequality

$$|f(\text{point } 1) - f(\text{point } 2)| \leq \psi(\text{distance})$$

where $\psi(x)$ is a positive continuous nondecreasing function of $x > 0$ which fulfills $\displaystyle\int_{\to +0} \{\psi(x)/x\} dx < \infty$. By the way, if the two points are (t_1, θ_1) and (t_2, θ_2) then the distance is $\sqrt{t_1^2 + t_2^2 - 2t_1t_2\cos(\theta_1 - \theta_2)}$.

Remark. This condition is a generalization of the uniform Hölder continuity, the latter corresponding to $\psi(x) = Kx^\alpha$.

Assumptions on a, b, c.

(i) $a, b, c \in C^1((r_0, \infty) \times \mathbf{S}^1)$, $a > 0$, $a/c \to 1$, and $b \to 0$ as $r \to \infty$ and there exist numbers k, l and r_1 $(k > 0,\ 0 < l < 2,\ r_1 \geq r_0)$ such that

$$g \geq k, \quad g_r/g \geq -l\rho'/\rho \quad (r \geq r_1,\ \theta \in \mathbf{S}^1).$$

(ii) $$g_\theta b/(g^2\rho') \to 0 \quad (r \to \infty),$$
$$g_\theta t/(g^2\rho') \to 0 \quad (r \to \infty).$$

(iii) As functions of t and θ,

$$\rho t^{-1}A_r, \quad \rho t^{-1}B_r, \quad \rho t^{-2}C_r, \quad t^{-1}A_\theta, \quad t^{-2}B_\theta, \quad t^{-1}C_\theta$$

have the limits at $t = 0$ (i.e., $r = \infty$), and satisfy the condition of Definition 3 near $t = 0$.

Our main theorem is as follows:

Theorem 3. *Under the above assumptions, the equation $-\triangle f = \lambda f$ on $\mathcal{M}$ has no non-trivial solution of integrable square, provided $\lambda > 0$.*

It should be noted that the conditions do not make reference to the second order derivatives of the metric.

This theorem is proved by combining the following two lemmas.

Lemma 1 (Estimate by isothermal coordinates).

Let the two-dimensional Riemannian manifold $\mathcal{M}$ admit a global system of coordinates u, v, $u_0 < u < \infty$, $v \in \mathbf{S}^1$ so that they are isothermal, that is, the metric has the form

$$ds^2 = \tau(u,v)\big(du^2 + dv^2\big)$$

by a positive function $\tau(u,v)$. We suppose that τ is absolutely continuous with respect to u for a.e. $v \in \mathbf{S}^1$ and of class C^1 with respect to v for a.e. u. Moreover let

$$\varphi(u) = \operatorname*{ess\,inf}_{v\in\mathbf{S}^1} \frac{\partial}{\partial u}\tau(u,v)$$

satisfy

$$\int_{u_0}^{\infty} \varphi(u)du = \infty.$$

Then for every non-trivial solution of $-\triangle f = \lambda f$ on $\mathcal{M}$ $(\lambda > 0)$, we can find numbers $C > 0$ and $u_1 \geq u_0$ such that

$$\int_{u_0<u<U} |f|^2 du \geq CU \quad (U \geq u_1)$$

holds ($d\mathcal{M} = \tau du dv$).

Lemma 2 (Existence of suitable isothermal coordinates).

If a two-dimensional Riemannian manifold satisfies the assumptions of Theorem 3, there exist a number r_1 and C^1-functions $u(r,\theta)$ and $v(r,\theta)$ defined for $r \geq r_1$, $\theta \in \mathbf{S}^1$, which satisfy

$$v_r = Bu_r - A\rho^{-1}u_\theta,$$

$$v_\theta = C\rho u_r - Bu_\theta.$$

Here (i) *for each θ, $u(r,\theta)$ is strictly increasing with r, $u_r(r,\theta)$ is absolutely continuous and $u(r,\theta) \to \infty$ as $r \to \infty$. On the other hand $v_\theta > 0$ and the value of $v(r,\theta)$ is determined up to the difference of $2k\pi$ $(k \in \mathbf{Z})$.*

(ii) *In terms of u and v, the metric is expressed as*

$$ds^2 = \tau(u,v)\big(du^2 + dv^2\big),$$

$$\tau = \frac{g}{Cu_r^2 - 2B\rho^{-1}u_r u_\theta + A\rho^{-2}u_\theta^2}.$$

(iii) $\varphi(u) = \operatorname*{ess\,inf}_{v\in\mathbf{S}^1} \partial\tau/\partial u$ *enjoys*

$$\int_{u_0}^{\infty} \varphi(u)du = \infty.$$

Lemma 1 together with Lemma 2 claims that if the solution f is square integrable over $\mathcal{M}$ then $f(r,\theta) \equiv 0$ for sufficiently large r, say, $r \geq r_1$. But, in our situation, we can easily verify the unique continuation property so that $f \equiv 0$ holds throughout $\mathcal{M}$. (So far as the unique continuation applies, $\mathcal{M}$ itself need not be of the shape described before. If only a part of $\mathcal{M}$ has that shape, we must have the same conclusion again.)

§3. Sketch of the proof of Lemma 2

The proof of Lemma 1 can be got by a standard argument. Therefore we will leave it to the full paper [5].

The main point of the proof of Lemma 2 is to obtain the solution of $\triangle u = 0$ which has the asymptotic form $u \sim \int dr/\rho$. To this end we change the variables from r,θ to t,θ and look for the solution of $\triangle u = 0$ having the form $u = -\log t + \xi(t,\theta)$, $\xi \in C^2$ in the neighborhood of $t = 0$. In fact, ξ enjoys the equation

$$(\tilde{C}\xi_x + \tilde{B}\xi_y)_x + (\tilde{B}\xi_x + \tilde{A}\xi_y)_y = t^{-1}C_t + t^{-2}B_\theta$$

where $x = t\cos\theta$, $y = t\sin\theta$ and $\tilde{A}$, $\tilde{B}$, and $\tilde{C}$ are quadratic forms of $\cos\theta$ and $\sin\theta$ whose coefficients are linear combinations of A, B and C. Thus we can apply the classical theory of Korn and Lichtenstein or its extended version by Hartman and Wintner. We cite here a part of their theorem.

Theorem (Hartman & Wintner [1]). *Suppose $A_1(x,y)$, $B_1(x,y)$, $B_2(x,y)$ and $C_1(x,y)$ are C^1-functions whose first order derivatives satisfy the condition of Definition* 3. *We assume $A_1C_1 - (B_1+B_2)^2/4 > 0$.*

Moreover, let $D(x,y)$ and $E(x,y)$ are functions which satisfy the condition of Definition 3. Then the equation

$$(C_1\xi_x + B_1\xi_y)_x + (B_2\xi_x + A_1\xi_y)_y + D\xi = E$$

has a C^2-solution in some neighborhood of $x = y = 0$.

From the assumed regularity of A, B and C, it is easy to see that the conditions of this theorem are fulfilled by putting $A_1 = \tilde{A}$, $B_1 = B_2 = \tilde{B}$, $C_1 = \tilde{C}$. Thus we obtain the desired ξ. Set

$$v(t,\theta) = \int_{\text{fixed point}}^{(t,\theta)} (Bu_t + At^{-1}u_\theta)dt - (Ctu_t + Bu_\theta)d\theta.$$

Then a straightforward calculation shows that the pair of $u = -\log t + \xi$ and v form a set of isothermal coordinates. The estimates for their derivatives up to second order are derived from the C^2 property of ξ with respect to t and θ.

What is left is to show $\int^\infty \varphi(u)du = \infty$, $\varphi(u)$ being $\operatorname{ess\,inf}_v \partial\tau/\partial u$. This calculation is somewhat involved, but eventually we are led to the conclusion that there exist constants $K > 0$ and $r_1 \geq r_0$ for which

$$\tau_u \geq K\rho^2\rho' \quad (r \geq r_1)$$

holds and that the contour $\{u = \text{const.}\}$ lies between the circles of radii r and $r + h(r;m)$, m being some constant not depending on r. We know that $\varphi(u)$ is the infimum of τ_u on the contour $\{u = \text{const.}\}$ while $\varphi(r;m)$ is the infimum of τ_u in the region between those circles. This fact establishes the lemma.

Example. Consider $\rho(r)$ which has the form $\rho(r) = \rho_0(r) - \rho_0'(r)\big(1 - k(r)\big)\sin r$ where $\rho_0(r)$ is a positive function having absolutely continuous derivative and $k(r)$ is an absolutely continuous function. We assume (i) $\rho_0(r) \to \infty$ (ii) $0 \leq \rho'(r) \leq 1$ (iii) $0 < k(r) \leq 1$ (iv) $k(r)^{-1}k'(r) \to 0$ (v) $\rho_0'(r)k(r)$ is nonincreasing (vi) $\rho_0'(r)k(r)\exp\big(\int_{r_0}^r [\rho_0(s)+1]^{-1}ds\big) \to \infty$ (vii) $\int_{r_0}^\infty \rho_0(r)\rho_0'(r)k(r)dr = \infty$ (viii) $\rho'(r)^{-1}\rho_0''(r)k(r)^{-1} \to 0$. Then we can show that $\rho(r)$ satisfies all the conditions. If we choose $\rho_0(r) = r^\alpha$ $(0 < \alpha \leq 1)$ or $\rho_0(r) = \log r$ then it fulfills (i)(ii)(iii). It also satisfies (vi)(vii) and (viii) if we choose a nondecreasing $k(r)$. In particular, by setting $k(r) = 1$, $\rho(r) = r^\alpha$ and $\rho(r) = \log r$ themselves meet the conditions.

Example. The following example shows how fast A, B, C should tend to their limits. Let $\rho(r) = r$ and put $a = 1 - r^{-\alpha}\cos\theta$, $b = r^{-\alpha}\sin\theta$

and $c = 1+r^{-\alpha}\cos\theta$ where $\alpha > 2$. Then $t = r^{-1}$ and $g = \sqrt{1 - r^{-2\alpha}}$. The crucial terms are $t^{-3}C_r$ and $t^{-2}B_\theta$. But they are close to $-\alpha t^{\alpha-2}\cos\theta$ and $-t^{\alpha-2}\cos\theta$ respectively. Therefore they fit the conditions.

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On a Backward Estimate for Solutions of Parabolic Differential Equations and its Application to Unique Continuation

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Abstract.

We prove a new backward estimate and a new strong unique continuation property for solutions $u \in \mathcal{C} = C^o((0,T);\ H^2(\mathbf{R}^n;\ e^{-\alpha|x|^2}\,dx)) \cap C^1((0,T);\ L^2(\mathbf{R}^n;\ e^{-\alpha|x|^2}\,dx))$ of parabolic differential equations $\frac{\partial u}{\partial t} = \Delta u + V(x,t)u$ under certain conditions on V, where $\alpha > 0$ is a fixed number.

§1. Main results

We consider the following parabolic differential equation:

$$\frac{\partial u}{\partial t} = \Delta u + V(x,t)u \quad in \quad \mathbf{R}^n \times (0,T), \tag{1.1}$$

where V is real-valued, $T > 0$, and $n \geq 3$. Let $\alpha > 0$ be a fixed number and let $w(x) = e^{-\alpha|x|^2}$. We denote by $L^2(\mathbf{R}^n; w(x)dx)$ the closure of $C_o^\infty(\mathbf{R}^n)$ under the norm $\|u\|_{L^2(w)} = (\int_{\mathbf{R}^n} |u(x)|^2 w(x)\,dx)^{1/2}$. We also denote by $H^2(\mathbf{R}^n; w(x)dx)$ the closure of $C_o^\infty(\mathbf{R}^n)$ under the norm $\|u\|_{H^2(w)} = \left(\sum_{0\leq|\beta|\leq 2} \|D^\beta u\|^2_{L^2(w)}\right)^{1/2}$, where $D^\beta = \partial_1^{\beta_1}\cdots\partial_n^{\beta_n}$, $\partial_j = \frac{\partial}{\partial x_j}$, $|\beta| = \sum_{j=1}^n \beta_j$ for $\beta = (\beta_1,\cdots,\beta_n)$. Put $\mathcal{C} = C^o((0,T); H^2(\mathbf{R}^n; w(x)\,dx)) \cap C^1((0,T); L^2(\mathbf{R}^n; w(x)\,dx))$. We say $u \in \mathcal{C}$ is a solution of (1.1) if u satisfies (1.1) in $L^2(\mathbf{R}^n; w(x)dx)$ for each $t \in (0,T)$.

For a point $z_o = (x_o, t_o) \in \mathbf{R}^n \times (0,T)$ and $0 < R < \sqrt{t_o}$, we set $S_R(t_o) = \{z = (x,t) \in \mathbf{R}^n \times (0,T) \mid t = t_o - R^2\}$. By using the backward heat kernel $G_{z_o}(z) = \dfrac{1}{(4\pi(t_o-t))^{n/2}} \exp(-\dfrac{|x-x_o|^2}{4(t_o-t)})$ which is defined

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for $t < t_o$, we define the weighted L^2 norm $H_{z_o}(R;u)$ and the weighted energy $I_{z_o}(R;u)$ over $S_R(t_o)$ as follows:

$$H_{z_o}(R;u) = \frac{1}{2}\int_{S_R(t_o)} u^2 G_{z_o}\,dx,$$

$$I_{z_o}(R;u) = \frac{1}{2}R^2\int_{S_R(t_o)} (|\nabla u|^2 - Vu^2)G_{z_o}\,dx.$$

Under certain assumptions on V we shall study the behaviour of $H_{z_o}(R;u)$ and $I_{z_o}(R;u)$ as $R \to 0$ and prove a 'monotonicity formula' for the weighted energy $I_{z_o}(R;u)$ (Lemma 3.1) and a doubling property for $H_{z_o}(R;u)$ (Theorem 1.3).

To state our assumptions on V, we first recall the definitions of the Fefferman-Phong class F_t and the Kato class K_n. $V \in L^1_{\text{loc}}(\mathbf{R}^n)$ is said to be of the Kato class K_n if

$$\lim_{r\to 0}\eta^K(r;V) = 0, \quad \eta^K(r;V) = \sup_{x\in\mathbf{R}^n}\int_{B_r(x)} \frac{|V(y)|}{|x-y|^{n-2}}\,dy,$$

where $B_r(x) = \{y \in \mathbf{R}^n | |x-y| < r\}$ for $r > 0$. For $1 \le t \le n/2$, $V \in L^t_{\text{loc}}(\mathbf{R}^n)$ is said to be of the Fefferman and Phong class F_t if

$$||V||_{F_t} = \sup_{x\in\mathbf{R}^n, r>0} r^2\left(\frac{1}{|B_r(x)|}\int_{B_r(x)} |V|^t\,dy\right)^{1/t} < +\infty.$$

We note that $F_{n/2} = L^{n/2}(\mathbf{R}^n) \subset F_t \subset F_s$ for $1 \le s \le t \le n/2$ and weak-$L^{n/2}(\mathbf{R}^n) \subset F_t$ for every $t \in [1, n/2)$; $V \in K_n$ implies $V \in F_1$; but $L^{n/2}(\mathbf{R}^n)$ and K_n are incomparable for $n \ge 3$.

For $1 < t \le n/2$, we define the function space Q_t by $Q_t = \{V = V_1 + V_2;\ V_1 \in K_n,\ V_2 \in F_t\}$ and for $V \in Q_t$ set

$$(1.2) \qquad ||V||_{Q_t} = ||V||^{R_o}_{Q_t} = \inf_{V=V_1+V_2\in Q_t}\{\eta^K(R_o;V_1) + ||V_2||_{F_t}\}$$

for $R_o > 0$. Throughout this paper we fix $R_o > 0$.

Definition 1.1. For $1 \le t \le n/2, p \ge 1$, we say V belongs to the class $Q_{t,p}(0,T)$, if V satisfies
(1) for each $t_o \in (0,T)$, there exist positive functions $W, U \ge 0$ and a compact set $K \subset \mathbf{R}^n$ such that

$$(1.3) \qquad |V(x,t_o - s)| \le W(x,s) + U(s), \quad \text{supp}_x W(\cdot,s) \subset K$$

for every $s \in (0,t_o)$,
(2) $|W(\cdot,s)|^p \in Q_t$ for every $s \in (0,t_o)$.

Now we state our assumptions for V.

Assumption (A). V satisfies the following conditions for $1 < t \leq n/2, p > 2$.
(i) $V \in Q_{t,p}(0,T)$ and $\tilde{V} = 2V + (x-x_o)\cdot\nabla_x V + 2(t-t_o)\partial_t V \in Q_{t,p}(0,T)$;
(ii) for the expression $|V| \leq W_1 + U_1$ and $|\tilde{V}| \leq W_2 + U_2$, put $f_j^{(t_o)}(s) = s^{2-4/p}\ \||W_j(s^2)|^p\|_{Q_t}^{1/p} + s^2 U_j(s^2), j = 1, 2$. Then there exists $s_1 > 0$ such that

$$(1.4) \qquad f_1^{(t_o)}(s) \to 0\ (s \to 0), \quad \int_0^{s_1} \frac{f_2^{(t_o)}(s)}{s}\, ds < +\infty$$

for every $t_o \in (0,T)$.

Example 1.2. (1) If $V \in C^1(\mathbf{R}^n \times (0,T))$ and $V, (1+|x|)|\nabla V|, |\partial_t V| \in L^\infty(\mathbf{R}^n \times (0,T))$ and have compact support for each $t \in (0,T)$, then V satisfies Assumption (A).
(2) Let $V(x,t) = V(x)$ be independent of time variable. If $|V|^p$ and $|\tilde{V}|^p$, $\tilde{V} = 2V + (x-x_o)\cdot\nabla_x V$, belong to the class Q_t for some $1 < t \leq n/2$ and $p > 2$ and have compact support, then V satisfies Assumption (A).

We state our main results.

Theorem 1.3 (Backward Estimate). *Suppose Assumption* (A). *Let $u \in \mathcal{C}$ be a solution of* (1.1). *Then for $z_o = (x_o, t_o) \in \mathbf{R}^n \times (0,T)$, there exist constants R^* and $C_o > 0$ such that*

$$(1.5) \qquad \int_{S_{2R}(t_o)} u^2 G_{z_o}\, dx \leq C_o \int_{S_R(t_o)} u^2 G_{z_o}\, dx$$

for every $0 < R < R^* (< \sqrt{t_o})$. Here C_o is a constant independent of R.

Theorem 1.3 implies

Theorem 1.4 (Unique Continuation). *Suppose Assumption* (A). *Let $u \in \mathcal{C}$ be a solution of* (1.1) *and let $0 \leq \gamma < 1$. If u satisfies, for some $z_o = (x_o, t_o) \in \mathbf{R}^n \times (0,T)$ and for arbitraly $N > 0$,*

$$(1.6) \qquad \int_{S_R(t_o) \cap \{|x-x_o| < R^\gamma\}} u^2 G_{z_o}\, dx = O(R^N) \quad as \quad R \to 0,$$

then $u(x,t) \equiv 0$ on $\mathbf{R}^n \times (t_o - (R^*)^2, t_o)$, where $R^* > 0$ is the number given in Theorem 1.3.

As a corollary of the proof of Theorem 1.3, we obtain backward uniqueness, if we assume

Assumption (A′). In addition to Assumption (A), V satisfies that the compact set K (associated with the definition $V \in Q_{t,p}(0,T)$) can be taken uniformly in $t_o \in (0,T)$ and $F_1(s) \equiv \sup_{t_o \in (0,T)} f_1^{(t_o)}(s) \to 0$ as $s \to 0$, where $f_1^{(t_o)}(s)$ is the function defined in Assumption (A) (ii).

Corollary 1.5 (Backward Uniqueness). *Suppose Assumption* (A′). *If the solution $u \in \mathcal{C}$ of* (1.1) *satisfies $u(\cdot, t_o) \equiv 0$ for some $t_o \in (0,T)$, then $u(\cdot,t) \equiv 0$ for every $t \in (0, t_o)$.*

We note that if the assumption (A′) is satisfied, then we can take $R^* = \min(1/\sqrt{8\alpha}, \sqrt{t_o}, R_*)$, R_* is independent of t_o. By this observation Corollary 1.5 follows easily (cf. [GL], [Ku]). As a direct consequence of Theorem 1.4 we have

Corollary 1.6 (Weak UCP). *Suppose the Assumption* (A). *If the solution $u \in \mathcal{C}$ of* (1.1) *vanishes in some open set $\omega \subset \mathbf{R}^n \times (0.T)$, then u vanishes in the horizontal component of ω in $\mathbf{R}^n \times (0.T)$.*

There are several results on backward uniqueness and unique continuation theorems (see e.g., [L], [M], [So], [SS], [LP]), but Theorem 1.3 is new even in the case $V \equiv 0$, and Theorem 1.4 yields the different type of strong unique continuation property for solutions of (1.1). Moreover, the method of this paper is different from the previous works. This work is a parabolic version of [Ku].

If $\Omega \subset \mathbf{R}^n$ is bounded, smooth, and convex, we can show the same results for solutions u of

$$(1.7)\quad \frac{\partial u}{\partial t} = \Delta u + V(x,t)u \quad in \quad \Omega \times (0,T), \quad u = 0 \quad on \quad \partial\Omega \times (0,T).$$

Recently we also proved similar results for weak solutions. However, we do not know whether the backward estimate of type (1.5) also holds or not for u satisfying (1.7) locally (that is, without boundary condition).

§2. Preliminaries

In this section we show an inequality which controls singularities of V in the proof of Theorems. Let $z_o = (x_o, t_o) \in \mathbf{R}^n \times (0,T)$ and put

$$\Phi_{z_o}(R;u) = \frac{1}{2}R^2 \int_{S_R(t_o)} |\nabla u|^2 G_{z_o}\, dx, \quad N_{z_o}(R;u) = \frac{I_{z_o}(R;u)}{H_{z_o}(R;u)}.$$

Then we have

Lemma 2.1. *Suppose $V \in Q_{t,p}(0,T)$ with $1 < t \le n/2, p \ge 1$. Then there exists a constant $C > 0$ such that*

$$\int_{S_R(t_o)} |V|u^2 G_{z_o}\,dx \le U(R^2)H_{z_o}(R;u) \qquad (2.1)$$
$$+ CR^{-4/p}\||W(R^2)|^p\|_{Q_t}^{1/p}(\Phi_{z_o}(R;u) + H_{z_o}(R;u))$$

for every $\sqrt{t_o} > R > 0$ and $u \in C^o((0,T);C_o^\infty(\mathbf{R}^n))$, where U and W are functions associated with V by Definition 1.1.

As an easy consequence of Lemma 2.1 we have

Lemma 2.2. *Suppose that $V \in Q_{t,p}(0,T)$ with $1 < t \le n/2, p > 2$ and that $f^{(t_o)}(s) \equiv s^2U(s^2) + s^{2-4/p}\||W(s^2)|^p\|_{Q_t}^{1/p} \to 0$ as $s \to 0$. Then there exist $C > 0$ and sufficiently small R_* such that*

$$C^{-1}\Phi_{z_o}(R:u) \le I_{z_o}(R:u) \le C\Phi_{z_o}(R;u) \qquad (2.2)$$

for every $0 < R < R_*$ satisfying $N_{z_o}(R;u) > 1$.

To prove Lemma 2.1, first we note that if $V \in K_n$,

$$\int_{\mathbf{R}^n} |V|u^2 \le C(n)\eta^K(r;V)\Big(\int_{\mathbf{R}^n} |\nabla u|^2\,dx + \frac{1}{r^2}\int_{\mathbf{R}^n} u^2\,dx\Big)$$

for every $r > 0$ and $u \in C_o^\infty(\mathbf{R}^n)$, and that if $V \in F_t$ with $1 < t \le n/2$,

$$\int_{\mathbf{R}^n} |V|u^2 \le C(n,t)\|V\|_{F_t}\int_{\mathbf{R}^n} |\nabla u|^2\,dx$$

for every $u \in C_o^\infty(\mathbf{R}^n)$ (see e.g. [F], [Si]). Hence if $V \in Q_t$ with $1 < t \le n/2$, we have

$$\int_{\mathbf{R}^n} |V|u^2 \le C(n,t,R_o)\|V\|_{Q_t}\Big(\int_{\mathbf{R}^n} |\nabla u|^2\,dx + \int_{\mathbf{R}^n} u^2\,dx\Big) \qquad (2.3)$$

for every $u \in C_o^\infty(\mathbf{R}^n)$, where $R_o > 0$ is a fixed constant.

Proof of Lemma 2.1. Let $t_o \in (0,T)$. We use the notaion $S_R = S_R(t_o)$ and $G = G_{z_o}$ for the sake of simplicity. Since $V \in Q_{t,p}(0,T)$, by the definition there exist $W, U \ge 0$ and a compact set $K \subset \mathbf{R}^n$ such that $|V(x, t_o - R^2)| \le W(x,R^2) + U(R^2)$ with $\mathrm{supp}_x W(\cdot, R^2) \subset K$ for every

$0 < R^2 < t_o$. Let $\eta \in C_o^\infty(\mathbf{R}^n)$ satisfy $\eta(x) \equiv 1$ on K, $0 \le \eta(x) \le 1$, and $|\nabla\eta(x)| \le C$. By Hölder's inequality, we have

$$\begin{aligned}
&\int_{S_R} |W(x,R^2)|u^2 G\,dx \\
&\qquad \le \Big(\int_{S_R} |W(x,R^2)|^p u^2 G\,dx\Big)^{1/p}\Big(\int_{S_R} u^2 G\,dx\Big)^{1/q} \\
&\qquad = (2H(R;u))^{1/q}\Big(\int_{S_R} |W(x,R^2)|^p u^2 G\,dx\Big)^{1/p},
\end{aligned}$$

where $1/p + 1/q = 1$. The inequality (2.3) yields

$$\begin{aligned}
&\int_{S_R} |W|^p u^2 G\,dx \\
&\qquad \le \int_{S_R} |W|^p (\eta u)^2 G\,dx \qquad (2.4)\\
&\qquad \le C(n,t)\||W|^p\|_{Q_t}\Big(\int_{S_R} |\nabla(\eta u G^{1/2})|^2\,dx + \int_{S_R} u^2 G\,dx\Big).
\end{aligned}$$

Since $|\nabla(\eta u G^{1/2})|^2 \le C(n)(u^2 + |\nabla u|^2)G + C(n,K)\frac{u^2}{R^4}G$, we obtain

$$\begin{aligned}
&\int_{S_R} |W|^p u^2 G\,dx \qquad (2.5)\\
&\qquad \le C(n,t,K)\||W|^p\|_{Q_t}\Big(\big(1 + \frac{1}{R^4}\big)H(R;u) + \int_{S_R} |\nabla u|^2 G\,dx\Big).
\end{aligned}$$

Hence it follows that

(2.6)

$$
\begin{aligned}
&\int_{S_R} |V|u^2 G\,dx \\
&\quad\le \int_{S_R} |W|u^2 G\,dx + U(R^2)\int_{S_R} u^2 G\,dx \\
&\quad\le 2U(R^2)H(R;u) \\
&\qquad + C(n,t,p)\||W|^p\|_{Q_t}^{1/p} H(R;u)^{1/q}\left(\frac{H(R;u)}{R^4}+\frac{\Phi(R;u)}{R^2}\right)^{1/p} \\
&\quad\le 2U(R^2)H(R;u) \\
&\qquad + C(n,t,p)R^{-4/p}\||W|^p\|_{Q_t}^{1/p}(H(R;u)+\Phi(R;u)).
\end{aligned}
$$

Q.E.D.

Proof of Lemma 2.2. Note that $I(R;u) = \Phi(R;u) - \Psi(R;u)$ and that $N(R;u)>1$ implies $H(R;u) < I(R;u)$ by definition. By Lemma 2.1 we have, for $R > 0$ satisfying $N(R;u) > 1$,

$$|\Psi(R;u)| \le Cf^{(t_o)}(R)\Phi(R;u). \tag{2.7}$$

Hence, by the assumption $f^{(t_o)}(s) \to 0$ as $s \to 0$, there exists $R_* > 0$ such that $Cf^{(t_o)}(R) < 1/2$ for every $0 < R < R_*$. Hence we obtain the desired estimate. Q.E.D.

§3. Proof of theorems

In this section we prove theorems. Suppose that V satisfies Assumption (A) and $u \in \mathcal{C}$ is a solution of (1.1) throughout this section. Without loss of generality, we may assume $z_o = (O,0)$ and consider (1.1) for $t < 0$. We write $S_R = S_R(0) = \{(x,t) | t = -R^2\}$, $G_o = G_{(O,0)}$, $H(R) = H_{(O,0)}(R;u)$, $I(R) = I_{(O,0)}(R;u)$ and $N(R) = N_{(O,0)}(R;u)$, and use the notation $P(u) = x\cdot\nabla u + 2t\partial_t u$. Let $R^* = \min(1/\sqrt{8\alpha}, R_*)$, where R_* is the number determined by Lemma 2.2 with respect to $t_o = 0$. Then we have

Lemma 3.1. *For* $0 < R < R^*$, $I(R)$ *is differentiable and satisfies*

$$
\begin{aligned}
I'(R) &= \frac{1}{2R}\int_{S_R} P(u)^2 G_o\,dx \\
&\quad - \frac{R}{2}\int_{S_R} (2V + x\cdot\nabla V + 2t\partial_t V)u^2 G_o\,dx.
\end{aligned} \tag{3.1}
$$

If $u(x,t) \not\equiv 0$ on $\mathbf{R}^n \times (-(R^*)^2, 0)$, then it follows that $H(R) > 0$ for every $0 < R < R^*$. We note that this fact can be proved by the similar argument as in [GL;p264] (see also the proof of Theorem 1.5 in [Ku]). Therefore we may assume that $H(R) > 0$ for every $0 < R < R^*$, and hence $N(R)$ is also differentiable on $(0, R^*)$. Let $\tilde{V} = 2V + x\cdot\nabla V + 2t\partial_t V$ have the expression $|\tilde{V}| \le W_2 + U_2$ by the assumption $\tilde{V} \in Q_{t,p}(0,T)$. Then by Lemma 3.1 we obtain the following differential inequality for $N(R)$.

Lemma 3.2. *There exists $C > 0$ such that*

(3.2)
$$\frac{N'(R)}{N(R)} \ge -C\left(\frac{R^2U_2(R^2) + R^{2-4/p}\||W_2(R^2)|^p\|_{Q_t}^{1/p}}{R}\right) \equiv -C\frac{f_2^{(0)}(R)}{R}$$

for $0 < R < R^*$ satisfying $N(R) > 1$.

Proof of Lemma 3.1. We follow the computation of Struwe [St]. Let $u_R(x,t) = u(Rx, R^2t)$. Then we have $\Phi(R;u) = \Phi(1, u_R)$. If u is a solution of $\dfrac{\partial u}{\partial t} = \Delta u + V(x,t)u$, then u_R is a solution of $\dfrac{\partial u_R}{\partial t} = \Delta u_R + V_R(x,t)u_R$, where $V_R(x,t) = R^2V(Rx, R^2t)$. By noting $\nabla G_o = -(x/2R^2)G_o$ on S_R, we obtain

$$\begin{aligned}
\Phi'(R;u) &= \frac{d\Phi(1;u_R)}{dR} \\
&= \int_{S_1} \nabla u_R \cdot \nabla\left(\frac{du_R}{dR}\right) G_o\,dx \\
&= -\int_{S_1} (\Delta u_R G_o + \nabla u_R \cdot \nabla G_o)\frac{du_R}{dR}\,dx \qquad (3.3)\\
&= \int_{S_R} \frac{P(u)}{R}\left(\frac{P(u)}{2} + R^2Vu\right)G_o\,dx \\
&= \frac{1}{2R}\int_{S_R} P(u)^2 G_o\,dx + R\int_{S_R} P(u)VuG_o\,dx.
\end{aligned}$$

On the other hand, since $\Psi(R;u) = \frac{1}{2}\int_{S_R} Vu^2G_o\,dx = \frac{1}{2}\int_{S_1} V_R u_R^2 G_o\,dx$, we have

$$\Psi'(R;u) = \frac{1}{2}\int_{S_1} \frac{dV_R}{dR} u_R^2 G_o\,dx + \int_{S_1} V_R \frac{du_R}{dR} u_R G_o\,dx$$
$$= \frac{R}{2}\int_{S_R} (2V + x\cdot\nabla V + 2t\partial_t V)u^2 G_o\,dx \tag{3.4}$$
$$+ R\int_{S_R} P(u)VuG_o\,dx.$$

Combining (3.3) with (3.4) we complete the proof. Q.E.D.

Proof of Lemma 3.2. Since $H(R;u) = H(1;u_R)$, we have

$$H'(R) = H'(R;u) = \int_{S_1} u_R \frac{du_R}{dR} G_o\,dx = \frac{1}{R}\int_{S_R} uP(u)G_o\,dx. \tag{3.5}$$

On the other hand, multiplying uG_o to (1.1) and integrating over S_R, we obtain

$$\int_{S_R} u\partial_t uG_o\,dx = -\int_{S_R} |\nabla u|^2 G_o\,dx - \int_{S_R} u\nabla u\cdot\nabla G_o\,dx + \int_{S_R} Vu^2 G_o\,dx.$$

Since $\nabla G_o = \frac{x}{2t}G_o$ on S_R, this implies

$$I(R) = \frac{1}{4}\int_{S_R} P(u)uG_o\,dx. \tag{3.6}$$

Hence we obtain $H'(R) = \frac{4}{R}I(R)$. Therefore, for $0 < R < R^*$, (3.1) and (3.7) yield

$$\frac{N'(R)}{N(R)} = \frac{I'(R)}{I(R)} - \frac{H'(R)}{H(R)}$$
$$= \frac{\int_{S_R} P(u)^2 G_o\,dx}{2RI(R)} - \frac{4I(R)}{RH(R)} \tag{3.7}$$
$$- \frac{R}{2I(R)}\int_{S_R} (2V + x\cdot\nabla V + 2t\partial_t V)u^2 G_o\,dx.$$

By Schwarz's inequality,

$$\frac{\int_{S_R} P(u)^2 G_o\,dx}{2RI(R)} - \frac{4I(R)}{RH(R)}$$
$$= \frac{\int_{S_R} P(u)^2 G_o\,dx}{\frac{R}{2}\int_{S_R} P(u)uG_o\,dx} - \frac{\int_{S_R} P(u)uG_o\,dx}{\frac{R}{2}\int_{S_R} u^2 G_o\,dx} \geq 0. \tag{3.8}$$

Thus we arrive at

$$\text{(3.9)} \qquad \frac{N'(R)}{N(R)} \geq -\frac{R}{2I(R)} \int_{S_R} (2V + x \cdot \nabla V + 2t\partial_t V) u^2 G_o \, dx$$

for $0 < R < R^*$. By Lemmas 2.1 and 2.2 we can conclude the desired esimate. Q.E.D.

Proof of Theorem 1.3. Note that the set $\{0 < R < R^* : N(R) > 1\}$ is open, because $N(R)$ is continuous. Hence there exist countable open disjoint intervals (R_j, R_{j+1}) such that $\{0 < R < R^* : N(R) > 1\} = \cup_{j=1}^{\infty}(R_j, R_{j+1})$. By Assumption (A) and Lemma 3.2, we have

$$\log(\frac{N(R_{j+1})}{N(R_j)}) \geq -C \int_0^{R^*} \frac{f_2(s)}{s} \, ds$$

for each $j = 1, 2, \cdots$. This implies

$$\text{(3.10)} \qquad N(R) \leq \max(1, N(R^*)) \exp(-C \int_0^{R^*} \frac{f_2(s)}{s} \, ds) (\equiv N_o)$$

for $0 < R < R^*$. Since $H'(R) = (4/R)I(R)$, we obtain

$$\text{(3.11)} \qquad H(2R) \leq H(R) \exp(4N_o \log 2), \; 0 < R < R^*.$$

This complete the proof of Theorem 1.3. Q.E.D.

Proof of Theorem 1.4. It is well-known that when the doubling estimate (1.5) in Theorem 1.3 holds, the condition that $H(R) = O(R^N)$ for every $N > 0$ as $R \to 0$ implies $H(R) \equiv 0$ for every $R \in (0, R^*)$ (see e.g.,[GL]). Hence it suffices to show $H(R) = O(R^N)$ for every $N > 0$. Let $0 \leq \gamma < 1$ and put

$$g(R) = \int_{S_R(t_o) \cap \{x; |x - x_o| \geq R^\gamma\}} u^2 G_{z_o} \, dx.$$

Then it is easy to see that there exists a constant M such that

$$g(R) \leq \frac{M}{R^n} \exp(-\frac{1}{8R^{2(1-\gamma)}}).$$

Actually we can take

$$M = \sup_{t \in [t_o - (R^*)^2, t_o]} \int_{\mathbf{R}^n} u^2(x,t) e^{-\frac{|x - x_o|^2}{8(R^*)^2}} \, dx < +\infty,$$

since $R^* \leq 1/\sqrt{8\alpha}$. Hence $g(R) = O(R^N)$ for every $N > 0$. By the assumption $f(R) = \int_{S_R(t_o)\cap\{x;|x-x_o|<R^\gamma\}} u^2 G_{z_o}\, dx = O(R^N)$, we can conclude that $H(R) = O(R^N)$ for every $N > 0$. Thus we complete the proof. Q.E.D.

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Large Atoms in the Magnetic Field of a Neutron Star

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Abstract.

The asymptotics of the ground state energy of large atoms as $Z \to \infty$ is given exactly by Thomas-Fermi theory. The introduction of a large magnetic field, B, changes the situation. If we set $B = cZ^p$ then, as $Z \to \infty$, there are 5 regions: $p < 4/3$, $p = 4/3$, $4/3 < p < 3$, $p = 3$, $p > 3$. The first three are described exactly by a modified TF theory. The fifth is describable exactly by a one-dimensional Hartree like theory. The fourth is describable exactly by a novel density matrix theory. A surprising conclusion is that although the magnetic field has a profound effect on the atomic energy in regions 2,3,4 and 5, the atom remains spherical (to leading order) in regions 2 and 3.

§1. Introduction

In this talk I shall discuss the effect on matter, specifically the ground state of atoms, of a very strong magnetic field. Results obtained in collaboration with J.P. Solovej and J. Yngvason will be summarized and details will appear elsewhere [LSY]. The physical motivation for studying extremely strong magnetic fields of the order of 10^{12}–10^{13} Gauss is that they are supposed to exist on the surface of neutron stars. This study was essentially begun in the early 70's with the work of Kadomtsev [K], Ruderman [R] and Mueller, Rau and Spruch [MRS]; see [FGP] and [FGPY] for further references. The argument given to explain these strong fields is that in the collapse, resulting in the neutron star, the magnetic field lines are trapped and thus become very dense. The structure of matter in strong magnetic fields is, therefore, a question of considerable interest in astrophysics. Mathematically, the problem turns out to involve an interesting exercise in semiclassical analysis.

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We use units in which $e = \hbar = 2m_e = 1$ and $c = c\hbar e^{-2} = \alpha^{-1} \approx 137$. The natural unit of length is $\hbar^2/2m_e e^2$, i.e., half the Bohr radius. The natural unit of magnetic field strength that we shall use is $(2m_e)^2 e^3 c/\hbar^3 = 9.4 \times 10^9$ Gauss. This is the field for which the magnetic length $\sqrt{c\hbar/eB}$ equals half the Bohr radius. Thus, in our units, $B \approx 10^2 - 10^3$ for some neutron stars.

The atomic nucleus of principal interest on the surface of a neutron star is presumably iron with $Z = 26$. This number is large and hence it is sensible to ask (rigorously) about the limit of the ground state energy of an atom as $Z \to \infty$. We shall calculate this limit exactly; its application to $Z = 26$ instead of $Z = \infty$ will entail some errors — for which we can give bounds.

§2. Main results

To give the quantum mechanical energy of a charged spin-$\frac{1}{2}$ particle in a magnetic field $\mathbf{B}$, we have to make a choice of vector potential $\mathbf{A}(x)$, satisfying $\mathbf{B} = \nabla \times \mathbf{A}$. The energy is then given by the Pauli Hamiltonian

$$H_{\mathbf{A}} = \left((\mathbf{p} - \mathbf{A}(x)) \cdot \sigma\right)^2 . \tag{2.1}$$

Here $\mathbf{p} = -i\nabla$ and $\sigma = (\sigma_1, \sigma_2, \sigma_3)$, are the Pauli matrices. We can also write $H_{\mathbf{A}} = (\mathbf{p} - \mathbf{A})^2 - \mathbf{B} \cdot \sigma$. We shall here concentrate on the case where $\mathbf{B}$ is constant, say $\mathbf{B} = (0, 0, B)$, with $B \geq 0$. We choose $\mathbf{A} = \frac{1}{2}\mathbf{B} \times x$.

The Hamiltonian describing an atom with N electrons and nuclear charge Z (with fixed nucleus) in a constant magnetic field $\mathbf{B}$ is

$$H_N = \sum_{i=1}^{N} (H_{\mathbf{A}}^{(i)} - Z|x_i|^{-1}) + \sum_{1 \leq i < j \leq N} |x_i - x_j|^{-1} . \tag{2.2}$$

H_N acts on the Hilbert space $\mathcal{H}_N = \bigwedge^N L^2(\mathbf{R}^3; \mathbf{C}^2)$ of antisymmetric (i.e., fermionic) spinor-valued functions. We are interested in $E(N, B, Z) = \inf \operatorname{spec}_{\mathcal{H}_N} H_N$, **the ground state energy** of H_N.

We want to let B and Z go to infinity. It is surprising, but true, that there are *five* different regimes in B and Z, depending on the relative magnitudes of B and Z. In the following $\rho(x)$ is the electron density in the ground state ψ:

$$\rho(x) = N \int \|\psi(x, x_2, \ldots, x_N)\|^2 d^3x_2 \ldots d^3x_N. \tag{2.3}$$

The five regions are the following.

1) $B \ll Z^{4/3}$, Z *large*:

The effect of the magnetic field is negligible. Standard Thomas-Fermi (TF) theory is exact as $Z \to \infty$, and therefore the electron density is spherical to leading order.

2) $B \sim Z^{4/3}$, Z *large*:

The magnetic field becomes important but the density is still almost spherical and stable atoms are almost neutral (see [Y]). A modified TF theory (depending on the constant $B/Z^{4/3}$), in which the energy, as in standard TF theory, is approximated by a functional of the density ρ alone, is exact as $Z \to \infty$. We call this functional the Magnetic Thomas-Fermi (MTF) functional (see Sect. IV below).

3) $Z^{4/3} \ll B \ll Z^3$, Z *large*:

The magnetic field is increasingly important. To leading order all electrons will be confined to the lowest Landau band. The modified TF theory is still exact as $Z \to \infty$. In fact, the modified TF theory simplifies somewhat in this region compared to the MTF functional from the previous region. we call the new functional the Strong Thomas-Fermi (STF) functional. The only difference between STF and standard TF theory is that the usual $\rho^{5/3}$ is replaced by ρ^3/B^2, while in the MTF theory from the previous region the function that replaces $\rho^{5/3}$ is more complicated (see (4.1) below). The density is almost spherical and stable atoms are almost neutral. Furthermore, the atom is getting smaller. The atomic radius behaves like $Z^{1/5}B^{-2/5} = Z^{-1/3}(B/Z^{4/3})^{-2/5}$. The energy behaves like $Z^{9/5}B^{2/5} = Z^{7/3}(B/Z^{4/3})^{2/5}$.

4) $B \sim Z^3$, Z *large*:

The modified TF theories are no longer applicable. Indeed, we shall in general not approximate the energy by functionals of the density ρ alone. The energy is approximated by a more complicated functional to be described below in Sect. IV depending on a one particle density matrix. We call this functional the Density Matrix (DM) functional. When B/Z^3 is large enough this functional again reduces to a density functional . For the first time the atom is no longer spherical to leading order. The length scale of the atom behaves like Z^{-1} and the energy like Z^3.

5) $B \gg Z^3$, *Z large*:

In this hyper-strong case the atom is essentially one-dimensional. We can find a new functional, the Hyper-Strong (HS) functional depending only on the one-dimensional density $\overline{\rho}$ obtained from ρ by integrating ρ over the directions perpendicular to the field $\mathbf{B}$, i.e., $\overline{\rho}(x_3) = \int\int \rho(x_1, x_2, x_3)dx_1dx_2$. The energy behaves like $Z^3[\ln(B/Z^3)]^2$ and the length scale along the magnetic field is $Z^{-1}[\ln(B/Z^3)]^{-1}$, while the radius perpendicular to the field is $Z^{-1}(B/Z^3)^{-1/2}$.

The mathematically more precise statements of these results involve two energy functions $E_{\rm MTF}(N,B,Z)$ and $E_{\rm DM}(N,B,Z)$. The energy $E_{\rm MTF}(N,B,Z)$ is obtained as the minimum of the magnetic Thomas-Fermi functional mentioned under 2) above, and $E_{\rm DM}(N,B,Z)$ is the minimum of the density matrix functional mentioned under 4). The exact definitions of these functionals are given in Sect. IV below.

The energies $E_{\rm MTF}$ and $E_{\rm DM}$ correspond to unique minimizers for the respective functionals. We denote the densities for these minimizers by $\rho_{\rm MTF}$ and $\rho_{\rm DM}$ respectively.

In the case when $B = 0$ the energy $E_{\rm MTF}(N,0,Z)$ is the energy of standard TF theory. It is known [LS] (see also [L]) that TF theory is asymptotically *exact* as $Z \to \infty$ with N/Z fixed, i.e.,

$$E_{\rm MTF}(N,0,Z)/E(N,0,Z) \to 1 \quad \text{as } Z \to \infty.$$

Is the same true when $B \neq 0$? The answer, surprisingly, depends on the relative magnitudes of B and Z, according to the 5 regions outlined above.

Theorem 1. *Let N/Z be fixed and suppose $B/Z^3 \to 0$ as $Z \to \infty$. Then*

$$E_{\rm MTF}(N,B,Z)/E(N,B,Z) \to 1 \qquad \textit{as } Z \to \infty.$$

This theorem covers the regions 1–3 above. For the regions 4 and 5 we have

Theorem 2. *Let N/Z be fixed and suppose $B/Z^{4/3} \to \infty$ as $Z \to \infty$. Then*

$$E_{\rm DM}(N,B,Z)/E(N,B,Z) \to 1 \qquad \textit{as } Z \to \infty.$$

Notice that there is an overlap of the regions of validity of the two theorems. In fact, both theorems cover region 3 above.

The energy functions satisfy the scalings

$$E_{\mathrm{MTF}}(N, B, Z) = Z^{7/3} E_{\mathrm{MTF}}(N/Z, B/Z^{4/3}, 1)$$

and

$$E_{\mathrm{DM}}(N, B, Z) = Z^3 E_{\mathrm{DM}}(N/Z, B/Z^3, 1).$$

In region 2 there is a non-trivial parameter $B/Z^{4/3}$. Likewise in region 4 there is B/Z^3. In the other three regions these parameters enter in trivial way since they are tending either to 0 or ∞.

Region 1 corresponds to $B/Z^{4/3} \to 0$ and $B/Z^3 \to 0$ in which case

$$E_{\mathrm{MTF}}(N/Z, B/Z^{4/3}, 1) \to E_{\mathrm{MTF}}(N/Z, 0, 1),$$

which is the energy of standard TF theory.

Region 3 corresponds to $B/Z^{4/3} \to \infty$, in which case we have the asymptotic expansion

$$E_{\mathrm{MTF}}(N/Z, B/Z^{4/3}, 1) \approx (B/Z^{4/3})^{2/5} E_{\mathrm{STF}}(N/Z) \quad \text{as} \quad B/Z^{4/3} \to \infty ,$$

where E_{STF} is an energy function obtained from the simplified TF theory described under 3) above.

The overlap of the regions of validity of Theorems 1 and 2 implies that

$$E_{\mathrm{DM}}(N/Z, B/Z^3, 1) \approx (B/Z^3)^{2/5} E_{\mathrm{STF}}(N/Z) \quad \text{as} \quad B/Z^3 \to 0 .$$

Finally, region 5 corresponds to $B/Z^3 \to \infty$, where the following asymptotic formula holds

$$E_{\mathrm{DM}}(N/Z, B/Z^3, 1) \approx [\ln(B/Z^3)]^2 E_{\mathrm{HS}}(N/Z) \quad \text{as} \quad B/Z^3 \to \infty ,$$

where E_{HS} is an energy function obtained from the one-dimensional functional mentioned in 5) above.

The energies E_{MTF}, E_{DM}, E_{STF} and E_{HS} correspond to unique minimizers for the respective functionals. We denote the densities for these minimizers by ρ_{MTF}, ρ_{DM}, ρ_{STF} and $\overline{\rho}_{\mathrm{HS}}$ respectively. We can prove that these densities approximate the quantum density ρ. However, to state these approximations we have to introduce different scalings in the different regions. In fact, the above approximating densities satisfy the following scaling relations

$$\rho_{\mathrm{MTF}}(x;N,B,Z)=Z^2\rho_{\mathrm{MTF}}\left(Z^{1/3}x\,;\frac{N}{Z},\frac{B}{Z^{4/3}},1\right)$$

$$\rho_{\mathrm{STF}}(x;N,B,Z)=Z^2\left(\frac{B}{Z^{4/3}}\right)^{6/5}\rho_{\mathrm{STF}}\left(\left(\frac{B}{Z^{4/3}}\right)^{2/5}Z^{1/3}x\,;\frac{N}{Z},1,1\right)$$

$$\rho_{\mathrm{DM}}(x;N,B,Z)=Z^4\rho_{\mathrm{DM}}\left(Zx\,;\frac{N}{Z},\frac{B}{Z^3},1\right)$$

$$\overline{\rho}_{\mathrm{HS}}(x_3;N,B,Z)=Z^2\ln\left(\frac{B}{Z^3}\right)\overline{\rho}_{\mathrm{HS}}\left(Z\ln\left(\frac{B}{Z^3}\right)x_3\,;\frac{N}{Z},1,1\right).$$

Theorem 3 (Convergence of the density). *In the five different regions the following relations hold as $Z\to\infty$. These limits are all in weak L^1_{loc} :*

(1–2) *If $B/Z^{4/3}\to\beta$, where $0\le\beta<\infty$ and if $N/Z=\lambda$ is fixed then*

$$Z^{-2}\rho(Z^{-1/3}x)\rightharpoonup\rho_{\mathrm{MTF}}(x;\lambda,\beta,1)\ .$$

(3) *If $B/Z^{4/3}\to\infty$ and $N/Z=\lambda$ is fixed then*

$$Z^{-2}\left(\frac{B}{Z^{4/3}}\right)^{-6/5}\rho\left(Z^{-1/3}\left(\frac{B}{Z^{4/3}}\right)^{-2/5}x\right)\rightharpoonup\rho_{\mathrm{STF}}(x;\lambda,1,1)\ .$$

(4) *If $B/Z^3\to\eta$, where $0<\eta<\infty$ and $N/Z=\lambda$ is fixed then*

$$Z^{-4}\rho_{\mathrm{DM}}(Z^{-1}x)\rightharpoonup\rho_{\mathrm{DM}}(x;\lambda,\eta,1)\ .$$

(5) *If $B/Z^3\to\infty$ and $N/Z=\lambda$ is fixed then*

$$\frac{1}{Z^2\ln(B/Z^3)}\overline{\rho}\left(\frac{x_3}{Z\ln(B/Z^3)}\right)\rightharpoonup\overline{\rho}_{\mathrm{HS}}(x_3;\lambda,1,1)\ .$$

§3. The one-body Hamiltonian

The spectrum of the one-body Hamiltonian $H_{\mathbf{A}}$ is described by the Landau bands $\varepsilon_{p\nu}=2B\nu+p^2$, where p is the momentum along the field and $\nu=0,1,2,\ldots$ is the index of the band. Owing to the spin degeneracy, the higher bands, $\nu \geq 1$, are twice as degenerate as the lowest band $\nu = 0$.

To calculate the energy of a large, complex atom one must first study the one-body Hamiltonian $H = H_{\mathbf{A}} + V(x)$, where V is an external potential. As usual, to calculate the ground state energy of a fermionic system we need to know the sum of the negative eigenvalues of the operator H (with $V \leq 0$ for simplicity).

In order to estimate accurately the sum of the negative eigenvalues of $H_{\mathbf{A}} + V(x)$ we need two things: (i) a lower bound for this quantity and (ii) an asymptotic (or semiclassical) limit formula for the quantity. These are provided by Theorems 4 and 5 below. The bound (i) is needed to control errors between the true answer and the semiclassical approximation. The semiclassical limit turns out to be relevant here (after some suitable scaling) because it is equivalent to the limit $Z \to \infty$.

There is an important difference between $H_{\mathbf{A}}$ and the operator $(\mathbf{p} - \mathbf{A})^2$ which has no spin dependence. While the spectrum of $(\mathbf{p} - \mathbf{A})^2$ is (B, ∞) the spectrum of $H_{\mathbf{A}}$ is $(0, \infty)$. Indeed, one can bound the sum of the negative eigenvalues of $(\mathbf{p} - \mathbf{A})^2 - V(x)$ by $-L \int |V(x)|^{5/2}\, dx$, (where L is some fixed constant) according to the standard Lieb-Thirring inequality (even with a magnetic field the proof of this inequality given in [LT] is still correct if one appeals to the diamagnetic inequality). However, in the case of $H_{\mathbf{A}} + V$ the question is somewhat more subtle. In fact, if $\int |V|^{3/2} < \infty$, the operator $(\mathbf{p} - \mathbf{A})^2 + V$ has a finite number of negative eigenvalues, while the operator $H_{\mathbf{A}} + V$ can have infinitely many negative eigenvalues (compare [I]). We can, however, prove [LSY] the following bound which is important in our proofs.

Theorem 1. *There exist universal constants $L_1, L_2 > 0$ such that if we let $e_j(B, V)$, $j = 1, 2, \ldots$ denote the negative eigenvalues of $H_{\mathbf{A}} + V$ with $V \leq 0$ then*

$$\sum_j |e_j(B, V)| \leq L_1 B \int |V(x)|^{3/2}\, d^3x + L_2 \int |V(x)|^{5/2}\, d^3x \,. \tag{3.1}$$

We can choose L_1 as close to $2/3\pi$ as we please, compensating with L_2 large.

The first term on the right side is a contribution from the lowest band, $\nu = 0$. For large B this is the leading term.

We now ask the question of a semiclassical analog of (3.1). Thus, consider the operator

$$\left[(h\mathbf{p} - b\mathbf{a}(x)) \cdot \sigma\right]^2 + v(x) \,, \tag{3.2}$$

where $\mathbf{a}(x) = \frac{1}{2}\hat{z} \times x$, $\hat{z} = (0,0,1)$ and $v \le 0$.

If one computes the leading term in h^{-1} of the sum of the negative eigenvalues of (3.2) for fixed b one finds as in [HR] that there is no b dependence. In our case, however, we shall not assume b fixed, or more precisely not assume that b is small compared with h^{-1}. The reason for this is that in the application to neutron stars it is not true, as we shall discuss below, that $b \ll h^{-1}$.

The interesting fact is, however, that we can prove ([LSY]) a semiclassical formula for the sum of the negative eigenvalues of the operator (3.2), which holds uniformly in b (even for large b).

Theorem 5. *Let $e_j(h,b,v)$, $j = 1,2,\ldots,$ denote the negative eigenvalues of the operator* (3.2), *with $v \le 0$. Then*

$$\lim_{h\to 0}\Big(\sum_j e_j(h,b,v)/E_{\text{scl}}(h,b,v)\Big) = 1,$$

uniformly in b, where E_{scl} is the semiclassical approximation defined by

$$\begin{aligned} &E_{\text{scl}}(h,b,v) \\ &= -\frac{1}{3\pi^2}h^{-2}b\int \Big(|v(x)|^{3/2} + 2\sum_{\nu=1}^{\infty}[|v(x)| - 2\nu bh]_+^{3/2}\Big)\,d^3x\,. \end{aligned} \tag{3.3}$$

Here $[t]_+ = t$ if $t > 0$, zero otherwise.

The formula (3.3) was already implicitly noted in [Y]. The integrand in (3.3) looks peculiar, but it has the following simple physical interpretation. Take a cubic box of volume L^3 in $\mathbb{R}^3$ and let the number $\mu > 0$ be some fixed Fermi level (or chemical potential). Then add together all the negative eigenvalues of $H_{\mathbf{A}} - \mu$. In the thermodynamic limit (large L) we can do this addition simply by using the known Landau levels, and the total energy per unit volume is the integrand in (3.3) in which $|v(x)|$ is set equal to μ.

For $bh \ll 1$, the right side of (3.3) reduces to the standard semiclassical formula from [HR],

$$-\frac{2}{15\pi^2}h^{-3}\int |v(x)|^{5/2}\,d^3x\,. \tag{3.4}$$

(Recall that we are counting the spin which accounts for the 2 in front of the sum in (3.3).) For $bh \gg 1$, the sum in (3.3) is negligible, and we are left with the first term.

Formula (3.3) (with h replaced by 1) can be compared with the Lieb-Thirring inequality (3.1), which holds even outside the semiclassical regime. The two terms in (3.1) correspond to respectively the $b \to \infty$ (first term) and $b \to 0$ (last term) asymptotics of (3.3).

As we know from elementary thermodynamics, the energy per unit volume as a function of the particle density ($\rho(x)$ in our case) is the Legendre transform of the pressure as a function of the chemical potential ($|v(x)|$). Thus, corresponding to $-(2/15\pi^2)|v(x)|^{5/2}$ in (3.4), there is the energy $(3/5)(3\pi^2)^{2/3}\rho(x)^{5/3}$, which is the usual kinetic energy expression in TF theory. Likewise, corresponding to (3.3) there is a kinetic energy which we call $w_B(\rho(x))$. It is no longer proportional to $\rho(x)^{5/3}$ but it is still a convex function of $\rho(x)$. It is proportional to $\rho(x)^3/B^2$ for small ρ, while it is asymptotically equal to $(3/5)(3\pi^2)^{2/3}\rho(x)^{5/3}$ as $\rho(x) \to \infty$.

§4. The many-electron atom

The essential ingredient in the study of the many-electron Hamiltonian H_N is to reduce it to a one-electron problem $H_{\mathbf{A}} + V_{\text{eff}}(x)$ with an effective mean field potential $V_{\text{eff}}(x) = -Z/|x| + \int |x-y|^{-1}\rho(y)d^3y$. This reduction involves approximating the repulsive energy

$$\int \|\psi(x_1, \dots, x_N)\|^2 \sum_{1 \le i < j \le N} |x_i - x_j|^{-1} d^3x_1 \dots d^3x_N \ ,$$

in the ground state ψ by

$$\int\int \rho(x)\rho(y)|x-y|^{-1}d^3x d^3y \ .$$

In standard TF theory the justification of this approximation is done by using the correlation inequality of Lieb and Oxford (see [L] and [LO]). This very same argument (and inequality) work in the presence of a magnetic field. If B is not too large compared with Z it continues to be effective. However, in the hyper-strong case $B \gg Z^3$ the argument is no longer effective, the reason being that the correlation estimate is three dimensional in nature, while the atom is now effectively one-dimensional. The proof of a correlation estimate applicable in the hyper-strong case is difficult and will appear elsewhere ([LSY]).

The density ρ appearing in the mean field potential V_{eff} will not be taken to be the exact (unknown) density of the ground state, but rather an approximation to the exact density obtained from the density functionals that we shall now define.

Armed with the foregoing, we introduce a (magnetic field dependent) TF theory by means of the following functional of the unknown electron density $\rho(x)$:

$$\begin{aligned}\mathcal{E}_{\rm MTF}(\rho) = \int w_B(\rho(x))d^3x - \int Z|x|^{-1}\rho(x)d^3x \\ + \frac{1}{2}\int\int \rho(x)|x-y|^{-1}\rho(y)d^3xd^3y.\end{aligned} \tag{4.1}$$

It differs from the usual TF functional only in the replacement of (const.)$\rho(x)^{5/3}$ by $w_B(\rho(x))$. We call this functional the **Magnetic Thomas-Fermi Functional**. It is studied in detail in [LSY]. The paper [TY] seems to be the earliest reference that uses a Thomas-Fermi theory that takes *all* Landau levels into account. This theory was also studied in [FGPY] and put on a rigorous basis in [Y] for the regime $B \sim Z^{4/3}$.

We now choose our density ρ to be the unique minimizer for $\mathcal{E}_{\rm MTF}$ constrained to the set $\int \rho \leq N$. We define the energy function that appears in Theorem 1 to be the infimum

$$E_{\rm MTF}(N,B,Z) = \inf_{\int \rho\leq N} \mathcal{E}_{\rm MTF}(\rho).$$

Theorems 4 and 5 play an essential role in the proof of Theorem 1. What makes the proof work when $B \ll Z^3$ is the fact that in the analysis of the mean-field, one-particle Hamiltonian, $H_{\mathbf{A}} + V_{\rm eff}(x)$, with $V_{\rm eff}(x) = -Z/|x| + \int |x-y|^{-1}\rho(y)d^3y$, and with ρ being the density that minimizes the TF energy, we are in the semiclassical regime. The potential $V_{\rm eff}(x)$ has the following behavior in Z and B

$$\begin{aligned} V_{\rm eff}(x) &= Z^{4/3}v(Z^{1/3}x) \quad \text{if } B \lesssim Z^{4/3} \\ V_{\rm eff}(x) &= Z^{4/5}B^{2/5}v(Z^{-1/5}B^{2/5}x) \quad \text{if } B \gtrsim Z^{4/3},\end{aligned} \tag{4.2}$$

where v is a function that does not depend significantly on B and Z.

Concentrating on the case $B \gtrsim Z^{4/3}$ we see, by a simple rescaling, that the Hamiltonian $H_A + V_{\rm eff}(x)$ is unitarily equivalent to the operator

$$Z^{4/5}B^{2/5}[((h\mathbf{p} - b\mathbf{a}(x))\cdot\sigma)^2 + v(x)], \tag{4.3}$$

where

$$h = (B/Z^3)^{1/5} \quad \text{and} \quad b = (B^2/Z)^{1/5}. \tag{4.4}$$

In the opposite case, when $B \lesssim Z^{4/3}$, we get $Z^{4/3}$ in place of $Z^{4/5}B^{2/5}$ in (4.3) and

$$h = Z^{-1/3} \quad \text{and} \quad b = B/Z. \tag{4.5}$$

When h is small we can study (4.3) by semiclassical methods.

If $B \gg Z^{4/3}$ we can replace $w_B(\rho)$ by its asymptotic form and we define the **Strong Thomas-Fermi** functional

$$\mathcal{E}_{\mathrm{STF}}(\rho) = \tfrac{4}{3}\pi^4 B^{-2} \int \rho(x)^3 d^3x - \int Z|x|^{-1}\rho(x)d^3x + \frac{1}{2}\int\int \rho(x)|x-y|^{-1}\rho(y)d^3xd^3y.$$

The analysis of E_{MTF} and E_{STF}, which is a separate story in itself, leads to the conclusions stated in 1), 2) and 3) of Section II. Conclusions 1) and 2) were proved by Yngvason [Y]; 3) is new. Since the TF energy functional has a unique minimizing $\rho(x)$ (because $\mathcal{E}_{\mathrm{MTF}}$ is strictly convex in ρ) this ρ must be spherically symmetric. Thus we are led to the following remarkable conclusion:

If $B/Z^3 \to 0$ as $Z \to \infty$, *the atom is always spherical* (*to leading order*) *despite the fact that* B *has a leading order effect on the ground state energy.*

In case 2, $B \approx Z^{4/3}$, we cannot say that all the electrons are in the lowest Landau band, but if $B \gg Z^{4/3}$, they are — as the following theorem states precisely.

Theorem 6. *If* Π_0^N *is the projection in the physical Hilbert space onto the subspace where all electrons are in the lowest Landau band, we can define the* **confined energy**

$$E_{\mathrm{conf}}(N,B,Z) \equiv \text{ ground state energy of } \Pi_0^N H_N \Pi_0^N. \tag{4.6}$$

Then, if $N < \lambda Z$ *for some fixed* $\lambda > 0$, *we have that*

$$E_{\mathrm{conf}}(N,B,Z)/E(N,B,Z) \to 1 \quad \text{if} \quad B \to \infty \quad \text{and if} \quad Z^{4/3}/B \to 0. \tag{4.7}$$

What happens if $B \approx Z^3$? Semiclassical analysis breaks down (in the sense of being no longer asymptotically exact as $Z \to \infty$). The atom is no longer spherical. However, the atom is so non-semiclassical (one person called it post-modern) that another analysis becomes possible.

This analysis, which we discuss next, is reminiscent of Hartree theory for bosons — even though it is relevant for fermionic electrons!

It is only the motion parallel to the magnetic field which can no longer be described semiclassically. The motion perpendicular to the field is still well approximated classically. To be more precise, the atom consists of a bundle of one dimensional quantum systems indexed by the position $x_\perp = (x_1, x_2)$ perpendicular to the field $\mathbf{B}$. The state of one of these one-dimensional systems is described by a finite family of orthogonal functions $e^{(j)}_{x_\perp}$, $j = 1, 2\ldots$ in $L^2(\mathbf{R})$ which are not normalized but satisfy $\|e^{(j)}_{x_\perp}\| \le B/2\pi$. This condition follows from the Pauli principle and the fact that the two-dimensional density of states in the lowest Landau band is exactly $B/2\pi$.

We can combine the functions $e^{(j)}_{x_\perp}$, $j = 1, 2, \ldots$ into a density matrix

$$\gamma : x_\perp \mapsto \gamma_{x_\perp}(x_3, y_3) = \sum_j e^{(j)}_{x_\perp}(x_3)\overline{e^{(j)}_{x_\perp}(y_3)} \ .$$

Then γ satisfies

(a) $0 \le \gamma_{x_\perp} \le (B/2\pi)I$ as an operator on $L^2(\mathbf{R})$
(b) $\int_{\mathbf{R}^2} \mathrm{Tr}_{L^2(\mathbf{R})}[\gamma_{x_\perp}]d^2x_\perp = N =$ the total number of electrons.

We can now approximate the energy by the functional

$$\begin{aligned}\mathcal{E}_{\mathrm{DM}}(\gamma) = &\int_{\mathbf{R}^2} \mathrm{Tr}_{L^2(\mathbf{R})}[(-\partial_3^2 - Z|x|^{-1})\gamma_{x_\perp}]d^2x_\perp \\ &+ \tfrac{1}{2}\int\int \rho_\gamma(x)\rho_\gamma(y)|x-y|^{-1}d^3x d^3y \ ,\end{aligned}$$

where $\rho_\gamma(x) = \gamma_{x_\perp}(x_3, x_3)$.

We denote

$$E_{\mathrm{DM}}(N, B, Z) = \inf\{\mathcal{E}(\gamma) : \gamma \text{ satisfies (a) and (b) above}\}.$$

This is the function appearing in Theorem 2. The Pauli principle comes into play in this theory only in condition (a). The proof of Theorem 2 is straightforward as soon as one has made the reduction to a one body problem and realized that condition (a) follows from the confinement to the lowest Landau band.

The Euler-Lagrange equation for the $\mathcal{E}_{\mathrm{DM}}$ minimization problem implies that the functions $e^{(j)}_{x_\perp}$ are eigenfunctions of the one-dimensional Schrödinger operator $h_{x_\perp} = -\frac{d^2}{dx_3^2} - V_{\mathrm{eff}}(x)$ where, as before, the effective potential is $V_{\mathrm{eff}}(x) = -Z/|x| + \int |x-y|^{-1}\rho_\gamma(y)d^3y$ with ρ_γ being the density corresponding to the minimizer γ for $\mathcal{E}_{\mathrm{DM}}$

§5. The super strong case $B \gg Z^3$

We shall present here the correct energy functional of the density when $B \gg Z^3$, and very briefly indicate what is involved in proving the correctness of the approximation.

The first step is to show that when B/Z^3 is larger than some critical value then the minimizing γ for $\mathcal{E}_{\mathrm{DM}}$ is rank one for every $x_\perp$. Since the eigenfunction of $\gamma_{x_\perp}$ must be the ground state of $h_{x_\perp}$ we can conclude that it is a positive function. In this case we can write $\gamma_{x_\perp}(x_3, y_3) = \sqrt{\rho(x_\perp, x_3)}\sqrt{\rho(x_\perp, y_3)}$ where $\rho(x) = \rho_\gamma(x)$.

The functional $\mathcal{E}_{\mathrm{DM}}$ thus becomes a density functional when B/Z^3 is large enough.

$$\begin{aligned}\mathcal{E}_{\mathrm{DM}}(\gamma) = \mathcal{E}_{\mathrm{SS}}(\rho) = \int \left(\frac{\partial}{\partial x_3}\sqrt{\rho(x)}\right)^2 d^3x - \int \frac{Z}{|x|}\rho(x)d^3x \\ + \frac{1}{2}\int \rho(x)|x-y|^{-1}\rho(y)d^3x d^3y,\end{aligned} \tag{5.1}$$

with the condition that

$$\int \rho(x_1, x_2, x_3)dx_3 \le \frac{B}{2\pi} \quad \text{for all } (x_1, x_2). \tag{5.2}$$

Then

$$\begin{aligned}E_{\mathrm{DM}}(N,B,Z) &= E_{\mathrm{SS}}(N,B,Z)) \\ &= \inf\left\{\mathcal{E}_{\mathrm{SS}}(\rho) \ : \ \int \rho \le N, \rho \text{ satisfies } (5.2)\right\}.\end{aligned} \tag{5.3}$$

We can now ask for the limit of $\mathcal{E}_{\mathrm{SS}}$ if $B/Z^3 \to \infty$, $Z \to \infty$ and N/Z is fixed. With some effort one can prove that $\mathcal{E}_{\mathrm{SS}}$ then simplifies to another functional, which we call the **hyper-strong functional** of a *one-dimensional* density $\rho_1(x), x \in \mathbb{R}$. That is, the atom is now so thin compared to its length that only the average density and its variation along the direction parallel to **B** matter.

It is convenient, in defining this average density, to rescale the variables. Thus, setting $\eta \equiv B/(2\pi Z^3)$, and taking $(Z\ln\eta)^{-1}$ as the unit of length, we define

$$\begin{aligned}\rho_1(x) &\equiv \frac{1}{Z^2\ln\eta}\overline{\rho}\left(\frac{1}{Z\ln\eta}x\right) \\ &\equiv \frac{1}{Z^2\ln\eta}\int \rho\left(x_1, x_2, \frac{1}{Z\ln\eta}x\right)dx_1dx_2,\end{aligned} \tag{5.4}$$

which has the normalization $\int \rho_1(x)dx = N/Z$. The hyper-strong functional is

$$(5.5) \qquad \mathcal{E}_{\mathrm{HS}}(\rho_1) = \int_{-\infty}^{\infty} \left(\frac{d}{dx}\sqrt{\rho_1(x)} \right)^2 dx - \rho_1(0) + \frac{1}{2}\int_{-\infty}^{\infty} \rho_1(x)^2 dx.$$

In other words, apart from some scalings, *the Coulomb potential is replaced by a Dirac delta function!* Using (5.5) we define a rescaled energy

$$(5.6) \qquad E_{\mathrm{HS}}(N/Z) \equiv \inf_{\int \rho_1 = N/Z} \mathcal{E}_{\mathrm{HS}}(\rho_1).$$

We assert that under the conditions stated above, $Z^3(\ln \eta)^2 E_{\mathrm{HS}}(N/Z) / E(N,B,Z) \to 1$ as $Z \to \infty, B/Z^3 \to \infty$ and N/Z is fixed.

A remarkable fact is that the minimizing ρ_1 can be evaluated exactly. The Euler-Lagrange equation is (with $\psi^2 \equiv \rho_1$ and Lagrange multiplier μ)

$$(5.7) \qquad -\ddot{\psi}(x) - \psi(0)\delta(x) + \psi^3(x) = -\mu\psi(x).$$

With $\lambda \equiv N/Z$, there are solutions only for $\lambda \leq 2$ (not $\lambda \leq 1$ as in TF theory):

$$(5.8) \qquad \begin{aligned} \psi(x) &= \frac{\sqrt{2}(2-\lambda)}{2\sinh[\frac{1}{4}(2-\lambda)|x| + c]} && \text{for } \lambda < 2 \\ \psi(x) &= \sqrt{2}(2+|x|)^{-1} && \text{for } \lambda = 2, \end{aligned}$$

with $\tanh c = (2-\lambda)/2$. The energy is

$$(5.9) \qquad E_{\mathrm{HS}}(\lambda) = \mathcal{E}_{\mathrm{HS}}(\psi^2) = -\frac{1}{4}\lambda + \frac{1}{8}\lambda^2 - \frac{1}{48}\lambda^3.$$

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Sufficient Condition for Non-uniqueness of the Positive Cauchy Problem for Parabolic Equations

Minoru Murata

Dedicated to Professor ShigeToshi Kuroda on the occasion of his 60th birthday

§1. Introduction

The purpose of this paper is to give a sufficient condition for non-uniqueness of non-negative solutions of the Cauchy problem

$$(1) \qquad (\partial_t - \Delta + V(x))u(x,t) = 0 \qquad \text{in} \quad R^n \times (0,\infty),$$

$$(2) \qquad u(x,0) = 0 \qquad \text{on} \quad R^n,$$

where V is a real-valued function in $L_{p,\mathrm{loc}}(R^n)$, $p > n/2$ for $n \geq 2$ and $p = 1$ for $n = 1$. We mean by a solution of (1)–(2) a function which belongs to

$$C^0(R^n \times [0,\infty)) \cap L_{2,\mathrm{loc}}([0,\infty); H^1_{\mathrm{loc}}(R^n_x))$$

and satisfies (1) and (2) in the weak sense and continuously, respectively (cf. [A]). We assume that

$$(3) \qquad |V(x) - W(|x|)| \leq C \qquad \text{on} \quad R^n$$

for some constant $C \geq 0$ and a measurable function W on $[0,\infty)$ with $\inf_{r\geq 0} W(r) > 0$. Our main result is the following

Theorem. *Suppose that*

$$(4) \qquad \int_1^\infty W(r)^{-1/2} dr < \infty.$$

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Then there exists a solution u of (1)–(2) *such that*

$$u(x,t) > 0 \qquad \textit{in} \quad R^n \times (0,\infty). \tag{5}$$

The proof of this theorem is given in Section 2.

In [M1], among other things, we have shown that:

Under some additional conditions on W, nonnegative solutions of (1)–(2) *are not unique if and only if* (4) *holds.*

The aim of this paper is to establish a half of this result without the additional conditions on W.

§2. Proof

In this section we prove the Theorem. A main idea of the proof is to exploit a relative version (see Lemmas 3 $\sim$ 6 below) of methods developed in connection with non-conservation of probability (cf. [D] and [Kh]). The proof is divided into several lemmas.

First, without loss of generality, we may and will assume that $W \geq 1$.

Consider the initial value problem

$$-g'' - [(n-1)/r]g' + W(r)g = 0 \qquad \text{in} \quad (0,\infty), \tag{6}$$

$$g(r) = 1 + o(r^\alpha) \qquad \text{as} \quad r \to 0, \tag{7}$$

where $\alpha = 1$ for $n = 1$ and $\alpha = 0$ for $n > 1$. A solution of (6)–(7) means a function g in $C^0([0,\infty)) \cap C^1((0,\infty))$ such that its derivative g' is absolutely continuous on any compact subinterval of $(0,\infty)$, and g satisfies (6) and (7). Let us see that (6)–(7) has a unique solution when $n > 2$. (When $n = 2$, it can be shown similarly; and it is clear if $n = 1$.) Since $W \in L_{p,\mathrm{loc}}(R^n)$, $p > n/2$, we have by Hölder's inequality

$$r^{2-n} \int_0^r s^{n-1} W(s)ds \leq Cr^{2-n/p} (\int_0^r W(s)^p s^{n-1} ds)^{1/p} < \infty \tag{8}$$

for any $r > 0$, where C is a positive constant independent of r. Thus a solution g of (6)–(7) satisfies

$$\lim_{r \to 0} rg'(r) = 0, \tag{9}$$

$$g'(r) = \int_0^r (s/r)^{n-1} W(s) g(s) ds, \qquad r > 0. \tag{10}$$

Putting

$$K(r,s) = [(s^{2-n} - r^{2-n})/(n-2)]W(s)s^{n-1}, \tag{11}$$

we have

$$\begin{aligned} &\int_0^r dt \int_0^t (s/t)^{n-1} W(s)ds = \int_0^r K(r,s)ds \\ &\le Cr^{2-n/p}\Big(\int_0^r W(s)^p s^{n-1} ds\Big)^{1/p} < \infty \end{aligned} \tag{12}$$

for any $r > 0$, where C is a positive constant independent of r. Thus g satisfies the integral equation

$$g(r) = 1 + \int_0^r K(r,s)g(s)ds \tag{13}$$

on $[0,\infty)$. Conversely, a solution of (13) in $C^0([0,\infty))$ is also a solution of the initial value problem (6)–(7). Now, in view of (12), the iteration method shows that (13) has a unique solution on $[0,\delta]$ for a sufficiently small positive number δ. The obtained solution is also a unique solution of (6)–(7) with $(0,\infty)$ replaced by $(0,\delta)$. By extending it, we get a unique solution g of (6)–(7). Furthermore, we see that $g > 0$ and $g' > 0$ in $(0,\infty)$.

With $f(r) = r^{(n-1)/2}g(r)$ and $w(r) = W(r) + (n-1)(n-3)/4r^2$, we have

$$f'' = w(r)f \qquad \text{in} \quad (0,\infty), \tag{14}$$

$$f(r) = r^{(n-1)/2}[1 + o(r^\alpha)] \qquad \text{as} \quad r \to 0. \tag{15}$$

The following Lemmas 1 and 2 play a technically main part in removing the additional conditions on W mentioned in the Introduction.

Lemma 1. *$f, f' > 0$ in $(0,\infty)$, $\inf_{r>1} f'(r)/f(r) > 0$, and*

$$\int_1^\infty (f/f')dr < \infty. \tag{16}$$

Proof. We have only to show the second and third assertions. With $F = f'/f$, we have from (14)

$$F' + F^2 = w \tag{17}$$

Let $a(r)$ be the solution of the initial value problem

$$a'' = (1/4)a \quad \text{in} \quad (1,\infty), \qquad a(1) = f(1), \qquad a'(1) = f'(1).$$

With $A = a'/a$,

$$(F-A)' + (F+A)(F-A) = w - 1/4 \geq 0 \qquad \text{in} \quad (1,\infty),$$
$$(F-A)(1) = 0.$$

Thus $F \geq A$, and so $\inf_{r>1} F(r) > 0$. We next show (16) simplifying an argument in [KN, 4.2 and 4.3]. We claim that

$$1/F + (1/2)(1/F^2)' \leq 2/w^{1/2} \tag{18}$$

in $(1,\infty)$. By (17),

$$(1/w)(F'/F^2) + 1/w = 1/F^2.$$

If $F' \geq 0$, then $F \leq w^{1/2}$; and so

$$1/F = F[1/w + (1/w)(F'/F^2)] \leq 1/w^{1/2} + F'/F^3.$$

If $F' < 0$, then $1/F \leq 1/w^{1/2}$ and

$$(1/2)(1/F^2)' = -F'/F^3 = 1/F - w/F^3 < 1/w^{1/2}.$$

Thus we get (18). Hence

$$\int_1^R F^{-1}dr + \frac{1}{2}[F(R)^{-2} - F(1)^{-2}] \leq \int_1^R 2w^{-1/2}dr \leq \int_1^\infty 4W^{-1/2}dr.$$

This together with (4) implies (16). Q.E.D.

Let f_1 be the solution of (14)–(15) with w replaced by $w+1$. Then we have

Lemma 2. *The function f_1/f is increasing and* $0 < \lim_{r\to\infty}(f_1/f)(r) < \infty$.

Proof. With $v = f_1/f$, we have

$$f^{-2}(f^2v')' = v \qquad \text{in} \quad (0,\infty), \tag{19}$$

$$v(r) = 1 + o(r^\alpha) \qquad \text{as} \quad r \to 0. \tag{20}$$

From (19)–(20) we get along the line in deriving (13) the equation

$$v(r) = 1 + \int_0^r [\int_s^r (f(s)/f(t))^2 dt] v(s) ds. \tag{21}$$

This implies that v is strictly increasing. Next, let us show the second assertion along the line given in [KN, 2.5]. With $u = \log(f_1/f)$ and $F = f'/f$, we have

$$u'' + (2F)u' + (u')^2 = 1. \tag{22}$$

This implies that $2u' \le 1/F - u''/F$. Thus, for any $R > 1$,

$$2\int_1^R u' dr \le \int_1^R (1/F) dr$$
$$- u'(R)/F(R) + u'(1)/F(1) + \int_1^R (-F'/F^2) u' dr.$$

Since $-F'/F^2 = 1 - w/F^2 < 1$ and $u' > 0$, we then have

$$2\int_1^R u' dr \le \int_1^R (1/F) dr + u'(1)/F(1) + \int_1^R u' dr.$$

Hence

$$u(R) \le \int_1^R (1/F) dr + u'(1)/F(1) + u(1).$$

This together with (16) implies that $\lim_{r\to\infty} f_1(r)/f(r) < \infty$.

Q.E.D.

Now put

$$H(x) = h(|x|) = (f_1/f)(|x|)[\lim_{s\to\infty} (f_1/f)(s)]^{-1}, \tag{23}$$

$$L = -g(|x|)^{-2} \sum_{j=1}^{n} (\partial/\partial x_j)(g(|x|)^2 \partial/\partial x_j), \tag{24}$$

where g is the solution of (6)–(7). Then we can easily obtain the following lemma.

Lemma 3. *H is a solution of the equation*

$$(L + 1)H = 0 \qquad in \quad R^n \tag{25}$$

such that $0 < H < 1$ *and* $\lim_{|x|\to\infty} H(x) = 1$.

Let $G(x,y)$ be the minimal Green function for $(L+1, R^n)$ (cf. [M3]). Then we have

Lemma 4. $0 < \int_{R^n} G(x,y)dy \leq 1 - H(x)$ *on* R^n.

Proof. Recall that $G = \lim_{R\to\infty} G_R$, where G_R is the Green function for $(L+1, B_R)$ with $B_R = \{x \in R^n; |x| < R)$. Put $U_R(x) = \int_{|y|<R} G_R(x,y)dy$. Then

$$(L+1)U_R = 1 \quad \text{in} \quad B_R, \qquad U_R = 0 \quad \text{on} \quad \partial B_R.$$

On the other hand,

$$(L+1)(1-H) = 1 \qquad \text{in} \quad B_R, \qquad 1 - H > 0 \quad \text{on} \quad \partial B_R.$$

Thus the maximum principle shows that $U_R < 1 - H$ in B_R. But

$$\lim_{R\to\infty} U_R(x) = \int_{R^n} G(x,y)dy.$$

This proves the lemma. Q.E.D.

Since Lemma 4 implies that $[(L+1)^{-1}1](x) < 1$, we can now apply a criterion for non-conservation of probability (cf. [D, Lemma 2.1]), which goes back to Khas'minskii [Kh]. Let $K(x,y,t)$ be the smallest fundamental solution for $(\partial_t + L, R^n \times (0,\infty))$ (cf. [M1, M2]), and put

$$v(x,t) = \int_{R^n} K(x,y,t)dy. \tag{26}$$

Then we have

Lemma 5. $v(x,0) = 1$, *and*

$$(\partial_t + L)v = 0 \quad \text{and} \quad 0 < v < 1 \qquad \text{in} \quad R^n \times (0,\infty). \tag{27}$$

Proof. For self-containedness, we briefly show that $0 < v < 1$. The maximum principle for a parabolic equation on a cylinder together with the semigroup property of the smallest fundamental solution implies that either $v = 1$ or $0 < v < 1$ in $R^n \times (0,\infty)$. On the other hand, by Lemma 4,

$$\int_0^\infty e^{-t} v(x,t)dt = \int_{R^n} G(x,y)dy < 1 \qquad \text{on} \quad R^n.$$

Hence $0 < v < 1$. Q.E.D.

The final step of the proof is the following

Lemma 6. *There exists a solution u having the desired properties of the Theorem.*

Proof. With v being the function given by (26), put

$$w(x,t) = g(x)(1 - v(x,t)). \tag{28}$$

Then we see that $w(x,0) = 0$, and

$$(\partial_t - \Delta + W)w = 0 \quad \textit{and} \quad 0 < w(x,t) < g(x) \quad \text{in} \quad R^n \times (0,\infty). \tag{29}$$

For $R > 0$, let u_R be the solution of the mixed problem

$$(\partial_t - \Delta + V)u_R = 0 \text{ in } B_R \times (0,\infty), \; u_R = w \text{ on } \partial(B_R \times (0,\infty))$$

(cf. [A]). Since $W - C \le V \le W + C$ by (3), the comparison theorem shows that

$$e^{-Ct} \le u_R(x,t)/w(x,t) \le e^{Ct} \qquad \text{in} \quad B_R \times (0,\infty).$$

We see that for some sequence $R_j \to \infty$, u_{R_j} converges uniformly on each compact subset of $R^n \times [0,\infty)$ to a solution u of (1) satisfying

$$e^{-Ct} \le u(x,t)/w(x,t) \le e^{Ct} \qquad \text{in} \quad R^n \times (0,\infty). \tag{30}$$

This proves the lemma. Q.E.D.

Remark. We can also prove the Theorem by using Theorem 5.5 of [M1] after establishing Lemma 2; because Lemma 2 and (21) imply that

$$\int_1^\infty ds \int_s^\infty (s/t)^{n-1}(g(s)/g(t))^2 dt < \infty.$$

But the proof given in this paper is more direct than the one based on Theorem 5.5 of [M1].

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Trudinger's Inequality and Related Nonlinear Elliptic Equations in Two-Dimension

Takayoshi Ogawa and Takashi Suzuki

§1. Introduction and results

We are concerned with the following nonlinear elliptic equations:

$$\begin{cases} -\Delta u = \lambda u e^{u^2}, & x \in B, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1}$$

where $B = B_1(0) \subset \mathbb{R}^2$ is a unit disk in $\mathbb{R}^2$ and λ is a positive parameter. We consider a family of solutions of (1) satisfying

$$\|u\|_{L^\infty} \to \infty \quad \text{as } \lambda \to 0. \tag{2}$$

The nonlinearity of the equation (1) is the Sobolev critical exponent in two-dimension. For any domain $\Omega \in \mathbb{R}^2$, It is well known that the Sobolev space $H_0^1(\Omega)$ is continuously imbedded in $L^p(\Omega)$ for any $p < \infty$ but is false in the case $p = \infty$. Trudinger [18] showed that for any $u \in H_0^1(\Omega)$ with $\|\nabla u\|_2 = 1$, there are two constants $\alpha > 0$ and $C > 0$ such that

$$\int_\Omega \exp\{\alpha u^2\} dx \le C|\Omega|. \tag{3}$$

Later, Moser [7] simplified the proof and improved that (3) is also valid for $\alpha \le 4\pi$. Here 4π is the constant of the isoperimetric inequality. The inequality (3) is also valid for any unbounded domain (Ogawa [9]). That is when Ω is any domain in $\mathbb{R}^2$, we have for all $u \in H_0^1(\Omega)$,

$$\int_\Omega \{\exp(u^2) - 1\} dx \le C\|u\|_2^2, \quad \|\nabla u\|_2 = 1. \tag{4}$$

(See also Ogawa-Ozawa [10] and Ozawa [12] for further extensions).

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These inequalities (3)–(4) indicates that the order of local singularities of H^1 functions are allowed as far as $\exp(u^2)$ is integrable. In other words e^{u^2} is the critical order of integrability for H^1-functions.

Concerning our problem (1), there are two different approaches. One is the variational method. When we consider the maximizing problem of the functional

$$\int_\Omega \exp\{\alpha u^2\}dx \quad \text{for } u \in H_0^1(\Omega), \quad \|\nabla u\|_2 = 1 \tag{5}$$

on a bounded domain. Then the extremal function (if it is achieved) becomes a solution of (1). Shaw [14] showed the existence of a positive solution of (1) for each parameter $\lambda > 0$ (see also Adimurti [1]). When the domain is a ball in $\mathbb{R}^n$, the maximum can be attained by some function even when $n = 2$ and $\alpha = 4\pi$ (Carleson-Chang [4]).

When the domain is a unit disk, all the positive smooth solution must be radially symmetric by Gidas-Ni-Nirenberg's result [5]. Therefore the Dirichlet problem may be written as the nonlinear ordinary differential equation:

$$\begin{cases} -u_{rr} - \dfrac{1}{r}u_r = \lambda u e^{u^2}, & x \in [0,1), \\ u(1) = 0, \quad u'(0) = 0. \end{cases} \tag{6}$$

By solving (6), we can obtain the details of the properties of the positive solution of (1), which is the second method. Atkinson-Peletier [2], [3] applied the shooting method to (6) and proved that the existence of radially symmetric solution of (1) satisfying

$$\|u\|_{L^\infty} \to \infty \text{ as } \lambda \to 0.$$

Our aim of this paper is to specify more precise behavior of the family of solutions $\{(u, \lambda)\}$ as $\lambda \to 0$. We have two results. First one states a *global behavior* of the solutions.

Theorem A. *Let u be a positive solution of* (1) *with the blow up condition* (2). *That is*

$$\|u\|_{L^\infty(B)} = u(0) \to \infty \text{ as } \lambda \to 0.$$

Then we have

$$u(x) \to 0 \text{ as } \lambda \to 0$$

for all $x \in B \setminus \{0\}$. *Moreover we have*

$$\lim_{\lambda \to 0} \lambda \int_B u e^{u^2} dx = 0, \tag{7}$$

$$\lim_{\lambda \to 0} \lambda \int_B (e^{u^2} - 1) dx = 0, \tag{8}$$

$$\underline{\lim}_{\lambda \to 0} \int_B |\nabla u|^2 dx \geq 4\pi. \tag{9}$$

This theorem says that the solution satisfying (2) must blow-up only at the origin. The inequality (9) shows the solution concentrates to the origin with its energy density $|\nabla u|^2$. The lower bound in (9) arise from the sharp exponent of the Trudinger inequality (3).

The second result is a *microscopic behavior* near the origin. When we rescale the solution by some sequence, then the solution has a limit function.

Theorem B. *There is a subsequence* $\{(u_m, \lambda_m)\}$ *of a family of solutions of* (1) *with* (2) *and a scaling sequance* $\{\gamma_m\}$ *such that* $\gamma_m \to 0$ *as* $\lambda_m \to 0$ *which satisfy*

$$u^2(\gamma_m x) - u^2(\gamma_m) \to 2\log\left(\frac{2}{1+|x|^2}\right) \quad \text{as } \lambda_m \to 0 \tag{10}$$

locally uniformly on $B \setminus \{0\}$.

The limit function of (10) is an exact solution of $-\Delta v = 2e^v$. Remark that since the nonlinearity of our problem is nonhomogeneous, the usual scaling $u \to \gamma^\mu u(\gamma x)$ does not work well. (For other nonlinearity or the higher dimensional case, see Nagasaki-Suzuki [8] and Itoh [6].)

The property (10) was firstly observed by Carleson-Chang in an implicit way. Later Struwe [15] obtained the similar result for the noncompact maximizing sequence for the variational problem (5) for the case $\alpha = 4\pi$. Our result Theorem B is, however, different from theirs, because in our case, the each factor of the sequence $\{u_m, \lambda_m)\}$ satisfies the equation (1). Moreover even the energy integral might blow up as $\lambda \to 0$ and therefore we can not obtain a priori estimate of $\{u_m\}$ from the Dirichlet integral. This is the crucial difference from the variational setting.

§2. Proof of Theorem A

We begin with the following lemma.

Lemma 1. *Let u be a positive, radially symmetric smooth solution of* (1). *We put $r = |x|$. Then we have*

$$r^2 u_r(r)^2 + 2\lambda r^2 (e^{u^2(r)} - 1) = \frac{\lambda}{2\pi} \int_{B_r} (e^{u^2} - 1) dx, \tag{11}$$

$$r u_r(r) = -\frac{\lambda}{2\pi} \int_{B_r} u e^{u^2} dx, \tag{12}$$

where $B_r = \{y \in \mathbb{R}^2, |y| < r\}$.

The first relation (11) is nothing else but the Pohozaev identity ([13]) associated to the equation (1).

Proof. Let u be a radially symmetric smooth solution of (1). Then u satisfies (6). Mutiplying (6) by $r u_r(r)$ and integrating on B_{r_0}, we have

$$-\int_0^{r_0} r^2 u_r u_{rr} dr - \int_0^{r_0} r u_r^2 dr = \lambda \int_0^{r_0} u e^{u^2} r^2 u_r dr.$$

Integrating by parts, we obtain

$$-\frac{1}{2} r_0^2 u_r(r_0)^2 = \frac{\lambda}{2} r_0^2 e^{u^2(r_0)} - \frac{\lambda}{2\pi} \int_{B_{r_0}} e^{u^2} dx,$$

which implies (11). The second relation (12) is a direct consequence of integration of the equation (6) on B_r. Q.E.D.

Proof of Theorem A. Combining (11) and (12) in Lemma 1 with choosing $r = 1$, we get

$$\frac{1}{4\pi} (\lambda \int_B u e^{u^2} dx)^2 = \lambda \int_B (e^{u^2} - 1) dx. \tag{13}$$

For any $k > 0$, we put

$$C_k = \max_{u \geq k} \frac{1 - e^{-u^2}}{u}.$$

Then we see $C_k \leq 1/k \to 0$ as $k \to \infty$. From (13)

$$\begin{aligned}\frac{1}{4\pi} (\lambda \int_B u e^{u^2} dx)^2 &= \lambda \int_{u \geq k} (e^{u^2} - 1) dx + \lambda \int_{u < k} (e^{u^2} - 1) dx \\ &\leq \lambda C_k \int_B u e^{u^2} dx + \lambda |B| \{e^{k^2} - 1\}.\end{aligned}$$

Accordingly we have

$$\overline{\lim_{\lambda \to 0}}(\lambda \int_B ue^{u^2} dx) \leq 4\pi C_k.$$

Since k is arbitrary, we can take k so large to obtain

$$\lim_{\lambda \to 0}(\lambda \int_B ue^{u^2} dx) = 0, \tag{14}$$

which shows (7) and therefore (8) by (13). Using (12) again, we have

$$ru_r \to 0 \quad \text{as } \lambda \to 0 \quad \text{uniformly on } B. \tag{15}$$

This proves that u vanishes except the origin, since

$$\begin{aligned} u(x) &= -\int_{|x|}^1 u_r dr \\ &\leq \frac{1}{\varepsilon} \int_\varepsilon^1 ru_r(r) dr \to 0. \end{aligned}$$

Finally, if

$$\underline{\lim}_{\lambda \to 0} \int_B |\nabla u|^2 dx < 4\pi,$$

then there is a subsequence $\{(u_m, \lambda_m)\}$ such that $\lim_{m\to\infty} \|\nabla u_m\|_2^2 = 4\pi - \delta$ for some $\delta > 0$. By virtue of the sharp version of Trudinger's inequality (3), we see

$$\int_\Omega \exp\{\alpha u_m^2\} dx \leq C|\Omega|.$$

with $\alpha = 1+\varepsilon$. Since $u \in L^p(B)$ for any $2 \leq p < \infty$, we have $\lambda_m u_m e^{u_m^2} \in L^{1+\varepsilon/2}$. By the standard elliptic regularity theorem, $\|\Delta u_m\|_{L^{1+\varepsilon/2}} \leq C$ and

$$\|u_m\|_{L^\infty(B)} \leq C \quad \text{(independent of } m\text{)},$$

which contradicts our assumption (2). Therefore we obtain (9). Q.E.D.

§3. Proof of Theorem B

By the transform $r = e^{-t/2}$ and $u(r) = w(t)$, we rewrite the equation (6) into the following:

$$
(16) \qquad \begin{cases} -w''(t) = \dfrac{\lambda}{4} w(t) e^{w(t)^2 - t} & \text{on } [0, \infty), \\ w(0) = 0, \\ w'(t) e^{t/2} \to 0 \qquad (t \to 0). \end{cases}
$$

For some scaling parameter τ such that $\tau \to \infty$, we define the rescaling function $v(t)$ as

$$
v(t) \equiv w^2(t+\tau) - w^2(\tau).
$$

Putting $w_\tau(t) \equiv w(t+\tau)$, we see that v satisfies

$$
(17.a) \qquad -v''(t) = k(w_\tau(t)) e^{v(t)-t} - \rho(w_\tau),
$$

$$
(17.b) \qquad v(0) = 0,
$$

$$
(17.c) \qquad \lim_{t \to \infty} \left(\frac{v'(t) e^{(t+\tau)/2}}{w_\tau(t)} \right) = 0,
$$

where we have put

$$
\begin{aligned} k(w_\tau) &= \frac{\lambda}{2} w_\tau(t)^2 e^{w(\tau)^2 - \tau}, \\ \rho(w_\tau(t)) &= 2 w_\tau'(t)^2. \end{aligned}
$$

We first show that;

Lemma 2. *Let $\tau > 0$ satisfies $w(t+\tau) \geq 1$ as $\lambda \to 0$ for all $t \in [-\delta, \infty)$ where $0 < \delta < \tau$. Then we have*

$$
(18) \qquad \rho(w_\tau(t)) \to 0 \text{ uniformly on } [-\tau, \infty),
$$

$$
(19) \qquad \frac{w_\tau(t)^2}{w(\tau)^2} \to 1 \quad \text{locally uniformly on } [-\delta, \infty)
$$

as $\lambda \to 0$.

Proof. Since from (15), we have for $\gamma = e^{-\tau/2}$,

$$
(20) \qquad \rho(w_\tau(t)) = 2 w_\tau'(t)^2 = \frac{1}{2} (\gamma r)^2 u_r(\gamma r)^2 \to 0
$$

uniformly for $r \in [0, 1/\gamma]$ and therefore $t \in [-\tau, \infty)$. This shows (18).

To show (19), we use

$$w_\tau(t)^2 = w(\tau)^2 + 2\int_0^t w_\tau(s)w'_\tau(s)ds.$$

We only show the case when $t \geq 0$. The other case is similar. Since $w_\tau(t)$ is increasing in t,

$$\begin{aligned} 1 \leq \frac{w_\tau(t)^2}{w(\tau)^2} &= 1 + \frac{2}{w(\tau)^2}\int_0^t w_\tau(s)w'_\tau(s)ds \\ &\leq 1 + 2\int_0^t \frac{w_\tau(s)^2}{w(\tau)^2}\frac{w'_\tau(s)}{w_\tau(s)}ds. \end{aligned}$$

By (20), we can choose λ small so that $|w'_\tau(s)| < \varepsilon$. Then since $w_\tau(s) > 1$,

$$1 \leq X(t) \equiv \frac{w_\tau(t)^2}{w(\tau)^2} \leq 1 + 2\varepsilon\int_0^t X(s)ds.$$

This yields

$$1 \leq X(t) \leq e^{2\varepsilon t} \qquad \text{for } t \in [0,\infty).$$

In particular,

$$X(t) \to 1 \quad \text{uniformly for } t \in [0,T] \quad \text{as } \lambda \to 0$$

for some fixed T. Q.E.D.

Proof of Theorem B. In the following, we shall omit the subscriptions for each subsequences.

We split the proof into two cases.

Case 1.

$$\max_{t>0} \lambda w(t)^2 e^{w(t)^2 - t} \to \infty \qquad (\lambda \to 0).$$

Since $w(0) = 0$, we can choose the scaling sequence $\{\tau\}$ as

$$\lambda w(\tau)^2 e^{w(\tau)^2 - \tau} = 1 \tag{21}$$

for the family of solutions $\{(u,\lambda)\}$. It is easy to see

$$\begin{aligned} &\tau \to \infty, \\ &w(t) \to \infty \quad \text{as } \lambda \to 0. \end{aligned}$$

Therefore we may assume $w_\tau(t) \geq w(\tau) > 1$ for $t > -\delta$ and from Lemma 2,

$$w_\tau(t)^2 \to 0 \text{ uniformly on } [-\tau, \infty), \tag{22}$$

$$\frac{w_\tau(t)^2}{w(\tau)^2} \to 1 \quad \text{locally uniformly on } [-\delta, \infty). \tag{23}$$

Next we claim that for any fixed $T > 0$,

$$\|v\|_{L^\infty(0,T)} \leq C$$

and there is a limit function $v_0(t)$ such that

$$v(t) \to v_0(t) \quad \text{locally uniformly on } [0, \infty).$$

For that purpose, we set $q(r) = v(t)$ with $r = e^{-t/2}$. Then the equation (17) can be written as follows:

$$\begin{cases} -\Delta q = 4\tilde{k}(u(\gamma r))e^{q(r)} - \tilde{\rho}(u(\gamma r))r^{-2} & \text{on } B_{\gamma^{-1}}, \\ \quad q = 0 \text{ on } \partial B, \end{cases} \tag{24}$$

where $B_a = \{y \in \mathbb{R}^2, \quad |y| < a\}$ and

$$\tilde{k}(u(\gamma r)) = \frac{\lambda}{2}\gamma^2 u(\gamma r)^2 e^{u(\gamma)^2},$$
$$\tilde{\rho}(u(\gamma r)) = 2\gamma^2 r^2 u(\gamma r)^2.$$

Since from (21), (22) and (23), we have for $r \in [\varepsilon, 1+\delta]$,

$$|\tilde{\rho}(u)r^{-2}| \leq C\frac{\eta^2}{\varepsilon^2} \to 0 \tag{25}$$

and

$$\tilde{k}(u(\gamma r) = \frac{\lambda}{2}w_\tau(t)^2 e^{w(\tau)^2 - \tau} = \frac{w_\tau(t)^2}{2w(\tau)^2} \to \frac{1}{2} \tag{26}$$

as $\lambda \to 0$. Therefore by the standard elliptic estimate, we have for fixed $\varepsilon > 0$,

$$|q_r(1)| \leq C, \tag{27}$$

$$\|q\|_{L^\infty(B_{1+\delta}\setminus B)} \leq C. \tag{28}$$

According to (24), (25) and (27),

(29)

$$\begin{aligned}
\|\tilde{k}(u)e^q\|_{L^1(B\setminus B_\varepsilon)} &= \int_{B\setminus B_\varepsilon} \tilde{k}(u)e^q dx \\
&= \int_{B\setminus B_\varepsilon} -\Delta q dx + \int_{B\setminus B_\varepsilon} \rho(u) r^{-2} dx \\
&= 2\pi \int_\varepsilon^1 -(rq_{rr} + q_r)dr + 4\pi \int_\varepsilon^1 (\gamma r)^2 u_r^2(\gamma r) r^{-1} dr \\
&\leq -2\pi q_r(1) + C\eta^2 \int_\varepsilon^1 r^{-1} dr \\
&\leq C - C\eta^2 \log\varepsilon \leq C.
\end{aligned}$$

Hence by (24), (25), (26) with (29), q satisfies

$$-\Delta q = 4\tilde{k}(u)e^q - \tilde{\rho} r^{-2} \leq 3e^q$$

with

$$\|3e^q\|_{L^1(B\setminus B_\varepsilon)} \leq C \quad \text{independent of } \lambda.$$

Then the nonlinear Harnack principle (Suzuki [16], [17]) implies the blow-up points of q in $B \setminus B_\varepsilon$ is finite. However q is radially symmetric, the blow-up points of q must be empty set. That is

$$\overline{\lim_{\lambda\to 0}} \|q\|_{L^\infty(B\setminus B_\varepsilon)} < \infty.$$

This proves

$$\|v\|_{L^\infty(0,T)} \leq C \quad \text{for small } \lambda.$$

By this a priori estimate with the equation (17) and Lemma 2, we obtain by Ascori-Arzela theorem, that there is a smooth function v_0 such that

$$v(t) \to v_0(t) \quad \text{locally uniformly on } [0, \infty)$$

with

$$-v_0''(t) = \frac{1}{2} e^{v_0(t) - t}. \tag{30}$$

We may solve (30) and conclude that

$$v(t) = u(\gamma x)^2 - u(\gamma)^2 \to v_0(t) = 2\log(\frac{2}{1+e^{-t}}) = 2\log(\frac{2}{1+|x|^2}).$$

This proves the theorem in the case 1.

Case 2.

$$\max_{t>0} \lambda w(t)^2 e^{w(t)^2 - t} < \infty \qquad (\lambda \to 0). \tag{31}$$

This case is rather simple. We choose $\{\tau\}$ as

$$\lim_{t\to\infty} w(t)^2 - w(\tau)^2 = 2\log 2. \tag{32}$$

This choice of τ assures us that

$$\tau \to \infty,$$
$$w(\tau)^2 \to \infty$$

and a priori estimate

$$0 \leq v(t) \leq 2\log 2. \tag{33}$$

By the assumption (31), we can choose a subsequence such that

$$\lambda w(\tau)^2 e^{w(\tau)^2 - \tau} \to 2\mu \quad \text{as } \lambda \to 0 \tag{34}$$

for some constant $\mu > 0$. Lemma 2 with (33) and (34) implies that

$$v(t) \to v_0(t) \quad \text{locally uniformly on } [0, \infty)$$

with

$$\begin{cases} -v_0''(t) = \dfrac{\mu}{2} e^{v_0(t) - t}, \\ v_0(0) = 0. \end{cases}$$

In fact, by the boundary condition at $t \to \infty$, we find that $\mu = 1$ and

$$v_0(t) = 2\log(\frac{2}{1 + e^{-t}}).$$

This proves our conclusion of Theorem B. Q.E.D

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Asymptotic Behavior of Solutions for the Coupled Klein-Gordon-Schrödinger Equations

Tohru Ozawa and Yoshio Tsutsumi

Dedicated to Professor S.T. Kuroda on his 60th birthday

§1. Introduction and theorem

In the present paper we consider the asymptotic behavior in time of solutions for the coupled Klein-Gordon-Schrödinger equations:

$$(1.1)\qquad i\frac{\partial}{\partial t}\psi + \frac{1}{2}\Delta\psi = \phi\psi, \quad t \in \mathbf{R}, \quad x \in \mathbf{R}^N,$$

$$(1.2)\qquad \frac{\partial^2}{\partial t^2}\phi - \Delta\phi + \phi = -|\psi|^2, \quad t \in \mathbf{R}, \quad x \in \mathbf{R}^N,$$

$$(1.3)\qquad \psi(0,x) = \psi_0(x), \quad \phi(0,x) = \phi_0(x), \quad \frac{\partial}{\partial t}\phi(0,x) = \phi_1(x).$$

Equations (1.1)–(1.2) describe a classical model of Yukawa's interaction of conserved complex nucleon field with neutral real meson field and the associated mass has been normalized as unity. Here ψ is a complex scalar nucleon field, and ϕ is a real scalar meson field. (1.1)–(1.2) are a semi-relativistic version of the coupled Klein-Gordon-Dirac equations (see, e.g., [2]).

Since the interaction above is only quadratic, the problems concerning asymptotic behavior of solutions are harder than the cases of higher interactions, especially in lower space dimensions. In order to examine the basic structure of nonlinearities of (1.1)–(1.2), it would be instructive to look at the decoupled case with self-interaction.

There are a large amount of papers concerning the asymptotic behavior in time of solutions for the nonlinear Schrödinger equation

$$(1.4)\qquad i\frac{\partial}{\partial t}u + \frac{1}{2}\Delta u = |u|^{p-1}u, \quad ,t \in \mathbf{R}, \quad x \in \mathbf{R}^N,$$

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and the nonlinear Klein-Gordon equation

$$\frac{\partial^2}{\partial t^2}u - \Delta u + u = -|u|^{p-1}u, \quad t \in \mathbf{R}, \quad x \in \mathbf{R}^N \tag{1.5}$$

(for the nonlinear Schrödinger equation, see [3], [6], [11]–[13], [16], [17], [19], [21], [23], [27]–[31] and for the nonlinear Klein-Gordon equation, see [4], [5], [14], [18], [20], [22], [24], [26]–[29]). When we consider the asymptotic behavior of solutions for (1.4) or (1.5), it is natural and important to investigate whether the wave operators $W_\pm$ exist or not. For (1.4), we define the wave operator W_+ as follows. Let $u_+(t)$ be the solution of the free Schrödinger equation

$$i\frac{\partial}{\partial t}u + \frac{1}{2}\Delta u = 0, \quad ,t \in \mathbf{R}, \quad x \in \mathbf{R}^N, \tag{1.6}$$

with $u_+(0) = \psi_+$. If one can look for the solution $u(t)$ of (1.4) with $u(0) = \psi_0$ such that $u(t)$ exists globally in time and

$$\|u_+(t) - u(t)\|_{L^2} \longrightarrow 0 \quad (t \to +\infty), \tag{1.7}$$

then the wave operator W_+ can be defined by a mapping from $u_+(0) = \psi_+$ to $u(0) = \psi_0$. Here ψ_+ and ψ_0 are called a scattered state and an interacting state, respectively. For the case of $t \to -\infty$, the wave operator W_- is defined in the same way. We can also consider the wave operators $W_\pm$ for (1.5). In [6], [26] and [31] it is proved that when $N = 3$ and the nonlinear term is quadratic, that is, $p = 2$, both in the cases (1.4) and (1.5) the wave operators $W_\pm$ can be defined for some data. On the other hand, in [3], [10], [14], [17] and [20] it is proved that when $N = 1, 2$ and $p = 2$, there exist no nontrivial asymptotically free solutions for (1.4) and (1.5), that is, the wave operators $W_\pm$ can not be defined for any nonzero data. This is because the time decay rate of solutions of (1.4) and (1.5) for $N = 2$ is worse than that for $N = 3$. Therefore, we have to consider the modified wave operators for (1.4) and (1.5) with $N = 2$ and $p = 2$ (see, e.g., [23]).

The unique global existence of solutions for (1.1)–(1.3) are already established (see [1], [2], [8] and [15]). Fukuda and M. Tsutsumi [9] and Strauss [29] studied the asymptotic behavior as $t \to \pm\infty$ of solutions for the coupled Klein-Gordon-Schrödinger equations with interactions higher than the quadratic order of (1.1)–(1.2). The results in [9] and [29] are similar to the results obtained for the decoupled nonlinear Klein-Gordon and Schrödinger equations.

If there is a complete analogy between the full system (1.1)–(1.2) and the decoupled system (1.4) and (1.5), it is natural to conjecture that when $N = 2$, the wave operators $W_\pm$ could not be defined for (1.1)–(1.2). But this conjecture is not true. The purpose in the present paper is to show that when $N = 2$, the wave operators $W_\pm$ for (1.1)–(1.2) can be defined for certain scattered data.

This is a sharp contrast to the decoupled case and gives a reason that the coupled Klein-Gordon-Schrödinger equations are not a simple superposition of the nonlinear Klein-Gordon and Schrödinger equations.

Before we state the theorem, we define several notations. Let $\omega = \sqrt{-\Delta + 1}$ and let $U(t) = e^{\frac{i}{2}t\Delta}$ be the evolution operator of the free Schrödinger equation. We denote by $\hat{f}$ the Fourier transform of f. For nonnegative integers m and s, we define H^m and $H^{m,s}$ as follows:

$$H^m = \{v \in \mathcal{S}'(\mathbf{R}^N); \|(1-\Delta)^{\frac{m}{2}} v\|_{L^2} < +\infty\},$$
$$H^{m,s} = \{v \in \mathcal{S}'(\mathbf{R}^N); \|(1+|x|^2)^{\frac{s}{2}}(1-\Delta)^{\frac{m}{2}} v\|_{L^2} < +\infty\}$$

with the norms

$$\|v\|_{H^m} = \|(1-\Delta)^{\frac{m}{2}} v\|_{L^2},$$
$$\|v\|_{H^{m,s}} = \|(1+|x|^2)^{\frac{s}{2}}(1-\Delta)^{\frac{m}{2}} v\|_{L^2},$$

respectively. For a multi-index $\alpha = (\alpha_1, \cdots \alpha_N)$ with nonnegative integers α_j, we put

$$|\alpha| = \alpha_1 + \cdots + \alpha_N,$$
$$(\frac{\partial}{\partial x})^\alpha = (\frac{\partial}{\partial x_1})^{\alpha_1} \cdots (\frac{\partial}{\partial x_N})^{\alpha_N}.$$

For $p \geq 1$ and a nonnegative integer k, we let

$$W^{k,p} = \{u \in L^p(\mathbf{R}^N); (\frac{\partial}{\partial x})^\alpha u \in L^p(\mathbf{R}^N), \quad |\alpha| \leq k\}$$

with the norm

$$\|u\|_{W^{k,p}} = \sum_{|\alpha| \leq k} \|(\frac{\partial}{\partial x})^\alpha u\|_{L^p}.$$

We now state the theorem.

Theorem 1.1. *Let $N = 2$ and $\varepsilon > 0$.*

(i) *Assume that $\psi_+ \in H^{2,4}$, $(1+|x|^2)^j (\frac{\partial}{\partial x})^\alpha \psi_+ \in L^1(\mathbf{R}^2)$ for $j + |\alpha| \leq 2$ and $\operatorname{supp} \hat{\psi}_+ \subset \{\xi; |\xi| \geq 1 + \varepsilon\}$. We put $u_+(t) = e^{\frac{i}{2}t\Delta}\psi_+$. Assume that $\phi_{+0} \in H^{4,2}$, $\phi_{+1} \in H^{3,2}$, for $|\alpha| \leq 4$ $(\frac{\partial}{\partial x})^\alpha \phi_{+0} \in L^1(\mathbf{R}^2)$,* *and for* $|\alpha| \le 3$ $(\frac{\partial}{\partial x})^\alpha \phi_{+1} \in L^1(\mathbf{R}^2)$. *We put* $v_+(t) = (\cos \omega t)\phi_{+0} + (\omega^{-1} \sin \omega t)\phi_{+1}$. *Then, there exists* $\eta > 0$ *such that if*

$$(1.8) \quad \|\psi_+\|_{H^{2,4}} + \sum_{j+|\alpha| \le 2} \|(1+|x|)^j (\frac{\partial}{\partial x})^\alpha \psi_+\|_{L^1} + \|\phi_{+0}\|_{H^{4,2}} + \|\phi_{+0}\|_{W^{4,1}} + \|\phi_{+1}\|_{H^{3,2}} + \|\phi_{+1}\|_{W^{3,1}} \le \eta,$$

(1.1)–(1.2) *have the unique solutions* (ψ, ϕ) *satisfying*

$$(1.9) \qquad \psi \in \bigcap_{j=0}^{1} C^j([0,\infty); H^{2-2j}),$$

$$(1.10) \qquad \phi \in \bigcap_{j=0}^{2} C^j([0,\infty); H^{2-j}),$$

$$(1.11) \qquad \|\psi(t) - u_+(t)\|_{H^2} + \|\phi(t) - v_+(t)\|_{H^2} + \|\frac{\partial}{\partial t}\phi(t) - \frac{\partial}{\partial t} v_+(t)\|_{H^1} = O(t^{-1}) \quad (t \to +\infty),$$

$$(1.12) \qquad (\int_t^{+\infty} \|\psi(s) - u_+(s)\|_{W^{2,4}}^4 ds)^{1/4} = O(t^{-1}) \quad (t \to +\infty),$$

where η *depends only on* ε.

(ii) *Assume that* $\psi_- \in H^{2,4}$, $(1+|x|^2)^j (\frac{\partial}{\partial x})^\alpha \psi_- \in L^1(\mathbf{R}^2)$ *for* $j + |\alpha| \le 2$ *and* $\operatorname{supp} \hat{\psi}_- \subset \{\xi; |\xi| \ge 1 + \varepsilon\}$. *We put* $u_-(t) = e^{\frac{i}{2}t\Delta}\psi_-$. *Assume that* $\phi_{-0} \in H^{4,2}$, $\phi_{-1} \in H^{3,2}$, *for* $|\alpha| \le 4$ $(\frac{\partial}{\partial x})^\alpha \phi_{-0} \in L^1(\mathbf{R}^2)$, *and for* $|\alpha| \le 3$ $(\frac{\partial}{\partial x})^\alpha \phi_{-1} \in L^1(\mathbf{R}^2)$. *We put* $v_-(t) = (\cos \omega t)\phi_{-0} + (\omega^{-1} \sin \omega t)\phi_{-1}$. *Then, there exists* $\eta > 0$ *such that if*

$$(1.13) \qquad \|\psi_-\|_{H^{2,4}} + \sum_{j+|\alpha| \le 2} \|(1+|x|)^j (\frac{\partial}{\partial x})^\alpha \psi_-\|_{L^1} + \|\phi_{-0}\|_{H^{4,2}} + \|\phi_{-0}\|_{W^{4,1}} + \|\phi_{-1}\|_{H^{3,2}} + \|\phi_{-1}\|_{W^{3,1}} \le \eta,$$

(1.1)–(1.2) *have the unique solutions* (ψ, ϕ) *satisfying*

$$(1.14) \qquad \psi \in \bigcap_{j=0}^{1} C^j([0,\infty); H^{2-2j}),$$

$$(1.15) \qquad \phi \in \bigcap_{j=0}^{2} C^j([0,\infty); H^{2-j}),$$

$$(1.16)\quad \|\psi(t)-u_-(t)\|_{H^2}+\|\phi(t)-v_-(t)\|_{H^2}+\|\frac{\partial}{\partial t}\phi(t)-\frac{\partial}{\partial t}v_-(t)\|_{H^1} = O(t^{-1}) \quad (t\to-\infty),$$

$$(1.17)\quad (\int_t^{+\infty}\|\psi(s)-u_-(s)\|_{W^{2,4}}^4 ds)^{1/4}=O(t^{-1}) \quad (t\to-\infty),$$

where η depends only on ε.

Remark. The unique global existence theorem for the Cauchy problem of (1.1)–(1.3) is already established (see [1], [2], [8] and [15]). In [1], [2], [8] and [15] only the case of $N=3$ is treated, but the proof of the unique global solutions for $N=2$ is easier than that for $N=3$. Therefore, the solutions of (1)–(2) given by (i) and (ii) of Theorem 1.1 can be extended to the whole time interval $(-\infty,+\infty)$.

The following corollary is an immediate consequence of Theorem 1.1.

Corollary 1.2. *Assume $N=2$. Let $\varepsilon>0$.*

(i) *By D_+ we denote the set of all scattered states $(\psi_+,\phi_{+0},\phi_{+1})$ such that* $\operatorname{supp}\hat{\psi}_+\subset\{\xi;|\xi|\geq 1+\varepsilon\}$ *and* (1.8) *holds. Then, for* (1.1)–(1.2) *the wave operator $W_+:(\psi_+,\phi_{+0},\phi_{+1})\mapsto(\psi(0),\phi(0),\frac{\partial}{\partial t}\phi(0))$ is well defined on D_+.*

(ii) *By D_- we denote the set of all scattered states $(\psi_-,\phi_{-0},\phi_{-1})$ such that* $\operatorname{supp}\hat{\psi}_-\subset\{\xi;|\xi|\geq 1+\varepsilon\}$ *and* (1.13) *holds. Then, for* (1.1)–(1.2) *the wave operator $W_-:(\psi_-,\phi_{-0},\phi_{-1})\mapsto(\psi(0),\phi(0),\frac{\partial}{\partial t}\phi(0))$ is well defined on D_-.*

The proofs in the previous papers [9] and [29] are the same as those used for (1.4) and (1.5) and do not have anything to do with the specific feature of quadratic nonlinearities. Our proof of Theorem 1.1 is based on the special property of the Yukawa interaction and on the improved decay estimates of the interaction term which take account of the difference between the propagation properties of the Schrödinger wave and the Klein-Gordon wave.

§2. Sketch of Proof of Theorem 1.1

We first summarize several lemmas needed for the proof of Theorem 1.1 without proofs.

Lemma 2.1. *Let $N\geq 1$.*

(i) *Let p and q be two positive constants such that* $1/p+1/q=1$ *and* $2 \leq p \leq +\infty$. *Then,*

$$\|U(t)v\|_p \leq (2\pi|t|)^{-N/2+N/p}\|v\|_q, \quad v \in L^q, \quad t \neq 0. \tag{2.1}$$

(ii) *Let k be a nonnegative integer. Suppose that for* $j+|\alpha| \leq k$ $(1+|x|)^{j+2}\big(\frac{\partial}{\partial x}\big)^{\alpha}\psi \in L^2$ *and* $(1+|x|)^{j+2}\big(\frac{\partial}{\partial x}\big)^{\alpha}\psi \in L^1$. *We put* $u_0(t,x) = e^{i|x|^2/(2t)}(it)^{-N/2}\hat{\psi}(\frac{x}{t})$. *Then, for some* $K>0$,

$$\sum_{2j+|\alpha|\leq k} \|\Big(\frac{\partial}{\partial x}\Big)^{\alpha}\Big(\frac{\partial}{\partial t}\Big)\{U(t)\psi - u_0(t)\}\|_2 \leq K|t|^{-1}\sum_{j+|\alpha|\leq k}\|(1+|x|)^{j+2}\Big(\frac{\partial}{\partial x}\Big)^{\alpha}\psi\|_2, \quad |t|\geq 1, \tag{2.2}$$

$$\sum_{2j+|\alpha|\leq k} \|\Big(\frac{\partial}{\partial x}\Big)^{\alpha}\Big(\frac{\partial}{\partial t}\Big)\{U(t)\psi - u_0(t)\}\|_\infty \leq K|t|^{-N/2-1}\sum_{j+|\alpha|\leq k}\|(1+|x|)^{j+2}\Big(\frac{\partial}{\partial x}\Big)^{\alpha}\psi\|_2, \quad |t|\geq 1, \tag{2.3}$$

where K depends only on k and N.

For the proof of Lemma 2.1, see, e.g., [33, Lemmma 2.1].

Lemma 2.2. *Assume* $N=2$. *Let k be a nonnegative integer. Then, for some* $K>0$,

$$\sum_{j+|\alpha|\leq k}\|\Big(\frac{\partial}{\partial x}\Big)^{\alpha}\Big(\frac{\partial}{\partial t}\Big)^{j}(\cos\omega t)v\|_\infty \leq K(1+|t|)^{-1}(\|v\|_{W^{2+k,1}}+\|v\|_{H^{2+k}}), \quad t\in\mathbf{R}, \tag{2.4}$$

$$\sum_{j+|\alpha|\leq k}\|\Big(\frac{\partial}{\partial x}\Big)^{\alpha}\Big(\frac{\partial}{\partial t}\Big)^{j}(\omega^{-1}\sin\omega t)v\|_\infty \leq K(1+|t|)^{-1}(\|v\|_{W^{1+k,1}}+\|v\|_{H^{1+k}}), \quad t\in\mathbf{R}, \tag{2.5}$$

where K depends only on k.

For the proof of Lemma 2.2, see, e.g., [5], [13] and [22].

We next state the decay estimate of solution for the Klein-Gordon equation outside of the light cone.

Lemma 2.3. *Assume $N \geq 1$. Let $\varepsilon > 0$ and let k be a nonnegative integer. Then, for some $L > 0$,*

$$\sum_{j+|\alpha|\leq k} \|(\frac{\partial}{\partial x})^{\alpha}(\frac{\partial}{\partial t})^{j}(\cos\omega t)v\|_{L^{\infty}(|x|>(1+\varepsilon)|t|)} \leq L(1+|t|)^{-2}\|v\|_{H^{k+[N/2]+1,2}}, \quad t \in \mathbf{R}, \tag{2.6}$$

$$\sum_{j+|\alpha|\leq k} \|(\frac{\partial}{\partial x})^{\alpha}(\frac{\partial}{\partial t})^{j}(\omega^{-1}\sin\omega t)v\|_{L^{\infty}(|x|>(1+\varepsilon)|t|)} \leq L(1+|t|)^{-2}\|v\|_{H^{k+[N/2],2}}, \quad t \in \mathbf{R}, \tag{2.7}$$

where $[N/2]$ is the largest integer that does not exceed $N/2$, and L depends only on ε, k and N.

The proof of Lemma 2.3 is based on the finite speed propagation of the Klein-Gordon wave. For the details, see, e.g., [25, Theorem XI. 17] and [33, Lemma 2.3].

We next consider the following problem: Given $h(t)$, find $u(t)$ such that

$$\frac{\partial^2}{\partial t^2}u - \Delta u + u = h(t), \quad t \geq 0, \quad x \in \mathbf{R}^N, \tag{2.8}$$

$$\|\frac{\partial}{\partial t}u(t)\|_2^2 + \|\nabla u(t)\|_2^2 + \|u(t)\|_2^2 \longrightarrow 0 \quad (t \to +\infty), \tag{2.9}$$

or

$$\frac{\partial^2}{\partial t^2}u - \Delta u + u = h(t), \quad t \leq 0, \quad x \in \mathbf{R}^N, \tag{2.10}$$

$$\|\frac{\partial}{\partial t}u(t)\|_2^2 + \|\nabla u(t)\|_2^2 + \|u(t)\|_2^2 \longrightarrow 0 \quad (t \to -\infty). \tag{2.11}$$

We assume that for some $M > 0$,

$$\sup_{t\in[0,\infty)} [(1+t)\|h(t)\|_2 + (1+t)^2 \sum_{1\leq j+|\alpha|\leq 3} \|(\frac{\partial}{\partial x})^{\alpha}(\frac{\partial}{\partial t})h(t)\|_2] \leq M, \tag{2.12}$$

$$\sup_{t\in(-\infty,0]} [(1-t)\|h(t)\|_2 + (1-t)^2 \sum_{1\leq j+|\alpha|\leq 3} \|(\frac{\partial}{\partial x})^{\alpha}(\frac{\partial}{\partial t})h(t)\|_2] \leq M. \tag{2.13}$$

We have the following lemma concerning the existence of solution for (2.8)–(2.9) and (2.10)–(2.11).

Lemma 2.4. *Let $N \geq 1$.*

(i) *Assume that $h \in \cap_{j=0}^{3} C^j([0,\infty); H^{3-j})$ and that $h(t)$ satisfies (2.12). Then, there exists a unique solution $u(t)$ of (2.8)–(2.9) such that*

$$u \in \bigcap_{j=0}^{4} C^j([0,\infty); H^{4-j}), \tag{2.14}$$

$$\sup_{t\in[0,\infty)} (1+t) \sum_{j+|\alpha|\leq 4} \|(\frac{\partial}{\partial x})^\alpha (\frac{\partial}{\partial t})^j u(t)\|_2 \leq C_0 M, \tag{2.15}$$

where M is defined (2.12) and C_0 is independent of h and u.

(ii) *Assume that $h \in \cap_{j=0}^{3} C^j((-\infty,0]; H^{3-j})$ and that $h(t)$ satisfies (2.13). Then, there exists a unique solution $u(t)$ of (2.10)–(2.11) such that*

$$u \in \bigcap_{j=0}^{4} C^j((-\infty,0]; H^{4-j}), \tag{2.16}$$

$$\sup_{t\in(-\infty,0]} (1-t) \sum_{j+|\alpha|\leq 4} \|(\frac{\partial}{\partial x})^\alpha (\frac{\partial}{\partial t})^j u(t)\|_2 \leq C_0 M, \tag{2.17}$$

where M is defined (2.13) and C_0 is independent of h and u.

For the proof of Lemma 2.4, see [33, Lemma 2.4].

Now we describe a sketch of proof of Theorem 1.1. We consider only the case of $t \to +\infty$, because the proof for the case of $t \to -\infty$ is quite similar to that for the case of $t \to +\infty$.

We seek the solutions to the final value problem for (1.1)–(1.2) around almost free solutions. We choose a function $z \in C^\infty([0,\infty))$ such that $z(t) = 1$ for $t \geq 2$ and $z(t) = 0$ for $0 \leq t \leq 1$. We put

$$u_1(t) = z(t)u_0(t) = z(t)e^{i|x|^2/(2t)}(it)^{-1}\hat{\psi}_+(\frac{x}{t}), \tag{2.18}$$

where $u_0(t,x)$ is defined in Lemma 2.1 (ii). Let $v_0(t,x)$ be a solution of (2.8)–(2.9) given by Lemma 2.4 (i) with $h = |u_1|^2$. We introduce the following almost free solutions.

$$u(t) = U(t)\psi_+, \quad v(t) = (\cos\omega t)\phi_{+0} + (\omega^{-1}\sin\omega t)\phi_{+1} + v_0(t). \tag{2.19}$$

We note that $u(t) = u_+(t)$. Furthermore, we put

$$\psi(t) = F(t) + u(t), \tag{2.20}$$

$$\phi(t) = N(t) + v(t). \tag{2.21}$$

We rewrite (1.1)–(1.2) as the following system of F and N:

$$\begin{aligned} i\frac{\partial}{\partial t}F + \frac{1}{2}\Delta F &= NF + N(u - u_1) + Nu_1 \\ &\quad + vF + f(t), \quad t \geq 0, \quad x \in \mathbf{R}^2, \end{aligned} \tag{2.22}$$

$$\begin{aligned} \frac{\partial^2}{\partial t^2}N - \Delta N + N &= |F|^2 + 2\Re(F(\bar{u} - \bar{u}_1)) \\ &\quad + 2\Re(F\bar{u}_1) + g(t), \quad t \geq 0, \quad x \in \mathbf{R}^2, \end{aligned} \tag{2.23}$$

$$\|F(t)\|_2 \to 0 \quad (t \to \infty), \tag{2.24}$$

$$\|\frac{\partial}{\partial t}N(t)\|_2^2 + \|\nabla N(t)\|_2^2 + \|N(t)\|_2^2 \to 0 \quad (t \to \infty), \tag{2.25}$$

where

$$f(t) = v(u - u_1) + vu_1, \tag{2.26}$$

$$g(t) = |u - u_1|^2 + 2\Re((u - u_1)\bar{u}_1). \tag{2.27}$$

If we have the solutions (F, N) of (2.22)–(2.25), then we obtain Theorem 1.1 (i) by (2.20)–(2.21). Lemmas 2.1–2.4 and the support condition of the Fourier image of ψ_+ show that $f(t)$ and $g(t)$ decay in L^2 fast enough as $t \to \infty$. Therefore, we can obtain the desired solutions (F, N) for (2.22)–(2.25). The details of the proof will be published elsewhere (see [33]).

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Inverse Iteration Method with a Complex Parameter II

Toshio Suzuki

§1. Introduction

Let A be a symmetric (n, n) matrix and let $\lambda_k, \phi_k, k = 1, \ldots, n$ be pairs of eigenvalues and the corresponding eigenvectors of A. The inverse iterarion process for the eigenvector ϕ_j is to solve the following linear equations with initial data $z^{(0)}$ under the conditions $|\lambda_j - \lambda| << c < |\lambda_k - \lambda|, (k \neq j)$:

$$(A - \lambda I)z^{(m+1)} = z^{(m)}, m = 0, 1, 2, \ldots \tag{1.1}$$

In the paper [1] we proposed the inverse iteration method with a complex parameter and showed some numerical results of our method. There we replaced λ in (1.1) by a complex parameter $\lambda + \sqrt{-1}\tau$ and managed to derive the utilities of the complex parameter with $|\tau| < \varepsilon$ under the following Assumption H.

Assumption H. Eigenvalues $\lambda_k, k = 1, 2, \ldots, n$ of A are known with the following accuracy: There are three numerical constants c,ε and λ such that $\inf_{k \neq j} |\lambda_j - \lambda_k| > 2c$ and $|\lambda_j - \lambda| < \varepsilon$ and $0 < 2\varepsilon < c$.

In the spectral theory, it is well known that the projection operator P_j to the eigenspace correspondong to the eigenvalue λ_j is represented as follows

$$P_j v = \frac{1}{2\pi\sqrt{-1}} \oint (A - \zeta I)^{-1} v d\zeta. \tag{1.2}$$

It can be considered that to solve the linear equation (1.1) is to execute the numerical integral of (1.2) with one point value. Since the result of our method is understood to be that with two point values, it will be taken for granted that our method is more effective than the standard traditional one.

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In this paper we show supplementary propositions to [1] and propose the more powerful version of our method in practical computations.

§2. Propositions and the iteration process

In the paper [1], we did not give the proofs of the propositions. The most important one of them can be improved as the following two propositions.

Consider the following equation with $\|z\| = 1$:

$$(A - \lambda I - \sqrt{-1}\tau I)w = z. \tag{2.1}$$

Let $z = \sum_{k=1}^{n} a_k\phi_k$, then we have

$$\begin{aligned} w &= \sum_{k=1}^{n} \frac{1}{\lambda_k - \lambda - \sqrt{-1}\tau} a_k\phi_k \\ &= \sum_{k=1}^{n} \frac{\lambda_k - \lambda}{(\lambda_k - \lambda)^2 + \tau^2} a_k\phi_k + \sqrt{-1}\sum_{k=1}^{n} \frac{\tau}{(\lambda_k - \lambda)^2 + \tau^2} a_k\phi_k. \end{aligned} \tag{2.2}$$

Put $x_k = \dfrac{\lambda_k - \lambda}{(\lambda_k - \lambda)^2 + \tau^2} a_k\phi_k$ and $y_k = \dfrac{\tau}{(\lambda_k - \lambda)^2 + \tau^2} a_k\phi_k$. Let $x = \sum_{k=1}^{n} x_k$ and $y = \sum_{k=1}^{n} y_k$.

Proposition 2.1. *Let x,y be the real and imaginary part of the solution of the equation* (2.1) *with $|\lambda_j - \lambda| < |\tau| \le \varepsilon$ under the Assumption* H *in which the inequality $|\lambda_j - \lambda| < |\varepsilon|$ is assumed. Put $\tilde{\lambda} = (Ay, y)/\|y\|^2$. If $3\|y\| > 2\|x\|$, then $|\lambda_j - \tilde{\lambda}| < |\tau|$.*

Proof. Put $\alpha_k = [(\lambda_k - \lambda)^2 + \tau^2]^{-1}$ and $T = \sum_{k=1}^{n} \alpha_k^2 a_k^2$. Since $4\varepsilon < 2c < \inf_{k\neq j} |\lambda_j - \lambda_k|$ by the assumption H. We have the following inequalities:

$$\sum_{k\neq j} |\lambda_k - \lambda|^2 \alpha_k^2 a_k^2 \ge 3\varepsilon \sum_{k\neq j} |\lambda_k - \lambda| \alpha_k^2 a_k^2$$

and

$$\sum_{k\neq j} |\lambda_k - \lambda|^2 \alpha_k^2 a_k^2 \ge 9\varepsilon^2 \sum_{k\neq j} \alpha_k^2 a_k^2.$$

Then from the assumption $3\|y\| \ge 2\|x\|$, we have

$$9\tau^2 T \ge 4\sum_{k=1}^{n} |\lambda_k - \lambda|^2 \alpha_k^2 a_k^2 \ge 4 \cdot 3\varepsilon \sum_{k\neq j} |\lambda_k - \lambda| \alpha_k^2 a_k^2,$$

that is,

$$\frac{3}{4}\frac{\tau^2}{\varepsilon}T \geq \sum_{k\neq j} |\lambda_k - \lambda| \alpha_k^2 a_k^2. \tag{2.3}$$

Similarly we also have the following inequality:

$$\frac{1}{4}\frac{\tau^2}{\varepsilon^2}T \geq \sum_{k\neq j} \alpha_k^2 a_k^2. \tag{2.4}$$

On the other hand, we have following estimates:

$$\begin{aligned}
|\lambda_j - \tilde{\lambda}| &= |\lambda_j - (Ay, y)/\|y\|^2| \\
&= |\lambda_j - (\sum_{k=1}^{n} \lambda_k \tau^2 \alpha_k^2 a_k^2 / \sum_{k=1}^{n} \tau^2 \alpha_k^2 a_k^2)| \\
&\leq \sum_{k\neq j} |\lambda_j - \lambda_k| \alpha_k^2 a_k^2 / T \\
&\leq \sum_{k\neq j} |\lambda_j - \lambda| \alpha_k^2 a_k^2 / T + \sum_{k\neq j} |\lambda - \lambda_k| \alpha_k^2 a_k^2 / T.
\end{aligned}$$

So we have the following results from (2.3) and (2.4):

$$|\lambda - \tilde{\lambda}| \leq |\tau| \frac{1}{4}\frac{\tau^2}{\varepsilon^2} + \frac{3}{4}\frac{\tau^2}{\varepsilon} \leq |\tau|.$$

Q.E.D.

The following proposition is easily derived by a similar argument used in the proof of the Proposition 2.1.

Proposition 2.2. *Under the same assumption of Proposition* 2.1, *if* $\|y\| \geq \|x\|$ *then* $|\lambda_j - \tilde{\lambda}| < \frac{\tau^2}{c}$.

These propositions bring the following more powerful version of the iteration process of our method in [1], where the step (2.7) and (2.8) are varied.

Let ξ be an initial vector and let $\tau^{(0)}$ be a real number whose absolute value is smaller than ε. Our iteration process consists of the following four steps (2.5)–(2.8), where $u^{(m)}$ and $v^{(m)}$ are real vectors.

$$(A - \lambda^{(m)} I - \sqrt{-1}\tau^{(m)} I) w^{(m)} = z^{(m)} \quad \text{where } z^{(0)} = \xi, \lambda^{(0)} = \lambda \tag{2.5}$$

$$z^{(m+1)} = \frac{v^{(m)}}{\|v^{(m)}\|} \quad \text{where } w^{(m)} = u^{(m)} + \sqrt{-1}v^{(m)} \tag{2.6}$$

$$\lambda^{(m+1)} = \begin{cases} (Az^{(m+1)}, z^{(m+1)}) & \text{if } 3\|v^{(m)}\| > 2\|u^{(m)}\| \\ \lambda^{(m)} & \text{otherwise.} \end{cases} \tag{2.7}$$

$$\tau^{(m+1)} = \frac{(\tau^{(m)})^2}{c} \quad \text{if } \|v^{(m)}\| > \|u^{(m)}\|. \tag{2.8}$$

Remark 2.3. The process (2.8) may be passed if $|\tau^{(m)}|$ is small enough.

§3. Applications

Propositions 2.1 and 2.2 show that even if we do not have a so accurate value of ε or even if the initial vector is not so well, $\lambda^{(m)}$ in the iteration process converges to the aimed eigenvalue efficiently by using better parameters in each iteration. So we can have an application of our method to get a rapid tool for computing eigen-pairs combining the bisection method. Its idea is such that: get rough estimates of eigenvalues by the bisection method, first, then, apply our iteration process. The computing time to improve the accuracy of an eigenvalue by 5 decimal digits with the aid of the bisection method is comparable to that of two times iterations of our method. So, for example, if, starting from the initial approximating value with the accuracy about 10^{-4}, we could have the eigenvalue with the accuracy 10^{-15} after two times iterations, this method is an improvement of the procedure done by only the bisection method. The test computations of this example and of the others of this kinds have shown satisfactory results. We do not have the optimal result as yet but the above example is at least one of the application of our method to get eigen-pairs more rapidly.

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Asymptotics for the Number of Negative Eigenvalues of Three–Body Schrödinger Operators with Efimov Effect

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Introduction

The Efimov effect is one of the most interesting results in the spectral analysis for three–body Schrödinger operators. Roughly speaking, it can be explained as follows: If all three two–body subsystems have no negative eigenvalues and if at least two of these subsystems have a resonance state at zero energy, then the three–body system under consideration has an infinite number of negative eigenvalues accumulating at zero. This remarkable spectral property was first discovered by Efimov [1] and the mathematically rigorous proof has been given by the works [4, 8, 10]. In the present note, we study the asymptotic distribution of these negative eigenvalues below the bottom zero of essential spectrum which is a three–cluster threshold energy. Let $N(E)$, $E > 0$, be the number of negative eigenvalues less than $-E$ with repetition according to their multiplicities. Then the result obtained here is, somewhat loosely stating, that $N(E)$ behaves like $|\log E|$ as $E \to 0$.

We first formulate precisely the main theorem and then make a brief comment on the recent related result obtained by Sobolev [7]. We consider a system of three particles with masses $m_j > 0$, $1 \leq j \leq 3$, which move in the three–dimensional space R^3 and interact with each other through a pair potential $V_{jk}(r_j - r_k)$, $1 \leq j < k \leq 3$, where $r_j \in R^3$ denotes the position vector of the j–th particle. For such a system, the energy Hamiltonian H (three–body Schrödinger operator) takes the form

$$(0.1) \qquad H = H_0 + V, \qquad V = \sum_{1 \leq j < k \leq 3} V_{jk}(r_j - r_k),$$

in the center–of–mass frame, where H_0 denotes the free Hamiltonian. Both the operators H_0 and H act on the space $L^2(R^6)$ and are repre-

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sented in various forms according to the choice of the Jacobi coordinates. The pair potential V_{jk} is assumed to satisfy the following condition:

$(V)_\rho$ $V_{jk}(x)$, $x \in R^3$, is real–valued and has the decay property

$$|V_{jk}(x)| \le C\,(1+|x|)^{-\rho} \quad \text{for some } \rho > 2.$$

By this assumption, the Hamiltonian H formally defined above admits a unique self–adjoint realization in $L^2(R^6)$. We denote by the same notation H this self–adjoint realization.

We use the letters α, β and γ to denote one of three pairs (j,k) with $1 \le j < k \le 3$. For a pair $\alpha = (j,k)$, we define the reduced mass m_α through the relation $1/m_\alpha = 1/m_j + 1/m_k$ and the two–body subsystem Hamiltonian H^α as

$$H^\alpha = -\Delta/2m_\alpha + V_\alpha, \quad V_\alpha(x) = V_{jk}(x), \quad \text{on } L^2(R^3_x).$$

We further assume that these subsystem Hamiltonians H^α have the following spectral properties:

$(H.1)$ H^α has no negative bound state energies for all pairs α.

$(H.2)$ H^α has a resonance state at zero energy for all pairs α.

Roughly speaking, the second assumption $(H.2)$ means that the equation $H^\alpha\varphi = 0$ has a solution $\varphi(x)$, $x \in R^3$, behaving like $\varphi(x) \sim |x|^{-1}$ at infinity. Such a solution is called a resonance state at zero energy. It should be noted that φ is not an eigenstate of H^α at zero energy. By the HVZ theorem ([5]), it follows from $(H.1)$ that H has essential spectrum beginning at zero and negative discrete spectrum. If, in addition, $(H.2)$ is satisfied, then H has an infinite number of negative eigenvalues accumulating at zero. In assumption $(H.1)$, we have assumed that any pair of two particles does not have bound states at negative energies. Nevertheless the three–body system has an infinite number of bound states at negative energies. As stated above, this spectral property is called the Efimov effect.

With the above notations and assumptions, we are now in a position to formulate the first theorem.

Theorem 1. *Assume that $(V)_\rho$, $(H.1)$ and $(H.2)$ are fulfilled. Let $N(E)$, $E > 0$, be the number of negative eigenvalues less than $-E$ of H with repetition according to their multiplicities. Then $N(E)$ obeys the following asymptotic formula:*

$$N(E) = C_0\,|\log E|\,(1 + o(1)), \qquad E \to 0,$$

for some $C_0 > 0$.

Remark 1. We should make some comments on the leading coefficient C_0 in the asymptotic formula. This constant C_0 does not depend on the pair potentials V_{jk} and is given as a positive function of only the ratios m_j/m_k between the masses. The constant is determined from an eigenvalue asymptotics for a certain compact integral operator and is in general difficult to write down in an explicit form. In the special case with identical masses, C_0 is determined as $C_0 = s/2\pi$ with the unique positive root $s > 0$ of the equation

$$s = 2^3 \cdot 3^{-1/2} (\sinh s\,\pi/6)/(\cosh s\,\pi/2).$$

Remark 2. (1) The following result can be also obtained in the course of proof: If at most one of two–body subsystem Hamiltonians H^α has a resonance state at zero energy, then H has only a finite number of negative eigenvalues; $N(E) = O(1)$, $E \to 0$. This result asserts the finiteness of discrete spectrum below the bottom of essential spectrum, even if the bottom coincides with a three–cluster threshold energy. (2) As previously stated, H has in general an infinite number of negative eigenvalues accumulating at zero except for a certain special case, if only two subsystem Hamiltonians have a resonance state at zero energy. Even in such a case, a similar asymptotic formula with another leading coefficient $C_0 > 0$ can be obtained.

The asymptotic formula for $N(E)$ has been first established by Sobolev [7] under the main assumption that pair potentials are non–positive $V_{jk} \leq 0$ and have the decay property $(V)_\rho$ with $\rho > 3$. The above properties of the leading coefficient C_0 has been also investigated in detail there. Theorem 1 is only a supplement of the interesting result obtained by Sobolev [7] and the proof is also based on the idea developed there. But the arguments undergo slight changes in many aspects, if the non–positivity assumption of pair potentials is not necessarily assumed.

The method here applies also to the problem on the eigenvalue asymptotics in the coupling limit. We consider the three–body Hamiltonian

$$H(\lambda) = H - \lambda V = H_0 + (1 - \lambda)V \qquad \text{on } L^2(R^6) \tag{0.2}$$

with a coupling constant λ, $0 < \lambda \ll 1$, small enough, where H is defined by (0.1) and is assumed to satisfy all the assumptions in Theorem 1. Let $N_0(\lambda)$ be the number of negative eigenvalues of $H(\lambda)$. For $\lambda > 0$, $H(\lambda)$

has only a finite number of negative eigenvalues but $N_0(\lambda) \to \infty$ as λ tends to the critical value 0. The theorem below gives the asymptotic formula as $\lambda \to 0$ for $N_0(\lambda)$.

Theorem 2. *Let the notations be as above. Suppose that the three–body Hamiltonian $H = H(0)$ fulfills the assumptions $(V)_\rho$, $(H.1)$ and $(H.2)$. Then $N_0(\lambda)$ behaves like*

$$N_0(\lambda) = 2\,C_0\,|\log\lambda|\,(1 + o(1)), \qquad \lambda \to 0,$$

with the same positive constant C_0 as in Theorem 1.

§1. Low energy analysis for two–body resolvents

The proof of the theorems above is based on the behavior at low energies of two–body resolvents with resonance at zero energy. We here make a brief review on this result. For details, see [2, 3].

Throughout the section, we work in the space $L^2 = L^2(R^3_x)$ and denote by $\langle\ ,\ \rangle$ the L^2 scalar product. We begin by defining precisely the resonance state at zero energy. Let $T = -\Delta + V_0$ be the two–body Schrödinger operator acting on L^2. We assume that the potential $V_0(x)$ has the decay property $(V)_\rho$ and that the operator T has the spectral properties $(H.1)$ and $(H.2)$. We now consider the equation $T\varphi = 0$. This equation can be put into the integral equation

$$(1.1) \qquad \varphi(x) = -(1/4\pi)\int |x-y|^{-1}V_0(y)\varphi(y)\,dy,$$

where the integration with no domain attached is taken over the whole space. Equation (1.1) is considered in the weighted L^2 space $L^2_{-s} = L^2(R^3_x; \langle x\rangle^{-2s}\,dx)$ with weight $\langle x\rangle^{-s} = (1+|x|^2)^{-s/2}$, $s > 1/2$ being taken close enough to $1/2$. If $\varphi \in L^2_{-s}$ solves the equation (1.1), then it is easily seen that φ behaves like

$$\begin{aligned}\varphi(x) &= -(1/4\pi)\langle V_0, \varphi\rangle |x|^{-1} + O(|x|^{-\rho+1}),\\ (\partial/\partial|x|)\varphi(x) &= (1/4\pi)\langle V_0, \varphi\rangle |x|^{-2} + O(|x|^{-\rho})\end{aligned}$$

as $|x| \to \infty$. We say that φ is a resonance state of T at zero energy, if $\langle V_0, \varphi\rangle \neq 0$ is satisfied. Thus the resonance state φ behaves like $\varphi(x) \sim |x|^{-1}$ as $|x| \to \infty$ and hence $\varphi \notin L^2$ is not a bound state at zero energy. On the other hand, if $\langle V_0, \varphi\rangle = 0$ is satisfied, then we obtain from (1.1) that $\varphi(x) = O(|x|^{-2})$, so that φ belongs to L^2 and becomes a bound state of T at zero energy. Conversely, if φ is a bound state at zero energy, then we can easily see that φ satisfies the relation $\langle V_0, \varphi\rangle = 0$. This implies that a resonance state at zero energy is non–degenerate. Under assumptions $(V)_\rho$, $(H.1)$ and $(H.2)$, it also follows from Theorem A.3.1 of [6] that T cannot have a bound state at zero energy (bottom of its spectrum) and hence T has only a resonance state.

Assumption $(V)_\rho$ enables us to choose a non–negative potential $U_0 \geq 0$ satisfying $(V)_\rho$ so that

$$W_0(x) = U_0(x) - V_0(x) \geq U_0(x)/2 \geq 0. \tag{1.2}$$

We define the Schrödinger operator S with potential U_0 by

$$S = -\Delta + U_0 \qquad \text{on } L^2(R^3_x) \tag{1.3}$$

and denote the resolvent of S as $R(d^2; S) = (S + d^2)^{-1}$ for $d > 0$. Since U_0 is non–negative, $R(0; S)$ can be also defined as a bounded operator from L^2_s into L^2_{-s} for any $s > 1$ and the generalized eigenfunction $\theta_0(x)$ of S at zero energy is obtained as a unique solution to the Lippmann–Schwinger equation. Let $A(d) : L^2 \to L^2$ be the operator defined by

$$A(d) = \mathrm{Id} - W_0^{1/2} R(d^2; S) W_0^{1/2}, \quad d \geq 0, \tag{1.4}$$

Id being the identity operator. It should be noted that this operator can be defined even for $d = 0$. Let Σ_1 be the kernel of $A(0)$. The kernel Σ_1 can be shown to be a one–dimensional space. Denote by $\psi_1 \in L^2$ the normalized function spanning Σ_1. Then we can show that $\psi_1(x)$ falls off with order $O(|x|^{-1-\rho/2})$ and satisfies $\langle \theta_0, W_0^{1/2}\psi_1\rangle \neq 0$. We decompose the space $L^2 = L^2(R^3_x)$ into the orthogonal sum $L^2 = \Sigma_1 \oplus \Sigma_2$ and we denote by P_j, $1 \leq j \leq 2$, the orthogonal projections onto Σ_j.

We study the behavior as $d \to 0$ of $A(d)$ defined above. To do this, we here introduce new notations. A bounded operator $T(d)$, $0 < d \ll 1$, acting on L^2 is said to be of class $Op(d^\nu)$, if its operator norm obeys the bound $\|T(d)\| = O(d^\nu)$ as $d \to 0$. When the difference $T_1(d) - T_2(d)$ is of class $Op(d^\nu)$, we denote this relation as $T_1(d) = T_2(d) + Op(d^\nu)$.

Lemma 1.1. *Let the notations be as above. Suppose that T fulfills $(V)_\rho$, $(H.1)$ and $(H.2)$. Then the operator $A(d)$ has the following properties.*

(i) *Let ϵ, $0 < \epsilon \ll 1$, be fixed arbitrarily. Then there exist positive constants c_ϵ and c'_ϵ such that*

$$c_\epsilon \,\mathrm{Id} \leq A(d) \leq c'_\epsilon \,\mathrm{Id}, \qquad d \geq \epsilon,$$

in the form sense.

(ii) *Define* $A_{jk}(d)$, $1 \le j,\ k \le 2$, *as* $A_{jk}(d) = P_j A(d) P_k$. *Then:*

(1) $A_{22}(d) \in Op(d^0)$ *and* $A_{22}(d) \ge c_2 P_2$ *for some* $c_2 > 0$.
(2) $A_{12}(d) \in Op(d^\nu)$ *for some* $\nu > 1/2$.
(3) $A_{11}(d) = \sigma_1 d P_1 + Op(d^\nu)$ *for some* $\nu > 1$, *where*

$$\sigma_1 = |\langle \theta_0, W_0^{1/2} \psi_1 \rangle|^2 / 4\pi > 0.$$

Remark 1.2. A similar argument applies to the Schrödinger operator $T = -\Delta/2m + V_0$ with reduced mass m. For such an operator, the constant σ_1 in the lemma is given as

$$\sigma_1 = 2^{-1/2} \pi^{-1} m^{3/2} \, |\langle \theta_0, W_0^{1/2} \psi_1 \rangle|^2,$$

where θ_0 is the generalized eigenfunction at zero energy of $S = -\Delta/2m + U_0$, U_0 being chosen to satisfy (1.2), and $\psi_1 \in L^2$ is the normalized function constructed for the operator S.

§2. Sketch of proof of Theorem 1

We here give a sketch for the proof of Theorem 1 only. See [9] for the detailed proof, including the proof of Theorem 2.

(0) We begin by introducing several basic notations used in the spectral analysis for three–body Schrödinger operators.

Let $\alpha = (j,k)$ be given pair and let l, $l \ne j,\ k$, be the index by which the third particle is labelled. Then the Jacobi coordinates associated with α are defined as

$$x_\alpha = r_j - r_k, \quad y_\alpha = r_l - (m_j r_j + m_k r_k)/(m_j + m_k). \tag{2.1}$$

We denote by $(p_\alpha, q_\alpha) \in R^{3\times 2}$ the coordinates dual to (x_α, y_α). In this coordinate system, the symbol $H_0(p_\alpha, q_\alpha)$ of the three–body free Hamiltonian H_0 is described as

$$H_0(p_\alpha, q_\alpha) = |p_\alpha|^2/2m_\alpha + |q_\alpha|^2/2n_\alpha,$$

where m_α again denotes the reduced mass associated with α and n_α is defined through the relation $1/n_\alpha = 1/m_l + 1/(m_j + m_k)$. Let $\beta \ne \alpha$ be another pair. Then a simple calculation yields

$$p_\alpha = \kappa^{\alpha\alpha} q_\alpha + \kappa^{\alpha\beta} q_\beta, \quad p_\beta = \kappa^{\beta\alpha} q_\alpha + \kappa^{\beta\beta} q_\beta, \tag{2.2}$$

where the coefficients $\kappa^{\alpha\alpha}$, $\kappa^{\beta\alpha}$, $\kappa^{\alpha\beta}$ and $\kappa^{\beta\beta}$ are explicitly expresssed in terms of the masses m_j, $1 \le j \le 3$, and, in particular, $\kappa^{\beta\alpha}$ and $\kappa^{\alpha\beta}$ satisfy $|\kappa^{\beta\alpha}| = |\kappa^{\alpha\beta}| = 1$. We also denote by $H_0(q_\alpha, q_\beta)$ the symbol representation for H_0 in the coordinate system (q_α, q_β). We further define the cluster Hamiltonian H_α as

$$H_\alpha = H_0 + V_\alpha, \quad V_\alpha = V_{jk}, \quad \text{on } L^2(R^6).$$

The base space $L^2(R^6)$ is decomposed as the tensor product

$$L^2(R^6) = L^2(R^3; dx_\alpha) \otimes L^2(R^3; dy_\alpha)$$

and hence the Hamiltonian H_α is represented as

$$H_\alpha = H^\alpha \otimes \mathrm{Id} + \mathrm{Id} \otimes T_\alpha \quad \text{on } L^2(R^3; dx_\alpha) \otimes L^2(R^3; dy_\alpha),$$

where H^α again denotes the two–body subsystem Hamiltonian associated with α and T_α is given as

$$T_\alpha = -\Delta/2n_\alpha \qquad \text{on } L^2(R^3; dy_\alpha). \tag{2.3}$$

We now choose a non–negative potential $U_\alpha = U_\alpha(x_\alpha) \ge 0$ to satisfy the property (1.2)

$$W_\alpha(x_\alpha) = U_\alpha(x_\alpha) - V_\alpha(x_\alpha) \ge U_\alpha(x_\alpha)/2 \ge 0$$

and define the Hamiltonians K^α and K_α as

$$\begin{aligned} K^\alpha &= -\Delta/2m_\alpha + U_\alpha \quad \text{on } L^2(R^3; dx_\alpha), \\ K_\alpha &= K^\alpha \otimes \mathrm{Id} + \mathrm{Id} \otimes T_\alpha \quad \text{on } L^2(R^3; dx_\alpha) \otimes L^2(R^3; dy_\alpha). \end{aligned} \tag{2.4}$$

We also define $A(d; K^\alpha) : L^2(R^3; dx_\alpha) \to L^2(R^3; dx_\alpha)$ as

$$A(d; K^\alpha) = \mathrm{Id} - W_\alpha^{1/2}(K^\alpha + d^2)^{-1} W_\alpha^{1/2}, \quad d \ge 0, \tag{2.5}$$

in a way similar to (1.4) and denote by P_j^α, $1 \le j \le 2$, the orthogonal projections associated with $A(0; K^\alpha)$, which are constructed in the same way as P_j in section 1. We further denote by $\theta_0^\alpha = \theta_0^\alpha(x_\alpha)$ the generalized eigenfunction of K^α at zero energy and by $\psi_1^\alpha \in L^2(R^3; dx_\alpha)$ the normalized function spanning the range of P_1^α, the range being a one–dimensional space. The operator $A(d; K^\alpha)$ defined above preserves the same properties as in Lemma 1.1 (see also Remark 1.2) and, in particular, we have

$$P_1^\alpha A(d; K^\alpha) P_1^\alpha = \sigma_\alpha \, d \, P_1^\alpha + Op(d^\nu), \quad d \to 0,$$

for some $\nu > 1$, where $\sigma_\alpha > 0$ is given as

$$\sigma_\alpha = 2^{-1/2}\pi^{-1} m_\alpha^{3/2} |\langle \theta_0^\alpha, W_\alpha^{1/2}\psi_1^\alpha\rangle|^2. \tag{2.6}$$

(1) We consider only E, $0 < E \ll 1$, small enough. For given self–adjoint operator A, we denote by $n(\mu; A)$ the number of eigenvalues greater than μ of A. Let $U = \sum_\alpha U_\alpha$ and $W = \sum_\alpha W_\alpha$, where the summation $\sum_\alpha$ is taken over all three pairs α. Define the Hamiltonian K by $K = H_0 + U = H + W$ and the bounded operator $M(E) : L^2(R^6) \to L^2(R^6)$ by

$$M(E) = (K+E)^{-1/2} W (K+E)^{-1/2} = \sum_\alpha M_\alpha(E)^* M_\alpha(E)$$

with $M_\alpha(E) = W_\alpha^{1/2}(K+E)^{-1/2}$. Then the quantity $N(E)$ in question coincides with $n(1; M(E))$ by the Birman–Schwinger principle. The next lemma is due to Sobolev [7].

Lemma 2.1. *Let $\mathcal{L}^2 = \sum \oplus L^2(R^6)$, three summands. Define the operator $\mathcal{M}(E) : \mathcal{L}^2 \to \mathcal{L}^2$ as*

$$\mathcal{M}(E) = \begin{pmatrix} M_\alpha(E)M_\alpha(E)^* & M_\alpha(E)M_\beta(E)^* & M_\alpha(E)M_\gamma(E)^* \\ M_\beta(E)M_\alpha(E)^* & M_\beta(E)M_\beta(E)^* & M_\beta(E)M_\gamma(E)^* \\ M_\gamma(E)M_\alpha(E)^* & M_\gamma(E)M_\beta(E)^* & M_\gamma(E)M_\gamma(E)^* \end{pmatrix},$$

where α, β and γ denote different three pairs. Then one has

$$N(E) = n(1; \mathcal{M}(E)).$$

(2) We denote by $\mathrm{Dia}\{B_\alpha, B_\beta, B_\gamma\}$ the 3×3 diagonal matrix with operators B_α, B_β and B_γ as diagonal entries. Let $\mathcal{M}(E)$ be as in Lemma 2.1. The off–diagonal entries of $\mathcal{M}(E)$ are all compact operators on $L^2(R^6)$ but the diagonal ones are not necessarily compact operators. Thus we look more carefully at the operator

$$M_\alpha(E)M_\alpha(E)^* = W_\alpha^{1/2}(K+E)^{-1}W_\alpha^{1/2}$$

in the diagonal entries of $\mathcal{M}(E)$.

Let K_α be defined by (2.4). We decompose the above operator into the sum $M_\alpha(E)M_\alpha(E)^* = M_{0\alpha}(E) + L_\alpha(E)$, where $M_{0\alpha}(E) = W_\alpha^{1/2}(K_\alpha + E)^{-1}W_\alpha^{1/2}$ and

$$L_\alpha(E) = W_\alpha^{1/2}((K+E)^{-1} - (K_\alpha+E)^{-1})W_\alpha^{1/2},$$

so that $\mathcal{M}(E)$ is represented as $\mathcal{M}(E) = \mathcal{M}_0(E) + \mathcal{M}_1(E)$ with

$$\mathcal{M}_0(E) = \mathrm{Dia}\{M_{0\alpha}(E), M_{0\beta}(E), M_{0\gamma}(E)\}.$$

We note that $\mathcal{M}_1(E) : \mathcal{L}^2 \to \mathcal{L}^2$ is a compact operator.

We now introduce a positive smooth function $\omega(s)$, $s > 0$, such that

$$\omega(s) = s \quad \text{for } 0 < s < 1, \qquad \omega(s) = 2 \quad \text{for } s > 2.$$

Let T_α be defined by (2.3) as an operator on $L^2(R^3; dy_\alpha)$. We define

$$\omega_\alpha(E) = \omega((T_\alpha + E)^{1/2}),$$

which is considered as an operator acting on $L^2(R^6)$ as well as on $L^2(R^3; dy_\alpha)$. We further define $A_\alpha(E) : L^2(R^6) \to L^2(R^6)$ as

$$A_\alpha(E) = \mathrm{Id} - M_{0\alpha}(E) = \mathrm{Id} - W_\alpha^{1/2}(K_\alpha + E)^{-1} W_\alpha^{1/2}.$$

By Lemma 1.1 (see also Remark 1.2), we can find strictly positive smooth bounded functions $f^\pm(s)$, $0 < c \le f^+(s) \le f^-(s)$, behaving like

$$f^\pm(s) = 1 + o(s^\nu), \qquad s \to 0,$$

for some $\nu > 0$ such that

$$A_\alpha(E) \ge f_\alpha^+(E)\omega_\alpha(E)P_1^\alpha + c_+ P_2^\alpha, \tag{2.7}$$

$$A_\alpha(E) \le f_\alpha^-(E)\omega_\alpha(E)P_1^\alpha + c_- P_2^\alpha \tag{2.8}$$

for some positive constants $c_\pm$, $0 < c_+ < c_-$, where

$$f_\alpha^\pm(E) = \sigma_\alpha f^\pm((T_\alpha + E)^{1/2})$$

with $\sigma_\alpha > 0$ given by (2.6), and the inequality relations are understood in the form sense. Denote by $F_\alpha^+(E)$ and $F_\alpha^-(E)$ the operators on the right side of (2.7) and (2.8), respectively, and define

$$\mathcal{F}_0^\pm(E) = \mathrm{Dia}\{F_\alpha^\pm(E), F_\beta^\pm(E), F_\gamma^\pm(E)\}.$$

Then it follows from (2.7) and (2.8) that

$$\mathcal{F}_0^+(E) \le \mathrm{Id} - \mathcal{M}_0(E) \le \mathcal{F}_0^-(E)$$

and hence we obtain from Lemma 2.1 that

$$n(1; \mathcal{Q}^-(E)) \le N(E) \le n(1; \mathcal{Q}^+(E)), \tag{2.9}$$

where

$$\mathcal{Q}^{\pm}(E) = \mathcal{F}_0^{\pm}(E)^{-1/2}\mathcal{M}_1(E)\mathcal{F}_0^{\pm}(E)^{-1/2}.$$

(3) We study the behavior as $E \to 0$ of Hilbert–Schmidt norm of the entry operators $Q^{\pm}_{\alpha\beta}(E)$ in $\mathcal{Q}^{\pm}(E)$. To do this, we here introduce new notations. Let $B(E)$, $0 < E \ll 1$, be a compact operator on $L^2(R^6)$. We say that $B(E)$ is of class $(HS)_\epsilon$, if for any $\epsilon > 0$ small enough, $B(E)$ has a decomposition $B(E) = B_1(E;\epsilon) + B_2(E;\epsilon)$ such that: (i) the Hilbert–Schmidt norm of $B_1(E;\epsilon)$ obeys the bound $\|B_1(E;\epsilon)\|_{HS} \le C_\epsilon$ for some C_ϵ independent of E; (ii) the operator norm of $B_2(E;\epsilon)$ obeys the bound $\|B_2(E;\epsilon)\| \le \epsilon$. If the difference between two operators $B_1(E)$ and $B_2(E)$ is of class $(HS)_\epsilon$, we denote this relation as $B_1(E) \sim B_2(E)$.

Lemma 2.2. $Q^{\pm}_{\alpha\alpha}(E) \sim 0$.

We analyse the operators $Q^{\pm}_{\alpha\beta}(E)$, $\alpha \ne \beta$, in the off–diagonal entries of $\mathcal{Q}^{\pm}(E)$. Recall that $\psi_1^\alpha \in L^2(R^3;dx_\alpha)$ is the normalized function spanning the range of P_1^α (one–dimensional space). Let $\chi(x)$, $x \in R^3$, be the characteristic function of the unit ball B_1 in R^3. We set

$$\zeta_\alpha(q_\alpha;E) = \chi(q_\alpha)\,(|q_\alpha|^2/2n_\alpha + E)^{-1/4}$$

and denote by $\Pi_{\alpha\beta}(E) : L^2(R^6;dx_\beta\,dq_\beta) \to L^2(R^6;dx_\alpha\,dq_\alpha)$, $\alpha \ne \beta$, the integral operator with the kernel $\psi_1^\alpha(x_\alpha)J_{\alpha\beta}(q_\alpha,q_\beta;E)\psi_1^\beta(x_\beta)$, where $J_{\alpha\beta}(q_\alpha,q_\beta;E)$ is defined by

$$J_{\alpha\beta}(q_\alpha,q_\beta;E) = \tau_{\alpha\beta}\,\zeta_\alpha(q_\alpha;E)(H_0(q_\alpha,q_\beta)+E)^{-1}\zeta_\beta(q_\beta;E) \tag{2.10}$$

with

$$\tau_{\alpha\beta} = 2^{-5/2}\,\pi^{-2}\,(m_\alpha m_\beta)^{-3/4}.$$

Let $\Psi_\alpha : L^2(R^3;dy_\alpha) \to L^2(R^3;dq_\alpha)$ be the Fourier transformation in y_α. We further define $S_{\alpha\beta}(E) : L^2(R^6) \to L^2(R^6)$ by $S_{\alpha\beta}(E) = \Psi_\alpha^*\Pi_{\alpha\beta}(E)\Psi_\beta$, $\alpha \ne \beta$.

Lemma 2.3. $Q^{\pm}_{\alpha\beta}(E) \sim S_{\alpha\beta}(E), \quad \alpha \ne \beta$.

Let $\mathcal{S}(E) : \mathcal{L}^2 \to \mathcal{L}^2$, $\mathcal{L}^2$ being as in Lemma 2.1, be the self–adjoint compact operator defined by

$$\mathcal{S}(E) = \begin{pmatrix} 0 & S_{\alpha\beta}(E) & S_{\alpha\gamma}(E) \\ S_{\beta\alpha}(E) & 0 & S_{\beta\gamma}(E) \\ S_{\gamma\alpha}(E) & S_{\gamma\beta}(E) & 0 \end{pmatrix}, \quad S_{\beta\alpha}(E) = S_{\alpha\beta}(E)^*.$$

Then Lemmas 2.2 and 2.3, together with (2.9), yield that

$$n((1+\epsilon); \mathcal{S}(E)) - C_\epsilon \le N(E) \le n((1-\epsilon); \mathcal{S}(E)) + C_\epsilon$$

for any $\epsilon > 0$ small enough, where $C_\epsilon > 0$ is independent of E. This relation can be easily obtained by use of the Weyl inequality

$$n(\lambda_1 + \lambda_2; A_1 + A_2) \le n(\lambda_1; A_1) + n(\lambda_2; A_2)$$

for the sum of compact operators A_1 and A_2.

(4) The proof of the theorem is completed in this step. Let

$$\mathcal{L}^2(B_1) = \sum_\alpha \oplus L^2(B_1; dq_\alpha), \quad \text{three summands.}$$

We denote by $J_{\alpha\beta}(E) : L^2(B_1; dq_\beta) \to L^2(B_1; dq_\alpha)$ the integral operator with the kernel $J_{\alpha\beta}(q_\alpha, q_\beta; E)$ defined by (2.10), and define the operator $\mathcal{J}_0(E) : \mathcal{L}^2(B_1) \to \mathcal{L}^2(B_1)$ as

$$\mathcal{J}_0(E) = \begin{pmatrix} 0 & J_{\alpha\beta}(E) & J_{\alpha\gamma}(E) \\ J_{\beta\alpha}(E) & 0 & J_{\beta\gamma}(E) \\ J_{\gamma\alpha}(E) & J_{\gamma\beta}(E) & 0 \end{pmatrix}.$$

Then it is easily seen that $n(\mu; \mathcal{S}(E)) = n(\mu; \mathcal{J}_0(E))$ for $\mathcal{S}(E)$ defined above and hence we have

$$n((1+\epsilon); \mathcal{J}_0(E)) - C_\epsilon \le N(E) \le n((1-\epsilon); \mathcal{J}_0(E)) + C_\epsilon. \tag{2.11}$$

The eigenvalue asymptotics for the integral operator $\mathcal{J}_0(E)$ has been in detail studied in Sobolev [7] by employing an argument used in the calculation of the canonical distribution of Toeplitz operators. We here summarize the results obtained there.

Lemma 2.4. *Let $n(\mu; \mathcal{J}_0(E))$ be as above. Then:*

(1) *There exists a limit*

$$\Theta_0(\mu) = \lim_{E \to 0} n(\mu; \mathcal{J}_0(E))/|\log E|$$

as a continuous function of $\mu > 0$.

(2) *The constant $C_0 = \Theta_0(1)$ depends only on the ratios between the masses of three particles under consideration and obeys the lower bound*

$$C_0 > \log 2/2\pi^2 > 0.$$

This lemma, together with relation (2.11), completes the proof of the theorem.

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