Advanced Studies in Pure Mathematics 25, 1997
CR-Geometry and Overdetermined Systems
pp. 1–40

Deformation Theory of CR-Structures and Its Application to Deformations of Isolated Singularities I

Takao Akahori

Introduction

Let (V, o) be a normal isolated singularity in C^N of complex dimension n. We would like to study a deformation theory of complex structures of (V, o). This problem is studied in several ways. For example, (1) Grauert's method (cf. [Gr1]), (2) Douady's method (cf. [Dou]), (3) Kuranishi's approach (cf. [Ku1], [Ku2]), etc. In this paper, we recall Kuranishi's approach and give a review of some contribution, done by T. Akahori and K. Miyajima (cf. [Ku1], [Ku2], [Ak1]-[Ak5], [Ak-My1], [My1]).

Now we set the intersection of V with the real hypersphere centered at o of radius ϵ, namely

$$M = V \cap S_\epsilon^{2N-1}.$$

This M is a non-singular real $2n-1$ dimensional C^∞ manifold, and over this M, a CR structure is induced from V. Namely, ${}^0T'' = C \otimes TM \cap T''N \mid_M$, where $N = V - o$. Conversely, this CR structure $(M, {}^0T'')$ determines the normal Stein space V, uniquely. Noting this result, in order to give a versal family of deformations of singularities, Kuranishi initiated his deformation theory of CR strucutres for a normal isolated singularity. To see Kuranishi's approach and to see our contribution, we recall Kodaira-Spencer's theory for deformation theory of complex structures of compact complex manifolds.

Let X be a complex manifold, and let $(X, T''X)$ denote the complex structure. Then, the deformation theory of complex structures proceeds as follows.

1) *Formulation.* Any deformation of the given complex structure $T''X$, can be parametrized by an element ϕ of $\Gamma(X, T'X \otimes (T''X)^*)$,

Received July 3, 1996
Revised October 31, 1996

which satisfies the deformation equation

$$\overline{\partial}^{(1)}_{T'X}\phi + R_2(\phi) = 0.$$

Here $\overline{\partial}_{T'X}$ means the Cauchy-Riemann operator associated with the holomorphic vector bundle $T'X$. And we have the deformation complex

$$0 \to \Gamma(X, T'X) \overset{\overline{\partial}_{T'X}}{\to} \Gamma(X, T'X \otimes (T''X)^*) \overset{\overline{\partial}_{T'X}}{\to} \cdots$$
$$\overset{\overline{\partial}_{T'X}}{\to} \Gamma(X, T'X \otimes \wedge^p(T''X)^*) \overset{\overline{\partial}_{T'X}}{\to} \Gamma(X, T'X \otimes \wedge^{p+1}(T''X)^*) \overset{\overline{\partial}_{T'X}}{\to} \cdots$$

on X (note that this is an elliptic complex).

Therefore our geometrical problem becomes a problem of a non-linear partial differential equations. To solve this, that is to say, to construct our solutions for this non-linear partial differential equation, there are two methods, namely, Kuranishi's method (see [Ku4]) and Kodaira-Spencer's method (see [Kod]). Kuranishi's method is to give a particular solution space by adding a new equation (Kuranishi's ingenious method; it is $\overline{\partial}^*\phi = 0$ in the compact complex manifold case). This method is applicable in many fields (for example, recent work of Donaldson's (see [Don])). For deformation theory of CR structures, by his method (adding some new equations), Kuranishi gave a special solution space, which is parametrized by $H^1(X, T'X)$ in [Ku1], [Ku2]. Actually, in order to make this special solution space clear, I started my research. On the other hand, Kodaira-Spencer's method is "so-called" power series method and obviously quite elementary. This method is divided into two parts.

2) *Formal Construction.* We construct the formal power series $\phi(t) = \sum \phi_\mu(t)t^\mu$ satisfying;

$$\phi_1(t) = \sum_{\lambda}^{q} \beta_\lambda t_\lambda,$$

where β_λ is a base of $H^1(X, T'X)$ and $q = \dim_C H^1(X, T'X)$, and

$$\overline{\partial}^{(1)}_{T'X}\phi_{\mu+1} + (\overline{\partial}^{(1)}_{T'X}\phi^\mu(t) + R_2(\phi^\mu)) \equiv 0 \bmod \quad t^{\mu+2}.$$

by using the Kodaira-Hodge decomposition theorem for the standard $\overline{\partial}_{T'X}$(this is elliptic).

3) *Convergence.* By using the ellipticity of the standard $\overline{\partial}_{T'X}$, we prove that our $\phi(t)$ converges on $\{t : t \in C^q, \mid t \mid< \epsilon\}$ where ϵ is chosen to be a sufficiently small positive number.

There are several similarities between our case (the case of CR structures over the link $M = V \cap S_\epsilon^{2N-1}$) and the above case (the complex structure case). For example, over the link, we have $\overline{\partial}_b$-operator and T' bundle (correspond to the standard $\overline{\partial}$-operator and the holomorphic tangent bundle). Therefore it is quite natural to try to construct deformation theory of CR structures over the link just like the compact complex manifold case. If we adopt Kodaira-Spencer's method in the CR case, the deformation complex should be

$$0 \to \Gamma(M, T') \overset{\overline{\partial}_{T'}}{\to} \Gamma(M, T' \otimes ({}^0T'')^*) \overset{\overline{\partial}_{T'}}{\to} \cdots$$
$$\overset{\overline{\partial}_{T'}}{\to} \Gamma(M, T' \otimes \wedge^p({}^0T'')^*) \overset{\overline{\partial}_{T'}}{\to} \Gamma(M, T' \otimes \wedge^{p+1}({}^0T'')^*) \overset{\overline{\partial}_{T'}}{\to} \cdots$$

(note that this complex is subellptic), where we denote

$$T' = \overline{{}^0T''} + C\zeta,$$

and ζ is a supplementary vector field of $\overline{{}^0T''} + {}^0T''$ (note that its choice is not canonical), and $\overline{\partial}_{T'}$ is the tangential Cauchy-Riemann operator associated with the holomorphic vector bundle T'. For this $\overline{\partial}_{T'}$, if $\dim_R M = 2n - 1 \geq 5$, we have the Neumann operator N satisfying for $\mu \in \Gamma(M, T' \otimes ({}^0T'')^*)$

$$\mu = \Box_{T'} N_{T'} \mu + H_{T'} \mu,$$

just like the Green operator for a compact complex manifold. However, there is one major difference between them. Even in the strongly pseudo-convex case, only 1/2 estimate holds for the $\overline{\partial}_b$-Neumann problem. Hence the Neumann operator gains only 1 derivative in the strongly pseudo-convex CR manifolds case, in contrast to the compact complex manifolds case where the Green operator gains 2 derivatives. Therefore in proving the convergence of the formal solution, we encounter severe difficulty.

To avoid this difficulty, Kuranishi [Ku4] proceeded as follows: He added a new equation (it resembled $\overline{\partial}^* \phi = 0$, but a complicated one) to the defomation equation, and fortunately this system of partial differential equations can be solved by using the Nash method (it is impossible to be solved by the Banach-inverse mapping theorem) (see [Ku1], [Ku2])). By this method, he obtained a versal family of CR structures. However, because of using Nash-Moser's inverse mapping theorem, he could not put a complex structure over the parameter space of this versal family. In order to improve this point, we proposed a new technique.

Our approach is to follow Kodaira-Spencer's method. Of course, we have to overcome the above analytical difficulty. Here is our approach. Even though the Neumann operator gains only 1 derivative, it still gains 2 derivatives in the direction ${}^0T'' + \overline{{}^0T''}$. So noting this fact, we follow the following line.

(Step 1) We establish the deformation theory of CR structures which vary in the direction ${}^0T'' + \overline{{}^0T''}$.

(Step 2) We obtain a new Neumann type operator which corresponds to Step 1 (obviously, we have to show a new a priori estimate).

This project was succesfully done in the case of $\dim_R M \geq 7$. Our result is that we find a suitable solution for the $\overline{\partial}_{T'}$-equation and fortunately it works well in the deformation theory of CR structures.

This work leads us to a study of the relation between Hodge theory of isolated singularities and deformations of CR structures (cf. [Ak-My2]). This will be discussed in Part II in this book. And there, Miyajima will give an idea about the application of the deformation theory of CR structures to deformations of normal isolated singularities, in the case of $\dim_R M \geq 5$.

§1. Kuranishi's original approach

We start with recalling Kuranishi's original approach to deformation theory of isolated singularities, and discuss several problems, which arose from his work. In the Introduction, we wrote that we improved his result, but from the beginning of our work, it seems that the point of view of Kuranishi is different from ours. Even though in his paper he wrote that he initiated his work in order to construct the versal family of CR structures, his main interest seems to be a geometry of real hypersurfaces (it reminds readers of Cartan's work). With this in mind, we briefly sketch his approach.

1.1. Deformation equation

Let N be a complex n-dimensional manifold. Let M be a real hypersurface of N. Then, a CR-structure ${}^0T''$ on M is induced from the complex structure of N. That is to say,

$$ {}^0T'' = C \otimes TM \cap T''N \mid_M . $$

By using a local coordinate of N, this is explicitly written as follows. We assume that, for a reference point p of M, we take a coordinate neighborhood U of p in N, and a system of complex coordinates $(z^1, \dots, z^n)$.

Let $r = 0$ be a C^∞ defining equation of M (we assume $dr \neq 0$ on M). We use the notation

$$r_j = \partial r/\partial z^j, \qquad r_{\overline{j}} = \partial r/\partial \overline{z}^j. \tag{1.1.1}$$

Then our ${}^0T''$ is written as follows.

$${}^0T'' = \{ \sum_{j=1}^n a^{\overline{j}} \partial/\partial \overline{z}^j \mid \sum_{j=1}^n a^{\overline{j}} r_{\overline{j}} = 0 \}. \tag{1.1.2}$$

We put a hermitian metric on N. With respect to this metric, we consider the dual vector field P' of ∂r (resp. the dual vector field P'' of $\overline{\partial} r$). We set a real supplementary vector field P to ${}^0T'' + \overline{{}^0T''}$ by

$$\sqrt{-1}P = P' - P''. \tag{1.1.3}$$

Now we set for $j = 1, \ldots, n$,

$$Z_{\overline{j}} = \partial/\partial \overline{z}^j - r_{\overline{j}} P''. \tag{1.1.4}$$

Then our ${}^0T''_{|U\cap M}$ is generated by $Z_{\overline{j}}, j = 1, \ldots, n$. If we set

$$P' = \sum_{j=1}^n p^j \frac{\partial}{\partial z^j}, \; P'' = \sum_{j=1}^n p^{\overline{j}} \frac{\partial}{\partial \overline{z}^j},$$

then we have

$$\sum_j p^j h_j = \sum_j p^{\overline{j}} h_{\overline{j}} = 1, \; p^{\overline{j}} = \overline{p^j},$$

and there is one relation among the $Z_{\overline{j}}$'s:

$$\sum_{j=1}^n r_{\overline{j}} Z_{\overline{j}} = 0.$$

Next let

$$Z^{\overline{k}} = i^* d\overline{z}^k - p^{\overline{k}} i^* d'' h$$

where $i : U \cap M \hookrightarrow U$ is the injection and $d''h = \sum_j \frac{\partial}{\partial \overline{z}^j} d\overline{z}^k$. Then $Z^{\overline{1}}, \ldots, Z^{\overline{n}}$ generate $({}^0T'')^*_{|U\cap M}$ and satisfy

$$\sum_{j=1}^n r_{\overline{k}} Z_{\overline{k}} = 0.$$

We set

$$T' = \overline{{}^0T''} + P,$$

and consider the natural isomorphism from T' to $T'N\,|_M$, induced from the inclusion map $T' \hookrightarrow CTM$ and the projection map $CTN\,|_M \to T'N\,|_M$. We use the notation τ for the inverse map of this isomorphism:

$$\tau : T'N\,|_M \to T'.$$

Then an element $\phi \in \Gamma(M, \mathrm{Hom}({}^0T'', (T'N)_{|M}))$ defines a subbundle ${}^\phi T''$ of CTM by

$${}^\phi T'' = \{X - \tau \circ \phi(X) \mid X \in {}^0 T''\}.$$

${}^\phi T''$ is an almost CR structure on M (cf. §2.1). The condition that ${}^\phi T''$ is a CR structure was described by Kuranishi as follows:

Theorem 1.1.1. *(see Theorem 3.1 in [Ku1]) Let $\phi \in \Gamma(M, (T'N)\,|_M \otimes ({}^0T'')^*)$ be sufficiently small so that ${}^\phi T''$ is defined. Let $z = (z_1, \ldots, z_n)$ be a chart of N. Write*

$$\phi = \sum_{k=1}^{k=n} \phi^k \partial/\partial z^k, \; \phi^k = \sum_{l=1}^{l=n} \phi_{\bar{l}}^k Z^{\bar{l}},$$

where $\sum_{l=1}^{k=n} p^{\bar{l}} \phi_{\bar{l}}^k = 0$. Then

$$P(\phi) = \overline{\partial}_b \phi - \sum_{j,k,l} (\partial^\tau \phi_{\bar{l}}^k / \partial z^j) \phi^j \wedge Z^{\bar{l}} (\partial/\partial z^k)$$
$$+ \sum_i h_i \phi^i \wedge \sum_{k,l} (\overline{\partial}_b p^{\bar{l}} - \sum_j (\partial^\tau p^{\bar{l}} / \partial z^j) \phi^j) \phi_{\bar{l}}^k (\partial/\partial z^k)$$

is independent of the choice of the chart z. ${}^\phi T''$ is integrable if and only if

$$P(\phi) = 0$$

(cf. §2.1 for $\overline{\partial}_b$).

1.2. Kuranishi's construction of the versal family of CR structures

As mentioned in the Introduction, our contribution is that we can apply Kodaira-Spencer's method to the deformation theory of CR structures. For the local structure of the "moduli space of CR structures", our contribution would be enough. However, in order to study a global "moduli", we surely have to adopt Kuranishi's line. But (1.2.1) below is not suitable. "A modified new construction" will be necessary, and this would lead to a kind of invariants as in Seiberg-Witten invariants ([Don]) for non-singular compact manifolds. If this is introduced, surely, this invariant must be an invariant of the isolated singularity (V, x). So in order to understand an isolated singularity, our CR geometrical method would give a very important device for isolated singularities. In any case, we recall the family which Kuranishi constructed. Note that an almost CR structure ${}^{\phi}T''$ induces the operator $\overline{\partial}_{{}^{\phi}T''} : \Gamma(M, C) \to \Gamma(M, ({}^{\phi}T'')^*)$. Kuranshi considered the operator $\overline{\partial}^{\phi}{}_b : \Gamma(M, C) \to \Gamma(M, ({}^0T'')^*)$ corresponding to $\overline{\partial}_{{}^{\phi}T''}$ under the natural isomorphism $\lambda^{\phi} :{}^0 T'' \to^{\phi} T''$. Then we consider the set of $\psi \in \Gamma(M, T' \otimes ({}^0T'')^*)$ satisfying

$$
(1.2.1) \qquad \begin{aligned} &\rho^{\psi}\psi = \rho^{\psi}t \\ &N^{\psi}(\overline{\partial}_b^{*\psi}P(\psi) + \overline{\partial}_b^{\psi}\overline{\partial}_b^{\sharp\psi}\psi) = 0. \end{aligned}
$$

Here ρ^{ψ} means the harmonic projection operator with respect to the $\overline{\partial}^{\psi}{}_b$-harmonic theory. We do not explain the notation in detail. See [Ku1] for the precise definitions. If ψ is so small, this set coincides with the set of ψ satisfying

$$
\begin{aligned} P(\psi) &= 0 \\ \overline{\partial}_b^{\sharp\psi}\psi &= 0, \end{aligned}
$$

where $\overline{\partial}_b^{\sharp\phi}$ means the "modified " adjoint operator of $\overline{\partial}^{\phi}$ with respect to the Levi metric.

Therefore the family constructed by Kuranishi seems to be a natural extension of Kuranishi's family in the compact complex manifolds case. We note that, in the compact complex manifold case, Kuranishi gave a complex analytic structure on the set of small $\phi \in A_M^{0,1}(T'M)$ satisfying

$$
\begin{aligned} P(\phi) &= 0 \\ \overline{\partial}_b^{*}\phi &= 0. \end{aligned}
$$

And as mentioned in the Introduction, this method (adding the new equation $\overline{\partial}_b^{*}\phi = 0$) is not available in the CR case (because Kohn's Neumann operator gains only 1 derivative, not like the Green operator). For this reason, even by the Nash-technique, we cannot solve the equation without modifying $\overline{\partial}_b^{*\psi}$. Here, "to solve " means that there is a finite dimensional Euclidean space $\mathcal{H}$ such that a small neighborhood of the origin parametrizes a neighborhood of the solution space of (1.2.1).

1.3. $\mathbf{C}^\infty$ parametrization of real hypersurfaces

We have to explain that the above family of solutions of (1.2.1) has a special geometrical meaning. For this, we must describe the "moduli space" of real hypersurfaces in a complex manifold, which are "very close" to the original real hypersurface M. If M and M' are both real hypersurfaces of the same complex manifold N and close in the C^∞ sense, then M' is called "very close" to M. Kuranishi showed that if a real hypersurface in N is close enough to M, then this real hypersurface corresponds to an element $\zeta \in \Gamma(M, T')$, a T'-valued global vector field. We can obtain this real hypersurface by wiggling the original real hypersurface M in N under a diffeomorphism generated by $\zeta \in \Gamma(M, T')$. In this way, we have a map from a small neighborhood of the origin of $\Gamma(M, T')$ into $\Gamma(M, T' \otimes ({}^0T'')^*)$ such that its linearization is $\overline{\partial}_{T'}$.

1.4. Versality (Equivalence problem)

Now we see the geometrical aspect of (1.2.1). Kuranishi proved the following property ([Ku1]): For any given family of deformations of complex manifold N, denoted N_ω, there is an embedding f of M into N_ω and an element t of $\mathcal{H}$ such that ${}^{\psi(t)}T''$ is "very close" to the structure induced by f. Namely, there is a complex manifold $\mathcal{N}$ with boundary such that there is a smooth map ρ from $\mathcal{N}$ to the interval [0,1] and the boundary $= \rho^{-1}(0) \cup \rho^{-1}(1)$, and

$$\rho^{-1}(0) = (M, {}^{\psi(t)}T'')$$
$$\rho^{-1}(1) = (M, \text{ the CR structure induced from the complex structure } N_\omega \text{ by } f).$$

There are several problems which should be considered in the spirit of the Kuranishi deformation theory of CR structures.

Problem 1. To determine holomorphic convex hulls.

For a subset M of N, in general, it is difficult to determine the holomorphic convex hull $\widetilde{M}$ of M in N. In fact, $\widetilde{M}$ is defined by

$$\widetilde{M} = \{p \ : \ p \in N, \ | f(p) | \leq \sup_{q \in M} | f(q) |, \text{ for any holomorphic function on } N\}.$$

Similarly, the local holomorphic convex hull is defined as follows. For a reference point p of M, and for a neighborhood U of p in N,

$$\widetilde{M \cap U} = \{p \; : \; p \in U, \; | f(p) | \leq \sup_{q \in U \cap M} | f(q) |,$$
$$\text{for any holomorphic function on } U\}.$$

So $\widetilde{M}, \widetilde{M \cap U}$ are defined by highly transcendental method. The problem is to construct holomorphic convex hulls $\widetilde{M}, \widetilde{M \cap U}$ by using deformation theory of M or $M \cap U$ (rather, it is better to say a replacement of M, $M \cap U$ in N respectively) and a geometry of ${}^0T''$ on M. Obviously, this problem is closely related to Kuranishi's problem "to prove Rossi's filling holes theorem by a geometrical method".

Problem 2. Equivalence problem.

The standard equivalence problem of real hypersurfaces were solved by Cartan, Tanaka, Chern-Moser. However, this equivalence is very strong. Namely, let M be a real hypersurface in a complex manifold N, and let M' be a real hypersurface in a complex manifold N'. In Chern-Moser's sense, in the real analytic category, the CR structure on M, induced from N is equivalent to the CR structure on M', induced from N' if and only if there is a biholomorphic map from a neighborhood of M in N to a neighborhood of M' in N'. Our equivalence differs from this. We assume that M, M' are both strongly pseudo-convex. Then, by Rossi's theorem with Stein factorization theorem, we have two Stein spaces, V, V'. The problem is that if V and V' are isomorphic to each other as germs of isolated singularities, is it possible to express this situation in the CR geometrical way? Furthermore, the Stein factorization procedure is highly transcendental. Is it possible to construct a Stein space V from CR geometry on M? So, Problem 2 is somewhat related to Problem 1.

Problem 3. Seiberg-Witten type invariants for isolated singularities.

The reason we posed this problem is that our approach is the so-called coordinate-free approach, so in this sense, our approach seems to be accessible to the introduction of a kind of "Seiberg-Witten invariants" for CR-structures just as in the differential geometric and topological way (though the Levi metric works in the construction). However, what we really need is an invariant for isolated singularities, which characterizes the "global moduli space of isolated singularities". At present, we cannot overcome this difficulty.

From the next section on, we recall our improvement. We hope that our setup could be of any help for the solution of the above mathematical problems.

§2. CR structures and $\overline{\partial}_b$

We begin with recalling the definition of $\overline{\partial}_b$. Sometimes, we use notation different from Kuranishi's one.

2.1. $\overline{\partial}_b$-operator

Let N be a complex manifold of complex dimension n. Let M be a smooth real hypersurface in N. This means that for every point p of M, there is a local defining function ρ of M over a neighborhood of p, satisfying $(d\rho)(p) \neq 0$. Then as is well known, over this M, we can introduce the tangential Cauchy-Riemann structure ${}^0T''$. Namely let

$$ {}^0T'' = C \otimes TM \cap T''N \mid_M . \tag{2.1.1} $$

Then, this ${}^0T''$ satisfies

$$ {}^0T'' \cap \overline{{}^0T''} = 0 \ , \ \dim_C \frac{C \otimes TM}{{}^0T'' + \overline{{}^0T''}} = 1 \tag{2.1.2} $$

$$ [\Gamma(M, {}^0T''), \Gamma(M, {}^0T'')] \subset \Gamma(M, {}^0T''). \tag{2.1.3} $$

And we can define the tangential Cauchy-Riemann operator $\overline{\partial}_b$. Namely, for any C^∞ function f in M, we set an element of $\Gamma(M, ({}^0T'')^*)$ by

$$ \overline{\partial}_b f(X) \ = \ Xf, \ X \ \in \ {}^0T''. $$

And we have a differential complex

$$ \begin{aligned} 0 \to \Gamma(M, C) \to \Gamma(M, ({}^0T'')^*) \to \Gamma(M, \wedge^2({}^0T'')^*) \to \cdots \\ \to \Gamma(M, \wedge^p({}^0T'')^*) \to \Gamma(M, \wedge^{p+1}({}^0T'')^*) \to \cdots . \end{aligned} $$

The explicit form of $\overline{\partial}_b$ is given by

$$ \overline{\partial}_b f = \sum_k \frac{\partial f}{\partial \overline{z}_k} Z^{\overline{k}} $$

in terms of the notation in Sect. 1.1.

This notion of ${}^0T''$ is generalized to an intrinsic structure on M as follows. Let M be a C^∞ manifold with real dimension $2n-1$. We assume that M is orientable. Let E be a subbundle of the complexified tangent bundle $C \otimes TM$ satisfying

$$E \cap \overline{E} = 0, \ \dim_C \frac{C \otimes TM}{E + \overline{E}} = 1, \tag{2.1.4}$$

$$[\Gamma(M, E), \Gamma(M, E)] \subset \Gamma(M, E), \tag{2.1.5}$$

where $\Gamma(M, E)$ denotes the space consisting of E-valued C^∞ sections. E is called a CR structure on M and the pair (M, E) a CR manifold. For a CR manifold (M, E), we can introduce a natural $\overline{\partial}_b$-operator in the same manner as above:

$$\overline{\partial}_b^{(p)} \ : \ \Gamma(M, \wedge^p E^*) \to \Gamma(M, \wedge^{p+1} E^*).$$

If there is no confusion, we abbreviate $\overline{\partial}_b$ for $\overline{\partial}_b^{(p)}$. And we have a differential complex

$$\begin{aligned} 0 \to \Gamma(M, C) \to \Gamma(M, E^*) \to \Gamma(M, \wedge^2 E^*) \to \cdots \\ \to \Gamma(M, \wedge^p E^*) \to \Gamma(M, \wedge^{p+1} E^*) \to \cdots. \end{aligned}$$

For an orientable CR manifold $(M, {}^0T'')$, we set a C^∞ vector bundle decomposition

$$C \otimes TM = {}^0T'' + {}^0T' + C \otimes F, \tag{2.1.6}$$

where ${}^0T' = \overline{{}^0T''}$ and F is a non-vanishing real C^∞ vector field on M satisfying for every point p of M,

$$F_p \notin {}^0T'_p + {}^0T''_p,$$

and $C \otimes F$ means the line bundle generated by F. For each point p of M, we define a hermitian form L_p on ${}^0T''_p$ by

$$L_p(X, Y)F_p = -\sqrt{-1}[X', \overline{Y'}]_{C \otimes F}(p) \qquad \text{for } X,\ Y \in {}^0T''_p \tag{2.1.7}$$

where X', Y' are in $\Gamma(M, {}^0T'')$ such that $X'_p = X$ and $Y'_p = Y$ hold, and $[X', \overline{Y'}]_{C \otimes F}$ denotes the projection of $[X', \overline{Y'}]$ to $C \otimes F$ according to the splitting (2.1.6). L_p is called the Levi-form at p and a CR manifold $(M, {}^0T'')$ is called strongly pseudo- convex if L_p has definite sign at every point p of M.

In the case that $(M, {}^0T'')$ is a CR manifold as in Sect.1.1, we can choose a local coordinate $(z_1, \dots, z_n)$ of N such that

$$r \ = \ 2\mathrm{Im}\ z_n \ - \ h(z_1, \dots, z_{n-1}, \overline{z}_1, \dots, \overline{z}_{n-1}, \mathrm{Re}\ z_n)$$

where h is a real valued C^∞ function satisfying grad $h(p) = 0$. Then $(M, {}^0T'')$ is strongly psudoconvex if and only if the complex Hessian $(\partial^2 h/\partial z_i \partial \overline{z}_j \ (p))_{1\le i,j\le n-1}$ is positive or negative definite.

2.2. T'-bundle and $\overline{\partial}_{T'}$-operator

Let $(M, {}^0T'')$ be an orientable CR manifold and fix the splitting (2.1.6). We set

$$T' = {}^0T' + C \otimes F.$$

Then, this T'-bundle admits a CR structure in the following sense. For u in $\Gamma(M, T')$, we set a first order differential operator

$$\overline{\partial}_{T'} \ : \ \Gamma(M, T') \ \to \ \Gamma(M, T' \otimes ({}^0T'')^*)$$

by $\overline{\partial}_{T'} u(X) = [X, u]_{T'}$ for $X \in {}^0T''$. We have to explain this definition more precisely. For each point p of M, for $X \in {}^0T''_p$, we take $X' \in \Gamma(M, {}^0T'')$ satisfying

$$X'_p = X.$$

$\overline{\partial}_{T'} u(X)$ is determined by

$$\overline{\partial}_{T'} u(X) = [X', u]_{T'},$$

where $[X', u]_{T'}$ means the T'-part of $[X', u]$ according to the splitting (2.1.6). Obviously, this definition makes sense. Because for any C^∞ function, for any $Z \in \Gamma(M, {}^0T'')$, and for any $u \in \Gamma(M, T')$,

$$\begin{aligned} [fZ, u]_{T'} &= (-u(f)Z + f[Z, u])_{T'} \\ &= f[Z, u]_{T'}. \end{aligned}$$

This means that our definition does not depend on the C^∞ extension of X. As for scalar valued differential forms, we can define $\overline{\partial}^{(p)}_{T'}$-operator. For example, for $\phi \in \Gamma(M, T' \otimes ({}^0T'')^*)$,

$$\overline{\partial}^{(1)}_{T'} \phi(X, Y) = [X, \phi(Y)]_{T'} - [Y, \phi(X)]_{T'} - \phi([X, Y]) \ for \ X, Y \in {}^0 T'' .$$

Then it satisfies $\overline{\partial}^{(1)}_{T'} \overline{\partial}_{T'} = 0$ (hence T' is a holomorphic vector bundle over a CR manifold in N. Tanaka's sense ([Ta])). And we have a differential complex

$$\begin{aligned} 0 \to \Gamma(M, T') \to \Gamma(M, T' \otimes ({}^0T'')^*) \to \Gamma(M, T' \otimes \wedge^2 ({}^0T'')^*) \to \cdots \\ \to \Gamma(M, T' \otimes \wedge^p ({}^0T'')^*) \to \Gamma(M, T' \otimes \wedge^{p+1} ({}^0T'')^*) \to \cdots . \end{aligned}$$

We note that the T'-bundle is a generalization of the holomorphic tangent bundle. In fact, if M is a real hypersurface in a complex manifold N, we consider

$$T'N\mid_M,$$

the restriction of the holomorphic tangent bundle $T'N$ to the real hypersurface M. Then the composite of the inclusion map of T' to $C\otimes TM \subset C\otimes TN$ and the projection of $C\otimes TN$ to T'', induces an isomorphism i from T' to $T'N\mid_M$ and preserves

$$\overline{\partial}_{T'N}i(u(X)) = i((\overline{\partial}_{T'}u)(X)), \qquad \text{for} \quad X \in {}^0T'',$$

if u satisfies some conditions (this will be discussed in §4.1), where $\overline{\partial}_{T'N}$ means the standard $\overline{\partial}$- operator on N, and for $X \in {}^0T''$, the left hand side makes sense.

§3. Geometry on deformations of CR manifolds

In this section, we briefly recall the deformation theory of strongly pseudo-convex CR-structures. Throughout this section, $(M,{}^0T'')$ is a strongly pseudo-convex compact CR manifold and we fix the splitting (2.1.6). For the detailed discussion, see [Ak1], [Ak2], [Ak3].

3.1. Almost CR manifolds

Let E be an almost CR structure on M. Then, by using the C^∞ vector bundle decomposition (2.1.6), we have a homomorphism from E to ${}^0T''$, the composite of the inclusion of E to $C\otimes TM$ and the projection of $C\otimes TM$ to ${}^0T''$.

Definition 3.1.1. *Let $(M,{}^0T'')$ be a CR manifold. An almost CR structure E is of finite distance from $(M,{}^0T'')$ if the above homomorphism is an isomorphism.*

Proposition 3.1.2. *Let $(M,{}^0T'')$ be a CR manifold and E an almost CR structure of finite distance from ${}^0T''$. Then there exists a $\phi \in \Gamma(M, T' \otimes ({}^0T'')^*)$ satisfying*

$$\begin{aligned} E =& {}^\phi T'' \\ =& \{X'; X' = X + \phi(X), X \in {}^0T''\}. \end{aligned}$$

Namely, ϕ defines a bundle homomorphism ${}^0T'' \to T'$ whose graph coincides with E. For the proof, see [Ak1].

3.2. Deformation equation

By Proposition 3.1,2, we see that for a given CR manifold $(M,{}^0T'')$, an almost CR manifold of finite distance from $(M,{}^0T'')$ is parametrized by $\Gamma(M,T'\otimes({}^0T'')^*)$. Now, in this section, we see under what condition this ${}^\phi T''$ is actually a CR structure. For this, we have to introduce notation. Let ϕ be an element of $\Gamma(M,T'\otimes({}^0T'')^*)$. We set elements,$R_2(\phi)$ and $R_3(\phi)$ of $\Gamma(M,T'\otimes\wedge^2({}^0T'')^*)$ by

(3.2.1)
$$R_2(\phi)(X,Y)=[\phi(X),\phi(Y)]_{T'}-\phi([X,\phi(Y)]_{{}^0T''}+[\phi(X),Y]_{{}^0T''})$$
(3.2.2)
$$R_3(\phi)(X,Y)=\phi([\phi(X),\phi(Y)]_{{}^0T''})$$

for X,Y in $\Gamma(M,{}^0T'')$.

We see that these $R_2(\phi)$,$R_3(\phi)$ make sense as elements of $\Gamma(M,T'\otimes\wedge^2({}^0T'')^*)$. In fact, for any C^∞ functions f and g, and $X',Y'\in\Gamma(M,{}^0T'')$, by a simple direct computation of brackets, we have

(3.2.3)
$$\begin{aligned}&R_2(\phi)(fX',gY')\\&\quad=[\phi(fX'),\phi(gY')]_{T'}-\phi([fX',\phi(gY')]_{{}^0T''}+[\phi(fX'),gY']_{{}^0T''}\\&\quad=fg\{[\phi(X'),\phi(Y')]_{T'}-\phi([X',\phi(Y')]_{{}^0T''})+[\phi(X'),Y']_{{}^0T''})\}.\end{aligned}$$

By (3.2.3), $R_2(\phi)(X'_p,Y'_p)$ depends only on X_p and Y_p. Hence $R_2(\phi)$ is an element of $\Gamma(M,T'\otimes\wedge^2({}^0T'')^*)$. Obviously $R_3(\phi)\in\Gamma(M,T'\otimes\wedge^2({}^0T'')^*)$ holds for the same reason.

In this notation, we have

Proposition 3.2.1. *(see Theorem 2.1 in [Ak1]) Let ϕ be an element of $\Gamma(M,T'\otimes({}^0T'')^*)$. Then an almost CR structure $(M,{}^\phi T'')$ is a CR structure if and only if ϕ satisfies the following non-linear equation:*

$$\begin{aligned}P(\phi)&=\overline{\partial}^{(1)}_{T'}\phi+R_2(\phi)+R_3(\phi)\\&=0.\end{aligned}\tag{3.2.4}$$

3.3. E_j-structures

Now we recall the subbundles E_j, which played quite successful roles in deformation theory of CR-structures. We set a subspace Γ_i of $\Gamma(M,T'\otimes\wedge^i({}^0T'')^*)$ by

$$\Gamma_i=\{\ u\ ;u\in\Gamma(M,{}^0T'\otimes\wedge^i({}^0T'')^*),(\overline{\partial}^{(i)}_{T'}u)_{C\otimes F}=0\},$$

where $(\overline{\partial}_{T'}^{(i)} u)_{C\otimes F}$ denotes the projection of $\overline{\partial}_{T'}^{(i)} u$ to $C \otimes F \otimes \wedge^{i+1}({}^0T'')$ according to (2.2.1). Then we have

Theorem 3.3.1. *(see Proposition 2.1 in [Ak3]) There is a subbundle E_i of $T' \otimes \wedge^i({}^0T'')^*$ satisfying*

$$\Gamma_i = \Gamma(M, E_i).$$

And there is a differential subcomplex

$$0 \to \Gamma(M,E)_1) \xrightarrow{\overline{\partial}_1} \Gamma(M,E_2) \xrightarrow{\overline{\partial}_2} \Gamma(M,E_3) \xrightarrow{\overline{\partial}_3} \cdots$$
$$\xrightarrow{\overline{\partial}_{i-1}} \Gamma(M,E_i) \xrightarrow{\overline{\partial}_i} \Gamma(M,E_{i+1}) \xrightarrow{\overline{\partial}_{i+1}} \cdots$$

where $\overline{\partial}_i$ means the restriction of $\overline{\partial}_{T'}^{(i)}$ to $\Gamma(M,E_i)$.

By Theorem 3.3.1 we have an injection $i : \text{Ker } \overline{\partial}_i \hookrightarrow \text{Ker } \overline{\partial}_{T'}^{(i)}$.

Theorem 3.3.2. *(see Theorems 2.3 and 2.4 in [Ak3]) The injection induces an isomorphism*

$$i: \text{ Ker } \overline{\partial}_i / \text{Im } \overline{\partial}_{i-1} \to \text{Ker } \overline{\partial}_{T'}^{(i)} / \text{Im } \overline{\partial}_{T'}^{(i-1)},$$

where $2 \leq i \leq n-1$, and the surjective map

$$i : \text{Ker } \overline{\partial}_1 \to \text{Ker } \overline{\partial}_{T'}^{(1)} / \text{Im } \overline{\partial}_{T'}^{(0)}.$$

3.4. Local expression for $\mathbf{E_j}$.

The explicit expression for the differential complex $(\Gamma(M,E_i), \overline{\partial}_i)$ is as follows. We briefly recall only the results. For the proof see [Ak2], [Ak3].

Let $\{U_k, h_k\}_{k\in K}$ be a local coordinate covering of M such that K is a finite set and U_k is homeomorphic to R^{2n-1}. And let $\{\rho_k\}_{k\in K}$ be a partition of unity subordinate to the coordinate covering of M. Since $(M, {}^0T'')$ is strongly pseudo-convex, there exists a moving frame $\{e_1^k, ., , e_{n-1}^k\}$ of ${}^0T'' \mid_{U_k}$ such that

$$[e_i^k, \overline{e}_j^k]_{C\otimes F} = \sqrt{-1}\delta_{i,j}F.$$

By using these frames, we have the following lemmas. For the proof, see Lemmas 3.1–3.4 in [Ak3] respectively.

Lemma 3.4.1. *Let ϕ be an element of $\Gamma(M, {}^0T' \otimes ({}^0T'')^*)$. Then ϕ belongs to $\Gamma(M, E_1)$ if and only if*

$$\phi^k_{i,j} - \phi^k_{j,i} = 0 \ for \ 1 \leq i, j \leq n-1,$$

where by $\phi^k_{i,j}$ we denote the C^∞-functions defined by

$$\phi(e^k_j) = \sum_i \phi^k_{i,j} \bar{e}^k_i.$$

As for the vector bundle E_2, we have the following lemma.

Lemma 3.4.2. *Let ϕ be an element of $\Gamma(M, {}^0T' \otimes \wedge^2({}^0T'')^*)$. Then ϕ belongs to $\Gamma(M, E_2)$ if and only if*

$$\phi^k_{i,(j,\alpha)} - \phi^k_{j,(i,\alpha)} + \phi^k_{\alpha,(i,j)} = 0$$

for all i,j,α satisfying $1 \leq i, j, \alpha \leq n-1$, where $\phi^k_{\alpha,(i,j)}$ denotes the C^∞-function defined by

$$\phi(e^k_i, e^k_j) = \sum_\alpha \phi^k_{\alpha,(i,j)} \bar{e}^k_\alpha.$$

Lemma 3.4.3. *For $\phi \in \Gamma(M, E_1)$, we have*

$$(\bar{\partial}_1 \phi)^k_{\alpha,(i,j)} = e^k_i \phi^k_{\alpha,j} - e^k_j \phi^k_{\alpha,i} + the \ terms \ of \ order \ zero \ of \ \phi$$

and for $\mu \in (M, E_2)$,

$$(\bar{\partial}_2 \mu)^k_{\alpha,(i,j,l)} = e^k_i \mu^k_{\alpha,(j,l)} - e^k_j \mu^k_{\alpha,(i,l)} + e^k_l \mu^k_{\alpha,(i,j)} \\ + the \ terms \ of \ order \ zero \ of \ \mu.$$

We put the inner product on $\Gamma(M, E_1)$, induced by the Levi metric. Let $\bar{\partial}^*_1$ denotes the adjoint operator of $\bar{\partial}_1$. Then the following lemma follows from these lemmas. Here we remark that $\{e^k_1, e^k_2, ., e^k_{n-1}\}$ are orthonormal with respect to this inner product.

Lemma 3.4.4. *For all ϕ in $\Gamma(M, E_1)$, $\bar{\partial}^*_1$ can be expressed by*

$$(\bar{\partial}^*_1 \phi)^k_{\alpha,i} = -\frac{1}{2} \sum_j \bar{e}^k_j \phi^k_{\alpha,(i,j)} - \frac{1}{2} \sum_j \bar{e}^k_j \phi^k_{i,(j,\alpha)} \\ + the \ terms \ of \ order \ 0 \ of \ \phi \ .$$

3.5. An a priori estimate.

First, we introduce new norms $\|\phi\|'_{(m)}$ and $\|\phi\|''_{(m)}$ on $\Gamma(M, E_p)$. Let $\{U_k, h_k\}_{k\in K}$ be a local coordinate covering of M, $\{\rho_k\}_{k\in K}$ be a partition of unity subordinate to this coordinate covering, and $\{e^k_1, \cdot, , e^k_{n-1}\}$ the moving frame of ${}^0T'' \mid_{U_k}$ as in §3.4. For $\phi \in \Gamma(M, E_p)$ and $I = (i_1, \ldots, i_p)$, a C^∞-funciton $\phi^k_{\alpha,I}$ on U_k is defined by

$$\phi(e^k_{i_1}, \ldots, e^k_{i_p}) = \sum_\alpha \phi^k_{\alpha,I}\bar{e}^k_\alpha.$$

Then we define the norms $\|\phi\|'_{(m)}$ and $\|\phi\|''_{(m)}$ respectively by

$$\begin{aligned}
\|\phi\|'^2_{(m)} = & \sum_{k\in K',i,\alpha,I} \|(\rho_k e^k_i \phi^k_{\alpha,I})h_k^{-1}\|^2_{(m)} \\
& + \sum_{k\in K',i,\alpha,I} \|(\rho_k \overline{e^k_i} \phi^k_{\alpha,I})h_k^{-1}\|^2_{(m)} \\
& + \|\phi\|^2_{(m)}, \\
\|\phi\|''^2_{(m)} = & \sum_{k\in K',i,j,\alpha,I} \|(\rho_k e^k_i e^k_j \phi^k_{\alpha,I})h_k^{-1}\|^2_{(m)} \\
& + \sum_{k\in K',i,j,\alpha,I} \|(\rho_k e^k_i \overline{e^k_j} \phi^k_{\alpha,I})h_k^{-1}\|^2_{(m)} \\
& + \sum_{k\in K',i,j,\alpha,I} \|(\rho_k \overline{e^k_i} e^k_j \phi^k_{\alpha,I})h_k^{-1}\|^2_{(m)} \\
& + \sum_{k\in K',i,j,\alpha,I} \|(\rho_k \overline{e^k_i e^k_j} \phi^k_{\alpha,I})h_k^{-1}\|^2_{(m)} \\
& + \|\phi\|^2_{(m+1)}.
\end{aligned}$$

By direct computation (using integration by parts), we can prove the following theorem. For the notation, for example, $\| \ \|'_{(m)}$-norms, and $\| \ \|''_{(m)}$-norms, see [Ak3].

Theorem 3.5.1. *(see Theorem 4.1(new estimate) in [Ak3]). Suppose that $(M, {}^0T'')$ is strongly pseudo convex and $\dim_R M = 2n-1 \geq 7$. Then the following estimate holds.*

$$\|\bar{\partial}_1^*\phi\|^2_{(0)} + \|\bar{\partial}_2\phi\|^2_{(0)} + \|\phi\|^2_{(0)} \geq C\|\phi\|'^2_{(0)}$$

for all $\phi \in \Gamma(M, E_2)$, where C is a positive constant.

Following the standard functional analysis method with Theorem 3.5.1, we have Theorem 3.5.2.

Theorem 3.5.2. *(see Theorem 4.1 in [Ak3]). Under the assumption of Theorem 3.5.1, we have a Neumann operator*

$$N : \Gamma_2(M, E_2) \to \Gamma_2(M, E_2)$$

such that

a) N is bounded,

b) if $\phi \in \Gamma(M, E_2)$, $N\phi$ is also in $\Gamma(M, E_2)$,

c) if $N\phi = 0$, $\phi \in \Gamma(M, E_2)$, then $\overline{\partial}\phi = 0$, and $\overline{\partial}^*\phi = 0$, and

d) if $\phi \in \Gamma(M, E_2)$, then $\phi = \overline{\partial}\overline{\partial}^* N\phi + \overline{\partial}^*\overline{\partial} N\phi + \alpha$, $\alpha \in \mathbf{H}$ where $\mathbf{H}$ is the null space of N.

Here we use the notation $\Gamma_2(M, E_2)$ for the Hilbert space obtained as the completion of $\Gamma(M, E_2)$ with respect to the L_2-norm.

3.6. Some estimates.

In this section, we recall some a priori estimates for the Neumann operator obtained in §3.5. By the standard argument, we have the following estimate from Theorem 3.5.1.

$$\|\phi\|'_{(m-1/2)} \leq C_m\{\|\overline{\partial}_2\phi\|_{(m)} + \|\overline{\partial}_1^*\phi\|_{(m)} + \|\phi\|_{(m)}\} \tag{3.6.1}$$

for all $\phi \in \Gamma(M, E_2)$, and

$$\|\phi\|'_{(m+1/2)} \leq C'_m\{\|\Box\phi\|_{(m-1/2)} + \|\phi\|_{(m-1/2)}\}, \tag{3.6.2}$$

for all $\phi \in \Gamma(M, E_2)$, where $\Box = \overline{\partial}_1\overline{\partial}_1^* + \overline{\partial}_2^*\overline{\partial}_2$.

More precisely, we have the following theorem.

Theorem 3.6.1. *(see Theorem 5.1 in [Ak3]). The following estimate holds:*

$$\|\phi\|''_{(m-1/2)} \leq C''_m\{\|\Box\phi\|_{(m-1/2)} + \|\phi\|_{(m-1/2)}\}$$

for all $\phi \in \Gamma(M, E_2)$, where m is a non-negative integer.

Thus we have

Corollary 3.6.2. *(see Corollary 5.2 in [Ak3]).*

$$\|N\mu\|''_{(m-1/2)} \leq C'_m \|\mu\|_{(m-1/2)}$$

for all $\mu \in \Gamma(M, E_2)$, where m is a non-negative integer.

Lemma 3.6.3. *(see Lemma 5.3 in [Ak3]). The following estimate holds.*

$$\|R_2(\phi)\|_{(m-1/2)} \leq C_m \|\phi\|'^2_{(m-1/2)},$$

for all $\phi \in \Gamma(M, E_2)$, where we assume $m \geq n+1$.

For the definition of $R_2(\phi)$, see (3.2.1) in this paper.

Proposition 3.6.4. *(see Proposition 5.4 in [Ak3]). Suppose that $R_2(\phi)$ is in $\Gamma(M, E_2)$. Then,*

$$\|\overline{\partial}_1^* N R_2(\phi)\|'_{(m-1/2)} \leq C_m \|\phi\|'^2_{(m-1/2)} \text{ for all } \phi \in \Gamma(M, E_2)$$

holds.

3.7. An application to the deformation theory of CR structures.

Let $(M, {}^0T'')$ be a compact strongly pseudo-convex CR manifold. By using the differential complex in Theorem 3.3.1, and a new Hogde type decomposition theorem in Theorem 3.5.2, we can discuss the deformation theory of CR structures.

Then, we have

Theorem 3.7.1. *(see Theorem 6.2 in [Ak3]). Under the assumption $\dim_R M = 2n-1 \geq 7$ and $H^2(M, T') = 0$, there is an E_1-valued C^2-class section $\phi(t)$, parametrized complex analytically by a neighborhood U of the origin in the Euclidean space $\mathcal{H}$, satisfying*

(1) $\phi(o) = 0$
(2) $P(\phi(t)) = \overline{\partial}_{T'}^{(1)} \phi(t) + R_2(\phi(t)) = 0$, *and*
(3) *the linear term of $\phi(t)$ is equal to $\sum_{\lambda=1}^q \beta_\lambda t_\lambda$, where $\{\beta_\lambda\}_{1 \leq \lambda \leq q}$ is a basic system of $\mathcal{H}$, $q = \dim_C \mathcal{H}$ and $\{t_i\}_{1 \leq i \leq q}$ are local coordinates of U.*

Here $\mathcal{H}$ is a subspace of $\Gamma(M, E_1)$ such that $\mathcal{H} \simeq \mathrm{Ker}\ \overline{\partial}_{T'}^{(1)} / \mathrm{Im}\ \overline{\partial}_{T'}^{(0)}$ holds (cf. Theorem 3.3.2) and m is a sufficiently large integer such that $m \geq n+2$ holds. (Note that $R_3(\phi) = 0$ holds for $\phi \in \Gamma(M, E_1)$.)

This theorem is proved by the standard Kodaira-Spencer deformation theory using Lemma 3.7.3 and Proposition 3.7.4 below. Since its argument is a prototype of the ones in Section 4 and in Part II, we will give a brief sketch of the proof.

A sketch of the proof. Let $\phi(t)$ be a $\Gamma(M, E_1)$-valued holomorphic function and

$$\phi(t) = \sum \phi_{k_1 k_2 \ldots k_q} t_1^{k_1} \ldots t_q^{k_q}$$

be the power series expansion of $\phi(t)$ with $\phi(0) = 0$. For simplicity, we abbreviate

$$\phi(t) = \sum_{\lambda=1}^{\infty} \phi_\lambda(t),$$

where $\phi_\lambda(t)$ is a homogeneous polynomial of degree λ in $(t_1, .., t_q)$. Let

$$\phi^\mu(t) = \sum_{\lambda=1}^{\mu} \phi_\lambda(t).$$

For any $\Gamma(M, E_1)$-valued holomorphic functions $\phi(t)$ and $\psi(t)$, we indicate by $\phi(t) \equiv_\mu \psi(t)$ that the power series expansion of $\phi(t) - \psi(t)$ in $(t_1, .., t_q)$ contains no term of degree $\lambda < \mu$.

Clearly the conditions (1) and (2) are equivalent to the system of congruence

$$(3.7.1)_\mu \qquad \overline{\partial}_{T'}^{(1)} \phi(t) + R_2(\phi(t)) \equiv_{\mu+1} 0 \ (\mu = 1, 2, \ldots).$$

Since $R_2(\phi(t)$ is of second order with respect to $\phi(t)$, we obtain

$$(3.7.2) \qquad R_2(\phi(t)) \equiv_{\mu+1} R_2(\phi^{\mu-1}(t)).$$

Hence we can rewrite $(3.7.1)_\mu$ as follows:

$$(3.7.3)_\mu \qquad \overline{\partial}_{T'}^{(1)} \phi_\mu(t) + R_2(\phi^{\mu-1}(t)) \equiv_{\mu+1} 0 \ (\mu = 1, 2, \ldots).$$

Further, these are equivalent to the following:

$$(3.7.4)_\mu \qquad \overline{\partial}_{T'}^{(1)} \phi_\mu(t) + P(\phi^{\mu-1}(t)) \equiv_{\mu+1} 0 \ (\mu = 1, 2, \ldots)$$

because of $\phi^\mu(t) = \phi_\mu(t) + \phi^{\mu-1}(t)$ and $P(\phi^{\mu-1}(t)) = \overline{\partial}_{T'}^{(1)} \phi^{\mu-1}(t) + R_2(\phi^{\mu-1}(t))$.

Now we shall construct $\phi(t)$ by induction on μ. We set $\phi_0 = 0$ and $\phi_1(t) = \sum_{\lambda=1}^{q} \beta_\lambda t_\lambda$. Then clearly $(3.7.5)_1$ holds.

Suppose that $\phi^{\mu-1}(t)$ is already determined and satisfies $(3.7.4)_{\mu-1}$. Then we will study the differential equation $(3.7.4)_\mu$.

$$(3.7.5)_\mu \qquad \overline{\partial}_{T'}^{(1)} \phi_\mu(t) + P(\phi^{\mu-1}(t)) \equiv_{\mu+1} 0.$$

We recall Theorem 4.10 in [Ak1]. (We note that this lemma holds for any twice continuously differentiable ϕ.)

Lemma 3.7.2. *(see Theorem 4.10 in [Ak1]). For any element $\phi \in \Gamma(M, T')$,*

$$\overline{\partial}^{\phi}_{T'} P(\phi) = 0.$$

From the assumption $P(\phi^{\mu-1}(t)) \equiv_\mu 0$ and Lemma 3.7.2, we obtain

$$\overline{\partial}^{(2)}_{T'} (P(\phi^{\mu-1}(t)) \equiv_{\mu+1} \overline{\partial}^{\phi^{\mu-1}(t)}_{T'} (P(\phi^{\mu-1}(t))) = 0. \tag{3.7.5}$$

Hence, under the assumption $H^2(M, T') = 0$, the partial differential equation $(3.7.4)_\mu$ has a solution in $\Gamma(M, T' \otimes ({}^0T'')^*)$. And the following proposition enables us to choose the solution relying on the Hodge decomposition in Theorem 3.5.2, that assures $\phi_\mu(t) \in \Gamma(M, E_1)$.

Proposition 3.7.3. *Given a $\Gamma(M, E_1)$-valued polynomial $\phi^{\mu-1}(t)$ in $(t_1, .., t_q)$ satisfying $P(\phi^{\mu-1}(t)) \equiv_\mu 0$, the homogeneous part of degree μ in $(t_1, \ldots, t_q)$ of $P(\phi^{\mu-1}(t))$ takes its value in $\Gamma(M, E_2)$.*

Hence, if we set

$$\phi_\mu(t) = -\overline{\partial}^* N\{\text{the } \mu\text{-th homogeous polynomial term of } P(\phi^{\mu-1}(t))\},$$

$\phi^{\mu-1}(t) + \phi_\mu(t)$ satisfies $(3.7.1)_\mu$, where N denotes the new Neumann operator obtained in Theorem 3.5.2.

The convergence of

$$\phi(t) = \phi_1(t) + \phi_2(t) + \ldots$$

is proved by the standard Kodaira-Spencer argument: For two powerseries

$$A(t) = \sum_{\nu=(\nu_1,\ldots,\nu_q)} a_\nu t_1^{\nu_1} \ldots t_q^{\nu_q}$$

and

$$B(t) = \sum_{\nu=(\nu_1,\ldots,\nu_q)} b_\nu t_1^{\nu_1} \ldots t_q^{\nu_q},$$

we denote

$$A(t) << B(t)$$

if $|a_\nu| \leq |b_\nu|$ holds for all ν. Let

$$A(t) := \frac{b}{16c} \sum_{\mu \geq 1} \frac{c^\mu}{\mu^2} (t_1 + \cdots + t_q)^\mu$$

be a convergent power series where b and c are positive constants. Then we have

$$||\phi_\mu(t)||'_{(m-\frac{1}{2})} << ||\overline{\partial}^* NR_2(\phi^{(\mu-1)}(t))||'_{(m-\frac{1}{2})} << C||\phi^{(\mu-1)}(t)||'^2_{(m-\frac{1}{2})}$$

by Proposition 3.6.4. Hence

$$||\phi_\mu(t)||'_{(m-\frac{1}{2})} << A(t) \qquad \text{follows from} \qquad ||\phi^{(\mu-1)}(t)||'_{(m-\frac{1}{2})} << A(t)$$

if we choose b and c sufficiently large at the beginning, because

$$||\phi_\mu(t)||'_{(m-\frac{1}{2})} << C||\phi^{(\mu-1)}(t)||'^2_{(m-\frac{1}{2})} << CA(t)^2 << \frac{b}{c}CA(t)$$

holds (remark that $A(t)^2 << \frac{b}{c}A(t)$ holds (cf. (5.116) in [Ko])).

We note that the assumption $H^2(M,T') = 0$ is not essential. We will discuss in Part II the case of $H^2(M,T') \neq 0$ and the completeness of $\phi(t)$ (which is called the Kuranishi versality). In any case, we have

Corollary 3.7.4. *The deformation $\phi(t)$ constructed in Theorem 3.7.1 is versal.*

§4. Geometry of the deformations of strongly pseudo-convex domains

4.1. $T'N$-valued complex.

Let N be a complex manifold and Ω be a relatively compact strongly pseudo-convex subdomain of N. We assume $\dim_C N \geq 4$. Let $T'N$ be the holomorphic tangent bundle on N. Then, there is a first order differential operator $\overline{\partial}_{T'N}$ from $\Gamma(\overline{\Omega}, T'N)$ to $\Gamma(\overline{\Omega}, T'N \otimes (T''N)^*)$, where $\Gamma(\overline{\Omega}, T'N)$ denotes the space of $T'N$-valued sections smooth up to the boundary $b\Omega$. Namely, for u in $\Gamma(\overline{\Omega}, T'N)$,

$$\overline{\partial}_{T'N} u(X) = [X, u]_{T'N},$$

where $X \in \Gamma(\overline{\Omega}, T''N)$, and $[X,u]_{T'N}$ denotes the projection to $T'N$ according to the decomposition of the vector bundle $C \otimes TN = T'N + T''N$. Then, as is well known, we can define a first order differential operator $\overline{\partial}^{(p)}_{T'N}$ from $\Gamma(\overline{\Omega}, T'N \otimes \wedge^p (T''N)^*)$ to $\Gamma(\overline{\Omega}, T'N \otimes \wedge^{p+1} (T''N)^*)$

in the same way as for scalar valued forms, and we have a differential complex.

$$0 \to \Gamma(\overline{\Omega}, T'N) \to \Gamma(\overline{\Omega}, T'N \otimes (T''N)^*) \to \Gamma(\overline{\Omega}, T'N \otimes \wedge^2(T''N)^*) \to$$
$$\to \Gamma(\overline{\Omega}, T'N \otimes \wedge^p(T''N)^*) \to \Gamma(\overline{\Omega}, T'N \otimes \wedge^{p+1}(T''N)^*) \to,$$

while we have the restriction map τ_p

$$\tau_p : \Gamma(\overline{\Omega}, T'N \otimes \wedge^p(T''N)^*) \ \to \Gamma(b\Omega, T' \otimes \wedge^p({}^0T'')^*),$$

given by

$$\tau_p\phi(X_1, .., X_p) = (i)^{-1}(\phi(X_1, .., X_p)) \quad \text{for} \quad X_j \in^0 T'',$$

where $i : T' \to T'N_{|M}$ denotes the isomorphism in §3.2. Henceforth, we abbreviate τ for τ_p. Then, we have

Lemma 4.1.1. *(see Lemma 1.1 in [Ak4]) Let ϕ be an element of $\Gamma(\overline{\Omega}, T'N \otimes \wedge^p(T''N)^*)$ satisfying*

$$\tau\phi \in \Gamma(b\Omega,{}^0 T' \otimes \wedge^p({}^0T'')^*) \ \textit{and} \ \overline{\partial}^{(p)}_{T'}\tau\phi \in \Gamma(b\Omega,{}^0 T' \otimes \wedge^{p+1}({}^0T'')^*).$$

Then,

$$\tau(\overline{\partial}^{(p)}_{T'}\phi) = \overline{\partial}^{(p)}_{T'}(\tau\phi).$$

Similarly, we have the following lemma.

Lemma 4.1.2. *(see Lemma 1.2 in [Ak4]) Let ϕ be an element of $\Gamma(\overline{\Omega}, T'N \otimes \wedge^p(T''N)^*)$ satisfying $\tau\phi \in \Gamma(b\Omega,{}^0 T' \otimes \wedge^p({}^0T'')^*)$ and $\tau(\overline{\partial}^{(p)}_{T'N}\phi) \in \Gamma(b\Omega,{}^0 T' \otimes \wedge^{p+1}({}^0T'')^*)$. Then,*

$$\tau(\overline{\partial}^{(p)}_{T'N}\phi) = \overline{\partial}^{(p)}_{T'}(\tau\phi).$$

4.2. Almost complex manifolds and deformation equation

In this section, we recall the deformation theory of complex structures and the deformation equation.

Let N be a C^∞ differentiable manifold of real dimension $2n$. Let E be a C^∞ subvector bundle of the complexified tangent bundle $C \otimes TN$ satisfying

$$C \otimes TN = E \oplus \overline{E}.$$

E is called an almost complex structure and the pair (N,E) an almost complex manifold. Now let $(N, T''N)$ be a complex manifold. Then, by using the canonical decomposition

$$C \otimes TN = T'N + T''N,$$

we have a homomorphism from E to $T''N$, the composite of the inclusion of E to $C \otimes TN$ and the projection of $C \otimes TN$ to $T''N$.

Definition 4.2.1.. *Let $(N, T''N)$ be a complex manifold and E an almost complex structure. E is of finite distance from $T''N$ if the above homomorhism is an isomorphism.*

Then, we have

Proposition 4.2.2. *If E is an almost complex manifold of finite distance from $T''N$, then there is a $\phi \in \Gamma(N, T'N \otimes (T''N)^*)$ satisfying*

$$\begin{aligned} E =& {}^{\phi}T''N \\ =&\{X'; X' = X + \phi(X), X \in T''N\}. \end{aligned}$$

By Proposition 4.2.2, we see that for a given CR manifold $(M, {}^0T'')$, almost CR manifolds of finite distance from $(M, {}^0T'')$ are parametrized by $\Gamma(N, T'N \otimes ({}^0T''N)^*)$. Now, in this section, we see when this ${}^{\phi}T''N$ is actually a complex manifold. For this, we must introduce notation. Let ϕ be an element of $\Gamma(N, T'N \otimes (T''N)^*)$. For this ϕ, we set an element $R_2(\phi)$ of $\Gamma(N, TN' \otimes \wedge^2(T''N)^*)$ by

$$(4.2.1) \quad R_2(\phi)(X, Y) = [\phi(X), \phi(Y)]_{T'N} - \phi([X, \phi(Y)]_{T''N} + [\phi(X), Y]_{T''N})$$

for X, Y in $\Gamma(N, T''N)$.

We remark that $R_2(\phi)$ makes sense as an element of $\Gamma(N, T'N \otimes \wedge^2(T''N)^*)$ for the same reason as in §3.2. In this notation, we have

Proposition 4.2.3. *Let ϕ be an element of $\Gamma(N, T'N \otimes (T''N)^*)$. Then an almost complex structure $(N, {}^{\phi}T''N)$ is a complex structure if and only if ϕ satisfies the following non-linear equation.*

$$(4.2.2) \qquad \begin{aligned} P(\phi) =& \overline{\partial}^{(1)}_{T'} \phi + R_2(\phi) \\ =& 0. \end{aligned}$$

4.3. $\mathcal{E}_j$-structures

As in the CR structure case, we introduce a subcomplex which satisfies a certain boundary condition.

We introduce a subspace $\mathcal{E}_p$ of $(\Gamma(\overline{\Omega}, T'N \otimes \wedge^p(T''N)^*)$ by

$$\mathcal{E}_p = \{ \phi \ : \ \phi \in \Gamma(\overline{\Omega}, T'N \otimes \wedge^p(T''N)^*), \ \tau\phi \in \Gamma(b\Omega, E_p)\}.$$

For $\mathcal{E}_p$, we show the following theorems.

Theorem 4.3.1. *(see Theorem 3.3 in [Ak4]). There is a differential subcomplex of* $(\Gamma(\overline{\Omega}, T'N \otimes \wedge^p(T''N)^*), \overline{\partial}^{(p)}_{T'N})$.

$$0 \to \mathcal{E}_0 \xrightarrow{\overline{\partial}} \mathcal{E}_1 \xrightarrow{\overline{\partial}_1} \mathcal{E}_2 \xrightarrow{\overline{\partial}_2} \cdots$$
$$\xrightarrow{\overline{\partial}_{p-1}} \mathcal{E}_p \xrightarrow{\overline{\partial}_p} \mathcal{E}_{p+1} \xrightarrow{\overline{\partial}_{p+1}} \cdots$$

where $\overline{\partial}_p$ *means the restriction of* $\overline{\partial}^{(p)}_{T'N}$ *to* $\mathcal{E}_p$.

For the proof, it is enough to show

$$\overline{\partial}^{(p)}_{T'N}\mathcal{E}_p \subset \mathcal{E}_{p+1}.$$

For ϕ in $\mathcal{E}_p$,

$$\tau\overline{\partial}^{(p)}_{T'N}\phi = \overline{\partial}^{(p)}_{T'}(\tau\phi) \text{ (by Lemma 4.1.1).}$$

By Theorem 3.3.1, we have

$$\overline{\partial}^{(p)}_{T'}(\tau\phi) \in \Gamma(b\Omega, E_{p+1}) \ (\tau\phi \text{ being in } \Gamma(b\Omega, E_p)).$$

So this completes the proof of Theorem 4.3.1.

Henceforth we write this complex by

$$(\mathcal{E}_p, \overline{\partial}_p).$$

For this complex, we have the following theorem.

Theorem 4.3.2. *(see Theorem 3.4 in [Ak4]). The injection* $\operatorname{Ker}\overline{\partial}_{(p)} \hookrightarrow \operatorname{Ker}\overline{\partial}^{(p)}_{T'N}$ *induces an isomorphism*

$$i: \ \operatorname{Ker}\overline{\partial}_p / \operatorname{Im}\overline{\partial}_{p-1} \to \operatorname{Ker}\overline{\partial}^{(p)}_{T'N} / \operatorname{Im}\overline{\partial}^{(p-1)}_{T'N} \ \ if \ p \geq 2$$

and in the case of $p = 1$, *the injection induces a surjective map*

$$\operatorname{Ker}\overline{\partial}_1 \to \operatorname{Ker}\overline{\partial}^{(1)}_{T'N} / \operatorname{Im}\overline{\partial}_{T'N} \to 0.$$

4.4. Estimates

In this section, we recall the new a priori estimate in [Ak4], [Ak5] for the subcomplex $(\mathcal{E}_p, \overline{\partial}_p)$. For this purpose, we make preparations.

Let $\{U_k, h_k\}_{k\in K}$ be a coordinate covering of N such that

$$\bigcup_{k\in K} U_k \supset \overline{\Omega} \qquad \text{and} \qquad K \text{ is finite.}$$

Let K' be a subset of K satisfying, for $k \in K'$,

$$U_k \cap b\Omega \neq 0.$$

Let $\{\rho_k\}_{k\in K}$ be a partition of unity subordinate to the above covering. In this paper, we use the Levi metric defined by Greiner and Stein (cf. Chapter 4 in [Gr-St]). Then, for a point $p \in M$, there are a coordinate open set U_k and an orthonormal basis $(e_1^k, .., e_{n-1}^k, e_n^k)$ of $T''N\mid_{U_k}$ satisfying

$$(e_j^k)_q \in^0 T_q'',$$

where $q \in b\Omega \cap U_k$ and $j = 1, \dots, n-1$,

$$\begin{aligned}[e_i^k, \overline{e}_j^k] = \sqrt{-1}(\delta_{i,j} + O(\rho))(e_n^k - \overline{e}_n^k)& \\ + \sum_{r=1}^{n-1} a_{i,j}^{k,r} e_r^k + \sum_{r=1}^{n-1} b_{i,j}^{k,r} \overline{e}_r^k&,\end{aligned}$$

on U_k, and e_n^k is globally defined in a neighborhood of the $b\Omega$, where $a_{i,j}^{k,r}$ and $b_{i,j}^{k,r}$ are C^∞-functions on U_k, ρ is the defining function of $b\Omega$ and $O(\rho)$ stands for a C^∞-function which vanishes on $b\Omega$ (therefore by using integration by parts, we can neglect the $O(\rho)$-term).

Now we put an L^2-norm and the $\| \|'$-norm on $\Gamma(\overline{\Omega}, T'N \otimes \wedge^p(T'')^*)$. I^p denotes the family of all ordered set $(i_1, .., i_p)$ of integers with $1 \leq i_1 < i_2 < \dots < i_p \leq n$. For any ϕ in $\Gamma(\overline{\Omega}, T'N \otimes \wedge^p(T'')^*)$, $I \in I^p$ and $l (1 \leq l \leq n)$, we define C^∞-functions $\phi_{l,I}^k$ on U_k by

$$\phi(e_{i_1}^k, .., e_{i_p}^k) = \sum_l \phi_{l,I}^k \overline{e}_l^k,$$

where $I = (i_1, .., i_p)$. Using these functions $(\rho_k \phi_{l,I}^k) h_k^{-1}$ in $C_o^\infty(R^{2n})$, we define the L^2 -norm on $\Gamma(\overline{\Omega}, T'N \otimes \wedge^p(T''N)^*)$ by

$$\|\phi\|^2 = \sum_{l,I,k} \|(\rho_k \phi_{l,I}^k) h_k^{-1}\|^2$$

where $\| \ \|^2$ means the L^2-norm on $C_o^\infty(R^{2n})$ and $C_o^\infty(R^{2n})$ means the space of C^∞-functions on R^{2n} with compact support. Next we introduce a $\| \ \|'$-norm on $\Gamma(\overline{\Omega}, T'N \otimes \wedge^p(T''N)^*)$ by

$$\begin{aligned}\|\phi\|'^2 = \textstyle\sum_{k\in K', i\neq n, l, I} &\|(\rho_k e_i^k \phi_{l,I}^k) h_k^{-1}\|^2 \\ + \sum_{k\in K', i\neq n, l, I} &\|(\rho_k \overline{e}_i^k \phi_{l,I}^k) h_k^{-1}\|^2 \\ + \sum_{k\in K', n\in I, l} &\|(\rho_k \overline{e}_n^k \phi_{l,I}^k) h_k^{-1}\|^2 \\ + \sum_{k\in K', n\notin I, l} &\|(\rho_k e_n^k \phi_{l,I}^k) h_k^{-1}\|^2 \\ + \sum_{k\neq K', I, l} &\|(\rho_k \phi_{l,I}^k) h_k^{-1}\|_{(1)}^2\end{aligned}$$

where $\| \ \|_{(1)}$ means the Sobolev 1-norm on $C_o^\infty(R^{2n})$.

Henceforth we omit h_k^{-1},ρ_k and the index k for brevity.

We set a vector field ξ on $b\Omega$ by

$$\tau(-\sqrt{-1}\overline{e}_n)$$

and fix the decomposition of the vector bundle $C \otimes T(b\Omega) = {}^0T'' + {}^0T' + C \otimes F$, where $F = \xi$.

With these preparations, we consider the following space B^2 of $\mathcal{E}_2$.

$$\begin{aligned}B^2 = \{\phi; \phi \in \Gamma(\overline{\Omega}, T'N \otimes \wedge^2(T''N)^*), \langle\sigma(\vartheta, d\rho)\phi, y\rangle = 0 \\ \text{on } b\Omega \text{ for any } y \text{ in } E_1 \text{ and } \tau\phi \in \Gamma(b\Omega, E_2)\},\end{aligned}$$

where $\langle , \rangle$ denotes the inner product defined by the Levi-metric, and ϑ denotes the formal adjoint operator of $\overline{\partial}_{T'}^{(1)}$.

On B^2, we have the following a priori estimate (the key estimate).

Theorem 4.4.1. *(see Theorem 4.3 in [Ak4] and Corollary 6.2 in [Ak5]). Assume that Ω is strongly pseudo-convex and* $\dim_C \Omega \geq 4$. *Then* *the following estimate holds.*

$$\begin{aligned}\|\vartheta\phi\|^2+&\|\overline{\partial}\phi\|^2 + \|\phi\|^2\\ \geq& c\{\sum_{i<j}\sum_{\alpha=1}^{n-1}\{\sum_{l=1}^{n-1}\|e_l\phi_{\alpha,(i,j)}\|^2 + \sum_{l=1}^{n-1}\|\overline{e}_l\phi_{\alpha,(i,j)}\|^2\}\}\\ &+\sum_{i<j}\{(\sum_{l\neq i,j}\|e_l\phi_{n,(i,j)}\|^2) + \|\overline{e}_i\phi_{n,(i,j)}\|^2 + \|\overline{e}_j\phi_{n,(i,j)}\|\}\\ =&c\|\phi\|'^2\end{aligned}$$

for all ϕ in B^2, where c is a positive constant independent of ϕ and for brevity, we write $\overline{\partial}$ for $\overline{\partial}^{(1)}_{T'N}$.

This was first proved by direct computation (see [Ak4]). Later, it was proved in a fairly wide framework in [Ak5]. The proof of this theorem in [Ak5] relied on estimates established in Theorem 3.5.1 and the following Proposition 4.4.2. In order to see Proposition 4.4.2, we have to recall some notation. We recall T'-bundle on M, and $\overline{\partial}_{T'}$ -operator. Let $\vartheta_{T'}$ be the formal adjoint operator of $\overline{\partial}_{T'}$ with respect to the Levi-metric. And we set a C^∞ vector bundle decomposition of $T'\otimes\wedge^p({}^0T'')^*$,

$$T'\otimes\wedge^p({}^0T'')^* = E_p + E_p^{\perp}. \tag{4.4.1}$$

Here $E_p^{\perp}$ is the complement of E_p with respect to the Levi metric. Then, our proposition is stated as follows.

Proposition 4.4.2. *(see Theorem 6.1 in [Ak5]). Suppose that* $\dim_R M = 2n-1 \geq 7$. *Then we have*

$$\|\vartheta_{T'}\psi\| + \|(\overline{\partial}_{T'}\psi)_{E_2^{\perp}}\| + \|\psi\| \geq c\|\psi\|'$$

for $\psi\in\Gamma(M,E_1^{\perp})$, where c is a positive constant, and $(\overline{\partial}_{T'}\psi)_{E_2^{\perp}}$ means the projection of $\overline{\partial}_{T'}\psi$ to $E_2^{\perp}$ according to (4.4.1).

4.5. The new Hodge decomposition theorem

Based on our estimate (Theorem 4.4.1), we discuss a new Hodge decomposition theorem, which differs from the standard one (see [Kohn]), and apply it in solving the Cauchy-Riemann equation in the subcomplex $(\mathcal{E}_p,\overline{\partial}_p)$. We note that our new Neumann operator preserves the boundary condition. Let G be a holomorphic vector bundle on N. Let D^p be the Cauchy-Riemann operator for G-valued p forms and D_b^p the induced operator over the boundary $b\Omega$. Let F_p be a subbundle of $G \otimes \wedge^p({}^0T'')^*$ over the boundary $b\Omega$. We set

$$\mathcal{F}^p = \{\phi; \phi \in \Gamma(\overline{\Omega}, G \otimes \wedge^p(T''N)^*),\ \tau\phi \in \Gamma(b\Omega, F_p)\},$$

where τ means the restriction map ϕ to the element of $\Gamma(b\Omega, G \otimes \wedge^p({}^0T'')^*)$. In the same way as in §4.4, we put the L^2-norm, the inner product and also $\|\ \|'$-norm on $\Gamma(\overline{\Omega}, G \otimes \wedge^p(T''N)^*)$. We set

$$\begin{aligned} B^p = \{\phi; \phi \in \Gamma(\overline{\Omega}, G \otimes \wedge^p(T''N)^*),\ \tau\phi \in \Gamma(b\Omega, F_p) \\ \text{and} \quad \langle \sigma(D^{*p-1}, d\rho)\phi, y\rangle = 0 \quad \text{for any} \quad y \in F_{p-1} \text{ on } b\Omega\}, \end{aligned}$$

where D^{*p-1} denotes the formal adjoint operator of D^{p-1}, $\sigma(D^{*p-1}, d\rho)$ means the symbol at $d\rho$, and ρ is the defining function for $b\Omega$ in N. For brevity, we write D^* for D^{*p} and D for D^p.

In this notation, our theorem is stated as follows.

Theorem 4.5.1. *(see Theorem 5.1 in [Ak4]) Suppose that*

$$D_b^{p-1}\Gamma(b\Omega, F_{p-1}) \subset \Gamma(b\Omega, F_p) \tag{A.1}$$

and

$$\|D^*\phi\|^2 + \|D\phi\|^2 + \|\phi\|^2 \geq c\|\phi\|'^2 \tag{A.2}$$

for all ϕ in B^p, where c is a positive constant independent of ϕ. Then, there are the new Neumann operator N; $\mathcal{L}_2^p \to \mathcal{L}_2^p$ and the new harmonic operator H; $\mathcal{L}_2^p \to \mathbf{H}$ satisfying

(1) *H and N are bounded,*
(2) *if ϕ is in $\Gamma(\overline{\Omega}, G \otimes \wedge^p(T'')^*)$, then $\mathbf{H}\phi$ and $N\phi$ are in $\Gamma(\overline{\Omega}, G \otimes \wedge^p(T'')^*)$,*
(3) *if ϕ is in $\Gamma(\overline{\Omega} \otimes \wedge^p(T'')^*)$, then $\phi = (DD^* + D^*D)N\phi + H\phi$*
(4) *$HN = NH$*
(5) *if ϕ is in $\Gamma(\overline{\Omega}, G \otimes \wedge^p(T'')^*)$, then $D^*N\phi$ is in $\mathcal{F}^{p-1}$, and in addition, if ϕ is in $\mathcal{F}^p$, $D\phi = 0$ and $H\phi = 0$, then $DD^*N\phi = \phi$ where L_2^p denotes the L^2-completion of $\Gamma(\overline{\Omega}, G \otimes \wedge^p(T'')^*)$ and $\mathbf{H} = \{\phi; \phi \in B^p, D, phi = 0$ and $D^*\phi = 0\}$.*

[Brief sketch of the proof]. We note that Kohn's standard Neumann operator relies on Morrey's estimate (in Kohn's case, the (A.1)

part is trivial). We briefly recall the proof, which is carried out in the standard functional analysis. First, we set a map T from B^p to

$$\Gamma(\overline{\Omega}, G \otimes \wedge^{p-1}(T'')^* \times \Gamma(\overline{\Omega}, G \otimes \wedge^{p+1}(T'')^*)$$

by $T\phi = (D^*\phi, D\phi)$. We complete T and use the same notation T for this. $D(T)$ denotes the domain on which this operator is defined. For $D(T)$, just by the standard argument, we have

Lemma 4.5.2. *$D(T)$ is dense in $\mathcal{L}_2^p$ and*

$$D(T) \cap \Gamma(\overline{\Omega}, G \otimes \wedge^p(T'')^*) = B^p.$$

And also we have the following lemma and proposition.

Lemma 4.5.3. *Let μ be in $\Gamma(\overline{\Omega}, G \otimes \wedge^{P+1}(T'')^*)$, and suppose that*

$$\langle \phi, \sigma(D^*, d\rho)\mu \rangle = 0 \ for\ all\ \phi\ in\ B^p\ on\ b\Omega.$$

Then $\tau\psi$ in $\Gamma(b\Omega, F_{p-1})$.

Proposition. 4.5.4.

$$\begin{aligned} D(T^*) \cap \{&\Gamma(\overline{\Omega}, G \otimes \wedge^{p-1}(''N)^*) \times \Gamma(\overline{\Omega}, G \otimes \wedge^{p+1}(T''N)^*)\} \\ &= \{(\psi, \mu) : \psi \in \Gamma(\overline{\Omega}, G \otimes \wedge^{p-1}(T''N)^*), \tau\psi \in \Gamma(b\Omega, F_{p-1}), \\ &\quad \mu \in \Gamma(\overline{\Omega}, G \otimes \wedge^{p+1}(T''N)^*) \ and\ \langle \sigma(D^*, d\rho)\mu, y \rangle = 0 \\ &\quad for\ any\ y\ in\ F_p\ on\ b\Omega\}, \end{aligned}$$

where $D(T^*)$ means the domain of T^*.

So, we obtain that for ϕ in B^p satisfying $T\phi$ in $D(T^*)$,

$$T^*T\phi = \Box\phi. \tag{4.5.1}$$

With these preparations, we prove Theorem 4.5.1. We follow Kohn-Nirenberg's approach in [K-N]. Namely, we first set

$$\mathbf{H} = \{\phi : \phi \in D(T), T\phi = 0\}.$$

Obviously, $\mathbf{H}$ is finite dimensional and so closed in $\mathcal{L}_2^p$. Next we study $B^p \cap H'$, where H' is the complement of $\mathbf{H}$ in $\mathcal{L}_2^p$. We consider the problem of finding a solution $\psi \in B^p \cap H'$

$$(T\psi, T\phi) = (\alpha, \phi) \quad \text{for} \quad \phi \in B^p \cap H',$$

where α is in $\Gamma(\overline{\Omega}, G \otimes \wedge^p(T''N)^*)$.

By the assumption (A.2), there is a unique element ψ in $B^p \cap H'$ (Theorem 4.1 in [Ak4]), And ψ satisfies the boundary condition

$$T\psi \in D(T^*). \tag{4.5.2}$$

That is to say, for any α in $\Gamma(\overline{\Omega}, G \otimes \wedge^p(T''N)^*)$, there is a ψ in B^p satisfying

$$T^*T\psi = \alpha - H\alpha,$$

where H is the projection of $\mathcal{L}_2^p$ to $\mathbf{H}$.

We set

$$N\alpha = \psi$$

and call N the new Neumann operator. We see that our new Neumann operator N satisfies the relation (5). We recall (4.5.2). Namely, for α in $\Gamma(\overline{\Omega}, G \otimes \wedge^p(T''N)^*)$,

$$TN\alpha \in D(T^*).$$

By the definition of T, $TN\alpha = (D^*N\alpha, DN\alpha)$. And by Proposition 4.5.4, we get

$$D^*N\alpha \in \mathcal{F}^{p-1}$$

and

$$\langle \sigma(D^*, d\rho)DN\alpha, y\rangle = 0 \quad \text{for all} \quad y \in F_p \quad \text{on} \quad b\Omega.$$

It remains to prove that under the assumptions $h\phi = 0$, $D\phi = 0$ and $\phi \in \mathcal{F}^p$, we obtain

$$DD^*N\phi = 0.$$

For this, we set

$$\mu = DD^*N\phi - \phi.$$

Then, form $D\phi = 0$, $D\mu = 0$ follows. In addition, we have

$$\begin{aligned} D^* &= D^*DD^*N\phi - D^*\phi \\ &= D^*(\phi - H\phi - D^*DN\phi) - D^*\phi \\ &= D^*\phi - D^*\phi \\ &= 0. \end{aligned}$$

Next we see that μ is in B^p, i.e., in $D(T)$. For this, we compute the following by integration by parts. For ψ in $\mathcal{F}^{p-1}$,

$$\begin{aligned} (\mu, D\psi) &= (DD^*N\phi - \phi, D\psi) \\ &= (DD^*N\phi, D\psi) - (\phi, D\psi) \\ &= (\phi - H\phi - D^*DN\phi, D\psi) - (\phi, D\psi) \\ &= -(H\phi, D\psi) - (D^*DN\phi, D\psi). \end{aligned}$$

We note that the boundary term vanishes, so we have

$$(H\phi, D\psi) = (D^* H\phi, \psi) = 0.$$

Similarly,

$$(D^* DN\phi, D\psi) = 0.$$

Because

$$\tau D\psi = D_b\psi \quad \text{is in} \quad \Gamma(b\Omega, F_p),$$

we have

$$\langle \sigma(D^*, d\rho) DN\phi, D\psi\rangle = 0 \quad \text{on} \quad b\Omega.$$

Thus,

$$(\mu, D\psi) = -(DN\phi, DD\psi) = 0.$$

On the other hand,

$$\begin{aligned}(\mu, D\psi) =& (D^*\mu, \psi) - \int_{b\Omega} \langle \sigma(D^*, d\rho)\mu, \psi\rangle d(b\Omega) \\ =& -\int_{b\Omega} \langle \sigma(D^*, d\rho)\mu, \psi\rangle d(b\Omega).\end{aligned}$$

Hence

$$\langle \sigma(D^*, d\rho)\mu, \psi\rangle = 0 \quad \text{on} \quad b\Omega \quad \text{for} \quad \psi \in \mathcal{F}^{p-1}.$$

By the definition of B^p and Lemma 4.5.2, we obtain

$$\mu \in D(T).$$

Combined with $D\mu = 0$ and $D^*\mu = 0$, we have

$$T\mu = 0.$$

Therefore

$$\mu \in \mathbf{H}.$$

Furthermore, for any $\alpha \in \mathbf{H}$,

$$(\mu, \alpha) = (DD^*N\phi - \phi, \alpha) = (DD^*N\phi, \alpha) \quad (\text{by} \quad H\phi = 0).$$

The boundary term vanishes, so we have

$$(DD^*N\phi, \alpha) = (D^*N\phi, D^*\alpha) = 0 \quad (\text{by} \quad \alpha \in \mathbf{H}).$$

So $\mu = 0$. This is the outline of the proof.

From this theorem with Theorem 4.4.1, we immediately obtain Corollary 4.5.5.

Corollary 4.5.5. *In the case of $G = T'N$, $F_p = E_p$, and $p = 2$, the new Neumann operator exists.*

4.6. Some estimates

In order to try to construct a versal family, we will review some estimates for the Neumann operator N obtained in §4.5.

First, we put the tangential Sobolev $(0, m)$-norm and $\| \ \|'$-norm on $\Gamma(\overline{\Omega}, T'N \otimes \wedge^p(T''N)^*)$. I^p denotes the family of all ordered sets $(i_1, .., i_p)$ of integers with $1 \leq i_1 < i_2 < ... < i_p \leq n$. For ϕ in $\Gamma(\overline{\Omega}, T'N \otimes \wedge^p(T''N)^*)$, $I \in I^p$ and $l(1 \leq l \leq n)$, we define C^∞ functions $\phi^k_{l,I}$ on U_k by

$$\phi(e^k_{i_1}, .., e^k_{i_p}) = \sum_l \phi^k_{l,I} \bar{e}^k_l, \text{ where } I = (i_1, .., i_p).$$

Using these functions, we define the tangential Sobolev $(0, m)$-norm on $\Gamma(\overline{\Omega}, T'N \otimes \wedge^p(T''N)^*)$ by

$$\|\phi\|^2_{(0,m)} = \sum_{l,I,k} \|(\rho_k \phi^k_{l,I}) h_k^{-1}\|^2_{(0,m)}$$

where for $k \in K'$, $\| \ \|^2_{(0,m)}$ means the tangential Sobolev $(0, m)$-norm on $C^\infty_0(R^{2n}_+)$ (here $C^\infty_0(R^{2n}_+)$ means the space of C^∞-functions on the upper half plane R^{2n}_+ with compact support), and for $k \notin K'$, $\| \ \|^2_{(0,m)}$ means the tangential Sobolev $(0, m)$-norm on $C^\infty_0(R^{2n})$ with compact support (for the definition of the tangential Sobolev norm, see Definition 2.5.1 in [Hö]). Next we introduce $\| \ \|'_{(0,m)}$-norm on $\Gamma(\overline{\Omega}, T'N \otimes \wedge^p(T''N)^*)$ by

$$\begin{aligned}
\|\phi\|'^2_{(0,m)} = & \sum_{k\in K', i\neq, l, I} \|(\rho_k e^k_i \phi^k_{l,I}) h_k^{-1}\|^2_{(0,m)} \\
& + \sum_{k\in K', i\neq, l, I} \|(\rho_k \bar{e}^k_i \phi^k_{l,I}) h_k^{-1}\|^2_{(0,m)} \\
& + \sum_{k\in K', n\in I, l} \|(\rho_k \bar{e}^k_n \phi^k_{l,I}) h_k^{-1}\|^2_{(0,m)} \\
& + \sum_{k\in K', n\notin I, l} \|(\rho_k e^k_n \phi^k_{l,I}) h_k^{-1}\|^2_{(0,m)} \\
& + \sum_{k\notin K', l, I} \|(\rho_k \phi^k_{l,I}) h_k^{-1}\|^2_{(1,m)}.
\end{aligned}$$

From now on, we assume $m > 2n$ unless we note otherwise. With these preparations, we show the following more precise estimate (cf. Theorem 4.4.1 in this paper).

Proposition 4.6.1. *(see Theorem 6.1 in [Ak4]) Assume that Ω is strongly pseudo-convex and $\dim_C \Omega \geq 4$. Then the following estimate holds.*

$$\|\vartheta\phi\|^2_{(0,m)} + \|\overline{\partial}\phi\|^2_{(0,m)} + \|\phi\|^2_{(0,m)} \geq c\|\phi\|'^2_{(0,m)}$$

for all ϕ in B^2, where c is a positive constant independent of ϕ and m is a non-negative integer.

For $\square = \overline{\partial}\vartheta + \vartheta\overline{\partial}$, we show some estimates by using this proposition. To do so, we must introduce a new norm. For μ in $\Gamma(\overline{\Omega}, T'N \otimes \wedge^p(T''N)^*)$, we set

$$\begin{aligned}
\|\mu\|''^2_{(0,m)} = & \sum_{k\in K', i<j<n, l, I} \|(\rho_k e^k_i e^k_j \mu^k_{l,I})h_k^{-1}\|^2_{(0,m)} \\
& + \sum_{k\in K', i<j<n, l, I} \|(\rho_k e^k_i \overline{e}^k_j \mu^k_{l,I})h_k^{-1}\|^2_{(0,m)} \\
& + \sum_{k\in K', i<j<n, l, I} \|(\rho_k \overline{e}^k_i e^k_j \mu^k_{l,I})h_k^{-1}\|^2_{(0,m)} \\
& + \sum_{k\in K', i<j<n, l, I} \|(\rho_k \overline{e}^k_i \overline{e}^k_j \mu^k_{l,I})h_k^{-1}\|^2_{(0,m)} \\
& + \sum_{k\in K', i<n, l, n\in I} \|(\rho_k \overline{e}^k_n \overline{e}^k_i \mu^k_{l,I})h_k^{-1}\|^2_{(0,m)} \\
& + \sum_{k\in K', i<n, l, n\in I} \|(\rho_k \overline{e}^k_n e^k_i \mu^k_{l,I})h_k^{-1}\|^2_{(0,m)} \\
& + \sum_{k\in K', i<n, l, n\notin I} \|(\rho_k e^k_n \overline{e}^k_i \mu^k_{l,I})h_k^{-1}\|^2_{(0,m)} \\
& + \sum_{k\in K', i<n, l, n\notin I} \|(\rho_k e^k_n e^k_i \mu^k_{l,I})h_k^{-1}\|^2_{(0,m)} \\
& + \sum_{k\in K', l, I} \|(\rho_k e^k_n \overline{e}^k_n \mu^k_{l,I})h_k^{-1}\|^2_{(0,m)} \\
& + \sum_{k\in K', l, I} \|(\rho_k \overline{e}^k_n e^k_n \mu^k_{l,I} h_k^{-1}\|^2_{(0,m)} + \|\mu\|^2_{(1,m)}.
\end{aligned}$$

After this, we omit the suffix k and the functions ρ_k, h_k. In this notation, our theorem is stated as follows.

Theorem 4.6.2. *(see Theorem 6.2 in [Ak4]) For μ in B^2,*

$$\|\mu\|''^2_{(0,m)} \leq c_m\{\|\Box\mu\|^2_{(0,m)} + \|\mu\|^2_{(0,m)}\}.$$

From this proposition, we immediately have

Proposition 4.6.3. *(see Theorem 6.3 in [Ak4]) For μ in $\Gamma(\overline{\Omega}, T'N \otimes \wedge^2(T''N)^*)$, we have*

$$\|\vartheta N\mu\|'_{(0,m)} \leq c_m\|\mu\|_{(0,m)},$$

where c_m is a positive constant.

Using this proposition, we can discuss an a priori estimate for $R_2(\phi)$. Combining this with the standard argument of functional analysis, we obtain the following main theorem.

Theorem 4.6.4. *(see Theorem 6.6 in [Ak4]) For ϕ in $\mathcal{E}_1$ satisfying $\phi_{n,n} = 0$ on $b\Omega$,*

$$\|\vartheta N R_2(\phi)\|'_{(0,m)} \leq c_m\|\phi\|'^2_{(0,m)}.$$

4.7. Construction

We construct a versal family of deformations of Ω, consisting of $T'N \otimes (T''N)^*$-valued $\mathcal{A}^m$-class elements. For this purpose, we must introduce a new subspace $\mathcal{E}'_1$ of $\mathcal{E}_1$.

First we set a linear map t from $\Gamma(\overline{\Omega}, T'N \otimes \wedge^p(T''N^*)$ to $\Gamma(b\Omega, T' \otimes \wedge^{p-1}({}^0T^{P\prime\prime})^*)$ by

$$t\phi(X_1, .., X_{p-1}) = \tau(\phi(e_n, X_1, .., X_{p-1})),$$

where X_j is an element of ${}^0T''$ and e_n is as introduced in §3.4. By using this t, we introduce

$$\mathcal{E}'_p = \{\phi : \phi \in \mathcal{E}_p,\ t\phi = 0\}.$$

Then for $\mathcal{E}'_p$, the following theorem holds.

Theorem 4.7.1. *(see Theorem 7.1 in [Ak4]). The injection $\mathcal{E}'_p \cap \operatorname{Ker}\overline{\partial}_p \hookrightarrow \operatorname{Ker}\overline{\partial}_p$ induces an isomorphism*

$$\{\mathcal{E}'_p \cap \operatorname{Ker}\overline{\partial}_p\}/\{\mathcal{E}'_p \cap \operatorname{Im}\overline{\partial}_{p-1}\} \backsimeq \operatorname{Ker}\overline{\partial}_p/\operatorname{Im}\overline{\partial}_{p-1}\ if\ p \geq 2,$$

and a surjective map

$$\mathcal{E}_1' \cap \operatorname{Ker} \overline{\partial}_1 \to \operatorname{Ker} \overline{\partial}^{(1)}_{T'N} / \operatorname{Im} \overline{\partial}_{T'N} \to 0.$$

Then from Theorem 4.7.1, we immediately have the following corollary.

Corollary 4.7.2. *(see Corollary 7.2 in [Ak4]). There is a finite dimensional sub-vector space $\mathcal{H}$ of $\mathcal{E}_1'$ such that the map in Theorem 4.3.2 induces an isomorphism*

$$\mathcal{H} \simeq \operatorname{Ker} \overline{\partial}^{(1)}_{T'N} / \operatorname{Im} \overline{\partial}_{T'N}.$$

With this corollary in mind, we will construct a versal family of $\mathcal{A}^m$ class. Our main theorem in this section is as follows.

Theorem 4.7.3. *(see Theorem 7.4 in [Ak4]). Under the assumptions $\dim_C \Omega = n \geq 4$ and $H^2(\Omega, T'N) = 0$, there is an $\mathcal{E}_1'$ valued $\mathcal{A}^m$ class element $\phi(t)$, parametrized complex analytically by a neighborhood U of the origin in the Euclidean space $\mathcal{H}$ satisfying*

(1) $\phi(0) = 0$,

(2) $\overline{\partial}^{(1)}_{T'N}\phi(t) + R_2(\phi(t)) = 0$, *and*

(3) *the linear term of $\phi(t)$ is equal to $\sum_\lambda^q \beta_\lambda t_\lambda$, where $\{\beta_\lambda\}_{1 \leq \lambda \leq q}$ is a system of bases of $\mathcal{H}$ and $\{t_i\}_{1 \leq i \leq q}$ are local coordinates of U. Here m is a sufficiently large integer satisfying $m \geq n + 2$.*

The construciton of $\phi(t)$ is the same as in the CR case (cf.Sect.3.7). However, in order to assumre the convergence, we need to construct $\phi(t)$ so that it is $\mathcal{E}_1'$-valued (cf. Theorem 4.6.4). Hence we introduce an operator $\mathbf{A} : \mathcal{E}_p \to \mathcal{E}_p'$ having the following properties:

$$\overline{\partial}^{(p)}_{T'N}\mathbf{A} = \overline{\partial}^{(p)}_{T'N},$$
$$\|\mathbf{A}\phi\|'_{(0,m)} \leq c_m \|\phi\|'_{(0,m)}.$$

See [Ak4] for the construction of $\mathbf{A}$.

By an argument parallel to that in the proof of Theorem 3.7.1, replacing $\Gamma(M, E_1)$ and the Hodge decomposition in Theorem 3.5.2 by $\mathcal{E}_1'$ and the Hodge decomposition in Theorem 4.5.1, we can trace the construciton in §3.7. The following lemma and proposition correspond to Lemma 3.7.2 and Proposition 3.7.3 in §3.7 respectively.

Lemma 4.7.4. *For any element ϕ in $\Gamma(\overline{\Omega}, T'N \otimes (T''N)^*)$,*

$$\overline{\partial}^{\phi}_{T'N} P(\phi) = 0.$$

Proposition 4.7.5. *Given $\mathcal{E}_1'$ valued holomorhic function $\phi^{\mu-1}(t)$ in $(t_1, .., t_q)$ satisfying $P(\phi^{\mu-1}(t)) \equiv_\mu 0$, the homogeneous part of degree μ in $(t_1, .., t_q)$ of $P(\phi^{\mu-1}(t))$ takes its value in $\mathcal{E}_2$.*

Only difference is that $\phi_\mu(t)$ is given by

$$\phi_\mu(t) = \mathbf{A}\{-\vartheta N\{\text{the } \mu \text{ th homogeous polynomial term of } P(\phi^{\mu-1}(t))\}\},$$

not by

$$\phi_\mu(t) = -\vartheta N\{\text{the } \mu \text{ th homogeous polynomial term of } P(\phi^{\mu-1}(t))\}.$$

This is required because $\vartheta N\{$the μ th homogeous polynomial term of $P(\phi^{\mu-1}(t))\}$ is not necessarily $\mathcal{E}_1'$-valued though it is certainly $\mathcal{E}_1$-valued. This adjustment is necessary for the convergence procedure. In order to carry out the convergence process in §3.2, using Theorem 4.6.4 instead of Proposition 3.6.4, we need the property that $\phi(t)$ is $\mathcal{E}_1'$-valued. Thus Theorem 4.7.4 is proved.

We note that the assumption $H^2(\Omega, T'N) = 0$ is not essential either in this case. The same modification as in the CR case is possible. And the proof of the Kuranishi versality is also the same as in the CR case.

References

[**Ak1**] Akahori, T., Intrinsic formula for Kuranishi's $\bar{\partial}^{\phi}_b$, Publ. RIMS, Kyoto Univ., **14** (1978), 615–641.

[**Ak2**] ———, Complex analytic construction of the Kuranishi family on a normal strongly pseudo convex manifold, Publ. RIMS, Kyoto Univ., **14** (1978), 789–847.

[**Ak3**] ———, The new estimate for the subbundles E_j and its application to the deformation of the boundaries of strongly pseudo convex domains, Invent. math., **63** (1981), 311–334.

[**Ak4**] ———, The new Neumann operator associated with deformations of strongly pseudo convex domains and its application to deformation theory, Invent. math., **68** (1982), 317–352.

[**Ak5**] ———, A criterion for the Neumann type problem over a differential complex on a strongly pseudo convex domain, Math. Ann., **264** (1983), 525–535.

[**Ak6**] ———, Complex analytic construction of the Kuranishi family on a normal strongly pseudo convex manifold with real dimension 5, Manuscripta math., **63** (1989), 29–43.

[**Ak7**] ———, A criterion for versality of deformations of tubular neighbourhoods of strongly pseudo convex boundaries, Can. J. Math., **44** (1992), 225–233.

[**Ak8**] ———, On the Kuranishi family of deformations of complex structures over a tubular neighbouhood of a strongly pseudo convex boundary with complex dimension 3, J. reine angew. Math., **433** (1992), 45–68.

[**Ak9**] ———, A survey of deformation theory of CR-structures, "Nilpotent geometry and analysis", Sūrikaisekikenkyūsho Kōkyūroku No. 875, 1994, pp. 117–128.

[**Ak-My1**] Akahori, T. and Miyajima, K., Complex analytic construction of the Kuranishi family on a normal strongly pseudo-convex manifold. II, Publ. RIMS, Kyoto Univ., **16** (1980), 811–834.

[**Ak-My2**] ———, An analogy of Tian-Todorov theorem on deformations of CR-structures, Compositio Math., **85** (1993), 57–85.

[**Ak-My3**] ———, A note on the analogue of the Bogomolov type theorem on deformations of CR-structures, Canad. Math. Bull., **37** (1994), 8-12.

[**Ak-My4**] ———, On the CR-version of the Tian-Todorov lemma, preprint.

[**An-Gr**] Andreotti, A. and Grauert, H., Théorèmes de finitude pour la cohomologie des espaces complexes, Bull. Soc. Math. Fr., **90** (1962), 193–259.

[**Bi-Ko**] Bingener, J. and Kosarew, S., Lokale Modulraume in der analytischen Geometrie, Aspects of Math., D2, D3, Vieweg-Verlag, Braunshweig.

[**Bl-Ep**] Bland, J. and Epstein, C.L., Embeddable CR-structures and deformations of pseudoconvex surfaces. Part I: Formal deformations, J. Alg. Geom., **5** (1996), 277–368.

[**Bu-Ml**] Buchweitz, R.O. and Millson, J.J., CR-Geometry and deformations of isolated singularities, to appear.

[**Don**] Donaldson,S.K., The Seiberg-Witten equations and 4-manifold topology, Bulletin of the AMS, **33** (1996), 45–70.

[**Dou**] Douady, A., Le problème des modules locaux pour les espaces $\mathbb{C}$-analytiques compacts, Ann. Sci. Ec. Norm. Supér., IV, **7** (1975), 569–602.

[**Fo-Ko**] Folland, G. B. and Kohn, J. J., "The Neumann problem for the Cauchy-Riemann complex", Annals of Mathematics Studies, Princeton Univ. Press, 1972.

[**Gr1**] Grauert, H., Über die Deformation isolierter Singularitäten analytischer Mengen, Inv. math, **15** (1972), 171–198.

[**Gr2**] Grauert, H., Der Satz von Kuranishi für kompakte komplexe Raüme, Inv. Math., **25** (1974), 107–142.

[**Gr-St**] Greiner, P. C. and Stein, E. M., "Estimates for the $\bar{\partial}$-Neumann problem", Mathematical Notes, Princeton Univ. Press, 1977.

[**Hö**] Hörmander, L., "Linear partial differential operators", Grundlehren der mathematischen Wissenshaften, Springer-Verlag, 1963.

[**Kod**] Kodaira, K., "Complex Manifolds and Deformation of Complex Structures", Grundlehren der mathematischen Wissenshaften, Springer-Verlag, 1981.

[**Ko-Ni-Sp**] Kodaira, K., Nirenberg, L. and Spencer, D.C., On the existence of deformations of complex analytic structures, Ann. of Math., **68** (1958), 450–459.

[**Kohn**] Kohn, J. J., "Boundaries of complex manifolds", Proc. Conf. Complex Analysis(Minneapolis) Springer, 1964, pp. 81–94.

[**Ko-Ro**] Kohn, J. J. and Rossi, H., On the extension of holomorphic functions from the boundary of a complex manifold, Ann. of Math., **81** (1965), 451–472.

[**Ku1**] Kuranishi, M., Deformations of isolated singularities and $\bar{\partial}_b$, Preprint, Columbia Univ..

[**Ku2**] ———, Application of $\bar{\partial}_b$ to deformation of isolated singularities, Proc. Symp. Pure Math., **30** (1977), 97-106.

[**Ku3**] Kuranishi, M., On the locally complete families of complex analytic structures, Ann. of Math., **75** (1962), 536–577.

[**Ku4**] Kuranishi, M., Deformations of compact complex manifolds, Montreal Univ..

[**L-M**] Lions,J.L., Magenes, "Non-homogeneous Boundary Value Problems and Applications", Springer, 1972.

[**N**] Nirenberg,L., "Lectures on linear partial differential equations", Regional conference series in Mathematics, Number 17(AMS), 1972.

[**My1**] Miyajima, K., Completion of Akahori's construction of the versal family of strongly pseudo-convex CR structures, Trans. A. M. S., **277** (1983), 163–172.

[**Oh1**] Ohsawa,T., A reduction theorem for cohomology groups of very strongly q-convex Kähler manifolds, Inv. math., **63** (1981), 335–354.

[**Oh2**] ———, Hodge spectral sequence on compact Kähler spaces, Publ. R.I.M.S., Kyoto Univ., **23** (1987), 265–274.

[**Oh-Ta**] Ohsawa,T. and Takegoshi,K., Hodge spectral sequence on pseudoconvex domains, Math. Z., **197** (1988), 1–12.

[**R**] Rumin,M., Formes differéntielles sur les varietes de contact, J.Differential Geometry, **39** (1994), 281–300.

[**Ta**] Tanaka, N., A differential-geometric study on strongly pseudo-convex manifolds, Lectures in Mathematics, Kyoto Univ., **9** (1975).

[**Ty**] Tyurina, G.N., Locally semi-universal flat deformations of isolated singularities of compelx spaces, Math. USSR, Izv., **3** (1969), 976–1000.

[**Ya**] Yau, S.-T., Kohn-Rossi cohomology and its application to the complex Plateau problem I, Ann. of Math., **113** (1981), 67–110.

Department of Mathematics
Himeji Institute of Technology
Himeji, 671-22
Japan

Advanced Studies in Pure Mathematics 25, 1997
CR-Geometry and Overdetermined Systems
pp. 41–59

Deformation Theory for the Hyperplane Line Bundle on $\mathbb{P}^1$

John Bland and Thomas Duchamp

Dedicated to Professor M. Kuranishi, on the occasion of his seventieth birthday.

Abstract.
We develop a correspondence between deformations of the standard CR structure on S^3 and deformations of formal neighbourhoods of the hyperplane bundle over $\mathbb{P}^1$; this correspondence leads to a geometric description of obstructions to the embeddability of CR structures.

§1. Introduction

In recent years, much work has been done on the imbeddability of CR structures on S^3. See, for example, [B], [BlEp], [BuEp], [CaLe], [Ep1], [Ep2], [Le1], [Le2]. In [B] and [Le1], a geometric description of sufficient conditions for embeddability was provided; moreover, it follows from a stability result in [Le1] that these conditions are also necessary for CR structures that are sufficiently close to the standard spherical CR structure. However, a geometric interpretation of the obstructions to embeddability was still lacking. In this paper, we look to providing such an interpretation.

Since $S^3 \subset \mathbb{C}^2 \subset \mathbb{P}^2$, we can view S^3 as bounding the complement of the unit ball in $\mathbb{P}^2$; call this complement U. We begin by surveying some results that relate the CR deformation theory of S^3 to the deformation theory for the pseudoconcave manifold U that it bounds, with particular emphasis on the embeddability question. We then show how the analysis of any sufficiently small deformation of the standard CR structure on S^3 can be localized to an analysis of the extended deformation of the complex structure in formal neighbourhoods of the hyperplane at infinity. Moreover, stable embeddability corresponds to the formal

Received January 17, 1996.
This research was partially supported by an NSERC grant.

neighbourhoods of the deformed structure being equivalent to formal neighbourhoods of the undeformed structure. One consequence is a new description of the *obstructions* to the embeddability of CR structures on S^3 in a neighbourhood of the standard CR structure.

Remark 1.1. *We mention here a notational convention. We will usually be working with expansions of various functions and tensors in powers of ζ. We will let $[\phi]_k$ denote the expansion of ϕ truncated at the k^{th} term, and let ϕ_k denote the coefficient of the k^{th} term itself; thus, for example, $\phi_k \zeta^k = [\phi]_k - [\phi]_{k-1}$.*

§2. Embeddability

In this section we will extend deformations of the CR structure on S^3 to deformations of the complex structure of a pseudoconcave manifold U, and indicate how the embeddability question for S^3 is related to certain properties of the deformed pseudoconcave manifold that it bounds.

We begin by introducing the notation and the framework. Let $\mathbf{z} = (\mathbf{z^1}, \mathbf{z^2})$ denote Euclidean coordinates on $\mathbb{C}^2$ with the Euclidean norm $||\mathbf{z}||^2 = |\mathbf{z^1}|^2 + |\mathbf{z^2}|^2$. Recall that $\mathbb{P}^2$ can be obtained from $\mathbb{C}^2$ by attaching a $\mathbb{P}^1$ at infinity, and that points on the hyperplane at infinity naturally correspond to lines through the origin in $\mathbb{C}^2$. We choose local coordinates in a neighbourhood of the hyperplane at infinity by setting $w := z^2/z^1, \zeta := 1/z^1$ for the lines on which $z^1 \neq 0$, and $\hat{w} := z^1/z^2, \hat{\zeta} := 1/z^2$ for the lines on which $z^2 \neq 0$. Let V_1 denote the open set on $\mathbb{P}^1$ on which $z^1 \neq 0$, and let V_2 denote the open set on $\mathbb{P}^1$ on which $z^2 \neq 0$.

Let $\pi : E \to \mathbb{P}^1$ denote the hyperplane bundle over $\mathbb{P}^1$. Recall that the total space E is naturally biholomorphic to the complement of the origin in $\mathbb{P}^2$, with the zero section of E corresponding to the hyperplane at infinity, and the fibres of E corresponding to the lines through the origin in $\mathbb{C}^2 \subset \mathbb{P}^2$. More precisely, we may represent the hyperplane line bundle using local coordinates $\{(w, \zeta) : w \in V_1\}$, $\{(\hat{w}, \hat{\zeta}) : \hat{w} \in V_2\}$ with transition functions

$$\hat{w} = \frac{1}{w} \qquad \hat{\zeta} = \frac{\zeta}{w} \qquad \text{on } V_1 \cap V_2 . \tag{2.1}$$

We can obtain a concrete embedding $\iota : E \hookrightarrow \mathbb{P}^2$ by setting

$$z^1 = 1/\zeta \qquad z^2 = w/\zeta \qquad \text{on } \pi^{-1}(V_1) \setminus \mathbb{P}^1 \tag{2.2}$$

and

$$z^2 = 1/\hat{\zeta} \qquad z^1 = \hat{w}/\hat{\zeta} \qquad \text{on } \pi^{-1}(V_2) \setminus \mathbb{P}^1 . \tag{2.3}$$

The inverse of the Euclidean norm $||\mathbf{z}||^{-2}$ restricted to the complement of the origin defines a hermitian metric $h = ||\mathbf{z}||^{-2}$ on E. In the local coordinates above, h is given by the formula

(2.4)
$$h(w,\zeta) = e^{-H(w)}\,|\zeta|^2 \text{ on } \pi^{-1}(V_1) \text{ and } h(\hat{w},\hat{\zeta}) = e^{-H(\hat{w})}\,|\hat{\zeta}|^2 \text{ on } \pi^{-1}(V_2)$$

where $e^{H(w)} = (1+|w|^2)$. Let U denote the total space of the (open) unit disk bundle of (E,h). Notice that U is biholomorphic to the complement of the closed unit ball in $\mathbf{P}^2$, and $\partial U = S^3$. The open pseudoconcave manifold U is covered by two coordinate charts, $U_1 = (\pi^{-1}V_1) \cap U \cap \{|w| < 4\}$, $U_2 = (\pi^{-1}V_2) \cap U \cap \{|\hat{w}| < 4\}$.

We next recall some basic facts of the CR deformation theory for $S^3 = \partial U$. Let η denote the connection for the hermitian metric h, and let $H_{(1,0)}U$ denote the space of the horizontal lifts of tangent vectors of type $(1,0)$ on $\mathbf{P}^1$; locally, $H_{(1,0)}U$ is spanned by the horizontal vector field

$$e = \frac{\partial}{\partial w} + \bar{w}\, e^{-H}\,\zeta \frac{\partial}{\partial \zeta}\,.$$

The holomorphic tangent bundle for S^3, $H_{(1,0)}S^3 := (T_{(1,0)}\mathbb{C}^2) \cap (\mathbb{C} \otimes TS^3)$, is simply $H_{(1,0)}U$ restricted to ∂U.

Remark 2.5. *The choice of a different hermitian norm on* $\mathbb{C}^2$ *induces a different hermitian metric* $\tilde{h}$ *on* E*, with the corresponding hermitian connection* $\tilde{\eta}$ *and horizontal* $(1,0)$ *vector field* $\tilde{e}$*. This choice can be interpreted as choosing a different circular domain in* $\mathbb{C}^2$*; note, however, that the circular domain still admits an* S^1 *action which preserves the holomorphic tangent space.*

A result of Kiremidjian [Kir] says that any small deformation of the CR structure on S^3 extends to define an integrable deformation of the complex structure on U. Moreover, in [B], an explicit extension is obtained in which the holomorphic structure on the hyperplane at infinity is left unchanged. For the convenience of the reader, we outline the argument here, and recall the precise statement of the result.

It is well known (see e.g. [B], [CL]) that every small deformation of the standard CR structure on S^3 is equivalent to one whose deformation tensor is of the form $\phi \in \Gamma\left(S^3, \mathrm{Hom}\left(H_{(0,1)}, H_{(1,0)}\right)\right)$. Moreover, by considering the action of the group of contact diffeomorphisms on deformations, one can show (see [B]) that, up to equivalence, ϕ is of the form (locally)

$$\phi = \sum_{k=1}^{\infty} \phi_k(w)\zeta^k d\bar{w} \otimes e\,,$$

where $\phi_k(w)$ are smooth functions. We refer to this as *exterior form.*

In the statement of the next theorem, and throughout the paper, we will use the anisotropic Folland Stein Γ^s norms [FS] to measure smoothness, and to introduce a topology on the various function spaces. These anisotropic norms measure L^2 derivatives only in the CR or conjugate CR directions; after fixing the connection form $\tilde{\eta}$, the span of these directions is precisely the span of the vector fields $\tilde{e}, \bar{\tilde{e}}$, or the distribution which is dual to the connection form.

Remark 2.6. *Throughout the paper, "smooth" objects will refer to objects with an appropriate degree of smoothness in some* Γ^s *norm.*

Theorem 2.7. *(Bland* [B]*) Let* ϕ *be a sufficiently small deformation of the standard CR structure on* $S^3 = \partial U$*, measured in the* Γ^s *norm relative to the standard framing of* S^3*,* $s \geq 6$*. Then* ϕ *is equivalent to a CR structure of the form*

$$\phi = \sum_{k=0}^{\infty} \phi_k(w)\zeta^k d\bar{w} \otimes e\,,$$

where $\phi_k(w)$ *are smooth functions of* w*; moreover, there exists a (possibly different) connection form* $\tilde{\eta}$ *with its corresponding horizontal lift* $\tilde{e}$ *of the basic vector field* $\partial/\partial w$ *such that* ϕ *is equivalent to a CR structure of the form*

$$\phi = \sum_{k=1}^{\infty} \tilde{\phi}_k(w)\zeta^k d\bar{w} \otimes \tilde{e}\,, \tag{2.8}$$

where $\tilde{\phi}_k(w)$ *are smooth functions of* w*.*

Throughout the remainder of the paper, we will assume that the deformation tensor is normalized according to equation (2.8). Moreover, we will drop the decoration "$\tilde{}$", and refer to the connection form as η and the corresponding horizontal lift of $\partial/\partial w$ as e. The Folland Stein Γ^s norms will be defined relative to the horizontal distribution in the tangent space on S^3 which is defined by η.

Using the ideas of [BD], one can show that ϕ extends in the obvious way to define an integrable deformation of the complex manifold U. Notice that ϕ vanishes along the zero section of E.

The following theorem summarizes the discussion. The extension result is a special case of the theorem of Kiremidjian [Kir], while the normalization procedure was contained in [B] (see also [BD]).

Theorem 2.9. *(Kiremidjian* [Kir], *Bland* [B]*) Let* ϕ *be a sufficiently small deformation of the standard CR structure on* $S^3 = \partial U$*, measured in the* Γ^s *norm relative to the standard framing of* S^3*,* $s \geq 6$*. Then* ϕ *extends to define an integrable deformation of the complex structure on* U*.*

Moreover, up to equivalence, ϕ *can be taken to be of the form*

$$\phi = \sum_{k=1}^{\infty} \phi_k(w)\zeta^k d\bar{w} \otimes e\,, \tag{2.10}$$

where $\phi_k(w)$ *are smooth functions of* w*.*

As a consequence of Kiremdjian's result and well known results of Harvey–Lawson [HL] and Folland and Kohn [FoKo], we have the following theorem, first obtained by Lempert in [Le1] .

Theorem 2.11. *(Lempert* [Le1]*) Let* ϕ *denote a sufficiently small deformation of the standard CR structure on* S^3*, as measure in the* Γ^s *norm,* $s \geq 6$*. Then* (S^3, ϕ) *is* C^1 *embeddable if and only if there exists a compact complex surface* X *and an embedding* $(S^3, \phi) \hookrightarrow X$ *for which* (S^3, ϕ) *disconnects* X *into two connected components.*

Proof. Suppose first that (S^3, ϕ) is embeddable. Then by Harvey–Lawson [HL], there is a normal Stein space V for which (S^3, ϕ) is the pseudoconvex boundary; resolve any singularities to obtain a smooth complex manifold $\tilde{V}$ for which (S^3, ϕ) is the pseudoconvex boundary. Kiremidjian's result implies that there is a complex manifold (U, ϕ) for which (S^3, ϕ) is the pseudoconcave boundary. Glue these two pieces along (S^3, ϕ); thus, we obtain a C^1 compact manifold X with an integrable complex structure, and (S^3, ϕ) disconnects X. By the Newlander–Nirenberg theorem, X is a smooth compact complex manifold.

Conversely, if there exists X and an embedding $(S^3, \phi) \hookrightarrow X$ which disconnects X, then (S^3, ϕ) is the pseudoconcave boundary of one component, and the pseudoconvex boundary of the other component. Let V denote the pseudoconvex component. By the results of Folland and Kohn on the solvability of $\bar{\partial}$ on compact complex manifolds with pseudoconvex boundary [FoKo], one can construct sufficiently many functions which are holomorphic on V and C^1 to the boundary to embed (S^3, ϕ). ∎

Using the analysis of Morrow and Rossi [MR], much more can be said about the manifold X; in fact, we obtain the following stability result of Lempert [Le1].

Theorem 2.12. *For $s \geq 6$, there is a neighbourhood of the standard CR structure on S^3 such that (S^3, ϕ) is embeddable if and only if (S^3, ϕ) is embeddable in $\mathbb{C}^2$.*

Proof. (Lempert [Le1]) In the previous theorem, we showed that if (S^3, ϕ) embedded, then it embedded into a complex surface X as a disconnecting hypersurface. In light of the normal form analysis of Theorem 2.9, we know that we can choose the pseudoconcave component of X to contain a rational curve $\mathbb{P}^1$ with the hyperplane bundle as its normal bundle. In this situation, a rigidity result of Morrow and Rossi [MR] states that X must be birational to $\mathbb{P}^2$, and the rational curve is a standard linear hyperplane; that is, we may choose X to be $\mathbb{P}^2$, and the pseudoconcave component is a neighbourhood of the hyperplane at infinity. ▌

The following corollary is immediate from the construction of the manifold X.

Corollary 2.13. *There is a neighbourhood of the standard CR structure on S^3 such that (S^3, ϕ) is embeddable if and only if (U, ϕ) is biholomorphic to a neighbourhood of the zero section of E.*

This relates the embeddability of (S^3, ϕ) to the deformation theory for the pseudoconcave complex manifold (U, ϕ) which it bounds. Moreover, we can infinitesimalize this result to arbitrarily small neighbourhoods of the rational curve $\mathbb{P}^1$. However, the analysis leads us to questions of convergence, and we will delay this result until the end of the next section.

§3. Formal Embeddability

In this section, we will relate the CR deformation theory for S^3 to the Morrow–Rossi deformation theory for formal neighbourhoods of the hyperplane at infinity.

Throughout this section, (S^3, ϕ) will denote a sufficiently small deformation of the standard CR structure in the Γ^s norm, $s \geq 6$, and (U, ϕ) will denote the extension of the deformation to the pseudoconcave side. We will assume that the deformation tensor has been placed in exterior form; that is, it can be expressed as $\phi = \sum_{k=1}^{\infty} \phi_k(w)\zeta^k d\bar{w} \otimes e$, where

ϕ_k are Γ^s functions which are constant on the fibres. Each graded piece $\phi_k(w)\zeta^k d\bar{w} \otimes e$ has a natural interpretation as a deformation tensor on $\mathbf{P}^1$ twisted by a positive power of the dual of the hyperplane line bundle, i.e. as a section of $\mathrm{Hom}(T_{(0,1)}\mathbf{P}^1, T_{(1,0)}\mathbf{P}^1) \otimes \otimes^k E^*$; moreover, since there are no zeroth order terms, the complex structure on $\mathbf{P}^1$ is left unchanged. We will henceforth refer to this rational curve as the $\mathbf{P}^1$.

Throughout this paper, we will be concerned only with objects which are holomorphic in the ζ variable; thus, we may identify them with the sum of sections of powers of the dual of the hyperplane bundle.

We now describe how to pass from the Dolbeault approach to deformation theory to the Cech approach in this situation. In brief, although the coordinate cover U_i is not a Stein cover, it is still sufficiently nice that we can pass from the deformation tensor to new coordinate functions (ξ, ρ) which are holomorphic in the deformed structure.

We can write down explicit formal expressions for local functions (ξ, ρ) and $(\hat{\xi}, \hat{\rho})$ which are holomorphic in the deformed structure and which converge on the chart $|w|^2 < 4$, (respectively, $|\hat{w}|^2 < 4$) (see, for instance, [B]). Then we look at the transition functions as expressed using the new coordinate systems (ξ, ρ) and $(\hat{\xi}, \hat{\rho})$. The next two propositions analyze these transition functions in a manner which is reminiscent of the formal neighbourhoods of $\mathbf{P}^1$ as studied by Morrow and Rossi [MR]; we will refer to this observation again after the statement and proofs of the propositions.

Proposition 3.1. *Let (S^3, ϕ) be a deformation of the standard CR structure of S^3 which is in exterior form and sufficiently small in the Γ^s norm, $s \geq 6$; let (U, ϕ) be its extension to U. Let U_1, U_2 be the standard coordinate cover of the neighbourhood U with coordinates (w, ζ) and their hatted counterparts.*

Then there exist local coordinates (ξ, ρ) on U_1 and their hatted counterparts on U_2 which are holomorphic to order k for the deformed complex structure.

Moreover, the new coordinates can be taken to be of the form

$$\rho = \zeta(1 + \sum_{j=1}^{k} \rho_j \zeta^j) + O(\zeta^{k+1}) \quad \textit{and} \quad \xi = w + \sum_{j=1}^{k} \xi_j \zeta^j + O(\zeta^{k+1})$$

where ξ_j, ρ_j are smooth functions of w.

Proof. We illustrate the approach in this case, introducing the formalism which we use in solving the $\bar{\partial}$ equation for the deformed structure (that is, $\bar{\partial}_\phi$), and the recursive algorithm.

As explained in [B], the $\bar\partial$ operator for the deformed structure is expressed as $\bar\partial_\phi = \bar\partial - \phi\circ\partial$. Let u be a function on U_1 that is holomorphic in the fibre directions; that is, u is a function of the form $u = \sum_j u_j\zeta^j$, where u_j is a function of w. Then

$$\phi\circ\partial(u) := (\sum_{l=1}^{\infty}\phi_l\zeta^l)\sum_j(e^{jH}\frac{\partial(e^{-jH}u_j)}{\partial w})\zeta^j\,d\bar w\,.$$

If we ask for a function which agrees with w when $\zeta = 0$, and is holomorphic to order k about $\mathbb{P}^1$, then we consider a function of the form $u = w + \sum_{j=1}^k u_j\zeta^j$, and solve inductively:

$$\bar\partial(u) = \phi\circ\partial(u)\,; \tag{3.2}$$

$$\sum_{j=1}^{k}\bar\partial u_j\zeta^j = (\sum_{l=1}^{\infty}\phi_l\zeta^l)\sum_{j=1}^{k}(e^{jH}\frac{\partial(e^{-jH}u_j)}{\partial w})\zeta^j\,d\bar w\,. \tag{3.3}$$

Since for each power of ζ we are solving a one variable $\bar\partial$ equation for u_k in terms of data which has been previously determined, we can obtain k^{th} order formal solutions for all k. Finally, it is a simple matter to observe that for any order k, we may obtain local functions of the form given in the proposition which are holomorphic to order k in the deformed structure.

∎

This proposition is a form of the statement that deformations of complex structures are locally trivial; moreover, the cover U_i is a "good" cover of U. The Cech data for the deformed complex manifold is given by the transition functions for the cover. We compute these in the next proposition.

Proposition 3.4. *Let (U,ϕ) and U_i be as above, and let*

$$\rho = \zeta(1+\sum_{j=1}^{k}\rho_j\zeta^j) \quad and \quad \xi = w + \sum_{j=1}^{k}\xi_j\zeta^j$$

be new coordinates which are holomorphic in the deformed complex structure as constructed in the last proposition. Then after possibly choosing new representative holomorphic functions for the deformed structure, which we still denote by (ξ,ρ), holomorphic transition functions for the deformed manifold (U,ϕ) can be taken to be of the form

$$\xi\hat\xi = 1 + \sum_{i=4}^{k}(\sum_{j=2}^{i-2}\frac{a_{ij}}{w^j})\zeta^i + O(\zeta^{k+1}) \tag{3.5}$$

and

$$\hat{\rho}\xi/\rho = 1 + \sum_{i=2}^{k}(\sum_{j=1}^{i-1} \frac{b_{ij}}{w^j})\zeta^i + O(\zeta^{k+1})\,; \tag{3.6}$$

that is, we can find holomorphic functions (ξ, ρ) *on* U_1 *and* $(\hat{\xi}, \hat{\rho})$ *on* U_2 *for the deformed complex structure* $\bar{\partial}_\phi$ *such that they satisfy the relations given above on* $U_1 \cap U_2$.

Remark 3.7. *Equations (3.5), (3.6) take on the more standard form of transition functions if we solve explicitly for* $\hat{\xi}, \hat{\rho}$ *respectively.*

Proof. Let ξ, ρ be of the form given in the previous proposition. Consider the first order expansion. The transition functions are given by

$$\begin{aligned} \xi\hat{\xi} &= (w + \xi_1\zeta)(\hat{w} + \hat{\xi}_1\hat{\zeta}) + O(\zeta^2) \\ &= 1 + \zeta(\hat{w}\xi_1) + \hat{\zeta}(w\hat{\xi}_1) + O(\zeta^2) \\ &= 1 + \zeta(\frac{1}{w}\xi_1 + \hat{\xi}_1) + O(\zeta^2)\,. \end{aligned} \tag{3.8}$$

Since the product $\xi\hat{\xi}$ is holomorphic on the intersection and the zeroth order term is constant, a simple calculation shows that the first order term $(\frac{1}{w}\xi_1 + \hat{\xi}_1)$ is holomorphic in the standard structure. Since $\xi, \hat{\xi}$ are only determined up to the addition of functions that are holomorphic in $w, \hat{w}$ respectively, we easily observe that we can choose these functions in such a way as to normalize the first order term to be zero.

Similarly, we consider the transition function for the fibre variable. In this case, we have

$$\begin{aligned} \frac{\hat{\rho}\xi}{\rho} &= \frac{\hat{\zeta}(1 + \hat{\rho}_1\hat{\zeta})(w + \xi_1\zeta)}{\zeta(1 + \rho_1\zeta)} + O(\zeta^2) \\ &= 1 + \hat{\zeta}(\hat{\rho}_1\hat{\zeta}w + \xi_1\zeta)/\zeta - \hat{\zeta}w\rho_1\zeta/\zeta + O(\zeta^2) \\ &= 1 + \zeta(\hat{\rho}_1/w + \xi_1/w - \rho_1) + O(\zeta^2) \end{aligned} \tag{3.9}$$

where as before, $(\hat{\rho}_1/w + \xi_1/w - \rho_1)$ is holomorphic on $U_1 \cap U_2$, ξ_1 is a smooth function determined by the previous step, and $\rho_1, \hat{\rho}_1$ are determined up to the addition of functions that are analytic in $w, \hat{w} = 1/w$ respectively. It is clear that we can normalize the expression in the brackets to be zero. This completes the first order normalization.

We now proceed to the inductive step. Assume that $\xi, \hat{\xi}, \rho, \hat{\rho}$ have been chosen to order $k - 1$ in such a way as to place the transition functions in normal form to order $k - 1$. Then

$$
\begin{aligned}
\xi\hat{\xi} &= (w+\sum_{i=1}^{k}\xi_i\zeta^i)(\hat{w}+\sum_{i=1}^{k}\hat{\xi}_i\hat{\zeta}^i)+O(\zeta^{k+1})\\
&= (w+\sum_{i=1}^{k-1}\xi_i\zeta^i+\xi_k\zeta^k)(\hat{w}+\sum_{i=1}^{k-1}\hat{\xi}_i\hat{\zeta}^i+\hat{\xi}_k\hat{\zeta}^k)+O(\zeta^{k+1})\\
&= (w+\sum_{i=1}^{k-1}\xi_i\zeta^i)(\hat{w}+\sum_{i=1}^{k-1}\hat{\xi}_i\hat{\zeta}^i)+w\hat{\xi}_k\hat{\zeta}^k+\hat{w}\xi_k\zeta^k+O(\zeta^{k+1}) \qquad (3.10)\\
&= (w+\sum_{i=1}^{k-1}\xi_i\zeta^i)(\hat{w}+\sum_{i=1}^{k-1}\hat{\xi}_i\hat{\zeta}^i)+\hat{\xi}_k\zeta^k/w^{k-1}+\xi_k\zeta^k/w\\
&+ O(\zeta^{k+1}).
\end{aligned}
$$

Using the fact that $\xi_k, \hat{\xi}_k$ are determined only up to the addition of a holomorphic function in $w, \hat{w}$ respectively, it is easy to observe that the normal form for the transition function is

$$
\xi\hat{\xi} = 1+\sum_{i=4}^{k}(\sum_{j=2}^{i-2}\frac{a_{ij}}{w^j})\zeta^i+O(\zeta^{k+1}). \tag{3.11}
$$

A similar argument for the fibre variable shows that

$$
\begin{aligned}
\hat{\rho}\xi/\rho &= \hat{\zeta}(1+\sum_{i=1}^{k}\hat{\rho}_i\hat{\zeta}^i)(w+\sum_{i=1}^{k}\xi_i\zeta^i)/\zeta(1+\sum_{i=1}^{k}\rho_i\zeta^i)+O(\zeta^{k+1})\\
&= \hat{\zeta}(1+\sum_{i=1}^{k-1}\hat{\rho}_i\hat{\zeta}^i)(w+\sum_{i=1}^{k-1}\xi_i\zeta^i)/\zeta(1+\sum_{i=1}^{k-1}\rho_i\zeta^i)+\hat{\zeta}w\hat{\rho}_k\hat{\zeta}^k/\zeta\\
&\quad +\hat{\zeta}\xi_k\zeta^k/\zeta-\hat{\zeta}w\rho_k\zeta^k/\zeta+O(\zeta^{k+1} \qquad (3.12)\\
&= \hat{\zeta}(1+\sum_{i=1}^{k-1}\hat{\xi}_i\hat{\zeta}^i)(w+\sum_{i=1}^{k-1}\xi_i\zeta^i)/\zeta(1+\sum_{i=1}^{k-1}\xi_i\zeta^i)\\
&\quad +\frac{\hat{\rho}_k}{w^k}\zeta^k+\xi_k\zeta^k/\omega-\rho_k\zeta^k+O(\zeta^{k+1}).
\end{aligned}
$$

Using the fact that $\rho_k, \hat{\rho}_k$ are determined only up to the addition of a holomorphic function in $w, \hat{w}$ respectively, and that ξ_k has been determined above, it is easy to observe that the normal form for the transition functions is

$$\hat{\rho}\xi/\rho = 1 + \sum_{i=2}^{k}(\sum_{j=1}^{i-1} \frac{b_{ij}}{w^j})\zeta^i . \tag{3.13}$$

∎

We now recall the Morrow–Rossi invariants. In [NirSp] (see also [MR]), Nirenberg and Spencer considered the deformation theory for embedded complex submanifolds. Their results in the current context are easy to describe. Two deformations (U, ϕ_1), (U, ϕ_2) are said to be formally k^{th} order equivalent along $\mathbb{P}^1$ if there is a diffeomorphism $(U, \phi_1) \to (U, \phi_2)$ which fixes $\mathbb{P}^1$ and is holomorphic to order $(k+1)$ along $\mathbb{P}^1$. We will call a deformation (U, ϕ) k^{th} *order standard* if (U, ϕ) is k^{th} order equivalent to the undeformed U along P^1. Nirenberg and Spencer showed that the obstruction to extending a k^{th} order formal equivalence to a $(k+1)^{st}$ order equivalence lies in the first cohomology of $\mathbb{P}^1$ with values in the tangent bundle of $\mathbb{P}^2$ restricted to $\mathbb{P}^1$, twisted by the $(k+1)$ power of the dual to the hyperplane line bundle. Since the tangent bundle to $\mathbb{P}^2$ restricted to $\mathbb{P}^1$ is $E^2 \oplus E$, their results in the current context can be stated as follows.

Theorem 3.14. *(Nirenberg–Spencer* [NirSp]*) Let (U, ϕ) be a deformation of U that is $(k-1)^{st}$ order standard along $\mathbb{P}^1$. The obstruction to (U, ϕ) being k^{th} order standard lies in $H^1(\mathbb{P}^1, (E^2 \oplus E) \otimes E^{-k})$.*

We may now cast the results of Proposition 3.4 in terms of the Morrow–Rossi invariants.

Corollary 3.15. *The deformed manifold (U, ϕ) is k^{th} order standard along $\mathbb{P}^1$ if and only if the coefficients a_{ij}, b_{ij} vanish for all $j \leq k$.*

Proof. As in Morrow and Rossi [MR], one can compute the invariants by considering a coordinate cover of $\mathbb{P}^1$ and computing normalized transition functions. A straightforward comparison shows that these are the same invariants as have been calculated in the previous proposition. ∎

The next proposition relates the invariants introduced above to the stable embeddability of the new structure. That such a relationship exists is clear, but it will be convenient to indicate an explicit algorithm for the procedure.

Our approach will be to obtain a deformation of the identity embedding $(z^1, z^2) : U \hookrightarrow \mathbb{P}^2$. Notice that while the functions (z^1, z^2)

are well defined on the complement of $\mathbb{P}^1$, they extend as meromorphic functions only to $\mathbb{P}^2$ minus a point on the hyperplane at infinity. (As a map, $(z^1, z^2) : U \hookrightarrow \mathbb{P}^2$ is well defined, but the components are not defined at the points $\zeta = w = 0$ in U_1 and $\hat{\zeta} = \hat{w} = 0$ in U_2.) We can, however, treat these functions as being defined on S^3. We will look for deformed functions (σ^1, σ^2) that "agree with (z^1, z^2) on $\mathbb{P}^1$".

To explain the meaning of this statement, notice that the circular action on S^3 induces a natural Fourier decomposition on the space of functions on S^3; any function with only negative Fourier components can be extended to U as a function that is holomorphic in the fibre variable ζ; conversely, functions that are holomorphic in the fibre variable restrict to S^3 as functions with only negative Fourier components. Thus, all functions are well defined on S^3, and the vanishing of the first k negative Fourier coefficients on S^3 corresponds to the vanishing to k^{th} order along $\mathbb{P}^1$ of the extended function. Moreover, k^{th} order formal neighbourhoods correspond to functions on S^3 with negative Fourier components up to order k. Thus, whenever the extension to $\mathbb{P}^1$ comes into question, we can view the analysis as taking place on S^3.

Proposition 3.16. *Let (S^3, ϕ), (U, ϕ) be as above.*

Suppose that there exist deformations (σ^1, σ^2) of (z^1, z^2) that are meromorphic to order $(k-1)$ along $\mathbb{P}^1$ in the deformed structure defined by ϕ and agree with (z^1, z^2) along $\mathbb{P}^1$; then the Morrow Rossi invariants vanish for all $j \leq k$.

Conversely, suppose that the Morrow Rossi invariants vanish for all $j \leq k$; then there exist deformations (σ^1, σ^2) of (z^1, z^2) that are meromorphic to order $(k-1)$ along $\mathbb{P}^1$ in the deformed structure defined by ϕ and agree with (z^1, z^2) along $\mathbb{P}^1$.

Proof. Let σ^1, σ^2 be the deformations of z^1, z^2 respectively that are holomorphic to order $(k-1)$ along $\mathbb{P}^1$ relative to the deformed complex structure. That is,

$$\sigma^1 = z^1 + O(\zeta^0) \qquad \sigma^2 = z^2 + O(\zeta^0)$$

and

$$\bar{\partial}_\phi \sigma^1 = O(\zeta^k) \qquad \bar{\partial}_\phi \sigma^2 = O(\zeta^k)$$

where the orders refer to the order of vanishing along $\mathbb{P}^1$. We can define local holomorphic coordinates by (notice that, while the functions may be defined to all orders, they are only holomorphic to the indicated order along $\mathbf{P}^1$):

$$\begin{array}{llll} \rho & = 1/\sigma^1 + O(\zeta^{k+2}) & \xi & = \sigma^2/\sigma^1 + O(\zeta^{k+1}) \quad \text{on } U_1 \\ \hat{\rho} & = 1/\sigma^2 + O(\zeta^{k+2}) & \hat{\xi} & = \sigma^1/\sigma^2 + O(\zeta^{k+1}) \quad \text{on } U_2 \end{array}$$

and it follows automatically that

$$\xi\hat{\xi} = 1 + O(\zeta^{k+1}) \qquad \hat{\rho}\xi/\rho = 1 + O(\zeta^{k+1}) \qquad \text{on } U_1 \cap U_2 .$$

In particular, the Morrow Rossi invariants vanish to order k.

Conversely, suppose that

$$\rho = \zeta(1 + \sum_{i=1}^{k} \rho_i \zeta^i) + O(\zeta^{k+1}) \qquad \xi = w + \sum_{i=1}^{k} \xi_i \zeta^i + O(\zeta^{k+1}) .$$

Then we can define sections σ^1, σ^2 by

$$\begin{array}{lllll} (3.17) & \sigma^1 = 1/\rho & \sigma^2 = \xi/\rho & \text{on } U_1 \\ (3.18) & \sigma^2 = 1/\hat{\rho} & \sigma^1 = \hat{\xi}/\hat{\rho} & \text{on } U_2 , \end{array}$$

where σ^1 is well defined to order $n, n \leq (k-1)$ if and only if

$$\begin{array}{lll} & 1/\rho - \hat{\xi}/\hat{\rho} = O(\zeta^{n+1}) & \text{on } U_1 \cap U_2 \\ (3.19) \quad \iff & \rho\hat{\xi}/\hat{\rho} = 1 + O(\zeta^{n+2}) & \text{on } U_1 \cap U_2 . \end{array}$$

Similarly, σ^2 is well defined to order n if and only if

$$\begin{array}{lll} & \hat{\rho}\xi/\rho = 1 + O(\zeta^{n+2}) & \text{on } U_1 \cap U_2 , \\ \iff & \rho/(\hat{\rho}\xi) = 1 + O(\zeta^{n+2}) & \text{on } U_1 \cap U_2 \\ (3.20) \iff & (\rho\hat{\xi}/\hat{\rho}) \cdot (1/\xi\hat{\xi}) = 1 + O(\zeta^{n+2}) & \text{on } U_1 \cap U_2 . \end{array}$$

Therefore, the pair of sections σ^1, σ^2 are well defined to order n if

$$(3.21) \qquad \hat{\rho}\xi/\rho = 1 + O(\zeta^{n+2}) \qquad \xi\hat{\xi} = 1 + O(\zeta^{n+2}) \qquad \text{on } U_1 \cap U_2 ;$$

that is, if $n \leq (k-1)$, and if the Morrow Rossi invariants vanish to order $(n+1)$. ▌

Remark 3.22. *We can expand σ_1, σ_2 in powers of ζ, and obtain*

$$(3.23) \qquad \sigma^1 = 1/\rho = \frac{1}{\zeta} \frac{1}{(1 + \sum_{i=1}^{k} \rho_i \zeta^i)} = \frac{1}{\zeta} \left(1 + \sum_{i=1}^{k} \sigma_i \zeta^i \right) + O(\zeta^k)$$

and

$$\sigma^2 = \xi/\rho = (w + \sum_{i=1}^{k} \xi_i \zeta^i)(1 + \sum_{i=1}^{k} \sigma_i \zeta^i)/\zeta + O(\zeta^k)\,. \tag{3.24}$$

Before stating the next theorem, we introduce the canonical solution operator for the $\bar{\partial}_b$ equation on (S^3, η). We denote it by $G\partial$, where $G\partial$ is defined by the properties (1) $G\partial\bar{\partial}u = u$ for all u orthogonal to the space of CR functions; (2) $G\partial\alpha = 0$ for all α orthogonal to the range of $\bar{\partial}$. Notice that $G\partial\bar{\partial}$ preserves the grading induced by the Fourier decomposition, and that the operator $G\partial$ gains one anisotropic derivative; that is, the operator $G\partial$ satisfies the regularity estimate in the Γ^s norms, $||G\partial u||_{k+1} \leq c||u||_k$ for some constant c.

In the paragraph above, a formal replacement of the operators $\bar{\partial}, \partial$ by $\bar{\partial}_b, \partial_b$ respectively expresses everything in terms of the boundary $\bar{\partial}$ operators; however, we choose this notation to emphasize the fact that formally, we may think in terms of the extended operators on the manifold U. Notice that while there is a formal means of passing between the two approaches, we have chosen a 'mixed' notation; that is, we compute L^2 inner products using the spherical volume form and the restriction of the functions to $S^3 = \partial U$, while we use the notation that is naturally associated with solving the $\bar{\partial}$ equation on U.

Theorem 3.25. *Let (S^3, ϕ), (U, ϕ) be as above. Then the deformed manifold (U, ϕ) is k^{th} order standard if and only if the functions $[\phi\partial(\sum_{j=0}^{n+1}(G\partial\phi\partial)^j z^i)]_n$, defined on S^3, are in the range of $\bar{\partial}_b$ for all $0 \leq n \leq k$ and $i = 1, 2$.*

Corollary 3.26. *Let (S^3, ϕ) , (U, ϕ) be as above. Then the deformed manifold (U, ϕ) is k^{th} order standard if and only if the functions $[\phi\partial(\sum_{j=0}^{k}(G\partial\phi\partial)^j z^i)]_k$, defined on S^3, are in the range of $\bar{\partial}_b$ for $i = 1, 2$.*

We will establish two preliminary lemmas before proving the theorem.

Lemma 3.27. *Suppose that $u = \sum_{j=-1}^{k} u_j \zeta^j$ and $v = \sum_{j=-1}^{k} v_j \zeta^j$ satisfy $\bar{\partial}_\phi u = O(\zeta^{k+1})$, $\bar{\partial}_\phi v = O(\zeta^{k+1})$, and $u_{-1} = v_{-1}$, $u_0 = v_0$; then*

$$1) \quad \bar{\partial} u_n \zeta^n = -[\bar{\partial}_\phi [u]_{(n-1)}]_n \quad \textit{for} \quad 0 \leq n \leq k \tag{3.28}$$

$$2) \quad u = v + O(\zeta^{k+1})\,. \tag{3.29}$$

Proof. The first observation follows directly from expanding the equation $[\bar{\partial}_\phi(u)]_k = 0$ in powers of ζ. Notice that the solution to equation 1, if it exists, is unique for $n \geq 1$, and unique up to a constant for $n = 0$.

The second statement follows by induction. It is true for $n = 0$. Assume it is true for $n-1$, $1 \le n \le k$; then for n, we have $\bar\partial(u_n\zeta^n) = -[\bar\partial_\phi[u]_{n-1}]_n = -[\bar\partial_\phi[v]_{n-1}]_n = \bar\partial(v_n\zeta^n)$. Thus, $\bar\partial(u_n\zeta^n) = \bar\partial(v_n\zeta^n)$ with $n \ge 1$ and uniqueness implies that $u_n = v_n$. ▮

Lemma 3.30. *Suppose that* $u = \sum_{j=-1}^{k} u_j\zeta^j$, $\bar\partial_\phi u = O(\zeta^{k+1})$, $u_{-1}\zeta^{-1} = z^1$ *and* $u_0 = [G\partial\phi\partial(z^1)]_0$*; then*

$$u_n\zeta^n = [\sum_{j=0}^{n+1}(G\partial\phi\partial)^j z^1]_n - [\sum_{j=0}^{n+1}(G\partial\phi\partial)^j z^1]_{n-1} \tag{3.31}$$

$$[u]_n = [\sum_{j=0}^{n+1}(G\partial\phi\partial)^j z^1]_n \tag{3.32}$$

for $1 \le n \le k$.

Proof. This also follows by induction. Suppose that the result is true for $n-1$, where $0 < n \le k$. Then

$$\begin{aligned}
\bar\partial u_n\zeta^n &= -\left[\bar\partial_\phi(\sum_{j=-1}^{n-1} u_j\zeta^j)\right]_n \\
&= -\left[(\bar\partial - \phi\partial)(\sum_{j=-1}^{n-1} u_j\zeta^j)\right]_n + \left[(\bar\partial - \phi\partial)(\sum_{j=-1}^{n-1} u_j\zeta^j)\right]_{(n-1)} \\
&= \left[\phi\partial(\sum_{j=-1}^{n-1} u_j\zeta^j)\right]_n - \left[\phi\partial(\sum_{j=-1}^{n-1} u_j\zeta^j)\right]_{(n-1)} \\
&= \left[\phi\partial(\sum_{j=0}^{n}(G\partial\phi\partial)^j z^1)\right]_n - \left[\phi\partial(\sum_{j=0}^{n}(G\partial\phi\partial)^j z^1)\right]_{n-1},
\end{aligned}$$

where we have taken advantage of the fact that $\left[\bar\partial_\phi(\sum_{j=-1}^{n-1} u_j\zeta^j)\right]_{n-1} = 0$ by adding it onto the second line.

Therefore,

$$\begin{aligned}
u_n\zeta^n &= \left[(\sum_{j=0}^{n}(G\partial\phi\partial)^{j+1}z^1)\right]_n - \left[(\sum_{j=0}^{n}(G\partial\phi\partial)^{j+1}z^1)\right]_{n-1} \\
&= \left[(\sum_{j=0}^{n+1}(G\partial\phi\partial)^{j}z^1)\right]_n - \left[(\sum_{j=0}^{n+1}(G\partial\phi\partial)^{j}z^1)\right]_{n-1}.
\end{aligned}$$

▮

We now prove the theorem.

Proof. Suppose that (σ^1, σ^2) defines a formal k^{th} order equivalence. Then $(\lambda\sigma^1 + a, \lambda\sigma^2 + b)$ also defines a k^{th} order equivalence for any constants λ, a, b. Choose a constant a such that

$$\sigma^1 + a = z^1 + G\partial\phi\partial(z^1) + O(\zeta^1)\,;$$

then $\sigma^1 + a$ satisfies the conditions in the last lemma, and

$$[\sigma^1 + a]_k = [\sum_{j=0}^{k+1}(G\partial\phi\partial)^j z^1]_k\,.$$

Moreover, since $\sigma^1 + a$ is meromorphic to order k, it follows that for all $0 \le n \le k$, $[\bar{\partial}_\phi([\sigma^1 + a]_k)]_n = [\bar{\partial}_\phi([\sigma^1 + a]_n)]_n = 0$, whence

$$\begin{aligned} \bar{\partial}([\sigma^1 + a]_n) &= [\phi\partial([\sigma^1 + a]_n)]_n \\ &= [\phi\partial([\sum_{j=0}^{n+1}(G\partial\phi\partial)^j z^1]_n)]_n\,; \end{aligned}$$

in particular, $[\phi\partial([\sum_{j=0}^{n+1}(G\partial\phi\partial)^j z^1]_n)]_n$ is in the range of $\bar{\partial}_b$ for all $0 \le n \le k$.

Conversely, suppose that $[\phi\partial([\sum_{j=0}^{n+1}(G\partial\phi\partial)^j z^1]_n)]_n$ is in the range of $\bar{\partial}_b$ for all $0 \le n \le k$; then define $\sigma^1 = ([\sum_{j=0}^{k+1}(G\partial\phi\partial)^j z^1]_k)$. We calculate

$$\begin{aligned} &(3.33)\quad \bar{\partial}_\phi(\sigma^1) \\ &= \bar{\partial}([\sum_{j=0}^{k+1}(G\partial\phi\partial)^j z^1]_k) - \phi\partial([\sum_{j=0}^{k+1}(G\partial\phi\partial)^j z^1]_k) \\ &= \bar{\partial}G\partial[\phi\partial\sum_{j=0}^{k}(G\partial\phi\partial)^j z^1]_k - \left[\phi\partial([\sum_{j=0}^{k+1}(G\partial\phi\partial)^j z^1]_k)\right]_k + O(\zeta^{k+1}) \\ &= [\phi\partial\sum_{j=0}^{k}(G\partial\phi\partial)^j z^1]_k - [\phi\partial\sum_{j=0}^{k+1}(G\partial\phi\partial)^j z^1]_k + O(\zeta^{k+1}) \\ &= O(\zeta^{k+1})\,. \end{aligned}$$

∎

Finally, we are able to state the main theorem.

Theorem 3.34. *Let (S^3, ϕ), and (U, ϕ) be as above. Then (S^3, ϕ) is embeddable if and only (U, ϕ) is formally standard; that is, if and only if all Morrow Rossi invariants vanish.*

Proof. First notice that if (S^3, ϕ) is embeddable, then (U, ϕ) is formally standard, and hence it is formally standard for all orders k.

Conversely, suppose that (U, ϕ) is formally standard for all k. Then by the last theorem, the functions $[\phi\partial(\sum_{j=0}^{n+1}(G\partial\phi\partial)^j z^i)]_n$, defined on S^3, are in the range of $\bar{\partial}_b$ for all $0 \leq n \leq k < \infty$ and $i = 1, 2$. The formal series $\sum_{j=0}^{\infty}(G\partial\phi\partial)^j z^i$ converges by standard operator estimates. Thus, these formal series define an actual smooth equivalence that is holomorphic to all orders along $\mathbf{P}^1$; by its construction, it is a holomorphic equivalence, and it restricts to S^3 as an embedding of the deformed structure. ∎

As an immediate consequence of the last theorem, we obtain explicit obstructions to embeddability. Recall that in [B], we showed that any sufficiently small deformation tensor can be put in interior normal form—that is, it is equivalent of one of the form $\phi = \mu\bar{\omega} \otimes Z$, where $Z = \bar{z}^2\partial/\partial z^1 - \bar{z}^1\partial/\partial z^2$, $\bar{\omega} = \bar{z}^2 d\bar{z}^1 - \bar{z}^1 d\bar{z}^2$, and $\mu = \mu_- + \mu_+$ where μ_+ corresponds to the part of μ with positive Fourier components, and μ_- is of the form $\mu_- = z^1\bar{h}_1 + z^2\bar{h}_2$ for CR functions h_1, h_2. Moreover, we showed that ϕ is embeddable if and only if $\mu_- = 0$, with a rather direct construction of the embeddability in this case. The results in this paper allow us to give a direct interpretation of the obstructions to embeddability as well.

Corollary 3.35. *If in the notation above,* $\phi = \mu_-\bar{\omega} \otimes Z$*, then* ϕ *is embeddable if and only if* $\mu_- = 0$.

Proof. In the statement, the deformation tensor ϕ is in exterior form. Consequently, Theorem 3.25 applies. Suppose ϕ is embeddable. If $[\phi\partial z^i]_n = 0$ for $i = 1, 2$, then $[\phi\partial z^i]_{n+1} = [(\mu_- Z(z^i)\bar{\omega})]_{n+1}$ is in the range of $\bar{\partial}_b$ for $i = 1, 2$. In particular, for any holomorphic function H

$$\int_{S^3} [(\mu_- Z(z^i))]_{n+1} H \, d\,\mathrm{vol} = 0$$

for $i = 1, 2$. Choosing the specific holomorphic functions h_1, h_2 for two separate choices for H, we find that a necessary condition for embeddability is

$$\int_{S^3} [(\mu_- Z(z^1))]_{n+1} h_2 - [(\mu_- Z(z^2))]_{n+1} h_1 \, d\,\mathrm{vol} = 0$$

or

$$\int_{S^3} [\mu_-]_n (\bar{z}^2 h_2 + \bar{z}^1 h_1) \, d\,\mathrm{vol} = 0\,.$$

Since $\mu_- = z^1\bar{h}_1 + z^2\bar{h}_2$, this implies that $[\mu_-]_n = 0$, and hence $[\phi\partial z^i]_{n+1} = [(\mu_- Z(z^i)\bar{\omega})]_{n+1} = 0$ for $i = 1, 2$. Thus, we are done by induction.

Conversely, if $\mu_- = 0$, then the structure is spherical, and hence embeddable. ▌

In fact, tracing through the arguments in this paper, one can identify the various terms in μ_- with nonvanishing Morrow–Rossi invariants; the non-embeddability of (S^3, ϕ) corresponds to a nontrivial twisting of the complex structure near the hyperplane at infinity.

In conclusion, we state the following infinitesimal version of the embedding result from the last section.

Theorem 3.36. *For $s \geq 6$, there is a Γ^s–neighbourhood of the standard CR structure on S^3 such that (S^3, ϕ) is embeddable if and only if some neighbourhood of $\mathbf{P}^1 \subset (U, \phi)$ is biholomorphic to a neighbourhood of the zero section of E.*

References

[B] J. Bland, *Contact geometry and CR-structures on S^3*, Acta Math. **172** (1994), 1–49.

[BD] J. Bland and T. Duchamp, *Moduli for pointed convex domains*, Invent. math. **104** (1991), 61-112.

[BlEp] J. Bland and C.L. Epstein, *Embeddable CR–structures and Deformations of Pseudoconvex Surfaces, Part I: Formal Deformations*, J. Alg. Geom., (to appear).

[BuEp] D. Burns and C. Epstein, *Embeddability for three-dimensional CR-manifolds*, JAMS, **3** (1990), 809-841.

[CaLe] D. Catlin and L. Lempert *A note on the instability of embeddings of Cauchy–Riemann manifolds*, J. Geom. Anal., **2** (1992), 99–104.

[CL] J.-H. Cheng and J.M. Lee, *A local slice theorem for 3-dimensional CR structures*, Amer. J. Math. **117** (1995), 1249–1298.

[Ep1] C. Epstein, *CR-structures on Three Dimensional Circle Bundles*, Invent. math. **109** (1992), 351-403.

[Ep2] C. Epstein, *A relative index on the space of embeddable CR structures, I and II*, preprint.

[FoKo] G.B. Folland and J.J. Kohn, *The Neumann Problem for the Cauchy–Riemann Complex*, Annals of Math. Studies **75** (1972),Princeton University Press.

[FS] G. B. Folland and E. M. Stein, *Estimates for the $\bar{\partial}_b$ complex and analysis on the Heisenberg group*, Comm. on Pure and Applied Math. **27** (1974), 429–522.

[HL] F. Reese Harvey and H. Blaine Lawson *On the boundaries of complex analytic varieties, I and II*, Annals of Math. **102** (1975) 223–290, and **106** (1977), 213–238.

[Kir] G.K. Kiremidjian, *A direct extension method for CR structures*, Math. Ann., **242** (1979), 1-19.

[Le1] L. Lempert, *On three dimensional Cauchy-Riemann manifolds*, JAMS **5** (1992), 1-50.

[Le2] L. Lempert, *Embeddings of three dimensional Cauchy-Riemann manifolds*, preprint.

[MR] J. Morrow and H. Rossi, *Some general results on equivalence of embeddings*, in "Recent Developments in Several Complex Variables", Ann. Math. Studies **100** (1981), 299–325.

[NirSp] L.Nirenberg and D.C.Spencer, *On the rigidity of holomorphic imbeddings*, Contributions to Function Theory, Tata Institute (1960), Bombay.

J. Bland
Department of Mathematics
University of Toronto
Toronto, M5S 1A1, Ontario
Canada

T. Duchamp
Department of Mathematics
University of Washington
Seattle, WA 98195
U. S. A.

Advanced Studies in Pure Mathematics 25, 1997
CR-Geometry and Overdetermined Systems
pp. 60–84

Aspects of Prescribing Ricci Curvature

Dennis DeTurck and Hubert Goldschmidt

Dedicated to Professor M. Kuranishi on his 70th birthday

§1. Introduction

This article concerns two problems involving the Ricci curvature of a Riemannian metric. In each of these problems, one seeks a metric whose Ricci curvature is prescribed in advance in some manner.

Let X be a manifold of dimension $n \geq 3$, whose tangent and cotangent bundles we denote by T and T^*, respectively. By $\bigotimes^m E$, $\bigwedge^k E$ and $S^l E$, we shall mean the m-th tensor power, k-th exterior product and the l-th symmetric product of a vector bundle E over X, respectively. Under the natural identification of $\mathrm{Hom}\,(T, T^*)$ with $T^* \otimes T^*$, we can view a symmetric 2-form R on X, that is, a section of S^2T^*, as a morphism $R^\flat : T \to T^*$; we shall consider the section $\det R$ of the line bundle $\mathrm{Hom}\,(\bigwedge^n T, \bigwedge^n T^*)$ which is induced by $R^\flat$.

The first problem consists in finding a Riemannian metric with prescribed Ricci tensor. We are given a section R of S^2T^* over X and we seek a Riemannian metric g in some neighborhood of a given point $x_0 \in X$ whose Ricci tensor $\mathrm{Ric}(g)$ is equal to R throughout this neighborhood. The first definitive results concerning the problem of prescribing the Ricci tensor were obtained in [4]. There it was shown that, if $R(x_0)$ is a non-degenerate symmetric quadratic form on T_{x_0}, then a solution of this problem always exists. Examples were also given showing that, when $R(x_0)$ is degenerate, a solution may or may not exist. In the present paper, our attention focuses on the problem of solving the equation $\mathrm{Ric}(g) = R$ when R is degenerate at *every* point of X, but has constant rank.

The second problem we consider here is the prescription of the principal Ricci curvatures of a Riemannian metric (without any prescription

Received April 20, 1996
The first author was supported in part by NSF Grant DMS 95-05175 and the second author was supported in part by NSF Grant DMS 92-03974.

of the directions in which these principal curvatures are attained). If $\{\lambda_1, \ldots, \lambda_n\}$ are given real-valued functions on X, we seek a metric g in a neighborhood of $x_0 \in X$ satisfying

$$\det(\mathrm{Ric}(g) - \lambda_i g) = 0, \qquad i = 1, \ldots, n,$$

on this neighborhood. If these equations hold, then these functions $\{\lambda_1, \ldots, \lambda_n\}$ are principal Ricci curvatures of g, i. e. if x is an arbitrary point of this neighborhood, the scalars $\lambda_1(x), \ldots, \lambda_n(x)$ are eigenvalues of $\mathrm{Ric}(g)$ with respect to g at x . This study of the principal Ricci curvatures in this sense has been proposed by many authors (in particular [3, p. 315] and [1, p. 180]). The reason for their interest in this question arises from the fact that, for generic Riemannian metrics, the principal Ricci curvatures provide canonical coordinates in which to express the metric. Such coordinates enable one to determine whether two Riemannian manifolds are locally (or even globally) isometric. In other words, the principal Ricci curvatures provide the key to obtaining a complete system of scalar invariants for a Riemannian manifold. Thus, it becomes desirable to know whether these important scalar invariants can be prescribed in advance. We will consider this problem in the generic case where the values $\{\lambda_1(x_0), \ldots, \lambda_n(x_0)\}$ are all distinct.

Both of our problems manifest themselves as systems of second-order partial differential equations for the metric g. The system corresponding to the second problem is underdetermined; in fact, it consists of n equations for the $n(n+1)/2$ unknown components of g. Thus we are able to solve the second problem using relatively "soft" techniques. Our main result is the following:

Theorem 1. *Let $\{\lambda_1, \ldots, \lambda_n\}$ be real-analytic real-valued functions on a real-analytic manifold X of dimension $n \geq 3$. Suppose that the values $\{\lambda_1(x_0), \ldots, \lambda_n(x_0)\}$ of these functions at a point $x_0 \in X$ are distinct. Then there exists a real-analytic Riemannian metric g on a neighborhood of x_0 whose principal Ricci curvatures are $\{\lambda_1, \ldots, \lambda_n\}$.*

In fact, under a slightly stronger hypothesis, the above theorem admits an elementary proof, which is given in §3; our precise result can be formulated as follows:

Theorem 2. *Let $\{\lambda_1, \ldots, \lambda_n\}$ be a set of C^∞-real-valued functions on X whose values $\{\lambda_1(x_0), \ldots, \lambda_n(x_0)\}$ at a point $x_0 \in X$ are distinct. If the differentials $\{d\lambda_1, \ldots, d\lambda_n\}$ of these functions are linearly independent at the point x_0, then there exists a C^∞-Riemannian metric g defined in a neighborhood of x_0 whose principal Ricci curvatures are $\{\lambda_1, \ldots, \lambda_n\}$.*

The proof of Theorem 1 (in the analytic case) relies on our finding a non-characteristic direction for the system of equations at each generic 2-jet of a Riemannian metric. This is in some ways similar to what happened for the non-degenerate case of our first problem in [4].

Our other problem, in which we wish to prescribe a degenerate Ricci tensor R, involves some fairly delicate analysis. We will work in the real-analytic category and attempt to construct a power series solution centered at $x_0 \in X$. As is well-known, the second-order equation $\mathrm{Ric}(g) = R$ always implies an additional system of first- order (in both g and R) equations. This is the so-called Bianchi identity for the Ricci curvature. In [4], it was shown that, in the non-degenerate case, the Bianchi identity is the only obstruction to the construction of the construction of the power series solution. When one wishes to extend a 2-jet of a solution g to the equation $\mathrm{Ric}(g) = R$ to a 3-jet of a solution, the Bianchi identity imposes a condition on the 1-jet of g. More generally, the k-jet of the Bianchi identity must be taken into account when specifying the $(k+1)$-jet of a solution to the equation $\mathrm{Ric}(g) = R$ in order to be able to extend this solution to one of order $k+2$. In §4, we explain how it is possible to overcome these obstructions.

However, when R is degenerate and but still has constant rank, additional constraints must be placed upon the unknown metric g, beyond those usually implied by the Bianchi identity. In particular, conditions must be imposed on the 0-jet of g. These make it more difficult (and in some cases, impossible) to satisfy the higher-order prolongations of the equation $\mathrm{Ric}(g) = R$. We denote by K the kernel of the morphism $R^\flat : T \to T^*$. In this paper, as in [6], we analyze the case where the distribution determined by the sub-bundle K of T is integrable and give a sufficient condition for local solvability of our equation. We are now in the midst of studying the case when this distribution is not integrable.

Our analysis of this degenerate case leads us to associate to each vector ξ of the kernel K_x of $R^\flat(x)$, with $x \in X$, a quadratic form Q_ξ on the tangent space T_x, which depends only on R and ξ. The obstruction to the local solvability of the equation $\mathrm{Ric}(g) = R$ can then be formulated as follows: At every point of X, the trace of each of these forms Q_ξ, with respect to a solution g of the equation $\mathrm{Ric}(g) = R$, must vanish. The first positive result we obtained (see [6, Theorem 6.1]) states that, if all these quadratic forms vanish at every point of X, then the equation $\mathrm{Ric}(g) = R$ admits local solutions in the real-analytic category; this includes the case when R is non-degenerate. More generally, we will also assume the space of quadratic forms $\mathcal{Q}_x = \{Q_\xi\}$, with $\xi \in K_x$, associated to a point $x \in X$ has constant dimension m, independent of the point x. For $x \in X$, the null-space of the pencil $\mathcal{Q}_x$ of quadratic forms contains K_x, and so its dimension $\nu(x)$ is $\geq m$.

Let $(x^1, \ldots, x^n)$ be a local coordinate system on a neighborhood U of $x \in X$ such that $\{\partial/\partial x^1, \ldots, \partial/\partial x^r\}$ is a frame for the integrable sub-bundle K of T. In terms of this coordinate system, we have $R_{ij} = 0$, for $1 \leq i \leq n$ and $1 \leq j \leq r$; the quadratic form Q^k corresponding to the section $\partial/\partial x^k$ of K over U, with $1 \leq k \leq r$, is given by

$$Q^k_{ij} = -\frac{1}{2}\frac{\partial R_{ij}}{\partial x^k},$$

for $1 \leq i, j \leq n$. According to our assumption, at every point x of U, the vector space of quadratic forms on T_x generated by $\{Q^1(x), \ldots, Q^r(x)\}$ is m-dimensional.

If $m = 0$, then on the open set U the tensor R depends only on the variables $x^{r+1}, \ldots, x^n$. Thus, R respects the local product structure induced on X by the leaves of K. In this case, if $n - r \geq 3$, or if $n - r = 2$ and R is semi-definite, then the results of [4] can be used to prove the local existence of a product metric which satisfies $\mathrm{Ric}(g) = R$. However, as indicated by corank-one examples on certain unimodular Lie groups (see [6] and [11]), there may be other solutions which do not respect the local product structure induced by R on X.

Our most general result for this problem can be stated as follows:

Theorem 3. *Let R be a real-analytic symmetric 2-form on a real-analytic manifold X of dimension $n \geq 3$. Suppose that the kernel K of $R^\flat$ is an integrable sub-bundle of T and that, for all $x \in M$, the space $\mathcal{Q}_x$ of quadratic forms $\{Q_\xi\}$, with $\xi \in K_x$, has constant dimension m. Let $x_0 \in X$ and suppose that there exists a Riemannian metric g_0 on X such that the trace (with respect to g_0) of the form Q_ξ vanishes, for all $\xi \in K_{x_0}$. Suppose that the following conditions do not hold:*

(i) *we have $m = 2$ and $n = 4$;*
(ii) *we have $m = 2$, $n = 5$ and $\nu(x_0) = 3$;*
(iii) *we have $m = 3$ and $n = 6$.*

Then there exists a real-analytic Riemannian metric g solution of the equation $\mathrm{Ric}(g) = R$ on a neighborhood of x_0.

This theorem tells us that, when the dimension of X is ≥ 7 and K is an integrable sub-bundle of T, our condition for local solvability is always sufficient. When the dimension of the spaces $\mathcal{Q}_x$ is either zero or one, then our proof is somewhat less complicated than in the case $m \geq 2$; the cases $m = 2$ or 3 require special attention.

In the special case when the kernel K of $R^\flat$ is a line bundle, the obstruction to local solvability described above and Theorem 3 lead to a condition, which is essentially necessary and sufficient, given by the following result:

Theorem 4. *Let R be a real-analytic symmetric 2-form on a real-analytic manifold X of dimension $n \geq 3$. Suppose that the kernel K of $R^\flat$ is a sub-bundle of T of rank 1. Let $x \in M$ and $\xi \in K_x$. If Q_ξ is non-zero, there exists a Riemannian metric g solution of the equation* $\mathrm{Ric}(g) = R$ *on a neighborhood of x if and only if Q_ξ is not semi-definite.*

The necessity of the condition is obvious, since Q_ξ must ultimately be traceless with respect to the metric g. We present an outline of the proof of these last two theorems in §4. For all the details, we refer the reader to [6].

In this paper, we shall use the theory of overdetermined partial differential equations of [9] or [2, Chapter IX] and the the notation and terminology introduced there.

We wish to thank R. Bryant and S. S. Chern for organizing a conference in March 1994 at the Mathematical Sciences Research Institute in Berkeley, where a preliminary version of Theorems 3 and 4 were presented. We thank M. Berger for suggesting the problem that led us to Theorem 1. Finally, we wish to thank M. Namba, K. Yamaguchi and the other organizers of the conference in honor of M. Kuranishi, for which this paper was written.

§2. The Ricci and Bianchi operators

If E is a fibered manifold over X, we denote by $J_k(E)$ the fibered manifold of k-jets of sections of E, and by $\pi_k : J_{k+l}(E) \to J_k(E)$ the natural projection. If s is a section of E over a neighborhood of $x \in X$, then $j_k(s)(x)$ is the k-jet of s at x; the projection $\pi : J_k(E) \to X$ sends $j_k(s)(x)$ into x. We identify $J_0(E)$ with E. If E is a vector bundle over X, we denote by $\mathcal{E}$ the sheaf of sections of E over X and we recall that there is a monomorphism of vector bundles

$$\varepsilon : S^kT^* \otimes E \to J_k(E);$$

if s is a section of E over a neighborhood of $x \in X$ whose $(k-1)$-jet vanishes at x, then $j_k(s)(x)$ is equal to the image under ε of a unique element $\varepsilon^{-1}j_k(s)(x)$ of $S^kT^* \otimes E$. If F is another vector bundle over X and

$$\varphi : S^2T^* \otimes E \to F$$

is a morphism of vector bundles, we denote by

$$\varphi^{(l)} : S^{k+l}T^* \otimes E \to S^lT^* \otimes F$$

the l-th prolongation of φ. If R is a non-degenerate section of S^2T^*, we consider the morphism $R^\sharp : T^* \to T$ which is the inverse of $R^\flat : T \to T^*$.

Let g be a Riemannian metric on X whose Levi-Civita connection and Ricci curvature we denote by ∇^g and $\mathrm{Ric}(g)$, respectively. We consider the inner product $\langle \ , \ \rangle_g$ on $\bigotimes^k T^*$ determined by g; then the trace, with respect to g, of an element h of S^2T^* is equal to $\mathrm{Tr}_g\, h = \langle h, g\rangle_g$. Let $(x^1, \dots, x^n)$ be a coordinate system on an open subset U of X. In expressions written in terms of such a local coordinate system, we shall use the summation convention. On U, the Levi-Civita connection ∇^g of g is determined by its Christoffel symbols

$$\Gamma^i_{jk} = \frac{1}{2} g^{is}\left(\frac{\partial g_{sj}}{\partial x^k} + \frac{\partial g_{sk}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^s}\right),$$

while the components of the Ricci tensor $\mathrm{Ric}(g)$ are given by

$$\mathrm{Ric}(g)_{ij} = \frac{\partial \Gamma^s_{ij}}{\partial x^s} - \frac{\partial \Gamma^s_{is}}{\partial x^j} + \Gamma^s_{ij}\Gamma^t_{st} - \Gamma^s_{it}\Gamma^t_{sj}.$$

The Bianchi operator

$$B_g : S^2T^* \to T^*$$

is the first-order linear differential operator defined by

$$(B_g h)(\xi) = \sum_{i=1}^n \big((\nabla^g h)(t_i, \xi, t_i) - \tfrac{1}{2}(\nabla^g h)(\xi, t_i, t_i)\big),$$

for a section h of S^2T^* over X and $\xi \in T_x$, with $x \in X$, where $\{t_1, \dots, t_n\}$ is an orthonormal basis of T_x. We also write $B_g h = \mathrm{Bian}(g, h)$; in fact, on U we have

$$B_g h = g^{ik}\left(\frac{\partial h_{ij}}{\partial x^k} - \frac{1}{2}\frac{\partial h_{ik}}{\partial x^j} - \Gamma^l_{ik} h_{lj}\right) dx^j. \tag{1}$$

The symbol

$$\sigma(B_g) : T^* \otimes S^2T^* \to T^*$$

of the differential operator B_g is given by

$$(\sigma(B_g)u)(\xi) = \sum_{i=1}^n \big(u(t_i, t_i, \xi) - \tfrac{1}{2} u(\xi, t_i, t_i)\big),$$

for $\xi \in T_x$, where $\{t_1, \dots, t_n\}$ is an orthonormal basis of T_x; we denote by $\sigma_l(B_g) : S^{l+1}T^* \otimes S^2T^* \to S^lT^* \otimes T^*$ the l-th prolongation of $\sigma(B_g)$.

The Ricci curvature of g satisfies the Bianchi identity

$$B_g \mathrm{Ric}(g) = 0. \tag{2}$$

Let $S^2_+T^*$ be the open fibered submanifold of S^2T^* consisting of the positive-definite symmetric 2-forms on X. Let E and F be vector bundles over X and let $\psi : E \to F$ be a morphism of vector bundles over $J_k(S^2_+T^*)$. If g is a Riemannian metric on X, we consider the morphism of vector bundles $\psi_g : E \to F$ over X sending $u \in E_x$ into $\psi(j_k(g)(x), u)$, where $x \in X$.

The morphism of fibered manifolds

$$\varphi : J_2(S^2_+T^*) \to S^2T^*,$$

sending the 2-jet $j_2(g)(x)$ at $x \in X$ of a Riemannian metric g on X into $\mathrm{Ric}(g)(x)$, is quasi-linear in the sense of [2, Chapter IX]. The symbol

$$\sigma(\varphi) : S^2T^* \otimes S^2T^* \to S^2T^*$$

of φ is the morphism of vector bundles over $J_1(S^2_+T^*)$ satisfying

$$\varphi(p + \varepsilon u) = \varphi(p) + \sigma(\varphi)_{\pi_1 p} u,$$

for all $p \in J_2(S^2_+T^*)$, $u \in S^2T^* \otimes S^2T^*$. If g is a Riemannian metric on X, the morphism of vector bundles

$$\sigma(\varphi)_g : S^2T^* \otimes S^2T^* \to S^2T^*$$

over X is given by

$$\begin{aligned}(\sigma(\varphi)_g u)(\xi, \eta) = \tfrac{1}{2} \sum_{i=1}^{n} \{&u(t_i, \xi, t_i, \eta) + u(t_i, \eta, t_i, \xi) \\ &- u(t_i, t_i, \xi, \eta) - u(\xi, \eta, t_i, t_i)\},\end{aligned}$$

for $u \in (S^2T^* \otimes S^2T^*)_x$ and $\xi, \eta \in T_x$, where $x \in X$ and $\{t_1, \dots, t_n\}$ is an orthonormal basis of T_x. The l-th prolongation

$$p_l(\varphi) : J_{l+2}(S^2_+T^*) \to J_l(S^2T^*)$$

of φ is quasi-linear; if $\sigma_l(\varphi)_g$ is the l-th prolongation of $\sigma(\varphi)_g$, then we have

$$p_l(\varphi)(q + \varepsilon u) = \varphi(p) + \sigma_l(\varphi)_g u,$$

for all $q \in J_{l+2}(S^2_+T^*)$ and $u \in S^{l+2}T^* \otimes S^2T^*$, with $\pi_1 q = j_1(g)(x)$.

We set $S^lT^* = 0$ for $l < 0$. Let g be a Riemannian metric on X and let

$$\mu_g : (S^lT^* \otimes S^2T^*) \oplus (S^{l+1}T^* \otimes T^*) \to S^{l-1}T^* \otimes T^*$$

be the morphism of vector bundles sending $u \oplus v$, with $u \in S^lT^* \otimes S^2T^*$ and $v \in S^{l+1}T^* \otimes T^*$, into $\sigma_{l-1}(B_g)u$.

From Lemma 3.1 of [6] (with $K = \{0\}$) (see also [4] and [2], Chapter IX), we obtain the following:

Lemma 1. *Let g be a Riemannian metric on X. The morphism of vector bundles $\sigma(\varphi)_g : S^2T^* \otimes S^2T^* \to S^2T^*$ is surjective and the sequences*

$$\tag{3} \begin{aligned} &S^{l+2}T^* \otimes S^2T^* \xrightarrow{\sigma_l(\varphi)_g \oplus \sigma_{l+1}(B_g)} \\ &\qquad (S^lT^* \otimes S^2T^*) \oplus (S^{l+1}T^* \otimes T^*) \xrightarrow{\mu_g} S^{l-1}T^* \otimes T^* \to 0 \end{aligned}$$

are exact for all $l \geq 0$.

From Lemma 1, we obtain the exact sequences

$$\tag{4} S^{l+2}T^* \otimes S^2T^* \xrightarrow{\sigma_l(\varphi)_g} S^lT^* \otimes S^2T^* \xrightarrow{\sigma_{l-1}(B_g)} S^{l-1}T^* \otimes T^* \to 0,$$

for all $l \geq 0$. If $\beta \in T^*_x$, with $x \in X$, we consider the morphisms

$$\sigma_\beta(\varphi)_g : S^2T^*_x \to S^2T^*_x, \qquad \sigma_\beta(B_g) : S^2T^*_x \to T^*_x$$

defined by

$$\sigma_\beta(\varphi)_g h = \sigma(\varphi)_g(\beta \otimes \beta \otimes h), \qquad \sigma_\beta(B_g)(\beta \otimes h),$$

for $h \in S^2T^*_x$. According to [8, §6], if β is a non-zero vector of T^*_x, with $x \in X$, the exactness of the sequences (4) gives us the exact sequence

$$\tag{5} S^2T^*_x \xrightarrow{\sigma_\beta(\varphi)_g} S^2T^*_x \xrightarrow{\sigma_\beta(B_g)} T^*_x \to 0.$$

Now let x_0 be an arbitrary point of X and ρ be an element of $S^2T^*_{x_0}$. From the surjectivity of the morphism $\sigma(\varphi)_g$, we infer that the equation $\mathrm{Ric}(g)(x_0) = \rho$ can always be solved. In fact, we may assume without loss of generality that $X = \mathbb{R}^n$ and that x_0 is the origin of $\mathbb{R}^n$. The surjectivity of the morphism $\sigma(\varphi)_{g_0}$ tells us that there is an element $u \in (S^2T^* \otimes S^2T^*)_{x_0}$ satisfying $\sigma(\varphi)_{g_0}u = \rho$. Then the symmetric tensor g given by

$$g_{ij} = \delta_{ij} + \tfrac{1}{2}u_{ij,kl}x^kx^l$$

is a Riemannian metric on a neighborhood of x_0 satisfying the desired condition.

It is obvious how to use this fact for the prescribed Ricci tensor problem; for our eigenvalue problem, in Lemma 2 below we shall choose ρ to be diagonal with respect to the coordinate system so that our metric will have the desired principal Ricci curvatures at one point.

We now turn to the relation of the Bianchi identity to our problems. Let R be a given section of S^2T^*. The morphism of fibered manifolds

$$\psi_R : J_1(S^2_+T^*) \to T^*,$$

sending the 2-jet $j_2(g)(x)$ at $x \in X$ of a Riemannian metric g on X into $\mathrm{Bian}(g, R)(x)$, is quasi-linear. The symbol

$$\sigma(\psi_R) : T^* \otimes S^2T^* \to T^*$$

of ψ_R is a morphism of vector bundles over $S^2_+T^*$ which satisfies

$$\psi_R(p + \varepsilon u) = \psi_R(p) + \sigma(\psi_R)_{\pi_0 p} u,$$

for all $p \in J_1(S^2_+T^*)$, $u \in T^* \otimes S^2T^*$; in fact, this morphism is determined by

$$\sigma(\psi_R)_g = -R^\flat \circ g^\sharp \circ \sigma(B_g), \tag{6}$$

where g is a Riemannian metric on X.

The most direct evidence that the Bianchi identity $\mathrm{Bian}(g, R) = 0$ provides us with an obstruction to finding solutions of the equation $\mathrm{Ric}(g) = R$ or solutions of the principal Ricci curvature problem arises from the exactness of the sequence (4), with $l = 1$, when one attempts to solve the equation $\mathrm{Ric}(g) = R$ to first-order at a point of X. For the principal Ricci curvature problem, in Lemma 2 below we are able to satisfy this obstruction easily, because we may choose R and g simultaneously. The implications of the Bianchi identity for the prescribed Ricci curvature problem are more subtle and shall be examined in §4.

§3. Prescribing the principal curvatures

This section is devoted to the proofs of Theorems 1 and 2 of the introduction. If g is a Riemannian metric on X and R is a section of S^2T^*, we denote by $\det_g R$ the real-valued function on X which is equal to the determinant of the endomorphism $g^\sharp \circ R^\flat$ of T. Then we easily see that the section $\det R$ of the line bundle $\mathrm{Hom}\,(\bigwedge^n T, \bigwedge^n T^*)$ vanishes at $x \in X$ if and only if the function $\det_g R$ vanishes at x. In particular, we know that $\lambda \in \mathbb{R}$ is an eigenvalue of $\mathrm{Ric}(g)$ with respect to g at x if and only if the function $\det_g(\mathrm{Ric}(g) - \lambda g)$ vanishes at $x \in X$, or equivalently if $\det(\mathrm{Ric}(g) - \lambda g)(x) = 0$.

We shall require the following lemma in our proof of Theorem 2.

Lemma 2. *Let g_0 be a Riemannian metric on X. Let $\lambda_1, \dots, \lambda_n$ be given distinct real numbers. Then there exists a Riemannian metric g on a neighborhood U of $x_0 \in X$, with $g(x_0) = g_0(x_0)$, such that the eigenvalues $\mu_1, \dots, \mu_n$ of $\mathrm{Ric}(g)$ with respect to g are C^∞-functions on U whose differentials are linearly independent at x_0 and which satisfy $\mu_j(x_0) = \lambda_j$, for $1 \leq j \leq n$.*

Proof. We may assume without loss of generality that $X = \mathbb{R}^n$, that x_0 is the origin of $\mathbb{R}^n$ and that

$$g_{0,ij}(x_0) = \delta_{ij}.$$

Consider the section ρ of S^2T^* determined by

$$\rho_{ij} = \lambda_j \delta_{ij} + (n\delta_{ij} - 1)(x_i + x_j).$$

As we have seen in §2, there is an element $u \in (S^2T^* \otimes S^2T^*)_{x_0}$ satisfying $\sigma(\varphi)_{g_0} u = \rho(x_0)$ and the symmetric tensor g given by

$$g_{ij} = \delta_{ij} + \tfrac{1}{2} u_{ij,kl} x^k x^l$$

is a Riemannian metric on a neighborhood U of x_0 which satisfies

$$\mathrm{Ric}(g)(x_0) = \rho(x_0).$$

Clearly $\lambda_1, \dots, \lambda_n$ are the eigenvalues of $\rho(x_0)$ with respect to the metric $g(x_0) = g_0(x_0)$. Since these scalars are distinct, there are C^∞-functions $\tilde{\lambda}_1, \dots, \tilde{\lambda}_n$ and an orthonormal frame $\{\xi_1, \dots, \xi_n\}$ for T, with respect to the metric g, over an neighborhood $U' \subset U$ of x_0 such that $\tilde{\lambda}_j(x_0) = \lambda_j$ and

$$\rho(\xi_j, \eta) = \tilde{\lambda}_j g(\xi_j, \eta), \tag{7}$$

for all vector fields η on U' and $1 \leq j \leq n$. Let ζ be a vector field on U'. From relation (7), with $\eta = \xi_j$ and $1 \leq j \leq n$, we obtain

$$(\nabla^g_\zeta \rho)(\xi_j, \xi_j) + 2\rho(\xi_j, \nabla^g_\zeta \xi_j) = \langle \zeta, d\tilde{\lambda}_j \rangle + \tilde{\lambda}_j g(\xi_j, \nabla^g_\zeta \xi_j). \tag{8}$$

Since ξ_j is a unitary vector field, we see that $g(\xi_j, \nabla^g_\zeta \xi_j) = 0$ and so by (7) we also have

$$\rho(\xi_j, \nabla^g_\zeta \xi_j) = \tilde{\lambda}_j g(\xi_j, \nabla^g_\zeta \xi_j) = 0.$$

Hence the equality (8) gives us

$$(\nabla^g_\zeta \rho)(\xi_j, \xi_j) = \langle \zeta, d\tilde{\lambda}_j \rangle.$$

Since $\xi_k = \partial/\partial x^k$ at x_0 and $\Gamma^k_{ij}(x_0) = 0$ for the metric g, from the definition of ρ we see that

$$(\nabla^g_{\xi_k} \rho)(\xi_i, \xi_j)(x_0) = \frac{\partial \rho_{ij}}{\partial x^k}(x_0) = (n - \delta_{ij})(\delta_{ik} + \delta_{jk}),$$

for $1 \leq i, j, k \leq n$. Thus we have $d\tilde{\lambda}_j = 2(n-1)dx^j$ at x_0, and so the differentials of the eigenvalues $\tilde{\lambda}_j$ are linearly independent at x_0. Using (1) and (2), it is easily verified that the section $h = \mathrm{Ric}(g) - \rho$, which vanishes at x_0, satisfies the Bianchi identity $B_g h = 0$ at x_0. Thus we know that

$$\sigma(B_g)\varepsilon^{-1} j_1(h)(x_0) = 0,$$

and so by the exactness of the sequence (4), with $l = 1$, there is an element v of $(S^3T^* \otimes S^2T^*)_{x_0}$ satisfying

$$\sigma_1(\varphi)_g v = -\varepsilon^{-1} j_1(h)(x_0).$$

Then a Riemannian metric $\tilde{g}$ on a neighborhood of x_0, whose 3-jet at x_0 is equal to $j_3(g)(x_0) + \varepsilon v$, satisfies the relations

$$j_2(\tilde{g})(x_0) = j_2(g)(x_0), \qquad j_1(\mathrm{Ric}(\tilde{g}) - \rho)(x_0) = 0. \tag{9}$$

Clearly there are C^∞-functions $\{\mu_1, \ldots, \mu_n\}$, which are eigenvalues of $\mathrm{Ric}(\tilde{g})$ with respect the metric $\tilde{g}$, such that $\mu_j(x_0) = \lambda_j$, for $1 \leq j \leq n$. From the equalities (9), we infer that $j_1(\mu_j)(x_0) = j_1(\tilde{\lambda}_j)(x_0)$; hence the differentials of the functions $\{\mu_1, \ldots, \mu_n\}$ are linearly independent at x_0.

Proof of Theorem 2. Let g_0 be a given Riemannian metric on X and let g be a Riemannian metric on a neighborhood of x_0 satisfying the assertions of Lemma 2 with respect to the distinct real numbers $\{\lambda_1(x_0), \ldots, \lambda_n(x_0)\}$ and g_0. Since their differentials are linearly independent at x_0, the eigenvalues $\mu_1, \ldots, \mu_n$ of $\mathrm{Ric}(g)$ with respect to g define a diffeomorphism $\mu = (\mu_1, \ldots, \mu_n)$ of an open neighborhood of x_0 onto an open neighborhood of $y_0 = (\lambda_1(x_0), \ldots, \lambda_n(x_0))$ in $\mathbb{R}^n$. Similarly by hypothesis, the mapping $\lambda = (\lambda_1, \ldots, \lambda_n)$ is a diffeomorphism of an open neighborhood of x_0 onto an open neighborhood of y_0 in $\mathbb{R}^n$. Thus $\mu^{-1} \circ \lambda$ is a local diffeomorphism Ψ of X defined on a neighborhood of x_0 such that $\mu \circ \Psi = \lambda$ on a neighborhood of x_0. Because

$$\det(\Psi^*(\mathrm{Ric}(g) - \mu_j g)) = 0$$

and $\lambda_j = \Psi^*\mu_j$, we see that Ψ^*g is a metric on a neighborhood of x_0 satisfying

$$\det\left(\mathrm{Ric}(\Psi^*g) - \lambda_j \Psi^*g\right) = 0,$$

for all $1 \leq j \leq n$.

The remainder of this section is devoted to the proof of Theorem 1. Let A_n be the affine space of all real monic polynomials in the variable λ of degree n, which is modeled on the vector space V_{n-1} of all real polynomials in the variable λ of degree $\leq n-1$. Let E and F be the trivial bundles over X whose fibers are equal to A_n and V_{n-1}, respectively. Then F is an affine bundle over X modeled on the vector bundle E. We consider the morphism of fibered manifolds

$$\Phi : J_2(S^2_+T^*) \to F,$$

sending the 2-jet $j_2(g)(x)$ at $x \in X$ of a Riemannian metric g on X into the monic polynomial $(-1)^n \det_g(\mathrm{Ric}(g) - \lambda g)(x)$ in the variable λ. The symbol of Φ is the morphism

$$\sigma(\Phi) : S^2T^* \otimes S^2T^* \to E$$

of vector bundles over $J_2(S^2_+T^*)$ defined as follows. If g is a Riemannian metric on X and h is a section of S^2T^* over a neighborhood of $x \in X$ satisfying $j_1(h)(x) = 0$, then $g + th$ is a Riemannian metric on neighborhood of x for $|t| < \epsilon$, with $\epsilon > 0$; the morphism $\sigma(\Phi)_g$ sends the element $\varepsilon^{-1} j_2(h)(x)$ of $(S^2T^* \otimes S^2T^*)_x$ into the vector $\frac{d}{dt}\Phi(g + th)_{|t=0}$ of E_x.

Let g be a Riemannian metric on X and $x \in X$. Let $\{\lambda_1, \dots, \lambda_n\}$ be the eigenvalues of $\mathrm{Ric}(g)$ with respect to g at x; then there is an orthonormal basis $\{\xi_1, \dots, \xi_n\}$ of T_x, with respect to the metric g, consisting of eigenvectors for $\mathrm{Ric}(g)$ satisfying

$$g^\sharp \cdot \mathrm{Ric}(g)^\flat \xi_j = \lambda_j \xi_j, \tag{10}$$

for $1 \leq j \leq n$. Let $\mathrm{Ric}'_g : S^2T^* \to S^2T^*$ be the linear differential operator of order 2 which is the linearization along g of the non-linear operator $h \mapsto \mathrm{Ric}(h)$, where h is a Riemannian metric on X. Let h be a section of S^2T^* over X. For $|t| < \epsilon$, we know that $g_t = g + th$ is a Riemannian metric on a neighborhood of x; by definition, we have

$$\mathrm{Ric}'_g(h) = \frac{d}{dt}\mathrm{Ric}_g(g + th)_{|t=0}.$$

For $|t| < \epsilon$ and $\lambda \in \mathbb{R}$, we consider the endomorphism

$$B_t^\lambda = g_t^\sharp \circ \mathrm{Ric}(g_t)^\flat - \lambda\,\mathrm{id}$$

of T_x; then it is easily seen that

$$B = \frac{d}{dt} B_t^\lambda{}_{|t=0} = \big(g^\sharp \circ \mathrm{Ric}'_g(h)^\flat - g^\sharp \circ h^\flat \circ g^\sharp \circ \mathrm{Ric}(g)^\flat\big)(x).$$

Clearly the vector

$$p(\lambda) = \frac{d}{dt} \det_{g_t}(\mathrm{Ric}(g_t) - \lambda g_t)(x)_{|t=0}$$

of V_{n-1} is determined by the relation

$$\sum_{i=1}^n B_0^\lambda \xi_1 \wedge \ldots \wedge B_0^\lambda \xi_{i-1} \wedge B\xi_i \wedge B_0^\lambda \xi_{i+1} \wedge \ldots \wedge B_0^\lambda \xi_n = p(\lambda)\, \xi_1 \wedge \ldots \wedge \xi_n,$$

for $\lambda \in \mathbb{R}$. According to (10), for $1 \leq i \leq n$, we have $B_0^\lambda \xi_i = (\lambda_i - \lambda)\xi_i$ and we see that

$$\begin{aligned} g(B\xi_i, \xi_i) &= \mathrm{Ric}'_g(h)(\xi_i, \xi_i) - \langle \xi_i, h^\flat \cdot g^\sharp \cdot \mathrm{Ric}(g)^\flat \xi_i \rangle \\ &= \mathrm{Ric}'_g(h)(\xi_i, \xi_i) - \lambda_i h(\xi_i, \xi_i). \end{aligned}$$

For $1 \leq i \leq n$, we consider the polynomial

$$P_i(\lambda) = \prod_{\substack{j=1 \\ j \neq i}}^n (\lambda_j - \lambda)$$

of V_{n-1}. From the previous relations, we obtain the equality

$$p(\lambda) = \sum_{i=1}^n \big(\mathrm{Ric}'_g(h) - \lambda_i h\big)(\xi_i, \xi_i) \cdot P_i(\lambda). \tag{11}$$

Since the symbol of the differential operator Ric'_g is equal to $\sigma(\varphi)_g$, from (11) we deduce that the morphism $\sigma(\Phi)_g$ at x is given by

$$\sigma(\Phi)_g u = (-1)^n \sum_{i=1}^n (\sigma(\varphi)_g u)(\xi_i, \xi_i) \cdot P_i(\lambda), \tag{12}$$

for $u \in (S^2T^* \otimes S^2T^*)_x$. If $\beta \in T^*_x$, let

$$\sigma_\beta(\Phi)_{j_2(g)(x)} : S^2T^*_x \to E_x \tag{13}$$

be the mapping sending the element v of $S^2T_x^*$ into $\sigma(\Phi)_g(\beta \otimes \beta \otimes v)$; thus we have

$$\sigma_\beta(\Phi)_{j_2(g)(x)}v = (-1)^n \sum_{i=1}^n (\sigma_\beta(\varphi)_g v)(\xi_i, \xi_i) \cdot P_i(\lambda), \tag{14}$$

for $v \in S^2T_x^*$.

Now assume that the eigenvalues $\{\lambda_1, \ldots, \lambda_n\}$ are distinct. Then the n polynomials $\{P_1, \ldots, P_n\}$ of degree $n-1$ form a basis for V_{n-1}; this fact follows from the relations $P_i(\lambda_i) \neq 0$ and $P_i(\lambda_j) = 0$ when $j \neq i$. Let β be a non-zero vector of T_x^*; we write $\beta_i = \beta(\xi_i)$, for $1 \leq i \leq n$. According to formula (14), the mapping (13) is surjective if and only if the mapping $S^2T_x^* \to \mathbb{R}^n$, which sends an element v of $S^2T_x^*$ into the n-tuple

$$\big((\sigma_\beta(\Phi)_g v)(\xi_i, \xi_i)\big)_{1 \leq i \leq n},$$

is surjective. From the exactness of the sequence (5), we infer that the mapping (13) is surjective if and only if, given an arbitrary element $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$, there exists $v \in S^2T_x^*$ satisfying $v(\xi_i, \xi_i) = y_i$ for all $1 \leq i \leq n$ and $\sigma_\beta(B_g)v = 0$. Hence the mapping (13) is surjective if and only if the following assertion holds: for all $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$, there exists an $n \times n$ symmetric matrix $A = (a_{ij})_{1 \leq i,j \leq n}$ satisfying $a_{ii} = 0$ and

$$\sum_{j=1}^n \beta_j a_{ij} = \beta_i z_i, \tag{15}$$

where

$$z_i = -y_i + \frac{1}{2} \sum_{j=1}^n y_j,$$

for all $1 \leq i \leq n$. If we set $m = n(n-1)/2$, we may view the equations (15) as a system of linear equations

$$(CA)_i = \beta_i z_i, \qquad i = 1, \ldots, n, \tag{16}$$

where the matrix A is viewed as a vector in $\mathbb{R}^m$ and C is a given $n \times m$-matrix each of whose entries is equal either to 0 or to one of the β_j's. We then see that our condition for the surjectivity of the mapping (13) can be reformulated as follows: for all $z = (z_1, \ldots, z_n) \in \mathbb{R}^n$, we can solve the system (16) for $A \in \mathbb{R}^m$. If $1 \leq i, j, k \leq n$ are distinct integers, it is easily verified that the matrix C possesses an $n \times n$-minor whose determinant is equal to $\pm 2\beta_i^{n-2}\beta_j\beta_k$. On the other hand, if there are at most two non-vanishing coefficients β_j of β, it is easily seen that this last condition for the surjectivity of (13) does not hold. Thus we have proved the following lemma:

Lemma 3. *Let g be a Riemannian metric on X. Assume that the eigenvalues $\{\lambda_1, \dots, \lambda_n\}$ of $\mathrm{Ric}(g)$ with respect to g at $x \in X$ are distinct. Let $\{\xi_1, \dots, \xi_n\}$ be an orthonormal basis of T_x, with respect to g, consisting of eigenvectors for $\mathrm{Ric}(g)$ and satisfying the relations (10), for $1 \leq j \leq n$. If β is a vector of T_x^*, then the mapping (13) is surjective if and only if there exist three distinct integers $1 \leq i, j, k \leq n$ such that $\beta(\xi_i) \cdot \beta(\xi_j) \cdot \beta(\xi_k) \neq 0$.*

Let N_2' be the subset of $J_2(S_+^2 T^*)$ consisting of all 2-jets $j_2(g)(x)$, where $x \in X$ and g is a Riemannian metric on X for which the eigenvalues of $\mathrm{Ric}(g)(x)$ with respect to $g(x)$ are distinct. According to Lemma 2, we see that N_2' is an open fibered submanifold of $J_2(S_+^2 T^*)$, with $\pi_0 N_2' = S_+^2 T^*$.

Let $\{\lambda_1, \dots, \lambda_n\}$ be given real-valued functions on X; assume that, for all $x \in X$, we have $\lambda_i(x) \neq \lambda_j(x)$, for $i \neq j$. We consider the A_n-valued function

$$P(\lambda) = \prod_{j=1}^{n} (\lambda - \lambda_j)$$

on X and the subset

$$N_2 = \{\, p \in N_2' \mid \Phi(p) = P(\lambda)(x), \text{ where } x = \pi(p) \,\}$$

of $J_2(S_+^2 T^*)$. Clearly, if a Riemannian metric g satisfies $\Phi(j_2(g)) = P(\lambda)$ at $x \in X$, then the eigenvalues of its Ricci tensor with respect to g at x are equal to $\{\lambda_1(x), \dots, \lambda_n(x)\}$. By Lemma 2, we know that $P(X) \subset \Phi(N_2')$. If $p \in N_2'$ and $x = \pi(p)$, according to Lemma 3 there exists an element β of T_x^* such that the mapping $\sigma_\beta(\Phi)_p : S^2 T_x^* \to E_x$ is surjective. From Lemma 1 of [7], we obtain:

Proposition 1. *Let $\{\lambda_1, \dots, \lambda_n\}$ be given real-valued functions on X; assume that, for all $x \in X$, we have $\lambda_i(x) \neq \lambda_j(x)$, for $i \neq j$. Then the subset N_2 of $J_2(S_+^2 T^*)$ is a formally integrable differential equation.*

Thus N_2 is an underdetermined differential equation. If the manifold X is real-analytic and if the functions $\{\lambda_1, \dots, \lambda_n\}$ are real-analytic, Theorem 2.2, Chapter IX of [9] provides us with the existence of local solutions of the equation N_2; in fact, if $x \in X$, it gives us a real-analytic

Riemannian metric g satisfying $\Phi(j_2(g)) = P(\lambda)$ on a neighborhood of $x \in X$. This completes the proof of Theorem 1.

§4. Prescribing the Ricci tensor

Let R be a given section of S^2T^* over X of constant rank. The expression $\mathrm{Bian}(g, R)$ is first-order in the metric g (as well as in R); in right-hand side of (1), the highest (first) derivatives of g are all multiplied by coefficients of R. This is reflected in formula (6) and, when R is degenerate, is the source of the problems which need to be overcome. We now indicate how to solve the prescribed Ricci curvature problem and why the Bianchi identity is the only obstruction to local solvability in the non-degenerate case.

We suppose that R is non-degenerate and we consider the quasi-linear morphism of fibered manifolds

$$\psi'_R : J_2(S^2_+T^*) \to T^*$$

determined by

$$\psi'_R(j_1(g)) = g^\flat \cdot R^\sharp \cdot \psi_R(j_1(g)),$$

where g is a Riemannian metric. According to (6) and Lemma 1, the symbol

$$\sigma(\psi'_R) : T^* \otimes S^2T^* \to T^*$$

of ψ'_R is the surjective morphism of vector bundles over $S^2_+T^*$ determined by the equality

$$\sigma(\psi'_R)_g = -\sigma(B_g)u,$$

where g is a Riemannian metric on X. Because of (2), a solution g to our original problem is also a solution of the system of second-order equations

$$\text{(17)} \qquad \mathrm{Ric}(g) = R, \qquad j_1(\psi'_R(g)) = 0.$$

The exactness of the sequences (3) is the main ingredient in the local solvability of this system of equations. In fact, since the morphism $\sigma(\psi'_R)$ is surjective, we obtain a Riemannian metric g on a neighborhood of $x_0 \in X$ satisfying $\psi'_R(g)(x_0) = 0$. Next, let $l \geq 0$ and let g be a Riemannian metric on a neighborhood of x_0 whose $(l+1)$-jet at x_0 satisfies

$$\text{(18)} \qquad j_{l-1}(\mathrm{Ric}(g) - R)(x_0) = 0, \qquad j_l(\psi'_R(g))(x_0) = 0.$$

Then we have $j_l(\mathrm{Bian}(g,R))(x_0)=0$. Thus if $u\in(S^lT^*\otimes S^2T^*)_{x_0}$ and $v\in(S^{l+1}T^*\otimes T^*)_{x_0}$ are the elements defined by

$$u=\varepsilon^{-1}j_l(\mathrm{Ric}(g)-R)(x_0),\qquad v=\varepsilon^{-1}j_{l+1}(\psi'_R(g))(x_0),$$

by (2) we easily see that $\mu_g(u,v)=0$. Hence by the exactness of the sequence (3), there exists an element $w\in(S^{l+2}T^*\otimes S^2T^*)_{x_0}$ such that

$$\sigma_l(\varphi)_g w=u,\qquad \sigma_{l+1}(B_g)w=v.$$

Since the morphisms φ and ψ'_R are quasi-linear, we see that a Riemannian metric on a neighborhood of x_0, whose $(l+2)$-jet at x_0 is equal to $j_{l+2}(g)(x_0)-\varepsilon w$, satisfies equations (18), with l replaced by $l+1$. Thus we obtain a formal solution at x_0 of our system (17), which has the special property that its jet of order 2 at x_0 is strongly prolongable. If R is a real-analytic section, a result of Malgrange [10] asserts that the equations (17) admit a convergent series solution at x_0, and yields a solution of our original equation.

We now turn to the case where R is degenerate. A solution g to the equation $\mathrm{Ric}(g)=R$ must satisfy some further equations which we now proceed to derive. The kernel K of the morphism $R^\flat: T\to T^*$ is a sub-bundle of T^*. We say that this bundle K is integrable if the sheaf $\mathcal{K}$ is stable under the Lie bracket.

Let ∇ be an arbitrary torsionless connection in T. We consider the section ∇R of $T^*\otimes S^2T^*$ and the Lie derivative $\mathcal{L}_\xi R$ of R along a vector field ξ on X.

The following lemma associates a section Q of $S^2T^*\otimes K^*$ to the section R of S^2T^*.

Lemma 4. *The section*

$$Q=(\lambda(\nabla R))_{|T\otimes T\otimes K}$$

of $S^2T^*\otimes K^*$ is independent of the torsionless connection ∇ and has the following properties:

(i) *If ξ is a section of K over X, we have*

$$Q(\eta,\zeta,\xi)=-\tfrac{1}{2}(\mathcal{L}_\xi R)(\eta,\zeta), \tag{19}$$

for all $\eta,\zeta\in T$.

(ii) *If $\eta\in T$, $\xi_1,\xi_2,\zeta\in K$, we have*

$$Q(\eta,\xi_1,\xi_2)+Q(\eta,\xi_2,\xi_1)=0,\qquad Q(\zeta,\xi_1,\xi_2)=0. \tag{20}$$

(iii) *The sub-bundle K of T is integrable if and only if*

$$Q(\eta, \xi_1, \xi_2) = 0,$$

for all $\xi_1, \xi_2 \in K$, $\eta \in T$.

Proof. If ∇, ∇' are torsionless connections in T, then there is a section L of $S^2T^* \otimes T$ such that

$$\nabla'_\xi \eta - \nabla_\xi \eta = L(\xi, \eta),$$

for $\xi, \eta \in \mathcal{T}$. It is easily verified that

$$(\lambda(\nabla' R - \nabla R))(\zeta_1, \zeta_2, \xi) = -R(L(\zeta_1, \zeta_2), \xi),$$

for $\xi, \zeta_1, \zeta_2 \in T$. If ξ belongs to K, the right-hand side of the above equation vanishes, and so we see that Q is independent of ∇. Since ∇ is torsionless, according to the definition of Q we have

$$Q(\eta, \zeta, \xi) = \tfrac{1}{2}(R([\xi, \eta], \zeta) + R(\eta, [\xi, \zeta]) - \xi \cdot R(\eta, \zeta)) = -\tfrac{1}{2}(\mathcal{L}_\xi R)(\eta, \zeta),$$

for all $\xi \in \mathcal{K}$, $\eta, \zeta \in \mathcal{T}$. We thus obtain (19) and see that

$$Q(\eta, \xi_1, \xi_2) = -\tfrac{1}{2}R(\eta, [\xi_1, \xi_2]),$$

for all $\xi_1, \xi_2 \in \mathcal{K}$, $\eta \in \mathcal{T}$. Assertions (ii) and (iii) follow directly from this equality.

If $\xi \in K$, we denote by Q_ξ the element of S^2T^* defined by

$$Q_\xi(\eta, \zeta) = Q(\eta, \zeta, \xi),$$

for $\eta, \zeta \in T$; from (20), we deduce that

$$Q_\xi(\xi, \eta) = 0, \tag{21}$$

for all $\eta \in T$. If ξ is a section of K over X, by (19) we have

$$Q_\xi = -\tfrac{1}{2}\mathcal{L}_\xi R.$$

If Q vanishes identically, then by Lemma 4,(iii) we see that K is integrable. Let K'' be the sub-bundle of T, with possibly varying fiber, consisting of all elements $\xi \in T$ satisfying

$$Q(\eta, \xi, \zeta) = 0,$$

for all $\eta \in T$, $\zeta \in K$. If K is integrable, then according to Lemma 4,(iii) we see that $K \subset K''$, and so the dimension $\nu(x)$ of K''_x is greater the rank of K. We shall consider the morphism of vector bundles $\iota : K \to S^2T^*$, which sends $\xi \in K$ into Q_ξ.

If g is a Riemannian metric on X, using ∇^g to define Q, we see that

$$(B_gR)(\xi) = \operatorname{Tr}_g Q_\xi, \tag{22}$$

for all $\xi \in K$. This provides us with a new obstruction to solvability; indeed, by (2) any solution g to the equation $\operatorname{Ric}(g) = R$ must also satisfy

$$\operatorname{Tr}_g Q_\xi = 0, \tag{23}$$

for all $\xi \in K$. If g is a Riemannian metric on X and ξ is a vector of K satisfying the relation (23), then we easily see that either Q_ξ vanishes or is not semi-definite. These remarks imply the following:

Theorem 5. *Let $x \in X$ and $\xi \in K_x$. If Q_ξ does not vanish and is semi-definite, then there does not exist a Riemannian metric g on any neighborhood of x such that* $\operatorname{Ric}(g) = R$.

In [6, Lemma 2.3], using (21) we prove the more precise version of the previous observation:

Lemma 5. *Let $x \in X$ and $\xi \in K_x$; assume that Q_ξ does not vanish. Then there exists a Riemannian metric g on X such that*

$$\operatorname{Tr}_g Q_\xi = 0$$

if and only if Q_ξ is not semi-definite.

By the preceding lemma, we see that Theorem 4 is a direct consequence of Theorem 3.

In terms of a local coordinate system $(x^1, \ldots, x^n)$ on an open subset U of X, using the flat connection ∇ on U satisfying $\nabla\partial/\partial x^j = 0$, for $1 \le j \le n$, to compute the section Q, we see that the section Q_ξ of S^2T^* over U corresponding to the section $\xi = \xi^j\partial/\partial x^j$ of K is given by

$$(Q_\xi)_{ij} = \frac{1}{2}\left(\frac{\partial R_{jk}}{\partial x^i} + \frac{\partial R_{ik}}{\partial x^j} - \frac{\partial R_{ij}}{\partial x^k}\right)\xi^k, \tag{24}$$

for $1 \le i, j \le n$. If the bundle K is integrable and if $\{\partial/\partial x^1, \ldots, \partial/\partial x^r\}$ is a frame for this bundle K over U, then we have $R_{ij} = 0$, for $1 \le i \le n$ and $1 \le j \le r$, and the section $Q^k = Q_{\xi_k}$ of S^2T^* over U corresponding to the section $\xi_k = \partial/\partial x^k$ of K, with $1 \le k \le r$, is given by

$$Q^k_{ij} = -\frac{1}{2}\frac{\partial R_{ij}}{\partial x^k}, \tag{25}$$

for $1 \leq i, j \leq n$. If the morphism of vector bundles $\iota : K \to S^2T^*$ has constant rank equal to m, then, at every point x of U, the vector space of elements of $S^2T^*_x$ generated by $\{Q^1(x), \ldots, Q^r(x)\}$ is m-dimensional.

We now consider some simple examples of degenerate Ricci candidates.

Example 1. A simple example of non-existence arising from Theorem 5 is given by the following. We consider the symmetric 2-form

$$R = dx^2 \otimes dx^2 + \cdots + dx^{n-1} \otimes dx^{n-1} + (1 + x^1)dx^n \otimes dx^n$$

on $\mathbb{R}^n$. Note that the rank of R is equal to $n-1$ in a neighborhood U of the origin and that the vector field $\xi = \partial/\partial x^1$ generates the sub-bundle K over U. According to (25), for $x \in U$, the element $Q^1(x)$ of $S^2T^*_x$ is determined by $Q^1_{ij}(x) = \lambda_i \delta_{ij}$, where $\lambda_i = 0$ for $1 \leq i \leq n-1$ and $\lambda_n = 1$; thus $Q^1(x)$ is non-zero and semi-definite. Hence we know that there does not exist a Riemannian metric g satisfying $\mathrm{Ric}(g) = R$ on any neighborhood of a point of U.

Example 2. The tensor $R = \pm\, dx^3 \otimes dx^3$ on $\mathbb{R}^3$ clearly has rank one everywhere. The vector fields $\{\partial/\partial x^1, \partial/\partial x^2\}$ generate the integrable sub-bundle K over $\mathbb{R}^3$, and by (25) the sections Q^1 and Q^2 of S^2T^* both vanish. As noted in the introduction, if x is an arbitrary point of $\mathbb{R}^3$, we can apply Theorem 3 to obtain the existence of a metric g satisfying $\mathrm{Ric}(g) = R$ on a neighborhood of x. However, it is worth noting that although R splits as a product, the same cannot be true for g, since then the one-dimensional factor of g would have to exhibit non-zero curvature, which is impossible. It is therefore interesting to exhibit an explicit solution g of the equation $\mathrm{Ric}(g) = R$. We first consider the case when $R = dx^3 \otimes dx^3$ and seek a solution g of the form

$$g = f(x^3)^2\,(dx^1 \otimes dx^1 + dx^2 \otimes dx^2) + h(x^3)^2\,dx^3 \otimes dx^3, \tag{26}$$

where f and h are non-vanishing functions of $t = x^3$. We shall use a prime to denote the derivative with respect to t, and will no longer indicate the dependence of f and h on t. This metric g is a warped product and is necessarily conformally flat. Its only possibly non-vanishing Christoffel symbols are determined by

$$\Gamma^1_{13} = \Gamma^2_{23} = f'/f, \qquad \Gamma^3_{11} = \Gamma^3_{22} = -ff'/h^2, \qquad \Gamma^3_{33} = h'/h.$$

An elementary calculation yields the components of its Ricci tensor:

$$\mathrm{Ric}(g)_{11} = \mathrm{Ric}(g)_{22} = -\left(\frac{f'}{h}\right)^2 - \frac{f}{h}\left(\frac{f'}{h}\right)', \quad \mathrm{Ric}(g)_{33} = -2\frac{h}{f}\left(\frac{f'}{h}\right)'$$

and $\mathrm{Ric}(g)_{ij} = 0$ for $i \neq j$. If we set $f(t) = e^{at}$ and $h(t) = Ce^{2at}$, where $a, C \in \mathbb{R}$ and $C > 0$, we find that the Ricci tensor of the metric g is equal to $2a^2 dx^3 \otimes dx^3$. With this choice of positive functions f and h, where $a = 1/\sqrt{2}$ and $C > 0$, the metric g given by (26) is a solution of the equation $\mathrm{Ric}(g) = dx^3 \otimes dx^3$ on $\mathbb{R}^3$. Finally, the only possibly non-vanishing Christoffel symbols of the metric

$$g' = e^{-2ax^3} dx^1 \otimes dx^1 + e^{2ax^3} dx^2 \otimes dx^2 + dx^3 \otimes dx^3$$

on $\mathbb{R}^3$, with $a \in \mathbb{R}$, are determined by:

$$\Gamma^1_{13} = -a, \quad \Gamma^2_{23} = a, \quad \Gamma^3_{11} = ae^{-2ax^3}, \quad \Gamma^3_{22} = -ae^{2ax^3}.$$

Then one easily verifies that the Ricci tensor of the metric g' is equal to $-2a^2 dx^3 \otimes dx^3$. Thus this metric g', with $a = 1/\sqrt{2}$, is a solution of the equation $\mathrm{Ric}(g') = -dx^3 \otimes dx^3$ on $\mathbb{R}^3$. It is interesting to examine the geometry of the metric g' on $\mathbb{R}^3$. For $c \in \mathbb{R}$, the hypersurface of $\mathbb{R}^3$ defined by $x^3 = c$ is a flat submanifold (but not complete), while the hypersurfaces of $\mathbb{R}^3$ defined by $x^1 = c$ and $x^2 = c$ have constant negative curvature (equal to $-a^2$) and are also not complete.

Example 3. The tensor

$$R = (dx^2 + x^3 dx^1) \otimes (dx^2 + x^3 dx^1) + dx^3 \otimes dx^3$$

on $\mathbb{R}^3$ has rank two and the kernel K is generated by the vector field $\xi = \partial/\partial x^1 - x^3 \partial/\partial x^2$. By (24), the section $Q = Q_\xi$ is non-zero at all points of $\mathbb{R}^3$; in fact, its non-zero coefficients are given by

$$Q_{13} = \frac{x^3}{2}, \qquad Q_{23} = \frac{1}{2}.$$

The Euclidean metric

$$g_0 = dx^1 \otimes dx^1 + dx^2 \otimes dx^2 + dx^3 \otimes dx^3$$

clearly satisfies $\mathrm{Tr}_{g_0} Q_\xi = 0$ on $\mathbb{R}^3$. Hence, if x is an arbitrary point of $\mathbb{R}^3$, Theorem 4 gives us a metric g satisfying $\mathrm{Ric}(g) = R$ on a neighborhood of x.

Example 4. The tensor

$$R = (dx^2 + x^3 dx^1) \otimes (dx^2 + x^3 dx^1)$$

on $\mathbb{R}^3$ has rank one and the kernel K is generated by the vector fields $\xi_1 = \partial/\partial x^1 - x^3 \partial/\partial x^2$ and $\xi_2 = \partial/\partial x^3$. Since $[\xi_1, \xi_2] = \partial/\partial x^2$, the distribution determined by K is *not* integrable. If g is a Riemannian metric on $\mathbb{R}^3$, then by (1) we see that

$$(B_g R)(\xi_1) = x^3 g^{13} + g^{23}, \qquad (B_g R)(\xi_2) = -(x^3 g^{11} + g^{12}).$$

It is easily seen that the integrability condition $B_g R = 0$ places two conditions of order zero on the metric g; namely, a solution g of the equation $\mathrm{Ric}(g) = R$ must satisfy the relations

$$g_{23} = 0, \qquad g_{12} = x^3 g_{22}.$$

This example shows that the singularity of the tensor R places substantial restrictions on the 0-jet of the solution metric g.

We now present an outline of the proof of Theorem 3. For simplicity, we assume that the morphism $\iota : K \to S^2T^*$ is injective and that its rank is equal to $m \geq 1$. We choose a fixed complement T' of K in T which allows us to split the equation $\mathrm{Bian}(g, R) = 0$, for the Riemannian metric g, into two pieces. The first one is a first-order equation similar to the equation $\psi'_R(j_1(g)) = 0$ considered above when R is non-degenerate. If $\rho : T^* \to T'^*$ is the natural restriction mapping, we denote by $R^\sharp$ the inverse of the isomorphism

$$\rho R^\flat : T' \to T'^*.$$

If $i : T' \to T$ is the inclusion mapping, we set

$$S = iR^\sharp \rho : T^* \to T.$$

Then the endomorphism $SR^\flat$ of T is equal to the projection π' of T onto T' corresponding to the decomposition $T = T' \oplus K$. We consider the quasi-linear morphism of fibered manifolds

$$\psi'' : J_2(S^2_+T^*) \to T^*,$$

determined by

$$\psi''(j_1(g)) = g^\flat \cdot S \cdot \psi_R(j_1(g)),$$

where g is a Riemannian metric on X. According to (6), the symbol

$$\sigma(\psi'') : T^* \otimes S^2T^* \to T^*$$

of ψ'' is the morphism of vector bundles over $S^2_+T^*$ determined by

$$\sigma(\psi'')_g = -g^\flat \pi' g^\sharp \sigma(B_g),$$

where g is a Riemannian metric on X; the l-th prolongation of the morphism $\sigma(\psi'')_g$ will be denoted by $\sigma_l(\psi'')_g$. The first part of the Bianchi identity is the first-order equation

$$\psi''(j_1(g)) = 0.$$

To a Riemannian metric g on X, we associate the section $\chi(g)$ of K^* determined by

$$\langle \xi, \chi(g) \rangle = \operatorname{Tr}_g Q_\xi,$$

for $\xi \in K$. The second equation $\chi(g) = 0$ arises from the relation (19) and imposes a condition of order zero on the metric g. We then attempt to apply the Cartan-Kähler theorem to the system

$$(27) \qquad \operatorname{Ric}(g) = R, \qquad j_1(\psi''(j_1(g))) = 0, \qquad j_2(\chi(g)) = 0$$

of partial differential equations of order 2 for the metric g. This approach fails; indeed, by prolonging the system (27), we find that a set of m additional identities must be taken into consideration. We now derive these scalar-valued identities and then study the new system consisting of (27) and these new identities.

Let g be a Riemannian metric on X. The morphism of vector bundles

$$\sigma(\chi)_g : S^2T^* \to K^*$$

over X obtained from the symbol $\sigma(\chi)$ of χ is given by

$$\langle \xi, \sigma(\chi)_g h \rangle = -\langle \xi, \sigma(\chi)_g h \rangle_g,$$

for $h \in S^2T^*$; its l-th prolongation will be denoted by $\sigma_l(\chi)_g$. We consider the sequence of vector bundles

$$(28) \quad S^2T^* \otimes S^2T^* \xrightarrow{\sigma_g} S^2T^* \oplus (T^* \otimes g^\flat(T')) \oplus (S^2T^* \otimes K^*) \xrightarrow{\nu_g} K^* \to 0,$$

where the morphism

$$\sigma_g = \sigma(\varphi)_g \oplus \sigma_1(\psi'')_g \oplus \sigma_2(\chi)_g$$

is the symbol of the system (27) associated to g, and where ν_g is the morphism of vector bundles determined by

$$\langle \xi, \nu_g(u \oplus v \oplus w) \rangle = \langle u + v, Q_\xi \rangle_g - \tfrac{1}{2} \operatorname{Tr}_g \langle \xi, w \rangle,$$

for $u \in S^2T^*$, $v \in T^* \otimes g^\flat(T')$, $w \in S^2T^* \otimes K^*$ and $\xi \in K$.

In [6, §4]), we first observe that the sequence (28) is a complex, because K is integrable, and then prove:

Lemma 6. *Let g be a Riemannian metric on X. If $\chi(g)$ vanishes at $x \in X$ and if T'_x is the orthogonal complement of K_x, then the sequence* (28) *is exact at x.*

Given the complexity of the morphism σ_g, it is remarkable that the cokernel morphism ν_g can be expressed in such a simple and natural manner. The non-surjectivity of the morphism σ_g leads us to our additional identities.

Let x_0 be a point of X and g_0 be a given Riemannian metric on X satisfying $\operatorname{Tr}_g Q_\xi = 0$, for all $\xi \in K_{x_0}$. Since we are interested in the local solvability, we may suppose that $X = \mathbb{R}^n$ with a system of linear coordinates $(x^1, \ldots, x^n)$ such that $\{\partial/\partial x^1, \ldots, \partial/\partial x^m\}$ is a frame for K over X. We set $\xi_j = \partial/\partial x^j$, for $1 \leq j \leq n$. Since K''_{x_0} contains K_{x_0}, without loss of generality, we may also suppose that $g_{0,ij}(x_0) = \delta_{ij}$ and that $\{\xi_1(x_0), \ldots, \xi_q(x_0)\}$ is a basis for K''_{x_0}, where $q = \nu(x_0) \geq m$. We choose T' to be the integrable complement of K in T generated by the vector fields $\{\xi_{m+1}, \ldots, \xi_n\}$ on X; thus T'_{x_0} is the orthogonal complement of K_{x_0} in T_{x_0} (with respect to g_0). We consider the flat connection ∇ in T for which all the vector fields ξ_j are horizontal. This connection preserves K and so, if g is a Riemannian metric on X, we may consider the section

$$P(g) = \nu_g\big((\operatorname{Ric}(g) - R) \oplus (\nabla \psi''(g)) \oplus \nabla^2 \chi(g)\big)$$

of K^*. A Riemannian metric g, solution of $\operatorname{Ric}(g) = R$, must also satisfy the equation $P(g) = 0$, which, in light of the complex (28), is in fact of first order. However, as $P(g)$ arises from the lower order terms of $\operatorname{Ric}(g)$, it is an expression which is quadratic in the first derivatives of g.

We then consider the new system M_2 of order 2 consisting of the equations (27) together with the second-order equation $j_1(P(g)) = 0$ for a Riemannian metric g. We construct a strongly prolongable 2-jet $j_2(g)(x_0)$ of a solution g of the equation M_2 at x_0. The main difficulty in proving the existence of such a formal solution at x_0 consists in finding a 1-jet $j_1(g)(x_0)$ of a Riemannian metric g at x_0, which is a solution of the first-order system

$$\psi''(g) = 0, \qquad j_1(\chi(g)) = 0, \qquad P(g) = 0$$

determined by our system M_2, such that the symbol of the system M_2 is involutive at $j_1(g)(x_0)$. Then, if R is a real-analytic section, a result of Malgrange [10] asserts that the system M_2 admits a convergent series solution at x_0, and thus yields a solution of our original equation. This completes our outline of the proof of Theorem 3, when m is equal to the rank of K and is ≥ 1.

References

[1] D. Alekseevskij, A. Vinogradov and V. Lychagin, "Basic Ideas and Concepts of Differential Geometry", Encyclopedia of Mathematical Sciences, Vol. 28, Springer-Verlag, Berlin, Heidelberg, New York, 1991.

[2] R. Bryant, S. S. Chern, R. Gardner, H. Goldschmidt and P. Griffiths, "Exterior differential systems", Math. Sci. Res. Inst. Publ., Vol. 18, Springer-Verlag, Berlin, Heidelberg, New York, 1991.

[3] E. Cartan, "Leçons sur la géométrie des espaces de Riemann", 2nd ed., Gauthier-Villars, Paris, 1946.

[4] D. DeTurck, Existence of metrics with prescribed Ricci curvature: local theory, Invent. Math., **65** (1981), 179–207.

[5] ———, Metrics with prescribed Ricci curvature, in "Seminar on differential geometry", edited by S. T. Yau, Ann. of Math. Studies, No. 102, Princeton University Press, University of Tokyo Press, Princeton, N.J., 1982, pp. 525–537.

[6] D. DeTurck and H. Goldschmidt, Metrics with prescribed curvature of constant rank. I. The integrable case, Adv. in Math., to appear.

[7] J. Gasqui, Sur l'existence locale d'immersions à courbure scalaire donnée, Math. Ann., **219** (1976), 123–126.

[8] H. Goldschmidt, Existence theorems for analytic partial differential equations, Ann. of Math., **86** (1967), 246–270.

[9] ———, Integrability criteria for systems of non-linear partial differential equations, J. Differential Geom., **1** (1967), 269–307.

[10] B. Malgrange, Équations de Lie. II, J. Differential Geom., **7** (1972), 117–141.

[11] J. Milnor, Curvatures of left invariant metrics on Lie groups, Adv. in Math., **21** (1976), 293–329.

D. DeTurk
Department of Mathematics
University of Pennsylvania
Philadelphia, PA 19104
U. S. A.

H. Goldschmidt
Department of Mathematics
Columbia University
New York, NY 10027
U. S. A.

Advanced Studies in Pure Mathematics 25, 1997
CR-Geometry and Overdetermined Systems
pp. 85–109

Deformations of Singularities, Complex Manifolds and CR–Structures

Charles L. Epstein

Dedicated to Prof. M. Kuranishi on the occasion of his seventieth birthday

Abstract.

This note is an expanded version of the author's lecture at the conference on Several Complex Variables held in Osaka, Dec. 1994. We consider deformations of complex spaces and their relationship to deformations of CR–structures. An invariant is introduced which measures the change in the algebra of CR–functions under a deformation. These issues are then considered in the context of the deformations of the total space of the tangent bundle of a Riemann surface. The last section contains problems in deformation theory.

§1. Introduction

In the 1960s and 1970s a great deal of progress was made in the study of the deformations of complex analytic spaces. A complex structure on a manifold has two fundamentally different descriptions as: 1. A holomorphic coordinate atlas, 2. A formally integrable subbundle of the complexified tangent bundle. We refer to 1. as the "holomorphic description" and 2. as the "real description." The equivalence of these descriptions is the content of the Newlander–Nirenberg theorem. The holomorphic description is more general as it can also be used for (possibly non–reduced) analytic spaces. These two representations lead to different descriptions for the deformations of the given structure. In the holomorphic description one fixes a coordinate cover, the deformations then appear as families of holomorphic gluing maps. In the second case one represents the deformed subbundle of the complexified tangent bundle as a graph over the reference structure. This is conveniently

Received December 20, 1995
Research supported in part by NSF grant DMS93–01088

parametrized by a vector valued $(0,1)$–form which satisfies a non–linear partial differential equation. In actual practice these descriptions are very difficult to directly compare. Instead one defines a universal object, the versal deformation, to which the various descriptions are compared.

The real description seems to be somewhat limited as it appears to require a smooth underlying space in order for the integrability conditions to be expressible in terms of a PDE. Kuranishi showed that this technique could be applied to study the deformations of an isolated singular point by considering the CR-structure induced on a (smooth) link of the singularity, see [Ku]. If the complex dimension of the singular space is at least 3 then there is a "real" description of the integrable deformations of this structure as the solution space of a non–linear PDE on the link. Once this space is obtained two problems remain in order to return to the original question of deforming a singularity: 1. One needs to show that the deformed CR–structures arise as the boundaries of complex spaces which are, in an appropriate sense, deformations of the initial singularity and 2. One needs to show that the solution space of the PDE has the structure of a complex space. Theorems of Boutet de Monvel and Harvey and Lawson assure that strictly pseudoconvex CR–structures on compact manifolds of dimension at least 5 do in fact arise as the boundaries of complex spaces, see [BdM, HaLa, Ro]. Additional geometric hypotheses on the initial singularity are needed in order to conclude that these spaces are deformations of the reference space. This problem has been treated in dimension greater than 3 by Buchwietz and Millson and Akahori and Miyajima, see [AkMi, BuMi, Mi].

If the dimension of initial variety is 2 then the situation is entirely different: there is no integrability condition and it is no longer the case that every CR–structure arises as the boundary of a complex space. This is equivalent to existence of an embedding of the manifold in $\mathbb{C}^n$ whose coordinate functions belong to the algebra of CR–functions. Such a CR–structure is called embeddable. The $\mathcal{C}^\infty$–generic structure is not embeddable, see [Ni, JT, Ep3]. Thus the central problem in obtaining a "real" description of the deformations of an isolated surface singularity is the problem of embeddability for CR–structures on compact 3–dimensional manifolds: give a criterion in terms of the deformation tensor of the CR–structure which characterizes the embeddable CR–structures. For the purposes of analysis one would like the criterion to be expressed in terms of a pseudodifferential equation. Of course a differential condition would be preferable but it is known from examples that the property of embeddability is non–local in nature. In general there may not be a simple condition which characterizes embeddability. Instead we seek a description of the general features of the set of embeddable structures intrinsically and as a subset of the space of all structures. For example are there situations when the embeddable structures form a submanifold or subvariety of the set of all structures? Does it make sense to say that the set of embeddable structures has infinite codimension? Indeed, with the current state of knowledge, it is not known in general if the set of embeddable structures is a closed subset in a reasonable topology.

In this lecture we give a description of the set of deformations of an embeddable CR–structure on a three dimensional manifold. We then define a stratification of the set of embeddable structures. This stratification has closed strata and is defined by formally analytic equations. We next consider the deformations of the total space of the tangent bundle of a Riemann surface and show how the real and holomorphic descriptions can be compared in this case. Finally we close with a collection of problems which bear on the problem of describing the space of embeddable CR–structures on a compact manifold.

Acknowledgments

Many of the ideas in this lecture were developed in collaborative efforts with Dan Burns, John Bland and Gennadi Henkin. I have also learned a lot in conversations with and from the papers of László Lempert. Several of the problems in §6 are motivated by his work, see [Le1-2]. I would also like to thank Ron Donagi for explaining the affine bundles which appear in §5 and Sascha Voronov for his help with algebraic geometry.

§2. Deformations of CR–structures

Let M denote a compact three manifold. A CR–structure on M is given by selecting a smoothly varying complex line $T_p^{0,1}M \subset T_pM \otimes \mathbb{C}$, $p \in M$. We require a non–degeneracy condition:

$$T_p^{0,1}M \cap \overline{T_p^{0,1}M} = \{0\} \text{ for every } p \in M. \tag{2.1}$$

As the fiber dimension in this case is one, the integrability condition is vacuous. The canonical example of a CR–structure arises on a real hypersurface in a complex manifold. If $M \hookrightarrow X$ is a real hypersurface then the CR–structure **induced** by the embedding is given by:

$$T_p^{0,1}M = T_p^{0,1}X \cap T_pM \otimes \mathbb{C}, \text{ for } p \in M.$$

The non–degeneracy condition implies that there is a smooth plane field $H \subset TM$ such that for every $p \in M$:

$$T_p^{1,0}M \oplus T_p^{0,1}M = H_p \otimes \mathbb{C}.$$

The plane field, H defines a contact structure on M if and only if the CR–structure is strictly pseudoconvex. We assume that this condition holds throughout the paper.

The CR–structure defines a pair of operators analogous to $\partial, \overline{\partial}$ by the formulæ :

$$\partial_b f = df \restriction_{T^{1,0}M}, \quad \bar{\partial}_b f = df \restriction_{T^{0,1}M} .$$

The operator $\bar{\partial}_b$ takes values in $\mathcal{C}^\infty(M; (T^{0,1}M)^*)$. This is a quotient of $(TM \otimes \mathbb{C})^*$; to represent $\bar{\partial}_b$ as a differential operator a non–canonical choice needs to be made. For example, selecting a one form defining H determines a representation of $\bar{\partial}_b$. We often use the notation $(M, \bar{\partial}_b)$ to denote the manifold M with the CR–structure which induces $\bar{\partial}_b$. The kernel of $\bar{\partial}_b$ is called the algebra of CR–functions. An embedding of the CR–manifold $(M, \bar{\partial}_b)$ is given by an embedding $F : M \to \mathbb{C}^N$ where the coordinate functions of F belong to $\ker \bar{\partial}_b$. The geometric description of a CR–embedding is that

$$F_* T^{0,1}M = T^{0,1}F(M),$$

where the CR–structure on the right hand side is that induced from the embedding.

The deformations of the CR–structure are given by sections of the endomorphism bundle:

$$\phi \in \mathcal{C}^\infty(M; \mathrm{End}(T^{0,1}M, T^{1,0}M)),$$

with

$${}^{\phi}T^{0,1}_p M = \{\bar{Z} + \phi_p(\bar{Z}) : \bar{Z} \in T^{0,1}_p M\}.$$

In order for (2.1) to hold we require:

$$\|\phi\|_{L^\infty} < 1. \tag{2.2}$$

We denote the set of endomorphisms satisfying (2.2) by $\mathrm{Def}(M, \bar{\partial}_b)$. Using the natural isomorphism:

$$\mathrm{End}(T^{0,1}M, T^{1,0}M) \simeq T^{1,0}M \otimes (T^{0,1}M)^*$$

the $\bar{\partial}_b$–operator defined by the deformation, ϕ can be represented as:

$$\bar{\partial}_b^{\phi} f = (\bar{\partial}_b + \phi \circ \partial_b) f. \tag{2.3}$$

Note that all such CR–structures have the same underlying plane field, *a priori* one might have expected that this should also be allowed to vary. However according to a theorem of A. Gray, contact fields are rigid and hence any small deformation of H is diffeomorphically equivalent to H by a diffeomorphism isotopic to the identity, see [Gy]. Thus no generality is lost in supposing that the underlying plane field is fixed.

The group Cont_H consists of diffeomorphisms which preserve the contact field, i.e.

$$\psi \in \mathrm{Cont}_H \iff \psi_* H_p = H_{\psi(p)},\ \forall p \in M.$$

This group has the topology of a smooth tame lie group as explained in [ChLe]. It acts on $\mathrm{Def}(M, \bar{\partial}_b)$ as follows: if $\phi \in \mathrm{Def}(M, \bar{\partial}_b)$, $\psi \in \mathrm{Cont}_H$ then we define $\psi \cdot \phi$ by

$$^{\psi\cdot\phi}T^{0,1}_{\psi(p)}M = \psi_* {}^{\phi}T^{0,1}_p M. \tag{2.4}$$

As $\ker \bar{\partial}^{\phi}_b = \psi^*(\ker \bar{\partial}^{\psi\cdot\phi}_b)$ the structures in a Cont_H–orbit should be considered geometrically equivalent. We define a moduli space for CR–structures as the quotient

$$\mathcal{M}(M, [\bar{\partial}_b]) = \mathrm{Def}(M, \bar{\partial}_b)/\,\mathrm{Cont}_H\,.$$

In analogy with the case of Riemann surfaces we define a "Teichmüller" space by

$$\mathcal{T}(M, [\bar{\partial}_b]) = \mathrm{Def}(M, \bar{\partial}_b)/\,\mathrm{Cont}^0_H,$$

where Cont^0_H denotes the identity component of Cont_H.

In a recent paper, Cheng and Lee have shown that one can construct a slice for the contact action on $\mathrm{Def}(M, \bar{\partial}_b)$ for any three manifold, see [ChLe]. Taking advantage of the S^1–action, Bland constructed a different slice for structures on S^3 near to the structure induced on the unit sphere, see [Bl]. The former slice is a real manifold whereas the latter has the structure of a smooth bundle with complex fibers over a real manifold. As $\mathrm{Def}(M, \bar{\partial}_b)$ is an open set in a complex Frechet manifold whereas Cont_H is a real group, it seems unlikely that either $\mathcal{T}(M, [\bar{\partial}_b])$ or $\mathcal{M}(M, [\bar{\partial}_b])$ has a natural structure as a complex space. In these proceedings Lempert has shown that the Teichmüller space of structures near to the structure induced on a strictly pseudoconvex hypersurface in $\mathbb{C}^2$ has a natural Frechet manifold structure. It should be emphasized that in the work of Cheng and Lee and Bland only the action of a neighborhood of the identity in Cont_H is considered whereas Lempert takes the quotient by the full identity component.

§3. The relative index and the stratification of the moduli space

The issue of embeddability is intimately connected with the stability properties of the $\ker \bar{\partial}_b^{\phi}$ under deformations. In order to better understand this question we recall a theorem of Kohn, see [Ko]:

Theorem [Kohn]. *A compact pseudoconvex CR–manifold $(M, \bar{\partial}_b)$ is embeddable if and only if the range $\bar{\partial}_b$ is closed in $L^2(M)$.*

In order to understand the content of this theorem it is useful to introduce a second order, self adjoint operator with the same kernel as $\bar{\partial}_b$. Once a contact form is fixed we can represent $\bar{\partial}_b$ as closed operator on L^2 and define its formal adjoint, $\bar{\partial}_b^*$. The operator

$$\Box_b f = \bar{\partial}_b^* \bar{\partial}_b f$$

has a self adjoint extension to $L^2(M)$. The range of $\bar{\partial}_b$ is closed if and only the range of $\Box_b$ is closed. Since $\Box_b$ is self adjoint its range is closed if and only if 0 is an isolated point in the $\operatorname{spec}(\Box_b)$.

If $\bar{\partial}_b$ defines an embeddable CR–structure then

$$\operatorname{spec}(\Box_b) = \{0 < \lambda_1 \leq \lambda_2 \leq \dots \quad \}$$

Zero is an eigenvalue of infinite multiplicity as this eigenspace is simply the L^2–closure of $\ker \bar{\partial}_b$; the sequence $\{\lambda_i\}$ is discrete and tends to infinity. If $\bar{\partial}_b$ is non–embeddable then in addition to the sequence, $\{\lambda_i\}$ tending to infinity $\operatorname{spec}(\Box_b) \supset \{\mu_i\}$ where

$$\mu_i > 0 \text{ and } \mu_i = O(i^{-N}),\ \forall N > 0.$$

If $\bar{\partial}_b'$ is a small embeddable deformation of an embeddable structure, $\bar{\partial}_b$ it is possible that $\Box_b'$ has a finite number of eigenvalues $\{\mu_1, \dots, \mu_m\}$ which satisfy: $0 < \mu_i << \lambda_1$. The eigenfunctions of $\Box_b'$ corresponding to the $\{\mu_i\}$ are small perturbations of functions in $\ker \Box_b$. The presence or absence of "small eigenvalues" is therefore a measure of the size of $\ker \bar{\partial}_b'$ relative to $\ker \bar{\partial}_b$.

Unfortunately this reasoning can only be carried out in a small neighborhood of the reference structure and appears to depend very strongly on the non–canonical choices made in the definition of $\bar{\partial}_b$. In order to obtain something more robust we need to modify our point of view. The starting point is the following theorem:

Theorem 3.1. *Let $(M, \bar{\partial}_b)$ denote a compact, strictly pseudoconvex, embeddable, 3–dimensional CR–manifold. Let $\phi \in \mathrm{Def}(M, \bar{\partial}_b)$ and let $\mathcal{S}$ denote the orthogonal projection onto $\ker \bar{\partial}_b$ relative to some choice of smooth volume form on M. The CR–manifold $(M, \bar{\partial}_b^{\phi})$ is embeddable if and only if*

$$\mathcal{S} : \ker \bar{\partial}_b^{\phi} \longrightarrow \ker \bar{\partial}_b$$

is a Fredholm operator.

Remark. The results in this section are proved in [Ep1-2]. Observe that this characterization of embeddability holds globally in $\mathrm{Def}(M, \bar{\partial}_b)$. Note also that if $\mathcal{S}'$ denotes the projection onto $\ker \bar{\partial}_b'$ $\mathcal{S} - \mathcal{S}'$ is a compact operator if and only if $\bar{\partial}_b = \bar{\partial}_b'$.

The proof of Theorem 3.1 uses a fortuitous representation for $\mathcal{S} \restriction_{\ker \bar{\partial}_b'}$, some functional analysis and the theorem of Kohn stated above. This result suggests that we define a relative invariant for a pair of embeddable structures:

Definition. *If $\bar{\partial}_b$ and $\bar{\partial}_b'$ are two embeddable structures with the same underlying plane field and orientation then define the relative index*

$$\mathrm{Ind}(\bar{\partial}_b, \bar{\partial}_b') = \mathrm{ind}(\mathcal{S} : \ker \bar{\partial}_b' \to \ker \bar{\partial}_b).$$

Unfortunately many non–canonical choices were made in the definition of the relative index. In order for $\mathrm{Ind}(\bar{\partial}_b, \bar{\partial}_b')$ to be an interesting invariant we need to show that its value is independent of these choices. Obviously a volume form was chosen to define the orthogonal projector. A little subtler is the fact that we would like the invariant to be geometric in nature and hence constant along orbits of $\mathrm{Cont}_H \times \mathrm{Cont}_H$. Independence of the choice of volume form is easily established, constancy along orbits of the contact group requires considerably more effort. Nonetheless we can prove the following result:

Theorem 3.2. *The relative index $\mathrm{Ind}(\bar{\partial}_b, \bar{\partial}_b')$ is independent of the choice of volume form and is constant along the orbits of $\mathrm{Cont}_H^0 \times \mathrm{Cont}_H^0$*

There are two results which are of interest in their own right used in the proof of Theorem 3.2

Proposition 3.1. *If $\mathcal{F} \subset \mathrm{Def}(M, \bar{\partial}_b)$ and there is a constant $C >$ 0 such that for $\phi \in \mathcal{F}$*

$$\lambda_1(\Box_b^{\phi}) > C$$

then the map $\phi \longrightarrow S^\phi$ is continuous from the $\mathcal{C}^\infty$–topology on $\mathcal{F}$ to the norm topology on bounded operators on $L^2(M)$.

Remark. Here $\lambda_1(\Box_b^\phi) = \inf\{\operatorname{spec}\Box_b^\phi \setminus \{0\}\}$.

The second result shows that the relative index defines, in a sense, a 1–cocycle on the space of embeddable structures:

Proposition 3.2. *If $\bar\partial_b^1$, $\bar\partial_b^2$, $\bar\partial_b^3$ are embeddable structures in* $\operatorname{Def}(M, \bar\partial_b)$ *such that there exists a continuous family of embeddable structures $\bar\partial_b^t$ joining $\bar\partial_b^2$ to $\bar\partial_b^3$ then:*

$$\operatorname{Ind}(\bar\partial_b^1, \bar\partial_b^3) = \operatorname{Ind}(\bar\partial_b^1, \bar\partial_b^2) + \operatorname{Ind}(\bar\partial_b^2, \bar\partial_b^3).$$

Remark. We believe that the cocycle relation should hold in complete generality for any triple of embeddable structures in $\operatorname{Def}(M, \bar\partial_b)$.

Using the relative index we can define a stratification of the space of embeddable structures:

$$\mathfrak{S}_n = \{\phi \in \operatorname{Def}(M, \bar\partial_b) : \operatorname{Ind}(\bar\partial_b, \bar\partial_b^\phi) \geq -n\}.$$

From the invariance under the action of Cont_H^0 it follows that the stratification descends to the Teichmüller space, $\mathcal{T}(M, \bar\partial_b)$. Large subsets of the strata are closed in the $\mathcal{C}^\infty$–topology:

Theorem 3.3. *For any $\epsilon > 0$ the set*

$$\mathfrak{S}_n \cap \{\phi : \|\phi\|_{L^\infty} \leq \frac{1}{2} - \epsilon\}$$

is closed in the $\mathcal{C}^\infty$–topology.

Remark. It follows from Theorem 3.3 that a strategy for showing that the set of embeddable CR–structures is closed is to show that there is a k, N and $\epsilon > 0$ so that, for $n \geq N$

$$\mathfrak{S}_n \cap \{\|\phi\|_{C^k} < \epsilon\} = \mathfrak{S}_N \cap \{\|\phi\|_{C^k} < \epsilon\}$$

In case the reference structure is that induced on a strictly pseudoconvex hypersurface in $\mathbb{C}^2$, the results in [Le2] and [Ep1–2] imply that this conjecture holds, with $N = 0$. Using a recent result of Eliashberg this has been improved for the case of S^3. In [Ep1] it is shown that the set of embeddable structures on S^3 coincides with $\mathfrak{S}_0$.

If V is an affine variety and M_1 and M_2 are nearby strictly pseudoconvex hypersurfaces bounding compact domains in V then it is reasonable that $\operatorname{Ind}(\bar\partial_b^1, \bar\partial_b^2) = 0$. Here $\{\bar\partial_b^i\}$ are the CR–structures induced by the embeddings $M_i \hookrightarrow V$.

Theorem 3.4. *If V is a variety and M_t, $t \in [0,1]$ is a continuous family of smooth strictly pseudoconvex hypersurfaces in $V \backslash \operatorname{sing}(V)$ which bound compact domains then*

$$\operatorname{Ind}(\bar\partial_b^t, \bar\partial_b^s) = 0, \ \forall s, t \in [0,1].$$

Here $\bar\partial_b^t$ is the CR–structure induced from the embedding $M_t \hookrightarrow V$.

Remark. We call the family $\{M_t : t \in [0,1]\}$ a **strictly pseudoconvex isotopy** of M_0 to M_1.

Recall the construction of the Kuranishi space for an isolated singular point, (V, p) : Intersect V with a sphere of small radius centered on the singular point to obtain a smooth, strictly pseudoconvex hypersurface $M \hookrightarrow V$. This embedding induces a CR–structure, $\bar\partial_b$ on M. One then considers (in higher dimensions) the integrable deformations of $(M, \bar\partial_b)$ modulo an equivalence relation. The equivalence relation is, roughly speaking that

$$(M, \bar\partial_b^1) \sim (M, \bar\partial_b^2), \tag{3.1}$$

if they are in the same strictly pseudoconvex isotopy class.

In the actual construction of the Kuranishi space one requires that this condition hold only to first order. From Theorem 3.4 it follows that if we define a space $\mathfrak{D}([M, \bar\partial_b])$ to be the set of embeddable deformations of $(M, [\bar\partial_b])$ modulo (3.1) then the stratification of $\operatorname{Def}(M, \bar\partial_b)$ actually descends to this space.

In [La] an interesting situation is considered. Let X be a compact strictly pseudoconvex domain in a smooth modification of a Stein space. Assume moreover that the maximal compact analytic set $A \subset X$, is a union of smooth curves with only normal crossings. Of course the deformation space for the complex manifold X is infinite dimensional, however if one considers the deformation space modulo the relation (3.1) it becomes finite dimensional. In fact Laufer showed that the base space for the versal deformation is a manifold parametrized by $H^1(X; \Theta)$. Here Θ is the sheaf of germs of holomorphic vector fields. In the next section we consider the example of X a domain in the $T^{1,0}\Sigma$, where Σ is a Riemann surface of genus at least 2. We compare the "holomorphic" description of the deformation space as $H^1(X,\Theta)$ with a real description derived from considering deformations of the CR–structure on ∂X. In particular we consider the stratification of $H^1(X,\Theta)$ defined by the relative index.

§4. The CR–geometry of the tangent bundle

Let Σ denote a Riemann surface of genus at least 2. The complex structure is defined by specifying a coordinate cover $\{(V_\alpha, z_\alpha)\}$ and holomorphic transition functions $\{f_{\alpha\beta}\}$ where

$$z_\alpha = f_{\alpha\beta}(z_\beta) \text{ on } z_\beta(V_\alpha \cap V_\beta). \tag{4.1}$$

Let $\pi : T^{1,0}\Sigma \to \Sigma$ denote the holomorphic tangent space of Σ. Using the coordinate vector fields, $\{\partial_{z_\alpha}\}$ to locally trivialize $T^{1,0}\Sigma$ we obtain coordinates, $\{U_\alpha, (z_\alpha, w_\alpha)\}$ for the tangent space where $U_\alpha = \pi^{-1}(V_\alpha)$ and

$$w_\alpha = f'_{\alpha\beta}(z_\beta)w_\beta$$

on the overlaps. The total space is compactified by adding the curve "at infinity." Denote this space by $\widehat{T}\Sigma$. A Riemannian metric on Σ defines a function, homogeneous of degree 2, on the fibers of $T^{1,0}\Sigma$. In a local coordinate system this function is represented by $h(z_\alpha, w_\alpha) = e^{2u^\alpha}|w_\alpha|^2$. Define a hypersurface $M \hookrightarrow T^{1,0}\Sigma$ by

$$M = \{p \in T^{1,0}\Sigma : h(p) = 1\}.$$

Let D denote the unit disk bundle in $T^{1,0}\Sigma$ bounded by M. It is clear from the form of h that M is invariant under the natural action of S^1 on $T^{1,0}\Sigma$:

$$U_\phi(z_\alpha, w_\alpha) = (z_\alpha, e^{i\phi}w_\alpha), \phi \in [0, 2\pi).$$

For $k \in \mathbb{Z}$ set

$$F_k = \{f \in \mathcal{C}^\infty(M) : U_\phi^* f = e^{ik\phi}\},$$

and

$$\mathcal{F}_k = \bigoplus_{j=k}^{\infty} F_j.$$

A contact form is defined on M by setting

$$\theta = -i\overline{\partial} \log h \restriction_M .$$

For the remainder of this section we suppose that h is defined by the constant curvature, -1 metric on Σ. A simple calculation shows that the one forms,

$$\theta^1 = \frac{\sqrt{2}dz_\alpha}{w_\alpha} \text{ on } M \cap U_\alpha,$$

piece together to give a globally defined one form, θ^1 on M which satisfies

$$d\theta = i\theta^1 \wedge \theta^{\bar{1}} \text{ and } U_\phi^* \theta^1 = e^{i\phi}\theta^1.$$

The section of $T^{0,1}M$ dual to $\theta^{\bar{1}}$ is of course globally defined and given in local coordinates by

$$\bar{Z} \restriction_{U_\alpha} = \frac{\bar{w}_\alpha}{\sqrt{2}}(\partial_{\bar{z}_\alpha} - 2u^\alpha_{\bar{z}_\alpha}\bar{w}_\alpha \partial_{\bar{w}_\alpha}).$$

The local coordinate representation of the $\bar{\partial}_b$–operator is:

$$\bar{\partial}_b f = \bar{Z}_\alpha f \theta^{\bar{1}}_\alpha.$$

A consequence of using the constant curvature metric to define h is that

$$\mathcal{L}_{\bar{Z}}\theta \wedge d\theta = 0,$$

thus the adjoint of $\bar{\partial}_b$, with these normalizations is

$$\bar{\partial}_b^*(g\theta^{\bar{1}}) = -Zg. \tag{4.2}$$

For details of these arguments see [Ep3]. The commutator $T = [\bar{Z}, Z]$ is a purely imaginary vector field that satisfies

$$T \restriction_{F_k} = -k. \tag{4.3}$$

Since $T^{0,1}M$ has a global non–vanishing section we identify $\mathrm{Def}(M, \bar{\partial}_b)$ with $\{\phi \in \mathcal{C}^\infty(M) : \|\phi\|_{L^\infty} < 1\}$ by setting

$${}^\phi T_p^{0,1}M = \{\lambda(\bar{Z}_p + \phi(p)Z_p) : \quad \lambda \in \mathbb{C}\}.$$

are S^1–invariant and so we say that deformations in $\mathcal{F}_{-2}$ ($\mathcal{F}_{-1}$) have non–negative (positive) Fourier coefficients. In [Ep3] it is shown that all deformations in $\mathcal{F}_{-2}$ are embeddable.

Using a formalism introduced by Bland and Duchamp in [BlDu] we can actually extend deformations lying in $\mathcal{F}_{-2}$ to integrable almost complex structures defined on the disk bundle in $T^{1,0}\Sigma$ bounded by M. In local coordinates $\phi \in \mathcal{F}_{-2}$ has a Fourier expansion

$$\phi(z_\alpha, w_\alpha) = \sum_{j=-2}^{\infty} a_j(z_\alpha, \bar{z}_\alpha) w_\alpha^j. \tag{4.4}$$

Define a vector field on $D \cap U_\alpha$ by:

$$\bar{\mathcal{Z}}_\alpha^\phi = (\partial_{\bar{z}_\alpha} - 2u^\alpha_{\bar{z}_\alpha} \bar{w}_\alpha \partial_{\bar{w}_\alpha}) + e^{2u^\alpha} \sum_{j=-2}^{\infty} a_j w_\alpha^{j+2} (\partial_{z_\alpha} - 2u^\alpha_{z_\alpha} w_\alpha \partial_{w_\alpha}). \tag{4.5}$$

Simple calculations show that

$$e^{2u^\alpha} w_\alpha (\bar{Z}_\alpha + \phi Z_\alpha) = \bar{\mathcal{Z}}_\alpha^\phi \restriction_M \text{ and}$$
$$\bar{\mathcal{Z}}_\beta^\phi = \bar{f}'_{\alpha\beta} \bar{\mathcal{Z}}_\alpha,$$

on the overlaps. As a consequence of the second relation it follows that $\{\bar{\mathcal{Z}}_\alpha d\bar{z}_\alpha\}$ is a globally defined vector valued $(0,1)$–form. In the next section we show that the $(1,0)$–part can be identified as the Dolbeault representative of a class in $H^1(D;\Theta)$. If we take $\bar{\mathcal{W}}_\alpha = \partial_{\bar{w}_\alpha}$ then

$$(\bar{\mathcal{Z}}_\alpha^\phi, \bar{\mathcal{W}}_\alpha)$$

is a globally defined, integrable, almost complex structure on the unit disk bundle which induces the CR–structure ${}^\phi T^{0,1}M$ on ∂D. This is a "real" representation of deformations of the complex structure on the disk bundle.

In [BlEp] a formalism is presented for studying the deformations of a surface singularity in terms of CR–structures on a link. Part of this program is to recognize when a deformation is, to first order, a wiggle of a hypersurface within a variety. A second order operator $\mathcal{P}$ is defined such that a deformation, ϕ is a wiggle, to first order if and only if $\phi = \mathcal{P}\varphi$. Results in [ChLe] show that $\mathcal{P}$ has a closed range and therefore it is reasonable to expect that the "versal deformation" of the isolated singular point will be found in $\ker \mathcal{P}^*$. Here the adjoint is defined relative to some choice of volume form. Note that the operator $\mathcal{P}^*$ is defined intrinsically on M. The analysis in [BlEp] applies to embedded CR–manifolds and the category of embedded deformations. So if we consider all embeddable families of deformations which lie in $\ker \mathcal{P}^*$ then we may be considering deformations of several families of singularities which share the reference CR–manifold as a link. A second caveat is that in the cited work it is assumed that the singularities are normal. In the next section some of the singularities to be considered are non–normal, so the results in [BlEp] cannot be applied directly but serve rather as motivation. In the case at hand one easily computes that $\mathcal{P}^* = Z^2$.

To complete our discussion of the geometry of M we relate the $\bar{\partial}_b$-operator to the $\overline{\partial}$–operator on Σ. Let κ denote the canonical bundle of Σ. A form of weight k is a section of $\kappa^{\otimes k}$, in local coordinates it is represented by $u = u^\alpha(z_\alpha, \bar{z}_\alpha)dz_\alpha^k$. A form of weight k defines a function $U \in F_k$, with local coordinate representation:

$$U(z_\alpha, w_\alpha) = u_a w_\alpha^k.$$

Moreover $\bar{\partial}_b U = 0$ if and only if $\overline{\partial} u = 0$. In this way we see that the $\ker \bar{\partial}_b \subset \mathcal{F}_0$ and the Fourier transform defines an isomorphism:

$$\ker \bar{\partial}_b = \bigoplus_{k=0}^{\infty} H^0(\Sigma; \kappa^k). \tag{4.6}$$

It is also useful to consider sections of $\overline{\kappa}^{-k}$; locally we have $v = v_\alpha d\bar{z}_\alpha^k$. In this case set $V \restriction_{U_\alpha \cap M} = v_\alpha e^{2ku^\alpha} w_\alpha^k$ to obtain a globally defined function on M. As before the equation $\partial_b V = 0$ is equivalent to $\partial v = 0$.

Finally we compute the $\overline{\partial}$–operator on $(1,0)$–vector fields defined in D. Let

$$p_{1,0} : TX \otimes \mathbb{C} \longrightarrow T^{1,0}X$$

denote the canonical projection defined by the complex structure. If V_α is a $(1,0)$–vector field defined in a subset of U_α then

$$\overline{\partial} V_\alpha = p_{1,0}[V_\alpha, \partial_{\bar{z}_\alpha}]d\bar{z}_\alpha + p_{1,0}[V_\alpha, \partial_{\bar{w}_\alpha}]d\bar{w}_\alpha. \tag{4.7}$$

§5. Deformations of the tangent bundle

In [La] the following theorem is proved:

Theorem [Laufer]. *Let X be a strictly pseudoconvex manifold with a one dimensional exceptional set A. Then there is a strictly pseudoconvex neighborhood of A in X and a deformation, $\omega : \mathcal{X} \to Q$ of $X = \omega^{-1}(0)$, with Q a manifold such that the Kodaira–Spencer map $\rho_0 : {}_QT_0 \to H^1(X, \Theta)$ is an isomorphism.*

We apply this theorem to study the deformations of the disk bundle $D \subset T^{1,0}\Sigma$, modulo the equivalence relation (3.1). Using the Fourier transform we reduce the computation of $H^1(D; \Theta)$ to computations on Σ. This introduces a grading on $H^1(D; \Theta)$. Using the Bland–Duchamp extension and the Dolbeault isomorphism, we identify elements of $H^1(D;\Theta)$ with elements of $\mathrm{Def}(M,\bar\partial_b)\cap\ker\mathcal{P}^*$. Using the CR–representation we then give a description of the geometry of the different types of deformations.

Choose coordinates, $\{(z_\alpha,V_\alpha)\}$ for Σ so that $\mathfrak{V}=\{V_\alpha\}$ is a Leray cover then $\{\pi^{-1}(V_\alpha)\}$ is a Leray cover of $T^{1,0}\Sigma$. Their intersections with D define a Leray cover of the disk bundle, which we denote by $\mathfrak{U}=\{U_\alpha\}$ and therefore

$$H^1(D;\Theta)\simeq H^1(\mathfrak{U};\Theta). \tag{5.1}$$

On the sets U_α the coordinate vector fields $\{\partial_{z_\alpha},\partial_{w_\alpha}\}$ trivialize the holomorphic tangent bundle. Locally, holomorphic sections take the form: $a_\alpha(z_\alpha,w_\alpha)\partial_{z_\alpha}+b_\alpha(z_\alpha,w_\alpha)\partial_{w_\alpha}$ where a_α and b_α have Taylor expansions

$$a_\alpha=\sum_{j=0}^\infty a_{\alpha j}(z_\alpha)w_\alpha^j,\quad b_\alpha=\sum_{j=0}^\infty b_{\alpha j}(z_\alpha)w_\alpha^j. \tag{5.2}$$

An elementary computation shows that on the overlaps:

$$b_{\alpha 0}=f'_{\alpha\beta}b_{\beta 0},\quad \begin{pmatrix} a_{\alpha j}\\ b_{\alpha(j+1)}\end{pmatrix}=[f'_{\alpha\beta}]^{-j}\begin{pmatrix} f'_{\alpha\beta} & 0\\ \frac{f''_{\alpha\beta}}{f'_{\alpha\beta}} & 1\end{pmatrix}\begin{pmatrix} a_{\beta j}\\ b_{\beta(j+1)}\end{pmatrix},\ j\geq 0. \tag{5.3}$$

Let $\mathcal{V}$ denote the vector bundle defined on Σ by the 2×2–matrix in (5.3). From (5.2) and (5.3) it follows that

$$H^1(D;\Theta)\simeq H^1(\Sigma;\kappa^{-1})\bigoplus_{j=0}^\infty H^1(\Sigma;\mathcal{V}\otimes\kappa^j). \tag{5.4}$$

The groups appearing on the right hand side can easily be computed. By Serre duality

$$H^1(\Sigma;\mathcal{V}\otimes\kappa^j)\simeq(H^0(\Sigma;\mathcal{V}'\otimes\kappa^{1-j}))'. \tag{5.5}$$

Here $\mathcal{V}'$ is the vector bundle dual to $\mathcal{V}$; it fits into a short exact sequence:

$$0\longrightarrow\kappa\longrightarrow\mathcal{V}'\longrightarrow\mathcal{O}\longrightarrow 0. \tag{5.6}$$

Since κ^{1-j} is locally free we can tensor in (5.6) to obtain the exact sequences of vector bundles

$$0\longrightarrow\kappa^{2-j}\longrightarrow\mathcal{V}'\otimes\kappa^{1-j}\longrightarrow\kappa^{1-j}\longrightarrow 0. \tag{5.7}$$

Using (5.5), (5.7) and the long exact sequence in cohomology one easily shows that

$$H^1(\Sigma;\mathcal{V}\otimes\kappa^j)=0\text{ for }j\geq 3. \tag{5.8}$$

The computations for $j \in \{0,1,2\}$ follow from the exact sequences

$$\begin{aligned}
j=0 \quad & 0 \longrightarrow H^0(\Sigma;\kappa^2) \longrightarrow H^0(\Sigma;\mathcal{V}'\otimes\kappa) \longrightarrow H^0(\Sigma;\kappa) \longrightarrow 0,\\
j=1 \quad & 0 \longrightarrow H^0(\Sigma;\kappa) \longrightarrow H^0(\Sigma;\mathcal{V}') \longrightarrow H^0(\Sigma;\mathcal{O}) \longrightarrow H^1(\Sigma;\kappa)\dots,\\
j=2 \quad & 0 \longrightarrow H^0(\Sigma;\mathcal{O}) \longrightarrow H^0(\Sigma;\mathcal{V}'\otimes\kappa^{-1}) \longrightarrow 0.
\end{aligned} \tag{5.9}$$

Only the case $j = 1$ requires further comment: a simple calculation shows that the generator of $H^0(\Sigma;\mathcal{O})$ is mapped by l to the 1–cocycle $\{\frac{f''_{\alpha\beta}}{f'_{\alpha\beta}}\}$ which is non–trivial in the one dimensional group $H^1(\Sigma;\kappa)$. Hence the sequence in this case can be replaced with

$$j=1 \quad 0 \longrightarrow H^0(\Sigma;\kappa) \longrightarrow H^0(\Sigma;\mathcal{V}') \longrightarrow 0. \tag{5.10}$$

Dualizing the sequences in (5.9) and (5.10) we obtain:

Proposition 5.1. *The S^1–action defines a grading of the cohomology group $H^1(D;\Theta)$ over $\{-1,0,1,\dots\}$. Denote the summands by H^1_j. We have $H^1_j = 0$ for $j \geq 3$ and*

$$\begin{aligned}
&0 \longrightarrow H^1(\Sigma;\kappa^{-1}) \longrightarrow H^1_{-1} \longrightarrow 0,\\
&0 \longrightarrow H^1(\Sigma;\mathcal{O}) \longrightarrow H^1_0 \longrightarrow H^1(\Sigma;\kappa^{-1}) \longrightarrow 0,\\
&0 \longrightarrow H^1_1 \longrightarrow H^1(\Sigma;\mathcal{O}) \longrightarrow 0\\
&0 \longrightarrow H^1_2 \longrightarrow H^1(\Sigma;\kappa) \longrightarrow 0.
\end{aligned} \tag{5.11}$$

Remark. A cohomology group occurring to the left of H^1_j in (5.11) corresponds to vector fields in the subbundle of Θ spanned by $\{\partial_{w_\alpha}\}$ whereas a group to the right corresponds to a section of the quotient of Θ by this subbundle. In what follows it is useful to observe that the quotient bundle has a non–holomorphic representation as the subbundle of $T^{1,0}(T^{1,0}\Sigma)$ spanned by $\{Z'_\alpha = \partial_{z_\alpha} - 2u^\alpha_{z_\alpha} w_\alpha \partial_{w_\alpha}\}$. This facilitates finding the Dolbeault representatives for the cohomology group $H^1(D;\Theta)$. We now show how these correspond to first order deformations in $\ker \mathcal{P}^*$.

Laufer's theorem states that all the first order deformations in $H^1(D;\Theta)$ are unobstructed and therefore correspond to genuine deformations of the complex structure on D. The Bland–Duchamp extension shows that CR–structures lying in $\mathcal{F}_{-2}$ are extensible as integrable complex structures on D. Thus for deformations in $H^1(D;\Theta)$, corresponding to CR–structures in $\ker \mathcal{P}^* \cap \mathcal{F}_{-2}$, the connection between the first order data and the actual complex structures is quite clear. As we shall see, there are first order deformations which correspond to CR–deformations in F_{-3}. In these cases the representation of the deformed CR–structure requires higher order data.

If $\phi \in \mathrm{Def}(M, \bar{\partial}_b)$ then, as observed in §4, we can extend ϕ to D as a mermorphic function in w_α. If $\phi \in \mathcal{F}_{-2}$ the extended function is actually holomorphic and the vector valued $(0,1)$–forms

$$\omega_\alpha^\phi = e^{2u^\alpha} \sum_{j=-2}^{\infty} a_j w_\alpha^{j+2}(\partial_{z_\alpha} - 2u_{z_\alpha}^\alpha w_\alpha \partial_{w_\alpha}) d\bar{z}_\alpha \tag{5.12}$$

patch together to give a globally defined $T^{1,0}$–valued, $(0,1)$–form. We denote this form by ω^ϕ.

With the normalizations given in §4 the operator $\mathcal{P}^* = Z^2$ and therefore

$$\ker \mathcal{P}^* \subset \bigoplus_{j=-\infty}^{0} F_j. \tag{5.13}$$

The $\ker \mathcal{P}^*$ has the following simple description:

Proposition 5.2. *The* $\ker \mathcal{P}^* \cap \mathcal{C}^\infty(M)$ *is equal to* $\ker Z \oplus \bar{Z} \ker Z$ *and the* $\ker Z = \overline{\ker \bar{Z}}$.

Proof. Using the L^2–inner product we can decompose

$$\ker Z^2 = (\ker Z^2 \ominus \ker Z) \oplus \ker Z. \tag{5.14}$$

From (4.2) we conclude that range $\bar{Z}$ is the orthogonal complement to $\ker Z$; to establish the first claim in the proposition it is only necessary to verify that

$$Z^2 \bar{Z} u = 0 \text{ for } u \in \ker Z. \tag{5.15}$$

We decompose $u \in \ker Z$ into its Fourier components:

$$u = \sum_{j=-\infty}^{0} u_j.$$

Since $u_j \in \ker Z$ it follows from (4.3) that

$$Z\bar{Z}u_j = [Z, \bar{Z}]u_j = ju_j.$$

This implies that (5.15) holds for each of the Fourier components of u and consequently for u itself. The second statement in the proposition is obvious.

Now we can give the correspondence between the "non–negative" classes in $H^1(D; \Theta)$ and first order deformations of the CR–structure on M.

Theorem 5.1. *The equivalence classes of "non–negative" first order deformations of the complex structure on D are represented by first order deformations of the CR–structure on M via the following correspondences:*

$$\begin{aligned} H_0^1 &\leftrightarrow \ker Z^2 \cap F_{-2}, \\ H_1^1 &\leftrightarrow \ker Z^2 \cap F_{-1} = \ker Z \cap F_{-1}, \\ H_2^1 &\leftrightarrow \ker Z^2 \cap F_0 = \ker Z \cap F_0. \end{aligned} \tag{5.16}$$

The $T^{1,0}$–valued, $(0,1)$–forms defined in (5.12) are Dolbeault representatives of the corresponding classes in $H^1(D;\Theta)$.

Remark. The remaining case of $H_{-1}^1 \leftrightarrow \bar{Z}(\ker Z \cap F_{-2})$ is discussed below.

Proof. Using Proposition 5.2 and (4.6) it is a simple matter to show that indicated pairs of vector spaces in (5.16) have the same dimensions and are therefore abstractly isomorphic. Because they depend holomorphically on w_α, the vector valued $(0,1)$–forms defined in (5.12) are $\overline{\partial}$–closed and hence define Dolbeault cohomology classes in $H^{0,1}(D;T^{1,0}D) \simeq H^1(D;\Theta)$. The isomorphism goes as follows: begin with a $\phi \in \mathrm{Def}(M,\bar{\partial}_b)$ and find local solutions to

$$\overline{\partial} V_\alpha = \omega_\alpha^\phi.$$

The 1–cocycle $\{V_\alpha - V_\beta\}$ represents the corresponding class in $H^1(D;\Theta)$. It is a simple matter to check that the vector fields $\{V_\alpha\}$ can be selected to respect the grading and hence define the maps in (5.16). The only point that remains is to show that the map from $\phi \to [\omega^\phi]$ is injective.

We give the details of this argument for some representative cases, beginning with the easiest case, $H_2^1 \simeq H^1(\Sigma;\kappa) \simeq \mathbb{C}$. The $\ker Z \cap F_0$ is easily seen to consist of exactly the constant functions. The class defined by $\lambda \in \mathbb{C}$ is trivial if and only if we can find $V \in \mathcal{C}^\infty(D;T^{1,0}D)$ such that

$$\overline{\partial} V = \omega^\lambda. \tag{5.17}$$

We let $V \restriction_{U_\alpha} = a_\alpha Z'_\alpha + b_\alpha \partial_{w_\alpha}$.

Using formula (4.7) we see that (5.17) is equivalent to:

$$\begin{aligned} \partial_{\bar{w}_\alpha} a_\alpha &= \partial_{\bar{w}_\alpha} b_\alpha = 0, \\ \partial_{\bar{z}_\alpha} a_\alpha &= -\lambda e^{2u^\alpha} w_\alpha^2, \\ \partial_{\bar{z}_\alpha} b_\alpha &= 2 w_\alpha a_\alpha u^\alpha_{z_\alpha \bar{z}_\alpha}. \end{aligned} \tag{5.18}$$

The w_α dependence follows immediately from (5.18):

$$a_\alpha = w_\alpha^2 A_\alpha(z_\alpha), \quad b_\alpha = w_\alpha^3 B_\alpha(z_\alpha).$$

This leaves the equations on Σ :

$$\begin{aligned} \partial_{\bar{z}_\alpha} A_\alpha &= -\lambda e^{2u^\alpha}, \\ \partial_{\bar{z}_\alpha} B_\alpha &= 2A_\alpha u^\alpha_{z_\alpha \bar{z}_\alpha}. \end{aligned} \tag{5.19}$$

In order for V to be globally defined it is necessary that $\{A_\alpha dz_\alpha\}$ piece together to define a global smooth section of κ. We rewrite the equation for A as

$$\bar{\partial}(A_\alpha dz_\alpha) = -\lambda e^{2u^\alpha} dz_\alpha \otimes d\bar{z}_\alpha. \tag{5.20}$$

$$d(A_\alpha dz_\alpha) = -\lambda e^{2u^\alpha} dz_\alpha \wedge d\bar{z}_\alpha. \tag{5.21}$$

It follows from Stokes theorem that this equation is solvable if and only if

$$\int_\Sigma \lambda e^{2u^\alpha} dz_\alpha \wedge d\bar{z}_\alpha = 0,$$

i.e. if and only if $\lambda = 0$. This completes the case $j = 2$. The case $j = 1$ is quite similar and is left to the reader.

We give the argument for one further case: $\phi \in \bar{Z}\psi$ where $\psi \in \ker Z \cap F_{-1}$. In local coordinates $\phi = w_\alpha^{-2} c_\alpha$, $\psi = w_\alpha^{-1} d_\alpha$, the equations satisfied by ϕ and ψ are

$$\begin{aligned} \partial_{\bar{z}_\alpha} d_\alpha &= e^{2u^\alpha} c_\alpha, \\ \partial_{z_\alpha}(e^{2u^\alpha} d_\alpha) &= 0. \end{aligned} \tag{5.22}$$

We need to show that there exists no vector field V such that

$$\bar{\partial} V = \omega^\phi. \tag{5.23}$$

If $V = a_\alpha Z'_\alpha + b_\alpha \partial_{w_\alpha}$, in local coordinates then (5.23) is equivalent to

$$\begin{aligned} \partial_{\bar{w}_\alpha} a_\alpha &= \partial_{\bar{w}_\alpha} b_\alpha = 0, \\ \partial_{\bar{z}_\alpha} a_\alpha &= -e^{2u^\alpha} c_\alpha, \\ \partial_{\bar{z}_\alpha} b_\alpha &= \frac{1}{2} w_\alpha a_\alpha e^{2u^\alpha}. \end{aligned} \tag{5.24}$$

The collection $\{a_\alpha(z_\alpha)\partial_{z_\alpha}\}$ patch together to define a global vector field on Σ. From (5.22) it is clear that $a_\alpha = -d_\alpha$ is a global solution to the second equation in (5.24). The solution is unique as $\ker \overline{\partial} = 0$ on $\mathcal{C}^\infty(\Sigma; T^{1,0}\Sigma)$. Setting $b_\alpha(z_\alpha, w_\alpha) = B_\alpha(z_\alpha)w_\alpha$, the last equation in (5.24) becomes:

$$\partial_{\bar{z}_\alpha} B_\alpha = -\frac{1}{2} d_\alpha e^{2u^\alpha}. \tag{5.25}$$

A simple calculation show that the $\{B_\alpha\}$ must patch together to define a function, B on Σ which satisfies

$$\overline{\partial} B = d_\alpha e^{2u^\alpha} d\bar{z}_\alpha.$$

This equation is not solvable as (5.22) implies that $\{d_\alpha e^{2u^\alpha} d\bar{z}_\alpha\}$ patch together to define a global section in $\ker \overline{\partial}^*$.

The remaining cases are left to the interested reader.

Before considering H^1_{-1} we give a brief geometric description of each of the types of deformations with non–negative Fourier coefficients. In [Ep3] it was shown that $\bar{Z} + \phi Z$ defines an embeddable deformation of $(M, \bar{\partial}_b)$ provided $\phi \in \mathcal{F}_{-2}$. However the possibility remains that $\operatorname{Ind}(\bar{\partial}_b, \bar{\partial}_b^\phi) \neq 0$. From the Bland–Duchamp extension it is evident that the zero section of $T^{1,0}\Sigma$ persists under these deformations as a negatively embedded curve diffeomorphic to Σ. Let D^ϕ denote the unit disk bundle with the complex structure defined by the Bland–Duchamp extension.

We begin with $\phi \in \ker \mathcal{P}^* \cap F_{-2}$. These structures are S^1–invariant and so can be embedded in the total space of a line bundle, L_ϕ of degree $2g-2$ over a Riemann surface, Σ_ϕ of genus g. If $\phi \in \ker Z \cap F_{-2} \simeq H^0(\Sigma; \kappa^2)$ then the CR–structure defined by ϕ can be realized as a hypersurface in the holomorphic tangent bundle of a Riemann surface. An easy calculation shows that for such ϕ,

$$\operatorname{Ind}(\bar{\partial}_b, \bar{\partial}_b^\phi) = 0.$$

For small enough deformations this implies that $\dim H^0(\Sigma_\phi; L_\phi^{-1}) = \dim H^0(\Sigma; \kappa) = g$. The only line bundles of the given degree which have a g–dimensional space of holomorphic sections are the canonical bundles. Thus L_ϕ^{-1} is the canonical bundle of Σ_ϕ. Of course the holomorphic quadratic differentials are isomorphic to the tangent space of the Teichmüller space of the Riemann surface Σ. Thus $\ker Z \cap F_{-2}$ provides a CR–representation for the local moduli of the deformations of Σ.

The other deformations in H^1_0 lie in $\bar{Z}(\ker Z \cap F_{-1}) \simeq H^0(\Sigma; \kappa)$. This group is classically identified with the holomorphic moduli for line bundles of a fixed degree over a Riemann surface. It is easy to show that for such $\phi \neq 0$ the relative index satisfies

$$\operatorname{Ind}(\bar{\partial}_b, \bar{\partial}_b^{\phi}) = -1.$$

Thus the line bundles obtained are not the tangent bundle as the $\dim(\Sigma_\phi ; L_\phi^{-1}) = g - 1$. It follows from Theorem 5.1 that all small deformations of the tangent bundle are found in this family. The complex structure on Σ_ϕ may also vary within this family though it varies $O(\phi^2)$ for small ϕ.

For the structures in $\ker Z \cap F_{-1}$ the complex structure on Σ as well as its normal bundle $[\Sigma] \restriction_\Sigma$ are unchanged. However D^ϕ is not a domain in the total space of a line bundle. This is proved as follows: one finds that the "first order" invariant defined in §2 of [MoRo] is non–zero. This is the obstruction to splitting the normal bundle sequence along Σ :

$$0 \longrightarrow \Theta_\Sigma \longrightarrow \Theta_{D^\phi} \longrightarrow N_\Sigma^\phi \longrightarrow 0.$$

Thus $\Sigma \hookrightarrow D^\phi$ is a holomorphic curve which is not the zero section of a line bundle. However if D_ϕ were a domain in a holomorphic line bundle then Σ would be homologous to the zero section. But this is impossible because Σ is negatively embedded and is therefore the unique holomorphic curve in its homology class. The structures in $\ker Z \cap F_0$ again cannot be embedded in the total space of a line bundle. In this case one computes that the "second order" obstruction defined in §3 of [MoRo] is non–vanishing and therefore the embedding of $\Sigma \hookrightarrow D^\phi$ is not equivalent to the embedding of Σ as the zero section in N_Σ^ϕ. The relative index in these cases satisfies:

$$\operatorname{Ind}(\bar{\partial}_b, \bar{\partial}_b^{\phi}) = 0.$$

We close with a discussion of $\phi \in \mathcal{F}_1$. Of course $\mathcal{F}_1 \subset \operatorname{range} \mathcal{P}$ so these are, to first order, wiggles of M within $T^{1,0}\Sigma$. Indeed it follows from the theorem of Grauert on lifting formal equivalences, [Gr] that there is a neighborhood of Σ in D^ϕ which is biholomorphic to a neighborhood of the zero section in $T^{1,0}\Sigma$. We conjecture that, for a reasonable notion of smallness, the small deformations in $\mathcal{F}_1$ can be realized by wiggling the hypersurface M within $T^{1,0}\Sigma$.

Finally we consider deformations in H^1_{-1}. Apparently the first order part should correspond to a CR–deformation in $\ker \mathcal{P}^* \cap F_{-3}$. The complex structures on D corresponding to deformations in H^1_{-1} are affine bundles which have no compact subvarieties. If the coefficients in the Bland–Duchamp extension were smooth on D then the zero section would persist as a holomorphic curve. This extension for structures in F_{-3} has a first order pole along the zero section of $T^{1,0}\Sigma$. In the present case, the singularity appears to be more in the nature of a "coordinate singularity" as opposed to an unfillable complex manifold.

Let $\phi \in \bar{Z}(\ker Z \cap F_{-2})$ and let ω^{ϕ} denote the vector valued $(0,1)$-form defined on $D \setminus \Sigma$ by (5.12). Clearly we can think of ω^{ϕ} as a first order deformation of the complex structure on the deleted space, $D \setminus \Sigma$. We show below that there is a Čech representative $\{\xi_{\alpha\beta}\} \in H^1(D\setminus\Sigma;\Theta)$ which extends to define a class in $H^1(D;\Theta)$. This provides an identification of the first order deformations of D in H^1_{-1} with these deformations of the CR–structure on M. The relationship in this case is not as transparent as in the previous cases, in part because the first order deformation of the CR–structure is not itself embeddable but requires higher order correction terms.

Let $\psi \in \ker Z \cap F_{-2}$ and set $\phi = \bar{Z}\psi$. In local coordinates $\phi = w_{\alpha}^{-3}c_{\alpha}$, $\psi = w_{\alpha}^{-2}d_{\alpha}$. To find the class in $H^1(D;\Theta)$ corresponding to ϕ we need to solve a system of equations,

$$\begin{aligned}
\partial_{\bar{w}_{\alpha}} a_{\alpha} &= \partial_{\bar{w}_{\alpha}} b_{\alpha} = 0,\\
\partial_{\bar{z}_{\alpha}} a_{\alpha} &= -w_{\alpha}^{-1} e^{2u^{\alpha}} c_{\alpha},\\
\partial_{\bar{z}_{\alpha}} b_{\alpha} &= \frac{1}{2} w_{\alpha} a_{\alpha} e^{2u^{\alpha}}.
\end{aligned} \tag{5.26}$$

Let $a_{\alpha} = w_{\alpha}^{-1}A_{\alpha}$. As before the fact that $\phi = \bar{Z}\psi$ implies that $A_{\alpha} = -d_{\alpha}$ piece together to give a global solution to the second equation in (5.26). This leaves only the equation:

$$\partial_{\bar{z}_{\alpha}} b_{\alpha} = -\frac{1}{2} d_{\alpha} e^{2u^{\alpha}}. \tag{5.27}$$

This can again be interpreted as a $\bar{\partial}$–equation on Σ which cannot be solved because the right hand side belongs to $\ker \bar{\partial}^*$. The vector fields $\{w_{\alpha}^{-1}A_{\alpha}Z'_{\alpha}\}$ piece together to define a global vector field on $D\setminus\Sigma$. The Čech representative of ω^{ϕ} is therefore

$$\{b_{\alpha}\partial_{w_{\alpha}} - b_{\beta}\partial_{w_{\beta}}\}.$$

These evidently extend to define a holomorphic 1–cocycle on D which represents a class in H^1_{-1}. Among the first order deformations of $(M, \bar{\partial}_b)$ in $\ker \mathcal{P}^* \cap \oplus_{\{k<-2\}} F_k$ only those in $\bar{Z}(\ker Z \cap F_{-2})$ correspond to classes which extend to D.

The elements in H^1_{-1} have a simple geometric description as the affine bundles over Σ with $T^{1,0}\Sigma$ as their underlying vector bundle. If

$\{\xi_{\alpha\beta}\}$ represents a class in $H^1(\Sigma;\kappa^{-1})$ then we can define coordinates $\{U_\alpha,(z_\alpha,v_\alpha)\}$ with transition functions:

$$z_\alpha = f_{\alpha\beta}(z_\beta), \quad v_\alpha = f'_{\alpha\beta}(z_\beta)v_\beta + \xi_{\alpha\beta}. \tag{5.28}$$

The cocycle condition for $H^1(\Sigma;\kappa^{-1})$ implies that these relations are consistent. We denote the total space of the affine bundle by A_ξ. It is a Stein manifold which can be compactified by adding a curve "at infinity," denote the compactified space by $\widehat{A}_\xi$. Let N be a small neighborhood of $0\in H^1(\Sigma;\kappa^{-1})$ and set:

$$\mathcal{A} = \bigcup_{\xi\in N}(\widehat{A}_\xi,\xi). \tag{5.29}$$

It is easy to see that $\mathcal{A}$ has a natural structure as a complex manifold and for sufficiently small N, $\pi:\mathcal{A}\to N$ is a deformation space of $\pi^{-1}(0)=\widehat{T}\Sigma$. Perhaps shrinking N further, the real analytic hypersurface $M\hookrightarrow\pi^{-1}(0)$ can be extended to a real analytic hypersurface $\mathcal{M}\hookrightarrow\mathcal{A}$, intersecting the fibers of π transversely.

Let $M_\xi=\pi^{-1}(\xi)\cap\mathcal{M}$ with the induced CR–structure denoted by $\bar\partial_b^\xi$. For $\xi\neq 0$ the hypersurface M_ξ bounds a domain, $D_\xi\subset A_\xi$. As A_ξ is a Stein manifold it follows from Hamilton's stability theorem that for $\xi\neq 0$ there is an open neighborhood $U_\xi\subset N$ such that for $\xi'\in U_\xi$ the complex manifold $D_{\xi'}$ can be realized as a small perturbation of D_ξ within A_ξ, see [Ha]. We can therefore apply Theorem 3.4 to conclude that

$$\mathrm{Ind}(\bar\partial_b^\xi,\bar\partial_b^{\xi'})=0 \text{ for } \xi'\in U_\xi. \tag{5.30}$$

Since $N\setminus\{0\}$ is connected we can apply Proposition 3.2 to conclude that (5.30) holds for any pair $\xi,\xi'\in N\setminus\{0\}$. In a forthcoming paper with Donagi a detailed analysis of the structure of the algebra of holomorphic functions on affine bundles will be given. A computation in that paper shows that if the genus of Σ is g then

$$\mathrm{Ind}(\bar\partial_b^0,\bar\partial_b^\xi)=-(g+1),\ \text{for } \xi\in N\setminus\{0\}. \tag{5.31}$$

This is a larger value than attained for any non–negative deformation.

In [BlEp] the analogous deformations are studied for line bundles over $\mathbb{P}^1$. In this case the linear part of the CR–representative is again of the form $\bar Z g$ for a homogeneous function $g\in\ker Z$. The corresponding embeddable deformation is simply $\bar Z g - g^2$. We conjecture that, in the present case, the embeddable deformation with first order part $\bar Z\psi$ is $\bar Z\psi+\frac{1}{2}\psi^2$.

§6. Problems in deformation theory

In this section we propose six problems in deformation theory which are related to the problem of characterizing the space of embeddable CR-structures.

Problem 1. Let $(M, \bar{\partial}_b)$ be a compact 3–dimensional, strictly pseudoconvex embeddable CR–manifold. Is there a generalization of Bland's construction of a slice for the action of Cont_H which has the structure of a complex fiber bundle over a real manifold, see [Bl]?

Problem 2. Suppose V is an affine variety and M_1, $M_2 \hookrightarrow V$ are two strictly pseudoconvex hypersurfaces in $V \setminus \mathrm{sing}(V)$ which bound compact domains and are smoothly isotopic in $V \setminus \mathrm{sing}(V)$. Are M_1 and M_2 in the same strictly pseudoconvex isotopy class, see Theorem 3.4 ?

Problem 3. If $(M, \bar{\partial}_b)$ is an embeddable, 3–dimensional, CR–manifold are there integers k and N and a positive ϵ so that

$$\mathfrak{S}_n \cap \{\phi : \|\phi\|_{C^k} < \epsilon\} = \mathfrak{S}_N \cap \{\phi : \|\phi\|_{C^k} < \epsilon\} \text{ for } n \geq N?$$

Problem 4. Is there an effective way to translate between the real and holomorphic descriptions of the deformations of complex (or CR) structures?

Problem 5. Suppose X is a smooth complex surface with a compact, maximal, exceptional analytic subvariety, A. Let $\mathcal{I}$ denote the ideal of A and $\mathcal{J} \subset \mathcal{I}$, a sub–ideal for which there exists a neighborhood $U_{\mathcal{J}}$ of A and proper embedding $F_{\mathcal{J}} : U_{\mathcal{J}} \setminus A \to \mathbb{C}^n \setminus \{0\}$ where the coordinate functions of $F_{\mathcal{J}}$ belong to $\mathcal{J}$. We can of course extend $F_{\mathcal{J}}$ to A by 0 and this defines the germ of a singularity $(F_{\mathcal{J}}(U_{\mathcal{J}}), 0) \subset (\mathbb{C}^n, 0)$. If $\mathcal{J}_2 \subset \mathcal{J}_1$ are two such sub–ideals of $\mathcal{I}$ then a deformation of $(F_{\mathcal{J}_1}(U_1), 0)$ induces a deformation of $(F_{\mathcal{J}_2}(U_2), 0)$. Is there a "universal" sub–ideal $\mathfrak{U} \subset \mathcal{I}$ such that for every ideal, $\mathcal{J}$ as above the versal deformation space of $(F_{\mathcal{J}}(U_{\mathcal{J}}), 0)$ can be realized as a subspace of the versal deformation of $(F_{\mathfrak{U}}(U_{\mathfrak{U}}), 0)$?

Problem 6. In [MoRo] the formal neighborhoods of a complex curve, Σ with fixed positive, normal bundle, N are considered. It is shown that there is an infinite dimensional space of inequivalent neighborhoods. Is the subspace of formal neighborhoods which can be realized by an embedding in a compact projective variety finite dimensional?

References

[AkMi] Takao Akahori and Kimio Miyajima, Complex analytic construction of the Kuranishi family on a normal strongly pseudoconvex manifold, II, Publ. Res. Inst. Math. Sci. (Kyoto University, Research Institute for Mathematical Sciences Publications), **16** (1980), 811–834.

[Bl] John Bland, Contact geometry and CR structures on S^3, Acta Math., **172** (1994), 1–49.

[BlDu] John Bland and T. Duchamp, Moduli for pointed convex domains, Invent. Math., **104** (1991), 61–112.

[BlEp] John Bland and C.L. Epstein, Embeddable CR–structures and deformations of psuedoconvex surfaces, Part I: Formal deformations, to appear Jour. of Alg. Geom., (1995), 1–81.

[BuMi] Ragnor-Olaf Buchweitz and John M. Millson, CR–geometry and Deformations of Isolated Singularities, to appear Memoirs of the AMS (June, 1992), 1–153.

[ChLe] Jih-Hsin Cheng and John M. Lee, A local slice theorem for 3–dimensional CR structures, to appear American Journal of Mathematics, 1–44.

[Ep1] C.L. Epstein, A relative index on the space of embeddable CR–structures, I, IHES preprint /M/95/28 (1995), 1–58.

[Ep2] ———, A relative index on the space of embeddable CR–structures, II, IHES preprint /M/95/29 (1995), 1–43.

[Ep3] ———, CR–structures on three dimensional circle bundles, Inv. Math., **109** (1992), 351–403.

[Gr] Hans Grauert, Uber Modifikationen und exzeptionelle analytische Mengen, Math. Ann., **146** (1962), 331–368.

[Gy] J.W. Gray, Some global properties of contact structures, Ann. of Math., **69** (1959), 421–450.

[Ha] Richard Hamilton, Deformation of complex structures on manifolds with boundary, I. The stable case, Jour. Diff Geo., **12** (1977), 1–45.

[HaLa] Reese Harvey and H. Blaine Lawson, On the boundaries of complex analytic varieties, I, Ann. of Math., **102** (1975), 223–290.

[JT] Howard Jacobowitz and Francois Treves, Aberrant CR structures, Hokkaido Math. J., **12** (1983), 276–292.

[Ko] ———, The range of the tangential Cauchy–Riemann operator, Duke Math. J., **53** (1986), 525–545.

[Ku] Masatake Kuranishi, Application of $\bar{\partial}_b$ to deformation of isolated singularities, Proc. of Symp. in Pure Math. of the AMS, **30** (1977), 97–106.

[La] Henry B. Laufer, Versal deformations for two-dimensional pseudoconvex manifolds, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4), **7** (1980), 511–521.

[Le1] László Lempert, On three dimensional Cauchy–Riemann manifolds, JAMS, **5** (1992), 1–50.

[Le2] ———, Embeddings of three dimensional Cauchy–Riemann manifolds, Math. Ann., **300** (1994), 1–15.

[Mi] John J. Millson, CR–geometry and Deformations of Isolated Singularities, Proceeding of Symposia in Pure Mathematics, **54, part 2** (1993), 411-433.

[MoRo] J. Morrow and H. Rossi, Some general results on equivalence of embeddings, in "Recent Developments in Several Complex Variables", Princeton University, pp. 299–325.

[Ni] Louis Nirenberg, On a problem of Hans Lewy, Uspekhi Math. Nauk, **29** (1974), 241-251.

[Ro] Hugo Rossi, Attaching analytic spaces to an analytic space along a pseudoconcave boundary, "Proc. Conf. on Complex Manifolds (A. Aeppli, E. Calabi, Röhrl eds.)", Springer Verlag, New York, Berlin, Heidelberg, 1965.

Department of Mathematics,
University of Pennsylvania
Philadelphia, PA 19104
U. S. A.

Advanced Studies in Pure Mathematics 25, 1997
CR-Geometry and Overdetermined Systems
pp. 110–121

Vector-Valued Forms and CR Geometry

Thomas Garrity and Robert Mizner

Dedicated to Professor M. Kuranishi on his 70th birthday

§1. Introduction.

Vector-valued forms arise in the study of various higher codimensional geometries. This note gives an overview of how the invariant theory of the Levi form (a vector-valued form) can be used to understand higher codimensional CR- structures.

Roughly speaking, the Levi form of a CR- structure of codimension c on a manifold M of dimension $2n+c$ can be interpreted as a map from M to the vector space consisting of c-tuples of $n \times n$ hermitian matrices (a vector space that we denote as Herm). However, this interpretation depends on a prior choice of moving coframesthat is, local sections of the cotangent bundle of M. Fortunately, there is a natural action of the group $G = GL(n, C) \times GL(c, R)$ on Herm that accounts for the effects of these choices. More precisely, there is a natural map from M to the quotient space Herm$/G$. Knowledge about the structure of this quotient space can be used to define canonical objects in higher codimensional CR- geometry. At present, the best developed example (discussed in §5) is a canonical connection for suitably generic CR- structures. The simplest examples, though, are functions defined on Herm$/G$, or equivalently, G-invariant functions defined on Herm : these lead one to explore the invariant theory of vector-valued forms as a tool in the study of CR-geometry. The history of invariant theory suggests two lines of approach. The first, discussed in §3, is to use methods of classical invariant theory to find explicit polynomial functions of vector-valued forms that are (relatively) invariant under the group action. While these techniques are quite old, the ensuing results for vector-valued forms are recent. The second, discussed in §4, is to use modern geometric invariant theory to study the quotient space directly. While the set Herm$/G$ has a standard quotient topology, it does not carry a globally defined differentiable

---

Received December 13, 1995

structure: to obtain a differentiable structure one must first eliminate certain non-generic points (the "unstable" ones). The conditions defining these points are essentially geometric in concept, but involve a fair amount of technical intricacy in practice intricacies that are rooted in the aforementioned classical techniques!

As we shall note in §4, the invariant theory of vector-valued forms has three technically distinct cases: codimension 1, codimension 2, and higher codimension. Essentially, the codimension 1 case can be understood using nothing more than the standard notions of rank and signature of a hermitian matrix, and is therefore quite easy. The codimension 2 case is considerably more involved, but still fairly elementary: it rests on the analysis of roots of polynomials in one unknown. However, the higher codimensional case involves the zero-locus of polynomials in many unknowns, and consequently shares much of the richness of classical algebraic geometry. In CR-geometry to date, the codimension 1 case is the only one to have received a great deal of attention (see [Bo] and [J] for introductions and bibliographies), so the invariant theory of forms has not been featured in the *CR* literature. The approach we describe here is carried out in detail in three papers: [M] treats both the invariant theory and the differential geometry for codimension 2, [GM1] develops the invariant theory for higher codimension, and [GM2] treats the corresponding differential geometry. An introduction to higher-codimensional *CR*-geometry, including the equivalence problem, is contained in [Tu]. Selected references to other approaches to *CR*- geometry of codimension greater than 1 include [Be], [Ta1] and [Ta2].

Both of the authors are honored to participate in this tribute to Professor Kuranishi, and acknowledge with gratitude the fundamental influence his ideas have had on their work. One of the authors (Mizner) would also like to take this opportunity to express his appreciation for the care and graciousness with which Professor Kuranishi supervised his doctoral work: as time passes, he realizes ever more fully how fortunate he was to have been a student of Kuranishi.

§2. Definitions.

CR- structures arise concretely in connection with real submanifolds of a complex space. For example, let M be the zero locus of c real valued functions $g^\alpha : \mathbb{C}^{2n+c} \to \mathbb{R}$. If the real differentials dg^α are linearly independent, then M is a real submanifold of codimension c and dimension $2n + c$. If the holomorphic differentials ∂g^α are linearly independent as well, then the complex structure of $\mathbb{C}^{2n+c}$ determines a complex rank n subbundle $\mathcal{H}$ of the complexified tangent bundle $\mathbb{C} \otimes TM$. This subbundle is an instance of a CR- structure of codimension c and dimension n. Each function g^α determines a $(2n+c)\times(2n+c)$ hermitian matrix with entries $\frac{\partial^2 g^\alpha}{\partial z^j \partial \bar{z}^k}$; together these matrices constitute a vector-valued hermitian form (i.e., a c-tuple of scalar hermitian forms) on the complexified tangent bundle $\mathbb{C}\otimes TM$. Note that this c-tuple is defined only up to a choice of the defining functions g^α and coordinates on $\mathbb{C}^{2n+c}$. The restriction of this form to the subbundle $\mathcal{H}$ is called the Levi form of the CR- structure.

The abstract notion of a CR- structure and its accompanying Levi form generalize this example.

2.1. Definition. Let M be a smooth (i.e. C^∞) manifold. A CR-structure of dimension n and codimension c is a rank n complex subbundle $\mathcal{H}\subset\mathbb{C}\otimes TM$ with the following properties:

(1) $\mathcal{H}\cap\bar{\mathcal{H}}$ is the zero subbundle;
(2) $[\mathcal{H},\mathcal{H}]\subset\mathcal{H}$. (This condition, called the integrability condition, is an important technicality that is automatically satisfied by CR-structures arising on real submanifolds.)

2.2. Definition. The Levi form of $\mathcal{H}$ is the bundle map

$$L:\mathcal{H}\times\mathcal{H}\to\mathbb{C}\otimes TM/(\mathcal{H}\oplus\bar{\mathcal{H}}),$$

defined by

$$L(X,Y)=i\pi[\bar{X},Y],$$

for all sections X and Y of $\mathcal{H}$, where $\pi:\mathbb{C}\otimes TM\to\mathbb{C}\otimes TM/(\mathcal{H}\oplus\bar{\mathcal{H}})$ is the natural projection. It is easy to verify that L is well-defined, and that L is hermitian: i.e., $L(Y,X)$ and $L(X,Y)$ are conjugate.

As indicated in the introduction, by choosing suitably adapted moving coframes (local sections of the complex cotangent bundle $\mathbb{C}\otimes TM$), one can express the Levi form as a locally defined map from M to the real vector space whose points are c-tuples of $n\times n$ hermitian matrices (denoted Herm(n,c), or Herm for short). However, this expression of L as a vector-valued hermitian form depends on the choice of sections. In order to escape this dependency, one defines a natural action of the group $G=GL(n,\mathbb{C})\times GL(c,\mathbb{R})$ on Herm , and verifies that the result of composing the locally defined Herm -valued map with the quotient projection Herm $\to$ Herm$/G$ is independent of choice of sections. Consequently, the locally defined Herm$/G$-valued maps piece together to yield a globally defined map $\mathcal{L}:M\to$ Herm$/G$.

2.3. Definition. The map $\mathcal{L}:M\to$ Herm$/G$ is called the Levi map .

Again as mentioned in the introduction, since every CR-manifold of dimension n and codimension c is mapped into the same quotient space Herm/G, information about Herm/G can be used to define canonical objects in CR- geometry. In §5 we provide examples, but in order to do this, we must first examine Herm/G in some detail.

§3. Classical invariant theory of vector-valued forms.

An absolute invariant of a vector-valued hermitian form is a function $f : \mathrm{Herm} \to \mathbb{C}$ that is constant on each orbit of the action of the group $G = GL(n, \mathbb{C}) \times GL(c, \mathbb{R})$, and hence equivalent to a function on the quotient space Herm/G: in symbols, $f(gB) = f(B)$. A relative invariant (of weight χ) is a function that satisfies the weaker condition $f(gB) = \chi(g)f(B)$, where $\chi : G \to \mathbb{C}^*$ is a homomorphism. (Of course, if χ is the trivial homomorphism, the relative invariant is in fact absolute.) From here on we use the word invariant to cover both cases. Although a relative invariant is not constant on each orbit, it does have the property that if it vanishes at any one point in an orbit then it vanishes at all points in that orbit. Therefore, although a relative invariant does not determine a function on the quotient space, it does nonetheless determine a zero-locus–a fact that has significant geometric repercussions.

The basic procedure of classical invariant theory in situations such as ours is to consider only homogeneous polynomial invariants, to note that these invariants constitute a ring, and to list the generators of this ring in a so-called First Fundamental Theorem. Next, the relations among the generators is given in a Second Fundamental Theorem. Continuing, one then seeks the relations among the relations, the relations among these new relations, etc., which is called computing the syzygies of the ring of invariants.

In [GM1], a first fundamental theorem for vector-valued hermitian forms is obtained by specializing a first fundamental theorem for sesquilinear forms, which in turn is obtained by adapting the proof of a first fundamental theorem for bilinear forms. The basic ideas of the proof stand out most clearly in the bilinear case, which we now describe.

A vector-valued bilinear form is a bilinear map $V \times V \to W$. For concreteness, we take V and W to be $\mathbb{C}^n$ and $\mathbb{C}^c$ respectively, and denote the vector space of all such forms as $Bil(n, c)$, or simply Bil. We view a point in Bil either as a map or as a c-tuple of $n \times n$ matrices as convenience dictates. The group $GL(n, \mathbb{C}) \times GL(c, \mathbb{C})$ acts on Bil, with the group element $g = (A, P)$ transforming the form $B = (B^1, \cdots, B^c)$ in stages: the matrix A acts on each component matrix of B, yielding an intermediate c-tuple

$$({}^t(A^{-1})B^1A^{-1}, \cdots, {}^t(A^{-1})B^cA^{-1});$$

the components of the final c-tuple are linear combinations of these intermediate components, with coefficients drawn from P. In short, the α-th component of $(A,P)B$ is

$$P_1^{\alpha}\,{}^t(A^{-1})B^1A^{-1} + \cdots + P_c^{\alpha}\,{}^t(A^{-1})B^cA^{-1}.$$

This action is natural in that the following diagram commutes:

$$\begin{array}{ccc} V \times V & \xrightarrow{B} & W \\ {\scriptstyle A\times A}\downarrow & & \downarrow{\scriptstyle P} \\ V \times V & \xrightarrow[gB]{} & W. \end{array}$$

For technical purposes, it is more convenient to use a compact indicial notation, in which each component matrix B^{α} is represented by its entries B^{α}_{jk}, and the new form $(A,P)B$ is represented by matrices D^{α}, where

$$D^{\alpha}_{jk} = P^{\alpha}_{\beta}(A^{-1})^r_j(A^{-1})^s_k B^{\beta}_{rs}.$$

Here the usual summation convention is in force: whenever an index appears in both a subscript and a superscript a summation is implied.

We note in passing that for both the sesquilinear and hermitian cases, all of these formulas are modified by conjugating each left-hand A. A precise statement of the first fundamental theorem for vector-valued bilinear forms (as given in [GM1]) involves some technical notation that while standard in invariant theory is not immediately transparent to the uninitiated. However, the basic idea can be paraphrased in familiar terms, at the price of succinctness.

3.1. Theorem (First Fundamental Theorem for vector-valued bilinear forms).

Part 1. If r is divisible by c and $2r$ is divisible by n, then the following construction will yield either an invariant homogeneous polynomial of degree r and weight

$$\chi(A,P) = (\det A)^{-4r/n}(\det P)^{r/c}$$

or the zero polynomial.

a) Consider a monomial of degree r in the components of B:

$$B^{j_1}_{i_1 i_2} B^{j_2}_{i_3 i_4} \cdots B^{j_r}_{i_{2r-1} i_{2r}}.$$

b) Select n of the subscript positions, and take an alternating sum, as in the computation of a determinant. That is, consider the $n!$ monomials obtained by successively replacing these n subscripts by each of the $n!$ permutations of the numbers $(1, 2, \cdots, n)$, attach a coefficient of $+1$ if the permutation is even or -1 if the permutation is odd, and sum the terms.

c) Select an additional n subscript positions, and compute a similar alternating sum, thereby obtaining a polynomial with $(n!)^2$ terms.

d) Continue in this way until each literal subscript has been replaced by one of the numbers $1, \cdots, n$. (This is possible because $2r$ is divisible by n.)

e) In an analogous sequence of steps, replace each literal superscript by one of the numbers $1, \cdots, c$. (This is possible because r is divisible by c.)

Part 2. The preceding construction depends on a partition of the $2r$ subscript positions into n-fold blocks and the r superscript positions into c-fold blocks. Each such partition determines either an invariant homogeneous polynomial of weight χ or the zero polynomial. Every non-zero linear combination of these polynomials is also an invariant homogeneous polynomial of weight χ.

Part 3. No other invariant homogeneous polynomials exist.

Note that by taking a ratio of two of these relatively invariant polynomials of equal weight we obtain absolutely invariant rational functions. Also note that the first fundamental theorem for sesquilinear and hermitian forms (given in [GM1]) is similar, except that r must be divisible by both n and c, only certain partitions of the subscripts are allowed, and the weight χ is replaced by the weight

$$\lambda(A, P) = (\det A)^{-r/n} (\det \bar{A})^{-r/n} (\det P)^{r/c}.$$

Appearances notwithstanding, this theorem has a straightforward proof. It is based on a translation into the language of representation theory, a trick, an invocation of a basic theorem about representations of the general linear group, and translation back into the language of bilinear forms.

The first translation proceeds as follows. The space Bil is isomorphic to the space $V^* \otimes V^* \otimes W$; a homogeneous polynomial of degree r defined on this space is equivalent to a linear function defined on the space $(V^* \otimes V^* \otimes W)^{\otimes r} = (V^*)^{\otimes 2r} \otimes (W)^{\otimes r}$; such a linear function is an element of the dual space $V^{\otimes 2r} \otimes (W^*)^{\otimes r}$. The group $GL(n, \mathbb{C})$ acts on $V = \mathbb{C}^n$ by matrix multiplication; similarly, $GL(c, \mathbb{C})$ acts on $W = \mathbb{C}^c$. These actions determine a standard representation of the group $GL(n, \mathbb{C}) \times GL(c, \mathbb{C})$ on the space $V^{\otimes 2r} \otimes (W^*)^{\otimes r}$. Routine unwinding of the definitions shows that the element of $V^{\otimes 2r} \otimes (W^*)^{\otimes r}$ corresponding to an invariant homogeneous polynomial of degree r is the basis of a 1-dimensional invariant subspace. Therefore the problem shifts to the description of all 1-dimensional invariant subspaces of $V^{\otimes 2r} \otimes (W^*)^{\otimes r}$.

The trick is to show that each such space is isomorphic to the tensor product of a 1-dimensional $GL(n, \mathbb{C})$-invariant subspace of $V^{\otimes 2r}$ with a 1-dimensional $GL(c, \mathbb{C})$-invariant subspace of $(W^*)^{\otimes r}$. The fact that these groups are reductive is essential.

The invocation refers to the classical description of the irreducible representations of the general linear groupa staple of both invariant theory and representation theory described repeatedly throughout the literature. It is at this point that all of the alternating sums described in the theorem make their entrance.

The retranslation basically reverses the first step.

§4. Quotient spaces.

From an algebraic-geometric perspective, the space Herm$/G$ can be understood in terms of the spectrum of the ring of invariant polynomials. Unfortunately, current knowledge of this ring is exhausted by the first fundamental theorem, which is by no means adequate to explicate the structure of its spectrum.

From a differential-geometric perspective, one would like to have a smooth structure on Herm$/G$ with respect to which the Levi map is smooth. Unfortunately, as is typical with such quotient space or moduli problems, certain "unstable" points in Herm get in the way. However, we do have the following theorem (Theorem 2.4 of [GM2]).

4.1. Theorem. *Let $G = GL(n, \mathbb{C}) \times GL(c, \mathbb{R})$, let $K \subset G$ be the subgroup consisting of all pairs $(zI_n, |z|^2 I_c)$ for z in the complex multiplicative group $\mathbb{C}^*$, and suppose that $c > 2$ and $n > c^2$. There exists a non-empty G-invariant open subset $Z \subset$ Herm whose image Z/G by the* *projection* ρ : Herm $\to$ Herm $/G$ *can be given a smooth structure in such a way that* $Z \to Z/G$ *is a principal bundle with structure group* G/K.

The proof of this theorem is much too long and technical to be satisfactorily summarized, but it is not difficult to develop the definition of the open set Z. First we note that every element of K fixes each point in Herm ; in order to be in Z, a form B must be fixed by no other elements of G. Moreover, B must have no non-zero null vectors $x \in \mathbb{C}^n$that is, in order to be in Z, B must have the property that

$$({}^t\bar{x}B^1x, \cdots, {}^t\bar{x}B^cx) = (0, \cdots, 0) \quad \text{implies} \quad x = 0.$$

The statement of the one remaining condition in the definition of Z, which is the most interesting, requires a few preliminaries.

Each hermitian form B determines a polynomial, namely

$$\det(x_1B^1 + \cdots + x_cB^c).$$

For some forms this polynomial vanishes identically, but for generic forms it is homogeneous of degree n and therefore has a zero-locus in the projective space $\mathbb{P}^{c-1}$, which we call the *associated hypersurface.* Thus, there is a map

$$\text{(4.1)} \qquad \text{Herm} - -- > (\text{degree } n \text{ hypersurfaces in } \mathbb{P}^{c-1}),$$

where the dotted arrow signifies that the map is densely, but not globally, defined. Let Y denote the set of those hypersurfaces that satisfy the natural geometric condition of having no non-trivial projective automorphism and no points of multiplicity greater than $c$that is, no points at which the defining polynomial vanishes along with all of its partial derivatives of order c or less. The final condition defining Z is that B is in Z only if its associated hypersurface is in Y.

Given an element (A, P) of G, one can use the matrix P to change coordinates in$\mathbb{P}^{c-1}$; that is, one can view P as an element of the projective linear group PGL and let it act on $\mathbb{P}^{c-1}$ accordingly. It is easy to show that the hypersurface associated to the form $(A, P)B$ differs from the hypersurface associated to B only by the action of the change of coordinates determined by P. Therefore, the map (4.1) determines a densely defined map of quotient spaces

$$\text{Herm}/G - -- > (\text{degree } n \text{ hypersurfaces in } \mathbb{P}^{c-1})/PGL$$

which restricts to a globally defined map

$$Z/G \to Y/PGL.$$

A major step in proving Theorem 4.1 is to show that Y/PGL can be given a smooth structure in such a way that $Y \to Y/PGL$ is a principal bundle with structure group PGL (see Theorem 5.1 of [GM2]). Basically, in studying Z/G by way of Y/PGL we are studying the action of the product group $GL(n, \mathbb{C}) \times GL(c, \mathbb{R})$ one factor at a time, which, in light of the two-step definition of the group action, is a natural approach.

Before going on to apply Theorem 4.1 to CR-geometry, we note that the polynomial $\det(x_1B^1 + \cdots + x_cB^c)$ explains the trichotomy mentioned in §2. If $c = 1$, it simply distinguishes the singular hermitian matrices from the non-singular; if $c = 2$, it is a homogeneous polynomial in two unknowns, which is essentially equivalent to an inhomogeneous polynomial in one unknown; if $c > 2$, algebraic geometry is clearly involved.

§5. CR geometry.

Recall that a CR-structure $\mathcal{H}$ of dimension n and codimension c on the smooth $2n + c$ dimensional manifold M determines the Levi map $\mathcal{L} : M \to \mathrm{Herm}/G$, where Herm is the vector space whose points are c-tuples of $n \times n$ matrices, $G = GL(n, \mathbb{C}) \times GL(c, \mathbb{R})$, and the action of G on Herm is defined so that the α-th component matrix of the c-tuple $(A, P)B$ is

$$P_1^{\alpha t}(\bar{A}^{-1})B^1A^{-1} + \cdots + P_c^{\alpha t}(\bar{A}^{-1})B^cA^{-1}.$$

As noted earlier, the Levi map furnishes a fundamental link between CR-geometry and the invariant theory of vector-valued forms, since it can be used to pull back any specified "structure" on Herm/G to produce a canonical CR- geometric object on M. For instance, we have already seen that each invariant function on Herm determines a zero-locus in Herm/G. The Levi map pulls this back to a subset of M, canonical in the sense that if $F : M \to M'$ is an isomorphism of CR manifolds, then F maps the specified subset of M bijectively to the specified subset of M'. In order to construct richer geometric objects, it seems necessary to restrict attention to those CR-structures that enjoy some type of homogeneity. As part of his general treatment of differential systems Tanaka [Ta1,Ta2] develops a full theory of CR-structures whose Levi maps are constant. Here we take a different approach and consider CR-structures whose Levi maps are valued in a specified open subset $\mathcal{U}$ of Herm/G. If the Levi map is constant, only one orbit of forms in Herm is connected with the CR-structure, and this orbit can be represented by a chosen canonical form. We would like to proceed similarly and choose a canonical form from each of the orbits corresponding to points in $\mathcal{U}$. Of course, in order to be useful for differential-geometric purposes, these choices must be smooth. Thus we need a smooth local section of Herm $\to$ Herm$/G$ defined on the open set $\mathcal{U}$.

In particular, we need $\mathcal{U}$ to have a smooth structure, so the set Z/G described in §4 is an obvious candidate. Moreover, Z has a sort of homogeneity since each of its points is fixed by the elements of K alone. Regretably, we do not know if $Z \to Z/G$ admits a smooth section. However, since $Z \to Z/G$ is a principal bundle, every point of Z/G has a neighborhood admitting a smooth section. Therefore, we take $\mathcal{U}$ to be such a neighborhood, and fix some section σ.

5.1. Definition. A CR-structure is *tractable of type* $\mathcal{U}$ if its Levi map $\mathcal{L} : M \to$ Herm$/G$ is valued in $\mathcal{U}$.

In standard differential-geometric fashion, every CR-structure, tractable or not, determines a subbundle of the coframe bundle of M, consisting of suitably "adapted" coframes. The structure group of this subbundle is unwieldy, but in the tractable case the subbundle can be reduced dramatically (Theorem 3.1 of [GM2]), yielding a subbundle of "better adapted" coframes with structure group K (which, we recall, is isomorphic to $\mathbb{C}^*$). The proof of this theorem uses a detailed analysis of the structure equations of moving coframes, but the core idea is simple. From the coefficients of these structure equations one can extract a c-tuple of hermitian matrices that represents the Levi form; the assumption of tractability allows one to single out those coframes whose structure equations give rise to canonical c-tuplesthat is, c-tuples in the image of the section σ.

Analysis of the structure equations of the reduced principal bundle associated to a tractable CR-structure leads to the construction of a canonical connection on this bundle (Theorem 4.1 of [GM2]). One immediate corollary (4.2 of [GM2]) is that the automorphisms of a tractable CR-structure constitute a Lie group. Another (4.3 of [GM2]) is that this connection can be canonically extended to an affine connection, thereby introducing an operation of covariant differentiation into the study of CR geometry. A third corollary (4.4 of [GM2]) is a canonical decomposition of the complexified tangent bundle of a tractable CR manifold as a direct sum of $2n + c$ complex line bundles, and a corresponding decomposition of the real tangent bundle as the direct sum of c real line bundle and n real plane bundles with complex structure.

§6. Conclusion.

The study of higher-codimensional CR-structures by way of the invariant theory of vector-valued hermitian forms has barely begun, and open questions abound.

As already noted in §3, the first fundamental theorem is just the first step in the classical approach to describing the ring of invariant polynomials. A second stepthe development of a second fundamental theorem and a description of the syzygiesis in progress ([G]). There remain numerous commutative-algebraic questions, along with the ultimate goal of a thorough understanding of the spectrum, and hence an algebraic-geometric understanding of the quotient space Herm$/G$. From a more practical point of view, there is the problem of computing invariants. The procedure described in Theorem 3.1 is constructive in principle, but hardly efficient, and significant improvements should be possible.

The invariant theory of vector-valued forms can be applied to branches of differential geometry apart from CR geometry. The second fundamental form of a Riemannian submanifold and the holomorphic second fundamental form of a complex submanifold are geometrically important vector-valued forms that are algebraically similar to the Levi form. Additionally, the geometry of a manifold with distribution involves a skew-symmetric bilinear form. Theorem 3.1 applies directly to the latter two cases. For Riemannian geometry, where one needs to consider the action of a product of orthogonal groups rather than general linear groups, the same methods apply, but the resulting formulas are more complex.

In CR geometry, there is the central issue of tractability. Is Z/G itself tractable? Are there tractable subsets with sections that can be explicitly described? Such a description would amount to a procedure for converting a given c-tuple of hermitian matrices to a specified canonical forma sort of "super Gram-Schmidt" process. The decomposition of the tangent bundle of a tractable CR manifold described in §5 shows that there are global obstructions to tractability. Can the invariant theory of forms elucidate any other aspects of global CR geometry?

Finally, to conclude on a note of sheer wishful thinking, might it be possible to use the approach described here, rooted in the teaching of Professor Kuranishi, to illuminate (or indeed, since this is wishful thinking, to solve) the embedding problem for higher codimensional CR structures?

References

[Be] V.K. Beloshapka, On holomorphic transformations of a quadric, Math. USSR Sbornik, **72 , No. 1** (1992), 189–205..

[Bo] A. Boggess, "CR manifolds and the tangential Cauchy-Riemann complex", CRC Press, Boca Raton, 1991.

[G] T. Garrity, On relations between invariants of vector-valued forms, in preparation.

[GM1] T. Garrity and R.I. Mizner, Invariants of vector-valued symmetric and hermitian forms, Linear Algebra and its Applications, **218** (1995), 225–237..

[GM2] ———, The equivalence problem for higher-codimensional CR structures, to appear in Pacific Journal of Mathematics..

[J] H. Jacobowitz, "An introduction to CR structures", AMS Mathematical Surveys and Monographs, Number 32, AMS, Providence, 1990.

[M] R. I. Mizner, CR structures of codimension 2, J. Diff. Geom., **30** (1989), 167–190..

[Ta1] N.Tanaka, On differential systems, graded Lie algebras and pseudogroups, J. Math. Kyoto Univ., **10** (1970), 1–82.

[Ta2] ———, On the equivalence problems associated with simple graded Lie algebras, Hokkaido Math. J., **8** (1979), 23–84.

[Tu] A. E. Tumanov, "The geometry of CR-manifolds", in Several Complex Variables III, G.M. Khenkin (ed.), Encyclopedia of Mathematical Sciences, Vol. 9, Springer- Verlag, Berlin, 1989, pp. 201–221..

Department of Mathematics
Williams College
Williamstown, MA 01267
U.S.A.

Advanced Studies in Pure Mathematics 25, 1997
CR-Geometry and Overdetermined Systems
pp. 122–140

The Infinitesimal Spectral Rigidity of the Real Grassmannians of Rank Two

Jacques Gasqui and Hubert Goldschmidt

Dedicated to Professor M. Kuranishi on his 70th birthday

Introduction

Let (X, g) be a Riemannian symmetric space of compact type. Consider a family of Riemannian metrics $\{g_t\}$ on X, for $|t| < \varepsilon$, with $g_0 = g$. We say that $\{g_t\}$ is an isospectral deformation of g if the spectrum of the Laplacian of the metric g_t is independent of t. We say that the space (X, g) is infinitesimally spectrally rigid (i.e., spectrally rigid to first order) if, for every such isospectral deformation $\{g_t\}$ of g, there is a one-parameter family of diffeomorphisms φ_t of X such that $g_t = \varphi_t^* g$ to first order in t at $t = 0$, or equivalently if the infinitesimal deformation $\frac{d}{dt} g_{t|t=0}$ of $\{g_t\}$ is a Lie derivative of the metric g.

In [13], V. Guillemin proves that the infinitesimal deformation h of an isospectral deformation of g satisfies the following integral condition: for every maximal flat totally geodesic torus Z contained in X and for all parallel vector fields ζ on Z, the integral

$$\int_Z h(\zeta, \zeta)\, dZ$$

vanishes, where dZ is the Riemannian measure of Z. If all of these integrals corresponding to a symmetric 2-form h on X vanish, we say that h satisfies the Guillemin condition. It is easily verified that a Lie derivative of the metric always satisfies the Guillemin condition. We say that the space (X, g) is rigid in the sense of Guillemin if the following Radon transform property holds on X: the only symmetric 2-forms on X satisfying the Guillemin condition are the Lie derivatives of the metric g. Thus according to [13], if the space (X, g) is rigid in the sense of Guillemin, it is infinitesimally spectrally rigid.

Received December 13, 1995
The second author was supported in part by NSF Grant DMS 92-03974.

We are interested in determining which irreducible symmetric spaces of compact type are infinitesimally spectrally rigid. Although much work has been done on the problem of isospectrality, there are still very few results for positively curved spaces.

In [9], we study the real Grassmannian $G_{2,n}$ of (unoriented) 2-planes in $\mathbb{R}^{n+2}$, with $n \geq 3$. Here, we present an outline of the proof of the following result of [9]:

Theorem 1. *For $n \geq 3$, the real Grassmannian $G_{2,n}$ is rigid in the sense of Guillemin.*

The real Grassmannian $G_{2,n}$, which is a space of rank 2, is therefore infinitesimally spectrally rigid. This provides us with the first examples of symmetric spaces (with positive curvature) of rank > 1 which are infinitesimally spectrally rigid.

All the previously known spectral rigidity results for symmetric spaces with positive curvature concern spaces of rank one. In fact, the Guillemin rigidity of the spaces of rank one which are not spheres was proved by Michel [14] for the real projective spaces $\mathbb{RP}^n$, and by Michel [14] and Tsukamoto [18] for the other projective spaces (see also [2], [5], [6]). On the other hand, for $2 \leq n \leq 6$, the spectral rigidity of the n-sphere S^n and the real projective space $\mathbb{RP}^n$ is established by Berger and Tanno (see [1] and [17]).

§1. The maximal flat Radon transform

Let X be a manifold whose tangent and cotangent bundles we denote by T and T^*. By S^kE and $\bigwedge^j E$, we shall mean the k-th symmetric product and the j-th exterior product of a vector bundle E over X, respectively. If E is a vector bundle over X, we denote by $E_{\mathbb{C}}$ its complexification, by $\mathcal{E}$ the sheaf of sections of E over X and by $C^\infty(E)$ the space of global sections of E over X.

Let g be a Riemannian metric on X and let ∇ be the Levi-Civita connection of (X, g). The Killing operator

$$D_0 : \mathcal{T} \to S^2\mathcal{T}^*$$

of (X, g), sending a vector field ξ on X into the Lie derivative of g along ξ, and the symmetrized covariant derivative

$$D^1 : \mathcal{T}^* \to S^2\mathcal{T}^*$$

defined by

$$(D^1\theta)(\xi, \eta) = \tfrac{1}{2}((\nabla\theta)(\xi, \eta) + (\nabla\theta)(\eta, \xi)),$$

for $\theta \in \mathcal{T}^*$, $\xi, \eta \in T$, are related by the formula

$$\tfrac{1}{2} D_0 \xi = D^1 g^\flat(\xi), \tag{1}$$

for $\xi \in \mathcal{T}$, where $g^\flat : T \to T^*$ is the isomorphism determined by the metric g.

Let R be the Riemann curvature tensor, as defined in [5, §4], which is a section of the bundle $\bigwedge^2 T^* \otimes \bigwedge^2 T^*$, and let $\widetilde{R}$ be the section of $\bigwedge^2 T^* \otimes T^* \otimes T$ determined by

$$g(\widetilde{R}(\xi_1, \xi_2, \xi_3), \xi_4) = R(\xi_1, \xi_2, \xi_3, \xi_4),$$

for $\xi_1, \xi_2, \xi_3, \xi_4 \in T$. Let

$$D_g : S^2\mathcal{T}^* \to \textstyle\bigwedge^2 \mathcal{T}^* \otimes \bigwedge^2 \mathcal{T}^*$$

be the linear differential operator defined by

$$\begin{aligned}
&(D_g h)(\xi_1, \xi_2, \xi_3, \xi_4) \\
&\qquad = \tfrac{1}{2}\{(\nabla^2 h)(\xi_1, \xi_3, \xi_2, \xi_4) + (\nabla^2 h)(\xi_2, \xi_4, \xi_1, \xi_3) \\
&\qquad\quad - (\nabla^2 h)(\xi_1, \xi_4, \xi_2, \xi_3) - (\nabla^2 h)(\xi_2, \xi_3, \xi_1, \xi_4) \\
&\qquad\quad - h(\widetilde{R}(\xi_1, \xi_2, \xi_3), \xi_4) + h(\widetilde{R}(\xi_1, \xi_2, \xi_4), \xi_3)\},
\end{aligned} \tag{2}$$

for $h \in S^2\mathcal{T}^*$ and $\xi_1, \xi_2, \xi_3, \xi_4 \in T$.

We now suppose that (X, g) is a Riemannian symmetric space of compact type. We write X as a homogeneous space G/H, where G is a compact, connected semi-simple Lie group, which acts on X by isometries, and H is a closed subgroup of G such that (G, H) is a symmetric pair. Since all the maximally totally geodesic flat tori of X are conjugate under the action of G on X, the space Ξ of all such tori of X may be regarded as a homogeneous space of G. The maximal flat Radon transform of X is a G-equivariant linear mapping from the space of functions on X to the space of functions on Ξ; it assigns to a function f on X the function $\hat{f}$ on Ξ, whose value at a torus Z of Ξ, is the integral

$$\int_Z f \, dZ$$

of f over Z, where dZ is the Riemannian measure of Z. Together with Guillemin's above-mentioned result, this leads us to define a maximal flat Radon transform for symmetric 2-forms as follows. Let L be the homogeneous G-vector bundle over Ξ whose fiber L_Z at a point $Z \in \Xi$ is the space of all parallel vector fields on the flat torus Z. We consider the space $C^\infty(\Xi, S^2L^*)$ of all sections of the vector bundle S^2L^* (of quadratic forms on L) over Ξ. The maximal flat Radon transform for symmetric 2-forms on X is the linear mapping

$$I : C^\infty(S^2T^*) \to C^\infty(\Xi, S^2L^*)$$

from the space of symmetric 2-forms on X to the space of quadratic forms on L, which assigns to a symmetric 2-form h on X the section $I(h)$ of S^2L^* whose value at the point $Z \in \Xi$ is determined by

$$I(h)(\zeta_1, \zeta_2) = \int_Z h(\zeta_1, \zeta_2)\, dZ,$$

where ζ_1, ζ_2 are elements of the fiber L_Z. The vector bundle S^2T^* is a homogeneous G-bundle over X and it is easily verified that the mapping I is G-equivariant. Clearly, a symmetric 2-form h on X satisfies the Guillemin condition if and only if $I(h)$ vanishes.

The adjoint space of X is the symmetric space which admits X as a Riemannian cover and which is itself not a Riemannian cover of another symmetric space. For example, the adjoint space of the n-sphere S^n is the real projective space $\mathbb{RP}^n$. The maximal flat tori of S^n are the closed geodesics (i.e. the great circles). A function on $\mathbb{RP}^n$ lifts to an even function on S^n, and all the even functions on S^n arise in this manner. The kernel of the maximal flat Radon transform of S^n is the space of the odd functions on S^n. In fact, the Radon transform is injective when restricted to the even functions on S^n; this is equivalent to the classic fact that the Radon transform for functions on $\mathbb{RP}^n$ is injective. In [11] and [12], Grinberg generalizes these results and proves that the maximal flat Radon transform for functions on X is injective if and only if the space X is equal to its adjoint space.

We say that a symmetric 2-form on X satisfies the zero-energy condition if all its integrals over the closed geodesics of X vanish. Lie derivatives of the metric always satisfy this condition. The space (X, g) is said to be infinitesimally rigid if the only symmetric 2-forms on X satisfying the zero-energy condition are the Lie derivatives of the metric g. For spaces of rank one, this notion of infinitesimal rigidity coincides with rigidity in the sense of Guillemin. Thus, as stated in the introduction, the infinitesimal rigidity of the projective spaces which are not spheres has been established.

In [9, §1], we show that:

Lemma 1. *A symmetric 2-form on X satisfying the zero-energy condition also satisfies the Guillemin condition.*

§2. The complex quadrics and the real Grassmannians

Let n be an integer ≥ 3. We henceforth suppose that X is the complex quadric Q_n, which is the complex hypersurface of complex projective space $\mathbb{CP}^{n+1}$ defined by the homogeneous equation

$$\zeta_0^2 + \zeta_1^2 + \cdots + \zeta_{n+1}^2 = 0,$$

where $\zeta = (\zeta_0, \zeta_1, \ldots, \zeta_{n+1})$ is the standard complex coordinate system of $\mathbb{C}^{n+2}$. Let g be the Kähler metric on Q_n induced by the Fubini-Study metric on $\mathbb{CP}^{n+1}$ of constant holomorphic curvature 4. The group $SU(n+2)$ acts on $\mathbb{CP}^{n+1}$ by holomorphic isometries. Its subgroup $SO(n+2)$ leaves the submanifold Q_n of $\mathbb{CP}^{n+1}$ invariant and acts transitively on Q_n; it is then easily verified that Q_n is the homogeneous space

$$SO(n+2)/SO(2) \times SO(n)$$

of the group $SO(n+2)$, which is an irreducible Hermitian symmetric space of compact type of rank 2.

We consider the functions

$$\begin{aligned} f_{0,1}(\zeta) &= (\zeta_0 + i\zeta_1)(\bar{\zeta}_2 + i\bar{\zeta}_3) - (\zeta_2 + i\zeta_3)(\bar{\zeta}_0 + i\bar{\zeta}_1), \\ f_{1,0}(\zeta) &= (\zeta_0 + i\zeta_1)(\bar{\zeta}_0 + i\bar{\zeta}_1) \end{aligned}$$

on $\mathbb{C}^{n+2}$. If r, s are integers ≥ 0, the function $f_{r,s} = f_{1,0}^r \cdot f_{0,1}^s$ on $\mathbb{C}^{n+2}$ is invariant under $U(1)$; its restriction to the unit sphere S^{2n+3} of $\mathbb{C}^{n+2}$ induces by passage to the quotient a function on $\mathbb{CP}^{n+1}$, whose restriction to Q_n we denote by $\tilde{f}_{r,s}$.

The complex conjugation of $\mathbb{C}^{n+2}$ induces an involutive isometry of $\mathbb{CP}^{n+1}$ which preserves the quadric Q_n. The induced isometry τ of Q_n commutes with the action of the group $SO(n+2)$. The group $\{\mathrm{id}, \tau\}$ of isometries of Q_n acts freely on Q_n and so we may consider the quotient Riemannian manifold Y of X by this group, with the metric induced by g, which is also a homogeneous space of $SO(n+2)$. The natural projection $\varpi : Q_n \to Y$ is a Riemannian submersion and a two-fold covering.

Let $\widetilde{G}_{2,n}$ be the real Grassmannian of oriented 2-planes in $\mathbb{R}^{n+2}$. It is easily verified that the mapping

$$\Psi : \widetilde{G}_{2,n} \to \mathbb{CP}^{n+1},$$

sending the oriented 2-plane of $\mathbb{R}^{n+2}$ determined by $x \wedge y$, where $\{x, y\}$ is an orthonormal system of vectors of $\mathbb{R}^{n+2}$, into the point of $\mathbb{CP}^{n+1}$ with homogeneous coordinates $\zeta = x + iy \in \mathbb{C}^{n+2}$, and that its image is contained in Q_n. This mapping Ψ allows us to identify the quadric Q_n with $\widetilde{G}_{2,n}$. If τ_0 is the involution of $\widetilde{G}_{2,n}$ corresponding to the change of orientation of a 2-plane, we see that

$$\tau \circ \Psi = \Psi \circ \tau_0.$$

Thus we may identify Y with the real Grassmannian $G_{2,n}$ and ϖ with the natural projection of $\widetilde{G}_{2,n}$ onto $G_{2,n}$, and then view the quadric Q_n as the double cover of the real Grassmannian $G_{2,n}$. In fact, the adjoint space of Q_n is the Grassmannian $G_{2,n}$.

This situation is entirely analogous to that of the sphere S^n viewed as the double cover of the real projective space $\mathbb{RP}^n$. If we consider the sphere S^n as the space of oriented lines in $\mathbb{R}^{n+1}$ and the real projective space $\mathbb{RP}^n$ as the set of lines in $\mathbb{R}^{n+1}$, the antipodal mapping σ of the sphere corresponds to the change of orientation of a line and the projective space $\mathbb{RP}^n$ is equal to the quotient of S^n by the group $\{\mathrm{id}, \sigma\}$ of isometries of S^n.

By analogy with the sphere, the involution τ of Q_n determines notions of even and odd tensors on Q_n: a symmetric p-form θ on Q_n is said to be even (resp. odd) if $\tau^*\theta = \theta$ (resp. $\tau^*\theta = -\theta$). The function $\tilde{f}_{r,s}$ on Q_n is even (resp. odd) if and only if the integer s is even (resp. odd). The space $C^\infty(S^pT^*_{\mathbb{C}})^{\mathrm{ev}}$ (resp. $C^\infty(S^pT^*_{\mathbb{C}})^{\mathrm{odd}}$) of all even (resp. odd) complex symmetric p-forms on X is an $SO(n+2)$-submodule of $C^\infty(S^pT^*_{\mathbb{C}})$, and we have the decomposition

$$C^\infty(S^pT^*_{\mathbb{C}}) = C^\infty(S^pT^*_{\mathbb{C}})^{\mathrm{ev}} \oplus C^\infty(S^pT^*_{\mathbb{C}})^{\mathrm{odd}}.$$

We now construct an explicit maximal totally geodesic flat torus of Q_n. Let

$$\pi : \mathbb{C}^{n+2} - \{0\} \to \mathbb{CP}^{n+1}$$

be the natural projection. We consider the submanifold Z_0 of X which is the image of the mapping $\sigma : \mathbb{R}^2 \to X$ defined by $\sigma(\theta, \varphi) = \pi\tilde{\sigma}(\theta, \varphi)$, where

$$\tilde{\sigma}(\theta, \varphi) = (\cos\theta, \sin\theta, 0, \dots, 0, -i\sin\varphi, i\cos\varphi) \in \mathbb{C}^{n+2},$$

for $(\theta, \varphi) \in \mathbb{R}^2$. This mapping σ satisfies

$$\sigma(\theta, \varphi) = \sigma(\theta + 2k\pi, \varphi + 2l\pi) = \sigma(\theta + k\pi, \varphi + k\pi),$$

for all $k, l \in \mathbb{Z}$ and $(\theta, \varphi) \in \mathbb{R}^2$. We consider the group of translations Γ of $\mathbb{R}^2$ generated by the vectors $(2\pi, 0)$ and (π, π) and the flat torus $\mathbb{R}^2/\Gamma$, which is the quotient of $\mathbb{R}^2$ by the group Γ. According to the preceding relations, we see that σ induces an imbedding

$$\bar{\sigma} : \mathbb{R}^2/\Gamma \to X.$$

Let (θ, φ) be the standard coordinate system on $\mathbb{R}^2$. It is easily verified that

$$\sigma^* g = \tfrac{1}{2}(d\theta \otimes d\theta + d\varphi \otimes d\varphi).$$

Therefore, if the quotient $\mathbb{R}^2/\Gamma$ is endowed with the flat metric induced by the metric $\frac{1}{2}(d\theta \otimes d\theta + d\varphi \otimes d\varphi)$ on $\mathbb{R}^2$, the mapping $\bar{\sigma}$ is a totally geodesic isometric imbedding. Thus Z_0 is a maximal flat totally geodesic torus of X.

The vector fields $\partial/\partial\theta$ and $\partial/\partial\varphi$ on $\mathbb{R}^2$ are σ-projectable; thus, there exist well-defined parallel vector fields ξ_0 and η_0 on Z_0 determined by $\partial/\partial\theta$ and $\partial/\partial\varphi$, respectively. In fact, $\{\xi_0, \eta_0\}$ is a basis for the space of parallel vector fields on Z_0.

Since the group $SO(n+2)$ acts transitively on Ξ, a symmetric 2-form h on X satisfies the Guillemin condition if and only if, for all $\phi \in SO(n+2)$, the section $I(\phi^* h)$ of $S^2 L^*$ vanishes at the torus Z_0 of Ξ, or equivalently if

$$\int_{Z_0} (\phi^* h)(\xi_0, \xi_0)\, dZ_0 = \int_{Z_0} (\phi^* h)(\eta_0, \eta_0)\, dZ_0 = \int_{Z_0} (\phi^* h)(\xi_0, \eta_0)\, dZ_0 = 0,$$

for all $\phi \in SO(n+2)$.

In [9, §3], by means of this torus Z_0 we prove that:

Proposition 1. (i) *The odd symmetric* 2*-forms on* X *satisfy the Guillemin condition.*

(ii) *A symmetric* 2*-form* h *on* Y *satisfies the Guillemin condition if and only if the even symmetric* 2*-form* $\varpi^* h$ *on* X *satisfies the Guillemin condition.*

On the symmetric space Q_n, according to [4, §7] the Lie derivatives of the metric can be characterized as the symmetric 2-forms h satisfying the condition $Q_g h = 0$, where Q_g is a non-trivial differential operator of order 3. Let x_0 be a given point of X; choose an open subset U of X containing x_0 such that $U \cap \tau(U) = \emptyset$. We now choose a symmetric 2-form on X whose support is contained in U and which satisfies $(Q_g h)(x_0) \neq 0$. We know that h is not a Lie derivative of the metric on any neighborhood of x_0. The symmetric 2-form $h - \tau^* h$ on X is odd and its restriction to U is equal to h. Thus we have proved the following:

Proposition 2. *There exist odd symmetric 2-forms on Q_n which are not Lie derivatives of the metric.*

From the preceding proposition, it follows that the quadric Q_n is not is rigid in the sense of Guillemin. This is analogous to the non-injectivity of the maximal flat Radon transform (for functions) on this space.

In order to prove the Guillemin rigidity of the adjoint space $G_{2,n}$ of the quadric Q_n, according to Proposition 1,(ii) it suffices to verify:

Theorem 2. *For $n \geq 3$, an even symmetric 2-form on Q_n satisfies the Guillemin condition if and only if it is the Lie derivative of the metric.*

From the infinitesimal rigidity of the quadrics Q_n, with $n \geq 4$, which we proved in [6], [7] and [8], we infer that it suffices to prove the preceding theorem when $n = 3$. Indeed, let h be an even symmetric 2-form on Q_n, with $n \geq 4$, satisfying the Guillemin condition. Let γ be a closed geodesic of Q_n. It is contained in a totally geodesic submanifold X' of Q_n isometric to the quadric Q_3. The restriction of h to X' is even and satisfies the Guillemin condition. (The demonstration of these last two assertions is given in the course of the proof of Theorem 10.1 of [9].) Then Theorem 2, with $n = 3$, tells us that the restriction of h to X' is a Lie derivative of the metric; thus the integral of h over γ vanishes. Hence the 2-form h on Q_n satisfies the zero-energy condition. The infinitesimal rigidity of the quadric Q_n now implies that h is a Lie derivative of the metric.

§3. Totally geodesic spheres

We now introduce decompositions of the bundle S^2T^* of symmetric 2-forms on X. The complex structure of X induces a decomposition

$$S^2T^* = (S^2T^*)^+ \oplus (S^2T^*)^-$$

of the bundle S^2T^* of symmetric 2-forms on X, where $(S^2T^*)^+$ is the sub-bundle of Hermitian forms and $(S^2T^*)^-$ is the sub-bundle of skew-Hermitian forms. We now use the differential geometry of X considered as a complex hypersurface of $\mathbb{CP}^{n+1}$, which has been studied by Smyth [15]. The components of the second fundamental form of X in $\mathbb{CP}^{n+1}$ generate a sub-bundle E of $(S^2T^*)^-$ of rank 2 and determine an involution of the bundle $(S^2T^*)^+$. If $(S^2T^*)^{++}$ and $(S^2T^*)^{+-}$ are the eigenbundles corresponding to the eigenvalues $+1$ and -1, respectively, of this involution, we therefore obtain a direct sum decomposition

$$(S^2T^*)^+ = (S^2T^*)^{++} \oplus (S^2T^*)^{+-}$$

of $(S^2T^*)^+$.

We consider the totally geodesic spheres of the quadric X of dimension 2 which are totally real. According to [3], these spheres have constant curvature 2 and are all conjugate under the action of the group $SO(n+2)$. Moreover, such a sphere is the double cover of a closed totally geodesic submanifold of the real Grassmannian $Y = G_{2,n}$ isometric to the projective plane $\mathbb{RP}^2$ with its metric of constant curvature 2.

Let B be the sub-bundle of S^2T^* consisting of the elements of S^2T^*, which vanish when restricted to the closed totally geodesic submanifolds of X isometric to the sphere S^2 with its metric of constant curvature 2. In [9, §5], we verify the following:

Lemma 2. *We have*

$$B = (S^2T^*)^{+-}.$$

From Lemma 2 and properties of the operator D_g, in [9, Lemma 5.2] we deduce that:

Lemma 3. *Let Z be a totally geodesic closed submanifold of X isometric to the 2-sphere S^2 with its metric of constant curvature 2. Let $x \in Z$ and let $\xi_1, \xi_2, \xi_3, \xi_4 \in T_x$ be tangent to Z. If h is a section of $(S^2T^*)^{+-}$ over X, then we have*

$$(D_g h)(\xi_1, \xi_2, \xi_3, \xi_4) = 0.$$

Using the injectivity of the Radon transform on the real projective plane, in [9, Lemma 5.3] we obtain the following result, which provides us with another relation between our two notions of rigidity:

Lemma 4. *Let h be a symmetric 2-form on the real Grassmannian Y satisfying the Guillemin condition. Then the restriction of h to a closed totally geodesic submanifold of Y isometric to the real projective plane with its metric of constant curvature 2 satisfies the zero-energy condition.*

From Lemma 4 and the infinitesimal rigidity of the real projective plane (see [14], [2] and [10, §2]), we immediately deduce:

Proposition 3. *Let h be a symmetric 2-form on Y satisfying the Guillemin condition and let Y' be a closed totally geodesic submanifold of Y isometric to the real projective plane with its metric of constant* *curvature* 2. *Then the restriction of* h *to* Y' *is a Lie derivative of the metric on* Y'.

This last proposition together with properties of the operator D_g can be used to show the following (see Proposition 5.4 of [9]):

Proposition 4. *Let* Z *be a totally geodesic closed submanifold of* X *isometric to the* 2*-sphere* S^2 *with its metric of constant curvature* 2. *Let* $x \in Z$ *and let* $\xi_1, \xi_2, \xi_3, \xi_4 \in T_x$ *be tangent to* Z. *An even symmetric* 2*-form* h *on* X *which satisfies the Guillemin condition also verifies the equation*

$$(D_g h)(\xi_1, \xi_2, \xi_3, \xi_4) = 0.$$

§4. The Radon transform on the complex quadric

As we have seen, the mapping I is $SO(n+2)$-equivariant. Thus the kernel of I is an $SO(n+2)$-submodule of $C^\infty(S^2T^*)$, and the space $\mathcal{N}_\mathbb{C}$ consisting of all sections of $S^2T^*_\mathbb{C}$ over X satisfying the Guillemin condition is a $SO(n+2)$-submodule of $C^\infty(S^2T^*_\mathbb{C})$. The space of all even complex symmetric 2-forms on X satisfying the Guillemin condition is the $SO(n+2)$-submodule

$$\mathcal{N}^{\mathrm{ev}}_\mathbb{C} = \mathcal{N}_\mathbb{C} \cap C^\infty(S^2T^*_\mathbb{C})^{\mathrm{ev}}$$

of $C^\infty(S^2T^*_\mathbb{C})$.

Let Γ be the set of equivalence classes of irreducible $SO(n+2)$-modules over $\mathbb{C}$. The vector bundle $F = S^pT^*_\mathbb{C}$, endowed with the Hermitian scalar product induced by the metric g, is homogeneous and unitary. The space $C^\infty(F)$ endowed with the Hermitian scalar product obtained from the Hermitian scalar product of F and the $SO(n+2)$-invariant Riemannian measure dX of X is a unitary $SO(n+2)$-module. We denote by $C^\infty_\gamma(F)$ the isotypic component of this $SO(n+2)$-module $C^\infty(F)$ corresponding to $\gamma \in \Gamma$.

The differential operator $D^1 : T^*_\mathbb{C} \to S^2T^*_\mathbb{C}$ is homogeneous, and so we have

$$D^1 C^\infty_\gamma(T^*_\mathbb{C}) \subset C^\infty_\gamma(S^2T^*_\mathbb{C}),$$

for all $\gamma \in \Gamma$. Since τ is an isometry and as the Lie derivatives of the metric satisfy the Guillemin condition, we see that

$$D^1 C^\infty(T^*_\mathbb{C})^{\mathrm{ev}} \subset \mathcal{N}^{\mathrm{ev}}_\mathbb{C}.$$

For $\gamma \in \Gamma$, we consider the $SO(n+2)$-submodule

$$\mathcal{N}^{\mathrm{ev}}_\gamma = \mathcal{N}^{\mathrm{ev}}_\mathbb{C} \cap C^\infty_\gamma(S^2T^*_\mathbb{C})$$

of the isotypic component $C^\infty_\gamma(S^2T^*_{\mathbb{C}})$.

Using the $SO(n+2)$-equivariance of the mapping I, the ellipticity of the operator D^1 and the results of [20, §5], we are able to reduce the assertion of Theorem 2 to a question involving the isotypic components of the $SO(n+2)$-module $C^\infty(S^2T^*_{\mathbb{C}})^{\mathrm{ev}}$ as follows:

Proposition 5. *The even symmetric 2-forms on Q_n satisfying the Guillemin condition are Lie derivatives of the metric if and only if the equality*

$$\mathcal{N}^{\mathrm{ev}}_\gamma = D^1 C^\infty_\gamma(T^*_{\mathbb{C}})^{\mathrm{ev}}$$

holds for all $\gamma \in \Gamma$.

In [9, §7], we consider an orthonormal basis $\{v_1, v_2, v_3, w_1, w_2, w_3\}$ of the tangent space T_{a_0} at a specific point $a_0 \in X$ which has the following property: for all $1 \le i < j \le 3$, the vectors $\{v_i, v_j\}$ (resp. $\{w_i, w_j\}$) are tangent to a totally geodesic closed submanifold of X isometric to the 2-sphere S^2 with its metric of constant curvature 2. We set

$$A_{ij}h = (D_g h)(v_i, v_j, v_i, v_j), \qquad A'_{ij}h = (D_g h)(w_i, w_j, w_i, w_j),$$

for $1 \le i < j \le 3$. Let

$$D : C^\infty(S^2T^*_{\mathbb{C}}) \to \mathbb{C}^5$$

be the linear mapping defined by

$$Dh = (A_{12}h, A_{23}h, A'_{12}h, A'_{13}h, A'_{23}h),$$

for $h \in C^\infty(S^2T^*_{\mathbb{C}})$. According to Proposition 4, the subspace $\mathcal{N}^{\mathrm{ev}}_{\mathbb{C}}$ of $C^\infty(S^2T^*_{\mathbb{C}})$ is contained in the kernel of D.

§5. The complex quadric of dimension three

In this section, we suppose that $n = 3$ and that X is the quadric Q_3 of dimension three.

We denote by $\mathfrak{g}$ the Lie algebra of the group $SO(5)$. We fix a specific Cartan subalgebra $\mathfrak{t}_{\mathbb{C}}$ of the complexification $\mathfrak{g}_{\mathbb{C}}$ of the Lie algebra $\mathfrak{g}$ and consider the linear forms λ_0 and λ_1 on $\mathfrak{t}_{\mathbb{C}}$ defined in [9, §9]. We choose a Weyl chamber of $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ for which the system of positive roots Δ^+ of $\mathfrak{g}_{\mathbb{C}}$ is equal to

$$\lambda_0 - \lambda_1, \quad \lambda_0 + \lambda_1, \quad \lambda_0, \quad \lambda_1;$$

then $\{\lambda_0 - \lambda_1, \lambda_1\}$ is a system of simple roots of $\mathfrak{g}_{\mathbb{C}}$. The highest weight of an irreducible $SO(5)$-module is a linear form $\Lambda = c_0\lambda_0 + c_1\lambda_1$ on $\mathfrak{t}_{\mathbb{C}}$, where c_0, c_1 are integers satisfying $c_0 \geq c_1 \geq 0$. The equivalence class of such an $SO(5)$-module is determined by this weight. We identify Γ with the set of all such linear forms on $\mathfrak{t}_{\mathbb{C}}$. In particular, we consider the elements

$$\gamma_{r,s} = (2r+s)\lambda_0 + s\lambda_1, \qquad \gamma'_{r,s} = (2r+s+1)\lambda_0 + s\lambda_1$$

of Γ, with $r, s \geq 0$. If $\gamma \in \Gamma$, an $SO(5)$-submodule W of $C^\infty_\gamma(S^p T^*_{\mathbb{C}})$ is isomorphic to k copies of an irreducible $SO(5)$-module with highest weight γ, where k is an integer called the multiplicity of W and denoted by $\operatorname{Mult} W$. In fact, the multiplicity of W is equal to the dimension of the weight subspace $\mathcal{C}(W)$ of W corresponding to the weight γ.

The decomposition of the bundle $S^2 T^*_{\mathbb{C}}$ of complex symmetric 2-forms on Q_3 into irreducible $SO(5)$-invariant sub-bundles described in [7] and the branching law of [19] allow us to compute the multiplicities of the isotypic components of $C^\infty(T^*_{\mathbb{C}})$ and $C^\infty(S^2 T^*_{\mathbb{C}})$, which are given by Proposition 9.1 of [9]. In particular, we obtain:

Proposition 6. *Let γ be an element of Γ. If the $SO(5)$-module $C^\infty_\gamma(T^*_{\mathbb{C}})$ (resp. $C^\infty_\gamma(S^2 T^*_{\mathbb{C}})$) is non-zero, then we may write γ in the form $\gamma = \gamma_{r,s}$ or $\gamma = \gamma'_{r,s}$, for some integers $r, s \geq 0$.*

If we define integers $d_{r,s}$ and $d'_{r,s}$ by

$$d_{r,s} = \begin{cases} 0 & \text{if } r = s = 0, \\ 2 & \text{if } r, s \geq 1, \\ 1 & \text{otherwise,} \end{cases} \qquad d'_{r,s} = \begin{cases} 1 & \text{if } s \geq 1, \\ 0 & \text{otherwise,} \end{cases}$$

for $r, s \geq 0$, then Proposition 9.1 of [9] tells us that

$$\operatorname{Mult} C^\infty_{\gamma_{r,s}}(T^*_{\mathbb{C}}) = 2d_{r,s}, \qquad \operatorname{Mult} C^\infty_{\gamma'_{r,s}}(T^*_{\mathbb{C}}) = 2d'_{r,s}, \tag{3}$$

$$\operatorname{Mult} C^\infty_{\gamma_{r,s}}(S^2 T^*_{\mathbb{C}}) \leq 13, \qquad \operatorname{Mult} C^\infty_{\gamma'_{r,s}}(S^2 T^*_{\mathbb{C}}) \leq 8, \tag{4}$$

for all $r, s \geq 0$.

According to Strichartz [16], the function $\tilde{f}_{r,s}$ is the highest weight vector of the irreducible $SO(5)$-module $C^\infty_{\gamma_{r,s}}(X) = C^\infty_{\gamma_{r,s}}(S^0 T^*_{\mathbb{C}})$ and an eigenfunction of the Laplacian of X, for $r, s \geq 0$. In [9], we use this result together with the equalities (3) to construct explicit highest weight vectors of the $SO(5)$-modules $C^\infty_{\gamma_{r,s}}(T^*_{\mathbb{C}})$ and $C^\infty_{\gamma'_{r,s}}(T^*_{\mathbb{C}})$ and to prove the following two lemmas.

Lemma 5. *The multiplicity of the $SO(5)$-module $C^\infty_{\gamma_{r,s}}(T^*_{\mathbb{C}})^{\mathrm{ev}}$ or of the $SO(5)$-module $C^\infty_{\gamma_{r,s}}(T^*_{\mathbb{C}})^{\mathrm{odd}}$ is equal to $d_{r,s}$. Moreover, the kernel of the operator $D^1 : C^\infty(T^*_{\mathbb{C}}) \to C^\infty(S^2T^*_{\mathbb{C}})$ is equal to $C^\infty_{\gamma_{0,1}}(T^*_{\mathbb{C}})^{\mathrm{ev}}$.*

Lemma 6. *For $r \geq 0, s \geq 1$, the $SO(5)$-modules $C^\infty_{\gamma'_{r,s}}(T^*_{\mathbb{C}})^{\mathrm{ev}}$ and $C^\infty_{\gamma'_{r,s}}(T^*_{\mathbb{C}})^{\mathrm{odd}}$ are irreducible.*

Thus from Proposition 5 and the two preceding lemmas, we deduce:

Proposition 7. *The even symmetric 2-forms on Q_3 satisfying the Guillemin condition are Lie derivatives of the metric if and only if the equalities*

$$\operatorname{Mult} \mathcal{N}^{\mathrm{ev}}_{\gamma_{r,s}} \leq d_{r,s}, \qquad \operatorname{Mult} \mathcal{N}^{\mathrm{ev}}_{\gamma'_{r,s}} \leq d'_{r,s}$$

hold for all $r, s \geq 0$, and if

$$\mathcal{N}^{\mathrm{ev}}_{\gamma_{0,1}} = \{0\}.$$

In [9], we introduce finite-dimensional subspaces $V_{r,s}$ and $W_{r,s}$ (resp. $V'_{r,s}$ and $W'_{r,s}$) of $C^\infty_{\gamma_{r,s}}(S^2T^*_{\mathbb{C}})$ (resp. $C^\infty_{\gamma'_{r,s}}(S^2T^*_{\mathbb{C}})$) all of whose elements are vectors of weight $\gamma_{r,s}$ (resp. $\gamma'_{r,s}$). We define generators of these subspaces in terms of the functions $\tilde{f}_{r,s}$ and explicit Hermitian symmetric 2-forms on $\mathbb{CP}^4$, and then verify that they are either odd or even and that

$$\begin{aligned} V_{r,2p} &\subset C^\infty(S^2T^*_{\mathbb{C}})^{\mathrm{ev}}, & V_{r,2p+1} &\subset C^\infty(S^2T^*_{\mathbb{C}})^{\mathrm{odd}}, \\ W_{r,2p+1} &\subset C^\infty(S^2T^*_{\mathbb{C}})^{\mathrm{ev}}, & W_{r,2p} &\subset C^\infty(S^2T^*_{\mathbb{C}})^{\mathrm{odd}}, \\ V'_{r,2p} &\subset C^\infty(S^2T^*_{\mathbb{C}})^{\mathrm{ev}}, & V'_{r,2p+1} &\subset C^\infty(S^2T^*_{\mathbb{C}})^{\mathrm{odd}}, \\ W'_{r,2p+1} &\subset C^\infty(S^2T^*_{\mathbb{C}})^{\mathrm{ev}}, & W'_{r,2p} &\subset C^\infty(S^2T^*_{\mathbb{C}})^{\mathrm{odd}}, \end{aligned} \tag{5}$$

for all $r, p \geq 0$.

In [9, §7 and §9], we simultaneously determine the dimension of the spaces $V_{r,s}$, $W_{r,s}$, $V'_{r,s}$ and $W'_{r,s}$ and prove the following lemma:

Lemma 7. *Let $r, s \geq 0$ be given integers.*
(i) If s is even, we have

$$\dim(\mathcal{N}_{\mathbb{C}} \cap V_{r,s}) \leq d_{r,s}, \qquad \dim(\mathcal{N}_{\mathbb{C}} \cap V'_{r,s}) \leq 1.$$

(ii) If s is odd, we have

$$\dim(\mathcal{N}_{\mathbb{C}} \cap W_{r,s}) \leq d_{r,s}, \qquad \dim(\mathcal{N}_{\mathbb{C}} \cap W'_{r,s}) \leq 1$$

(iii) We have

$$\mathcal{N}_{\mathbb{C}} \cap V'_{r,0} = \{0\}.$$

Then using Proposition 9.1 of [9], we verify that

$$\begin{aligned} \operatorname{Mult} C^{\infty}_{\gamma_{r,s}}(S^2T^*_{\mathbb{C}}) &= \dim V_{r,s} + \dim W_{r,s}, \\ \operatorname{Mult} C^{\infty}_{\gamma'_{r,s}}(S^2T^*_{\mathbb{C}}) &= \dim V'_{r,s} + \dim W'_{r,s}, \end{aligned} \tag{6}$$

for all $r, s \geq 0$. From the inclusions (5) and the equalities (6), we obtain directly the following:

Lemma 8. *For $r, s \geq 0$, we have*

$$\begin{aligned} \mathcal{C}(C^{\infty}_{\gamma_{r,2s}}(S^2T^*_{\mathbb{C}})^{\mathrm{ev}}) &= V_{r,2s}, & \mathcal{C}(C^{\infty}_{\gamma_{r,2s+1}}(S^2T^*_{\mathbb{C}})^{\mathrm{odd}}) &= V_{r,2s+1}, \\ \mathcal{C}(C^{\infty}_{\gamma_{r,2s}}(S^2T^*_{\mathbb{C}})^{\mathrm{odd}}) &= W_{r,2s}, & \mathcal{C}(C^{\infty}_{\gamma_{r,2s+1}}(S^2T^*_{\mathbb{C}})^{\mathrm{ev}}) &= W_{r,2s+1}, \\ \mathcal{C}(C^{\infty}_{\gamma'_{r,2s}}(S^2T^*_{\mathbb{C}})^{\mathrm{ev}}) &= V'_{r,2s}, & \mathcal{C}(C^{\infty}_{\gamma'_{r,2s+1}}(S^2T^*_{\mathbb{C}})^{\mathrm{odd}}) &= V'_{r,2s+1}, \\ \mathcal{C}(C^{\infty}_{\gamma'_{r,2s}}(S^2T^*_{\mathbb{C}})^{\mathrm{odd}}) &= W'_{r,2s}, & \mathcal{C}(C^{\infty}_{\gamma_{r,2s+1}}(S^2T^*_{\mathbb{C}})^{\mathrm{ev}}) &= W'_{r,2s+1}. \end{aligned}$$

In fact, we have $W_{0,1} = 0$ and $W'_{0,r} = 0$, for $r \geq 0$. Thus by Lemma 8, we see that

$$C^{\infty}_{\gamma_{0,1}}(S^2T^*_{\mathbb{C}})^{\mathrm{ev}} = \{0\}, \qquad \mathcal{N}_{\mathbb{C}} \cap W'_{r,0} = \{0\}, \tag{7}$$

for $r \geq 0$. Since

$$\operatorname{Mult} \mathcal{N}^{\mathrm{ev}}_{\gamma} = \dim(\mathcal{N}_{\mathbb{C}} \cap \mathcal{C}(C^{\infty}_{\gamma}(S^2T^*_{\mathbb{C}})^{\mathrm{ev}})),$$

for $\gamma \in \Gamma$, from the equalities (7), Proposition 7 and Lemmas 7 and 8, we deduce Theorem 2 for the quadric Q_3. We recall that this last result implies both Theorems 1 and 2.

§6. Integration over flat tori of the quadric

This section is devoted to some aspects of the proof of Lemma 7. It is obtained by integrating the symmetric 2-forms of the spaces $V_{r,s}$, $W_{r,s}$, $V'_{r,s}$ and $W'_{r,s}$ over flat totally geodesic tori of the quadric Q_3; for the first equality of Lemma 7,(i) we shall also consider the restriction of the operator D to $V_{r,s}$.

We assume that X is again the quadric Q_n. The restriction of the mapping σ to the subset $\widetilde{Z}_0 = [0, 2\pi] \times [0, \pi]$ of $\mathbb{R}^2$ is a diffeomorphism from $\widetilde{Z}_0$ to Z_0. Therefore if f is a function on Z_0, then we see that

$$(8) \quad \int_{Z_0} f \, dZ_0 = \frac{1}{2} \int_{\widetilde{Z}_0} (\sigma^* f)(\theta, \varphi) \, d\theta \, d\varphi = \frac{1}{2} \int_0^{\pi} \int_0^{2\pi} (\sigma^* f)(\theta, \varphi) \, d\theta \, d\varphi.$$

The following formulas relate the decomposition of the bundle S^2T^* and the parallel vector fields on flat totally geodesic tori and play a fundamental role in our computations of integrals of symmetric 2-forms over these tori. If J is the complex structure of X and ζ is a vector field on Z_0, we define complex vector fields on X along Z_0 by

$$\zeta' = \tfrac{1}{2}(\zeta - iJ\zeta), \qquad \zeta'' = \tfrac{1}{2}(\zeta + iJ\zeta) = \overline{\zeta'},$$

which are of type $(1,0)$ and $(0,1)$, respectively. Let

$$\pi_+ : S^2T^* \to (S^2T^*)^{++},$$
$$\pi_{++} : S^2T^* \to (S^2T^*)^{++}, \qquad \pi_{+-} : S^2T^* \to (S^2T^*)^{+-}$$

be the orthogonal projections. If h is a section of $(S^2T^*)^+$ over X, we have

$$(9) \quad \begin{gathered} (\pi_{++}h)(\xi_0, \xi_0) = h(\xi_0, \xi_0), \qquad (\pi_{++}h)(\eta_0, \eta_0) = h(\eta_0, \eta_0), \\ (\pi_{++}h)(\xi_0, \eta_0) = (\pi_{+-}h)(\xi_0, \xi_0) = (\pi_{+-}h)(\eta_0, \eta_0) = 0, \\ (\pi_{+-}h)(\xi_0, \eta_0) = h(\xi_0, \eta_0). \end{gathered}$$

If $\operatorname{Hess} f$ denotes the Hessian of a complex-valued function f on X, because X is a Kähler manifold, from Lemma 1.1 of [7] and (9) we obtain the relations

$$(10) \quad (\pi_{++}\operatorname{Hess} f)(\eta_0, \eta_0) = 2(\pi_+\operatorname{Hess} f)(\eta_0', \eta_0'') = 2(\partial\bar{\partial} f)(\eta_0', \eta_0'')$$

and

$$(11) \quad \begin{aligned} (\pi_{+-}\operatorname{Hess} f)(\xi_0, \eta_0) &= (\pi_+\operatorname{Hess} f)(\xi_0', \eta_0'') + (\pi_+\operatorname{Hess} f)(\eta_0', \xi_0'') \\ &= (\partial\bar{\partial} f)(\xi_0', \eta_0'') + (\partial\bar{\partial} f)(\eta_0', \xi_0''). \end{aligned}$$

In [9, §3], we give explicit expressions for the vector fields ξ_0' and η_0' on the intersection of Z_0 and an open dense subset V of $\mathbb{CP}^{n+1}$ in terms of a system of holomorphic coordinates on V.

We now again suppose that X is the quadric Q_3. For $r \geq 0$ and $s \geq 1$, we consider the section

$$h_{r,s} = \tilde{f}_{r,s-1}\pi_{+-}\mathrm{Hess}\,\tilde{f}_{0,1}$$

of $(S^2T^*)^{+-}_{\mathbb{C}}$; for $r \geq 0$, we set $h_{r,0} = 0$. For $r, s \geq 0$, we define a subspace $\widetilde{V}_{r,s}$ of $C^\infty_{\gamma_{r,s}}(S^2T^*_{\mathbb{C}})$ generated by explicit complex symmetric 2-forms h_j, with $1 \leq j \leq 7$, which are sections of $(S^2T^*)^{++}_{\mathbb{C}}$ or of $(S^2T^*)^{-}_{\mathbb{C}}$. The subspace $V_{r,s}$ of $C^\infty_{\gamma_{r,s}}(S^2T^*_{\mathbb{C}})$ is generated by $\widetilde{V}_{r,s}$ and the section $h_{r,s}$ of $(S^2T^*)^{+-}_{\mathbb{C}}$. According to Lemma 3, we have

$$Dh_{r,s} = 0. \tag{12}$$

At the end of this section, we shall indicate how to prove the following:

Lemma 9. *Let $s \geq 2$ be an even integer and let $r \geq 0$ be an arbitrary integer. Then we have*

$$I(h_{r,s}) \neq 0.$$

The restriction $D_{r,s} : \widetilde{V}_{r,s} \to \mathbb{C}^5$ of the operator D to the space $\widetilde{V}_{r,s}$ is determined by a 5×7 matrix, which is given by formula (7.5) of [9]. In [9, §7], we use this result to compute the dimension of the space $\widetilde{V}_{r,s}$ and show that the rank of $D_{r,s}$ is equal to $\dim \widetilde{V}_{r,s} - d_{r,s}$ when $s \neq 1$; in particular, when $r, s \geq 2$ we prove that the 2-forms $\{h_j\}$, with $1 \leq j \leq 7$, are linearly independent and that the mapping $D_{r,s}$ is surjective. Since $\mathcal{N}^{\mathrm{ev}}_{\mathbb{C}}$ is contained in the kernel of D, from these last results, Lemma 9 and the equality (12), we easily deduce the first equality of Lemma 7,(i).

For $\alpha \in \mathbb{R}$, let ψ_α be the element of $SO(5)$ defined by

$$\psi_\alpha(\zeta)_0 = \sin\alpha \cdot \zeta_4 + \cos\alpha \cdot \zeta_2, \qquad \psi_\alpha(\zeta)_4 = \cos\alpha \cdot \zeta_4 - \sin\alpha \cdot \zeta_2,$$
$$\psi_\alpha(\zeta)_1 = \zeta_1, \qquad \psi_\alpha(\zeta)_2 = \zeta_3, \qquad \psi_\alpha(\zeta)_3 = \zeta_0,$$

where $\zeta \in \mathbb{C}^5$.

Let $r \geq 0$ and $s \geq 1$ be given integers. We set

$$q_{r,s}(t,\theta,\varphi) = (t^2\cos^2\varphi - \sin^2\theta)^r \cdot (t\cos\theta\cos\varphi + \sin\theta\sin\varphi)^{s-1}$$

and

$$p_{r,s}(t,\theta,\varphi) = (\cos\theta\cos\varphi + t\sin\theta\sin\varphi) \cdot q_{r,s}(t,\theta,\varphi),$$

for $t, \theta, \varphi \in \mathbb{R}$. We consider the polynomial

$$P_{r,s}(t) = \int_{\widetilde{Z}_0} p_{r,s}(t, \theta, \varphi)\, d\theta\, d\varphi$$

in t. Using the expressions for the functions $\psi_\alpha^* \tilde{f}_{r,s-1}$ and $\psi_\alpha^* \tilde{f}_{0,1}$ on V and for the vector fields ξ_0' and η_0' on $Z_0 \cap V$, according to (12) we verify the equality

$$(\psi_\alpha^* h_{r,s})(\xi_0, \eta_0)(\sigma(\theta, \varphi)) = \frac{(-1)^s}{2^r}\, p_{r,s}(\sin\alpha, \theta, \varphi), \tag{13}$$

holds for all $\theta, \varphi \in \mathbb{R}$.

Lemma 10. *If $s \geq 2$ is an even integer, then there exists $\alpha_0 \in \mathbb{R}$ such that the integral*

$$\int_{Z_0} (\psi_{\alpha_0}^* h_{r,s})(\xi_0, \eta_0)\, dZ_0$$

does not vanish.

Proof. The coefficient of t^{2r+s-1} of the polynomial $P_{r,s}(t)$ is equal to the integral

$$\int_{\widetilde{Z}_0} \cos^{2r}\varphi\, (\cos\theta\cos\varphi)^{s-2} \cdot ((s-1)\sin^2\theta\sin^2\varphi + \cos^2\theta\cos^2\varphi)\, d\theta\, d\varphi,$$

which is clearly positive. Thus the polynomial $P_{r,s}$ is non-zero and so there exists a real number α_0 such that $P_{r,s}(\sin\alpha_0)$ does not vanish. From (8) and (13), we infer that the integral of the lemma corresponding to this element $\alpha_0 \in \mathbb{R}$ does not vanish.

Since ψ_{α_0} induces an isometry of X, we see that Lemma 9 is a direct consequence of Lemma 10. The constant term of the polynomial $P_{r,s}(t)$ is easily seen to vanish, and hence so does the integral

$$\int_{Z_0} (\psi_0^* h_{r,s})(\xi_0, \eta_0)\, dZ_0.$$

Therefore our proof really does require the variation of the family of integrals

$$\int_{Z_0} (\psi_\alpha^* h_{r,s})(\xi_0, \eta_0)\, dZ_0,$$

with $\alpha \in \mathbb{R}$.

The other assertions of Lemma 7 are proved by performing various integrations over flat tori. They are either done directly or, as in the case of Lemma 9, by computing the variation of the integrals over a one or two parameter family of flat totally geodesic tori.

References

[1] M. Berger, "Le spectre d'une variéte riemannienne", Lect. Notes in Math., Vol. 194, Springer-Verlag, Berlin, Heidelberg, New York, 1971.

[2] A. Besse, "Manifolds all of whose geodesics are closed", Ergeb. Math. Grenzgeb., Bd. 93, Springer-Verlag, Berlin, Heidelberg, New York, 1978.

[3] Y. Dieng, Quelques résultats de rigidité infinitésimale pour les quadriques complexes, C. R. Acad. Sci. Paris Sér. I Math., **304** (1987), 393–396.

[4] J. Gasqui and H. Goldschmidt, Déformations infinitésimales des espaces riemanniens localement symétriques. I, Adv. in Math., **48** (1983), 205–285.

[5] ———, Déformations infinitésimales des espaces riemanniens localement symétriques. II. La conjecture infinitésimale de Blaschke pour les espaces projectifs complexes, Ann. Inst. Fourier (Grenoble), **34** 2 (1984), 191–226.

[6] ———, Rigidité infinitésimale des espaces projectifs et des quadriques complexes, J. Reine Angew. Math., **396** (1989), 87–121.

[7] ———, On the geometry of the complex quadric, Hokkaido Math. J., **20** (1991), 279–312.

[8] ———, The infinitesimal rigidity of the complex quadric of dimension four, Amer. Math. J., **116** (1994), 501–539.

[9] ———, Radon transforms and spectral rigidity on the complex quadrics and the real Grassmannians of rank two, J. Reine Angew. Math., to appear.

[10] H. Goldschmidt, On the infinitesimal rigidity of the complex quadric, Contemp. Math., **140** (1992), 49–63.

[11] E. Grinberg, Aspects of flat Radon transform, Contemp. Math., **140** (1992), 73–85.

[12] ———, Flat Radon transforms on compact symmetric spaces with application to isospectral deformations.

[13] V. Guillemin, On micro-local aspects of analysis on compact symmetric spaces, in "Seminar on micro-local analysis", by V. Guillemin, M. Kashiwara and T. Kawai, Ann. of Math. Studies, No. 93, Princeton University Press, University of Tokyo Press, Princeton, N.J., 1979, pp. 79–111.

[14] R. Michel, Problèmes d'analyse géométrique liés à la conjecture de Blaschke, Bull. Soc. Math. France, **101** (1973), 17–69.

[15] B. Smyth, Differential geometry of complex hypersurfaces, Ann. of Math., **85** (1967), 246–266.

[16] R. Strichartz, The explicit Fourier decomposition of $L^2(SO(n)/SO(n-m))$, Can. J. Math., **27** (1975), 294–310.

[17] S. Tanno, Eigenvalues of the Laplacian of Riemannian manifolds, Tôhoku Math. J., **25** (1973), 391–403.

[18] C. Tsukamoto, Infinitesimal Blaschke conjectures on projective spaces, Ann. Sci. École Norm. Sup. (4), **14** (1981), 339–356.

[19] ———, Spectra of Laplace-Beltrami operators on $SO(n+2)/SO(2)\times SO(n)$ and $Sp(n+1)/Sp(1)\times Sp(n)$, Osaka J. Math., **18** (1981), 407–426.

[20] N. Wallach, "Harmonic analysis on homogeneous spaces", Marcel Dekker, New York, 1973.

J.Gasqui
Institut Fourier,
Université de Grenoble I,
B. P. 74,
38402 Saint-Martin d'Hères Cedex,
France
Jacques.Gasqui@ujf-grenoble.fr

H.Goldschmidt
Department of Mathematics,
Columbia University,
New York, NY 10027,
U. S. A.
hg@math.columbia.edu

Advanced Studies in Pure Mathematics 25, 1997
CR-Geometry and Overdetermined Systems
pp. 141–157

On Deformations of Self-Dual Vector Bundles over Quaternionic Manifolds

James F. Glazebrook and Duraiswamy Sundararaman [1]

Abstract.

In this paper we survey a number of results concerned with the deformations of quaternionic structures on classes of quaternionic manifolds and the deformation theory of Hermitian bundles with self-dual connections. The deformations in question are shown to correspond to the deformation theory of complex structures and holomorphic vector bundles over an associated complex manifold referred to as a twistor space. Results related to hypercomplex and hyperkähler manifolds are also discussed.

§0. Introduction.

In this paper we consider classes of quaternionic manifolds M and deformations of Hermitian vector bundles on M that possess self-dual connections in the sense of **[18]**, and survey a number of results in this area. We also describe how the Kuranishi deformation theory for general G-structures is applied in this context. The deformation theory is applied firstly in the quaternionic category and then secondly in the holomorphic category of associated (almost) complex manifolds which are often referred to as *twistor spaces.*

The original problems can be traced back to the classification of $SU(2)$–bundles with self–dual connection on 4–manifolds (see e.g., **[1]** **[7]**). A broad generalization of the latter case to that of quaternionic Kähler manifolds was the subject of **[18]** (see also **[10]**, **[21]**). The classification problem becomes much more difficult and results are only known at present for special quaternionic bundles which carry a self–dual connection (see §2).

Received October 19, 1995
Revised August 19, 1996
[1] Supported by a Howard University Research Grant I92-07.

Our references to progress on this problem, are [**5**], [**6**], [**15**], [**10**], [**18**], [**21**], [**22**], [**23**]. In a different direction, the case of foliated quaternionic structures has been considered in [**11**], [**12**].

The first named author would like to thank the Department of Mathematics of George Washington University for providing computer services during the preparation of this paper and to A. Swann for providing several references.

§1. G–structures and their deformations.

Following [**13**], let M be a smooth manifold, $\dim_{\mathbb{R}} M = n$, and let $\mathbf{G} \subset GL(n, \mathbb{R})$ be a connected Lie group. A **G**–*structure* on M is given by a reduction of the principal tangent frame bundle of TM from $GL(n, \mathbb{R})$ to $\mathbf{G}$. If $GL(n, \mathbb{R}) \to B \to M$ denotes the principal tangent frame bundle, then let $\mathbf{G} \to B_{\mathbf{G}} \to M$ be the principal bundle resulting from this reduction. Consider a local diffeomorphism $f : M \to M$ lifting to a bundle automorphism $f_* : B \to B$. We say that f is a local **G**–automorphism if $f_*(B_{\mathbf{G}}) \subset B_{\mathbf{G}}$. For an open set U in M, let $X \in C^{\infty}_U(TM)$ be a vector field which generates a local 1-parameter group $f(t) = \exp(tX)$ of local diffeomorphisms. Let $U \subset \mathbb{R}^{\nu}$ be an open neighbourhood of O in $\mathbb{R}^{\nu}$ with parameter $t = (t_1, \dots, t_{\nu})$ and let $\mathcal{W} \xrightarrow{\omega} U$ be a smooth fibre bundle with fibre M. The structure group of $T\mathcal{W}$ may be defined as follows: consider the group of all matrices $\begin{bmatrix} a & c \\ 0 & b \end{bmatrix}$ where $a \in GL(n, \mathbb{R})$, $b \in GL(\nu, \mathbb{R})$ and $c \in \mathrm{Hom}(\mathbb{R}^{\nu}, \mathbb{R}^n)$. Let $\mathbf{G}^*$ be the group of all matrices of this type where $a \in \mathbf{G}$ (one can take $\mathbf{G}' \subset \mathbf{G}^*$ to consist of the subgroups where $c = 0$). For a given $\mathbf{G}^*$–structure on $\mathcal{W}$ there exists on the fibre $M_t = \omega^{-1}(t)$ an induced **G**–structure

$$G \longrightarrow B_{\mathbf{G}}(t) \longrightarrow M_t$$

For an open set $\mathcal{U} \subset M$, there is a natural **G**–structure on $\mathcal{W} \times \mathcal{U}$ induced from that on $\mathcal{W}$. If $\mathcal{W}$ possesses a $\mathbf{G}^*$–structure, then $\omega : \mathcal{W} \to \mathcal{U}$ is a deformation of the **G**–structure on M if:

i) there exists a **G**–diffeomorphism between M and $M_0 = \omega^{-1}(0)$, and
ii) the bundle $\mathcal{W} \xrightarrow{\omega} \mathcal{U}$ is locally trivial.

§2. Quaternionic manifolds and their twistor spaces.

Let us now take $n = 4m \quad (m > 1)$. We shall say that M is *almost quaternionic* if the principal tangent frame bundle $B_{\mathbf{G}}$ of M is equipped with a

$$GL(m,\mathbb{H})\,GL(1,\mathbb{H}) := GL(m,\mathbb{H}) \times_{\mathbb{R}} GL(1,\mathbb{H})$$

connection (see **[25]**, **[26]**). We shall denote this $\mathbf{G}$-structure on M by $\mathbf{G}_M$. Observe that $\mathbf{G}_M$ is the same group as $GL(m,\mathbb{H})\,Sp(1) := GL(m,\mathbb{H}) \times_{\mathbb{Z}_2} Sp(1)$. Equivalently, there is a distinguished rank 3 subbundle $\mathbb{G} \subset \mathrm{End}(TM)$ having a local basis $\{I, J, K\}$ satisfying the usual quaternion identities

$$I^2 = J^2 = K^2 = -1, \quad IJ = K = -JI, \ etc.$$

Let E and H respectively denote the vector bundles associated to the fundamental representations of $GL(m,\mathbb{H})$ and $Sp(1)$ respectively on $\mathbb{C}^{2m}$ and $\mathbb{C}^2$. Then taking the following tensor product over $\mathbb{C}$, we have

$$T^*_{\mathbb{C}}M \cong E \otimes H$$

and

$$\Lambda^2 T^*_{\mathbb{C}}M \cong S^2E + \wedge^2 E \otimes S^2 H$$

If $g_M(IX, IY) = g_M(X, Y)$ for any local section I of $\mathbb{G}$ satisfying $I^2 = -1$, then M is said to be *quaternionic Hermitian* whereby $\mathbf{G}_M$ reduces to the group $Sp(m)\,Sp(1)$ and the above decomposition of 2-forms is refined to

$$\Lambda^2 T^*_{\mathbb{C}}M \cong S^2H + S^2E + \Lambda^2_0 E \otimes S^2 H$$

where $\Lambda^2 E \cong \mathbb{R} + \Lambda^2_0 E$ is the decomposition into irreducible $Sp(m)$-modules. The *fundamental 4-form* of M is a global 4-form Ω defined locally by

$$\Omega = \omega_I \wedge \omega_I + \omega_J \wedge \omega_J + \omega_K \wedge \omega_K$$

for I, J, K a local basis as before and $\omega_I(X, Y) = g_M(X, IY)$ is the local 2-form associated to I, etc. A 2-form F on M is said to be *self–dual* if

$$*F = c_i F \wedge \Omega^{m-1}$$

where * denotes the Hodge star operator on M and for $1 \leq i \leq 3$, the c_i are constants corresponding to the eigenspaces of $\Lambda^2 T^*_{\mathbb{C}}M$ (see **[9]**). We shall be interested in complex vector bundles $V \to M$ for which the connection ∇ on V has a curvature 2-form R_∇ which is $c_2 = c_{S^2E}$-self-dual; specifically, $R_\nabla \in \Omega^2(M, \mathrm{End}\,V \otimes S^2E)$. It will be convenient for us to call such a connection simply *self-dual*, generalizing the situation in dimension 4 (see e.g., **[1]**, **[7]**).

The twistor space Z associated to M is the 2-sphere bundle

$$Z = S^2(\mathbb{G}) = \{I \in \mathbb{G} : \|I\| = 1\}$$

giving rise to the twistor fibration $\pi : Z \to M$ with fibre S^2. Since M is almost quaternionic, it can be shown that Z is an almost complex manifold with almost complex structure $\mathcal{I}$ for each choice of a $\mathbf{G}_M$-connection $\nabla_{\mathbf{G}}$ on M ([**25**]). This connection is determined by a choice of a horizontal distribution on the principal frame bundle $B_{\mathbf{G}}$. Then Z can be regarded as the bundle associated to the structure $\mathbf{G}_M$ via the adjoint action of the $GL(1,\mathbb{H})$ factor on S^2. Equivalently, $Z = \mathbb{P}_{\mathbb{C}}(H)$. Note that there is a canonical $GL(2m+1,\mathbb{C})$–structure $\mathbf{G}_Z$ on Z whereby the horizontal distribution on $B_{\mathbf{G}}$ determines that on Z. The almost complex structure $\mathcal{I}$ at $I \in Z$ is taken to be I on horizontal tangent vectors and multiplication by ι on vertical tangent vectors with respect to π. Now $\mathcal{I}$ depends on the torsion of $\nabla_{\mathbf{G}_M}$ alone where the integrability of $\mathcal{I}$ is equivalent to $\nabla_{\mathbf{G}}$ being torsion-free. In the terminology of [**25**], M is (integrable) *quaternionic* if M admits a torsion–free $\nabla_{\mathbf{G}}$-connection. Thus the 'integrability' of M as a quaternionic manifold is equivalent to Z being a complex manifold (with $\mathcal{I}$ integrable). For a given parameter value t as in §1, we shall denote by $\mathcal{D}$ the space of torsion–free $\nabla_{\mathbf{G}}$–connections on M. Each $I \in Z_x$ determines a decomposition into complex types

$$\Lambda^r(T_x^*M)_{\mathbb{C}} = \mathcal{A}_I^{r,0} \oplus \mathcal{A}_I^{r-1,1} \oplus \cdots \oplus \mathcal{A}_I^{0,r}$$

with $\mathcal{A}_{-I}^{r,0} = \mathcal{A}_I^{0,r} = \overline{\mathcal{A}_I^{0,r}}$. In particular for $r = 2$,

$$\Lambda^2 T_{\mathbb{C}}^* M = \mathcal{A}_I^{2,0} + \mathcal{A}_I^{1,1} + \mathcal{A}_I^{0,2}$$

we have

$$S^2E = \bigcap_{I \in Z} \mathcal{A}_I^{1,1}$$

Recalling that at $I \in Z$, the complex structure $\mathcal{I}$ on Z is equivalent to I on horizontal vectors, it can be shown that if $V \to M$ is a complex vector bundle with connection ∇ whose curvature R_∇ is a self-dual 2–form, then π^*V is a holomorphic vector bundle on Z [**18**]. Here, the complex structure $\tilde{\mathcal{I}}$ on π^*V is obtained by taking $\pi^*\nabla$ to give the local splitting

$$T(\pi^*V) = TZ \oplus \mathbb{C}^r \ (r = \operatorname{rank} V)$$

and then take $\tilde{\mathcal{I}} = (\mathcal{I}, \iota)$ where ι denotes the usual almost complex structure on $\mathbb{C}^r$. Let

$$A^r = \Lambda^r E \oplus S^r H \subset (\Lambda^r T^* M)_{\mathbb{C}}$$

be associated to the irreducible $\mathbf{G}_M$ $(= GL(m,\mathbb{H})\, Sp(1))$–submodule of $\Lambda^r T^* M$ of highest weight in the $Sp(1)$ factor. It is sometimes convenient to write the decomposition of A^r in the following way: let B^r denote the subbundle of $\Lambda^r T^* M$ formed by the sum of $\mathbf{G}_M$–components distinct from A^r. For $2 \leq r \leq 2m$, we shall set

$$\Lambda^r (T^* M)_{\mathbb{C}} = A^r \oplus B^r$$

With regards to the above decomposition of 2–forms, we see that $B^2 = S^2 E$ and thus if V has a self–dual connection ∇, its curvature R_∇ is B^2-valued. Likewise, if V has a c_1–self–dual connection implying that R_∇ is valued in $S^2 H$, then we say that ∇ is *anti-self-dual.* We shall restrict our attention mainly to those connections which are *self–dual.* For $z \in \pi^{-1}(x)$, we have

$$(A^r)_x = \sum_x \mathcal{A}_x^{0,q}$$

where $A_x^0 \cong \mathbb{C}$, $A_x^1 = (T_x^* M)_{\mathbb{C}}$, $A_x^2 = (S^2 H + \Lambda_0^2 E \otimes S^2 H)_x$, etc.

Let $\eta^r : (\Lambda^r T^* M)_{\mathbb{C}} \to A^r$ be the projection and set $D = \eta \circ d$. Then if M is quaternionic, the complex

$$0 \longrightarrow A^0 \overset{D=d}{\longrightarrow} A^1 \overset{D}{\longrightarrow} A^2 \longrightarrow \cdots \longrightarrow A^{2m}$$

is an elliptic complex on M (i.e. $D^2 = 0$). There is a direct relationship with the Dolbeault complex

$$0 \longrightarrow \mathcal{A}^{0,0} \overset{\bar\partial}{\longrightarrow} \mathcal{A}^{0,1} \overset{\bar\partial}{\longrightarrow} \mathcal{A}^{0,2} \overset{\bar\partial}{\longrightarrow} \cdots \longrightarrow \mathcal{A}^{0,2m+1} \longrightarrow 0$$

on Z. Specifically, we have a short exact sequence of vector bundles

$$0 \longrightarrow \mathcal{A}_{\text{hor}}^{0,1} \longrightarrow \mathcal{A}^{0,1} \longrightarrow \mathcal{A}_{\text{ver}}^{0,1} \longrightarrow 0$$

involving $(0,1)$–forms horizontal and vertical with respect to π. Taking exterior powers leads to

$$0 \longrightarrow \mathcal{A}_{\text{hor}}^{0,q} \overset{\alpha}{\longrightarrow} \mathcal{A}^{0,q} \overset{\beta}{\longrightarrow} \mathcal{A}_{\text{ver}}^{0,1} \otimes \mathcal{A}_{\text{hor}}^{0,q-1} \longrightarrow 0$$

and the restriction of $\mathcal{A}^{0,q}$ to each fibre $\pi^{-1}(x)$ has a holomorphic structure such that:

a) $\bar\partial_{\text{ver}} = \beta \circ \bar\partial \circ \alpha$ restricted to $\pi^{-1}(x)$ is the usual $\bar\partial$–operator with coefficients in $\mathcal{A}_{\text{hor}}^{0,q}$

b) α induces an isomorphism

$$\alpha^\# : (A^r)_x \overset{\cong}{\longrightarrow} H^0(\pi^{-1}(x), \mathcal{O}(\mathcal{A}_{\text{hor}}^{0,r})) = \ker(\bar\partial_{\text{ver}} \mid \pi^{-1}(x))$$

c) if $\omega \in C^\infty(A^r)$, then $\alpha^\#(\omega) = (\pi^*\omega)^{0,r}$ and $\alpha^\# \circ D = \bar\partial \circ \alpha^\#$.

Theorem 2.1. [18] *Let $V \to M$ be a complex vector bundle with self-dual connection. On extending D to $A^r(V) = A^r \otimes V$, the complex*

$$0 \longrightarrow A^0(V) \xrightarrow{D} A^1(V) \xrightarrow{D} A^2(V) \cdots \longrightarrow A^{2m}(V) \longrightarrow 0$$

is elliptic and

$$H^r(Z, \mathcal{O}(\pi^* V)) = \begin{cases} \dfrac{\ker(D \mid A^r(V))}{D(A^{r-1}(V))} & \text{for} \quad 0 \leq r \leq 2m \\ 0 \quad \text{for} \quad r = 2m+1. \end{cases}$$

§3. Quaternionic deformations of vector bundles with self–dual connections.

Let M be a compact quaternionic manifold ($\dim_{\mathbb{R}} M = 4m$). Recall from §2 that we have the rank 3 subbundle $\mathbb{G} \subset \operatorname{End} TM$. Let G be a connected Lie subgroup of $GL(N, \mathbb{H})\, Sp(1)$, for some N. Of interest to us are the cases for which G is one of $GL(N, \mathbb{H})\, Sp(1)$, $GL(N, \mathbb{H})$, $Sp(N)$ or $Sp(N)\, Sp(1)$. Henceforth, we assume that G is one of these listed subgroups. Let us now consider a principal G–bundle $G \to P \to M$. In this case, the adjoint bundle of P, denoted $\operatorname{Ad} P = P \times_{\operatorname{Ad}} \mathfrak{g}$, contains a rank 3 subbundle $\mathbb{G}_P$ corresponding to the adjoint representation of $Sp(1)$ which descends to $\mathbb{G}$.

We shall consider those G–bundles on M whose associated vector bundle $V \to M$ has a self–dual connection ∇_V (recall with curvature $R_\nabla \in \Omega^2(M, S^2 E \otimes \operatorname{Ad} P)$). In the terminology of [**25**], $(V, \nabla_V) \to M$ is said to be a *quaternionic vector bundle.*

Definition 3.1. *Let $V \to M$ be a quaternionic vector bundle associated to the principal G-bundle $G \to P \to M$. A quaternionic deformation of the bundle $G \to P \to G$ is specified by a deformation of the self-dual connection ∇_V within the space of G–connections together with a deformation M_t of the $\mathbf{G}_M$–structure of M through torsion-free $\mathbf{G}_M$ connections in $\mathcal{D}$.*

Note that since a self–dual connection on V induces the same on $\mathbb{G}_P$, we obtain a family of complexes $(A^*(\mathbb{G}_P), D_t)$ for each torsion–free connection $D_t \in \mathcal{D}$.

Recalling that the pull–back of V by π^* to Z gives a holomorphic vector bundle $(\tilde{V}, \tilde{J}) \to (Z, J)$ associated to a holomorphic principal bundle $\tilde{P}$ on Z. We can implement the simultaneous deformations of $(\tilde{V}, \tilde{J})$ and (Z, J) in the holomorphic category. This was studied in

[27] **[28]** for complex principal bundles and we shall outline one of the main results. We commence by considering the (complexified) Atiyah sequence

$$0 \to \mathrm{Ad}_{\mathbb{C}}\, \tilde{P} \to T_{\mathbb{C}}\tilde{P}/G_{\mathbb{C}} \to T_{\mathbb{C}}Z \to 0$$

At the level of sheaves, let us agree to write the corresponding exact sequence as

$$0 \to \mathrm{Ad}\, \tilde{P} \to \tilde{Q} \to TZ \to 0$$

Tensoring this sequence with $\mathcal{A}^{0,r}$ on Z, we set

$$T_1^r = \mathrm{Ad}\, \tilde{P} \otimes \mathcal{A}^{0,r},\ T_2^r = \tilde{Q} \otimes \mathcal{A}^{0,r},\ T_3^r = TZ \otimes \mathcal{A}^{0,r}$$

and summing each term over $r \geq 0$, we obtain the exact sequence

$$0 \longrightarrow T_1 \longrightarrow T_2 \xrightarrow{h} T_3 \longrightarrow 0$$

where the operators $\bar{\partial}$ and $[\ ,\ ]$ are extended to each T_i in the usual way.

Definition 3.2. *An almost complex principal bundle structure on $G_{\mathbb{C}} \to \tilde{P} \to Z$ is a pair $(\tilde{J}, J)$ where $\tilde{J}$ (respectively J) is an almost complex structure on $\tilde{P}$ (respectively Z), such that*

i) *$\tilde{J}$ is $G_{\mathbb{C}}$–invariant*
ii) *the almost complex structure on $\tilde{P}/G_{\mathbb{C}}$ induced by $\tilde{J}$ is J, and*
iii) *$\tilde{J}$ restricted to each fibre gives the integrable almost complex structure on $G_{\mathbb{C}}$.*

Proposition 3.3. **[27]**, **[28]** *There is a bijective correspondence between the almost complex structures $(\tilde{J}, J)$ on $G_{\mathbb{C}} \to \tilde{P} \to Z$ which are sufficiently close to given fixed almost complex structures $(\tilde{J}_0, J_0)$ and elements $\psi \in T_1^1$ close to 0 and satisfying $h(\psi) = \varphi$ where φ is taken relative to J. The integrability condition is*

$$\bar{\partial}\psi - \frac{1}{2}[\psi, \psi] = 0.$$

To see how Proposition 3.3 can be applied to this situation, we recall from **[11]**, **[12]** that a vector field X on M is said to be a *quaternionic vector field* if via its infinitesimal automorphisms it preserves the quaternionic structure $\mathbf{G}_M$ of M. For $m = 1$, $GL(1,\mathbb{H})\, GL(1,\mathbb{H})$ is the conformal group $CO(4)$ on a 4–manifold. The twistor correspondence yields the following (see e.g., **[11]**, **[24]**):

Proposition 3.4. *If X is a quaternionic vector field on M then X induces a holomorphic vector field Y on Z such that $\pi_*(Y) = X$. Conversely, a projectable holomorphic vector field on Z induces a quaternionic vector field on M.*

Lemma 3.5. *An infinitesimal $\mathbf{G}_M$–automorphism of M lifts via π to an infinitesimal $\mathbf{G}_Z$–holomorphic automorphism of Z and conversely on π–related vector fields.*

Following the discussion in §1, let $\mathcal{W} \xrightarrow{\omega} \mathcal{U}$ be a $\mathbf{G}_M$–deformation and $\tilde{\mathcal{W}} \xrightarrow{\tilde{\omega}} \tilde{\mathcal{U}}$ be a $\mathbf{G}_Z$–deformation.

Proposition 3.6. *A $\mathbf{G}_M$–deformation induces a $\mathbf{G}_Z$–deformation and conversely on π–related vector fields. In particular, the diagram below is commutative:*

$$\begin{array}{ccc} \tilde{\mathcal{W}} & \xrightarrow{\tilde{\omega}} & \tilde{\mathcal{U}} \\ {\scriptstyle w}\downarrow & & \downarrow{\scriptstyle u} \\ \mathcal{W} & \xrightarrow{\omega} & \mathcal{U} \end{array}$$

Denoting the $\tilde{V}$–valued forms on Z by $\mathcal{A}^*(\tilde{V})$ we let $\nabla''(\tilde{V})$ denote the set of $\mathbb{C}$–linear maps

$$\nabla'' : \mathcal{A}^0(\tilde{V}) \to \mathcal{A}^{0,1}(\tilde{V})$$

satisfying

$$\nabla''(fs) = (d''f)s + f \cdot \nabla'' s$$

for $s \in \mathcal{A}^0(\tilde{V})$, $f \in \mathcal{A}^0$. Each ∇'' extends uniquely to a $\mathbb{C}$–linear map

$$\nabla'' : \mathcal{A}^{p,q}(\tilde{V}) \to \mathcal{A}^{p,q+1}(\tilde{V}),\ p, q \geq 0$$

satisfying

$$\nabla''(\psi\nu) = d''\psi \wedge \nu + (-1)^{r+s}\psi \wedge \nabla''\nu$$

for $\nu \in \mathcal{A}^{p,q}(\tilde{V})$, $\psi \in \mathcal{A}^{r,s}$. The set $\nabla''(\tilde{V})$ is an affine space which can be identified with the infinite dimensional vector space $\mathcal{A}^{0,1}(\operatorname{End}\tilde{V}) \cong \mathcal{A}^{0,1}(\operatorname{Ad}\tilde{P})$. Let $\mathcal{H}''(\tilde{V}) \subset \nabla''(\tilde{V})$ be the set of those ∇'' satisfying the integrability condition $\nabla'' \circ \nabla'' = 0$. The set $\mathcal{H}''(\tilde{V})$ can be regarded as the set of holomorphic bundle structures on $\tilde{V}$. The group $GL(\tilde{V})$ of C^∞ bundle automorphisms of $\tilde{V}$ (inducing the identity transformation on Z) acts on $\nabla''(\tilde{V})$ and maps $\mathcal{H}''(\tilde{V})$ to itself. Two holomorphic structures ∇_1'' and ∇_2'' of $\tilde{V}$ are said to be equivalent if they lie on the same $GL(\tilde{V})$–orbit. The moduli space of holomorphic structures on $\tilde{V}$ is the quotient space $\mathcal{H}''(\tilde{V})/\,GL(\tilde{V})$.

For a given parameter s say, the family of such connections $\{\nabla_V\}_s$, as a deformation space, is $H^1(\mathbb{G}_P)$ **[18]** . Thus for given parameter values s and t, we may consider the space

$$\{\{(\nabla_V)_s\}, \{D_t\}\} := \{\mathcal{Q}_{s,t}\} \cong H^1(\mathbb{G}_P) \times \mathcal{D}$$

Let $\mathcal{J}$ denote the space of almost complex structures on Z and let $\mathcal{R} = \mathcal{H}''(\tilde{V})/\,GL(\tilde{V}) \times \mathcal{J}$.

Proposition 3.7. *For ψ, φ as in Proposition 3.3 and $J \in \mathcal{J}$ integrable, the space $\{\mathcal{Q}_{s,t}\}$ injects into*

$$T_{(\nabla'',J)}\mathcal{R} = \{\psi \in T_1^1 : h(\psi) = \varphi,\ \bar{\partial}\psi - \frac{1}{2}[\psi, \psi] = 0\}.$$

§4. Complex vector bundles over quaternionic Kähler manifolds.

Henceforth, we assume that M is compact and further, assume that M is *quaternionic Kähler* which by definition means that the linear holonomy of M is contained in the subgroup $Sp(m)\,Sp(1) \subset SO(4m)$. Note then that E becomes the bundle associated to the fundamental representation of $Sp(m)$ on $\mathbb{C}^{2m}$. For a complex vector bundle $V \to M$ with connection ∇, one often considers the *Yang-Mills* functional

$$\mathrm{YM}(\nabla) = \frac{1}{2}\int_M \|R_\nabla\|^2 d\nu_M$$

along with the following topological invariant (or 'instanton number')

$$k = \frac{1}{8\pi^2}\int_M \mathrm{Tr}(R_\nabla \wedge R_\nabla) \wedge \Omega^{m-1}$$

which is equal to $\langle p_1(V), [\Omega^{m-1}]\rangle$. With regards to the decomposition in §2, we can write the above as

$$k = -\frac{1}{8\pi^2}\int_M \sum_{i=1}^{3} c_i \|R_i\|^2 \Omega^m.$$

When $R_\nabla = R_i$, we have $\mathrm{YM}(\nabla) = 4\pi^2|k/c_i|$ and the functional YM is minimized when ∇ is c_1 or c_2–self–dual (anti–self–dual or self–dual respectively) (see **[9]**, **[18]**). In proceeding, we consider two classes of vector bundles relevant to the previous discussion:

$\mathbb{V}_M$:= {Pairs (V, ∇_V) where $V \to M$ is a complex vector bundle and ∇_V is a self–dual Hermitian connection on V.}

$\mathbb{V}_Z$:= {Pairs $(\tilde{V}, \nabla_{\tilde{V}})$ where $\tilde{V} \to Z$ is a holomorphic vector bundle with Hermitian $(1,0)$–connection $\nabla_{\tilde{V}}$ and Hermitian metric $h(\ ,\ )$. The bundle $\tilde{V}$ is flat restricted to the fibres of π and is assumed to be endowed with a 'real'-structure $\tau : Z \to Z$ (see e.g. [**1**]) which lifts to a bundle automorphism $\tilde{\tau} : \tilde{V} \to \tilde{V}$. A bundle map $\sigma : \tilde{V} \to \tilde{V}$ is then defined fibrewise by

$$f \in \tilde{V}_z \to \sigma(f) \in \tilde{V}^*_{\tau(z)}$$

where $\sigma(f)(s) := h(f, \tilde{\tau}(f))$ for each $s \in \tilde{V}_{\tau(z)}$. The map σ is an antiholomorphic bundle automorphism.}

A fundamental result is the following:

Theorem 4.1. *The assignment*

$$\mathbb{V}_M \ni (V, \nabla_V) \to (\pi^* V, \pi^* \nabla_V) \in \mathbb{V}_Z$$

defines a bijective correspondence between $\mathbb{V}_M$ *and* $\mathbb{V}_Z$.

Following [**26**], if M has positive scalar curvature, then Z admits a Kähler-Einstein metric of positive scalar curvature (the model example is to take $M = \mathbb{H}P^m$, m-dimensional quaternionic projective space with corresponding $Z = \mathbb{C}P^{2m+1}$). When M has positive scalar curvature, a pair $(\pi^* V, \pi^* \nabla_V)$ arising from Theorem 4.1 on Z, is a holomorphic vector bundle with Ricci-flat Hermitian-Einstein connection [**21**]. Let $\mathbb{V}^h_Z$ denote elements of $\mathbb{V}_Z$ endowed with these extra properties. Then we have:

Corollary 4.2. *Let M be a compact quaternionic Kähler manifold with positive scalar curvature. Then the assignment*

$$\mathbb{V}_M \ni (V, \nabla_V) \to (\pi^* V, \pi^* \nabla_V) \in \mathbb{V}^h_Z$$

defines a bijective correspondence between $\mathbb{V}_M$ *and* $\mathbb{V}^h_Z$.

§5. Hermitian–Einstein vector bundles and the Kuranishi space

Continuing from the end of the last section, let M be a compact quaternionic –Kähler manifold of positive scalar curvature and $\tilde{V} \to Z$ a holomorphic vector bundle. For a Hermitian (metric) structure h on

$\tilde{V}$, let $\mathcal{D}(\tilde{V},h)$ denote the set of connections on $\tilde{V}$ preserving h and let $U(\tilde{V},h)$ be the subgroup of $GL(\tilde{V})$ consisting of unitary automorphisms of $(\tilde{V},h)$. We now consider the set

$$\mathcal{H}(\tilde{V},h)=\{\nabla\in\mathcal{D}(\tilde{V},h):\nabla=\nabla^{'}+\nabla^{''},\ \nabla^{''}\in\mathcal{H}^{''}(\tilde{V})\}$$

If $\nabla\in\mathcal{H}(\tilde{V},h)$, then $\nabla^{''}$ defines a unique holomorphic structure in $\tilde{V}$ such that $\nabla^{''}$ is the $\bar{\partial}$–operator on $\tilde{V}$–valued (p,q)–forms. Accordingly, ∇ is the Hermitian connection of $(\tilde{V},h)$ with respect to this holomorphic structure.

Consider now the subset $\mathcal{E}(\tilde{V},h)$ of Hermitian–Einstein connections on $\tilde{V}$:

$$\mathcal{E}(\tilde{V},h)=\{\nabla\in\mathcal{H}(\tilde{V},h):\iota\Lambda R(\nabla)=cI_{\tilde{V}}\}$$

where $R(\nabla)$ denotes the curvature of ∇ and $\Lambda:\mathcal{A}^{p,q}\to\mathcal{A}^{p-1,q-1}$ is $(1,1)$contraction relative to the Kähler form. The space $\mathcal{E}(\tilde{V},h)/U(\tilde{V},h)$ is the moduli space of Hermitian-Einstein structures on $\tilde{V}$ and we have an injective map

$$\mathcal{E}(\tilde{V},h)\longrightarrow\mathcal{H}^{''}(\tilde{V})/\,GL(\tilde{V})$$

Let us now consider the tangent spaces to these moduli spaces at given connections. Firstly, the tangent space to $\mathcal{H}^{''}(\tilde{V})/\,GL(\tilde{V})$ at $\nabla^{''}$, is given by

$$H^{0,1}(Z,\mathrm{End}\,\tilde{V}^{\nabla^{''}})=\frac{\{\alpha\in\mathcal{A}^{0,1}(\mathrm{End}\,\tilde{V}):\nabla^{''}\alpha=0\}}{\{\nabla''f:f\in\mathcal{A}^{0}(\mathrm{End}\,\tilde{V})\}}.$$

The tangent space of $\mathcal{E}(\tilde{V},h)/U(\tilde{V},h)$ at ∇ is given by

$$\mathbf{H}^1=\frac{\{\alpha\in\mathcal{A}^1(\mathrm{End}(\tilde{V},h):\nabla\alpha\in\mathcal{A}^{1,1}(\mathrm{End}(\tilde{V},h))\ \mathit{and}\ \Lambda\nabla\alpha=0\}}{\{\nabla f:f\in\mathcal{A}^0(\mathrm{End}(\tilde{V},h))\}}$$

$$\cong\{\alpha\in\mathcal{A}^1(\mathrm{End}(\tilde{V},h):\nabla\alpha\in\mathcal{A}^{1,1}(\mathrm{End}(\tilde{V},h),\ \Lambda\nabla\alpha=0\ \mathit{and}\ \nabla^*\alpha=0\}$$

Let $\alpha=\alpha^{'}+\alpha^{''}\in\mathcal{A}^1(\mathrm{End}(\tilde{V},h))$. The assignment $\alpha\to\alpha^{''}$ gives an isomorphism of $\mathbf{H}^1$ onto the space of harmonic $(0,1)$–forms with values in $\mathrm{End}(\tilde{V},h)$, leading to

$$\begin{aligned}\mathbf{H}^1&\cong H^{0,1}(Z,\mathrm{End}(\tilde{V}^{\nabla^{''}}))\\&=\{\alpha\in\mathcal{A}^{0,1}(\mathrm{End}\,\tilde{V}):\nabla^{''}\alpha^{''}=0\ \mathit{and}\ \nabla^{''*}\alpha^{''}=0\}.\end{aligned}$$

To proceed, let us define:

$$\mathcal{B}^p = \mathcal{A}^p_{\mathbb{R}}(\mathrm{End}(\tilde{V},h))$$

$$\mathcal{B}^{p,q} = \mathcal{A}^{p,q}(\mathrm{End}(\tilde{V},h)) = \mathcal{A}^{p,q} \otimes_{\mathbb{R}} \mathcal{B}^0 = \mathcal{A}^{p,q} \otimes_{\mathbb{R}} \mathcal{A}^0_{\mathbb{R}}(\mathrm{End}(\tilde{V},h))$$

$$\mathcal{B}^2_+ = \mathcal{B}^2 \cap (\mathcal{B}^{2,0} + \mathcal{B}^{0,2} + \mathcal{B}^0\omega) = \{\alpha + \bar{\alpha} + \beta\omega : \alpha \in \mathcal{B}^{2,0} \textit{ and } \beta \in \mathcal{B}^0\}$$

$$\mathcal{C}^{0,q} = \mathcal{A}^{0,q}(\mathrm{End}\,\tilde{V}) = \mathcal{A}^{0,q} \otimes_{\mathbb{C}} \mathcal{A}^{0,0}(\mathrm{End}\,\tilde{V})\ .$$

Consider now the complex

$$\begin{array}{ccccccccccc}
0 \to \mathcal{B}^0 & \xrightarrow{\ \nabla\ } & \mathcal{B}^1 & \xrightarrow{\ \nabla_+\ } & \mathcal{B}^2_+ & \xrightarrow{\ \nabla_2\ } & \mathcal{B}^{0,3} & \xrightarrow{\ \nabla\ } & \ldots & \mathcal{B}^{0,n} \to 0 \\
\big\downarrow j_0 & & \big\downarrow j_1 & & \big\downarrow j_2 & & \big\downarrow j_3 & & & \big\downarrow j_n \\
0 \to \mathcal{C}^{0,0} & \xrightarrow{\ \nabla''\ } & \mathcal{C}^{0,1} & \xrightarrow{\ \nabla''\ } & \mathcal{C}^{0,2} & \xrightarrow{\ \nabla''\ } & \mathcal{C}^{0,3} & \xrightarrow{\ \nabla''\ } & \ldots & \mathcal{C}^{0,n} \to 0
\end{array}$$

where $(\mathcal{C}^*)$ is elliptic if $\nabla \in \mathcal{H}(\tilde{V},h)$ and $(\mathcal{B}^*)$ is elliptic if $\nabla \in \mathcal{E}(\tilde{V},h)$. We can decompose $\mathcal{B}^2$ as $\mathcal{B}^2 = \mathcal{B}^2_+ \oplus \mathcal{B}^2_-$ where

$$\mathcal{B}^2_- = \{\alpha \in \mathcal{A}^{1,1}(\mathrm{End}(\tilde{V},h)) : \alpha = \bar{\alpha} \textit{ and } \Lambda\alpha = 0\}$$

and we have the projections

$$p_+ : \mathcal{B}^2 \to \mathcal{B}^2_+ \qquad p_- : \mathcal{B}^2 \to \mathcal{B}^2_-$$

$$p^{2,0} : \mathcal{B}^2 \to \mathcal{B}^{2,0} \qquad p^{0,2} : \mathcal{B}^2 \to \mathcal{B}^{0,2}$$

where we set $\nabla_+ = p_+ \circ \nabla$ and $\nabla_2 = \nabla'' \circ p^{0,2}$. Then $\mathcal{E}(\tilde{V},h)$ can be expressed as

$$\mathcal{E}(\tilde{V},h) = \{\nabla + \alpha : \alpha \in \mathcal{B}^1 \textit{ and } \nabla_+\alpha + p_+(\alpha \wedge \alpha) = 0\}$$

Consider a slice $\nabla + \mathcal{S}_\nabla$ in $\mathcal{E}(\tilde{V},h)$ in which

$$\mathcal{S}_\nabla = \{\alpha \in \mathcal{B}^1 : \nabla_+\alpha + p_+(\alpha \wedge \alpha) = 0 \textit{ and } \nabla^*\alpha = 0\}$$

and the condition $\nabla^*\alpha = 0$ states that the slice is orthogonal to the $U(\tilde{V},h)$–orbit of ∇ . The *Kuranishi map* $k : \mathcal{B}^1 \to \mathcal{B}^1$ is defined by

$$k(\alpha) = \alpha + \nabla_+^* \circ G \circ p_+(\alpha \wedge \alpha)$$

where G denotes the Green's operator. If $\nabla + \alpha \in \mathcal{E}(\tilde{V}, h)$, then it can be shown that $\nabla_+(k(\alpha)) = 0$. On the other hand, we have $\nabla^*(k(\alpha)) = \nabla^*\alpha$. Taking the harmonic forms in $\mathcal{B}^q$,

$$\mathbf{H}^q = \{\beta \in \mathcal{B}^q : \Delta\beta = 0\}$$

we have

$$k(\mathcal{S}_\nabla) \subset \mathbf{H}^1 = \{\beta \in \mathcal{B}^1 : \nabla_+\beta = 0 \text{ and } \nabla^*\beta = 0\}.$$

Letting End^0 denote trace–free skew Hermitian–endomorphisms, we arrive at

Theorem 5.1. [17] *Let $\nabla \in \mathcal{E}(\tilde{V}, h)$. If*

$$H^0(Z, \mathrm{End}^0(\tilde{V}^{\nabla''})) = 0 \qquad \text{and} \qquad H^2(Z, \mathrm{End}^0(\tilde{V}^{\nabla''})) = 0$$

then the Kuranishi map k defines a homeomorphism of a neighborhood of 0 in the slice $\mathcal{S}_D$ onto a neighborhood of 0 in $\mathbf{H}^1 \cong H^1(Z, \mathrm{End}(\tilde{V}^{\nabla''}))$.

Remark 5.2. *A holomorphic vector bundle $\tilde{V} \to Z$ is said to be simple if every holomorphic endomorphism is a constant. The second named author and independently Miyajima* [28] [19] *have shown for an algebraic manifold Z there is an isomorphism of (not necessarily reduced) complex vector spaces*

$$\mathcal{M}_{an} \cong \mathcal{M}_{alg}$$

between the moduli spaces of holomorphic simple vector bundles and algebraic simple bundles on Z.

§6. Remarks on Hyperkähler structures and the moduli of hyperholomorphic vector bundles.

A $4n$–dimensional Riemannain manifold is *hyperkähler* if its holonomy group is contained in $Sp(n) \subset SO(4n)$. Equivalently, M is quaternionic Hermitian with I, J and K globally defined and

$$d\omega_I = d\omega_J = d\omega_K = 0.$$

Since $Sp(n) \subset SU(2n)$, a hyperkähler manifold is Ricci–flat [3], [4].

Theorem 6.1. [2] *Let (M, I) be a compact Kähler manifold with (complex) symplectic form ω. Then for any Kähler class $\alpha \in H^2(M, \mathbb{R})$, there exists on M a unique Riemannian metric g_M such that*

(1) *g_M is hyperkähler;*
(2) *I is a parallel almost–complex structure;*
(3) *the Kähler class of (g_M, I) is α.*

For a hyperkähler manifold M, there is an S^2 of almost complex structures

$$\{aI + bJ + cK \ : a^2 + b^2 + c^2 = 1\}$$

and the twistor space

$$Z = M \times \mathbb{C}P^1 \xrightarrow{\pi} M$$

fibres holomorphically over M. Let $p : Z \to \mathbb{C}P^1$ be the natural projection. The antipodal map of the fibres defines a *real* structure on Z (see e.g. [**1**]). Using the three Kähler forms $\omega_I, \omega_J, \omega_K$ along with a stereographic complex coordinate ξ on $\mathbb{C}P^1$, the form

$$\omega = (\omega_I + \iota\omega_K) + 2\xi\omega_I + (\omega_J - \iota\omega_K)$$

is a complex symplectic form on the fibres of p taking values in the line bundle $\mathcal{O}(2)$. The converse is true for a complex manifold Z of complex dimension $2n + 1$ which fibres holomorphically $p : Z \to \mathbb{C}P^1$ with the above properties tenable: the parametrization of the (real) holomorphic structures is a (real) $4n$–dimensional hyperkähler manifold whose twistor space is Z (see [**14**]).

The Uhlenbeck–Yau theorem [**29**] says that an indecomposable holomorphic vector bundle V over a compact Kähler manifold M admits a Yang–Mills metric connection if and only if V is stable, and this metric is unique. Applying this to such a bundle $V \to M$ with Hermitian connection Θ over a hyperkähler manifold M yields:

Proposition 6.2. *The metric connection Θ is hyperholomorphic (that is, holomorphic with respect to each of I, J and K) if and only if its curvature R_Θ is of type $(1,1)$ with respect to any complex structure induced by the hyperkähler structure of M.*

It can be shown that such a hyperholomorphic connection Θ is Yang–Mills. In fact, $\Lambda(R_\Theta) = 0$ where Λ denotes contraction with each Kähler $(1,1)$–form. For instance, on a hyperkähler surface a stable holomorphic bundle $\tilde{V}$ with $\deg \tilde{V} = 0$ always admits a unique hyperholomorphic connection [**20**] [**8**].

Let $\mathcal{S} \to M$ a be locally–free sheaf on and consider the composition of maps:

$$[\ ,\ ] \circ \varpi \ : H^1(\operatorname{End}\mathcal{S}) \times H^1(\operatorname{End}\mathcal{S}) \longrightarrow H^2(\operatorname{End}\mathcal{S})$$

where

$$\varpi \ : H^1(\operatorname{End}\mathcal{S}) \times H^1(\operatorname{End}\mathcal{S}) \longrightarrow H^2(\operatorname{End}\mathcal{S} \otimes \operatorname{End}\mathcal{S})$$

and

$$[\ ,\ ]\ : \operatorname{End}\mathcal{S} \times \operatorname{End}\mathcal{S} \longrightarrow \operatorname{End}\mathcal{S}$$

denotes the commutator map. The above composition gives defines the Yoneda pairing

$$\lambda\ : \operatorname{Ext}^1(\mathcal{S},\mathcal{S}) \times \operatorname{Ext}^1(\mathcal{S},\mathcal{S}) \longrightarrow \operatorname{Ext}^2(\mathcal{S},\mathcal{S})$$

which is an obstruction to the existence of a deformation (see e.g. **[16]**) . Specifically, suppose that $\varrho \in \operatorname{Ext}^1(\mathcal{S},\mathcal{S})$ satisfies $\lambda(\varrho,\varrho) \neq 0$ then there exists no deformation parametrized by a complex space $\mathcal{M}$ such that the image of the Kodaira–Spencer map

$$T_x\mathcal{M} \to \operatorname{Ext}^1(\operatorname{End}\mathcal{S})$$

is proportional to λ. In this way the Yoneda pairing turns out to be the only obstruction to the existence of a deformation of a hyperholomorphic vector bundle over a hyperkähler manifold.

For a stable bundle $\tilde{V} \to M$, let $\mathcal{M}_{st,\ \tilde{V}}$ be the coarse moduli space of $\tilde{V}$. This exists by virtue of **[20]** and is non–reduced and non–separated in general. Then

$$T\mathcal{M}_{st,\ \tilde{V}}|V' \cong \operatorname{Ext}^1(V',V').$$

Theorem 6.3. *The Kuranishi space of a hyperholomorphic vector bundle over a hyperkähler manifold is isomorphic as a complex space with the intersection of an open ball in* $\operatorname{Ext}^1(V,V)$ *with the quadric cone*

$$\{\varrho \in \operatorname{Ext}^1(V,V)\ : \lambda(\varrho,\varrho) = 0\}.$$

The complete details of this result will appear elsewhere.

References

[1] M. F. Atiyah: *Geometry of Yang–Mills fields,* Lezioni Fermiane, Scuola Normale Superiore, Pisa 1979.

[2] A. Beauville: *Variétés Kählériennes dont la premiere classe de Chern est nulle,* J. Differ. Geom. 18 (1983) 755-782.

[3] A. Besse: *Einstein Manifolds*, Springer Verlag, Berlin– Heidelberg– New York.

[4] E. Calabi: *Métriques Kählériennes et fibrés holomorphes,* Ann. scient. Éc. Norm. Sup., 4^{e} série, t. 12, (1979), 269–294.

[5] H. Doi and T. Okai: *Moduli space of 1–instantons on a quaternionic projective space* $\mathbb{HP}^n$, Hiroshima Math. J. **19**, (1989) 251–258.

[6] H. Doi and T. Okai: *Compactifying the moduli space of 1– instantons on* $\mathbb{HP}^n$, in 'Geometry of Manifolds', 35th Symp. in Differ. Geom., Matsumoto Japan 1988, Perspect. Math. **8**, 447 (1989).

[7] S. K. Donaldson and P. B. Kronheimer: *The Geometry of Four–manifolds,* Oxford Math. Monographs, Clarendon Press, Oxford, 1990.

[8] A. Fujiki: *Hyperkähler structure on the moduli space of flat bundles,* 'Prospects in Complex Geometry, Lecture Notes in Math. **1468**, Springer Verlag, Berlin–Heidelberg–New York (1991) 1–83.

[9] K. Galicki and Y. S. Poon: *Duality on quaternionic manifolds,* J. Math. Phys. 5, (1991), 1263–1268.

[10] J. F. Glazebrook: *On linear systems and higher–dimensional gauge fields,* Reps. Math. Phys. **25**, 141–147 (1988) (Preprint entitled: 'A twistor description of Ward's class A_k').

[11] J. F. Glazebrook, F. W. Kamber, H. Pedersen and A. Swann: *Foliation reduction and self-duality,* in *Geometric Study of Foliations*, World Scientific Press, 1994, 219–249.

[12] J. F. Glazebrook and D. Sundararaman: *Deformations of foliated quaternionic structures and the twistor correspondence,* in The Adem Memorial Vol., Bol. Soc. Mat. Mex. **37** 177–187

[13] P. A. Griffiths: *On the existence of a locally complete germ of deformation of certain G-structures,* Math. Ann. 1598, (1965), 151–171.

[14] N. J. Hitchin: *Monopoles, Minimal Surfaces and Algebraic Curves,* Sém de Math. Sup., **105** , Les Presses de l'Université de Montreal 1987.

[15] H. J. Kim: *Moduli of Hermite–Einstein vector bundles,* Math. Zeit. **195**, (1987), no. 1, 143–150.

[16] K. Kodaira: *Complex Manifolds and Deformation of Complex Structures,* Springer Verlag, Berlin–Heidelberg–New York.

[17] S. Kobayashi: *Differential Geometry of Complex Vector Bundles,* Publ. Math. Soc. Japan **15**, Iwanami Shoten, Publ. and Princeton University Press, 1987.

[18] M. Mamone–Capria and S. M. Salamon: *Yang–Mills fields on quaternionic spaces,* Nonlinearity **1**, (1988), 517–530.

[19] K. Miyajima: *Kuranishi family of vector bundles and algebraic description of the moduli space of Einstein-Hermitian connections,* Publ. RIMS, Kyoto Univ. **25**, (1989), 301–320.

[**20**] S. Mukai: *Hyperkähler structure on the moduli space of Yang–Mills connections*, Math. Ann. **276** (1987).

[**21**] T. Nitta: *Vector bundles over quaternionic Kähler manifolds*, Tohoku Math. J. **40**, No. 3, (1988), 425–440.

[**22**] T. Nitta: *Moduli spaces of Yang–Mills connections over quaternionic Kähler manifolds*, Osaka J. Math., **26**, 287 (1989).

[**23**] T. Nitta: *Compactifications of moduli spaces of Einstein–Hermitian connections for null–correlated bundles*, Adv. Studies in Pure Math 18–I 1990, 'Recent Topics in Differential and Analytic Geometry', 397–416.

[**24**] P. Piccinni: *On the infinitesimal automorphisms of quaternionic structures*, J. Math. Pure Appl. **72**, (1993) 593–605.

[**25**] S. M. Salamon: *Differential geometry of quaternionic manifolds*, Ann. Éc. Norm. Sup. Paris **19**, (1985), 31–55.

[**26**] S. M. Salamon: *Quaternionic Kähler manifolds*, Invent. Math. **67**, (1982), 143–171.

[**27**] D. Sundararaman: *Moduli, deformations and classifications of compact complex manifolds*, Pitman Research Notes in Mathematics **45**, Harlow–New York: Longman–Wiley 1980.

[**28**] D. Sundararaman: *On the Kuranishi space of a holomorphic principal bundle over a compact complex manifold*, Studies in Analysis, Adv. in Math. Suppl. Studies, Vol. 4, (1979), 233–239.

[**29**] K. Uhlenbeck and S.T Yau: *On the existence of Hermitian –Yang–Mills connections in stable vector bundles*, Commun. Pure and Appl. Math. **39**, (1986), 257–293.

J.F.Glazebrook
Department of Mathematics
Eastern Illinois University
Charleston, IL 61920,
U.S.A.
cfjfg@ux1.cts.eiu.edu

D.Sundararaman
Department of Mathematics
Howard University
Washington, D.C. 20059,
U.S.A.
alama41@humain.bitnet

Advanced Studies in Pure Mathematics 25, 1997
CR-Geometry and Overdetermined Systems
pp. 158–166

A Method of Prolongation of Tangential Cauchy-Riemann Equations

Chong-Kyu Han and Jae-Nyun Yoo

This article is not available due to permission restrictions.

This article is not available due to permission restrictions.

This article is not available due to permission restrictions.

This article is not available due to permission restrictions.

This article is not available due to permission restrictions.

This article is not available due to permission restrictions.

This article is not available due to permission restrictions.

This article is not available due to permission restrictions.

Advanced Studies in Pure Mathematics 25, 1997
CR-Geometry and Overdetermined Systems
pp. 167–220

Invariant Theory of the Bergman Kernel

Kengo Hirachi and Gen Komatsu

Dedicated to Professor M. Kuranishi on his 70th birthday

Introduction

This article is a brief report of recent developments in Fefferman's program, proposed and initiated in [F3], concerning invariant expression of the singularity of the Bergman kernel K^{B} on the diagonal of a strictly pseudoconvex domain $\Omega \subset \mathbb{C}^n$ with smooth boundary. It was proved by Fefferman in [F1] that

$$K^{\mathrm{B}} = \frac{\varphi^{\mathrm{B}}}{r^{n+1}} + \psi^{\mathrm{B}} \log r \quad \text{with } \varphi^{\mathrm{B}},\ \psi^{\mathrm{B}} \in C^\infty(\overline{\Omega}), \tag{0.1}$$

where $r \in C^\infty$ is a defining function of the boundary $\partial\Omega$ such that $r > 0$ in Ω and $dr \neq 0$ on $\partial\Omega$. The problem is to choose r appropriately and express φ^{B} modulo $O^{n+1}(r)$ and ψ^{B} modulo $O^\infty(r)$ invariantly in the sense of local biholomorphic geometry. This can be compared with the asymptotic expansion of the heat kernel associated with the diagonal of a compact Riemannian manifold, where the time variable corresponds to the function r in (0.1). The boundary $\partial\Omega$ is approximated at every point by a sphere (hyperquadric), and carries a differential-geometric structure, called the CR (or pseudo-conformal) structure.

Let us employ an extrinsic approach due to Chern and Moser in [CM], [M], and put the boundary $\partial\Omega$ (formally) in Moser's normal form $N(A)$ with $A = (A^\ell_{\alpha\overline{\beta}})$ given by

$$2\,\mathrm{Re}\, z_n = |z'|^2 + \sum_{|\alpha|,|\beta|\geq 2} \sum_{\ell=0}^{\infty} A^\ell_{\alpha\overline{\beta}}\, z'_\alpha\, \overline{z'_\beta}\, (\mathrm{Im}\, z_n)^\ell,$$

where $z = (z', z_n) = (z_1, \dots, z_{n-1}, z_n) \in \mathbb{C}^n$. (For the notation z'_α and $|\alpha|$ with ordered multi-indices α, see Subsection 1.1, (B) below.)

Received July 99, 1996

Then CR invariants of weight $w \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$ are defined as polynomials $P = P(A)$ satisfying the transformation law

$$P(A) = P(\widetilde{A})\,|\det \Phi'(0)|^{2w/(n+1)} \tag{0.2}$$

for local (or formal) biholomorphic mappings Φ such that $\Phi(N(A)) = N(\widetilde{A})$ and $\Phi(0) = 0$. We wish to express the asymptotic expansions

$$\begin{aligned} \varphi^{\mathrm{B}} &= \sum_{k=0}^{n} \varphi_k\, r^k \mod O^{n+1}(r), &\qquad \varphi_k \in C^\infty(\overline{\Omega}), \\ \psi^{\mathrm{B}} &= \sum_{k=0}^{\infty} \psi_k\, r^k \mod O^{\infty}(r), &\qquad \psi_k \in C^\infty(\overline{\Omega}), \end{aligned} \tag{0.3}$$

of φ^{B} and ψ^{B} in (0.1) in terms of CR invariants. We thus consider local (or localizable) domain functionals $K = K_\Omega$ near a reference point at the boundary $\partial\Omega$ satisfying a transformation law of weight $w \in \mathbb{Z}$:

$$K_{\Omega_1} = K_{\Omega_2} \circ \Phi\,|\det \Phi'|^{2w/(n+1)} \tag{0.4}$$

for local biholomorphic mappings $\Phi : \Omega_1 \to \Omega_2$ preserving the reference points, cf. (0.2).

The Bergman kernel K^{B} satisfies (0.4) with $w = n+1$. If one could find a defining function r satisfying (0.4) with $w = -1$, then there would be a hope to have expansions as in (0.3) such that φ_k for $k \le n$ and ψ_{k-n-1} for $k \ge n+1$ satisfy (0.4) with $w = k$. According to Hörmander [Hö], the boundary value of φ^{B} agrees with that of the Levi determinant

$$J[r] = (-1)^n \det \begin{pmatrix} r & \partial r/\partial \overline{z}_k \\ \partial r/\partial z_j & \partial^2 r/\partial z_j\, \partial \overline{z}_k \end{pmatrix}$$

multiplied by $n!/\pi^n$. Thus we are led to the zero Dirichlet boundary value problem for the complex Monge-Ampère equation

$$J[u] = 1 \quad \text{and} \quad u > 0 \quad \text{in } \Omega; \qquad u = 0 \quad \text{on } \partial\Omega. \tag{0.5}$$

According to Fefferman [F2], any solution of $J[u] = 1$ satisfies (0.4) with $w = -1$. However, the solution of (0.5), of which the unique existence is guaranteed by Cheng and Yau in [CY], has a finite differentiability up to the boundary. This fact is seen from the asymptotic expansion below due to Lee and Melrose in [LM] (cf. also Graham [G2]):

$$u = r \sum_{k=0}^{\infty} \eta_k \cdot (r^{n+1} \log r)^k, \qquad \eta_k \in C^\infty(\overline{\Omega}), \tag{0.6}$$

with a C^∞ defining function r as before.

There are C^∞ approximate solutions $r = r^{\mathrm{F}}$ of (0.5) satisfying

$$J[r^{\mathrm{F}}] = 1 + O^{n+1}(r) \quad \text{near } \partial\Omega \qquad (r = r^{\mathrm{F}} > 0 \quad \text{in } \Omega).$$

In [F2], Fefferman gave an explicit algorithm of constructing such a function r^{F}. Let us refer to these r^{F} as Fefferman's defining functions. After reviewing quickly in Section 1 the background of the problem which contains expositions of CR invariants, the Bergman kernel and the complex Monge-Ampère boundary value problem, we state in Section 2 Fefferman's main results in [F3], which were supplemented recently by Bailey, Eastwood and Graham in [BEG], on the expansion of φ^{B} in (0.3) by using Fefferman's defining function $r = r^{\mathrm{F}}$. Local domain functionals, called Weyl invariants, of weight $\leq n$ are defined by using the curvature of the Lorentz-Kähler metric with potential function $|z_0|^2 r^{\mathrm{F}}(z)$ on a bundle $\mathbb{C}^* \times \overline{\Omega}$ (or a neighborhood of $\mathbb{C}^* \times \partial\Omega$) with an extra variable $z_0 \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$. It is proved in [F3] and [BEG] that any CR invariant of weight $\leq n$ is realized as the boundary value of a Weyl invariant and that the expansion of φ^{B} in (0.3) with $r = r^{\mathrm{F}}$ is valid, where each φ^{B}_k is a Weyl invariant of weight k. Proofs of these results are outlined in Section 5.

The two dimensional case is exceptional and it is possible to obtain a very precise result by using Fefferman's defining function $r = r^{\mathrm{F}}$. We overview in Section 3 the work of Graham in [G1] and [G2] supplemented by the authors' joint work with Nakazawa in [HKN1] and [HKN2]. There are no nonzero CR invariants of weight 1, 2, and the expansion of φ^{B} in (0.3) with $r = r^{\mathrm{F}}$ is trivial, that is, $\varphi^{\mathrm{B}} = 2/\pi^2 + O^3(r)$. For ψ^{B} in (0.1), it is shown in [G1] and [HKN2] that

$$\psi^{\mathrm{B}} = \frac{2}{\pi^2}\left(-3\,\eta_1 + W_4\, r + W_5\, r^2\right) + O^3(r) \quad \text{with} \quad r = r^{\mathrm{F}},$$

where W_k for $k = 4, 5$ are Weyl invariants of weight k and η_1 is that in (0.6) with $r = r^{\mathrm{F}}$. This result is best possible as far as Fefferman's defining function is used. Explicit determination of W_4 and W_5 is also done in [HKN2] (partial results are found in [G1] and [HKN1]). In order to identify universal constants appearing in W_4 and W_5, it is necessary to express the singularity of the Bergman kernel in terms of Moser's normal form coefficients. This is done in [HKN1] and [HKN2] by using microlocal calculus due to Kashiwara in [Kas] and Boutet de Monvel in [B1]–[B3]. We explain this method in Section 4.

In order to get a complete expansion of ψ^{B} as in (0.3), it is necessary to take account of the ambiguity of $r = r^{\mathrm{F}}$. In [Hi], a special family of Fefferman's defining functions parametrized by $C^\infty(\partial\Omega)$ (or rather the space of formal power series) is so defined as to satisfy (0.4) with $w = -1$. This family leads to the definition of Weyl invariants with ambiguity measured by $C^\infty(\partial\Omega)$. It is proved in [Hi] that the space CR invariants of arbitrary weight exactly corresponds to that of Weyl invariants without ambiguity and that the expansion of ψ^{B} in (0.3) with a Fefferman's defining function r parametrized by $C^\infty(\partial\Omega)$ is valid, where each ψ_k^{B} is a Weyl invariant, with ambiguity, of weight $k+n+1$. This expansion of ψ^{B} is invariant in the sense that each Weyl invariant with ambiguity measured by $C^\infty(\partial\Omega)$ is a universal polynomial of $A = (A^\ell_{\alpha\overline{\beta}})$ and $C = (C^\ell_{\alpha\overline{\beta}})$, where $A^\ell_{\alpha\overline{\beta}}$ are Moser's normal form coefficients and $C^\ell_{\alpha\overline{\beta}}$ appear as the coefficients of the power series expansion of an element $f \in C^\infty(\partial\Omega)$, that is,

$$f(z', \overline{z'}, \operatorname{Im} z_n) = \sum_{|\alpha|,|\beta|\geq 0} \sum_{\ell=0}^{\infty} C^\ell_{\alpha\overline{\beta}}\, z'_\alpha\, \overline{z'_\beta}\, (\operatorname{Im} z_n)^\ell.$$

In Section 6, we state these results more precisely and outline the proofs.

In this article, we restrict ourselves to the local analysis of the Bergman kernel associated with a general strictly pseudoconvex domain, and do not refer to related topics. Here we only mention two of these. The first one is an analogue of Fefferman's program above for the Szegö kernel associated with an invariant surface element on the boundary of a strictly pseudoconvex domain. This problem was also posed in [F3], and the analysis of the Bergman kernel presented in this article applies to the Szegö kernel as well, after a slight modification (cf. [HKN1], [HKN2]). Another topic is a conformal analogue of the construction of CR invariants in terms of Weyl invariants. This problem was posed by Fefferman and Graham in [FG]. For recent progress of this topic, the reader should see the papers by Bailey-Eastwood-Graham [BEG] and by Eastwood-Graham [EG]; there are also comprehensive survey articles by Graham [G3] and by Bailey [Ba].

§1 Backgrounds

1.1 CR invariants

(A) Local boundary equivalence problem. A remarkable phenomenon in Several Complex Variables is the existence of a domain Ω (in fact, many domains) such that all holomorphic functions in Ω extend holomorphically across a part of the boundary $\partial\Omega$ to a larger domain simultaneously. If such a phenomenon does never occur for Ω, then Ω is called a *domain of holomorphy.* Assume for simplicity that Ω is a domain in $\mathbb{C}^n$ with C^∞ boundary. That is, $\Omega = \{r > 0\}$, where $r \in C^\infty(\mathbb{C}^n, \mathbb{R})$ is a *defining function* of the boundary $\partial\Omega$ and thus $|dr| > 0$ on $\partial\Omega$. A well-known theorem of Oka states that Ω is a domain of holomorphy if and only if it is *pseudoconvex* at every boundary point. The pseudoconvexity at $z \in \partial\Omega$ is by definition the non-negativity of the eigenvalues of the *Levi form* of r at $z = (z_1, \dots, z_n)$ given by

$$L_{r,z}(\xi, \overline{\xi}) = -\sum_{j,k=1}^{n} \frac{\partial^2 r(z)}{\partial z_j \partial \overline{z}_k}\, \xi_j\, \overline{\xi}_k \quad \text{for } \xi = (\xi_1, \dots, \xi_n) \in T_z^{1,0}(\partial\Omega),$$

where $T_z^{1,0}(\partial\Omega) = \big\{\xi \in \mathbb{C}^n;\ \sum_{j=1}^n \xi_j\, \partial r(z)/\partial z_j = 0\big\}$, and thus each element $\xi \in T_z^{1,0}(\partial\Omega)$ is identified with a $(1,0)$-vector $\sum \xi_j\, \partial/\partial z_j$ which is tangential to $\partial\Omega$ at z. If the Levi form is positive-definite on $\partial\Omega$, then Ω is said to be *strictly pseudoconvex.* The notion of (strict) pseudoconvexity is defined independently of the choice of r.

Let Ω_1 and Ω_2 be strictly pseudoconvex domains in $\mathbb{C}^n$ with C^∞ boundaries. If there exists a biholomorphic mapping $\Phi : \Omega_1 \to \Omega_2$, then Ω_1 and Ω_2 are said to be *holomorphically equivalent.* When are Ω_1 and Ω_2 holomorphically equivalent? A necessary condition is formulated via a theorem of Fefferman [F1] which states that if Φ as above exists then Φ extends to a C^∞ diffeomorphism from $\overline{\Omega}_1$ to $\overline{\Omega}_2$. (If the boundaries are real analytic, then Φ extends biholomorphically across the boundaries, cf. Lewy [L2].) Thus one can compare the boundaries. The boundary value of Φ is a diffeomorphism $\Phi_0 : \partial\Omega_1 \to \partial\Omega_2$ such that the components are CR functions, those functions which are annihilated by differentiation with respect to sections of the bundle $T^{0,1}(\partial\Omega) = \overline{T^{1,0}(\partial\Omega)}$. Suppose now we are given Φ_0, a CR diffeomorphism. If the boundaries are real analytic, then Φ_0 has an analytic extension to a full neighborhood of $\partial\Omega_1$. In general, Φ_0 extends holomorphically to Ω_1 according to a theorem of Lewy [L1]. These are in fact local results, and one is led to a *local boundary equivalence problem* of comparing open portions M_j of $\partial\Omega_j$ $(j = 1, 2)$, which are strictly pseudoconvex real hypersurfaces. That is, one asks when there exists a CR diffeomorphism $\Phi_0 : M_1 \to M_2$ such that $\Phi_0(p_1) = p_2$, where the pairs (M_j, p_j) with $p_j \in M_j$ are prescribed. In what follows, we mainly consider the real analytic case, and identify Φ_0 with its holomorphic extension Φ. More precisely, we regard Φ as a germ of mapping between germs of surface (M_j, p_j). In the C^∞ case, we regard Φ as a formal mapping given by formal power series between C^∞ surfaces (M_j, p_j), and thus we are only concerned with the Taylor expansions of defining functions of M_j about the reference points

$p_j \in M_j$.

(B) Moser's normal form. Let $M \subset \mathbb{C}^n$ be a strictly pseudoconvex real hypersurface containing the origin $0 \in \mathbb{C}^n$ as a reference point, and assume first that M is real analytic. To study the biholomorphic equivalence problem of M in the previous subsection, Moser [M], [CM] introduced the notion of normal form of M defined as follows.

For the standard coordinate system $z = (z_1, \dots, z_n)$ in $\mathbb{C}^n$, we write $z = (z', z_n)$ and set $z'_\alpha = z_{\alpha_1} \cdots z_{\alpha_a}$, where $\alpha = (\alpha_1, \dots, \alpha_a)$ is an ordered multi-index of length $|\alpha| = a$, that is, $\alpha_j \in \{1, \dots, n-1\}$ for $j = 1, \dots, a$. After a holomorphic change of coordinates, M is locally written near the origin as

$$2u = |z'|^2 + F_A(z', \overline{z'}, v), \qquad z_n = u + i\,v, \tag{1.1}$$

where F_A is a real analytic function having the Taylor expansion

$$F_A(z', \overline{z'}, v) = \sum_{|\alpha|+|\beta|+2\ell \geq 3} A^{\ell}_{\alpha\overline{\beta}}\, z'_\alpha\, \overline{z'_\beta}\, v^\ell = \sum_{\alpha,\beta} A_{\alpha\overline{\beta}}(v)\, z'_\alpha\, \overline{z'_\beta}\,.$$

(The meaning of the subscript A in F_A will be made clear just after the definition of Moser's normal form.) We say that M given by (1.1) is in *pre-normal form* if $\overline{A_{\alpha\overline{\beta}}(v)} = A_{\beta\overline{\alpha}}(v)$ hold for all α, β and each $A_{\alpha\overline{\beta}}(v)$ is unchanged under permutation of α and that of β. These normalizations are always possible.

By another change of coordinates, M in pre-normal form is made to satisfy $A_{\alpha\overline{\beta}}(v) = 0$ when $|\alpha| < 2$ or $|\beta| < 2$, and thus

$$F_A(z', \overline{z'}, v) = \sum_{|\alpha|,|\beta| \geq 2} A_{\alpha\overline{\beta}}(v)\, z'_\alpha\, \overline{z'_\beta}, \quad A_{\alpha\overline{\beta}}(v) = \sum_{\ell=0}^{\infty} A^{\ell}_{\alpha\overline{\beta}}\, v^\ell. \tag{1.2}$$

Definition. A surface M in pre-normal form given by (1.1) is said to be in *Moser's normal form* if (1.2) holds and the following trace conditions are fulfilled:

$$\operatorname{tr} A_{2\overline{2}}(v) = 0, \quad (\operatorname{tr})^2 A_{2\overline{3}}(v) = 0, \quad (\operatorname{tr})^3 A_{3\overline{3}}(v) = 0. \tag{1.3}$$

Here, $A_{a\overline{b}}(v) = (A_{\alpha\overline{\beta}}(v))_{|\alpha|=a, |\beta|=b}$, and $(\operatorname{tr})^m A_{a\overline{b}}(v)$ for $m = 1, 2, 3$ means that the contractions with respect to Kronecker's delta $\delta^{j\overline{k}}$ are taken m times for the indices α, β in $A_{\alpha\overline{\beta}}(v)$ with $|\alpha| = a$, $|\beta| = b$.

If M is a surface in Moser's normal form, we write $M = N(A)$ and $A \in \mathcal{N}$, where $A = (A^{\ell}_{\alpha\overline{\beta}})$ is a collection of the coefficients in (1.2). Thus

$\mathcal{N}$ is the vector space of all collections A giving Moser's normal forms. We may identify a surface $N(A)$ with $A \in \mathcal{N}$.

The existence of Moser's normal form is guaranteed as follows.

Theorem 1.1 ([CM], [M]). *For any M in pre-normal form, there exists a holomorphic change of coordinates $w = \Phi(z)$ such that $\Phi(M)$ is in Moser's normal form. The mapping Φ is unique under the conditions*

$$\Phi(0) = 0, \quad \Phi'(0) = \text{identity}, \quad \operatorname{Im}\left(\partial^2 w_n(0)/\partial z_n^2\right) = 0,$$

where Φ' denotes the holomorphic differential of Φ.

According to Theorem 1.1, there exists a holomorphic coordinate system $z = (z', z_n)$ such that M is in Moser's normal form $N(A)$. We refer to z', z_n as *Moser's normal coordinates.* These give "real" coordinates z', $\overline{z'}$, u, v with $z_n = u + i\,v$. We rather use coordinates z', $\overline{z'}$, ρ_A, v, where

$$\rho_A = 2u - |z'|^2 - F_A(z', \overline{z'}, v),$$

so that $N(A)$ is given by the equation $\rho_A = 0$.

In general, Moser's normal form of a surface M is not unique; M has a unique normal form if and only if M is locally equivalent to a sphere, in which case the normal form is given by

$$M_0 = \partial\Omega_0 = \{2u = |z'|^2\}, \quad \text{where } \Omega_0 = \{2u > |z'|^2\}.$$

The model domain Ω_0 is a Siegel domain which is biholomorphic to a ball. Elements of $\mathrm{Aut}(\Omega_0)$, the group of holomorphic automorphisms of Ω_0, are linear fractional transformations. The non-uniqueness of the normal form is measured by using the *isotropy group* H of $\mathrm{Aut}(\Omega_0)$ at the origin 0 defined by $H = \{h \in \mathrm{Aut}(\Omega_0);\ h(0) = 0\}$; elements of H are biholomorphic at 0. In fact, there is a group action

$$(1.4) \qquad H \times \mathcal{N} \ni (h, A) \mapsto h.A \in \mathcal{N}$$

such that equivalence classes of $\mathcal{N}$ are realized by H-orbits of $\mathcal{N}$. The action (1.4) is defined by $N(h.A) = M$ with $M = h(N(A))$ when M is in Moser's normal form. In general, M is merely in pre-normal form, but Theorem 1.1 guarantees the unique existence of a local biholomorphic mapping Φ such that $\Phi(M)$ is close to M and in Moser's normal form. Then the action (1.4) is defined by $N(h.A) = \Phi(M)$. That is,

$$(1.4)' \qquad N(h.A) = E_{h,A}(N(A)), \quad \text{where} \quad E_{h,A} = \Phi \circ h.$$

Observe that $E'_{h,A}(0) = h'(0)$.

Let us finally give remarks on the case where the original real hypersurface $M \subset \mathbb{C}^n$, being strictly pseudoconvex, is not real analytic but merely C^∞. In the category of formal power series, the notions of pre-normal form and Moser's normal form make sense. After a formal change of variables, M can be always put in pre-normal form, and Theorem 1.1 has an obvious analogue. We continue to use the notations $N(A)$ and $A \in \mathcal{N}$. (We have a larger class $\mathcal{N}^\infty \supsetneqq \mathcal{N}$ but abuse notation by writing both $\mathcal{N}$ and $\mathcal{N}^\infty$ as $\mathcal{N}$.) Then the action (1.4) remains well-defined.

Remark 1.1. Let a surface M with a reference point $p \in M$ be real analytic or C^∞. Then by Theorem 1.1, there exists a (formal) biholomorphic mapping Φ_p such that $\Phi_p(p) = 0$ and $\Phi_p(M) = N(A)$ for some $A = (A^\ell_{\alpha\overline{\beta}}) \in \mathcal{N}$. We now regard each $A^\ell_{\alpha\overline{\beta}}$ as a function of $p \in M$. Then a family $\{\Phi_p\}_{p \in M}$ can be chosen in such a way that $A^\ell_{\alpha\overline{\beta}}$ is real analytic or C^∞. This fact is contained in the proof of Theorem 1.1.

(C) Local scalar invariants. Given a surface M with a reference point $p \in M$, local scalar invariants of M at p are defined as follows. For $A = (A^\ell_{\alpha\overline{\beta}}) \in \mathcal{N}$, we regard components $A^\ell_{\alpha\overline{\beta}}$ as variables and consider functions of A.

DEFINITION. A polynomial $P(A)$ in $A \in \mathcal{N}$ is called a *CR invariant of weight* $w \in \mathbb{N}_0$ if

$$P(A) = |\det h'(0)|^{2w/(n+1)}\, P(h.A) \qquad \text{for any} \quad h \in H. \tag{1.5}$$

We denote by I^{CR}_w the totality of CR invariants of weight w, and thus I^{CR}_w is the complexification of a real vector space.

Each $P(A) \in I^{\mathrm{CR}}_w$ determines a functional $M \mapsto P_M$ defined by

$$P_M(p) = |\det \Phi_p'(p)|^{2w/(n+1)}\, P(A) \quad \text{with} \quad \Phi_p(M) = N(A),$$

where Φ_p is a mapping in Remark 1.1. The function P_M is real analytic or C^∞ according to the regularity assumption on M, and the value $P_M(p)$ is independent of the choice of Φ_p. We have a transformation law under biholomorphic mappings Φ:

$$P_M(p) = |\det \Phi'(p)|^{2w/(n+1)}\, P_{\Phi(M)}(\Phi(p)) \qquad (p \in M).$$

Conversely, given a functional $P_M(p)$ of a pair (M, p) satisfying the law above, if $P_{N(A)}(0)$ is a polynomial in $A \in \mathcal{N}$ then $P_{N(A)}(0) \in I^{\mathrm{CR}}_w$.

Every $P(A) \in I_w^{\mathrm{CR}}$ is a polynomial in $A \in \mathcal{N}$ of homogeneous weight w, if we define the *weight* of $A^\ell_{\alpha\overline{\beta}}$ by

$$\mathrm{w}(A^\ell_{\alpha\overline{\beta}}) = \mathrm{w}(\alpha\overline{\beta}\ell) = (|\alpha| + |\beta|)/2 + \ell - 1.$$

This fact is seen by using dilations $\phi_r \in H$ defined by $\phi_r(z', z_n) = (rz', r^2 z_n)$ for $r > 0$. We have $P(A) = r^{2w} P(\phi_r.A)$, while the action $\phi_r.A = \widetilde{A}$ is given by $\widetilde{A}^\ell_{\alpha\overline{\beta}} = r^{-|\alpha|-|\beta|-2\ell+2} A^\ell_{\alpha\overline{\beta}}$.

1.2 The Bergman kernel

For a general domain $\Omega \subset \mathbb{C}^n$, we denote by $H^{\mathrm{B}}(\Omega)$ the Hilbert space of L^2 holomorphic functions in Ω with the norm $\|\cdot\|_{\mathrm{B}}$. Then the *Bergman kernel* associated with Ω is defined by

$$K^{\mathrm{B}}(z) = K^{\mathrm{B}}(z, \overline{z}) = \sum_j |h_j(z)|^2 \quad \text{for} \quad z \in \Omega,$$

where $\{h_j\}_j$ is an arbitrary complete orthonormal system of $H^{\mathrm{B}}(\Omega)$. The series $\sum |h_j(z)|^2$ converges uniformly on every compact subset ω of Ω, by virtue of the following inequality with a constant $C_\omega > 0$:

$$|h(z)| \le C_\omega \|h\|_{\mathrm{B}} \quad \text{for} \quad z \in \omega, \quad h \in H^{\mathrm{B}}(\Omega).$$

(In fact, $\sum |h_j(z)|^2$ is the square of the norm of the evaluation functional $h \mapsto h(z)$ on $H^{\mathrm{B}}(\Omega)$.) Thus, a complex extension of $K^{\mathrm{B}}(z) = K^{\mathrm{B}}(z, \overline{z})$ is given by

$$K^{\mathrm{B}}(z, \overline{w}) = \sum_j h_j(z)\, \overline{h_j(w)} \quad \text{for} \quad z, w \in \Omega,$$

which is holomorphic in $(z, \overline{w})$. This function $K^{\mathrm{B}}(z, \overline{w})$, which is also referred to as the *Bergman kernel*, is the *reproducing kernel* associated with the Hilbert space $H^{\mathrm{B}}(\Omega)$ in the sense that

$$K^{\mathrm{B}}(\,\cdot\,, \overline{w}) \in H^{\mathrm{B}}(\Omega) \quad \text{for} \quad w \in \Omega \ \text{ fixed},$$

$$\overline{K^{\mathrm{B}}(z, \overline{w})} = K^{\mathrm{B}}(w, \overline{z}) \quad \text{for} \quad z, w \in \Omega,$$

$$h(z) = \int_\Omega K^{\mathrm{B}}(z, \overline{w})\, h(w)\, dV(w) \quad \text{for} \quad h \in H^{\mathrm{B}}(\Omega), \ z \in \Omega,$$

where $dV(w)$ denotes the standard volume element of $\mathbb{C}^n$ at w.

When we wish to emphasize the dependence on Ω, we write $K^{\mathrm{B}}(z,\overline{w})$ as $K^{\mathrm{B}}_{\Omega}(z,\overline{w})$. Recall that each element $h \in H^{\mathrm{B}}(\Omega)$ is identified with a holomorphic n-form $\omega_h(z) = h(z)\, dz_1 \wedge \cdots \wedge dz_n$, and

$$\frac{i^{n^2}}{2^n} \int_{\Omega} \omega_h \wedge \overline{\omega_h} = \|h\|^2_{\mathrm{B}} < +\infty.$$

Thus the Bergman kernel $K^{\mathrm{B}}_{\Omega}(z,\overline{w})$ is defined for a complex manifold Ω. (This fact will not be used explicitly, since we shall mainly work locally near a boundary point.) Also, the *transformation law* for the Bergman kernel under a biholomorphic mapping $\Phi : \Omega_1 \to \Omega_2$ is given as follows:

$$(1.6) \qquad K^{\mathrm{B}}_{\Omega_1}(z,\overline{z}) = K^{\mathrm{B}}_{\Omega_2}(\Phi(z), \overline{\Phi(z)}\,)\, |\det \Phi'(z)|^2 \quad \text{for } \ z \in \Omega_1,$$

a relation which can be complexified.

Example. If $\Omega \subset \mathbb{C}^n$ is the unit ball, then

$$K^{\mathrm{B}}(z,\overline{w}) = \frac{n!/\pi^n}{(1 - z\cdot\overline{w})^{n+1}}\,, \quad \text{where} \quad z\cdot\overline{w} = \sum_{j=1}^{n} z_j\,\overline{w}_j\,.$$

For our model domain $\Omega_0 = \bigl\{ z = (z', z_n) \in \mathbb{C}^n;\ z_n + \overline{z}_n > |z'|^2 \bigr\}$,

$$(1.7) \qquad K^{\mathrm{B}}_{\Omega_0}(z,\overline{w}) = \frac{n!}{\pi^n} \left(z_n + \overline{w}_n - z'\cdot\overline{w'}\,\right)^{-n-1}.$$

Remark 1.2. (1°) If Ω is a domain in $\mathbb{C}$, then

$$K^{\mathrm{B}}(z,\overline{w}) = -\frac{\partial^2 G(z,w)}{\partial z\, \partial\overline{w}} \qquad \text{for } \ z, w \in \Omega,$$

where $G(z,w)$ denotes the Green function normalized by multiplying a constant (cf. Schiffer [Scr]). An operator version is given by using the $\overline{\partial}$-operator and its L^2 adjoint $\overline{\partial}^*$ as $\mathbf{K}^{\mathrm{B}} = 1 - \overline{\partial}^* \mathbf{G}\, \overline{\partial}$, where $\mathbf{G}$ denotes the Green operator and $\mathbf{K}^{\mathrm{B}}$, called the *Bergman projector*, stands for the orthogonal projector of $L^2(\Omega)$ to the closed subspace $H^{\mathrm{B}}(\Omega)$.

(2°) An analogous formula is available for a domain Ω in $\mathbb{C}^n$ as far as the complex Laplacian $\Box = \overline{\partial}\,\overline{\partial}^* + \overline{\partial}^*\overline{\partial}$ for $(0,1)$-forms on Ω has a closed range in L^2. The generalized inverse $\mathbf{N}$, called the *$\overline{\partial}$-Neumann operator*, satisfies $\mathbf{K}^{\mathrm{B}} = 1 - \overline{\partial}^* \mathbf{N}\, \overline{\partial}$. If, for instance, Ω is a strictly pseudoconvex domain with C^∞ boundary, then $\mathbf{N}$ is defined and C^∞ pseudolocal at every point of the closure $\overline{\Omega}$ (cf. Folland-Kohn [FK]). Then, the Bergman kernel $K^{\mathrm{B}}(z,\overline{w})$ as a function of (z,w) is C^∞ on $\overline{\Omega}\times\overline{\Omega}$ off the diagonal of $\partial\Omega\times\partial\Omega$ (cf. Kerzman [Ke]).

From now on, we assume that $\Omega=\{z;\ r(z)>0\}\subset\mathbb{C}^n$ is a strictly pseudoconvex domain, where r is a smooth (C^∞ or real analytic) defining function of the boundary. It has been known that the Bergman kernel $K^{\mathrm{B}}(z)=K^{\mathrm{B}}(z,\overline{z})$ tends to $+\infty$ as z approaches to a boundary point. The magnitude of divergence is measured by virtue of a theorem of Hörmander [Hö] as follows:

$$\lim_{z\to p} r(z)^{n+1}\,K^{\mathrm{B}}(z)=\frac{n!}{\pi^n}\,J[r](p)\quad\text{for}\ p\in\partial\Omega, \tag{1.8}$$

where $J[r]$ denotes the *Levi determinant* of r given by

$$J[r]=(-1)^n\det\begin{pmatrix} r & \partial r/\partial\overline{z}_k\\ \partial r/\partial z_j & \partial^2 r/\partial z_j\,\partial\overline{z}_k\end{pmatrix}. \tag{1.9}$$

We shall rather refer to $J[\,\cdot\,]$ as the (complex) *Monge-Ampère operator.* A far-reaching refinement of (1.8) is given as follows.

Theorem 1.2 ([F1]). *Let $\Omega=\{z\in\mathbb{C}^n;\ r(z)>0\}$ be a strictly pseudoconvex domain, where r is a C^∞ defining function of $\partial\Omega$. Then there exist $\varphi^{\mathrm{B}},\psi^{\mathrm{B}}\in C^\infty(\overline{\Omega})$ such that*

$$K^{\mathrm{B}}(z,\overline{z})=K^{\mathrm{B}}(z)=\frac{\varphi^{\mathrm{B}}(z)}{r(z)^{n+1}}+\psi^{\mathrm{B}}(z)\,\log r(z). \tag{1.10}$$

In particular, $\varphi^{\mathrm{B}}=(n!/\pi^n)J[r]$ on $\partial\Omega$.

Remark 1.3. If $\partial\Omega$ with r is real analytic, then φ^{B} and ψ^{B} are real analytic too, so that (1.10) is complexified (cf. Kashiwara [Kas]):

$$K^{\mathrm{B}}(z,\overline{w})=\frac{\varphi^{\mathrm{B}}(z,\overline{w})}{r(z,\overline{w})^{n+1}}+\psi^{\mathrm{B}}(z,\overline{w})\,\log r(z,\overline{w}). \tag{1.10$'$}$$

Even when $\partial\Omega$ is C^∞, the above equality (1.10)$'$ remains valid with C^∞ functions $r(z,\overline{w})$, $\varphi^{\mathrm{B}}(z,\overline{w})$, $\psi^{\mathrm{B}}(z,\overline{w})$ of $(z,w)\in\overline{\Omega}\times\overline{\Omega}$ which are regarded as almost analytic extensions of $r(z)=r(z,\overline{z})$, ... in the sense that $\partial r(z,\overline{w})/\partial\overline{z}$, ... and $\partial r(z,\overline{w})/\partial w$, ... vanish to infinite order at $z=w$ (cf. Boutet de Monvel-Sjöstrand [BS]).

Remark 1.4. The singularities (1.10) and (1.10)$'$ are localizable to a neighborhood of a boundary point as follows. If Ω_1 and Ω_2 are strictly pseudoconvex domains with smooth (C^∞ or real analytic) boundaries such that $\overline{\Omega}_1 \cap V = \overline{\Omega}_2 \cap V$ for a neighborhood V of a point $p \in \partial\Omega_1 \cap \partial\Omega_2$, then there exists a smaller neighborhood V_0 of p such that the difference $K^{\mathrm{B}}_{\Omega_1}(z,\overline{w}) - K^{\mathrm{B}}_{\Omega_2}(z,\overline{w})$ are smooth for $z, w \in \overline{\Omega}_1 \cap V_0 = \overline{\Omega}_2 \cap V_0$.

Remark 1.5. (1°) An elementary property of the Bergman kernel is the monotonicity with respect to the domain:

$$K^{\mathrm{B}}_{\Omega_1}(z) \geq K^{\mathrm{B}}_{\Omega_2}(z) \quad \text{when} \quad z \in \Omega_1 \subset \Omega_2.$$

In the proof of (1.8), this fact and the model case formula (1.7) are used together with a localization argument, after a scaling of the coordinates (cf. Hörmander [Hö]).

(2°) Fefferman's original proof of Theorem 1.2 requires a more precise approximation of Ω from inside at a boundary point by a domain Ω_{ball} which is locally biholomorphic to a ball. Roughly speaking, starting from an explicit approximation of the decomposition $1 = \mathbf{K}^{\mathrm{B}} + \overline{\partial}^* \mathbf{N}\, \overline{\partial}$, the Bergman kernel is obtained as a Neumann series, where successive integrations over a thin domain given locally by $\Omega \setminus \Omega_{\mathrm{ball}}$ are involved. The estimates are extremely hard (cf. [F1]).

(3°) An alternative proof of Theorem 1.2 is given by Boutet de Monvel and Sjöstrand [BS], where the singularity of the Bergman kernel is written as a Fourier integral distribution with complex phase:

$$K^{\mathrm{B}}(z,\overline{w}) \sim \int_0^\infty e^{-t\, r(z,\overline{w})}\, p^{\mathrm{B}}(z,\overline{w},t)\, dt \qquad \mathrm{mod}\ C^\infty,$$

where $p^{\mathrm{B}}(z,\overline{w},t)$ is a symbol admitting an asymptotic expansion

$$p^{\mathrm{B}}(z,\overline{w},t) \sim \sum_{j=0}^{\infty} t^{n-j} p^{\mathrm{B}}_j(z,\overline{w}), \qquad p^{\mathrm{B}}_j(\,\cdot\,,\overline{\cdot}\,) \in C^\infty(\overline{\Omega} \times \overline{\Omega}).$$

This expression yields $(1.10)'$ via the following formulas for the Laplace transforms, which are valid for $p \in \mathbb{C}$ with $\mathrm{Re}\, p > 0$:

$$\begin{aligned} \int_0^\infty t^m\, e^{-pt}\, dt &= \frac{m!}{p^{m+1}} && \text{for } m \geq 0, \\ \mathrm{pf} \int_0^\infty t^{-m}\, e^{-pt}\, dt &= \frac{(-1)^m\, p^{m-1}}{(m-1)!}\, (\log p + C_m) && \text{for } m \geq 1, \end{aligned}$$

where C_m are constants and pf stands for the Hadamard finite part.

(4°) For Kashiwara's proof [Kas] of (1.10) in the real analytic case and its application, see Section 4 below.

The equality (1.10) in Theorem 1.2 is referred to as an asymptotic expansion. A reason is that if the boundary $\partial\Omega$ is locally flattened by a real change of coordinates $z = \Psi(s, r)$ with $s \in \mathbb{R}^{2n-1}$ then

$$K^{\mathrm{B}}(\Psi(s,r)) = \frac{\varphi^{\mathrm{B}}(\Psi(s,r))}{r^{n+1}} + \psi^{\mathrm{B}}(\Psi(s,r)) \log r,$$

and the Taylor expansions about $r = 0$ of $\varphi^{\mathrm{B}}(\Psi(s,r))$ modulo $O^{n+1}(r)$ and $\psi^{\mathrm{B}}(\Psi(s,r))$ provide an asymptotic expansion of $K^{\mathrm{B}}(\Psi(s,r))$. This is analogous to that of the heat kernel. However, the biholomorphic invariance is lost, for the expansion depends on the choices of the real coordinate system (s, r) and the defining function r. Instead, we make the following tentative definition.

DEFINITION. A domain functional $K(z) = K_\Omega(z)$ is said to satisfy a (biholomorphic) transformation law of *weight* $w \in \mathbb{Z}$ if

$$K_{\Omega_1}(z) = K_{\Omega_2}(\Phi(z))\,|\det \Phi'(z)|^{2w/(n+1)} \tag{1.11}$$

for any biholomorphic mapping $\Phi \colon \Omega_1 \to \Omega_2$. This definition extends to *local domain functionals* defined only near a boundary point.

The equality (1.6) means that the Bergman kernel satisfies a transformation law of weight $n+1$. If there would exist a defining function r satisfying a transformation law of weight -1, then we could speak of an invariant expansion of the Bergman kernel given by the expansions

$$\begin{aligned} \varphi^{\mathrm{B}} &= \sum_{j=0}^{n} \varphi_j\, r^j \qquad\quad \mod O^{n+1}(r), \\ \psi^{\mathrm{B}} &= \sum_{j=0}^{\infty} \varphi_{n+1+j}\, r^j \quad \mod O^{\infty}(r), \end{aligned} \tag{1.12}$$

with $\varphi_j \in C^\infty(\overline{\Omega})$ for $j \in \mathbb{N}_0$ satisfying transformation laws of weight j. Here, the first relation in (1.12) means that the difference between both sides is smoothly divisible by r^{n+1}, and the second relation means that

$$\psi^{\mathrm{B}} = \sum_{j=0}^{m} \varphi_{n+1+j}\, r^j \mod O^{m+1}(r) \qquad \text{for any } \; m \in \mathbb{N}.$$

In fact, the situation is not so simple. Nevertheless, this is approximately the case, as we shall see in the next subsection.

1.3 The Monge-Ampère boundary value problem

Recall the (complex) Monge-Ampère operator $J[\,\cdot\,]$ defined in (1.9). If $\Phi : \Omega_1 \to \Omega_2$ is a biholomorphic mapping, then

$$J[u_1] = J[u_2] \circ \Phi \quad \text{with} \quad u_1 = |\det \Phi'|^{-2/(n+1)}\, u_2 \circ \Phi$$

for any function u_2 in Ω_2 (cf. Fefferman [F2]). In particular, every solution u of the Monge-Ampère equation $J[u] = 1$ satisfies a transformation law of weight -1 in the sense of (1.11). This fact motivates us to consider the zero Dirichlet boundary value problem

$$(1.13) \qquad J[u^{\mathrm{MA}}] = 1 \;\text{ and }\; u^{\mathrm{MA}} > 0 \;\text{ in }\; \Omega; \qquad u^{\mathrm{MA}} = 0 \;\text{ on }\; \partial\Omega.$$

The problem (1.13) has a unique solution but it has only a finite degree of smoothness up to the boundary (cf. Cheng-Yau [CY]):

$$(1.14) \qquad u^{\mathrm{MA}} \in C^\infty(\Omega) \cap C^{n+3/2-\varepsilon}(\overline{\Omega}) \qquad \text{for any } \varepsilon > 0.$$

The solution u^{MA} admits an asymptotic expansion, with an arbitrary defining function r of $\partial\Omega$ such that $\Omega = \{r > 0\}$ (cf. Lee-Melrose [LM]):

$$(1.15) \qquad u^{\mathrm{MA}} \sim r \sum_{k=0}^{\infty} \eta_k \cdot (r^{n+1} \log r)^k, \qquad \eta_k \in C^\infty(\overline{\Omega}).$$

In particular, (1.14) is improved as follows: $u^{\mathrm{MA}} \in C^{n+2-\varepsilon}(\overline{\Omega})$ for any $\varepsilon > 0$ small. In the expansion (1.15) considered near a reference point at the boundary, the function η_0 depends globally on the choice of r, whereas the Taylor expansions of η_k for $k \geq 1$ are determined locally by those of η_0 and r (cf. [LM]).

Though the solution u^{MA} of (1.13) is a defining function of $\partial\Omega$ and satisfies a transformation law of weight -1, it is not C^∞ smooth up to the boundary. Thus we cannot use u^{MA} in an invariant expansion of the Bergman kernel of the form (1.12). Instead, we confine ourselves to a C^∞ defining function $r = r^{\mathrm{F}}$ of $\partial\Omega$ satisfying (1.13) approximately in the sense that

$$(1.16) \qquad J[r^{\mathrm{F}}] = 1 + O^{n+1}(r) \;\text{ near }\; \partial\Omega \qquad (r = r^{\mathrm{F}} > 0 \;\text{ in }\; \Omega).$$

Fefferman [F2] considered r^{F} precedent to the above stated works of Cheng-Yau [CY] and Lee-Melrose [LM]. In [F2], an explicit algorithm of constructing r^{F} is given locally near a boundary point (cf. Subsection 3.2 below). We refer to r^{F} as a *Fefferman's defining function.* For later use, we summarize properties of r^{F}:

(1^{F}) r^{F} is unique modulo $O^{n+2}(r)$, or the ambiguity of r^{F} is $O^{n+2}(r)$;

(2^{F}) r^{F} satisfies a transformation law of weight -1 modulo $O^{n+2}(r)$;
(3^{F}) r^{F} makes sense locally near a reference point at the boundary.

By (1^{F}), we mean that if r_1^{F} and r_2^{F} satisfy (1.16) then $r_1^{\mathrm{F}} - r_2^{\mathrm{F}} = O^{n+2}(r)$ and that if r_1^{F} satisfies (1.16) so does $r_2^{\mathrm{F}} = r_1^{\mathrm{F}} + \delta$ whenever $\delta = O^{n+2}(r)$. The fact (1^{F}) follows from the condition (1.16); and (1^{F}) implies (2^{F}), because if $\Phi : \Omega_1 \to \Omega_2$ is biholomorphic then

$$J[r_1^{\mathrm{F}}] = J[r_2^{\mathrm{F}}] \circ \Phi \quad \text{with } r_1^{\mathrm{F}} = |\det \Phi'|^{-2/(n+1)}\, r_2^{\mathrm{F}} \circ \Phi$$

for any Fefferman's defining function r_2^{F} of Ω_2. By (3^{F}), we mean that the properties (1^{F}) and (2^{F}) are valid locally near a reference point at the boundary.

By continuing Fefferman's construction beyond r^{F}, Graham [G2] constructed a local asymptotic solution u^{G} of (1.13) in the form

$$u^{\mathrm{G}} = r \sum_{k=0}^{\infty} \eta_k^{\mathrm{G}} \cdot (r^{n+1} \log r)^k, \qquad \eta_k^{\mathrm{G}} \in C^\infty(\overline{\Omega}). \tag{1.17}$$

Theorem 1.3 ([G2]). *Let $r = r^{\mathrm{F}}$ be a Fefferman's defining function of Ω. Then, for any $a \in C^\infty(\partial\Omega)$, there exists a unique asymptotic solution $u = u^{\mathrm{G}}$ of the form* (1.17) *to the problem*

$$J[u] = 1 + O^\infty(r) \quad \textit{near } \partial\Omega, \quad \eta_0^{\mathrm{G}} = 1 + a\, r^{n+1} + O^{n+2}(r). \tag{1.18}$$

Furthermore, η_k^{G} for each $k \geq 1$ has the following properties:

(1^{G}) *η_k^{G} modulo $O^{n+1}(r)$ is independent of the choice of a and r^{F};*
(2^{G}) *η_k^{G} has a transformation law of weight $k(n+1)$ modulo $O^{n+1}(r)$;*
(3^{G}) *η_k^{G} modulo $O^{n+1}(r)$ makes sense locally near a boundary point.*

The asymptotic solution u^{G} is a formal series of the form (1.17). The first relation of (1.18) means that $J[u^{\mathrm{G}}] - 1$ is formally flat on $\partial\Omega$ in the sense that for any $m \in \mathbb{N}$ there exists a finite sum u_m^{G} corresponding to (1.17) such that $J[u_m^{\mathrm{G}}] - 1$ is continuously divisible by r^m. The meanings of (1^{G})–(3^{G}) are similar to those of (1^{F})–(3^{F}), except for the fact that u^{G} is uniquely determined by a and r^{F}, where a is prescribed in a neighborhood of a reference point at the boundary.

Let us return to the problem mentioned at the end of the previous subsection. We wish to realize an invariant expansion of the Bergman kernel of the form (1.12) with $r = r^{\mathrm{F}}$. Because of the ambiguity of r^{F}, the invariance becomes approximate and the expansion of ψ^{B}, even if possible, only makes sense as a finite sum, say,

$$(1.12)'_N \qquad \psi^{\mathrm{B}} = \varphi_{n+1} + \varphi_{n+2}\, r + \cdots + \varphi_N\, r^{N-n-1} \mod O^{N-n}(r).$$

Suppose we are given vector subspaces $I_j^{\mathrm{W}} \subset C^\infty(\overline{\Omega})$ $(0 \le j \le N)$ with the following properties:

- (1^{W}) Elements of I_j^{W} make sense modulo $O^{N-j+1}(r)$ (we regard these as equivalence classes modulo $O^{N-j+1}(r)$);
- (2^{W}) Each element of I_j^{W} satisfies the transformation law of weight j modulo $O^{N-j+1}(r)$;
- (3^{W}) The boundary value of each element $\varphi \in I_j^{\mathrm{W}}$ is a CR invariant, and the resulting mapping $I_j^{\mathrm{W}} \to I_j^{\mathrm{CR}}$ is surjective. In addition, if the boundary $\partial\Omega$ is in normal form $N(A)$ near the origin, then $\partial_z^\alpha \partial_{\overline{z}}^\beta \varphi(0)$ $(|\alpha| + |\beta| \le N - j)$ for $\varphi \in I_j^{\mathrm{W}}$ are polynomials in A.

The latter condition in (3^{W}) is referred to as the *polynomial dependence* of $\varphi \in I_j^{\mathrm{W}}$ on the boundary. The functions φ^{B}, ψ^{B} and r^{F} have a similar property, as we shall see in Sections 3 and 4. If $N \ge n$, then the conditions (1^{W})–(3^{W}) yield the expansion of φ^{B} in (1.12) as follows. Since the boundary value of φ^{B} is an element of I_0^{CR} being a constant, (3^{W}) implies the existence of $\varphi_0 \in I_0^{\mathrm{W}}$ such that $\varphi^{\mathrm{B}} = \varphi_0 + O^1(r)$. Then the approximate invariance of the smooth function $\widetilde{\varphi}_1 := (\varphi^{\mathrm{B}} - \varphi_0)/r$ makes sense. By virtue of (1^{W})–(3^{W}) and the polynomial dependence of φ^{B} and r^{F}, the boundary value of $\widetilde{\varphi}_1$ belongs to I_1^{CR}, and thus (3^{W}) implies as before the existence of $\varphi_1 \in I_1^{\mathrm{W}}$ such that $\varphi^{\mathrm{B}} = \varphi_0 + \varphi_1\, r + O^2(r)$. Then induction yields the expansion of φ^{B} as in (1.12).

A construction of I_j^{W} for $0 \le j \le n$ is discussed in the next section.

The same argument applies to the expansion of ψ^{B} as in $(1.12)'_N$, but the approximate invariance of the right side of $(1.12)'_N$ only makes sense modulo $O^{n+1}(r)$ by the ambiguity of $r = r^{\mathrm{F}}$. Consequently, we have invariant expressions of φ_j for $0 \le j \le \min(N, 2n+1)$ whenever I_j^{W} for $0 \le j \le N$ are constructed. In Section 3, we consider the case $n = 2$ and realize the optimal case $N = 5$, that is, we express φ_j for $0 \le j \le 5$ explicitly by constructing I_j^{W} for $0 \le j \le 5$.

§2 Weyl invariants

Elements of the spaces I_j^{W} $(0 \le j \le n)$ in Subsection 1.3 are realized by Weyl invariants in the sense of Fefferman [F3]. This notion was introduced in [F3] as an analogy of that in Riemannian Geometry, where the Bergman kernel is compared with the heat kernel. Reviewing quickly the heat kernel asymptotics in Subsection 2.1, we give the definition of Weyl invariants in Subsection 2.2. Then in Subsection 2.3, we state the main results of this section, due to Fefferman [F3] and Bailey-Eastwood-Graham [BEG], on Weyl invariants and the invariant expansion of the Bergman kernel.

2.1 Heat kernel on a Riemannian manifold

Let (M, g) be an n-dimensional compact Riemannian manifold. We denote by Δ_g the (negative) Laplacian acting on functions on M, and consider the initial value problem for the heat equation:

$$\partial u/\partial t - \Delta_g u = 0 \quad \text{on} \quad M \times [0, \infty), \qquad u|_{t=0} = f,$$

where $f \in C^\infty(M)$ is prescribed arbitrarily. Then there exists a unique solution, which has the form $u(x,t) = \int_M H_t(x,y)\, f(y)\, dV(y)$, where dV stands for the volume element on M. The function $H_t(x,y)$ for $x, y \in M$ and $t > 0$ is called the *heat kernel* (for functions) associated with Δ_g. Let us consider the restriction $H_t(x,x)$ to the diagonal of $M \times M$. This is a smooth function as far as $t > 0$, but becomes singular as $t \to +0$. More precisely, the following asymptotic expansion holds:

$$H_t(x,x) \sim t^{-n/2} \sum_{m=0}^{\infty} a_m(x)\, t^m \quad \text{with } a_m \in C^\infty(M).$$

The coefficient functions a_m are determined locally by the metric g. In addition, these are Riemannian invariants defined as follows. Let us take a normal coordinate system $x = (x_1, \ldots, x_n)$ about a reference point $p \in M$. The choice of normal coordinate systems has freedom corresponding to the action of the isotropy group $O(n)$, and an action of $O(n)$ is induced on jets of the metric, $g_{jk,ab\cdots c} = \partial_{x_a}\partial_{x_b} \cdots \partial_{x_c} g_{jk}(p)$, where $\partial_{x_a} = \partial/\partial x_a$, etc. A universal polynomial $P_m = P_m(g_{jk,ab\cdots c})$ is called a (local) *Riemannian invariant* if it is invariant under this action of $O(n)$.

For the curvature tensor R of g, we consider its successive covariant derivatives and denote the components by $R_{ijkl,ab\cdots c}$. Then each $g_{jk,ab\cdots c}$ is a polynomial of $(R_{ijkl,ab\cdots c})$, and thus each Riemannian invariant is written as an $O(n)$-invariant polynomial of $(R_{ijkl,ab\cdots c})$, where $O(n)$ acts tensorially on $(R_{ijkl,ab\cdots c})$. According to Weyl's invariant theory, the vector space of all Riemannian invariants is generated by complete contractions of the form

$$\operatorname{contr}\left(\nabla^{p_1} R \otimes \cdots \otimes \nabla^{p_s} R\right),$$

where the contractions are taken over all indices. Consequently, each a_m in the heat kernel is expressed as a linear combination of these complete contractions such that $2m = p_1 + \cdots + p_s + s$. This equality is seen by scaling the metric.

2.2 Definition of Weyl invariants

CR invariants can be compared with Riemannian invariants with Moser's normal coordinates in place of Riemannian normal coordinates. A substitute for the Riemannian curvature is the curvature of the ambient metric, which is defined as follows.

Let $r = r^{\mathrm{F}}$ be a Fefferman's defining function of the domain Ω. Introducing an extra variable $z_0 \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$, we consider a function $r_{\#}(z_0, z) = |z_0|^2 r(z)$ on $\mathbb{C}^* \times \overline{\Omega}$. Then, a tensor of $(1,1)$-type

$$g = \sum_{j.k=0}^{n} g_{j\overline{k}}\, dz_j\, d\overline{z}_k = \sum_{j.k=0}^{n} \frac{\partial^2 r_{\#}}{\partial z_j\, \partial \overline{z}_k}\, dz_j\, d\overline{z}_k$$

defines a Lorentz-Kähler metric in a neighborhood of $\mathbb{C}^* \times \partial\Omega$. This metric g is called an *ambient metric* associated with $\partial\Omega$. Due to the ambiguity of $r = r^{\mathrm{F}}$ modulo $O^{n+2}(r)$, the ambient metric is well-defined only up to the n-th jets along $\mathbb{C}^* \times \partial\Omega$.

As in the Riemannian case, scalar invariants are constructed from the metric g as follows. For the curvature tensor R of g, we consider successive covariant derivatives $R^{(p,q)} = \overline{\nabla}^{q-2} \nabla^{p-2} R$ and complete contractions of the form

$$W_{\#} = \mathrm{contr}\left(R^{(p_1,q_1)} \otimes \cdots \otimes R^{(p_s,q_s)}\right). \tag{2.1}$$

These are functions in a neighborhood of $\mathbb{C}^* \times \partial\Omega \subset \mathbb{C}^* \times \overline{\Omega}$, and the restrictions $W = W_{\#}|_{z_0=1}$ are defined near $\partial\Omega$. The *weight* of $W_{\#}$ in (2.1) is defined by $w = \sum_{j=1}^{s} (p_j + q_j)/2 - s$.

DEFINITION. A *Weyl invariant* of *weight* w is a linear combination of complete contractions of the form (2.1) of the weight w.

By definition, a Weyl invariant $W_{\#}$ is a functional of r. Nevertheless, we also use this terminology for the composite function $(z_0, z) \mapsto W_{\#}$ or the equivalence class modulo the ambiguity of $r = r^{\mathrm{F}}$. For a Weyl invariant $W_{\#}$ of weight w, we set $W = W_{\#}|_{z_0=1}$. Then

$$W_{\#}(z_0, z) = |z_0|^{-2w}\, W(z).$$

Accordingly, we still call W a *Weyl invariant* of weight w.

Let us recall that r^{F} is a local domain functional of weight -1 in the sense of (1.12) but with error of $O^{n+2}(r)$. Likewise, Weyl invariants W of weight w are local domain functionals of weight w with some error. The argument involving the error is somewhat technical, and we postpone it until the next subsection. Instead, we give here a transformation law under a biholomorphic mapping $\Phi\colon \Omega_1 \to \Omega_2$ for representatives of Weyl invariants defined by a Fefferman's defining function r_2 of Ω_2 and its pull-back $r_1 = |\det \Phi'|^{-2/(n+1)}\, r_2 \circ \Phi$ to Ω_1. To emphasize the dependence on $r = r^{\mathrm{F}}$, we write $g = g[r]$, $W_\# = W_\#[r]$, $W = W[r]$. Then

$$W[r_1] = |\det \Phi'|^{2w/(n+1)} W[r_2] \circ \Phi. \tag{2.2}$$

This is seen as follows. We lift Φ to a bundle map $\Phi_\# : \mathbb{C}^* \times \Omega_1 \to \mathbb{C}^* \times \Omega_2$ defined by

$$\Phi_\#(z_0, z) = (z_0 \cdot (\det \Phi'(z))^{-1/(n+1)}, \Phi(z)). \tag{2.3}$$

Then $(r_1)_\# = (r_2)_\# \circ \Phi_\#$, and $\Phi_\#$ is an isometry with respect to the metrics $g[r_1]$ and $g[r_2]$. Thus $W_\#[r_2] \circ \Phi_\# = W_\#[r_1]$, which implies (2.2).

2.3 Results of Fefferman and Bailey-Eastwood-Graham

We begin with a consideration of the dependence on the choice of Fefferman's defining function.

Proposition 2.1. *If $W[r]$ is a Weyl invariant of weight $w \le n$, then $W[r]$ modulo $O^{n-w+1}(r)$ is independent of the choice of $r = r^{\mathrm{F}}$.*

The proof of Proposition 2.1 is done by using Moser's normal coordinates. If the boundary $\partial\Omega$ is locally in Moser's normal form $N(A)$, then $W[r]$ is written in terms of the coordinate system $(z', \overline{z'}, \rho_A, v)$ as

$$W[r] = \sum_{m=0}^{n-w} \sum_{\alpha,\beta,\ell} P^{\ell m}_{\alpha\overline{\beta}}(A)\, z'_\alpha\, \overline{z'_\beta}\, v^\ell \rho_A^m + O^{n-w+1}(\rho_A),$$

where $P^{\ell m}_{\alpha\overline{\beta}}(A)$ are polynomials in A (cf. the statement (#) in Subsection 3.2, (B) below.) The desired result then follows, since the main part of the expression of $W[r]$ above is independent of the choice of $r = r^{\mathrm{F}}$.

By Proposition 2.1 above and (2.2) in the previous subsection, we have an approximate transformation law corresponding to (2.2), but for arbitrary Fefferman's defining functions $r_j = r_j^{\mathrm{F}}$ of Ω_j $(j = 1, 2)$:

$$W[r_1] = |\det \Phi'|^{2w/(n+1)} W[r_2] \circ \Phi \mod O^{n-w+1}(r_1). \tag{2.4}$$

In particular, the boundary value of a Weyl invariant of weight $w \leq n$ gives a CR invariant of weight w. The converse is the first main result of this section.

Theorem 2.1 ([F3], [BEG]). *Every CR invariant of weight $\leq n$ is given by the boundary value of a Weyl invariant.*

The statement of Theorem 2.1 was first proved by Fefferman [F3] for CR invariants of weight $\leq n-19$. The weight restriction was removed recently by Bailey-Eastwood-Graham [BEG]. We outline the proof of Theorem 2.1 in Section 5.

Let I_w^{W} denote the totality of Weyl invariants of weight w. By virtue of Proposition 2.1 and Theorem 2.1 above, the spaces I_w^{W} for $0 \leq w \leq n$ satisfy the conditions (1^{W}), (2^{W}) and (3^{W}) in Subsection 1.3 with $N = n$. Consequently, the argument given there is valid, and we have:

Theorem 2.2 ([F3],[BEG]). *For φ^{B} in the expression* (1.10) *of the Bergman kernel, the following expansion holds:*

$$\varphi^{\mathrm{B}} = \sum_{k=0}^{n} W_k\, r^k + O^{n+1}(r) \quad \textit{with} \quad W_k \in I_w^{\mathrm{W}}.$$

§3 Explicit computation in the two dimensional case

For domains in $\mathbb{C}^2$, it is possible to refine Theorems 2.1 and 2.2, as we mentioned at the end of Section 1. We also get explicit results, which are stated in Subsection 3.1. These results are obtained with the aid of asymptotic calculi of the Monge-Ampère equation and the Bergman kernel, where explicit algorithms are necessary. Postponing the calculus of the Bergman kernel until the next section, we discuss that of the Monge-Ampère equation in Subsection 3.2.

3.1 The two dimensional case

(A) Results. Consider for a domain Ω in $\mathbb{C}^2$ the approximate invariant expansions of φ^{B} and ψ^{B} expressing the singularity of the Bergman kernel

$$K^{\mathrm{B}} = \varphi^{\mathrm{B}}\, r^{-3} + \psi^{\mathrm{B}} \log r \quad \text{with} \quad r = r^{\mathrm{F}}$$

in terms of Fefferman's defining function r^{F}. To write down explicit results, it is convenient to normalize φ^{B} and ψ^{B} by writing

$$K^{\mathrm{B}} = \frac{2}{\pi^2}\left(\widetilde{\varphi}^{\mathrm{B}}\, r^{-3} + \widetilde{\psi}^{\mathrm{B}} \log r\right),$$

so that $\widetilde{\varphi}^{\mathrm{B}} = 1$ on $\partial\Omega$, cf. (1.8). As we shall see below, we can completely determine $\widetilde{\varphi}^{\mathrm{B}}$ and $\widetilde{\psi}^{\mathrm{B}}$ both modulo $O^3(r)$. The results are optimal and better than those in the higher dimensional case.

As before, let M be a portion of $\partial\Omega$, and assume that M is in Moser's normal form $N(A)$, where $A = (A^{\ell}_{\alpha\overline{\beta}}) \in \mathcal{N}$. Changing notation slightly, we write $A^{\ell}_{p\overline{q}}$ in place of $A^{\ell}_{\alpha\overline{\beta}}$ with $|\alpha| = p$ and $|\beta| = q$, since $(\alpha, \beta) \mapsto (p, q)$ is bijective. Then the trace conditions on A take the form

$$A_{2\overline{2}}(v) = A_{2\overline{3}}(v) = A_{3\overline{3}}(v) = 0,$$

so that $A^{\ell}_{p\overline{q}} = 0$ for $p + q + \ell \leq 5$. That is, M can be approximated by a sphere to order 5, though in the higher dimensional case the third order approximation is optimal. By this fact, the two dimensional case is exceptional in the sense that the Weyl invariants are less ambiguous (cf. Lemma 3.3 and Remark 3.2 below).

To state the main results of this section, we begin by presenting bases of the vector spaces I^{CR}_w of CR invariants of weight $w \leq 5$.

Lemma 3.1. *$I^{\mathrm{CR}}_1 = I^{\mathrm{CR}}_2 = \{0\}$ and*

$$\dim I^{\mathrm{CR}}_3 = \dim I^{\mathrm{CR}}_4 = 1, \qquad \dim I^{\mathrm{CR}}_5 = 2.$$

The spaces I^{CR}_3 and I^{CR}_4 are generated by $A^0_{4\overline{4}}$ and $|A^0_{2\overline{4}}|^2$, respectively. The space I^{CR}_5 is spanned by $F^{\mathrm{CR}}_5(1,0)$ and $F^{\mathrm{CR}}_5(0,1)$, where

$$F^{\mathrm{CR}}_5(a,b) = F\big(a, b, -2a + (10/9)\, b, -a + b/3\big)$$

with $F(a,b,c,d) = a\,|A^0_{5\overline{2}}|^2 + b\,|A^0_{4\overline{3}}|^2 + \mathrm{Re}\,\big\{\big(c\,A^0_{3\overline{5}} - i\,d\,A^1_{2\overline{4}}\big)\,A^0_{4\overline{2}}\big\}$.

For the proof, see [G1] for $w \leq 4$ and [HKN2] for $w = 5$.

As a consequence of Lemma 3.1, the expansion of $\widetilde{\varphi}^{\mathrm{B}}$ is trivial:

$$\widetilde{\varphi}^{\mathrm{B}} = \text{constant} + O^3(r) \qquad (\text{constant} = 1). \tag{3.1}$$

To proceed further, it is necessary to extend $A^0_{4\overline{4}} \in I^{\mathrm{CR}}_3$ approximate invariantly to the domain Ω. This is done by using the first coefficient function η^{G}_1 of the asymptotic series u^{G} in Subsection 1.3. It is proved by Graham [G2] that:

Lemma 3.2. *The boundary value of η^{G}_1 is a CR invariant of weight 3. Specifically, $\eta^{\mathrm{G}}_1 = 4\,A^0_{4\overline{4}}$ on M.*

Let us proceed further to describe $\widetilde{\psi}^{\mathrm{B}}$ modulo $O^3(r)$. As we stated in Subsection 1.3, Fefferman's defining function $r = r^{\mathrm{F}}$ makes invariant sense modulo $O^4(r)$, and η_1^{G} modulo $O^3(r)$ is independent of the choices of r^{F} and the data $a \in C^\infty(M)$ determining u^{G}. Consequently, it suffices to extend $|A^0_{2\overline{4}}|^2 \in I_4^{\mathrm{CR}}$ to Ω in such a way that the extension satisfies an approximate transformation law of weight 4 modulo $O^2(r)$. Such an extension is realized by a Weyl invariant. (The Weyl invariant of weight ≥ 3 are subject to a restriction stronger than that in Subsection 2.2, because Proposition 2.1 is irrelevant to the case $n = 2$. See Remark 3.2 below.) Specifically, we consider complete contractions of weight $w = p + q - 2$ of the form:

$$\|R^{(p,q)}\|^2 = \sum g^{\alpha_1\overline{\alpha'_1}} \cdots g^{\alpha_p\overline{\alpha'_p}}\, g^{\beta'_1\overline{\beta}_1} \cdots g^{\beta'_q\overline{\beta}_q}\, R_{\alpha\overline{\beta}}\, \overline{R_{\beta'\overline{\alpha'}}}\,,$$

where the sum runs over ordered multi-indices α, α', β, β' of lengths $|\alpha| = |\alpha'| = p$, $|\beta| = |\beta'| = q$, e.g. $\alpha = (\alpha_1, \ldots, \alpha_p) \in \{0, 1, 2\}^p$, and

$$R_{\alpha_1\cdots\alpha_q\overline{\beta}_1\cdots\overline{\beta}_q} = R_{\alpha_1\overline{\beta}_1\alpha_2\overline{\beta}_2;\alpha_3\cdots\alpha_q\overline{\beta}_3\cdots\overline{\beta}_q}\,.$$

As before, we restrict $\|R^{(p,q)}\|^2$ to $z_0 = 1$ and regard it as a function on the base domain Ω. It is shown in [HKN2] that (cf. Remark 3.2 below):

Lemma 3.3. *If $w = p+q-2 = 4, 5$ then $\|R^{(p,q)}\|^2$ modulo $O^{6-w}(r)$ is independent of the ambient metric. The boundary values are given by*

$$\begin{aligned} 3\,\|R^{(4,2)}\|^2\big|_M &= 7\,\|R^{(3,3)}\|^2\big|_M = 2^8 \cdot 21\,|A^0_{4\overline{2}}|^2, \\ \|R^{(5,2)}\|^2\big|_M &= -4 \cdot (5!)^2 F_5^{\mathrm{CR}}(1, 18), \\ \|R^{(4,3)}\|^2\big|_M &= -4 \cdot (5!)^2 F_5^{\mathrm{CR}}(4/3, 57/5). \end{aligned}$$

Using these three lemmas, we get:

Theorem 3.1. *There exist universal constants c_0, c_1, c_2, c_3, c'_1, c'_2, c'_3 independent of $A \in \mathcal{N}$ such that*

$$\begin{aligned} \widetilde{\psi}^{\mathrm{B}} + c_0\,\eta_1^{\mathrm{G}} &= c_1\|R^{(3,3)}\|^2\,r + \left(c_2\|R^{(5,2)}\|^2 + c_3\|R^{(4,3)}\|^2\right) r^2 + O^3(r) \\ &= c'_1\|R^{(4,2)}\|^2\,r + \left(c'_2\|R^{(5,2)}\|^2 + c'_3\|R^{(4,3)}\|^2\right) r^2 + O^3(r). \end{aligned}$$

The constant c_0 was determined in Graham [G1], where he proved

$$\widetilde{\psi}^{\mathrm{B}} = -12A^0_{4\overline{4}} \quad \text{on} \quad M, \quad \text{so that} \quad c_0 = 3. \tag{3.2}$$

It is shown in [HKN2] that:

Theorem 3.2. *For other universal constants in Theorem 3.1 above,*

$$c_1=\frac{1}{160},\quad c_2=\frac{1}{20160},\quad c_3=\frac{1}{560};$$
$$c_1'=\frac{3}{1120},\quad c_2'=\frac{61}{141120},\quad c_3'=\frac{3}{7840}.$$

Theorems 3.1 and 3.2 together with (3.1) and (3.2) are the main results of this section.

Remark 3.1. For the two dimensional analysis of φ^{B} and ψ^{B} stated above, Graham [G1] originally proved (3.1) and

$$\widetilde{\psi}^{\mathrm{B}}+3\,\eta_1^{\mathrm{G}}=(\mathrm{constant})\,|A^0_{2\bar{4}}|^2\,r+O^2(r)\qquad(\mathrm{constant}=24/5),$$

where the determination of the constant is due to [HKN1]. This result on $\widetilde{\psi}^{\mathrm{B}}$ is refined one step further in [HKN2] to get Theorems 3.1 and 3.2, where the first statement of Lemma 3.3 concerning the ambiguity of the Weyl invariants is crucial.

Remark 3.2. In the argument above, we have only considered the complete contractions of the form $\|R^{(p,q)}\|^2$, because these generate all Weyl invariants of weight $w\le 5$, $w\neq 3$ (see [HKN2]). To state it more precisely, let I_w^{W} denote the vector space of all Weyl invariants of weight w which are well-defined modulo $O^{6-w}(r)$, and set $\widetilde{I}_w^{\mathrm{W}}=I_w^{\mathrm{W}}/\sim$, where $\sim$ stands for the equivalence relation of having the same boundary value. Then $\dim\widetilde{I}_1^W=\dim\widetilde{I}_2^W=\dim\widetilde{I}_3^W=0$, $\dim\widetilde{I}_4^W=1$ and $\dim\widetilde{I}_5^W=2$. Bases of $\widetilde{I}_4^W$ and $\widetilde{I}_5^W$ are given by the boundary values of

$$\|R^{(4,2)}\|^2\ \ (\text{or}\ \ \|R^{(3,3)}\|^2)\quad\text{and}\quad\{\|R^{(5,2)}\|^2,\ \|R^{(4,3)}\|^2\},$$

respectively. Consequently, there are isomorphisms $\widetilde{I}_w^{\mathrm{W}}\cong\widetilde{I}_w^{\mathrm{CR}}$ for $w\le 5$, $w\neq 3$. In the exceptional case $w=3$, the CR invariant $A^0_{4\bar{4}}$ generating the space I_3^{CR} is realized by the boundary value of a linear complete contraction, but the contraction is defined only up to $O^1(r)$ (see [HKN2]).

(B) Determination of the universal constants. We first write down $\widetilde{\psi}^{\mathrm{B}}$ explicitly in terms of Moser's normal coordinate system $z=(z_1,z_2)$. It is sufficient to consider an expansion of $\widetilde{\psi}^{\mathrm{B}}$ along the half-line $p_t=(0,t/2)\in\mathbb{C}^2$ $(t>0)$. Let $F(a,b,c,d)$ be as in Lemma 3.1. Using a method which will be explained in Section 4, We have:

Proposition 3.1. *As $t \to +0$ along the half-line $p_t = (0, t/2)$,*

$$\begin{aligned}\widetilde{\psi}^{\mathrm{B}} =& -12\, A^0_{4\bar{4}} - (216\,|A^0_{2\bar{4}}|^2 + a_1\, A^0_{5\bar{5}} + a_2\, A^1_{4\bar{4}})\, t \\ &+ \big(F(660, 1116, a_3, a_4) + a_5\, A^0_{6\bar{6}} + a_6\, A^1_{5\bar{5}} + a_7\, A^2_{4\bar{4}}\big)\, t^2 + O^3(t),\end{aligned}$$

where a_j for $j = 1, 2, \ldots, 7$ are constants independent of $A \in \mathcal{N}$.

We next refine Lemmas 3.2 and 3.3. It is rather easy to see that

$$r^{\mathrm{F}} = t + O^3(t) \quad \text{as} \quad t \to +0 \quad \text{along} \quad p_t = (0, t/2). \tag{3.3}$$

We have the following two propositions.

Proposition 3.2. *As $t \to +0$ along the half-line $p_t = (0, t/2)$,*

$$\begin{aligned}\eta^{\mathrm{G}}_1 =& 4\, A^0_{4\bar{4}} + \Big(\frac{368}{5}\, |A^0_{2\bar{4}}|^2 + b_1\, A^0_{5\bar{5}} + b_2\, A^1_{4\bar{4}}\Big)\, t \\ &- \Big(F\Big(\frac{680}{3}, \frac{1956}{5}, b_3, b_4\Big) + b_5\, A^0_{6\bar{6}} + b_6\, A^1_{5\bar{5}} + b_7\, A^2_{4\bar{4}}\Big)\, t^2 + O^3(t),\end{aligned}$$

where b_j for $j = 1, 2, \ldots, 7$ are constants independent of $A \in \mathcal{N}$.

Proposition 3.3. *As $t \to +0$ along the half-line $p_t = (0, t/2)$,*

$$\begin{aligned}\|R^{(4,2)}\|^2 &= 2^8 \cdot 7\, |A^0_{4\bar{2}}|^2 + 2^8\, F(50, 936, d_1, d_2)\, t + O^2(t), \\ \|R^{(3,3)}\|^2 &= 2^8 \cdot 3\, |A^0_{4\bar{2}}|^2 + 2^8 \cdot 3\, F(25, 243, d_3, d_4)\, t + O^2(t),\end{aligned}$$

where d_1, d_2, d_3, d_4 are constants independent of $A \in \mathcal{N}$.

Using these three propositions together with Lemma 3.3, (3.2) and (3.3), we can determine all universal constants in Theorem 3.1 and get Theorem 3.2.

3.2 The complex Monge-Ampère asymptotics

The proofs of the results stated in Section 2 and Subsection 3.1 require knowledge of the construction and properties of the asymptotic solutions of the complex Monge-Ampère boundary value problem (1.13). In this subsection, we summarize these. In particular, we present the method of proving Proposition 3.2. After reviewing in the part (A) Graham's construction of his asymptotic solutions as in Theorem 1.3, we consider in the part (B) its expansion with respect to Moser's normal form coefficients $A = (A^{\ell}_{\alpha\bar{\beta}})$. We are then required to write down the linearization with respect to A, and this is done finally in the part (C).

(A) Construction of the asymptotic solution. We first recall Fefferman's construction [F2] of his defining functions r^{F} of $M \subset \partial\Omega$, which are locally defined smooth approximate solutions of (1.13). Starting from an arbitrary smooth defining function ρ of M, we define r_s for $s = 1, \dots, n+1$ successively by

$$(3.4) \qquad r_1 = J[\rho]^{-1/(n+1)}\,\rho, \qquad r_s = \left(1 + c_s^{-1}(1 - J[r_{s-1}])\right) r_{s-1},$$

where $c_s = s(n+2-s)$. Then r_s are smooth defining functions satisfying

$$(3.5)_s \qquad J[r_s] = 1 + O^s(\rho) \qquad (s = 1, \dots, n+1),$$

and thus we may set $r^{\mathrm{F}} = r_{n+1}$. In fact, $(3.5)_1$ holds, since $J[\phi\,\rho] = \phi^{n+1}J[\rho] + O^1(\rho)$ whenever ϕ is smooth. Furthermore, $(3.5)_s$ implies $(3.5)_{s+1}$ for $1 \le s \le n$, since

$$(3.6) \quad J[r + \phi\, r^{s+1}] = J[r] + c_{s+1}\phi\, r^s + O^{s+1}(\rho) \quad (s = 1, \dots, n+1)$$

whenever r is a smooth defining function of M satisfying $J[r] = 1 + O^s(\rho)$. Note that $c_{n+2} = 0$ and thus r_{n+2} cannot be defined by (3.4). Instead, the above equality (3.6) for $s = n+1$ yields the uniqueness of r^{F} modulo $O^{n+2}(\rho)$.

We next recall Graham's construction [G2] of his asymptotic solutions u^{G} of (1.13), which are formal series of the form

$$r + r\sum_{k=0}^{\infty} \eta_k \cdot (r^{n+1}\log r)^k \qquad \text{with } r = r^{\mathrm{F}},$$

where η_k are functions of $(z, \overline{z})$ smooth up to M. Starting from a Fefferman's defining function $r = r^{\mathrm{F}}$ with the initial defining function ρ arbitrarily chosen, we set $u_{n+1} = r$ and define u_s for $s \ge n+2$ successively in such a way that each u_s is a formal series as above (in fact, we can choose u_s to be a finite sum) and satisfies

$$(3.7)_s \qquad J[u_s] = 1 + O^{s-0}(r) \qquad (s \ge n+2),$$

where $O^{s-0}(r)$ stands for an error term of the form $r^s \sum_{k=0}^{\infty} \eta_k \cdot (\log r)^k$. Obviously, $(3.7)_{n+1}$ follows from $(3.5)_{n+1}$. For the ambient metric $g = (g_{j\overline{k}})$ with potential $r_\#$, we define an approximate Laplacian by

$$\Delta[g] = \sum_{j,k=0}^{n} g^{j\overline{k}}\, \frac{\partial^2}{\partial z_j \partial \overline{z}_k}, \qquad \text{where } \left(g^{j\overline{k}}\right) = (g_{j\overline{k}})^{-1}.$$

Using this, we define a linear differential operator $L = L[r]$ by

$$L[r]f = \Delta[g]\left(|z_0|^2 f\right)\big|_{z_0=1}.$$

It follows that if u_s satisfies $(3.7)_s$ then

$$J[u_s + \phi_{s+1}\, r^{s+1}] = J[u_s] + L(\phi_{s+1}\, r^{s+1}) + O^{s+1-0}(r),$$

where ϕ_{s+1} is a formal series of the form $\sum_{k=0}^{\infty} \eta_k \cdot (\log r)^k$. Thus $(3.7)_{s+1}$ is satisfied by $u_{s+1} = u_s + \phi_{s+1}\, r^{s+1}$ if ϕ_{s+1} is subject to

$$(3.8)_s \qquad L(\phi_{s+1}\, r^{s+1}) = 1 - J[u_s] + O^{s+1-0}(r) \qquad (s \geq n+1),$$

which is regarded as a linearized equation of $J[u] = 1$. If $(3.8)_s$ is solved for all s, then an asymptotic solution u^{G} is given by the formal limit of u_s as $s \longrightarrow \infty$.

To solve $(3.8)_s$ for $s \geq n+1$, we use the coordinate system $(z', \overline{z'}, r, v)$ and try to determine successively the coefficients of the expansion

$$\phi_{s+1}\, r^{s+1} = \sum_{j \geq s} \sum_{k \geq 0} c_{j,k}[\phi_{s+1}]\, r^j (\log r)^k,$$

where $c_{j,k}[\phi_{s+1}]$ are smooth functions of $(z', \overline{z'}, v)$. Setting

$$L = I + E \quad \text{with } I = \partial_r\,(r\partial_r - n - 2),$$

we see that E is a tangential operator in the sense that it does not contain differentiation with respect to r. Consequently, if we write $(3.8)_s$ as

$$(3.9)_s \qquad I(\phi_{s+1}\, r^{s+1}) = 1 - J[u_s] + O^{s+1-0}(r) \quad (s \geq n+1),$$

then the right side belongs to $O^{s-0}(r)$. Dropping the error term $O^{s+1-0}(r)$ in $(3.9)_s$ and regarding the result as an ordinary differential equation of the form $If = g$, we can determine all the coefficients $c_{j,k}[\phi_{s+1}]$ uniquely provided $c_{n+2,0}[\phi_{n+2}]$ is prescribed, a condition which exactly corresponds to the ambiguity of u^{G}. Therefore, u^{G} is obtained as desired.

(B) Dependence of the asymptotic solution on the normal form coefficients. For a surface in Moser's normal form $N(A)$, let us use the real coordinate system $(z', \overline{z'}, \rho_A, v)$. If we consider the Taylor expansions with respect to this coordinate system, then

(#) the Taylor coefficients of r^{F} modulo $O^{n+2}(\rho_A)$ and those of η_k^{G} modulo $O^{n+1}(\rho_A)$ are polynomials in A.

This can be seen as follows. Starting from the defining function $\rho = \rho_A$, we construct $r = r^{\mathrm{F}}$ and u^{G} with $c_{n+2}[\phi_{n+1}] = 0$ by the algorithm given in the part (A) above. Then (#) holds without error terms, and thus we may write $r^{\mathrm{F}} = r^{\mathrm{F}}_A$ and $u^{\mathrm{G}} = u^{\mathrm{G}}_A$. The statement (#) for general r^{F} and u^{G} follows from (1^{F}) and (1^{G}) in Subsection 1.3.

To prove Proposition 3.2, we need to know the explicit dependence of r^{F}_A and u^{G}_A on A. We thus expand u^{G}_A in powers of A as follows (the expansion of r^{F}_A in powers of A will be discussed in the part (C) below):

$$u^{\mathrm{G}}_A = \sum_{s=0}^{\infty} \psi_s \quad \text{with } \psi_s = \sum_{j\geq 1} \sum_{k \geq 0} \eta_{j,k}[\psi_s]\, \rho_A^j (\log \rho_A)^k, \tag{3.10}$$

where $\eta_{j,k}[\psi_s] = \eta_{j,k}[\psi_s](z', \overline{z'}, v; A)$ are homogeneous polynomials of degree s in A such that the coefficients are polynomials in $(z', \overline{z'}, v)$. Regarding (3.10) as an asymptotic series in powers of A, we have:

Proposition 3.4. *There exists a unique asymptotic series u^{G}_A of the form* (3.10) *such that $J[u^{\mathrm{G}}_A] = 1$ and $\eta_{n+2,0} := \sum_{s=0}^{\infty} \eta_{n+2,0}[\psi_s] = 0$.*

Proposition 3.4 is proved by constructing $u^{\mathrm{G}}_{A,s} := \sum_{m\leq s} \psi_m$ for $s \in \mathbb{N}_0$, and the algorithm is actually used in the proof of Proposition 3.2 (cf. [HKN2]). The construction is similar to that of u_s in the part (A) above, and done as follows. First, $u^{\mathrm{G}}_{A,0} = \psi_0 = \rho_A$ follows from the condition $\eta_{n+2,0}[\psi_0] = 0$. For $s > 0$, we have by induction on s that

$$J[u^{\mathrm{G}}_A] = J[u^{\mathrm{G}}_{A,s-1}] + L[\rho_A]\, \psi_s + O^{s+1}(A),$$

where $O^{s+1}(A)$ stands for a term which does not contain polynomials of degree $\leq s$ in A. (Here, ρ_A is regarded as an independent variable, and the dependence of ρ_A on A is not taken into account.) The above equality is written as a linear equation for ψ_s (cf. $(3.8)_s$ in the part (A)):

$$L[\rho_A]\, \psi_s = 1 - J[u^{\mathrm{G}}_{A,s-1}] + O^{s+1}(A).$$

Therefore, ψ_s and thus $u^{\mathrm{G}}_{A,s}$ are determined inductively by solving this equation under the condition $\eta_{n+2,0}[\psi_s] = 0$.

(C) First variation of the Monge-Ampère equation. Let us next consider the dependence of r^{F}_A on $A \in \mathcal{N}$. To prove Proposition 3.3 in the previous subsection, we need to know r^{F}_A modulo $O^2(A)$ explicitly. Less precise information is required also in the proof of Theorem 2.1 (see Section 5 below). We thus consider $r^{\mathrm{F}}_{\varepsilon A}$ for a real parameter ε, and seek an approximate boundary value problem which characterizes the first variation $\widetilde{r}^{\mathrm{F}}_A = (d/d\varepsilon)r^{\mathrm{F}}_{\varepsilon A}\big|_{\varepsilon=0}$.

We begin with a heuristic argument for an exact asymptotic solution u^{G}_A in place of a smooth approximate one r^{F}_A, disregarding the difficulty due to the ambiguity of Fefferman's defining functions. Supposing as if $u^{\mathrm{G}}_{\varepsilon A}$ were smoothly depending on $\varepsilon \in \mathbb{R}$ small and had no singularity even on the boundary, we set $\widetilde{u}^{\mathrm{G}}_A = (d/d\varepsilon)u^{\mathrm{G}}_{\varepsilon A}\big|_{\varepsilon=0}$. Then, a relation characterizing $\widetilde{u}^{\mathrm{G}}_A$ is obtained by taking the first variation of the formal boundary value problem

$$(3.11) \qquad J[u^{\mathrm{G}}_{\varepsilon A}] = 1 \quad \text{in} \ \ \Omega_\varepsilon, \qquad u^{\mathrm{G}}_{\varepsilon A} = 0 \quad \text{on} \ \ \partial\Omega_\varepsilon = N(\varepsilon A),$$

where Ω_ε is a pseudoconvex side of $N(\varepsilon A)$. The first equality yields

$$(3.12) \qquad L[\rho_0]\,\widetilde{u}^{\mathrm{G}}_A = 0 \quad \text{in} \ \ \Omega_0.$$

The second equality of (3.11) is written as $u^{\mathrm{G}}_{\varepsilon A}(z', \overline{z}', u, v) = 0$ evaluated at $u = (|z'|^2 + \varepsilon\, F_A(z', \overline{z'}, v))/2$. Differentiating both sides of this equality with respect to ε and evaluating the result at $\varepsilon = 0$, we have

$$\widetilde{u}^{\mathrm{G}}_A(z', \overline{z'}, |z'|^2/2, v) = -\frac{1}{2}\,\frac{\partial u^{\mathrm{G}}_0}{\partial u}(z', \overline{z'}, |z'|^2/2, v)\, F_A(z', \overline{z'}, v).$$

Recalling that $u^{\mathrm{G}}_0 = \rho_0 = 2u - |z'|^2$, we get

$$(3.13) \qquad \widetilde{u}^{\mathrm{G}}_A = -F_A \qquad \text{on} \ \ \partial\Omega_0 = N(0) = \{\rho_0 = 0\}.$$

The function $\widetilde{u}^{\mathrm{G}}_A$ is obtained by solving the linear equation (3.12) under the boundary condition (3.13).

Returning to the original problem of expressing the first variation of r^{F}_A, we have:

Proposition 3.5. *The first variation $\widetilde{r}^{\mathrm{F}}_A = (d/d\varepsilon)r^{\mathrm{F}}_{\varepsilon A}\big|_{\varepsilon=0}$ exists and satisfies the approximate boundary value problem*

$$(3.14) \quad L[\rho_0]\,\widetilde{r}^{\mathrm{F}}_A = O^{n+1}(\rho_0) \quad in \ \ \Omega_0, \qquad \widetilde{r}^{\mathrm{F}}_A = -F_A \quad on \ \ \partial\Omega_0 = N(0).$$

The problem (3.14) *has a formal power series solution which is unique modulo $O^{n+2}(\rho_0)$.*

The proof of the latter part of Proposition 3.5 above is done similarly to that of Proposition 3.4.

To give an explicit representation of $\widetilde{r}^{\mathrm{F}}_A$, it is convenient to lift the problem (3.14) to $\mathbb{C}^* \times \Omega_0$. Setting $(\rho_0)_\# = |z_0|^2\rho_0$, $(F_A)_\# = |z_0|^2 F_A$ and $(\widetilde{r}^{\mathrm{F}}_A)_\# = |z_0|^2\widetilde{r}^{\mathrm{F}}_A$, we write (3.14) as

$$(3.15)\quad \Delta_0(\widetilde{r}^{\mathrm{F}}_A)_\# = O^{n+1}\left((\rho_0)_\#\right), \qquad (\widetilde{r}^{\mathrm{F}}_A)_\# = -(F_A)_\# + O^1\left((\rho_0)_\#\right),$$

where $\Delta_0 = \Delta[g_0]$, which is the (negative) Laplacian with respect to the ambient metric g_0 with potential $(\rho_0)_\#$. Solutions of (3.15) are given by

$$(3.16)\quad (\widetilde{r}^{\mathrm{F}}_A)_\# = -(F_A)_\# - \sum_{s=1}^{n+1} \frac{(-\rho_0)^s_\# \Delta_0^s (F_A)_\#}{c_1 \cdots c_s} \quad \bmod O^{n+2}\left((\rho_0)_\#\right),$$

where $c_s = s(n+2-s)$, which are the same constants as those in (3.4). To see that the right side of (3.16) gives a solution of (3.15), we use the projective coordinates $z_0 = \zeta_0$, $z_j = \zeta_j/\zeta_0$ $(j = 1, \dots, n)$. Then

$$(\rho_0)_\# = \zeta_0\,\overline{\zeta}_n + \zeta_n\,\overline{\zeta}_0 - \sum_{j=1}^{n-1} |\zeta_j|^2 \quad \text{and} \quad g_0 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -I_{n-1} & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

where I_{n-1} is the identity matrix. Noting that $(g_0)^{-1} = g_0$, we have

$$\Delta_0 = \frac{\partial^2}{\partial\zeta_0\,\partial\overline{\zeta}_n} + \frac{\partial^2}{\partial\zeta_n\,\partial\overline{\zeta}_0} - \sum_{j=1}^{n-1} \frac{\partial^2}{\partial\zeta_j\,\partial\overline{\zeta}_j}.$$

This expression permits us to compute the commutator

$$[\Delta_0, (\rho_0)^s_\#] = s\,(\rho_0)^{s-1}_\#(Z + \overline{Z} + n + s),$$

where $Z = \sum_{j=0}^n \zeta_j\,\partial/\partial\zeta_j$. Consequently,

$$\Delta_0\left((\rho_0)^s_\#\Delta_0^s(F_A)_\#\right) = (\rho_0)^s_\#\Delta_0^{s+1}(F_A)_\# + c_s(\rho_0)^{s-1}_\#\Delta_0^s(F_A)_\#.$$

Therefore, $(\widetilde{r}^{\mathrm{F}}_A)_\#$ in (3.16) satisfies (3.15).

Remark 3.3. In proving Lemma 3.2 stated in the previous subsection, Graham [G2] uses essentially the same expression for $\widetilde{r}^{\mathrm{F}}_A$ as that for $(\widetilde{r}^{\mathrm{F}}_A)_\#$ given by (3.16).

§4 Microlocal calculus of the Bergman kernel

4.1 Outline

Proposition 3.1 is proved by using a method of Boutet de Monvel [B1]–[B3] of computing explicitly the singularity of the Bergman kernel.

In this section, we briefly explain his method which remains valid in the n dimensional case (cf. Theorem 4.2 below). To get an alternative proof of Theorem 1.2, Boutet de Monvel and Sjöstrand constructed in [BS] a Fourier integral operator $\mathbf{A}^{\mathrm{FIO}}$ with complex phase, which transforms the Bergman kernel of a strictly pseudoconvex domain $\Omega \subset \mathbb{C}^n$ to that of a model domain Ω_0 (cf. Example in Subsection 1.2). It might be difficult to derive information we need from $\mathbf{A}^{\mathrm{FIO}}$. It would seem, for a general strictly pseudoconvex domain Ω, that there is no known system of differential equations which characterizes the Bergman kernel, and that this is a reason why the computation of the Bergman kernel was not so easy.

Kashiwara discovered in [Kas] a system of microdifferential equations (i.e. pseudodifferential equations in the real analytic category or its complexification) which characterizes the Bergman kernel $K^{\mathrm{B}}(z, \overline{z})$ up to a multiplicative constant. This system arises as the formal adjoint of a system which characterizes the singularity of the Heaviside function of the domain Ω (i.e. the characteristic function of Ω or its complexification) up to a multiplicative constant (cf. Theorem 4.1 below). The Heaviside function of the model domain Ω_0 is transformed to that of Ω by a shift (or translation) operator $\mathbf{A}^{\mathrm{shift}}(z, \partial_z)$, and consequently, the operator $\mathbf{A}^{\mathrm{B}}(z, \partial_z)$ which transforms the Bergman kernel of Ω_0 to that of Ω is given by

$$(4.1) \qquad \mathbf{A}^{\mathrm{B}} = \mathbf{A}^{*-1} = \sum_{j=0}^{\infty} (1 - \mathbf{A}^*)^j \qquad \text{for } \mathbf{A} = \mathbf{A}^{\mathrm{shift}},$$

where $\mathbf{A}^* = \mathbf{A}^*(z, \partial_z)$ is the formal adjoint of the shift operator $\mathbf{A} = \mathbf{A}(z, \partial_z)$. This formula, due to Boutet de Monvel, remains valid formally in the C^∞ category.

The operator $\mathbf{A}^{\mathrm{B}}$ is much simpler than the Fourier integral operator $\mathbf{A}^{\mathrm{FIO}}$, because the shift operator $\mathbf{A}^{\mathrm{shift}}$ is completely explicit. However, we have to be careful with two points. We are now in a complexified world, so that z and $\overline{z}$ are independent variables. A point in (4.1) is that $\mathbf{A}^{\mathrm{shift}} = \mathbf{A}^{\mathrm{shift}}(z, \partial_z)$ is realized as a holomorphic operator, and it is convenient to regard $\mathbf{A}^{\mathrm{shift}}$ as a (formal) microdifferential operator of infinite order. For such operators, usual definitions of the composition, the formal adjoint and the asymptotic expansion should be modified. Another point in (4.1) is that $\mathbf{A}^{\mathrm{B}}$ acts on functions on Ω_0, while $\mathbf{A}^*$ with $\mathbf{A} = \mathbf{A}^{\mathrm{shift}}$ acts on functions on Ω. Though we only consider as operands special types of functions related to the asymptotic expansion of the Bergman kernel as in the part (B) of Subsection 3.2, we need to expand these functions in powers of A. Then the singularity on $N(A)$ loses its role in the asymptotic expansions of operators and operands. We thus need to introduce the notion of weight for formal operators and operands. The formal setting in this sense is necessary even under the assumption that the boundary is real analytic.

The proof of Proposition 3.1 is done by computing explicitly the necessary terms in the right side of (4.1). In the remaining part of this section, we describe briefly the justification of (4.1) and its application to the proof of Proposition 3.1, after a quick overview of the theory of hyperfunctions.

4.2 Quick review of hyperfunction theory

For a mild function f on $\mathbb{R}$, say, in the Schwartz class, let us consider the Cauchy integrals

$$F^{\pm}(z) := \frac{1}{2\pi i}\int_{\mathbb{R}} \frac{f(t)}{t-z}\,dt \qquad \text{for} \quad z \in \mathbb{C}^{\pm},$$

where $\mathbb{C}^{\pm} = \{z \in \mathbb{C};\ \pm\mathrm{Im}\, z > 0\}$. According to the Plemelj formula, the boundary values $f^{\pm}(x) = F^{\pm}(x \pm i\,0)$ for $x \in \mathbb{R}$ exist and satisfy $2\,f^{\pm} = \pm f + i\,\mathcal{H}[f]$, where $\mathcal{H}$ is the Hilbert transformation. In particular,

$$f^{+} - f^{-} = f \quad \text{in } \mathbb{R}. \tag{4.2}$$

More generally, for a Schwartz distribution $f \in \mathcal{D}'(\mathbb{R})$, there exist $F^{\pm} \in \mathcal{O}(\mathbb{C}^{\pm})$ such that the boundary values $f^{\pm}$ on $\mathbb{R}$ exist in $\mathcal{D}'(\mathbb{R})$ and satisfy (4.2). Consequently, f is realized by the pair $F = (F^{+}, F^{-})$ regarded as a holomorphic function in a disconnected open set $\mathbb{C}^{+} \cup \mathbb{C}^{-}$. We identify $F_1, F_2 \in \mathcal{O}(\mathbb{C}^{+} \cup \mathbb{C}^{-})$ if $F_1 - F_2$ extends holomorphically to $\mathbb{C}$, and denote the quotient space by $\mathcal{B}(\mathbb{R})$. Thus $\mathcal{D}'(\mathbb{R}) \subset \mathcal{B}(\mathbb{R})$. Elements of $\mathcal{B}(\mathbb{R})$ are called *hyperfunctions* on $\mathbb{R}$. For $F \in \mathcal{O}(\mathbb{C}^{+} \cup \mathbb{C}^{-})$, we regard (4.2) as a formal expression and write $f \in \mathcal{B}(\mathbb{R})$. Differentiation of $f \in \mathcal{B}(\mathbb{R})$ is then defined by that of $F \in \mathcal{O}(\mathbb{C}^{+} \cup \mathbb{C}^{-})$, and the definition is compatible with that on $\mathcal{D}'(\mathbb{R})$.

The space $\mathcal{B}(X)$ of hyperfunctions on an arbitrary open set $X \subset \mathbb{R}$ is defined similarly by taking an open set $U \subset \mathbb{C}$ such that $X \subset U$ is relatively closed. Each element $f \in \mathcal{B}(X)$ is realized by a function $F \in \mathcal{O}(U \setminus X)$, and two functions $F_1, F_2 \in \mathcal{O}(U \setminus X)$ are identified when $F_1 - F_2$ extends holomorphically to U. The space $\mathcal{B}(X)$ is independent of the choice of U. Multiplication of $f \in \mathcal{B}(X)$ by a real analytic function g on X is then defined by that on $F \in \mathcal{O}(U \setminus X)$ by the complex extension of g to a suitable U, and the definition is again compatible with that on $\mathcal{D}'(\mathbb{R})$. It is remarkable that the restriction mapping $\mathcal{B}(\mathbb{R}) \to \mathcal{B}(X)$ is surjective.

Example. The Heaviside function $Y \in \mathcal{D}'(\mathbb{R})$ is realized by a function $F \in \mathcal{O}(\mathbb{C} \setminus [0, \infty))$ satisfying $f^+(x) = (-1/2\pi i) \log x$ for $x > 0$. Thus, the Delta measure $\delta \in \mathcal{D}'(\mathbb{R})$ is realized by the function $-1/2\pi i z$. More generally, a class of distributions on an open set $X \subset \mathbb{R}$ containing the origin is given by holomorphic functions on $U \setminus X$, with U as above, of the form

$$\frac{\varphi(z)}{z^\ell} + \psi(z) \log z \quad \text{with } \ell \in \mathbb{N}_0 \text{ and } \varphi, \psi \in \mathcal{O}(U).$$

For $f \in \mathcal{B}(X)$, its *support* $\operatorname{supp} f$ is defined by the complement of the largest open subset of X on which $f = 0$. For a compact set $K \subset X$, we denote by $\mathcal{B}_K(X)$ the totality of $f \in \mathcal{B}(X)$ such that $\operatorname{supp} f \subset K$. Then $\mathcal{B}_K(X)$ is identified with the dual of the space $C^\omega(K)$ of real analytic functions near K. Thus, elements of $\mathcal{B}_K(X)$ are regarded as analytic functionals. Each element $f \in \mathcal{B}(X)$ is expressed as a locally finite sum $f = \sum f_j$ such that $\operatorname{supp} f_j \subset X$ are compact. This gives an alternative definition of $\mathcal{B}(X)$, which remains valid in the higher dimensional case.

For a Schwartz distribution f on an open set $X \subset \mathbb{R}^n$, there exist open convex cones $\Gamma_j \subset \mathbb{R}^n$ with vertices at the origin and functions $F_j \in \mathcal{O}(X + i\Gamma_j)$ for $j = 1, \dots, N$ such that

$$f(x) = \sum_{j=1}^{N} F_j(x + i\Gamma_j 0) \quad \text{for} \quad x \in X, \tag{4.3}$$

where $F_j(x + i\Gamma_j 0)$ denote the limits of $F_j(x + iy)$ as $y \to 0$ with $y \in \Gamma_j$. Similarly for $f \in \mathcal{B}(X)$, and this property can be used as a definition of $\mathcal{B}(X)$, in which an arbitrary list of holomorphic functions $(F_1, \dots, F_N)$ is considered. Let $WF_A(f)$ denote the analytic wave front set of $f \in \mathcal{D}'(X)$. Then for $(x_0, y) \in T^*X \setminus 0$, we have $(x_0, y) \notin WF_A(f)$ if and only if there exists a representation of f of the form (4.3) for x near x_0 such that $y \notin \bigcup \Gamma_j^\circ$, where Γ_j° denote the (open) dual cones of Γ_j. The *microanalyticity* of $f \in \mathcal{B}(X)$ is defined by this condition, and the *singular spectrum* of f is defined by $\mathrm{S.S.}\, f = \{(x, y) \in T^*X \setminus 0;\ f \notin \mathcal{A}_{(x,y)}\}$, where $\mathcal{A}_{(x,y)}$ denotes the set of germs of hyperfunctions which are microanalytic in the direction (x, y). Thus $\mathrm{S.S.}\, f = WF_A(f)$ for $f \in \mathcal{D}'(X)$.

A microlocal singularity (in the analytic category) of a hyperfunction is called a *microfunction.* That is, for $f \in \mathcal{B}(X)$, a microfunction at $(x, y) \in T^*X \setminus 0$ is defined by f modulo $\mathcal{A}_{(x,y)}$. The equivalence class is denoted by $[f]$, and the totality of such equivalence classes is denoted by $\mathcal{C}_{(x,y)}$. Given a microfunction $[f] \in \mathcal{C}_{(x,y)}$, there exists $F \in \mathcal{O}(X + i\Gamma)$

with an open convex cone Γ such that $y \in \Gamma^\circ$ and that $f(x) - F(x + i\,\Gamma 0)$ is microanalytic in the direction (x, y). Thus $[f]$ is identified with the equivalence class $[F]$ of $F(x + i\,\Gamma 0)$.

Differentiation of a microfunction $[f] \in \mathcal{C}_{(x,y)}$ is defined by using a holomorphic function F such that $f(x) - F(x + i\,\Gamma 0)$ is microanalytic, and similarly for multiplication by a real analytic function. These define the action of linear differential operators with analytic coefficients on microfunctions. It is also possible to define indefinite integration of $[f]$ with respect to a variable, say $\partial_{x_1}^{-1}$ at (x, y) with $y_1 \neq 0$. The analogue of pseudodifferential operator in analytic category, acting on microfunction, is called *microdifferential operator.* The symbol of a microdifferential operator of order m is a formal series $P(z, \xi) = \sum_{j=-\infty}^{m} p_j(z, \xi)$ of holomorphic functions on a conic open set $\Omega \subset T^*\mathbb{C}^n \setminus 0$ such that each p_j is homogeneous of degree j in ξ and satisfies

$$|p_j(z, \xi)| \leq C_K^{-j}(-j)! \quad \text{for } j < 0 \tag{4.4}$$

on each compact set $K \subset \Omega$, where $C_K > 0$ is a constant. Near a point $(x, y) \in \Omega \cap T^*\mathbb{R}^n$ with $y_n \neq 0$, each $p_j(z, \xi)$ admits an expansion

$$p_j(z, \xi) = \sum_{k=-\infty}^{j} \sum_{|\alpha| = j-k} a_{k\alpha}(z) \xi'^{\alpha} \xi_n^k.$$

Thus replacing ξ by ∂_z we may define $P(z, \partial_z)F(z)$ as a convergent series for each holomorphic function $F(z)$ on a wedge $X + i\Gamma$ such that $X + i\Gamma^\circ \subset \Omega$. In this action the ambiguity of the indefinite integral $\partial_{z_n}^{-1}$ causes only a difference by a function that extends holomorphically to $z = x$. Thus the action of $P(z, \partial_z)$ to $[F(x + i\Gamma 0)] \in \mathcal{C}_{(x,y)}$ can be defined by the modulo class of $P(z, \partial_z)F(z)$.

Remark 4.1. A microdifferential operator of infinite order $P(z, \partial_z)$ is also defined by giving the symbol

$$P(z, \xi) = \sum_{j=-\infty}^{\infty} p_j(z, \xi) \qquad (p_j \in \mathcal{O}(\Omega)),$$

where each p_j is homogeneous of degree j in ξ. In addition to (4.4), it is required that

$$|p_j(z, \xi)| \leq C_{K,\varepsilon}\, \varepsilon^j / j! \qquad (j \in \mathbb{N}_0,\ \varepsilon > 0),$$

where $C_{K,\varepsilon} > 0$ is a constant. Thus $P(z, \partial_z)$ is a local operator. In Subsection 4.4 below, we shall be concerned with a shift operator $\mathbf{A} =$

$\mathbf{A}(z,\partial_z)$. Though $\mathbf{A}$ is not a local operator, we regard it as a formal microdifferential operator of infinite order.

A far more precise description of the matters in this subsection is found in a book by Kaneko [Kan].

4.3 Kashiwara's characterization of the Bergman kernel

Let $\Omega \subset \mathbb{C}^n$ be a strictly pseudoconvex domain with a local defining function ρ which is positive in Ω and real analytic near a point $p \in \partial\Omega$, and let $M \subset \partial\Omega$ be a small neighborhood of p. Setting $X = \mathbb{C}^n$ and denoting by X' the complex conjugate of X, we regard $X \times X'$ as the complexification of X identified with $\mathbb{R}^{2n}$. Then ρ extends holomorphically to a neighborhood $U \subset X \times X'$ of M, and the complexification of M is given by $N = \{\rho(z,\overline{z}) = 0\} \subset U$. We also have $\Omega = \{\rho(z,\overline{z}) > 0\} \subset X \times X'$ and $\Omega \times \Omega' \subset \{\operatorname{Re}\rho(z,\overline{z}) > 0\}$ near M.

The Bergman kernel $K^{\mathrm{B}} = \varphi^{\mathrm{B}}\rho^{-n-1} + \psi^{\mathrm{B}}\log\rho$ near M has a multi-valued holomorphic extension to $U \setminus N$ (cf. Remark 1.3). Thus, setting

$$U^{\pm} = \{(z,\overline{z}) \in U;\ \pm\operatorname{Im}\rho(z,\overline{z}) > 0\},$$

we have $K^{\mathrm{B}} \in \mathcal{O}(U^+)$. Another multi-valued function on $U \setminus N$ is defined by $Y(\rho) = -(1/2\pi i)\log\rho$, and we have $Y(\rho) \in \mathcal{O}(U^+ \cup U^-)$ which represents the characteristic function of Ω near M. Let us regard K^{B} and $Y(\rho)$ as elements of $\mathcal{O}(U^+)$. Then these define hyperfunctions with the same singular spectrum

$$T^*_M X = \{(x, \lambda\, d\rho(x)) \in T^*X;\ x \in M,\ 0 \neq \lambda \in \mathbb{R}\},$$

the conormal bundle of M. Similarly for multi-valued functions on $U \setminus N$ of the form

$$K = \sum_{\ell=1}^{m} \varphi_\ell\, \rho^{-\ell} + \psi \log\rho \quad \text{with } \varphi_\ell,\ \psi \in \mathcal{O}(U),\ \ m \in \mathbb{N}. \tag{4.5}$$

For $(x,y) = (x, d\rho(x)) \in T^*_M X$, elements $[K] \in \mathcal{C}_{(x,y)}$ defined by K of the form (4.5) are called *holomorphic microfunctions*, and the totality of these is denoted by $(\mathcal{C}_{N|X\times X'})_{(x,y)}$. In what follows, we omit the bracket in $[K]$ and regard K as a holomorphic microfunction.

Action of microdifferential operators on $\mathcal{C}_{(x,y)}$ preserves the subspace $(\mathcal{C}_{N|X\times X'})_{(x,y)}$. Let $K \in (\mathcal{C}_{N|X\times X'})_{(x,y)}$ such that $\varphi \neq 0$ in (4.5). Then, for a microdifferential operator of the form $P(z,\partial_z)$, there exists a unique microdifferential operator of the form $Q(\overline{z},\partial_{\overline{z}})$ such that

$$P(z,\partial_z)K = Q(\overline{z},\partial_{\overline{z}})K. \tag{4.6}$$

Such an operator P is generated by z_j and $\partial/\partial z_j$ for $j = 1, \ldots, n$. Using these generators, we get a system of equations of the form (4.6), and this system characterizes K in $(\mathcal{C}_{N|X\times\overline{X}})_{(x,y)}$ up to a constant multiple. For a general theory including these facts, see Sato-Kawai-Kashiwara [SKK] and Schapira [Sca].

A system characterizing the Bergman kernel can be written down explicitly. The following theorem is due to Kashiwara [Kas].

Theorem 4.1 ([Kas]). *The Bergman kernel K^{B} satisfies*

$$P^*(z, \partial_z)\, K^{\mathrm{B}} = Q^*(\overline{z}, \partial_{\overline{z}})\, K^{\mathrm{B}}, \tag{4.7}$$

whenever $P(z,\partial_z)\,Y(\rho) = Q(\overline{z},\partial_{\overline{z}})\,Y(\rho)$ with $Y(\rho) = (-1/2\pi i)\log\rho$, where $P^* = P^*(z,\partial_z)$ and $Q^* = Q^*(\overline{z},\partial_{\overline{z}})$ are the formal adjoints of P and Q, respectively.

In the next subsection, we shall give a procedure of constructing the solution to this system of equation by using Moser's normal coordinates.

4.4 A formula of Boutet de Monvel

In the previous subsection, we fixed a domain and considered microdifferential operators of finite order. To study the shift operator $\mathbf{A}$ mentioned in Subsection 4.1, we need to define formal microdifferential operators of infinite order. These operators act on holomorphic microfunctions of infinite order defined by setting $m = \infty$ in (4.5).

It is non-trivial to define such operators, via the symbols, carrying the operations of taking composition, formal adjoint and inverse. We need to introduce the notion of *weight* for the variable $z = (z_1, \ldots, z_n)$ by setting

$$\mathrm{w}(z_j) = -1/2 \quad (j = 1, \ldots, n-1), \qquad \mathrm{w}(z_n) = -1,$$

and extend it to ∂_z and the dual variable $\xi = (\xi_1, \ldots, \xi_n)$ of z by

$$\mathrm{w}(\partial/\partial z_j) = \mathrm{w}(\xi_j) = -\mathrm{w}(z_j) \qquad (j = 1, \ldots, n).$$

(We do not consider the notion of weight for polynomials in $A \in \mathcal{N}$ in this subsection.) Then we may say polynomials $P_{w/2} = P_{w/2}(z, \xi, \xi_n^{-1})$ to be of homogeneous weight $w/2$. By a formal sum of such polynomials $P_{w/2}$ with respect to $w \in \mathbb{Z}$ bounded above, we define the (total) symbol of a *formal microdifferential operator of infinite order*. In other words, we regard the symbol as an asymptotic series of decreasing weight. For these operators, operations of taking the composition, the formal adjoint and the inverse are defined, as usual, by using weight in place of order. These operations are compatible with those for microdifferential operators of finite order.

We next define holomorphic microfunctions of infinite order. Again, it is necessary to introduce the notion of weight for holomorphic microfunctions with support $N(0) = \{\rho_0 = 0\} \subset X \times \overline{X}$ by setting

$$\mathrm{w}(\overline{z}_j) = \mathrm{w}(z_j), \quad \mathrm{w}(\log \rho_0) = 0, \quad \mathrm{w}(\rho_0^{-\ell}) = \ell.$$

We consider asymptotic series of decreasing weight:

$$K = \sum_{j=-\infty}^{m} K_{j/2} \quad \text{with } \mathrm{w}(K_{2/j}) = 2/j, \tag{4.8}$$

where $K_{j/2} \in (\mathcal{C}_{N(0)|X\times\overline{X}})_{(0,dz_0)}$. Then we can define an action of formal operators of infinite order $P(z,\partial_z) = \sum_{j=-\infty}^{m'} P_{j/2}(z,\partial_z)$ to K of the form (4.8) by setting

$$P(z,\partial_z)K = \sum_{j=-\infty}^{m+m'} K'_{j/2} \quad \text{with } K'_{j/2} = \sum_{k+\ell=j} P_{k/2}(z,\partial_z)K_{\ell/2}.$$

We refer to a series of the form (4.8) as a *holomorphic microfunction of infinite order.*

Let us restrict ourselves to real analytic surfaces in Moser's normal form $N(A)$ $(A \in \mathcal{N})$. To define the shift operator $\mathbf{A}$ by giving its symbol, we need the following:

Lemma 4.1 ([B1]). *There exists a unique complex-valued defining function of $N(A)$ of the form $\rho_B^{\mathrm{BM}}(z,\overline{z}) = \rho_0(z,\overline{z}) - H_B(z,\overline{z'})$, where $H_B(z,\overline{z'})$ are convergent power series of the form*

$$H_B(z,\overline{z'}) = \sum_{|\alpha|,\,|\beta|\geq 2} B_{\alpha\overline{\beta}}(z_n)\, z'_\alpha\, \overline{z'_\beta}, \quad B_{\alpha\overline{\beta}}(z_n) = \sum_{\ell=0}^{\infty} B^{\ell}_{\alpha\overline{\beta}}\, z_n^{\ell}.$$

The coefficients $B = (B^{\ell}_{\alpha\overline{\beta}})$ are polynomials in $A = (A^{\ell}_{\alpha\overline{\beta}})$, and the trace conditions (1.3) *are valid for $B_{\alpha\overline{\beta}}(z_n)$ in place of $A_{\alpha\overline{\beta}}(v)$.*

With the defining function ρ_B^{BM} in Lemma 4.1, any holomorphic microfunction with support $N(A)$ is written as

$$\varphi\,\rho^{-m} + \psi\log\rho \quad \text{with } \rho = \rho_B^{\mathrm{BM}}. \tag{4.9}$$

Let us expand (4.9) by using

$$\left(\rho_B^{\mathrm{BM}}\right)^{-m} = \rho_0^{-m}\left(1 - \frac{H_B}{\rho_0}\right)^{-m} = \sum_{\ell=0}^{\infty}\binom{m}{\ell}(-H_B)^{\ell}\rho_0^{-m-\ell},$$

$$\log \rho_B^{\mathrm{BM}} = \log\rho_0 + \log\left(1 - \frac{H_B}{\rho_0}\right) = \log\rho_0 - \sum_{\ell=1}^{\infty}\frac{1}{\ell}\left(\frac{H_B}{\rho_0}\right)^{\ell}.$$

The right sides are asymptotic series of decreasing weight, since H_B consists of terms of weight ≤ -2. Consequently, we obtain an expression of (4.9) as a formal sum of holomorphic microfunctions with support $N(0)$. The asymptotic series thus obtained uniquely determines the original holomorphic microfunction (4.9). We thus identify (4.9) with its asymptotic expansion of the form (4.8).

Lemma 4.2 ([B1]). *Let $\mathbf{A}(z, \partial_z)$ be a formal microdifferential operator of infinite order defined by the symbol*

$$\mathbf{A}(z,\xi) = \exp[-H_B(z, -\xi'/\xi_n)\,\xi_n] \quad \textit{with } \xi = (\xi', \xi_n).$$

Then

$$Y(\rho_B^{\mathrm{BM}}) = \mathbf{A}Y(\rho_0).$$

Lemma 4.2 is proved by direct computation using the relations

$$(\partial/\partial z_j)\,(\partial/\partial z_n)^{-1}\log\rho_0 = \overline{z}_j\log\rho_0 \qquad (j = 1, \ldots, n-1).$$

Changing the notation slightly, we denote by K_A^{B} the Bergman kernel associated with the domain bounded by $N(A)$. The singularity of its complex extension is again denoted by $K_A^{\mathrm{B}} = K_A^{\mathrm{B}}(z, \overline{z})$. Regarding it as a holomorphic microfunction, we can state the following theorem of Boutet de Monvel [B1], which is used in the proof of Proposition 3.1.

Theorem 4.2 ([B1]). *Let $\mathbf{A}(z, \partial_z)$ be as in Lemma 4.2. Then the formal adjoint $\mathbf{A}^*$ is invertible as a formal microdifferential operator of infinite order, and the following equality holds:*

$$K_A^{\mathrm{B}} = \mathbf{A}^{*-1}K_0^{\mathrm{B}}. \tag{4.10}$$

The invertibility of $\mathbf{A}^*$ is a consequence of the fact that the symbol expansion of $1 - \mathbf{A}^*$ consists of terms of negative weight. In fact, the inverse $\mathbf{A}^{*-1}$ is given by (4.1), because the right side of (4.1) makes sense as an asymptotic series of decreasing weight. The formula (4.10) follows from Theorem 4.1, since $K = \mathbf{A}^{*-1}K_0^{\mathrm{B}}$ satisfies the microdifferential equation $P^*K = Q^*K$ whenever $P = P(z, \partial_z)$ and $Q = Q(\overline{z}, \partial_{\overline{z}})$ satisfy

$$PY(\rho_B^{\mathrm{BM}}) = QY(\rho_B^{\mathrm{BM}}). \tag{4.11}$$

In fact, by virtue of Lemma 4.2 and the commutation relation

$$Q(\overline{z}, \partial_{\overline{z}}) \circ \mathbf{A}(z, \partial_z) = \mathbf{A}(z, \partial_z) \circ Q(\overline{z}, \partial_{\overline{z}}),$$

it follows from (4.11) that

$$\mathbf{A}^{-1} \circ P \circ \mathbf{A}Y(\rho_0) = QY(\rho_0). \tag{4.12}$$

Then Theorem 4.1 yields $\mathbf{A}^* \circ P^* \circ \mathbf{A}^{*-1}K_0^{\mathrm{B}} = Q^*K_0^B$, so that $P^*K = Q^*K$. A point is that $\mathbf{A}^{-1} \circ P \circ \mathbf{A}$ is an operator of finite order. This fact automatically follows from the relation (4.12).

Let us next sketch the proof of Proposition 3.1 by using the formula (4.10) in Theorem 4.2. We consider the expansion of $\widetilde{\psi}^{\mathrm{B}}$ along the half-line $p_t = (0, t/2) \in \mathbb{C}^2$ $(t > 0)$:

$$\widetilde{\psi}^{\mathrm{B}}(p_t) = F_3(A) + F_4(A)\,t + F_5(A)\,t^2 + \cdots.$$

Then each F_j depends only on the terms in $\mathbf{A}^{*-1}$ of the form

$$F_{jk}(A)\,z_2^k\,(\partial/\partial z_2)^{k-j} \qquad (k = 0, 1, \ldots, j-3). \tag{4.13}$$

In addition, if we write $\mathbf{A}^{*-1} = 1 + \sum_{j=-\infty}^{-1} Q_{j/2}$ the expansion of $\mathbf{A}^{*-1}$ of decreasing weight, then F_j is determined by Q_{-j}. On the other hand, if we set $\mathbf{A} = 1 + \sum_{j=-\infty}^{-1} P_{j/2}$, then the trace conditions (1.3) for H_B yield $P_{j/2} = 0$ for $j \geq -4$, and thus $\mathbf{A} = 1 + \sum_{j=-\infty}^{-5} P_{j/2}$. Using these facts, we can show that Q_{-3}, Q_{-4} and Q_{-5} are written as

$$\begin{aligned} Q_{-3} &= -P^*_{-3}, \qquad Q_{-4} = -P^*_{-4} + P^*_{-2} \circ P^*_{-2} \\ Q_{-5} &= -P^*_{-5} + P^*_{-5/2} \circ P^*_{-5/2} + P^*_{-2} \circ P^*_{-3} + P^*_{-3} \circ P^*_{-2}. \end{aligned}$$

The identification of $F_j(A)$ given in Proposition 3.1 is done by computing explicitly the terms of the form (4.13) in each Q_{-j}.

§5 Parabolic invariant theory

In this section, we outline the proof of Theorem 2.1. This amounts to reviewing the invariant theory of Fefferman [F3] supplemented by Bailey-Eastwood-Graham [BEG].

5.1 $H_\#$-invariants of the curvature

Recall that CR invariants are H-invariants of $A \in \mathcal{N}$ (cf. (1.5)). To compare the boundary values of Weyl invariants with CR invariants, it is convenient to represent linear fractional transformations $h \in H$ by matrices $h_\#$ with respect to the projective coordinates $\zeta = (\zeta_0, \ldots, \zeta_n)$ used in Subsection 3.2, (C). Let $H_\#$ denote the (parabolic) subgroup of $\mathrm{SU}(g_0)$ given by

$$H_\# = \{h_\# \in \mathrm{SU}(g_0);\ h_\# e_0 = \lambda e_0 \text{ with some } \lambda \in \mathbb{C}^*\},$$

where $e_0 = {}^t(1, 0, \ldots, 0) \in \mathbb{C}^{n+1}$. Then each element $h_\# \in H_\#$ defines $h \in H$ such that $\lambda = \big(\det h'(0)\big)^{-1/(n+1)}$. An $H_\#$-action on $\mathcal{N}$ is given by $h_\#.A = h.A$, and the definition of CR invariants (1.5) is written as

$$P(h_\#.A) = |\lambda|^{2w} P(A) \qquad \text{for } h_\# \in H_\#,\ \ A \in \mathcal{N}.$$

To regard the boundary values of Weyl invariants as $H_\#$-invariants, we need to define $H_\#$-invariants of the curvature R of the ambient metric by using the $H_\#$-action on $A \in \mathcal{N}$. We first identify R with its Taylor expansion about e_0 with respect to the coordinates $\zeta = (\zeta_0, \ldots, \zeta_n)$. That is, given a domain with boundary in Moser's normal form $N(A)$, we write $R = (R_{\alpha\overline{\beta}})_{|\alpha|,|\beta| \geq 2}$ for the components

$$R_{\alpha\overline{\beta}} = R_{\alpha_1\overline{\beta}_1\alpha_2\overline{\beta}_2;\alpha_3\cdots\alpha_p\overline{\beta}_3\cdots\overline{\beta}_q}$$

of the covariant derivatives of the curvature R evaluated at e_0, where $\alpha = \alpha_1 \cdots \alpha_p$ and $\beta = \beta_1 \cdots \beta_q$ are lists of holomorphic indices $0, 1, \ldots, n$. We now introduce the notion of *weight* for the components $R_{\alpha\overline{\beta}}$, as a generalization of that for Weyl invariants, by setting

$$\mathrm{w}(R_{\alpha\overline{\beta}}) = \mathrm{w}(\alpha\overline{\beta}) = \frac{\|\alpha\overline{\beta}\|}{2} - 1 \quad \text{with } \|\alpha\overline{\beta}\| = \sum_{j=1}^{p} \|\alpha_j\| + \sum_{j=1}^{q} \|\beta_j\|,$$

where $\|0\| = 0$, $\|j\| = 1$ for $j = 1, \ldots, n-1$ and $\|n\| = 2$.

Let us next restrict ourselves to the components $R_{\alpha\overline{\beta}}$ of weight $\leq n$. We then see, as in the proof of Proposition 2.1, that $R_{\alpha\overline{\beta}}$ is a polynomial in A, so that we may write $R_{\alpha\overline{\beta}}$ as $R_{\alpha\overline{\beta}}(A)$. Furthermore,

$(5.1)_1$ $\quad R_{\alpha\overline{\beta}}(A)$ is a polynomial in A of homogeneous weight $\mathrm{w}(\alpha\overline{\beta})$,

$(5.1)_2$ $$R_{\alpha\overline{\beta}}(A) = 0 \qquad (-1 \leq \mathrm{w}(\alpha\overline{\beta}) < 1).$$

These are seen as follows. Given $h_\# = (h_i{}^j) \in H_\#$, we consider the curvature corresponding to $h_\#.A \in \mathcal{N}$. Then the components of weight $\leq n$ and of type (p,q) are transformed by

$$(5.2) \qquad R_{\alpha\overline{\beta}}(h_\#.A) = \lambda^{p-1}\overline{\lambda}^{q-1} \sum_{|\alpha'|=p,|\beta'|=q} h_\alpha{}^{\alpha'} \overline{h_\beta{}^{\beta'}} R_{\alpha'\overline{\beta'}}(A)$$

with $\lambda \in \mathbb{C}$ defined by $h_\# e_0 = \lambda\, e_0$, where $h_\alpha{}^{\alpha'} = h_{\alpha_1}{}^{\alpha'_1} \cdots h_{\alpha_p}{}^{\alpha'_p}$ and $h_\beta{}^{\beta'} = h_{\beta_1}{}^{\beta'_1} \cdots h_{\beta_q}{}^{\beta'_q}$. The transformation law is thus weighted by the factor $\lambda^{p-1}\overline{\lambda}^{q-1}$. If in particular $h_\#$ corresponds to a dilation ϕ_r, then $R_{\alpha\overline{\beta}}(h_\#.A) = r^{-2\mathrm{w}(\alpha\overline{\beta})} R_{\alpha\overline{\beta}}(A)$. Thus $(5.1)_1$ is obtained. The proof of $(5.1)_2$ is simple. Since components of $A \in \mathcal{N}$ satisfy $\mathrm{w}(A^\ell_{\alpha\overline{\beta}}) \geq 1$, it follows that each $R_{\alpha\overline{\beta}}(A)$ with $\mathrm{w}(\alpha\overline{\beta}) < 1$ is a constant, which is 0 because $R_{\alpha\overline{\beta}}(0) = 0$.

Regarding $R_{\alpha\overline{\beta}} \in \mathbb{C} = \mathbb{R} + i\mathbb{R}$ with $|\alpha|, |\beta| \geq 2$ as independent variables, we denote by $\mathcal{R}^{\mathrm{aux}}$ the totality of the points $R = (R_{\alpha\overline{\beta}})_{|\alpha|,|\beta|\geq 2}$ satisfying

$$(5.3) \qquad R_{\alpha\overline{\beta}} = 0 \qquad (-1 \leq \mathrm{w}(\alpha\overline{\beta}) < 1).$$

Thus $\mathcal{R}^{\mathrm{aux}}$ is a countable dimensional real vector space. Truncating components of $R = (R_{\alpha\overline{\beta}}) \in \mathcal{R}^{\mathrm{aux}}$ by $\mathrm{w}(\alpha\overline{\beta}) \leq n$, we obtain an infinite dimensional vector space $\mathcal{R}^{\mathrm{aux}}_n$ as the quotient space of $\mathcal{R}^{\mathrm{aux}}$. This space $\mathcal{R}^{\mathrm{aux}}_n$ admits an $H_\#$-action

$$H_\# \times \mathcal{R}^{\mathrm{aux}}_n \ni (h_\#, R_n) \mapsto h_\#.R_n \in \mathcal{R}^{\mathrm{aux}}_n$$

given by the right side of (5.2) with $R_{\alpha'\overline{\beta'}}$ in place of $R_{\alpha'\overline{\beta'}}(A)$. In fact, since $h_i{}^j = 0$ for $\|i\| < \|j\|$, it follows that the $H_\#$-action on $\mathcal{R}^{\mathrm{aux}}_n$ above is well-defined.

Returning to the components of the curvature $R = (R_{\alpha\overline{\beta}})_{|\alpha|,|\beta|\geq 2}$, we write $R_n(A) = (R_{\alpha\overline{\beta}}(A))_{\mathrm{w}(\alpha\overline{\beta})\leq n}$ and denote by $\mathcal{R}_n$ the image of the map $\mathcal{N} \ni A \mapsto R_n(A) \in \mathcal{R}^{\mathrm{aux}}_n$. It then follows from (5.2) and the definition of the $H_\#$-action on $\mathcal{R}^{\mathrm{aux}}_n$ that

$$h_\#.(R_n(A)) = R_n(h_\#.A) \in \mathcal{R}^{\mathrm{aux}}_n.$$

That is, the map $A \mapsto R_n(A)$ is $H_\#$-equivariant and $\mathcal{R}_n$ is an $H_\#$-invariant subset of $\mathcal{R}^{\mathrm{aux}}_n$. In what follows, we sometimes abbreviate the variable $R_n \in \mathcal{R}^{\mathrm{aux}}_n$ by writing R.

DEFINITION. A polynomial $P = P(R)$ in $R \in \mathcal{R}_n^{\mathrm{aux}}$ is called an *$H_\#$-invariant* of $\mathcal{R}_n$ of *weight* $w \leq n$ if

$$P(h_\#.R) = |\lambda|^{2w} P(R) \quad \text{for any} \quad (h_\#, R) \in H_\# \times \mathcal{R}_n.$$

Two $H_\#$-invariants are identified if these are identical as functions on $\mathcal{R}_n$. The totality of $H_\#$-invariants of $\mathcal{R}_n$ is denoted by $I_w(\mathcal{R}_n)$.

For $R \in \mathcal{R}^{\mathrm{aux}}$, let us consider complete contractions

$$W(R) = \mathrm{contr}\left(R^{(p_1,q_1)} \otimes \cdots \otimes R^{(p_s,q_s)}\right) \tag{5.4}$$

of the tensors $R^{(p,q)} = (R_{\alpha\overline{\beta}})_{|\alpha|=p,|\beta|=q}$ with respect to the flat metric g_0. Then $W(R)$ is a polynomial in $R \in \mathcal{R}^{\mathrm{aux}}$ of homogeneous weight. If $\mathrm{w}(W(R)) \leq n$, then $W(R)$ depends only on $R \in \mathcal{R}_n^{\mathrm{aux}}$ because of (5.3), and thus $W(R)$ gives an $H_\#$-invariant of $\mathcal{R}_n$. We define *Weyl invariants of* $\mathcal{R}_n$ as linear combinations of the complete contractions of the form (5.4) which are of homogeneous weight $\leq n$. Denoting by $I_w^{\mathrm{W}}(\mathcal{R}_n)$ the totality of Weyl invariants of weight w, we have $I_w^{\mathrm{W}}(\mathcal{R}_n) \subset I_w(\mathcal{R}_n)$ for $w \leq n$.

The surjection $\mathcal{N} \ni A \mapsto R(A) \in \mathcal{R}_n$ induces a map

$$I_w(\mathcal{R}_n) \ni P(R) \mapsto P(R(A)) \in I_w^{\mathrm{CR}} \qquad (w \leq n). \tag{5.5}$$

Therefore, Theorem 2.1 follows from:

Theorem 2.1′. (I) *The map* (5.5) *is surjective (and thus bijective).*

(II) $I_w^{\mathrm{W}}(\mathcal{R}_n) = I_w(\mathcal{R}_n)$ *for* $w \leq n$.

We outline the proofs of (I) and (II) in Subsections 5.2 and 5.3, respectively.

5.2 Bijectivity of (5.5)

The proof of the part (I) in Theorem 2.1′ is done by giving the inverse of the map (5.5). We first note by $w \leq n$ that any $Q(A) \in I_w^{\mathrm{CR}}$ depends only on

$$A_n = (A^\ell_{\alpha\overline{\beta}})_{\mathrm{w}(\alpha\overline{\beta}\ell)\leq n} \quad \text{for } A = (A^\ell_{\alpha\overline{\beta}}) \in \mathcal{N},$$

so that one may write $Q(A) = Q(A_n)$. Let $\mathcal{N}_n$ denote the totality of such A_n, that is, $\mathcal{N}_n = \{A_n;\ A \in \mathcal{N}\}$. Then, $R(A) \in \mathcal{R}_n$ for $A \in \mathcal{N}$

depends only on $A_n \in \mathcal{N}_n$, and thus the map $\mathcal{N} \ni A \mapsto R(A) \in \mathcal{R}_n$ induces a surjection

$$\mathcal{F} : \mathcal{N}_n \ni A_n \mapsto R(A_n) \in \mathcal{R}_n, \quad \text{where } R(A_n) = R(A). \tag{5.6}$$

This surjection is $H_\#$-equivariant, where the $H_\#$-action

$$H_\# \times \mathcal{N}_n \ni (h_\#, A_n) \mapsto h_\#.A_n \in \mathcal{N}_n$$

is well-defined from the $H_\#$-action on $\mathcal{N}$. We have:

Theorem 5.1. *The surjection $\mathcal{F}$ in* (5.6) *is bijective and the inverse $\mathcal{G} = \mathcal{F}^{-1}$ extends to a polynomial map $\mathcal{R}_n^{\mathrm{aux}} \to \mathcal{N}_n$, in the sense that the components are polynomials in $R \in \mathcal{R}_n^{\mathrm{aux}}$. (The map $\mathcal{G}$ is automatically $H_\#$-equivariant.)*

Assuming for a while the validity of Theorem 5.1, let us first prove the bijectivity of the map (5.5). Given $Q(A_n) \in I_w^{\mathrm{CR}}$ arbitrarily, we set $P(R) = Q(\mathcal{G}(R))$ for $R \in \mathcal{R}_n$. Then

$$P(\mathcal{F}(A_n)) = Q(\mathcal{G} \circ \mathcal{F}(A_n)) = Q(A_n),$$

and the $H_\#$-equivariance of $\mathcal{G}$ implies $P(R) \in I_w(\mathcal{R}_n)$. Conversely, given $P(R) \in I_w(\mathcal{R}_n)$ arbitrarily, we set $Q(A_n) = P(\mathcal{F}(A_n))$ for $A_n \in \mathcal{N}_n$. Then

$$Q(\mathcal{G}(R)) = P(\mathcal{F} \circ \mathcal{G}(R)) = P(R),$$

and the $H_\#$-equivariance of $\mathcal{F}$ implies $Q(A_n) \in I_w^{\mathrm{CR}}$. Consequently, the pull-back by $\mathcal{G}$ gives the inverse map of (5.5), and thus (I) in Theorem 2.1′ is proved.

To prove Theorem 5.1, we extend the target space $\mathcal{R}_n$ of the map $\mathcal{F}$ in (5.6) to $\mathcal{R}_n^{\mathrm{aux}}$. That is, if we denote this new map again by $\mathcal{F}$,

$$\mathcal{F} : \mathcal{N}_n \to \mathcal{R}_n^{\mathrm{aux}} \qquad (\text{and } \mathcal{F}(\mathcal{N}_n) = \mathcal{R}_n)\,. \tag{5.7}$$

Now note that $\mathcal{F}$ is finite dimensional in the sense that $\mathcal{N}_n$ is a finite dimensional vector space. Then the injectivity of $\mathcal{F}$ follows from the following proposition.

Proposition 5.1. *The differential $\mathcal{F}'(0) : \mathcal{N}_n \to \mathcal{R}_n^{\mathrm{aux}}$ of $\mathcal{F}$ in* (5.7) *at the origin is injective. Consequently, $\mathcal{F}$ is an embedding and $\mathcal{R}_n \subset \mathcal{R}_n^{\mathrm{aux}}$ is a finite dimensional manifold. (We are always working near the origin.)*

To complete the proof of Theorem 5.1, it remains to show that $\mathcal{G}$ extends to a polynomial map. By Proposition 5.1, we get an extension of $\mathcal{G}$,

$$\mathcal{R}_n^{\mathrm{aux}} \ni R \mapsto A(R) = (A^{\ell}_{\alpha\overline{\beta}}(R)) \in \mathcal{N}_n,$$

such that each component $A^{\ell}_{\alpha\overline{\beta}}(R)$ is a formal power series in R of homogeneous weight $\mathrm{w}(\alpha\overline{\beta}\ell)$. In addition, the series $A^{\ell}_{\alpha\overline{\beta}}(R)$ depends only on a finite number of components of R and is convergent near the origin. Using $(5.1)_2$, we can remove monomials of degree $> \mathrm{w}(\alpha\overline{\beta}\ell)$ from $A^{\ell}_{\alpha\overline{\beta}}(R)$ without changing the value on $\mathcal{R}_n$. The resulting polynomials give a polynomial extension of $\mathcal{G}$.

We conclude this subsection by sketching the proof of Proposition 5.1. Setting $R_n = \mathcal{F}'(0)A_n$, we wish to show that $R_n = 0$ implies $A_n = 0$. To express $R_n = (R_{\alpha\overline{\beta}})$ explicitly, we take Fefferman's defining functions $r_{\varepsilon A}$ of $N(\varepsilon A)$ given in Subsection 3.2, (B), and denote by $\widetilde{r}^{\mathrm{F}}_A$ the first variation at $\varepsilon = 0$. Then

$$(5.8) \qquad R_{\alpha\overline{\beta}} = \partial_\zeta^\alpha \partial_{\overline{\zeta}}^\beta (\widetilde{r}^{\mathrm{F}}_A)_\# |_{e_0}, \quad \text{where } (\widetilde{r}^{\mathrm{F}}_A)_\#(\zeta,\overline{\zeta}) = |z_0|^2 \widetilde{r}^{\mathrm{F}}_A(z,\overline{z}).$$

Turning from ∂_ζ and $\partial_{\overline{\zeta}}$ to ∂_z and $\partial_{\overline{z}}$, we see that the assumption $R_n = 0$ is equivalent to

$$(5.9) \qquad \partial_z^\alpha \partial_{\overline{z}}^\beta (\widetilde{r}^{\mathrm{F}}_A)_\#(0,0) = 0 \qquad (\mathrm{w}(\alpha\overline{\beta}) \le n,\ |\alpha|, |\beta| \ge 2).$$

On the other hand, we have seen in Subsection 3.2, (C) that $\widetilde{r}^{\mathrm{F}}_A$ is uniquely determined modulo $O^{n+2}(\rho_0)$ as a solution of the linear equation

$$(5.10) \quad L_{\rho_0}(\widetilde{r}^{\mathrm{F}}_A)_\# = O^{n+1}(\rho_0), \quad \widetilde{r}^{\mathrm{F}}_A|_{2u=|z'|^2} = -\sum_{\mathrm{w}(\alpha\overline{\beta}\ell)\le n} A^{\ell}_{\alpha\overline{\beta}}\, z'_\alpha\, \overline{z'_\beta}\, v^\ell.$$

Now $A_n = 0$ follows from (5.9) via (5.10). The proof is similar to that of the uniqueness of Moser's normal form, where the trace conditions (1.3) are used crucially.

5.3 $H_\#$-invariants of $\mathcal{R}_n$ are Weyl invariants.

Let $T_0\mathcal{R}_n$ denote the tangent space of $\mathcal{R}_n$ at the origin, and thus

$$T_0\mathcal{R}_n = \mathcal{F}'(0)\, T_0\, \mathcal{N}_n \subset \mathcal{R}_n^{\mathrm{aux}} \qquad (T_0\mathcal{N}_n = \mathcal{N}_n \text{ as a set}).$$

Then the $H_\#$-action on $\mathcal{R}_n$ induces an $H_\#$-action on $T_0\mathcal{R}_n$, which agrees with that on $\mathcal{R}_n^{\mathrm{aux}}$ restricted to $T_0\mathcal{R}_n$. The $H_\#$-invariants of $T_0\mathcal{R}_n$ is defined as in the definition of those of $\mathcal{R}_n$, in which $\mathcal{R}_n$ is literally replaced by $T_0\mathcal{R}_n$. Now the proof of the part (II) in Theorem 2.1′ is reduced to:

Theorem 5.2. *Every $H_\#$-invariant of $T_0\mathcal{R}_n$ of weight $\leq n$ is a Weyl invariant.*

Assuming the validity of Theorem 5.2 above for a moment, we first prove the statement (II). Let $P(R)$ be an $H_\#$-invariant of $\mathcal{R}_n$ of weight $\leq n$. We denote by $p(R)$ the lowest degree part of $P(R)$. Then $p(R)$ is an $H_\#$-invariant on $T_0\mathcal{R}_n$. It then follows from Theorem 5.2 that there exists a Weyl invariant $W(R)$ such that $p(R) = W(R)$ on $T_0\mathcal{R}_n$. Though $W(R)$ differs $p(R)$ from on $\mathcal{R}_n$, the difference consists of terms of higher degree. Thus we can repeat this procedure and write $P(R)$ as a sum of Weyl invariants. This proves (II).

The proof of Theorem 5.2 requires a defining system of equations of $T_0\mathcal{R}_n$. In view of (5.8), we have

$$T_0\mathcal{R}_n = \{(R_{\alpha\overline{\beta}}) \in \mathcal{R}_n^{\mathrm{aux}}\,;\ R_{\alpha\overline{\beta}} = \partial_\zeta^\alpha \partial_{\overline{\zeta}}^\beta (\widetilde{r}_A^{\mathrm{F}})_\#|_{e_0},\ A \in \mathcal{N}_n\}.$$

From this expression, we obtain a defining system of $T_0\mathcal{R}_n$ in terms of the variables $(R_{\alpha\overline{\beta}}) \in \mathcal{R}_n^{\mathrm{aux}}$:

$$(5.11)_1 \qquad R_{\alpha\overline{\beta}} = \overline{R_{\beta'\overline{\alpha'}}} \quad \text{(for any permutation } \alpha'\,\overline{\beta'} \text{ of } \alpha\,\overline{\beta}\,),$$

$$(5.11)_2 \qquad R_{0\alpha\overline{\beta}} = (1-|\alpha|)R_{\alpha\overline{\beta}}, \qquad R_{\alpha\overline{0\beta}} = (1-|\beta|)R_{\alpha\overline{\beta}},$$

$$(5.11)_3 \qquad \sum_{j,k=0}^{n} g_0^{j\overline{k}} R_{j\alpha\overline{k\beta}} = R_{0\alpha\overline{n\beta}} + R_{n\alpha\overline{0\beta}} - \sum_{j=1}^{n-1} R_{j\alpha\overline{j\beta}} = 0.$$

Here $(g_0^{j\overline{k}}) = (g_0)^{-1} = g_0$. The Hermitian symmetry $(5.11)_1$ is equivalent to the fact that $(\widetilde{r}_A^{\mathrm{F}})_\#$ is real. The reduction rule $(5.11)_2$ is a consequence of the homogeneity of $(\widetilde{r}_A^{\mathrm{F}})_\#$ in ζ and $\overline{\zeta}$. (Here we have set $R_{\alpha\overline{\beta}} = 0$ if $|\alpha| \leq 1$ or $|\beta| \leq 1$.) The relation $(5.11)_3$ comes from $\Delta_0(\widetilde{r}_A^{\mathrm{F}})_\# = O^{n+1}((\rho_0)_\#)$ of (3.15).

Disregarding the weight restriction, we consider $H_\#$-invariants of

$$\mathcal{H} = \big\{(R_{\alpha\overline{\beta}})_{|\alpha|,|\beta|\geq 2};\ R_{\alpha\overline{\beta}} \text{ satisfy } (5.11)_1,\ (5.11)_2 \text{ and } (5.11)_3\big\}.$$

As far as $H_\#$-invariants of weight $\leq n$ are concerned, an invariant of $T_0\mathcal{R}_n$ is an invariant of $\mathcal{H}$, and vice versa. Consequently, Theorem 5.2 is contained in a more general:

Theorem 5.3. *Every $H_\#$-invariant of $\mathcal{H}$ is a Weyl invariant.*

Fefferman [F3] proved this result for invariants of weight $\leq n - 19$. The weight restriction was later removed by Bailey-Eastwood-Graham

[BEG]. The proof in [BEG] is constructive and gives an algorithm of writing the given $H_{\#}$-invariant as a linear combination of complete contractions. In what follows, we explain this algorithm.

Let $R = (R_{\alpha\overline{\beta}}) \in \mathcal{H}$. Then each $R_{\alpha\overline{\beta}}$ is written as a linear combination of

$$\widetilde{A}^{\ell}_{\alpha'\overline{\beta'}} = R_{\alpha' n \cdots n \overline{\beta'}}$$

where α' and β' are lists of $\{1, \ldots, n-1\}$ of length ≥ 2, and the number of n is ℓ. In fact, indices 0, $\overline{0}$, $\overline{n}$ in $R_{\alpha\overline{\beta}}$ can be deleted by repeated use of $(5.11)_2$ and $(5.11)_3$. Setting $\widetilde{A}^{\ell}_{p\overline{q}} = (\widetilde{A}^{\ell}_{\alpha\overline{\beta}})_{|\alpha|=p, |\beta|=q}$, we regard it as a symmetric tensor on $\mathbb{C}^{n-1}$ of type (p,q). Then the $U(n-1)$-action on $\widetilde{A}^{\ell}_{p\overline{q}}$ is the restriction of the $H_{\#}$-action to $U(n-1) \subset H_{\#}$. Thus an $H_{\#}$-invariant $P(R)$ can be regarded as a $U(n-1)$-invariant $P(\widetilde{A})$ of $\widetilde{A}^{\ell}_{\alpha\overline{\beta}}$. (This procedure amounts to rewriting polynomials in R as those in $A^{\ell}_{\alpha\overline{\beta}}$.) Using Weyl's invariant theory for $U(n-1)$, we can write $P(\widetilde{A})$ as a linear combination of complete contractions on $\mathbb{C}^{n-1}$, that is, those with respect to $(\delta^{j\overline{k}})$:

$$\text{contr}\left(\widetilde{A}^{\ell_1}_{p_1\overline{q}_1} \otimes \cdots \otimes \widetilde{A}^{\ell_d}_{p_d\overline{q}_d}\right). \tag{5.12}$$

In addition, we can make so that these contractions do not contain

$$\text{tr}\widetilde{A}^{\ell}_{2\overline{2}},\ (\text{tr})^2\widetilde{A}^{\ell}_{2\overline{3}},\ (\text{tr})^2\widetilde{A}^{\ell}_{3\overline{2}},\ (\text{tr})^3\widetilde{A}^{\ell}_{3\overline{3}}.$$

From these contractions on $\mathbb{C}^{n-1}$, we manufacture complete contractions with respect to the ambient metric g_0, depending on the degree d of the polynomial $P(\widetilde{A})$, as follows.

At first, let $d < n$. From the linear combination $P(\widetilde{A})$ of complete contractions of the form (5.12), we make a partial sum consisting of terms corresponding to $\ell_1 = \cdots = \ell_d = 0$, and replace the complete contractions there formally by those with respect to g_0:

$$\text{contr}\left(R^{(p_1,q_1)} \otimes \cdots \otimes R^{(p_d,q_d)}\right).$$

Then we get a Weyl invariant, which agrees with the given $H_{\#}$-invariant $P(R)$. The proof of this fact requires careful examination of the $H_{\#}$-action on complete contractions of the form (5.12).

When $d \geq n$, we cannot expect this. In fact, if for instance an $H_{\#}$-invariant $P(R)$ of degree $d \geq n$ contains an alternating sum of n indices, then $P(R)$ is not manufactured by the procedure above. We thus proceed as follows. Let us first recall that, in the case $d < n$, we have formally replaced complete contractions with respect to $(\delta^{j\overline{k}})$ by those with respect to g_0. That is, we have ignored the right side of

$$\sum_{j,k=1}^{n-1} \delta^{j\overline{k}} T_{j\overline{k}} + \sum_{j,k=0}^{n} g_0^{j\overline{k}} T_{j\overline{k}} = T_{0\overline{n}} + T_{n\overline{0}}$$

for an arbitrary tensor $(T_{j\overline{k}})$ of type $(1,1)$. Taking account of the right side, we now express complete contractions of the form (5.12) in terms of partial contractions with respect to g_0. That is, we manufacture tensors $(T_{\alpha\overline{\beta}})$ by partially contracting $R^{(p_1,q_1)} \otimes \cdots \otimes R^{(p_s,q_s)}$ in such a way that (5.12) is given by a linear combination of components of $(T_{\alpha\overline{\beta}})$ of the form $T_{0\cdots 0 n \cdots n \overline{0} \cdots \overline{0} \overline{n} \cdots \overline{n}}$. The indices 0 and $\overline{0}$ can be eliminated by repeated use of $(5.11)_2$. We then get an expression of the given $H_{\#}$-invariant $P(R)$ as a linear combination of components $T_{n\cdots n \overline{n} \cdots \overline{n}}$ of $(T_{\alpha\overline{\beta}})$. Let us recall by the definition of $T_0\mathcal{R}_n$ that $R_{\alpha\overline{\beta}} = \partial_\zeta^\alpha \partial_{\overline{\zeta}}^\beta (\widetilde{r}_A^{\mathrm{F}})_{\#}|_{e_0}$. Likewise, $T_{\alpha\overline{\beta}}$ are given by the "values" of formal power series at e_0. Then $T_{\alpha\overline{\beta}}$ are extended to jets at e_0, and the partial derivatives of the extensions of $T_{\alpha\overline{\beta}}$ make sense. If these are used as substitutes for the covariant derivatives, then a scalar is obtained by the complete contraction. We do this procedure after some algebraic manipulations which are technical. Then the scalar above is a Weyl invariant, which coincides with the original $H_{\#}$-invariant $P(R)$ up to a multiple. It turns out that components of $(T_{\alpha\overline{\beta}})$ other than $T_{n\cdots n \overline{n} \cdots \overline{n}}$ do not contribute to the resulting Weyl invariant.

§6 Full invariant expansion of the Bergman kernel

So far, we have considered an invariant expression of the singularity of the Bergman kernel $K^{\mathrm{B}} = \varphi^{\mathrm{B}} r^{-n-1} + \psi^{\mathrm{B}} \log r$ by using Fefferman's defining function $r = r^{\mathrm{F}}$. Because of the ambiguity of r^{F}, it was only possible to express φ^{B} modulo $O^{n+1}(r)$ in general (Theorem 2.2), and ψ^{B} modulo $O^2(r)$ even in the case $n = 2$ (Theorems 3.1 and 3.2). In this section, we express ψ^{B} modulo $O^\infty(r)$ invariantly by using a special family of Fefferman's defining functions. (The details are found in [Hi].) That family, which we denote by $\mathcal{R}_{\partial\Omega}^{\mathrm{F}}$, is parametrized by $C^\infty(\partial\Omega)$ and satisfies

$$(6.1) \qquad r_1 := |\det \Phi'|^{-2/(n+1)} r_2 \circ \Phi \in \mathcal{R}_{\partial\Omega_1}^{\mathrm{F}} \quad \text{for} \quad r_2 \in \mathcal{R}_{\partial\Omega_2}^{\mathrm{F}}$$

for biholomorphic mappings $\Phi : \Omega_1 \to \Omega_2$. This can be regarded as an exact transformation law of weight -1 without error.

In Subsection 6.1 below, we lift the Monge-Ampère boundary value problem (1.12) to a $\mathbb{C}^*$-bundle over Ω. Then the lifted problem admits asymptotic solutions which are similar to those of the original problem (1.12) in Theorem 1.3. Elements of $\mathcal{R}^{\mathrm{F}}_{\partial\Omega}$ are obtained as the "smooth parts" of these asymptotic solutions. Using $r \in \mathcal{R}^{\mathrm{F}}_{\partial\Omega}$, we define as before Weyl invariants, which inherit the ambiguity measured by $\mathcal{R}^{\mathrm{F}}_{\partial\Omega}$. These Weyl invariants with ambiguity, together with r, are used in expressing a full expansion of ψ^{B} in the Bergman kernel.

6.1 A special family of Fefferman's defining functions

Given a strictly pseudoconvex domain Ω with C^∞ boundary, we take a thin one-sided neighborhood $V \subset \overline{\Omega}$ of $\partial\Omega$ and consider the following equation for a function $U = U(z_0, z)$ on $\mathbb{C}^* \times V$:

$$(6.2) \qquad (-1)^n \det\left(\partial^2 U/\partial z_j \partial \overline{z}_k\right)_{j,k=0,\dots,n} = |z_0|^{2n}.$$

In terms of differential forms, (6.2) is intrinsically written as

$$(6.3) \qquad (-1)^n (\partial\overline{\partial} U)^{n+1} = dv,$$

where $dv = (n+1)!|z_0|^{2n} dz_0 \wedge d\overline{z}_0 \wedge \cdots \wedge dz_n \wedge d\overline{z}_n$. If U is of the form $U(z_0, z) = |z_0|^2 u(z)$, then (6.2) is reduced to the equation $J[u] = 1$. That is, (6.2) is a lift of the complex Monge-Ampère equation to the $\mathbb{C}^*$-bundle $\mathbb{C}^* \times V$. The bundle structure on $\mathbb{C}^* \times V$ is given by $\Phi_\#$ in (2.3), where Φ is a local (or formal) biholomorphic change of coordinates near a point of $\partial\Omega$. The transition function $\Phi_\#$ preserves dv. Thus $\Phi_\#$ preserves the equation (6.3) in the sense that if U_2 satisfies (6.3) so does $U_1 = U_2 \circ \Phi_\#$.

We consider asymptotic solutions to (6.2) of the form

$$(6.4) \qquad U = r_\# + r_\# \sum_{k=1}^{\infty} \eta_k \cdot (r^{n+1} \log r_\#)^k \quad \text{with } r_\# = |z_0|^2 r,$$

where $\eta_k \in C^\infty(V)$, and r is a defining function of Ω, $r > 0$ in Ω. Let us identify two such formal series if the corresponding r and η_k agree modulo $O^\infty(\partial\Omega)$. Then the totality of such asymptotic solutions is parametrized by $C^\infty(\partial\Omega)$ as follows.

Proposition 6.1. *Let X be a C^∞ vector field on V which is transversal to $\partial\Omega$. Then, for any $f \in C^\infty(\partial\Omega)$, there exists a unique asymptotic solution U of the form* (6.4) *to the equation* (6.2) *such that*

$$X^{n+2} r|_{\partial\Omega} = f.$$

If U_2 is an asymptotic solution of the form (6.4) in $V_2 \supset \partial\Omega_2$ so is $U_1 = U_2 \circ \Phi_\#$ in $V_1 \supset \partial\Omega_1$, where $\Phi : V_1 \to V_2$ is a biholomorphic mapping satisfying $\Phi(\partial\Omega_1) = \partial\Omega_2$. This transformation law is rewritten as $r_1 = |\det \Phi'|^{-2/(n+1)} r_2 \circ \Phi$ and $\eta_{1,k} = |\det \Phi'|^{2k} \eta_{2,k} \circ \Phi$, where

$$U_j = (r_j)_\# + (r_j)_\# \sum_{k=1}^{\infty} \eta_{j,k} \cdot \left(r_j^{n+1} \log (r_j)_\#\right)^k \qquad (j = 1, 2).$$

For an asymptotic solution U to (6.2) of the form (6.4), we call r in (6.4) the *smooth part* of U, and denote by $\mathcal{R}^{\mathrm{F}}_{\partial\Omega}$ the totality of the smooth parts. Then the transformation law (6.1) for $\mathcal{R}^{\mathrm{F}}_{\partial\Omega}$ is valid. In addition, each smooth part r is a Fefferman's defining function, that is, r satisfies $J[r] = 1 + O^{n+1}(r)$.

Remark 6.1. If we drop the subscript # from $r_\#$ in (6.4), then we get Graham's asymptotic solutions (1.17) with $\eta_0^{\mathrm{G}} = 1$. However, the transformation law (6.1) breaks down. Similarly, if we add the subscript # to r in $r^{n+1} \log r_\#$, then again (6.1) breaks down.

6.2 A refinement of Theorem 2.2

Starting from a Fefferman's defining function $r \in \mathcal{R}^{\mathrm{F}}_{\partial\Omega}$, we construct Weyl invariants as in Subsection 2.2. That is, for the Lorentz-Kähler metric g with potential $r_\#$ in a thin neighborhood $\mathbb{C}^* \times V \subset \mathbb{C}^* \times \overline{\Omega}$ of $\mathbb{C}^* \times \partial\Omega$, we consider the curvature R of g and successive covariant derivatives $R^{(p,q)} = \overline{\nabla}^{q-2} \nabla^{p-2} R$. Then a *Weyl invariant* of weight w is defined as a linear combination of the complete contractions of the form

$$\operatorname{contr}\left(R^{(p_1,q_1)} \otimes \cdots \otimes R^{(p_s,q_s)}\right) \quad \text{with} \quad \frac{1}{2} \sum_{j=1}^{s} (p_j + q_j) - s = w.$$

By definition, a Weyl invariant $W_\#$ is a functional of $r \in \mathcal{R}^{\mathrm{F}}_{\partial\Omega}$, and thus we write $W_\# = W_\#[r]$. As in Section 2, we also use this terminology for the composite function $(z_0, z) \mapsto W_\#[r]$. We denote the restriction of $W_\#[r]$ to $z_0 = 1$ by $W[r]$, and still call it a *Weyl invariant.* It follows from the construction that the transformation law (6.1) for $\mathcal{R}^{\mathrm{F}}_{\partial\Omega}$ implies

$$W[r_1] = |\det \Phi'|^{2w/(n+1)} W[r_2] \circ \Phi$$

for a Weyl invariant of weight w, cf. (2.4) in Subsection 2.3.

With $r \in \mathcal{R}^{\mathrm{F}}_{\partial\Omega}$, let us consider the expression (1.10) in Theorem 1.2 for the Bergman kernel. Observe that ψ^{B} is uniquely determined modulo $O^\infty(r)$ and independent of the choice of r. Nevertheless, we regard ψ^{B} as a functional of $r \in \mathcal{R}^{\mathrm{F}}_{\partial\Omega}$ and write $\psi^{\mathrm{B}} = \psi^{\mathrm{B}}[r]$. Then we have:

Theorem 6.1. *For each $j \geq n+1$, there exists a Weyl invariant W_j of weight j such that if $r \in \mathcal{R}^{\mathrm{F}}_{\partial\Omega}$ then*

$$\psi^{\mathrm{B}}[r] = \sum_{k=0}^{\infty} W_{k+n+1}[r]\, r^k \mod O^{\infty}(r).$$

That is, for each $m > 0$, $\psi^{\mathrm{B}}[r] = \sum_{k=0}^{m} W_{k+n+1}[r]\, r^k \mod O^{m+1}(r)$.

This theorem refines Theorem 2.2 (cf. Remark 6.2 below).

6.3 Generalization of the CR invariant

Recall that Theorem 2.2 follows from Theorem 2.1. In order to refine Theorem 2.1, we need to generalize the notion of CR invariant taking account of the ambiguity described by $\mathcal{R}^{\mathrm{F}}_{\partial\Omega}$. Let us begin by recalling that Proposition 6.1 gives a bijection $C^{\infty}(\partial\Omega) \to \mathcal{R}^{\mathrm{F}}_{\partial\Omega}$ as far as a vector field X is specified. For a reference point $p \in \partial\Omega$, this parametrization is localizable to a neighborhood of p, but we rather consider formally. We have a bijection $C^{\infty}_{\partial\Omega,p} \to \mathcal{R}^{\mathrm{F}}_{\partial\Omega,p}$, where $C^{\infty}_{\partial\Omega,p}$ and $\mathcal{R}^{\mathrm{F}}_{\partial\Omega,p}$ denote the spaces of all Taylor expansions about p of elements of $C^{\infty}(\partial\Omega)$ and $\mathcal{R}^{\mathrm{F}}_{\partial\Omega}$, respectively. Thus $C^{\infty}_{\partial\Omega,p}$ and $\mathcal{R}^{\mathrm{F}}_{\partial\Omega,p}$ consist of formal power series, though the notation $C^{\infty}_{\partial\Omega,p}$ might be somewhat confusing.

The family $\mathcal{R}^{\mathrm{F}}_{\partial\Omega,p}$ satisfies a formal transformation law corresponding to (6.1), and this transformation law is transplanted to $C^{\infty}_{\partial\Omega,p}$. To write it down explicitly, we assume that $\partial\Omega$ near p is in Moser's normal form $N(A)$, and take $X = \partial/\partial\rho_A$ with respect to the coordinate system $(z', \overline{z'}, \rho_A, v)$. Each element $f \in C^{\infty}_{\partial\Omega,p}$ is written in the form

$$f(z', \overline{z'}, v) = \sum_{\alpha,\beta,\ell} C^{\ell}_{\alpha\overline{\beta}}\, z'_{\alpha}\, \overline{z'_{\beta}}\, v^{\ell}.$$

We denote by $\mathcal{C}$ the totality of collections of the coefficients $C = (C^{\ell}_{\alpha\overline{\beta}})$. Thus $C^{\infty}_{\partial\Omega,p}$ is identified with $\mathcal{C}$. If $r \in \mathcal{R}^{\mathrm{F}}_{\partial\Omega,p}$ is in the image of f under the bijection $C^{\infty}_{\partial\Omega,p} \to \mathcal{R}^{\mathrm{F}}_{\partial\Omega,p}$, then

$$r = \sum_{\alpha,\beta,\ell,m} P^{\ell m}_{\alpha\overline{\beta}}(A, C)\, z'_{\alpha}\, \overline{z'_{\beta}}\, v^{\ell} \rho_A^m,$$

where $P^{\ell m}_{\alpha\overline{\beta}}(A, C)$ are polynomials in $(A, C) \in \mathcal{N} \times \mathcal{C}$. We thus write $r = r(A, C)$, and use the notation $\mathcal{R}^{\mathrm{F}}_{N(A)}$ for the totality of $r = r(A, C)$ with $(A, C) \in \mathcal{N} \times \mathcal{C}$. Thus $\mathcal{R}^{\mathrm{F}}_{\partial\Omega,p}$ is identified with $\mathcal{R}^{\mathrm{F}}_{N(A)}$, and we have a bijection $\mathcal{C} \to \mathcal{R}^{\mathrm{F}}_{N(A)}$ as far as $A \in \mathcal{N}$ is specified.

The H-action (1.4) on $\mathcal{N}$ extends to that on $\mathcal{N} \times \mathcal{C}$ as follows. For $(A, C) \in \mathcal{N} \times \mathcal{C}$ and $h \in H$, we define $(\widetilde{A}, \widetilde{C}) = h.(A, C)$ by $\widetilde{A} = h.A$ and $r(\widetilde{A}, \widetilde{C}) = |\det E'_{h,A}|^{-2/(n+1)} r(A, C) \circ h$, where $E_{h,A}$ is defined by (1.4)$'$. Then we have, as a generalization of (1.4), a group action

$$H \times \mathcal{N} \times \mathcal{C} \ni (h, A, C) \mapsto h.(A, C) \in \mathcal{N} \times \mathcal{C}, \tag{6.5}$$

which is regarded as a transformation law for $C^{\infty}_{\partial\Omega,p}$ parametrizing $\mathcal{R}^{\mathrm{F}}_{\partial\Omega,p}$.

We now recall that CR invariants are defined by (1.5). This notion is generalized as follows. Let $I^{\mathrm{CR}}_w(\mathcal{C})$ denote the totality of polynomials P in $(A, C) \in \mathcal{N} \times \mathcal{C}$ such that

$$P(A, C) = |\det h'(0)|^{2w/(n+1)} P(h.(A, C)) \quad \text{for any} \quad h \in H.$$

Then $I^{\mathrm{CR}}_w = I_w(\mathcal{N}) \subset I_w(\mathcal{N} \times \mathcal{C}) = I^{\mathrm{CR}}_w(\mathcal{C})$, where $I_w(\mathcal{N} \times \mathcal{C})$ stands for the space of *H-invariants* of $\mathcal{N} \times \mathcal{C}$, and similarly for $I_w(\mathcal{N})$. As in the case of CR invariants, elements of $I^{\mathrm{CR}}_w(\mathcal{C})$ can be identified with smooth (C^{∞} or real analytic) functions on $\partial\Omega$.

6.4 Boundary values of $\mathcal{C}$-dependent Weyl invariants

We want to refine Theorem 2.1 in such a way that the refinement implies Theorem 6.1. As in the previous subsection, let us consider a surface in Moser's normal form $N(A)$, and take $X = \partial/\partial\rho_A$ with respect to the coordinate system $(z', \overline{z'}, \rho_A, v)$. For a Weyl invariant $W = W[r]$ of weight w, the value at the origin is a polynomial in (A, C). We thus write it as $W(A, C)$, and denote the totality of these polynomials by $I^{\mathrm{W}}_w(\mathcal{N} \times \mathcal{C})$. Let $I^{\mathrm{W}}_w(\mathcal{N})$ denote the totality of $W(A, C) \in I^{\mathrm{W}}_w(\mathcal{N} \times \mathcal{C})$ which do not contain the variable $C \in \mathcal{C}$. Then Proposition 2.1 implies $I^{\mathrm{W}}_w(\mathcal{N} \times \mathcal{C}) = I^{\mathrm{W}}_w(\mathcal{N})$ for $w \leq n$, and Theorem 2.2 is restated as

$$I^{\mathrm{W}}_w(\mathcal{N} \times \mathcal{C}) = I^{\mathrm{W}}_w(\mathcal{N}) = I^{\mathrm{CR}}_w \qquad \text{for} \quad w \leq n.$$

Improving this, we have:

Theorem 6.2. *For any $w \in \mathbb{N}_0$, $I^{\mathrm{W}}_w(\mathcal{N} \times \mathcal{C}) = I^{\mathrm{CR}}_w(\mathcal{C})$ and thus $I^{\mathrm{W}}_w(\mathcal{N}) = I^{\mathrm{CR}}_w$.*

Theorem 6.3. *If $n \geq 3$, then $I^{\mathrm{W}}_w(\mathcal{N} \times \mathcal{C}) = I^{\mathrm{W}}_w(\mathcal{N})$ for $w \leq n+2$ and $I^{\mathrm{W}}_{n+3}(\mathcal{N} \times \mathcal{C}) \neq I^{\mathrm{W}}_{n+3}(\mathcal{N})$. If $n = 2$, then $I^{\mathrm{W}}_w(\mathcal{N} \times \mathcal{C}) = I^{\mathrm{W}}_w(\mathcal{N})$ for $w \leq 5$ and $I^{\mathrm{W}}_6(\mathcal{N} \times \mathcal{C}) \neq I^{\mathrm{W}}_6(\mathcal{N})$.*

In the case $n = 2$, these theorems imply $I^{\mathrm{W}}_w(\mathcal{N} \times \mathcal{C}) = I^{\mathrm{W}}_w(\mathcal{N}) = I^{\mathrm{CR}}_w$ for $w \leq 5$ (cf. Remark 3.2).

Remark 6.2. By direct computation, we can show that if $n = 2$ then $W_6(A, C) \notin I_6^{\mathrm{W}}(\mathcal{N})$ for the Weyl invariant W_6 in Theorem 6.1. This fact will be published elsewhere.

Theorem 6.1 is proved by using Theorem 6.2 if we recall the proof of Theorem 2.2 which uses Theorem 2.1. In the next subsection, we outline the proof of Theorem 6.2, which is analogous to that of Theorems 2.1. We omit the proof of Theorem 6.3, which is technical and consists of careful inspection of the proof of Theorem 6.2.

6.5 $\mathcal{C}$-dependent invariant theory

Recalling that Theorem 2.1 follows from Theorem 2.1′ at the end of Subsection 5.1, let us first formulate a substitute for Theorem 2.1′. We have to remove the weight restriction by using $\mathcal{N} \times \mathcal{C}$ in place of $\mathcal{N}$.

For a surface in Moser's normal form $N(A)$ with $X = \partial/\partial\rho_A$, we take $r = r(A, C) \in \mathcal{R}^{\mathrm{F}}_{N(A)}$ and consider the curvature R of the Lorentz-Kähler metric g with potential $r_\#$. As in Subsection 5.1, we identify R with the collection of the components $R_{\alpha\overline{\beta}}$ of the covariant derivatives at e_0, and write $R = (R_{\alpha\overline{\beta}})$. Then each $R_{\alpha\overline{\beta}}$ is a polynomial in $(A, C) \in \mathcal{N} \times \mathcal{C}$. We thus write $R_{\alpha\overline{\beta}} = R_{\alpha\overline{\beta}}(A, C)$ and define a map

$$\mathcal{F} : \mathcal{N} \times \mathcal{C} \ni (A, C) \mapsto R(A, C) \in \mathcal{R}^{\mathrm{aux}},$$

where $R(A, C) = (R_{\alpha\overline{\beta}}(A, C))$, and set $\mathcal{R} = \mathcal{F}(\mathcal{N} \times \mathcal{C})$. This map $\mathcal{F}$ and $\mathcal{R}$ are refinements of the map in (5.7) and $\mathcal{R}_n$.

Let us recall that the $H_\#$-action on $\mathcal{N}$ induces that on $\mathcal{R}_n$ via (5.2). Likewise, the $H_\#$-action on $\mathcal{N} \times \mathcal{C}$, defined by $h_\#.(A, C) = h.(A, C)$, induces that on $\mathcal{R}$. Thus we can define *$H_\#$-invariants* of weight w on $\mathcal{R}$, and we denote the totality of these by $I_w(\mathcal{R})$. The map $\mathcal{F}$ is $H_\#$-equivariant and induces an injection

$$\mathcal{F}^*: I_w(\mathcal{R}) \ni P(R) \mapsto P(\mathcal{F}(A, C)) \in I_w(\mathcal{N} \times \mathcal{C}) = I_w^{\mathrm{CR}}(\mathcal{C}),$$

which corresponds to the map in (5.5). Let $I_w^{\mathrm{W}}(\mathcal{R})$ denote the subspace of $I_w(\mathcal{R})$ consisting of elements which are given by linear combinations of complete contractions of the form (5.4) of weight w. Then we can state a substitute for Theorem 2.1′ as follows.

Theorem 6.2′. (I) *The map $\mathcal{F}^*$ is bijective.*
(II) $I_w(\mathcal{R}) = I_w^{\mathrm{W}}(\mathcal{R})$ *for each* $w \in \mathbb{N}_0$.

Theorem 6.2 follows from Theorem 6.2′.

As in the case of Theorem 2.1′, the proof of (I) is reduced to proving the injectivity of $\mathcal{F}'(0)$.

The statement (II) for $w \le n$ is equivalent to that in Theorem 2.1′, and most parts of the proof work as well for the case $w > n$. The point is to show

$$I_w^{\mathrm{W}}(T_0\mathcal{R}) = I_w(T_0\mathcal{R}), \tag{6.6}$$

where $T_0\mathcal{R} \subset \mathcal{R}^{\mathrm{aux}}$ is the tangent space of $\mathcal{R}$ at 0. In Subsection 5.3, we outlined the proof of (6.6) for $w \le n$, where $(5.11)_3$ was used crucially. The equality $(5.11)_3$, stating that $(R_{\alpha\overline{\beta}})$ is trace-free, follows from the equation

$$\Delta_0(\widetilde{r}_A^{\mathrm{F}})_\# = O^{n+1}((\rho_0)_\#). \tag{6.7}$$

where Δ_0 and $(\rho_0)_\#$ are those in (3.15). To prove (6.6) in the case $w > n$, we need to compute explicitly the error term $O^{n+1}((\rho_0)_\#)$ of (6.7) when $\widetilde{r}_A^{\mathrm{F}}$ is replaced by

$$\widetilde{r}_{A,C} = \frac{d}{d\varepsilon} r(\varepsilon A, \varepsilon C)\bigg|_{\varepsilon=0}.$$

The result is:

$$\Delta_0 \widetilde{r}_{A,C} = c_n\, \mu^{n+1} \Delta_0^{n+2} \widetilde{r}_{A,C}, \quad \text{where } c_n = \frac{(-1)^{n+1}}{(n+1)!^2}.$$

Using this equality in place of (6.7), we can remove the restriction $w \le n$ in the argument of Subsection 5.3, and obtain (6.6) with the aid of the invariant theory of [BEG].

Remark 6.3. In general, the Weyl invariants W_k in Theorem 6.1 are not uniquely determined, since there are linear relations among the boundary values of complete contractions of the form (2.1). For instance, in the case $n = 2$, the boundary values of $\|R^{(4,2)}\|^2$ and $\|R^{(3,3)}\|^2$ are linearly dependent (and, accordingly, Theorem 3.1 includes two expressions for W_4 and W_5). Under the terminology of this section, $\|R^{(3,3)}\|^2$ and $\|R^{(2,4)}\|^2$ are polynomials on $\mathcal{R}^{\mathrm{aux}}$ such that the restrictions to the submanifold $\mathcal{R}$ are linearly dependent functions.

The situation is similar for the Weyl invariants W_k in Theorem 2.2, though we do not know specific examples of non-uniqueness. (Note that $\|R^{(3,3)}\|^2$ and $\|R^{(2,4)}\|^2$ for $n = 2$ are irrelevant to Theorem 2.2 because of the weight restriction.) It should be mentioned that a basis of Weyl invariants of degree $d < n$ is given in [BEG]; in particular, it is shown that, if $d = 2 < n$, then $\|R^{(p,q)}\|^2$ $(p \geq q \geq 2)$ form a basis of quadratic Weyl invariants.

References

[Ba] T. N. Bailey, Parabolic invariant theory and geometry, in "The Penrose Transform and Analytic Cohomology in Representation Theory", Contemp. Math., 154, pp. 169–180, Amer. Math. Soc., 1993.

[BEG] T. N. Bailey, M. G. Eastwood and C. R. Graham, Invariant theory for conformal and CR geometry, Ann. of Math., **139** (1994), 491–552.

[B1] L. Boutet de Monvel, Complément sur le noyau de Bergman, Séminaire EDP, École Polytech. (1985–86), Exposé n° XX.

[B2] L. Boutet de Monvel, Le noyau de Bergman en dimension 2, Séminaire EDP, École Polytech. (1987–88), Exposé n° XXII.

[B3] L. Boutet de Monvel, Singularity of the Bergman kernel, in "Complex Geometry", Lecture Notes in Pure and Appl. Math., 143, pp. 13–29, Dekker, 1993.

[BS] L. Boutet de Monvel and J. Sjöstrand, Sur la singularité des noyaux de Bergman et de Szegö, Astérisque, **34–35** (1976), 123–164.

[CY] S.-Y. Cheng and S.-T. Yau, On the existence of a complete Kähler metric on non-compact complex manifolds and the regularity of Fefferman's equation, Comm. Pure Appl. Math., **33** (1980), 507–544.

[CM] S. S. Chern and J. K. Moser, Real hypersurfaces in complex manifolds, Acta Math., **133** (1974), 219–271.

[EG] M. G. Eastwood and C. R. Graham, Invariants of conformal densities, Duke Math. J., **63** (1991), 633–671.

[F1] C. Fefferman, The Bergman kernel and biholomorphic mappings of pseudoconvex domains, Invent. Math., **26** (1974), 1–65.

[F2] C. Fefferman, Monge-Ampère equations, the Bergman kernel, and geometry of pseudoconvex domains, Ann. of Math., **103** (1976), 395–416; *Correction*, ibid., **104** (1976), 393–394.

[F3] C. Fefferman, Parabolic invariant theory in complex analysis, Adv. in Math., **31** (1979), 131–262.

[FG] C. Fefferman and C. R. Graham, Conformal invariants, in "Élie Cartan et les Mathématiques d'Aujourd'hui", Astérisque, hors série (1985), 95–116.

[FK] G. B. Folland and J. J. Kohn, "The Neumann problem for the Cauchy-Riemann complex", Ann. of Math. Stud., 75, Princeton Univ. Press, 1972.

[G1] C. R. Graham, Scalar boundary invariants and the Bergman kernel, in "Complex Analysis II", Lecture Notes in Math., 1276, pp. 108–135, Springer, 1987.

[G2] C. R. Graham, Higher asymptotics of the complex Monge-Ampère equation, Compositio Math., **64** (1987), 133–155.

[G3] C. R. Graham, Invariant theory of parabolic geometries, in "Complex Geometry", Lecture Notes in Pure and Appl. Math., 143, pp. 53–66, Dekker, 1993.

[Hi] K. Hirachi, Construction of boundary invariants and the logarithmic singularity of the Bergman kernel, preprint.

[HKN1] K. Hirachi, G. Komatsu and N. Nakazawa, Two methods of determining local invariants in the Szegö kernel, in "Complex Geometry", Lecture Notes in Pure and Appl. Math., 143, pp. 77–96, Dekker, 1993.

[HKN2] K. Hirachi, G. Komatsu and N. Nakazawa, CR invariants of weight five in the Bergman kernel, preprint.

[Hö] L. Hörmander, L^2 estimates and existence theorems for the $\overline{\partial}$ operator, Acta Math., **113** (1965), 89–152.

[Kan] A. Kaneko, "Introduction to Hyperfunctions", Kluwer, 1988.

[Kas] M. Kashiwara, Analyse micro-locale du noyau de Bergman, Séminaire Goulaouic-Schwartz, École Polytech. (1976–77), Exposé n° VIII.

[Ke] N. Kerzman, The Bergman kernel function. Differentiability at the boundary, Math. Ann., **195** (1972), 149–158.

[L1] H. Lewy, On the local character of the solutions of an atypical linear differential equation in three variables and a related theorem for regular functions of two complex variables, Ann. of Math., **64** (1956), 514–522.

[L2] H. Lewy, On the boundary behavior of holomorphic mappings, Contrib. Centro Linceo Inter. Sc. Mat. e Loro Appl., 35, pp. 1–8, Acad. Naz. dei Lincei, 1977.

[LM] J. Lee and R. Melrose, Boundary behaviour of the complex Monge-Ampère equation, Acta Math., **148** (1982), 159–192.

[M] J. K. Moser, Holomorphic equivalence and normal forms of hypersurfaces, Proc. Sympos. Pure Math., 27, 2, pp. 109–112, Amer. Math. Soc., 1975.

[SKK] M. Sato, T. Kawai and M. Kashiwara, Microfunctions and pseudo-differential equations, in "Hyperfunctions and Pseudo-Differential Equations", Lecture Notes in Math., 287, pp. 265–529,Springer, 1973.

[Sca] P. Schapira, "Microdifferential Systems in the Complex Domain", Grundlehren Math. Wiss., 269, Springer, 1985.

[Scr] M. Schiffer, The kernel function of an orthonormal system, Duke Math. J., **13** (1946), 529–540.

Department of Mathematics
Osaka University
Toyonaka Osaka 560
Japan

Advanced Studies in Pure Mathematics 25, 1997
CR-Geometry and Overdetermined Systems
pp. 221–236

Spaces of Cauchy-Riemann Manifolds

László Lempert

§1. Introduction.

This work deals with the embeddability problem for three dimensional, compact, strongly pseudoconvex Cauchy-Riemann (CR) manifolds. Such a CR manifold is given by a compact manifold M without boundary, $\dim M = 3$; a rank two subbundle $H \subset TM$; and an endomorphism $J : H \to H$ that satisfies $J^2 = -\mathrm{id}$. (All manifolds, bundles etc. in this paper are C^∞ smooth.) Strong pseudoconvexity means that for any nonzero local section X of H the vector field $[X, JX]$ is transverse to H; or, equivalently, H defines a contact structure on M. By declaring the frame $X, JX, [JX, X]$ positively oriented, M acquires a canonical orientation.

A C^1 function $f : M \to \mathbb{C}$ is CR if it satisfies the tangential Cauchy-Riemann equations

$$Xf + iJXf = 0, \qquad X \in H. \tag{1.1}$$

A central problem of the theory is to understand how many solutions (1.1) has; in particular, if there are sufficiently many C^∞ solutions $f_1, \ldots, f_k$ to give rise to a smooth embedding $f = (f_j) : M \to \mathbb{C}^k$ into some Euclidean space. If this is so, the CR manifold (M, H, J) is called embeddable. In contrast with the higher dimensional case (see [3]) there may be very few CR functions on a three dimensional CR manifold; in fact, typically, the only CR functions are the constants, see [4,8,10,20,21].

We would like to describe the space of all (three dimensional, compact, strongly pseudoconvex) CR manifolds (M, H, J); the subspace of embeddable manifolds; and also to understand how many non isomorphic embeddable CR manifolds there are. Here two CR manifolds

---

Received September 21, 1995
Revised November 16, 1995
Research partially supported by the NSF

(M, H, J), (M', H', J') are isomorphic if there is a diffeomorphism $\Phi : M \to M'$ such that $\Phi_* H = H'$ and $\Phi_* J = J'$.

The above classification problems have two components. One is to classify all contact manifolds (M, H). This is a problem of differential topology, and we shall not consider it here. Instead, we shall deal with the other component of classification: given a contact manifold (M, H) describe the space of (embeddable) CR structures J on it. Thus, given a compact three dimensional (M, H), let $\mathcal{S}_0 = \mathcal{S}_0(M, H)$ denote the set of smooth CR structures $J : H \to H$ on it. It is easy to endow this set with the structure of a smooth Fréchet manifold (in the sense of [9,19]). Further, let $\mathcal{B}_0 \subset \mathcal{S}_0$ denote the set of embeddable CR structures, and $\mathcal{M}_0 = \mathcal{B}_0/\sim$ the moduli space of CR structures on M, where $J \sim J'$ if (M, H, J) and (M, H, J') are isomorphic. The Fréchet-Lie group Cont of contact diffeomorphisms of (M, H) acts on $\mathcal{B}_0$, and two CR structures are isomorphic if they are on the same Cont orbit; thus $\mathcal{M}_0 = \mathcal{B}_0/\mathrm{Cont}$.

The problem of describing the moduli space $\mathcal{M}_0$ is complicated by the circumstance that the action of Cont is not free: this is due to the fact that most CR manifolds have no CR automorphisms other than the identity while some, such as the standard sphere in $\mathbb{C}^2$, have. As a result, even if $\mathcal{B}_0$ turns out to be a smooth submanifold of $\mathcal{S}_0$, one will expect $\mathcal{M}_0$ to have complicated singularities. To get around this, we will endow our CR manifolds with a *marking*, a device comparable to passing from the moduli space of Riemann surfaces to the Teichmüller space. It is quite likely that for different CR manifolds different types of marking will be convenient; the markings we will introduce work very well for CR manifolds that are close to the simplest CR manifold, the sphere in $\mathbb{C}^2$. Thus, a marking μ will consist of an ordered pair of distinct points $p_1, p_2 \in M$ and vectors $v_i \in T_{p_i}M$ transverse to H_{p_i}. We will also require that v_i point to the *positive side* of H_{p_i}, i.e., X, JX, v_2 should be a positively oriented frame for nonzero $X \in H_{p_i}$. Given (M, H), we let $\mathcal{S} = \mathcal{S}(M, H)$ denote the Fréchet manifold of pairs (J, μ), where J is a CR structure on (M, H) and μ is a marking; and $\mathcal{B} = \mathcal{B}(M, H) \subset \mathcal{S}$ the subset corresponding to embeddable structures. As contact diffeomorphisms act on markings, we have an action of Cont on $\mathcal{B}$, and we denote $\mathcal{M} = \mathcal{M}(M, H) = \mathcal{B}/\mathrm{Cont}$. The spaces $\mathcal{S}$, $\mathcal{B}$, $\mathcal{M}$ are not very different from $\mathcal{S}_0$, $\mathcal{B}_0$, $\mathcal{M}_0$. Indeed, the mappings $\mathcal{S} \to \mathcal{S}_0$ etc. obtained by forgetting the marking are surjective and have finite (twelve) dimensional fibers. On the other hand, as we shall see, sometimes Cont acts on $\mathcal{B}$ freely, and this means that the structure of $\mathcal{M}$ is easier to describe than that of $\mathcal{M}_0$.

One can in general conjecture that $\mathcal{B}$ (resp $\mathcal{B}_0$) is a closed subset of $\mathcal{S}$ (resp $\mathcal{S}_0$), and even an analytic subset. Further, when $\mathcal{B}$ is a submanifold, $\mathcal{M}$ should also be a manifold. At present there are some results which point in this direction, that we will survey below; but overall, the conjectures are very much open.

Most available results pertain to CR manifolds that are embeddable into $\mathbb{C}^2$. Thus, suppose (M, H, J_0) is embeddable into $\mathbb{C}^2$.

Theorem 1.1. *$J_0 \in \mathcal{S}_0$ has a neighborhood $\mathcal{N}_0$ such that $\mathcal{N}_0 \cap \mathcal{B}_0$ is closed in $\mathcal{N}_0$. Similarly, for any marking μ_0, $(J_0, \mu_0) \in \S$ has a closed neighborhood $\mathcal{N}$ such that $\mathcal{N} \cap \mathcal{B}$ is closed in $\mathcal{S}$.*

This was first proved by Epstein, see [7]. The proof given in Section 2 is based on the stability theorem of [17], also used by Epstein; but our proof avoids the very precise spectral analysis of the tangential Cauchy-Riemann operator from [7]. Applying the more general stability results of H.-L. Li, see [18], the same theorem can be proved e.g. for (M, H, J_0) that is embeddable into the total space of a line bundle over $\mathbb{P}_1$ as the boundary of a neighborhood of the zero section.

For stronger results we will need to assume that (M, H, J_0) is (S^3, H_0, J_0), the CR structure inherited by the unit sphere $\{z \in \mathbb{C}^2 : |z| = 1\}$ from $\mathbb{C}^2$.

Theorem 1.2. *(Bland [1]) If $(M, H, J_0) = (S^3, H_0, J_0)$ then $J_0 \in \mathcal{S}_0$ has a neighborhood $\mathcal{N}_0$ such that $\mathcal{N}_0 \cap \mathcal{B}_0$ is a submanifold of $\mathcal{N}_0$.*

Given a marking $\mu = (p_i, v_i)$ on $(S^3, H_0) \subset \mathbb{C}^2$, look at the complex lines $L_i \subset \mathbb{C}^2$ that pass through p_i and whose tangent space contains v_i. If L_1 and L_2 intersect each other in one point, and this point is an interior point of the unit ball, we say that the marking is elliptic.

Theorem 1.3. *If μ_0 is an elliptic marking of (S^3, H_0) then $(J_0, \mu_0) \in \mathcal{S}$ has a* Cont *invariant neighborhood $\mathcal{N}$ such that*
(a) $\mathcal{U} = \mathcal{N} \cap \mathcal{B}$ is a submanifold of $\mathcal{N}$.
(b) $\mathcal{U} \to \mathcal{U}/\mathrm{Cont}(\subset \mathcal{M})$ is a trivial smooth principal Cont *bundle for some smooth structure on $\mathcal{U}/\mathrm{Cont}$.*

Thus a nonempty open piece of the moduli space $\mathcal{M}$ is an infinite dimensional Fréchet manifold. It is very likely that the neighborhood $\mathcal{N}$ in the above theorem can be chosen to contain all CR structures that admit an embedding into $\mathbb{C}^2$ as a strongly convex hypersurface, with arbitrary "elliptic" markings (ellipticity of a marking in this case can also be defined in terms of the Kobayashi metric, see section 5). What is missing from the proof is an improvement on Bland's Theorem 1.2

to the effect that $\mathcal{N}_0$ can be chosen to consist of all CR manifolds that embed into $\mathbb{C}^2$ as strongly convex hypersurfaces.

As to its form, Theorem 1.3 is related to the slice theorem of Cheng and Lee in [6], but the content is rather different. Indeed, Cheng and Lee construct a local slice for the action of Cont on the space of all CR structures (not just embeddable ones). Also, their approach is more abstract, and the slice is obtained by the application of the implicit function theorem, while in our approach the moduli space is represented in rather concrete terms.

§2. Proof of Theorem 1.1

We first observe that there is a positive integer k such that if a CR manifold (M, H, J) admits a CR embedding into $\mathbb{C}^2$ of class C^k then it also admits a CR embedding of class C^∞, i.e., is embeddable in our terminology. To check this we recall Boutet de Monvel's criterion, see [3], that (M, H, J) is embeddable if the tangential Cauchy-Riemann operator $\bar{\partial}_J : L^2(M) \to L^2_{0,1}(M)$ has closed range (to define the above L^2 spaces we endow M with a continuous Riemann metric). On the other hand if (M, H, J) is C^k embeddable into $\mathbb{C}^2$ then it can be regarded as the C^k boundary of a strongly pseudoconvex domain $D \subset \mathbb{C}^2$, and $\bar{\partial}_J$ becomes the boundary operator $\bar{\partial}_b$. Now Kohn in [12] shows that in this case $\bar{\partial}_b : L^2(\partial D) \to L^2_{0,1}(\partial D)$ has closed range. Strictly speaking Kohn assumes that D has C^∞ boundary, but his proof uses only finitely many derivatives of a defining function of D; whence the theorem is true as soon as ∂D is of class C^k with k sufficiently large. Putting these two results together we obtain our claim.

Next choose a CR embedding $f_0 : (M, H, J_0) \to \mathbb{C}^2$ of class C^∞, a k as in the above observation, and an $\epsilon > 0$ with the property that any C^k mapping $f : M \to \mathbb{C}^2$ whose C^k-distance to f_0 is $\leq \epsilon$ is a (differentiable) embedding. In [17] we proved J_0 has a neighborhood $\mathcal{N}_0 \subset \mathcal{S}_0$ such that for any $J \in \mathcal{N}_0 \cap \mathcal{B}_0$ the CR manifold (M, H, J) admits a CR embedding f into $\mathbb{C}^2$ with $|f - f_0|_{C^{k+1}} < \epsilon$. To verify $\mathcal{N}_0 \cap \mathcal{B}_0$ is closed in $\mathcal{N}_0$, let $J_\nu \in \mathcal{N}_0 \cap \mathcal{B}_0$ be a sequence converging to $J \in \mathcal{N}_0$. Choose CR embeddings $f_\nu : (M, H, J_\nu) \to \mathbb{C}^2$ with $|f_\nu - f_0|_{C^{k+1}} < \epsilon$. In view of the Arzelà-Ascoli theorem there is a subsequence f_{ν_j} that converges to some $f : M \to \mathbb{C}^2$ in the C^k topology; as $|f - f_0|_{C^k} \leq \epsilon$, f is an embedding. It is in fact a CR embedding of (M, H, J) of class C^k. According to our initial observation this implies J is embeddable, $J \in \mathcal{N}_0 \cap \mathcal{B}_0$, and we are done.

The second claim of Theorem 1.1 follows from the first if we note that the mapping $\mathcal{N} \to \mathcal{N}_0$ defined by forgetting the marking is smooth and $\mathcal{B}$ is the preimage of $\mathcal{B}_0$.

§3. Embedding families of CR structures.

In this section we will consider a CR manifold (M, H, J_0) with a CR embedding $f_0 : M \to \mathbb{C}^2$, and a smooth family $j : \mathcal{T}_1 \to \mathcal{S}_0$ of CR structures, parametrized by some neighborhood $\mathcal{T}_1$ of 0 in a Fréchet space $\mathcal{T}$, $j(0) = J_0$. We will also assume that j takes values in the subspace of embeddable CR structures $\mathcal{B}_0 \subset \mathcal{S}_0$. We will prove

Theorem 3.1. *There are a neighborhood $\mathcal{T}_2 \subset \mathcal{T}_1$ of 0 and a smooth family of CR embeddings $f(t) : (M, H, j(t)) \to \mathbb{C}^2$, $t \in \mathcal{T}_2$, such that $f(0) = f_0$.*

Above $f(t)$ being a smooth family means that $\mathcal{T}_2 \ni t \mapsto f(t) \in C^\infty(M, \mathbb{C}^2)$ is a C^∞ mapping. A closely related result is given in [7, part II, Theorem 8.1]. There a real analytic family of embeddable CR structures parametrized by an interval is considered; it is not assumed, though, that the manifolds embed into $\mathbb{C}^2$. The conclusion is then as above (with $\mathbb{C}^2$ replaced by some $\mathbb{C}^n$, and without $f(0) = f_0$). It is also indicated how to extend that theorem to higher dimensional parameter spaces and smooth families if a certain relative index vanishes. This latter condition is known to be satisfied when $(M, H, j(0))$ embeds into $\mathbb{C}^2$.

Observe that Theorem 3.1 is a counterpart of Theorem 1.1 in [17], but while [17] is about embeddings of *perturbations* J_1 of J_0, Theorem 3.1 is about embeddings of *deformation* families. Neither result is a consequence of the other, but their proofs use similar tools. For this reason we start by recalling some results from [17].

Let $\bar{Y}$ be a compact complex manifold with smooth boundary ∂Y and interior Y; we will denote the complex structure of $\bar{Y}$ by $\mathcal{J}_0$. Let $L \to (\bar{Y}, \mathcal{J}_0)$ be a smooth line bundle, holomorphic on Y. The Cauchy-Riemann operator $C^\infty(L) \to C^\infty_{0,1}(L)$ will be denoted $\bar{D}$. In [17] we introduced a scale of anisotropic Sobolev norms $\| \ \|_s$, $s = 1, 2, \dots$ on $C^\infty(L)$ resp. $C^\infty_{0,1}(L)$ whose basic properties we will list below.

Proposition 3.2. *C^k Hölder norms ($k = 1, 2, \dots$) are dominated by $\| \ \|_s$ if $s > 2(n + k)$, $n = \dim_{\mathbb{C}} Y$.*

The norms $\| \ \|_s$ come from inner products $(\ , \ )_s$. It follows that if we denote the completion of $C^\infty(L)$, $C^\infty_{0,1}(L)$ with respect to $\| \ \|_s$ by $\mathcal{H}^s$, $\mathcal{H}^s_{0,1}$, these spaces are Hilbert spaces with inner product (the extension of) $(\ ,\ )_s$, and for $s > 2n+2$ they are continuously embedded into $C^1(L)$ resp. $C^1_{0,1}(L)$. Also, [17, Proposition 2.2] implies that $\bar{D} : \mathcal{H}^s \to \mathcal{H}^{s-1}_{0,1}$ is continuous.

From now on we will also assume that the Levi form of ∂Y has at least one negative eigenvalue in every point of ∂Y. Then we have

Proposition 3.3. *There are constants C_s such that for $u \in \mathcal{H}^s$*

$$\|u\|_s \leq C_s(\|\bar{D}u\|_{s-1} + \|u\|_0). \tag{3.1}$$

Proof. (3.1) is proved in [17, Theorem 3.1] for $u \in C^\infty(L)$; since $C^\infty(L)$ is dense in $\mathcal{H}^s$ it is true for $u \in \mathcal{H}^s$ as well.

Now assume that $(Y, \mathcal{J}_0)$ contains a nonsingular compact complex hypersurface Z without boundary, and that $j(t)$ is a smooth family of complex structures on $\bar{Y}$ parametrized by some open neighborhood $\mathcal{T}'$ of 0 in a Fréchet space T. We will also assume that $\mathcal{J}(0) = \mathcal{J}_0$, and that the tangent bundle TZ is invariant under all $\mathcal{J}(t)$; thus Z is a complex submanifold of all $(Y, \mathcal{J}(t))$. Let $L(t) \to (\bar{Y}, \mathcal{J}(t))$ denote the line bundles determined by the divisor Z, $L(0) = L$, and $\bar{D}(t) : C^1(L(t)) \to C_{0,1}(L(t))$ the corresponding Cauchy-Riemann operators, $\bar{D}(0) = \bar{D}$.

Proposition 3.4. *There are a neighborhood $\mathcal{T}'' \subset \mathcal{T}'$ of 0, a family of smooth bundle isomorphisms $\Phi_t : L \to L(t)$, and a smooth family of first order linear partial differential operators $\Lambda_t : C^\infty(L) \to C^\infty_{0,1}(L)$, $t \in \mathcal{T}''$, such that*

(i) $\Lambda_0 = 0$,

(ii) for every $s = 1, 2, \ldots$, Λ_t extends to a smooth family of operators $\mathcal{H}^s \to \mathcal{H}^{s-1}_{0,1}$;

(iii) for $u \in C^\infty(L)$ the Cauchy-Riemann equation $\bar{D}(t)\Phi_t \circ u = 0$ is equivalent to $\bar{D}u + \Lambda_t u = 0$.

The proof of this parallels the proof of [17, Lemma 4.2], so we omit it. Here we record that according to the proof in [17], Φ_t, Λ_t satisfy

$$\bar{D}(t)\Phi_t \circ u = \Phi_t \circ \bar{D}u + \Phi_t \circ \Lambda_t u. \tag{3.2}$$

It follows that for $t \in \mathcal{T}''$, $s = 1, 2, \ldots$ there are constants $C_{s,t}$ such that for $u \in \mathcal{H}^s$

$$\|u\|_s \leq C_{s,t}(\|\bar{D}u + \Lambda_t u\|_{s-1} + \|u\|_0). \tag{3.3}$$

Indeed, this is just (3.1) applied to $\bar{D}(t)$, $L(t)$ instead $\bar{D}$, L (here we assume $\mathcal{T}''$ is so small that the Levi form of $\partial(Y, \mathcal{J}(t))$ has a negative eigenvalue for $t \in \mathcal{T}''$).

Proposition 3.5. *Given s, the constant $C_{s,t}$ in (3.3) is locally uniform in $t \in \mathcal{T}''$.*

Proof. Fix $\tau \in \mathcal{T}''$. With $t \in \mathcal{T}''$ we have

$$\begin{aligned}\|u\|_s &\leq C_{s,\tau}(\|(\bar{D} + \Lambda_\tau)u\|_{s-1} + \|u\|_0) \\ &\leq C_{s,\tau}(\|(\bar{D} + \Lambda_t)u\|_{s-1} + \|u\|_0) + C_{s,\tau}\|(\Lambda_\tau - \Lambda_t)u\|_{s-1}.\end{aligned}$$

If t is sufficiently close to τ, Proposition 3.4 (ii) implies that the last term here is $\leq \|u\|_s/2$. It then follows that $C_{s,t} = 2C_{s,\tau}$ can be chosen in (3.3).

Proposition 3.6. *There is a neighborhood $\mathcal{T}''' \subset \mathcal{T}''$ of 0 such that given $s > 2n + 4$ we have with constants $\bar{C}_{s,t}$ $(t \in \mathcal{T}''')$*

$$\|u\|_s \leq \bar{C}_{s,t}\|\bar{D}u + \Lambda_t u\|_{s-1},$$

whenever $u \in \mathcal{H}^s$ is orthogonal to $H^0(L)$ with respect to $(\ ,\ )_0$. Here $\bar{C}_{s,t}$ can be chosen locally uniform in $t \in \mathcal{T}'''$.

Proof. Let $\tau \in \mathcal{T}''$. If we can not find uniform constants $\bar{C}_{s,t}$ in any neighborhood of τ then in view of Proposition 3.5 there exist a sequence $t_i \to \tau$ and $u_i \in \mathcal{H}^s$ orthogonal to $H^0(L)$ with $\|u_i\|_0 = 1$, $\|\bar{D}u_i + \Lambda_{t_i}u_i\|_{s-1} \to 0$. By Proposition 3.5, $\|u_i\|_s$ is bounded, hence by Proposition 3.2 and the Arzelà-Ascoli theorem a subsequence u_{ν_i} converges to $u \in C^1(L)$ in C^1-norm. Then $\|u\|_0 = 1$. Also $\bar{D}u + \Lambda_\tau u = 0$, so that $\phi_\tau \circ u \in H^0(L(\tau))$, and u is orthogonal to $H^0(L)$. But by [17, Proposition 5.2] this implies $u = 0$ if τ is in a sufficiently small neighborhood $\mathcal{T}'''$ of 0, which is a contradiction.

Corollary 3.7. *The operator*

$$\bar{D} + \Lambda_t : \mathcal{H}^s \to \mathcal{H}^{s-1}_{0,1}$$

has closed range if $s > 2n + 4$, $t \in \mathcal{T}'''$.

Proof of Theorem 3.1. We can assume that M is a hypersurface in $\mathbb{C}^2$ and its CR structure J_0 is inherited from $\mathbb{C}^2$; further, that f_0 is the inclusion $M \subset \mathbb{C}^2$. M divides $\mathbb{P}_2 \supset \mathbb{C}^2$ in two parts, let Y denote the pseudoconcave one, and $\bar{Y} = Y \cup M$. Denote the complex structure on $\bar{Y}$ inherited from $\mathbb{P}_2$ by $\mathcal{J}_0$. Also, let $Z \subset Y$ be the line at infinity. In

[16] we pointed out that Kiremidjian's theorem in [11] implies that for $t \in \mathcal{T}_1$ near 0 $\bar{Y}$ admits a complex structure $\mathcal{J}(t)$ that induces the CR structure $j(t)$ on (M, HM), and which agrees with $\mathcal{J}_0$ in points of Z. In fact Kiremidjian's proof constructs a unique such $\mathcal{J}(t)$, and one checks that $\mathcal{J}(t)$ depends smoothly on t. Also $\mathcal{J}(0) = \mathcal{J}_0$.

Let now $L(t)$, $\bar{D}(t)$, $\mathcal{H}^s$, $\mathcal{H}^{s-1}_{0,1}$ be as above, and Φ_t, Λ_t as in Proposition 3.4. The homogeneous coordinates on $\mathbb{P}_2$ give rise to three sections $u_0^i \in H^0(L)$, $i = 0, 1, 2$ that span $H^0(L)$. In [17, Section 5] we proved that with a sufficiently small neighborhood $\mathcal{T}_2$ of 0 there are unique sections $u_t^i \in C^\infty(L)$, $t \in \mathcal{T}_2$, such that $\Phi_t \circ u_t^i \in H^0(L(t))$ and $u_t^i - u_0^i$ is orthogonal to $H^0(L)$ with respect to $(\ ,\ )_0$, $i = 0, 1, 2$. We will assume $\mathcal{T}_2 \subset \mathcal{T}'''$ from Proposition 3.6, and proceed to show that $u_t^i \in C^\infty(L)$ depends smoothly on t. It will suffice to show that for every $s > 8$ every $\tau \in \mathcal{T}_2$ has a neighborhood on which the mapping $t \mapsto u_t^i \in \mathcal{H}^s$ is smooth.

Observe that $\Phi_t \circ u_t^i \in H^0(L(t))$ implies $(\bar{D} + \Lambda_t)u_t^i = 0$, hence also

$$(\bar{D} + \Lambda_\tau)(u_t^i - u_\tau^i) + (\Lambda_t - \Lambda_\tau)(u_t^i - u_\tau^i) = (\Lambda_t - \Lambda_\tau)u_\tau^i. \tag{3.4}$$

Introduce a left inverse Q of $\bar{D} + \Lambda_\tau$ as follows. Given $s > 8$ and $\alpha \in \mathcal{H}^{s-1}_{0,1}$, write $\alpha = \beta + \gamma$ with β in the range of $\bar{D} + \Lambda_\tau : \mathcal{H}^s \to \mathcal{H}^s_{0,1}$ and γ orthogonal to this range *with respect to* $(\ ,\ )_{s-1}$. By virtue of Corollary 3.7 this can be done. Put $Q\alpha = u \in \mathcal{H}^s$ if $(\bar{D} + \Lambda_\tau)u = \alpha$ and u is orthogonal to $H^0(L)$ with respect to $(\ ,\ )_0$. Such a u can be found because dim $H^0(L) =$ dim $\operatorname{Ker}(\bar{D} + \Lambda_\tau) = 3$, see [17, Proposition 5.1, also Proposition 5.2]. Also note that $Q : \mathcal{H}^{s-1}_{0,1} \to \mathcal{H}^s$ is a bounded operator.

Putting $(\bar{D} + \Lambda_t)(u_t^i - u_\tau^i) = \alpha_t$ we can write (3.4) as

$$\alpha_t + (\Lambda_t - \Lambda_\tau)Q\alpha_t = (\Lambda_t - \Lambda_\tau)u_\tau^i.$$

If t is sufficiently close to τ, the norm of the operator $(\Lambda_t - \Lambda_\tau)Q : \mathcal{H}^{s-1}_{0,1} \to H^{s-1}_{0,1}$ is less than one, whence it follows that

$$\alpha_t = (I + (\Lambda_t - \Lambda_\tau)Q)^{-1}(\Lambda_t - \Lambda_\tau)u_\tau^i \in \mathcal{H}^{s-1}_{0,1}$$

depends smoothly on t. Therefore the same holds for $u_t^i = u_\tau^i + Q\alpha_t \in \mathcal{H}^s$, and this proves that $t \mapsto u_t^i \in C^\infty(L)$ is smooth.

Assuming $\mathcal{T}_2$ is sufficiently small we now obtain the smooth family $f(t)$ of the Theorem in the form

$$f(t) = \left(\frac{u_t^1}{u_t^0}, \frac{u_t^2}{u_t^0}\right)\bigg|_M = \left(\frac{\Phi_t \circ u_t^1}{\Phi_t \circ u_t^0}, \frac{\Phi_t \circ u_t^2}{\Phi_t \circ u_t^0}\right)\bigg|_M.$$

§4. Moduli spaces of convex domains

Our approach to Theorem 1.3 will be through moduli spaces of convex domains. In this section we will review the relevant facts.

We will denote by $\mathcal{X}_0$ the space of strongly convex smoothly bounded domains $D \subset \mathbb{C}^2$ that contain the origin. As explained in [15] (where the notation was $\mathcal{X}$) this is a Fréchet manifold, in fact an open convex cone in some Fréchet space. The set of those $D \in \mathcal{X}_0$ that are invariant under the circle action

$$\gamma_t : z \mapsto e^{it} z \qquad (t \in \mathbb{R}), \tag{4.1}$$

i.e. the circular domains, will be denoted $\mathcal{C}_0$. We will work with marked domains, too. A marking for $D \in \mathcal{X}_0$ consists of a pair η of linearly independent vectors $\eta_1, \eta_2 \in T_0^{1,0}D$. The space of marked domains (D, η) with $D \in \mathcal{X}_0$ will be denoted $\mathcal{X}$; this is again a Fréchet manifold. We will say that two marked domains (D, η), $(D', \eta') \in \mathcal{X}$ are equivalent, $(D, \eta) \sim (D', \eta')$, if there is a biholomorphism $\Phi : D \to D'$ that fixes 0 and maps the marking $\eta = (\eta_1, \eta_2)$ to $\eta' = (\eta_1', \eta_2')$. The moduli space $\mathcal{X}/\sim$ of strongly convex smooth domains was described in [2] and [15] in terms of invariants associated with the Kobayashi metric. We will briefly recall how this can be done, mostly following [2], though not its notation.

Given any strongly convex domain D, a point $a \in D$ and a vector $v \in T_a^{1,0}D$, consider holomorphic mappings f of the unit disc

$$\Delta = \{\zeta \in \mathbb{C} : |\zeta| < 1\}$$

into D such that $f(0) = a$ and $f_*(0)\, \partial/\partial\zeta = \lambda v$ with some $\lambda \geq 0$. There is a unique mapping f that maximizes the value of λ, called extremal map (determined by a and v); this map is smooth on $\bar{\Delta}$ and maps the circle $\partial\Delta$ into ∂D, see [13]. If, for fixed a we let v vary, the vectors $f_*(0)\partial/\partial\zeta$ for the corresponding extremal maps f will trace the smooth boundary of a strongly convex circular domain in $T_a^{1,0}D$, called the Kobayashi indicatrix.

Now the first invariant of $(D, \eta) \in \mathcal{X}$ is obtained by looking at the Kobayashi indicatrix $I^* \subset T_0^{1,0}D$ of D at 0. There is a unique linear map $A : T_0^{1,0}D \to \mathbb{C}^2$ that sends the marking (η_1, η_2) to the standard basis $(1, 0)$, $(0, 1)$ of $\mathbb{C}^2$; the image of I^* under A will be denoted $I = I(D, \eta)$.

An exponential-like mapping $r : \partial I \to \partial D$, called the circular representation, can be defined as follows. Given $v \in \partial I$, let $f : \Delta \to D$ be the extremal map determined by $0 \in D$, $A^{-1}v \in T_0^{1,0}D$. Then putting

$r(v) = f(1)$ a diffeomorphism $r : \partial I \to \partial D$ is obtained. This diffeomorphism has a natural extension to a homeomorphism $\bar{I} \to \bar{D}$, smooth off 0, which is also called circular representation, but we will not deal with this extension.

The hypersurfaces ∂D, $\partial I \subset \mathbb{C}^2$ inherit a CR structure from $\mathbb{C}^2$, denoted $(\partial D, H(\partial D), J_D)$ resp. $(\partial I, H(\partial I), J_I)$, and it turns out that the circular representation r maps $H(\partial I)$ to $H(\partial D)$, i.e. r is a contact diffeomorphism. However, in general it does not intertwine J_D, J_I - in other words, it is not a CR isomorphism - and one can measure the extent to which it distorts the CR structure by looking at the complex line bundles over ∂D resp. ∂I

$$\begin{aligned} H^{0,1}(\partial D) &= \{\xi + iJ_D\xi : \xi \in H(\partial D)\} \subset \mathbb{C} \otimes H(\partial D), \\ H^{0,1}(\partial I) &= \{\xi + iJ_I\xi : \xi \in H(\partial I)\} \subset \mathbb{C} \otimes H(\partial I), \\ H^{1,0}(\partial I) &= \overline{H^{0,1}(\partial I)}, \end{aligned}$$

and the CR deformation tensor which is a bundle map

$$\Phi_{D,\eta} : H^{0,1}(\partial I) \to H^{1,0}(\partial I)$$

with the property that the pull back

$$r_*^{-1} H^{0,1}(\partial D) \subset \mathbb{C} \otimes H(\partial I) = H^{0,1}(\partial I) \oplus H^{1,0}(\partial I)$$

is the graph of $\Phi_{D,\eta}$. By [2], the pair $(I(D,\eta), \Phi_{D,\eta})$ depends only the equivalence class of $(D,\eta) \in \mathcal{X}$, and conversely, the knowledge of $(I(D,\eta), \Phi_{D,\eta})$ allows one to reconstruct the equivalence class of (D,η). Furthermore, the range of the invariants $I(D,\eta)$, $\Phi_{D,\eta}$ can also be described to some extent.

To this end notice that the circle action (4.1) decomposes any tensor $\Phi : H^{0,1}(\partial I) \to H^{1,0}(\partial I)$ into homogeneous tensors: $\Phi = \sum_{-\infty}^{\infty} \Phi_\nu$; here $\gamma_t^* \Phi_\nu = e^{i\nu t}\Phi_\nu$. Denote by $\mathcal{D}_I$ the Fréchet space of smooth tensors $\Phi : H^{0,1}(\partial I) \to H^{1,0}(\partial I)$ whose decomposition contains homogeneous terms Φ_ν with $\nu > 0$ only. These spaces patch together to form a smooth Fréchet bundle $\mathcal{D} = \bigcup_{I \in \mathcal{C}_0} \mathcal{D}_I \to \mathcal{C}_0$. By [2] for any $(D,\eta) \in \mathcal{X}$ the CR deformation tensor $\Phi_{D,\eta}$ is in $\mathcal{D}_{I(D,\eta)}$, whence we obtain a (smooth) mapping $\widetilde{h} : \mathcal{X} \to \mathcal{D}$ that associates with $(D,\eta) \in \mathcal{X}$ the pair of invariants $(I(D,\eta), \Phi_{D,\eta})$. By the above discussion $\widetilde{h}$ descends to a mapping $h : \mathcal{X}/\sim \to \mathcal{D}$, and it turns out that h is a homeomorphism onto an open neighborhood of the zero section in the bundle $\mathcal{D} \to \mathcal{C}_0$. This is essentially contained in [2], although Bland and Duchamp consider only domains with fixed indicatrix I. The proof, however, carries over for variable indicatrices; see also çquivalent result. As in [15, Theorem 10.2] one can prove that $\tilde{h}$ has smooth local right inverses. That is, given $(D,\eta) \in \mathcal{X}$, there are a neighborhood $\mathcal{V} \subset \mathcal{D}$ of $\tilde{h}(D,\eta)$ and a smooth mapping $k : \mathcal{V} \to \mathcal{X}$ with $\tilde{h} \circ k = \mathrm{id}_{\mathcal{V}}$ and $k(\tilde{h}(D,\eta)) = (D,\eta)$.

In the sequel we will endow $\mathcal{X}/\sim$ with the smooth structure that is induced from $\mathcal{D}$ by the homeomorphism h. Thus we have

Proposition 4.1. *The canonical projection $\mathcal{X} \to \mathcal{X}/\sim$ is smooth and has smooth local right inverses.*

In addition to the circular representation $r : \partial I \to \partial D$ discussed above, later we will also need a canonical contact diffeomorphism between $(\partial D, H(\partial D))$ and (S^3, H_0). We end this section by describing how such a diffeomorphism can be constructed. This construction is not holomorphically invariant (unlike the circular representation); it could be made invariant for marked domains (D,η), but invariance will not be the issue in our discussion.

Our first observation is the following. Let H_t $(t \in [0,1])$ be a smooth family of contact structures on a manifold M. Then one can canonically associate with this family a contact diffeomorphism $g : (M, H_0) \to (M, H_1)$.

We will justify this observation under the assumption that H_t are orientable, hence given by a smooth family α_t of one forms on $M : H_t = \mathrm{Ker}\ \alpha_t$. The forms $d\alpha_t$ restricted to H_t are nondegenerate, whence there is a unique smooth family V_t of vector fields tangent to H_t such that

$$V_t \lrcorner d\alpha_t \bigg|_{H_t} = \frac{d\alpha_t}{dt}\bigg|_{H_t}.$$

Denoting Lie derivative by $\mathcal{L}$, these vector fields therefore satisfy

$$\mathcal{L}_{V_t}\alpha_t = d(V_t \lrcorner \alpha_t) + V_t \lrcorner d\alpha_t = V_t \lrcorner\ d\alpha_t \equiv \frac{d\alpha_t}{dt} \mod \alpha_t.$$

This implies that the flow g_t of the time dependent field V_t pulls back α_t to some multiple of α_0, in particular $g = g_1$ is a contact diffeomorphism between (M, H_0) and (M, H_1).

If now $D \in \mathcal{X}_0$, the mapping $\rho(z) = z/\|z\|$ defines a diffeomorphism of ∂D to S^3, although this is not in general a contact diffeomorphism between $(\partial D, H(\partial D))$ and (S^3, H_0). To remedy this, denoting the unit ball of $\mathbb{C}^2$ by B, and putting $D_t = tD + (1-t)B$, $\rho|_{\partial D_t}$ will push forward the contact structures $H(\partial D_t)$ to a smooth family of contact structures H_t on S^3. Our observation above now supplies a canonical contact diffeomorphism $g : (S^3, H_0) \to (S^3, H_1)$, whence also a canonical contact diffeomorphism

$$g^{-1} \circ \rho : (\partial D, H(\partial D)) \to (S^3, H_0). \tag{4.2}$$

§5. Proof of Theorem 1.3

The previous section described the spaces $\mathcal{X}$, $\mathcal{X}/\sim$ of convex domains; we shall now connect those spaces with the spaces $\mathcal{B}(S^3, H_0)$, $\mathcal{M}(S^3, H_0)$ of CR manifolds. Thus, let $(D, \eta) \in \mathcal{X}$ be a marked domain. Its boundary ∂D inherits a CR structure from $\mathbb{C}^2$, $(\partial D, H(\partial D), J_D)$. The marking η of D determines a marking μ' of this CR manifold as follows. For $i = 1, 2$, consider the extremal mapping $e^i : \Delta \to D$ determined by η_i, and put

$$p_i = e^i(1), \qquad v_i = -\frac{1}{\lambda}\, e^i_*(1) \mathrm{Im}\, \frac{\partial}{\partial \zeta}.$$

The marking $\mu' = (p_i, v_i)$ defines a marked CR manifold $(\partial D, H(\partial D), J_D, \mu')$. Via the contact diffeomorphism (4.2) constructed in Section 4 this marked CR manifold can be identified with a marked CR manifold $(J, \mu) \in \mathcal{B}(S^3, H_0)$, which we will also denote $\Theta(D, \eta)$. Thus Θ is a mapping from $\mathcal{X}$ to $\mathcal{B}$, and indeed a smooth mapping that descends to a mapping $\theta : \mathcal{X}/\sim \to \mathcal{M}$.

Conversely, consider a marked CR manifold $(J, \mu) \in \mathcal{B}$, where J is sufficiently close to the standard CR structure J_0 of the sphere. By [17] this implies there is a smooth CR embedding

$$f : (S^3, H_0, J) \to \mathbb{C}^2 \tag{5.1}$$

with image a strongly convex hypersurface. Denote the domain bounded by this hypersurface by D; the marking $\mu = (p_i, v_i)$ then defines points $p_i' = f(p_i) \in \partial D$ and vectors $v_i' = f_* v_i \in T'_{p_i} \partial D$.

By [5] there are extremal mappings $e^i : \bar{\Delta} \to \bar{D}$ with

$$e^i(1) = p_i', \qquad e^i_*(1) \mathrm{Im}\, \partial/\partial\zeta = -\lambda_i v_i' \qquad (\lambda_i > 0).$$

These extremal mappings are unique up to composition by holomorphic automorphisms of Δ that fix 1. When $J = J_0$ and $\mu = \mu_0$ is an elliptic marking, the extremal discs $e^1(\Delta)$, $e^2(\Delta)$ intersect in one point inside

D. More generally, if a CR manifold (S^3, H_0, J) admits a strongly convex embedding into $\mathbb{C}^2$, we will say that a marking μ is elliptic if the two extremal discs $e^1(\Delta)$, $e^2(\Delta)$ determined by the marking as above, intersect (in which case they intersect in exactly one point). The set of elliptically marked CR manifolds will be denoted $\mathcal{E} \subset \mathcal{B}$.

Below we will need the fact that the extremal discs $e^i(\Delta)$ depend smoothly on D and p_i', v_i'. This does not seem to have been published anywhere, but a proof can be easily obtained by a small modification of the arguments in [14]. A slightly weaker theorem is proved in [5], where D is, however, kept fixed. In any case, this shows that the subset $\mathcal{E}$ of elliptically marked CR manifolds is open in $\mathcal{B}$; in particular, if the marking μ above was close to an elliptic marking μ_0 of (S^3, H_0, J_0) then itself is elliptic, that is, the extremal discs $e^1(\Delta)$, $e^2(\Delta)$ above intersect in some point. By modifying the embedding of (S^3, H_0, J) into $\mathbb{C}^2$ we can assume that the point of intersection is 0; and also that $e^1(0) = e^2(0) = 0$. With this normalization the λ_i in (4.2) are uniquely determined, and we can define a marking η of $D \in \mathcal{X}_0$ by

$$\eta_i = \frac{1}{\lambda_i}\; e^i(0)\frac{\partial}{\partial \zeta} \in T_0^{1,0}D \qquad (i = 1, 2).$$

We have thus associated a marked domain $(D, \eta) \in \mathcal{X}$ with an elliptically marked CR structure $(J, \mu) \in \mathcal{E}$, in particular, with (J, μ) close to (J_0, μ_0). This association is not unique, for it depends on the CR embedding (5.1) we choose. However, Bland's theorem (Theorem 1.2) implies that (J_0, μ_0) has a neighborhood $\mathcal{N} \subset \mathcal{S}$ such that $\mathcal{U} = \mathcal{N} \cap \mathcal{B}$ is a submanifold of $\mathcal{N}$, and then Theorem 3.1 implies that (after a possible shrinking) the mapping f in (5.1) can be chosen to depend smoothly on (J, μ). This then makes the passage from $(J, \mu) \in \mathcal{U}$ to (D, η) a smooth mapping $\Psi : \mathcal{U} \to \mathcal{X}$. The construction was such that for $u \in \mathcal{U}$ $\Theta(\Psi(u))$ is on the Cont orbit of u, and for $x \in \mathcal{X}$ the marked domains $\Psi(\Theta(x))$ and x are equivalent. By replacing $\mathcal{N}$ by its Cont orbit we can assume that $\mathcal{N}$, hence $\mathcal{U}$ are Cont invariant; then Ψ descends to a continuous open mapping $\psi : \mathcal{U}/\text{Cont} \to \mathcal{X}/\sim$ and $\theta \circ \psi = \text{id}_{\mathcal{U}/\text{Cont}}$.

Ideas like the ones employed in the construction of the mappings Θ, Ψ also let one understand the action of Cont on the open set $\mathcal{E} \subset \mathcal{B}$.

Proposition 5.1. *If two elliptically marked CR manifolds are CR diffeomorphic then there is a unique CR diffeomorphism between them. Moreover, if $\mathcal{T}_1$ is an open set in some Fréchet space and $F, G : \mathcal{T}_1 \to \mathcal{E} \subset \mathcal{S}$ are smooth mappings (as mappings into $\mathcal{S}$) such that for every $t \in \mathcal{T}_1$, $F(t)$ and $G(t)$ are CR diffeomorphic, then the CR diffeomorphism between them (an element of Cont) depends smoothly on t.*

Proof. If the elliptically marked CR manifolds (S^3, H_0, J, μ) and $(S^3, H_0, \hat{J}, \hat{\mu})$ are CR diffeomorphic, the unique CR diffeomorphism between them can be constructed as follows. Taking suitable convex embeddings of these CR manifolds into $\mathbb{C}^2$ the image hypersurfaces will bound strongly convex domains D, $\hat{D}$ and the markings μ, $\hat{\mu}$ will induce markings η, $\hat{\eta}$ on them, as explained above. It will suffice to show that there is a unique biholomorphism H between (D, η) and $(\hat{D}, \hat{\eta})$, this latter being a consequence of the biholomorphic invariance of extremal discs. Indeed, let the linear map $L : T_0^{1,0} D \to T_0^{1,0} \hat{D}$ map η to $\hat{\eta}$, then for any $z \in \bar{D} \backslash \{0\}$ $H(z) \in \bar{\hat{D}}$ can be obtained as follows. Let $e : \Delta \to \Delta$ be the unique extremal mapping such that $e(0) = 0$, $e(\alpha) = z$ with some α, $0 < \alpha \leq 1$, and let $\hat{e} : \Delta \to \hat{D}$ be the unique extremal mapping such that $\hat{e}_*(0)\partial/\partial\zeta = \lambda L e_*(0)\partial/\partial\zeta$, with some $\lambda > 0$. Then $H(z)$ is given by $\hat{e}(\alpha)$; in particular H is unique.

The second half of the Proposition is proved using the above passage from $\mathcal{E}$ to $\mathcal{X}$, and in addition Theorem 3.1 and the smooth dependence of extremal maps on the data (the target domain, base points, resp. tangent vector).

At this point we are ready to prove Theorem 1.3. Let π, ω denote the canonical projections $\mathcal{B} \to \mathcal{B}/\mathrm{Cont}$ resp. $\mathcal{X} \to \mathcal{X}/\sim$, see the diagrams:

$$
\begin{tikzcd}
\mathcal{B} \arrow[d, "\pi"'] & \mathcal{X} \arrow[l, "\Theta"'] \arrow[d, "\omega"] \\
\mathcal{B}/\mathrm{Cont} & \mathcal{X}/\sim \arrow[l, "\theta"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{B} \supset \mathcal{U} \arrow[r, "\Psi"] \arrow[d, "\pi"'] & \mathcal{X} \arrow[d, "\omega"] \\
\mathcal{U}/\mathrm{Cont} \arrow[r, "\psi"] & \mathcal{X}/\sim .
\end{tikzcd}
$$

The pullback of the smooth structure of $\mathcal{X}/\sim$ by ψ defines a smooth structure on $\mathcal{U}/\mathrm{Cont}$. Thus ψ and its inverse $\theta|_{\psi(\mathcal{U}/\mathrm{Cont})}$ are diffeomorphisms. We need to show that $\pi : \mathcal{U} \to \mathcal{U}/\mathrm{Cont}$ is a trivial smooth principal bundle with structure group Cont. First, $\pi = \theta \circ \omega \circ \Psi$ is smooth. Second, assuming $\mathcal{U}$ is sufficiently small, a section of π can be gotten by looking at a smooth local right inverse σ of ω defined near $\omega(\Psi(J_0, \mu_0))$ (cf. Proposition 4.1); then $\kappa = \Theta \circ \sigma \circ \psi$ is a smooth section of π. Finally, denoting the action of a contact diffeomorphism $\gamma \in \mathrm{Cont}$ on $\mathcal{U}$ by a superscript, we define a smooth Cont equivariant mapping

$$\mathrm{Cont} \times \mathcal{U}/\mathrm{Cont} \ni (\gamma, \bar{u}) \mapsto (\kappa(\bar{u}))^\gamma \in \mathcal{U}.$$

This has a smooth inverse

$$\mathcal{U} \ni u \mapsto (\Gamma(\kappa(\pi(u)), u), \pi(u)),$$

where $\Gamma(v, u)$ denotes the unique CR diffeomorphism between $v \in \mathcal{U}$ and $u \in \mathcal{U}$, cf. Proposition 5.1. Hence the Cont bundles $\mathcal{U} \to \mathcal{U}/$ Cont and Cont $\times \mathcal{U}/$Cont $\to \mathcal{U}/$Cont are isomorphic, whence the theorem follows.

References

[1] J. Bland, Contact geometry and CR structures on S^3, Acta Math, **172** (1994), 1–49.

[2] J. Bland and T. Duchamp, Moduli for pointed convex domains, Invent. Math., **104** (1991), 61–112.

[3] L. Boutet de Monvel, Intégration des equations de Cauchy-Riemann induites formelles, Séminaire Goulaoic-Lions-Schwartz, Exposé IX (1974-1975).

[4] D. Burns, Global behavior of some tangential Cauchy-Riemann equations, Partial Differential Equations and Geometry (Proc. Conf., Park City, Utah), Marcel Dekker, New York (1979).

[5] C.-H. Chang, M. C. Hu, H.-P. Lee, Extremal discs with prescribed boundary data, Trans. Amer. Math. Soc., **310** (1988), 355–369.

[6] T. H. Cheng, J. M. Lee, A local slice theorem for 3-dimensional CR structures, preprint.

[7] C.Epstein, A relative index on the space of embeddable CR structures, I-II, preprint.

[8] E. Falbel, Non-embeddable CR manifolds and surface singularities, Invent. Math., **108** (1992), 49–65.

[9] R. Hamilton, The inverse function theorem of Nash and Moser, Bull. Amer. Math. Soc., **7** (1982), 65–222.

[10] H. Jacobowitz, F. Treves, Non-realizable CR structures, Invent. Math., **66** (1982), 231–249.

[11] G. K. Kiremidjian, A direct extension method for CR structures, Math. Ann., **242** (1979), 1–19.

[12] J. J. Kohn, Harmonic integrals on strongly pseudo-convex manifolds I, II, Ann. of Math. (2), **78** (1963), 112–148;,**79** (1964), 450–472.

[13] L. Lempert, La métrique de Kobayashi et la représentation des domaines sur la boule, Bull. Soc. Math. France, **109** (1981), 427–474.

[14] ——, Intrinsic distances and holomorphic retracts, Proc. Conf. Complex Analysis and Applications '81 Sofia, Bulg. Acad. Sci. (1984), 341–364.

[15] ——, Holomorphic invariants, normal forms and the moduli space of convex domains, Ann. of Math (2), **128** (1988), 43–78.

[16] ——, On three dimensional Cauchy-Riemann manifolds, J. Amer. Math. Soc., **5** (1992), 923–969.

[17] ———, Embeddings of three dimensional Cauchy-Riemann manifolds, Math. Ann., **300** (1994), 1–15.

[18] H. L. Li, The stability of embeddings of Cauchy-Riemann manifolds, Thesis, Purdue University (1995).

[19] J. W. Milnor, Remarks on infinite dimensional Lie groups, in Relativity, Groups, and Topology II Les Houches Session (B. de Witt and R. Stora, eds); North Holland, Amsterdam .

[20] L. Nirenberg, On a problem of Hans Lewy, Uspekhi Mat. Nauk, **29** (1974), 241–251.

[21] H. Rossi, Attaching analytic spaces to an analytic space along a pseudoconcave boundary, Proc. Conf. Complex Analysis (Minneapolis), Springer-Verlag, Berlin-Heidelberg-New York (1965).

Department of Mathematics
Purdue University
West Lafayette, IN 47907
U. S. A.

Advanced Studies in Pure Mathematics 25, 1997
CR-Geometry and Overdetermined Systems
pp. 237–246

Some Remarks on Compact Strongly Pseudoconvex CR Manifolds

Hing Sun LUK and Stephen S. T. YAU

Dedicated to Professor M. Kuranishi on his 70th birthday

§0. Introduction.

In this note, we make some remarks on compact strongly pseudoconvex CR manifolds. These remarks are related to the problem of minimal embedding dimension of a compact strongly pseudoconvex CR manifold in complex Euclidean space and the classification problem of compact strongly pseudoconvex CR manifolds. Most are contained in [LuY1-3] and in our joint paper with Yung Yu [LuYY]. We hope that this expository note would be of interest for the study of the relationship between the geometry of a compact strongly pseudoconvex CR manifold and the singularities that it may bound, much in the spirit of Kuranishi's application of $\overline{\partial}_b$ to the deformation of isolated singularities [Ku1]. The first author most gratefully recalls the years at Columbia when he studied with Professor Kuranishi and was brought into the fascinating field of CR geometry, being inspired by Professor Kuranishi's mathematical power and depth.

§1 Preliminary

Definition 1.1. *Let X be a connected real manifold of dimension $2n-1$, $n \geq 2$. A CR structure on X is an $(n-1)$-dimensional subbundle S of the complexified tangent bundle $\mathbb{C}TX$ such that*

(a) $S \cap \overline{S} = \{0\}$,

(b) *If L, L' are local sections of S, then so is $[L, L']$.*

Definition 1.2. *Let $L_1, \ldots, L_{n-1}$ be a local frame of S. Then $\overline{L}_1, \ldots, \overline{L}_{n-1}$ is a local frame of $\overline{S}$ and one may choose a local section N*

Received September 25, 1995

of TX such that $L_1, \ldots, L_{n-1}, \overline{L}_1, \ldots, \overline{L}_{n-1}, N$ is a local frame of $\mathbb{C}TX$. The matrix (c_{ij}) defined by

$$[L_i, \overline{L}_j] = \sum a_{ij}^k L_k + \sum b_{ij}^k \overline{L}_k + \sqrt{-1} c_{ij} N$$

is Hermitian and is called the Levi form. The CR manifold X is called strongly pseudoconvex if its Levi form is definite at each point of X. This condition is independent of the choice of local frame and it implies the orientability of X.

A fundamental invariant in CR geometry is the $\overline{\partial}_b$ cohomology introduced by Kohn-Rossi [KR]. The formulation below follows Tanaka [T].

Definition 1.3. *Let $A^k(X) = \wedge^k \mathbb{C}TX^*$ and $\mathcal{A}^k(X) = \Gamma(A^k(X))$. Then $\{\mathcal{A}^k(X), d\}$ is the deRham complex. With the notation in Definition 1.1, let*

$$A^{p,q}(X) = \{\varphi \in A^{p+q}(X) : \varphi(Y_1, \ldots, Y_{p-1}, \overline{Z}_1, \ldots, \overline{Z}_{q+1}) = 0 \ , \\ \text{for all } Y\text{'s in } \mathbb{C}TX \text{ and } Z\text{'s in } S\}$$

and $\mathcal{A}^{p,q}(X) = \Gamma(A^{p,q}(X))$. Then

$$A^{p+1,q-1}(X) \subset A^{p,q}(X) \quad \text{and} \quad d\mathcal{A}^{p,q}(X) \subset \mathcal{A}^{p,q+1}(X) \ .$$

Hence let $C^{p,q}(X) = A^{p,q}(X)/A^{p+1,q-1}(X)$ and $\mathcal{C}^{p,q}(X) = \Gamma(C^{p,q}(X))$. Then

$$\begin{array}{ccc} \mathcal{A}^{p,q}(X) & \xrightarrow{d} & \mathcal{A}^{p,q+1}(X) \\ \cup & & \cup \\ \mathcal{A}^{p+1,q-1}(X) & \xrightarrow{d} & \mathcal{A}^{p+1,q}(X) \end{array} \quad \text{induces} \quad \mathcal{C}^{p,q}(X) \xrightarrow{\overline{\partial}_b} \mathcal{C}^{p,q+1}(X).$$

The cohomology groups of the resulting complex $\{\mathcal{C}^{p,q}(X), \overline{\partial}_b\}$ will be denoted by $H^{p,q}(X)$.

The harmonic theory for the $\overline{\partial}_b$ complex on compact strongly pseudoconvex CR manifolds was developed by Kohn [Ko]. The theory of harmonic integrals on strongly pseudoconvex CR structures over small balls was due to Kuranishi [Ku2]. Using the former theory, Boutet de Monvel [B] proved that if X is a compact strongly pseudoconvex CR manifold of dimension $2n-1$ and $n \geq 3$, then there exist C^∞ functions $f_1, \ldots, f_N$ on X such that each $\overline{\partial}_b h_j = 0$ and $f = (f_1, \ldots, f_N)$ defines an embedding of X in $\mathbb{C}^N$. Thus, any compact strongly pseudoconvex CR manifold of dimension ≥ 5 can be CR embedded in some complex

Euclidean space. Using the theory of harmonic integrals over small balls of special type, Kuranishi [Ku2] proved that any strongly pseudoconvex CR manifold of dimension $2n-1$ with $n \geq 5$ can be locally CR embedded as a real hypersurface in $\mathbb{C}^n$. For $n = 4$, Akahori [Ak] proved that Kuranishi's local embedding theorem is also true.

§2 Concerning the minimal embedding dimension in complex Euclidean space

Let us first consider a compact strongly pseudoconvex manifold X of dimension $2n-1$ where $n \geq 3$. As mentioned above, X can be CR embedded in some $\mathbb{C}^N$. It is therefore of interest to study the minimal dimensional complex Euclidean space in which X CR embeds. Our starting point is the following two theorems.

Theorem 2.1. *(Harvey-Lawson [HL], see also Yau [Y]) Let X be a compact strongly pseudoconvex CR manifold of dimension $2n-1$, $n \geq 2$, in $\mathbb{C}^N$. Then there exists a unique bounded complex analytic subvariety V of dimension n in $\mathbb{C}^N \setminus X$ such that X is the boundary of V in the C^∞ sense. Further, V is smooth except at finitely many isolated singular points.*

Theorem 2.2. *(Yau [Y]) Let X be a compact strongly pseudoconvex CR manifold of dimension $2n-1$, $n \geq 3$, which is the boundary of a Stein space V with isolated singularities $p_1, \ldots, p_m$. Then for $1 \leq q \leq n-2$,*

$$H^{p,q}(X) \cong \bigoplus_{i=1}^{m} H^{q+1}_{p_i}(V, \Omega^p_V)$$

where Ω^p_V is the sheaf of germs of holomorphic p-forms on V. If $p_1, \ldots, p_m$ are hypersurface singularities, then

$$\dim H^{p,q}(X) = \begin{cases} 0 & p+q \leq n-2 \quad 1 \leq q \leq n-2 \\ \displaystyle\sum_{i=1}^{m} \tau_i & p+q = n-1, n \quad 1 \leq q \leq n-2 \\ 0 & p+q \geq n+1 \quad 1 \leq q \leq n-2 \end{cases}$$

where τ_i is the number of moduli of V at p_i.

Theorem 2.2 provides a solution to the Kohn-Rossi conjecture [KR] that "in general, either there is no boundary cohomology (in degree

(p,q), $q \neq 0$, $n-1$) or it must result from the interior singularities". Moreover it provides us with obstructions to CR embedding:

Theorem 2.3. *Let X be a compact strongly pseudoconvex CR manifold of dimension $2n-1$, $n \geq 3$. Then X cannot be CR embedded in $\mathbb{C}^n$ unless all $H^{p,q}(X) = 0$, $1 \leq q \leq n-2$. Further, X cannot be CR embedded in $\mathbb{C}^{n+1}$ if one of the following does not hold:*

(1) $H^{p,q}(X) = 0$ *for* $p+q \leq n-2$ *and* $1 \leq q \leq n-2$

(2) $\dim H^{p,q}(X) = \dim H^{p',q'}(X)$ *for* $\left.\begin{matrix} p+q \\ p'+q' \end{matrix}\right\} = n-1, n$ *and* $1 \leq q, q' \leq n-2$

(3) $H^{p,q}(X) = 0$ *for* $p+q \geq n+1$ *and* $\leq q \leq n-2$

We next consider an interesting class of CR manifolds.

Definition 2.4. *Let X be a CR manifold with structure bundle S. Let α be a smooth S^1-action on X and $\mathbf{v}$ be its generating vector field. The S^1-action α is called holomorphic if $\mathcal{L}_{\mathbf{v}}\Gamma(S) \subset \Gamma(S)$. It is called transversal if $\mathbf{v}$ is transversal to $S \oplus \overline{S}$ in $\mathbb{C}TX$ at every point of X.*

For a CR manifold X which admits a transversal holomorphic S^1-action, the invariant Kohn-Rossi cohomology is defined as follows:

Definition 2.5. *With the notation in Definition 2.4, consider first the differential operator on k forms $N : \mathcal{A}^k(X) \to A^k(X)$ defined by $N\varphi = \sqrt{-1}\mathcal{L}_{\mathbf{v}}\varphi$, $\varphi \in \mathcal{A}^k(X)$. Observe that N leaves invariant the spaces $\mathcal{A}^{p,q}(X)$ and $\mathcal{C}^{p,q}(X)$, and commutes with the operators d and $\overline{\partial}_b$. Hence N acts on the cohomology groups $H^{p,q}(X)$. Now define the invariant Kohn-Rossi cohomology by $\tilde{H}^{p,q}(X) = \{c \in H^{p,q}(X) : Nc = 0\}$.*

For a compact strongly pseudoconvex CR manifold X of dimension $2n-1$, $n \geq 3$, which admits a transversal holomorphic S^1-action, the invariant Kohn-Rossi cohomology $\tilde{H}^{p,q}(X)$, for $1 \leq p+q \leq 2n-N-1$, are obstructions to CR embedding in $\mathbb{C}^N$. This is implied by the following theorem:

Theorem 2.6. *Let X be a compact strongly pseudoconvex CR manifold of dimension $2n-1$, $n \geq 3$, which admits a transversal holomorphic S^1-action. Suppose that X is CR embeddable in $\mathbb{C}^N$. Then $\tilde{H}^{p,q}(X) = 0$ for all $1 \leq p+q \leq 2n-N-1$.*

The proof of Theorem 2.6 contains two main parts. The first part depends heavily on the work of Lawson-Yau [LY], which provides us with topological restrictions on X. In particular it can be shown that the de Rham cohomology groups $H^k(X) = 0$ for $1 \leq k \leq 2n - N - 1$. The second part follows Tanaka's differential geometric study on the $\overline{\partial}_b$ cohomology groups [T]. The existence of the vector field $\mathbf{v}$ with $[\mathbf{v}, \Gamma(S)] \subset \Gamma(S)$ entails a formalism analogous to Kähler geometry linking the various cohomology groups via harmonic forms. The details of the proof of Theorem 2.6 are contained in [LuY2].

For 3 dimensional compact strongly pseudoconvex CR manifolds, global CR embedding in complex Euclidean space may fail and much work has been done recently on this phenomenon. See for example [BlE], [BuE], [L]. We only remark that as a consequence of the global invariants to be discussed in the next section, we find obstructions to CR embedding in $\mathbb{C}^3$, assuming that the 3 dimensional strongly pseudoconvex CR manifold is CR embeddable in some $\mathbb{C}^N$ to begin with. These obstructions provide us with numerous examples of such 3 dimensional CR manifolds not CR embeddable in $\mathbb{C}^3$.

§3 Concerning invariants of compact strongly pseudoconvex CR manifolds

As a first step towards the difficult classification problem of compact strongly pseudoconvex CR manifolds, it would be useful to understand the following notion of equivalence which is weaker than CR equivalence.

Definition 3.1. *Assume that X_1, X_2 are compact strongly pseudoconvex CR manifolds of dimension $2n-1$, $n \geq 2$, which are CR embeddable in some $\mathbb{C}^{N_1}$, $\mathbb{C}^{N_2}$ respectively. X_1, X_2 are called algebraically equivalent if the corresponding varieties V_1, V_2, which are bounded by X_1, X_2 in $\mathbb{C}^{N_1}$, $\mathbb{C}^{N_2}$ according to Theorem 2.1, have isomorphic singularities Y_1, Y_2, i.e., $(V_1, Y_1) \cong (V_2, Y_2)$ as germs of varieties.*

Thus, for $n = 2$, we are restricting ourselves to embeddable compact strongly pseudoconvex CR manifolds. It is not difficult to show that CR equivalence implies algebraic equivalence.

In case a compact strongly pseudoconvex CR manifold X of dimension $2n-1$ embeds in $\mathbb{C}^{n+1}$, $n \geq 2$, it is the boundary of a complex hypersurface V with isolated singularities $p_1, \ldots, p_m$. In this case, an Artinian algebra can be associated to X as follows.

Definition 3.2. *With the above notation, let f_i be a defining function of the germ (V, p_i), $1 \leq i \leq m$. Then the $\mathbb{C}$-algebra $A_i =$*

$\mathcal{O}_{n+1}\Big/\left(f_i, \frac{\partial f_i}{\partial z_0}, \ldots, \frac{\partial f_i}{\partial z_n}\right)$ *is a commutative local Artinian algebra called the moduli algebra of* (V, p_i). *The moduli algebra is independent of the choice of defining function. We associate to the CR manifold* X *the Artinian algebra* $A(X) = \bigoplus_{i=1}^{m} A_i$.

By the work of Mather-Yau [MY] on isolated hypersurface singularities, it can be shown that the associated Artinian algebras are complete algebraic CR invariants in the following sense.

Theorem 3.3. *[LuY3] Two compact strongly pseudoconvex real codimension 3 CR manifolds* X_1, X_2 *are algebraically equivalent if and only if the associated Artinian algebras* $A(X_1)$, $A(X_2)$ *are isomorphic* $\mathbb{C}$ *algebras.*

We remark that there are Torelli type examples in which the Artinian algebras $A(X_t)$ associated to a family of compact strongly pseudoconvex real codimension 3 CR manifolds X_t suffice to distinguish CR equivalence. For example, in the family $X_t = \{(x, y, z) \in \mathbb{C}^3 : x^6 + y^3 + z^2 + tx^4y = 0 \text{ and } |x|^2 + |y|^2 + |z|^2 = \varepsilon^2\}$ where $\varepsilon > 0$ is a small fixed number and $t \in \mathbb{C}$ with $4t^2 + 27 \neq 0$, X_{t_1}, X_{t_2} are CR equivalent if and only if $A(X_{t_1})$, $A(X_{t_2})$ are isomorphic $\mathbb{C}$ algebras.

For the rest of this section, we consider embeddable 3 dimensional compact strongly pseudoconvex CR manifolds. By taking resolutions of the singularities of the subvariety V bounded by such a CR manifold X in complex Euclidean space, numerical invariants under algebraic equivalence may be defined, as follows.

Definition 3.4. *Let* $\pi : M \to V$ *be a resolution of the singularities* Y *of* V *such that the exceptional set* $A = \pi^{-1}(Y)$ *has normal crossing, i.e., the irreducible components* A_i *of* A *are nonsingular, they intersect transversally and no three meet at a point. According to Artin [Ar], there exists a unique minimal positive divisor* Z, *called the fundamental cycle, with support on* A, *such that* $Z \cdot A_i \leq 0$ *for all* i. *For any positive divisor* $D = \sum d_i A_i$, *let* $\mathcal{O}_M(-D)$ *be the sheaf of germs of holomorphic functions on* M *vanishing to order* d_i *on* A_i, *let* $\mathcal{O}_D = \mathcal{O}_M/\mathcal{O}_M(-D)$ *and let* $\chi(\mathcal{O}_D) = \sum_{i=0}^{2}(-1)^i \dim H^i(M, \mathcal{O}_D)$. *It can be proved that* $p_f(X) \stackrel{\text{def}}{=} 1 - \chi(\mathcal{O}_Z)$, $p_a(X) \stackrel{\text{def}}{=} \sup(1 - \chi(\mathcal{O}_D))$ *where* D *ranges over all positive divisors with support on* A *and* $p_g(X) \stackrel{\text{def}}{=} \dim H^1(M, \mathcal{O})$ *are defined independent of the resolution* π *and are invariants of* X *under algebraic equivalence. The detailed proofs are contained in [LuYY]. We refer to*

$p_f(X)$, $p_a(X)$ and $p_g(X)$ as the fundamental genus, arithmetic genus and geometric genus of X respectively.

The following facts are known

- $0 \le p_f(X) \le p_a(X) \le p_g(X)$
- $p_f(X) = 0 \Leftrightarrow p_a(X) = 0 \Leftrightarrow p_g(X) = 0$.

Further numerical invariants under algebraic equivalence are given by $m_Z(X) \stackrel{\text{def}}{=} Z \cdot Z$, $q(X) \stackrel{\text{def}}{=} \dim H^0(M - A, \Omega^1)/H^0(M, \Omega^1)$, $\chi(X) \stackrel{\text{def}}{=} K \cdot K + \chi_T(A)$ and $\omega(X) \stackrel{\text{def}}{=} K \cdot K + \dim H^1(M, \Omega^1)$, where Ω^1 is the sheaf of germs of holomorphic 1-form on M, $\chi_T(A)$ is the topological Euler characteristic of A and K is the canonical divisor on M. These invariants are defined independent of the choice of the resolution π. Since K is a divisor with rational coefficient, $\chi(X)$ and $\omega(X)$ are in general rational numbers.

Using the above invariants, one may attempt a rough algebraic classification of embeddable 3 dimensional compact strongly pseudoconvex CR manifolds. In particular,

Definition 3.5. *An embeddable 3 dimensional compact strongly pseudoconvex CR manifold X is called a rational (resp. elliptic) CR manifold if $p_a(X) = 0$ (resp. $p_a(X) = 1$).*

If X is a rational or an elliptic CR manifold embeddable in $\mathbb{C}^3$ and M_0 is the minimal good resolution of the subvariety V bounded by X in $\mathbb{C}^3$, then the weighted dual graph for the exceptional set of M_0 is completely classified. The same also holds for those X embeddable in $\mathbb{C}^3$ and has $p_g(X) = 1$. With the weighted dual graphs classified, the topology of the embedding of the exceptional set in M_0 is well understood.

As an application, one obtains obstructions to embedding in $\mathbb{C}^3$ for the above three classes of CR manifolds when their weighted dual graphs fail to have the required forms. For example, a rational CR manifold whose weighted dual graph is not a direct sum of the graphs A_k, D_k, E_6, E_7, E_8 is not embeddable in $\mathbb{C}^3$.

Similarly in view of the following theorem, one obtains numerical obstructions to embedding in $\mathbb{C}^3$ for those CR manifolds failing the conditions in the theorem.

Theorem 3.6. *Let X be a compact strongly pseudoconvex 3 dimensional CR manifold embeddable in $\mathbb{C}^3$. Then*

(1) *$\chi(X)$ and $\omega(X)$ are integers.*

(2) $10p_g(X) + \omega(X) \geq 0$.

(3) *If* $p_a(X) = 1$, *then* $\chi(X) \geq -3$.

(4) *If* X *admits a transversal holomorphic* S^1*-action, then* $6p_g(X) + \chi(X) > 0$.

The proof of Theorem 3.6 is contained in [LuY1], [LuYY]. We remark that (4) depends on the Durfee conjecture which is solved by Xu and the second author in [XY].

It is interesting to note that there are compact strongly pseudoconvex 3 dimensional CR manifolds with arbitrarily large minimal embedding dimensions. For any positive integer N, take any 2 dimensional strongly pseudoconvex complex manifold with maximal compact analytic set A which is a smooth rational curve having self intersection number $-N$. The corresponding weighted dual graph is hence $\overset{\bullet}{-N}$. On blowing down A, one gets a 2 dimensional rational singularity (V, p). The minimal embedding dimension of (V, p) is $-A \cdot A + 1 = N + 1$. Let X be the intersection of V with a small sphere centered at p. Then the minimal embedding dimension of X is $N + 1$.

Work is in progress on determining the weighted dual graph associated to X as above, in terms of the CR manifold X intrinsically. This however is a difficult problem.

References

[Ak] Akahori, T., A new approach to the local embedding theorem of CR structures for $n \geq 4$ (the local solvability for the operator $\overline{\partial}_b$ in the abstract sense), Memoirs Amer. Math. Soc., 366, 1987.

[Ar] Artin, M., On isolated rational singularities of surfaces, Amer. J. Math. 88 (1966), 129-136.

[B] Boutet de Monvel, L., Intégration des équations de Cauchy-Riemann, Séminaire Goulaouic-Lions-Schwartz, Ex ré IX, 1974-1975.

[BlE] Bland, J., and Epstein, C.L., Embeddable CR-structures and deformations of pseudoconvex surfaces Part I: Formal deformations, to appear in Jour. of Alg. Geom.

[BuE] Burns, D., and Epstein, C.L., Embeddability for three dimensional CR-manifolds, J. Amer. Math. Soc. 3 (1990), 809-841.

[HL] Harvey, F.R., and Lawson, H.B., On boundaries of analytic varieties I, Ann. of Math. 102 (1975), 223-290.

[Ko] Kohn, J.J., Boundaries of complex manifolds, Proc. Conference on Complex Manifolds (Minneapolis), Springer-Verlag, New York, 1965.

[KR] Kohn, J.J., and Rossi, H., On the extension of holomorphic functions from the boundary of a complex manifold, Ann. of Math. 81 (1965), 451-472.

[Ku1] Kuranishi, M., Application of $\overline{\partial}_b$ to deformation of isolated singularities, Proc. of Symp. in Pure Math. 30 (1977), 97-106.

[Ku2] Kuranishi, M., Strongly pseudoconvex CR structures over small balls I, II, III, Ann. of Math. (2) 115 (1982), 451-500; ibid 116 (1982), 1-64; ibid 116 (1982), 249-330.

[L] Lempert, L., Embeddings of three dimensional Cauchy Riemann manifolds, Math. Ann. 300 (1994), 1-15.

[LY] Lawson, H.B., and Yau, S.S.-T, Holomorphic symmetries, Ann. Sci. École Norm. Sup. 4^e Série, t.20 (1987), 557-577.

[LuY1] Luk, H.S., and Yau, S.S.-T, Obstructions to embedding of real $(2n-1)$-dimensional compact CR manifolds in $\mathbb{C}^{n+1}$, Proc. of Sympos. Pure Math. 52 (1991), Part 3, 261-276.

[LuY2] Luk, H.S., and Yau, S.S.-T., Invariant Kohn-Rossi cohomology and obstructions to embedding of compact real $(2n-1)$-dimensional CR manifolds in $\mathbb{C}^N$, to appear in J. Math. Soc. Japan.

[LuY3] Luk, H.S., and Yau, S.S.-T., Complete algebraic CR invariants of real codimension 3 strongly pseudoconvex CR manifold, to appear.

[LuYY] Luk, H.S., Yau, S.S.-T., and Yu, Y., Algebraic classification and obstructions to embedding of strongly pseudoconvex compact 3-dimensional CR manifolds in $\mathbb{C}^3$, Math. Nachr. 170 (1994), 183-200.

[MY] Mather, J., and Yau, S.S.-T., Classification of isolated hypersurface singularities by their moduli algebras, Invent. Math. 69 (1982), 243-251.

[T] Tanaka, N., A differential geometric study on strongly pseudoconvex manifolds, Lectures in Mathematics, Kyoto University, 9, Kinokuniya Book-Store Co., Ltd., 1975.

[XY] Xu, Y.-J., and Yau, S.S.-T., Durfee conjecture and coordinate free characterization of homogeneous singularities, J. Differential Geom. 37 (1993), 375-396.

[Y] Yau, S.S.-T., Kohn-Rossi cohomology and its application to the complex Plateau problem I, Ann. of Math. 113 (1981), 67-110.

Hing Sun Luk
Department of Mathematics,
The Chinese University of Hong Kong,
Shatin, N.T.,
Hong Kong
hsluk@math.cuhk.hk

Stephen S.-T. Yau
Department of Mathematics, Statistics and Computer Science,
University of Illinois at Chicago,
Box 4348, M/C 249,
Chicago, IL 60680,
U.S.A.
U32790@UICVM.BITNET

Advanced Studies in Pure Mathematics 25, 1997
CR-Geometry and Overdetermined Systems
pp. 247–275

Deformation Theory of CR-Structures and Its Application to Deformations of Isolated Singularities II

Kimio Miyajima

Introduction

Deformations of an analytic variety with only isolated singular points induce deformations of strongly pseudo-convex CR structures on its link. It is M. Kuranishi who initiated to consider deformations of compact strongly pseudo-convex CR structures expecting to describe deformations of isolated singular points of analytic varieties. Since non-equivalent CR manifolds can bound the same isolated singular point, we consider deformations of CR structures up to equivalence weaker than the CR-equivalence. This equivalence is induced from wiggling in a complex manifold and we will call the deformation theory of CR structures under that equivalence the *Kuranishi deformation theory* of CR structures. In [Ku3], [Ku4], M. Kuranishi obtained a C^∞-family of deformations of the CR structure on a compact strongly pseudo-convex CR manifold of real dimension five or higher, continuing his early works on deformations of compact complex structures ([Ku1], [Ku2]). We consider holomorphic families of CR structures. In the first half of this survey, we will review the holomorphically parametrized deformation theory of strongly pseudo-convex CR structures developped by T. Akahori et al. ([Ak1], [Ak2], [Ak3], [Ak4], [Ak-My1], [Ak-My2], [Ak-My3], [Ak-My4], [Bu-Ml], [My1], [My2], [My3]) and its relationship with algebraic deformation theory of isolated singularities ([Do], [Gr], [Tj]).

The relationship between compact strongly pseudo-convex CR manifolds and isolated singularities is based on the fact that an embeddable compact strongly pseudo-convex CR manifold bounds a unique normal Stein complex space ([Ha-La]) and all compact strongly pseudo-convex

Received June 30, 1996
Partially supported by Grant-in-Aid for Scientific Research (No.07640128) Ministry of Education and Culture of Japan and The Sumitomo Foundation

CR manifolds of real dimension five or higher are embeddable ([BM]). In contrast with the higher dimensional case, embeddable three dimensional CR manifolds are rare. (Embeddability is a major problem in the study of three dimensional compact strongly pseudo-convex CR manifolds. Refer to the references of [Bl-Du], [Ep] and [Lm] for papers about that problem.) Recently, J. Bland and C. Epstein generalized the Kuranishi deformation theory to embeddable compact strongly pseudo-convex three dimensional CR structure case and show that the stably embeddable formal deformation theory of a strongly pseudo-convex three dimensional CR structure is isomorphic to the formal deformation theory of the normal isolated surface singularity it bounds ([Bl-Ep]). In the latter half of this survey, we will develop their deformation theory to the actual deformation level in the higher dimensional case and compare it with the deformation theory of normal isolated singularities. By this argument, we will see that the stably embeddable deformation theory of strongly pseudo-convex CR structures fits to the flat deformation theory of normal isolated singularities and then we will complete Kuranishi's program describing the semi-uinversal family of normal isolated singularities of complex dimension three or higher in terms of the CR language.

Other developments in the study of isolated singularities by CR geometry are done in [Lu-Ya] and [Ya]. Refer to [Oh3] for other results in the study of isolated singularities by transcendental methods. The moduli of compact strongly pseudo-convex CR manifolds under CR-equivalence is of natural interest from CR-geometry. It is treated in [Ch-Le] and [Lm] in three dimensional cases. The moduli space of strongly pseudo-convex CR structures on a compact real three-fold divides into two parts; the part of embeddable CR structures and the part of non-embeddable ones. In [Ep], moduli of embeddable CR structures is considered in connection with deformations of isolated singularities, and in [Bl-Du], moduli of non-embeddable ones on S^3 is considered.

In this survey, we consider only the case of real dimension five or higher. In Sections 1–4, we will review the construction of the Kuranishi semi-universal family of compact strongly pseudo-convex CR structures and its relationship with the semi-universal family of normal isolated singularities. The main part of the construction of the Kuranishi semi-universal family was presented in Part I under the assumption $H^2_{\bar\partial_{T'}}(T') = 0$. Hence we will only give a modification needed for treating the general case. In Sections 5 and 6, we will work on the stably embeddable deformation theory of strongly pseudo-convex CR structures on a link of a normal isolated singularity.

Notation

(N.1) Let V be a closed normal subvariety of a ball

$$B(c^*) := \{w \in \mathbf{C}^N \mid \sum_{\beta=1}^{N} |w^\beta|^2 < c^*\}$$

such that the origin $0 \in \mathbf{C}^N$ is the only singular point of V and V and $\partial B(c)$ intersect transversally for any $0 < c < c^*$. We denote by U the regular part of V and by ι the natural inclusion map $U \hookrightarrow \mathbf{C}^N$. We will use the following notations throughout this survey:

$$G(a,b) := \{w \in \mathbf{C}^N \mid a < \sum_{\beta=1}^{N} |w^\beta|^2 < b\} \ (0 < a < b < c^*),$$

$$S_c := \{w \in \mathbf{C}^N \mid \sum_{\beta=1}^{N} |w^\beta|^2 = c\} \ (0 < a < c \leq b),$$

$$\Omega(a,b) := V \cap G(a,b), \qquad M_c := V \cap S_c.$$

(N.2) For a holomorphic vector bundle over a CR manifold M, we denote by $A_b^{0,q}(E)$ (resp. $\Gamma(U,E)$) the space of E-valued tangential (0,q)-forms (resp. the space of C^∞-sections of E over a domain $U \subset M$). $A_{b,k}^{0,q}(E)$ (resp. $\Gamma_k(U,E)$ and $\Gamma_k^{'}(U,E)$) the completion of $A_b^{0,q}(E)$ with respect to the Sobolev k-norm (resp. of $\Gamma(U,E)$ with respect to the Sobolev k-norm and the Folland-Stein k-norm).

(N.3) Let S be a germ of a (not necessarily reduced) complex space at the distinguished point $s_o \in S$ and X be a (not necessarily compact) complex manifold (in our argument below, X is a neighbourhood of M_c in U). By a family of deformations of X over (S, s_o), we mean a smooth holomorphic map (in Grothendieck's sense) of complex spaces $\pi : \mathcal{X} \to S$ with $\pi^{-1}(s_o) \simeq X$, that is, for any $x \in \mathcal{X}$ there exist neighbourhoods $\mathcal{W}$ of x in $\mathcal{X}$, W of the origin in $\mathbf{C}^n$ ($n = \dim_{\mathbf{C}} X$) and an isomorphism q so that the diagram

$$\begin{array}{ccc} \mathcal{W} & \xrightarrow{q} & W \times S \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle p_2} \\ S & = & S \end{array}$$

commutes where p_2 denotes the projection onto the second factor. As a local trivialization of a family of deformations of X, we always take a local trivialization of this type. Then we always have a local coordinate $(\zeta^1, \ldots, \zeta^n, s_1, \ldots, s_d)$ of $\mathcal{X}$ such that $(\zeta^1, \ldots, \zeta^n)$ (resp. $(s_1, \ldots, s_d)$) is the coordinate of W (resp. of an ambient space $\mathbf{C}^d$ of S) in the above trivialization.

(N.4) Suppose that X is a locally closed sub-manifold of $\mathbf{C}^N$. By a family of displacements of X in $\mathbf{C}^N$ over a germ (S, s_o), we mean a family of deformations of X over (S, s_o), $\pi : \mathcal{X} \to S$, together with an embedding $\Phi : \mathcal{X} \to \mathbf{C}^N \times S$ such that $\pi = p_2 \circ \Phi$ holds where $p_2 : \mathbf{C}^N \times S \to S$ denotes the projection onto the second factor.

(N.5) There are two approaches to the CR manifolds; the extrinsic approach and the intrinsic one (i.e. treatments as a real submanifold of a complex manifold and as a real manifold equipped with an abstract CR structure, respectively). These approaches are equivalent in the case of compact strongly pseudo-convex CR manifolds of real dimension greater than or equals to five, while there are differences between them in real three dimensional case. Our treatment of deformations of CR structures is based on the intrinsic approach. Refer to [Ta] for the systematic study of this approach. In order to compare deformations of CR structures and that of singularities, we need to take account of the extrinsic approach as well. Refer to [Fo-Ko] or [Ko-Ro] for the extrinsic approach.

Our approach to deformations of normal isolated singularities from deformation theory of CR structures will be done through the following three steps. In each step, we use several fundamental theories. First step: We construct a family of CR structures by a generalized Kodaira-Spencer construction. Refer to [Ko] for the Kodaira-Spencer construction in the case of deformations of compact complex manifolds. The Kodaira-Spencer construction heavily depends on the harmonic theory. Refer to [Fo-Ko] for the standard harmonic theory on a CR manifold. Second step: We prove that the family constructed in the first step is Kuranishi versal. In order to carry out the ideal theoretic argument in the proof, we use the Grauert division theorem. Refer to [Gr] or [Fo-Kn] for the Grauert division theorem. Third step: We compare the family constructed in the first step with the semi-universal family of normal isolated singularities. By the comparison using the Kuranishi semi-universality of the family of CR structures and the semi-versality of the family of isolated singularities, we have a formal isomorphism of their parameter spaces. In order to reach the actual isomorphism, we use the Artin approximation theorem. Refer to [Ar] for the Artin approximation theorem.

We remark that W. Goldmann and J. Millson established a general comparison method. We can compare the above two families directly (without Step 2) using this general method. Refer to [Go-Ml1] and [Go-Ml2] for this general comparison method and to [Bu-Ml] for the approach using this method.

§1. Kuranishi deformation theory of CR structures

Let $M := M_c$ for some $0 < c < c^*$ (cf. (N.1)) and ${}^\circ T''$ be the strongly pseudo-convex CR structure on M induced from the complex structure of V (cf. Part I, sections 2.1 for the notion of a strongly pseudo-convex CR structures). In this section, we formulate some fundamental notions of Kuranishi deformation theory of CR structures on M.

A holomorphic family of CR structures is a notion analogous to a family of complex structures. We fix a splitting Part I, (2.1.6);

$$\mathbf{C}TM = {}^\circ T' + {}^\circ T'' + \mathbf{C}F$$

and denote by $T' = {}^\circ T' + \mathbf{C}F$ the holomorphic tangent bundle of M.

Definition 1.1. *Let T be a germ of a complex subspace of $\mathbf{C}^d$ at the origin defined by an ideal $\mathfrak{I}_T \subset \mathbf{C}\{t_1, \dots, t_d\}$. A holomorphic family of deformations of the CR structure ${}^\circ T''$ over $(T, 0)$ is $\phi(t) \in A_b^{0,1}(T')[[t_1, \dots, t_d]] \cap \bigcap_{k>0} A_{b,k}^{0,1}(T')\{t_1, \dots, t_d\}$ satisfying*

(1) $\phi(0) = 0$,

(2) $P(\phi(t)) \in \mathfrak{I}_T A_{b,k-1}^{0,2}(T')\{t_1, \dots, t_d\}$ *for all* $k >> 0$

where $P(\phi) = 0$ is the integrability condition (cf. Part I, Section 3.2).

We will simply denote it by $\phi(t)$ $(t \in (T, 0))$.

An embedding of M into a family of complex manifolds induces a family of CR structures.

Definition 1.2. *Let S be a germ of a complex subspace of $\mathbf{C}^{d'}$ at the origin defined by an ideal $\mathfrak{I}_S \subset \mathbf{C}\{s_1, \dots, s_{d'}\}$. Let $\pi : \mathcal{U} \to S$ be a family of complex manifolds. A holomorphic family of embeddings of M into that family is a mapping $F : M \times S \to \mathcal{U}$ with $\pi \circ F = p_2$ where p_2 denotes the projection onto the second factor, which is described locally as follows: Let $\{(\mathcal{W}_i, (\zeta_i^1, \dots, \zeta_i^n, s_1, \dots, s_{d'}))\}_{i\in\Lambda}$ be a system of local coordinates of $\mathcal{U}$ as in (N.3). Let $\{U_i\}_{i\in\Lambda}$ be an open covering of M such that $F(U_i \times S) \subset \mathcal{W}_i$. If F is described by $\zeta_i^\alpha = F_i^\alpha(x_i, s)$ $(\alpha = 1, \dots, n)$ on $U_i \times S$ with respect to the above local coordinate of $\mathcal{W}_i$ and the local coordinate $(x_1^1, \dots, x_i^{2n-1})$ of U_i, then*

(1) $F_i^\alpha(s) \in \Gamma(U_i, 1)[[s_1, \dots, s_{d'}]] \cap \Gamma_k(U_i, 1)\{s_1, \dots, s_{d'}\}$ $(\alpha = 1, \dots, n)$ *for all* $k >> 0$,

(2) $F_i^\alpha(s) - f_{ij}^\alpha(F_j(s), s) \in \mathfrak{I}_S\Gamma_k(U_i \cap U_j, 1)\{s_1, \dots, s_{d'}\}$ $(\alpha = 1, \dots, n)$ *for all* $k >> 0$,

where $\zeta_i^\alpha = f_{ij}^\alpha(\zeta_j, s)$ $(\alpha = 1, \dots, n)$ *is the coordinate transformation on* $\mathcal{W}_i \cap \mathcal{W}_j$.

The holomorphic family $\phi(s)$ $(s \in (S, 0))$ of CR structures induced by F is characterized by

$$(\bar{\partial}_b - \phi(s))F_i^\alpha(s) \in \mathfrak{I}_S A_b^{0,1}(U_i, 1)[[s_1, \dots, s_{d'}]].$$

We call this $\phi(s)$ $(s \in (S, s_o))$ the family of CR structures induced by F.

Since the Kuranishi CR deformation theory is arranged suitable for deformation theory of normal isolated singularities, we do not consider CR structures up to CR isomorphism but consider them up to wiggling in an ambient complex manifold. The following notion of the versality is reasonable for our deformation theory.

Definition 1.3. *A holomorphic family* $\phi(t)$ $(t \in (T, 0))$ *of deformations of* $^\circ T''$ *is Kuranishi versal if it has the following property: For any family of deformations of complex manifolds* $\pi : \mathcal{U} \to S$ *over* (S, s_o) *such that* $\pi^{-1}(s_o)$ *is a neighbourhood of* M *in* U, *there exist a holomorphic map of germs* $\tau : (S, s_o) \to (T, 0)$ *and a holomorphic family of embeddings* $F : M \times S \to \mathcal{U}$ *such that* $F_{|M \times s_o} = \iota$ *holds and the holomorphic family of CR structures induced by* F *coincides with* $\phi(\tau(s))$ $(s \in (S, s_o))$.

We will examine the fist derivative of a family of CR structures. Let $\phi(t)$ $(t \in (T, 0))$ be a holomorphic family of deformations of $^\circ T''$. Since the linear term of $P(\phi)$ is $\bar{\partial}_{T'}\phi$, we have $\bar{\partial}_{T'}v(\phi(t)) = 0$ for $v \in T_0T$ where we denote by T_0T the Zariski tangent space of T at 0. Next, if $\phi(t)$ $(t \in (T, 0))$ and $\psi(t)$ $(t \in (T, 0))$ are holomorphic families of CR structures induced by holomorphic families of embeddings into a family of complex manifolds $F, G : M \times T \to \mathcal{U}$ respectively, then we have

$$\{\sum_{\alpha=1}^n v(F_i^\alpha(t) - G_i^\alpha(t))\frac{\partial}{\partial \zeta_i^\alpha}\} \in A_b^0(T'U_{|M})$$

and

$$\rho^{1,0}(v(\phi(t) - \psi(t))) = \bar{\partial}_b(\sum_{\alpha=1}^n v(F_i^\alpha(t) - G_i^\alpha(t))\frac{\partial}{\partial \zeta_i^\alpha})$$

where $\rho^{1,0} : \mathbf{C}TU_{|M} \to T^{'}U_{|M}$ denotes the projection onto the (1,0)-part. Hence, it is natural to call the $\bar{\partial}_{T'}$ cohomology class of $v(\phi(t))$ the infinitesimal deformation class of the family $\phi(t)$ $(t \in (T,0))$ along the direction $v \in T_0T$.

Definition 1.4. *For a holomorphic family $\phi(t)$ $(t \in (T,0))$ of deformations of CR structures, the infinitesimal deformation map is a linear map $\rho : T_0T \to H^1_{\bar{\partial}_{T'}}(T^{'})$ given by $\rho(v) :=$ the cohomology class of $v(\phi(t))$. A holomorphic family is called effective if its infinitesimal deformation map is injective.*

An effective and Kuranishi versal family is called a Kuranishi semi-universal family.

§2. Construction of the Kuranishi semi-universal family of CR structures

In this section, we consider how to construct the Kuranishi semi-universal family of CR structures on M.

(I) First, we consider the case of $\dim_{\mathbf{R}} M \geq 7$ and will review the construction of the Kuranishi semi-universal family in [Ak3] and [Ak-My1]. However we modify the ideal theoretic argument in [Ak-My1] by using the Grauert division theorem instead of a small tric there, because the adaptation of the division theorem is the most relevant way to treat the case of non-reduced parameter spaces. We first try to construct it using J. J. Kohn's solution of the $\bar{\partial}_b$-Neumann problem (cf. [Fo-Ko]). Though it works only on the formal family level, we will try it, because this is a straightforward analogue of the standard Kodaira-Spencer construction in the case of deformations of complex structures on a compact complex manifold and, by this consideration, we will well understand the naturality of the adaptation of the sub-complex $(\Gamma(M, E_q), \bar{\partial}_q)$ in [Ak3] (cf. Part I, Section 3). The $\bar{\partial}_b$-Neumann Hodge decomposition which we will use is

$$(2.1) \qquad \eta = \rho_{T'}\eta + \bar{\partial}_{T'}\bar{\partial}^*_{T'}N_{T'}\eta + \bar{\partial}^*_{T'}\bar{\partial}_{T'}N_{T'}\eta \quad \text{for } \eta \in A^{0,2}_b(T^{'}),$$

where $\rho_{T'}$ denotes the orthogonal projection onto the harmonic space $\mathbf{H}^2_b(T^{'})$ (cf. [Fo-Ko, Theorem 5.4.12]). Let $d = \dim H^1_{\bar{\partial}_{T'}}(T^{'})$ and $\phi_1, \ldots, \phi_d$ be $\bar{\partial}_{T'}$-closed forms which give cohomology basis of $H^1_{\bar{\partial}_{T'}}(T^{'})$.

Set

$$\phi_1(t_1,\dots,t_d) := \sum_{\sigma=1}^{d} \phi_\sigma t_\sigma, \tag{2.2}$$

$$\phi_\mu(t_1,\dots,t_d) := \mu\text{-th homogeneous term of}$$

$$-\bar\partial^*_{T'} N_{T'} P(\phi^{(\mu-1)}(t_1,\dots,t_d))$$

(cf. Part I, section 3.2 for the definition of $P(\phi)$),

$$\phi^{(\mu)}(t_1,\dots,t_d) := \phi^{(\mu-1)}(t_1,\dots,t_d) + \phi_\mu(t_1,\dots,t_d)$$

and

$$\hat\phi(t_1,\dots,t_d) := \lim_{\mu\to\infty}\phi^{(\mu)}(t_1,\dots,t_d).$$

Thus we have $\hat\phi(t_1,\dots,t_d) \in A_b^{0,1}(T')[[t_1,\dots,t_d]]$ (we will denote it simply by $\hat\phi(t)$) satisfying

$$\hat\phi(t) + \bar\partial^*_{T'} N_{T'} R(\hat\phi(t)) = \phi_1(t) \tag{2.3}$$

where $R(\phi) := R_2(\phi) + R_3(\phi)$. Take a basis $e_1,\dots,e_\ell$ of $\mathbf{H}_b^2(T')$, then we have $\hat b_1(t),\dots,\hat b_\ell(t) \in \mathbf{C}[[t_1,\dots,t_d]]$ such that

$$\rho_{T'} P(\hat\phi(t)) = \sum_{\gamma=1}^{\ell} \hat b_\gamma(t) e_\gamma.$$

Denote by $\hat{\mathfrak{I}}$ an ideal of $\mathbf{C}[[t_1,\dots,t_d]]$ generated by $\hat b_1(t),\dots,\hat b_r(t)$.

Proposition 2.1.

$$P(\hat\phi(t)) \in \hat{\mathfrak{I}} A_b^{0,2}(T')[[t_1,\dots,t_d]].$$

Proof. By (2.3) and using the Hodge decomposition (2.1), we have

$$P(\hat\phi(t)) = \rho_{T'} R(\hat\phi(t)) + \bar\partial^*_{T'} \bar\partial_{T'} N_{T'} R(\hat\phi(t)).$$

Using the fact $\bar\partial_b^{\phi} P(\phi) = 0$ (cf. Part I, Lemma 3.7.2), we can prove by induction on μ that

$$\bar\partial_{T'} R(\hat\phi(t)) \in (\hat{\mathfrak{I}} + \mathfrak{m}^{\mu+1}) A_b^{0,3}(T')[[t_1,\dots,t_d]]$$

holds for all $\mu \geq 1$ where $\mathfrak{m}$ denotes the maximal ideal of $\mathbf{C}\{t_1, \ldots, t_d\}$. Q.E.D.

Proposition 2.2. *$\hat{\phi}(t)$ is formally Kuranishi versal, that is, for any family of deformations of a neighborhood of M in U there exist τ and F in Definition 1.3 as formal power series in s.*

Proof. Let S be a germ of an analytic sub-space of $\mathbf{C}^{d'}$ defined by an ideal $\mathfrak{I}_S \subset \mathbf{C}\{s_1, \ldots, s_{d'}\}$ and let $\pi : \mathcal{U} \to S$ be a family of complex manifolds such that $\pi^{-1}(0)$ is a neighbourhood of M in U and $\{(\zeta_i^1, \ldots, \zeta_i^n, s_1, \ldots, s_{d'})\}_{i \in \Lambda}$ be a system of local coordinates of $\mathcal{U}$, as in (N.3), with the coordinate transformation $\zeta_i^\alpha = f_{ij}^\alpha(\zeta_j, s)$ $(\alpha = 1, \ldots, n)$. Then we will construct $\{\hat{F}_i^\alpha(s)\}$ $(\alpha = 1, \ldots, n)$ and $\hat{\tau}(s)$ by solving the following equations inductively, where we denote by $F_{i|\mu}^\alpha(s)$ and $\tau_\mu(s)$ the homogeneous terms of $\hat{F}_i^\alpha(s)$ and $\hat{\tau}(s)$ of degree μ respectively and denote $F_i^{(\mu)\,\alpha}(s) = F_{i|0}^\alpha(s) + \cdots + F_{i|\mu}^\alpha(s)$ and $\tau^{(\mu)}(s) = \tau_0(s) + \cdots + \tau_\mu(s)$.

$$(2.4)_\mu \quad F_i^{(\mu-1)\,\alpha}(s) - f_{ij}^\alpha(F_j^{(\mu-1)}(s), s) \in (\mathfrak{I}_S + \mathfrak{m}^\mu)\Gamma(U_i \cap U_j, 1)[[s_1, \ldots, s_{d'}]],$$

$$(2.5)_\mu \quad (\bar{\partial}_b - \hat{\phi}(\tau^{(\mu-1)}(s)))F_i^{(\mu-1)\,\alpha}(s) \in (\mathfrak{I}_S + \mathfrak{m}^\mu)\Gamma(U_i, 1)[[s_1, \ldots, s_{d'}]],$$

$$(2.6)_\mu \quad \hat{b}_\gamma(\tau^{(\mu-1)}(s)) \in (\mathfrak{I}_S + \mathfrak{m}^\mu)\mathbf{C}[[s_1, \ldots, s_{d'}]] \ (\gamma = 1, \ldots, \ell).$$

Let $F_i^{(0)\,\alpha}(s) = z_i^\alpha$ $(\alpha = 1, \ldots, n)$ and $\tau_0(s) = 0$. Suppose that $F_i^{(\mu-1)\,\alpha}(s)$ and $\tau^{(\mu-1)}(s)$ are obtained such that $(2.4)_{\mu-1}$–$(2.6)_{\mu-1}$ hold.

Let $F_{i|\mu}^{'\,\alpha}(s)$ be the solution of

$$(2.7) \quad -\sum_{\alpha=1}^n \{F_{i|\mu}^{'\,\alpha}(s) - \sum_{\gamma=1}^n \frac{\partial f_{ij}^\alpha}{\partial z_j^\gamma}(\zeta_j, 0) F_{j|\mu}^{'\,\gamma}(s)\} \frac{\partial}{\partial z_i^\alpha} \equiv \sum_{\alpha=1}^n \{F_i^{(\mu-1)\,\alpha}(s) - f_{ij}^\alpha(F_j^{(\mu-1)\,\alpha}(s), s)\} \frac{\partial}{\partial z_i^\alpha}$$

$$\mod (\mathfrak{I}_S + \mathfrak{m}^{\mu+1})\Gamma(U_i, T'X_{|M})[[s_1, \ldots, s_{d'}]].$$

Then, there exists $\theta(s) \in A_b^{0,1}(T'X_{|M})[[s_1, \ldots, s_{d'}]]$ such that

$$\theta(s) - (\bar{\partial}_b - \hat{\phi}(\tau^{(\mu-1)}(s))) \left(\sum_{\alpha=1}^n F_i^{(\mu-1)\,\alpha}(s) \frac{\partial}{\partial z_i^\alpha} \right) \in (\mathfrak{I}_S + \mathfrak{m}^{\mu+1}) A_b^{0,1}(U_i, T'X_{|M})[[s_1, \ldots, s_{d'}]].$$

By a direct calculation, we have

Lemma 2.3.

$$(\bar{\partial}_b - \phi)^2 = -P(\phi) - Q(\phi)$$

where $Q(\phi) \in A_b^{0,2}({}^{\phi}T'')$ *is given by*

$$Q(\phi)(\bar{X}, \bar{Y})f := (\bar{\partial}_b - \phi)f([\bar{X}, \phi(\bar{Y})]_{\circ T''} + [\phi(\bar{X}), \bar{Y}]_{\circ T''} - [\phi(\bar{X}), \phi(\bar{Y})]_{\circ T''})$$

for a function f *on* M.

Lemma 2.4.

(1) $\hat{b}_\gamma(\tau^{(\mu-1)}(s)) \in (\hat{\mathfrak{I}}_S + \mathfrak{m}^{\mu+1})\mathbf{C}[[s_1, \ldots, s_{d'}]]$ $(\gamma = 1, \ldots, r)$,

(2) $\bar{\partial}_b \theta(s) \in (\hat{\mathfrak{I}}_S + \mathfrak{m}^{\mu+1})A_b^{0,2}(U_i, T'X_{|M})[[s_1, \ldots, s_{d'}]]$.

Proof. (1) By Lemma 2.3, we have

$$\bar{\partial}_b \theta(s) + \rho^{1,0} P(\hat{\phi}(\tau^{(\mu-1)}(s))) \in (\hat{\mathfrak{I}}_S + \mathfrak{m}^{\mu+1})A_b^{0,2}(T'X_{|M})[[s_1, \ldots, s_{d'}]]$$

where $\rho^{1,0} : \mathbf{C}TU_{|M} \to T'U_{|M}$ denotes the projection onto the (1,0)-part. Hence

$$\rho_{T'} P(\hat{\phi}(\tau^{(\mu-1)}(s))) \in (\hat{\mathfrak{I}}_S + \mathfrak{m}^{\mu+1})A_b^{0,2}(T')[[s_1, \ldots, s_{d'}]].$$

(2) follows from (1) Q.E.D.

Let $F''^{\ \alpha}_{\mu}(s)$ and $\tau_\mu(s)$ be the solutions of

$$\bar{\partial}_b\{\sum_{\alpha=1}^{n} F''^{\ \alpha}_{\mu}(s)\frac{\partial}{\partial z_i^\alpha}\} - (\sum_{\sigma=1}^{d} \phi_\sigma \tau_\mu^\sigma(s)) \tag{2.8}$$

$$\equiv -(\bar{\partial}_b - \hat{\phi}(\tau^{(\mu-1)}(s))\left(\sum_{\alpha=1}^{n}(F_i^{(\mu-1)\ \alpha}(s) + F'^{\ \alpha}_{i|\mu}(s))\frac{\partial}{\partial z_i^\alpha}\right)$$

$$\mathrm{mod}\ (\mathfrak{I}_S + \mathfrak{m}^{\mu+1})A_b^{0,1}(T'X_{|M})[[s_1, \ldots, s_{d'}]].$$

Then it is clear that

$$\begin{cases} F_i^{(\mu)\ \alpha}(s) := F_i^{(\mu-1)\ \alpha}(s) + F'^{\ \alpha}_{i|\mu}(s) + F''^{\ \alpha}_{\mu}(s) \\ \tau^{(\mu)}(s) := \tau^{(\mu-1)}(s) + \tau_\mu(s) \end{cases}$$

satisfy $(2.4)_\mu$–$(2.6)_\mu$. Since the solvability of the equation (2.8) is assured by Lemma 2.4, we have $\{F_i^{(\mu)\,\alpha}(s)\}$ and $\tau^{(\mu)}(s)$ for all $\mu \geq 0$. Q.E.D.

In the proof of Proposition 2.2, we used the Grauert division theorem (cf. [Gr]) in order to solve the linear equations (2.7) and (2.8).

In order to show the convergence of $\hat{\phi}(t)$, $\{\hat{b}_\gamma(t)\}_{1\leq\gamma\leq\ell}$, $\hat{\tau}(s)$ and $\{\hat{F}_i^\alpha(s)\}$ with respect to the Sobolev norm, we need the coercive estimate for the Neumann problem. Though $\bar{\partial}_b$-Neumann problem is not the case, we remark that the following weak-coercive estimate

$$(2.9) \qquad ||\vartheta_{T'} N_{T'}\xi||_k' \leq c||\xi||_k \text{ for } \xi \in \Gamma(M, T')$$

is enough for the convergence of them with respect the Folland-Stein norm $||\ ||_k'$ (cf. Part I, section 3.5 for the definition of the Folland-Stein norm), as long as $\hat{\phi}(t)$ is $^\circ T'$-valued.

In fact, if $\hat{\phi}(t) \in A_b^{0,1}(^\circ T')[[t_1, \ldots, t_d]]$ is assured in the above construction, we have

$$||\phi_\mu(t)||_k' << ||\vartheta_{T'} N_{T'} R(\phi^{(\mu-1)}(t))||_k' << C||\phi^{(\mu-1)}(t)||_k'^2$$

by (2.9) and Part I, Lemma 3.6.3 (we should remark that the estimate in Part I, Lemma 3.6.3 holds for all $\phi \in A_b^{0,1}(^\circ T')$). Where we use the same notation $A(t) << B(t)$ as in Part I, Section 3.7.

Taking account of the following Lemma together with Part I, Theorems 3.3.2 and 3.5.2, we can trace the above construciton relying on the complex $(\Gamma(M, E_q), \bar{\partial}_q)$ instead of $(A_b^{0,q}(T'), \bar{\partial}_{T'})$ and obtain $\hat{\phi}(t)$ which is $A_b^{0,1}(^\circ T')$-valued and satisfies (2.2) and (2.3). (This is the reson why the sub-complex $(\Gamma(M, E_q), \bar{\partial}_q)$ of $(A_b^{0,q}(T'), \bar{\partial}_{T'})$ was introduced in [Ak3].)

Lemma 2.5. *([My1, Proposition 1.1]) For $\phi \in \Gamma(M, E_1)$, $P(\phi)$ is in $\Gamma(M, E_2)$.*

Hence, we obtain convergent $\phi(t)$ and $\{b_\gamma(t)\}_{1\leq\gamma\leq\ell}$, by modifying the construction as above using $\big(\Gamma(M, E_q), \bar{\partial}_q\big)$ and the Hodge decomposition in Part I, Theorem 3.5.2 instead of $\left(A_b^{0,q}(T'), \bar{\partial}_{T'}\right)$ and the standard $\bar{\partial}_b$-Neumann Hodge decomposition (2.1) respectively. If we set

$$\mathfrak{I} := (b_1(t), \ldots, b_\ell(t)) \subset \mathbf{C}\{t_1, \ldots, t_d\}.$$

then the Grauert division theorem ([Gr]) says that Proposition 2.1 implies

$$P(\phi(t)) \in \mathfrak{I}A_{b,k}^{0,2}(T')\{t_1, \ldots, t_d\} \text{ for all } k >> 0.$$

The proof of convergence of $\hat{\tau}(s)$ and $\{\hat{F}_i^\alpha(s)\}$ is done by the same calculation as in [My3, Note], since $\hat{\phi}(t) \in A_b^{0,1}({}^\circ T')[[t_1, \dots, t_d]]$ holds. And we have, by $(2.4)_\mu$–$(2.6)_\mu$ $(\mu \geq 0)$,

$$F_i^\alpha(s) - f_{ij}^\alpha(F_j(s), s) \in \mathfrak{I}_S \Gamma_k(U_i \cap U_j, 1)\{s_1, \dots, s_{d'}\} \ (\alpha = 1, \dots, n)$$

for all $k >> 0$,

$$(\bar{\partial}_b - \phi(\tau(s)))F_i^\alpha(s) \in \mathfrak{I}_S \Gamma_k(U_i, 1)\{s_1, \dots, s_{d'}\} \ (\alpha = 1, \dots, n)$$

for all $k >> 0$,

$$b_\gamma(\tau(s)) \in \mathfrak{I}_S \ (\gamma = 1, \dots, \ell).$$

Hence we have a Kuranishi semi-universal family of deformations of ${}^\circ T''$.

The parameter space of that semi-universal family is described as $b^{-1}(0)$ by means of the holomorphic map $b : H^1_{\bar{\partial}_b}(T') \supset D \to \mathbf{H}^2 \simeq H^2_{\bar{\partial}_b}(T')$ given by $h(t) = \rho P(\phi(t))$ where $\rho : \Gamma(M, E_2) \to \mathbf{H}^2$ is the orthogonal projection onto the harmonic space $\mathbf{H}^2 \subset \Gamma(M, E_2)$ (cf. Part I, section 2.5).

(II) Next, we consider the case of $\dim_{\mathbf{R}} M = 5$. In this case, $H^2_{\bar{\partial}_{T'}}(T')$ may be infinite dimensional. However, the $\bar{\partial}_b$-Neumann harmonic space $\mathbf{H}^2_b(T')$ is a closed subspace of the L^2-completion $A^{0,2}_{b,0}(T')$ of $A_b^{0,2}(T')$ and the projection operator onto it makes sense. The Hodge decompositions at degree 2 are obtained as follows using the $\bar{\partial}_b$-Neumann operators at degree 1

$$\eta = \rho_{T'}\eta + \bar{\partial}_{T'} N_{T'} \bar{\partial}^*_{T'} \eta \ \text{ for } \eta \in A_b^{0,2}(T'), \tag{2.10}$$

where $\rho_{T'}$ denotes the orthogonal projection onto $\mathbf{H}^2_b(T')$. The construction of $\hat{\phi}(t) \in A_b^{0,1}(T')[[t_1, \dots, t_d]]$ in part (I) can be carried out using the decomposition (2.10) as follows: Let

$$\phi_1(t_1, \dots, t_d) := \sum_{\sigma=1}^d \phi_\sigma t_\sigma,$$

$$\phi_\mu(t) := \mu\text{-th homogeneous term of } -N_{T'}\bar{\partial}^*_{T'} P(\phi^{(\mu-1)}(t)),$$

$$\phi^{(\mu)}(t_1, \dots, t_d) := \phi^{(\mu-1)}(t_1, \dots, t_d) + \phi_\mu(t_1, \dots, t_d)$$

and

$$\hat{\phi}(t_1, \dots, t_d) := \lim_{\mu \to \infty} \phi^{(\mu)}(t_1, \dots, t_d).$$

And let $\{\hat{b}_\lambda(t)\}_{\lambda\in\Lambda}$ be given by

$$\hat{b}_\lambda(t) = \Big(\rho_{T'} P(\hat{\phi}(t)), e_\lambda\Big)$$

for an orthonormal basis $\{e_\lambda\}_{\lambda\in\Lambda}$ of $A^{0,2}_{b,0}(T')$. Let $\hat{\mathfrak{J}}$ be an ideal of $\mathbf{C}[[t_1,\ldots,t_d]]$ generated by $\{\hat{b}_\lambda(t)\}_{\lambda\in\Lambda}$. If we note that $P(\hat{\phi}(t)) \in \hat{\mathfrak{J}}A^{0,2}_b(T')[[t_1,\ldots,t_d]]$ is equivalent to $\Big(P(\hat{\phi}(t)), e_\lambda\Big) \in \hat{\mathfrak{J}}$ for all $\lambda \in \Lambda$, Proposition 2.1 also holds and Proposition 2.2 can be proved by the same argument. Therefore the construction of a formally Kuranishi semi-universal formal family in part (I) of this section is also valid for the case of $\dim_{\mathbf{R}} M = 5$.

In the case of normal strongly pseudo-convex CR manifolds of real-dimension 5, T. Akahori constructed $\hat{\phi}(t) \in A^{0,1}_b(T')[[t_1,\ldots,t_d]]$ which is convergent with respect to the $\| \|''_k$-norm (cf. [Ak4]).

§3. Smoothness of the Kuranishi semi-universal families

In this section, we consider the problem of when the parameter space of the Kuranishi semi-universal family of CR structures on M is smooth. We denote the parameter space of the Kuranishi semi-universal family by T_{CR} (in five dimensional case, we denote the parameter space of the formally Kuranishi semi-universal formal family by $\hat{T}_{CR}$).

(I) The Kodaira-Spencer-type smoothness. By the construction of T_{CR} or $\hat{T}_{CR}$ in §2, it is clear that if $H^2_{\bar{\partial}_{T'}}(T') = 0$ then T_{CR} ($\hat{T}_{CR}$ in five dimensional case) is smooth.

(II) The Bogomolov-type smoothness. The Bogomolov smoothness theorem is a smoothness theorem based on the other principle: In the case of deformations of a compact Kähler manifolds, if the canonical bundle K_X is trivial, then by the inner product with a non-vanishing holomorphic $(n,0)$-form, the integrability condition $P(\phi)$ is converted to an equation of ordinary differential $(n-1,1)$-forms. Using the pure Hodge structure on a compact Kähler manifold, the converted equation is solved without obstructions. Hence the parameter space of the semi-universal family is smooth.

On a strongly pseudo-convex CR manifold, there does not exists a natural pure Hodge structure much less a $(\partial_b, \bar{\partial}_b)$-double complex. In [Ak-My2], we introduced a sub-space $F^{p,q} \subset A^{p,q}_b(T')$. Let θ be a real contact form (that is, a non-vanishing real 1-form which annihilates

${}^{\circ}T' + {}^{\circ}T''$) and let

$$F^{p,q} := \{\theta \wedge \alpha \in \theta \wedge \Gamma(M, \wedge^{p-1}({}^{\circ}T')^* \wedge \wedge^q({}^{\circ}T'')^*) \mid d\theta \wedge \alpha = 0\}.$$

Then a double-complex $(F^{p,q}; \partial, \bar{\partial})$ is naturally induced. The higher part of the total simple complex of $(F^{p,q}; \partial, \bar{\partial})$ coincides with (the higher part of) the Rumin complex (cf. [Ru] for the Rumin complex). If there exists a $\bar{\partial}_b$-closed non-vanishing $(n,0)$-form (i.e. there exists a non-vanishing $\omega \in \Gamma(M, \wedge^n(T')^*)$ satisfying $\bar{\partial}_{\wedge^n(T')^*}\omega = 0$ or equivalently there exists a non-vanishing $\omega \in \Gamma(M, \wedge^n(T'U_{|M})^*)$ satisfying $\bar{\partial}_b\omega = 0$), the inner product with that $(n,0)$-form induces an isomorphism of complexes

$$\iota : (\Gamma(M, E_q), \bar{\partial}_q) \simeq (F^{n-1,q}, \bar{\partial}),$$

where $(\Gamma(M, E_q), \bar{\partial}_q)$ is the sub-complex of $(A_b^{0,q}(T'), \bar{\partial}_{T'})$ introduced by T. Akahori (cf. Part I, Section 3). Hence, the only difference from the compact Kähler case is the lack of the pure Hodge structure on $(F^{p,q}; \partial, \bar{\partial})$. Because of this lack, the analogue of the Bogomolov smoothness does not necessarily hold in deformations of CR structures (cf. [Ak-My3]). Hence, we consider unobstructedness of a subspace of $H^1_{\bar{\partial}_{T'}}(T')$, where we call a subspace *unobstructed* if there exists a holomorphic family of CR structures whose infinitesimal deformation space coincides with that space.

Let $I^{p,q} := Z^{p,q}_{\partial} \cap Z^{p,q}_{\bar{\partial}} / Z^{p,q}_{\partial} \cap \bar{\partial}F^{p,q-1}$ and $J^{p,q} := \partial F^{p-1,q} \cap Z^{p,q}_{\bar{\partial}} / \partial F^{p-1,q} \cap \bar{\partial}F^{p,q-1}$ with denoting $Z^{p,q}_{\partial} := \mathrm{Ker}\partial \cap F^{p,q}$ and $Z^{p,q}_{\bar{\partial}} := \mathrm{Ker}\bar{\partial} \cap F^{p,q}$.

Theorem 3.1. *([Ak-My2], [Ak-My4]) Suppose that $\dim_{\mathbf{R}} M \geq 7$. If $J^{n-1,2} = 0$ then $\iota^{-1}(I^{n-1,1})$ is unobstructed.*

Further developments in connection with deformations of isolated singularities are done in [My5] and [My6] using the Hodge structure on a strongly pseudo-convex domain ([De], [Oh1], [Oh2], [Oh-Ta]).

§4. Deformation theory of normal isolated singularities

In this section, we review briefly deformation theory of normal isolated singularities. Refer to [Tj] and [Gr] for details.

Let V be a germ of an analytic variety with a unique singular point o. In this article, we assume that V is a normal complex space.

Definition 4.1. *A family of deformations of V over a germ (S, s_o) is a flat holomorphic mapping of germs $f : \mathcal{V} \to S$ with $f^{-1}(s_o) \simeq V$.*

The equivalence of two families are defined by the equivalence of the two flat holomorphic mappings and the notion of versality is defined in a usual manner. The infinitesimal deformation map is a map $\rho : T_{s_o}S \to \mathrm{Ext}^1(\Omega^1_V, \mathcal{O}_V)$ and a holomorphic family is called effective if the infinitesimal deformation map is injective. An effective and versal family is called a semi-universal family. It is shown in [Tj] that the obstruction space is $\mathrm{Ext}^2(\Omega^1_V, \mathcal{O}_V)$ and H. Grauert ([Gr]) proved the existence of the semi-universal family.

We may assume that V is a closed subvariety of a ball $B(c^*)$ in $\mathbf{C}^N$ defined by $\tilde{h}_1 = \cdots = \tilde{h}_{m_1} = 0$ and o is the origin of $\mathbf{C}^N$. Denote $B := B(c^*)$, $\Omega := \Omega(a, b)$, and $M := M_c$ for some fixed $0 < a < c \leq b < c^*$.

We recall Tjurina's description of $\mathrm{Ext}^q(\Omega^1_V, \mathcal{O}_V)$ (cf. [Tj]): The sheaf of germs of Kähler differentials Ω^1_V is given by $\Omega^1_V := \Omega^1_B/\Omega'_{|V}$ where Ω' is the sub-sheaf of Ω^1_B consisting of germs of forms ω such that $\omega = \sum_\lambda f_\lambda d\tilde{h}_\lambda + \sum_\lambda \tilde{h}_\lambda \phi_\lambda$ with $f_\lambda \in \mathcal{O}_B$ and $\phi_\lambda \in \Omega^1_B$. Hence, we have a free resolution of Ω^1_V,

$$0 \leftarrow \Omega^1_V \leftarrow \Omega^1_B \otimes \mathcal{O}_V \overset{d_0}{\leftarrow} \mathcal{O}_V^{m_1} \overset{d_1}{\leftarrow} \mathcal{O}_V^{m_2} \overset{d_2}{\leftarrow} \mathcal{O}_V^{m_3} \overset{d_3}{\leftarrow} \cdots$$

where $d_0(u_1, \ldots, u_{m_1}) := \sum_\lambda u_\lambda d\tilde{h}_\lambda$.

$\mathrm{Ext}^*(\Omega^1_V, \mathcal{O}_V)$ is the cohomology groups of the following complex:

$$0 \to H^0(V, \Theta_B \otimes \mathcal{O}_V) \overset{d_0^*}{\to} H^0(V, \mathcal{O}_V^{m_1}) \overset{d_1^*}{\to} H^0(V, \mathcal{O}_V^{m_2}) \overset{d_2^*}{\to} H^0(V, \mathcal{O}_V^{m_3}) \to .$$

Since V is normal, this complex is quasi-isomorphic to the following complex:

$$0 \to H^0(\Omega, \Theta_B \otimes \mathcal{O}_\Omega) \overset{d_0^*}{\to} H^0(\Omega, \mathcal{O}_\Omega^{m_1}) \overset{d_1^*}{\to} H^0(\Omega, \mathcal{O}_\Omega^{m_2}) \overset{d_2^*}{\to} H^0(\Omega, \mathcal{O}_\Omega^{m_3}) \to$$

where we note that $d_0^*(v) = (v(\tilde{h}_1), \ldots, v(\tilde{h}_{m_1}))$ $v \in H^0(\Omega, \Theta_B \otimes \mathcal{O}_\Omega)$. Using the commutative diagram,

$$\begin{array}{ccccccccc} 0 & \to & \Theta_\Omega & \overset{F}{\to} & \Theta_B \otimes \mathcal{O}_\Omega & \to & N_{\Omega/B} & \to & 0 \\ & & \| & & \| & & \downarrow & & \\ 0 & \to & \Theta_\Omega & \overset{F}{\to} & \Theta_B \otimes \mathcal{O}_\Omega & \overset{d_0^*}{\to} & \mathcal{O}_\Omega^{m_1} & \to & \cdots \end{array}$$

where $F : \Theta_\Omega \to \Theta_B \otimes \mathcal{O}_\Omega$ denotes the differential of the natural embedding $\iota : \Omega \to G$ and $N_{\Omega/G}$ is the normal bundle of Ω in G, we have

Proposition 4.1.

(1) $\mathrm{Ext}^0(\Omega^1_V, \mathcal{O}_V) \simeq H^0(\Omega, \Theta_\Omega)$,

(2) $\mathrm{Ext}^1(\Omega^1_V, \mathcal{O}_V) \simeq \mathrm{Ker}\{H^1(\Omega, \Theta_\Omega) \to H^1(\Omega, \Theta_B \otimes \mathcal{O}_\Omega)\}$,

(3) $\mathrm{Ext}^2(\Omega^1_V, \mathcal{O}_V) \simeq \mathrm{Ker}\{H^1(\Omega, N_{\Omega/B}) \to H^1(\Omega, \mathcal{O}_\Omega^{m_1})\}$.

Using [Ya, pp.81-82] and [Hö, Theorem 3.4.9] and noting that depth $\mathcal{O}_{V,o} \geq r$ is equivalent to $H^q(V \setminus o, \mathcal{O}_{V \setminus o}) = 0$ $(1 \leq q \leq r-2)$ (cf. [Ba]), we have

Theorem 4.2. *If* depth $\mathcal{O}_{V,o} \geq 3$ *and* $\dim_{\mathbf{C}} V \geq 4$,

(1) $\mathrm{Ext}^1(\Omega^1_V, \mathcal{O}_V) \simeq H^1_{\bar\partial_b}(T'U_{|M})$,

(2) $\mathrm{Ext}^2(\Omega^1_V, \mathcal{O}_V) \simeq H^1_{\bar\partial_b}(N_{\Omega/G|M}) \subset H^2_{\bar\partial_b}(T'U_{|M})$.

Remark. (Cf. [Bl-Ep], Propositions 6.1 and 6.2 below.) In the case of $\dim_{\mathbf{C}} V = 2$:

(1) $\mathrm{Ext}^1(\Omega^1_V, \mathcal{O}_V)$ is a finite dimensional subspace of $H^1_{\bar\partial_b}(T'U_{|M})$, though the latter space is infinite dimensional.

(2) $\mathrm{Ext}^2(\Omega^1_V, \mathcal{O}_V)$ is a finite dimensional subspace of $H^1_{\bar\partial_b}(N_{\Omega/G|M})$.

Suppose that $\dim_{\mathbf{C}} V \geq 4$. Take a model of the semi-universal family of deformations of V, say $f : \mathcal{V} \to S$, such that $f^{-1}(s_o) \simeq V$ $(s_o \in S)$ as germs at the singular point o. We may assume that $\Omega \subset f^{-1}(s_o)$ and let $\phi(t)$ $(t \in (T_{CR}, 0))$ be the Kuranishi semi-universal family of deformations of CR structures on M constructed in §2. Based on Theorem 4.2, the following comparison theorem is proved.

Theorem 4.3. *([Bu-Ml], [My2]) If $\dim_{\mathbf{C}} V \geq 4$ and depth $\mathcal{O}_{V,o} \geq 3$, then $(T_{CR}, 0) \simeq (S, s_o)$ and the holomorphic family $\phi(t)$ $(t \in (T_{CR}, 0))$ is induced by a holomorphic family of embeddings F:*

$$\begin{array}{ccc} M \times T_{CR} & \overset{F}{\hookrightarrow} & \mathcal{V} \\ \downarrow p_2 & & \downarrow f \\ T_{CR} & \simeq & S. \end{array}$$

§5. Stably embeddable deformations of CR structures

A compact strongly pseudo-convex CR manifold arises as a boundary of a Stein space if and only if it is embedded in a complex Euclidean space ([Ha-La]). Such CR manifolds are called embeddable CR manifolds. While all compact strongly pseudo-convex CR manifold of real dimension five or higher are embeddable ([BM]), embeddable CR manifolds are rare in the three dimensional case. In [Bl-Ep], J. Bland and C. Epstein formulated deformation theory of embeddable three dimensional CR structures and showed that it is equivalent on the formal deformation level to the deformation theory of normal isolated surface singularities.

In the higher dimensional case, though all compact strongly pseudo-convex CR manifolds bound Stein spaces, there are differences between the Kuranishi deformation theory of CR structures and the deformation theory of normal isolated singularities, unless depth$\mathcal{O}_{V,o} \geq 3$ (cf. [Bu-Ml, §10]). The Kuranishi deformation theory of CR structures would correspond to the non-flat deformation theory (cf. [Es]), while the flat deformation theory corresponds to a special deformation theory of CR structures. Recently, [My6] generalizes the stably embeddable deformation theory of three dimensional CR structures in [Bl-Ep] to the higher dimensional complex structure case, and shows that it fits to the deformation theory of normal isolated singularities. In this section and the next one, we consider the CR-version of [My6] and complete the Kuranishi program describing the semi-universal family of normal isolated singularities in terms of the CR-language, in the case of complex dimension three or higher.

Let V be as at the beginning of §4 and use the same notation about M, ${}^\circ T''$, B and Ω as in §4.

Definition 5.1. *Let T be a germ of a complex subspace of $\mathbf{C}^d$ at the origin defined by an ideal $\mathfrak{I}_T \subset \mathbf{C}\{t_1, \ldots, t_d\}$. A stably embeddable family of deformations of the CR structure ${}^\circ T''$ in $\mathbf{C}^N$ over $(T, 0)$ is a holomorphic family $\phi(t)$ of deformations of ${}^\circ T''$ over $(T, 0)$ such that there exists*

$$g(t) \in A_b^0(T'\mathbf{C}^N{}_{|M}))[[t_1, \ldots, t_d]] \cap \cap_{k>0} A_{b,k}^0(T'\mathbf{C}^N{}_{|M}))\{t_1, \ldots, t_d\}$$

satisfying

$$(\bar\partial_b - \phi(t))(\iota + g(t)) \in \mathfrak{I}_T A_{b,k}^{0,1}(T'\mathbf{C}^N{}_{|M})\{t_1, \ldots, t_d\}$$

for all $k >> 0$.

We consider a stably embeddable family of CR structures up to wiggling in an ambient complex manifold. Hence, we take the following notion of versality.

Definition 5.2. *A stably embeddable family $\phi(t)$ $(t \in (T,0))$ of deformations of ${}^\circ T''$ in $\mathbf{C}^N$ is Kuranishi versal if it has the following property: For any family of displacements (in $\mathbf{C}^N$) of a neighbourhood of M in U, over a germ (S, s_o), say*

$$\begin{array}{ccc} \mathcal{U} & \hookrightarrow & \mathbf{C}^N \times S \\ \downarrow \pi & & \downarrow p_2 \\ S & = & S, \end{array}$$

there exist a holomorphic map $\tau : (S, s_o) \to (T, 0)$ and a holomorphic family of embeddings $F : M \times S \to \mathcal{U}$ such that $F_{|M \times 0} = \iota$ and the holomorphic family of CR structures induced by F coincides with $\phi(\tau(s))$ $(s \in (S, s_o))$.

Let $(\phi(t)$ $(t \in (T,0))$ be a stably embeddable family of deformations of ${}^\circ T''$ with which $g(t)$ is associated. Since

$$P(\phi(t)) = \bar{\partial}_{T'}\phi(t) + R_2(\phi(t)) + R_3(\phi(t)) = 0 \text{ and } (\bar{\partial}_b - \phi(t))(\iota + g(t)) = 0,$$

we have

$$\bar{\partial}_{T'} v(\phi(t)) = 0 \qquad \text{and} \qquad \bar{\partial}_b v(g(t)) - d\iota v(\phi(t)) = 0$$

for $v \in T_0 T$. Hence $v(\phi(t))$ is $\bar{\partial}_{T'}$-closed and $d\iota v(\phi(t))$ is $\bar{\partial}_b$-exact.

Definition 5.3. *For a stably embeddable family $\phi(t)$ $(t \in (T,0))$ of deformations of a CR structure, the infinitesimal deformation map is the linear map*

$$\rho : T_0 T \to \mathrm{Ker}\{H_b^1(T') \to H_b^1(T'\mathbf{C}^N{}_{|M})\}$$

given by $\rho(v) :=$ the cohomology class of $v(\phi(t))$. A holomorphic family is called effective if its infinitesimal deformation map is injective.

An effective and Kuranishi versal family is called a Kuranishi semi-universal family.

§6. Construction of the Kuranishi semi-universal family of stably embeddable deformations of CR structures

In this section, we will use the same notation as in §5. In [M6], we introduced a double-complex $K_{\bar{\Omega}}^{\bullet,\bullet}$:

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & 0 & \rightarrow & H^0(\bar{\Omega}, T^{'}B_{|U}) & \xrightarrow{H} & H^0(\bar{\Omega}, \oplus^{m_1} 1_U) & \rightarrow & 0 \\
 & & \downarrow & & \downarrow i & & \downarrow i & & \\
0 & \rightarrow & A^0_{\bar{\Omega}}(T^{'}U) & \xrightarrow{F} & A^0_{\bar{\Omega}}(T^{'}B_{|U}) & \xrightarrow{H} & A^0_{\bar{\Omega}}(\oplus^{m_1} 1) & \rightarrow & 0 \\
 & & \downarrow \bar{\partial} & & \downarrow \bar{\partial} & & \downarrow \bar{\partial} & & \\
0 & \rightarrow & A^{0,1}_{\bar{\Omega}}(T^{'}U) & \xrightarrow{F} & A^{0,1}_{\bar{\Omega}}(T^{'}B_{|U}) & \xrightarrow{H} & A^{0,1}_{\bar{\Omega}}(\oplus^{m_1} 1) & \rightarrow & 0 \\
 & & \downarrow \bar{\partial} & & \downarrow \bar{\partial} & & \downarrow \bar{\partial} & & \\
0 & \rightarrow & A^{0,2}_{\bar{\Omega}}(T^{'}U) & \xrightarrow{F} & A^{0,2}_{\bar{\Omega}}(T^{'}B_{|U}) & \xrightarrow{H} & A^{0,2}_{\bar{\Omega}}(\oplus^{m_1} 1) & \rightarrow & 0 \\
 & & \downarrow \bar{\partial} & & \downarrow \bar{\partial} & & \downarrow \bar{\partial} & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}
$$

where $K_{\bar{\Omega}}^{0,0} = A^0_{\bar{\Omega}}(T^{'}U)$ and we denote $H^0(\bar{\Omega}, E) := \{u \in A^0_{\bar{\Omega}}(E) \mid \bar{\partial}u = 0\}$ for a holomorphic vector bundle E over U, i denotes the inclusion map and F (resp. H) is the differential of the natural embedding $\iota : U \rightarrow B$ (resp. the homomorphism given by $H(v) = (v(\tilde{h}_1), \ldots, v(\tilde{h}_{m_1}))$ for $v \in T^{'}B_{|U}$).

Proposition 6.1 ([My7]).

$$\operatorname{Ext}^q(\Omega^1_V, \mathcal{O}_V) \simeq H^q(K_{\bar{\Omega}}^{\bullet,\bullet}) \ (q = 1, 2).$$

As the CR-version of $K_{\bar{\Omega}}^{\bullet,\bullet}$, we consider the following double complex $K_M^{\bullet,\bullet}$;

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & 0 & \rightarrow & H^0_b(T^{'}B_{|M}) & \xrightarrow{H} & H^0_b(\oplus^{m_1} 1) & \rightarrow & 0 \\
 & & \downarrow & & \downarrow i & & \downarrow i & & \\
0 & \rightarrow & A^0_b(T^{'}U_{|M}) & \xrightarrow{F} & A^0_b(T^{'}B_{|M}) & \xrightarrow{H} & A^0_b(\oplus^{m_1} 1) & \rightarrow & 0 \\
 & & \downarrow \bar{\partial}_b & & \downarrow \bar{\partial}_b & & \downarrow \bar{\partial}_b & & \\
0 & \rightarrow & A^{0,1}_b(T^{'}U_{|M}) & \xrightarrow{F} & A^{0,1}_b(T^{'}B_{|M}) & \xrightarrow{H} & A^{0,1}_b(\oplus^{m_1} 1) & \rightarrow & 0 \\
 & & \downarrow \bar{\partial}_b & & \downarrow \bar{\partial}_b & & \downarrow \bar{\partial}_b & & \\
0 & \rightarrow & A^{0,2}_b(T^{'}U_{|M}) & \xrightarrow{F} & A^{0,2}_b(T^{'}B_{|M}) & \xrightarrow{H} & A^{0,2}_b(\oplus^{m_1} 1) & \rightarrow & 0 \\
 & & \downarrow \bar{\partial}_b & & \downarrow \bar{\partial}_b & & \downarrow \bar{\partial}_b & &
\end{array}
$$

where $K_M^{0,0} = A_b^0(T'U_{|M})$, $H_b^0(E)$ denotes the space of all CR-sections of a holomorphic vector bundle E on M and we denote by the same symbol F the composite of the projection $\rho^{1,0} : T' \to T'U_{|M}$ and $F : T'U_{|M} \to T'B_{|M}$, and i and H are the same as above.

The analytic restrictions $\tau : A_{\bar{\Omega}}^{0,q}(T'U) \to A_b^{0,q}(T'U_{|M})$ and $\tau : A_{\bar{\Omega}}^{0,q} \to A_b^{0,q}$ induce a homomorphism of double complexes

$$\tau : K_{\bar{\Omega}}^{\bullet,\bullet} \to K_M^{\bullet,\bullet}.$$

Proposition 6.2. *τ induces an isomorphism*

$$H^q(K_{\bar{\Omega}}^{\bullet,\bullet}) \simeq H^q(K_M^{\bullet,\bullet})$$

for $q = 1, 2$.

For the proof, we use the following lemma.

Lemma 6.3. *Let M be a real hypersurface of a complex manifold U and $\gamma : E_2 \to E_3$ a surjective homomorphism of C^∞-vector bundles over U. We suppose that there exists a splitting $j : E_3 \to E_2$. Then, for any $u_3 \in \Gamma(U, E_3)$ and $\mathbf{u}_2 \in \Gamma(M, E_{2|M})$ with $\gamma(\mathbf{u}_2) = u_{3|M}$, there exists $u_2 \in \Gamma(U, E_2)$ such that $\gamma(u_2) = u_3$ and $u_{2|M} = \mathbf{u}_2$.*

Proof. We may assume that $\Omega = M \times (-\epsilon, \epsilon)$. Let $u_2(x,t) := j(u_3(x,t)) - j(u_3(x,0)) + \mathbf{u}_2(x)$. Then $u_2(x,0) := \mathbf{u}_2(x)$ and $\gamma(u_2(x,t)) = u_3(x,t) - u_3(x,0) + \gamma(\mathbf{u}_2(x)) = u_3(x,t)$. Q.E.D.

Proof of Proposition 6.2. The case of $q = 2$: First, we prove the surjectivity. Let

$$(\varphi_2, \mathbf{g}_1, \mathbf{h}_0) \in A_b^{0,2}(T'U_{|M}) \oplus A_b^{0,1}(T'B_{|M}) \oplus A_b^0(\oplus^{m_1} 1_M)$$

satisfies $d(\varphi_2, \mathbf{g}_1, \mathbf{h}_0) = (0,0,0)$. We will find

$$(\bar{\phi}_2, \bar{g}_1, \bar{h}_0) \in A_{\bar{\Omega}}^{0,2}(T'U) \oplus A_{\bar{\Omega}}^{0,1}(T'B_{|U}) \oplus A_{\bar{\Omega}}^0(\oplus^{m_1} 1_U)$$

satisfying $d(\bar{\phi}_2, \bar{g}_1, \bar{h}_0) = (0,0,0)$ and $\tau(\bar{\phi}_2, \bar{g}_1, \bar{h}_0) = (\varphi_2, \mathbf{g}_1, \mathbf{h}_0)$. Applying [Ko-Ro, Theorem 7.5] to $\mathbf{k}_0 := \gamma \mathbf{h}_0 \in H_b^0(E_{|M})$, there exists $\bar{k}_0 \in H^0(\bar{\Omega}, E)$ such that $\bar{k}_0 = \mathbf{k}_0$ where we denote by E the quotient bundle $\oplus^{m_1} 1_U / N_{U/B}$. By Lemma 6.3, there exists $h_0 \in A_{\bar{\Omega}}^0(\oplus^{m_1} 1_U)$ such that $\gamma(h_0) = k_0$ and $h_{0|M} = \mathbf{h}_0$. Since $\gamma(\bar{\partial} h_0) = \bar{\partial} k_0 = 0$, by Lemma 6.3, there exists $g_1 \in A_{\bar{\Omega}}^{0,1}(T'B_{|U})$ such that $\alpha(g_1) = \beta^{-1}(\bar{\partial} h_0)$ and $\tau g_1 = \mathbf{g}_1$. Since $\beta \circ \alpha(\bar{\partial} g_1) = 0$, by Lemma 6.3, there exists

$\phi_2 \in A^{0,2}_{\bar\Omega}(T'U)$ such that $F\phi_2 = \bar\partial g_1$ and $\tau\phi_2 = \varphi_2$. Next, we prove the injectivity. Let

$$(\bar\phi_2, \bar g_1, \bar h_0) \in A^{0,2}_{\bar\Omega}(T'U) \oplus A^{0,1}_{\bar\Omega}(T'B_{|U}) \oplus A^{0}_{\bar\Omega}(\oplus^{m_1} 1_U)$$

and suppose $\tau(\bar\phi_2, \bar g_1, \bar h_0) = d(\varphi_1, \mathbf{g}_0, \mathbf{h}_{-1})$ where

$$(\varphi_1, \mathbf{g}_0, \mathbf{h}_{-1}) \in A^{0,1}_b(T'U_{|M}) \oplus A^0_b(T'B_{|M}) \oplus H^0_b(\oplus^{m_1} 1_M).$$

By the Lewy extension theorem, we have $\bar h_{-1} \in H^0(\bar\Omega, \oplus^{m_1}\mathcal{O}_U)$ such that $\bar h_{-1|M} = \mathbf{h}_{-1}$. Since $\bar\partial\gamma(h_0 - h_{-1}) = \gamma\bar\partial h_0 = \gamma\beta\alpha(g_1) = 0$ and $\gamma(h_0 - h_{-1})_{|M} = \gamma\beta\alpha(\mathbf{g}_0) = 0$, $\gamma(h_0 - h_{-1}) = 0$. Hence, by Lemma 6.3, there exists $g_0 \in A^0_{\bar\Omega}(T'B_{|U})$ such that $\alpha(g_0) = h_0 - h_{-1}$ and $g_{0|M} = \mathbf{g}_0$. Since $\beta\alpha(g_1 - \bar\partial g_0) = \bar\partial h_0 - \bar\partial(h_0 - h_{-1}) = 0$, $g_1 - \bar\partial g_0 \in A^{0,1}_{\bar\Omega}(T'U)$. Hence, if we set $F\phi_1 := -g_1 + \bar\partial g_0$, then $\bar\partial\phi_1 = \phi_2$ because $F\bar\partial\phi_1 = -\bar\partial g_1 = F\phi_2$.

The case of $q = 1$: First, we prove the surjectivity. Let

$$(\varphi_1, \mathbf{g}_0, \mathbf{h}_{-1}) \in A^{0,1}_b(T'U_{|M}) \oplus A^0_b(T'B_{|M}) \oplus H^0_b(\oplus^{m_1} 1_M)$$

satisfies $d(\varphi_1, \mathbf{g}_0, \mathbf{h}_{-1}) = (0, 0, 0)$. We will find

$$(\bar\phi_1, \bar g_0, \bar h_{-1}) \in A^{0,1}_{\bar\Omega}(T'U) \oplus A^0_{\bar\Omega}(T'B) \oplus H^0(\bar\Omega, \oplus^{m_1}\mathcal{O}_U)$$

satisfying $d(\bar\phi_1, \bar g_0, \bar h_{-1}) = (0, 0, 0)$ and $\tau(\bar\phi_1, \bar g_0, \bar h_{-1}) = (\varphi_1, \mathbf{g}_0, \mathbf{h}_{-1})$. By the Lewy extension theorem, there exists $\bar h_{-1} \in H^0(\bar\Omega, \oplus^{m_1}\mathcal{O}_U)$ such that $\bar h_{-1|M} = \mathbf{h}_{-1}$. If we set $\bar k_{-1} := \gamma(\bar h_{-1}) \in H^0(\bar\Omega, E)$, $\bar k_{-1} = 0$ because $k_{-1|M} = 0$. Hence $\bar h_{-1} \in H^0(\bar\Omega, N_{U/B})$ and by Lemma 6.3, there exists $\bar g_0 \in A^0_{\bar\Omega}(T'B)$ such that $\beta\alpha(\bar g_0) = \bar h_{-1}$ and $\bar g_{0|M} = \mathbf{g}_0$. Since $\beta\alpha(\bar\partial\bar g_0) = \bar\partial\bar h_{-1} = 0$, $\bar\partial\bar g_0 \in A^{0,1}_{\bar\Omega}(T'U)$. Hence, there exists $\bar\phi_1 \in A^{0,1}_{\bar\Omega}(T'U)$ such that $F\bar\phi_1 = \bar\partial\bar g_0$ and $\tau\bar\phi_1 = \varphi_1$. Next, we prove the injectivity. Let

$$(\bar\phi_1, \bar g_0, \bar h_{-1}) \in A^{0,1}_{\bar\Omega}(T'U) \oplus A^0_{\bar\Omega}(T'B) \oplus H^0(\bar\Omega, \oplus^{m_1}\mathcal{O}_U)$$

and suppose $\tau(\bar\phi_1, \bar g_0, \bar h_{-1}) = d(\varphi_0, \mathbf{g}_{-1})$ where $(\varphi_0, \mathbf{g}_{-1}) \in A^0_b(T'U_{|M}) \oplus H^0_b(T'B_{|M})$. We have $\bar g_{-1} \in H^0(\bar\Omega, T'B)$ such that $\bar g_{-1|M} = \mathbf{g}_{-1}$ by the Lewy extension theorem. Since $(\bar h_{-1} - H(\bar g_{-1}))_{|M} = 0$, $\bar h_{-1} - H(\bar g_{-1}) = 0$. Since $\beta\alpha(\bar g_0 - \bar g_{-1}) = \bar h_{-1} - H(\bar g_{-1}) = 0$, there exists $\bar\phi_0 \in A^0_{\bar\Omega}(T'U)$ such that $F\bar\phi_0 = \bar g_0 - \bar g_{-1}$ and $\bar\phi_{0|M} = \varphi_0$. Since $F\bar\phi_1 = \bar\partial\bar g_0 = F\bar\partial\bar\phi_0$, we have $\bar\phi_1 = \bar\partial\bar\phi_0$. Q.E.D

Let $(K^\bullet_M, d)$ be the total simple complex of the double complex $K^{\bullet,\bullet}_M$.

Proposition 6.4. *If* $\dim_{\mathbf{R}} M \geq 5$*, there exist operators* $Z : K_M^q \to \operatorname{Ker} d$ *and* $Q : \operatorname{Ker} d \to K_M^{q-1}$ $(q = 1, 2)$*, satisfying*

(1) $Z_{|\mathrm{Ker}d} = id_{\mathrm{Ker}d}$,

(2) $d \circ Q \circ d = d$.

Hence, if we set $\mathcal{H}_M^q := (1 - d \circ Q) \circ Z(K_M^q)$ and $\rho_{\mathcal{H}} := (1 - d \circ Q) \circ Z : K_M^q \to \mathcal{H}_M^q$, then we have

Corollary 6.5. *For* $q = 1, 2$,

(1) *The natural homomorphism* $\mathcal{H}_M^q \to H^q(K_M^{\bullet,\bullet})$ *is an isomorphism,*

(2) *a homotopy formula* $u = \rho_{\mathcal{H}} u + d \circ Q \circ Zu + (1 - Z)u$ *holds for* $u \in K_M^q$.

The existence of Z and Q is proved by a parallel argument of [My6, §4] with the $\bar{\partial}$-analysis on $\bar{\Omega}$ replaced by the $\bar{\partial}_b$-analysis on M, where we use the standard $\bar{\partial}_b$-Neumann Hodge decompositions at $A_b^{0,1}(T')$, $A_b^{0,1}(N_{U/B|M})$ and $A_b^{0,1}$; say $\eta = \rho\eta + \bar{\partial}_b \bar{\partial}_b^* N_b \eta + \bar{\partial}_b^* \bar{\partial}_b N_b \eta$, and the ones at $A_b^0(T' B_{|M})$ and A_b^0; say $\eta = \rho\eta + \bar{\partial}_b^* N_b \bar{\partial}_b \eta$. These Hodge decompositions are all possible if $\dim_{\mathbf{R}} M \geq 5$ (cf. [Fo-Ko]). At the same time, we have the following estimates. We denote by $|| \, ||_k$ and $|| \, ||_k'$ the Sobolev norm and the Folland-Stein norm respectively of order k.

Proposition 6.6.

(1) *For* $(a_1, b_0, c_{-1}) \in K_M^{0,1} \oplus K_M^{1,0} \oplus K_M^{2,-1}$*, let* $Z(a_1, b_0, c_{-1}) = (a_1', b_0', c_{-1}')$ *and* $Q(a_1', b_0', c_{-1}') = (a_0'', b_{-1}'')$*. Then*

$$||a_0''||_k' \leq C||a_1'||_k \leq C'||a_1||_k$$

holds.

(2) *For* $(a_2, b_1, c_0) \in K_M^{0,2} \oplus K_M^{1,1} \oplus K_M^{2,0}$*, let* $Z(a_2, b_1, c_0) = (a_2', b_1', c_0')$ *and* $Q(a_2', b_1', c_0') = (a_1'', b_0'', c_{-1}'')$*. Then*

$$||a_1''||_k + ||b_0''||_k' \leq C||b_1'||_k \leq C'||b_1||_k$$

holds.

Here C *and* C' *denote constants independent of* (a_1, b_0, c_{-1}) *nor* (a_2, b_1, c_0).

Furthermore, the same adjustments as [My6] are possible.

Let

$$
\begin{aligned}
{}^{\circ}K_M^{0,q} &:= \{a_q \in A_b^{0,q}({}^{\circ}T') \mid \bar{\partial}_b a_q \in A_b^{0,q+1}({}^{\circ}T')\},\\
{}^{\circ}K_M^{1,q} &:= \{b_q \in A_b^{0,q}({}^{\circ}\tilde{T}'B_{|\partial\Omega}) \mid \bar{\partial}_b b_q \in A_b^{0,q+1}({}^{\circ}\tilde{T}'B_{|\partial\Omega})\}
\end{aligned}
$$

where ${}^{\circ}\tilde{T}'B$ is the subbundle of $T'B$ given by

$$
{}^{\circ}\tilde{T}'B := \{v \in T'B \mid v(\sum_{\beta=1}^{N} |w^{\beta}|^2) = 0\}.
$$

Then by the parallel argument as in the latter part of [My6, §3], we can construct Z and Q so that the following proposition holds.

Proposition 6.7. *For any cohomology class in $H^1(K_M^{\bullet})$ has a representative in ${}^{\circ}K_M^{0,1} \oplus {}^{\circ}K_M^{1,0} \oplus K_M^{2,-1}$.*

Proposition 6.8.

$$
\begin{aligned}
&(1)\quad Z(a_2, b_1, c_0) \in {}^{\circ}K_M^{0,2} \oplus {}^{\circ}K_M^{1,1} \oplus K_M^{2,0},\\
&\qquad\qquad\qquad\qquad \textit{if } (a_2, b_1, c_0) \in {}^{\circ}K_M^{0,2} \oplus {}^{\circ}K_M^{1,1} \oplus K_M^{2,0},\\
&(2)\quad Q(a_2, b_1, c_0) \in {}^{\circ}K_M^{0,1} \oplus {}^{\circ}K_M^{1,0} \oplus K_M^{2,-1},\\
&\qquad\qquad\qquad\qquad \textit{if } (a_2, b_1, c_0) \in {}^{\circ}K_M^{0,2} \oplus {}^{\circ}K_M^{1,1} \oplus K_M^{2,0}.
\end{aligned}
$$

Using Z and Q, we construct the Kuranishi semi-universal family of stably embeddable deformations of ${}^{\circ}T''$ by the argument in §2. Though a stably embeddable deformation of ${}^{\circ}T''$ is represented by $\phi \in K_M^{0,1}$ with which a $g \in K_M^{1,0}$ satisfying $(\bar{\partial} - \phi)(\iota + g) = 0$ is associated, we consider a triple $(\phi, g, k) \in K_M^{0,1} \oplus K_M^{1,0} \oplus K_M^{2,-1}$ satisfying

$$
\begin{aligned}
&P(\phi, g, k) :=\\
&\Big(\bar{\partial}_b\phi + R_2(\phi) + R_3(\phi), (\bar{\partial} - \phi)(\iota + g), (\tilde{h} + \tilde{k}) \circ (\iota + g)\Big) = (0,0,0)
\end{aligned}
$$

where $\tilde{k}$ denotes a holomorphic extension of k over $\bar{B}(c)$. Note that the holomorphic extension of k is possible in a unique way (cf. [Bl-Ep, Theorem A.1]), and that $\tilde{k} \circ (\iota + g)$ is considered as a Taylor series. We remark that the last term concerns the equation of the image $(\iota + g)(M)$; that is, $\tilde{h} + \tilde{k}$ is the defining equation of the subvariety which $(\iota + g)(M)$ bounds.

The construction of the Kuranishi semi-universal family using the complex $(K_M^{\bullet}, d)$ is parallel to the argument in §2. In fact, by the argument with $(A_b^{0,\bullet}(T'), \bar\partial_b)$, $\rho + \bar\partial_b\bar\partial_b^* N$ and $\bar\partial_b^* N$ replaced by $(K_M^\bullet, d)$, Z and Q respectively, we can prove the existence of

$$(\hat\phi(t), \hat g(t), \hat k(t)) \in (K_M^{0,1} \oplus K_M^{1,0} \oplus K_M^{2,-1})[[t_1, \dots, t_d]]$$

such that

(6.1) $(\hat\phi(0), \hat g(0), \hat k(0)) = (0,0,0)$,

(6.2) $(\hat\phi(t), \hat g(t), \hat k(t)) \equiv \sum_{\sigma=1}^d (\phi_\sigma, g_\sigma, k_\sigma) t_\sigma \bmod \mathfrak{m}^2$ where $\{[(\phi_\sigma, g_\sigma, k_\sigma)]\}_{\sigma=1}^d$ is a cohomology basis of $H^1(K_M^{\bullet,\bullet})$ and $\mathfrak{m}$ denotes the maximal ideal of $\mathbf{C}\{t_1, \dots, t_d\}$,

(6.3) there exists an extension $\tilde k(t) \in H^0(\bar B, \mathcal{O}_B)$ of $\hat k(t)$ such that

$$P(\hat\phi(t), \hat g(t), \tilde k(t)) :=$$
$$\left(\bar\partial\hat\phi(t) - \frac{1}{2}[\hat\phi(t), \hat\phi(t)], (\bar\partial - \hat\phi(t))(\iota + \hat g(t)), (\tilde h + \tilde k(t)) \circ (\iota + \hat g(t))\right)$$
$$\in \hat{\mathfrak{I}}(K_M^{0,2} \oplus K_M^{1,1} \oplus K_M^{2,0})[[t_1, \dots, t_d]],$$

where $\hat{\mathfrak{I}}$ is an ideal of $\mathbf{C}[[t_1, \dots, t_d]]$ generated by $\hat b_1(t), \dots, \hat b_\ell(t)$ and $\rho_{\mathcal{H}} P(\hat\phi(t), \hat g(t), \tilde k(t)) = \sum_{\beta=1}^\ell \hat b_\beta(t) e_\beta$ with respect to a basis $e_1, \dots, e_\ell$ of $\mathcal{H}^2$,

(6.4) it is formally Kuranishi versal, that is, there exists τ and F in Definition 5.2 as formal power series in s.

The proof of (6.4) needs to treat an extra term other than the argument in the proof of Proposition 2.2. Let $\pi : \mathcal{U} \to S$ together with an embedding $\Psi : \mathcal{U} \hookrightarrow \mathbf{C}^N \times S$ be a family of displacements (in $\mathbf{C}^N$) of a neighborhood of M in U. Suppose that Ψ is expressed by $w^\beta = \Psi_i^\beta(\zeta_i, s)$ $(\beta = 1, \dots, N)$ with respect to a local coordinate $(\zeta_i^1, \dots, \zeta_i^n, s_1, \dots, s_{d'})$ of $\mathcal{U}$ as in (N.3) and the coordinate $(w^1, \dots, w^N)$ of $\mathbf{C}^N$. By the argument parallel to the proof of Proposition 2.2, we can prove the existence of

$$\hat\tau(s) \in \mathbf{C}^d[[s_1, \dots, s_{d'}]]$$
$$\hat F_i^\alpha(s) \in \Gamma(U_i, T'U_{|M})[[s_1, \dots, s_{d'}]] \ (\alpha = 1, \dots, n)$$
$$\hat\eta^\beta(s) \in H^0(\bar B(c), \mathcal{O}_B)[[s_1, \dots, s_{d'}]] \ (\beta = 1, \dots, N)$$

satisfying

$$(1) \qquad \hat{\tau}(0) = 0,\ \hat{F}_i^\alpha(0) = id_M,\ \hat{\eta}^\beta(0) = w^\beta$$

$$(2) \qquad \hat{F}_i^\alpha(s) - f_{ij}^\alpha(\hat{F}_j(s), s) \equiv 0 \bmod \mathfrak{I}_S$$

$$(3) \qquad \left(\bar{\partial} - \hat{\phi}(\hat{\tau}(s))\right) \hat{F}_i^\alpha(s) \equiv 0 \bmod \mathfrak{I}_S$$

$$(4) \qquad \hat{\eta}^\beta(\Psi_i(\hat{F}_i(s), s), s) - \hat{g}_i^\beta(\hat{\tau}(s)) \equiv 0 \bmod \mathfrak{I}_S$$

$$(5) \qquad b_\gamma(\hat{\tau}(s)) \equiv 0 \bmod \mathfrak{I}_S,$$

where $\zeta^\alpha = f_{ij}^\alpha(\zeta_j, s)$ $(\alpha = 1, \ldots, n)$ is the coordinate transformation and $\mathfrak{I}_S$ denotes the defining ideal of S in $\mathbf{C}\{s_1, \ldots, s_{d'}\}$. Then (6.4) follows from the existence of the above $\{\hat{F}_i^\alpha(s)\}$ and $\hat{\tau}(s)$.

In order to assure the convergence of $\hat{\phi}(t)$, $\hat{g}(t)$, $\hat{b}_\gamma(t)$ $(\gamma = 1, \ldots, \ell)$, $\hat{\tau}(s)$ and $\{\hat{F}_i^\alpha(s)\}$, we need the adjustment of Z and Q as above. Indeed, by these adjustment and by starting the construction with the initial term

$$(\hat{\phi}_1(t), \hat{g}_1(t), \hat{k}_1(t)) = \sum_{\sigma=1}^{d} (\phi_\sigma, g_\sigma, k_\sigma) t_\sigma$$

such that

$$(\phi_\sigma, g_\sigma, k_\sigma) \in {}^\circ K_M^{0,1} \oplus {}^\circ K_M^{1,0} \oplus K_M^{2,-1} \ (\sigma = 1, \ldots, d)$$

holds (it is possible by Proposition 6.7),we have $\hat{\phi}(t) \in {}^\circ K_M^{0,1}[[t_1, \ldots, t_d]]$ which assures the convergence of $\hat{\phi}(t)$ and $\hat{g}(t)$ for the same reason as in §2, using the estimate of Z and Q (cf. Proposition 6.6). The convergence of $\hat{b}_\gamma(t)$ follows from the fact that $P_{|\mathcal{H}} : \mathcal{H} \to K_M^{0,2} \oplus K_M^{1,1}$ is injective where P denotes the projection operator of $K_M^{0,2} \oplus K_M^{1,1} \oplus K_M^{2,0}$ onto the first two factors. The convergence of $\hat{\tau}(s)$ and $\{\hat{F}_i^\alpha(s)\}$ is proved by the same calculation as in [My3].

Hence we have

Theorem 6.9. *Let V be a locally closed normal Stein subvariety in $\mathbf{C}^N$ and M a link of one singular point V. If $dim_{\mathbf{C}} V \geq 3$, then there exists a Kuranishi semi-universal family of stably embeddable deformations of CR structures on M.*

Let V and M be as in Theorem 6.9 and o the normal isolated singular point which M bounds. Let $f : \mathcal{V} \to S$ be the semi-universal family of flat deformations of the germ (V, o). We may assume that $\mathcal{V} \subset \mathbf{C}^N \times S$ is a subspace and $M \subset f^{-1}(s_o)$ $(s_o \in S)$.

Theorem 6.10. *Let V and M be as in Theorem 6.9 and $f : \mathcal{V} \to S$ as above. Let $\phi(t)$ $(t \in (T, 0))$ be the Kuranishi semi-universal family of stably embeddable deformations of CR structures on M (obtained in Theorem 6.9). Then $(T, 0) \simeq (S, s_o)$ and there exists a holomorphic family of embeddings of M into the family $\mathcal{V} \to S$ such that $\phi(t)$ $(t \in (T, 0))$ is induced from this family of embeddings.*

Outline of the proof. A formal family $\tilde{h}_1 + \tilde{k}_1(t), \ldots, \tilde{h}_{m_1} + \tilde{k}_{m_1}(t)$ of holomorphic functions on $\bar{B}(c)$ (obtained in Theorem 6.9) defines a formal family of subvarieties of $\bar{B}(c)$, say $\hat{\mathcal{V}} \subset \bar{B}(c) \times \hat{T}$, and we can prove that it is a flat family by the same argument as [Bl-Ep, Theorem 5.1]. Hence, we can compare $\phi(t)$ $(t \in (T, 0))$ with $f : \mathcal{V} \to S$ using their Kuranishi semi-universality and formally semi-universality respectively. Theorem 6.10 follows from this comparison taking account of Propositions 6.1 and 6.2.

In the case of $\dim_{\mathbf{C}} V = 2$, our notion of stably embeddable deformations of CR structures is nothing but the one of three dimensional embeddable CR structures in [Bl-Ep]. In fact, $H^1(K_M^{\bullet})$ coincides with $\mathrm{Def}_1(M, \bar{Z}, X_0)$ (the space of first order embeddable deformations) in [Bl-Ep], where $\bar{Z}$ denotes the original CR structure on M and X_0 coincides with the embedding ι. However, the construction of the (convergent) semi-universal family of stably embeddable deformations of three dimensional CR structures on M is still open due to the difficulty of the analysis at K_M^2.

References

[**Ak1**] Akahori, T., Intrinsic formula for Kuranishi's $\bar{\partial}_b^{\phi}$, Publ. RIMS, Kyoto Univ., **14** (1978), 615–641.

[**Ak2**] ———, Complex analytic construction of the Kuranishi family on a normal strongly pseudo convex manifold, Publ. RIMS, Kyoto Univ., **14** (1978), 789–847.

[**Ak3**] ———, The new estimate for the subbundles E_j and its application to the deformation of the boundaries of strongly pseudo convex domains, Invent. math., **63** (1981), 311–334.

[**Ak4**] ———, Complex analytic construction of the Kuranishi family on a normal strongly pseudo convex manifold with real dimension 5, Manuscripta math., **63** (1989), 29–43.

[**Ak5**] ———, A survey of deformation theory of CR-structures, "Nilpotent geometry and analysis", Sūrikaisekikenkyūsho Kōkyūroku No. 875, 1994, pp. 117–128.

[**Ak-My1**] Akahori, T. and Miyajima, K., Complex analytic construction of the Kuranishi family on a normal strongly pseudo-convex manifold. II, Publ. RIMS, Kyoto Univ., **16** (1980), 811–834.

[**Ak-My2**] ———, An analogy of Tian-Todorov theorem on deformations of CR-structures, Compositio Math., **85** (1993), 57–85.

[**Ak-My3**] ———, A note on the analogue of the Bogomolov type theorem on deformations of CR-structures, Canad. Math. Bull., **37** (1994), 8-12.

[**Ak-My4**] ———, On the CR-version of the Tian-Todorov lemma, preprint.

[**An-Gr**] Andreotti, A. and Grauert, H., Théorèmes de finitude pour la cohomologie des espaces complexes, Bull. Soc. Math. Fr., **90** (1962), 193–259.

[**Ba-St**] Bănică, C. and Stănăşilvă, O., "Algebraic Methods in the Global Theory of Complex Spaces", John Wiley & Sons, 1976.

[**Bl-Du**] Bland, J. and Dechamp, T., Deformation theory for the hyperplane line bundle on $\mathbf{P}^1$, in this volume.

[**Bl-Ep**] Bland, J. and Epstein, C.L., Embeddable CR-structures and deformations of pseudoconvex surfaces. Part I: Formal deformations, J. Alg. Geom., **5** (1996), 277–368.

[**BM**] Boutet de Monvel, Intégration des equations de Cauchy-Riemann induites formelles, Séminaire Goulaoic-Lions-Schwartz, Exposé (1974-1975).

[**Bu-Ml**] Buchweitz, R.O. and Millson, J.J., CR-Geometry and deformations of isolated singularities, to appear.

[**Ch-Le**] Chen, J. and Lee, J., A local slice theorem for 3-dimensional CR structures, Amer. J. Math., **117** (1995), 1249–1298.

[**De**] Demailley,J.-P., Cohomology of q-convex spaces in top degrees, Math. Z., **204** (1990), 283–295.

[**Do**] Donin, I.F., Complete families of deformations of germs of complex spaces, Math. USSR, Sb., **18** (1972), 397–406.

[**Ep**] Epstein, C. L., Deformations of singularities, complex manifolds and CR-structures, in this volume.

[**Es**] von Essen, H., Nonflat deformations of modules and isolated singularities, Math. Ann., **287** (1990), 413–427.

[**Fo-Kn**] Forster,O. and Knorr,K., Über die Deformationen von Vektorraumbündeln auf kompakten komplexen Raümen, Math. Ann., **209** (1974), 291–346.

[**Fo-Ko**] Folland, G. B. and Kohn, J. J., "The Neumann problem for the Cauchy-Riemann complex", Annals of Mathematics Studies, Princeton Univ. Press, 1972.

[**Go-Ml1**] Goldman, W.M. and Millson, J.J., The deformation theorey of representations of fundamental groups of compact Kähler manifolds, Publ. Math. IHES, **67** (1988), 43-96.

[**Go-Ml2**] ———, The homotopy invariance of the Kuranishi space, Ill. J. Math., **34** (1990), 337–367.

[Gr] Grauert, H., Über die Deformation isolierter Singularitäten analytischer Mengen, Inv. math, **15** (1972), 171–198.

[Ha-La] Harvey, F. and Lawson, H., On the boundaries of complex analytic varieties, I, Ann. of Math., **102** (1975), 223–290; II, Ann. of Math., **106** (1977), 213–238.

[Hö] Hörmander, L., L^2 estimates and existence theorems for the $\bar{\partial}$ operator, Acta Math. J., **113** (1965), 89–152.

[Ko] Kodaira, K., "Complex Manifolds and Deformation of Complex Structures", Grundlehren der mathematischen Wissenshaften, Springer-Verlag, 1981.

[Ko-Ro] Kohn, J. J. and Rossi, H., On the extension of holomorphic functions from the boundary of a complex manifold, Ann. of Math., **81** (1965), 451–472.

[Ku1] Kuranishi, M., On the locally complete families of complex analytic structures, Ann. of Math., **75** (1962), 536–577.

[Ku2] ———, Deformations of compact complex manifolds, Montreal University.

[Ku3] ———, Deformations of isolated singularities and $\bar{\partial}_b$, Preprint, Columbia University.

[Ku4] ———, Application of $\bar{\partial}_b$ to deformation of isolated singularities, Proc. Symp. Pure Math., **30** (1977), 97-106.

[Le] Lee, J., Pseudo-Einstein structures on CR manifolds, Amer. J. Math., **110** (1988), 157–178.

[Lm] Lempert, L., Spaces of Cauchy-Riemann Manifolds, in this volume.

[Lu-Ya] Luk, H. and Yau, S.S.-T., Some remarks on compact strongly pseudoconvex CR manifolds, in this volume.

[My1] Miyajima, K., Completion of Akahori's construction of the versal family of strongly pseudo-convex CR structures, Trans. A. M. S., **277** (1983), 163–172.

[My2] ———, Deformation of a complex manifold near a strongly pseudoconvex real hypersurface and a realization of the Kuranishi family of strongly pseudo-convex CR-structures, Math. Z. (1990), 593–602.

[My3] ———, On realization of families of strongly pseudo-convex CR structures, Tokyo J. Math., **15** (1992), 153–170.

[My4] ———, A Bogomolov-type smoothness on deformations of quasi-Gorenstein cone singularities of dim ≥ 4, in "Geometric Complex Analysis, ed. J.Noguchi et al.", World Scientific, 1996, pp. 461–465.

[My5] ———, A note on the Bogomolov-type smoothness on deformations of the regular parts of isolated singularities, to appear in the Proc. of A. M. S..

[My6] ———, Non-algebraic description of flat deformations of normal isolated singularities, preprint.

[**Oh1**] Ohsawa,T., A reduction theorem for cohomology groups of very strongly q-convex Kähler manifolds, Inv. math., **63** (1981), 335–354.

[**Oh2**] ———, Hodge spectral sequence on compact Kähler spaces, Publ. R.I.M.S., Kyoto Univ., **23** (1987), 265–274.

[**Oh3**] ———, A report on isolated singularities by transcendental methods, in this volume.

[**Oh-Ta**] Ohsawa,T. and Takegoshi,K., Hodge spectral sequence on pseudoconvex domains, Math. Z., **197** (1988), 1–12.

[**Ru**] Rumin, M., Un complexe de formes différentielles sur les variétes de contact, C. R. Acad. Sci. Paris, **310** (1990), 401–404.

[**Ta**] Tanaka, N., A differential-geometric study on strongly pseudoconvex manifolds, Lectures in Mathematics, Kyoto Univ. (1975).

[**Tj**] Tjurina, G.N., Locally semi-universal flat deformations of isolated singularities of complex spaces, Mth. USSR, Izv., **3** (1969), 976–1000.

[**Ya**] Yau, S.S.-T., Kohn-Rossi cohomology and its application to the complex Plateau problem I, Ann. of Math., **113** (1981), 67–110.

Mathematical Institute
College of Liberal Arts
Kagoshima University
Kagoshima 890
Japan

Advanced Studies in Pure Mathematics 25, 1997
CR-Geometry and Overdetermined Systems
pp. 276–284

A Report on Isolated Singularities by Transcendental Methods

Takeo Ohsawa

Dedicated to Professor M. Kuranishi on his 70th birthday

1. An *isolated singularity* (of a complex analytic space) is by definition a germ of a reduced and irreducible complex analytic space at an isolated singular point. By a *model* of an isolated singularity we shall mean an irreducible complex analytic subset V of $\mathbb{C}^N$ containing the origin as the unique singular point. To any model V of an isolated singularity (V, o), one can associate three manifolds of completely different nature.

i) A nonsingular model of V: By Hironaka's desingularization theorem, there exists a complex manifold $\tilde{V}$ and a proper holomorphic map $\pi : \tilde{V} \to V$ such that $\pi|\tilde{V} \setminus \pi^{-1}(o)$ is a biholomorphism. In virtue of the existence of $\tilde{V}$, equivalence questions between the isolated singularities can be transferred to more geometric ones (cf. [G-2], [H-R]). Moreover a lot of work has been done on the classification of isolated singularities by manipulating the invariants on $\tilde{V}$ (cf. [I]).

ii) $V \cap S_\varepsilon$, where $S_\varepsilon = \{z \in \mathbb{C}^N \mid \|z\| = \varepsilon\}$ and ε is so chosen that $S_{\varepsilon'}$ and V intersect transversally for all $\varepsilon' \in (0, 2\varepsilon)$: As a differentiable manifold, $V \cap S_\varepsilon$ falls into the class of strongly pseudoconvex CR manifolds, or spc manifolds. Since the spc structure naturally yields L^2 estimates for the tangential Cauchy-Riemann operators, the method of PDE in the theory of deformation of complex manifolds is carried over to spc manifolds. As a result, several fundamental questions including the construction of the versal family of singularities have been solved by this method (cf. [A-1], [A-M] and [N-O]).

iii) $V \setminus \{o\}$: Manifolds of this type appear as the ends of locally symmetric varieties of rank one, and they were studied in a general framework by Andreotti-Grauert [A-G] as the ends of "pseudoconcave"

Received September 25, 1995
Revised December 8, 1995

spaces. Afterwards it was noticed that some questions on isolated singularities are tractable as a sort of boundary value problem on $V \setminus \{o\}$ for the $\bar{\partial}$-operator. In fact, analysis of this type turned out to be useful when one wants to understand the intersection cohomology of projective varieties (cf. [O-4,5,9]).

The purpose of this note is to report miscellaneous results on these three types of manifolds that are obtained since Professor M. Kuranishi gave a series of inspiring lectures at RIMS in 1976.

2. Given any desingularization $\pi : \tilde{V} \to V$, $\pi^{-1}(o)$ is called the *exceptional set.* Topology of the exceptional set is somewhat restrictive in the following sense.

Theorem 1. $\dim_{\mathbb{C}} H^r(\pi^{-1}(o), \mathbb{C}) \equiv 0 \pmod 2$ *if* $r \geq n$ *and* $r \equiv 1 \pmod 2$.

This result seems to have been a sort of folklore in the late seventies (cf. [L-R], [F]), but rigorous proofs appeared only gradually in [O-1,2,6] and [O-T]. Although a complete proof of Theorem 1 is only available via Artin's algebraization theorem and Deligne's mixed Hodge theory, it may be worthwhile to note that the following constitutes already a significant part of the proof, which motivated further generalization of the Hodge theory to noncompact manifolds.

Theorem 2 (cf. [O-1,2], [O-T]). *Let X be a Kähler manifold of dimension n and let $D \subset X$ be a strongly pseudoconvex domain with C^2-smooth boundary. Then there exists a complete Kähler metric ω on D such that $H^r(D, \mathbb{C})$ are canonically isomorphic to the space of L^2 harmonic forms of degree r with respect to ω for all $r > n$.*

The range $r > n$ is optimal. In fact, $\dim_{\mathbb{C}} H^n(D, \mathbb{C}) < \infty$ but the space of L^2 harmonic n-forms is infinite dimensional for any strongly pseudoconvex domain. As the above ω one may take any complete Kähler metric which is quasi-isometrically equivalent to the Levi form of a C^∞ exhaustion function with bounded gradient outside a compact subset of D. A typical example of such a metric is the Bergman metric on strongly pseudoconvex domains in $\mathbb{C}^n$ (cf. [D-F]). For the Bergman metric, the L^2 cohomology groups of type (p, q) are known to be infinite dimensional for $p + q = n$ (cf. [D-F], [O-8]). Recently the boundary values of these cohomology classes are studied for the unit ball (cf. [J-K]).

Applying Theorem 2, one can deduce the following Hartogs type theorem.

Theorem 3 (cf. [O-2]). *The natural restriction map*

$$\rho^{p,q} : H^{p,q}(\tilde{V}) \longrightarrow H^{p,q}(\tilde{V} \setminus \pi^{-1}(0))$$

is surjective if $p + q < n - 1$.

The range $p+q < n-1$ is also optimal. In fact, since $\tilde{V} \setminus \pi^{-1}(o) \cong V \setminus \{o\}$ and $H^{n,0}(\tilde{V})$ is naturally identified with the set of L^2 holomorphic n-forms on $V \setminus \{o\}$, $\dim_{\mathbb{C}} \operatorname{Coker} \rho^{n,0}$ does not depend on the choice of the nonsingular model $\tilde{V}$. It is easy to see that $\dim_{\mathbb{C}} \operatorname{Coker} \rho^{n,0} = \dim_{\mathbb{C}} H^1(\tilde{V}, \mathcal{O}_{\tilde{V}})$ and to verify that $H^1(\tilde{V}, \mathcal{O}_{\tilde{V}}) \neq \{o\}$ if $\dim V = 2$ and $\tilde{V}$ contains a nonrational curve.

Van Straten [V-S] discovered a remarkable application of Theorem 3 to Zariski-Lipman conjecture by showing for the case $\dim V \geq 3$ that the germ (V, o) is nonsingular if the tangent sheaf of V is locally free.

As for the de Rham cohomology classes on $\tilde{V} \setminus \pi^{-1}(o)$, we have the same extendability result for the degrees less than $n - 1$. This range is also optimal in general, although one can prove the following by analysing a spectral sequence that abuts to $H^r(\pi^{-1}(o), \mathbb{C})$.

Theorem 4 (cf. [O-6]). *If the inclusion map* $\pi^{-1}(0) \hookrightarrow \tilde{V}$ *is a homotopy equivalence, the natural restriction map*

$$H^r(\tilde{V}, \mathbb{C}) \longrightarrow H^r(\tilde{V} \setminus \pi^{-1}(0), \mathbb{C})$$

is surjective for $r \leq n - 1$.

Corollary. *In the above situation, every cohomology class in* $H^n(\tilde{V}, \mathbb{C})$ *can be represented by a compactly supported closed form.*

Thus we are naturally led to the following

Question. *Is every closed holomorphic* $(n-1)$*-form on* $\tilde{V} \setminus \pi^{-1}(o)$ *holomorphically extendable to* $\tilde{V}$*?*

Note that this is certainly true if $n = 1$, since closed 0-forms are locally constant functions. The first nontrivial case $n = 2$ was solved affirmatively by T. Ueda [U]. Mentioning further a partial result, we have that if $\tilde{V}$ is a Zariski open subset of a nonsingular projective variety Z and the given form is extendable to $Z \setminus \pi^{-1}(o)$ then it is extendable also across $\pi^{-1}(o)$. In fact one can prove the following.

Theorem 5 (cf. [F1], [O-10]). *Let X be an irreducible projective variety with singular locus Y, and let p be a nonnegative integer satisfying $p < \operatorname{codim} Y$. Then a holomorphic p-form f is extendable holomorphically to a nonsingular model of X if and only if f is closed.*

After [O-10] was written down, S. Kosarew settled the question affirmatively by an algebraic method (personal communication).

As another question on $\tilde{V}$ we would like to mention the following which was asked by S. Nakano around 1976.

Problem. *Is any d-exact $(1,1)$-form on $\tilde{V}$ of the form $\partial\bar{\partial}\varphi$?*

In case the canonical bundle of $\tilde{V}$ is trivial, one may employ the above mentioned L^2 Hodge theory (cf. Theorem 2 and the remark) to solve it affirmatively. As a result, one has the smoothness of Kuranishi spaces for the deformation of certain isolated singularities (cf. [M]). It should be noted, however, that the answer to Nakano's question is negative in general, because it is not necessarily true that all the topologically trivial line bundles over $\tilde{V}$ arise as flat $U(1)$ bundles.

3. Applying Theorem 4, one can describe some topological properties of $V \cap S_\varepsilon$.

Theorem 6. *Let $\alpha_i \in H^{r_i}(V \cap S_\varepsilon, \mathbb{C})$, $i = 1, 2, \dots, m$. Then the cup product $\alpha_1 \cup \alpha_2 \cup \cdots \cup \alpha_m$ is zero whenever $\sum_{i=1}^m r_i \geq n$ and $r_i \leq n-1$ for all i.*

Proof. Let $\tilde{V}_\varepsilon := \{w \in \tilde{V} \mid \|\pi(w)\| < 2\varepsilon\}$. Then, by Theorem 4 there exist $\tilde{\alpha}_i \in H^{r_i}(\tilde{V}_\varepsilon, \mathbb{C})$ such that $\tilde{\alpha}_i | V \cap S_\varepsilon = \alpha_i$. Since $\alpha_1 \cup \cdots \cup \alpha_m = (\tilde{\alpha}_1 \cup \cdots \cup \tilde{\alpha}_m) | V \cap S_\varepsilon$, we obtain the conclusion from the corollary of Theorem 4.

Corollary. *If $n \geq 2$, there does not exist an isolated singularity (V, o) for which $V \cap S_\varepsilon$ is homotopically equivalent to $\underbrace{S^1 \times \cdots \times S^1}_{2n-1}$.*

Such a phenomenon was first noticed by D. Sullivan for hypersurface singularities of dimension 2 (cf. [Ka]).

As we have mentioned earlier, there is a natural abstract notion of spc manifolds. Recall that a $(2n-1)$-dimensional differentiable manifold M of class C^∞ is called a CR manifold if there are subbundles T, T', F of the complexified tangent bundle $T_M \otimes \mathbb{C}$ such that T is involutive, $T = \overline{T'}$, $\operatorname{rank}_{\mathbb{C}} F = 1$ and $T_M \otimes \mathbb{C} = T \oplus T' \oplus F$. For any local frame

$\{\omega_1, \ldots, \omega_{n-1}\}$ for T and a local frame θ for F with $\bar{\theta} = -\theta$, one has a matrix valued function (c_{ij}) defined by $[\omega_i, \bar{\omega}_j] = c_{ij}\theta \pmod{T \oplus T'}$. We say M is a strongly pseudoconvex CR manifold, or shortly an spc manifold, if one can choose the frames $\{\omega_i\}$ and θ around each point of M so that (c_{ij}) is positive definite. $V \cap S_\varepsilon$ is then an spc manifold if one puts $T = (T_{V \cap S_\varepsilon} \otimes \mathbb{C}) \cap T^{1,0}_{\mathbb{C}^N}$, $T' = \bar{T}$ and $F = (T \oplus T')^\perp$. In the same manner, the boundary of any strongly pseudoconvex domain is, if it is of class C^∞, regarded as an spc manifold. The following result ensures that passage from $V \cap S_\varepsilon$ to the class of spc manifolds is a good abstraction.

Theorem 7(cf. [B], [O-3]). *Every connected compact spc manifold of dimension ≥ 5 is the boundary of a strongly pseudoconvex domain in a complex manifold.*

Therefore, combining Theorem 7 with the remark preceding Theorem 4, one has the following by the same argument as in the proof of Theorem 6.

Theorem 8. *Let M be a connected compact spc manifold of dimension $2n-1$ with $n \geq 3$. Then the cup product $\alpha_1 \cup \cdots \cup \alpha_m$ of $\alpha_i \in H^{r_i}(M, \mathbb{C})$ is zero whenever $\sum_{i=1}^m r_i \geq n+1$ and $r_i \leq n-2$ for all i.*

Question. *Is there any direct proof of Theorem 8 that does not use Theorem 7 and the Hodge theory on strongly pseudoconvex domains?*

A recent work of T. Akahori [A-2] may lead to an answer to it.

For three dimensional spc manifolds, it is well known that they are even locally not embeddable as a real hypersurface of a complex manifold (cf. [Ni]). As for the recent embeddability and non-embeddability results, the reader is referred to articles of C. Epstein [E-1,2].

4. Although the compactness of $V \cap S_\varepsilon$ is a great advantage for using analytic tools, one might also be inclined to study analytic objects on the manifold $V \setminus \{o\}$ because it carries a complete Kähler metric by a theorem of Grauert (cf. [G1]). Since Grauert's Kähler metric on $V \setminus \{o\}$ is of the form $\partial\bar{\partial}\varphi$, where φ is bounded near o, one can immediately deduce from Bochner-Nakano's formula that the $\bar{\partial}$-equation $\bar{\partial} f = g$ has an L^2 solution near o for any $\bar{\partial}$-closed L^2 (p,q)-form g on V, provided that $p+q > n$. In order to proceed further, we need the following observation due to Donnelly and Fefferman [D-F] (See [O-T] for a simplified proof).

Theorem 9. *Let (M,ω) be a connected complete Kähler manifold of dimension n such that there exists a C^∞ strictly plurisubharmonic function φ with bounded gradient on M such that $\partial\bar\partial\varphi$ is quasi-isometrically equivalent to ω. Then the $L^2\bar\partial$-cohomology group of (M,ω) of type (p,q) vanishes if $p+q\neq n$.*

Metrics satisfying the hypothesis of Theorem 9 arise very naturally. Instances are the metric $\partial\bar\partial(-\log(-\log\|z\|))$ on the punctured unit ball $\mathbb{B}_* := \{z \in \mathbb{C}^N \mid 0 < \|z\| < 1\}$ and its restriction to $V \cap \mathbb{B}_*$. As another immediate instance, one can mention the Bergman metric on a strongly pseudoconvex domain in $\mathbb{C}^n$. A remarkable fact attached to the L^2 cohomology vanishing in Theorem 9 is that the L^2 estimates

$$\|u\| \leq C(\|\bar\partial u\| + \|\bar\partial^* u\|)$$

hold for $C = 3\sup|\partial\varphi|_\omega$. This allows us to study the L^2 cohomology of $V \cap \mathbb{B}_*$ with respect to non-complete metrics that are the limits of $\partial\bar\partial\varphi_t$ satisfying the uniformity condition $\partial\bar\partial\varphi_t \geq \partial\varphi_t \cdot \bar\partial\varphi_t$. Among such metrics is the restriction of the Euclidean metric $\partial\bar\partial\|z\|^2$ to $V \cap \mathbb{B}_*$. To state results of this kind, let us denote by $H^{p,q}_{(2)}(U)$, for any neighbourhood U of o in V, the L^2 $\bar\partial$-cohomology group of $U \setminus \{o\}$ of type (p,q). For the unit ball $\mathbb{B} = \{z \in \mathbb{C}^N \mid \|z\| < 1\}$, it has long been known that $H^{0,q}_{(2)}(\mathbb{B}) = \{0\}$ for $q \geq 1$ (cf. [Hö]). By the above mentioned argument one can show that $H^{p,q}_{(2)}(\mathbb{B}\cap V) = \{0\}$ if $p+q > n$ (cf. [O-4]). If one denotes by $H^{p,q}_{(2),0}(\mathbb{B}\cap V)$ the L^2 $\bar\partial$-cohomology of $\mathbb{B}_*\cap V$ with respect to $\partial\bar\partial\|z\|^2$ with supports contained in compact subsets of V, one has also the dual vanishing $H^{p,q}_{(2),0}(\mathbb{B}\cap V) = \{0\}$, $p+q < n$ (cf. [O-7]). Similarly one has also the vanishing of the L^2 de Rham cohomology groups for the corresponding degrees. Moreover, with an additional technical effort one can manage to prove

Theorem 10 (cf. [O-7,9]).

$$H^r_{(2)}(V \cap \mathbb{B}) = \{0\} \quad \textit{for} \quad r \geq n$$

and

$$H^r_{(2),0}(V \cap \mathbb{B}) = \{0\} \quad \textit{for} \quad r < n.$$

Here $H^r_{(2)}(V\cap\mathbb{B})$ (resp. $H^r_{(2),0}(V\cap\mathbb{B})$) denotes the r-th L^2 de Rham cohomology group of $V\cap\mathbb{B}$ with respect to $\partial\bar\partial\|z\|^2$ (resp. that with relatively compact supports in V).

Corollary. *For any projective variety $X \subset \mathbb{P}^N$ whose singular points are isolated, the L^2 de Rham cohomology group of X is canonically isomorphic to the intersection cohomology group of X in the sense of Goresky-MacPherson.*

As a concluding remark we would like to indicate a next interesting topic in the analysis of isolated singularities. This will be a question of estimating $\dim H^{p,q}_{(2)}(V \cap \mathbb{B}_*)$ or $\dim H^r_{(2)}(V \cap \mathbb{B}_*)$ with respect to complete Kähler metrics on $V \setminus \{o\}$ that does not satisfy the condition of Theorem 9. Such metrics arise naturally by adding Kähler metrics on $\tilde{V}$. Therefore it seems that something like the following must have an answer.

Question. *Let ω_1 and ω_2 be complete Kähler metrics on $V \setminus \{o\}$. Is it true that $\omega_1 \geq \omega_2$ implies $\dim H^r_{(2)}(V \cap \mathbb{B}_*)_{\omega_1} \geq \dim H^r_{(2)}(V \cap \mathbb{B}_*)_{\omega_2}$?*

References

[A-1] Akahori, T., The new estimate for the subbundles E_j and its application to the deformation of boundaries of strongly pseudo-convex domains, Invent. math., **63** (1981), 311–334.

[A-2] ———, On Hodge theory over strongly pseudoconvex boundaries, *Geometric Complex Analysis*, edited by Junjiro Noguchi et al., World Scientific, Singapore, 1995.

[A-M] Akahori, T. and Miyajima, K., An analogy of Tian-Todorov theorem on deformations of CR-structures, Comp. Math., **85** (1993), 57–85.

[A-G] Andreotti, A. and Grauert, H., Algebraische Körper von automorphen Funktionen, Nachr. Akad. Wiss. Göttingen, II. Math.-Phys., **Kl** (1961), 39–48.

[B] Boutet de Monvel, L., Intégration des équations de Cauchy-Riemann induites formelles, *Séminaire Goulaouc-Lions-Schwartz*, 1974–75: *Équations aux derivées partielles linéaires et non linéaires*, Exp. No. 9, 14pp.

[D-F] Donnelly, H. and Fefferman, C., L^2-cohomology and index theorem for the Bergman metric, Ann. of Math., **118** (1983), 593–619.

[E-1,2] Epstein, C.L., A relative index on the space of embeddable CR structures, I, II, preprint.

[Fl] Flenner, H., Extendabillity of differential forms on non-isolated singularities, Invent. math., **94** (1988), 317–326.

[F] Fujiki, A., Hodge to de Rham spectral sequence on a strongly pseudoconvex manifold, unpublished.

[G-1] Grauert, H., Characterisierung der Holomorphiegebiete durch die vollständige Kählersche Metrik, Math. Ann., **131** (1956), 38–75.

[G-2] ———, Über Modifikationen und exzeptionelle analytische Mengen, Math. Ann., **146** (1962), 331–368.

[H-R] Hironaka, H. and Rossi, H., On the equivalence of imbeddings of exceptional complex spaces, Math. Ann., **156** (1964), 313–333.

[Hö] Hörmander, L., L^2-estimates and existence theorems for the $\bar{\partial}$-operator, Acta Math., **113** (1965), 89–152.

[I] Ishii, S., Theory of singularities (In Japanese), Springer Verlag Tokyo, to appear.

[J-K] Julg, P. and Kasparov, G., Operator K-theory for the group $\mathrm{SU}(n,1)$, J. Reine angew. Math., **463** (1995), 99–152.

[Ka] Kato, M., Some problems in topology, Manifolds-Tokyo 1973, Proceedings of the International Conference on Manifolds and Related Topics in Topology, edited by A. Hattori, Univ. of Tokyo Press, 421–432.

[L-R] Lieberman, D. and Rossi, H., Deformations of strongly pseudo-convex manifolds, Rencontre sur l'analyse complexe à plusieurs variables et les systèmes surdéterminés (Textes Conf., Univ. Montréal, Montréal, Que., 1974), 119–165 Presses Univ. Montréal, Montréal, Que., 1975.

[M] Miyajima, K., A Bogomolov-type smoothness on deformations of quasi-Gorenstein cone singularities of dim 4, *Geometric Complex Analysis*, edited by Junjiro Noguchi et al., World Scientific, Singapore, 1995.

[N-O] Nakano, S. and Omoto, H., Local deformations of isolated singularities associated with negative line bundle sover abelian varieties, Nagoya Math. J., **75** (1979), 41–70.

[Ni] Nirenberg, L, On a problem of Hans Lewy, Uspeki Math. Nauk., **292** (1974), 241–251.

[O-1] Ohsawa, T., A reduction theorem for cohomology groups of very strongly q-convex Kähler manifolds, Invent. math., **63** (1981), 335–354.

[O-2] ———, Addendum to: A reduction theorem for cohomology groups of very strongly q-convex Kähler manifolds, Invent. math., **66** (1982), 391–393.

[O-3] ———, Global realization of strongly pseudoconvex CR manifolds, Publ. RIMS Kyoto Univ., **20** (1984), 599–605.

[O-4] ———, Hodge spectral sequences on compact Kähler spaces, Publ RIMS Kyoto Univ., **23** (1987), 265–274.

[O-5] ———, Cheeger-Goresky-MacPherson's conjecture for the varieties with isolated singularities, Math. Z., **206** (1991), 219–224.

[O-6] ———, An extension of Hodge theory to Kähler spaces with isolated singularities of restricted type, Publ. RIMS Kyoto Univ., **24** (1988), 253–263.

[O-7] ———, Supplement to: "Hodge spectral sequence on compact Kähler spaces", Publ. RIMS Kyoto Univ., **27** (1991), 505–507.

[O-8] ———, On the infinite dimensionality of the middle L^2 cohomology of complex domains, Publ. RIMS Kyoto Univ., **25** (1989), 499–502.

[O-9] ———, On the L^2 cohomology group of isolated singularities, Adv. Stud. in Pure Math., **22**, Progress in Differential Geometry, 1993, 247–263.

[O-10] ———, On the removable singularity of differential forms on algebraic varieties, preprint.

[O-T] Ohsawa, T. and Takegoshi, K., Hodge spectral sequence on pseudoconvex domains, Math. Z., **197** (1988), 1–12.

[U] Ueda, T., Neighbourhoods of a rational curve with a node, Publ. RIMS Kyoto Univ., **27** (1991), 681–693.

[V-S] Van Straten, D. and Steenbrink, J., Extendability of holomorphic differential forms near isolated hypersurface singularities, Abh. Math. Sem. Univ. Hamburg, **55** (1985), 97–110.

Graduate School of Polymathematics
Nagoya University
Chikusa-ku, Nagoya 464-01
Japan

Advanced Studies in Pure Mathematics 25, 1997
CR-Geometry and Overdetermined Systems
pp. 285–317

The Fusion Matrix and the Verlinde Loop Operators in Conformal Field Theory

Duong H. Phong and Roberto Silvotti

This article is not available due to permission restrictions.

This article is not available due to permission restrictions.

This article is not available due to permission restrictions.

This article is not available due to permission restrictions.

This article is not available due to permission restrictions.

This article is not available due to permission restrictions.

This article is not available due to permission restrictions.

This article is not available due to permission restrictions.

This article is not available due to permission restrictions.

This article is not available due to permission restrictions.

This article is not available due to permission restrictions.

This article is not available due to permission restrictions.

This article is not available due to permission restrictions.

This article is not available due to permission restrictions.

This article is not available due to permission restrictions.

This article is not available due to permission restrictions.

This article is not available due to permission restrictions.

This article is not available due to permission restrictions.

This article is not available due to permission restrictions.

This article is not available due to permission restrictions.

This article is not available due to permission restrictions.

This article is not available due to permission restrictions.

This article is not available due to permission restrictions.

This article is not available due to permission restrictions.

This article is not available due to permission restrictions.

This article is not available due to permission restrictions.

This article is not available due to permission restrictions.

This article is not available due to permission restrictions.

This article is not available due to permission restrictions.

This article is not available due to permission restrictions.

This article is not available due to permission restrictions.

Advanced Studies in Pure Mathematics 25, 1997
CR-Geometry and Overdetermined Systems
pp. 318–354

On the Rigidity of Differential Systems modelled on Hermitian Symmetric Spaces and Disproofs of a Conjecture concerning Modular Interpretations of Configuration Spaces

Takeshi Sasaki, Keizo Yamaguchi and Masaaki Yoshida

Dedicated to Professor M. Kuranishi on his 70th birthday

Let $E(k,n;\alpha)$ be the hypergeometric system of differential equations of type (k,n) defined on the configuration space $X(k,n)$ of n hyperplanes in general position of the projective space $\mathbb{P}^{k-1}$, where α is a system of parameters:

$$\alpha = (\alpha_1, \dots, \alpha_n), \quad \alpha_1 + \cdots + \alpha_n = n - k.$$

The space $X(k,n)$ is an affine set of dimension

$$m = (n-k-1)(k-1),$$

and the rank (the dimension of the linear space of solutions at a generic point) of the system $E(k,n;\alpha)$ is

$$r = \binom{n-2}{k-1}.$$

A projective solution $\varphi : X(k,n) \longrightarrow \mathbb{P}^{r-1}$ is defined by $x \mapsto u_1(x) : \cdots : u_r(x)$, where the u_j's are linearly independent solutions of the system. Note that φ is multi-valued.

When $k = 2$, we have

$$r = m + 1;$$

so the dimension of the source space and that of the target space of the map φ agree.

Received October 23, 1995
Revised March 9, 1996

When $(k,n) = (3,6)$, we have

$$r = m + 2 \ (= 6);$$

so the image of φ is a hypersurface of $\mathbb{P}^5$.

These exhaust all the cases when the codimension of the image $\mathrm{Im}(\varphi)$ of the projective solution φ does not exceed 1.

Consider the following integral

$$u_\Delta(x) = \int_\Delta \prod_{j=1}^{n-1} l_j(x,t)^{\alpha_j - 1} dt_1 \wedge \cdots \wedge dt_{k-1},$$

where $l_j(x,t)$ are defining equations of the n hyperplanes (l_n is the hyperplane at infinity) of $\mathbb{P}^{k-1}$ representing $x \in X(k,n)$, and Δ is a real $(k-1)$-dimensional twisted cycle. If $\alpha_j \notin \mathbb{Z}$, there are r cycles Δ_ν such that the u_{Δ_ν}'s are linearly independent solutions.

Notice that when $n = 2k$, the most symmetric system of parameters is given by

$$\left\{\frac{1}{2}\right\} = \left(\frac{1}{2}, \ldots, \frac{1}{2}\right).$$

When $(k,n;\alpha) = (2,4;\{1/2\})$, the following facts are classical: The integrals above are elliptic integrals, i.e., periods of elliptic curves, the equation describes the family of elliptic curves (double covers of $\mathbb{P}^1 - \{4 \text{ points}\}$), the image $\mathrm{Im}(\varphi)$ of the projective solution φ is the upper half plane $H \subset \mathbb{P}^1$, and the map φ has a single-valued inverse so that we have the isomorphism

$$X(2,4) \cong H/\Gamma(2),$$

where $\Gamma(2) \subset SL(2,\mathbb{Z})$ is the principal congruence subgroup of level 2.

When $(k,n;\alpha) = (3,6;\{1/2\})$, the following is known ([MSY1]): The integrals above give periods of K3 surfaces (double covers of $\mathbb{P}^2 - \{6 \text{ lines}\}$), the equation describes a 4-dimensional family of such K3 surfaces, the image $\mathrm{Im}(\varphi)$ of the projective solution φ lies in a non-singular quadratic hypersurface Q of $\mathbb{P}^5$; indeed it is an open dense subset of the non-compact dual $D \subset Q$ of Q, and that φ has a single-valued inverse map so that we have an isomorphism

$$X(3,6) \cong (D - \{\text{fixed points of } \Gamma\})/\Gamma,$$

where Γ is an arithmetic subgroup of the group of automorphisms of D.

Since Q can be regarded as the Grassmannian variety $\mathrm{Gr}_{2,4}$, and since the Grassmannian $\mathrm{Gr}_{k-1,n-2}$ can be equivariantly and minimally embedded in $\mathbb{P}^{r-1}$, we are very happy if $\mathrm{Im}(\varphi)$ might lie in $\mathrm{Gr}_{k-1,n-2} \subset \mathbb{P}^{r-1}$.

Especially when $(k, n; \alpha) = (4, 8; \{1/2\})$, many mathematicians are expecting that $\mathrm{Im}(\varphi)$ would lie in $\mathrm{Gr}_{3,6} \subset \mathbb{P}^{20-1}$, and that we get a nice isomorphism like the examples above. Because the system describes a 9-dimensional family of Calabi-Yau 3-folds (double covers of $\mathbb{P}^3 - \{8 \text{ planes}\}$), it is a hot topic now. Notice that the integral above gives periods of such 3-folds.

We are very sorry to declare the following

Theorem 1. *If $k \geqq 3$, $n - k \geqq 3$ and $(k, n) \neq (3, 6)$, then the image $\mathrm{Im}(\varphi)$ of the projective solution of the system $E(k, n; \alpha)$ does not lie in $\mathrm{Gr}_{k-1,n-2} \subset \mathbb{P}^{r-1}$ for any α_j.*

The proof is given by showing that the system $E(k, n)$ is not equivalent to the system of differential equations defining the Plücker embedding of $\mathrm{Gr}_{k,1,n-2}$. The actual key to proving inequivalence is the computation of certain Lie algebra cohomology, which due to Se-ashi reduces the problem to the comparison of the symbols of both systems.

In Sections 1 and 2 we review the equivalence problem of differential systems and prove a general result on rigidity of differential systems modelled on equivariant projective embedding of the hermitian symmetric spaces (Corollary 3). The comparison of the symbols will be given in Section 3. In Section 4 we provide a much simpler proof of inequivalence valid for $E(4, 8)$.

Acknowledgment: When the first and the third authors were preparing the paper [MSY1], they dreamed about the story of $E(4, 8; \{1/2\})$ analogous to $E(3, 6; \{1/2\})$. It was disproved soon; they were disappointed and had no idea to publish this negative fact. After Professor Y. Se-ashi's unexpected death, his notes were completed by the second author, who pointed out that the conjecture could be disproved generally by following the line of the completed note. Meanwhile several mathematicians asked the third author whether the image of the projective solution of $E(4, 8; \{1/2\})$ is in $\mathrm{Gr}_{3,6}$, moreover some of them showed him (sketchy) proofs. So we decided to publish this negative result.

1. Projective embedding of hermitian symmetric spaces

As we explained in [MSY2], it is classically well known that a system R in m variables of rank r is nothing but an m-dimensional submanifold

M in $\mathbb{P}^{r-1}$; more precisely, two such systems are said to be equivalent if one is transformed into the other by a change of independent variables and by the replacement of the unknown by its product with a non-zero function and we have the bijective correspondence

$$\{\text{germs of systems in } m \text{ variables of rank } r\}/\text{equivalence}$$

$$\leftrightarrow \{\text{germs of } m\text{-dimensional submanifolds in } \mathbb{P}^{r-1}\} \;/\; PGL(r)$$

by associating to a system R the image M of its projective solution.

As for the system $E(3,6;\{1/2\})$, we checked in [MSY1] that the image of the projective solution lies in a non-singular quadratic hypersurface Q by utilizing the projective hypersurface theory in $\mathbb{P}^5$.

Our concern in this paper is the Grassmannian variety $\mathrm{Gr}_{k-1,n-2}$ in $\mathbb{P}^{r-1}$ embedded as the image of the Plücker embedding, on the lower side of the above correspondence. Hence, in this section, we would like to construct group-theoretically a system $R(k,n)$ in m variables of rank r, which corresponds to $\mathrm{Gr}_{k-1,n-2}$ in $\mathbb{P}^{r-1}$ in the above diagram, where $m = (n-k-1)(k-1)$ and $r = \binom{n-2}{k-1}$, and we discuss the inequivalence of $E(k,n)$ and $R(k,n)$ in §3 by virtue of Se-ashi's theory for the equivalence of integrable linear differential equations of finite type.

For this purpose and also as a motivation to introduce Se-ashi's theory in §2, which in fact enables us to construct $R(k,n)$ a little generally, we will consider here projective embedding of hermitian symmetric spaces.

Group-theoretically, a compact irreducible hermitian symmetric space M corresponds to a simple graded Lie algebra of the first kind as follows: Let $\mathfrak{l} = \mathfrak{l}_{-1} \oplus \mathfrak{l}_0 \oplus \mathfrak{l}_1$ be a simple graded Lie algebra of the first kind, i.e.,

(i) $\mathfrak{l}$ is a simple Lie algebra over $\mathbb{C}$.

(ii) $\mathfrak{l} = \mathfrak{l}_{-1} \oplus \mathfrak{l}_0 \oplus \mathfrak{l}_1$ is a vector space direct sum such that $\mathfrak{l}_{-1} \neq \{0\}$.

(iii) $[\mathfrak{l}_p, \mathfrak{l}_q] \subset \mathfrak{l}_{p+q}$, where $\mathfrak{l}_p = \{0\}$ for $|p| \geqq 2$.

Let L be the simply connected Lie group with Lie algebra $\mathfrak{l}$ and L' be the analytic subgroup of L with Lie algebra $\mathfrak{l}' = \mathfrak{l}_0 \oplus \mathfrak{l}_1$. Then $M = L/L'$ is a compact (irreducible) hermitian symmetric space and every compact irreducible hermitian symmetric space is obtained in this manner from a simple graded Lie algebra of the first kind. M is called the *model space* associated with $\mathfrak{l} = \mathfrak{l}_{-1} \oplus \mathfrak{l}_0 \oplus \mathfrak{l}_1$. For example, when $M = \mathrm{Gr}_{k-1,n-2}$, we have $\mathfrak{l} = \mathfrak{sl}(n-2, \mathbb{C})$ and the gradation $\mathfrak{l} = \mathfrak{l}_{-1} \oplus \mathfrak{l}_0 \oplus \mathfrak{l}_1$ is given by subdividing matrices as follows:

(1.1)
$$\mathfrak{l}_{-1} = \left\{ \begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix} \mid C \in M(p,i) \right\}, \ \mathfrak{l}_1 = \left\{ \begin{pmatrix} 0 & D \\ 0 & 0 \end{pmatrix} \mid D \in M(i,p) \right\},$$
$$\mathfrak{l}_0 = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \mid A \in M(i,i), \ B \in M(p,p) \text{ and } \mathrm{tr}A + \mathrm{tr}B = 0 \right\}.$$

where $i = k-1$, $p = n-k-1$ and $M(a,b)$ denotes the set of $a \times b$ matrices.

An equivariant projective embedding of the model space $M = L/L'$ can be obtained from an irreducible representation of L as follows: Let $\tau : L \to GL(T)$ be an irreducible representation of L with the highest weight Λ. Let t_Λ be a maximal vector in T of the highest weight Λ. Then a stabilizer of the line $[t_\Lambda]$ spanned by v_Λ in T is a parabolic subgroup of L. When this stabilizer coincides with L', we obtain an equivariant projective embedding of $M = L/L'$ by taking the L-orbit passing through $[t_\Lambda]$ in the projective space $P(T)$ consisting of all lines in T passing through the origin. For example, when $M = \mathrm{Gr}_{k-1,n-2}$, we take the exterior representation τ_0 of $L = SL(n-2, \mathbb{C})$ on $T = \bigwedge^{k-1} \mathbb{C}^{n-2}$:

$$\tau_0 : SL(n-2, \mathbb{C}) \to GL(\wedge^{k-1}\mathbb{C}^{n-2}),$$

where $\tau_0(a)(v_1 \wedge \cdots \wedge v_{k-1}) = a(v_1) \wedge \cdots \wedge a(v_{k-1})$ for $a \in SL(n-2, \mathbb{C})$ and $v_i \in \mathbb{C}^{n-2}$ $(i = 1, 2, \dots, n-1)$. Let $\{e_1, \dots, e_{n-2}\}$ be the natural basis of $\mathbb{C}^{n-2}$. Then τ_0 is an irreducible representation of $SL(n-2, \mathbb{C})$ with the maximal vector $e_1 \wedge \cdots \wedge e_{k-1}$ for a suitable choice of a Cartan subalgebra and a simple root system of $\mathfrak{sl}(n-2, \mathbb{C})$. From (1.1), we see that the stabilizer of the line $[e_1 \wedge \cdots \wedge e_{k-1}]$ coincides with L'. Thus we see that the Plücker embedding of $\mathrm{Gr}_{k-1,n-2}$ is obtained from the irreducible representation τ_0 of $SL(n-2, \mathbb{C})$.

Next, for an irreducible representation $\tau : L \to GL(T)$, we will construct a (positive) line bundle F over M such that the above orbit is obtained as an embedding of M by global sections of F. To construct F, let us take the dual representation $\rho : L \to GL(S)$ of τ, i.e., $S = T^*$ is the dual space of T and $\rho = \tau^*$ is defined by

$$\langle \rho(g)(\xi), t \rangle = \langle \xi, \tau(g^{-1})(t) \rangle,$$

for $g \in L, t \in T, \xi \in T^*$ and $\langle \ , \ \rangle$ is the canonical pairing between T^* and T. Then, when τ is an irreducible representation with the highest weight

Λ (for a fixed choice of a Cartan subalgebra and a simple root system of $\mathfrak{l}$), ρ is the irreducible representation with the lowest weight $-\Lambda$. Let us take a basis $\{t_1, \ldots, t_r\}$ of T consisting of weight vectors of τ such that $t_1 = t_\Lambda$. Then the dual basis $\{s_1, \ldots, s_r\}$ of $\{t_1, \ldots, t_r\}$ in $S = T^*$ consists of weight vectors of ρ and s_1 is a weight vector corresponding to $-\Lambda$. Let W and W' be the subspaces of S spanned by a vector s_1 and by vectors $s_2, \ldots, s_r$, respectively. Since L' is the stabilizer of the line $[t_1]$, W' is preserved by L'. Hence we get the representation ρ_W of L':

$$\rho_W : L' \to GL(W),$$

through the projection $\pi_0 : S = W \oplus W' \to W$.

Relative to the representation ρ_W, L' acts on $L \times W$ on the right by

$$(g, w)g' = (gg', \rho_W(g')^{-1}(w)),$$

for $g \in L, w \in W$ and $g' \in L'$. Then $F = L \times W/L'$ is the line bundle over $M = L/L'$.

As is well known, the space $\Gamma(F)$ of global sections of F is identified with the space $\mathcal{F}(L, W)_{L'}$ of all W-valued functions f on L satisfying

$$f(gg') = \rho_W(g')^{-1} f(g),$$

for $g \in L$ and $g' \in L'$, via the correspondence $f \in \mathcal{F}(L, W)_{L'} \mapsto \sigma_f \in \Gamma(F)$ given by

$$\sigma_f(\pi_1(g)) = \pi_2(g, f(g)),$$

where $\pi_1 : L \to M = L/L'$ and $\pi_2 : L \times W \to F$ denote the natural projections. Then each $s \in S$ defines an element $\sigma_s \in \Gamma(F)$ via the above correspondence by

$$f_s(g) = \pi_0(\rho(g^{-1})s)$$

for $g \in L$.

Now let us check that global sections of F give the desired embedding of M into $P(T)$. We utilize the above basis $\{t_1, \ldots, t_r\}$ and $\{s_1, \ldots, s_r\}$ of T and $S = T^*$. Let us consider a map $\hat{\varphi}$ of L into T defined by

$$\hat{\varphi}(g) = \sum_{i=1}^{r} \langle f_{s_i}(g), t_1 \rangle t_i \tag{1.2}$$

for $g \in L$. Then, from $\langle f_{s_i}(g), t_1 \rangle = \langle \rho(g^{-1}) s_i, t_1 \rangle$, $\hat{\varphi}$ induces a map φ

of M into $P(T)$ satisfying the commutative diagram

$$\begin{array}{ccc} L & \xrightarrow{\hat{\varphi}} & T\backslash\{0\} \\ \downarrow & & \downarrow \\ M = L/L' & \xrightarrow{\varphi} & P(T). \end{array}$$

For $g \in L$, if we represent $\tau(g)$ as a matrix A with respect to the basis $\{t_1, \dots, t_r\}$, $\rho(g^{-1})$ is represented by the transposed matrix ${}^tA$ of A with respect to the basis $\{s_1, \dots, s_r\}$.From (1.2), $\hat{\varphi}(g)$ corresponds to the first row vector of ${}^tA$. Hence we obtain

$$\hat{\varphi}(g) = \tau(g)(t_1).$$

Thus the image of φ coincides with the L-orbit passing through $[t_1]$ in $P(T)$.

Owing to Se-ashi's theory, which will be discussed in the next section, we can construct a system R_ρ of linear differential equations of rank r on F such that every local solution of R_ρ is a restriction of σ_s for some $s \in S$ as in the following: Let $J^p(F)$ be the bundle of p-jets of F. The fiber $J^p_x(F)$ of $J^p(F)$ over a point x of M is the quotient of the space of germs of sections of F at x by the subspace of germs which vanish to order $p+1$ at x. Let $\pi^p_q : J^p(F) \to J^q(F)$ denote the natural projection for $p > q$. At each point $x \in M = L/L'$, let $(R^\rho_p)_x$ be the subspace of $J^p_x(F)$ defined by

$$(R^\rho_p)_x = \{ j^p_x(\sigma_s) \mid s \in S \},$$

where $j^p_x(\sigma_s)$ is the p-jet at x of the section σ_s. Let R^ρ_p be the subbundle of $J^p(F)$ defined by

$$R^\rho_p = \bigcup_{x \in M} (R^\rho_p)_x.$$

Then there exists a natural number p_0 such that π^p_{p-1} induces a bundle isomorphism of R^ρ_p onto R^ρ_{p-1} for every $p \geqq p_0$ (for more detail, see §2.2). Putting $R^\rho = R^\rho_{p_0}$, we see that R^ρ has the desired property. In fact, R^ρ is the model equation for the typical symbol of type $(\mathfrak{l}, \rho)$ in Se-ashi's theory (see Proposition in §2.3).

We denote by $R(k, n)$ the system constructed as above from the exterior representation ρ_0 of $L = SL(n-2, \mathbb{C})$ on $S = \bigwedge^{n-k-1} \mathbb{C}^{n-2}$, which is dual to the representation τ_0. Then, from the construction, the projective solution of $R(k,n)$ coincides with the Plücker embedding of $M = \mathrm{Gr}_{k-1,n-2}$. Thus we obtain the system in m variables of rank r corresponding to $\mathrm{Gr}_{k-1,n-2}$ in $\mathbb{P}^{r-1}$ in the bijective correspondence given at the beginning of this section. We shall examine the symbol of $R(k,n)$ in detail and discuss the inequivalence of $E(k,n)$ and $R(k,n)$ in §3.

2. Se-ashi's Theorem

Se-ashi's theory on the equivalence of integrable linear differential equations of finite type deals with the special classes of equations characterized by their symbols, namely, with those equations having the typical symbol of type $(\mathfrak{l}, \rho)$, where ρ is an irreducible representation of a (semi-)simple graded Lie algebra $\mathfrak{l}$ of the first kind. We will briefly review his theory and also prove a theorem on the Lie algebra cohomology, which was left unpublished in his note. We will confine ourselves in the holomorphic category and take $\mathfrak{l}$ to be a simple Lie algebra over $\mathbb{C}$ in the following argument, although his theory applies also in the real C^∞ category and for semi-simple Lie algebras over $\mathbb{R}$.

2.1. Linear differential equations of finite type. Let us begin with recalling some generalities on jet bundles. Let M be a manifold of dimension m. We denote by T and T^* the tangent and the cotangent bundle of M respectively. For a vector bundle E over M, we denote by $J^p(E)$ the bundle of p-jets of E. The fibre of $J^p(E)$ over a point x of M is the quotient of the space of germs of sections of E at x by the subspace of germs which vanish to order $p+1$ at x. We identify $J^0(E)$ with E and put $J^{-1}(E) = M$ for convention. Let π^p_q denote the natural projection of $J^p(E)$ onto $J^q(E)$ for $p > q$. For a section s of E, its p-th jet at x is denoted by $j^p_x(s)$. There exist the natural vector bundle morphism $\varepsilon_p : S^pT^* \otimes E \to J^p(E)$ and the exact sequence

$$0 \longrightarrow S^pT^* \otimes E \xrightarrow{\varepsilon_p} J^p(E) \xrightarrow{\pi^p_{p-1}} J^{p-1}(E) \longrightarrow 0,$$

where S^pT^* denotes the p-th symmetric product of T^*.

A subbundle R of $J^p(E)$ is called a system of (homogeneous) linear differential equations of order p on E. A solution of R is a (local) section s of E satisfying $j^p_x(s) \in R_x$ at each $x \in M$. Let $R_r = \pi^p_r(R)$ be the image of the projection of R into $J^r(E)$ and put $\mathfrak{g}_r = R_r \cap (S^rT^* \otimes E)$ for $r \leqq p$, which is called the r-th *symbol* of R. We have an exact sequence

$$0 \longrightarrow \mathfrak{g}_r \xrightarrow{\varepsilon_r} R_r \xrightarrow{\pi^r_{r-1}} R_{r-1} \longrightarrow 0.$$

The direct sum $S_x = \bigoplus_{r=0}^{p}(\mathfrak{g}_r)_x$ is called the (total) symbol of R at $x \in M$, where $(\mathfrak{g}_r)_x \subset S^rT_x^* \otimes E_x$ denotes the fibre of $\mathfrak{g}_r$ over x.

A system R of order p is said to be of finite type if $\mathfrak{g}_p = 0$, i.e., if $\pi_{p-1}^p : R \to R_{p-1}$ is an isomorphism. A system R of finite type is said to be integrable if, for each $\eta \in R$, there is a (local) solution s for which $j_x^p(s) = \eta$, where $x = \pi_{-1}^p(\eta)$. In this case, such a solution s is uniquely determined by the initial condition $\eta \in R_x$. Thus, by a continuation of solutions along a curve $x_t, t \in [0,1]$ on M, we get a parallel displacement $\tau : R_{x_0} \to R_{x_1}$. Namely, for each $\eta_0 \in R_{x_0}$, we take a local solution s of R such that $j_{x_0}^p(s) = \eta_0$, continue this solution along x_t and put $\tau(\eta_0) = \eta_1 = j_{x_1}^p(s) \in R_{x_1}$. In this manner, we obtain a connection ∇ in the vector bundle R over M. Since the above parallel displacement is independent of curves joining x_0 and x_1 in a neighborhood of x_0, ∇ is a flat connection. In fact, ∇ is induced from the Spencer operator acting on $J^p(E)$ (Proposition 1.5.1 [S]).

Let E and E' be vector bundles over M. Let R and R' be systems of order p on E and E', respectively. Then a bundle isomorphism $\phi : E \to E'$ is called an isomorphism of R onto R' if $J^p(\phi)$ maps R onto R', where $J^p(\phi) : J^p(E) \to J^p(E')$ is the lift of ϕ. In this case we denote by $R^p(\phi)$ the restriction of $J^p(\phi)$ to R. Obviously, $R^p(\phi)$ is a vector bundle isomorphism of R onto R', which preserves the flat connections in R and R'.

2.2. Typical symbol of type $(\mathfrak{l}, \rho)$. Let R be a system of linear differential equations of order p on E and let $\mathfrak{g}_r$ be the r-th symbol of R for $r = 0, \dots, p$. We fix vector spaces V and W over $\mathbb{C}$ such that dim $V =$ dim M and dim $W = \mathrm{rank} E$, respectively. Let $S = \bigoplus_{r=0}^{p} S_r$ be a graded vector subspace of $\bigoplus_{r=0}^{p} S^rV^* \otimes W$. Then the system R is said to be of type S if, for each $x \in M$, there exist linear isomorphisms $z_T : V \cong T_x$ and $z_E : W \cong E_x$ such that the induced isomorphism $({}^tz_T^{-1}) \otimes z_E : S^rV^* \otimes W \cong S^rT_x^* \otimes E_x$ sends S_r onto $(\mathfrak{g}_r)_x$ for every r. In this case, S is called the *typical symbol* of R.

Now we introduce the important classes of typical symbols for integrable systems of linear differential equations of finite type in the following.

Let $\mathfrak{l} = \mathfrak{l}_{-1} \oplus \mathfrak{l}_0 \oplus \mathfrak{l}_1$ be a simple graded Lie algebra over $\mathbb{C}$ of the first kind and $\rho : \mathfrak{l} \to \mathfrak{gl}(S)$ an irreducible representation of $\mathfrak{l}$ on a vector space S.

As is well-known, there exists a unique element $Z \in \mathfrak{l}_0$ (Lemma

4.1.1. [S]) such that

$$\mathfrak{l}_p = \{ X \in \mathfrak{l} \mid [Z, X] = pX \} \qquad (p = -1, 0, 1).$$

Z is called the characteristic element of $\mathfrak{l} = \mathfrak{l}_{-1} \oplus \mathfrak{l}_0 \oplus \mathfrak{l}_1$. Since $ad(Z)$ is a semi-simple endomorphism with eigenvalues -1, 0 and 1, $\rho(Z)$ is a semi-simple endomorphism of S (Corollary 6.4 [Hu]) with real eigenvalues (see the arguments in §2.5). Moreover, putting $S_{(\mu)} = \{ s \in S \mid \rho(Z)(s) = \mu s \}$, we have

$$\rho(\mathfrak{l}_p) S_{(\mu)} \subset S_{(\mu+p)} \qquad \text{for} \quad p = -1, 0, 1.$$

Let λ_0 be the minimum eigenvalue of $\rho(Z)$ and put $S_r = S_{(\lambda_0 + r)}$ for $r \geqq 0$. Then, since ρ is irreducible, there exists a natural number p_0 (Proposition 4.2.1 [S]) such that $S_r \neq \{0\}$ for $r = 0, 1, \ldots, p_0 - 1$ and

$$S = \bigoplus_{r=0}^{p_0 - 1} S_r.$$

For each integer q $(0 \leqq q < p_0)$ put $S_q(q) = \{ s \in S_q \mid \rho(\mathfrak{l}_{-1})(s) = 0 \}$. Then $S_0(0) = S_0$ and $S_q(q)$ is a $\rho(\mathfrak{l}_0)$-invariant subspace of S_q. We define a linear subspace $S(q) = \bigoplus_{q \leqq r < p_0} S_r(q)$ of S inductively by

$$S_{r+1}(q) = \rho(\mathfrak{l}_1)(S_r(q)) \subset S_{r+1}.$$

One can easily check that $S_r(q)$ is $\rho(\mathfrak{l}_0)$-invariant and $\rho(\mathfrak{l}_{-1})(S_{r+1}(q)) \subset S_r(q)$ by induction on $r \geqq q$. Thus $S(q)$ is a $\rho(\mathfrak{l})$-submodule of S. Since ρ is irreducible, we get $S(0) = S$ and $S(q) = 0$ for $q > 0$. Hence, putting $S_r = \{0\}$ for $r \geqq p_0$, we obtain

$$S_0 = \{ s \in S \mid \rho(\mathfrak{l}_{-1})(s) = 0 \}, \tag{2.1}$$

and

$$S_{r+1} = \rho(\mathfrak{l}_1)(S_r) \qquad \text{for} \quad r \geqq 0. \tag{2.2}$$

Now we put $V = \mathfrak{l}_{-1}$ and $W = S_0$. Then we have a linear isomorphism ι_r of S_r into $S^r V^* \otimes W$ $(r = 1, \ldots, p_0 - 1)$ defined by

$$\iota_r(s)(X_1, \ldots, X_r) = (-1)^r \rho(X_1) \cdots \rho(X_r)(s).$$

Since $\mathfrak{l}_{-1}$ is abelian, ι_r is well-defined. In this manner, $S = \bigoplus_{r \geqq 0} S_r$ is regarded as a graded vector subspace of $\bigoplus_{r \geqq 0} S^r V^* \otimes W$, which is called the *typical symbol of type* $(\mathfrak{l}, \rho)$.

As an example, we construct the typical symbol of type $(\mathfrak{l}, \rho)$, when $\mathfrak{l} = \mathfrak{sl}(n-2, \mathbb{C})$ is endowed with the gradation given in (1.1) and $\rho = \rho_0$ is the exterior representation on $S = \bigwedge^{n-k-1} \mathbb{C}^{n-2}$:

$$\rho : \mathfrak{sl}(n-2, \mathbb{C}) \to \mathfrak{gl}(\bigwedge^{n-k-1} \mathbb{C}^{n-2}),$$

where

$$\rho(X)(v_1 \wedge \cdots \wedge v_{n-k-1}) = \sum_{i=1}^{n-k-1} v_1 \wedge \cdots \wedge X(v_i) \wedge \cdots \wedge v_{n-k-1}$$

for $X \in \mathfrak{sl}(n-2, \mathbb{C})$ and $v_i \in \mathbb{C}^{n-2}$ $(i = 1, 2, \ldots, n-k-1)$.

Let $\{e_1, \ldots, e_{n-2}\}$ be the natural basis of $\mathbb{C}^{n-2}$. Then $\mathfrak{l}' = \mathfrak{l}_0 \oplus \mathfrak{l}_1$ is the isotropy (stabilizer) algebra of the line $[e_1 \wedge \cdots \wedge e_{k-1}]$ in $\bigwedge^{k-1} \mathbb{C}^{n-2}$. We denote by $E_{ab} \in \mathfrak{gl}(n-2, \mathbb{C})$ $(1 \leqq a, b \leqq n-2)$ the matrix whose (a, b)-component is 1 and all of whose other components are 0. From (1.1), we have the following basis for $V = \mathfrak{l}_{-1}$ and $\mathfrak{l}_1$:

$$\begin{aligned} V = \mathfrak{l}_{-1} &= \langle E_{pi} \mid 1 \leqq i \leqq k-1,\ k \leqq p \leqq n-2 \rangle \\ \mathfrak{l}_1 &= \langle E_{ip} \mid 1 \leqq i \leqq k-1,\ k \leqq p \leqq n-2 \rangle \end{aligned}$$

Since $E_{pi}(e_j) = \delta_{ij} e_p$ for $1 \leqq j \leqq k-1$ and, $E_{pi}(e_q) = 0$ for $k \leqq q \leqq n-2$, we have from (2.1)

$$W = S_0 = \langle e_k \wedge \cdots \wedge e_{n-2} \rangle.$$

For $1 \leqq i_1 < \cdots < i_r \leqq k-1$ and $k \leqq p_1 < \cdots < p_r \leqq n-2$, we put

$$e(p_1, \ldots, p_r) = e_k \wedge \cdots \wedge \widehat{e}_{p_1} \wedge \cdots \wedge \widehat{e}_{p_r} \wedge \cdots \wedge e_{n-2} \in \bigwedge^{n-k-r-1} \mathbb{C}^{n-2},$$

and consider the following element of S:

$$s(i_1, \ldots, i_r, p_1, \ldots, p_r) = e_{i_1} \wedge \cdots \wedge e_{i_r} \wedge e(p_1, \ldots, p_r) \in S = \bigwedge^{n-k-1} \mathbb{C}^{n-2}.$$

Then, from (2.2) and $E_{ip}(e_j) = 0$, $E_{ip}(e_q) = \delta_{pq} e_i$ for $1 \leqq j \leqq k-1$, $k \leqq q \leqq n-2$, we get

$$\begin{aligned} S_r = \langle s(i_1, \ldots, i_r, p_1, \ldots, p_r) \mid \\ 1 \leqq i_1 < \cdots < i_r \leqq k-1, k \leqq p_1 < \cdots < p_r \leqq n-2 \rangle, \end{aligned}$$

for $r = 1, 2, \ldots, p_0 - 1$ and

$$S_r = \{0\},$$

for $r \geqq p_0 = \min\{k, n-k\}$. Moreover, for $X = \sum_{ip} X_{ip} E_{pi} \in V$, we have

$$\begin{aligned}
&\iota_r(s(i_1, \ldots, i_r, p_1, \ldots, p_r))(X, \ldots, X) \\
=&r!(-1)^r X(e_{i_1}) \wedge \cdots \wedge X(e_{i_r}) \wedge e(p_1, \ldots, p_r) \\
=&r!(-1)^r (\sum_\sigma \operatorname{sgn} \sigma \, X_{i_1 p_{\sigma(1)}} \cdots X_{i_r p_{\sigma(r)}}) \, e_{p_1} \wedge \cdots \wedge e_{p_r} \wedge e(p_1, \ldots, p_r).
\end{aligned}$$

Thus, by fixing a basis of W and identifying SV^* with the ring of polynomials on V, we see that $S_1 = V^*$ and $S_r \subset S^r V^*$ is spanned by the minor determinants of degree r of the matrix (X_{ip}), which are the linear coordinates of V.

2.3. Model systems. Starting from the typical symbol $S = \bigoplus_{r=0}^p S_r \subset \bigoplus_{r=0}^p S^r V^* \otimes W$ with the properties $S_0 = W$ and $S_p = 0$, we now explain a recipe to construct an integrable system of differential equations of finite type of order p modeled after S.

The construction of the model system R_S is preceded by the consideration of the Lie algebra $\mathfrak{g}$ of infinitesimal automorphisms of the constant coefficient differential equations modeled after S.

Let $E_0 = V \times W$ be the trivial bundle over the vector space V. Then the fibre $J_0^p(E_0)$ of $J^p(E_0)$ at the origin $0 \in V$ is identified with $\bigoplus_{r=0}^p S^r V^* \otimes W$, where $S^r V^* \otimes W$ can be regarded as the set of W-valued homogeneous polynomials of degree r on V. Thus, starting from the typical symbol $S = \bigoplus_{r=0}^p S_r \subset \bigoplus_{r=0}^p S^r V^* \otimes W$, our first (local) model is the constant coefficient differential equations given as the subbundle $\hat{R}_S = V \times S$ of $J^p(E_0)$, whose solutions consist of W-valued polynomials contained in $S \subset SV^* \otimes W$.

Let us consider an infinitesimal bundle automorphism of E_0 preserving $\hat{R}_S$. An infinitesimal bundle automorphism of E_0 has a form

$$\sum_i \xi^i(x) \frac{\partial}{\partial x^i} + \sum_{\alpha, \beta} A_{\alpha,\beta}(x) y^\beta \frac{\partial}{\partial y^\alpha},$$

where (x^i) and (y^α) are linear coordinates of V and W, respectively. Thus the Lie algebra $\mathfrak{a}$ of (formal) infinitesimal bundle automorphisms of E_0 can be expressed as a graded Lie algebra $\mathfrak{a} = \bigoplus_{r \geqq -1} \mathfrak{a}_r$ by putting

$$\mathfrak{a}_r = S^{r+1} V^* \otimes V \oplus S^r V^* \otimes \mathfrak{gl}(W),$$

where $\mathfrak{a}_{-1} = V$ corresponds to constant coefficient vector fields on V. The bracket operation in $\mathfrak{a}$ is given by

$$\begin{aligned}
[f \otimes v,\, g \otimes w\,] &= -f(i(v)g) \otimes w + g(i(w)f) \otimes v, \\
[f \otimes A, g \otimes w\,] &= g(i(w)f) \otimes A, \\
[f \otimes A, g \otimes B] &= fg \otimes [A, B],
\end{aligned}$$

where $f, g \in SV^*$, $v, w \in V$ and $A, B \in \mathfrak{gl}(W)$; $i(v)$ denotes the inner multiplication. The Lie algebra $\mathfrak{a}$ acts naturally on the space $SV^* \otimes W$ that is regarded as the space of cross sections of E_0:

$$(f \otimes v + g \otimes A)(h \otimes w) = -f(i(v)h) \otimes w + gh \otimes A(w),$$

where $f, g, h \in SV^*$, $v, w \in V$ and $A \in \mathfrak{gl}(W)$.

Then the Lie algebra $\mathfrak{g}$ of infinitesimal automorphisms of $\hat{R}_S$ is given by

$$\mathfrak{g} = \{\, X \in \mathfrak{a} \mid X(S) \subset S\,\}.$$

$\mathfrak{g}$ is a graded subalgebra of $\mathfrak{a} = \bigoplus_{r \geqq -1} \mathfrak{a}_r$, i.e., $\mathfrak{g} = \bigoplus_{r \geqq -1} \mathfrak{g}_r$, where $\mathfrak{g}_r = \mathfrak{g} \cap \mathfrak{a}_r$. The Lie algebra $\mathfrak{gl}(S)$ has also the gradation given by

$$\mathfrak{gl}(S)_r = \{\, X \in \mathfrak{gl}(S) \mid X(S_l) \subset S_{l+r} \quad \text{for any } l\,\}.$$

Referring the action above we have a restriction homomorphism: $\mathfrak{g} \to \mathfrak{gl}(S)$, which sends $\mathfrak{g}_r$ into $\mathfrak{gl}(S)_r$. Assume here the following two conditions for S, which are satisfied by the typical symbol of type $(\mathfrak{l}, \rho)$:

$(A1)$ The action of $\mathfrak{a}_{-1} = V$ leave S invariant.

$(A2)$ The action of $\mathfrak{a}_{-1} = V$ on S is faithful.

Then this homomorphism turns out to be injective and we can characterize $\mathfrak{g}_r$ as a subspace of $\mathfrak{gl}(S)_r$ as follows:

$$\begin{aligned}
&\mathfrak{g}_{-1} = V, \\
&\mathfrak{g}_r = \{\, X \in \mathfrak{gl}(S)_r \mid [\mathfrak{g}_{-1}, X] \subset \mathfrak{g}_{r-1}\,\} \qquad \text{for} \quad r \geqq 0.
\end{aligned} \tag{2.3}$$

Put $\mathfrak{u}_r = S^r V^* \otimes \mathfrak{gl}(W) \subset \mathfrak{a}_r$. Then $\mathfrak{u} = \bigoplus_{r \geqq 0} \mathfrak{u}_r$ is an ideal of $\mathfrak{a}$ and $\mathfrak{n} = \mathfrak{u} \cap \mathfrak{g}$ is an ideal of $\mathfrak{g}$. We can see

$$\mathfrak{n}_r = \{\, X \in \mathfrak{gl}(S)_r \mid [\mathfrak{g}_{-1}, X] \subset \mathfrak{n}_{r-1}\,\} \qquad \text{for} \quad r \geqq 0, \tag{2.4}$$

where we put $\mathfrak{n}_{-1} = \{0\}$ for convention.

In the case of the typical symbol of type $(\mathfrak{l}, \rho)$, we have the following: We identify $\mathfrak{l}$ with its image $\rho(\mathfrak{l})$ in $\mathfrak{gl}(S)$ as follows. Let $\mathfrak{c}$ denote the centralizer of $\mathfrak{l}$ in $\mathfrak{gl}(S)$ and $\mathfrak{g}^{\perp}$ the orthogonal complement of $\mathfrak{g}$ in $\mathfrak{gl}(S)$ with respect to the non-degenerate bilinear form Tr given by $\mathrm{Tr}(X, Y) = \text{trace } XY$ for $X, Y \in \mathfrak{gl}(S)$. Then, from (2.3) and (2.4), we have (Proposition 4.4.1 [S])

$$(2.5) \qquad \mathfrak{g} = \mathfrak{l} \oplus \mathfrak{c}, \quad \mathfrak{n} = \mathfrak{c}, \qquad \mathfrak{gl}(S) = \mathfrak{l} \oplus \mathfrak{n} \oplus \mathfrak{g}^{\perp} \quad (\text{Tr-orthogonal}).$$

In fact, since ρ is irreducible, $\mathfrak{c}$ coincides with the center of $\mathfrak{gl}(S)$ in our case.

Now let $S = \bigoplus_{r=0}^{p} S_r$ be a typical symbol satisfying $S_0 = W$, $S_p = 0$, and the above conditions $(A.1)$ and $(A.2)$. Then the model system R_S is constructed as follows: We filtrate the space S by subspaces $S^r = \bigoplus_{l=r}^{p} S_l$. Notice that the group $GL(V) \times GL(W)$ acts on $\mathfrak{a}$ by the adjoint action: for $a \in GL(V) \times GL(W)$ and $X \in \mathfrak{a}$, the action is $(aX)(s) = (a \cdot X \cdot a^{-1})(s)$ for $s \in S$. Let us define groups

$$G_0 = \{\, a \in GL(V) \times GL(W) \mid a(S) \subset S \,\},$$
$$GL^{(0)}(S) = \{\, g \in GL(S) \mid g(S^r) \subset S^r \quad \text{for any r} \,\}.$$

Let $\widetilde{G}$ be the analytic subgroup of $GL(S)$ with Lie algebra $\mathfrak{g} \in \mathfrak{gl}(S)$ and put

$$G = \widetilde{G} \cdot G_0,$$
$$G' = G \cap GL^{(0)}(S).$$

We see that the groups G_0 and G' are Lie subgroups of $GL(S)$ with Lie algebras $\mathfrak{g}_0$ and $\mathfrak{g}' = \bigoplus_{r \geqq 0} \mathfrak{g}_r$ respectively. Since G' preserves the filtration $\{S^r\}_{r \geqq 0}$ of S, we get the representation ρ_W of G':

$$\rho_W : G' \to GL(W),$$

through the projection $\pi_0 : S = \bigoplus_{r=0}^{p} S_r \to S_0 = W$.

Let E_S be the vector bundle over $M = G/G'$ associated with the representation $\rho_W : G' \to GL(W)$; G' acts on $G \times W$ on the right by

$$(g, w)g' = (gg', \rho_W(g')^{-1}(w)),$$

for $g \in G, w \in W$ and $g' \in G'$. Then E_S is the vector bundle over $M = G/G'$ defined by $E_S = G \times W/G'$. As in §1, each $s \in S$ defines an element $\sigma_s \in \Gamma(E_S)$ by considering the equivalence class of $(g, \rho_W(g^{-1})(s)) \in G \times W$.

At each point $x \in M = G/G'$, let $(R_S)_x$ be the subspace of $J^p_x(E_S)$ defined by

$$(R_S)_x = \{\, j^p_x(\sigma_s) \mid s \in S \,\}.$$

Let R_S be the subbundle of $J^p(E_S)$ defined by

$$R_S = \bigcup_{x \in M} (R_S)_x.$$

Then we have

Proposition. (Proposition 2.4.1 [S].) *R_S is an integrable system of linear differential equations of finite type of order p of type S and every local solution of R_S is a restriction of σ_s for some $s \in S$.*

We call R_S the system of equations *modeled after S*. In the case when S is the typical symbol of type $(\mathfrak{l}, \rho)$, it follows from (2.5) that $G/G' = L/L'$. Moreover, when ρ is the irreducible representation of $\mathfrak{l}$ given in §1, we see that R^ρ coincides with the system of equations modeled after S.

2.4. Normal reduction. Let R be an integrable system of linear differential equations of finite type of order p of type S on E. Then R is a vector bundle over the base manifold M with typical fibre S. A frame z of R at $x \in M$ is a linear isomorphism of S onto R_x. Let $F(R)$ be the frame bundle of R:

$$F(R) = \bigcup_{x \in M} F_x(R),$$

where $F_x(R)$ denotes the set of all frames of R at $x \in M$. $F(R)$ is a principal $GL(S)$-bundle over M. The flat connection ∇ in R induces the connection and the connection form $\tilde{\omega}$ on $F(R)$ is a $\mathfrak{gl}(S)$-valued 1-form. Se-ashi's theorem (Theorem A below) asserts the existence of a good reduction of the pair $(F(R), \tilde{\omega})$ for a system R with the typical symbol of type $(\mathfrak{l}, \rho)$. This reduction is carried out in several steps.

First, let $\{S^r\}_{r \geqq 0}$ be the filtration of S. The associated graded vector space $\mathrm{gr}(S) = \bigoplus_{r \geqq 0} S^r/S^{r+1}$ can be naturally identified with $S = \bigoplus_{r \geqq 0} S_r$. Let $GL^{(0)}(S)$ denote the subgroup of $GL(S)$ consisting of all elements $a \in GL(S)$ which preserve the filtration $\{S^r\}_{r \geqq 0}$ of S. For $a \in GL^{(0)}(S)$, we denote by $\mathrm{gr}(a) \in GL(S)$ the induced automorphism of the graded vector space $S = \bigoplus_{r=0}^{p} S_r$. Define

$$G^{(0)} = \{\, a \in GL^{(0)}(S) \mid \mathrm{gr}(a) \in G_0 \,\}.$$

The Lie algebra of $G^{(0)}$ is given by $\mathfrak{g}^{(0)} = \mathfrak{g}_0 \oplus \bigoplus_{r=1}^{p-1} \mathfrak{gl}(S)_r$. Then we have the natural reduction of the structure group $GL(S)$ of $F(R)$ to $G^{(0)}$ as follows : At each $x \in M$, R_x has a filtration $\{R_x^r\}_{r \geqq 0}$ given by

$$R_x^r = \text{Ker } (\pi_{r-1}^p : R_x \to J_x^{r-1}(E))$$

Put

$$\widehat{P}_x(R) = \{ z \in F_x(R) \mid z(S^r) \subset R_x^r \quad \text{for any } r \}.$$

Obviously, $\widehat{P}(R) = \bigcup_{x \in M} \widehat{P}_x(R)$ is a principal

$GL^{(0)}(S)$-subbundle of $F(R)$. Since $\mathfrak{g}_r = R_r \cap (S^k T^* \otimes E)$ denotes the r-th symbol of R, each frame $z \in \widehat{P}_x(R)$ induces a graded map $\text{gr}(z) : S_r \to (\mathfrak{g}_r)_x$. We put

$$\begin{aligned} P_x(R) = \{ z \in \widehat{P}_x(R) \mid \\ &\text{gr}(z) \text{ is the extension of isomorphisms } V \cong T_x \text{ and } W \cong E_x \}. \end{aligned}$$

Then $P(R) = \bigcup_{x \in M} P_x(R)$ is a principal $G^{(0)}$-subbundle of $F(R)$. Let $\pi : P(R) \to M$ be the bundle projection and let ω be the restriction to $P(R)$ of the connection form $\tilde{\omega}$ on $F(R)$. According to the decomposition $\mathfrak{gl}(S) = \bigoplus_{r=-p+1}^{p-1} \mathfrak{gl}(S)_r$, the form ω is decomposed as

$$\omega = \sum_r \omega_r.$$

It has the following properties (Proposition 3.2.2 [S]):

$$(2.6) \quad \begin{cases} (1) \quad d\omega + \dfrac{1}{2}\omega \wedge \omega = 0, \\ (2) \quad \omega_r = 0 \quad \text{for } r \leqq -2, \\ (3) \quad \omega_{-1} \text{ is a } \mathfrak{g}_{-1}\text{-valued basic form, that is, } \quad \omega_{-1} \text{ gives the} \\ \qquad \text{isomorphism } T_z(P(R))/\text{ Ker } \pi \cong \mathfrak{g}_{-1} \text{ at each } z \in P(R). \end{cases}$$

The pair $(P(R), \omega)$ characterizes the equivalence class of the system R (Proposition 3.3.1 [S]). Namely, let R and R' be integrable systems of type S. Then an isomorphism ϕ of R onto R' induces the bundle isomorphism $P(\phi) : (P(R), \omega) \to (P(R'), \omega')$, i.e., $P(\phi)$ is a bundle isomorphism of $P(R)$ onto $P(R')$ satisfying $P(\phi)^*\omega' = \omega$. Conversely, for any isomorphism $\Psi : (P(R), \omega) \to (P(R'), \omega')$, there exists a unique isomorphism ϕ of R onto R' such that $\Psi = P(\phi)$.

Second, in order to state the normality condition for G'-reduction of $P(R)$, we prepare the Spencer cohomology associated with the adjoint representation of $\mathfrak{l}_{-1}$ on $\mathfrak{gl}(S)$.

On the space $C = \bigoplus C^{p,q}$ of cochains

$$C^{p,q} = \bigwedge^{q}(\mathfrak{l}_{-1})^* \otimes \mathfrak{gl}(S)_{p-1},$$

we define the coboundary operator $\partial : C^{p,q} \to C^{p-1,q+1}$ by

$$\partial c(X_0, \ldots, X_q) = \sum_j (-1)^j [\rho(X_j), c(X_o, \ldots, \hat{X}_j, \ldots, X_q)].$$

The cohomology group $H^q(\mathfrak{l}_{-1}, \mathfrak{gl}(S)) = \bigoplus_p H^{p,q}(\mathfrak{l}_{-1}, \mathfrak{gl}(S))$ of this cochain complex (C, ∂) is called the Spencer cohomology group associated with the adjoint representation of $\mathfrak{l}_{-1}$ on $\mathfrak{gl}(S)$. Moreover, the adjoint operator $\partial^* : C^{p-1,q+1} \to C^{p,q}$ is given by

$$\partial^* c(X_1, \ldots, X_q) = \sum_i [\rho(E^i), c(E_i, X_1, \ldots, X_q)],$$

where $\{E_i\}$ is a basis of $\mathfrak{l}_{-1}$ and $\{E^i\}$ is the dual basis of $\mathfrak{l}_1$ relative to the Killing form B. Let τ be the complex conjugation relative to a compact real form of $\mathfrak{l}$ such that $\tau(\mathfrak{l}_1) = \mathfrak{l}_{-1}$ and $\tau(\mathfrak{l}_0) = \mathfrak{l}_0$. We have a (hermitian) inner product given by $\{X, Y\} = -B(X, \tau(Y))$. Moreover, since $\mathfrak{l}$ is simple, we can find an inner product $\langle , \rangle$ on S such that $\langle \rho(X)(s), s' \rangle + \langle s, \rho(\tau(X))(s') \rangle = 0$ for $s, s' \in S$ and $X \in \mathfrak{l}$. Then we define the inner product $(,)$ on $\mathfrak{gl}(S)$ by $(u, v) =$trace (uv^*), where $u, v \in \mathfrak{gl}(S)$ and v^* is the adjoint of v relative to $\langle , \rangle$. These inner products induce naturally an inner product on $C^{p,q}$. Then, relative to this inner product, ∂^* is seen to be the adjoint of ∂. Thus we can develop a harmonic theory for (C, ∂), using the laplacian $\Delta = \partial\partial^* + \partial^*\partial$. In fact, we will apply the harmonic theory of Kostant to compute $H^{p,1}(\mathfrak{l}_{-1}, \mathfrak{g}^\perp)$ in §2.5. We denote by $\mathcal{H}$ the harmonic projection. For $\mathfrak{l}$-submodule $\mathfrak{g}^\perp$ of $\mathfrak{gl}(S)$, we put $C(\mathfrak{g}^\perp) = \wedge(\mathfrak{l}_{-1})^* \otimes \mathfrak{g}^\perp$. Then $(C(\mathfrak{g}^\perp), \partial)$ is a subcomplex of (C, ∂).

Let $(Q(R), \chi)$ be a G'-reduction of $(P(R), \omega)$; i.e., $Q(R)$ is a G'-principal subbundle of $P(R)$ and χ is the restriction of ω to $Q(R)$. According to the decomposition $\mathfrak{gl}(S) = \mathfrak{g} \oplus \mathfrak{g}^\perp$, the form χ is decomposed as

$$\chi = \chi_{\mathfrak{g}} + \chi_{\mathfrak{g}^\perp}.$$

Since Tr is $Ad(G')$-invariant, we have $R_a^* \chi_{\mathfrak{g}} = \mathrm{Ad}(a^{-1}) \chi_{\mathfrak{g}}$ and $R_a^* \chi_{\mathfrak{g}^\perp} = \mathrm{Ad}(a^{-1}) \chi_{\mathfrak{g}^\perp}$ for any $a \in G'$. For $X \in \mathfrak{g}'$, $\chi_{\mathfrak{g}^\perp}(X^*) = 0$ since $\chi(X^*) = X$.

From (2) and (3) of (2.6), we have $(\chi_{\mathfrak{g}^\perp})_p = 0$ for $p \leqq -1$. Moreover, $\chi_{\mathfrak{g}}$ gives an isomorphism between $T_u(Q(R))$ and $\mathfrak{g}$ at each point $u \in Q(R)$. Namely, we have (Proposition 5.1.1 [S]) the following.

(1) $(Q(R), \chi_{\mathfrak{g}})$ is a Cartan connection of type G/G' over M.

(2) $\chi_{\mathfrak{g}^\perp}$ is a tensorial 1-form on $Q(R)$.

We now define a $C^1(\mathfrak{g}^\perp)(=\mathrm{Hom}(\mathfrak{l}_{-1}, \mathfrak{g}^\perp))$-valued function c on $Q(R)$ by

$$c(u)(X) = \chi_{\mathfrak{g}^\perp}(X_u^*) \qquad \text{for } u \in Q(R), X \in \mathfrak{l}_{-1}.$$

c is called the *structure function* on $Q(R)$. For each p, c^p denotes the $C^{p,1}(\mathfrak{g}^\perp)$-component of c, i.e., $c^p(u)(X) = (\chi_{\mathfrak{g}^\perp})_{p-1}(X_u^*)$. Then

$$c^p = 0 \qquad \text{for} \quad p \leqq 0. \tag{2.7}$$

We note here that, if c vanishes identically, we have $\chi = \chi_{\mathfrak{g}}$ and, from (1) of (2.6), $(Q(R), \chi)$ is a flat Cartan connection of type G/G'.

A G'-reduction $(Q(R), \chi)$ is said to be *normal* if the function c is ∂^*-closed. Now we can state Se-ashi's Theorem (Theorem 5.1.2, Theorem 5.2.2 [S]) as follows.

Theorem A. (1) *For every integrable system R of differential equations of type $(\mathfrak{l}, \rho)$, there exists a unique normal reduction $(Q(R), \chi)$ of $(P(R), \omega)$.*

(2) *Let R and R' be integrable systems of type $(\mathfrak{l}, \rho)$. Then an isomorphism ϕ of R onto R' induces the isomorphism $Q(\phi) : (Q(R), \chi) \to (Q(R'), \chi')$, i.e., $Q(\phi)$ is a bundle isomorphism of $Q(R)$ onto $Q(R')$ satisfying $Q(\phi)^*\chi' = \chi$. Conversely, for an isomorphism $\Psi : (Q(R), \chi) \to (Q(R'), \chi')$, there exists a unique isomorphism ϕ of R onto R' such that $\Psi = Q(\phi)$.*

(3) *If the structure function c vanishes identically, then R is locally isomorphic with the model system of type $(\mathfrak{l}, \rho)$. Furthermore, the harmonic part $\mathcal{H}c$ of c gives a fundamental system of invariants of R, i.e., c vanishes if and only if $\mathcal{H}c$ vanishes.*

2.5. Vanishing theorem on $H^1(\mathfrak{l}_{-1}, \mathfrak{g}^\perp)$. Let us recall some facts on simple graded Lie algebras $\mathfrak{l} = \mathfrak{l}_{-1} \oplus \mathfrak{l}_0 \oplus \mathfrak{l}_1$ of the first kind, following [Y], which are necessary in the subsequent discussion.

Let Z be the characteristic element of $\mathfrak{l} = \mathfrak{l}_{-1} \oplus \mathfrak{l}_0 \oplus \mathfrak{l}_1$. Since $\mathrm{ad}(Z)$ is a semi-simple endomorphism of $\mathfrak{l}$, we can take a Cartan subalgebra $\mathfrak{t}$

of $\mathfrak{l}$ containing Z. Let Φ be the set of roots of $\mathfrak{l}$ relative to $\mathfrak{t}$. Then we have the root space decomposition of $\mathfrak{l}$:

$$\mathfrak{l} = \mathfrak{t} \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{l}_\alpha,$$

where $\mathfrak{l}_\alpha = \{ X \in \mathfrak{l} \mid [H, X] = \alpha(H)X \quad \text{for all} \quad H \in \mathfrak{t}\}$ is the root space for $\alpha \in \Phi$. We have by definition $\alpha(Z) = -1, 0$ or 1 for any $\alpha \in \Phi$. Let us choose a simple root system $\Delta = \{\alpha_1, \ldots, \alpha_l\}$ of Φ such that $\alpha(Z) \geqq 0$ for all $\alpha \in \Delta$. Then there exists a unique simple root $\alpha_{i_0} \in \Delta$ such that $\alpha_{i_0}(Z) = 1$, $\alpha_i(Z) = 0$ for $i \neq i_0$ and the gradation is given by

$$\mathfrak{l}_0 = \mathfrak{t} \oplus \bigoplus_{\alpha\in\Phi_0^+} (\mathfrak{l}_\alpha \oplus \mathfrak{l}_{-\alpha}), \tag{2.8}$$

$$\mathfrak{l}_{-1} = \bigoplus_{\alpha\in\Phi_1^+} \mathfrak{l}_{-\alpha}, \ \mathfrak{l}_1 = \bigoplus_{\alpha\in\Phi_1^+} \mathfrak{l}_\alpha,$$

where $\Phi_p^+ = \{\alpha \in \Phi^+ \mid \alpha(Z) = p\}$ and Φ^+ is the set of positive roots. Because of the partition $\Phi^+ = \Phi_0^+ \cup \Phi_1^+$, we see that $n_{i_0}(\theta) = 1$ for the highest root $\theta = \sum_{i=1}^l n_i(\theta)\alpha_i$ and that

$$\Phi_p^+ = \{\alpha = \sum_{i=1}^{l} n_i(\alpha)\alpha_i \in \Phi^+ \mid n_{i_0}(\alpha) = p\} \qquad \text{for} \quad p = 0, 1. \tag{2.9}$$

Conversely, let $\mathfrak{l}$ be a simple Lie algebra over $\mathbb{C}$. Let us fix a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{l}$ and a simple root system $\Delta = \{\alpha_1, \ldots, \alpha_l\}$ of Φ. Choose a simple root α_{i_0} such that $n_{i_0}(\theta) = 1$ for the highest root $\theta = \sum_{i=1}^l n_i(\theta)\alpha_i$, and define the partition $\Phi^+ = \Phi_0^+ \cup \Phi_1^+$ by (2.9). Then we can construct the gradation of $\mathfrak{l}$ of the first kind by (2.8), i.e., by defining the characteristic element $Z \in \mathfrak{t}$ by

$$\alpha_i(Z) = \begin{cases} 1 & \text{if} \quad i = i_0, \\ 0 & \text{if} \quad i \neq i_0. \end{cases} \tag{2.10}$$

We denote the simple graded Lie algebra $\mathfrak{l} = \mathfrak{l}_{-1} \oplus \mathfrak{l}_0 \oplus \mathfrak{l}_1$ obtained in this manner by $(X_l, \{\alpha_{i_0}\})$, when $\mathfrak{l}$ is a simple Lie algebra of type X_l. Here X_l stands for the Dynkin diagram of $\mathfrak{l}$ representing Δ and α_{i_0} is a vertex of X_l with the coefficient 1 for the highest root. It is known [Y, §3] that simple graded Lie algebras of the first kind are completely classified by the diagram automorphism of $(X_l, \{\alpha_{i_0}\})$. For example, the gradation of $\mathfrak{l} = \mathfrak{sl}(n-2, \mathbb{C})$ given in (1.1) corresponds to $(A_{n-3}, \{\alpha_{k-1}\})$. We refer the reader to [Y, §4.4] for the detail.

Let $\tau : \mathfrak{l} \to \mathfrak{gl}(T)$ be an irreducible representation with the highest weight Λ. Let t_Λ be a maximal vector in T of the highest weight Λ. Then an isotropy algebra at $[t_\Lambda] \in P(T)$ coincides with $\mathfrak{l}' = \mathfrak{l}_0 \oplus \mathfrak{l}_1$ if and only if $(\Lambda, \alpha_{i_0}) \neq 0$ and $(\Lambda, \alpha_i) = 0$ for simple roots α_i other than α_{i_0}, where $(\,,\,)$ denotes the inner product in $(\mathfrak{t}_{\mathbb{R}})^*$.

Let $\rho : \mathfrak{l} \to \mathfrak{gl}(S)$ be the dual representation of τ ; i.e., $S = T^*$ is the dual space of T and $\rho = \tau^*$ is defined by

$$\langle \rho(X)(\xi), t\rangle + \langle \xi, \tau(X)(t)\rangle = 0,$$

for $X \in \mathfrak{l}, t \in T, \xi \in T^*$ and $\langle\ ,\ \rangle$ is the canonical pairing between T^* and T. Then ρ is an irreducible representation with the lowest weight $\Gamma = -\Lambda$. Hence the minimum eigenvalue λ_0 of $\rho(Z)$ is given by $\lambda_0 = \Gamma(Z)$. From (2.10), we see that the eigenvalues of $\rho(Z)$ are of the form ; $\lambda_0, \lambda_0 + 1, \ldots, \lambda_0 + p_0 - 1 = \widehat{\Lambda}(Z)$, where $\widehat{\Lambda}$ is the highest weight of ρ. When $\mathfrak{l}' = \mathfrak{l}_0 \oplus \mathfrak{l}_1$ is the isotropy algebra at $[t_\Lambda]$, the λ_0-eigenspace of $\rho(Z)$ coincides with the weight space for Γ, i.e., $S_0 = \langle s_1\rangle$ in the notation of §1.

Given an irreducible representation $\rho : \mathfrak{l} \to \mathfrak{gl}(S)$ on S, consider the adjoint representation $\mathrm{ad} \circ \rho : \mathfrak{l} \to \mathfrak{gl}(\mathfrak{gl}(S))$ on $\mathfrak{gl}(S)$. Then, from $[\rho(Z), Y](s) = \rho(Z)Y(s) - rY(s)$ for $s \in S_r$, we have

$$Y(S_r) \subset S_{l+r} \quad \text{for all } r \quad \text{if and only if} \quad [\rho(Z), Y] = lY.$$

Thus $\rho(Z) \in \mathfrak{gl}(S)$ is the characteristic element of the gradation of $\mathfrak{gl}(S) = \bigoplus_r \mathfrak{gl}(S)_r$.

To state the theorem of Kostant, we prepare the notation for the Weyl group W of the root system Φ. For an element $\sigma \in W$, we put $\Phi^- = -\Phi^+$ and $\Phi_\sigma = \sigma(\Phi^-) \cap \Phi^+$. Then $\sigma(\delta) = \delta - \langle\Phi_\sigma\rangle$, where $\delta = \frac{1}{2}\sum_{\alpha\in\Phi^+}\alpha$ and $\langle\Phi_\sigma\rangle$ denotes the sum of all elements in Φ_σ. For a fixed $(X_l, \{\alpha_{i_0}\})$, we define the subset W^0 of W by putting

$$W^0 = \{\sigma \in W \mid \Phi_\sigma \subset \Phi_1^+\}.$$

Moreover, we put

$$W(q) = \{\sigma \in W \mid n(\sigma) = q\} \quad \text{and} \quad W^0(q) = W^0 \cap W(q),$$

where $n(\sigma)$ is the number of roots in Φ_σ. For an element $\sigma \in W^0(q)$, we put $x_{\Phi_\sigma} = x_{\beta_1} \wedge \cdots \wedge x_{\beta_q}$ where $\Phi_\sigma = \{\beta_1, \ldots, \beta_q\} \subset \Phi_1^+$ and x_{β_i} is a root vector for the root $\beta_i \in \Phi_1^+$.

The theorem due to Kostant that we utilize is the following.

Theorem B. (Proposition 10.1 [MM], Theorem (Kostant) [Y, §5.1].) *Let $\mathfrak{l} = \mathfrak{l}_{-1} \oplus \mathfrak{l}_0 \oplus \mathfrak{l}_1$ be a simple graded Lie algebra over $\mathbb{C}$ represented by $(X_l, \{\alpha_{i_0}\})$ as above. Let $\tau : \mathfrak{l} \to \mathfrak{gl}(T)$ be an irreducible representation of $\mathfrak{l}$ on T with the lowest weight Γ.*

Then the harmonic space $\mathcal{H}^q$ of the cochain complex $C^q = T \otimes \wedge^q(\mathfrak{l}_{-1})^*$ can be decomposed into the irreducible $\mathfrak{l}_0$-module as follows:

$$\mathcal{H}^q = \bigoplus_{\sigma \in W^0(q)} \mathcal{H}^{\xi_\sigma},$$

where $\mathcal{H}^{\xi_\sigma}$ is the irreducible $\mathfrak{l}_0$-module with the lowest weigh $\xi_\sigma = \sigma(\Gamma - \delta) + \delta = \sigma(\Gamma) + \langle \Phi_\sigma \rangle$ generated by the lowest weight vector

$$t_{\sigma(\Gamma)} \otimes x_{\Phi_\sigma},$$

where $t_{\sigma(\Gamma)}$ is a weight vector in T with weight $\sigma(\Gamma)$ and
$x_{\Phi_\sigma} = x_{\beta_1} \wedge \cdots \wedge x_{\beta_q} \in \bigwedge^q \mathfrak{l}_1 \cong \bigwedge^q(\mathfrak{l}_{-1})^*$.

We apply this theorem to our case when $q = 1$. In this case we have $W^0(1) = \{\sigma_{i_0}\}$, where $\sigma_{i_0} = \sigma_{\alpha_{i_0}}$ is the reflection corresponding to the simple root α_{i_0}. Hence $\mathcal{H}^1$ is an irreducible $\mathfrak{l}_0$-module with the lowest weight $\xi_{i_0} = \sigma_{i_0}(\Gamma) + \alpha_{i_0}$.

Now we show the following vanishing theorem for $H^{p,1}(\mathfrak{l}_{-1}, \mathfrak{g}^\perp)$.

Theorem 2. *Let $\mathfrak{l} = \mathfrak{l}_{-1} \oplus \mathfrak{l}_0 \oplus \mathfrak{l}_1$ be a simple graded Lie algebra over $\mathbb{C}$ and let $M = L/L'$ be the model space associated with $\mathfrak{l} = \mathfrak{l}_{-1} \oplus \mathfrak{l}_0 \oplus \mathfrak{l}_1$. Let $\rho : \mathfrak{l} \to \mathfrak{gl}(S)$ be an irreducible representation on S and $H^1(\mathfrak{l}_{-1}, \mathfrak{g}^\perp)$ be the first Lie algebra cohomology associated with the adjoint representation of $\mathfrak{l}_{-1}$ on $\mathfrak{g}^\perp$ induced from $\mathrm{ad} \circ \rho : \mathfrak{l}_{-1} \to \mathfrak{gl}(\mathfrak{gl}(S))$, where $\mathfrak{gl}(S) = \mathfrak{g} \oplus \mathfrak{g}^\perp$.*

Then, for each $\rho : \mathfrak{l} \to \mathfrak{gl}(S)$,

$$H^{p,1}(\mathfrak{l}_{-1}, \mathfrak{g}^\perp) = \{0\} \quad \textit{for all } p \geqq 1,$$

except when M is a projective space $\mathbb{P}^m$ or a hyperquadric Q^m.

Proof. The adjoint representation $\mathrm{ad} \circ \rho : \mathfrak{l} \to \mathfrak{gl}(\mathfrak{gl}(S))$ on $\mathfrak{gl}(S)$ is decomposable according to the decomposition

$$\mathfrak{gl}(S) = \mathfrak{g} \oplus \mathfrak{g}^\perp,$$

and the gradation $\mathfrak{gl}(S) = \oplus_r \mathfrak{gl}(S)_r$ coincides with the eigenspace decomposition of $\mathrm{ad} \circ \rho(Z)$. To utilize Theorem B, we further decompose

$\mathfrak{g}^{\perp}$ into direct sum of irreducible $\mathfrak{l}$-modules

$$\mathfrak{g}^{\perp} = \bigoplus m_{\Gamma} T_{\Gamma},$$

where T_{Γ} is an irreducible $\mathfrak{l}$-submodule with the lowest weight Γ. Then we have

$$H^1(\mathfrak{l}_{-1}, \mathfrak{g}^{\perp}) = \bigoplus m_{\Gamma} H^1(\mathfrak{l}_{-1}, T_{\Gamma}).$$

By Theorem B, the harmonic space $\mathcal{H}^1_{\Gamma}$ representing $H^1(\mathfrak{l}_{-1}, T_{\Gamma})$ is an irreducible $\mathfrak{l}_0$-module in $T_{\Gamma} \otimes \mathfrak{l}_1$ generated by

$$t_{\sigma_{i_0}(\Gamma)} \otimes x_{\alpha_{i_0}},$$

where $t_{\sigma_{i_0}(\Gamma)}$ is the weight vector with weight $\sigma_{i_0}(\Gamma)$ and $x_{\alpha_{i_0}}$ is a root vector for $\alpha_{i_0} \in \Phi^+_1$. Thus $\mathcal{H}^1_{\Gamma} \subset C^{p,1}(\mathfrak{g}^{\perp})$, if $t_{\sigma_{i_0}(\Gamma)} \in \mathfrak{gl}(S)_{p-1}$. Hence p is given by

$$p - 1 = \sigma_{i_0}(\Gamma)(Z).$$

Let us compute the integer $\sigma_{i_0}(\Gamma)(Z)$. For each $\alpha \in \mathfrak{t}^*$, we denote by t_α and h_α the elements of $\mathfrak{t}$ defined by

$$B(t_\alpha, h) = \alpha(h) \quad \text{for } h \in \mathfrak{t} \qquad \text{and} \qquad h_\alpha = \frac{2t_\alpha}{(\alpha, \alpha)},$$

where $(\alpha, \alpha) = B(t_\alpha, t_\alpha)$ and B is the Killing form of $\mathfrak{l}$. Moreover, we put

$$\langle \mu, \alpha \rangle = \frac{2(\mu, \alpha)}{(\alpha, \alpha)} = \mu(h_\alpha) \qquad \text{for } \mu \in \mathfrak{t}^*.$$

Thus, for the simple root system $\{\alpha_1, \dots, \alpha_l\}$ of Φ, $\{h_{\alpha_1}, \dots, h_{\alpha_l}\}$ forms a basis of $\mathfrak{t}$. With respect to this basis, we put

$$Z = \sum_{i=1}^{l} a_i h_{\alpha_i}.$$

Then we compute

$$\begin{aligned} \sigma_{i_0}(\Gamma)(Z) &= (\Gamma - \langle \Gamma, \alpha_{i_0} \rangle \alpha_{i_o})(Z) = \Gamma(Z) - \langle \Gamma, \alpha_{i_0} \rangle \\ &= (a_{i_0} - 1)\langle \Gamma, \alpha_{i_0} \rangle + \sum_{i \neq i_0} a_i \langle \Gamma, \alpha_i \rangle \end{aligned} \tag{2.11}$$

Since Γ is the lowest weight, we have $\langle \Gamma, \alpha_i \rangle \leqq 0$ for $i = 1, \dots, l$ and $\langle \Gamma, \alpha_j \rangle < 0$ for some j. Let us now check the sign of $(a_{i_0} - 1)$ and a_i.

From (2.10), we have

$$\alpha_i(Z) = \sum_{j=1}^{l} \langle \alpha_i, \alpha_j \rangle a_j = \begin{cases} 1 & \text{if} \quad i = i_0, \\ 0 & \text{if} \quad i \neq i_0. \end{cases}$$

Hence, we see that $(a_1, \ldots, a_l)$ coincides with the i_0-th column vector of the inverse matrix C^{-1} of the Cartan matrix $C = (\langle \alpha_i, \alpha_j \rangle)$ of $\mathfrak{l}$. It is a well-known fact that all entries of C^{-1} are positive numbers (see, e.g., Table 1 [Hu, p.69]). Moreover, if $a_{i_0} > 1$, we see, from (2.11), that $\sigma_{i_0}(\Gamma)(Z) < 0$ for every Γ, i.e., $p < 1$ for every $\mathcal{H}^1_\Gamma \subset C^{p,1}(\mathfrak{g}^\perp)$. Hence we get $H^{p,1}(\mathfrak{l}_{-1}, \mathfrak{g}^\perp) = \{0\}$ for all $p \geqq 1$ in this case. Thus our task is to list up those $(X_l, \{\alpha_{i_0}\})$ for which $a_{i_0} \leqq 1$. In fact, from Table 1 [Hu, p.69], we obtain the following list of $(X_l, \{\alpha_{i_0}\})$ for which $a_{i_0} \leqq 1$:

$$\begin{array}{llllll} (A_l, \{\alpha_1\}) & a_1 = \dfrac{l}{l+1} & (l \geqq 1), & (A_3, \{\alpha_2\}) & a_2 = 1 & \\ (B_l, \{\alpha_1\}) & a_1 = 1 & (\mathfrak{l} \geqq 2), & (D_l, \{\alpha_1\}) & a_1 = 1 & (\mathfrak{l} \geqq 4), \end{array}$$

Here we identify $(B_2, \{\alpha_1\}) \cong (C_2, \{\alpha_2\})$, $(D_4, \{\alpha_1\}) \cong (D_4, \{\alpha_3\}) \cong (D_4, \{\alpha_4\})$ and $(A_l, \{\alpha_1\}) \cong (A_l, \{\alpha_l\})$ by diagram automorphisms. One can easily check (cf. [Y, §4.4]) that, when $(X_l, \{\alpha_{i_0}\})$ coincides with one of the above list, the model space $M = L/L'$ corresponds to $\mathbb{P}^l$ $(l \geqq 1)$, $Q^4 = Gr_{2,4}$, Q^{2l-1} $(l \geqq 2)$ and $Q^{2(l-1)}$ $(l \geqq 4)$. This completes the proof of Theorem C.

Now, combining Theorem A (3), Theorem C and (2.7), we obtain

Corollary 3. *Let $\mathfrak{l} = \mathfrak{l}_{-1} \oplus \mathfrak{l}_0 \oplus \mathfrak{l}_1$ be a simple graded Lie algebra over $\mathbb{C}$ and let $M = L/L'$ be the model space associated with $\mathfrak{l} = \mathfrak{l}_{-1} \oplus \mathfrak{l}_0 \oplus \mathfrak{l}_1$. Let $\rho : \mathfrak{l} \to \mathfrak{gl}(S)$ be an irreducible representation of $\mathfrak{l}$. Then, except when $M = \mathbb{P}^m$ or Q^m, every integrable system R of differential equations of type $(\mathfrak{l}, \rho)$ is locally isomorphic with the model system R^ρ of type $(\mathfrak{l}, \rho)$.*

3. Proof of Theorem 1

In this section we will show the inequivalence of $E(k, n)$ and $R(k, n)$ for $(k, n) \neq (3, 6)$ and prove Theorem. Recall that $R(k, n)$ is the model system of type $(\mathfrak{l}, \rho_0)$, where $\mathfrak{l} = \mathfrak{sl}(n-s, \mathbb{C})$ with the gradation $\mathfrak{l} = \mathfrak{l}_{-1} \oplus \mathfrak{l}_0 \oplus \mathfrak{l}_1$ given by (1.1) and ρ_0 is the exterior representation of $\mathfrak{sl}(n-2, \mathbb{C})$ on $\bigwedge^{n-k-1} \mathbb{C}^{n-2}$. By the argument in §2.2 and §2.3, we see that $R(k, n)$ is an integrable system of order $p_0 = \min\{k, n-k\}$ over the model space $M = Gr_{k-1,n-2}$. Hence, by Corollary D, $R(k,n)$ is characterized solely by its symbol. Thus, to prove Theorem, we need only to show that $E(k,n)$ is not of type $(\mathfrak{l}, \rho_0)$ for $(k,n) \neq (3,6)$, i.e., the symbol of $E(k,n)$ at a generic point is not equivalent to the typical symbol of $R(k,n)$ discussed in §2.2 for $(k,n) \neq (3,6)$.

3.1. The symbol of the Plücker embedding

We recall the calculations in §2.2. Let us take the following basis for $V = \mathfrak{l}_{-1}$ and S_r,

$$\begin{aligned} V =& \mathfrak{l}_{-1} = \langle E_{pi} \mid 1 \leqq i \leqq k-1, k \leqq p \leqq n-2 \rangle, \\ S_r =& \langle s(i_1, \dots, i_r, p_1, \dots, p_r) \mid \\ & \quad 1 \leqq i_1 < \cdots < i_r \leqq k-1, k \leqq p_1 < \cdots < p_r \leqq n-2 \rangle, \end{aligned}$$

where

$$s(i_1, \dots, i_r, p_1, \dots, p_r) = e_{i_1} \wedge \cdots \wedge e_{i_r} \wedge e(p_1, \dots, p_r) \in S = \bigwedge^{n-k-1} \mathbb{C}^{n-2}.$$

Then we have

$$\begin{aligned} &\iota_r(s(i_1, \dots, i_r, p_1, \dots, p_r))(X, \dots, X) = \\ &r!(-1)^r (\sum_\sigma \operatorname{sgn} \sigma \, X_{i_1 p_{\sigma(1)}} \cdots X_{i_r p_{\sigma(r)}}) \, e_{p_1} \wedge \cdots \wedge e_{p_r} \wedge e(p_1, \dots, p_r), \end{aligned}$$

for $X = \sum_{ip} X_{ip} E_{pi} \in V$. Thus, by fixing a basis of $W = S_0$ and identifying SV^* with the ring of polynomials on V, we see that $S_1 = V^*$ and $S_r \subset S^r V^*$ is spanned by the minor determinants of degree r of the matrix (X_{ip}). By construction of $R(k,n)$,

$$S = \bigoplus_{r=0}^{p_0} S_r$$

is the typical symbol of $R(k,n)$. Hence, putting $R_r(k,n) = \pi^{p_0}_r(R(k,n))$, the symbol $\mathfrak{g}_r = R_r(k,n) \cap (S^r T^* \otimes E)$ of $R_r(k,n)$ is of type $S_r \subset S^r V^*$ at each point of $M = \mathrm{Gr}_{k-1,n-2}$.

Now let us first show that $R(k,n)$ is essentially a second order system. More precisely, we claim

$R(k,n)$ is the $(p_0 - 2)$-th prolongation of $R_2(k,n)$

Namely p_0-th order system $R(k,n)$ is obtained from the second order system $R_2(k,n)$ by adding successive (partial) derivatives to $R_2(k,n)$. In order to show this, since $\pi^r_{r-1} : R_r(k,n) \to R_{r-1}(k,n)$ is onto by construction, we need only to show that the symbol $\mathfrak{g}_r$ of $R_r(k,n)$ is the $(r-2)$-th prolongation of $\mathfrak{g}_2$. In fact we have

Lemma 3.1. *The space $S_r \subset S^r V^*$ is equal to the $(r-2)$-th prolongation $p^{(r-2)}(S_2)$ of $S_2 \subset S^2 V^*$.*

Here we recall that s-th (algebraic) prolongation $p^{(s)}(S_2)$ of S_2 is given by

$$p^{(s)}(S_2) = S_2 \otimes \otimes^s V^* \cap S^{s+2} V^*.$$

Proof. Let T_r be the annihilator of S_r in $S^r V$, where we identify $S^r V$ with the dual space of $S^r V^*$. Then T_2 is generated by the following vectors ;

$$\begin{array}{cl} E_{pi} \cdot E_{qj} + E_{qi} \cdot E_{pj} & (1 \leqq i < j \leqq k-1, k \leqq p < q \leqq n-2) \\ E_{pj} \cdot E_{qj} & (1 \leqq i \leqq k-1, k \leqq p < q \leqq n-2) \\ E_{qi} \cdot E_{qj} & (1 \leqq i < j \leqq k-1, k \leqq q \leqq n-2) \\ E_{qj}^2 & (1 \leqq j \leqq k-1, k \leqq q \leqq n-2) \end{array}$$

where $\cdot$ denotes the symmetric product. Let $T_2^{(s)}$ denote the annihilator of $p^{(s)}(S_2)$ in $S^{s+2}V$. Then we have

$$T_2^{(s)} = \langle f \cdot g \mid f \in S^s V \quad \text{and} \quad g \in T_2 \rangle.$$

Moreover, since S_{s+2} is generated by the minor determinants of degree $s+2$ of the matrix (X_{ip}), we have

$$T_2^{(s)} \subset T_{s+2}. \tag{3.1}$$

We observe here that each monomial $E_{p_1 i_1} \cdot E_{p_2 i_2} \cdot \cdots \cdot E_{p_{s+2} i_{s+2}}$ in $S^{s+2}V$ belongs to $T_2^{(s)}$ if there is a repetition among the indices $i_1, \ldots, i_{s+2}$ or $p_1, \ldots, p_{s+2}$. On the other hand, given indices $i_1, \ldots, i_{s+2}$ and $p_1, \ldots, p_{s+2}$ such that $1 \leqq i_1 < \cdots < i_{s+2} \leqq k-1$ and $k \leqq p_1 < \cdots < p_{s+2} \leqq n-2$, we see that $(s+2)!$ monomials

$$E_{p_1 i_{\sigma(1)}} \cdot E_{p_2 i_{\sigma(2)}} \cdot \cdots \cdot E_{p_{s+2} i_{\sigma(s+2)}},$$

where σ runs for all permutations of degree $s+2$, span (at most) 1-dimensional subspace modulo $T_2^{(s)}$. In fact, to see this, it is enough to line up all the permutations of degree $(s+2)$ in one row so that each permutation $(l_1, \ldots, l_{s+2})$, where $l_i = \sigma(i)$ $(i = 1, 2, \ldots, s+2)$, is obtained by a transposition from the former permutation in this row. Then the dimension count shows

$$\operatorname{codim} T_2^{(s)} \leqq \binom{k-1}{s+2} \times \binom{n-k-1}{s+2} = \dim S_{s+2},$$

which, together with (3.1), implies $T_2^{(s)} = T_{s+2}$. This completes the proof of Lemma.

In view of this lemma, we will discuss the inequivalence of second order systems $E(k,n)$ and $R_2(k,n)$ in §3.3. Here the symbol $\mathfrak{g}_2 = R_2(k,n) \cap (S^2T^* \otimes E)$ of $R_2(k,n)$ is of type $S_2 \subset S^2V^*$ at each point of $M = Gr_{k-1,n-2}$. Let $\{e_{ip}\}$ denote the dual basis of $\{E_{pi}\}$ in V^*. Then recall that $S_2 \subset S^2V^*$ is generated by the following elements of S^2V^* ;

$$S_{ijpq} = e_{ip} \cdot e_{jq} - e_{iq} \cdot e_{jp}, \qquad (1 \leqq i < j \leqq k-1, k \leqq p < q \leqq n-2).$$

3.2. The symbol of $E(k,n)$

For a set of parameters

$$\alpha = (\alpha_1, \ldots, \alpha_n), \quad \sum_{j=1}^{n} \alpha_j = n-k,$$

the hypergeometric system of type (k,n) is the system of linear differential equations:

$$\begin{aligned} &\sum_{j=1}^{n} x_{lj} \frac{\partial u}{\partial x_{ij}} + \delta_{il} u = 0, \\ &\sum_{i=1}^{k} x_{ij} \frac{\partial u}{\partial x_{ij}} - (\alpha_j - 1) u = 0, \\ &\frac{\partial^2 u}{\partial x_{ip} \partial x_{jq}} - \frac{\partial^2 u}{\partial x_{iq} \partial x_{jp}} = 0, \end{aligned}$$

where

$$(x_{ij}) \in M^*(k,n) = \{k \times n\text{-matrices such that no } k\text{-minor vanishes}\}.$$

The configuration space $X(k,n)$ of distinct n points on the projective $(k-1)$-space is by definition given as

$$X(k,n) = GL(k) \backslash M^*(k,n) / H(n),$$

where $H(n)$ is the group consisting of diagonal non-singular n-matrices. Though the above system is not defined on $X(k,n)$, its projective solutions are defined on it. So instead of transforming the system into a $GL(k)\times H(n)$-invariant form, we restrict this system to the "subset" of $M^*(k,n)$ defined as follows:

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 \\ 0 & 1 & \cdots & 0 & 1 & x_{2\ k+2} & \cdots & x_{2n} \\ \vdots & & \ddots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 & 1 & x_{k\ k+2} & \cdots & x_{kn} \end{pmatrix}.$$

Note that any element of $M^*(k,n)$ can be taken to this form by $GL(k)\times H(n)$, in other words, this is a section of the projection $M^*(k,n)\to X(k,n)$. So in the following, we identify this subset with $X(k,n)$, i.e., we regard $(x_{ip})\in X(k,n)$.

The restricted system $E(k,n)=E(k,n;\alpha_1,\dots,\alpha_n)$ consists of the following differential equations relative to the variables x_{ip}, $2\leqq i\leqq k$, $k+2\leqq p\leqq n$.

$$\begin{aligned}(\alpha-1+\theta)\theta_{jq}u &= x_{jq}(\theta^q-\alpha_q+1)(\theta_j+\alpha_j)u,\\ x_{jp}(\theta^p-\alpha_p+1)\theta_{jq}u &= x_{jq}(\theta^q-\alpha_q+1)\theta_{jp}u,\\ x_{iq}(\theta_i+\alpha_i)\theta_{jq}u &= x_{jq}(\theta_j+\alpha_j)\theta_{iq}u,\\ x_{iq}x_{jp}\theta_{ip}\theta_{jq}u &= x_{ip}x_{jq}\theta_{iq}\theta_{jp}u,\end{aligned}\tag{3.2}$$

where

$$\theta_{ip}=x_{ip}\frac{\partial}{\partial x_{ip}},\quad \theta_i=\sum_{p=k+2}^{n}\theta_{ip},\quad \theta^p=\sum_{i=2}^{k}\theta_{ip},\quad \theta=\sum_{i=2}^{k}\sum_{p=k+2}^{n}\theta_{ip}.$$

and

$$\alpha=\alpha_2+\cdots+\alpha_{k+1}.$$

Refer to [MSY1]. Here and in the following, the indices i and j run from 2 to k, and the indices p and q from $k+2$ to n.

Now let us calculate the symbol of $E(k,n)$. In the spirit of §2, we regard $E(k,n)$ as the subbundle of $J^2(E)$ defined by (3.2), where $E=\mathbb{C}\times X(k,n)$ is the trivial line bundle over the configuration space $X(k,n)$. Let $S_2(x)=E(k,n)\cap(S^2T_x^*\otimes\mathbb{C})$ be the symbol of $E(k,n)$ at $x=(x_{ip})\in X(k,n)$. We regard $S_2(x)$ as a subspace of $S^2T_x^*$. Then, from (3.2), we see that the annihilator $T_2(x)$ of $S_2(x)$ in S^2T_x is generated by the following elements:

$$\begin{aligned}
A_{jq} &= \sum_{i,p}(x_{ip}\xi_{ip}x_{jq}\xi_{jq} - x_{jq}x_{iq}\xi_{iq}x_{jp}\xi_{jp}),\\
B_{jpq} &= x_{jp}\sum_i x_{ip}\xi_{ip}x_{jq}\xi_{jq} - x_{jq}\sum_i x_{iq}\xi_{iq}x_{jp}\xi_{jp},\\
C_{ijq} &= x_{iq}\sum_p x_{ip}\xi_{ip}x_{jq}\xi_{jq} - x_{jq}\sum_p x_{iq}\xi_{iq}x_{jp}\xi_{jp},\\
D_{ijpq} &= x_{iq}x_{jp}x_{ip}\xi_{ip}x_{jq}\xi_{jq} - x_{ip}x_{jq}x_{iq}\xi_{iq}x_{jp}\xi_{jp}.
\end{aligned}$$

where we put $\xi_{ip} = \frac{\partial}{\partial x_{ip}}$, and $\{\xi_{ip}\}$ forms a basis of T_x. Since

$$\begin{aligned}
B_{jpq} &= x_{jp}x_{jq}((\sum_i x_{ip}\xi_{ip})\xi_{jq} - (\sum_i x_{iq}\xi_{iq})\xi_{jp}),\\
C_{ijq} &= x_{iq}x_{jq}((\sum_p x_{ip}\xi_{ip})\xi_{jq} - (\sum_p x_{jp}\xi_{jp})\xi_{iq}),\\
D_{ijpq} &= x_{ip}x_{iq}x_{jp}x_{jq}(\xi_{ip}\xi_{jq} - \xi_{iq}\xi_{jp}),
\end{aligned}$$

and

$$A_{jq} \equiv x_{jq}(\sum_p x_{jp}\xi_{jp})(\sum_i(1-x_{iq})\xi_{iq}) \quad \text{modulo} \quad D_{ijpq},$$

$T_2(x)$ is generated by

$$\begin{aligned}
A'_{jq} &= \eta_j\eta^q,\\
B'_{jpq} &= \eta^p\xi_{jq} - \eta^q\xi_{jp},\\
C'_{ijq} &= \eta_i\xi_{jq} - \eta_j\xi_{iq},\\
D'_{ijpq} &= \xi_{ip}\xi_{jq} - \xi_{iq}\xi_{jp},
\end{aligned}$$

where

$$\eta_j = \sum_p x_{jp}\xi_{jp}, \quad \eta^q = \sum_i(1-x_{iq})\xi_{iq}.$$

Furthermore, the first three are equal to the following, respectively, modulo the generator D'_{ijpq}.

$$\hat{A}_{jq} = (\sum_{i,p}(x_{ip} - x_{iq}x_{jp})\xi_{ip})\xi_{jq},$$
$$\hat{B}_{jpq} = (\sum_{i}(x_{iq} - x_{ip})\xi_{ip})\xi_{jq},$$
$$\hat{C}_{ijq} = (\sum_{p}(x_{ip} - x_{jp})\xi_{ip})\xi_{jq}.$$

Let us now compute the generators of $S_2(x)$. We denote by $\{e_{ip}\}$ the dual basis of $\{\xi_{ip}\}$. Since any elements of $S_2(x)$ are annihilated by above elements of $T_2(x)$, we look for the elements of the form

$$E_{ijpq} = e_{ip} \cdot e_{jq} + e_{iq} \cdot e_{jp} + \sum_{\ell<m,s} P^{ijpq}_{\ell ms} e_{\ell s} \cdot e_{ms} + \sum_{m,r<s} Q^{ijpq}_{mrs} e_{mr} \cdot e_{ms} + \sum_{m,s} R^{ijpq}_{ms} {e_{ms}}^2.$$

Obviously, this satisfies $D'_{\ell mrs}(E_{ijpq}) = 0$. By requiring E_{ijpq} to be annihilated by $\hat{C}_{\ell ms}$ and by $\hat{B}_{mrs}$, we can determine the coefficients P's and Q's as follows:

$$E_{ijpq} = e_{ip} \cdot e_{jq} + e_{iq} \cdot e_{jp} - \frac{x_{iq} - x_{jq}}{x_{ip} - x_{jp}} e_{ip} \cdot e_{jp} - \frac{x_{ip} - x_{jp}}{x_{iq} - x_{jq}} e_{iq} \cdot e_{jq} - \frac{x_{jq} - x_{jp}}{x_{iq} - x_{ip}} e_{ip} \cdot e_{iq} - \frac{x_{iq} - x_{ip}}{x_{jq} - x_{jp}} e_{jp} \cdot e_{jq} + \sum_{m,s} R^{ijpq}_{ms} {e_{ms}}^2.$$

The condition $\hat{A}_{ms}(E_{ijpq}) = 0$ is a little complicated; a calculation shows

$$\begin{aligned}
R^{ijpq}_{ip} &= -\frac{x_{jq} - x_{jp}x_{iq}}{(1 - x_{ip})x_{ip}} + \frac{x_{jp}}{x_{ip}} \frac{x_{iq} - x_{jq}}{x_{ip} - x_{jp}} + \frac{x_{iq}}{x_{ip}} \frac{x_{jq} - x_{jp}}{x_{iq} - x_{ip}}, \\
R^{ijpq}_{iq} &= -\frac{x_{jp} - x_{jq}x_{ip}}{(1 - x_{iq})x_{iq}} + \frac{x_{jq}}{x_{iq}} \frac{x_{ip} - x_{jp}}{x_{iq} - x_{jq}} + \frac{x_{ip}}{x_{iq}} \frac{x_{jq} - x_{jp}}{x_{iq} - x_{ip}}, \\
R^{ijpq}_{jp} &= -\frac{x_{iq} - x_{ip}x_{jq}}{(1 - x_{jp})x_{jp}} + \frac{x_{ip}}{x_{jp}} \frac{x_{iq} - x_{jq}}{x_{ip} - x_{jp}} + \frac{x_{jq}}{x_{jp}} \frac{x_{iq} - x_{ip}}{x_{jq} - x_{jp}}, \\
R^{ijpq}_{jq} &= -\frac{x_{ip} - x_{iq}x_{jp}}{(1 - x_{jq})x_{jq}} + \frac{x_{iq}}{x_{jq}} \frac{x_{ip} - x_{jp}}{x_{iq} - x_{jq}} + \frac{x_{jp}}{x_{jq}} \frac{x_{iq} - x_{ip}}{x_{jq} - x_{jp}}, \\
R^{ijpq}_{ms} &= 0 \quad \text{otherwise.}
\end{aligned}$$

We put

$$R_{ip} = R^{ijpq}_{ip};$$

then, we see that

(3.3)

$$\begin{aligned}E_{ijpq} &= e_{ip}\cdot e_{jq} + e_{iq}\cdot e_{jp} - \frac{x_{iq}-x_{jq}}{x_{ip}-x_{jp}} e_{ip}\cdot e_{jp} - \frac{x_{ip}-x_{jp}}{x_{iq}-x_{jq}} e_{iq}\cdot e_{jq} \\ &\quad - \frac{x_{jq}-x_{jp}}{x_{iq}-x_{ip}} e_{ip}\cdot e_{iq} - \frac{x_{iq}-x_{ip}}{x_{jq}-x_{jp}} e_{jp}\cdot e_{jq} \\ &\quad + R_{ip}{e_{ip}}^2 + R_{iq}{e_{iq}}^2 + R_{jp}{e_{jp}}^2 + R_{jq}{e_{jq}}^2.\end{aligned}$$

Here we note that E_{ijpq} is a quadratic polynomial in four variables e_{ip}, e_{iq}, e_{jp} and e_{jq}. Thus, the space $S_2(x)$ is generated by these elements E_{ijpq} $\quad(2 \leqq i < j \leqq k, k+2 \leqq p < q \leqq n)$.

In the following, we use R_{ip} written in the form

$$\begin{aligned}R_{ip} =& \frac{x_{iq}x_{jp} - x_{iq} - x_{jp} + x_{jq}}{x_{ip}} + \frac{x_{iq}x_{jp} - x_{jq}}{1 - x_{ip}} \\ &+ \frac{x_{iq}-x_{jq}}{x_{ip}-x_{jp}} + \frac{x_{jp}-x_{jq}}{x_{ip}-x_{iq}}.\end{aligned} \tag{3.4}$$

3.3. Proof

By summarizing the discussion in the above subsections, our task is now to show the inequivalence of the symbol spaces $S_2(x)$ and S_2 for a generic point x of $X(k,n)$. More precisely, we need to show that, for a generic point $x \in X(k,n)$, there does not exist a linear isomorphism ϕ of V onto T_x such that $\phi^* : S^2T_x^* \to S^2V^*$ sends $S_2(x)$ onto S_2. In other words our task is to show, for a generic point $x \in X(k,n)$, the projective inequivalence of the varieties $V(S_2(x))$ and $V(S_2)$, where $V(S_2(x))$ and $V(S_2)$ are varieties in the projective spaces PT_x^* and PV^*, which are defined by the quadratic generators of $S_2(x)$ and S_2, respectively.

Here we note that, since the generators of S_2 are minor determinants of degree 2 of the matrix (e_{ip}), $V(S_2)$ is called the *Segre variety* and coincides with the image of $\mathbb{P}^{k-2} \times \mathbb{P}^{n-k-2}$ under the *Segre embedding*. Especially, we see that $V(S_2)$ is a smooth projective variety of dimension $n-4$. Referring to this fact, we will check the above inequivalence by looking at the most primitive invariants of varieties, i.e., by counting the dimension of $V(S_2(x))$. In fact we can check that

$$\dim V(S(x)) < n - 4,$$

at a generic point $x = (x_{ip}) \in X(k,n)$ as in the following.

Let us first examine the typical and easiest case when $(k,n)=(3,7)$. The dimension of S_2 is 3; the space S_2 is generated by

$$E_{2356},\ E_{2357},\ E_{2367}.$$

For ease of reference we index the coordinates as follows:

$$\begin{pmatrix} x_{25} & x_{26} & x_{27} \\ x_{35} & x_{36} & x_{37} \end{pmatrix} = \begin{pmatrix} x & \alpha_1 & \alpha_2 \\ \beta & \gamma_1 & \gamma_2 \end{pmatrix}.$$

Each E_* is a homogeneous polynomial of e_{ip}. We introduce inhomogeneous coordinates by

$$Y_1=e_{26}/e_{25},\ Y_2=e_{27}/e_{25},\ Z=e_{35}/e_{25},\ W_1=e_{36}/e_{25},\ W_2=e_{37}/e_{25}.$$

Then the E_*'s, more precisely the quotients $E_*/{e_{25}}^2$, are functions of the inhomogeneous coordinates. The explicit forms are given by (3.3) and (3.4) as follows:

$$\begin{aligned} E_{2356} =& W_1 + Y_1 Z + A_1 + B_1 {Y_1}^2 + C_1 Z^2 + D_1 {W_1}^2 \\ & - \frac{\alpha_1-\gamma_1}{x-\beta} Z - \frac{x-\beta}{\alpha_1-\gamma_1} Y_1 W_1 - \frac{\gamma_1-\beta}{\alpha_1-x} Y_1 - \frac{\alpha_1-x}{\gamma_1-\beta} Z W_1, \end{aligned}$$

where

$$\begin{aligned} A_1 &= \frac{\alpha_1\beta-\alpha_1-\beta+\gamma_1}{x} + \frac{\alpha_1\beta-\gamma_1}{1-x} + \frac{\alpha_1-\gamma_1}{x-\beta} + \frac{\beta-\gamma_1}{x-\alpha_1}, \\ B_1 &= \frac{x\gamma_1-x-\gamma_1+\beta}{\alpha_1} + \frac{x\gamma_1-\beta}{1-\alpha_1} + \frac{x-\beta}{\alpha_1-\gamma_1} + \frac{\gamma_1-\beta}{\alpha_1-x}, \\ C_1 &= \frac{\gamma_1 x-\gamma_1-x+\alpha_1}{\beta} + \frac{\gamma_1 x-\alpha_1}{1-\beta} + \frac{\gamma_1-\alpha_1}{\beta-x} + \frac{x-\alpha_1}{\beta-\gamma_1}, \\ D_1 &= \frac{\beta\alpha_1-\beta-\alpha_1+x}{\gamma_1} + \frac{\beta\alpha_1-x}{1-\gamma_1} + \frac{\beta-x}{\gamma_1-\alpha_1} + \frac{\alpha_1-x}{\gamma_1-\beta}; \end{aligned}$$

$$\begin{aligned} E_{2357} =& W_2 + Y_2 Z + A_2 + B_2 {Y_2}^2 + C_2 Z^2 + D_2 {W_2}^2 \\ & - \frac{\alpha_2-\gamma_2}{x-\beta} Z - \frac{x-\beta}{\alpha_2-\gamma_2} Y_2 W_2 - \frac{\gamma_2-\beta}{\alpha_2-x} Y_2 - \frac{\alpha_2-x}{\gamma_2-\beta} Z W_2, \end{aligned}$$

where

$$
\begin{aligned}
A_2 &= \frac{\alpha_2\beta - \alpha_2 - \beta + \gamma_2}{x} + \frac{\alpha_2\beta - \gamma_2}{1-x} + \frac{\alpha_2 - \gamma_2}{x-\beta} + \frac{\beta - \gamma_2}{x - \alpha_2}, \\
B_2 &= \frac{x\gamma_2 - x - \gamma_2 + \beta}{\alpha_2} + \frac{x\gamma_2 - \beta}{1-\alpha_2} + \frac{x-\beta}{\alpha_2 - \gamma_2} + \frac{\gamma_2 - \beta}{\alpha_2 - x}, \\
C_2 &= \frac{\gamma_2 x - \gamma_2 - x + \alpha_2}{\beta} + \frac{\gamma_2 x - \alpha_2}{1-\beta} + \frac{\gamma_2 - \alpha_2}{\beta - x} + \frac{x - \alpha_2}{\beta - \gamma_2}, \\
D_2 &= \frac{\beta\alpha_2 - \beta - \alpha_2 + x}{\gamma_2} + \frac{\beta\alpha_2 - x}{1-\gamma_2} + \frac{\beta - x}{\gamma_2 - \alpha_2} + \frac{\alpha_2 - x}{\gamma_2 - \beta};
\end{aligned}
$$

$$
\begin{aligned}
E_{2367} &= Y_1W_2 + Y_2W_1 + AY_1{}^2 + BY_2{}^2 + CW_1{}^2 + DW_2{}^2 \\
&\quad - \frac{\alpha_2 - \gamma_2}{\alpha_1 - \gamma_1}Y_1W_1 - \frac{\alpha_1 - \gamma_1}{\alpha_2 - \gamma_2}Y_2W_2 - \frac{\gamma_2 - \gamma_1}{\alpha_2 - \alpha_1}Y_1Y_2 - \frac{\alpha_2 - \alpha_1}{\gamma_2 - \gamma_1}W_1W_2,
\end{aligned}
$$

where

$$
\begin{aligned}
A &= \frac{\alpha_2\gamma_1 - \alpha_2 - \gamma_1 + \gamma_2}{\alpha_1} + \frac{\alpha_2\gamma_1 - \gamma_2}{1-\alpha_1} + \frac{\alpha_2 - \gamma_2}{\alpha_1 - \gamma_1} + \frac{\gamma_1 - \gamma_2}{\alpha_1 - \alpha_2}, \\
B &= \frac{\alpha_1\gamma_2 - \alpha_1 - \gamma_2 + \gamma_1}{\alpha_2} + \frac{\alpha_1\gamma_2 - \gamma_1}{1-\alpha_2} + \frac{\alpha_1 - \gamma_1}{\alpha_2 - \gamma_2} + \frac{\gamma_2 - \gamma_1}{\alpha_2 - \alpha_1}, \\
C &= \frac{\gamma_2\alpha_1 - \gamma_2 - \alpha_1 + \alpha_2}{\gamma_1} + \frac{\gamma_2\alpha_1 - \alpha_2}{1-\gamma_1} + \frac{\gamma_2 - \alpha_2}{\gamma_1 - \alpha_1} + \frac{\alpha_1 - \alpha_2}{\gamma_1 - \gamma_2}, \\
D &= \frac{\gamma_1\alpha_2 - \gamma_1 - \alpha_2 + \alpha_1}{\gamma_2} + \frac{\gamma_1\alpha_2 - \alpha_1}{1-\gamma_2} + \frac{\gamma_1 - \alpha_1}{\gamma_2 - \alpha_2} + \frac{\alpha_2 - \alpha_1}{\gamma_2 - \gamma_1}.
\end{aligned}
$$

Thus, on the Zariski open subset ($D_1 \neq 0$ and $D_2 \neq 0$) of $X(3,7)$, from the equations $E_{2356} = E_{2357} = 0$, we can solve W_1 and W_2 in terms of Y_1, Z and Y_2, Z, respectively. Substituting these into $E_{2367} = 0$, we get a non-trivial equation for Y_1, Y_2 and Z. Thus we see that $\dim V(S_2(x)) = 2$ at a generic point of $X(3,7)$, whereas $\dim V(S_2) = 3$. More precisely, we observe this fact from the following computation of the differentials:

$$
\begin{aligned}
dE_{2356} &= \left(1 - \frac{x-\beta}{\alpha_1 - \gamma_1}Y_1 - \frac{\alpha_1 - x}{\gamma_1 - \beta}Z + 2D_1W_1\right)dW_1 \\
&\quad + \left(Z - \frac{x-\beta}{\alpha_1 - \gamma_1}W_1 - \frac{\gamma_1 - \beta}{\alpha_1 - x} + 2B_1Y_1\right)dY_1 \\
&\quad + \left(Y_1 - \frac{\alpha_1 - \gamma_1}{x-\beta} - \frac{\alpha_1 - x}{\gamma_1 - \beta}W_1 + 2C_1Z\right)dZ,
\end{aligned}
$$

$$\begin{aligned} dE_{2357} = & \left(1 - \frac{x-\beta}{\alpha_2-\gamma_2}Y_2 - \frac{\alpha_2 - x}{\gamma_2-\beta}Z + 2D_2W_2\right)dW_2 \\ & + \left(Z - \frac{x-\beta}{\alpha_2-\gamma_2}W_2 - \frac{\gamma_2-\beta}{\alpha_2-x} + 2B_2Y_2\right)dY_2 \\ & + \left(Y_2 - \frac{\alpha_2-\gamma_2}{x-\beta} - \frac{\alpha_2-x}{\gamma_2-\beta}W_2 + 2C_2Z\right)dZ, \end{aligned}$$

$$\begin{aligned} dE_{2367} = & \left(Y_2 - \frac{\alpha_2-\gamma_2}{\alpha_1-\gamma_1}Y_1 - \frac{\alpha_2-\alpha_1}{\gamma_2-\gamma_1}W_2 + 2CW_1\right)dW_1 \\ & + \left(W_2 - \frac{\alpha_2-\gamma_2}{\alpha_1-\gamma_1}W_1 - \frac{\gamma_2-\gamma_1}{\alpha_2-\alpha_1}Y_2 + 2AY_1\right)dY_1 \\ & + \left(Y_1 - \frac{\alpha_1-\gamma_1}{\alpha_2-\gamma_2}Y_2 - \frac{\alpha_1-\alpha_2}{\gamma_1-\gamma_2}W_1 + 2DW_2\right)dW_2 \\ & + \left(W_1 - \frac{\alpha_1-\gamma_1}{\alpha_2-\gamma_2}W_2 - \frac{\gamma_1-\gamma_2}{\alpha_1-\alpha_2}Y_1 + 2BY_2\right)dY_2. \end{aligned}$$

In the general case, we take the following inhomogeneous coordinates;

$$\begin{aligned} & Y_p = e_{2\,p}/e_{2\,k+2} \quad (k+2 < p \leqq n), \quad Z_i = e_{i\,k+2}/e_{2\,k+2} \quad (2 < i \leqq k), \\ & W_{ip} = e_{i\,p}/e_{2\,k+2} \quad (2 < i \leqq k,\ k+2 < p \leqq n). \end{aligned}$$

Then, similarly as in the case of $(k,n) = (3,7)$, from the quadratic equation $E_{2ik+2p} = 0$, we can solve W_{ip} $(2 < i \leqq k,\ k+2 < p \leqq n)$ in terms of Y_p and Z_i on the Zariski open subset of $X(k,n)$. Substituting these into $E_{ijpq} = 0$, we get non-trivial equations for Y's and Z's. Thus, at a generic point $x \in X(k,n)$, we obtain

$$\dim V(S_2(x)) < n - 4 = \dim V(S_2),$$

which completes the proof of Theorem.

4. Disproof of a dream on $E(4,8;\{1/2\})$

The authors are afraid that the reader would not be satisfied by the argument in the previous section based on [S] and [Y], which are hardly elementary. So in this section we give an elementary proof for $E(4,8;\{1/2\})$ that $\mathrm{Im}(\varphi)$ does not lie in $\mathrm{Gr}_{3,6} \subset \mathbb{P}^{19}$.

The idea is as follows: Assume the contrary. Then the *restriction* of the projective solution to any stratum consisting of degenerate 8-plane arrangements in P^3 has its image in quadratic hypersurfaces in a projective space, since Grassmannians can be defined only by quadratic equations. If we choose a 1-dimensional stratum, the restricted equation is an ordinary differential equation; so we can know whether its image lies in a quadric by the vanishing of the Laguerre-Forsyth invariant.

Let us carry out the above program. We consider the degenerate stratum given by the following matrix:

$$\begin{pmatrix} 1 & & & & 1 & & & 1 \\ & 1 & & & -1 & 1 & & \\ & & 1 & & & -1 & 1 & \\ & & & 1 & & & -1 & -x \end{pmatrix},$$

where each column defines a hyperplane. The integral belonging to the stratum is of the form

$$\int t_1^{\alpha_1-1}t_2^{\alpha_2-1}t_3^{\alpha_3-1}(1-t_1)^{\alpha_4-1}(t_1-t_2)^{\alpha_5-1}(t_2-t_3)^{\alpha_6-1}(1-xt_3)^{\alpha_7-1}dT,$$

where $dT = dt_1 \wedge dt_2 \wedge dt_3$. The associated ordinary differential equation in x is of fourth order and coincides with the so-called generalized hypergeometric differential equation ${}_4E_3(a_1, a_2, a_3, a_4; b_1, b_2, b_3)$:

$$\theta(\theta+b_1-1)(\theta+b_2-1)(\theta+b_3-1)z-x(\theta+a_1)(\theta+a_2)(\theta+a_3)(\theta+a_4)z = 0,$$

where $\theta = xd/dx$ (refer to [E]), which admits the solution given by the following power series:

$${}_4F_3(a_1, a_2, a_3, a_4; b_1, b_2, b_3; x) = \sum_{n=0}^{\infty} \frac{(a_1, n)(a_2, n)(a_3, n)(a_4, n)}{(b_1, n)(b_2, n)(b_3, n)(1, n)} x^n,$$

where

$$a_1 = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_5 + \alpha_6 - 2, \quad a_2 = \alpha_2 + \alpha_3 + \alpha_6 - 1, \quad a_3 = \alpha_3,$$
$$a_4 = 1 - \alpha_7, \quad b_1 = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 - 2,$$
$$b_2 = \alpha_2 + \alpha_3 + \alpha_5 + \alpha_6 - 1, \quad b_3 = \alpha_3 + \alpha_6,$$

and $(a, n) = a(a + 1) \cdots (a + n - 1)$.

Now, consider the case where all α_i are equal to $1/2$; the corresponding parameters are $a_1 = a_2 = a_3 = a_4 = 1/2$ and $b_1 = b_2 = b_3 = 1$. The question is to see if the curve in $\mathbb{P}^3$ defined by the ${}_4E_3$ lies on quadratic surfaces for this special choice of parameters. To proceed further, we need to recall a bit of the Laguerre-Forsyth theory. We start with an ordinary differential equation of the form

$$y^{\cdot\cdot\cdot\cdot} + 4p_1y^{\cdot\cdot\cdot} + 6p_2y^{\cdot\cdot} + 4p_3y^{\cdot} + p_4y = 0,$$

where y is the indeterminate of the variable x and the dot denotes the derivation relative to x. We can find a non-vanishing function λ and a new variable t so that the function $z = \lambda y$ relative to the coordinate t satisfies the ordinary differential equation

$$z'''' + 4r_3 z' + r_4 z = 0, \tag{4.1}$$

where r_3 and r_4 are differential polynomials of p_i, and $'$ denotes the derivation relative to t. The Laguerre-Forsyth theory (refer to, say, [MSY2], [W]) tells us that

$$\theta_3 = r_3 dt^3 \qquad \text{and} \qquad \theta_4 = (r_4 - 2r_3')dt^4$$

are projective invariants; that is, independent of the choice of such a coordinate t. For the case ${}_4E_3(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}; 1, 1, 1)$, a calculation shows $r_3 = 0$.

On the other hand, for the ordinary differential equation

$$z'''' + rz = 0,$$

we can check that

$$I = \frac{(8rr'' - 9(r')^2)^2}{r^5}$$

is an absolute invariant; in our case it is equal to

$$-\frac{16(125x^6 - 4650x^5 + 3075x^4 - 38572x^3 + 3075x^2 - 4650x + 125)^2}{x(5x+1)^5(x+5)^5}.$$

In particular, I is not constant.

We next consider the case where the projective curve defined by the equation (4.1) is on a nondegenerate quadratic surface, say, $\zeta_1\zeta_4 = \zeta_2\zeta_3$ in $\mathbb{P}^3(\zeta_1, \zeta_2, \zeta_3, \zeta_4)$. Then around a generic point, we can choose a coordinate t so that the set of independent solutions is $\{1, t, f, tf\}$ for a function f. This means that the equation (4.1) is the tensor product of two differential equations

$$y_1'' = 0 \qquad \text{and} \qquad y_2'' = \frac{f''}{f'} y_2';$$

namely, $y_1 y_2$ are general solutions of (4.1). Such an ordinary differential equation is studied by [Ha] and its general form is known to be

$$z'''' - 2gz''' - 2g'z' + (g^2 - g'' - c^2)z = 0,$$

where g is a function and c is a constant. The invariants r_3 and r_4 of this equation are given by

$$r_3 = \frac{1}{2}g', \quad r_4 = 4c^2 - \frac{1}{5}g'' - \frac{36}{25}g^2.$$

If the image curve of a projective solution of the equation ${}_4E_3(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}; 1, 1, 1)$ lies on a nondegenerate quadratic surface, since $r_3 = 0$, the function g must be constant, and so r_4 should also be constant, which implies $I = 0$. Therefore, our curve does not lie on any nondegenerate quadratic surface.

Suppose that the image $\mathrm{Im}\varphi$ is on the Grassmannian $\mathrm{Gr}_{3,6}$, then the image of a projective solution of the restricted system ${}_4E_3$ would be in the intersection $\mathrm{Gr}_{3,6} \cap L$ of $\mathrm{Gr}_{3,6}$ and a 3-dimensional linear subvariety L of $\mathbb{P}^{20-1}$. Since Grassmannians can be defined only by quadrics, the curve $\mathrm{Gr}_{3,6} \cap L$ in L must be the intersection of two quadric surfaces. If the pencil generated by two quadric surfaces consists of degenerate quadrics only, the intersection must be linear, which contradicts that the projective solution is defined by linearly independent solutions.

References

[**E**] A. Erdélyi, "Higher Transcendental Functions, vol. 1", New York Toronto London, McGraw-Hill Book Co., 1953.

[**Ha**] G.H. Halphen, Sur les invariants des équations différentielles linéaires du quatrième ordre, Acta Math., **3** (1883), 325–380.

[**Hu**] J.E. Humphreys, "Introduction to Lie Algebras and Representation Theory", Springer-Verlag, New York, 1972.

[**MM**] Y. Matsushima and S. Murakami, On certain cohomology groups attached to hermitian symmetric spaces, Osaka J. Math., **2** (1965), 1–35.

[**MSY1**] K. Matsumoto, T. Sasaki and M. Yoshida, The monodromy of the period map of a 4-parameter family of K3 surfaces and the hypergeometric function of type (3,6), Internat. J. Math., **3** (1992), 1–164.

[**MSY2**] ———, Recent progress of Gauss-Schwarz theory and related geometric structures, Memoirs of the Faculty of Science, Kyushu University, Ser A, **47** (1993), 283–381.

[**S**] Y. Se-ashi, On differential invariants of integrable finite type linear differential equations, Hokkaido Math. J., **17** (1988), 151–195.

[**W**] E. J. Wilczynski, "Projective Differential Geometry of Curves and Ruled Surfaces", reprinted by Chelsea Publ. Co., Teubner, 1906;.

[**Y**] K. Yamaguchi, Differential systems associated with simple graded Lie algebras, Adv. Studies in Pure Math., **22** (1993), 413–494.

Takeshi Sasaki
Department of Mathematics, Faculty of Science,
Kobe University, Kobe 657,
Japan

Keizo Yamaguchi
Department of Mathematics, Faculty of Science,
Hokkaido University, Sapporo 060,
Japan

Masaaki Yoshida
Graduate School of Mathematics,
Kyushu University, Fukuoka 812,
Japan

Advanced Studies in Pure Mathematics 25, 1997
CR-Geometry and Overdetermined Systems
pp. 355–363

Infinitesimal CR Automorphisms

Nancy K. Stanton

To Masatake Kuranishi on his seventieth birthday

Let M be a real hypersurface through the origin in $\mathbf{C}^n$ or, more generally, an integrable CR manifold of hypersurface type. A smooth vector field X on M is called an *infinitesimal CR automorphism* of M if the local one-parameter group it generates is a local group of CR automorphisms of M. Fix $p \in M$ and let $\operatorname{aut}(M,p)$ denote the space of infinitesimal CR automorphisms of M which are defined in a neighborhood of p.

Throughout this paper, M will denote a connected analytic real hypersurface in $\mathbf{C}^n$. For $p \in M$, there is a distinguished subspace $\operatorname{hol}(M,p) \subset \operatorname{aut}(M,p)$ defined as follows. If Z is a holomorphic vector field defined in a neighborhood of $p \in \mathbf{C}^n$ and $X = \operatorname{Re} Z$, then the local one-parameter group of X is a group of biholomorphic transformations [KN, remarks preceding Proposition IX.2.10]. Here, by holomorphic vector field, I mean a vector field of type $(1,0)$ with holomorphic coefficients. Hence, if X is tangent to M, then $X \in \operatorname{aut}(M,p)$. Let $\operatorname{hol}(M,p)$ denote the space of all infinitesimal CR automorphisms X of M defined in some neighborhood of p which are of the form $X = \operatorname{Re} Z$ for some holomorphic vector field Z, $\operatorname{hol}(M,p) \subset \operatorname{aut}(M,p)$. Let $\operatorname{hol}(M) = \operatorname{hol}(M,0)$ and $\operatorname{aut}(M) = \operatorname{aut}(M,0)$.

Infinitesimal CR automorphisms are useful in the study of hypersurfaces with degenerate Levi form. I will survey some recent results about $\operatorname{hol}(M)$ and $\operatorname{aut}(M)$ and their applications. In Section 1, I use infinitesimal CR automorphisms to characterize homogeneous hypersurfaces. Section 2 describes applications of holomorphic nondegeneracy to finite dimensionality of $\operatorname{hol}(M)$ and to mappings of algebraic hypersurfaces. I will discuss some conditions for equality of $\operatorname{hol}(M)$ and $\operatorname{aut}(M)$ in Section 3.

Received October 2, 1995
Research supported in part by NSF grant DMS 93-01345

1. Homogeneous hypersurfaces Following the terminology of Baouendi, Rothschild and Trèves ([BRT]), a real hypersurface in $\mathbf{C}^n$ is called *rigid* if there are coordinates $(z_1, \ldots, z_{n-1}, w = u + iv)$ such that M is given by an equation of the form

$$v = F(z, \bar{z}),$$

a *rigid equation.* Tanaka [T] called these *regular* and D'Angelo [DA] called them *T-regular.*

Among rigid hypersurfaces, the simplest ones are the homogeneous hypersurfaces. A rigid hypersurface is *homogeneous* if it is locally biholomorphically equivalent to

$$v = p(z, \bar{z}) \tag{1.1}$$

with p a homogeneous polynomial. This terminology comes from the fact that (1.1) is invariant under the nonisotropic dilations

$$(z, w) \to (tz, t^m w) = \delta_t(z, w) \tag{1.2}$$

where m is the degree of the polynomial p.

How can you tell if a rigid hypersurface is homogeneous? This problem was first posed by Linda Rothschild. The problem is local, so I will assume that $0 \in M$ and will work locally in a neighborhood of 0. Equivalences will preserve the origin. I can make a biholomorphic change of coordinates so that either M is the hyperplane $v = 0$ or M is given by an equation of the form

$$v = p(z, \bar{z}) + O(m + 1)$$

where p is a nontrivial homogeneous polynomial of degree m with no pure terms in z or $\bar{z}$. In this case, m is an invariant, the *type* of M at the origin, and M is of *finite type* at the origin. Suppose that the origin is a point of type m. A vector field Y is *homogeneous* of weight j if

$$Y(f \circ \delta_t) = t^{-j}(Yf) \circ \delta_t$$

where δ_t is the nonisotropic dilation (1.2).

If M is homogeneous, given by

$$v = p(z, \bar{z})$$

with p homogeneous of degree m, then

$$Y_0 = 2\,\mathrm{Re}\left(\sum_{j=1}^{n-1} z_j \frac{\partial}{\partial z_j} + mw \frac{\partial}{\partial w}\right)$$

is in $\operatorname{hol}(M)$ and is homogeneous of weight 0. It is the infinitesimal generator of the dilations δ_{e^t}. Call a vector field $Y \in \operatorname{hol}(M)$ an *approximate infinitesimal dilation* if

$$Y = Y_0 + \text{terms of weight} \geq 1.$$

Theorem 1.3 ([S5, Theorem 4.1]). *Let M be a rigid analytic real hypersurface through the origin in $\mathbf{C}^n$. Suppose M is given by a rigid equation of the form*

$$v = p(z, \overline{z}) + O(m+1)$$

with p a nontrivial polynomial homogeneous of degree m having no pure terms. Then M is homogeneous if and only if M has an approximate infinitesimal dilation.

This theorem was first proved in $\mathbf{C}^2$ ([S1], [S2], [S3]), then in $\mathbf{C}^n$ under the additional hypothesis that $\dim \operatorname{hol}(M) < \infty$ ([S4]).

Theorem 1.3 can be generalized to characterize weighted homogeneous hypersurfaces. Fix positive integers $m_1, \ldots, m_n$. Now I will use $(z_1, \ldots, z_n)$ as coordinates. The non-isotropic group of dilations determined by $(m_1, \ldots, m_n)$ is the group $\{\delta_t : t > 0\}$ where

$$\delta_t(z) = (t^{m_1} z_1, \ldots, t^{m_n} z_n).$$

A function h is *homogeneous of weight j* if $h \circ \delta_t = t^j h$. A vector field Y is *homogeneous of weight j* if

$$Y(f \circ \delta_t) = t^{-j}(Yf) \circ \delta_t.$$

Let

$$Y_0 = 2 \operatorname{Re} \sum_{j=1}^{n} m_j z_j \frac{\partial}{\partial z_j}.$$

The one-parameter group generated by Y_0 is the group of non-isotropic dilations $\{\delta_{e^t} : t \in \mathbf{R}\}$. An analytic real hypersurface M is *weighted homogeneous* (with respect to the non-isotropic group of dilations) if it is locally equivalent, via a biholomorphic map which preserves the origin, to a hypersurface given by an equation of the form

$$P(z, \overline{z}) = 0$$

where P a polynomial which is homogeneous with respect to the non-isotropic group of dilations.

As before, call a vector field $Y \in \operatorname{hol}(M)$ an *approximate infinitesimal dilation* if

$$Y = Y_0 + \text{terms of weight} \geq 1.$$

Theorem 1.4 ([S5, Theorem 4.1]). *Let M be an analytic real hypersurface through the origin in $\mathbf{C}^n$ and suppose there is an approximate infinitesimal dilation $Y \in \operatorname{hol}(M)$. Then M is weighted homogeneous.*

This theorem does not require the hypothesis that M be rigid and there is no nondegeneracy hypothesis or finite type hypothesis on M.

The theorem can be proved by a technique used by Poincaré in his thesis [P] and generalized by Dulac [Du]. One linearizes Y, that is, one finds a change of coordinates so that in the new coordinates $\widetilde{z}$,

$$Y = 2 \operatorname{Re} \sum_{j=1}^{n} m_j \widetilde{z}_j \frac{\partial}{\partial \widetilde{z}_j}.$$

To do this, one first finds a formal change of variables, then one applies Poincaré's by now standard domination argument to prove that the formal change converges.

Now, after reordering the coordinates and multiplying $\widetilde{z}_n$ by i if necessary, I can assume M is given by an equation of the form

$$\operatorname{Im} \widetilde{z}_n = \widetilde{F}(\widetilde{z}', \overline{\widetilde{z}'}, \operatorname{Re} \widetilde{z}_n) \tag{1.5}$$

where $\widetilde{z}' = (\widetilde{z}_1, \dots, \widetilde{z}_{n-1})$. Applying Y to this equation shows that the right side of this equation is a weighted homogeneous polynomial and hence M is homogeneous.

By replacing $\widetilde{z}_n$ with $a\widetilde{z}_n$ for an appropriate $a \in \mathbf{C}$, one may assume that (1.5) is a rigid equation. This yields the following proposition.

Proposition 1.6 ([S5, Proposition 4.3]). *If M is weighted homogeneous then M is rigid.*

2. Holomorphic nondegeneracy How can one tell whether $\operatorname{hol}(M)$ is finite dimensional? In $\mathbf{C}^2$ it is for any hypersurface M of finite type. The example

$$v = |z_1|^2$$

in $\mathbf{C}^n$, $n \geq 3$, shows that some stronger nondegeneracy hypothesis is required in higher dimensions. In this example, $\operatorname{Re} f(z,w) \frac{\partial}{\partial z_2} \in \operatorname{hol}(M)$ for any holomorphic function f.

Definition. *Let M be an analytic real hypersurface in $\mathbf{C}^n$. A nontrivial holomorphic vector field W is called a holomorphic tangent to M at p if W is defined in a neighborhood of p and $W\big|_M$ is tangent to M. The hypersurface M is holomorphically nondegenerate at p if M*

has no holomorphic tangent at p. If M has a holomorphic tangent at p, M is holomorphically degenerate at p.

Theorem 2.1 ([S4, Theorem 4.3]). *Let M be an analytic real hypersurface through the origin in $\mathbf{C}^2$. The following are equivalent.*

(1) $\operatorname{hol}(M)$ *is finite dimensional;*
(2) *M is not flat;*
(3) *the Levi form of M is somewhere nondegenerate;*
(4) *M is holomorphically nondegenerate at the origin.*

In higher dimensions holomorphic nondegeneracy is not the same as nonflat, finite type, essentially finite or somewhere Levi nondegenerate. (See [BJT] for the definition of essentially finite.)

Theorem 2.2 ([BR2, Theorem 2, Proposition 4.2], [S6, Corollaries 3.3, 3.4]). *Let M be an analytic real hypersurface through the origin in $\mathbf{C}^n$. The following are equivalent.*

(1) *M is holomorphically nondegenerate at the origin.*
(2) *M is everywhere holomorphically nondegenerate.*
(3) *M is essentially finite on an open dense set.*

In general, and even for many simple examples of hypersurfaces with polynomial defining equations, it is very difficult to compute $\operatorname{hol}(M)$. If M is rigid with a rigid defining equation which is a polynomial, in principle—and often in fact—it is easy to check whether M is holomorphically nondegenerate at the origin.

Holomorphic nondegeneracy is a natural condition to introduce in connection with finite dimensionality of $\operatorname{hol}(M)$. Suppose M is a holomorphically degenerate real hypersurface, with holomorphic tangent Z. Then for all multi-indices α, $X_\alpha = \operatorname{Re} z^\alpha Z \in \operatorname{hol}(M)$ so $\dim \operatorname{hol}(M) = \infty$. This gives one direction of the following theorem.

Theorem 2.3 ([S4, Theorem 4.16], [S6, Theorem 1.7]). *Let M be an analytic real hypersurface through the origin in $\mathbf{C}^n$. Then the space* $\operatorname{hol}(M)$ *is finite dimensional if and only if M is holomorphically nondegenerate.*

In $\mathbf{C}^2$ the theorem follows easily from Theorem 2.1. Theorem 2.3 was first proved in the case of rigid hypersurfaces [S4]. In the rigid case the proof is long and technical; much of the work goes into proving an approximate version of the theorem, which requires a polynomial hypersurface to approximate M and an approximate version of $\operatorname{hol}(M)$. In dimensions greater than 2, the approximating hypersurface must include some higher order terms; the homogeneous part may not give a good approximation. The proof gives a bound on $\dim \mathrm{hol}(M)$ which depends on the type at the origin and the defining equation. To prove the theorem in the general case, one shows that if M is holomorphically nondegenerate and $\dim \mathrm{hol}(M) \geq 1$, then there is an open dense set $U \subset M$ and an integer ℓ (computable in terms of an appropriate defining function for M) such that if $p \in U$, then M is rigid, essentially finite and of type 2 at p, and $\dim \mathrm{hol}(M,p) \leq \ell$.

The following theorem of Baouendi and Rothschild gives an application of holomorphic nondegeneracy to mappings of algebraic hypersurfaces. A real hypersurface is *algebraic* if it is contained in the zero set of a nontrivial real valued polynomial. A holomorphic map is *algebraic* if its components satisfy polynomial equations with polynomial coefficients.

Theorem 2.4 ([BR2, Theorem 1]). *Let M be a holomorphically nondegenerate algebraic real hypersurface in $\mathbf{C}^n$ and let M' be an algebraic real hypersurface in $\mathbf{C}^n$. If f is a biholomorphic map taking M to M' then f is algebraic. Conversely, if M is a holomorphically degenerate algebraic real hypersurface which contains the origin, then there is a nonalgebraic biholomorphic map f defined in a neighborhood of the origin, with $f(0) = 0$, which takes M to itself.*

3. Analyticity of infinitesimal CR automorphisms For any analytic real hypersurface M and any $p \in M$, $\mathrm{hol}(M,p) \subset \mathrm{aut}(M,p)$. The two spaces are not always equal.

Example 3.1 ([S4, Example 7.11]). Let $M = \{v = 0\} \subset \mathbf{C}^2$. Then
$$X = e^{-1/u^2}\frac{\partial}{\partial u} \in \mathrm{aut}(M).$$
However, $X \notin \mathrm{hol}(M)$ so $\mathrm{hol}(M) \subsetneq \mathrm{aut}(M)$.

There is a sufficient condition for equality of $\mathrm{hol}(M)$ and $\mathrm{aut}(M)$.

Proposition 3.2 ([S3, Remark 2.5]). *Let M be an analytic real hypersurface through the origin in $\mathbf{C}^n$. Suppose every CR diffeomorphism on M is analytic. Then* $\mathrm{hol}(M) = \mathrm{aut}(M)$.

The next theorem summarizes what is known about equality of $\mathrm{hol}(M)$ and $\mathrm{aut}(M)$ in the case that $\mathrm{hol}(M)$ is finite dimensional.

Theorem 3.3. *Let M be an analytic real hypersurface through the origin in $\mathbf{C}^n$. Suppose that one of the following holds.*

(1) *M is essentially finite;*

(2) *M is rigid and every neighborhood U of 0 contains a point $p \in M$ such that the Levi form of M is nondegenerate at p;*
(3) *M is algebraic and holomorphically nondegenerate.*

Then $\mathrm{aut}(M)$ *is finite dimensional and* $\mathrm{aut}(M) = \mathrm{hol}(M)$.

Theorem 3.3 was proved for hypersurfaces satisfying (1) and (2) in [S4, Theorem 6.1]. For hypersurfaces satisfying (3) it follows from Proposition 3.2 and the following theorem of Baouendi, Huang and Rothschild.

Theorem 3.4 [BHR, Theorem 1]. *Let M and M' be algebraic real hypersurfaces in $\mathbf{C}^n$ and suppose that M is holomorphically nondegenerate. If H is a smooth CR map from M to M' and the Jacobian determinant of H is not everywhere 0, then H extends holomorphically to a neighborhood of M.*

To describe additional results on the question of when $\mathrm{hol}(M) = \mathrm{aut}(M)$, I need a characterization of infinitesimal CR automorphisms analogous to the definition of $\mathrm{hol}(M)$.

Proposition 3.5. *Let M be a real hypersurface through the origin in $\mathbf{C}^n$ and let X be a smooth tangent vector field defined in a neighborhood of the origin on M. Then $X \in \mathrm{aut}(M)$ if and only if*

$$X = \mathrm{Re} \sum_{j=1}^{n} f_j \frac{\partial}{\partial z_j} \tag{3.6}$$

where each f_j is a CR function on a neighborhood of the origin in M.

Proof. Let X be a $\mathcal{C}^\infty$ real vector field tangent to M. By Theorem 1 of [BR1], it suffices to show that X is of the form (3.6) if and only if for every smooth section Y of $T^{0,1}(M)$ on a neighborhood of the origin,

$$[X, Y] \in T^{0,1}(M). \tag{3.7}$$

Now $X = (Z + \overline{Z})\big|_M$ for some smooth vector field $Z = \sum_{j=1}^n f_j \frac{\partial}{\partial z_j}$ defined in a neighborhood of the origin. Let $Y = \sum_{j=1}^n g_j \frac{\partial}{\partial \overline{z}_j} \in \mathcal{C}^\infty(T^{0,1}(M))$. Then Y extends to a $\mathcal{C}^\infty$ vector field $\widetilde{Y}$ of type $(0,1)$ defined in a neighborhood of the origin. Now

$$\begin{aligned}
[X, Y] &= ([Z, \widetilde{Y}] + [\overline{Z}, \widetilde{Y}])\big|_M \\
&= \Big(\sum_{j=1}^{n} (Z g_j) \frac{\partial}{\partial \overline{z}_j} - \sum_{j=1}^{n} (Y f_j)\, \frac{\partial}{\partial z_j} + [\overline{Z}, \widetilde{Y}]\Big)\Big|_M .
\end{aligned}$$

The first and last terms are of type $(0,1)$. Hence (3.7) holds for all Y if and only if $Yf_j \equiv 0$ for all smooth sections Y of $T^{0,1}(M)$, so if and only if f_j is a CR function for each j.

Baouendi, Huang and Rothschild proved the following theorem about failure of analyticity of CR diffeomorphisms for holomorphically degenerate hypersurfaces.

Theorem 3.8 ([BHR, Theorem 4]). *Let M be an analytic holomorphically degenerate real hypersurface through the origin in $\mathbf{C}^n$. If there is a germ at 0 of a smooth CR function on M which does not extend to be holomorphic in any neighborhood of 0, then there is a germ of a smooth CR diffeomorphism from M to itself, fixing 0, which does not extend holomorphically to any neighborhood of 0.*

This result is closely related to the question of when $\mathrm{hol}(M) = \mathrm{aut}(M)$ in the holomorphically degenerate case.

Theorem 3.9. *Let M be a holomorphically degenerate analytic real hypersurface through the origin in $\mathbf{C}^n$. Then $\mathrm{hol}(M) = \mathrm{aut}(M)$ if and only if every CR function defined on a neighborhood of the origin in M extends to be holomorphic on a neighborhood of the origin in $\mathbf{C}^n$.*

Proof. Suppose every CR function on a neighborhood of the origin in M extends to be holomorphic. Let $X \in \mathrm{aut}(M)$. Then X is given by (3.6) for some CR functions f_j. There is a neighborhood U of the origin in $\mathbf{C}^n$ such that f_j, $j = 1, \dots, n$, extends to a holomorphic function F_j on U. Hence, $X = \mathrm{Re}\, Z\big|_M$ where $Z = \sum F_j \frac{\partial}{\partial z_j}$, and $X \in \mathrm{hol}(M)$.

Suppose $\mathrm{hol}(M) = \mathrm{aut}(M)$. Let Z be a holomorphic tangent to M at the origin, $Z = \sum f_j \frac{\partial}{\partial z_j}$, for some holomorphic functions f_j. Let f be a CR function defined on a neighborhood of the origin in M. Then, by Proposition 3.5,

$$X = \mathrm{Re} \sum_{j=1}^{n} f f_j \frac{\partial}{\partial z_j}$$

is in $\mathrm{aut}(M)$, so $X \in \mathrm{hol}(M)$. Because $X \in \mathrm{hol}(M)$, the proof of Theorem 3.8 shows that f extends to be holomorphic in a neighborhood of the origin, so every CR function extends.

References

[BHR] M. S. Baouendi, X. Huang and L. Rothschild, Regularity of CR mappings between algebraic hypersurfaces, preprint.

[BJT] M. S. Baouendi, H. Jacobowitz and F. Trèves, On the analyticity of CR mapping, Annals of Math., **122** (1985), 365-400.

[BR1] M. S. Baouendi and L. P. Rothschild, Transversal Lie group actions on abstract CR manifolds, Math. Annalen, **287** (1990), 19-33.

[BR2] ———, Mappings of real algebraic hypersurfaces, preprint.

[BRT] M. S. Baouendi, L. P. Rothschild and F. Trèves, CR structures with group action and extendability of CR functions, Inventiones Math., **82** (1985), 359-396.

[DA] J. P. D'Angelo, Defining equations for real analytic real hypersurfaces in C^n, Trans. A.M.S., **294** (1986), 71-84.

[Du] H. Dulac, Solutions d'un système d'équations différentielles dans le voisinage de valeurs singulières, Bull. Soc. Math. de France, **40** (1912), 324-383.

[KN] S. Kobayashi and K. Nomizu, "Foundations of Differential Geometry", II, Wiley Interscience, New York, 1969.

[P] H. Poincaré, Sur les propriétés des fonctions définies par les équations aux différences partielles, Première Thèse (1879); "Œuvres", I, IL-CXXIX, Gauthier-Villars, Paris, 1928.

[S1] N. K. Stanton, Rigid hypersurfaces in C^2, in "Proc. Symposia Pure Math.", Vol. 52, Part 3, Amer. Math. Soc., Providence, 1991, pp. 347-354.

[S2] ———, A normal form for rigid hypersurfaces in C^2, Amer. J. Math., **113** (1991), 877-910.

[S3] ———, Infinitesimal CR automorphisms of rigid hypersurfaces in $\mathbf{C}^2$, J. Geometric Analysis, **1** (1991), 231-267.

[S4] ———, Infinitesimal CR automorphisms of rigid hypersurfaces, Amer. J. Math., **117** (1995), 141-167.

[S5] ———, Homogeneous real hypersurfaces, Mathematical Research Letters, **2** (1995), 311-319.

[S6] ———, Infinitesimal CR automorphisms of real hypersurfaces, Amer. J. Math., to appear.

[T] N. Tanaka, On the pseudo-conformal geometry of hypersurfaces of the space of n complex variables, J. Math. Soc. Japan, **14** (1962), 397-429.

Department of Mathematics,
University of Notre Dame,
Notre Dame, IN 46556
U.S.A.
Nancy.K.Stanton@nd.edu

Advanced Studies in Pure Mathematics 25, 1997
CR-Geometry and Overdetermined Systems
pp. 364–385

Lie-Drach-Vessiot Theory

Infinite dimensional differential Galois theory

Hiroshi Umemura

Introduction

Despite trials of several authors since the 19-th century, at least to our taste, infinite dimensional differential Galois theory is unfinished. We propose an infinite dimensional differential Galois theory based on a rigorous foundation. This note is prepared for the non-specialists as an introduction to our papers [U5], [U6], where interested readers can find details. After we briefly recall the history and the principle of Galois theory, we show the marvelous ideas of the classical authors on infinite dimensional differential Galois theory as well as the problems which their ideas give rise to. We can avoid all these difficulties and attach to an ordinary differential field extension L/K of finite type, or intuitively to an ordinary algebraic differential equation, a formal group **Inf-gal** of infinite dimension. **Inf-gal** is a new invariant of an ordinary algebraic differential equation. In fact, no such invariants were known. We explain an application to be expected of the invariant **Inf-gal** to the Painlevé equations in §6. A brief account on the formal group of infinite dimension and the construction of **Inf-gal** is also given.

All the rings that we consider are commutative and unitary $\mathbb{Q}$ -algebras.

§1. History

Galois (1811–32) and Abel (1802–29) invented Galois theory of algebraic equations. Their purpose was proving the impossibility of solving a general algebraic equation of degree 5 by extraction of radicals. This historical problem is the origin of Galois theory but the significance of Galois theory is prominent in later developments of number theory. We cannot speak of algebraic number theory, class field theory ... etc. without Galois theory.

Received September 29, 1995

It was Lie (1842–99) who, inspired by the works of Galois and Abel, had a dream of applying their rich ideas to differential equations. This dream was one of the ultimate goals of his career (cf. [L]). The differential Galois theory that Lie had in his mind is essentially infinite dimensional. However, he had to begin by constructing finite dimensional theories such as, for example, theory of Lie groups. In the history of differential Galois theory, we have to carefully distinguish the infinite dimensional theories from the finite dimensional theories.

Picard (1856–1941) was the first who realized a part of Lie's dream. He published in 1887 Galois theory of linear differential equations that we can find now in vol. 3 of his cours d'analyse [Pi]. The theory, nowadays called Picard-Vessiot theory, is finite dimensional.

In 1898 a young French mathematician Drach (1871–1941) published an ambitious thesis [D1]. In his thesis he tried to construct a Galois theory of non-linear ordinary differential equations, which is infinite dimensional. Being the first trial of differential Galois theory of infinite dimension, the thesis is remarkable but it is troublesome too. For, soon after Drach had got his degree, Vessiot pointed out important errors in his thesis. We learn from letters in Pommaret [Po2] how Vessiot's comments embarrassed the judges of the thesis.

Vessiot not only discovered the defects of Drach's thesis but he also devoted himself in establishing Drach's work on a rigorous foundation. He was awarded the Grand Prix of the Academy of the Sciences of Paris in 1902 for a series of papers [V1], [V2], [V3]. To his regret, today we remember Vessiot in Picard-Vessiot theory, which is finite dimensional, and not in infinite dimensional theory to which he was deeply attached. In spite of Vessiot's works, it seems, at least for our taste, general Galois theory is not achieved. After Vessiot, the theory was left untouched for several decades until Pommaret wrote the monograph [Po1] in 1983.

Kolchin (1916–91) is famous for his differential Galois theory. His major contributions [K] to differential Galois theory are as follows.

(1) He made finite dimensional differential Galois theory complete. The theory existed since the end of the 19-th century and he constructed the theory on a rigorous foundation using the language of algebraic geometry of Weil.
(2) He founded differential algebra.

The second contribution should be as important as the first.

In the 60's Jacobson, Sweedler, Bourbaki et al. established Galois theories of inseparable field extensions. The idea is to replace a finite group by a finite group scheme or more generally by a bialgebra (cf. [Wi]). Compared with the epoch of Vessiot and Drach, the evolution of algebraic geometry is remarkable. It allows us to propose an infinite dimensional differential Galois theory.

§2. Ideal theory

We have two ideal Galois theories: (I) Classical Galois theory of field extensions or Galois theory of algebraic equations, (II) Kolchin theory.

(I) Classical Galois theory of field extensions.

Let L/K be a Galois extension and $G = \mathrm{Gal}(L/K)$ the Galois group. So G is the group of K-automorphisms of the field L.

(i) Galois correspondence.

We have a $1:1$-correspondence between the elements of the following two sets.

(1) The set of intermediate fields of the extension L/K.

(2) The set of subgroups of G.

For an intermediate field $L \supset M \supset K$, the corresponding subgroup $G(M)$ is

$$\{g \in G | \text{The } K\text{-automorphism } g : L \to L \text{ leaves every element of } M \text{ invariant}\}.$$

To a given subgroup H, there corresponds the intermediate field

$$L^H = \{z \in L | z \text{ is invariant for every } g \in H\}.$$

(ii) Surjectivity.

If M is an intermediate field of L/K such that M/K is Galois, then $G(M) = \mathrm{Gal}(L/M)$ is a normal subgroup of $\mathrm{Gal}(L/K)$ and we have an exact sequence

$$1 \to \mathrm{Gal}(L/M) \to \mathrm{Gal}(L/K) \to \mathrm{Gal}(M/K) \to 1.$$

(II) Kolchin theory.

In the Kolchin theory, we consider a differential field extension. Namely an ordinary differential field is a pair (L, δ) consisting of a field L and a derivation $\delta : L \to L$ so that we have $\delta(ab) = \delta(a)b + a\delta(b)$ for every $a, b \in L$. We often denote the differential field (L, δ) by L when there is no danger of confusion. We say that an element $y \in L$ is a constant if $\delta(y) = 0$. The set C_L of constants of L forms a subfield of L. Similarly a partial differential field

$$(L, \{\delta_1, \delta_2, \ldots, \delta_n\})$$

consists of a field L and derivations $\delta_i : L \to L$ for $1 \le i \le n$ that are commutative: $\delta_i\delta_j = \delta_j\delta_i$ for $1 \le i, j \le n$.

Kolchin introduces the notion of strongly normal extension that generalizes classical Galois extension. Let L/K be a differential field extension that is strongly normal. The differential Galois group $G = \mathrm{Gal}(L/K)$ of the extension L/K is the group of differential K-automorphisms of L. We can show that the Galois group G is an algebraic group defined over the field C_K of constants of K.

(i) Galois correspondence.

We have a 1:1-correspondence of the following two sets.

(1) The set of differential intermediate fields of the extension L/K.

(2) The set of closed subgroups of the differential Galois group $\mathrm{Gal}(L/K)$.

The correspondence is given as in classical Galois theory.

(ii) Surjectivity.

If M is as intermediate field of the extension L/K such that M/K is strongly normal, then $G(M) = \mathrm{Gal}(L/M)$ is a closed normal subgroup of $\mathrm{Gal}(L/K)$ and we have an exact sequence

$$1 \to \mathrm{Gal}(L/M) \to \mathrm{Gal}(L/K) \to \mathrm{Gal}(M/K) \to 1.$$

In an ideal theory, we have on the one hand a field extension (resp. abstract, differential, ...) and on the other hand a group like object (resp. abstract group, algebraic group, ...) such that we have (i) the Galois correspondence and (ii) the surjectivity. Moreover the group like object should be simpler than the field extension.

Example (Jacobson, Sweedler et al.). *In Galois theory of inseparable field extension, we replace a finite group by a finite group scheme. In this theory, we cannot expect an ideal theory and the Galois group G is not uniquely determined. For a field extension L/K that generalizes classical Galois extension, we have a 1:1-correspondence between the elements of the following two sets.*

(1) *The set of certain type of intermediate fields of L/K.*

(2) *The set of certain type of subalgebras of the Hopf algebra G.*

Here the adjective certain depends on the theory and the choice of the Hopf algebra G, which is not uniquely determined in general when the extension L/K is given. See [Wi].

§3. Principle of Galois theory

Let us see the principle of Galois theory.

(I) Classical Galois theory or Galois theory of algebraic equations.

Let K be a ground field. We consider an algebraic equation

$$a_0x^n + a_1x^{n-1} + \cdots + a_n = 0 \qquad (a_0 \neq 0) \tag{3.1}$$

with coefficients in K so that $a_i \in K$ for $0 \leq i \leq n$. We assume that all the roots of (3.1) are simple. Let S be the set of vectors $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ of distinct solutions x_i of the algebraic equation (3.1). The symmetric group S_n of degree n naturally operates on S: For $s \in S_n$ and $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in S$, we define

$$s(x_1, x_2, \ldots, x_n) := (x_{s(1)}, x_{s(2)}, \ldots, x_{s(n)}).$$

If we take a fixed element $\mathbf{x} \in S$, then

$$S_n \to S \qquad s \mapsto s(x)$$

is a bijection, i.e. (S_n, S) is a principal homogeneous space. Evidently

$$K(\mathbf{x}) = K(x_1, x_2, \ldots, x_n)$$

coincides with

$$K(s(\mathbf{x})) = (x_{s(1)}, x_{s(2)}, \ldots, x_{s(n)}).$$

So a certain $s \in S_n$ defines a K-automorphism of the field extension $K(\mathbf{x})/K$.

(II) Kolchin theory

Let us take for example the differential field $(\mathbb{C}(x), d/dx)$ of rational functions as a ground field. We consider a linear differential equation.

$$a_0\frac{d^ny}{dx^n} + a_1\frac{d^{n-1}y}{dx^{n-1}} + \cdots a_ny = 0 \qquad (a_0 \neq 0)$$

with coefficients in $\mathbb{C}(x)$ so that $a_i \in \mathbb{C}(x)$ for $0 \leq i \leq n$. Let now S be the set of vectors $\mathbf{y} = {}^t(y_1, y_2, \ldots, y_n)$ of linearly independent solutions y_i of (3.1) $(1 \leq i \leq n)$. Then the general linear group $\mathrm{GL}_n(\mathbb{C})$ of degree n operates on S: For $A \in \mathrm{GL}_n(\mathbb{C})$ and $\mathbf{y} \in S$, we have $A\mathbf{y} \in S$. Moreover

$$(\mathrm{GL}_n(\mathbb{C}), S)$$

is a principal homogeneous space. Namely if we take an element $\mathbf{y} \in S$, then

$$\mathrm{GL}_n(\mathbb{C}) \to S, \qquad A \mapsto A\mathbf{y}$$

is a bijection. Since we have $K(\mathbf{y}) = K(A\mathbf{y})$ for $A \in \mathrm{GL}_n(\mathbb{C})$, a certain $A \in \mathrm{GL}_n(\mathbb{C})$ induces a K-automorphism of the differential field extension $K(\mathbf{y})/K$.

§4. Ideas of classical authors in infinite dimensional Galois theories

We take for simplicity $(\mathbb{C}(x), d/dx)$ as a ground field and consider a non-linear differential equation

$$y^{(n)} = A(x, y, y', \ldots, y^{(n-1)}) \tag{4.1}$$

with coefficients in $\mathbb{C}(x)$ such that

$$A \in \mathbb{C}(x)(y, y', \ldots, y^{(n-1)}) = \mathbb{C}(x, y, y', \ldots, y^{(n-1)}).$$

Let us recall that a function

$$F(X, Y_0, Y_1, \ldots, Y_{n-1})$$

of $n+1$ variables

$$X, Y_0, Y_1, \ldots, Y_{n-1}$$

is a first integral of the differential equation (4.1) if $F(x, y, y', \ldots, y^{(n-1)})$ is constant for a every solution y of (4.1). As is well-known, the function

$$F(X, Y_0, Y_1, \ldots, Y_{n-1})$$

is a first integral of (4.1) if and only if it satisfies a partial linear differential equation

$$LF = 0, \tag{4.2}$$

where

$$L = \partial/\partial X + Y_1 \partial/\partial Y_0 + \cdots + Y_{n-1}\partial/\partial Y_{n-2} \\ + A(X, Y_0, Y_1, \ldots, Y_{n-1})\partial/\partial Y_{n-1}.$$

If we have a vector $(F_1, F_2, \ldots, F_n)$ of independent first integrals, i.e. the Jacobian $|\partial F_i/\partial Y_j| \neq 0$, then by the inverse function theorem, for arbitrary constants $c_1, c_2, \ldots, c_n \in \mathbb{C}$, we get functions

$$y_j(x, c_1, c_2, \ldots, c_n), \qquad 0 \leq j \leq n-1$$

satisfying

$$F_i(x, y_0(x,c), y_1(x,c), \cdots, y_n(x,c)) = c_i, \qquad 1 \leq i \leq n. \tag{4.3}$$

Differentiating (4.3) with respect to x, we get

$$\partial F_i/\partial x + \sum_{l=1}^{n} \partial F_i/\partial Y_l \cdot \partial y_l/\partial x = 0. \tag{4.4}$$

Since $F_1, F_2, \ldots, F_n$ are independent, substituting

$$Y_0 = y_0, \quad Y_1 = y_1, \quad \ldots, Y_{n-1} = y_{n-1}$$

in (4.2), we get from (4.4)

$$\partial^j y(x,c)/\partial x^j = y_j(x,c) \quad \text{for} \quad 0 \leq j \leq n-1$$

and

$$y^{(n)} = A(x, y, y', \ldots, y^{(n-1)}),$$

where in the latter equality the derivative is taken with respect to x. So $y(x,c)$ is a general solution of (4.1), i.e. a solution that depends on n parameters. So transcendentally or more precisely modulo the inverse function theorem, looking for a general solution $y(x, c_1, c_2, \ldots, c_n)$ of the non-linear ordinary differential equation (4.1) is equivalent to finding n independent first integrals $F_1 F_2, \ldots, F_n$.

There are two procedures in infinite dimensional Galois theory of the classical authors:

(1) Linearization. They replace the given ordinary non-linear differential equation (4.1) by the linear partial equation (4.2).
(2) Galois theory of linear partial equations. They look for a Galois theory of the partial linear equation (4.2).

We explain why they preferred the linear partial equation to the non-linear ordinary equation. Let

$$u = (u_1, u_2, \ldots, u_n) \mapsto \varphi = (\varphi_1(u), \varphi_2(u), \ldots, \varphi_n(u))$$

be a coordinate transformation of n variables and

$$F = (F_1, F_2, \ldots, F_n)$$

be a vector of independent solutions of (4.2). Then

$$\varphi(F) = (\varphi_1(F), \varphi_2(F), \ldots, \varphi_n(F))$$

is again a vector of independent solutions of (4.2). Hence if we set

$$S = \{(F_1, F_2, \ldots, F_n) \mid F_1, F_2, \ldots, F_n \text{ are independent solutions of } (4.2)\},$$

then the pseudo-group Γ_n of coordinate transformations (which we may regard as an infinite dimensional Lie group) operates on S in such a way that

$$(\Gamma_n, S)$$

is a principal homogeneous space! So we are just as in the ideal theories studied in §3. Of course since Γ_n is not a group, we must clarify the definition that

$$(\Gamma_n, S)$$

is a principal homogeneous space. This is a beautiful idea but there are several difficulties to overcome. Let us study the idea more closely. Galois theory of algebraic equations teaches us that the Galois group is not attached to an algebraic equation but to a field extension. Therefore one has to clarify the ground field, which the classical authors call le domaine de rationalité. So let K be the ground field of the differential equation (4.1) so that K is a differential field such that

$$x \in K \qquad \text{and} \qquad A(x, y, y', \ldots, y^{(n-1)}) \in K(y, y', \ldots, y^{(n-1)}).$$

We take a fixed solution y of (4.1) and we consider a differential field extension $K\langle y\rangle = K(y, y', \ldots, y^{(n-1)})/K$ generated by y over K.

Problem 1. *We start from the ground field and the particular solution y of (4.1). When we pass from the non-linear ordinary to the linear partial, it is not evident at all how to choose a ground field for the linear partial equation (4.2).*

Aside from Problem 1, in the Galois theory of the linear partial equation, we have to choose a vector $(F_1, F_2, \ldots, F_n)$ of independent solutions of (4.1).

Problem 2. *Even if we can choose canonically the ground field $\mathcal{K}$ of the linear partial equation (4.2), namely even if we can solve Problem 1, there is no canonical choice of a vector $(F_1, F_2, \ldots, F_n)$ of independent solutions of (4.2).*

More precisely, let $F' = (F'_1, F'_2, \ldots, F'_n)$ be another vector of independent solutions of (4.2). We denote by

$$\mathcal{K}\langle F\rangle = \mathcal{K}\langle F_1, F_2, \ldots, F_n\rangle$$

$$(\text{ resp. } \mathcal{K}\langle F'\rangle = \mathcal{K}\langle F'_1, F'_2, \ldots, F'_n\rangle)$$

the partial differential field generated over $\mathcal{K}$ by

$$F_1, F_2, \ldots, F_n(\text{ resp. } F'_1, F'_2, \ldots, F'_n).$$

Then $\mathcal{K}\langle F\rangle$ is not $\mathcal{K}$-isomorphic to $\mathcal{K}\langle F'\rangle$. So there is no chance that we have

$$\mathcal{K}\langle F\rangle = \mathcal{K}\langle F'\rangle$$

contrary to the ideal theories. Consequently when $F' = \varphi(F)$, we can not hope that φ induces a $\mathcal{K}$-automorphism of the differential field $\mathcal{K}\langle F\rangle$.

Problem 3. *In the Galois theory of the linear partial equation (4.2), there are always obscurities related with pseudo-groups.*

For example, we have to make clear the definition of a principal homogeneous space of a pseudo-group. As they deal with pseudo-groups, there are also uncomfortable question about domain of convergence.

Among these Problems 1, 2, 3, Problem 3 is less serious. Some authors do not touch Problem 1. Just asserting that Galois theory of the non-linear ordinary differential equation (4.1) is equivalent to Galois theory of the linear partial differential equation (4.2), they devote themselves to Galois theory of the linear partial equation (4.2). Problem 2 annoyed the classical authors very much. Their efforts are concentrated on overcoming this difficulty.

§5. Our theory

Inspired of an idea of Vessiot [V4] published in 1946, which is one of his last articles, we propose a Galois theory of infinite dimension. Thanks to theory of schemes, we can avoid all the problems in §4. Let L/K be an ordinary differential field extension such that the field L is finitely generated over K as an abstract field. Intuitively this is equivalent to considering a non-linear algebraic differential equation with coefficients in K. We attach to the extension L/K a formal group

$$\textbf{Inf-gal}(L/K)$$

of infinite dimension in general. **Inf-gal** is short for infinitesimal Galois group. Here are properties of the formal group.

(1) For a differential intermediate field M of L/K, we have a canonical surjective morphism

$$\textbf{Inf-gal}(L/K) \to \textbf{Inf-gal}(M/K)$$

of formal groups.

(2) Kolchin introduced a strongly normal extension as a differential counter part of a Galois extension in classical Galois theory. Let L/K be a strongly normal extension with Galois group G in the

sense of Kolchin so that G is an algebraic group defined over the field C_K of constants of K. Then the formal group

$$\textbf{Inf-gal}(L/K)$$

is isomorphic to the formal group $\hat{G}$ associated to the differential Galois group G of L/K.

(3) If L is finite algebraic over K, then

$$\textbf{Inf-gal}(L/K) = 0$$

(4) If L is generated by constants over K, then

$$\textbf{Inf-gal}(L/K) = 0$$

(3) and (4) says that the invariant $\textbf{Inf-gal}(L/K)$ ignores finite algebraic difference and constant difference. Since the extensions in (3) and (4) are trivial in general study of differential equations, an invariant may vanish for these types of extensions. Moreover examples show that we can not expect the Galois correspondence.

§6. An application to be expected

The Painlevé equations $(\mathrm{P}_1, \mathrm{P}_2, \ldots, \mathrm{P}_6)$ were discovered around 1900 :

P_1 $\quad y'' = 6y^2 + x;$

P_2 $\quad y'' = 2y^3 + xy + \alpha, \alpha \in \mathbb{C}$ being a parameter;

$\ldots,$

where the derivation is taken with respect to x. The motivation of the discovery was the research of special functions that generalize the Weierstraß $\wp$-function. Since the $\wp$-function is uniform on $\mathbb{C}$ and satisfies an algebraic differential equation

$$\wp'^2 = 4\wp^3 - g_2\wp - g_3, \qquad \text{with} \qquad g_2, g_3 \in \mathbb{C}$$

of the first order. They had to study an algebraic differential equation

$$y'' = R(x, y, y'), \tag{6.1}$$

of the second order whose solutions are uniform on $\mathbb{C}$, where $R(x, y, y')$ is a rational function of x, y, y' with coefficients in $\mathbb{C}$. Since it is difficult to characterize uniformity in terms of differential equation (6.1), they replaced uniformity by an assumption on (6.1) that it has no moving singular points. Painlevé determined all such differential equations and then he threw away those that he could integrate by the so far known functions. This refining led him to the Painlevé equations. So it was natural to expect that the Painlevé equations are irreducible to the classical functions or they define new functions.

Theorem-Conjecture (6.2) (Painlevé 1902). *The first Painlevé equation* P_1 *is irreducible?*

There was a controversy between Painlevé and Liouville on this theorem- conjecture (1902/3)(cf. vol. 3, [P]). At the end of the dispute, Painlevé had resort to Drach's Galois theory. Painlevé knew that the Drach theory [D1] is wrong but he believed that one could sooner or later correct the errors. He was too optimist in this opinion (cf. Painlevé:Sur l'irréductibilité de l'équation $y" = 6y^2 + x$, pp 104-109, vol. 3, [P]). Finally in 1988 Nishioka proved

Theorem (6.3) ([N], [U]). *The first Painlevé equation* P_1 *is irreducible.*

Contrary to Painlevé's guess, Nishioka's proof does not depend on infinite dimensional differential Galois theory. In fact, we had not yet such a theory in 1988! To explain Theorem (6.3), let us recall the definition of classical functions

Definition (6.4) ([U1], [U2]). *We start from the field* $\mathbb{C}(x)$ *of rational functions of one variable and construct recursively the field of classical functions of one variable by iteration of the following permissible operations:*

(1) *The derivation* d/dx*;*

(2) *The four rules of arithmetics:*$+, -, \times, \div$*;*

(3) *Solution of a homogeneous linear differential equation*

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots a_n y = 0 \qquad (a_0 \neq 0),$$

where the a_i*'s are so far recursively constructed classical functions;*

(4) *Substitution in an Abelian function. We can substitute so far constructed functions into Abelian functions. For example, the Weierstrass* $\wp$*-function is an abelian function so that for a classical function* $f(x)$, $\wp(f(x))$ *is a new classical function.*

Definition (6.4) is a practical definition. Namely the meaning of the permissible operations (1), (2), (3)and (4) should be explained. The operations (1) and (2) are equivalent to allowing construction of a differential field. There is a theoretical definition of the permissible operations that unifies (3) and (4) (cf. [U1]). The precise form of theorem (6.2) is

Theorem (6.4). *No solution of* P_1 *is classical.*

As its assertion shows, theorem (6.2) is of negative character. We can illustrate the assertion by the following picture. We compare functions to stars and the adjective *classical* to *observable* by an old telescope. We live on the earth and observe stars by telescope. So a classical function is a star observable by an old telescope. The set of classical functions forms a world around the earth, whereas a first Painlevé transcendent, namely a solution of the first Painlevé equation P_1, twinkles far away from our planet. Thus we can rephrase theorem (6.2) in the following manner.

Theorem (6.5). *We cannot observe any solution of the first Painlevé equation* P_1 *by an old telescope.*

The formal group **Inf-gal** offers us a new invariant of a non-linear ordinary differential equation. So far we had no invariant of a non-linear differential equation. The formal group **Inf-gal** allows us to observe solutions of P_1.

Problem (6.6). *For a solution* y *of* $\mathrm{P}_1 : y'' = 6y^2 + x$, *calculate*

$$\mathbf{Inf\text{-}gal}(\mathbb{C}(x, y, y')/\mathbb{C}(x)).$$

If we can calculate $\mathbf{Inf\text{-}gal}(\mathbb{C}(x, y, y')/\mathbb{C}(x))$, then it will be the first positive result on the nature of a solution of P_1, which implies in particular the irreducibility of P_1. There is an old conjectural result due to Drach [D2] in 1915 which depends on his incomplete theory.

Theorem-Conjecture (6.6) (Drach). *The Galois group of* P_1 *is the Lie pseudo-group of coordinate transformations of 2 variables that leave the area invariant:*

$$\{(u_1, u_2) \mapsto \varphi(u) = (\varphi_1(u), \varphi_2(u)) | \varphi(u) \textit{ is a coordinate transformation with the Jacobian } \partial(\varphi_1, \varphi_2)/\partial(u_1, u_2) = 1\}.$$

For formal groups and Lie pseudo-groups, see §7.

§7. Lie pseudo-group and formal group

Let us recall the definition of a formal group. General references for formal groups are Serre [S] and Hazewinkel [H].

Definition (7.1). *A formal group of dimension n over a commutative ring R is an n-tuple $F = (f_i)$ of formal power series*

$$f_i(u, v) \in R[[u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n]] = R[[u, v]]$$

such that

$$\begin{aligned} &(1) \quad F(u, 0) = u, \qquad F(0, v) = v, \\ &(2) \quad F(u, F(v, w)) = F(F(u, v,), w), \end{aligned}$$

where

$$u = (u_1, u_2, \ldots, u_n), \qquad 0 = (0, 0, \ldots, 0), \ldots \ \textit{etc.}.$$

We can show that there exists an n-tuple

$$\theta(u) = (\theta_1(u), \theta_2(u), \ldots, \theta_n(u)) \in R[[u]]^n = R[[u_1, u_2, \ldots, u_n]]^n$$

such that

$$\theta(0) = 0, \qquad \text{and} \qquad F(u, \theta(u)) = F(\theta(u), u) = 0.$$

Example (7.2). *A formal group arises from a Lie group. Let G be a real or complex analytic Lie group of dimension n. Writing the group law*

$$G \times G \to G$$

locally at $1 \in G$, we get a formal group $\hat{G}$ of dimension n over $\mathbb{R}$ or over $\mathbb{C}$ associated to G. For the additive group $\mathbb{R}$ of real numbers, $\hat{\mathbb{R}}$ is $F(u, v) = u + v$. For the multiplicative group $\mathbb{R}^*$ of non-zero real numbers, $\hat{\mathbb{R}}^*$ is $F(u, v) = u + v + uv$.

Definition (7.3). *A morphism*

$$\varphi : F = (f_1, f_2, \ldots, f_m) \to G = (g_1, g_2, \ldots, g_n)$$

of formal groups over R is an n-tuple

$$\varphi = (\varphi_1, \varphi_2, \ldots, \varphi_n) \in R[[u_1, u_2, \ldots, u_m]]^n$$

such that

$$\varphi(0) = 0 \qquad \textit{and} \qquad \varphi(F(u, v)) = G(\varphi(u), \varphi(v)).$$

Example (7.4). *A morphism of Lie groups gives rise to a morphism of the associated formal groups. Let*

$$\varphi : G_1 \to G_2$$

be a morphism of analytic Lie groups. Writing φ locally at 1, we get a morphism $\hat{\varphi} : \hat{G}_1 \to \hat{G}_2$ of formal groups. Particularly if we consider a morphism

$$\varphi : \mathbb{R} \to \mathbb{R}^*, \qquad u \mapsto \exp u,$$

then we get

$$\hat{\varphi} : \hat{\mathbb{R}} \to \hat{\mathbb{R}}^*$$

given by a power series $\exp u - 1$.

Let now A be an R-algebra. We denote by $N(A)$ the ideal of nilpotent elements of the ring A:

$$N(A) = \{x \in A| \\ x^m = 0 \text{ for a certain positive integer } m \text{ depending on } x\}.$$

A formal group F of dimension n over R defines a group structure on

$$N(A)^n = \{(a_1, a_2, \ldots, a_n) | a_i \in N(A) \quad 1 \le i \le n\},$$

which we denote by $\mathbf{F}(A)$. In fact, we define the product of two elements $a, b \in N(A)^n$ by

$$a \cdot b = F(a, b) = (f_1(a, b), f_2(a, b), \ldots, f_n(a, b)).$$

Then the product is associative, $0 = (0, 0, \ldots, 0)$ is the neutral element and the inverse a^{-1} is given by $\theta(a)$. Since the construction is functorial on A, we get a group functor

$$\mathbf{F} : \text{Category of } R\text{-algebras} \to \text{Category of groups}.$$

Let $\varphi : F \to G$ be a morphism of formal groups over R. Then the morphism φ induces a morphism $\varphi(A) : \mathbf{F}(A) \to \mathbf{G}(A)$ of groups, Since $\varphi(A)$ is functorial on A, we get a morphism $\varphi : \mathbf{F} \to \mathbf{G}$ of group functors. We can recover the formal group F from the group functor $\mathbf{F}$. More precisely we have,

Proposition (7.5). *For formal groups F, G over R, we have*

$$\mathrm{Hom}_{\text{formal group}}(F, G) \xrightarrow{\sim} \mathrm{Hom}_{\text{group functor}}(\mathbf{F}, \mathbf{G}).$$

Now we treat coordinate transformations of dimension n. We assume for simplicity $n = 1$. Let A be a ring and

$$\varphi(x) = a_0 + a_1x + a_2x^2 + \cdots,$$
$$\psi(x) = b_0 + b_1x + b_2x^2 + \cdots$$

two power series with coefficients in A:$\varphi(x), \psi(x) \in A[[x]]$. If we calculate formally the composite, we get

$$\psi \circ \varphi = b_0b_1a_0 + b_2a_0^2 + \cdots + (b_1a_1 + 2b_2a_0a_1 + 3b_3a_0^2a_1 + \cdots)x + \cdots.$$

Since there is a problem of convergence of coefficients, $\psi \circ \varphi$ is not an element of $A[[x]]$. If a_0 is nilpotent or if there exists a positive integer n with $a^n = 0$, then $\psi \circ \varphi$ is an element of $A[[x]]$. We set

$$\begin{aligned}\Gamma(A) = \{\varphi(x) = a_0 + a_1x + a_2x^2 + \cdots \in A[[x]]|\\ \varphi(x) \text{ is almost identity or } \varphi(x) \equiv x \text{ modulo nilpotent element,}\\ \text{in other words } a_0, a_1 - 1, a_2, a_3, \ldots \in N(A)\}\end{aligned}$$

So if

$$\varphi(x), \psi(x) \in \Gamma(A),$$

then

$$\psi \circ \varphi \in \Gamma(A).$$

We can show that for $\varphi(x) \in \Gamma(A)$ the inverse function $\varphi^{-1}(x) \in \Gamma(A)$ so that $\Gamma(A)$ is a group.

Remark. *Another natural way of introducing the group $\Gamma(A)$ is the group of infinitesimal automorphisms of the topological ring $A[[x]]$. Namely it is easy to see*

$$\begin{aligned}\Gamma(A) = \{\varphi : A[[x]] \to A[[x]]|\varphi \textit{ is an } A\textit{-automophism of the ring } A[[x]]\\ \textit{continous with respect to the } (x)\textit{-adic topology of which}\\ \textit{the reduction}\bar{\varphi} : (A/N(A))[[x]] \to (A/N(A))[[x]] \textit{ is the identity.}\}.\end{aligned}$$

Since $\Gamma(A)$ is functorial on A, we get the group functor

$$\Gamma_1 : \text{Category of } \mathbb{Q}\text{-algebras} \to \text{Category of groups}, \quad A \mapsto \Gamma(A)$$

of infinitesimal coordinate transformations of 1-variable. We can regard the group functor Γ_1 as a formal group over $\mathbb{Q}$. In fact, let us consider two formal power series

$$\varphi(x) = u_0 + (1 + u_1)x + u_2x^2 + \cdots,$$
$$\psi(x) = v_0 + (1 + v_1)x + v_2x^2 + \cdots,$$

where

$$u_0, u_1, u_2, \ldots, v_0, v_1, v_2, \ldots$$

are variables over $\mathbb{Q}$. Then formally we have

$$\begin{aligned}\psi \circ \varphi(x) =& v_0(1+v_1)u_0 + v_2 u_0^2 + \ldots \\ &+ (1 + u_1 + v_1 + 2b_2 u_0(1+u_1) + 3v_3 u_0^2(1+u_1) + \ldots)x + \ldots \\ =& f_0(u,v) + (1 + f_1(u,v))x + f_2(u,v)x^2 + \ldots.\end{aligned}$$

We set

$$F = (f_0(u,v), f_1(u,v), \ldots) \in \mathbb{Z}[[u,v]]^\infty.$$

Then we have

$$\begin{aligned}&(1) \quad F(u,0) = u, \qquad F(0,v) = v,\\ &(2) \quad F(u, F(v,w)) = F(F(u,v,),w).\end{aligned}$$

Namely $F(u,v)$ is a formal group of infinite dimension such that the associated group functor $\mathbf{F}$ is the group functor Γ_1. Ritt [R] had a similar idea of introducing the formal group $F = (f_0(u,v), f_1(u,v), \ldots)$ of infinite dimension (See also [W]). Similarly we introduce the group functor

$$\Gamma_n : \text{Category of } \mathbb{Q}\text{-algebras} \to \text{Category of groups}$$

of infinitesimal coordinate transformations of n-variables. Let us now consider the group subfunctor

$$G_1(A) = \{\varphi(x) \in \Gamma_1(A) | a_1 = a_2 = \ldots = 0, \text{i.e. } \varphi(x) = a_0 + x\}$$

for every algebra A. We can define the group subfunctor G_1 by a differential equation:

$$G(A) = \{\varphi(x) \in \Gamma_1(A) | d\varphi/dx = 1\}$$

for every algebra A. Similarly if we set

$$G_3(A) = \{\varphi(x) \in \Gamma(A) | \{\varphi(x); x\} = 0\},$$

for every algebra A, then G_3 is a group subfunctor of Γ_1. Here we denote by $\{y; x\}$ the Schwarzian derivative

$$(d^3y/dx^3)/(dy/dx) - (3/2)[(d^2y/dx^2)/(dy/dx)]^2.$$

G_1 and G_3 are subgroup functors of Γ_1 defined by a differential equation.

Definition (7.6). *A Lie-Ritt functor is a group subfunctor of* Γ_n *defined by differential equations.*

G_1 and G_3 are examples of Lie-Ritt functor.

Remark (7.7). *We should clarify the coefficients. Namely we should say that a Lie-Ritt functor defined over a ring* R *is the group subfunctor of* Γ_{nR} *consisting of solutions of differential equations defined over* $R[[x]]$. *Here* Γ_{nR} *is the restriction of the group functor* Γ_n *to the category of* R*-algebras, which is a subcategory of the category of* $\mathbb{Q}$*-algebras.*

Definition (7.8). *Let* G, H *be Lie-Ritt functors defined over a ring* R. *Then a morphism* $G \to H$ *of Lie-Ritt functors is a morphism* $G \to H$ *of group functors.*

Example (7.9). *The Lie-Ritt functor* G_1 *is isomorphic to* $\hat{\mathbb{R}}$ *that is isomorphic to* $\hat{\mathbb{R}}^*$. *The Lie-Ritt functor* G_3 *is isomorphic to* $\widehat{\mathrm{SL}}_{2\mathbb{R}}$. *Since* G_1 *is defined over* $\mathbb{Q}$ *and* $\hat{\mathbb{R}}$ *is defined over* $\mathbb{R}$, *precisely speaking, we have to say either that the restriction* $G_{1\mathbb{R}}$ *of the functor* G_1 *to the category of* $\mathbb{R}$*-algebras is isomorphic to* $\hat{\mathbb{R}}$ *or that the Lie-Ritt functor* G_1 *is isomorphic to the formal group* $\hat{\mathbf{G}}_{\mathbf{a}\mathbb{Q}}$ *associated to the additive group scheme* $\mathbf{G}_{\mathbf{a}\mathbb{Q}}$. *A similar remark should be done for* G_3.

Definition (7.8) of morphism seems more natural than the traditional definition using prolongations.

Question (7.10). *Let* G, H *be Lie-Ritt functors defined over* $\mathbb{C}$, *which are traditionally called Lie pseudo-groups. Then*

$$\mathrm{Hom}_{\text{Lie pseudo-group}}(G, H) = \mathrm{Hom}_{\text{Lie-Ritt}}(G, H)\ ?$$

§8. Construction of Inf-gal

Let (A, δ) be a differential $\mathbb{Q}$-algebra. We denote the abstract $\mathbb{Q}$-algebra A by $A^\natural$, when we emphasize that we consider the abstract algebra. We have a morphism

$$i : A \to A^\natural[[t]] \qquad a \mapsto \sum_{n=0}^{\infty} \frac{\delta^n a}{n!} t^n$$

of rings. In fact, this is a morphism

$$(A, \delta) \to (A^\natural[[t]], d/dt)$$

of differential rings. We call i the universal Taylor morphism. For, i is universal among Taylor morphisms. A Taylor morphism is a differential ring morphism

$$(A, \delta) \to (B[[t]], d/dt),$$

where B is an abstract $\mathbb{Q}$-algebra. Let now L/K be an ordinary differential field extension such that $L^\natural$ is finitely generated over $K^\natural$. We have a commutative diagram

$$\begin{array}{ccc} L & \xrightarrow{\ i\ } & L^\natural[[t]] \\ \uparrow & & \uparrow \\ K & \longrightarrow & K^\natural[[t]]. \end{array}$$

Let us now take a transcendence basis

$$u_1, u_2, \ldots, u_n$$

of $L^\natural/K^\natural$ so that we have derivations

$$\frac{\partial}{\partial u_i} : K^\natural(u_1, u_2, \ldots, u_n) \to K^\natural(u_1, u_2, \ldots, u_n) \qquad \text{for} \qquad 1 \le i \le n.$$

Since $L^\natural/K^\natural(u)$ is algebraic, the derivations

$$\partial/\partial u_i : K^\natural(u) \to K^\natural(u)$$

extends to derivations $L^\natural \to L^\natural$, which we also denote by $\partial/\partial u_i$. So we get a partial differential field

$$(L^\natural[[t]][t^{-1}], \{d/dt, \partial/\partial u_1, \partial/\partial u_2, \ldots, \partial/\partial u_n\}).$$

Here the $\partial/\partial u_i$ operate on the coefficients of a formal Laurent series:

$$\frac{\partial}{\partial u_i} \sum_{n>>-\infty} a_n t^n = \sum_{n>>-\infty} \frac{\partial a_n}{\partial u_i} t^n.$$

We define $\mathcal{L}$ as the partial differential subfield of $L^\natural[[t]][t^{-1}]$ generated by $i(L)$ and the field $L^\natural$ of constant Laurent series. We denote by $\mathcal{K}$ the partial differential subfield of $L^\natural[[t]][t^{-1}]$ generated by $i(K)$ and $L^\natural$. So we get a partial differential field extension $\mathcal{L}/\mathcal{K}$. The definition of the extension $\mathcal{L}/\mathcal{K}$ involves the K-derivations $\partial/\partial u_i : L^\natural \to L^\natural$. But since we added $L^\natural$ in construction, the extension $\mathcal{L}/\mathcal{K}$ is independent of the choice of the K-derivations $\partial/\partial u_i$ or of the transcendence basis

$$u_1, u_2, \ldots, u_n.$$

So we constructed $\mathcal{L}/\mathcal{K}$ canonically from L/K. This is the key point to avoid Problems 1 and 2 of §4.

We have the universal Taylor morphism

$$j : (L^{\natural}, \{\partial/\partial u_1, \partial/\partial u_2, \dots, \partial/\partial u_n\} \to L^{\natural}[[w_1, w_2, \dots, w_n]]$$

sending an element $a \in L$ to

$$\sum_{m \in \mathbb{N}^n} \frac{1}{m!} \frac{\partial^{|m|}}{\partial u_1^{m_1} \partial u_2^{m_2} \dots \partial u_n^{m_n}} w_1^{m_1} w_2^{m_2} \dots w_n^{m_n} \in L^{\natural}[[w_1, w_2, \dots, w_n]].$$

Here we use a usual notation: For $m = (m_1, m_2, \dots, m_n) \in \mathbb{N}^n$,

$$m! = m_1! m_2! \dots m_n!, \qquad |m| = \sum_{i=1}^{n} m_i.$$

So we get a differential morphism

$$h : L^{\natural}[[t]][t^{-1}] \to L^{\natural}[[w, t]][t^{-1}]$$

of expanding the coefficients:

$$h(\sum_{i>>-\infty} a_i t^i) = \sum_{i>>-\infty} j(a_i) t^i)$$

for $\sum_{i>>-\infty} a_i t^i \in L^{\natural}[[t]][t^{-1}]$. Hence by restriction to the subalgebra $\mathcal{L}$, we obtain a differential morphism

$$\mathcal{L} \to L^{\natural}[[w, t]][t^{-1}]$$

which we denote again by h.

We now consider infinitesimal deformations of h. Namely we set

$$\mathcal{F}_{\mathcal{L}/\mathcal{K}}(A) = \{f : \mathcal{L} \to A[[w, t]][t^{-1}] | f \text{ is a } \mathcal{K}\text{-differential morphism such that } f \equiv h \text{ modulo nilpotent elements of } A\}$$

for an $L^{\natural}$-algebra A so that $\mathcal{F}_{\mathcal{L}/\mathcal{K}}(A)$ is the set of infinitesimal deformations of h in A. Let $(\mathrm{Alg}/L^{\natural})$ be the category of $L^{\natural}$-algebras. We get a functor

$$\mathcal{F}_{\mathcal{L}/\mathcal{K}} : (\mathrm{Alg}/L^{\natural}) \to (\mathrm{Set}),$$

where we denote by (Set) the category of sets. We can show that there exists a Lie-Ritt functor (or Lie pseudo-group in the traditional language) $\mathcal{G}$ defined over $L^{\natural}$ that operates on the functor $\mathcal{F}_{\mathcal{L}/\mathcal{K}}$ in such a way that

$$(\mathcal{G}, \mathcal{F}_{\mathcal{L}/\mathcal{K}})$$

is a principal homogeneous space. The Lie-Ritt functor $\mathcal{G}$ is by definition the infinitesimal Galois group $\mathbf{Inf\text{-}gal}(L/K)$ of the given extension L/K.

Example (8.1). *Let us see what we have done by one of the simplest examples. Let L be the differential subfield of $(\mathbb{C}[[x]][x^{-1}], d/dx)$ generated by*

$$y = \exp x$$

over $K = \mathbb{C}(x)$. So $L = (\mathbb{C}(x, y), d/dx)$ and we have $dy/dx = y$. It follows from the definition of the universal Taylor morphism

$$i : L \to L^\natural[[t]], \qquad i(y) = y \exp t \in L^\natural[[t]].$$

Since L/K is a transcendental extension generated by y, we take $u_1 = y$ as a transcendence basis. So

$$i(L).L^\natural = L^\natural(t, \exp t) \subset L^\natural[[t]][t^{-1}]$$

is closed under d/dt and $\partial/\partial y$ and hence

$$\mathcal{L} = L^\natural(t, \exp t).$$

We have evidently $\mathcal{K} = L^\natural(t)$. For a $L^\natural$-algebra A, a $\mathcal{K}$-morphism

$$f : \mathcal{L} \to A[[t]][t^{-1}]$$

is defined by sending the generator $\exp t$ *over $\mathcal{K}$ to $c.\exp t$ with $c \in A$. So*

$$\mathcal{F}_{\mathcal{L}/\mathcal{K}}(A) = \{f : \mathcal{L} \to A[[t]][t^{-1}]| \\ \text{There exists } c \in A \text{ such that } c \equiv 1 \bmod N(A), \quad f(\exp t) = c.\exp t\}.$$

The formal group

$$\hat{\mathbf{G}}_{\mathbf{m}L^\natural}(A) = \{c \in A | c \equiv 1 \bmod N(A)\}$$

operates on $\mathcal{F}_{\mathcal{L}/\mathcal{K}}(A)$. Namely for $f \in \mathcal{F}_{\mathcal{L}/\mathcal{K}}(A)$ with $f(\exp t) = c.\exp t$ and

$$c' \in \hat{\mathbf{G}}_{\mathbf{m}L^\natural}\{c \in A | c \equiv 1 \bmod N(A)\},$$

$$c'f \in \mathcal{F}_{\mathcal{L}/\mathcal{K}}(A)$$

is the $\mathcal{K}$-morphism that sends $\exp t$ *to $c'f(\exp t) = (c'c).\exp t$:*

$$(c'f)(\exp t) = (c'c).\exp t.$$

So

$$(\hat{\mathbf{G}}_{\mathbf{m}L^\natural}(A), \mathcal{F}_{\mathcal{L}/\mathcal{K}}(A))$$

is a principal homogeneous space and hence $\mathbf{Inf}\text{-}\mathbf{gal}(L/K) = \hat{\mathbf{G}}_{\mathbf{m}L^\natural}$.

The argument above allows us to prove in general that for a strongly normal extension L/K with Galois group G, we have

$$\mathbf{Inf}\text{-}\mathbf{gal}(L/K) = \hat{G}_{L^\natural}.$$

References

[D1] Drach, J., Essai sur une théorie générale de l'intégation et sur la classification des transcendentes, Ann. sci. de l'École normale Sup. (3), **15** (1898), 243-384.

[D2] ———, Sur le groupe de rationalité des équations du second ordre de M. Painlevé, Bull. Sci. Math., **39** (1915), 1-ère partie, 149-166.

[H] Hazewinkel, M., "Formal groups and applications", Academic Press, New York, San Francisco, London, 1978.

[K] Kolchin, E., "Differential algebra and algebraic groups", Academic Press, New York, San Francisco, London, 1973.

[L] Lie, S., Influence de Galois sur le développement des mathématiques, in "Vol 6, Gesammelte Abhandlugen", Johnson reprint corporation, New York, London, 1973, pp. 562-601.

[N] Nishioka, K., A note on the transcendency of Painlevé's first transcedent, Nagoya Math. J., **109** (1988), 63 - 67.

[P] Painlevé, P., "Oeuvres", Édition du CNRS, Paris, 1973-1975.

[Pi] Picard, E., "Traité d'analyse", Gauthier-Villars, Paris, 1896.

[Po1] Pommaret, J. F., "Differential Galois theory", Gordon and Breach, New York, London, Paris, 1983.

[Po2] ———, "Lie pseudogroup and mechanics", Gordon and Breach, New York, London, Paris, 1987.

[R] Ritt, J. F., Associative differential operations, Ann. Math., **51** (1950), 756-765.

[S] Serre, J.-P., "Lie algebras and Lie groups", Benjamin, New York, 1965.

[U1] Umemura, H., On the irreducibility of the first differential equation of Painlevé, in "Algebraic geometry and commutative algebra in honor of Masa- yoshi NAGATA", Kinokuniya, Tokyo, 1987, pp. 771-789.

[U2] ———, On the irreducibility of the first Painlevé equation, Sugaku Expositions, **2** (1989), 231-252.

[U3] ———, On a class of numbers generated by differential equations related with algebraic groups, Nagoya Math. J., **133** (1994),.

[U4] ———, Galois Theory of algebraic and differential equations, Nagoya Math. J., to appear.

[U5] ———, Differential Galois theory of infinite dimension, Nagoya math. J., to appear.

[V1] Vessiot, E., Sur la théorie des groupes continus, Ann. Sci. École Norm. Sup., (3), **20** (1903), 411-451.

[V2] ———, Sur l'intégration des systèmes différentiels qui admettent des groupes continus de transformations,, Acta math., **28** (1904), 307-349.

[V3] ———, Sur la théorie de Galois et ses diverses généralisations, Ann. Sci. École Norm. Sup. (3), **21** (1904), 9-85.

[V4] ———, Sur une généralisation de la réductibilité des équations et systèmes d'équations finies ou différentielles, Ann. Sci. École Norm. Sup. (3), **63** (1946), 1-22.

[W] Weisfeiler, On Lie algebras of differential formal groups of Ritt, Bull. Amer. Math. Soc., **84** (1978), 127-130.

[Wi] Winter, D. J., "The structure of fields, GTM 16", Springer-Verlag, New York, Heiderberg, Berlin, 1974.

Graduate school of Polymathematics
Nagoya University
Chikusa-ku, Nagoya 464-01
Japan

Advanced Studies in Pure Mathematics 25, 1997
CR-Geometry and Overdetermined Systems
pp. 386–405

Prolongation Projection Commutativity Theorem

Jose M. Veloso

Abstract.

If the symbol g_k of a SPDE R_k is 2-acyclic, then the operations of prolongation and projection on R_k commute

$$\rho_{k+l+1}^{k+l+2}((R_k)_{+l+2}) = \left(\rho_{k+l}^{k+l+1}((R_k)_{+l+1})\right)_{+1}.$$

We apply this to study contact of three-dimensional CR-manifolds.

§1. Introduction

S. Chern and J. Moser [2] proved that two real hypersurfaces of $\mathbf{C}^2$ have a contact of fifth order and in the non-umbilic case of sixth order. The G-structure associated to a real hypersurface is of order two but their definition involves fifth order derivatives. Studying these facts through the SPDE of jets of biholomorphic functions between the real hypersurfaces, we found the following theorem:

Theorem 1.1 (Prolongation projection commutativity theorem). *Let $R_k \subset J_k(M, N, \rho)$ be a system of partial differential equations such that*

(i) $\alpha : R_k \to N$ *is a submersion*
(ii) the symbol g_k of R_k is 2-acyclic
(iii) g_{k+1} is a vector bundle on $(\rho_k^{k+1})^{-1}(R_k)$

Then, for every $l \geq 0$,

$$\rho_{k+l+1}^{k+l+2}((R_k)_{+l+2}) = \left(\rho_{k+l}^{k+l+1}((R_k)_{+l+1})\right)_{+1}.$$

Theorem 1.2 (Formal integrability theorem [4]). *Under the hypothesis of the above theorem and the assumption that*

$$\rho_k^{k+1}((R_k)_{+1}) = R_k,$$

Received October 2, 1995.

we get that

$$\rho_{k+l}^{k+l+1} : (R_k)_{+l+1} \to (R_k)_{+l}$$

is a submersion for every $l \geq 1$.

The formal integrability theorem for linear PDE systems was first proved by Quillen [6] and with weak assumptions by Goldschmidt [3], who also proved it in the non-linear case [4]. A version of this theorem using involutivity is in Kuranishi [5]. All these publications used the set R_k of integral jets of the PDE system to prove the theorem. Ruiz [7, 8, 9] utilizes the sheaf I_k of functions which are null on R_k; this approach seems to us more natural and we follow this approach.

In Section 2 we present the basic facts following [7, 8, 9]. Section 3 contains the proof of Theorem 1.1. In Section 4 we apply the theorem to study contact of three-dimensional CR-manifolds. Corollary 4.1 shows that the G-structure associated to a CR-manifold M is the projection in order two of fifth order jets which have fifth-order contact with the hyperquadric $\mathrm{Im}w = z\bar{z}$. Theorem4.3 relates the normal form of M [2] with the invariants of Cartan [1].

I'd like to thank to A.M.Rodrigues and J.Verderesi for several discussions on this subject. I dedicate this paper to prof.M.Kuranishi, in occasion of his seventieth birthday. He was very kind to me and my contacts with him were always stimulating.

§2. Basic definitions

Let M, N be manifolds, $T = TN$ the tangent bundle of N, $\rho : M \to N$ a submersion, and $J_k = J_k(M, N, \rho)$ the manifold of k-jets of local sections of $\rho : M \to N$. Denoting by $\rho_l^k : J_k \to J_l$, $k > l$, the canonical projections, and by $\rho_0^k = \beta_k : J_k \to M$ and $\rho_{-1}^k = \alpha_k : J_k \to N$ the projections to *target* and *source* respectively, the sheaf of algebras of C^∞-functions on J_k will be denoted by F_k. If $Z_k \in J_k$, let be $Z_l = \rho_l^k(Z_k)$, for $l \leq k$. In particular, $\beta_k(Z_k) = Z$ and $\alpha_k(Z_k) = z$.

We identify Z_k with the linear application (cf. [7])

$$Z_k = (Z_k)_* : T_z N \to T_{Z_{k-1}} J_{k-1}$$

given by

$$(Z_k)_* = (j^{k-1}\sigma)_* v$$

where $Z_k = j_z^k \sigma$.

If θ is a vector field on N, we define the *formal derivative*

$$\partial_\theta : F_k \to F_{k+1}$$

by

$$(\partial_\theta f)(Z_{k+1}) = df(Z_{k+1})_*(\theta_z))$$

where $f \in F_k$. This derivative has the properties
(i) $\partial_{a.\theta} f = a.\partial_\theta f$
(ii) $\partial_{[\theta,\eta]} = [\partial_\theta, \partial_\eta]$
where a is a real function on N, and η is a vector field on N. Let $x = (x^1, \cdots, x^n)$ be a chart on $U \subset N$, $(x, y) = (x^1, \cdots, x^n, y^1, \cdots, y^m)$ a chart on $\rho^{-1}(U)$, and $(x, y^j_\alpha, 0 \leq j \leq m, 0 \leq \mid \alpha \mid \leq k)$ a chart on $(\rho^k_0)^{-1}(U)$, where

$$y^j_\alpha(j^k_z \sigma) = \frac{\partial^{|\alpha|}\sigma^j}{\partial x^\alpha}(z)$$

and $\sigma = (\sigma^1, \cdots, \sigma^m)$ is a section of ρ on U.

In this coordinate system

$$(Z_{k+1})_*(\frac{\partial}{\partial x^i}) = \frac{\partial}{\partial x^i} + \sum_{|\alpha| \leq k} y^j_{\alpha+1_i}(Z_{k+1}) \frac{\partial}{\partial y^j_\alpha}$$

and

$$\partial_i f = \frac{\partial}{\partial x^i} + \sum_{|\alpha| \leq k} \frac{\partial f}{\partial y^j_\alpha} y^j_{\alpha+1_i}$$

where $f \in F_k$, and ∂_i denotes ∂_θ when $\theta = \partial/\partial x^i$.

Let $Q_k = \mathrm{Ker}(\rho^k_{k-1})_*$ be the vector bundle on J_k of vertical tangent vectors with respect to ρ^k_{k-1}. The fiber of Q_k on Z_k is denoted by Q_{Z_k}. The dual bundle of Q_k is denoted by Q^*_k. If $f \in F_k$, then $d(\partial_\theta f) \mid_{Q_{Z_{k+1}}}$ depends only on $df \mid_{Q_{Z_k}}$ and $\theta(z)$. So we have a map

$$d_K : T_z N \otimes Q^*_{Z_k} \to Q^*_{Z_{k+1}}$$

defined by

$$d_K(\theta_z \otimes df \mid_{Q_{Z_k}}) = d(\partial_\theta f) \mid_{Q_{Z_{k+1}}} .$$

In coordinates

$$d_K(\frac{\partial}{\partial x^i} \otimes dy^j_\alpha \mid_{Q_{Z_k}}) = dy^j_{\alpha+1_i} \mid_{Q_{Z_{k+1}}} .$$

If $Q^*_{Z_\infty} = \sum_{k\geq 0} Q^*_{Z_k}$, we define *Koszul's complex* $(\Lambda T_z \otimes Q^*_{Z_\infty}, d_K)$ by

$$d_K(v_1 \wedge \cdots \wedge v_l \otimes \mu) = \\ \sum_{i=1}^{l} (-1)^{i+1} v_1 \wedge \cdots \wedge v_{i-1} \wedge v_{i+1} \wedge \cdots \wedge v_l \otimes d_K(v_i \otimes \mu).$$

Definition 2.1. *A system of partial differential equations (SPDE) is a subsheaf of ideals I_k of F_k locally finitely generated. The subset R_k of J_k,*

$$R_k = \{Z_k \in J_k : f(Z_k) = 0, \forall f \in I_k\}$$

is the set of integral jets of I_k. In case (R_k, N, α_k) is a submersion, I_k(or R_k) is said to be regular. The subsheaf of ideals of F_{k+1} generated by

$$(\rho_k^{k+1})^* I_k \cup \{\partial_\theta f : f \in I_k, \theta \in \Gamma(TN)\}$$

is called the prolongation $(I_k)_{+1}$ of I_k.

We shall write I_{k+1} instead of $(I_k)_{+1}$. The subsheafs $I_{k+l}, l \geq 2$ are defined inductively. Suppose $(x^1, \cdots, x^n)$ is a chart on N, and $f_p, 1 \leq p \leq r$ a system of (local) generators of I_k, then a system of (local) generators of I_{k+l} is given by $\{\partial_\alpha f_p : 1 \leq p \leq r, 0 \leq\mid \alpha \mid\leq l\}$, where $\alpha = (\alpha_1, \cdots, \alpha_n)$ and $\partial_\alpha f_p = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n} f_p$. We will assume that F_k is contained in F_{k+l}, through the inclusion $(\rho_k^{k+l})^* : F_k \to F_{k+l}$.

Definition 2.2. *The symbol h_{Z_k} at the integral jet Z_k of I_k is the subset of $Q^*_{Z_k}$ defined by*

$$h_{Z_k} = \{df \mid_{Q_{Z_k}} : f \in I_k\}$$

The family of symbols on R_k is denoted by h_k, i.e. $(h_k)_{Z_k} = h_{Z_k}$.

If $Z_{k+1} \in (\rho_k^{k+1})^{-1}(Z_k)$, with $Z_k \in R_k$, put $h_{Z_{k+1}} = d_K(T_z \otimes h_{Z_k})$, and $h_{Z_{k+l+1}} = d_K(T_z \otimes h_{Z_{k+l}}), l \geq 1$ for every $Z_{k+l+1} \in (\rho_k^{k+l+1})^{-1}(Z_k)$. Also, we put

$$h_{k+l} = \{h_{Z_{k+l}} : Z_{k+l} \in (\rho_k^{k+l})^{-1}(R_k)\}.$$

In case $Z_{k+l} \in R_{k+l}, h_{Z_{k+l}}$ coincides with the symbol of I_{k+l} at Z_{k+l}, i.e. $h_{Z_{k+l}} = dI_{k+l} \mid_{Q_{Z_{k+l}}}$. Let us put $h_{Z_\infty} = \sum_{l\geq 0} h_{Z_{k+l}}$. Then $h_{Z_\infty} \subset Q^*_{Z_\infty}$, and from $d_K(T_z \otimes h_{Z_\infty}) \subset h_{Z_\infty}$ it follows that $(\Lambda T_z \otimes h_{Z_\infty}, d_K)$ is a subcomplex of Koszul's complex. The *(j,k+l+1)-th homology group* of this subcomplex is

$$H_{(j,k+l+1)}(Z_k) = \frac{\ker(d_K : \Lambda^j T_z \otimes h_{Z_{k+l+1}} \to \Lambda^{j-1} T_z \otimes h_{Z_{k+l+2}})}{d_K(\Lambda^{j+1} T_z \otimes h_{Z_{k+l}})}$$

for $l \geq 0$.We say that h_{Z_k} is *r-acyclic* if $H_{j,k+l+1}(Z_k) = 0$, for $0 \leq j \leq r$, $l \geq 0$ and h_k is *r-acyclic* if h_{Z_k} is r-acyclic for every $Z_k \in R_k$. Clearly , h_k is 0-acyclic. If $g_{Z_k} \subset Q_{Z_k}$ is defined by $g^{\perp}_{Z_k} = h_{Z_k}$, then g_{Z_k} is also called the *symbol* of R_k at Z_k. It is proved in [9] that h_{Z_k} is 1-acyclic if and only if g_{Z_k} is 2-acyclic in the sense of [4].

§3. The prolongation projection commutativity theorem

Let us put

$$I^{k+l+1}_{k+l} = \{f \in F_{k+l} : (\rho^{k+l+1}_{k+l})^* f \in I_{k+l+1}\}$$

for $l \geq 0$. It is clear that $I_{k+l} \subset I^{k+l+1}_{k+l}$. If R^{k+l+1}_{k+l} denotes the set of integral jets of I^{k+l+1}_{k+l}, then $\rho^{k+l+1}_{k+l}(R_{k+l+1}) \subset R^{k+l+1}_{k+l}$. In general the equality doesn't hold. The following proposition gives a condition for this.

Proposition 3.1. *If I_k is a regular SPDE, and h_{k+l+1} is a vector bundle on $(\rho^{k+l+1}_k)^{-1}(R_k)$,then*

$$R^{k+l+1}_{k+l} = \rho^{k+l+1}_{k+l}(R_{k+l+1}).$$

Furthermore, if $f_p, 1 \leq p \leq r$ are local independent generators of I_k, then I^{k+l+1}_{k+l} is generated by

$$\{\partial_\alpha f_p, 1 \leq p \leq r, \mid \alpha \mid \leq l; g_t, 1 \leq t \leq s\},$$

where $g_t = \sum_{p=1}^{r}(\sum_{|\beta|=l+1} a^{\beta,p}_t \partial_\beta f_p + b^p f_p)$ *with* $a^{\beta,p}_t \in F_k, b^p \in F_{k+l+1}$.

Proof: Let U_k be an open set in J_k, where $f_p, 1 \leq p \leq r$ are defined, and $U_{k+j} = (\rho^{k+j}_k)^{-1}(U_k), j \geq 1$. By hypothesis, $df_1, \cdots, df_r$ are linearly independents at every $Z_k \in U_k$, then

$$U_k \cap R_k = \{Z_k \in J_k : f_p(Z_k) = 0, 1 \leq p \leq r\}.$$

Let us put $V_{k+j} = U_{k+j} \cap (\rho^{k+j}_k)^{-1}(R_k), j \geq 0$. Since h_{k+l+1} is a vector bundle on V_{k+l+1}, for every p,$1 \leq p \leq r$, there exist $\Lambda_p \subset \{\alpha \in \mathbf{N}^n :$

$|\alpha| = l+1\}$ such that $\{d(\partial_{\alpha_p} f_p) \mid_{Q_{Z_{k+l+1}}}, 1 \le p \le r, \alpha_p \in \Lambda_p\}$ is a basis of $h_{Z_{k+l+1}}, Z_{k+l+1} \in V_{k+l+1}$ (eventually shrinking U_k). Then, given q and α , $1 \le q \le r, \mid \alpha \mid = l+1$, there exist functions $A^{p,\alpha_p}_{q,\alpha}$ on V_k, such that

$$d(\partial_\alpha f_q) \mid_{Q_{Z_{k+l+1}}} + \sum_{p=1}^{r} \sum_{\alpha_p \in \Lambda_p} A^{p,\alpha_p}_{q,\alpha}(Z_k) d(\partial_{\alpha_p} f_p) \mid_{Q_{Z_{k+l+1}}} = 0$$

for every $Z_{k+l+1} \in V_{k+l+1}$. This is so since $d(\partial_\alpha f_p) \mid_{Q_{Z_{k+l+1}}}$ depends only on Z_k. Let $a^{p,\alpha_p}_{q,\alpha}$ be extensions to U_k of functions $A^{p,\alpha_p}_{q,\alpha}$. Then $\partial_\alpha f_p + \sum a^{p,\alpha_p}_{q,\alpha} \partial_{\alpha_p} f_p$ are constant on the fibers of the submersion $\rho^{k+l+1}_{k+l} \mid_{V_{k+l+1}}: V_{k+l+1} \to V_{k+l}$. This implies that, given q and α,$1 \le q \le r, |\alpha| = l+1$, there are functions $g_{q,\alpha}$ on U_{k+l} such that $\partial_\alpha f_q + \sum a^{p,\alpha_p}_{q,\alpha} \partial_{\alpha_p} f_p - g_{q,\alpha}$ are identically zero on V_{k+l+1}, which is the zero set of functions $(\rho^{k+l+1}_{k+l})^* f_p$,$1 \le p \le r$. By implicit function theorem there are functions $b^p_{q,\alpha} \in U_{k+l+1}$ such that

$$\partial_\alpha f_q + \sum a^{p,\alpha_p}_{q,\alpha} \partial_{\alpha_p} f_p - g_{q,\alpha} + \sum b^p_{q,\alpha} f_p = 0$$

in U_{k+l+1}.Then I_{k+l+1} is generated by

$$\{\partial_\beta f_p, \mid \beta \mid \le l; g_{p,\alpha}, \mid \alpha \mid = l+1; \partial_{\alpha_p} f_p, \alpha_p \in \Lambda_p; 1 \le p \le r\}$$

where $g_{p,\alpha} \in F_{k+l}$. So I^{k+l+1}_{k+l} is generated by

$$\{\partial_\beta f_p, |\beta| \le l; g_{p,\alpha}, |\alpha| = l+1; 1 \le p \le r\}$$

which proves the second part of Proposition. Since $\partial_{\alpha_p} f_p$ are independents, given $Z_{k+l} \in R^{k+l+1}_{k+l} \cap U_{k+l}$, there exist $Z_{k+l+1} \in U_{k+l+1}$ such that $\partial_{\alpha_p} f_p(Z_{k+l+1}) = 0$. This implies $Z_{k+l+1} \in R_{k+l+1}$, so $R^{k+l+1}_{k+l} \subset \rho^{k+l+1}_{k+l}(R_{k+l+1})$, which completes the proof.

R_{k+l+1} is not necessarily a manifold, nor R^{k+l+1}_{k+l}. To guarantee this, we need the following Proposition, which is dual of a result in [3].

Proposition 3.2. *If I_k is a regular SPDE such that*
(i) h_k is 1-acyclic;
(ii) h_{k+1} is a vector bundle on $(\rho^{k+1}_k)^{-1}(R_k)$
then h_{k+l+1} is a vector bundle on $(\rho^{k+l+1}_k)^{-1}(R_k)$ for every $l \ge 0$.

Proof: By induction on l, suppose h_{k+l+1} is a vector bundle . For every $Z_{k+l+2} \in (\rho^{k+l+2}_k)^{-1}(R^k)$ the sequence

$$\Lambda^2 T_z \otimes h_{Z_{k+l}} \overset{d_K}{\to} T_z \otimes h_{Z_{k+l+1}} \overset{d_K}{\to} h_{Z_{k+l+2}}$$

is exact by (i), then $\dim(T_z \otimes h_{Z_{k+l+1}}) = \dim h_{Z_{k+l+2}} + \dim(d_K(\Lambda^2 T_z \otimes h_{Z_{k+l}}))$. If I_k is generated by $f_1, \cdots, f_r$, then h_{k+l} is generated by the restrictions to $(\rho_k^{k+l})^{-1}(R_k)$ of $d(\partial_\alpha f_p) \mid_{Q_{k+l}}, 1 \leq p \leq r, |\alpha| = l$, and similarly, $d_K(\Lambda^2 T \otimes h_{k+l})$ and h_{k+l+2} are generated by a finite number of C^∞-sections. Since the rank of a linear system with variables coefficients is a lower semicontinuous function, $\dim h_{k+l+2}$ and $\dim(\Lambda^2 T \otimes h_{k+l})$ are lower semicontinuous functions, so by induction hypothesis and the above equality, it follows $\dim h_{k+l+2}$ and $\dim d_K(\Lambda^2 T \otimes h_{k+l})$ are constant functions, which proves h_{k+l+2} is a vector bundle.

Theorem 3.1 (Prolongation projection commutativity theorem). *If I_k is a SPDE such that*

(i) h_k is 1-acyclic;

(ii) h_{k+1} is a vector subbundle on $(\rho_k^{k+1})^{-1}(R_k)$;

then

$$(I_{k+l}^{k+l+1})_{+1} = I_{k+l+1}^{k+l+2}$$

or equivalently

$$(R_{k+l}^{k+l+1})_{+1} = R_{k+l+1}^{k+l+2}$$

for all $l \geq 0$.

Proof: Let $f_p, 1 \leq p \leq r$ be a set of independent generators of I_k. It follows from Proposition3.2 that h_{k+l+1} is a vector bundle for every $l \geq 0$, and applying Proposition3.1, I_{k+l+1}^{k+l+2} is generated by $\partial_\beta f_p, 1 \leq p \leq r, \mid \beta \mid = l+1$, and functions

$$g_t = \sum_{p=1}^{r} \sum_{i,j=1}^{r} \sum_{|\alpha|=l} a_{i,j,t}^{\alpha,p} \partial_i \partial_j \partial_\alpha f_p + b_t^p f_p,$$

where $a_{i,j,t}^{\alpha,p} \in F_k, b_t^p \in F_{k+l+2}, a_{i,j,t}^{\alpha,p} = a_{j,i,t}^{\alpha,p}$ and $1 \leq t \leq s$. To show $I_{k+l+1}^{k+l+2} \subset (I_{k+l}^{k+l+1})_{+1}$, we must prove that $g_t \in (I_{k+l}^{k+l+1})_{+1}$, for every $1 \leq t \leq s$. If $Z_{k+l+2} \in (\rho_k^{k+l+2})^{-1}(R_k)$, then

$$0 = dg_t \mid_{Q_{Z_{k+l+2}}} = \sum a_{i,j,t}^{\alpha,p} d(\partial_i \partial_j \partial_\alpha f_p) \mid_{Q_{Z_{k+l+2}}} \tag{3.1}$$

by $f_p(Z_{k+l+2}) = 0$, $1 \leq p \leq r$. Put

$$(w_{p,\alpha})_{Z_{k+l}} = d(\partial_\alpha f_p) \mid_{Q_{Z_{k+l}}},$$

$$(w_{p,\alpha,j})_{Z_{k+l+1}} = d(\partial_j \partial_\alpha f_p) \mid_{Q_{Z_{k+l+1}}}$$

and

$$(w_{p,\alpha,j,i})_{Z_{k+l+2}} = d(\partial_i\partial_j\partial_\alpha f_p)\mid_{Q_{Z_{k+l+2}}}.$$

Then (3.1) can be written as $\sum a^{\alpha,p}_{i,j,t} w_{p,\alpha,j,i} = 0$ on $(\rho^{k+l+2}_k)^{-1}(R_k)$, which is equivalent to

$$d_K(\sum a^{\alpha,p}_{i,j,t}\frac{\partial}{\partial x^i}\otimes w_{p,\alpha,j}) = 0.$$

From (i) there exist functions $B^{\alpha,p}_{i,j,t}$ on R_k, with $B^{\alpha,p}_{i,j,t} = -B^{\alpha,p}_{j,i,t}$, such that

$$d_K(\frac{1}{2}\sum B^{\alpha,p}_{i,j,t}\frac{\partial}{\partial x^j}\wedge\frac{\partial}{\partial x^i}\otimes w_{p,\alpha}) = \sum\frac{\partial}{\partial x^i}\otimes a^{\alpha,p}_{i,j,t}w_{p,\alpha,j}.$$

Then $\sum\frac{\partial}{\partial x^i}\otimes(B^{\alpha,p}_{i,j,t} - a^{\alpha,p}_{i,j,t})w_{p,\alpha,j} = 0$, so $\sum(B^{\alpha,p}_{i,j,t} - a^{\alpha,p}_{i,j,t})w_{p,\alpha,j} = 0$. Let be $b^{\alpha,p}_{i,j,t}$ extensions of $B^{\alpha,p}_{i,j,t}$ to U_k so that

$$b^{\alpha,p}_{i,j,t} = -b^{\alpha,p}_{j,i,t}. \tag{3.2}$$

Then

$$\sum_{j,\alpha,p}(b^{\alpha,p}_{i,j,t} - a^{\alpha,p}_{i,j,t})d(\partial_j\partial_\alpha f_p)\mid_{Q_{k+l+1}} = 0$$

on $(\rho^{k+l+1}_k)^{-1}(R^k)$. This means $\sum_{j,\alpha,p}(b^{\alpha,p}_{i,j,t} - a^{\alpha,p}_{i,j,t})\partial_j\partial_\alpha f_p$ is constant on the fibers of $(\rho^{k+l+1}_{k+l})^{-1}(R_k)$ over $(\rho^{k+l}_k)^{-1}(R_k)$, so there exist functions $H_{i,t}\in F_{k+l}$ such that

$$\sum_{j,\alpha,p}(b^{\alpha,p}_{i,j,t} - a^{\alpha,p}_{i,j,t})\partial_j\partial_\alpha f_p - H_{i,t} = 0$$

on $(\rho^{k+l+1}_k)^{-1}(R_k)$. This set is the null set of $f_1,\cdots,f_r$, then there exist functions $c^p_{i,t}\in F_{k+l+1}$ which satisfies

$$\sum_{j,\alpha,p}(b^{\alpha,p}_{i,j,t} - a^{\alpha,p}_{i,j,t})\partial_j\partial_\alpha f_p - H_{i,t} = \sum_p c^p_{i,t}f_p.$$

It follows that $H_{i,t}\in I^{k+l+1}_{k+l}$ and

$$\begin{aligned}\sum_i \partial_i H_{i,t} &= \sum_{i,j,\alpha,p}(b^{\alpha,p}_{i,j,t} - a^{\alpha,p}_{i,j,t})\partial_i\partial_j\partial_\alpha f_p \\ &+ \sum_{i,j,\alpha,p}\partial_i(b^{\alpha,p}_{i,j,t} - a^{\alpha,p}_{i,j,t})\partial_j\partial_\alpha f_p - \sum_{i,p}(c^p_{i,t}\partial_i f_p - \partial_i(c^p_{i,t})f_p).\end{aligned}$$

From (3.2), $\sum_{i,j} b^{\alpha,p}_{i,j,t}\partial_i\partial_j\partial_\alpha f_p = 0$, so

$$\sum_i \partial_i H_{i,t} + g_t = 0, \operatorname{mod} I_{k+l+1}.F_{k+l+2}.$$

But the left side is in F_{k+l+1} so $\sum_i \partial_i H_{i,t} + g_t \in I_{k+l+1}$, and consequently $g_t \in (I^{k+l+1}_{k+l})_{+1}, 1 \leq t \leq s$, which completes the proof.

Corollary 3.1. *Under the hypothesis of the preceding Theorem and* $I^{k+1}_k = I_k$ *we have* $I^{k+l+1}_{k+l} = I_{k+l}$*, for all* $l \geq 0$.

Proof of Theorem1.2: From Theorem 1.1(i) we have $I_k = \{f \in F_k : f(R_k) = 0\}$ is regular, from (ii) $h_k = g_k^\perp$ is 1-acyclic [9], from (iii) h_{k+1} is a vector bundle and $I^{k+1}_k = I_k$. Applying Corollary3.1, $R^{k+l+1}_{k+l} = R_{k+l}, l \geq 0$, and from Proposition 3.1 we get $\rho^{k+l+1}_{k+l} : R_{k+l+1} \to R_{k+l}$ is onto, for every $l \geq 0$. The g_{k+l+1} are vector bundles, so these projections are submersions.

§4. Contact of hypersurfaces of $\mathbf{C}^2$

A three dimensional manifold M with a codimension one distribution $\Delta \subset TM$, an operator J on Δ such that $J^2 = -I$, and an one form θ such that $\theta^\perp = \Delta$ and $\theta \wedge d\theta \neq 0$ is a *Cauchy-Riemann manifold.* A real hypersurface of $\mathbf{C}^2$ has a natural structure of CR-manifold with $\Delta = TM \cap J(TM)$. From now on, M and M' will denote CR-manifolds. A diffeomorphism $f : M \to M'$ is a *CR-diffeomorphism* if $f_*(\Delta) = \Delta'$, and $f_*(J) = J'$. If $\Delta_{\mathbf{C}}$ is the complexification of Δ, then $\Delta_{\mathbf{C}} = \Delta^{1,0} \oplus \Delta^{0,1}$, and f is a CR-diffeomorphism if and only if

$$f_*(\Delta^{1,0}) = (\Delta')^{1,0}. \tag{4.3}$$

Let U be an open set of M, Z_1 a no null section of $\Delta^{1,0}|_U$, $Z_{\bar{1}} = \overline{Z_1}$, and

$$Z_0 = -i[Z_1, Z_{\bar{1}}]. \tag{4.4}$$

Then $Z_0, Z_1, Z_{\bar{1}}$ is a basis of $T_{\mathbf{C}}M|_U$. If h is a complex valued function on U, we will write $h_i = Z_i(h), i = 0, 1, \bar{1}$. Let a, b, c be the complex valued functions defined by

$$[Z_1, Z_0] = aZ_1 + \bar{b}Z_{\bar{1}} + cZ_0, \tag{4.5}$$

which satisfy, as a consequence of Jacobi's identity

$$b_1 - a_{\bar{1}} + a\bar{c} - bc = 0$$

$$\bar{c}_1 - c_{\bar{1}} + i(a + \bar{a}) = 0.$$

Let U' be an open set of M', $Z'_i, i = 0, 1, \bar{1}$ as above with the corresponding functions a', b', c'. We denote by D_k the open set of $J_k = J_k(M \times M', M, \pi_1)$ corresponding to k-jets of local diffeomorphisms of M in M', where π_1 is the canonical projection of $M \times M'$ on M. Put $D_k(U, U') = (\beta_k)^{-1}(U \times U') \subset D_k$. On $D_1(U, U')$ we introduce the coordinates system

$$p^i_j : D_1(U, U') \to \mathbf{C}, i, j = 0, 1, \bar{1},$$

defined by

$$f_*(Z_j(x)) = \sum_{i=0,1,\bar{1}} p^i_j(j^1_x f) Z'_i(f(x)).$$

These coordinates are not independent, and satisfy the relations

$$\overline{p^i_j} = p^{\bar{i}}_{\bar{j}}, i, j = 0, 1, \bar{1},$$

where $\bar{0} = 0, \bar{\bar{1}} = 1$ by convention. The coordinates on $D_2(U, U')$ are defined by

$$p^i_{jk}(j^2_x f) = Z_j(p^i_k(j^1 f))(x), i, j, k = 0, 1, \bar{1}.$$

Again $\overline{p^i_{jk}} = p^{\bar{i}}_{\bar{j}\bar{k}}$. If $[Z_i, Z_j] = \sum a^k_{ij} Z_k$, it follows from $f_*[Z_i, Z_j] = [f_* Z_i, f_* Z_j]$ that

$$\sum_k a^k_{ij}(x) p^m_k(j^1_x f) = p^m_{ij}(j^2_x f) - p^m_{ji}(j^2_x f) + \sum_{r,s} p^r_i(j^1_x f) p^s_j(j^1_x f) a'^m_{rs}(f(x)).$$

For instance

(4.6)
$$p^0_{1\bar{1}} - p^0_{\bar{1}1} = i(p^0_0 - p^1_1 p^{\bar{1}}_{\bar{1}} + p^1_{\bar{1}} p^{\bar{1}}_1) + c'(p^1_1 p^0_{\bar{1}} - p^0_1 p^1_{\bar{1}}) + \bar{c}'(p^{\bar{1}}_1 p^0_{\bar{1}} - p^{\bar{1}}_{\bar{1}} p^0_1).$$

Coordinates in $D_3(U, U')$ are defined by

$$p^i_{mjk}(j^3_x f) = Z_m(p^i_{jk}(j^2 f))(x), i, j, k = 0, 1, \bar{1},$$

and successively.

Equation (4.3) in coordinates is

$$f_* Z_1 = p^1_1(j^1 f) Z'_1$$

or

$$p_1^{\bar{1}}(j^1 f) = p_1^0(j^1 f) = 0.$$

Let I_1 be the SPDE generated on $D_1(U, U')$ by

$$I_1 : \{p_1^{\bar{1}} = p_1^0 = 0 \text{ and conjugated equations.}$$

The solutions of I_1 are (local) CR-diffeomorphisms from M to M'. The prolongation I_2 of I_1 is generated by

$$(4.7) \qquad I_2 : \begin{cases} p_1^{\bar{1}} = p_1^0 = 0 \\ p_{11}^{\bar{1}} = p_{\bar{1}1}^{\bar{1}} = p_{01}^{\bar{1}} = p_{11}^0 = p_{\bar{1}1}^0 = p_{01}^0 = 0 \\ \text{and conjugated equations.} \end{cases}$$

It follows from (4.6) and (4.7) that

$$(4.8) \qquad p_0^0 - p_1^1 p_{\bar{1}}^{\bar{1}} = 0.$$

If $I_1^2 = \tilde{I}_1$, then $\tilde{I}_1$ is generated as

$$\tilde{I}_1 : \begin{cases} p_1^{\bar{1}} = p_1^0 = p_0^0 - p_1^1 p_{\bar{1}}^{\bar{1}} = 0 \\ \text{and conjugated equations.} \end{cases}$$

Proposition 4.1. *h_1 is 1-acyclic.*

Proof: Put $\alpha = (\alpha_0, \alpha_1, \alpha_{\bar{1}}) \in \mathbf{N}^3$, and write $p_\alpha^j = p_{0\cdots01\cdots1\bar{1}\cdots\bar{1}}^j$, where the index i appears α_i−times. Then h_k is generated by

$$h_k = \left[dp_\alpha^1, dp_{\bar{\alpha}}^{\bar{1}}, \alpha_{\bar{1}} \neq 0; dp_\alpha^0, \alpha_1 + \alpha_{\bar{1}} \neq 0; \mid \alpha \mid = k\right]$$

and

$$n_k = \dim h_k = 2\left\{\frac{(k+2)!}{k!2!} - \frac{(k+1)!}{k!1!}\right\} + \left\{\frac{(k+2)!}{k!2!} - 1\right\} = \frac{3k^2 + 5k}{2}.$$

We will show that the sequence

$$(4.9) \qquad 0 \to \Lambda^3 T \otimes h_{k-2} \xrightarrow{d_K} \Lambda^2 \otimes h_{k-1} \xrightarrow{d_K} T \otimes h_k \xrightarrow{d_K} h_{k+1} \to 0$$

is exact in $T \otimes h_k$, for $k \geq 2$. As we know $d_K(T \otimes h_k) = h_{k+1}$, it is enough to show that $\dim d_K(\Lambda^2 T \otimes h_{k-1}) = 3n_k - n_{k+1}$, for $k \geq 2$. But

$$0 \to \Lambda^3 T \otimes Q_{k-2} \xrightarrow{d_K} \Lambda^2 T \otimes Q_{k-1} \xrightarrow{d_K} T \otimes Q_k \xrightarrow{d_K} Q_{k+1} \to 0$$

is exact, so if $\omega \in \Lambda^2 T \otimes h_{k-1}$ is such that $d_K \omega = 0$, then there exists $\eta \in \Lambda^3 T \otimes Q_{k-2}$ such that $d_K \eta = \omega$. If $\eta = Z_0 \wedge Z_1 \wedge Z_{\bar{1}} \otimes \theta$, with

$\theta \in Q_{k-2}$, then $d_K\eta = Z_1 \wedge Z_{\bar{1}} \otimes \partial_0\theta - Z_0 \wedge Z_{\bar{1}} \otimes \partial_1\theta + Z_0 \wedge Z_1 \otimes \partial_{\bar{1}}\theta$,where $\partial_i\theta = d_K(Z_i \otimes \theta)$. Consequently , $\partial_i\theta \in h_{k-1}$, for $i = 0, 1, \bar{1}$, so $\theta \in h_{k-2}$. Then $\eta \in \Lambda^3 T \otimes h_{k-2}$, and this shows (4.9) is exact at $\Lambda^2 T \otimes h_{k-1}$, so

$$\begin{aligned} &\dim d_K(\Lambda^2 T \otimes h_{k-1}) \\ &\qquad = \dim \Lambda^2 T \otimes h_{k-1} - \dim \Lambda^3 T \otimes h_{k-2} = 3n_{k-1} - n_{k-2}. \end{aligned}$$

The equality $3n_k - n_{k+1} = 3n_{k-1} - n_{k-2}$ is a simple verification, which shows (4.9) is exact.

Proposition 4.2. *For every $k \geq 1$,*

$$I_k^{k+1} = \tilde{I}_k$$

Proof: This follows from Theorem3.1 and Proposition4.1

The same way as above, we verify $\tilde{I}_2$ is generated by

$$\tilde{I}_2 : \begin{cases} \text{equations(4.7)(4.8)} \\ p^0_{00} - p^{\bar{1}}_{\bar{1}} p^1_{01} - p^1_1 p^{\bar{1}}_{0\bar{1}} = 0 \\ \frac{p^{\bar{1}}_{\bar{1}\bar{1}}}{p^{\bar{1}}_{\bar{1}}} - 2i \frac{p^1_0}{p^1_1} - (\bar{c} - \bar{c}')p^{\bar{1}}_{\bar{1}} = 0 \\ \text{and conjugated equations.} \end{cases} \tag{4.10}$$

Then

$$\tilde{I}_1^2 = \tilde{I}_1$$

and if we define

$$\breve{I}_2 = \tilde{I}_2^3$$

then $\breve{I}_2$ is generated [10] by

$$\breve{I}_2 : \begin{cases} \text{equations(4.10)} \\ \frac{p^{\bar{1}}_{0\bar{1}}}{p^{\bar{1}}_{\bar{1}}} - \frac{p^0_{00} + 3i p^1_0 p^{\bar{1}}_0}{2p^0_{00}} + \frac{i}{2}(d - d' p^0_0) - \frac{1}{2}(c p^1_0 - c' p^{\bar{1}}_0) = 0 \\ \text{and conjugated equations} \end{cases} \tag{4.11}$$

where

$$d = \frac{1}{2}(c_{\bar{1}} + i(a - 2\bar{a}). \tag{4.12}$$

It follows from (4.11) that

$$\breve{I}_1^2 = \tilde{I}_1.$$

Proposition 4.3. *$\tilde{h}_2$ is 1-acyclic.*

Proof: It is easy to see that $\tilde{h}_k$ is generated by

$$\begin{aligned}\tilde{h}_k &= \Big[dp^1_\alpha, dp^{\bar{1}}_{\bar{\alpha}}, \alpha \neq (k-1,1,0),(k,0,0); dp^0_\alpha, \alpha \neq (k,0,0);\\ &\qquad dp^0_{(k,0,0)} - p^{\bar{1}}_{\bar{1}} dp^1_{(k-1,1,0)} - dp^{\bar{1}}_{(1,k-1,0)}\Big]\end{aligned}$$

and $\tilde{n}_k = \dim \tilde{h}_k = 3\frac{(k+2)!}{k!2!} - 4$. As in the proof of Proposition4.1, $3\tilde{n}_k - \tilde{n}_{k+1} = 3\tilde{n}_{k-1} - \tilde{n}_{k-2}$ for $k \geq 3$. Observe that equality doesn't hold for $k = 2$, so $\tilde{h}_1$ is not 2-acyclic.

Proposition 4.4. *For every $k \geq 2$,*

$$\tilde{I}^{k+1}_k = \check{I}_k.$$

Proof: The same as Proposition4.2.
Let be now

$$\hat{I}_2 = \check{I}^3_2.$$

Then (cf. [10]) $\hat{I}_2$ is generated by

$$\text{(4.13)} \qquad \hat{I}_2 : \begin{cases} \text{equations (4.11)} \\ \dfrac{p^{\bar{1}}_{00}}{p^{\bar{1}}_{\bar{1}}} - \dfrac{p^{\bar{1}}_0 p^0_{00}}{p^{\bar{1}}_{\bar{1}} p^0_0} + (\kappa - \kappa' p^1_1 p^0_0) - \dfrac{i p^1_0 (p^{\bar{1}}_0)^2}{p^{\bar{1}}_{\bar{1}} p^0_0} \\ \qquad - c' \dfrac{p^1_0 p^{\bar{1}}_0}{p^{\bar{1}}_{\bar{1}}} - b' p^1_1 p^1_0 + (id' - \overline{a}') p^1_1 p^{\bar{1}}_0 = 0 \\ \text{and conjugated equations} \end{cases}$$

with

$$\text{(4.14)} \qquad \kappa = -\frac{i}{3}(c_0 - id_1 + icd + ac - \bar{b}\bar{c}).$$

Proposition 4.5. *$\check{h}_2$ is 1-acyclic.*

Proof: The fiber bundle $\check{h}_k, k \geq 2$ is generated by

$$\begin{aligned}\check{h}_k &= \big[dp^1_\alpha, dp^{\bar{1}}_{\bar{\alpha}}, \alpha \neq (k-1,1,0),(k,0,0); dp^0_\alpha, \alpha \neq (k,0,0);\\ &\qquad 2p^{\bar{1}}_{\bar{1}} dp^1_{(k-1,1,0)} - dp^0_{(k,0,0)}; 2p^1_1 dp^{\bar{1}}_{(k-1,0,1)} - dp^0_{(k,0,0)}\big].\end{aligned}$$

Define $\check{h}_1$, doing $k = 1$ above. If $\check{n}_k = \dim \check{h}_k = \frac{3(k+2)!}{k!2!} - 3$, then $3\check{n}_k - \check{n}_{k+1} = 3\check{n}_{k-1} - \check{n}_{k-2}$ for $k \geq 3$, and the proof are in the same lines of Proposition4.1.

Proposition 4.6. *For every $k \geq 2$*

$$\breve{I}_k^{k+1} = \hat{I}_k.$$

Proof: As in Proposition4.2.

Proposition 4.7. *$\hat{h}_2$ is 1-acyclic.*

Proof: We have

$$\begin{aligned}\hat{h}_2 &= \Big[dp_{ij}^0, (i,j) \neq (0,0); dp_{ij}^1, (i,j) \neq (0,0),(1,0); dp_{00}^1 - \frac{p_0^1}{p_0^0}dp_{00}^0; \\ &\qquad dp_{01}^1 - \frac{1}{2p_{\bar{1}}^{\bar{1}}}dp_{00}^0; \text{ and conjugated elements}\Big]\end{aligned}$$

and $\hat{h}_k = Q_k^*$, for $k \geq 3$. It is enough to show $d_K(\Lambda^2 T \otimes \hat{h}_2) = d_K(\Lambda^2 T \otimes Q_2^*)$, or, $d_K(\Lambda^2 T \otimes [dp_{00}^0]) \subset d_K(\Lambda^2 T \otimes \hat{h}_2)$, and this is consequence of

$$d_K(e_1 \wedge e_{\bar{1}} \otimes dp_{00}^0) = d_K(e_0 \wedge e_{\bar{1}} \otimes dp_{01}^0 - e_0 \wedge e_1 \otimes dp_{0\bar{1}}^0)$$

and

$$\begin{aligned} d_K(e_0 \wedge e_1 \otimes dp_{00}^0) &= \frac{2}{p_{\bar{1}}^1} d_K \Bigg[e_0 \wedge e_{\bar{1}} \otimes \left(dp_{01}^{\bar{1}} - \frac{p_0^{\bar{1}}}{p_0^0} dp_{01}^0 \right) \\ &\quad + e_1 \wedge e_0 \left(dp_{0\bar{1}}^{\bar{1}} - \frac{1}{2} p_{\bar{1}}^1 dp_{00}^0 - \frac{p_0^{\bar{1}}}{p_0^0} dp_{0\bar{1}}^0 \right) \\ &\quad - e_1 \wedge e_{\bar{1}} \left(dp_{00}^{\bar{1}} - \frac{p_0^{\bar{1}}}{p_0^0} dp_{00}^0 \right) \Bigg]. \end{aligned}$$

The SPDE $\hat{I}_2^3$ is generated by (cf[10])

$$\text{(4.15)} \qquad \hat{I}_2^3 : \begin{cases} \text{equations (4.13)} \\ p_{\bar{1}}^{\bar{1}} r - p_{\bar{1}}^1 (p_0^0)^2 r' = 0 \\ \text{and conjugated equations} \end{cases}$$

where

$$\text{(4.16)} \qquad r = \kappa_1 - \bar{b}_0 - 2c\kappa - \bar{b}(a + \bar{a} - id);$$

If we define

$$R\,|_U = rZ_{\bar{1}}^* \wedge Z_0^* \otimes Z_0^* \otimes Z_{\bar{1}} + \bar{r} Z_{\bar{1}}^* \wedge Z_0^* \otimes Z_0^* \otimes Z_1$$

then R is a tensor on M,i.e., $R \in \Gamma(\Lambda^2 T^* \otimes T^* \otimes T)$.

Definition 4.1. *The tensor R is the curvature tensor of the CR-manifold M. We say M is umbilic at $x \in M$ if $R(x) = 0$, otherwise M is said non-umbilic at $x \in M$; M is said umbilic(non-umbilic) if M is umbilic (non-umbilic) at every $x \in M$*

Example: The *quadric* Q is defined by $Q = \{(z, w) \in \mathbf{C}^2 : w - \bar{w} = 2iz\bar{z}\}$. If $Z_1 = \frac{i}{2}\frac{\partial}{\partial z} - \bar{z}\frac{\partial}{\partial w}$ then $Z_0 = -\frac{1}{2}(\frac{\partial}{\partial w} + \frac{\partial}{\partial \bar{w}})$. Then $a = b = c = 0$ and $R = 0$, so Q is umbilic.

Proposition 4.8. *The diagram*

$$
\begin{array}{ccccccccc}
I_6 & \rightarrow & \tilde{I}_5 & \rightarrow & \check{I}_4 & \rightarrow & \hat{I}_3 & \rightarrow & \hat{I}_2^3 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
I_5 & \rightarrow & \tilde{I}_4 & \rightarrow & \check{I}_3 & \rightarrow & \hat{I}_2 & \rightarrow & \tilde{I}_1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
I_4 & \rightarrow & \tilde{I}_3 & \rightarrow & \check{I}_2 & \rightarrow & \tilde{I}_1 & & \\
\downarrow & & \downarrow & & \downarrow & & & & \\
I_3 & \rightarrow & \tilde{I}_2 & \rightarrow & \tilde{I}_1 & & & & \\
\downarrow & & \downarrow & & & & & & \\
I_2 & \rightarrow & \tilde{I}_1 & & & & & & \\
\downarrow & & & & & & & & \\
I_1 & & & & & & & &
\end{array} \tag{4.17}
$$

is commutative, with horizontal arrows surjective and the arrows representing the projection of projectable functions.

Proof: It is a consequence of the above propositions.

Theorem 4.1. *Given CR-manifolds M and M' and points $x \in M$ and $x' \in M'$ there exist a fifth order jet of CR-diffeomorphism doing a fifth order contact between M and M' at points x and x'.*

Proof: Proposition4.8 says that $\beta_5 : I_5 \rightarrow M \times M'$ is surjective, then there exists $X \in I_5$ such that $\beta_5(X) = (x, x')$.

Theorem 4.2. *If M' is umbilic, then it is locally CR-diffeomorphic to the hyperquadric Q.*

Proof: Let be $M = Q$; then r and r' are 0, and from (4.15) we get $\hat{I}_3$ is onto $\hat{I}_2$. As $\hat{h}_2$ is 1-acyclic, Corollary3.1 says $\hat{I}_2$ is formally integrable. But $\hat{h}_3 = Q_3^*$, then $\hat{I}_2$ is completely integrable (cf[5]), so there exists a neighborhood U of $x \in Q$ and a CR-diffeomorphism $f : U \rightarrow f(U) \subset M'$ solution of $\hat{I}_2$.

Corollary 4.1. *If M'=Q , then* $\hat{I}_2 \cap \beta_2^{-1}(M, 0)$ *is a G-structure associated to M, where the group G is the group of CR-automorphisms of Q.*

Suppose now that M and M' are non-umbilic. Then $\hat{I}_2^3$ is a regular SEDP , and in (4.15) we can replace the new equation by

$$p_1^1 = \epsilon \frac{\lambda}{\lambda'}, \epsilon = \pm 1$$

where

$$\lambda = \frac{\sqrt{r}}{\sqrt[8]{r\bar{r}}} \tag{4.18}$$

($\sqrt[8]{r\bar{r}}$ taken as positive root) and λ' defined similarly. Then

$$\hat{I}_2^3 : \begin{cases} \text{equations (4.13)} \\ p_1^1 = \epsilon\lambda/\lambda', \epsilon = \pm 1 \\ \text{and conjugated equations} \end{cases} \tag{4.19}$$

Defining

$$\bar{I}_2 = \hat{I}_2^3$$

we can verify

$$\bar{I}_2^3 : \begin{cases} \text{equations (4.19)} \\ \alpha = \alpha' \\ \beta = \beta' \\ \text{and conjugated equations} \end{cases} \tag{4.20}$$

where

$$\alpha = \frac{\bar{\lambda}_1}{\bar{\lambda}} + \frac{\lambda_1}{2\lambda} - \frac{c}{2} \tag{4.21}$$

and

$$\beta = 2\left(\alpha - \frac{\bar{\lambda}_1}{\bar{\lambda}}\right)\left(\bar{\alpha} - \frac{\lambda_{\bar{1}}}{\lambda}\right) + \frac{\lambda_{\bar{1}}\bar{\lambda}_1}{\lambda\bar{\lambda}} - i\left(\frac{\lambda_0}{\lambda} - \frac{\bar{\lambda}_0}{\bar{\lambda}}\right) - d. \tag{4.22}$$

As $\bar{h}_2 = \hat{h}_2$, we obtain in non-umbilic case an extension of (4.17):

$$
\begin{array}{ccccccccccc}
I_7 & \rightarrow & \tilde{I}_6 & \rightarrow & \check{I}_5 & \rightarrow & \hat{I}_4 & \rightarrow & \bar{I}_3 & \rightarrow & \bar{I}_2^3 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
I_6 & \rightarrow & \tilde{I}_5 & \rightarrow & \check{I}_4 & \rightarrow & \hat{I}_3 & \rightarrow & \bar{I}_2 & & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & & & \\
I_5 & \rightarrow & \tilde{I}_4 & \rightarrow & \check{I}_3 & \rightarrow & \hat{I}_2 & & & & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & & & \\
I_4 & \rightarrow & \tilde{I}_3 & \rightarrow & \check{I}_2 & \rightarrow & \tilde{I}_1 & & & & \\
\downarrow & & \downarrow & & \downarrow & & & & & & \\
I_3 & \rightarrow & \tilde{I}_2 & \rightarrow & \tilde{I}_1 & & & & & & \\
\downarrow & & \downarrow & & & & & & & & \\
I_2 & \rightarrow & \tilde{I}_1 & & & & & & & & \\
\downarrow & & & & & & & & & & \\
I_1 & & & & & & & & & &
\end{array} \tag{4.23}
$$

where all horizontal arrows are onto.

Proposition 4.9. *There exists a sixth order contact between two CR-manifolds at two non-umbilic points.*

Proof: It follows from $\rho_2^6(I_6) = \bar{I}_2$
The following theorem is in [2]:

Theorem 4.3. *There exists a seventh order contact between a real hypersurface of $\mathbf{C}^2$ at a non-umbilic point and the hypersurface defined by*

$$v = z\bar{z} + 2Re\left\{ z^4\bar{z}^2[1 + \frac{16}{5}\alpha(0)z + i(\frac{275}{128}\alpha(0)\bar{\alpha}(0) - \beta(0))u] \right\}$$

where α, β are the functions defined in (4.21),(4.22)

Proof: Let be $M = \{(z, w) \in \mathbf{C}^2 : v = F(z, \bar{z}, u), \text{with}\, w = u + iv\}$. Choosing coordinates (z, u) on M, take

$$Z_1 = \frac{\partial}{\partial z} - A\frac{\partial}{\partial u}; \tag{4.24}$$

then from (4.4)

$$Z_0 = \frac{2B}{(1 + f_u^2)}\frac{\partial}{\partial u} \tag{4.25}$$

where $A = f_z/(f_u+i)$ and $B = -f_{z\bar z}+\bar A f_{uz}+Af_{u\bar z}-A\bar A f_{uu}$. It follows from (4.5)

$$(4.26)\qquad a=b=0, c=A_u-2\frac{f_u f_{uz}-Af_u f_{uu}}{1+f_u^2}+\frac{B_z-AB_u}{B}$$

$$(4.27)\qquad f(z,\bar z,u)=\sum a_{jkl}z^j\bar z^k u^l$$

with $a_{jkl}=0, j+k+l\le 5,(j,k,l)\neq(1,1,0); a_{1,1,0}=1; a_{kjl}=\overline{a_{jkl}}$. From (4.12),(4.14),(4.16),(4.24),(4.26)

$$(4.28)\qquad r=-\frac{1}{6}c_{11\bar 1}-\frac{i}{3}c_{10}+c\left(\frac{2}{3}ic_0+\frac{i}{2}c_{1\bar 1}-\frac{1}{3}cc_{\bar 1}\right)+\frac{1}{6}c_1c_{\bar 1}.$$

From (4.24), (4.25), (4.27), (4.28)

$$Z_1(0)=\frac{\partial}{\partial z}; Z_0(0)=-2\frac{\partial}{\partial u}$$

$$c(0)=c_1(0)=c_{\bar 1}(0)=c_0(0)=c_{1\bar 1}(0)=c_{10}(0)=0$$

$$c_{11\bar 1}(0)=4!2!a_{420}$$

$$r(0)=a_{420}.$$

As $r(0)\neq 0$, by Proposition4.9, we can choose $a_{420}=a_{240}=1$, and all others coefficients of sixtieth-order nulls, so $r(0)=1$, and

$$(4.29)\qquad v=z\bar z+2Re(z^4\bar z^2)+o(7)$$

Again from (4.24),(4.25),(4.28),(4.29)

$$(4.30)\qquad c_{111\bar 1}(0)=5!2!a_{520}; c_{\bar 1 11\bar 1}(0)=4!3!a_{430}; c_{011\bar 1}(0)=-4!2!2a_{421}$$

and from (4.28);(4.30)

$$(4.31)\qquad r_1(0)=5a_{520}; r_{\bar 1}(0)=3a_{430}; r_0(0)=-2a_{421}.$$

From (4.17)

$$(4.32)\qquad \lambda_j=\frac{\lambda}{8}\left(3\frac{r_j}{r}-\frac{\bar r_j}{\bar r}\right), j=0,1,\bar 1$$

and from (4.18),(4.31)

$$\lambda(0) = 1; \lambda_1(0) = \frac{1}{8}(15a_{520} - 3a_{430});$$

$$\lambda_{\bar{1}}(0) = \frac{1}{8}(9a_{430} - 5a_{520}); \lambda_0(0) = \frac{1}{4}(a_{241} - a_{421}).$$

From (4.20), (4.21), (4.32)

$$\alpha(0) = \frac{5}{16}(3a_{340} + a_{520})$$

$$\begin{aligned} \beta(0) &= \frac{9}{128}(5\alpha(0) - 16a_{340})(5\bar{\alpha}(0) - 16a_{430}) \\ &+ \frac{1}{64}(5\alpha(0) - 24a_{340})(5\bar{\alpha}(0) - 24a_{430} + \mathrm{Im}(a_{241}). \end{aligned}$$

Therefore we can choose

$$a_{520} = \frac{16}{5}\alpha(0); a_{421} = -i(\beta(0) - \frac{275}{128}\alpha(0)\bar{\alpha}(0))$$

$$a_{250} = \overline{a_{520}}; a_{241} = \overline{a_{421}}$$

all the others coefficients nulls, and the theorem follows.

References

[1] E. Cartan,*Sur la géométrie pseudo-conforme des hypersurfaces de l'espace de deux variables complexes*, Oeuvres Complètes, partieII, Éditions du CNRS, 1984, 1231-1304

[2] S.S.Chern;J.Moser,*Real hypersurfaces in complex manifolds*,Acta Math.-**133**(1974)219-271

[3] H.Goldschmidt,*Existence theorems for analytic linear partial differential equations*,Ann. of Math.**86**(1967)246-270

[4] H.Goldschmidt,*Integrability criteria for systems of non-linear partial differential equations*,J.Differential Geometry**1**(1967)269-307

[5] M.Kuranishi,*Lectures on Involutive Systems of Partial Differential Equations*,Publ. Soc. Matemática São Paulo,1967

[6] D.G.Quillen,*Formal properties of over-determined systems of linear partial differential equations* ,Ph.D.Thesis,Harvard University,1964

[7] C.Ruiz,*Prolongement formel des systèmes différentiels extérieur d'ordre supérieur*,C.R.Acad.Sc.Paris**285**(1977)1077-1080

[8] C.Ruiz,*Complexe de Koszul du symbole d'un systéme différentiel extérieur*,C.R.Acad.Sc.Paris**286**(1978)55-58

[9] C.Ruiz,*Propriétètes de dualité du prolongement formel des systémes différentiels extérieurs***286** (1978)99-101

[10] J.M.Veloso;J.Verderesi,*Three- dimensional Cauchy-Riemann manifolds*, Dinamical Systems and Partial Differential Equations: Proceedings VII ELAM,Equinoccio,1986

Departamento de Matemática
Universidade Federal do Pará
CEP 66075-110 Belém-PA
Brasil
veloso@marajo.ufpa.br

Advanced Studies in Pure Mathematics 25, 1997
CR-Geometry and Overdetermined Systems
pp. 406–418

A Complex Frobenius Problem

Sidney M. Webster

Dedicated to Professor Kuranishi

Introduction

A complex Frobenius structure on a smooth (real) manifold M is a smooth complex vector sub-bundle E of the complexified tangent bundle $T(M)\otimes\mathbf{C}$ which satisfies the integrability condition $[E,E]\subseteq E$ (i.e. the set of local sections of E is closed under Lie bracket). Such a structure is also termed formally integrable, or involutive. The bracket of a section of E with a section of the complex conjugate bundle $\overline{E}$, taken mod $E+\overline{E}$, gives the Levi-form of the structure. If $E\cap\overline{E}=0$, then E is a CR structure; it is an almost complex structure if also $E+\overline{E}=T(M)\otimes\mathbf{C}$. The integrability problem is to find independent functions, the differentials of which span the sub-bundle $E^{\perp}$ of complex covectors annihilating E. The problem of local solvability is to establish a Poincaré lemma in the natural de Rham-Dolbeault complex associated to the differential ideal generated by sections of $E^{\perp}$.

The case of identically vanishing Levi-form was already treated in works by Nirenberg [13] and Hörmander [7]. The Mizohata operator [5], [12] on $\mathbf{R}^2$ gives perhaps the simplest complex Frobenius structure for which local solvability fails (it is also important for the canonical transformation theory of partial differential equations [8]). Nirenberg [14] has shown that local integrability fails for small perturbations of this structure. Certain interesting higher dimensional analogues have been studied by Trèves [18]. These are structures on $\mathbf{R}^{n+1}$ induced by (local) maps $f:\mathbf{R}^{n+1}\to\mathbf{C}$, $df\neq 0$. The topology of the fibers $f^{-1}(a)$ plays a key role in the questions of local solvability and of local integrability for small perturbations of these structures. In his recent book [19] Trèves also treats the integrability and solvability problem for a variety of important structures.

Received May 9, 1995.
Partially supported by NSF Grant DMS-9201966.

Here we consider local structures which are modeled on those induced by *generic folds* $F : M \to \mathbf{C}^n$, where M is an open subset of $\mathbf{R}^{2n}$. The fiber is always a pair of points which coalesce along a smooth hypersurface $N \subset M$. For $n = 1$, F essentially induces the Mizohata structure [17]. For $n \geq 2$ the case where $F(N)$ is part of the boundary of a strongly pseudoconvex domain $D \subset \mathbf{C}^n$ is of special interest. F is an *interior fold* if $F(M) \subseteq \overline{D}$, and an *exterior fold* if $F(M) \cap D = \emptyset$.

Our **main result** implies that *complex Frobenius structures which are small perturbations of strongly pseudoconvex interior folds are locally integrable, if* $n \geq 2$. It is false for $n = 1$ by Nirenberg's example. We show by counterexample ((1.7) below) that local integrability also fails for small perturbations of strongly pseudoconvex *exterior* folds ($n = 2$). Thus, fiber topology does not suffice to determine the main properties of fold-like structures. The integrability results given here for them are more similar to those known for CR structures of hypersurface type. We refer specifically to the positive embedding results of Kuranishi [11] and Akahori [1], and the counterexamples of Nirenberg [14] and Jacobowitz and Trèves [10]. Though we point out that there is no unresolved dimension as there is (presently) for CR structures.

Originally we had hoped that the current problem would be more similar to the Kuranishi embedding problem and be amenable to the methods of [20]; but we were unable to construct an exact homotopy formula, i. e. one valid without shrinking the domain. However, it turned out that one could establish the above integrability result, and much more easily, by reducing it to a theorem of Hanges and Jacobowitz [6]. One drawback to this method is that it only yields C^∞ regularity. Thus we were unable to address the question of C^k regularity, one of out original aims. However, we hope that the work will shed some further light of the integrability problem.

The main result is proved also without the aid of the Poincaré lemma. In fact we use our integrability theorem to reduce it to the case of an interior fold, which we carry out in section 1, for the admissible degrees. We are indebted to Cordaro and Trèves [3] for a helpful remark in this respect. In section 2 we characterize intrinsically the various formal generic fold-like structures and give a useful normalization for them. This is used in section 3 together with [6] to derive the main result.

§1. Generic folds

Let $z = (z', z^n = x^n + iy^n)$ be complex coordinates on $\mathbf{C}^n$ and consider a (local) domain with boundary

$$D : y^n > h(z', x^n)\ ,\ \partial D : y^n = h(z', x^n)\ , \tag{1.1}$$

where h is a smooth real function with $h(0) = 0$, $dh(0) = 0$. We also use (z', t, s) as coordinates on $\mathbf{R}^{2n} \cong \mathbf{C}^{n-1} \times \mathbf{R}^2$ and define a map $F : M \to \mathbf{C}^n$ by

$$F(z', t, s) = (z', z^n = t + i(\frac{1}{2}s^2 + h(z', t))), \tag{1.2}$$

where M is a suitable neighborhood of 0 in $\mathbf{R}^{2n}$ which is symmetric about N,

$$N = M \cap \{s = 0\}. \tag{1.3}$$

F is a *generic fold* of M onto $\overline{D}$ with $F(N) = \partial D$.

The complex one-forms

$$\begin{aligned} \theta^\alpha = dz^\alpha\ ,\theta^n = dz^n, \\ dz^n = (1 + ih_t)dt + i(sds + h_\alpha dz^\alpha + h_{\overline{\alpha}} d\overline{z}^\alpha), \end{aligned} \tag{1.4}$$

($h_\alpha = \partial h/\partial z^\alpha$, etc.) span an n-dimensional sub-bundle $E^\perp$ of the complex cotangent bundle. (We use the index ranges

$$1 \le \alpha, \beta, \gamma \le n - 1\,;\, 1 \le i, j, k \le n\,, \tag{1.5}$$

and the summation convention for repeated indices). We let E be the sub-bundle of complex vectors annihilated by $E^\perp$. With $\{\theta^\alpha, \theta^n, d\overline{z}^\alpha, ds\}$ as a basis of complex covectors, we get dual complex complex vector fields $X_{\overline{\alpha}}$, $X_{\overline{n}}$ spanning E, for which

$$df \equiv X_{\overline{\alpha}} f d\overline{z}^\alpha + X_{\overline{n}} f ds, \text{ mod } (\theta^j),$$

for any smooth function f. We readily compute

$$X_{\overline{\alpha}} = \partial_{\overline{\alpha}} - i\frac{h_{\overline{\alpha}}}{1 + ih_t}\partial_t\,,\ X_{\overline{n}} = \partial_s - i\frac{s}{1 + ih_t}\partial_t\,. \tag{1.6}$$

These vectors together with their complex conjugates,

$$X_j \equiv \overline{X}_{\overline{j}},$$

are linearly independent except along N, where $X_{\overline{n}}$ becomes real and transverse and spans $E \cap \overline{E}$. Along N we have the bracket relations

$$[X_\alpha, X_{\overline{\beta}}]|_N = -ig_{\alpha\overline{\beta}}\partial_t\,, [X_\alpha, X_{\overline{n}}]|_N = 0\,, [X_n, X_{\overline{n}}]|_N = -ig_{n\overline{n}}\partial_t\,.$$

Here $g_{n\overline{n}} = [2(1 + h_t^2)]^{-1}$, and $g_{\alpha\overline{\beta}}$ corresponds to the Levi-form of ∂D under the equivalence $F|_N$. The full matrix $g_{i\overline{j}}$, defined along N, represents the Levi-form of the complex Frobenius structure E. It is positive definite if D is strongly pseudoconvex, in which case F is an interior fold. F is an exterior fold relative to $\mathbf{C}^n - D$ if the latter domain is strongly pseudoconvex.

Next we consider some abstractly defined structures E on $\mathbf{R}^4$ with coordinates $(z = x + iy, t, s)$. E is the span of the complex vector fields

$$X_{\overline{1}} = \partial_{\overline{z}}\,, X_{\overline{2}} = \partial_s - is(1 + s\xi + z)\partial_t\,, \tag{1.7}$$

where $\xi(t, s)$ is a smooth real valued function defined near 0. Clearly, $[X_{\overline{1}}, X_{\overline{2}}] = 0$, $[X_1, X_{\overline{1}}] = 0$, and $[X_1, X_{\overline{2}}] = -i\partial_t$. Thus E is formally integrable, and $E \cap \overline{E}$ is non-zero only along the hypersurface $N : x = -s - s^2\xi(t, s)$, where it is spanned by the transverse vector $X_{\overline{2}} = \mathrm{Re}(X_{\overline{2}})$. The Levi-form is non-degenerate indefinite along N. In [14] Nirenberg has constructed a function ξ for which $(X_{\overline{2}}|_{z=0})u = 0$ has no non-constant solution $u(t, s)$. Thus, if $X_{\overline{j}}F = 0$, $j = 1, 2$, then $(dF \wedge dz)(0) = 0$, so that $(dF_1 \wedge dF_2)(0) = 0$. Hence, the structure E, which is a formally integrable strictly pseudo convex *exterior* fold is *not* actually integrable near 0 for this choice of ξ.

We return to the fold structure (1.2),(1.6) and consider the problem of local solvability. We assume that the smooth one-form

$$\varphi = \varphi_{\overline{\alpha}}d\overline{z}^\alpha + \varphi_{\overline{n}}ds\,, \tag{1.8}$$

satisfies the compatibility condition $\overline{\partial}_E\varphi = 0$, or

$$X_{\overline{i}}\varphi_{\overline{j}} - X_{\overline{j}}\varphi_{\overline{i}} = 0\,, 1 \le i, j \le n. \tag{1.9}$$

The problem is to find a smooth function g with $\overline{\partial}_E g = \varphi$, i.e.

$$X_{\overline{j}}g = \varphi_{\overline{j}}\,, 1 \le j \le n. \tag{1.10}$$

We assume, in addition, that φ satisfies

$$\varphi_{\overline{n}} = O(s^\infty)\,, \tag{1.11}$$

along N. More intrinsically, $\varphi_{\overline{n}}$ is the interior product of φ with a smooth non-vanishing section of E which becomes real along N. Thus, (1.11) says that φ is *flat on the real characteristics of* E. By a standard argument (see below) one can always achieve (1.11) by adding an exact form to φ. Grushin's example shows that, even with this extra condition, (1.10) may not have a solution for $n = 1$. However, if $n \geq 2$ and F is a strongly pseudoconvex interior fold, we shall show that (1.9) is sufficient for the existence of a solution to (1.10).

We define open sets and maps

$$M^{\pm} = \{(z', t, s) \in M : \pm s > 0\}, F_{\pm} = F|_{M^{\pm}},$$

so that $F_{\pm} : M^{\pm} \to D$ are isomorphisms of Frobenius structures. We set $\varphi^{\pm} = (F_{\pm}^{-1})^* \varphi$, so that

$$\begin{aligned} \varphi^{\pm} &= b_{\overline{\alpha}}^{\pm} d\overline{z}^{\alpha} \pm [2(y^n - h)]^{-1/2} b_{\overline{n}}^{\pm} d(y^n - h) \\ &\equiv (b_{\overline{\alpha}}^{\pm} \pm \frac{-h_{\overline{\alpha}} b_{\overline{n}}^{\pm}}{\sqrt{2(y^n - h)}}) d\overline{z}^{\alpha} \pm \frac{(i - h_{x^n}) b_{\overline{n}}^{\pm}}{2\sqrt{2(y^n - h)}} d\overline{z}^n , \end{aligned} \tag{1.12}$$

where $b_{\overline{j}}^{\pm} = \varphi_{\overline{j}} \circ F_{\pm}^{-1}$, and $\equiv$ is congruence mod dz^j. These forms will blow up along ∂D, unless $\varphi_{\overline{n}}$ vanishes along N. This is the motivation for the condition (1.11).

Lemma 1.1. *If (1.9) and (1.11) hold, then the forms $\varphi^{\pm}$ are smooth $\overline{\partial}$-closed (0,1)-forms on the closure $\overline{D}$.*

proof: By (1.9) they are closed, and the $b_{\overline{n}}^{\pm}$ are clearly smooth on $\overline{D}$ and vanish to infinite order on ∂D by (1.11). By the chain rule and (1.9) we have

$$-2i\partial_{\overline{z}^n} b_{\overline{\alpha}}^{+} = (1 + ih_{x^n})(X_{\overline{\alpha}} \varphi_{\overline{n}}) \circ F_{+}^{-1} . \tag{1.13}$$

But each $X_{\overline{\alpha}} \varphi_{\overline{n}}$ also vanishes to infinite order along N, so these functions are also smooth up to the boundary. For each fixed z' we apply the Cauchy formula to $b_{\overline{\alpha}}^{+}$ on the domain with counter-clockwise boundary

$$\begin{aligned} (1.14)\ D(z') &= \{z_n : |x^n| \leq \delta\ ,\ h(z', x^n) \leq y^n \leq h(z', x^n) + \rho\}, \\ \partial D(z') &= a_0(z') - a_\rho(z') + c_\delta(z') - c_{-\delta}(z') , \\ a_\sigma(z') &= \{x^n + i(h(z', x^n) + \sigma) :\ -\delta \leq x^n \leq +\delta\} , \\ c_\sigma(z') &= \{\sigma + iy^n :\ h(z', \sigma) \leq y^n \leq h(z', \sigma) + \rho\} , \end{aligned}$$

where $\delta > 0$, $\rho > 0$ are sufficiently small. We have ($\zeta^n = \xi^n + i\eta^n$)

(1.15)

$$2\pi i b_{\overline{\alpha}}^{+}(z', z^n) = \int_{\partial D(z')} \frac{b_{\overline{\alpha}}^{+}(z', \zeta^n)}{\zeta^n - z^n} d\zeta^n + \int_{D(z')} \frac{\partial_{\overline{z}^n} b_{\overline{\alpha}}^{+}(z', \zeta^n)}{\zeta^n - z^n} d\zeta^n \wedge d\overline{\zeta}^n .$$

In the double integral we extend the numerator of the integrand smoothly by zero across ∂D. Then we may change the domain of integration $D(z')$ to one whose boundary is independent of z', except for the upper curve $a_\rho(z')$, which varies smoothly with z'. It's then clear that the double integral gives a function of (z', z^n) which is smooth up to the boundary. In the line integral over $c_\delta(z')$, we make the substitution $\sigma = \eta^n - h(z', \delta)$ to get

$$\int_{\sigma=0}^{\rho} \frac{\varphi_{\overline{\alpha}}(z', \delta, \sqrt{2\sigma})}{\delta + i(\sigma + h(z', \delta)) - z^n} i d\sigma \ .$$

This is smooth in (z', z^n), as z^n crosses the boundary curve $a_0(z')$. Similarly for $c_{-\delta}(z')$. For the line integral over $a_\rho(z')$ we have

$$\int_{\xi^n=-\delta}^{+\delta} \frac{\varphi_{\overline{\alpha}}(z', \xi^n, \sqrt{2\rho})}{\xi^n + i(h(z', \xi^n) + \rho) - z^n} (1 + ih_{x^n}(z', \xi^n)) d\xi^n \ .$$

This is also clearly smooth in (z', z^n) across the boundary. For the integral over $a_0(z')$, we set $\rho = 0$ in the above. Then $\varphi_{\overline{\alpha}}(z', \xi^n, 0)$ is a smooth function. Thus we have a Cauchy integral of a smooth function over a smooth curve, both depending smoothly on the parameter z'. By a well-known argument this gives a function of (z', z^n) which is smooth up to the boundary. The same argument for φ^- completes the proof.

To achieve (1.11) we replace φ by $\varphi - \overline{\partial}_E g$, where g is chosen so that $\varphi_{\overline{n}} - X_{\overline{n}} g = O(s^\infty)$. Formally, we set

$$\varphi_{\overline{n}} = \sum_{k=0}^{\infty} w_k(z', t) s^k, \ g = \sum_{k=1}^{\infty} g_k(z', t) s^k,$$

where $k! w_k = \partial_s^k \varphi_{\overline{n}}(z', t, 0)$. We determine the functions g_k successively by $g_1 = w_0$, $2g_2 = w_1$, and

$$\frac{-i}{1 + ih_t} \partial_t g_k + (k+2) g_{k+2} = w_{k+1}.$$

By the theorem of E. Borel (see [8]) there is a smooth function g with these prescribed s-derivatives along $s = 0$.

Now we assume that D is strongly pseudoconvex, so that we may invoke Kohn's solution of the $\overline{\partial}$-problem [4]. Alternately, and in a more elementary vein, we may employ the local solution operators of Range and Lieb [15]. Thus, there exist functions $g^\pm$ smooth on $\overline{D}$ with $\overline{\partial} g^\pm = \varphi^\pm$. It follows that on ∂D we have the tangential Cauchy-Riemann equations $\overline{\partial}_b (g^+ - g^-) = 0$. By the H. Lewy extension theorem (smooth version), there is a function g^0, holomorphic on D and smooth on $\overline{D}$, with $g^0 = g^+ - g^-$ on ∂D. We replace g^- by $g^- + g^0$, so that $g^+ = g^-$ on ∂D. Then we define g on M by: $g = g^+ \circ F$ on M^+ and $g = g^- \circ F$ on M^-. Clearly, g is continuous on M and smooth on $M^+ \cup M^-$, where it satisfies (1.10). Also, all tangential derivatives $\partial^K_{(z,t)} g$ are continuous on $M^+ \cup N$ and on $M^- \cup N$. If we repeatedly differentiate the equation $X_{\overline{n}} g = \varphi_{\overline{n}}$ and use (1.6), we see that all derivatives of g are continuous across N. Thus, in contrast to the example of Grushin [5], [14] we have the following.

Proposition 1.2 (Poincaré lemma). *Let the smooth one-form φ (1.8) satisfy the compatibility condition (1.9) relative to the fold structure E induced by the map F (1.2). If $n \geq 2$ and if F is a strongly pseudoconvex interior fold, then there exists a smooth function g with $\overline{\partial}_E g = \varphi$.*

The Poincaré lemma also holds for $(0, q)$-forms (mod $E^\perp$) for $2 \leq q \leq n-1$. (That it does not hold for top degree forms follows from a general result of Cordaro and Hounie [2]). We sketch the argument. Relative to the coframe (1.4) the $(0, q)$-form φ has the representation

$$\varphi = \varphi' + ds \wedge \varphi'',$$

where φ', φ'' are q- and $q-1$-forms in $d\overline{z}'$ with coefficients in (z', t, s). In an obvious notation

$$\overline{\partial}_E \varphi = \overline{\partial}'_E \varphi' + ds \wedge (X_{\overline{n}} \varphi' - \overline{\partial}'_E \varphi'').$$

By a change $\varphi \mapsto \varphi - \overline{\partial}_E \psi'$, where ψ' is a $(0, q-1)$-form without ds, we can achieve $\varphi'' = O(s^\infty)$ as above. Suppose that $\overline{\partial}_E \varphi = 0$. Then the analogue of lemma (1.1) shows that the transplanted forms $\varphi^\pm$ are smooth on $\overline{D}$ and agree along ∂D. There exist smooth $(0, q-1)$-forms $\psi^\pm$ on $\overline{D}$ with $\overline{\partial} \psi^\pm = \varphi^\pm$. We have $\overline{\partial}(\psi^+ - \psi^-) = 0$ along ∂D. If we are below the Lewy unsolvability degree, i. e. $q - 1 < n - 1$, then there is a smooth $(0, q-2)$-form η with $\overline{\partial}\eta = \psi^+ - \psi^-$ along ∂D. We replace ψ^- by $\psi^- + \overline{\partial}\eta$. We patch together the forms $F^*\psi^\pm$ on $M^\pm$ as above to get a smooth form ψ satisfying $\overline{\partial}_E \psi = \varphi$.

§2. Local normalization

We consider n independent smooth (C^∞) complex vector fields $X_{\overline{j}}$, defined on an open set M of $\mathbf{R}^{2n}$ containing 0, and denote their complex conjugates by X_j as before. At each point $x \in M$, E_x is the complex vector space spanned by the $X_{\overline{j}}(x)$. We assume that $E_0 \cap \overline{E}_0$ is one-dimensional. After changing the frame, we may assume that $X_{\overline{n}}(0) =$

$X_n(0)$ spans $E_0 \cap \overline{E}_0$, and that $X_{\overline{n}}(0)$, $X_{\overline{\alpha}}(0)$, $X_\alpha(0)$ (see (1.5)) span $E_0 + \overline{E}_0$. We then choose coordinates $(z^\alpha = x^\alpha + iy^\alpha, t, s)$ so that $X_{\overline{\alpha}}(0) = \partial_{\overline{\alpha}}$, $X_{\overline{n}} = \partial_s$, and ∂_t is transverse to $E + \overline{E}$ at 0. Then we may write

$$X_{\overline{j}} = A_{\overline{j}}^{\overline{\beta}} \partial_{\overline{\beta}} + A_{\overline{j}}^{\overline{n}} \partial_s + A_{\overline{j}}^{\beta} \partial_\beta + B_{\overline{j}} \partial_t \,,$$

where $A_{\overline{j}}^{\overline{i}}(0) = \delta_{\overline{j}}^{\overline{i}}$. After changing the frame via the inverse matrix of $A_{\overline{j}}^{\overline{i}}$, we get

$$(2.1) \qquad \begin{aligned} X_{\overline{\alpha}} &= \partial_{\overline{\alpha}} + A_{\overline{\alpha}}^{\beta} \partial_\beta + B_{\overline{\alpha}} \partial_t \,, \\ X_{\overline{n}} &= \partial_s + A_{\overline{n}}^{\beta} \partial_\beta + B_{\overline{n}} \partial_t \,, \end{aligned}$$

where $A_{\overline{i}}^{\beta}(0) = 0$, $B_{\overline{j}}(0) = 0$. In this form (2.1) the frame is uniquely determined by the coordinate system (z, t, s), which we refer to as adapted to our structure at 0.

The *degeneracy locus*,

$$(2.2) \qquad N = \{x \in M : E_x \cap \overline{E}_x \neq 0\} \,,$$

is the set of points where some non-trivial linear combination of the $X_{\overline{j}}$ is real, or equivalently, where the vectors $X_{\overline{j}}, X_j$ are dependent. If we write out the condition $0 = a^j X_{\overline{j}} + b^j X_j$ and first eliminate a^n, we get

$$(2.3) \qquad N = \{r = 0\} \,, r = \det \begin{bmatrix} \delta_{\beta}^{\alpha} & A_{\overline{\beta}}^{\alpha} & -iA_{\overline{n}}^{\alpha} \\ A_{\beta}^{\overline{\alpha}} & \delta_{\overline{\beta}}^{\overline{\alpha}} & iA_{n}^{\overline{\alpha}} \\ B_\alpha & B_{\overline{\alpha}} & 2\mathrm{Im}(B_{\overline{n}}) \end{bmatrix} .$$

The shift in the bars over indices reflects complex conjugation, and a factor of i has been inserted in the last column of the determinant to make r real. Our *genericity* assumption on the degeneracy is the transversality condition

$$(2.4) \qquad X_{\overline{n}} r(0) \neq 0 \,.$$

It implies that N is a smooth hypersurface in M. Alternately, we may state it as follows: $dr(0)$, extended to $T_0(M) \otimes \mathbf{C}$ and then restricted to E_0 , is non-zero. It follows that $E \cap (T(N) \otimes \mathbf{C})$ gives a CR structure of real hypersurface type on N. If we additionally restrict the initially chosen frame above so that the $X_{\overline{\alpha}}$ span the (0,1)-vectors of this CR structure, then $\partial_{\overline{\alpha}} r(0) = 0$. We then make the coordinate change

$$z'^\alpha = z^\alpha \,, t' = t \,, s' = r(z, t, s) \,,$$

and the corresponding frame change

$$X'_{\overline{n}} = (r_s + A^{\beta}_{\overline{n}} r_\beta + B_{\overline{n}} r_t)^{-1} X_{\overline{n}}\,, X'_{\overline{\alpha}} = X_{\overline{\alpha}} - (r_{\overline{\alpha}} + A^{\beta}_{\overline{\alpha}} r_\beta + B_{\overline{\alpha}} r_t) X'_{\overline{n}}\,.$$

After this we may assume that N has the form (1.3), and that the vectors $X_{\overline{j}}$ still have the form (2.1).

To preserve these normalizations, we restrict our coordinate changes to the form

$$z'^{\alpha} = z^{\alpha} + f^{\alpha}\ , t' = t + f^0\ , s' = s + f^n\ , \tag{2.5}$$

where

$$f^{\alpha} = O(2)\ , f^0 = O(2)\ , f^n = s\hat{f}^n\ , \hat{f}^n = O(1).$$

In the prime system there is a unique adapted E-frame

$$\begin{aligned} X'_{\overline{\alpha}} &= \partial'_{\overline{\alpha}} + A'_{\overline{\alpha}}{}^{\beta} \partial'_\beta + B'_{\overline{\alpha}} \partial_{t'}\ , \\ X'_{\overline{n}} &= \partial_{s'} + A'_{\overline{n}}{}^{\beta} \partial'_\beta + B'_{\overline{n}} \partial_{t'}\ . \end{aligned} \tag{2.6}$$

As in Section 1 of [20] we have the relations

$$\begin{aligned} A^{\beta}_{\overline{\alpha}} + X_{\overline{\alpha}} f^{\beta} &= (\delta^{\overline{\gamma}}_{\overline{\alpha}} + X_{\overline{\alpha}} f^{\overline{\gamma}}) A'_{\overline{\gamma}}{}^{\beta} + X_{\overline{\alpha}} f^n A'_{\overline{n}}{}^{\beta}, \\ B_{\overline{\alpha}} + X_{\overline{\alpha}} f^0 &= (\delta^{\overline{\gamma}}_{\overline{\alpha}} + X_{\overline{\alpha}} f^{\overline{\gamma}}) B'_{\overline{\gamma}} + X_{\overline{\alpha}} f^n B'_{\overline{n}}, \\ A^{\beta}_{\overline{n}} + X_{\overline{n}} f^{\beta} &= X_{\overline{n}} f^{\overline{\gamma}} A'_{\overline{\gamma}}{}^{\beta} + (1 + X_{\overline{n}} f^n) A'_{\overline{n}}{}^{\beta}, \\ B_{\overline{n}} + X_{\overline{n}} f^0 &= X_{\overline{n}} f^{\overline{\gamma}} B'_{\overline{\gamma}} + (1 + X_{\overline{n}} f^n) B'_{\overline{n}}. \end{aligned} \tag{2.7}$$

Exactly as in [20] we may choose $f^{\alpha}(z,t)$, $f^0(z,t)$ to achieve

$$[A^{\beta}_{\overline{\alpha}}]|_N = O(|(z,t)|^2), \tag{2.8}$$

$$[B_{\overline{\alpha}}]|_N = -ib_{\beta\overline{\alpha}} z^{\beta} + B^*_{\overline{\alpha}}(z,t),\ B^*_{\overline{\alpha}} = O(|(z,t)|^2),$$

where the hermitian matrix $b_{\beta\overline{\alpha}}$ represents the Levi-form at 0 of the CR structure on N.

For all further normalizations, we restrict to changes (2.5) with $f = s\hat{f}$, so that N and the functions (2.8) on it remain unchanged. The last two equations of (2.7) give

$$[A^{\beta}_{\overline{n}} + \hat{f}^{\beta} - \hat{f}^{\overline{\gamma}} A^{\beta}_{\overline{\gamma}}]|_N = [(1 + \hat{f}^n) A'_{\overline{n}}{}^{\beta}]|_N,$$

$$[B_{\overline{n}} + \hat{f}^0 - \hat{f}^{\overline{\gamma}} B_{\overline{\gamma}}]|_N = [(1+\hat{f}^n)B'_{\overline{n}}]|_N.$$

We can choose $\hat{f}^\beta(z,t)$, $\hat{f}^0(z,t)$ uniquely so that the left side of the first equation, and the real part of the left side of the second vanish. After this change we may assume

$$A^\beta_{\overline{n}} = s\hat{A}^\beta_{\overline{n}} \, , B_{\overline{n}} = s\hat{B}_{\overline{n}} \, , \mathrm{Im}(\hat{B}_{\overline{n}}) \neq 0 \, ,$$

since $r = 0$ and $\partial_s r \neq 0$ on N in (2.3). Now we restrict to changes (2.5) with

$$f^\beta = s^2 \hat{f}^\beta \, , f^0 = s^2 \hat{f}^0 \, , f^n = s\hat{f}^n \, . \tag{2.9}$$

Substituting into (2.7), dividing by s, and letting $s \to 0$ gives

$$[\hat{A}^\beta_{\overline{n}} + 2\hat{f}^\beta - 2\hat{f}^{\overline{\gamma}} A^\beta_{\overline{\gamma}}]|_N = [(1+\hat{f}^n)\hat{A}'^{\,\beta}_{\overline{n}}]|_N \, ,$$

$$[\hat{B}_{\overline{n}} + 2\hat{f}^0 - 2\hat{f}^{\overline{\gamma}} B_{\overline{\gamma}}]|_N = [(1+\hat{f}^n)\hat{B}'_{\overline{n}}]|_N \, .$$

Thus we can choose $\hat{f}(z,t)$ to make

$$[\hat{A}'^{\,\beta}_{\overline{n}}]|_N = 0 \, , [\mathrm{Re}(\hat{B}'_{\overline{n}})]|_N = 0 \, , [\mathrm{Im}(\hat{B}'_{\overline{n}})]|_N = -1 \, .$$

After these preliminary normalizations we make the coordinate change (2.5),(2.9) with

$$\hat{f}(z,t,s) = \sum_{j=1}^{\infty} \frac{1}{j!} s^j g(s/\varepsilon_j) \hat{f}_j(z,t) \, , \tag{2.10}$$

where $\hat{f} = (\hat{f}^\beta, \hat{f}^0, \hat{f}^n)$. Following [8], vol. I, p. 16, we choose $g(s)$ to be a fixed smooth real valued function of suitably small compact support with $g(s) - 1$ vanishing to infinite order at $s = 0$. The functions $\hat{f}_j$ are successively chosen, depending on the previous choices, to achieve

$$A^\beta_{\overline{n}} = O(s^{j+1}) \, , B_{\overline{n}} + is = O(s^{j+1}) \, . \tag{2.11}$$

This is independent of $\varepsilon_j > 0$ which is then chosen so small that

$$|\partial^K_{(z,t,s)}[s^j g(s/\varepsilon_j)\hat{f}_j(z,t)]| \leq j! 2^{-j} \, , |K| \leq j-1 \, .$$

The transformation so constructed is smooth, and in the final coordinate system (2.11) holds for every j. Hence, we have established the following.

Lemma 2.1. *There exists a smooth adapted coordinate system (z,t,s) and corresponding frame (2.1) so that (1.3), (2.2) and (2.8) hold, and*

$$A_{\overline{n}}^{\beta} = O(s^{\infty}) , B_{\overline{n}} + is = O(s^{\infty}) .$$

For the case $n = 1$, see Sjörstrand [16] and Trèves [17].

§3. Structures with positive definite Levi-form

With the normalizations of lemma (2.1) we have the bracket relations at 0

$$(3.1) \qquad [X_i, X_{\overline{j}}]_0 = -2ib_{i\overline{j}}\partial_t , b_{\alpha\overline{n}} = 0 , b_{n\overline{n}} = 1/2 .$$

$b_{i\overline{j}}$ represents the Levi-form of the structure E at 0. It will be positive definite precisely when $b_{\alpha\overline{\beta}}$, which represents the Levi-form of the CR structure on N, is positive definite.

Theorem 3.1. *Let E be a complex Frobenius structure of rank n on a neighborhood of 0 in $\mathbf{R}^{2n}$ with $n \geq 2$. Suppose that E has a generic degeneracy with positive definite Levi-form at 0. Then there exists a neighborhood M of 0 and a strongly pseudoconvex interior fold $F : M \to \mathbf{C}^n$ inducing the structure E on M.*

For the proof we assume the normalizations of lemma (2.1) and consider the transformation

$$T : (z, t, s) \mapsto (z, t, y = \frac{1}{2}s^2) ,$$

and its restrictions $T_{\pm}$to $M^{\pm}$. On $U = T_{\pm}(M^{\pm}) \subseteq \{y \geq 0\}$ we have the vector fields

$$X'_{\overline{\alpha}} = (T_+)_*(X_{\overline{\alpha}}) , X'_{\overline{n}} = (T_+)_*(\frac{1}{s}X_{\overline{n}}) .$$

Clearly $X'_{\overline{n}}$ is smooth up to the boundary $y = 0$.

By a Cauchy integral argument similar to the one given in the proof of lemma (1.1), it follows that the $X'_{\overline{\alpha}}$'s are also smooth up to the boundary of U. In fact, we just set $h \equiv 0$ in (1.13),(1.14), and replace $b_{\overline{\alpha}}^{+}$ by $A_{\overline{\alpha}}^{\beta} \circ (T_+)^{-1}$, or $B_{\overline{\alpha}} \circ (T_+)^{-1}$ in (1.15). Hence,we have a smooth almost complex structure with strongly pseudoconvex boundary on U. By the theorem of Hanges and Jacobowitz [6] there is a holomorphic coordinate system $G_+ : U \to \mathbf{C}^n$ which is smooth up to the boundary for a perhaps smaller U containing 0. Similarly, we have a smooth holomorphic

$G_- : U \to \mathbf{C}^n$ for the almost complex structure induced by T_-. The map

$$G_+^{-1} \circ G_- : G_-(\partial U) \to G_+(\partial U)$$

is a CR equivalence. By the smooth version of the H. Lewy extension theorem, it extends to a biholomorphic equivalence $H : G_+(U) \to G_-(U)$, which is smooth up to the boundary. We replace G_- by $H \circ G_-$, and then define

$$F = \begin{cases} G_+ \circ T & on \quad M^+ \\ G_- \circ T & on \quad M^- \end{cases}$$

An argument strictly analogous to that given in the proof of proposition (1.2) shows that F is smooth on M. Since F is an embedding of N, its coordinate functions are independent on M also. Hence, F satisfies the requirements of the theorem.

The hypotheses on the Levi-form can clearly be weakened, since they may be so in the Hanges-Jacobowitz theorem (Catlin), and in the H. Lewy extension theorem (Trepreau).

References

[1] T. Akahori, A new approach to the local embedding theorem of CR structures for $n \geq 4$, Memoirs AMS 67 (1987).
[2] P. Cordaro and J. Hounie, On the local solvability of underdetermined systems of vector fields, Amer. J. Math. 112(1990)243-270.
[3] P. Cordaro and F. Treves, personal communication.
[4] G. B. Folland and J. J. Kohn, The Neumann Problem for the Cauchy-Riemann Complex, Ann. of Math. Studies 75, Princeton University Press (1972).
[5] V. V. Grushin, A certain example of a differential equation without solutions, Math. Notes 10(1971)499-501 (from Mat. Zametki 10 (1971)125-128).
[6] N. Hanges and H. Jacobowitz, A remark on almost complex structures with boundary, Amer. Jour. Math. 111(1989)53-64.
[7] L. Hörmander, The Frobenius-Nirenberg theorem, Ark. Mat.29 (1964) 425-432.
[8] ——, The Analysis of Linear Partial Differential Operators, Vol.1-4, Springer-Verlag, New York (1983-85).
[9] H. Jacobowitz, Global Mizohata structures, J. Geom. Anal. 3 (1992) 153-193.
[10] ——, and F. Treves, Aberrant CR structures, Hokkaido Math. J. 22 (1983) 276-292.

[11] M. Kuranishi, Strongly pseudoconvex CR structures over small balls III, Ann. of Math. 116 (1982) 249-330.

[12] S. Mizohata, Solutions nulles et solutions non analytic, J. Math. Kyoto Univ. 1(1962)271-302.

[13] L. Nirenberg, A complex Frobenius theorem, Seminars on analytic functions I, Princeton (1957)172-189.

[14] ——, Lectures on linear partial differential equations, CBMS Studies No. 17, AMS(1973).

[15] M. Range and I. Lieb, Lösungsoperatoren für den Cauchy-Riemann-Kpmplex mit C^k-Abschätzungen, Math. Ann. 253 (1980) 145-164.

[16] J. Sjörstrand, Note on a paper of F. Treves concerning Mizohata type operators, Duke Math. Jour. 47(1980)601-608.

[17] F. Treves, Remarks about certain first-order linear PDE in two variables, Comm. PDE 5(4) (1980)381-425.

[18] ——, On the local solvability and the local integrability of systems of vector fields, Acta Math. 151(1983)1-38.

[19] F. Treves, Hypo-Analytic Structures: Local Theory, Princeton University Press (1992).

[20] S. M. Webster, On the proof of Kuranishi's embedding theorem, Ann. Inst. H. Poincaré, Anal. non-lin., 6(1989)183-207.

Department of Mathematics
University of Chicago
Chicago, IL 60637
U. S. A.