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Deformations of Coxeter hyperplane arrangements and their characteristic polynomials

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Abstract.

Let $\mathcal{A}$ be a Coxeter hyperplane arrangement, that is the arrangement of reflecting hyperplanes of an irreducible finite Coxeter group. A deformation of $\mathcal{A}$ is an affine arrangement each of whose hyperplanes is parallel to some hyperplane of $\mathcal{A}$. We survey some of the interesting combinatorics of classes of such arrangements, reflected in their characteristic polynomials.

§1. Introduction

Much of the motivation for the study of arrangements of hyperplanes comes from Coxeter arrangements. Because of their importance in algebra, Coxeter arrangements have been studied a great deal in the context of representation theory of semisimple Lie algebras (where they arose), invariant theory of reflection groups, combinatorics of root systems and Coxeter groups, combinatorics of convex polytopes and oriented matroids and within the general theory of hyperplane arrangements [42]. From a geometric, combinatorial and algebraic point of view, they are fairly well understood in terms of their classification, facial structure, intersection posets, characteristic polynomials and freeness; see [17, §2.3] and [42, Chapter 6].

A *deformation* of a Coxeter arrangement $\mathcal{A}$ is an affine arrangement each of whose hyperplanes is parallel to some hyperplane of $\mathcal{A}$. Interesting examples of such arrangements first arose in the study of affine Weyl groups by Shi [53, 54] and have appeared since then in various mathematical contexts. Their combinatorics was first investigated systematically by Stanley [59] and relates to objects studied classically in enumeration such as trees, set partitions and partially ordered sets. A

[1]The present article was written while the author was a Hans Rademacher Instructor at the University of Pennsylvania.

major role in this study has been played by the characteristic polynomial.

In the present exposition we describe some of the lively work on deformations of Coxeter arrangements that has been carried out in the recent past. We emphasize the combinatorial and algebraic properties related to their characteristic polynomials, a topic which we find rich and interesting enough to stand on its own. We discuss some of the relevant motivation and include a number of open questions which are often suggested naturally by the results.

Acknowledgement. *The author has benefited from the work of Alexander Postnikov and Richard Stanley* [44, Chapter 1][45, 59], *some of which was carried out in parallel with his own thesis work* [3][4, Part II], *as well as from various discussions with Anders Björner, Victor Reiner, Gian-Carlo Rota, Bruce Sagan, Hiroaki Terao and Günter Ziegler. He is indebted to Victor Reiner and Richard Stanley for their valuable comments.*

§2. Background

The characteristic polynomial. Let $\mathbb{K}$ be a field. A hyperplane arrangement $\mathcal{A}$ in $\mathbb{K}^\ell$ is a finite collection of affine hyperplanes in $\mathbb{K}^\ell$, i.e. affine subspaces of $\mathbb{K}^\ell$ of codimension one. We will mostly be interested in arrangements over the reals, so that $\mathbb{K} = \mathbb{R}$. We call $\mathcal{A}$ *central* if all hyperplanes in $\mathcal{A}$ are linear. The *characteristic polynomial* of $\mathcal{A}$ is defined as

$$\chi(\mathcal{A}, q) = \sum_{x \in L_{\mathcal{A}}} \mu(\hat{0}, x)\, q^{\dim x}, \tag{1}$$

where $L_{\mathcal{A}} = \{\cap \mathcal{F} : \mathcal{F} \subseteq \mathcal{A}\}$ (partially ordered by reverse inclusion) is the *intersection poset* of $\mathcal{A}$, $\hat{0} = \mathbb{K}^\ell$ is the unique minimal element of $L_{\mathcal{A}}$ (which corresponds to $\mathcal{F}$ being empty) and μ stands for its Möbius function [60, §3.7].

The characteristic polynomial $\chi(\mathcal{A}, q)$ is a fundamental combinatorial and topological invariant of $\mathcal{A}$ and plays a significant role throughout the theory of hyperplane arrangements [42]. If $\mathbb{K} = \mathbb{R}$ then $\chi(\mathcal{A}, q)$ gives valuable enumerative information about the cell decomposition of the space $\mathbb{R}^\ell$, induced by $\mathcal{A}$ [17, §2.1]. The cells in this decomposition are the *faces* of $\mathcal{A}$. The faces of dimension ℓ are simply the connected components of the space obtained from $\mathbb{R}^\ell$ by removing the hyperplanes of $\mathcal{A}$ and are called the *regions* or *chambers* of $\mathcal{A}$.

2.1. Theorem (Zaslavsky [69]). *The number of regions into which $\mathcal{A}$ dissects $\mathbb{R}^\ell$ is equal to $(-1)^\ell\chi(\mathcal{A},-1)$.*

In particular, for any k, the number of faces of $\mathcal{A}$ of dimension k depends only on $L_\mathcal{A}$ and ℓ.

On the other hand, if $\mathbb{K}=\mathbb{C}$ then $\chi(\mathcal{A},q)$ gives topological information about the complement $M_\mathcal{A}=\mathbb{C}^\ell-\bigcup_{H\in\mathcal{A}}H$. The following result was proved by Orlik and Solomon in the context of their fundamental work [41] on the cohomology algebra $H^*(M_\mathcal{A},\mathbb{Z})$ of $M_\mathcal{A}$.

2.2. Theorem (Orlik–Solomon [41]). *If $\mathcal{A}$ is a central arrangement in $\mathbb{C}^\ell$ then*

$$\sum_{i\geq 0}\operatorname{rank} H^i(M_\mathcal{A},\mathbb{Z})\,q^i=(-q)^\ell\chi(\mathcal{A},-1/q).$$

For the cohomological significance of $\chi(\mathcal{A},q)$ when $\mathcal{A}$ is a *subspace arrangement* we refer to Björner [15, §7] and Björner and Ekedahl [16]. The following corollary of Theorem 2.2 continues to hold when $\mathcal{A}$ is a subspace arrangement, see [15, §8.3].

2.3. Corollary. *If $\mathcal{A}$ is an arrangement in $\mathbb{R}^\ell$ then*

$$\sum_{i\geq 0}\operatorname{rank} H^i(M_\mathcal{A},\mathbb{Z})=\sum_{i\geq 0}\operatorname{rank} H^i(M_{\mathcal{A}^\mathbb{C}},\mathbb{Z}),$$

where $M_\mathcal{A}$ is the complement of $\mathcal{A}$ in $\mathbb{R}^\ell$ and $M_{\mathcal{A}^\mathbb{C}}$ is the complement of its complexification $\mathcal{A}^\mathbb{C}$ in $\mathbb{C}^\ell$.

Freeness. Let $\mathcal{A}$ be central and $\mathbb{S}:=\mathbb{K}[x_1,x_2,\ldots,x_\ell]$ be the polynomial ring over $\mathbb{K}$ in ℓ variables. Let Q be the product of the linear forms in $\mathbb{S}$ defining the hyperplanes of $\mathcal{A}$, so that Q is unique up to multiplication by an element of $\mathbb{K}^*$, and let $Q\,\mathbb{S}$ be the principal ideal in $\mathbb{S}$ generated by Q. The *module of derivations* $D(\mathcal{A})$ of $\mathcal{A}$ is the set of all derivations $\theta:\mathbb{S}\to\mathbb{S}$ such that $\theta(Q)\in Q\,\mathbb{S}$. $D(\mathcal{A})$ is actually a module over $\mathbb{S}$. The arrangement $\mathcal{A}$ is called *free* [63] if $D(\mathcal{A})$ is a free $\mathbb{S}$-module. One can associate to $\mathcal{A}$ a multiset of ℓ nonnegative integers, called the *exponents* of $\mathcal{A}$. They are the degrees of the elements in any basis of the free $\mathbb{S}$-module $D(\mathcal{A})$.

2.4. Theorem (Terao [65][41, Theorem 4.137]). *If $\mathcal{A}$ is free with exponents $e_1,e_2,\ldots,e_\ell$ then*

$$\chi(\mathcal{A},q)=\prod_{i=1}^{\ell}(q-e_i).$$

Theorem 2.4 is one of a number of results which explain factorization phenomena for $\chi(\mathcal{A}, q)$. Other approaches include supersolvability [58] and its generalizations [14, 19], inductive freeness [63], recursive freeness [75], factorization of rooted complexes [18], factorization [26, 67] and inductive factorization [34]. For background we refer to these sources, [42, Chapter 4] and the survey article [50]. A purely algebraic–combinatorial proof of Theorem 2.4 was given in Solomon and Terao [56]; see also [42, Chapter 4].

Coxeter arrangements. Let Φ be an irreducible root system in $\mathbb{R}^\ell$ [33, §1.2], equipped with the standard inner product. We rely on [33] for basic background and terminology on root systems. The *Coxeter arrangement* $\mathcal{A}_\Phi$ corresponding to Φ is the arrangement of the linear hyperplanes

$$(\alpha, x) = 0$$

orthogonal to the roots $\alpha \in \Phi$, i.e. the reflecting hyperplanes of the associated finite Coxeter group W. See [42, Chapter 6] and [17, §2.3] for expositions of Coxeter arrangements from algebraic-topological and geometric-combinatorial points of view, respectively. The following result will be of interest here.

2.5. Theorem (Arnol'd [1, 2], Saito [51, 52]). *The Coxeter arrangement $\mathcal{A}_\Phi$ is free with exponents the exponents of the root system Φ.*

In fact, explicit bases for the modules of derivations were constructed in terms of the *basic invariants* [33, §3.5] of the algebra of W-invariant polynomials by Saito [51] and Terao [64]. The analogue of Theorem 2.5 for complex reflection groups and a generalization to all reflection arrangements appear in Terao [64, 66].

2.6. Corollary. *If $e_1, e_2, \ldots, e_\ell$ are the exponents of Φ then*

$$\chi(\mathcal{A}_\Phi, q) = \prod_{i=1}^{\ell} (q - e_i).$$

§3. Deformations of Coxeter arrangements

We now assume that Φ is crystallographic [33, §2.9], so that W is a Weyl group. We let Φ^+ be a choice of positive roots. When we give equations for the hyperplanes of deformations of $\mathcal{A}_\Phi$ we will choose Φ and

Φ^+ as in [33, §2.10] and denote the dimension of the ambient space by n. The *braid arrangement* $\mathcal{A}_n$, for instance, consists of the hyperplanes in $\mathbb{R}^n$ of the form $x_i - x_j = 0$. In this notation we prefer to consider $\mathcal{A}_n$ as an arrangement in $\mathbb{R}^n$ (so that its characteristic polynomial has q as a factor), even though it corresponds to the Coxeter arrangement $\mathcal{A}_\Phi$ for $\Phi = A_{n-1}$, which is an arrangement in $\mathbb{R}^{n-1}$ (so that $\ell = n-1$). We extend this convention to deformations of $\mathcal{A}_n$ as well.

We begin with three motivating classes of examples to which we will come back in the next section.

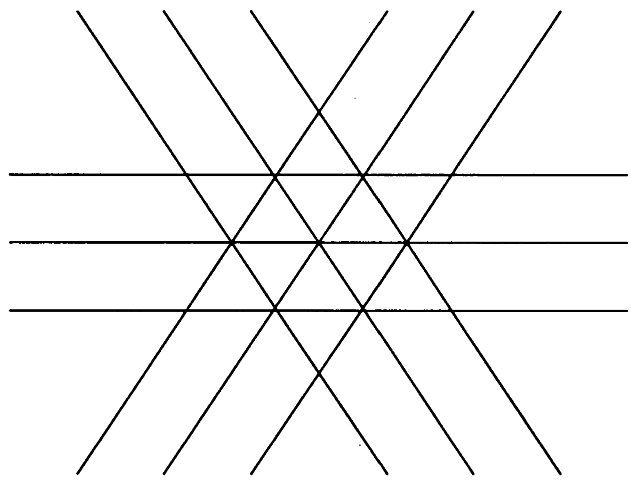

FIG 1. The Catalan arrangement for A_2.

The Catalan arrangement. The *Catalan arrangement* Cat_Φ consists of the hyperplanes

$$\begin{array}{ll} (\alpha, x) = -1, & \\ (\alpha, x) = 0, & \alpha \in \Phi^+ \\ (\alpha, x) = 1, & \end{array}$$

in $\mathbb{R}^\ell$. It is invariant under the action of the Weyl group W and is shown in Figure 1 for $\Phi = A_2$. For $\Phi = A_{n-1}$ the hyperplanes are

$$x_i - x_j = -1, 0, 1 \ \text{ for } \ 1 \le i < j \le n.$$

We denote this arrangement in $\mathbb{R}^n$ by Cat_n. The terminology "Catalan arrangement" comes from the fact that the number of regions of Cat_n, divided by $n!$, is equal to the nth Catalan number. It was observed by Stanley [59, §2] that the regions of Cat_n within the fundamental Weyl chamber of $\mathcal{A}_n$ are in bijection with *unit interval orders* with n elements, i.e. partial orders which come from unit intervals $I_1, I_2, \ldots, I_n$ on the real line by letting $I_i < I_j$ if I_i lies entirely to the left of I_j. To see this, it suffices to let the ith interval be $[x_i, x_i + 1]$, where $x_1 < x_2 < \cdots < x_n$, and observe that the partial order defined by these n intervals depends only on the region of Cat_n in which the point $(x_1, x_2, \ldots, x_n)$ lies. For a treatment of the theory of interval orders see [27].

In another direction, it was observed by Postnikov (see Remark 2 in [47, §6] and [7]) that the regions of Cat_Φ within the fundamental Weyl chamber of $\mathcal{A}_\Phi$ are in bijection with *nonnesting partitions* on Φ, i.e. antichains in the *root order* of Φ, defined on Φ^+ by $\alpha \le \beta$ if $\beta - \alpha$ is a linear combination of positive roots with nonnegative coefficients. The following theorem is a special case of Theorem 4.6 for the classical root systems and has also been verified for $\Phi = G_2, F_4$ and E_6 (see [24, §3]).

3.1. Theorem ([3, 4]). *Let Φ be of type A, B, C or D. We have $\chi(\mathrm{Cat}_\Phi, q) = \chi(\mathcal{A}_\Phi, q - h)$, where h is the Coxeter number of Φ. In particular, the number of regions of* Cat_Φ *is equal to*

$$\prod_{i=1}^{\ell} (e_i + h + 1)$$

and the number of nonnesting partitions on Φ is equal to

$$\prod_{i=1}^{\ell} \frac{e_i + h + 1}{e_i + 1},$$

where $e_1, e_2, \ldots, e_\ell$ are the exponents of Φ.

The Shi arrangement. The *Shi arrangement* $\mathcal{S}_\Phi$ consists of the hyperplanes

$$\begin{array}{ll} (\alpha, x) = 0, & \\ & \alpha \in \Phi^+ \\ (\alpha, x) = 1, & \end{array}$$

in $\mathbb{R}^\ell$. This is shown in Figure 2 for $\Phi = A_2$. For $\Phi = A_{n-1}$ the hyperplanes are

$$\begin{array}{l} x_i - x_j = 0 \ \text{ for } \ 1 \le i < j \le n, \\ x_i - x_j = 1 \ \text{ for } \ 1 \le i < j \le n. \end{array}$$

We denote this arrangement in $\mathbb{R}^n$ by $\mathcal{S}_n$. The arrangement $\mathcal{S}_\Phi$ was first considered by Shi in his investigation of the affine Weyl group $\widetilde{A}_{n-1}$ of type A_{n-1} [53, §7]. The regions of $\mathcal{S}_n$ correspond to certain equivalence classes of elements of $\widetilde{A}_{n-1}$, called "admissible sign types", which were shown by Shi to play a significant role in the Kazhdan–Lusztig theory of cells [36] for this group.

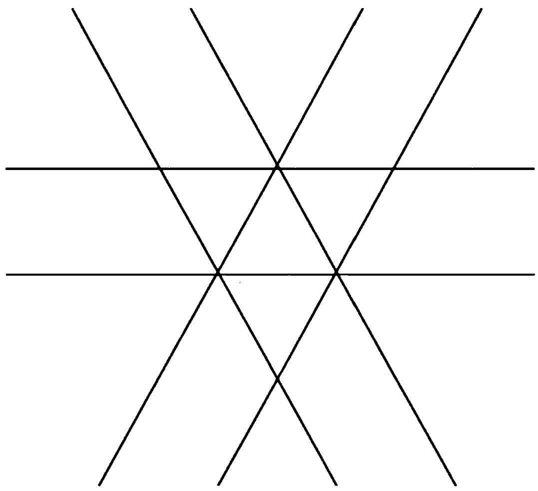

FIG 2. The Shi arrangement for A_2.

Since Shi's work, the arrangement $\mathcal{S}_\Phi$ has continued to appear as related to affine Weyl groups in Headley [30, 31, 32], invariant theory of finite Coxeter groups in Solomon and Terao [57], and representations of affine Hecke algebras in Ram [46], as an object of independent interest in enumerative combinatorics in the type A case [5, 10, 59, 61], and as a particularly nice example where techniques from the theory of hyperplane arrangements apply [3, 4, 6, 32, 44, 45]. Much of the interest initially attracted by the Shi arrangement is due to the following surprising result.

3.2. Theorem (Shi [54], [53, Corollary 7.3.10] for $\Phi{=}A_{n-1}$). *The number of regions of $\mathcal{S}_\Phi$ is $(h+1)^\ell$, where h is the Coxeter number of Φ. In particular, the number of regions of $\mathcal{S}_n$ is $(n+1)^{n-1}$.*

Shi gave a constructive proof of this fact for $\mathcal{S}_n$ [53] by considering the elements of the affine Weyl group of type A_{n-1} which correspond to the regions and a uniform but lengthy proof in the general case [54] using his notion of "sign type" for affine Weyl groups. More direct combinatorial proofs in the type A case can be found in Headley [31], Stanley [59] and Athanasiadis and Linusson [10, §2]. The proof in [59] yields an interesting refinement of the enumeration of the regions by a certain distance statistic; see Theorem 6.13.

The following stronger result, via Theorem 2.1, on the characteristic polynomial of $\mathcal{S}_\Phi$ was proved by Headley, whose argument relied on Theorem 3.2 and induction.

3.3. Theorem (Headley [30, 31, 32]). *We have* $\chi(\mathcal{S}_\Phi, q) = (q-h)^\ell$, *where* h *is the Coxeter number of* Φ. *In particular, we have* $\chi(\mathcal{S}_n, q) = q(q-n)^{n-1}$.

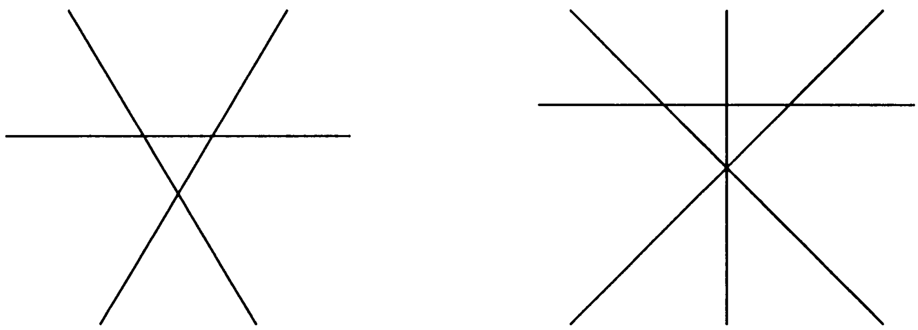

FIG 3. The Linial arrangements for A_2 and B_2.

The Linial arrangement. In the rest of the paper we allow Φ to be the non-reduced system BC_n, which is the union of B_n and C_n in the standard choice of [33, §2.10].

The *Linial arrangement* $\mathcal{L}_\Phi$ consists of the hyperplanes

$$(\alpha, x) = 1, \quad \alpha \in \Phi^+$$

in $\mathbb{R}^\ell$. It is shown in Figure 3 for $\Phi = A_2$ and B_2. For $\Phi = A_{n-1}$ the hyperplanes are

$$x_i - x_j = 1 \text{ for } 1 \leq i < j \leq n.$$

We denote this arrangement in $\mathbb{R}^n$ by $\mathcal{L}_n$. Interest in the arrangement $\mathcal{L}_n$ came from a surprising conjecture of Linial, Ravid and Stanley (see [59, §4]) stating that the number of regions of $\mathcal{L}_n$ is equal to the number f_n of *alternating trees* on $n+1$ vertices, i.e. trees on the vertex set $\{1, 2, \ldots, n+1\}$ such that no $i < j < k$ are consecutive vertices of a path in the tree, in the order i, j, k. Alternating trees first appeared in [28]. The explicit formula

$$f_n = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} (k+1)^{n-1} \tag{2}$$

was found by Postnikov [43, Theorem 1], who later proved the conjecture about $\mathcal{L}_n$ as follows.

3.4. Theorem (Postnikov [44, Theorem 1.4.5][45, Theorem 8.1]). *The number of regions of the Linial arrangement $\mathcal{L}_n$ is equal to f_n.*

There is no bijective proof of the Linial–Ravid–Stanley conjecture at present. Postnikov's theorem naturally suggests the problem of finding directly an explicit formula for the characteristic polynomial of $\mathcal{L}_n$. Such a formula was first given in [3, 4]; see also [44, §1.5][45, §9] and [59, Corollary 4.2]. The proof in [3, 4] was simplified in [9, §3]; see also Section 4.

3.5. Theorem ([3, Theorem 4.2][4, Theorem 6.4.2]). *The Linial arrangement $\mathcal{L}_n$ has characteristic polynomial*

$$\chi(\mathcal{L}_n, q) = \frac{q}{2^n} \sum_{k=0}^{n} \binom{n}{k} (q-k)^{n-1}.$$

This expression implies Theorem 3.4, via Theorem 2.1. For results on the asymptotic behaviour of $\chi(\mathcal{L}_n, q)$ for large n, see Postnikov [44, §1.6.3]. The analogous problem to compute $\chi(\mathcal{L}_\Phi, q)$ in general is also suggested by a conjecture of Postnikov and Stanley [45, §9] which, in a special case, states that all roots of $\chi(\mathcal{L}_\Phi, q)$ have the same real part; see Conjecture 3.6.

The affine Weyl arrangement. As is apparent from the previous examples, interesting deformations of $\mathcal{A}_\Phi$ often occur as subarrangements of the *affine Weyl arrangement* $\widetilde{\mathcal{A}}_\Phi$

$$(\alpha, x) = k, \quad \alpha \in \Phi^+, \quad k \in \mathbb{Z},$$

the arrangement of reflecting hyperplanes of the *affine Coxeter group* $\widetilde{W}$. For integers $a \leq b$ we denote by $\mathcal{A}_\Phi^{[a,b]}$ the subarrangement of hyperplanes

$$(\alpha, x) = k, \quad \alpha \in \Phi^+, \quad k = a, a+1, \ldots, b.$$

These include $\mathcal{A}_\Phi$, Cat_Φ, $\mathcal{S}_\Phi$ and $\mathcal{L}_\Phi$ and, more generally, the *extended Catalan arrangements* $\mathcal{A}_\Phi^{[-a,a]}$

$$(\alpha, x) = -a, -a+1, \ldots, a, \quad \alpha \in \Phi^+,$$

the *extended Shi arrangements* $\mathcal{A}_\Phi^{[-a+1,a]}$

$$(\alpha, x) = -a+1, -a+2, \ldots, a, \quad \alpha \in \Phi^+$$

and the *extended Linial arrangements* $\mathcal{A}_\Phi^{[1,b]}$

$$(\alpha, x) = 1, 2, \ldots, b, \quad \alpha \in \Phi^+.$$

These extended analogues have similar properties with those of Cat_Φ, $\mathcal{S}_\Phi$ and $\mathcal{L}_\Phi$, respectively; see Section 4. The connection between interval orders and deformations of $\mathcal{A}_n$, for instance, was extended in [59, §2] by considering labeled *marked intervals* with arbitrary prescribed lengths. As an example, suppose that the ith interval $I_i = [x_i, x_i + \lambda_i - 1]$ has integral length $\lambda_i - 1$ and is marked at all its points $x_i + k$ which are an integral distance k from the endpoint x_i. The number of inequivalent orders for placing these marked intervals on a line such that no two marks coincide is equal to the number of regions of the deformation of $\mathcal{A}_n$ with hyperplanes

$$(3) \quad x_i - x_j = -\lambda_i + 1, \ldots, -1, 0, 1, \ldots, \lambda_j - 1 \text{ for } 1 \le i < j \le n,$$

since comparing the marks $x_i + k$ and $x_j + l$ amounts to choosing one of the halfspaces determined by the hyperplane $x_i + k = x_j + l$. These placements correspond to nonnesting set partitions [7] whose blocks are labeled and have sizes $\lambda_1, \lambda_2, \ldots, \lambda_n$ (a set partition π of $[m] := \{1, 2, \ldots, m\}$ is *nonnesting* if whenever $a < b < c < d$ and a, d are *consecutive* elements of a block B of π, b and c are not both contained in a block B' of π). They have also appeared in a geometric context related to monotone paths on polytopes [8]. The characteristic polynomials of the arrangements (3), which include the extended Catalan arrangements of type A, and those of root system analogues of (3) have turned out to be useful for the enumeration of nonnesting partitions by block size; see Proposition 4.7 and [7].

The family of arrangements in the following conjecture includes the extended Shi and Linial arrangements.

3.6. Conjecture (Postnikov–Stanley [45, §9]). *If a, b are nonnegative integers, not both zero, satisfying $a \le b$ then all roots of the polynomial $\chi(\mathcal{A}_\Phi^{[-a+1,b]}, q)$ have the same real part.*

For a semi-generic deformation of $\mathcal{A}_n$, see [59, §3][45, §6]. Other deformations of Coxeter arrangements appear in [59, §2], [4, Chapters 6–7].

§4. The characteristic polynomial

The examples in the previous section make it clear that tools to compute the characteristic polynomial explicitly are desirable. Such tools have traditionally included the following.

Definition: the defining equation (1) [32] or the equivalent expression, given in [42, Lemma 2.55],

$$\chi(\mathcal{A}, q) = \sum_{\mathcal{F} \subseteq \mathcal{A}} (-1)^{\#\mathcal{F}}\, q^{\dim \mathcal{F}}, \tag{4}$$

where the sum is over all subarrangements $\mathcal{F}$ of $\mathcal{A}$ and $\dim \mathcal{F}$ is the dimension of the intersection of the hyperplanes in $\mathcal{F}$; see, e.g., [44, 45].

Deletion-Restriction: this powerful technique in the theory of arrangements yields the formula ([42, Theorem 2.56])

$$\chi(\mathcal{A}, q) = \chi(\mathcal{A}', q) - \chi(\mathcal{A}'', q),$$

where $\mathcal{A}'$ and $\mathcal{A}''$ are obtained from $\mathcal{A}$ by deleting or restricting on a hyperplane $H \in \mathcal{A}$ [42, p. 14]; see, e.g., [23, 24, 6].

Chromatic Polynomials: the "signed chromatic polynomial" interpretation of Zaslavsky [70] (if $\mathcal{A}$ consists of some of the reflecting hyperplanes of Coxeter type B) and its generalization to "gain graph coloring" [73, §4]; see, e.g., [70, 71, 20], [42, §2.4] and [29, 73, 74], respectively.

Factorizations: the theory of supersolvable [58], inductively free [63] or, more generally, free arrangements [63] [42, Chapter 4], when $\chi(\mathcal{A}, q)$ factors; see, e.g., [42, §4.3] and [23, 24, 6, 12, 35].

For a discussion from a matroid theoretic point of view we refer to Kung [38, §5]; see also Zaslavsky [72]. Here we mention that Crapo's identity [21] [38, p. 49], which, in the language of arrangements, expresses the characteristic polynomial of a subarrangement of $\mathcal{A}$ in terms of those of its restrictions, has been of use in this context; see, e.g., [39].

Useful tools have resulted recently by interpreting the right hand sides of (1) or (4) using Möbius inversion or inclusion-exclusion, respectively. This is easily done when $\mathcal{A}$ is defined over a finite field $\mathbb{F}_q$, since then $q^{\dim x}$ is the cardinality of x. The following theorem appeared, in a dual formulation, as early as 1970 in the work of Crapo and Rota [22, §16] (see the discussion in [3, §1]) and was stated in the language of arrangements by Terao [66, Proposition 4.10]; see also [42, Theorem 2.69]. The proof is an easy application of Möbius inversion [49] [60, §3.7]. We denote by $V_{\mathcal{A}}$ the union of the hyperplanes of $\mathcal{A}$.

4.1. Theorem (Crapo–Rota [22], Terao [66]). *If $\mathcal{A}$ is an arrangement in $\mathbb{F}_q^n$ then the cardinality of $\mathbb{F}_q^n - V_{\mathcal{A}}$ is equal to the value $\chi(\mathcal{A}, q)$ of the characteristic polynomial of $\mathcal{A}$ at q.*

Let $\mathcal{A}$ be an arrangement in $\mathbb{K}^n$ where $\mathbb{K}$ has characteristic zero, say $\mathbb{K} = \mathbb{R}$ for simplicity, and let $\mathbb{Z}_q$ denote the abelian group of integers modulo q. We call $\mathcal{A}$ a *$\mathbb{Z}$-arrangement* if its hyperplanes are given by equations with integer coefficients. Such equations define subsets of the finite set $\mathbb{Z}_q^n$ if we reduce their coefficients modulo q. We still denote by $V_{\mathcal{A}}$ the union of these subsets, supressing q in the notation. If q avoids a finite set of prime factors, which depends on $\mathcal{A}$, then the intersection poset of the reduced arrangement in $\mathbb{Z}_q^n$ is isomorphic to that of $\mathcal{A}$ and Theorem 4.1 gives a combinatorial interpretation to the value $\chi(\mathcal{A}, q)$. This idea was first used for the purpose of computing the characteristic polynomial in [3][4, Part II] and allows for a variety of techniques from enumerative combinatorics to be employed.

The next theorem, stated as in [9, Theorem 2.1], generalizes easily to subspace arrangements [3, Theorem 2.2][4, Theorem 5.2.1] [16]. It was given independently by Björner and Ekedahl in their recent work [16] on the cohomology of subspace arrangements over finite fields; see Proposition 3.2 and Lemma 5.1 in [16].

4.2. Theorem (Athanasiadis [3, 4, 9], Björner–Ekedahl [16]). *Let $\mathcal{A}$ be a $\mathbb{Z}$-arrangement in $\mathbb{R}^n$. There exist positive integers m, k which depend only on $\mathcal{A}$, such that for all q relatively prime to m with $q > k$,*

$$\chi(\mathcal{A}, q) = \#\left(\mathbb{Z}_q^n - V_{\mathcal{A}}\right).$$

For subarrangements of the Coxeter arrangement of type B, Theorem 4.2 specializes to Zaslavsky's chromatic polynomial interpretation [70] or its generalization to subspace arrangements by Blass and Sagan [20, Theorem 2.1]. For a different generalization of Theorem 4.1 in the context of the Tutte polynomial see Reiner [48]. Finally, an interesting point of view and interpretation to (1) and (4) in terms of *valuations* appears in Ehrenborg and Readdy [25], who give several applications to classes of complex arrangements.

Theorem 4.2 has been quite useful for classes of deformations of Coxeter arrangements [3, 4, 7, 8, 9] [68, §4]. In the remainder of this section we give applications related to the examples in Section 3. For an illustration, we give a proof of Theorem 3.3 in the case $\Phi = A_{n-1}$, taken from [3, 4].

Proof of Theorem 3.3 for $\Phi = A_{n-1}$. Theorem 4.2 implies that, for large primes q, $\chi(\mathcal{S}_n, q)$ counts the number of n-tuples $x = (x_1, x_2, \ldots, x_n) \in \mathbb{F}_q^n$ which satisfy

$$x_i - x_j \neq 0, 1$$

in $\mathbb{F}_q$ for all $1 \le i < j \le n$. Since x satisfies these conditions if and only if $x+m := (x_1+m, \dots, x_n+m)$ does so, this number is the number of such x with $x_n = 0$, multiplied by q. These n-tuples x are in bijection with linear orderings of the integers $1, 2, \dots, n$ and $q-n$ indistinguishable objects such that n is first in the ordering and no two integers $i < j$ occur consecutively in the order j, i. Indeed, let i be in position $k+1$ if $x_i = k \bmod q$, to get such an ordering.

To construct these orderings, one can place the $q-n$ objects along a line, place n first from the left and then insert $1, \dots, n-1$ in $(q-n)^{n-1}$ ways, so that between any two consecutive objects or to the right of the rightmost one, the integers are ordered in increasing order. This shows that $\chi(\mathcal{S}_n, q) = q(q-n)^{n-1}$ for infinitely many values of q and proves the result. Q.E.D.

The next few results can be proved by variations of the argument in the previous proof. The proofs of Theorems 4.3 and 4.6 are case by case. The next result was also obtained by Postnikov and Stanley [44, p. 39] [45, §9.2] for $\Phi = A_{n-1}$ (see also [9, Proposition 5.3]) and generalizes Theorem 3.3 for the classical root systems.

4.3. Theorem ([4, §7.1–7.2]). *Let Φ be of type A, B, C, D or BC. For the extended Shi arrangement $\mathcal{A} = \mathcal{A}_\Phi^{[-a+1,a]}$ we have*

$$\chi(\mathcal{A}, q) = (q - ah)^\ell,$$

where for $\Phi = BC_n$ the Coxeter number is defined as $h = 2n+1$. In particular, the number of regions of $\mathcal{A}_\Phi^{[-a+1,a]}$ is $(ah+1)^\ell$.

An application in the spirit of [23, 35] comes from considering arrangements between the braid and Catalan arrangement $\mathcal{A}_n$ and Cat_n. For $G \subseteq \mathcal{E}_n := \{(i,j) \in [n] \times [n] : i \ne j\}$, let $\mathcal{A}_{n,G}$ be the arrangement of hyperplanes

$$\begin{aligned} x_i - x_j &= 0 \quad \text{for } 1 \le i < j \le n, \\ x_i - x_j &= 1 \quad \text{for } (j,i) \in G. \end{aligned} \tag{5}$$

Note that if G is empty, $G = \{(j,i) \in \mathcal{E}_n : i < j\}$, or $G = \mathcal{E}_n$, then $\mathcal{A}_{n,G}$ specializes to $\mathcal{A}_n$, $\mathcal{S}_n$, or Cat_n, respectively, to which the next proposition applies. For a generalization and analogous results for other root systems see [4, Theorem 6.2.10 and §6.3][3, §3].

4.4. Proposition ([3, Theorem 3.9][4, Theorem 6.2.7]). *Suppose that the set $G \subseteq \mathcal{E}_n$ has the following properties:*

(i) *If $i, j < k$, $i \ne j$ and $(i,j) \in G$, then $(i,k) \in G$ or $(k,j) \in G$.*

(ii) *If $i, j < k$, $i \neq j$ and $(i,k) \in G$, $(k,j) \in G$, then $(i,j) \in G$.*
Then

$$\chi(\mathcal{A}_{n,G}, q) = q \prod_{1<j\leq n} (q - c_j),$$

where $c_j = n + a_j - j + 1$ and a_j is the number of (j,i) or (i,j) in G with $i < j$.

The conditions in Proposition 4.4 become simpler if $\mathcal{A}_{n,G}$ contains hyperplanes of the form $x_i - x_j = 1$ only for $i < j$, i.e. if it lies between $\mathcal{A}_n$ and $\mathcal{S}_n$. We state this special case for later reference.

4.5. Corollary ([3, Theorem 3.4][4, Theorem 6.2.2]). *Suppose that the set $G \subseteq \{(j,i) \in \mathcal{E}_n : i < j\}$ has the following property: if $1 \leq i < j < k \leq n$ and $(j,i) \in G$ then $(k,i) \in G$. Then*

$$\chi(\mathcal{A}_{n,G}, q) = q \prod_{1<j\leq n} (q - c_j),$$

where $c_j = n - \#\{i < j : (j,i) \notin G\}$.

Recall from Theorem 2.6 that the characteristic polynomial of $\mathcal{A}_\Phi$ factors with roots the exponents of Φ. The following result was also obtained in [44, Proposition 1.5.8][45, Theorem 9.8] for $\Phi = A_{n-1}$; see also [3, Theorem 5.5] and [9, Proposition 5.3].

4.6. Theorem ([4, Corollary 7.2.3 and Theorem 7.2.6]). *Let Φ be of type A, B, C, D or BC. For the extended Catalan arrangement $\mathcal{A} = \mathcal{A}_\Phi^{[-a,a]}$ we have*

$$\chi(\mathcal{A}, q) = \chi(\mathcal{A}_\Phi, q - ah),$$

if Φ has type A, B, C or D and

$$\chi(\mathcal{A}, q) = \begin{cases} \chi(\mathcal{A}_\Phi, q - (2n+1)a), & \text{if } a \text{ is even,} \\ \chi(\mathcal{A}_\Phi, q - (2n+1)a - 1), & \text{if } a \text{ is odd,} \end{cases}$$

if Φ has type BC.

The arrangement (3) reduces to the extended Catalan arrangement of type A for $\lambda_1 = \lambda_2 = \cdots = \lambda_n = a + 1$. Its characteristic polynomial can be computed by an easy application of Theorem 4.2.

4.7. Proposition ([8, §4]). *If $\mathcal{A}$ is the arrangement* (3) *and m is the sum of the positive integers λ_i for $1 \le i \le n$, then*

$$\chi(\mathcal{A}, q) = q \prod_{j=m-n+1}^{m-1} (q-j).$$

In particular, the number of nonnesting partitions of $[m]$ *with block sizes* $\lambda_1, \lambda_2, \ldots, \lambda_n$ *is equal to*

$$\frac{m!}{r_\lambda\,(m-n+1)!},$$

where $r_\lambda = r_1!r_2!\cdots$ *and r_j is the number of indices i with $\lambda_i = j$.*

Root system analogues of Proposition 4.7 for $\Phi = B_n$ and C_n appear in [7].

The expression for $\chi(\mathcal{L}_n, q)$ in Theorem 3.5 was obtained in [3, 4] by a similar but less straightforward argument, based on Theorem 4.2. It generalizes easily to the extended Linial arrangements. Let S be the *shift operator*, acting on polynomials in y by

$$Sf(y) := f(y-1).$$

We state the next result in the elegant form given in [44, 45]. For a relatively short proof based on Theorem 4.2 see [9, §3].

4.8. Theorem ([3, §4][4, §6.4] [45, Theorem 9.7]). *For $\Phi = A_{n-1}$ and $a \ge 1$, the extended Linial arrangement has characteristic polynomial*

$$\chi(\mathcal{A}_\Phi^{[1,a]}, q) = \frac{1}{(a+1)^n}\,(1+S+S^2+\cdots+S^a)^n\, q^{n-1}.$$

Theorem 4.8 implies the fact that all roots of $\chi(\mathcal{A}_\Phi^{[1,a]}, q)$ have the same real part. Indeed, if the polynomial f has this property then so does $(S+\zeta)f$, if $\zeta \in \mathbb{C}$ satisfies $|\zeta| = 1$; see [44, Lemma 1.5.12][45, Lemma 9.12] for an elegant, short proof. Using this reasoning, Postnikov and Stanley settled their Conjecture 3.6 in the type A case.

4.9. Theorem ([44, Theorem 1.5.11] [45, Theorem 9.11]). Conjecture 3.6 *is true for* $\Phi = A_{n-1}$.

Explicit formulae for the characteristic polynomials of the arrangements in Conjecture 3.6 were obtained in [9, §4–5] for the other classical root systems. The proofs follow the ones for the type A case in [9, §3] but are more involved. We give the formulae for the extended Linial arrangements.

4.10. Theorem ([9, §4]). *For the extended Linial arrangement $\mathcal{A} = \mathcal{A}_\Phi^{[1,a]}$ and for a even or odd, respectively, $\chi(\mathcal{A},q)$ is equal to*

$$\begin{cases} \frac{1}{(a+1)^{n+1}} \, (1+S^2+S^4+\cdots+S^{2a})^{n-1} \, (1+S+S^2+\cdots+S^a) \; q^n, \\ \frac{4S}{(a+1)^{n+1}} \, (1+S^2+S^4+\cdots+S^{2a})^{n-1} \, (1+S^2+S^4+\cdots+S^{a-1}) \; q^n \end{cases}$$

if $\Phi = B_n$ or C_n,

$$\begin{cases} \frac{1}{(a+1)^{n+1}} \, (1+S^2+S^4+\cdots+S^{2a})^{n-2} \, (1+S+S^2+\cdots+S^a)^4 \; q^n, \\ \frac{8S}{(a+1)^{n+1}} \, (1+S^2) \, (1+S^2+\cdots+S^{2a})^{n-1} \, (1+S^2+\cdots+S^{a-1})^4 \; q^n \end{cases}$$

if $\Phi = D_n$ and

$$\begin{cases} \frac{1}{(a+1)^{n+1}} \, (1+S^2+S^4+\cdots+S^{2a})^{n} \, (1+S+S^2+\cdots+S^a) \; q^n, \\ \frac{2S}{(a+1)^{n+1}} \, (1+S^2+S^4+\cdots+S^{2a})^{n} \, (1+S^2+S^4+\cdots+S^{a-1}) \; q^n \end{cases}$$

if $\Phi = BC_n$.

The next result follows as in the type A case; see [9].

4.11. Theorem ([9, Theorem 1.2]). Conjecture 3.6 *is true for all root systems of type A, B, C, D or BC.*

§5. Freeness

Recall from Theorem 2.4 that the characteristic polynomial of a free arrangement factors completely over the nonnegative integers and from Theorem 2.5 that the Coxeter arrangement $\mathcal{A}_\Phi$ is free with exponents the exponents $e_1, e_2, \ldots, e_\ell$ of Φ. In view of the numerous instances in Sections 3 and 4 in which $\chi(\mathcal{A},q)$ factors, it is natural to ask whether various deformations of $\mathcal{A}_\Phi$ are free, when homogenized to central arrangements by the cone operation [42, Definition 1.15].

Freeness of the cones of the extended Catalan and Shi arrangements was conjectured in [24] and remains unsettled, except for the type A case [24, §3] [6, §3]. We continue to denote by h the Coxeter number of Φ.

5.1. Conjecture (Edelman–Reiner [24, Conjecture 3.3]). *The cone of the extended Catalan arrangement $\mathcal{A}_\Phi^{[-a,a]}$ is free with exponents $1, e_1 + ah, e_2 + ah, \ldots, e_\ell + ah$.*

5.2. Conjecture (Edelman–Reiner [24, Conjecture 3.3]). *The cone of the extended Shi arrangement $\mathcal{A}_\Phi^{[-a+1,a]}$ is free with exponents* 1 *with multiplicity one, and ah with multiplicity ℓ.*

Edelman and Reiner have stated these conjectures for an irreducible crystallographic root system Φ. In view of Theorems 4.3 and 4.6, it is natural to include the non-reduced system BC_n. The conjectures are not true in general in the non-crystallographic case; see the comments after Conjecture 3.3 in [24].

Except for Theorems 4.3 and 4.6, evidence in support of the conjectures is provided by the fact that they have been verified in the case of type A; see the proof of [24, Theorem 3.2] and [6, Corollary 3.4], respectively. Moreover, in the case of Conjecture 5.2, additional evidence is provided by work of Solomon and Terao [57] on the double Coxeter arrangement, which we will briefly describe.

Suppose $\mathcal{A}$ is central in $\mathbb{K}^\ell$ and that α_H is the linear form which defines $H \in \mathcal{A}$, so that $H = \ker(\alpha_H)$. Let $\mathbb{S} = \mathbb{K}[x_1, x_2, \ldots, x_\ell]$ be the polynomial ring, as in Section 2, and $\mathrm{Der}_\mathbb{S}$ be its module of derivations. In his theory of *free multiarrangements* [76], Ziegler has defined the $\mathbb{S}$-module

$$E(\mathcal{A}) = \{\theta \in \mathrm{Der}_\mathbb{S} : \theta(\alpha_H) \in \mathbb{S}\,\alpha_H^2 \text{ for } H \in \mathcal{A}\},$$

which is a submodule of $D(\mathcal{A})$. Note that the restriction of the cone of $\mathcal{S}_\Phi$ to the hyperplane at infinity $x_0 = 0$ is the *double Coxeter arrangement*, i.e. $\mathcal{A}_\Phi$ with each hyperplane having multiplicity two. Thus by Ziegler's [76, Theorem 11], the $a = 1$ case of Conjecture 5.2 implies that the double Coxeter arrangement is free, in the sense of the following theorem.

5.3. Theorem (Solomon–Terao [57]). *Let Φ be any irreducible root system. The module $E(\mathcal{A}_\Phi)$ is free with all degrees of the elements in a basis equal to the Coxeter number h.*

Moreover, Solomon and Terao [57, Theorem 1.4] construct an explicit basis of $E(\mathcal{A}_\Phi)$ in terms of the invariant theory of the Coxeter group W. This raises naturally the following question.

5.4. Question. *Is there a basis of the module of derivations of the cone of $\mathcal{S}_\Phi$ which can be described explicitly in terms of the invariant theory of the Weyl group W?*

Beginning with work of Stanley [58] on subarrangements of the braid arrangement $\mathcal{A}_n$, called *graphical arrangements*, classes of subarrangements of Coxeter arrangements have been studied [35] and characterized

[23, 13] from the point of view of freeness; see also [24], [11, §7][12]. It was shown by Stanley [58] that the supersolvable – or free – graphical arrangements correspond to *chordal* graphs and by Edelman and Reiner [23] that the free arrangements between $\mathcal{A}_n$ and the Coxeter arrangement of type B_n correspond to *threshold* graphs. For interesting classes of free or non-free subarrangements, in particular for non-free graphical arrangements whose characteristic polynomials factor completely over the integers, see Kung [38].

Various deformations of $\mathcal{A}_n$ were studied in this sense in [6]. We mention a complete characterization for the family of arrangements $\mathcal{A}_{n,G}$, defined in (5), which lie between $\mathcal{A}_n$ and $\mathcal{S}_n$. The class of arrangements in this family with free cones turns out to be, essentially, the class which appears in Corollary 4.5. The condition in Corollary 4.5 has also appeared in the characterization of freeness in a different family; see Bailey [11, Theorem 7.3][12].

5.5. Theorem ([6, Theorem 4.1]). *Let $G \subseteq E_n := \{(j,i) \in [n] \times [n] : i < j\}$. The following are equivalent:*

(i) *$\mathcal{A}_{n,G}$ is inductively free.*
(ii) *The cone of $\mathcal{A}_{n,G}$ is free.*
(iii) *There is a permutation $w = w_1 w_2 \cdots w_n$ of $[n]$ such that*

$$w^{-1} \cdot G = \{(j,i) : (w_j, w_i) \in G\}$$

is contained in E_n and satisfies the condition in Corollary 4.5.

A similar characterization for the family of arrangements between $\mathcal{S}_n$ and Cat_n is given in [6, Theorem 4.3]. Specifically, if $E_n \subseteq G \subseteq \mathcal{E}_n$ and $\overline{G} = \{(j,i) : (i,j) \in \mathcal{E}_n - G\} \subseteq E_n$, then $\mathcal{A}_{n,G}$ has free cone if and only if so does $\mathcal{A}_{n,\overline{G}}$. In contrast with the situation in [23], most of the free arrangements of Theorem 5.5 are not supersolvable; see [6, Theorem 4.2]. For characterizations of supersolvability for deformations of $\mathcal{A}_n$, see Zaslavsky [74, §3].

§6. Remarks and open questions

In this section we include a number of questions other than Conjecture 3.6 (which is still open for the exceptional root systems), Conjectures 5.1 and 5.2 and Question 5.4. Our main objective is to point out that from many perspectives, the classes of deformations of Coxeter arrangements we have discussed are still not well understood.

All known proofs of Theorem 3.3 proceed with a case by case verification. A positive answer to Question 5.4 would give a uniform proof, via Terao's factorization theorem. The fact that Theorem 3.3 is stated in an elementary, uniform way suggests the following question.

6.1. Question. *Is there an elementary, case-free proof of* Theorem 3.3*?*

Similar questions can be asked about Theorems 4.3 and 4.6 and the curious property of $\chi(\mathcal{A}, q)$ in Conjecture 3.6, which Postnikov and Stanley refer to as the "Riemann hypothesis" for $\mathcal{A}$ [44, 45]. In particular, it is natural to ask whether Theorems 4.3, 4.6 and 4.11 extend to the exceptional root systems.

6.2. Question. *Are there case-free proofs of* Theorems 4.3 *and* 4.6*? In particular, are these theorems valid for the exceptional crystallographic root systems?*

6.3. Question. *Is there a case-free, conceptual proof of* Conjecture 3.6*?*

It would also be desirable to find simpler derivations of the formulae in Theorem 4.10 than those of [9], which may not give the best insight possible, especially in the case of the root system D_n. In particular, there is no conceptual explanation to the fact that these formulae coincide for the root systems B_n and C_n.

The Riemann hypothesis for $\mathcal{A}$ does not apply exclusively to the arrangements of Conjecture 3.6, as the following example shows.

6.4. Example ([9, Proposition 6.1]). *The arrangement with hyperplanes*

$$\begin{aligned} 2x_i &= 0, 1, 2, \ldots, 2a \quad \textit{for} \ \ 1 \le i \le n, \\ x_i - x_j &= 0, 1, \ldots, a \quad \textit{for} \ \ 1 \le i < j \le n, \\ x_i + x_j &= 0, 1, \ldots, a \quad \textit{for} \ \ 1 \le i < j \le n \end{aligned}$$

has characteristic polynomial

$$\frac{1}{a^{n+1}} \, S^{2n+1} \, (1 + S^2 + S^4 + \cdots + S^{2a-2})^{n+1} \, q^n$$

and hence satisfies Conjecture 3.6.

By analogy with the numerous theories built to explain the phenomenon of complete factorization of $\chi(\mathcal{A}, q)$ over the integers (see Section 2), we ask the following.

6.5. Question. *Is there a natural algebraic condition on $\mathcal{A}$ which implies the Riemann hypothesis of* Conjecture 3.6 *for $\chi(\mathcal{A}, q)$?*

In various characterizations of freeness, such as those in [23, 24, 6, 11], the families of arrangements under consideration are indexed by undirected graphs on n vertices. Classes of arrangements which correspond to pairs of graphs seem to be more challenging to analyze from the point of view of freeness. It is not known, for instance, which the free subarrangements of the Coxeter arrangement of type B_n are; see [23]. Proposition 4.4 suggests an explicit characterization of the arrangements between $\mathcal{A}_n$ and Cat_n with free cones.

6.6. Conjecture. *For $G \subseteq \mathcal{E}_n$, the cone of $\mathcal{A}_{n,G}$ is free if and only if G satisfies the two conditions in* Proposition 4.4.

Motivated by the fact that Coxeter arrangements are $K(\pi, 1)$ [42, Chapter 6] we ask the following about the topology of the complexifications of $\mathcal{S}_\Phi$ and Cat_Φ.

6.7. Question. *Is the Shi arrangement $\mathcal{S}_\Phi$ a $K(\pi, 1)$ arrangement? Is the Catalan arrangement* Cat_Φ *a $K(\pi, 1)$ arrangement?*

Finally, we collect some questions and facts about the combinatorics of the face structure of the arrangements in Section 3.

Direct bijective proofs of Theorem 3.2 for the type A case can be found in [61, §2] [10, §2]; see also [31] and Remark 1 in [10, §4] for a proof by deletion-restriction. The bijections in [61, 10] generalize to the extended Shi arrangements. The one in [10] generalizes also to the family of arrangements between $\mathcal{A}_n$ and $\mathcal{S}_n$ [10, Theorem 1.2].

6.8. Question. *Are there simple bijective proofs of* Theorem 3.2 *for cases other than that of type A?*

For the braid arrangement $\mathcal{A}_n$, it is well known that faces of a fixed dimension k correspond to ordered partitions of the set $[n]$ with k blocks. In the case of type A, Shi's formula for the number of regions of $\mathcal{S}_n$ was generalized to k-dimensional faces in [3, Theorem 6.5][4, Corollary 8.2.2] as follows.

6.9. Theorem ([3, 4]). *For $0 \leq k \leq n$, the number of faces of $\mathcal{S}_n$ of dimension k is given by*

$$f_k(\mathcal{S}_n) = \binom{n}{k} \sum_{i=0}^{n-k} (-1)^i \binom{n-k}{i} (n-i+1)^{n-1}.$$

Equivalently,

$$(6) \qquad f_k(\mathcal{S}_n) = \binom{n}{k} \# \{f : [n-1] \to [n+1] \mid [n-k] \subseteq \mathrm{Im} f\},$$

where Imf *is the image of the map* f.

6.10. Question. *Is there a simple bijective proof of* Theorem 6.9*? Can the poset of faces of* $\mathcal{S}_n$*, partially ordered by inclusion of their closures, be described in terms of the maps in* (6)*?*

It is plausible that such a bijection will specialize to the one between regions of $\mathcal{S}_n$ and parking functions given in [10, §2] for $k = n$. Theorem 6.9 generalizes to the extended Shi arrangements [4, Theorem 8.2.1].

The "coincidence" of the formulae for the number f_n of regions of the Linial arrangement $\mathcal{L}_n$ and alternating trees on $n+1$ vertices suggests the following question.

6.11. Question. *Is there a bijective proof of* Theorem 3.4*?*

We refer to [59, §4] for a number of combinatorial interpretations and expressions for f_n. In particular, Postnikov [43, §4][44, Theorem 1.4.3] has given a bijection between alternating trees on $n+1$ vertices and *local binary search trees* on n vertices. Here we remark that f_n is also equal to the number of n-tuples $x = (x_1, x_2, \ldots, x_n) \in \mathbb{Z}_{n+1}^n$ which satisfy $x_i - x_j \neq 1$ in $\mathbb{Z}_{n+1}$ for $1 \leq i < j \leq n$ and $x_1 = 0$ or, in other words, to the number of ways to distribute $1, 2, \ldots, n$ in $n+1$ boxes arranged cyclically, with repetitions allowed, such that no j is immediately followed clockwise by an $i < j$. This follows from the proof of Theorem 3.5 in [3, 4, 9] by letting $q = n+1$.

The regions of Cat_n, $\mathcal{S}_n$ and $\mathcal{L}_n$ are in bijection with certain classes of posets that can be characterized in terms of forbidden induced subposets, see [44, §1.3.1][45, §7], [5] and [44, §1.4.6][45, §8.2], respectively. It would be interesting to find other instances of this phenomenon.

The enumeration of regions by the "distance statistic" has been of interest in the context of deformations of Coxeter arrangements. The *distance* $\rho_{R_0}(R)$ of a region R of $\mathcal{A}$ from a fixed base region R_0 is the number of hyperplanes of $\mathcal{A}$ which separate R from R_0. The following result for Coxeter arrangements is classical.

6.12. Theorem (Solomon [55]). *For any irreducible root system Φ we have*

$$\sum_R q^{\rho_{R_0}(R)} = \prod_{i=1}^{\ell} (1 + q + q^2 + \cdots + q^{e_i}),$$

where R runs through all regions of $\mathcal{A}_\Phi$, R_0 is any fixed region and $e_1, e_2, \ldots, e_\ell$ are the exponents of Φ.

For the Shi arrangement $\mathcal{S}_n$ the distance enumerator, for a suitably chosen base region R_0, turns out to be the *inversion enumerator* for trees [40]. Indeed, let R_0 be the region defined by the inequalities $x_1 > x_2 > \cdots > x_n$ and $x_1 - x_n < 1$. An *inversion* of a tree T on the vertex set $\{0, 1, \ldots, n\}$ is a pair (i, j) with $1 \leq i < j \leq n$ such that vertex j lies on the path in T from 0 to i. The bijection described in [59, §5] and one due to Kreweras [37] yield the following result. A proof and generalization to the extended Shi arrangements is given in [61].

6.13. Theorem (Pak–Stanley [59, Theorem 5.1][61]). *For each $m = 0, 1, \ldots, \binom{n}{2}$, the number of regions R of $\mathcal{S}_n$ with distance m from R_0 is equal to the number of trees on $\{0, 1, \ldots, n\}$ with $\binom{n}{2} - m$ inversions.*

It would be interesting to find a simpler and more direct proof of this theorem. See the notes in [61, §3] for related open questions.

6.14. Question (Stanley [62]). *Are there analogues of* Theorem 6.13 *for root systems other than those of type A? Is there an analogue for the Linial arrangement $\mathcal{L}_n$?*

It was observed by Stanley [62] that the distance enumerator for the Catalan arrangement Cat_n is

$$\sum_R q^{\rho_{R_0}(R)} = C_n(q) \prod_{i=1}^{n-1} (1 + q + q^2 + \cdots + q^i),$$

where R_0 is as in the case of $\mathcal{S}_n$ and

$$C_n(q) = \sum_\lambda q^{|\lambda|},$$

with $\lambda = (\lambda_1, \lambda_2, \ldots)$ running over all partitions with $\lambda_i \leq n - i$.

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On the Cohomology of Discriminantal Arrangements and Orlik-Solomon Algebras

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For Peter Orlik on the occasion of his sixtieth birthday.

Abstract.

We relate the cohomology of the Orlik-Solomon algebra of a discriminantal arrangement to the local system cohomology of the complement. The Orlik-Solomon algebra of such an arrangement (viewed as a complex) is shown to be a linear approximation of a complex arising from the fundamental group of the complement, the cohomology of which is isomorphic to that of the complement with coefficients in an arbitrary complex rank one local system. We also establish the relationship between the cohomology support loci of the complement of a discriminantal arrangement and the resonant varieties of its Orlik-Solomon algebra.

Introduction

Let $\mathcal{A}$ be an arrangement of N complex hyperplanes, and let $M(\mathcal{A})$ be its complement. For each hyperplane H of $\mathcal{A}$, let f_H be a linear polynomial with kernel H, and let λ_H be a complex number. Each point $\boldsymbol{\lambda} = (\dots, \lambda_H, \dots) \in \mathbb{C}^N$ determines an integrable connection $\nabla = d + \Omega_{\boldsymbol{\lambda}}$ on the trivial line bundle over $M(\mathcal{A})$, where $\Omega_{\boldsymbol{\lambda}} = \sum_{H \in \mathcal{A}} \lambda_H d \log f_H$, and an associated complex rank one local system $\mathcal{L}$ on $M(\mathcal{A})$. Alternatively, if $\mathbf{t} \in (\mathbb{C}^*)^N$ is the point in the complex torus corresponding to $\boldsymbol{\lambda}$, then the local system $\mathcal{L}$ is induced by the representation of the fundamental group of $M(\mathcal{A})$ which sends any meridian about $H \in \mathcal{A}$ to $t_H = \exp(-2\pi \mathrm{i} \lambda_H)$.

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Due largely to its various applications, the cohomology of $M(\mathcal{A})$ with coefficients in $\mathcal{L}$ has been the subject of considerable recent interest. These applications include representations of braid groups, generalized hypergeometric functions, and the Knizhnik-Zamolodchikov equations from conformal field theory. See, for instance, the works of Aomoto, Kita, Kohno, Schechtman, and Varchenko [1, 2, 21, 28, 30], and see Orlik and Terao [25] as a general reference for arrangements. Of particular interest in these applications are the discriminantal arrangements of [28], the complements of which may be realized as configuration spaces of ordered points in $\mathbb{C}$ punctured finitely many times. (Note that our use of the term "discriminantal" differs from that of [25].)

The local system cohomology $H^*(M(\mathcal{A});\mathcal{L})$ may be studied from a number of points of view. For instance, if $\mathcal{A}$ is real, that is, defined by real equations, the complement $M(\mathcal{A})$ is homotopy equivalent to the Salvetti complex X of $\mathcal{A}$, see [26]. In this instance, the complex X may be used in the study of local systems on $M(\mathcal{A})$. This approach is developed by Varchenko in [30], to which we also refer for discussion of the applications mentioned above, and has been pursued by Denham and Hanlon [13] in their study of the homology of the Milnor fiber of an arrangement.

If $\mathcal{A}$ is $K(\pi,1)$, that is, the complement $M(\mathcal{A})$ is a $K(\pi,1)$-space, then local systems on $M(\mathcal{A})$ may be studied from the point of view of cohomology of groups. Any representation of the fundamental group G of the complement of a $K(\pi,1)$ arrangement gives rise to a G-module L, and a local system of coefficients $\mathcal{L}$ on $M(\mathcal{A})$. Since $M(\mathcal{A})$ is a $K(\pi,1)$-space, we have $H_*(M(\mathcal{A});\mathcal{L}) = H_*(G;L)$ and $H^*(M(\mathcal{A});\mathcal{L}) = H^*(G;L)$, see for instance Brown [8]. The class of $K(\pi,1)$ arrangements includes the discriminantal arrangements noted above, as they are examples of fiber-type arrangements, well-known to be $K(\pi,1)$, see e.g. Falk and Randell [17].

For any arrangement $\mathcal{A}$, let $B(\mathcal{A})$ denote the Brieskorn algebra of $\mathcal{A}$, generated by 1 and the closed differential forms $d\log f_H$, $H \in \mathcal{A}$. As is well-known, the algebra $B(\mathcal{A})$ is isomorphic to $H^*(M(\mathcal{A});\mathbb{C})$, and to the Orlik-Solomon algebra $A(\mathcal{A})$, so is determined by the lattice of $\mathcal{A}$, see [7, 24, 25]. If $\mathcal{L}$ is a local system on $M(\mathcal{A})$ determined by "weights" $\boldsymbol{\lambda}$ which satisfy certain Aomoto non-resonance conditions, work of Esnault, Schechtman, and Viehweg [14], extended by Schechtman, Terao, and Varchenko [27], shows that $H^*(M(\mathcal{A});\mathcal{L})$ is isomorphic to the cohomology of the complex $(B(\mathcal{A}),\Omega_{\boldsymbol{\lambda}}\wedge)$. Thus for non-resonant weights, the local system cohomology may be computed by combinatorial means, using the Orlik-Solomon algebra equipped with differential $\mu(\boldsymbol{\lambda})$, given by left-multiplication by $\omega_{\boldsymbol{\lambda}}$, the image of $\Omega_{\boldsymbol{\lambda}}$ under the isomorphism $B(\mathcal{A}) \to A(\mathcal{A})$.

For arbitrary (resonant) weights, one has

$$\dim H^k(A(\mathcal{A}), \mu(\boldsymbol{\lambda})) \leq \dim H^k(M(\mathcal{A}); \mathcal{L}) \leq \dim H^k(M(\mathcal{A}); \mathbb{C})$$

for each k. See Libgober and Yuzvinsky [23] for the first of these inequalities. The second is obtained using stratified Morse theory in [9], and resolves a question raised by Aomoto and Kita in [2]. For resonant weights, the precise relation between $H^*(A(\mathcal{A}), \mu(\boldsymbol{\lambda}))$ and $H^*(M(\mathcal{A}); \mathcal{L})$ is not known.

However, recent results suggest that $H^k(A(\mathcal{A}), \mu(\boldsymbol{\lambda}))$ may be viewed as a "linear approximation" of $H^k(M(\mathcal{A}); \mathcal{L})$, at least for small k. The resonant varieties, $\mathcal{R}_k^m(A(\mathcal{A})) = \{\lambda \in \mathbb{C}^N \mid \dim H^k(A(\mathcal{A}), \mu(\boldsymbol{\lambda})) \geq m\}$, of the Orlik-Solomon algebra were introduced by Falk in [16]. For $k = 1$ and any arrangement $\mathcal{A}$, it is known that $\mathcal{R}_1^m(A(\mathcal{A}))$ coincides with the tangent cone of the cohomology support locus of the complement, $\Sigma_m^1(M(\mathcal{A})) = \{\mathbf{t} \in (\mathbb{C}^*)^N \mid \dim H^1(M(\mathcal{A}); \mathcal{L}) \geq m\}$, at the point $(1, \ldots, 1)$, see [11, 22, 23]. For certain arrangements, we present further "evidence" in support of this philosophy here.

If $\mathcal{A}$ is a fiber-type arrangement, the fundamental group G of the complement $M(\mathcal{A})$ may be realized as an iterated semidirect product of free groups, and $M(\mathcal{A})$ is a $K(G, 1)$-space, see [17, 25]. For any such group, we construct a finite, free $\mathbb{Z}G$-resolution, $C_\bullet(G)$, of $\mathbb{Z}$ in [10]. This resolution may be used to compute the homology and cohomology of G with coefficients in any G-module L, or equivalently, that of $M(\mathcal{A})$ with coefficients in any local system $\mathcal{L}$. We have $H_*(M(\mathcal{A}); \mathcal{L}) = H_*(C_\bullet(G) \otimes_G L)$ and $H^*(M(\mathcal{A}); \mathcal{L}) = H^*(\mathcal{H}om_G(C_\bullet(G), L))$, see [8].

Briefly, for a fiber-type arrangement $\mathcal{A}$, the relationship between the cohomology theories $H^*(A(\mathcal{A}), \mu(\boldsymbol{\lambda}))$ and $H^*(M(\mathcal{A}); \mathcal{L})$ is given by the following assertion. *For any $\boldsymbol{\lambda}$, the complex $(A(\mathcal{A}), \mu(\boldsymbol{\lambda}))$ is a linear approximation of the complex $\mathcal{H}om_G(C_\bullet(G), L)$.* We prove a variant of this statement in the case where $\mathcal{A}$ is a discriminantal arrangement here. We also establish the relationship between the resonant varieties $\mathcal{R}_k^m(A(\mathcal{A}))$ and cohomology support loci $\Sigma_m^k(M(\mathcal{A}))$ of these arrangements, analogous to that mentioned above in the case $k = 1$.

The paper is organized as follows. The Orlik-Solomon algebra of a discriminantal arrangement admits a simple description, which fascilitates analysis of the differential of the complex $(A(\mathcal{A}), \mu(\boldsymbol{\lambda}))$. We carry out this analysis, which is elementary albeit delicate, in section 1, and obtain an explicit (inductive) description of the differential $\mu(\boldsymbol{\lambda})$. In section 2, we recall the construction of the resolution $C_\bullet(G)$ from [10] in the instance where G is the fundamental group of the complement of a discriminantal arrangement, and exhibit a complex $(\mathbf{C}^\bullet, \boldsymbol{\delta}^\bullet(\mathbf{t}))$ which computes the cohomology $H^*(M(\mathcal{A}), \mathcal{L})$ for an arbitrary rank one local system. We then study in section 3 a linear approximation of $(\mathbf{C}^\bullet, \boldsymbol{\delta}^\bullet(\mathbf{t}))$, and relate it, for arbitrary $\boldsymbol{\lambda}$, to the complex $(A(\mathcal{A}), \mu(\boldsymbol{\lambda}))$. We conclude by realizing the resonant varieties of the Orlik-Solomon algebra of a discriminantal arrangement as the tangent cones at the identity of the cohomology support loci of the complement in section 4.

§1. Cohomology of the Orlik-Solomon Algebra

Let $M_n = \{(x_1, \dots, x_n) \in \mathbb{C}^n \mid x_i \neq x_j \text{ if } i \neq j\}$ be the configuration space of n ordered points in $\mathbb{C}$. Note that M_n may be realized as the complement of the braid arrangement $\mathcal{A}_n = \{x_i = x_j, 1 \leq i < j \leq n\}$ in $\mathbb{C}^n$. Classical work of Fadell and Neuwirth [15] shows the projection $\mathbb{C}^n \to \mathbb{C}^\ell$ defined by forgetting the last $n - \ell$ coordinates gives rise to a bundle map $p : M_n \to M_\ell$. From this it follows that M_n is a $K(P_n, 1)$-space, where $P_n = \pi_1(M_n)$ is the pure braid group on n strands.

The typical fiber of the bundle of configuration spaces $p : M_n \to M_\ell$ may be realized as the complement of an arrangement in $\mathbb{C}^{n-\ell}$, a discriminantal arrangement in the sense of Schechtman and Varchenko, see [28, 30]. The fiber over $\mathbf{z} = (z_1, \dots, z_\ell) \in M_\ell$ may be realized as the complement, $M_{n,\ell} = M(\mathcal{A}_{n,\ell})$, of the arrangement $\mathcal{A}_{n,\ell}$ consisting of the $N = \binom{n}{2} - \binom{\ell}{2}$ hyperplanes

$$H_{i,j} = \begin{cases} \ker(x_j - x_i) & \ell + 1 \leq i < j \leq n, \\ \ker(x_j - z_i) & 1 \leq i \leq \ell,\ \ell + 1 \leq j \leq n, \end{cases}$$

in $\mathbb{C}^{n-\ell}$ (with coordinates $x_{\ell+1}, \dots, x_n$). Note that $M_{n,\ell}$ is the configuration space of $n - \ell$ ordered points in $\mathbb{C} \setminus \{z_1, \dots, z_\ell\}$, and that the topology of $M_{n,\ell}$ is independent of $\mathbf{z}$, see [15, 5, 20]. We first record some known results on the cohomology of $M_{n,\ell}$.

1.1. The Orlik-Solomon Algebra

The fundamental group of the configuration space $M_{n,\ell}$ may be realized as $P_{n,\ell} = \pi_1(M_{n,\ell}) = \ker(P_n \to P_\ell)$, the kernel of the homorphism from P_n to P_ℓ defined by forgetting the last $n - \ell$ strands. From the homotopy exact sequence of the bundle $p : M_n \to M_\ell$, we see that $M_{n,\ell}$ is a $K(P_{n,\ell}, 1)$-space. The cohomology of this space, and hence of this group, may be described as follows.

Let $\mathcal{E} = \bigoplus_{q=0}^{N} \mathcal{E}^q$ be the graded exterior algebra over $\mathbb{C}$, generated by $e_{i,j}$, $\ell + 1 \leq j \leq n$, $1 \leq i < j$. Let $\mathcal{I}$ be the ideal in $\mathcal{E}$ generated, for

$1 \le i < j < k \le n$, by

$$e_{i,j} \wedge e_{i,k} - e_{i,j} \wedge e_{j,k} + e_{i,k} \wedge e_{j,k} \text{ if } j \ge \ell + 1, \quad \text{and} \quad e_{i,k} \wedge e_{j,k} \text{ if } j \le \ell.$$

Note that $\mathcal{E}^q \subset \mathcal{I}$ for $q > n - \ell$. The Orlik-Solomon algebra of the discriminantal arrangement $\mathcal{A}_{n,\ell}$ is the quotient $A = \mathcal{E}/\mathcal{I}$.

Theorem 1.2. *The cohomology algebra $H^*(M_{n,\ell};\mathbb{C}) = H^*(P_{n,\ell};\mathbb{C})$ is isomorphic to the Orlik-Solomon algebra $A = A(\mathcal{A}_{n,\ell})$.*

The grading on $\mathcal{E}$ induces a grading $A = \bigoplus_{q=0}^{n-\ell} A^q$ on the Orlik-Solomon algebra $A = A(\mathcal{A}_{n,\ell})$. Let $a_{i,j}$ denote the image of $e_{i,j}$ in A, and note that these elements form a basis for A^1 and generate A. From the description of the ideal $\mathcal{I}$ above, it is clear that all relations among these generators are consequences of the following:

$$a_{i,k} \wedge a_{j,k} = \begin{cases} a_{i,j} \wedge (a_{j,k} - a_{i,k}) & \text{if } j \ge \ell + 1, \\ 0 & \text{if } j \le \ell, \end{cases} \tag{1.1}$$

for $1 \le i < j < k \le n$.

This observation leads to a natural choice of basis for the algebra A. For $m \le n$, write $[m,n] = \{m, m+1, \dots, n\}$. If $I = \{i_1, \dots, i_q\}$ and $J = \{j_1, \dots, j_q\}$ satisfy the conditions $J \subseteq [\ell + 1, n]$ and $1 \le i_p < j_p$ for each p, let $a_{I,J} = a_{i_1,j_1} \wedge \cdots \wedge a_{i_q,j_q}$. If $|J| = 0$, set $a_{I,J} = 1$.

Proposition 1.3. *For each q, $0 \le q \le n - \ell$, the forms $a_{I,J}$ with $|J| = q$ and I as above form a basis for the summand A^q of the Orlik-Solomon algebra A of the discriminantal arrangement $\mathcal{A}_{n,\ell}$. Furthermore, the summand A^q decomposes as a direct sum, $A^q = \bigoplus_{|J|=q} A_J$, where $A_J = \bigoplus_I \mathbb{C}\, a_{I,J}$.*

Remark 1.4. These results are well-known. For instance, if $\mathcal{A} = \mathcal{A}_n$ is the braid arrangement, Theorem 1.2 follows from results of Arnol'd [4] and Cohen [12], which show that $H^*(M_n;\mathbb{C})$ is generated by the forms $a_{i,j} = d\log(x_j - x_i)$, with relations (1.1) (with $\ell = 1$). For any discriminantal arrangement $\mathcal{A}_{n,\ell}$, Theorem 1.2 is a consequence of results of Brieskorn and Orlik-Solomon, see [7, 24, 25].

As mentioned in the Introduction, the discriminantal arrangements $\mathcal{A}_{n,\ell}$ are examples of (affine) fiber-type or supersolvable arrangements. The structure of the Orlik-Solomon algebra of any such arrangement $\mathcal{A}$ was determined by Terao [29]. The basis for the algebra $A(\mathcal{A}_{n,\ell})$ exhibited in Proposition 1.3 above is the **nbc**-basis (with respect to a natural ordering of the hyperplanes of $\mathcal{A}_{n,\ell}$), see [25]. The Orlik-Solomon algebra of any supersolvable arrangement admits an analogous basis, see Björner-Ziegler [6], and see Falk-Terao [18] for affine supersolvable arrangements.

1.5. The Orlik-Solomon Algebra as a Complex

Recall that $N = \binom{n}{2} - \binom{\ell}{2}$, and consider $\mathbb{C}^N$ with coordinates $\lambda_{i,j}$, $\ell+1 \leq j \leq n$, $1 \leq i < j$. Each point $\boldsymbol{\lambda} \in \mathbb{C}^N$ gives rise to an element $\omega = \omega_{\boldsymbol{\lambda}} = \sum \lambda_{i,j} \cdot a_{i,j}$ of A^1. Left-multiplication by ω induces a map $\mu^q(\boldsymbol{\lambda}) : A^q \to A^{q+1}$, defined by $\mu^q(\boldsymbol{\lambda})(\eta) = \omega \wedge \eta$. Clearly, $\mu^{q+1}(\boldsymbol{\lambda}) \circ \mu^q(\boldsymbol{\lambda}) = 0$, so $(A^\bullet, \mu^\bullet(\boldsymbol{\lambda}))$ is a complex.

We shall obtain an inductive formula for the boundary maps of the complex $(A^\bullet, \mu^\bullet(\boldsymbol{\lambda}))$. The projection $\mathbb{C}^{n-\ell} \to \mathbb{C}$ onto the first coordinate gives rise to a bundle of configuration spaces, $M_{n,\ell} \to M_{\ell+1,\ell}$, with fiber $M_{n,\ell+1}$, see [15, 5, 20]. The inclusion of the fiber $M_{n,\ell+1} \hookrightarrow M_{n,\ell}$ induces a map on cohomology which is clearly surjective. This yields a surjection $\pi : A(\mathcal{A}_{n,\ell}) \to A(\mathcal{A}_{n,\ell+1})$.

Write $A = A(\mathcal{A}_{n,\ell})$ and $\widehat{A} = A(\mathcal{A}_{n,\ell+1})$, and denote the generators of both A and $\widehat{A}$ by $a_{i,j}$. In terms of these generators, the map π is given by $\pi(a_{i,\ell+1}) = 0$, and $\pi(a_{i,j}) = a_{i,j}$ otherwise. Let $\widehat{\omega} \in \widehat{A}$ denote the image of $\omega \in A^1$ under π. If we write $\omega = \sum_{k=\ell+1}^{n} \omega_k$, where $\omega_k = \sum_{i=1}^{k-1} \lambda_{i,k} \cdot a_{i,k}$, then $\widehat{\omega} = \sum_{k=\ell+2}^{n} \omega_k$. As above, left-multiplication by $\widehat{\omega}$ induces a map $\widehat{\mu}^q(\boldsymbol{\lambda}) : \widehat{A}^q \to \widehat{A}^{q+1}$, and $(\widehat{A}^\bullet, \widehat{\mu}^\bullet(\boldsymbol{\lambda}))$ is a complex. The following is straightforward.

Lemma 1.6. *The map $\pi : (A^\bullet, \mu^\bullet(\boldsymbol{\lambda})) \to (\widehat{A}^\bullet, \widehat{\mu}^\bullet(\boldsymbol{\lambda}))$ is a surjective chain map.*

Let $(B^\bullet, \mu_B^\bullet(\boldsymbol{\lambda}))$ denote the kernel of the chain map π. The terms are of the form $B^q = \bigoplus A_K^q$, where $\ell+1 \in K$ and $|K| = q$. In particular, $B^0 = 0$. We now identify the differential $\mu_B^\bullet(\boldsymbol{\lambda})$. If $k < m \leq n$ and $J \subseteq [m,n]$, let $\{k,J\}$ denote the (ordered) subset $\{k\} \cup J$ of $[k,n]$. For a linear map F, write $[F]^k$ for the map $\oplus_1^k F$.

Proposition 1.7. *The complex $(B^\bullet, \mu_B^\bullet(\boldsymbol{\lambda}))$ decomposes as the direct sum of ℓ copies of the complex $\widehat{A}^\bullet$, shifted in dimension by one, with the sign of the boundary map reversed. In other words, $(B^\bullet, \mu_B^\bullet(\boldsymbol{\lambda})) \cong ((\widehat{A}^{\bullet-1})^\ell, -[\widehat{\mu}^{\bullet-1}(\boldsymbol{\lambda})]^\ell)$.*

Proof. For $1 \leq q \leq n-\ell$, we have $B^q = \bigoplus A^q_{\{\ell+1,J\}}$, where the sum is over all $J \subseteq [\ell+2, n]$ with $|J| = q-1$. Each summand may be written as $A^q_{\{\ell+1,J\}} = \bigoplus_{i=1}^{\ell} a_{i,\ell+1} \wedge A_J^{q-1}$. Thus, $B^q = \bigoplus_{i=1}^{\ell} a_{i,\ell+1} \wedge \widehat{A}^{q-1}$ is isomorphic to the direct sum of ℓ copies of $\widehat{A}^{q-1}$ via the map $B^q \to [\widehat{A}^{q-1}]^\ell$, $a_{i,\ell+1} \wedge a_{I,J} \mapsto (0, \ldots, a_{I,J}, \ldots, 0)$.

Now consider the boundary map $\mu_B^q(\boldsymbol{\lambda}) : B^q \to B^{q+1}$ of the complex $B^\bullet$, induced by left-multiplication by $\omega = \sum_{k=\ell+1}^n \omega_k$. Let $\eta = a_{i,\ell+1} \wedge a_{I,J}$ be a generator for B^q. Since $a_{i,k} \wedge a_{j,k} = 0$ for all $i, j < k$, we have $\omega_{\ell+1} \wedge \eta = 0$. Thus,

$$\mu_B^q(\boldsymbol{\lambda})(\eta) = \omega \wedge \eta = (\omega - \omega_{\ell+1}) \wedge \eta = -a_{i,\ell+1} \wedge (\omega - \omega_{\ell+1}) \wedge a_{I,J}.$$

Write $(\omega - \omega_{\ell+1}) \wedge a_{I,J} = \xi_1 + \xi_2$ in terms of the basis for A specified in Proposition 1.3, where $\xi_1 \in \bigoplus_{\ell+1 \in K} A_K^q$ and $\xi_2 \in \bigoplus_{\ell+1 \notin K} A_K^q$. Then we have $\omega \wedge \eta = -a_{i,\ell+1} \wedge (\xi_1 + \xi_2) = -a_{i,\ell+1} \wedge \xi_2$. Checking that $\widehat{\omega} \wedge a_{I,J} = \xi_2$ in $\widehat{A}$, we have $\mu_B^q(\boldsymbol{\lambda})(a_{i,\ell+1} \wedge a_{I,J}) = -a_{i,\ell+1} \wedge \widehat{\mu}^{q-1}(\boldsymbol{\lambda})(a_{I,J})$. Thus, with the change in sign, the boundary map $\mu_B^\bullet(\boldsymbol{\lambda})$ respects the direct sum decomposition $B^\bullet \cong (\widehat{A}^{\bullet-1})^\ell$. Q.E.D.

1.8. Boundary Maps

We now study the differential of the complex $(A^\bullet, \mu^\bullet(\boldsymbol{\lambda}))$. The direct sum decompositions of the terms of the complexes $A^\bullet$, $\widehat{A}^\bullet$, and $B^\bullet$ exhibited above yield

$$A^q = \bigoplus_{|J|=q} A_J^q = \left(\bigoplus_{\ell+1 \in J} A_J^q\right) \oplus \left(\bigoplus_{\ell+1 \notin J} A_J^q\right) = B^q \oplus \widehat{A}^q.$$

Let $\pi_B : A^q \to B^q$ denote the natural projection. With respect to the direct sum decomposition of the terms $A^q = B^q \oplus \widehat{A}^q$, the boundary map $\mu^\bullet(\boldsymbol{\lambda})$ of the complex $A^\bullet$ is given by $\mu^q(\boldsymbol{\lambda})(v_1, v_2) = (\mu_B^q(\boldsymbol{\lambda})(v_1) + \Psi^q(\boldsymbol{\lambda})(v_2), \widehat{\mu}^q(\boldsymbol{\lambda})(v_2))$, where $\Psi^q(\boldsymbol{\lambda}) = \pi_B \circ \mu^q(\boldsymbol{\lambda}) : \widehat{A}^q \to B^{q+1}$. In matrix form, we have

$$\mu^q(\boldsymbol{\lambda}) = \begin{pmatrix} \mu_B^q(\boldsymbol{\lambda}) & 0 \\ \Psi^q(\boldsymbol{\lambda}) & \widehat{\mu}^q(\boldsymbol{\lambda}) \end{pmatrix}. \tag{1.2}$$

Since $\widehat{A}^\bullet$ is the complex associated to the discriminantal arrangement $\mathcal{A}_{n,\ell+1}$ in $\mathbb{C}^{n-\ell-1}$ and $B^\bullet \cong (\widehat{A}^{\bullet-1})^\ell$ decomposes as a direct sum by Proposition 1.7, we inductively concentrate our attention on the maps $\Psi^q(\boldsymbol{\lambda})$. Fix $J \subseteq [\ell+2, n]$, and denote the restriction of $\Psi^q(\boldsymbol{\lambda})$ to the summand A_J^q of $\widehat{A}^q$ by $\Psi_J^q(\boldsymbol{\lambda})$. For $\eta \in A_J^q$, since $\pi_B(\omega_k \wedge \eta) = 0$ if $k \notin \{\ell+1, J\}$, we have $\Psi_J^q(\boldsymbol{\lambda})(\eta) = \omega_{\ell+1} \wedge \eta + \sum_{j \in J} \pi_B(\omega_j \wedge \eta)$. Thus, $\Psi_J^q(\boldsymbol{\lambda}) : A_J^q \to A_{\{\ell+1,J\}}^{q+1} = \bigoplus_{m=1}^\ell a_{m,\ell+1} \wedge A_J^q$.

For $1 \leq m \leq \ell$, let $\pi_{m,\ell+1} : A_{\{\ell+1,J\}}^{q+1} \to a_{m,\ell+1} \wedge A_J^q$ denote the natural projection. Then (the matrix of) $\Psi_J^q(\boldsymbol{\lambda}) : A_J^q \to \left(A_J^q\right)^\ell$ may be expressed as

(1.3)

$$\Psi_J^q(\boldsymbol{\lambda}) = \begin{pmatrix}\pi_{1,\ell+1} \circ \Psi_J^q(\boldsymbol{\lambda}) & \cdots & \pi_{m,\ell+1} \circ \Psi_J^q(\boldsymbol{\lambda}) & \cdots & \pi_{\ell,\ell+1} \circ \Psi_J^q(\boldsymbol{\lambda})\end{pmatrix},$$

and we focus our attention on one such block, that is, on the composition

$$\pi_{m,\ell+1} \circ \Psi^q_J(\boldsymbol{\lambda}) : A^q_J \longrightarrow A^{q+1}_{\{1,J\}} \longrightarrow a_{m,\ell+1} \wedge A^q_J. \tag{1.4}$$

Write $J = \{j_1, \dots, j_q\}$ and for $1 \le p \le q$, let $J_p = \{j_1, \dots, j_p\}$ and $J^p = J \setminus J_p$. If $p = 0$, set $J_0 = \emptyset$ and $J^0 = J$. Then for $a_{I,J} \in A^q_J$, it is readily checked that $\pi_{m,\ell+1} \circ \Psi^q_J(\boldsymbol{\lambda})(a_{I,J}) = \pi_{m,\ell+1} \circ \pi_B(\omega \wedge a_{I,J})$ is given by

(1.5)
$$\pi_{m,\ell+1} \circ \Psi^q_J(\boldsymbol{\lambda})(a_{I,J}) = \sum\nolimits_{p=0}^{q} \pi_{m,\ell+1} \circ \pi_B(\omega_{j_p} \wedge a_{I_p,J_p}) \wedge a_{I^p,J^p},$$

where $j_0 = \ell + 1$. In light of this, we restrict our attention to $\pi_{m,\ell+1} \circ \pi_B(\omega_{j_q} \wedge a_{I,J})$. We describe this term using the following notion.

Definition 1.9. Fix $J = \{j_1, \dots, j_q\} \subseteq [\ell+2, n]$ and $m \le \ell$. If $I = \{i_1, \dots, i_q\}$ and $1 \le i_p < j_p$ for each p, a set $K = \{k_{s_1}, \dots, k_{s_t}, k_{s_{t+1}}\}$ is called *I-admissible* if

1. $\{i_{s_1}, \dots, i_{s_t}\} \subseteq I \setminus \{i_q\}$ and $i_{s_{t+1}} = i_q$;
2. $\{k_{s_1}, i_{s_1}\} = \{m, \ell + 1\}$; and
3. $\{k_{s_p}, i_{s_p}\} = \{k_{s_{p-1}}, j_{s_{p-1}}\}$ for $p = 2, \dots, t + 1$.

Note that the last condition is vacuous if K is of cardinality one. Note also that $1 \le k_{s_p} < j_{s_p}$ and $k_{s_p} \ne i_{s_p}$ for each p.

Lemma 1.10. *We have*

$$\pi_{m,\ell+1} \circ \pi_B(\omega_{j_q} \wedge a_{I,J}) = \sum\nolimits_K \lambda_{k_q,j_q} a_{m,\ell+1} \wedge b_{j_1} \wedge \cdots \wedge b_{j_q},$$

where the sum is over all I-admissible sets $K = \{k_{s_1}, \dots, k_{s_t}, k_{s_{t+1}} = k_q\}$, and

$$b_{j_p} = \begin{cases} a_{i_p,j_p} - a_{k_p,j_p} & \text{if } p \in \{s_1, \dots, s_t, q\}, \\ a_{i_p,j_p} & \text{if } p \notin \{s_1, \dots, s_t, q\}. \end{cases}$$

Proof. Let $a_{i,j}$ and $a_{k,j}$ be elements of $A^1_{\{j\}}$. Write $r = \min\{i, k\}$ and $s = \max\{i, k\}$. From (1.1), we have either $a_{i,j} \wedge a_{k,j} = a_{r,s} \wedge (a_{k,j} - a_{i,j})$ if $s \ge \ell + 1$, or $a_{i,j} \wedge a_{k,j} = 0$ if $s \le \ell$. It follows from these considerations, and a routine exercise to check the sign, that summands $\lambda_{k_q,j_q} a_{m,\ell+1} \wedge b_{j_1} \wedge \cdots \wedge b_{j_q}$ of $\pi_{m,\ell+1} \circ \pi_B(\omega_{j_q} \wedge a_{I,J})$ arise only from I-admissible sets K. Q.E.D.

Now write $\pi_{m,\ell+1} \circ \pi_B(\omega_{j_q} \wedge a_{I,J}) = \sum_R \lambda^J_{R,I} a_{m,\ell+1} \wedge a_{R,J}$, where the sum is over all $R = \{r_1, \dots, r_q\}$, $1 \le r_p < j_p$, $1 \le p \le q$, and $\lambda^J_{R,I} \in \mathbb{C}$.

Proposition 1.11. *The coefficient* $\lambda^J_{R,I}$ *of* $a_{m,\ell+1} \wedge a_{R,J}$ *in* $\pi_{m,\ell+1} \circ \pi_B(\omega_{j_q} \wedge a_{I,J})$ *is given by*

$$\lambda^J_{R,I} = (-1)^{|R\setminus R\cap I|} \sum\nolimits_K \lambda_{k_q,j_q}$$

where the sum is over all I*-admissible sets* K *such that* $R \setminus R \cap I \subseteq K$.

Proof. Let $K = \{k_{s_1}, \dots, k_{s_t}, k_q\}$ be an I-admissible set. Associated with K, we have the term $\lambda_{k_q,j_q} a_{m,\ell+1} \wedge b_{j_1} \wedge \cdots \wedge b_{j_q}$ of $\pi_{m,\ell+1} \circ \pi_B(a_{I,J} \wedge \omega_{j_q})$ from Lemma 1.10. If $R \setminus R \cap I \not\subseteq K$, it is readily checked that this term contributes nothing to the coefficient $\lambda^J_{R,I}$ of $a_{m,\ell+1} \wedge a_{R,J}$. On the other hand, if $R \setminus R \cap I \subseteq K$, then the above term contributes the summand $(-1)^{|R\setminus R\cap I|}\lambda_{k_q,j_q}$ to the coefficient $\lambda^J_{R,I}$. Q.E.D.

We now obtain a complete description of the map $\pi_{m,\ell+1} \circ \Psi^q_J(\boldsymbol{\lambda}) : A^q_J \to a_{m,\ell+1} \wedge A^q_J$ from (1.4). Write

$$\pi_{m,\ell+1} \circ \Psi^q_J(\boldsymbol{\lambda})(a_{I,J}) = \sum\nolimits_R \Lambda^J_{R,I} a_{m,\ell+1} \wedge a_{R,J},$$

where, as above, the sum is over all $R = \{r_1, \dots, r_q\}$, $1 \le r_p < j_p$, $1 \le p \le q$, and $\Lambda^J_{R,I} \in \mathbb{C}$. Let $\epsilon_{R,I} = 1$ if $R = I$, and $\epsilon_{R,I} = 0$ otherwise.

Theorem 1.12. *The coefficient* $\Lambda^J_{R,I}$ *of* $a_{m,\ell+1} \wedge a_{R,J}$ *in* $\pi_{m,\ell+1} \circ \Psi^q_J(\boldsymbol{\lambda})$ *is given by*

$$\Lambda^J_{R,I} = (-1)^{|R\setminus R\cap I|}\Big(\epsilon_{R,I}\lambda_{m,\ell+1} + \sum\nolimits_{j\in J} \sum\nolimits_K \lambda_{k,j}\Big),$$

where, if $j = j_p$, *the second sum is over all* I_p*-admissible sets* $K = \{k_{s_1}, \dots, k_{s_t}, k\}$ *for which* $R \setminus R \cap I \subseteq K$.

Proof. From (1.5), we have

$$\pi_{m,\ell+1} \circ \Psi^q_J(a_{I,J}) = \sum\nolimits_{p=0}^q \pi_{m,\ell+1} \circ \pi_B(\omega_{j_p} \wedge a_{I_p,J_p}) \wedge a_{I^p,J^p},$$

and the summand corresponding to $p = 0$ is simply $\lambda_{m,\ell+1} a_{m,\ell+1} \wedge a_{I,J}$. For $p \ge 1$, write $\pi_{m,\ell+1} \circ \pi_B(\omega_{j_p} \wedge a_{I_p,J_p}) = \sum_{R_p} \lambda^{J_p}_{R_p,I_p} a_{m,\ell+1} \wedge a_{R_p,J_p}$, where the sum is over all $R_p = \{r_1, \dots, r_p\}$. For a fixed R, the coefficient of $a_{m,\ell+1} \wedge a_{R,I}$ in $\pi_{m,\ell+1} \circ \pi_B(\omega \wedge a_{I,J})$ may then be expressed as

$$\Lambda^J_{R,I} = \epsilon_{R,I}\lambda_{m,\ell+1} + \sum\nolimits_{p=1}^q \lambda^{J_p}_{R_p,I_p},$$

where $R = R_p \cup I^p$. Note that we have $R \setminus R \cap I = R_p \setminus R_p \cap I_p$ for such R.

By Proposition 1.11, we have $\lambda^{J_p}_{R_p,I_p} = (-1)^{|R_p \setminus R_p \cap I_p|} \sum_K \lambda_{k_p,j_p}$, where the sum is over all I_p-admissible sets K with $R_p \setminus R_p \cap I_p \subseteq K$. Thus,

$$\Lambda^J_{R,I} = \epsilon_{R,I} \lambda_{m,\ell+1} + \sum_{p=1}^{q} (-1)^{|R_p \setminus R_p \cap I_p|} \sum_K \lambda_{k_p,j_p},$$

and since $R = R_p \cup I^p$, we have $R \setminus R \cap I = R_p \setminus R_p \cap I_p \subseteq K$. Q.E.D.

Remark 1.13. In light of the decomposition of the boundary maps of the complex $(A^\bullet, \mu^\bullet(\boldsymbol{\lambda}))$ given by (1.2) and (1.3), the above theorem, together with the "initial conditions"

$$\mu^0(\boldsymbol{\lambda}) : A^0 \to A^1,\ 1 \mapsto \sum_{k=\ell+1}^{n} \omega_k = \sum_{k=\ell+1}^{n} \sum_{i=1}^{k-1} \lambda_{i,k} a_{i,k},$$

provides a complete description of the boundary maps $\mu^\bullet(\boldsymbol{\lambda})$.

§2. Resolutions and Local Systems

The fundamental group of the complement of a discriminantal arrangement, and more generally that of any fiber-type arrangement, may be realized as an iterated semidirect product of free groups. For any such group G, in [10] we construct a finite free $\mathbb{Z}G$-resolution $C_\bullet(G)$ of the integers. We recall the construction of this resolution in notation consonant with that of the previous section.

Denote the standard generators of the pure braid group P_n by $\gamma_{i,j}$, $1 \le i < j \le n$, and for each j, let G_j be the free group on the $j-1$ generators $\gamma_{1,j}, \dots, \gamma_{j-1,j}$. Then the pure braid group may be realized as $P_n = G_n \rtimes \cdots \rtimes G_2$. More generally, for $1 \le \ell \le n$, the group $P_{n,\ell} = \ker(P_n \to P_\ell)$ may be realized as $P_{n,\ell} = G_n \rtimes \cdots \rtimes G_{\ell+1}$. Note that $P_n = P_{n,1}$. For $\ell < j$, the monodromy homomorphisms $P_{j-1,\ell} \to \mathrm{Aut}(G_j)$ are given by the (restriction of the) Artin representation. For $s < j$, we shall not distinguish between the braid $\gamma_{r,s}$ and the corresponding (right) automorphism $\gamma_{r,s} \in \mathrm{Aut}(G_j)$. The action of $\gamma_{r,s}$ on G_j is by conjugation: $\gamma_{r,s}(\gamma_{i,j}) = \gamma_{r,s}^{-1} \cdot \gamma_{i,j} \cdot \gamma_{r,s} = z_i \cdot \gamma_{i,j} \cdot z_i^{-1}$, where

$$z_i = \begin{cases} \gamma_{r,j}\gamma_{s,j} & \text{if } i = r \text{ or } i = s, \\ [\gamma_{r,j}, \gamma_{s,j}] & \text{if } r < i < s, \\ 1 & \text{otherwise.} \end{cases} \tag{2.1}$$

See Birman [5] and Hansen [20] for details, and as general references on braids.

2.1. Some Fox Calculus

We first establish some notation and record some elements of the Fox Calculus [19, 5], and results from [10] necessary in the construction.

Denote the integral group ring of a (multiplicative) group G by $\mathbb{Z}G$. We regard modules over $\mathbb{Z}G$ as left modules. Elements of the free module $(\mathbb{Z}G)^n$ are viewed as row vectors, and $\mathbb{Z}G$-linear maps $(\mathbb{Z}G)^n \to (\mathbb{Z}G)^m$ are viewed as $n \times m$ matrices which act on the right. For such a map F, denote the transpose by $F^\top$, and recall that $[F]^k$ denotes the map $\oplus_1^k F$. Denote the $n \times n$ identity matrix by $\mathbb{I}_n$.

For the single free group $G_j = \langle \gamma_{i,j} \rangle$, a free $\mathbb{Z}G_j$-resolution of $\mathbb{Z}$ is given by

$$0 \to (\mathbb{Z}G_j)^{j-1} \xrightarrow{\Delta_j} \mathbb{Z}G_j \xrightarrow{\epsilon} \mathbb{Z} \to 0, \tag{2.2}$$

where $\Delta_j = \begin{pmatrix}\gamma_{1,j} - 1 & \cdots & \gamma_{j-1,j} - 1\end{pmatrix}^\top$, and ϵ is the augmentation map, given by $\epsilon(\gamma_{i,j}) = 1$. For each element $\gamma \in P_{j-1,\ell}$, conjugation by γ induces an automorphism $\gamma : G_j \to G_j$, and a chain automorphism $\gamma_\bullet$ of (2.2), which by the "fundamental formula of Fox Calculus," can be expressed as

$$\begin{array}{ccc}
(\mathbb{Z}G_j)^{j-1} & \xrightarrow{\Delta_j} & \mathbb{Z}G_j \\
\Big\downarrow{\scriptstyle \mathcal{J}(\gamma)\circ\tilde{\gamma}} & & \Big\downarrow{\scriptstyle \tilde{\gamma}} \\
(\mathbb{Z}G_j)^{j-1} & \xrightarrow{\Delta_j} & \mathbb{Z}G_j
\end{array} \tag{2.3}$$

where $\mathcal{J}(\gamma) = \left(\frac{\partial \gamma(\gamma_{i,j})}{\partial \gamma_{k,j}}\right)$ is the $(j-1) \times (j-1)$ Jacobian matrix of Fox derivatives of γ, and $\tilde{\gamma}$ denotes the extension of γ to the group ring $\mathbb{Z}G_j$, resp., to $(\mathbb{Z}G_j)^{j-1}$. For a second element β of $P_{j-1,\ell}$, we have $(\gamma \cdot \beta)_\bullet = (\beta \circ \gamma)_\bullet = \beta_\bullet \circ \gamma_\bullet$ by the "chain rule of Fox Calculus": $\mathcal{J}(\beta \circ \gamma) = \tilde{\beta}(\mathcal{J}(\gamma)) \cdot \mathcal{J}(\beta)$. In particular, $\mathcal{J}(\gamma^{-1}) = \tilde{\gamma}^{-1}(\mathcal{J}(\gamma)^{-1})$.

Now fix ℓ, $1 \le \ell \le n$, and consider the group $P_{n,\ell} = G_n \rtimes \cdots \rtimes G_{\ell+1}$. Let $\mathcal{R} = \mathbb{Z}P_{n,\ell}$ denote the integral group ring of $P_{n,\ell}$, For $\gamma \in P_{j-1,\ell}$ as above, define $m_\gamma : \mathcal{R} \to \mathcal{R}$ by $m_\gamma(r) = \gamma \cdot r$. From (2.3) and extension of scalars, we obtain

$$\begin{array}{ccc}
\mathcal{R} \otimes_{\mathbb{Z}G_j} (\mathbb{Z}G_j)^{j-1} & \xrightarrow{\mathrm{id} \otimes \Delta_j} & \mathcal{R} \otimes_{\mathbb{Z}G_j} \mathbb{Z}G_j \\
\Big\downarrow{\scriptstyle m_\gamma \otimes \mathcal{J}(\gamma)\circ\tilde{\gamma}} & & \Big\downarrow{\scriptstyle m_\gamma \otimes \tilde{\gamma}} \\
\mathcal{R} \otimes_{\mathbb{Z}G_j} (\mathbb{Z}G_j)^{j-1} & \xrightarrow{\mathrm{id} \otimes \Delta_j} & \mathcal{R} \otimes_{\mathbb{Z}G_j} \mathbb{Z}G_j
\end{array}$$

The map $m_\gamma \otimes \mathcal{J}(\gamma) \circ \tilde{\gamma}$ and the canonical isomorphism $\mathcal{R} \otimes_{\mathbb{Z}G_j} (\mathbb{Z}G_j)^{j-1} \cong \mathcal{R}^{j-1}$ define an $\mathcal{R}$-linear automorphism $\rho_j(\gamma) : \mathcal{R}^{j-1} \to \mathcal{R}^{j-1}$, whose matrix is $\gamma \cdot \mathcal{J}(\gamma)$, see [10, Lemma 2.4]. Furthermore, we have the following.

Lemma 2.2 ([10, Lemma 2.6]). *For each j, $2 \leq j \leq n$, the action of the group $P_{j-1,\ell}$ on the free group G_j gives rise to a representation $\rho_j : P_{j-1,\ell} \to \mathrm{Aut}_{\mathcal{R}}(\mathcal{R}^{j-1})$ with the property that $\rho_j(\gamma) = m_\gamma \otimes \mathcal{J}(\gamma) \circ \tilde{\gamma}$ for every $\gamma \in P_{j-1,\ell}$.*

Remark 2.3. Via the convention $\rho_j(\gamma_{p,q}) = \mathbb{I}_{j-1}$ for $q \geq j$, the above extends to a representation $\rho_j : P_{n,\ell} \to \mathrm{Aut}_{\mathcal{R}}(\mathcal{R}^{j-1})$ of the entire group $P_{n,\ell}$. We denote by $\tilde{\rho}_j : \mathcal{R} \to \mathrm{End}_{\mathcal{R}}(\mathcal{R}^{j-1})$ the extension of ρ_j to the group ring $\mathcal{R}$. We also use $\tilde{\rho}_j$ to denote the homomorphism $\mathrm{Hom}_{\mathcal{R}}(\mathcal{R}^m, \mathcal{R}^n) \to \mathrm{Hom}_{\mathcal{R}}(\mathcal{R}^{m(j-1)}, \mathcal{R}^{n(j-1)})$ defined by replacing each entry x of an $m \times n$ matrix by $\tilde{\rho}_j(x)$.

2.4. The Resolution

We now recall the construction of the free resolution $\epsilon : C_\bullet = C_\bullet(G) \to \mathbb{Z}$ over the ring $\mathcal{R} = \mathbb{Z}G$ from [10], in the case where $G = P_{n,\ell}$ is the fundamental group of the complement of the discriminantal arrangement $\mathcal{A}_{n,\ell}$. If $J = \{j_1, \ldots, j_q\} \subseteq [\ell+1, n]$, recall that for $p < q$, $J_p = \{j_1, \ldots, j_p\}$ and $J^p = J \setminus J_p$. For such a set J, let C_q^J be a free $\mathcal{R}$-module of rank $(j_1 - 1) \cdots (j_q - 1)$.

Let $C_0 = \mathcal{R}$, and, for $1 \leq q \leq n - \ell$, let $C_q = \bigoplus_{|J|=q} C_q^J$, where the sum is over all $J \subseteq [\ell+1, n]$. The augmentation map, $\epsilon : C_0 \to \mathbb{Z}$, is the usual augmentation of the group ring, given by $\epsilon(\gamma) = 1$, for $\gamma \in P_{n,\ell}$. We define the boundary maps of $C_\bullet$ by recursively specifying their restrictions Δ^J to the summands C_q^J as follows:

If $J = \{j\}$, we define $\Delta_J : C_1^J = \mathcal{R}^{j-1} \to \mathcal{R} = C_0$ as in the resolution (2.2), by $\Delta_J = \begin{pmatrix} \gamma_{1,j} - 1 & \cdots & \gamma_{j-1,j} - 1 \end{pmatrix}^\top$.

In general, if $J = \{j_1, \ldots, j_q\}$, then $J^1 = \{j_2, \ldots, j_q\}$ and $J_{q-1} = \{j_1, \ldots, j_{q-1}\}$, and we define $\Delta_J : C_q^J \to C_{q-1}^{J^1}$ by $\Delta_J = -\tilde{\rho}_{j_q}(\Delta_{J_{q-1}})$

Now define $\Delta^J : C_q^J \to \bigoplus_{p=1}^q C_{q-1}^{J \setminus \{j_p\}}$ by

$$\Delta^J = \left(\Delta_J, [\Delta_{J^1}]^{d_1}, \ldots, [\Delta_{J^p}]^{d_p}, \ldots, [\Delta_{J^{q-1}}]^{d_{q-1}}\right),$$

where $d_p = (j_1 - 1) \cdots (j_p - 1)$.

Finally, define $\partial_q : C_q \to C_{q-1}$ by $\partial_q = \sum_{|J|=q} \Delta^J$.

Theorem 2.5 ([10, Theorem 2.10]). *Let $\mathcal{R} = \mathbb{Z}P_{n,\ell}$ be the integral group ring of the group $P_{n,\ell}$. Then the system of $\mathcal{R}$-modules and homomorphisms $(C_\bullet, \partial_\bullet)$ is a finite, free resolution of $\mathbb{Z}$ over $\mathcal{R}$.*

Remark 2.6. The proof of this result in [10] makes use of a mapping cone decomposition of the complex $(C_\bullet, \partial_\bullet)$. This decomposition may be described as follows. Let $(\widehat{C}_\bullet, \widehat{\partial}_\bullet)$ denote the subcomplex of $(C_\bullet, \partial_\bullet)$ with terms $\widehat{C}_q = \bigoplus_{\ell+1 \notin J} C_q^J$, and boundary maps $\widehat{\partial}_q = \partial_q|_{\widehat{C}_q}$ given by restriction. The complex $\widehat{C}_\bullet$ may be realized as $\widehat{C}_\bullet = C_\bullet(P_{n,\ell+1}) \otimes_{P_{n,\ell}} \mathcal{R}$, where $\epsilon : C_\bullet(P_{n,\ell+1}) \to \mathbb{Z}$ is the resolution over $\mathbb{Z}P_{n,\ell+1}$ obtained by applying the above construction to the group $P_{n,\ell+1} < P_{n,\ell}$.

Let $(D_\bullet, \partial_\bullet^D)$ denote the direct sum of ℓ copies of the complex $\widehat{C}_\bullet$, with the sign of the boundary map reversed. That is, $D_q = (\widehat{C}_q)^\ell$ and $\partial_q^D = -[\widehat{\partial}_q]^\ell$. The terms of this complex may be expressed as $D_q = \bigoplus_{\ell+1 \in K} C_{q+1}^K$, where $|K| = q$. Using this description, define a map $\Xi_\bullet : D_\bullet \to \widehat{C}_\bullet$ by setting the restriction of Ξ_q to the summand C_{q+1}^K of D_q to be equal to $\Delta_K : C_{q+1}^K \to C_q^J \subset \widehat{C}_q$, where $K = \{\ell+1\} \cup J$.

As shown in [10], the map $\Xi_\bullet : D_\bullet \to \widehat{C}_\bullet$ is a chain map, and the original complex $(C_\bullet, \partial_\bullet)$ may be realized as the mapping cone of $\Xi_\bullet$. Explicitly, the terms of $C_\bullet$ decompose as $C_q = D_{q-1} \oplus \widehat{C}_q$. With respect to this decomposition, the boundary map $\partial_{q+1} : C_{q+1} \to C_q$ is given by $\partial_{q+1}(u, v) = (-\partial_q^D(u), \Xi_q(u) + \widehat{\partial}_{q+1}(v))$.

2.7. Rank One Local Systems

The abelianization of the group $P_{n,\ell}$ is free abelian of rank $N = \binom{n}{2} - \binom{\ell}{2}$. Let $(\mathbb{C}^*)^N$ denote the complex torus, with coordinates $t_{i,j}$, $\ell+1 \leq j \leq n$, $1 \leq i < j$. Each point $\mathbf{t} \in (\mathbb{C}^*)^N$ gives rise to a rank one representation $\nu_\mathbf{t} : P_{n,\ell} \to \mathbb{C}^*$, $\gamma_{i,j} \mapsto t_{i,j}$, an associated $P_{n,\ell}$-module $L = L_\mathbf{t}$, and a rank one local system $\mathcal{L} = \mathcal{L}_\mathbf{t}$ on the configuration space $M_{n,\ell}$. The homology and cohomology of $P_{n,\ell}$ with coefficients in L (resp., that of $M_{n,\ell}$ with coefficients in $\mathcal{L}$) are isomorphic to the homology and cohomology of the complexes $\mathbf{C}_\bullet := C_\bullet \otimes_{P_{n,\ell}} L$ and $\mathbf{C}^\bullet := \mathcal{H}om_{P_{n,\ell}}(C_\bullet, L)$ respectively, see [8].

The terms, $\mathbf{C}_q = C_q \otimes_\mathcal{R} \mathbb{C}$ and $\mathbf{C}^q = \mathrm{Hom}_{P_{n,\ell}}(C_q, L)$, of these complexes are finite dimensional complex vector spaces. Notice that $\dim \mathbf{C}_q = \dim \mathbf{C}^q = \dim A^q = \sum_{|J|=q} (j_1 - 1) \cdots (j_q - 1)$, where the sum is over all $J \subseteq [\ell+1, n]$. Denote the boundary maps of $\mathbf{C}_\bullet$ and $\mathbf{C}^\bullet$ by $\boldsymbol{\partial}_q(\mathbf{t}) : \mathbf{C}_q \to \mathbf{C}_{q-1}$ and $\boldsymbol{\delta}^q(\mathbf{t}) : \mathbf{C}^q \to \mathbf{C}^{q+1}$. As we follow [8] in our definition of $\mathbf{C}^\bullet$, these maps are related by

$$\boldsymbol{\delta}^q(\mathbf{t})(u)(x) = (-1)^q u(\boldsymbol{\partial}_{q+1}(\mathbf{t})(x)) \tag{2.4}$$

for $u \in \mathbf{C}^q$ and $x \in C_{q+1}$. To describe these maps further, we require some notation.

Consider the evaluation map $\mathcal{R} \times (\mathbb{C}^*)^N \to \mathbb{C}$, which takes an element f of the group ring, and a point $\mathbf{t}$ in $(\mathbb{C}^*)^N$ and yields $f(\mathbf{t}) := \tilde{\nu}_{\mathbf{t}}(f)$, the evaluation of f at $\mathbf{t}$. Fixing $f \in \mathcal{R}$ and allowing $\mathbf{t} \in (\mathbb{C}^*)^N$ to vary, we get a holomorphic map $\mathbf{f} : (\mathbb{C}^*)^N \to \mathbb{C}$. More generally, we have the map $\mathrm{Mat}_{r\times s}(\mathcal{R}) \times (\mathbb{C}^*)^N \to \mathrm{Mat}_{r\times s}(\mathbb{C})$, $(F, \mathbf{t}) \mapsto F(\mathbf{t}) := \tilde{\nu}_{\mathbf{t}}(F)$. For fixed $F \in \mathrm{Mat}_{p\times q}(\mathcal{R})$, we get a map $\mathbf{F} : (\mathbb{C}^*)^N \to \mathrm{Mat}_{r\times s}(\mathbb{C})$. With these conventions, if $\dim \mathbf{C}_q = r$ and $\dim \mathbf{C}_{q+1} = s$, the boundary maps of the complexes $\mathbf{C}_\bullet$ and $\mathbf{C}^\bullet$ may be viewed as evaluations, $\boldsymbol{\partial}_q(\mathbf{t})$ and $\boldsymbol{\delta}^q(\mathbf{t})$, of maps $\boldsymbol{\partial}_q : (\mathbb{C}^*)^N \to \mathrm{Mat}_{r\times s}(\mathbb{C})$ and $\boldsymbol{\delta}^q : (\mathbb{C}^*)^N \to \mathrm{Mat}_{s\times r}(\mathbb{C})$.

We shall subsequently be concerned with the derivatives of these maps at the identity element $\mathbf{1} = (1, \dots, 1)$ of $(\mathbb{C}^*)^N$. The (holomorphic) tangent space of $H^1(M_{n,\ell}; \mathbb{C}^*) = (\mathbb{C}^*)^N$ at $\mathbf{1}$ is $H^1(M_{n,\ell}; \mathbb{C}) = \mathbb{C}^N$, with coordinates $\lambda_{i,j}$. The exponential map $\mathrm{T}_{\mathbf{1}}\,(\mathbb{C}^*)^N \to (\mathbb{C}^*)^N$ is the coefficient map $H^1(M_{n,\ell}; \mathbb{C}) \to H^1(M_{n,\ell}; \mathbb{C}^*)$ induced by $\exp : \mathbb{C} \to \mathbb{C}^*$, $\lambda_{i,j} \mapsto e^{\lambda_{i,j}} = t_{i,j}$. For an element f of $\mathcal{R}$, the derivative of the corresponding map $\mathbf{f} : (\mathbb{C}^*)^N \to \mathbb{C}$ at $\mathbf{1}$ is given by $\mathbf{f}_* : \mathbb{C}^N \to \mathbb{C}$, $\mathbf{f}_*(\boldsymbol{\lambda}) = \frac{d}{dx}\big|_{x=0} f(\dots e^{x\lambda_{i,j}} \dots)$. More generally, for $F \in \mathrm{Mat}_{r\times s}(\mathcal{R})$, we have $\mathbf{F}_* : \mathbb{C}^N \to \mathrm{Mat}_{r\times s}(\mathbb{C})$.

§3. A Complex of Derivatives

We now relate the cohomology theories $H^*(A^\bullet, \mu^\bullet(\boldsymbol{\lambda}))$ and $H^*(M_n; \mathcal{L})$ by relating the complexes $(A^\bullet, \mu^\bullet(\boldsymbol{\lambda}))$ and $(\mathbf{C}^\bullet, \boldsymbol{\delta}^\bullet(\mathbf{t}))$. As above, let $(\boldsymbol{\partial}_q)_*$ and $\boldsymbol{\delta}^q_*$ denote the derivatives of the maps $\boldsymbol{\partial}_q$ and $\boldsymbol{\delta}^q$ at $\mathbf{1} \in (\mathbb{C}^*)^N$.

Theorem 3.1. *The complex $(A^\bullet, \mu^\bullet(\boldsymbol{\lambda}))$ is a linear approximation of the complex $(\mathbf{C}^\bullet, \boldsymbol{\delta}^\bullet(\mathbf{t}))$. For each $\boldsymbol{\lambda} \in \mathbb{C}^N$, the system of complex vector spaces and linear maps $(\mathbf{C}^\bullet, \boldsymbol{\delta}^\bullet_*(\boldsymbol{\lambda}))$ is a complex. For each q, we have $A^q \cong \mathbf{C}^q$, and, under this identification, $\mu^q(\boldsymbol{\lambda}) = \boldsymbol{\delta}^q_*(\boldsymbol{\lambda})$.*

From the discussions in sections 1.1 and 2.7, it is clear that $A^q \cong \mathbf{C}^q$. In light of the sign conventions (2.4) used in the construction of the complex $(\mathbf{C}^\bullet, \boldsymbol{\delta}^\bullet(\mathbf{t}))$ and the fact that $(A^\bullet, \mu^\bullet(\boldsymbol{\lambda}))$ is a complex, to show that $(\mathbf{C}^\bullet, \boldsymbol{\delta}^\bullet_*(\boldsymbol{\lambda}))$ is a complex, and to prove the theorem, it suffices to establish the following.

Proposition 3.2. *For each q, we have $\mu^q(\boldsymbol{\lambda}) = (-1)^q \left[(\boldsymbol{\partial}_{q+1})_*(\boldsymbol{\lambda})\right]^\top$.*

The maps $\mu^q(\boldsymbol{\lambda})$ were analyzed in section 1.5. We now carry out a similar analysis of the maps $(\boldsymbol{\partial}_{q+1})_*(\boldsymbol{\lambda})$.

3.3. Some Calculus

We first record some facts necessary for this analysis. Recall that $\mathcal{R}$ denotes the integral group ring of the group $P_{n,\ell}$. For $f, g \in \mathcal{R}$, the Product Rule yields $(\mathbf{f} \cdot \mathbf{g})_*(\boldsymbol{\lambda}) = \mathbf{f}_*(\boldsymbol{\lambda}) \cdot \mathbf{g}(\mathbf{1}) + \mathbf{f}(\mathbf{1}) \cdot \mathbf{g}_*(\boldsymbol{\lambda})$. Similarly, for $F \in \mathrm{Mat}_{p\times q}(\mathcal{R})$ and $G \in \mathrm{Mat}_{q\times r}(\mathcal{R})$, matrix multiplication and the differentiation rules yield

$$(3.1) \qquad (\mathbf{F} \cdot \mathbf{G})_*(\boldsymbol{\lambda}) = \mathbf{F}_*(\boldsymbol{\lambda}) \cdot \mathbf{G}(\mathbf{1}) + \mathbf{F}(\mathbf{1}) \cdot \mathbf{G}_*(\boldsymbol{\lambda}).$$

As an immediate consequence of the Product Rule, for $\gamma, \zeta \in P_{n,\ell}$ and $\tau = [\zeta, \gamma]$ a commutator, we have $(\boldsymbol{\gamma}^{-1})_* = -\boldsymbol{\gamma}_*$, and $\boldsymbol{\tau}_* = 0$. Consequently, $(\boldsymbol{\zeta} \cdot \boldsymbol{\gamma} \cdot \boldsymbol{\zeta}^{-1})_* = \boldsymbol{\gamma}_*$.

Now recall the representations ρ_j defined in Lemma 2.2, and used in the construction of the resolution $C_\bullet$. Associated to each $\gamma \in P_{j-1,\ell}$, we have a map $\boldsymbol{\rho}_j(\gamma) : (\mathbb{C}^*)^N \to \mathrm{Aut}(\mathbb{C}^{j-1})$. Since γ acts on the free group by conjugation, we have $\boldsymbol{\rho}_j(\gamma)(\mathbf{1}) = \mathbb{I}_{j-1}$. Identify $\mathrm{End}(\mathbb{C}^{j-1})$ as the tangent space to $\mathrm{Aut}(\mathbb{C}^{j-1})$ at the identity, and denote the derivative of the map $\boldsymbol{\rho}_j(\gamma)$ at $\mathbf{1}$ by $\boldsymbol{\rho}_j(\gamma)_* : \mathbb{C}^N \to \mathrm{End}(\mathbb{C}^{j-1})$.

Define $(\boldsymbol{\rho}_j)_* : P_{j-1,\ell} \to \mathrm{Hom}(\mathbb{C}^N, \mathrm{End}(\mathbb{C}^{j-1}))$ by $(\boldsymbol{\rho}_j)_*(\gamma) = \boldsymbol{\rho}_j(\gamma)_*$. The chain rule of Fox Calculus and a brief computation reveal that $(\boldsymbol{\rho}_j)_*$ is a homomorphism, and is trivial on the commutator subgroup $P'_{n,\ell}$. This yields a map $\mathbb{C}^N \to \mathrm{Hom}(\mathbb{C}^N, \mathrm{End}(\mathbb{C}^{j-1}))$, $\lambda_{r,s} \mapsto \boldsymbol{\rho}_j(\gamma_{r,s})_*$, which we continue to denote by $(\boldsymbol{\rho}_j)_*$. For $\gamma \in P_{n,\ell}$, view the derivative, $\boldsymbol{\gamma}_*(\boldsymbol{\lambda}) = \sum c_{r,s}\lambda_{r,s}$, of the corresponding map $\boldsymbol{\gamma}$ as a linear form in the $\lambda_{r,s}$. Then we have the following "chain rule":

$$(3.2) \qquad \boldsymbol{\rho}_j(\gamma)_*(\boldsymbol{\lambda}) = \sum c_{r,s}(\boldsymbol{\rho}_j)_*(\lambda_{r,s}) = (\boldsymbol{\rho}_j)_*(\boldsymbol{\gamma}_*(\boldsymbol{\lambda})).$$

In particular, $\boldsymbol{\rho}(\gamma_{r,s})_* = \boldsymbol{\rho}_*(\lambda_{r,s})$, which we now compute.

Lemma 3.4. *For $r < s < j$, the derivative of the map $\boldsymbol{\rho}_j(\gamma_{r,s})$ is given by $\boldsymbol{\rho}_j(\gamma_{r,s})_*(\boldsymbol{\lambda}) =$*

$$\begin{pmatrix} \lambda_{r,s} \cdot \mathbb{I}_{r-1} & 0 & 0 & 0 & 0 \\ 0 & \lambda_{r,s} + \lambda_{s,j} & 0 & -\lambda_{r,j} & 0 \\ 0 & 0 & \lambda_{r,s} \cdot \mathbb{I}_{s-r-1} & 0 & 0 \\ 0 & -\lambda_{s,j} & 0 & \lambda_{r,s} + \lambda_{r,j} & 0 \\ 0 & 0 & 0 & 0 & \lambda_{r,s} \cdot \mathbb{I}_{j-s-1} \end{pmatrix}.$$

Proof. The matrix of $\rho_j(\gamma_{r,s})$ is $\gamma_{r,s} \cdot \mathcal{J}(\gamma_{r,s})$, where $\mathcal{J}(\gamma_{r,s})$ is the Fox Jacobian. Thus, $\boldsymbol{\rho}_j(\gamma_{r,s})(\mathbf{t}) = t_{r,s} \cdot \boldsymbol{\mathcal{J}}(\gamma_{r,s})(\mathbf{t}) = (t_{r,s} \cdot \mathbb{I}_{j-1}) \cdot \boldsymbol{\mathcal{J}}(\gamma_{r,s})(\mathbf{t})$, where $\boldsymbol{\mathcal{J}}(\gamma_{r,s})(\mathbf{t})$ is the map induced by the Fox Jacobian. By the Product Rule (3.1), we have

$$\boldsymbol{\rho}_j(\gamma_{r,s})_*(\boldsymbol{\lambda}) = (\lambda_{r,s} \cdot \mathbb{I}_{j-1}) \cdot \boldsymbol{\mathcal{J}}(\gamma_{r,s})(\mathbf{1}) + \boldsymbol{\mathcal{J}}(\gamma_{r,s})_*(\boldsymbol{\lambda}).$$

The action of $\gamma_{r,s}$ on the free group $G_j = \langle \gamma_{i,j} \rangle$ is recorded in (2.1). Computing Fox derivatives and evaluating at $\mathbf{t}$ yields the familiar Gassner matrix of $\gamma_{r,s}$ (see [5]), $\boldsymbol{\mathcal{J}}(\gamma_{r,s})(\mathbf{t}) =$

$$\begin{pmatrix} \mathbb{I}_{r-1} & 0 & 0 & 0 & 0 \\ 0 & 1 - t_{r,j} + t_{r,j}t_{s,j} & 0 & t_{r,j}(1 - t_{r,j}) & 0 \\ 0 & \vec{u} & \mathbb{I}_{s-r-1} & -\vec{u} & 0 \\ 0 & 1 - t_{s,j} & 0 & t_{r,j} & 0 \\ 0 & 0 & 0 & 0 & \mathbb{I}_{j-s-1} \end{pmatrix},$$

where $\vec{u} = \begin{pmatrix}(1 - t_{r+1,j})(1 - t_{r,j}) & \cdots & (1 - t_{s-1,j})(1 - t_{r,j})\end{pmatrix}^{\top}$. Since $\boldsymbol{\mathcal{J}}(\gamma_{r,s})(\mathbf{1}) = \mathbb{I}_{j-1}$, the result follows upon differentiating $\boldsymbol{\mathcal{J}}(\gamma_{r,s})(\mathbf{t})$. Q.E.D.

3.5. Boundary Map Derivatives

We now obtain an inductive formula for the derivatives of the boundary maps of the complex $(\mathbf{C}_\bullet, \boldsymbol{\partial}_\bullet(\mathbf{t}))$. The mapping cone decomposition of the resolution $(C_\bullet, \partial_\bullet)$ discussed in Remark 2.6 gives rise to an analogous decomposition of the complex $(\mathbf{C}_\bullet, \boldsymbol{\partial}(\mathbf{t}))$. Specifically, the terms decompose as $\mathbf{C}_q = \mathbf{D}_{q-1} \oplus \widehat{\mathbf{C}}_q$, and with respect to this decomposition, the matrix of the boundary map $\boldsymbol{\partial}_{q+1}(\mathbf{t}) : \mathbf{C}_{q+1} \to \mathbf{C}_q$ is given by

$$\boldsymbol{\partial}_{q+1}(\mathbf{t}) = \begin{pmatrix} -\boldsymbol{\partial}_q^D(\mathbf{t}) & \boldsymbol{\Xi}_q(\mathbf{t}) \\ 0 & \widehat{\boldsymbol{\partial}}_{q+1}(\mathbf{t}) \end{pmatrix}. \tag{3.3}$$

Up to sign, the complex $(\mathbf{D}_\bullet, \boldsymbol{\partial}_\bullet^D(\mathbf{t}))$ is a direct sum of ℓ copies of the complex $(\widehat{\mathbf{C}}_\bullet, \widehat{\boldsymbol{\partial}}_\bullet(\mathbf{t}))$, which arises from the group $P_{n,\ell+1} < P_{n,\ell}$. In light of this, we restrict our attention to the chain map $\Xi_\bullet$ and its components $\Delta_{\{\ell+1,J\}}$, their evaluations $\boldsymbol{\Xi}_\bullet(\mathbf{t})$ and $\boldsymbol{\Delta}_{\{\ell+1,J\}}(\mathbf{t})$, and the derivatives of these evaluations at $\mathbf{1}$.

For $J = \{j_1, \dots, j_q\} \subseteq [\ell + 2, n]$, let $\rho_J = \tilde{\rho}_{j_q} \circ \cdots \circ \tilde{\rho}_{j_1}$ and $d_J = (j_1 - 1) \cdots (j_q - 1)$. Then $\Delta_{\{\ell+1,J\}} = (-1)^q \rho_J(\Delta_{\ell+1})$, where $\Delta_{\ell+1} = \begin{pmatrix}\gamma_{1,\ell+1} - 1 & \cdots & \gamma_{\ell,\ell+1} - 1\end{pmatrix}^{\top}$, and the matrix of $\Delta_{\{\ell+1,J\}}$ is $\ell \cdot d_J \times d_J$ with $d_J \times d_J$ blocks $(-1)^q \rho_J(\gamma_{m,\ell+1} - 1)$, $1 \leq m \leq \ell$. We concentrate our attention on one such block.

Fix m, $1 \leq m \leq \ell$, and let M denote the matrix of $\rho_J(\gamma_{m,\ell+1} - 1)$. Similarly, let M' denote the matrix of $\rho_{J_{q-1}}(\gamma_{m,\ell+1} - 1)$. Then M is the matrix of $\tilde{\rho}_{j_q}(M')$. Since M is $d_J \times d_J$, its rows and columns are naturally indexed by sets $R = \{r_1, \ldots, r_q\}$ and $I = \{i_1, \ldots, i_q\}$, $1 \leq r_p, i_p \leq j_p - 1$. We thus denote the entries of M by $M_{R,I}$. With these conventions, we have

$$M_{R,I} = \left[\tilde{\rho}_{j_q}(M'_{R',I'})\right]_{r_q,i_q} \tag{3.4}$$

where, for instance, $I' = I_{q-1} = I \setminus \{i_q\}$.

Now consider the block $\mathbf{M}(\mathbf{t})$ of $\mathbf{\Delta}_{\{\ell+1,J\}}(\mathbf{t})$ arising from the block M of the matrix of $\Delta_{\{\ell+1,J\}}$ above. Recall the notion of an I-admissible set from Definition 1.9, and recall that $\epsilon_{R,I} = 1$ if $R = I$, and $\epsilon_{R,I} = 0$ otherwise.

Theorem 3.6. *Let $J \subseteq [\ell+2, n]$ and let M denote the matrix of $\rho_J(\gamma_{m,\ell+1} - 1)$. Then the entries of the derivative, $\mathbf{M}_*(\boldsymbol{\lambda})$, of the evaluation $\mathbf{M}(\mathbf{t})$ are given by*

$$[\mathbf{M}_*(\boldsymbol{\lambda})]_{R,I} = (-1)^{|R \setminus R \cap I|}\Big(\epsilon_{R,I}\lambda_{m,\ell+1} + \sum\nolimits_{j \in J} \sum\nolimits_{K} \lambda_{k,j}\Big),$$

where, if $j = j_p$, the second sum is over all I_p-admissible sets $K = \{k_{s_1}, \ldots, k_{s_t}, k\}$ for which $R \setminus R \cap I \subseteq K$.

Proof. The proof is by induction on $|J|$.

If $J = \{j\}$, then $M = \gamma_{m,\ell+1} \cdot \mathcal{J}(\gamma_{m,\ell+1}) - \mathbb{I}_{j-1}$ is the matrix of $\tilde{\rho}_j(\gamma_{m,\ell+1} - 1)$, so $\mathbf{M}(\mathbf{t}) = \boldsymbol{\gamma}_{m,\ell+1}(\mathbf{t}) \cdot \boldsymbol{\mathcal{J}}(\gamma_{m,\ell+1})(\mathbf{t}) - \mathbb{I}_{j-1}$. Since the derivative of the constant $\mathbb{I}_{j-1}$ is zero, the entries of $\mathbf{M}_*(\boldsymbol{\lambda})$ are given by Lemma 3.4 (with $r = m$ and $s = \ell+1$). In this instance, we have $I = \{i\}$, and a set $K = \{k\}$ is I-admissible if $k \neq i$ and $\{k, i\} = \{m, \ell+1\}$. It follows that the case $|J| = 1$ is a restatement of Lemma 3.4.

In general, let $J = \{j_1, \ldots, j_{q-1}, j_q\}$ and, as in (3.4) above, write $M_{R,I} = \left[\tilde{\rho}_{j_q}(M'_{R',I'})\right]_{r_q,i_q}$. Then we have

$$[\mathbf{M}(\mathbf{t})]_{R,I} = \left[\tilde{\boldsymbol{\rho}}_{j_q}(M'_{R',I'})(\mathbf{t})\right]_{r_q,i_q} \quad \text{and}$$
$$[\mathbf{M}_*(\boldsymbol{\lambda})]_{R,I} = \left[\tilde{\boldsymbol{\rho}}_{j_q}(M'_{R',I'})_*(\boldsymbol{\lambda})\right]_{r_q,i_q}.$$

By induction, the entries of the matrix $\mathbf{M}'_*(\boldsymbol{\lambda})$ are given by

$$[\mathbf{M}'_*(\boldsymbol{\lambda})]_{R',I'} = (-1)^{|R' \setminus R' \cap I'|}\Big(\epsilon_{R',I'}\lambda_{m,\ell+1} + \sum\nolimits_{j_p \in J'} \sum\nolimits_{K} \lambda_{k_p,j_p}\Big),$$

where $J' = J_{q-1} = J \setminus \{j_q\}$, and for $j_p \in J'$, the second sum is over all I_p-admissible sets K for which $R' \setminus R' \cap I' \subseteq K$.

By the chain rule (3.2), the entries of $\mathbf{M}_*(\boldsymbol{\lambda})$ are given by

(3.5)
$$[\mathbf{M}_*(\boldsymbol{\lambda})]_{R,I} = \Big[\tilde{\rho}_{j_q}(M'_{R',I'})_*(\boldsymbol{\lambda})\Big]_{r_q,i_q} = \Big[(\rho_{j_q})_*\big((\mathbf{M}'_{R',I'})_*(\boldsymbol{\lambda})\big)\Big]_{r_q,i_q}$$
$$= \Big[S\Big(\epsilon_{R',I'}(\rho_{j_q})_*(\lambda_{m,\ell+1}) + \sum\nolimits_{j_p \in J'} \sum\nolimits_{K} (\rho_{j_q})_*(\lambda_{k_p,j_p})\Big)\Big]_{r_q,i_q},$$

where $S = (-1)^{|R' \setminus R' \cap I'|}$. By Lemma 3.4, for $r < s < j_q$, we have

(3.6)
$$\Big[(\rho_{j_q})_*(\lambda_{r,s})\Big]_{r_q,i_q} = \begin{cases} \lambda_{r,s} + \lambda_{k_q,j_q} & \text{if } i_q = r_q \text{ and } \{k_q, i_q\} = \{r,s\}, \\ -\lambda_{r_q,j_q} & \text{if } i_q \neq r_q \text{ and } \{r_q, i_q\} = \{r,s\}, \\ 0 & \text{otherwise.} \end{cases}$$

The entries of $\mathbf{M}_*(\boldsymbol{\lambda})$ may be calculated from (3.5) using (3.6), yielding the formula in the statement of the theorem. We conclude the proof by making several observations which elucidate this calculation.

First consider the case $R' = I'$. Then $S = 1$ and $\epsilon_{R',I'} = 1$. If $r_q = i_q$, then the first case of (3.6) yields a contribution of $\lambda_{m,\ell+1} + \lambda_{k_q,j_q}$ to $[\mathbf{M}_*(\boldsymbol{\lambda})]_{I,I}$, provided that $\{k_q, i_q\} = \{m, \ell+1\}$. Note that this condition implies that the set $K = \{k_q\}$ is I-admissible (and that $k_q \neq i_q$). Note also that in this instance we have $R = I$, $R \setminus R \cap I = \emptyset \subset K$, $\epsilon_{R,I} = 1$, and $|R \setminus R \cap I| = 0$.

If $R' = I'$ and $r_q \neq i_q$, then the second case of (3.6) contributes $-\lambda_{r_q,j_q}$ to $[\mathbf{M}_*(\boldsymbol{\lambda})]_{R,I}$ if $\{r_q, i_q\} = \{m, \ell+1\}$. In this instance, the set $\{r_q\}$ is I-admissible. Since $R' = I'$ and $r_q \neq i_q$, we have $|R \setminus R \cap I| = 1$.

For general R' and I', suppose that $S \cdot \lambda_{k_p,j_p}$ is a summand of $[\mathbf{M}'_*(\boldsymbol{\lambda})]_{R',I'}$ for some $p \leq q-1$. Then, by the inductive hypothesis, this summand arises from an I_p-admissible set $K = \{k_{s_1}, \ldots, k_{s_t}, k_p\}$ with $R' \setminus R' \cap I' \subseteq K$. If $r_q = i_q$, then the first case of (3.6) yields a contribution of $S \cdot (\lambda_{k_p,j_p} + \lambda_{k_q,j_q})$ to $[\mathbf{M}_*(\boldsymbol{\lambda})]_{R,I}$, provided that $\{k_q, i_q\} = \{k_p, j_p\}$. For such k_q, it is readily checked that the set $K \cup \{k_q\}$ is I-admissible. Also, since $r_q = i_q$, we have $R \setminus R \cap I = R' \setminus R' \cap I' \subseteq K$.

If, as above, $S \cdot \lambda_{k_p,j_p}$ is a summand of $[\mathbf{M}'_*(\boldsymbol{\lambda})]_{R',I'}$ and $r_q \neq i_q$, then the second case of (3.6) contributes $-S \cdot \lambda_{r_q,j_q}$ to $[\mathbf{M}_*(\boldsymbol{\lambda})]_{R,I}$ provided $\{r_q, i_q\} = \{k_p . j_p\}$. In this instance, the set $K \cup \{r_q\}$ is I-admissible, and since $r_q \neq i_q$, we have $R \setminus R \cap I = (R' \setminus R' \cap I') \cup \{r_q\} \subseteq K \cup \{r_q\}$, and $|R \setminus R \cap I| = |R' \setminus R' \cap I'| + 1$.

Applying these observations to (3.5) above completes the proof. Q.E.D.

3.7. Proof of Proposition 3.2

We now use Theorems 1.12 and 3.6 to show that the differential of the complex $(A^\bullet, \mu^\bullet(\boldsymbol{\lambda}))$ is given by $\mu^q(\boldsymbol{\lambda}) = (-1)^q\big[(\boldsymbol{\partial}_{q+1})_*(\boldsymbol{\lambda})\big]^\top$, where $(\boldsymbol{\partial}_q)_*(\boldsymbol{\lambda})$ is the derivative of the boundary map of the complex $(\mathbf{C}_\bullet, \boldsymbol{\partial}_\bullet(\mathbf{t}))$, thereby proving Proposition 3.2 and hence Theorem 3.1 as well.

The proof is by induction on $d = n - \ell$, the cohomological dimension of the group $P_{n,\ell}$, (resp., the rank of the discriminantal arrangement $\mathcal{A}_{n,\ell}$).

In the case $d = 1$, the complexes $A^\bullet$ and $\mathbf{C}_\bullet$ are given by

$$A^0 \xrightarrow{\mu^0(\boldsymbol{\lambda})} A^1 \quad \text{and} \quad \mathbf{C}_1 \xrightarrow{\boldsymbol{\partial}_1(\mathbf{t})} \mathbf{C}_0$$

respectively, where $A^0 = \mathbf{C}_0 = \mathbb{C}$, $A^1 = \oplus_{i<n}\mathbb{C}a_{i,n}$, and $\mathbf{C}_1 = \mathbb{C}^{n-1}$. The boundary maps are $\mu^0(\boldsymbol{\lambda}) : 1 \mapsto \sum_{i<n} \lambda_{i,n} \cdot a_{i,n}$ and $\boldsymbol{\partial}_1(\mathbf{t}) = \begin{pmatrix} t_{1,n} - 1 & \cdots & t_{n-1,n} - 1\end{pmatrix}^\top$. Identifying A^1 and $\mathbf{C}_1$ in the obvious manner, we have $\mu^0(\boldsymbol{\lambda}) = (-1)^0\big[(\boldsymbol{\partial}_1)_*(\boldsymbol{\lambda})\big]^\top$.

In the general case, we identify A^q and $\mathbf{C}_q$ in an analogous manner. In particular, the rows and columns of the matrix of the boundary map $\boldsymbol{\partial}_{q+1}(\mathbf{t}) : \mathbf{C}_{q+1} \to \mathbf{C}_q$ are indexed by basis elements $a_{I,J}$ of A^{q+1} and A^q, or simply by the underlying sets I and J, respectively. To show that $\mu^q(\boldsymbol{\lambda}) = (-1)^q\big[(\boldsymbol{\partial}_{q+1})_*(\boldsymbol{\lambda})\big]^\top$, we make use of the decomposition of the complex $A^\bullet$ established in Proposition 1.7, and that of $\mathbf{C}_\bullet$ stemming from the mapping cone decomposition of the resolution $C_\bullet$ described in Remark 2.6. Recall from (1.2) and (3.3) that with respect to these decompositions, the boundary maps may be expressed as

$$\mu^q(\boldsymbol{\lambda}) = \begin{pmatrix} \mu^q_B(\boldsymbol{\lambda}) & 0 \\ \Psi^q(\boldsymbol{\lambda}) & \widehat{\mu}^q(\boldsymbol{\lambda}) \end{pmatrix} \quad \text{and} \quad \boldsymbol{\partial}_{q+1}(\mathbf{t}) = \begin{pmatrix} -\boldsymbol{\partial}^{\mathbf{D}}_q(\mathbf{t}) & \Xi_q(\mathbf{t}) \\ 0 & \widehat{\boldsymbol{\partial}}_{q+1}(\mathbf{t}) \end{pmatrix}.$$

The maps $\widehat{\mu}^q(\boldsymbol{\lambda})$ and $\widehat{\boldsymbol{\partial}}_{q+1}(\mathbf{t})$ are the boundary maps of the complexes $\widehat{A}^\bullet$ and $\widehat{\mathbf{C}}_\bullet$ arising from the cohomology algebra $A(\mathcal{A}_{n,\ell+1})$ and fundamental group $P_{n,\ell+1}$ of the complement of the discriminantal arrangement $\mathcal{A}_{n,\ell+1}$. So by induction, we have $\widehat{\mu}^q(\boldsymbol{\lambda}) = (-1)^q\big[(\widehat{\boldsymbol{\partial}}_{q+1})_*(\boldsymbol{\lambda})\big]^\top$ for each q. Since the complexes $B^\bullet \cong (\widehat{A}^\bullet)^\ell$ and $\mathbf{D}_\bullet \cong (\widehat{\mathbf{C}}_\bullet)^\ell$ decompose as direct sums, with boundary maps $\mu^q_B(\boldsymbol{\lambda}) = -[\widehat{\mu}^{q-1}(\boldsymbol{\lambda})]^\ell$ and $\boldsymbol{\partial}^{\mathbf{D}}_q(\mathbf{t}) = -[\widehat{\boldsymbol{\partial}}_q(\mathbf{t})]^\ell$ the inductive hypothesis also implies that

$$\begin{aligned} \mu^q_B(\boldsymbol{\lambda}) &= -\Big[(-1)^{q-1}\big[(\widehat{\boldsymbol{\partial}}_q)_*(\boldsymbol{\lambda})\big]^\top\Big]^\ell \\ &= (-1)^q\Big[\big[(\widehat{\boldsymbol{\partial}}_q)_*(\boldsymbol{\lambda})\big]^\ell\Big]^\top = (-1)^q\big[-(\boldsymbol{\partial}^{\mathbf{D}}_q)_*(\boldsymbol{\lambda})\big]^\top. \end{aligned}$$

Thus it remains to show that $\Psi^q(\boldsymbol{\lambda}) = (-1)^q\big[(\Xi_q)_*(\boldsymbol{\lambda})\big]^\top$. For this, it suffices to show that the restriction $\Psi^q_J(\boldsymbol{\lambda}) : A^q_J \to A^{q+1}_{\{\ell+1,J\}}$ of $\Psi^q(\boldsymbol{\lambda})$ is dual to the derivative of the summand $\boldsymbol{\Delta}_{\{\ell+1,J\}}(\mathbf{t}) : \mathbf{C}^{\{\ell+1,J\}}_{q+1} \to \mathbf{C}^J_q$ of $\Xi_q(\mathbf{t})$ for each $J = \{j_1, \dots, j_q\} \subseteq [\ell+2, n]$. As noted in (1.3), the matrix of $\Psi^q_J(\boldsymbol{\lambda})$ is $d_J \times \ell \cdot d_J$ with $d_J \times d_J$ blocks $\pi_{m,\ell+1} \circ \Psi^q_J(\boldsymbol{\lambda})$, where $d_J = (j_1 - 1) \cdots (j_q - 1)$. Similarly, from the discussion in section 3.5, we have that the matrix of $\boldsymbol{\Delta}_{\{\ell+1,J\}}(\mathbf{t})$ is $\ell \cdot d_J \times d_J$ with $d_J \times d_J$ blocks $(-1)^q\big(\boldsymbol{\rho}_J(\gamma_{m,\ell+1})(\mathbf{t}) - \mathbb{I}_{d_J}\big)$. Comparing the formulas obtained in Theorem 1.12 and Theorem 3.6, we see that $\pi_{m,\ell+1} \circ \Psi^q_J(\boldsymbol{\lambda}) = \big[\boldsymbol{\rho}_J(\gamma_{m,\ell+1})_*(\boldsymbol{\lambda})\big]^\top$. It follows readily that

$$\Psi^q_J(\boldsymbol{\lambda}) = (-1)^q\big[\big(\boldsymbol{\Delta}_{\{\ell+1,J\}}\big)_*(\boldsymbol{\lambda})\big]^\top,$$

completing the proof.

§4. Cohomology Support Loci and Resonant Varieties

In an immediate application of Theorem 3.1, we establish the relationship between the cohomology support loci of the complement of the discriminantal arrangement $\mathcal{A}_{n,\ell}$ and the resonant varieties of its Orlik-Solomon algebra.

Recall that each point $\mathbf{t} \in (\mathbb{C}^*)^N$ gives rise to a local system $\mathcal{L} = \mathcal{L}_\mathbf{t}$ on the complement $M_{n,\ell}$ of the arrangement $\mathcal{A}_{n,\ell}$. For sufficiently generic $\mathbf{t}$, the cohomology $H^k(M_{n,\ell}, \mathcal{L}_\mathbf{t})$ vanishes (for $k < n - \ell$), see for instance [21, 10]. Those $\mathbf{t}$ for which $H^k(M_{n,\ell}; \mathcal{L}_\mathbf{t})$ does not vanish comprise the cohomology support loci

$$\Sigma^k_m(M_{n,\ell}) = \{\mathbf{t} \in (\mathbb{C}^*)^N \mid \dim H^k(M_{n,\ell}; \mathcal{L}_\mathbf{t}) \geq m\}.$$

These loci are algebraic subvarieties of $(\mathbb{C}^*)^N$, which, since $M_{n,\ell}$ is a $K(P_{n,\ell}, 1)$-space, are invariants of the group $P_{n,\ell}$.

Similarly, each point $\boldsymbol{\lambda} \in \mathbb{C}^N$ gives rise to an element $\omega = \omega_{\boldsymbol{\lambda}} \in A^1$ of the Orlik-Solomon algebra of the arrangement $\mathcal{A}_{n,\ell}$. For sufficiently generic $\boldsymbol{\lambda}$, the cohomology $H^k(A^\bullet, \mu^\bullet(\boldsymbol{\lambda}))$ vanishes (for $k < n - \ell$), see [31, 16]. Those $\boldsymbol{\lambda}$ for which $H^k(A^\bullet, \mu^\bullet(\boldsymbol{\lambda}))$ does not vanish comprise the resonant varieties

$$\mathcal{R}^m_k(A) = \{\boldsymbol{\lambda} \in \mathbb{C}^N \mid \dim H^k(A^\bullet, \mu^\bullet(\boldsymbol{\lambda})) \geq m\}.$$

These subvarieties of $\mathbb{C}^N$ are invariants of the Orlik-Solomon algebra A of $\mathcal{A}_{n,\ell}$.

Recall that $\mathbf{1} = (1, \dots, 1)$ denotes the identity element of $(\mathbb{C}^*)^N$.

Theorem 4.1. *Let $\mathcal{A}_{n,\ell}$ be a discriminantal arrangement with complement $M_{n,\ell}$ and Orlik-Solomon algebra A. Then for each k and each m, the resonant variety $\mathcal{R}_k^m(A)$ coincides with the tangent cone of the cohomology support locus $\Sigma_m^k(M_{n,\ell})$ at the point $\mathbf{1}$.*

Proof. For each $\mathbf{t} \in (\mathbb{C}^*)^N$, the cohomology of $M_{n,\ell}$ with coefficients in the local system $\mathcal{L}_{\mathbf{t}}$ is isomorphic to that of the complex $(\mathbf{C}^\bullet, \boldsymbol{\delta}^\bullet(\mathbf{t}))$. So $\mathbf{t} \in \Sigma_m^k(M_{n,\ell})$ if and only if $\dim H^k(\mathbf{C}^\bullet, \boldsymbol{\delta}^\bullet(\mathbf{t}) \geq m$. An exercise in linear algebra shows that

$$\Sigma_m^k(M_{n,\ell}) = \{\mathbf{t} \in (\mathbb{C}^*)^N \mid \operatorname{rank} \boldsymbol{\delta}^{k-1}(\mathbf{t}) + \operatorname{rank} \boldsymbol{\delta}^k(\mathbf{t}) \leq \dim \mathbf{C}^k - m\}.$$

For $\boldsymbol{\lambda} \in \mathbb{C}^N$, we have $\boldsymbol{\lambda} \in \mathcal{R}_k^m$ if $\dim H^k(A^\bullet, \mu^\bullet(\boldsymbol{\lambda})) \geq m$. So, as above,

$$\mathcal{R}_k^m(A) = \{\boldsymbol{\lambda} \in \mathbb{C}^N \mid \operatorname{rank} \mu^{k-1}(\boldsymbol{\lambda}) + \operatorname{rank} \mu^k(\boldsymbol{\lambda}) \leq \dim A^k - m\}.$$

By Theorem 3.1, $\dim A^k = \dim \mathbf{C}^k$ and $\mu^k(\boldsymbol{\lambda}) = \boldsymbol{\delta}_*^k(\boldsymbol{\lambda})$ for each k. Thus,

$$\mathcal{R}_k^m(A) = \{\boldsymbol{\lambda} \in \mathbb{C}^N \mid \operatorname{rank} \boldsymbol{\delta}_*^{k-1}(\boldsymbol{\lambda}) + \operatorname{rank} \boldsymbol{\delta}_*^k(\boldsymbol{\lambda}) \leq \dim \mathbf{C}^k - m\},$$

and the result follows. Q.E.D.

The cohomology support loci are known to be unions of torsion-translated subtori of $(\mathbb{C}^*)^N$, see [3]. In particular, all irreducible components of $\Sigma_m^k(M_{n,\ell})$ passing through $\mathbf{1}$ are subtori of $(\mathbb{C}^*)^N$. Consequently, all irreducible components of the tangent cone are linear subspaces of $\mathbb{C}^N$. So we have the following.

Corollary 4.2. *For each k and each m, the resonant variety $\mathcal{R}_k^m(A)$ is the union of an arrangement of subspaces in $\mathbb{C}^N$.*

Remark 4.3. For $k = 1$, Theorem 4.1 and Corollary 4.2 hold for any arrangement $\mathcal{A}$, see [11, 22, 23]. In particular, as conjectured by Falk [16, Conjecture 4.7], the resonant variety $\mathcal{R}_1^m(A(\mathcal{A}))$ is the union of a subspace arrangement. Thus, Corollary 4.2 above may be viewed as resolving positively a strong form of this conjecture in the case where $\mathcal{A} = \mathcal{A}_{n,\ell}$ is a discriminantal arrangement.

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On the number of Bounding Cycles for Nonlinear Arrangements

James Damon*

To Peter Orlik on his sixtieth birthday

For a real hyperplane arrangement $A \subset \mathbb{R}^n$, among the first invariants that were determined for A were the number of chambers in the complement $\mathbb{R}^n \backslash A$ by Zavslavsky [Za] and the number of bounded chambers by Crapo [Cr]. In the consideration of certain classes of hypergeometric functions, there also arise arrangements of hypersurfaces which need not be hyperplanes (see e.g. Aomoto [Ao]). In this paper we will obtain a formula for the number of bounded regions (i.e. chambers) in the complement of a nonlinear arrangement of hypersurfaces. For example, for the general position arrangements of quadrics in Figure 1, we see the number of bounded regions in the complement are respectively 1, 5, and 13.

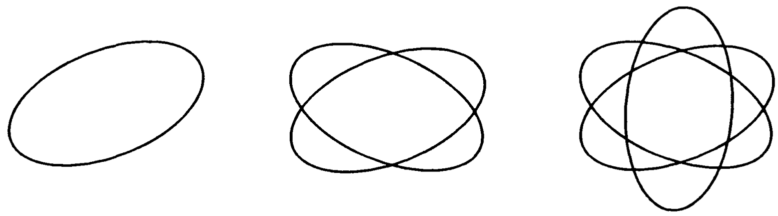

Figure 1

A computation of the number of bounded regions in the complement depends on the degrees of the hypersurfaces as well as the combinatorial structure of the arrangement. Hence, the form such a formula should take is less obvious, even given the answer for hyperplane arrangements. Moreover, in the real case for hypersurfaces of degree > 1 there is the added complication that the number depends upon the specific hypersurfaces (another choice of real quadrics could have fewer real intersections).

In the case of real hyperplane arrangements, the number of bounded regions in the complement represents an intrinsic invariant for the associated complex arrangements. Each bounded region has a bounding

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cycle, and these cycles represent the nontrivial cycles in the associated complex hyperplane arrangement. For arrangements of hypersurfaces of degree > 2, there is the added complication that the number of bounded regions does not accurately count the number of "bounding cycles" for the complexification. For example, the arrangement in figure 2 of a quadric and elliptic curve has a maximum of 6 bounded regions while by [D1, §6], the number of "bounding cycles" for the complexification is 8. In fact, it is the number of "bounding cycles" for arrangements of smooth complex hypersurfaces $A \subset \mathbb{C}^n$ which is intrinsic and we shall refer to these cycles as the "bounding cycles".

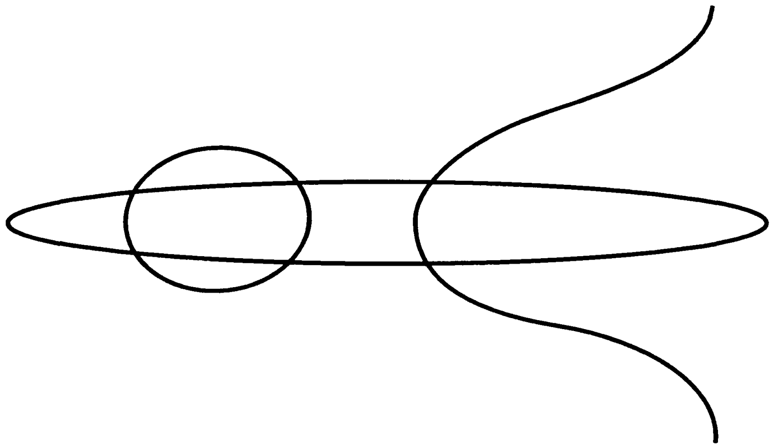

Figure 2

In order to obtain a formula for the number of bounding cycles, we are led to consider more generally a nonlinear arrangement of hypersurfaces A on a smooth complete intersection $X \subset \mathbb{C}^n$, and consider the corresponding number of "relative bounding cycles" for (X, A). Moreover, we consider nonlinear arrangements which are the analogues of arbitrary hyperplane arrangements rather than just general position arrangements. Then, in Theorem 1, we shall give a general formula (5.13) for the number of "relative bounding cycles" for a nonlinear arrangement A of q-tic hypersurfaces on a smooth complete intersection X of multidegree $\boldsymbol{d}$, where the nonlinear arrangement is generally based on any central hyperplane arrangement $\mathcal{A}$. The formula is valid for general nonlinear arrangements provided that the arrangement and complete intersection are "nondegenerate at infinity" (see §1). It has the form

$$d \cdot \left(\sum_{j=0}^{r} \alpha_j \cdot q^{r-j} \mu_{r-j}(\mathcal{A}) \right)$$

Here, d is the product of the multidegrees and the coefficient α_j involves s_j, the j–th complementary function to the elementary symmetric functions, applied to the multidegree $\boldsymbol{d}$ and common arrangement degree q. Also, $\mu_{r-j}(\mathcal{A})$ are the higher multiplicities of the arrangement $\mathcal{A}$ [D2, §4]. The higher multiplicities are certain intrinsic geometric invariants of the central arrangement, which are the analogues of those originally introduced by Teissier [Te] for hypersurfaces (see §2).

Several key ideas play crucial roles in obtaining such a formula. In §1 we reduce computation of global invariants to local invariants of an appropriate mapping (§1). Then, the number of "relative bounding cycles" turns out to be a "relative singular Milnor number" which, in turn, is sum of the usual Milnor number for an isolated complete intersection singularity and the "singular Milnor number" for the intersection of the arrangement and the complete intersection. In the special case that such a nonlinear arrangement is based on a free arrangement, formulas were given in [D1] for the special case where $X = \mathbb{C}^n$ or X is a smooth complete intersection but the arrangement consists of hyperplanes. We recall these formulas in §5. To generalize these formulas for all central arrangements rather than just free arrangements, we introduce a version of "nonlinear deletion–restriction" (§3). The version we give does not yield analogues of the complete results obtained by Orlik–Terao [OT] for the topology of complements of hyperplane arrangements; however, it suffices for counting the number of bounding cycles. It leads to functional equations (4.1) which such a formula for the number of bounding cycles must satisfy (§4). The form of the solution to these functional equations is obtained in terms of the higher multiplicities of the central arrangement and the multidegree of the complete intersection X. It is obtained by expressing the formulas valid for free central arrangements in terms of higher multiplicities (§5). The proof that the formula satisfies the functional equations is given in §6. As a consequence, it follows that the formulas which originally were obtained for the special case of nonlinear arrangements based on free arrangements, when reexpressed in terms of higher multiplicities, are seen to hold for all nonlinear arrangements.

In all that follows, we shall use standard notation and terminology for hyperplane arrangements as given in [OT], especially chapters 1 and 2.

Contents

(5) §5 Formula for the Number of Relative Bounding Cycles
(6) §6 Proof of the General Formula

§1. Nonlinear Arrangements Nondegenerate at ∞

To define a nonlinear arrangement exhibiting intersection properties of a linear arrangement, we begin by defining a nonlinear arrangement based on a central hyperplane arrangement. We modify the approach to nonlinear arrangements given in [D2]. Consider a smooth complete intersection $X \subset \mathbb{C}^n$ of dimension r defined by a polynomial map $g = (g_1, \dots, g_{n-r}) : \mathbb{C}^n \to \mathbb{C}^{n-r}$. We let $\deg(g_i) = d_i$ and refer to X as having multidegree $\boldsymbol{d} = (d_1, \dots, d_{n-r})$. Also, consider a central hyperplane arrangement $\mathcal{A}\,(= \cup H_i) \subset \mathbb{C}^p$ (for which all of the hyperplanes H_i contain 0). Let $\varphi : \mathbb{C}^n \to \mathbb{C}^p$ be a polynomial mapping.

Definition 1.1. A nonlinear (affine) arrangement of smooth hypersurfaces $A \subset X$ based on a central hyperplane arrangement $\mathcal{A} \subset \mathbb{C}^p$ is defined by $A = \varphi'^{-1}(\mathcal{A})$ where $\varphi : \mathbb{C}^n \to \mathbb{C}^p$ is a polynomial map and $\varphi' = \varphi|X$ is transverse to $\mathcal{A}$ (i.e. φ' is transverse to each flat of $\mathcal{A}$).

In the special case that $X = \mathbb{C}^n$, we obtain a nonlinear arrangement $A \subset \mathbb{C}^n$. If $A' \subset \mathbb{C}^n$ is a nonlinear arrangement and X is a smooth complete intersection which is transverse to A' (i.e. to the "nonlinear flats"of A'), then $A = X \cap A'$ is a nonlinear arrangement in X.

Example 1.2. If $A_n \subset \mathbb{C}^n$ denotes the Boolean arrangement of coordinate hyperplanes and $\varphi : \mathbb{C}^2 \to \mathbb{C}^n$ is a polynomial mapping whose coordinate functions are generic quadratic polynomials, let $A = \varphi^{-1}(A_n)$. Then A is a general position arrangement of quadrics as in figure 1 for $n = 1, 2, 3$.

Example 1.3. If in place of A_n in example 1.2, we consider the braid arrangement $B_3 \subset \mathbb{C}^3$ and let $\varphi : \mathbb{C}^2 \to \mathbb{C}^3$ be given by

$$\varphi(z_1, z_2) = (2z_1^2 + 6z_2^2 - 8, z_1^2 + 4z_2^2 - 5, 4z_1^2 + 7z_2^2 - 11)$$

then A consists of the three quadrics whose common intersection exhibits the triple intersection of the braid arrangement B_3. This nonlinear arrangement has 9 bounding cycles as we see in figure 3

Example 1.4. Again let $A_n \subset \mathbb{C}^n$ denote the Boolean arrangement of coordinate hyperplanes and let $\varphi : \mathbb{C} \to \mathbb{C}^n$ be a polynomial mapping whose coordinate functions are generic polynomials of degrees $(q_1, \dots, q_n)$. Then, set–theoretically, $A = \varphi^{-1}(A_n)$ is a collection of

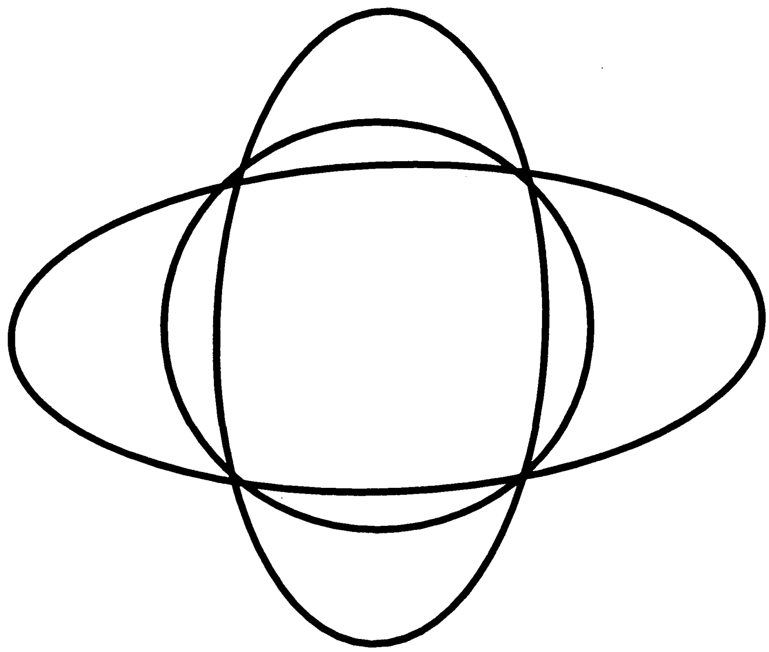

Figure 3

$q = \sum q_i$ points in $\mathbb{C}$, and appears identical to a general position arrangement of points in $\mathbb{C}$. However, A is a nonlinear general position arrangement of 0–dimensional varieties each consisting of q_i points. It is analogous to a colored braid arrangement where sets of q_i points share the same color and are indistinguishable.

Example 1.5. Lastly, we let $X \subset \mathbb{C}^3$ denote a quadric surface and we consider the nonlinear arrangement A on X obtained as the intersection $A = X \cap B_3$. A real version of this is given in figure 4. Although $H_1(A)$ is generated by 5 cycles, there are 6 relative bounding cycles corresponding to the 6 regions in the ellipsoid.

The conditions we have given for nonlinear arrangements are not in themselves sufficient to allow us to determine the number of bounding cycles. We must also control the behavior at infinity for both the complete intersection X and the nonlinear arrangement A.

(1.6) We do this by viewing both as the intersection of singular complete intersections and nonlinear arrangements in $\mathbb{C}^{n+1}$ with the affine space $\mathbb{C}^n \times \{1\}$. We extend both polynomial mappings g and φ by homogenization. However, we view the homogenized maps as germs at 0. For example, from $g = (g_1, \ldots, g_{n-r}) : \mathbb{C}^n \to \mathbb{C}^{n-r}$ defining X, we define $G = (G_1, \ldots, G_{n-r}) : \mathbb{C}^{n+1}, 0 \to \mathbb{C}^{n-r}, 0$ where

$$G_i(z_1, \ldots, z_{n+1}) = (z_{n+1})^{d_i} \cdot g_i(z_1/z_{n+1}, \ldots, z_n/z_{n+1})$$

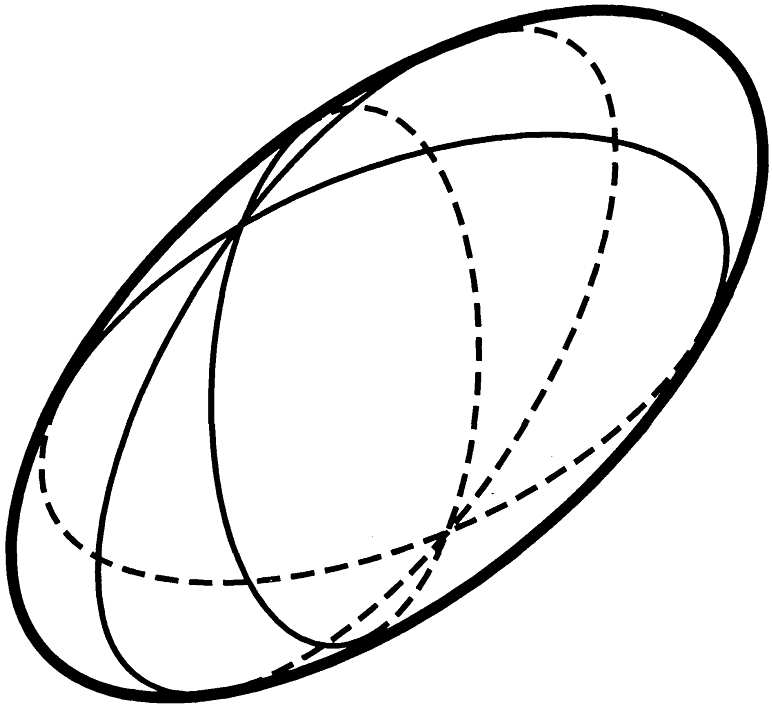

Figure 4

with $d_i = \deg(g_i)$. We similarly define $\Phi : \mathbb{C}^{n+1}, 0 \to \mathbb{C}^p, 0$ from φ. We let $\mathcal{X} = G^{-1}(0)$, $\Phi' = \Phi|\mathcal{X}$, and $\tilde{\mathcal{A}} = \Phi'^{-1}(\mathcal{A})$.

The properties of X and A at infinity are given by the properties of $\mathcal{X}$ and $\tilde{\mathcal{A}}$.

Definition 1.7. First, we say that X is smooth including ∞ if both G and $G|\mathbb{C}^n$ define isolated complete intersection singularities (ICIS) $\mathcal{X} = G^{-1}(0)$ and $X_0 = \mathcal{X} \cap \mathbb{C}^n$ (the conditions imply that $\mathcal{X}$ and $\mathbb{C}^n$ are transverse off 0). Second, we say that the nonlinear arrangement $A \subset X$ is nondegenerate at ∞ if both $\Phi' = \Phi|\mathcal{X}$ and $\Phi|X_0$ are transverse to $\mathcal{A}$ in a punctured neighborhood of 0.

In figure 5, We observe the relation between $\mathcal{X}$, A, etc.

The results that we obtain will apply to a nonlinear arrangement $A \subset X$ which is nondegenerate at ∞ and X is smooth including ∞. Observe that if $\tilde{\mathcal{A}} = \Phi'^{-1}(\mathcal{A})$, $\mathcal{X}$, and $\mathbb{C}^n$ are in general position off 0, then $A' = A \cap X \subset X$ is a nonlinear arrangement which is nondegenerate at ∞ (this was the definition used in [D2]).

We should mention that we add the descriptive term "affine" in the referring to a nonlinear arrangement to distinguish from the case of a central nonlinear arrangement such as $\tilde{\mathcal{A}} = \Phi^{-1}(\mathcal{A})$ where the defining map Φ is only required to be transverse to $\mathcal{A}$ off 0 (see [D1, §6]).

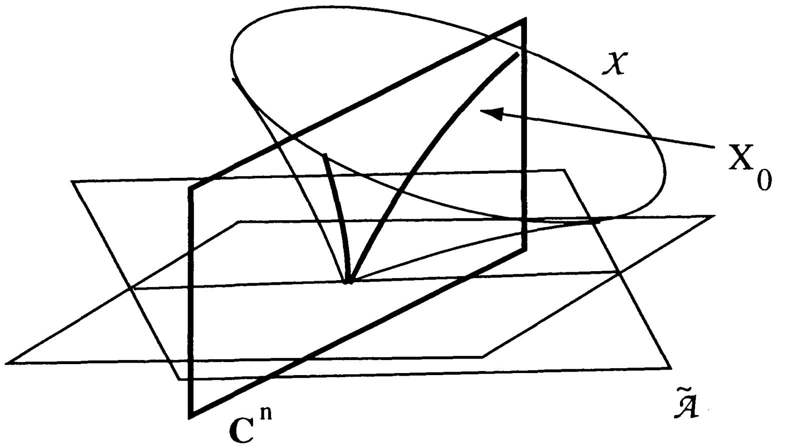

Figure 5

In order to apply deletion–restriction later to nonlinear arrangements, we shall make use of the following lemma.

Lemma 1.8. *Suppose $X \subset \mathbb{C}^n$ is smooth including ∞, and $A \subset X$ is a nonlinear arrangement nondegenerate at ∞ (defined as $\varphi^{-1}(\mathcal{A})$). Let K be a flat of $\mathcal{A}$ and $\mathcal{A}^K$ denote the restriction of $\mathcal{A}$ to K. Then, $X_K = \varphi^{-1}(K)$ is smooth including ∞, and $A^K = \varphi^{-1}(\mathcal{A}^K) \subset X_K$ is a nonlinear arrangement nondegenerate at ∞.*

Proof. By assumption, both $\Phi' : \mathcal{X} \to \mathbb{C}^p$ and $\Phi'|X_0$ are transverse to the central hyperplane arrangement $\mathcal{A} \subset \mathbb{C}^p$ in a punctured neighborhood of 0. In particular, they are transverse to K in a punctured neighborhood of 0. Then, by a straighforward fiber square argument, $\mathcal{X}_K = \Phi'^{-1}(K)$ and $X_{0K} = \Phi'^{-1}(K) \cap X_0$ are ICIS, and both $\Phi'|\mathcal{X}_K : \mathcal{X}_K \to K$ and $\Phi'|X_{0K} : X_{0K} \to K$ are transverse to $\mathcal{A}^K$ in a punctured neighborhood of 0. Hence, $A^K \subset X_K$ is a nonlinear arrangement nondegenerate at ∞. Q.E.D.

Reduction from Global to Local Properties:

Suppose $A \subset X$ is a nonlinear affine arrangement nondegenerate at ∞ and that X is smooth including ∞. We constuct as in (1.6) the associated homogeneous objects $\mathcal{X}$, etc. There is a basic relation between the local properties of the homogeneous objects and the corresponding affine ones. First, both $\mathcal{X}$ and $\tilde{A}$ are transverse to $\mathbb{C}^n$ off 0. Second, since $\mathbb{C}^n$ is transverse to $\mathcal{X}$ off 0, it follows that $\mathbb{C}^n \times \{t\}$ is transverse to $\mathcal{X}$ and $X = \mathcal{X} \cap (\mathbb{C}^n \times \{1\})$ is the smooth complete intersection in $\mathbb{C}^n \times \{1\} \simeq \mathbb{C}^n$. We let $X_t = \mathcal{X} \cap (\mathbb{C}^n \times \{t\})$. Likewise, $A = \tilde{A} \cap (\mathbb{C}^n \times \{1\})$ is the nonlinear affine arrangement in X. Third, $\varphi = \Phi|\mathbb{C}^n \times \{1\}$.

Then, by a theorem of Hamm [Ha], for a sufficiently small ball B_ϵ and $0 < |t| \ll \epsilon$, $X_t \cap B_\epsilon$ is the Milnor fiber of the ICIS X_0 and is homotopy equivalent to a bouquet of spheres of (real) dimension r. The number of such spheres is the Milnor number, which we denote by $\mu(X_0)$.

Also, by [D1, §7] (and see [D3]), for ϵ sufficiently small and $0 < |t| \ll \epsilon$, $X_t \cap \tilde{\mathcal{A}} \cap B_\epsilon$ is the singular Milnor fiber of the nonisolated complete intersection singularity $A_0, 0$ viewed as a nonlinear section of $\{0\} \times \mathcal{A}, 0 \subset \mathbb{C}^{n-r+1+p}, 0$. Using a theorem of Lê [Lê1], it is also proven that $X_t \cap \tilde{\mathcal{A}} \cap B_\epsilon$ is homotopy equivalent to a bouquet of spheres of (real) dimension $r-1$. Then, $X_t \cap \tilde{\mathcal{A}} \cap B_\epsilon$ is called the *singular Milnor fiber* of A_0, and the number of such spheres is the singular Milnor number, denoted by $\mu(A_0)$ ([DM, §4], [D1, §7] and [D3]). Strictly speaking, these (singular) Milnor fibers are associated to the mappings $g' = (g, z_{n+1})$ and (g', Φ) defining X_0 and A_0 as an ICIS or a nonlinear section of a complete intersection; however, by [D1, §6, 7] the description given here is consistent with the definition for the mappings.

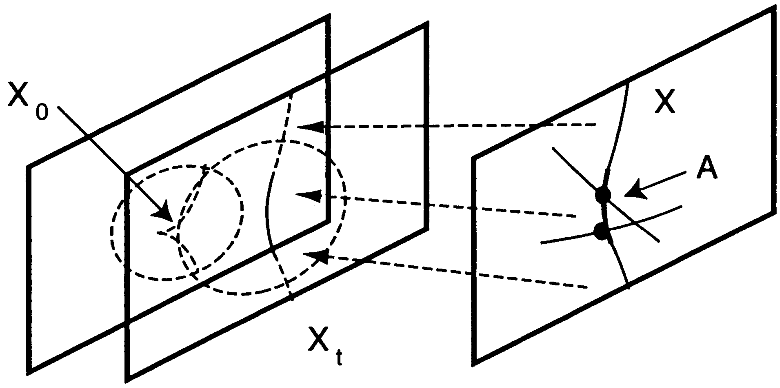

Figure 6

The global affine spaces X and A are related to the (singular) Milnor fibers by the following result, which is a slight variant of Prop. 2.5 in [D2], but whose proof is virtually identical.

Proposition 1.9. *Suppose $A \subset X$ is a nonlinear (affine) arrangement nondegenerate at ∞ (with X smooth including ∞). Then, X is homeomorphic to the Milnor fiber $X_t \cap B_\epsilon$ of X_0 via a homeomorphism ψ which can be chosen to send A to $X_t \cap \tilde{\mathcal{A}} \cap B_\epsilon$ (see fig. 6).*

Number of Bounding Cycles as Relative Singular Milnor Numbers:

In light of the preceding discussion of the relation between nonlinear arrangements and singular Milnor fibers, we define the number of bounding cycles in terms of (relative) homology of the nonlinear arrangement.

Definition 1.10. If the nonlinear arrangement $A \subset \mathbb{C}^n$ is nondegenerate at ∞, then the *number of bounding cycles* is defined to be $\dim H_{n-1}(A)$. If $A \subset X$ is a nonlinear (affine) arrangement nondegenerate at ∞ (with X smooth including ∞), then the number of *relative bounding cycles* is defined to be $\dim H_r(X, A)$.

We observe that if $X = \mathbb{C}^n$, then the number of relative bounding cycles for A is the same as the number of bounding cycles. Because the relative Milnor fiber $(X_t \cap B_\epsilon, X_t \cap \tilde{\mathcal{A}} \cap B_\epsilon)$ is a relative CW–complex of dimension r, by the homotopy properties of (singular) Milnor fibers we see that

$$\begin{aligned}\dim H_r(X_t \cap B_\epsilon, X_t \cap \tilde{\mathcal{A}} \cap B_\epsilon)\\ = \dim H_r(X_t \cap B_\epsilon) + \dim H_{r-1}(X_t \cap \tilde{\mathcal{A}} \cap B_\epsilon)\end{aligned}$$

By proposition 1.9 and the exact sequence of a pair, this implies for the affine spaces that $H_k(X, A)$ is only nonzero when $k = r$ and

$$\dim H_r(X, A) = \dim H_r(X) + \dim H_{r-1}(A) \tag{1.11}$$

We shall refer to $\dim H_r(X_t \cap B_\epsilon, X_t \cap \tilde{\mathcal{A}} \cap B_\epsilon)$ as the relative singular Milnor number of A_0. Via proposition 1.9 and 1.11, we can summarize the discussion by

(1.12) the number of relative bounding cycles for $(X, A) =$
the relative singular Milnor number of (X_0, A_0)

Remark 1.13. The singular Milnor numbers can be explicitly computed in the case that $\mathcal{A}$ is a free arrangement [D1, §6], then A_0 is called an *almost free arrangement* and $\mu(A_0)$ can be computed as the length of a determinantal module, see [DM, thms 5, 6] and [D1, §4]. This is further extended in [D1, §7, 8] to *almost free complete intersections*, intersections of almost free divisors which are the transverse off 0. This includes nonlinear arrangements such as A_0.

It was the formulas for almost free nonlinear arrangements A_0 which suggested the existence and form for a general formula given in §5.

§2. Higher Multiplicities of Central Arrangements

A general formula for the number of bounding cycles must be expressed in terms on intrinsic invariants of arrangements. We recall just such a set of intrinsic geometric invariants of central arrangements, viewed as nonisolated singularities. These are the higher multiplicities.

For the case of isolated hypersurface singularities, Teissier [Te] introduced a series of higher multiplicities, namely the μ_i appearing in his μ^*–sequence $\mu^* = (\mu_0, \dots, \mu_n)$. Specifically, given $f_0 : \mathbb{C}^n, 0 \to \mathbb{C}, 0$, if Π is a generic s–dimensional subspace in $\mathbb{C}^n$ then $f_0|\Pi$ has an isolated singularity and Teissier defines $\mu_s(f_0) = \mu(f_0|\Pi)$, where $\mu(\cdot)$ denotes the usual Milnor number. This was extended to arbitrary singularities $(V, 0)$ by Lê and Teissier [LêT] considering instead generic projections $V, 0 \to \Pi, 0$ for linear subspaces Π of varying dimensions. They consider the Euler characteristics of Milnor fibers of such projections.

In [D1, §4], we considered higher multiplicities for nonisolated complete intersection singularities $V, 0 \subset \mathbb{C}^n, 0$ using the analogue of Teissier's original definition. A Zariski open subset of s-dimensional subspaces $\Pi \subset \mathbb{C}^n$ are (geometrically) transverse to V off 0. We view the inclusion $i : \Pi \to \mathbb{C}^n$ as a section of V. For simplicity we assume $P = \mathbb{C}^s$ so that we have a map germ $i : \mathbb{C}^s, 0 \to \mathbb{C}^n, 0$ which is then transverse to V off 0. By the s–th higher multiplicity $\mu_s(V)$ we mean the singular Milnor number of the generic nonlinear section i (and $\mu_0(V) \overset{\text{def}}{=} 1$). For a central arrangement $\mathcal{A} \subset \mathbb{C}^p$, if i_t is a perturbation of i which is transverse to $\mathcal{A}$, then the affine arrangement $i_t(\mathbb{C}^s) \cap \mathcal{A}$ is homotopy equivalent to a bouquet of $s-1$–spheres and $\mu_s(\mathcal{A})$ counts the number of such spheres.

We summarize the main properties on the higher multiplicities of central hyperplane arrangements.

Proposition 2.1. *Suppose that $\mathcal{A} \subset \mathbb{C}^p$ is a central arrangement.*

(1) *If $r = r(\mathcal{A})$ is the rank of $\mathcal{A} = \cup H_i$ $(= \operatorname{codim}(\cap H_i))$, then $\mu_k(\mathcal{A}) = 0$ if $k \geq r$ and*

$$\mu_{r-1}(\mathcal{A}) = |\mu_{\text{Möb}}(\mathcal{A})|$$

($\mu_{\text{Möb}}(\mathcal{A})$ *denotes the Möbius function of the lattice* $L(\mathcal{A})$, [OT, Chap. 2]);

(2) *If $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$ is a deletion–restriction triple, then*

$$\mu_k(\mathcal{A}) = \mu_k(\mathcal{A}') + \mu_{k-1}(\mathcal{A}'')$$

(3) *The k-th Betti number of the complement $M(\mathcal{A}) = \mathbb{C}^p \backslash \mathcal{A}$ is given by*

$$b_k(M(\mathcal{A})) = \mu_k(\mathcal{A}) + \mu_{k-1}(\mathcal{A})$$

(in particular, $b_r(M(\mathcal{A})) = \mu_{r-1}(\mathcal{A})$ as $\mu_r(\mathcal{A}) = 0$).

(4) *If $\mathcal{A}$ is a free arrangement, then*

$$\mu_k(\mathcal{A}) = \sigma_k(\exp'(\mathcal{A}))$$

If $\exp(\mathcal{A}) = (e_0, e_1, \ldots, e_{p-1})$ *with* $e_0 = 1$*, then* $\exp'(\mathcal{A}) = (e_1, \ldots, e_{p-1})$ *and* $\sigma_k(\cdot)$ *denotes the* k*-th elementary symmetric function.*

(5) *If* $\mathcal{A}$ *is the complexification of a real, essential arrangement* $\mathcal{A}_{\mathbb{R}}$*, then the* β *invariant of Crapo is given by*

$$\beta(\mathcal{A}) = \mu_{p-1}(\mathcal{A})$$

Proof. The proofs of these results essentially follow from [D1, §5]. 1), 3), and 4) are respectively Proposition 5.6, Lemma 5.6, and Proposition 5.2 of [D1]. 5) follows from 1) and the equality of $\beta(\mathcal{A})$ and $\mu_{\text{Möb}}(\mathcal{A})$. Lastly, for 2) if Π is a generic k-dimensional subspace, then $(\mathcal{A} \cap \Pi, \mathcal{A}' \cap \Pi, \mathcal{A}'' \cap \Pi)$ is still a deletion–restriction triple. Then, if $k + 1 = \dim(\Pi) > r(\mathcal{A})$ then $\mu_k(\mathcal{A})$, $\mu_k(\mathcal{A}')$, and $\mu_{k-1}(\mathcal{A}'')$ are zero. If $k < r(\mathcal{A})$, then by 1) these multiplicities are $|\mu_{\text{Möb}}(\mathcal{A} \cap \Pi)|$, $|\mu_{\text{Möb}}(\mathcal{A}' \cap \Pi)|$, and $|\mu_{k-1}(\mathcal{A}'' \cap \Pi)|$. Then, the result follows from Corollary 2.3.12 of [OT] (if $H \cap \Pi$ is not a separator, then $r(\mathcal{A}' \cap \Pi) < k + 1$ so $\mu_k(\mathcal{A}') = 0$). Q.E.D.

We give several corollaries.

Corollary 2.2. *For a central hyperplane arrangement* $\mathcal{A} \subset \mathbb{C}^p$*, the Poincare polynomial of the complement* $M(\mathcal{A})$ *is given by*

$$P(\mathcal{A}, t) = (1 + t) \cdot \mu(\mathcal{A}, t) \qquad \textit{where} \quad \mu(\mathcal{A}, t) = \sum_{j=0}^{p-1} \mu_j(\mathcal{A}) t^j$$

is the "multiplicity polynomial" of $\mathcal{A}$*.*

We should note that in [OT], $P(\mathcal{A}, t)$ is equivalently given by $\pi(\mathcal{A}, t)$.

As a second corollary we obtain a formula for the multiplicities of a "product of arrangements" $\mathcal{A} \subset \mathbb{C}^{p_1}$ and $\mathcal{B} \subset \mathbb{C}^{p_2}$. This is the arrangement $(\mathcal{A} \times \mathbb{C}^{p_2}) \cup (\mathbb{C}^{p_1} \times \mathcal{B}) \subset \mathbb{C}^p$, where $p = p_1 + p_2$. This same construction appears in other situations in singularity theory where the term "product" is misleading, so we refer to it more generally as the "product union" of $\mathcal{A}$ and $\mathcal{B}$, and denote it by $\mathcal{A} \bowtie \mathcal{B}$ [D1, §3]. Let

$$\lambda_k(\mathcal{A}, \mathcal{B}) = \sum_{j=0}^{k} \mu_j(\mathcal{A}) \mu_{k-j}(\mathcal{B}).$$

Corollary 2.3. *For a central hyperplane arrangements* $\mathcal{A}$ *and* $\mathcal{B}$

$$\mu_k(\mathcal{A} \bowtie \mathcal{B}) = \lambda_k(\mathcal{A}, \mathcal{B}) + \lambda_{k-1}(\mathcal{A}, \mathcal{B}) \tag{2.4}$$

Proof. By [OT, Lemma 2.3.3]

$$P(\mathcal{A} \bowtie \mathcal{B}, t) = P(\mathcal{A}, t) \cdot P(\mathcal{B}, t)$$

Then, using Corollary 2.2 we conclude

$$\mu(\mathcal{A} \bowtie \mathcal{B}, t) = (1 + t)\mu(\mathcal{A}, t) \cdot \mu(\mathcal{B}, t) \tag{2.5}$$

Equating coefficients of t^k in (2.5) yields the corollary Q.E.D.

Remark 2.6. The expression in Corollary 2.3 does not obviously depend upon $\mathcal{A}$ nor $\mathcal{B}$ being arrangements. This suggests the conjecture that Formula 2.4 is valid for arbitrary germs of hypersurfaces $\mathcal{A}$ and $\mathcal{B}$.

§3. Nonlinear Deletion–Restriction

We consider a central arrangement $\mathcal{A} \subset \mathbb{C}^p$ with H a hyperplane in $\mathcal{A}$. Let $(H \subset \mathcal{A}, \mathcal{A}', \mathcal{A}'')$ be a deletion–restriction triple for the hyperplane H [OT, def. 1.2.14]. Recall it consists of arrangements $\mathcal{A}' = \{H' \in \mathcal{A} : H' \neq H\}$, and $\mathcal{A}'' = \mathcal{A}^H$ $(= H \cap (\cup H')$, where the union is over $H' \in \mathcal{A}')$.

Definition 3.1. Let $A \subset X$ be a nonlinear arrangement nondegenerate at ∞ defined as $\varphi'^{-1}(\mathcal{A})$ for $\varphi' = \varphi | X$ (with X smooth including ∞). The associated *nonlinear deletion–restriction triple* $(H' \subset A, A', A'')$ consists of the smooth hypersurface $H' = X_H$, and the nonlinear arrangements A, $A' = \varphi'^{-1}(\mathcal{A}')$, and $A'' = \varphi'^{-1}(\mathcal{A}'')$.

By proposition 1.8, H' is smooth including ∞ and both $A' \subset X$ and $A'' \subset H'$ are nondegenerate at ∞.

If $\chi(Y)$ denotes the Euler characteristic of Y, then nonlinear deletion--restriction takes the following simple form.

Proposition 3.2 (Nonlinear Deletion-Restriction). *For the nonlinear deletion–restriction triple $(H' \subset A, A', A'')$,*

$$\chi(X, A) = \chi(X, A') - \chi(H', A'')$$

Proof. This is a simple application of Euler characteristic arguments for exact sequences. As $A'' = A' \cap H'$ and $A = A' \cup H'$, Mayer–Vietoris implies

$$\chi(A) = \chi(H') + \chi(A') - \chi(A'') \tag{3.3}$$

Subtracting both sides of (3.3) from $\chi(X)$, and using that for $Y \subset X$, $\chi(X, Y) = \chi(X) - \chi(Y)$, we obtain the conclusion. Q.E.D.

Normally, Proposition 3.2 would not yield strong conclusions. However, in our case all of the arrangements are nondegenerate at ∞. We recall by proposition 1.9 and the discussion following that X is homeomorphic to the Milnor fiber of an ICIS of dimension r, and A is homeomorphic to the singular Milnor fiber for a nonlinear section of a (nonisolated) complete intersection. Then, (X, A) is a relative CW–complex of dimension r, and X and A are homotopy equivalent to bouquets of spheres of dimensions r, respectively $r-1$. Hence, by (1.11)

$$\begin{aligned} \chi(X,A) &= (-1)^r \dim H_r(X,A) \\ &= (-1)^r (\dim H_r(X) + \dim H_{r-1}(A)) \\ &= (-1)^r(\mu(X_0) + \mu(A_0)) \end{aligned} \tag{3.4}$$

(here $\mu(X_0)$ and $\mu(A_0)$ denote the (singular) Milnor numbers of the (singular) Milnor fibers).

Thus, nonlinear deletion–restriction (3.2) takes the form

$$\dim H_r(X,A) = \dim H_r(X,A') + \dim H_{r-1}(H',A'') \tag{3.5}$$

Our next goal is to find a formula for $\dim H_r(X, A)$ which satisfies (3.5).

Remark. Even if we only wanted a formula for the number of bounding cycles for a nonlinear arrangement in $\mathbb{C}^n$, we see that deletion–restriction leads us to consider via (3.5) the relative number of bounding cycles of A'' on the nonlinear hypersurface H'. Hence, it is really necessary to establish a general result of the form we obtain.

§4. Functional Equations for the Number of Bounding Cycles

The formula for the number of relative bounding cycles is a formula for $H^r(X, A)$ which must satisfy nonlinear deletion–restriction in the form (3.5). At this point *we restrict to the case where all of the coordinate functions of Φ are homogeneous of the same degree q* so that A is a nonlinear arrangement of q–tic hypersurfaces, as will be any nonlinear arrangement obtained by deletion or restriction. In the special case when $\mathcal{A}$ consists of a single hyperplane H, we also have A is the Milnor fiber of the homogeneous ICIS $\mathcal{X} \cap \Phi^{-1}(H) \cap \mathbb{C}^n$ which has multidegree $\boldsymbol{d} = (d_1, \ldots, d_{n-r}, q)$. Thus, as in (3.4) $\dim H_r(X, A)$ is the sum of two Milnor numbers of homogeneous ICIS, and hence is given by the formulas of Greuel–Hamm [GH] and Giusti [Gi] which only involve the multidegree. We denote these formulas for Milnor numbers by $\mu(\boldsymbol{d})$ and $\mu(\boldsymbol{d}, q)$.

Also, in the case when $\mathcal{A}$ is a free arrangement and Φ is homogeneous, we may apply the formula for the singular Milnor number $\mu(A)$ in [D1, Thm 2] together with the formula from [DM, Thm 5], together with proposition 2.3 of [D4] to conclude that for $\mathcal{A}$ fixed, $\mu(A)$ only depends on the multidegree $\boldsymbol{d} = (d_1, \dots, d_{n-r})$, and the degree q. Thus, we seek a formula for $\dim H_r(X, A)$ in the form $p(\boldsymbol{d}, q, \mathcal{A})$ which satisfies the equation (3.5) so that when $\mathcal{A}$ is a single hyperplane, it becomes the sum of the Milnor numbers for the pair of ICIS (X, A). These equations become the following functional equations.

(4.1) *Functional Equations for a Nonlinear Deletion–Restriction Triple*: $(H \subset \mathcal{A}, \mathcal{A}', \mathcal{A}'')$

(1)

$$p(\boldsymbol{d}, q, \mathcal{A}) = p(\boldsymbol{d}, q, \mathcal{A}') + p((\boldsymbol{d}, q), q, \mathcal{A}'')$$

(2)

$$p(\boldsymbol{d}, q, \{H\}) = \mu(\boldsymbol{d}) + \mu(\boldsymbol{d}, q)$$

Remark. To reduce excessive notation, it will be understood in the functional equations (4.1) that the ambient space for the nonlinear arrangements is $\mathbb{C}^n$.

Proposition 4.2. *Suppose $p(\boldsymbol{d}, q, \mathcal{A})$ satisfies the functional equations* (4.1) *for all deletion–restriction triples $(H \subset \mathcal{A}, \mathcal{A}', \mathcal{A}'')$. If X is a homogeneous hypersurface of multidegree $\mathbf{d}$, smooth including ∞ and $A \subset X$ is a nonlinear arrangement of smooth q–tic hypersurfaces nondegenerate at ∞, then*

$$\dim H_r(X, A) = p(\boldsymbol{d}, q, \mathcal{A})$$

Proof. This is proven by induction on the number $|\mathcal{A}|$ of hyperplanes in $\mathcal{A}$. For one hyperplane, it follows by (4.1-2). Then, by the induction hypothesis, if it holds for arrangements $\mathcal{A}'$ with $|\mathcal{A}'| < m$ and $|\mathcal{A}| = m$, then by (4.1-1) and (3.5) we obtain the result for $\mathcal{A}$. Q.E.D.

§5. Formula for the Number of Relative Bounding Cycles

To find a candidate for $p(\boldsymbol{d}, q, \mathcal{A})$, we examine special cases obtained in [D2] for the case that $\mathcal{A}$ is a free arrangement. The special cases compute the singular Milnor number of central nonlinear arrangements based on free arrangements. The (relative) singular Milnor number computes the the number of (relative) bounding cycles for the associated affine nonlinear arrangement.

First for hyperplane arrangements, we recall ([D1, §5] or [D3, §7]) that if $\Phi : \mathbb{C}^{n+1} \to \mathbb{C}^p$ and $\Phi|(\mathbb{C}^n \times \{0\})$ are linear embeddings transverse to $\mathcal{A}$ off 0, then $\tilde{\mathcal{A}} = \Phi^{-1}(\mathcal{A})$ is called an almost free arrangement and $A = \tilde{\mathcal{A}} \cap (\mathbb{C}^n \times \{1\})$ is called an *almost free affine arrangement* (based on $\mathcal{A}$). We also refer to A as being $\mathcal{A}$–generic. For example, if $\mathcal{A}$ is a Boolean arrangement, then an affine $\mathcal{A}$–generic arrangement is a general position arrangement.

The almost free affine arrangement A is the singular Milnor fiber of a generic hyperplane section of the almost free arrangement $\tilde{\mathcal{A}} = \Phi^{-1}(\mathcal{A})$. The singular Milnor number $\mu(\tilde{\mathcal{A}})$ also gives the higher multiplicity $\mu_n(\mathcal{A})$ [D1, §4,5]. Also, for an almost free arrangement A based on $\mathcal{A}$, $\mu_k(\mathcal{A}) = \sigma_k(\exp'(\mathcal{A}))$ where $\exp'(\mathcal{A}) = (e_1, \dots, e_{p-1})$ and $\sigma_k(\boldsymbol{x})$ denotes the k–th elementary symmetric function in $\boldsymbol{x} = (x_1, \dots, x_{p-1})$.

Notation. In the formulas that follow, in addition to the elementary symmetric functions $\sigma_k(\boldsymbol{x})$, we shall also need the collection of related functions $s_k(\boldsymbol{x})$. Here $s_k(\boldsymbol{x})$ is defined to be the polynomial defined as the sum of all monomials of degree k in $\boldsymbol{x}$ (here $s_0(\boldsymbol{x}) \stackrel{\text{def}}{=} 1$). These functions naturally complement the elementary symmetric functions, have analogous expansions as well as other properties listed in [D4, §2.]. In using these functions we will have occasion to evaluate $\sigma_k(\boldsymbol{x})$ where $x_j = a$ for say the last ℓ values of j. We indicate this by $\sigma_k(x_1, \dots, x_{p-\ell}, a^\ell)$. We may do this for several different a_i, and as well for the functions s_k.

Two special cases of the general formula we seek are given by the following.

Proposition 5.1 ([D1, prop. 6.12]). *Let A be an $\mathcal{A}$–generic affine nonlinear arrangement of hypersurfaces each of degree q (with $\mathcal{A}$ free). Then,*

(5.2) *the number of bounding cycles of* $A = \sigma_n((q-1)^n, qe_1, \dots, qe_{p-1})$

(*where again* $\exp'(\mathcal{A}) = (e_1, \dots, e_{p-1})$).

In [D1], we used a special form of nonlinear arrangement $A = X \cap A'$ where X is the transverse to the nonlinear arrangement A' including points at ∞ (see also [D2, def. 2.6]).

Proposition 5.3 ([D1, Theorem 8.19]). *Suppose that A' is an $\mathcal{A}$–generic affine hyperplane arrangement with $\mathcal{A}$ free. Let $A = X \cap A'$ where X is smooth of multidegree $\mathbf{d}$ including ∞ and transverse to A' including ∞. Then*

(5.4) *the number of relative bounding cycles of* (X, A)

$$= d \cdot \left(\sum_{j=0}^{r} s_j(\boldsymbol{d} - \mathbf{1}) \mu_{r-j}(\mathcal{A}) \right)$$

where $d = \prod_{i=1}^{n-r} d_i$ *and* $\boldsymbol{d} - \mathbf{1} = (d_1 - 1, \ldots, d_{n-r} - 1)$.

Comparing these two results we first notice that (5.4) is given in terms of the multidegree $\boldsymbol{d}$ and the higher multiplicities $\mu_j(\mathcal{A})$, while that (5.2) is not. Second, in the special case of a smooth hypersurface X, (5.4) can be reexpressed as follows.

Corollary 5.5. *Suppose A' is an $\mathcal{A}$–generic affine arrangement (with $\mathcal{A}$ free), and that X is an r–dimensional smooth hypersurface of degree $d + 1$ which is smooth and transverse to A' including ∞. Then, for $A = X \cap A'$*

(5.6) *the number of relative bounding cycles of* $(X, A) = d^n \cdot P(\mathcal{A}, d^{-1})$

Remark. It follows from results of Orlik-Terao [OT2] that when X is a homogeneous hypersurface, the relative Euler characteristic equals the RHS of (5.6) for *arbitrary arrangements* $\mathcal{A}$. This suggests that Proposition 5.3 should hold without the condition on $\mathcal{A}$. To compare (5.2) and (5.4) we first restate (5.2) in a form involving the higher multiplicities as follows.

Proposition 5.7. *Let A be an $\mathcal{A}$-generic affine nonlinear arrangement of hypersurfaces each of degree q (with $\mathcal{A}$ free). Then,*

(5.8) *the number of bounding cycles of* A

$$= \sum_{j=0}^{n} \binom{n}{j} (q-1)^j q^{n-j} \mu_{n-j}(\mathcal{A})$$

Proof. Using properties of elementary symmetric functions, we expand

(5.9) $\sigma_n((q-1)^n, qe_1, \ldots, qe_{p-1})$

$$= \sum_{j=0}^{n} \sigma_j((q-1)^n) \sigma_{n-j}(qe_1, \ldots, qe_{p-1})$$

Also,

$$\sigma_j((q-1)^n) = (q-1)^j \sigma_j(1^n) = (q-1)^j \binom{n}{j} \tag{5.10}$$

Similarly,

(5.11) $\sigma_{n-j}(qe_1, \ldots, qe_{p-1}) = q^{n-j}\sigma_{n-j}(e_1, \ldots, e_{p-1}) = q^{n-j}\mu_{n-j}(\mathcal{A})$

Here the last equality follows from (2.1-4) as $\mathcal{A}$ is a free arrangement. Substituting (5.10) and (5.11) into (5.9) gives the result. Q.E.D.

The formulas in Proposition 5.3 and 5.8 show a greater resemblance if we observe that by properties of the functions s_j (see [D1, §2]),

$$\binom{n}{j}(q-1)^j = s_j((q-1)^{n-j-1})$$

so that (5.8) can be written

$$\sum_{j=0}^{n} s_j((q-1)^{n-j-1})q^{n-j}\mu_{n-j}(A') \tag{5.12}$$

Then, the form of (5.4) and (5.12) suggest the following candidate as a general formula.

$$p(\boldsymbol{d}, q, \mathcal{A}) = d \cdot \left(\sum_{j=0}^{r} s_j(\boldsymbol{d}-\boldsymbol{1}, (q-1)^{r-j+1}) \cdot q^{r-j}\mu_{r-j}(\mathcal{A})\right) \tag{5.13}$$

where $d = \prod_{i=1}^{n-r} d_i$ and $\boldsymbol{d}-\boldsymbol{1} = (d_1 - 1, \ldots, d_{n-r} - 1)$.

We shall show that this is correct.

Theorem 1. *Suppose $X \subset \mathbb{C}^n$ is an r–dimensional complete intersection of multidegree $\boldsymbol{d}$, which is smooth including ∞. Let $A \subset X$ be a nonlinear arrangement of smooth q–tic hypersurfaces based on central arrangement $\mathcal{A}$ and nondegenerate at ∞. Then,*

the number of relative bounding cycles of (X, A)

$= p(\boldsymbol{d}, q, \mathcal{A})$ *given by* (5.13)

Because (5.4) and (5.8) are special cases of this result, we obtain as a corollary.

Corollary 2. *In the special cases of propositions* 5.3 *or* 5.7, *except that we allow $A \subset \mathbb{C}^n$ to be a nonlinear arrangement based on any central arrangement $\mathcal{A}$, then the number of bounding cycles of A, respectively the number relative bounding cycles of (X, A), is given by* (5.8), *respectively* (5.4).

Also, because the number of relative bounding cycles is also a relative singular Milnor number, we can also deduce as a corollary both the singular Milnor number and higher multiplicities.

Corollary 3. *Let X_0 be a homogeneous r-dimensional ICIS of multidegree $\mathbf{d}$. Also, let $A_0 = \Phi'^{-1}(\mathcal{A}) \subset X_0$ be a nonlinear central arrangement consisting of hypersurfaces of degree q, where $\Phi' : X_0 \to \mathbb{C}^p$ is transverse to $\mathcal{A}$ off 0. Then,*

(1)

$$\text{the singular Milnor number } \mu(A_0) = p(\mathbf{d}, q, \mathcal{A}) - \mu(X_0)$$

(2) *Likewise, the k-th higher multiplicity is given by*

$$\mu_k(A_0) = p((\mathbf{d}, 1^{n-k}), q, \mathcal{A}) - \mu_k(X_0)$$

Example 5.14. We return to the examples (1.2) and (1.3) of nonlinear arrangements of quadrics $A \subset \mathbb{C}^2$ based on a central arrangement $\mathcal{A}$. By Corollary 2,

$$\text{(5.15)} \qquad \begin{aligned} \text{number of bounding cycles of } A &= 1 + 4(\mu_1(\mathcal{A}) + \mu_2(\mathcal{A})) \\ &= 1 + 4b_2(\mathcal{A}) \end{aligned}$$

where $b_2(\mathcal{A})$ denotes the second Betti number of $M(\mathcal{A})$. For $\mathcal{A}$ the Boolean arrangement $A_p \subset \mathbb{C}^p$, $\mu_k(A_p) = \binom{p-1}{k}$ where $\binom{p-1}{k} = 0$ if $k > p - 1$. After simplifying (5.15), we obtain for general position arrangements of p quadrics in general position in $\mathbb{C}^2$, $1 + 4\binom{p}{2}$ bounding cycles. This yields the numbers 1, 5, and 13 for the first three cases in Fig. 1 (providing an alternate formula to Corollary 5.1).

For example (1.3), we have by (4) of proposition 2.1, for the braid arrangement $B_p \subset \mathbb{C}^p$, $\mu_k(B_p) = \sigma_k(2, \ldots, p-1)$. So for a nonlinear braid arrangement in example (1.3), we have by (5.15) the number of bounding cycles equals $1 + 4(\binom{3}{2}) = 9$.

Example 5.16. Second, consider as in example (1.5) a nonlinear hyperplane braid arrangement A on the hypersurface X of degree d in $\mathbb{C}^n$. By Corollary 2 and Proposition 5.3, the number of relative bounding cycles equals

$$\text{(5.17)} \qquad \begin{aligned} &d((d-1)^{n-1} + (d-1)^{n-2} \cdot \sigma_1(2, \ldots, n-1) + \cdots + \\ &\quad (d-1) \cdot \sigma_{n-2}(2, \ldots, n-1) + \sigma_{n-1}(2, \ldots, n-1) \end{aligned}$$

For example, on $\mathbb{C}^3$ when $d = 2$, we obtain $2(1^2 + 1 \cdot 2 + 0) = 6$ relative bounding cycles as shown in Fig. 4.

§6. Proof of the General Formula

We note that Theorem 1 is an immediate consequence of Proposition 4.2, provided we can show that $p(\boldsymbol{d}, q, \mathcal{A})$ satisfies the functional equations (4.1).

For the first functional equation, we may write $p(\boldsymbol{d}, q, \mathcal{A}') + p((\boldsymbol{d}, q), q, \mathcal{A}'')$ as

$$\text{(6.1)} \quad d \cdot \left(\sum_{j=0}^{r} s_j(\boldsymbol{d} - \mathbf{1}, (q-1)^{r-j+1}) \cdot q^{r-j} \mu_{r-j}(\mathcal{A}') \right)$$
$$+ d \cdot q \cdot \left(\sum_{j=0}^{r-1} s_j((\boldsymbol{d}, q) - \mathbf{1}, (q-1)^{r-1-j+1}) \cdot q^{r-1-j} \mu_{r-1-j}(\mathcal{A}'') \right)$$

The first sum can be written

$$\text{(6.2)} \quad d \cdot s_r(\boldsymbol{d} - \mathbf{1}, q-1) \mu_0(\mathcal{A}')$$
$$+ d \cdot \left(\sum_{j=0}^{r-1} s_j(\boldsymbol{d} - \mathbf{1}, (q-1)^{r-j+1}) \cdot q^{r-j} \mu_{r-j}(\mathcal{A}') \right)$$

In the second sum, we see

$$\text{(6.3)} \quad s_j((\boldsymbol{d}, q) - \mathbf{1}, (q-1)^{r-1-j+1}) = s_j(\boldsymbol{d} - \mathbf{1}, (q-1)^{r-j+1})$$

Using (6.3) and taking the factor q inside, the second sum becomes

$$\text{(6.4)} \quad d \cdot \left(\sum_{j=0}^{r-1} s_j(\boldsymbol{d} - \mathbf{1}, (q-1)^{r-j+1}) \cdot q^{r-j} \mu_{r-1-j}(\mathcal{A}'') \right)$$

Thus, if we add (6.2) to (6.4) we obtain

$$\text{(6.5)} \quad d \cdot s_r(\boldsymbol{d} - \mathbf{1}, q-1) \mu_0(\mathcal{A}')$$
$$+ d \cdot \left(\sum_{j=0}^{r-1} s_j(\boldsymbol{d} - \mathbf{1}, (q-1)^{r-j+1}) \cdot q^{r-j} \left(\mu_{r-j}(\mathcal{A}') + \mu_{r-1-j}(\mathcal{A}'') \right) \right)$$

Then, $\mu_0(\mathcal{A}') = 1 = \mu_0(\mathcal{A}))$ and by (2) of Proposition 2.1,

$$\mu_{r-j}(\mathcal{A}) = \mu_{r-j}(\mathcal{A}') + \mu_{r-1-j}(\mathcal{A}'')$$

Hence, (6.5) becomes

$$d \cdot \left(\sum_{j=0}^{r} s_j(\boldsymbol{d} - \mathbf{1}, (q-1)^{r-j+1}) q^{r-j} \mu_{r-j}(\mathcal{A}) \right) = p(\boldsymbol{d}, q, \mathcal{A})$$

this establishes the first functional equation.

For the second equation, we use the formulas of Greuel-Hamm [GH] and Giusti [Gi] for the Milnor number of a homogeneous ICIS. If it has multidegree $\boldsymbol{d} = (d_1, \ldots, d_{n-r})$. We write it in an equivalent form as in the remark following Theorem 8.10 in [D2]

$$(6.6) \qquad \mu(\boldsymbol{d}) = (-1)^{r+1} + d \cdot \sum_{j=0}^{r} \sigma_{r-j}((-1)^n) s_j(\boldsymbol{d})$$

where again $d = \prod_{i=1}^{n-r} d_i$ and $\sigma_k((-1)^n) = \sigma_k(-1, \ldots, -1)$ with n factors -1. Now, we may apply the τ function in [D3] and write the sum in (6.6) as $\tau(\boldsymbol{D})$ where $\boldsymbol{D}$ is the $r \times (n-r+1)$ matrix

$$\boldsymbol{D} = \begin{pmatrix} d_1 - 1 & d_2 - 1 & \ldots & d_{n-r} - 1 & -1 \\ d_1 - 1 & d_2 - 1 & \ldots & d_{n-r} - 1 & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ d_1 - 1 & d_2 - 1 & \ldots & d_{n-r} - 1 & -1 \end{pmatrix}$$

Then, by the definition of s_r in [D3, §2],

$$(6.7) \qquad \begin{aligned} \tau(\boldsymbol{D}) &= s_r(d_1 - 1, d_2 - 1, \ldots, d_{n-r} - 1, -1) \\ &= s_r(\boldsymbol{d} - \mathbf{1}, -1) \end{aligned}$$

Then, we let $d_{n-r+1} = q$, and $\boldsymbol{d}' = (d_1, \ldots, d_{n-r}, d_{n-r+1})$. Using (6.7)

$$(6.8) \qquad \begin{aligned} \mu(\boldsymbol{d}) + \mu((\boldsymbol{d}, q)) = (-1)^{r+1} &+ d \cdot s_r(\boldsymbol{d} - \mathbf{1}, -1) \\ &+ (-1)^{r-1+1} + d \cdot d_{n-r+1} \cdot s_{r-1}(\boldsymbol{d}' - \mathbf{1}, -1) \end{aligned}$$

Then, (6.8) equals

$$(6.9) \qquad d \cdot \big(s_r(\boldsymbol{d} - \mathbf{1}, -1) + d_{n-r+1} \cdot s_{r-1}(\boldsymbol{d}' - \mathbf{1}, -1)\big)$$

We may rewrite the second term of (6.9)

$$(6.10) \qquad \begin{aligned} d_{n-r+1} \cdot s_{r-1}(\boldsymbol{d}' - \mathbf{1}, -1) = (d_{n-r+1} - 1) &\cdot s_{r-1}(\boldsymbol{d}' - \mathbf{1}, -1) \\ &+ s_{r-1}(\boldsymbol{d}' - \mathbf{1}, -1) \end{aligned}$$

Also, by the "Generalized Pascal Relation"for s_j (see [D3, §2])

$$(6.11) \, s_r(\boldsymbol{d}' - \mathbf{1}, -1) = s_r(\boldsymbol{d} - \mathbf{1}, -1) + (d_{n-r+1} - 1) \cdot s_{r-1}(\boldsymbol{d}' - \mathbf{1}, -1)$$

Thus, we may apply (6.10) and (6.11) to rewrite (6.9) as

$$(6.12) \qquad d \cdot \big(s_r(\boldsymbol{d}' - \mathbf{1}, -1) + s_{r-1}(\boldsymbol{d}' - \mathbf{1}, -1)\big)$$

We may apply the expansion property of the s_j functions (again see [D3, §2]) to obtain

(6.13) $s_r(\boldsymbol{d'} - \mathbf{1}, -1) = s_r(\boldsymbol{d'} - \mathbf{1}) + (-1)s_{r-1}(\boldsymbol{d'} - \mathbf{1}, -1)$

Thus, substituting (6.13) into (6.12), we obtain for (6.9)

$$\begin{aligned} d \cdot s_r(\boldsymbol{d'} - \mathbf{1}) &= d \cdot s_r(\boldsymbol{d} - \mathbf{1}, d_{n-r+1} - 1) \\ &= d \cdot s_r(\boldsymbol{d} - \mathbf{1}, q - 1) = p(\boldsymbol{d}, q, \{H\}) \end{aligned}$$

The last equation results from $\mu_j(\{H\}) = 0$ for all $j > 0$.

This completes the proof of the theorem.

Corollary 2 is an immediate consequence of Theorem 1. For Corollary 3, we need only observe that the relative singular Milnor number

$$\mu(X_0, A_0) = \mu(X_0) + \mu(A_0)$$

is exactly the number of relative bounding cycles, which by (5.13) yields the formula for the singular Milnor number.

Also, for the k-th higher multiplicities we also have for a generic k-plane Π

(6.10) $$\mu(\Pi \cap X_0, \Pi \cap A_0) = \mu(\Pi \cap X_0) + \mu(\Pi \cap A_0)$$

while

(6.11) $\mu_k(X_0) = \mu(\Pi \cap X_0)$ and $\mu_k(A_0) = \mu(\Pi \cap A_0)$

Thus, combining (6.10) and (6.11) with the result for the singular Milnor fiber gives the result for higher multiplicities.

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Embedding the graphs of regular tilings and star-honeycombs into the graphs of hypercubes and cubic lattices*

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Abstract.

We review the regular tilings of d-sphere, Euclidean d-space, hyperbolic d-space and Coxeter's regular hyperbolic honeycombs (with infinite or star-shaped cells or vertex figures) with respect of possible embedding, isometric up to a scale, of their skeletons into a m-cube or m-dimensional cubic lattice. In section 2 the last remaining 2-dimensional case is decided: for any odd $m \geq 7$, star-honeycombs $\{m, m/2\}$ are embeddable while $\{m/2, m\}$ are not (unique case of non-embedding for dimension 2). As a spherical analogue of those honeycombs, we enumerate, in section 3, 36 Riemann surfaces representing all nine regular polyhedra on the sphere. In section 4, non-embeddability of all remaining star-honeycombs (on 3-sphere and hyperbolic 4-space) is proved. In the last section 5, all cases of embedding for dimension $d > 2$ are identified. Besides hyper-simplices and hyper-octahedra, they are exactly those with bipartite skeleton: hyper-cubes, cubic lattices and 8, 2, 1 tilings of hyperbolic 3-, 4-, 5-space (only two, $\{4,3,5\}$ and $\{4,3,3,5\}$, of those 11 have compact both, facets and vertex figures).

§1. Introduction

We say that given tiling (or honeycomb) T has a l_1-*graph* and embeds up to *scale* λ into m-cube H_m (or, if the graph is infinite, into cubic lattice $\mathbf{Z}_m$), if there exists a mapping f of the vertex-set of the skeleton graph of T into the vertex-set of H_m (or $\mathbf{Z}_m$) such that

$$\begin{aligned}\lambda d_T(v_i, v_j) &= ||f(v_i), f(v_j)||_{l_1} \\ &= \sum_{1 \leq k \leq m} |f_k(v_i) - f_k(v_j)| \text{ for all vertices } v_i, v_j,\end{aligned}$$

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where d_T denotes the graph-theoretical distance in contrast to the normed-space distance l_1. The smallest such number λ is called *minimal scale.*

Denote by $T \to H_m$ (by $T \to \mathbf{Z_m}$) isometric embedding of the skeleton graph of T into m-cube (respectively, into m-dimensional cubic lattice); denote by $T \to (1/2)H_m$ and by $T \to (1/2)\mathbf{Z_m}$ isometric up to scale 2 embedding.

Call an embeddable tiling *l_1-rigid*, if all its embeddings as above are pairwise equivalent. All, except hyper-simplexes and hyper-octahedra (see Remark 4 below), embeddable tilings in this paper turn out to be l_1-rigid and so, by a result from [Shp93], having scale 1 or (only for non-bipartite planar tilings) 2. Those embeddings were obtained by constructing a complete system of *alternated zones*; see [CDG97], [DSt96], [DSt97].

Actually, a tiling is a special case of a honeycomb, but we reserve the last term for the case when the cell or the vertex figure is a star-polytope and so the honeycomb covers the space several times; the multiplicity of the covering is called its *density.*

Embedding of Platonic solids was remarked in [Kel75] and precised, for the dodecahedron, in [ADe80]. Then [Ass81] showed that $\{3,6\}, \{6,3\}$, and $\{m,k\}$ (for even $m \geq 8$ and $m = \infty$) are embeddable. The remaining case of odd m and limit cases of $m = 2, \infty$ was decided in [DSt96]; all those results are put together in the Theorem 1 below.

All four star-polyhedra are embeddable. The great icosahedron $\{3,5/2\}$ of Poinsot and the great stellated dodecahedron $\{5/2,3\}$ of Kepler have the skeleton (and, moreover, the surface) of, respectively, icosahedron and dodecahedron; each of them has density 7. All ten star-4-polytopes are not embeddable: see the Theorem 3 below.

The case of Archimedean tilings of 2-sphere and of Euclidean plane was decided in [DSt96]; it turns out that for any such tilings (except $Prism_3$ and its dual, both embeddable) exactly one of two (a tiling and its dual) is embeddable. For 3-sphere and 3-space it was done in [DSt98b]; for example, Gosset's semiregular 4-polytope snub 24-cell turns out to be embeddable into half-12-cube. All 92 *regular-faced* 3-polytopes were considered in [DGr97b] and, for all higher dimensions, in [DSt96]. The *truncations* of regular polytopes were considered in [DSt97]. Another large generalization of Platonic solids - *bifaced* polyhedra - were considered in [DGr97b]. (Some generalizations of Archimedean plane tilings, *2-uniform* ones and *equi-transitive* ones, were treated in [DSt96], [DSt97], respectively.) Finally, skeletons of (Delaunay and Voronoi tilings of) lattices were dealt with in [DSt98a].

Embeddable ones, among all regular tilings of all dimensions, having compact facets and vertex fugures, were identified in [DSt96], [DSt97].

Coxeter (see [Cox54]) extended the concept of regular tiling, permitting infinite cells and vertex figures, but with the fundamental region of the symmetry group of a finite content. His second extension was to permit *honeycombs*, i.e. star-polytopes can be cells or vertex figures. For the 2-dimensional case, on which we are focusing in the next Section, above extensions produced only following new honeycombs - $\{m/2, m\}$ and $\{m, m/2\}$ for any odd $m \geq 7$ - which are hyperbolic analogue of spherical star-polyhedra $\{5/2, 5\}$ (the small stellated dodecahedron of Kepler) and $\{5, 5/2\}$ (the great dodecahedron of Poinsot). Both $\{5/2, 5\}$ and $\{5, 5/2\}$ have the skeleton of the icosahedron. For any odd m above honeycombs cover the space (2-sphere for $m = 5$) 3 times. The skeleton of $\{m, m/2\}$ is, evidently, the same as of $\{3, m\}$, because it can be seen as $\{3, m\}$ with the same vertices and edges forming m-gons instead of triangles. The faces of $\{m/2, m\}$ are stellated faces of $\{m, 3\}$ and it have the same vertices as $\{3, m\}$.

We adopt here classical definition of the regularity: the transitivity of the group of symmetry on all faces of each dimension. But, as remarked the referee, the modern concept of regularity, which requires transitivity on flags, would not necessitate any change in the results.

The following *5-gonal* inequality (see [Dez60]):

$d_{ab} + (d_{xy} + d_{xz} + d_{yz}) \leq (d_{ax} + d_{ay} + d_{az}) + (d_{bx} + d_{by} + d_{bz})$

for distances between any five vertices a, b, c, x, y, is an important necessary condition for embedding of graphs, which will be used in proofs of Theorems 3,4 below.

This paper is a continuation of general study of l_1-graphs and l_1-metrics, surveyed in the book [DLa97], where many applications and connections of this topic are given. In addition, we tried here to extract from purely geometric, affine structures, considered below, their new, purely combinatorial (in terms of metrics of their graphs) properties.

§2. Planar tilings and hyperbolic honeycombs

They are presented in the Table 1 below; we use the following notation:

1. The row indicates the facet (cell) of the tiling (or honeycomb), the column indicates its vertex figure. The tilings and honeycombs are denoted usually by their Schläfli notation, but in the Tables 1, 3-5 below we omit the brackets and commas for convenience (in order to fit into page).

2. By m we denote m-gon, by $m/2$ star-m-gon (if m is odd). By α_3,

β_3, γ_3, Ico, Do and δ_2 we denote regular ones tetrahedron $\{3,3\}$, octahedron $\{3,4\}$, cube $\{4,3\}$, icosahedron $\{3,5\}$, dodecahedron $\{5,3\}$ and the square lattice $Z_2 = \{4,4\}$. The numbers are: any $m \geq 7$ in 8th column, row and any *odd* $m \geq 7$ in 9th column, row.

3. We consider that: $\{2,m\}$ is a 2-vertex multi-graph with m edges; $\{m,2\}$ can be seen as a m-gon; all vertices of $\{m,\infty\}$ are on the absolute conic at infinity (it has an infinite degree); the faces ∞ of $\{\infty,m\}$ are inscribed in *horocycles* (circles centered in ∞).

Table 1. 2-dimensional regular tilings and honeycombs.

	2	3	4	5	6	7	m	∞	$\frac{m}{2}$	$\frac{5}{2}$
2	22	23	24	25	26	27	$2m$	2∞		
3	32	α_3	β_3	Ico	**36**	37	$3m$	3∞		$3\frac{5}{2}$
4	42	γ_3	δ_2	45	46	47	$4m$	4∞		
5	52	Do	54	55	56	57	$5m$	5∞		$5\frac{5}{2}$
6	62	**63**	64	65	66	67	$6m$	6∞		
7	72	73	74	75	76	77	$7m$	7∞		
m	$m2$	$m3$	$m4$	$m5$	$m6$	$m7$	mm	$m\infty$	$m\frac{m}{2}$	
∞	$\infty 2$	$\infty 3$	$\infty 4$	$\infty 5$	$\infty 6$	$\infty 7$	∞m	$\infty\infty$		
$\frac{m}{2}$							$\frac{m}{2}m$			
$\frac{5}{2}$		$\frac{5}{2}3$		$\frac{5}{2}5$						

Theorem 1. *All 2-dimensional tilings $\{m,k\}$ are embeddable, namely:*

(i) *if $1/m + 1/k > 1/2$ (2-sphere), then*

$\{2,m\} \to H_1$ *for any m,* $\{m,2\} \to (1/2)H_m$ *for odd m and* $\{m,2\} \to H_{m/2}$ *for even m;*

$\{3,3\} = \alpha_3 \to (1/2)H_3$, $(1/2)H_4$; $\{4,3\} = \gamma_3 \to H_3$; $\{3,4\} = \beta_3 \to (1/2)H_4$;

$\{3,5\} = Ico(\sim \{3,5/2\} \sim \{5,5/2\} \sim \{5/2,5\}) \to (1/2)H_6$ *and* $\{5,3\} = Do(\sim \{5/2,3\}) \to (1/2)H_{10}$;

(ii) *if $1/m + 1/k = 1/2$ (Euclidean plane), then*

$\{2,\infty\} \to H_1$, $\{\infty,2\} \to Z_1$; $\{4,4\} = \delta_2 \to \mathbf{Z}_2$, $\{3,6\} \to (1/2)\mathbf{Z}_3$, $\{6,3\} \to \mathbf{Z}_3$;

(iii) *if $1/2 > 1/m + 1/k$ (hyperbolic plane), then*

$\{m,k\} \to (1/2)\mathbf{Z}_\infty$ *if m is odd, $k \leq \infty$ and* $\{m,k\} \to \mathbf{Z}_\infty$ *is m is even or ∞, $k \leq \infty$.*

Remark 1 (notation and terms here are from [Cox37], [Cro97]).

(i) The embedding of the icosahedron $\{3,5\}$ into $(1/2)H_6$ was mentioned in [Cox50] on pages 450–451, as *regular skew icosahedron.* There are 5 proper regular-faced fragments of $\{3,5\}$: 5-pyramid, 5-antiprism, para-bidiminished $\{3,5\}$, meta-bidiminished $\{3,5\}$, and tridiminished $\{3,5\}$; 5-pyramid embeds into $(1/2)H_5$, all others into $(1/2)H_6$.

(ii) The antipodal quotients of (embeddable, see Theorem 1 (i) above) cube, icosahedron, dodecahedron are regular maps $\{4,3\}_3$, $\{3,5\}_5$, $\{5,3\}_5$ on the projective plane, which are K_4, K_6, the Petersen graph; they embed into $(1/2)H_m$ for $m = 4$, 6, 6, respectively.

(iii) Besides $\{4,4\}$, $\{3,6\}$, $\{6,3\}$ (embeddable, see Theorem 1 (ii) above), there are exactly three other infinite regular polyhedra. They are *regular skew* polyhedra $\{4,6|4\}$, $\{6,4|4\}$, $\{6,6|3\}$, which can be obtained by deleting of cells from the tilings of 3-space by cubes ($\mathbf{Z}_3$), by truncated octahedra (the Voronoi tiling for the lattice A_3^*), by regular tetrahedra and truncated tetrahedra (Föppl uniform tiling). They are, respectively: embeddable into $\mathbf{Z}_3$, embeddable into $\mathbf{Z}_6$, not 5-gonal. All finite regular skew 4-polytopes are: the family $\{4,4|m\}$ of self-dual quadringulations of the torus (it is the product of two m-cycles and so embeddable into $(1/2)H_{2m}$ for odd m or into H_m for even m), not 5-gonal $\{6,4|3\}$, $\{4,6|3\}$, $\{8,4|3\}$ and its undecided dual $\{4,8|3\}$.

(iv) Examples of other interesting regular maps are the Dyck map $\{3,8\}_6$ (8-valent map with 12 vertices and 32 triangular faces), the Klein map $\{3,7\}_8$ (7-valent map with 24 vertices and 56 triangular faces) and $\{4,5\}_5$ (5-valent map with 16 vertices and 20 quadrangular faces). Those maps (all of oriented genus 3) come from the hyperbolic tilings $\{3,8\}$, $\{3,7\}$, $\{5,4\}$, respectively (which are embeddable; see Theorem 1 (iii) above) by identification of some vertices of the unit cell. Those three maps and their duals are all not 5-gonal. But, for example, the 3-valent partition of the torus into 4 hexagons is embeddable: it is the cube on the torus.

Remark 2 (notation and terms here are from [Cox73], [Wen71] and [Cro97]). With V.P. Grishukhin we considered embeddability of following non-convex polyhedra:

(i) All non-Platonic facetings of Platonic solids (see [Cox73], page 100) are: 4 star-polyhedra $\{5/2,5\}$, $\{5,5/2\}$, $\{5/2,3\}$, $\{3,5/2\}$ and 4 regular compounds $2\alpha_3$ (Kepler's *stella octangula*), $5\gamma_3$, $5\alpha_3$, $10\alpha_3$. The remaining regular compound is $5\beta_3$, which is dual to $5\gamma_3$. In this Remark only, contrary to Theorem 1 (i), we consider all visible "vertices" of polyhedra, not only those of their shells. Then it turns out, that $\{5/2,5\}$, $\{5,5/2\}$, $\{5/2,3\}$, $\{3,5/2\}$, $2\alpha_3$, $5\beta_3$ have the same skeletons as dual truncated, respectively, $\{3,5\}$, $\{5,3\}$, $\{5,3\}$, truncated $\{3,5\}$, γ_3, icosidodecahedron. $5\alpha_3$ has the same skeleton as dual snub dodecahedron.

Among all 4 star-polyhedra, 5 regular compounds and their 9 duals, all embeddable ones are:

$\{5/2,5\} \to (1/2)H_{10}$, $\{5,5/2\}$ ($\sim \{5/2,3\}$) $\to (1/2)H_{26}$, $\{3,5/2\} \to (1/2)H_{70}$, $2\alpha_3 \to (1/2)H_{12}$, dual $5\beta_3$ ($\sim 5\gamma_3$) $\to H_{15}$, dual $5\alpha_3 \to (1/2)H_{15}$.

(ii) Among 8 stellations $A - H$ of $\{3,5\}$ (the main sequence, see [Cro97], page 272), all embeddable ones are $A = \{3,5\}$, $B \sim \{5,5/2\}$ and $G \sim H \sim \{3,5/2\}$. Also the dual of the stellation De_2f_2 of $\{3,5\}$ has the same skeleton as the rhombicosidodecahedron and it embeds into $(1/2)H_{16}$. The stellations $De_1 \sim Fg_2 \sim C = 5\alpha_3$ and Fg_1, De_2f_2 are not embeddable.

(iii) Among the compounds of two dual Platonic solids and dual compounds, all embeddable ones are $2\alpha_3$ and, into $(1/2)H_{28}$, the dual of $\{3,5\}+\{5,3\}$. Among all 53 non-convex non-regular uniform polyhedra (Nos. 67–119 in [Wen71]), two are quasi-regular: the dodecadodecahedron and the great icisidodecahedron (see [Cox73], page 101 and Nos. 73, 94 in [Wen71]). Again we consider all visible "vertices" and see a pentagram 5/2 as *pentacle* (10-sided non-convex polygon). Then both above polyhedra and their duals are not embeddable. But, for example, the ditrigonal dodecahedron (No. 80 in [Wen71], a relative of No. 73) embeds into $(1/2)H_{20}$.

The following theorem gives the family of all non-embeddable regular planar cases.

Theorem 2. *For any odd $m \geq 7$ we have*
(i) $\{m/2, m\}$ *is not embeddable*;
(ii) $\{m, m/2\}$ ($\sim \{3,m\}$) $\to (1/2)\mathbf{Z}_\infty$.

The assertion (ii) is trivial. The proof of (i) will be preceded by 3 lemmas and first two of them are easy but of independent interest for embedding of (not necessary planar) graphs. Lemma 1 can be extended on the isometric cycles.

Let G be a graph, scale λ embeddable into $\mathbf{Z}_m$, let C be an oriented circuit of length t in G and let e be an arc in C. Then there are λ *elementary vectors*, i.e. steps in the cubic lattice $\mathbf{Z}_m$, corresponding to the arc e; denote them by $x_1(e), ..., x_{\lambda(e)}$. Clearly, the sum of all vectors $x_i(e)$ by all i and all arcs e of the circuit, is the zero-vector.

Now, if t is even, denote by e^* the arc opposite to e in the circuit C; if t is odd, denote by e', e'' two arcs of C opposite to e. For even t, call the arc e *balanced* if the set of all its vectors $x_i(e)$ coincides with the set of all $x_i(e^*)$, but the vectors of arc e^* go in opposite direction *on the circuit* C to the vectors of e. For odd t, call the arc e *balanced* if a half of vectors of e' together with a half of vectors of the second opposite arc e'' form a partition of the set of vectors of e and those vectors go in opposite direction (on C) to those of arc e.

Remind, that the *girth* of the graph is the length of its minimal circuit.

Lemma 1. *Let G be an embeddable graph of girth t. Then*

(i) *any arc of a circuit of length t is balanced;*

(ii) *if t is even, then any arc of a circuit of length $t+1$ is also balanced.*

Lemma 2. *Let G be an embeddable graph of girth t and let P be an isometric oriented path of length at most $\lfloor t/2 \rfloor$ in G. Then there are no two arcs on this path having vectors, which are equal, but have opposite directions on the path.*

Lemma 3. *The girth of the skeleton of $\{m/2, m\}$ is 3 for $m = 5$ and $m - 1$ for any odd $m \geq 7$.*

Proof of Lemma 3.

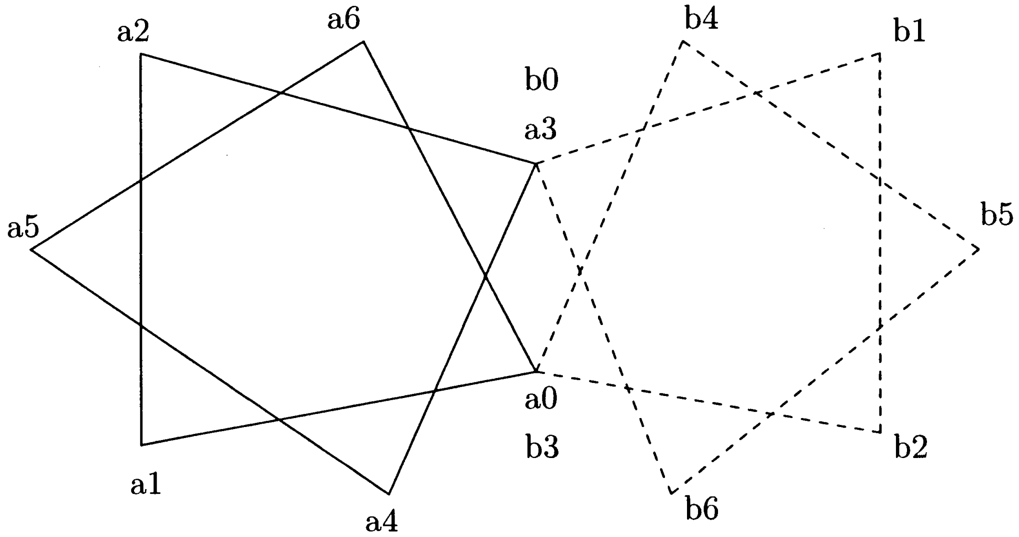

Fig . 1a. A fragment of $\{7/2, 7\}$

Consider Fig. 1a. Take a cell $A = (a_0, ..., a_m = a_0)$ of the $\{m/2, m\}$, i.e. a star m-gon, seen as an oriented cycle of length $m = 2k + 1$. Consider following automorphism of the honeycomb: a turn by 180 degrees around the mid-point of the segment $[a_0, a_k]$. The image of A is the oriented star m-gon $B = (b_0, ..., b_m = b_0)$ with $b_0 = a_k$, $b_k = a_0$. Consider now oriented cycle $C = (a_0, a_1, ..., a_k = b_0, ..., b_k = a_0)$ of even length $m - 1 = 2k$. In order to prove the Lemma 3, we will show that C is a cycle of minimal length.

First we show that the graph distance $d(a_0, a_k) = k$, i.e. the path $P := (a_0, a_1, ..., a_k)$ is a shortest path from a_0 to a_k. It will imply that $d(a_0, c(A)) = d(a_k, c(A)) = k$, where $c(A)$ is the center of the cell A, because all vertices of $\{m/2, m\}$ are vertices of regular triangles of $\{3, m\}$.

Let Q be a shortest path from a_0 to a_k. Then it goes around the vertex $c(A)$ or the center $c(B)$ of the cell B, because otherwise Q goes through at least one of the vertices a_{k+1}, a_{2k}, b_{k+1}, b_{2k} and so Q contains at least one of the pairs of vertices (a_0, a_{k+1}), $(a_0 = b_k, b_{2k})$, $(b_k = a_0, a_{2k})$, $(a_k = b_0, b_{k+1})$. But each of those pairs has, by the symmetry of our honeycomb $\{m/2, m\}$, same distance between them as (a_0, a_k); it contradicts to the supposition that Q is a shortest path. So, we can suppose that Q goes around $c(A)$ (the argument is the same if it goes around $c(B)$). Now, to each edge (s, t), corresponds, from the center $c(A)$ of A, the angle $(s, c(A), t)$. The $2k + 1$ edges of A are only edges, for which this angle is $4k\pi/(2k + 1)$; for any other edge, the angle is smaller, since it is more far from $c(A)$. So, if Q contains an edge, other than one from A, then, in order to reach a_k from a_0, it should be of length more than k. Therefore, any shortest path from a_0 to a_k, should consist only of edges of A and then it is of length k. So, $d(a_0, c(A)) = k$ also, as well as for any edge of $\{3, m\}$. Same holds for $m = 5$.

We will show now that:

(i) any path R of length $2k - 2$ is not closed and

(ii) R cannot be closed by only one edge.

But C is a closed path of length $2k$; so (i), (ii) will imply that $2k$ (respectively, $2k+1$) is the minimal length of any (respectively, any odd) simple isometric cycle in the graph. For $m = 5$ (ii) does not holds.

Suppose that R is closed; let as see it as a $2k - 2$-gon on hyperbolic plane. Any angle of R is a multiple $i(2\pi/m)$, but $i > 1$ for at least one angle, because $(2k - 2)(2\pi/m) < 2\pi$. Suppose that a angle has $1 < i \leq k$; the argument will be the same if $k + 1 \leq i < m - 1$, but for the complementary angle $(m - i)(2\pi/m)$ with $1 < m - i \leq k$.

See Fig. 1b for the following argument. Fix an angle r, s, t between two adjacent edges (r, s) and (s, t) of R. Let $s*$ be the opposite vertex to s on R, let (s, r'), (s, t') be the edges such that the angles r, s, r', t, s, t' are $2\pi/m$. Let A, B be the cells $m/2$, defined by pairs (r, s), (s, r') and (t, s), (s, t') of their adjacent edges and $c(A)$, $c(B)$ are their centers. The vertex $c(A)$ not belongs to the path from s to $s*$ of length $k-1$, since we proved above that $d(s, c(A) = k)$; so this path should go around $c(A)$. Let p be the vertex of A, reachable from s by $k - 1$ steps on A, starting by r; let q be the vertex of B, reachable from s by $k - 1$ steps on B, starting by t. By mirror on (r, s) (respectively, (s, t)) we obtain the cells

A', B', their centers $c(A')$, $c(B')$ and vertices p', q', which are reflections of p, q. Call *A-domain*, the part of the hyperbolic plane, bounded by half-lines $(c(A), p, \infty)$, $(c(A'), p'\infty)$ and the angle $(c(A), s, c(A'))$; call *B-domain*, the part, bounded similarly for B. Actually, B-domain is the reflection of A-domain by the bisectrisse of the angle (r, s, t).

We will show now that the vertex $s*$ should belong to both A-and B-domains. But they do not have common points, besides s. This contradiction will show that our R, a closed path of length $2k-2$, do not exists. Any edge of the path $(s, t, ..., s*)$ of length $k-1$ is seen from $c(A)$ under angle at most $4\pi/m$ with equality if and only if this edge belongs to A (as, for example, the edge (r, s)). Summing up those angles along the path $(st, ..., s*)$, we get less than $(k-1)(4\pi/m)$, obtained for the path of length $k-1$ from s to p, going along A. It implies that $s*$ belongs to A-domain and also, by reflection, to B-domain.

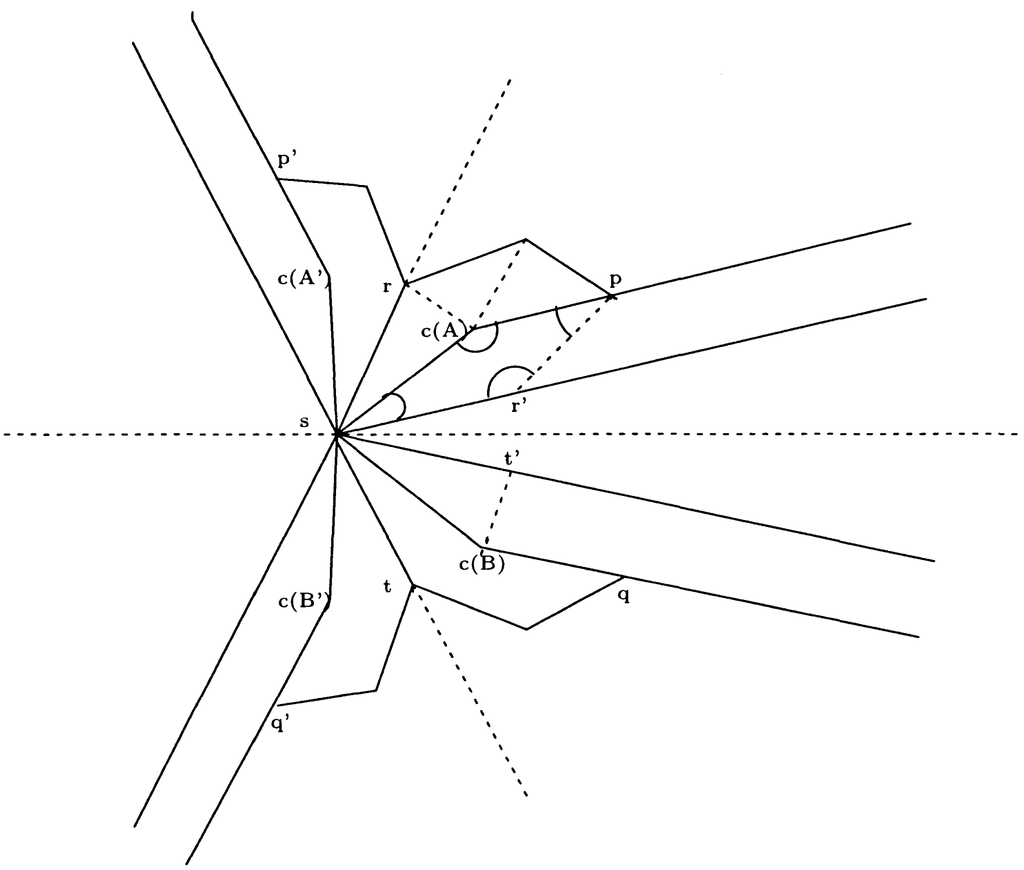

Fig. 1b. A fragment of $\{9/2, 9\}$

But A-and B-domains intersect only in point s, because the lines through $(c(A), p)$ and (s, r') diverge on the hyperbolic plane. In fact, denote by α_1, α_2, β_1, β_2 the angles $(p, c(A), s)$, $(c(A), s, r')$, $(c(A), p, r')$,

(p, r', s), respectively. They are equal to $4\pi/m+2\pi/m$, π/m, $\pi/m+\pi/m$, $2\pi/m + \pi/m$, respectively. So $\alpha_1 + \alpha_2 = 7\pi/m \leq \pi$, since $m \geq 7$ and the lines, if they converge or parallel, do it on the right side of Fig. 1b. Now, $\beta_1 + \beta_2 = 5\pi/m < \pi$ and the lines, if they converge or parallel, do it on the left. So, they diverge.

We demonstrated ad absurdum, the non-existence of the vertex $s*$ and so, of the *closed* path R. So, a path R of length $2k-2$ is not closed. But p, q is never an edge; so we need at least two edges in order to close R. If two edges are enough, then points r', t' coincide, i.e. $i = 2$; actually, two edges will be enough in the case $m = 7$. The proof of Lemma 3 is completed. Q.E.D.

Proof of Theorem 2. Consider star-m-gons A, B and the circuit C as in beginning of the proof of Lemma 3 above. Take the arc $e = (a_0, a_1)$ on the circuit C; by Lemma 1 (i), e is balanced, i.e. the vectors $x_i(e^*)$ of the opposite arc $e^* = (b_0, b_1)$ are the same, as of the arc e, but they have opposite directions with respect of the circuit C. The same arc e, seen as an arc of the circuit B of length m, is opposite to two arcs in this odd circuit and, in particular, to the arc (a_k, a_{k+1}). The last arc has, by Lemma 1 (ii), $\lambda/2$ vectors, coinciding with vectors of e, but with opposite direction on the circuit B. Finally, consider the oriented path $(a_{k+1}, a_k = b_0, b_1)$ of length 2 in our $\{m/2, m\}$. Its two arcs have vectors, coinciding, but going in opposite direction on this path. But it contradicts to Lemma 2, because $2 < k$. Q.E.D.

§3. Spherical analogue of Coxeter's honeycombs

In this Section we consider, for any pair (i, m) of integers, such that $1 \leq i < m/2$ and g.c.d.$(i, m) = 1$, star-polygons m/i. Clearly, $m/1$ denotes now a convex m-gon; so we see star-polygons as a generalization of convex ones. We will allow further extension: star-polygons m/i with $m/2 < i < m$, let us call them *large* star-polygons. They cannot be represented on Euclidean or hyperbolic plane, because they have there the same representation as $m/(m-i)$. But they can be represented on the sphere by the following way; see Fig. 2 for the simplest 3/1 and 3/2. Let $a_0, ..., a_{m-1}$ be m points, placed in this order, on a great circle of the sphere, in order to form a regular m-gon. Then the spherical (great circle) distance $d(a_0, a_i)$ is $2\pi i/m$, but on m/i, the length of the way is $d(a_0, a_i)$ for $i < m/2$ and $2\pi - d(a_0, a_i)$ otherwise. Using this larger set of polygons, we will look for spherical representations of regular (i.e. with a group of symmetry acting transitively on all j-faces, $0 \leq j \leq 2$) polyhedra.

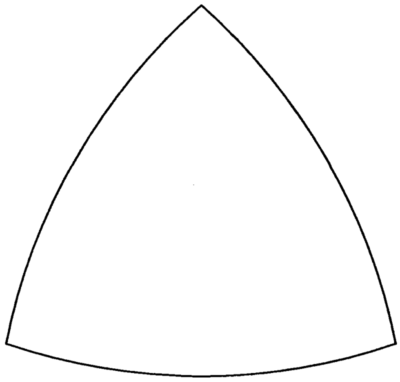

Fig. 2a. 3/1

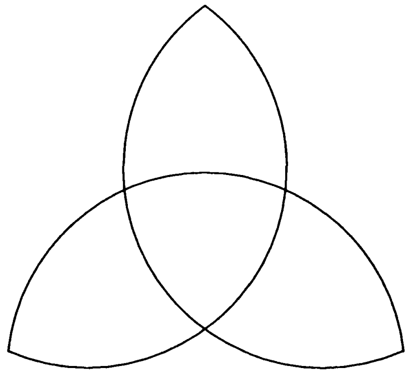

Fig. 2b. 3/2

In the Table 2 below, the rows (columns) denote a cell (respectively, a vertex figure) of would-be representations. If the representation, corresponding to a given pair of $\{m/i, n/j\}$ of polygons, exists, we denote it by this pair and write its density in corresponding cell of the Table 2. The densities were counted directly, by superposing the representation on corresponding regular polyhedron. But the expression of the density, given in the formula 6.41 of [Cox73] for multiply-covered sphere is valid for our representations, i.e. the density of $\{m/i, n/j\}$ is $N_1(i/m + j/n - 1/2)$, where N_1 is the number of edges. (Above expression is equivalent to Cayley's generalization of Euler's Formula, given as the formula 6.42 in [Cox73].) Our representations are *Riemann surfaces*, i.e. d-sheeted spheres (or d almost coincident, almost spherical surfaces) with the sheets connected in certain branch-points.

We see a m/i as a representation of the m-cycle on the sphere, together with a bi-partition of i-covering of the sphere. Call *interior* the part with angle, which is less than π. For representations below, the vertex figure selects uniquely the part of the cell: namely, the vertex figure n/j gives the value $2\pi j/n$ for the angle of the cell. It takes interior of the cell if $j < n/2$ and exterior otherwise.

The Table 2 shows that each of all nine regular polyhedra (seen as abstract surfaces) admits four such Riemann surfaces and we checked, case by case, that all 36 are different and that remaining 28 possible representations do not exist. Each of four representations for each regular polyhedron has same genus as corresponding abstract surface; so the genus is four for 8 representations of the form $\{5/i, 5/j\}$ and zero for all others.

In the Table 2, the column with 2/1 corresponds to doubling of regular polygons. Alexandrov ([Ale58]) considered, for other purpose, the doubling of any convex polygon as an abstract sphere, realized as a degenerated (i.e. with volume 0) convex polyhedron. $\{m, 2\}$ and $\{2, n\}$

on the plane and the sphere appeared also in Section 7 of [FTo64]. By analogy, we will do such doubling for star-polygons m/i with $i < m/2$. But for large star-polygons we should do doubling on the sphere. The row and the column with m/i correspond to any pair of mutually prime integers (i, m), $1 \leq i < m$. As Table 2 shows, there exist all representations $\{2/1, m/i\}$ and $\{m/i, 2/1\}$ and each of them has density i (and the genus 0).

An infinity of other representations can be obtained by permitting polygons $m/(i{+}tm)$ for any integer $t \geq 0$; the way on the edge (a_0, a_{i+tm}) will be $2\pi t - d(a_0, a_{i+tm})$.

Table 2. 36 representations of regular polyhedra on the sphere.

	$\frac{2}{1}$	$\frac{3}{1}$	$\frac{3}{2}$	$\frac{4}{1}$	$\frac{4}{3}$	$\frac{5}{1}$	$\frac{5}{4}$	$\frac{5}{2}$	$\frac{5}{3}$	$\frac{m}{i}$	$\frac{m}{m-i}$
$\frac{2}{1}$	1	1	2	1	3	1	4	2	3	i	$m-i$
$\frac{3}{1}$	1	1	3	1	7	1	19	7	13		
$\frac{3}{2}$	2	3	5	5	11	11	29	17	23		
$\frac{4}{1}$	1	1	5								
$\frac{4}{3}$	3	7	11								
$\frac{5}{1}$	1	1	11					3	9		
$\frac{5}{4}$	4	19	29					21	27		
$\frac{5}{2}$	2	7	17			3	21				
$\frac{5}{3}$	3	13	23			9	27				
$\frac{m}{i}$	i										
$\frac{m}{m-i}$	$m-i$										

§4. Star-honeycombs

Besides star-polygons and four regular star-polyhedra on 2-sphere, which are all embeddable (last four are isomorphic to Ico or Do), there are ([Cox54]) only following regular star-honeycombs: ten regular star-polytopes on 3-sphere and four star-honeycombs in hyperbolic 4-space; see the Tables 1, 3-5. In this Section we show that none of last 14 is embeddable. Consider first the case of 3-sphere.

There are six regular 4-polytopes (4-simplex α_4, 4-cross-polytope β_4, 4-cube γ_4, self-dual 24-cell and the pair of dual 600-cell and 120-cell) and ten star-4-polytopes; see the Chapter 14 in [Cox73]. [Ass81] showed non-embeddability of 24-and 600-cell; [DGr97c] did it for 120-cell. Clearly, γ_4 and β_4 are H_4 and $(1/2)H_4$ themselves.

Embeddable ones among Archimedean tilings of 3-sphere and 3-space, were identified in [DSt98b]; for example, *snub* 24-cell (semi-regular Gosset's 4-polytope $s\{3,4,3\}$) embeds into $(1/2)H_{12}$ while the Grand Antiprism of [Con67] is not embeddable.

The isomorphisms among ten star-4-polytopes, see [vOs15] and pages 266–267 of [Cox73], preserve all incidencies and imply, of course, isomorphisms of the skeletons of those polytopes. Using Schläfli notation, those isomorphisms *of graphs* are:

(i) $\{5/2,5,3\} \sim \{5,5/2,3\}$;

(ii) $\{5/2,3,3\} \sim$ 120-cell (remind the isomorphism of $\{5/2,3\}$ and $\{5,3\}$);

(iii) all remaining seven skeletons are isomorphic with the skeleton of 600-cell (moreover, $\{3,5,5/2\}$ has same faces; remind the isomorphism of $\{3,5/2\}$ and $\{3,5\}$).

So eight star-polytopes from (ii) and (iii) above are not embeddable. Remaining case (i) is decided by the Theorem 3 below, using 5-gonal inequality.

Theorem 3. *None of ten star-4-polytopes is embeddable.*

Proof of Theorem 3. In view of above isomorphisms, it will be enough to show that (the skeleton of) 4-polytope $P := \{5/2,5,3\}$ is not 5-gonal. P is the stellated 120-cell and $\{5/2,5\}$ is the (small) stellated dodecahedron, i.e. all face-planes are extended until their intersections form a pyramid on each face. P has 120 vertices, as 600-cell; namely, the centers of all 120 (dodecahedral) cells of 120-cell. For any vertex s of P, denote by Do(s) the corresponding dodecahedron. P has (as well as 120-cell) 1200 edges, 720 faces and 120 cells; its density is 4. Any edge (s,t) of P goes through interiors of $Do(s)$, $Do(t)$ and the edge of 120-cell, linking those dodecahedra; (s,t) is a continuation of this edge in both directions till the centers of dodecahedra $Do(s)$, $Do(t)$.

Consider now Fig. 3. Take as vertices a and b (for future contre-example for 5-gonal inequality) some two vertices of $\{5/2,5\}$ (a cell of P), which are centers of two face-adjacent dodecahedral cells of 120-cell. Let $Q := (q_1,q_2,q_3,q_4,q_5)$ be this common face of adjacency, presented by the 5-cycle of its vertices. For any q_i there is unique star-5-gon (a,d_i,b,d'_i,d''_i), such that sides (b,d'_i) and (d''_i,a) intersect in the point q_i. Now, $D := (d_1,d_2,d_3,d_4,d_5)$ is a 5-cycle in P, because each (d_{i-1},d_i) is an edge in one of five cells $\{5/2,5\}$ of P, containing vertices a and b. Put $x := d_1$, $y := d_2$, $z := d_4$ and check that the 5-gonal inequality for five vertices a,b,x,y,z of P, does not hold.

In fact, $d_{xy} = 1 = d_{ax} = d_{ay} = d_{az} = d_{bx} = d_{by} = d_{bz}$, because of the presence of corresponding edges in P. Therefore, d_{xz}, d_{yz} and

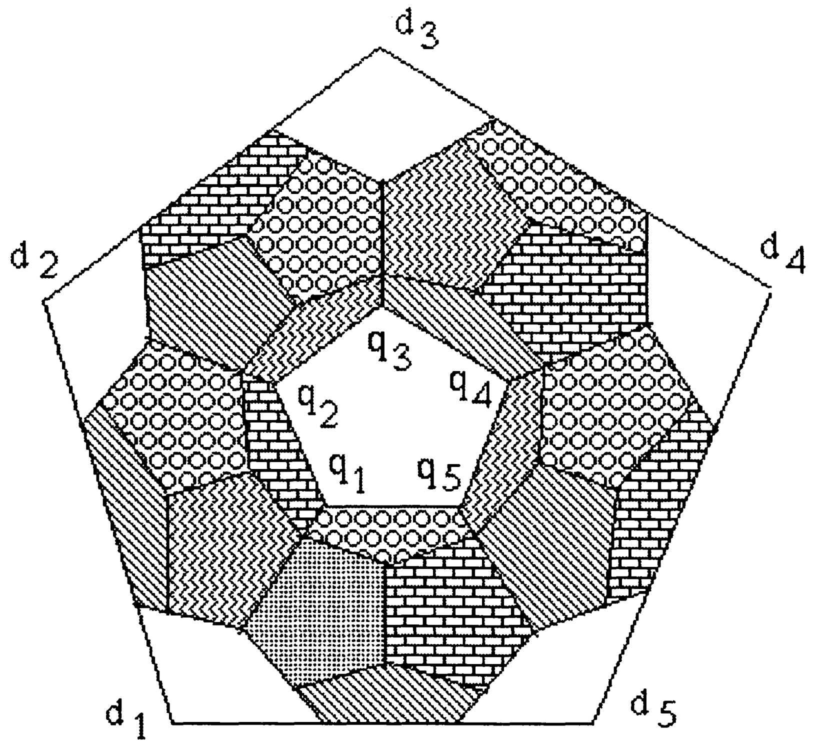

Fig. 3. A fragment of 5/2 5 3

d_{ab} are at most 2. So, the absence of edges (x, z), (y, z) and (a, b) will complete the proof of the Theorem 3. The edge (a, b) does not exist, because $Do(a)$ is face-adjacent to $Do(b)$. The edge (x, z) does not exists, because the line, linking vertices x and z, goes, besides $Do(x)$ and $Do(z)$, through two other dodecahedra (such that their stellations are $\{5/2, 5\}$, containing vertices a, b, d_2, d_3 or a, b, d_3, d_4). By symmetry, the edge (y, z) does not exist also. We are done. Q.E.D.

Corollary. *None of four star-honeycombs in hyperbolic 4-space is embeddable.*

Proof of Corollary. In fact, $\{5/2, 5, 3, 3\}$ has cell which contains (because of the Theorem 3), as an induced subgraph, non-5-gonal graph K_5–K_3. But any induced graph of diameter 2 is isometric; so $\{5/2, 5, 3, 3\}$ is not 5-gonal. $\{3, 3, 5, 5/2\}$ has cell $\{3, 3, 5\}$ = 600-cell. Two other have cells which are isomorphic to 600-cell. But 600-cell (seen by Gosset's construction as capping of all 24 icosahedral cells of snub 24-cell) contains also a forbidden induced graph of diameter 2: pyramid on icosahedron (it violates *7-gonal* inequality, which is also necessary for embedding; see [Dez60], [DSt96]). So, three other star-4-polytopes are also non-7-gonal and non-embeddable. Q.E.D.

§5. Regular tilings of dimension $d \geq 3$

The Tables 3-5 below present all of them and also all regular honeycombs in the dimensions 3, 4, 5; for higher dimensions, $(d+1)$-simplices α_{d+1}, $(d+1)$-cross-polytopes β_{d+1}, $(d+1)$-cubes γ_{d+1} and cubic lattices δ_d are only regular ones.

In those Tables, 24-, 600-, 120- are regular spherical 4-polytopes $\{3,4,3\}$, $\{3,3,5\}$, $\{5,3,3\}$ with indicated number of cells and $De(D_4)$, $Vo(D_4)$ are regular partitions $\{3,3,4,3\}$, $\{3,4,3,3\}$ of Euclidean 4-space, which are also Delaunay (Voronoi, respectively) partitions associated with the (point) lattice D_4.

All cases of embeddability are marked be the star * in the Tables. As in Table 1 above, we omit in Tables 3-5 (in order to fit them in the page) the brackets and commas in Schläfli notation.

Table 3. 3-dimensional regular tilings and honeycombs.

	α_3	γ_3	β_3	Do	Ico	δ_2	63	36	$3\frac{5}{2}$	$\frac{5}{2}3$	$5\frac{5}{2}$	$\frac{5}{2}5$
α_3	α_4*		β_4*		600−			336	$33\frac{5}{2}$			
β_3		24−				344						
γ_3	γ_4*		δ_3*		435*			436*				
Ico				353							$35\frac{5}{2}$	
Do	120−		534		535			536	$53\frac{5}{2}$			
δ_2		443*				444*						
36							363					
63	633*		634*		635*			636*				
$\frac{5}{2}3$	$\frac{5}{2}33$				$\frac{5}{2}35$							
$3\frac{5}{2}$												$3\frac{5}{2}5$
$\frac{5}{2}5$				$\frac{5}{2}53$							$\frac{5}{2}5\frac{5}{2}$	
$5\frac{5}{2}$										$5\frac{5}{2}3$		$5\frac{5}{2}5$

Table 4. 4-dimensional regular tilings and honeycombs.

Table 5. 5-dimensional regular tilings and honeycombs.

	α_5	γ_5	β_5	$Vo(D_4)$	$De(D_4)$	δ_4
α_5	α_6*		β_6*			
β_5					33343	
γ_5	γ_6*		δ_5*			
$De(D_4)$				33433		
$Vo(D_4)$		34333				34334
δ_4					43343*	

Theorems 1, 2 above show that all regular 2-dimensional tilings and star-honeycombs are embeddable except $\{m/2, m\}$ for all odd $m \geq 7$. The following Theorem decides all remaining regular cases.

Theorem 4. *All embeddable regular tilings and honeycombs of dimension $d \geq 3$ are tilings:*

(i) *either α_{d+1}, or β_{d+1}, or*

(ii) *all with bipartite skeleton:*

(ii-1) *all with cell γ_d: γ_{d+1}, δ_d and 3 hyperbolic ones: $\{4,3,5\}$, $\{4,3,3,5\}$, non-compact $\{4,3,6\}$;*

(ii-2) *all 4 with cell δ_{d-1}: hyperbolic non-compact $\{4,4,3\}$, $\{4,4,4\}$, $\{4,3,4,3\}$ and $\{4,3,3,4,3\}$;*

(ii-3) *all 4 with cell $\{6,3\}$: hyperbolic non-compact $\{6,3,3\}$, $\{6,3,4\}$, $\{6,3,5\}$, $\{6,3,6\}$.*

All l_1-rigid regular tilings are the bipartite ones; all bipartite ones (except γ_{d+1} and δ_d themselves) embed into $\mathbf{Z}_\infty$.

Proof of Theorem 4. In fact, we review all cases of Tables 3-5. All compact cases (on first 5 rows, columns of Table 3 and first 6 rows, columns of Table 4) were decided in [DSt97]. Non-embeddability for all 14 star-polytopes and star-honeycombs (in Tables 3, 4) was established in Section 4. It remains 11, 2, 5 non-compact tilings of hyperbolic 3-, 4-, 5-space; we will show that 7, 1, 1, respectively, of them are embeddable into $\mathbf{Z}_\infty$, while 8 others are not 5-gonal.

The tilings $\{3,4,3,4\}$, $\{3,4,3,3,3\}$, $\{3,3,4,3,3\}$, $\{3,4,3,3,4\}$ have non-5-gonal graph $K_5 - K_3$ as induced subgraph of the cell. $\{3,6,3\}$ (respectively, $\{3,4,4\}$) contain induced $K_5 - K_3$, because each its edge is common to 3 (respectively, to 4) triangles. $\{3,3,6\}$ is a simplicial manifold with 6 triangles on an edge; taking 1-st, 3-rd and 5-th of them, we get again induced $K_5 - K_3$. A particularity of $T := \{3,3,3,4,3\}$ is that the cell β_4 of its vertex figure $De(D_4)$ is also the equatorial section of the cell β_5 of T. All neighbors of a vertex s of T form $De(D_4)$. Take an isometric subgraph $K_5 - K_3$ in $De(D_4)$, given in [DSt98a]. The vertex s

is a neighbor of each of its five vertices; obtained 6-vertex graph is non-5-gonal graph of diameter 2, which is, using above particularity of T, is an induced subgraph of T. (Compare with embeddable tiling $\{4,3,3,4,3\}$ by γ_5, having the same vertex figure.) All seven above tilings are not 5-gonal, because any induced graph of diameter 2 is isometric. Finally, each edge of $\{5,3,6\}$ is common to 6 disjoint pentagons; taking 1-st, 3-rd and 5-th of them we obtain non-5-gonal 11-vertex induced subgraph of diameter 4 of $\{5,3,6\}$; a routine check shows that it is isometric.

Other hyperbolic tilings embed into $\mathbf{Z}_\infty$, because of Lemma 4 below; it is easy to find reflections, required by Lemma 4 in each case. It is easy to check l_1-rigidity for all (except of Tetrahedron, which is not l_1-rigid) cases of embedding for dimension 2. Now, any bipartite embeddable graph is l_1-rigid, because it has scale 1. The proof is complete. Q.E.D.

Let T be any (not necessary regular) convex d-polytope or tiling of Euclidean or hyperbolic d-space by convex polytopes, such that the skeleton is a bipartite graph. (We admit infinite cells and, if regular, infinite vertex figures.) Then the set of its edges can be partitioned into *zones*, i.e. sequences of edges, such that any edge of a sequence is the opposite to the previous one on a 2-face (which should, therefore, be even).

Lemma 4. *Let T is as above; suppose that the mid-points of edges of each zone lie on hyperplanes, different for each zone, which are (some of) reflection hyperplanes of T and perpendicular to edges of their zones. Then T embeds into $\mathbf{Z}_m$ with m no more than the number of zones.*

Proof of Lemma 4. It follows directly from the fact that each geodesic path (in the skeleton of T) intersects any zone in at most one edge. Q.E.D.

Remark 3. Embedding of any bipartite regular tiling can be obtained, using Lemma 4. The reflections, required by Lemma 4 (let us call them *zonal* reflections) generate, because of simple connectedness of T, a vertex-transitive group of automorphisms of T (call it *zonal* group); so T is uniform and the zonal group is generated by the zonal reflections of all edges incident to a fixed vertex of T. For any fixed 2-face of T, which is a $2k$-gon, let $m_1,\dots,m_k$ be the zonal reflections of its edges, considered in the cyclic order. Then the product $m_1\dots m_k m_1\dots m_k = \langle 1\rangle$ (i.e. $m_1\dots m_k$ is an involution) and those relations, for all 2-faces around a fixed vertex of T, are all defining relations for the zonal group of T. So, the zonal group is not 2-transitive on vertices. For example, the zonal group of Archimedean truncated β_3 is an 1-transitive subgroup of index 2 of the octahedral group $\mathrm{Aut}(T) = O_h$, which is 2-transitive. Also, a polytope in the conditions of Lemma 4 is not necessary zonotope. For example, any centrally-symmetric non-Archimedean (by choice of the length of truncation) truncated β_3 fits in it; it is a zonohedron in original sense of Fedorov, but not in usual sense of Minkowski (with all edges of each zone having same length).

Remark 4. All infinite families of regular tilings are embeddable. In fact, m-gons, $\delta_{n-1} = \mathbf{Z}_n$, $\gamma_n = H_n$, α_n, β_n are embeddable and, moreover, first three are l_1-rigid. But embeddings of skeletons of α_n and, for $n \geq 4$, β_n, is more complicate. It is considered in detail (in terms of corresponding complete graph K_{n+1} and Cocktail-Party graph $K_{n\times 2}$ in Chapter 23 [DLa97] and Section 4 of Chapter 7 [DLa97], respectively. Any α_n, $n \geq 3$, is not l_1-rigid, i.e. it admits at least two different embeddings. We give now two embeddings of α_n into m-cubes with scale λ , realizing, respectively, maximum and minimum of m/λ. The first one is $\alpha_n \to (1/2)H_{n+1}$. Now define $m_n = 2n/(n+1)$ for odd n and $= (2n+2)/(n+2)$ for even n; define λ_n be the minimal even positive number t such that tm_n is an integer. Then α_n embeds into tm_n-cube with scale λ_n; for example, α_4 embeds into 10-cube with scale 6. Any β_n, $n \geq 4$, is not l_1-rigid. All embeddings of β_n are into 2λ-cube with any such even scale λ that α_{n-1} embeds into m-cube, $m \leq 2\lambda$, with scale λ. For minimal such scale, denote it μ_n, the following is known: $n > \mu_n \geq 2\lceil n/4 \rceil$ with equality in the lower bound for any $n \leq 80$ and, in the case of n divisible by 4, if and only if there exists an Hadamard matrix of order n. In particular, $\beta_3 \to (1/2)H_4$, $\beta_4 \to (1/2)H_4$ (in fact, they coincide as 4-polytopes, but there are two embeddings), and β_5 embeds only with scale 4 (into H_8).

Remark 5. This note finalizes the study of embeddability for regular tilings done in [DSt96], [DSt97]; we correct now following misprints there: a) in the sentence "Any l_1-graph, not containing K_n, is l_1-rigid" on p.1193 [DSt96], should be K_4 instead of K_n; b) in the sentence, on p.1194 [DSt96], about partitions of Euclidean plane, embeddable into $\mathbf{Z}_m$, $m < \infty$, should be $\leq$ instead of $<$; c) in the sentence about Föppl partition on p.1292 [DSt97], should be α_3 and truncated α_3 instead of α_3.

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On the homotopy theory of arrangements, II

Michael Falk and Richard Randell

Abstract.

In "On the homotopy theory of arrangements" published in 1986 the authors gave a comprehensive survey of the subject. This article updates and continues the earlier article, noting some key open problems.

Let M be the complement of a complex arrangement. Our interest here is in the topology, and especially the homotopy theory of M, which turns out to have a rich structure. In the first paper of this name [37], we assembled many of the known results; in this paper we wish to summarize progress in the intervening years, to reiterate a few key unsolved questions, and propose some new problems we find of interest.

In the first section we establish some terminology and notation, and discuss general homotopy classification problems. We introduce the matroid-theoretic terminology that has become more prevalent in the subject in recent years. In this section we also sketch Rybnikov's construction of arrangements with the same matroidal structure but non-isomorphic fundamental groups. In Section 2 we consider some algebraic properties of the fundamental group of the arrangement. Properties of interest include the lower central series, the Chen groups, the rational homotopy theory of the complement, and the cohomology of the group. At the time of our first paper many questions in this area were in flux, so we make a special effort here to clarify the situation. The group cohomology is naturally of interest in the third section as well, which focuses on when or if the complement is aspherical. It is this property which fostered much of the initial interest in arrangements (in the guise of the pure braid space); it is of interest that the determination of when

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the complement is aspherical is far from settled. Finally, in the fourth section we consider what one might call the topology of the fundamental group. We describe group presentations that have been discovered since the publication of [37], including the recent development of braided wiring diagrams. We also sketch the considerable progress in the study of the Milnor fiber associated with an arrangement.

In 1992 the long-awaited book *Arrangements of Hyperplanes,* by Peter Orlik and Hiroaki Terao appeared, to the delight of all of us working in arrangements. We refer the reader to this text as a general reference on arrangements, and adopt their notation and terminology except where specified. We also mention that perhaps the most interesting development in arrangements in the last ten years involves the deep and fascinating connections with hypergeometric functions. We are pleased to refer the reader to the lecture notes of Orlik and Terao [64] from the 1998 Tokyo meeting for a comprehensive exposition of this material.

$\mathcal{S}$1. Combinatorial and topological structure

One significant change in the study of the homotopy theory of arrangements since the publication of [37] has been the introduction of matroid-theoretic terminology and techniques into the subject. In this section we review this approach and describe progress toward the topological classification of hyperplane complements. Refer to [89, 66] for further details on matroids.

1.1. The matroid of an arrangement

Let $V = \mathbb{C}^\ell$ and let $\mathcal{A} = \{H_1, \dots, H_n\}$ be a central arrangement of hyperplanes in V. For each hyperplane H_i choose a linear form $\alpha_i \in V^*$ with $H_i = \ker(\alpha_i)$. The product $Q(\mathcal{A}) = \prod_{i=1}^n \alpha_i$ is the *defining polynomial* of the arrangement.

The *underlying matroid* $G(\mathcal{A})$ of $\mathcal{A}$ is by definition the collection of subsets of $[n] := \{1, \dots, n\}$ given by

$$G(\mathcal{A}) = \{S \subseteq [n] \mid \{\alpha_i \mid i \in S\} \text{ is linearly dependent}\}.$$

Elements of $G = G(\mathcal{A})$ are called *dependent sets.* Minimal dependent sets are called *circuits.* Independent sets and bases are defined in the obvious way. The *rank* $\mathrm{rk}(S)$ of a set $S \subseteq [n]$ is the size of a maximal independent subset of S. The rank of G (or $\mathcal{A}$) is $\mathrm{rk}([n])$. The *closure* $\overline{S}$ of a set S is defined by

$$\overline{S} = \bigcup \{T \subseteq [n] \mid T \supseteq S \text{ and } \mathrm{rk}(T) = \mathrm{rk}(S)\}.$$

A set S is *closed* if $\overline{S} = S$. Closed sets are also called *flats*. The collection of closed sets, ordered by inclusion, forms a geometric lattice $L(G)$ which is isomorphic to the intersection lattice $L(\mathcal{A})$ defined and studied in [65]. The isomorphism $L(G) \to L(\mathcal{A})$ is given by $S \mapsto \bigcap_{i \in S} H_i$.

Thus the matroid $G(\mathcal{A})$ contains the same information as the intersection lattice $L(\mathcal{A})$. One of the simple advantages of the matroid-theoretic approach is the fact that the matroid $G(\mathcal{A})$ is determined uniquely by any of a number of different pieces of data besides the set of flats. For instance, the set of circuits, the rank function, or the set of bases, each determine the matroid, and thus the intersection lattice. Besides giving a nice conceptual framework for the combinatorial structure of arrangements, techniques and deep results from the matroid theory literature have been applied with some benefit in the study of the topology of arrangements.

The line generated by α_i in V^* depends only on H_i, and thus $\mathcal{A}$ determines a unique point configuration $\mathcal{A}^*$ in the projective space $\mathbb{P}(V^*) \cong \mathbb{C}P^{\ell-1}$. The dual point configuration $\mathcal{A}^*$ can be used to depict the combinatorial structure of an arrangement in case $\mathrm{rk}(\mathcal{A}) \leq 4$ if the defining forms α_i have real coefficients. (In this case $\mathcal{A}$ is called a *complexified* arrangement.) One merely plots the points α_i in a suitably chosen affine chart $\mathbb{R}^{\ell-1}$ in the real projective space $\mathbb{R}P^{\ell-1}$, for instance by scaling the α_i so that the coefficient of x_1 in each is equal to 1, and then ignoring this coefficient. Dependent flats of rank two (or three) are seen in these affine configurations as lines (or planes) containing more than two (or three) points. These lines and planes are usually explicitly indicated in the picture. This is especially useful for arrangements of rank four. Since the hyperplanes are indicated by points in $\mathbb{R}^3$, they don't obscure the internal structure as a collection of affine planes in $\mathbb{R}^3$ would (see Figure 5). These depictions of projective point configurations are generalized to give *affine diagrams* of arbitrary matroids. Dependent flats are again explicitly indicated with "lines" or "planes", which in the general case may not be straight or flat in the euclidean sense. It is common to refer to flats of rank one, two, or three in an arbitrary matroid as points, lines, or planes respectively. These diagrams are useful for the study of arrangements which are not complexified real arrangements (see Figures 1 and 2).

1.2. Basic topological results

The seminal result in the homotopy theory of arrangements is the calculation of the cohomology algebra of the complement $M = M(\mathcal{A}) := \mathbb{C}^\ell - \bigcup_{i=1}^n H_i$ by Orlik and Solomon [63]. Motivated by work of Arnol'd

[1], and using tools established by Brieskorn [10], they gave a presentation of $H^*(M)$ in terms of generators and relations. The presentation $A(\mathcal{A})$ depends only on the underlying matroid $G = G(\mathcal{A})$, and is now called the *Orlik-Solomon (or OS) algebra* of G. Henceforth we will refer to the OS algebra $A(\mathcal{A})$ rather than the cohomology ring $H^*(M)$. The algebra $A(\mathcal{A})$ is defined as the quotient of the exterior algebra on generators $e_1, \dots, e_n$ by the ideal I generated by "boundaries" of dependent sets of G. See [65] for a precise definition.

This result of [63] gave rise to a collection of "homotopy type" conjectures, which assert that various homotopy invariants of the complement depend only on $G(\mathcal{A})$. A great deal of research in the homotopy theory of arrangements has been focused on conjectures of this type. Note that such conjectures may have "weak" or "strong" solutions: one may show that the invariant depends only on the matroid, or one may give an algorithm to compute the invariant from matroidal data.

The major positive result in this direction is the lattice-isotopy theorem, proved by the second author in [77]. It asserts that the homotopy type, indeed the diffeomorphism type of the complement remains constant through a "lattice-isotopy," that is, a one-parameter family of arrangements in which the intersection lattice, or equivalently, the underlying matroid remains constant.

This result is often recast in terms of matroid realization spaces, which are related to the well-known "matroid stratification" of the Grassmannian. We describe this connection. The defining forms α_i of $\mathcal{A}$ can be identified with row vectors, and thus the arrangement $\mathcal{A}$ can be identified with an $n \times \ell$ matrix R over $\mathbb{C}$. This matrix is called a *realization* of the underlying matroid. Two realizations R and R' are equivalent if there is a nonsingular diagonal $n \times n$ matrix S and a nonsingular $\ell \times \ell$ matrix T such that $R' = SRT$. The corresponding arrangements will then be linearly isomorphic. The set of equivalence classes of realizations of a fixed matroid G is called the (projective) realization space $\mathcal{R}(G)$ of G. Now assume the matrix R has rank ℓ, i.e., that $\mathcal{A}$ is an essential arrangement. Then the column space of R is an ℓ-plane P_R (sometimes denoted $P_{\mathcal{A}}$) in $\mathbb{C}^n$. Note that an isomorphic copy of the arrangement $\mathcal{A}$ inside P_R is formed by the intersection of P_R with the coordinate hyperplanes in $\mathbb{C}^n$. Postmultiplying A by a nonsingular matrix doesn't affect P_R. Thus we see that the realization space $\mathcal{R}(G)$ can be identified with a subset $\Gamma(G)$ of the space of orbits of the diagonal $(\mathbb{C}^*)^n$ action on the Grassmanian $\mathcal{G}_\ell(\mathbb{C}^n)$ of ℓ-planes in $\mathbb{C}^n$. The subsets $\widehat{\Gamma}(G) = \{P_R \mid R \text{ is a realization of } G\} \subseteq \mathcal{G}_\ell(\mathbb{C}^n)$ are called *matroid strata*, although they do not comprise a stratification in the usual sense, since the closure of a stratum may not be a union of strata [85]. These strata play a central role in the theory of generalized hypergeometric functions, especially when the original arrangement $\mathcal{A}$ is generic. The topology of the strata themselves can be as complicated as arbitrary affine varieties over $\mathbb{Q}$ even for matroids of rank three, by a celebrated theorem of Mnëv [59]. These strata are connected by "deletion maps," whose fibers are themselves complements of arrangements [5, 30].

Realizations in $\Gamma(G)$ correspond to arrangements which have the same underlying matroid G, as determined by the arbitrary ordering of the hyperplanes. Thus, for the study of homotopy type as a function of intrinsic combinatorial structure (i.e., without regard to labelling), the true "moduli space" for arrangements should be the quotient of $\mathcal{G}_\ell(\mathbb{C}^n)$ by the action of the $S_n \times (\mathbb{C}^*)^n$. Then linear isomorphism classes of arrangements with isomorphic underlying matroids (or isomorphic intersection lattices) correspond to points of the orbit space $\Gamma(G)/\operatorname{Aut}(G)$.

Randell's lattice-isotopy theorem can be reformulated as follows: two arrangements which are connected by a path in $\widehat{\Gamma}(G)$ (or $\Gamma(G)$) have diffeomorphic complements. Thus one is led to the difficult problem of understanding the set of path components of $\Gamma(G)/\operatorname{Aut}(G)$.

More detailed combinatorial data will suffice to uniquely determine the homotopy type of the complement. For instance, in the case of complexified real arrangements, the defining forms $\alpha_i, 1 \leq i \leq n$ determine an underlying *oriented matroid.* This is most easily described in terms of bases: the matroid $G(\mathcal{A})$ is determined by the collection $\mathcal{B}$ of maximal independent subsets $B \subseteq [n]$. These can naturally be identified with ordered subsets of $[n]$. The *oriented* matroid $\widehat{G}(\mathcal{A})$ is then a partition $\mathcal{B} = \mathcal{B}_+ \cup \mathcal{B}_-$ of the set of ordered bases of $G(\mathcal{A})$ into positive and negative bases, corresponding to the sign of the (nonzero) determinant of the corresponding ordered sets of linear forms. The work of Salvetti [81], as refined by Gelfand and Rybnikov [39], shows that the underlying oriented matroid of a complexified real arrangement uniquely determines the homotopy type of the complement. In fact one can construct a partially ordered set $\mathcal{K}(\widehat{G})$ directly from the oriented matroid $\widehat{G}$ whose "nerve", or collection of linearly ordered subsets, forms a simplical complex homotopy equivalent to the complement. In subsequent work, Björner and Ziegler [7] (see also Orlik [61]) generalized the construction to arbitrary arrangements (or arrangements of subspaces), in terms of combinatorial structures called *2-matroids* [7] or *complex oriented matroids* [93]. They showed that this detailed combinatorial data determines the complement up to piecewise-linear homeomorphism.

The relation between Randell's lattice-isotopy theorem and the combinatorial complexes of [81, 39, 7, 61] has not been fully explored. In particular, it would be interesting to cast the notion of lattice-isotopy in combinatorial terms, i.e., as a sequence of elementary "isotopy moves" on the posets $\mathcal{K}(\widehat{G})$ which leave the homotopy type of the nerve unchanged. A first step in this direction was accomplished in [29]. We pose this as our first open problem.

Problem 1.1. *Prove a combinatorial lattice-isotopy theorem, that "isotopic" (complex) oriented matroids (with the same underlying matroid) determine homotopy equivalent cell complexes.*

1.3. Homotopy classification

The fundamental question whether the homotopy type of $M(\mathcal{A})$ is uniquely determined by $G(\mathcal{A})$ was answered in the negative by Rybnikov in [80]. The basic building block of his construction is the MacLane matroid, whose affine diagram is pictured in Figure 1. For this matroid

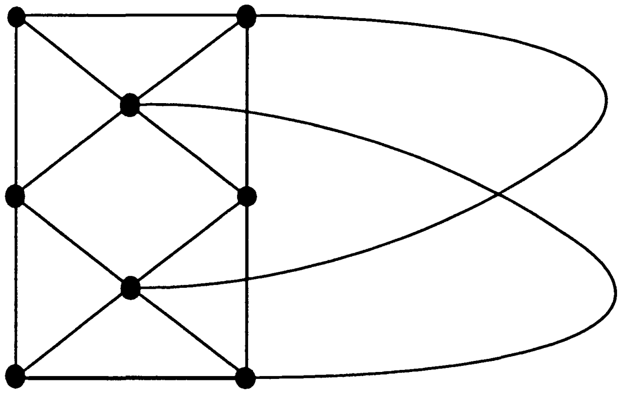


FIGURE 1. The MacLane matroid

G, the realization space $\mathcal{R}(G)$ consists of two conjugate complex realizations R and $\overline{R}$, corresponding to arrangements $\mathcal{A}$ and $\overline{\mathcal{A}}$. One can "amalgamate" these realizations along one of the three-point lines (rank-two flats) to form arrangements $\mathcal{A}*\mathcal{A}$ and $\mathcal{A}*\overline{\mathcal{A}}$ of rank four with thirteen hyperplanes. These arrangements have the same underlying matroid, of rank four on 13 points, pictured in Figure 2. Rybnikov establishes some special properties of this matroid, for instance, that any automorphism of the OS algebra arises from a matroid automorphism, which must preserve or interchange the factors of the amalgamation. Using these he is able to show that the arrangements $\mathcal{A}*\mathcal{A}$ and $\mathcal{A}*\overline{\mathcal{A}}$ have nonisomorphic fundamental groups, since the first has an automorphism which switches the factors preserving orientations of the natural generators, while the only automorphism of the second which switches factors must reverse orientations. Refer to Section 4.1 for a more detailed description of the

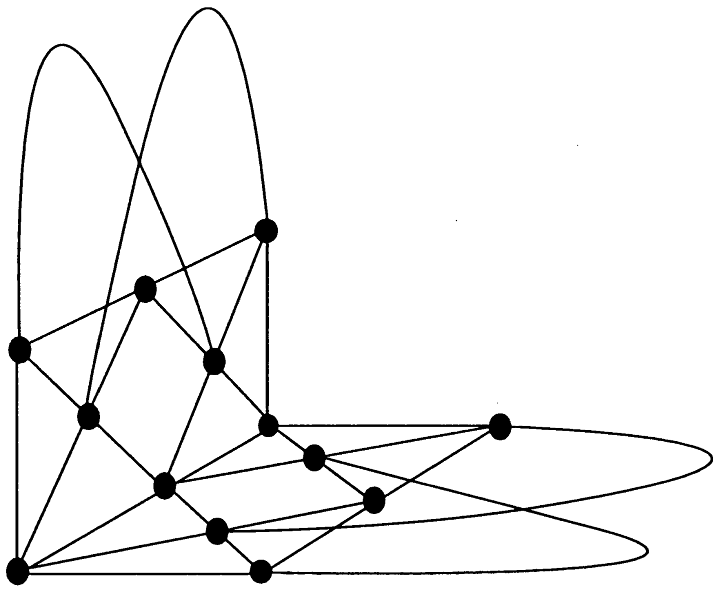


FIGURE 2. The Rybnikov matroid

fundamental group. Rybnikov actually uses the rank-three truncation of this matroid, and 3-dimensional generic sections of these arrangements, but this operation does not affect the fundamental group.

The last part of Rybnikov's argument is quite delicate and very specialized. None of the known invariants of fundamental groups, for instance those described elsewhere in this paper, will distinguish these two groups.

Problem 1.2. *Find a general invariant of arrangement groups that distinguishes the two Rybnikov arrangements, and generalize his construction.*

To date this is the only known example of this phenomenon. In particular it is not known if this behavior is exhibited by complexified arrangements.

Problem 1.3. *Prove that the underlying matroid of a complexified arrangement determines the homotopy type, or find a counter-example.*

Partial results along these lines were obtained by Jiang and Yau [44] and Cordovil [18]. In [44] a condition on the underlying matroid G is given which implies that the realization space of G is path-connected, so that any two arrangements realizing G have diffeomorphic complements by the lattice-isotopy theorem. In [18] it is shown that complexified arrangements whose underlying matroids are isomorphic via a correspondence which preserves a (geometrically defined) "shelling order" will have identical braid-monodromy groups.

The extent to which arrangements with non-isomorphic matroids can have homotopy equivalent complements has also been studied (see, e.g., [28, 29, 13, 24, 32]) with some degree of success. One approach to this problem is purely combinatorial, namely to classify OS algebras up to graded algebra isomorphism. This approach is adopted in [28, 32, 24]. A powerful invariant is developed in [32], sufficient to distinguish all known non-trivial examples which are not already known to be isomorphic.

At this point all known examples of matroids with isomorphic OS algebras can be explained by two simple operations [35, 72]. The first of these is a construction involving a well-known equivalence of affine arrangements arising from the "cone-decone" construction [65, Prop. 5.1], along with the trivial fact that the complement of the direct sum of affine arrangements, denoted $\coprod$ in [65], is diffeomorphic to the cartesian product of the complements of the factors. In fact this construction can be applied to arbitrary pairs of matroids to yield central arrangements with non-isomorphic matroids and diffeomorphic complements [24, 35]. This construction always yields arrangements with non-connected (i.e., nontrivial direct sum) matroids. Jiang and Yau [45] show that this phenomenon cannot occur in rank three, that is, the diffeomorphism type of the complement of a rank-three arrangement uniquely determines the underlying matroid. Thus the rank-three examples of [29], which have non-isomorphic underlying matroids, have complements which are homotopy equivalent but not diffeomorphic.

The second operation which yields isomorphic OS algebras is truncation. It is shown in [72] that the truncations of two matroids with isomorphic OS algebras will have the same property. (It is not known if truncation preserves homotopy equivalence). These two "moves" suffice to explain the examples produced in [65, 29], indeed all known examples of this phenomenon. Thus it seems an orderly classification of OS algebras may be within reach.

Problem 1.4. *Classify OS algebras up to graded isomorphism.*

In the alternative, we suggest the following.

Problem 1.5. *Find a pair of arrangements with homotopy equivalent complements and whose underlying matroids are non-isomorphic, connected, and inerectible (i.e., not truncations).*

Cohen and Suciu in [12, 13, 14] approach this same problem of homotopy classification using invariants of the fundamental group. Their approach has the advantage that it may also be used to distinguish the complements of arrangements with the same underlying matroid. Some of this work is described elsewhere in this paper. Here we merely remark on the surprising connection described in [14, 55, 54] between the characteristic varieties of [53] arising from the Alexander invariant of the fundamental group, and the resonant varieties of [32], which arise from the OS algebra.

$\mathcal{S}2$. Algebraic properties of the group of an arrangement

The topology of hyperplane complements seems to be to a large extent controlled by the fundamental group. These "arrangement groups" have relatively simple global structure, being pieced together out of free groups in a fairly straightforward way (see Sections 4.1 and 3.3), but have surprisingly delicate fine structure. At the time of the writing of [37] there was a great deal of activity around the study of the lower central series of these groups, and connections with rational homotopy theory and Chen's theory of iterated integrals. In this section we report on progress in these areas in the intervening years.

2.1. The LCS formula, quadratic algebras, rational $K(\pi, 1)$ and parallel arrangements

Discoveries of Kohno [48] and the authors [36] showed that Witt's formula for the lower central series of finitely generated free Lie algebras (or, equivalently, free groups) generalized to a wide class of hyperplane complements. The so-called LCS formula reads

$$\prod_{n\geq 1}(1-t^n)^{\phi_n} = \sum_{i\geq 0} b_i(-t)^i,$$

relating the ranks ϕ_n of factors in the lower central series of the fundamental group $\pi_1(M)$ to the betti numbers $b_i = \dim(A^i(\mathcal{A}))$ of M. In [36, 43] it is shown that this formula holds for all fiber-type arrangements. These are arrangements whose underlying matroids are supersolvable [87]. This result was ostensibly extended to *rational* $K(\pi, 1)$ arrangements in [26, 47]. (See also Section 2.2.) We refer the reader to [26, 65] for a precise definition of rational $K(\pi, 1)$ arrangement. Briefly, if $\mathcal{S}$ is the 1-minimal model of M (or, equivalently, of $A(\mathcal{A})$), then $\mathcal{A}$ is rational $K(\pi, 1)$ if $H^*(\mathcal{S}) \cong A(\mathcal{A})$. It is shown in [26] that fiber-type arrangements are rational $K(\pi, 1)$.

The technical results of [36] were used in [38] to show that fundamental groups of fiber-type arrangements (in particular, the pure braid group) are residually nilpotent. This result turned out to be important for the theory of knot invariants of finite type [84].

The situation surrounding the LCS formula was very much in flux during the preparation of [37], a fact reflected in the equivocal footnotes in the table of implications in that paper. The situation has been clarified somewhat in the meantime. Our purpose here is to briefly summarize the current understanding of these issues.

Recall that an arrangement of rank three is *parallel* if for any four hyperplanes of $\mathcal{A}$ in general position, there is a fifth hyperplane in $\mathcal{A}$ containing two of the six pairwise intersections. The OS algebra $A(\mathcal{A})$ is *quadratic* if the relation ideal I (defined in Section 1.2) is generated by its elements of degree two. We will sometimes say $\mathcal{A}$ is quadratic. This is a combinatorial condition, which will be discussed in further detail in Section 3.2. In general the quotient of the exterior algebra $\Lambda(e_1, \dots, e_n)$ by the ideal generated by the degree two elements of I is called the *quadratic closure* of $A(\mathcal{A})$, denoted $\overline{A}(\mathcal{A})$. Here is a summary of cogent results established in [26, 27].

(i) If $\mathcal{A}$ is a rational $K(\pi, 1)$ arrangement, then $\mathcal{A}$ is quadratic.
(ii) Every parallel arrangement is quadratic.
(iii) Every rational $K(\pi, 1)$ arrangement satisfies the LCS formula.
(iv) Every quadratic arrangement satisfies the LCS formula at least to third degree.

In [37] we cited an unpublished note which claimed that every parallel arrangement is a rational $K(\pi, 1)$. Using the construction of [26], in 1994 Falk wrote a *Mathematica* program to compute ϕ_4, and checked the smallest example of a parallel, non-fiber-type arrangement of rank 3. This arrangement, labelled X_2 in [37], consists of the planes $x \pm z = 0, y \pm z = 0, x + y \pm 2z = 0$, and $z = 0$, and is pictured in Figure 3. We obtained the result $\phi_4 = 15$, whereas the LCS formula would predict $\phi_4 = 10$.

So the implications

$$\begin{aligned} \text{parallel} &\implies \text{rational } K(\pi, 1), \\ \text{quadratic} &\implies \text{rational } K(\pi, 1), \\ \text{parallel} &\implies \text{LCS}, \end{aligned}$$

and

$$\text{quadratic} \implies \text{LCS}$$

recorded in [37] are all false.

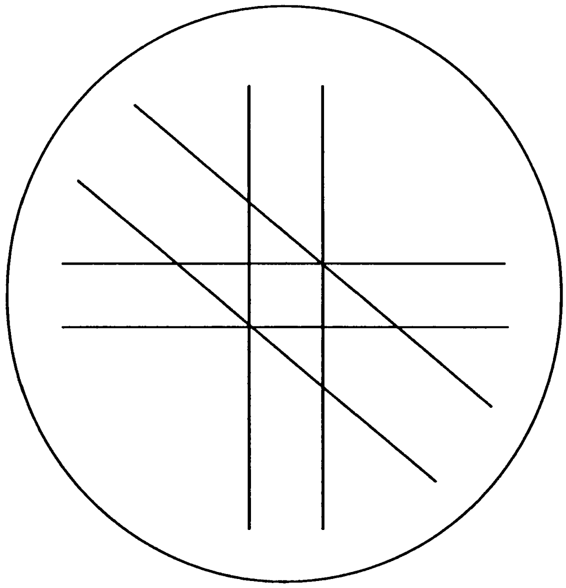


FIGURE 3. The arrangement X_2

Subsequently, work of Shelton-Yuzvinsky [82], and Papadima-Yuzvinsky [67] provided further clarification. Let $\mathcal{L}$ denote the holonomy Lie algebra of M, the quotient of the free Lie algebra on generators $x_1, \ldots x_n$ by the image of the map $H_1(M) \to \Lambda^2(H_1(M))$ dual to the cup product. Let $U = U(\mathcal{A})$ be its universal enveloping algebra, a dual object to the 1-minimal model $\mathcal{S}$. The Hilbert series of U is $\prod_{n\geq 1}(1 - t^n)^{-\phi_n}$. Kohno constructs a chain complex (R, δ) which, when exact, forms a resolution of $\mathbb{Q}$ as a trivial U-module. In this case $\mathcal{A}$ is a rational $K(\pi, 1)$ arrangement, and the LCS formula holds.

Shelton and Yuzvinsky [82] realized that $U(\mathcal{A})$ is the Koszul dual of the quadratic closure of $A(\mathcal{A})$. We refer the reader to [82] for a precise definition; loosely speaking, the defining relations for the Koszul dual U form the orthogonal complement to those of $\overline{A}(\mathcal{A})$ inside the tensor product $T_2(A^1(\mathcal{A}))$. They observed that the Aomoto-Kohno complex (R, δ) is the usual Koszul complex of U, and thus is exact if and only if U is a Koszul algebra — U is *Koszul* iff $\mathrm{Ext}_U^{p,q}(\mathbb{Q}, \mathbb{Q}) = 0$ unless $p = q$. It follows from this that $A(\mathcal{A})$ is a quadratic algebra. (This observation was also made by Hain [41].) The LCS formula is then a consequence of Koszul duality. They give a combinatorial proof that $A(\mathcal{A})$ is quadratic and that $U(\mathcal{A})$ is Koszul if $\mathcal{A}$ is a supersolvable arrangement.

The results of [82] were strengthened and extended in [67] to give a description of $H^*(\mathcal{S})$ in terms of Koszul algebra theory, for more general spaces. In particular, it is shown in [67] that $\mathcal{A}$ is rational $K(\pi, 1)$ if only if the OS algebra is Koszul. In addition, Papadima and Yuzvinsky gave an alternate proof that the arrangement X_2 above fails the LCS formula. Finally, using a "central-to-affine" reduction argument, they were able to prove the following.

Theorem 2.1 ([67]). *For arrangements of rank three, the LCS formula holds if and only if the arrangement is fiber-type.*

Peeva [71] applies techniques of commutative algebra and Gröbner basis theory to obtain a short proof that supersolvable arrangements satisfy the LCS formula, in addition to other related computational results.

In research closely related to the lower central series of arrangement groups, Kohno used the iterated integral/holonomy Lie algebra approach to construct representations of the (pure) braid group, and more generally to study the monodromy of local systems over hyperplane complements. This work is also closely tied to the theory of generalized hypergeometric functions. See [49] for a description of these developments. Cohen and Suciu pursued similar ideas using methods more closely connected to those of [36] in [15].

2.2. The D_n reflection arrangements

The fundamental groups of the reflection arrangements of type D_n have been studied using some of the technical machinery of [36]. Note that these arrangements, for $n > 3$, are not supersolvable. The author of [58] constructs a presentation which he claims presents these fundamental groups as "almost direct products" in the sense of [36, 15]. He used this to show that these groups are residually nilpotent. In 1994 we tried to use this presentation to get more precise calculations for the lower central series of these groups, at least for $n = 4$. In fact we found that the presentation in [58] is not correct. Even for the D_3 arrangement, which is supersolvable, the results one deduces from [58] do not jibe with the LCS formula, which is known to hold for D_3. In [56] Liebman and Markushevich adopt a different approach and derive a different presentation to show that the D_n arrangement groups are residually nilpotent.

It was in the course of this research that we started computing ϕ_4 by machine. In addition to finding the counterexample X_2 described above, we also computed $\phi_4 = 183$ for the D_4 reflection arrangement. The LCS formula yields $\phi_4 = 186$. So the D_4 arrangement fails the LCS formula, contrary to another assertion [46] reported on in [37].

The work of Shelton and Yuzvinsky [82] make it clear why the argument of [46] for the LCS formula for the D_n reflection arrangements fails: these arrangements, for $n > 3$, do not have quadratic OS algebras, by [26]. Hence the Aomoto-Kohno complex $R.$ cannot be exact for these arrangements.

So we are left with no examples of arrangements which are not supersolvable, yet are rational $K(\pi, 1)$, and no examples of arrangements satisfying the LCS formula which are not rational $K(\pi, 1)$.

Problem 2.2. *Find examples of non-supersolvable or non-rational $K(\pi, 1)$ arrangements satisfying the LCS formula, or prove that such examples do not exist.*

2.3. Work of Cohen and Suciu on the Chen groups

As noted above, the ranks of the quotients in the lower central series of fiber-type arrangements are determined by the betti numbers of the complement. From this point of view, the pure braid groups look like products of free groups (though they are not; see [38].) In the last few years, Cohen and Suciu have introduced the Chen groups into the study of arrangements, providing a computable tool for distinguishing similar arrangements.

The Chen groups of a group G are the lower central series quotients of G modulo its second commutator subgroup G''. If for any group G we let $\Gamma_k(G)$ denote the k^{th} lower central series subgroup, then the homomorphism $G \to G/G''$ induces an epimorphism

$$\frac{\Gamma_k(G)}{\Gamma_{k+1}(G)} \to \frac{\Gamma_k(G/G'')}{\Gamma_{k+1}(G/G'')} = k^{th} \text{ Chen group}$$

Thus the ranks ϕ_k of quotients of lower central series groups are no less than the corresponding ranks θ_k of Chen groups. In the case of the pure braid group, the ranks θ_k are determined in [12]; they are given by the generating function

$$\sum_{k=2}^{\infty} \theta_k t^{k-2} = \binom{n+1}{4} \cdot \frac{1}{(1-t)^2} - \binom{n}{4}$$

In particular, these numbers differ from those for the product of free groups, providing a tidy proof that the pure braid groups are not such products.

Cohen and Suciu [11] provide a detailed study of these groups including a method for their computation from a presentation of the Alexander invariant (see the discussion of presentations of the fundamental group below.) It is interesting that while these groups are very effective in distinguishing similar groups, there is not yet an example of combinatorially equivalent arrangements with different Chen ranks. In particular, they do not distinguish the examples of Rybnikov [80] of combinatorially equivalent, homotopically different arrangements (see Section 1.3).

2.4. Cohomological properties of the fundamental group

In 1972 Deligne [21] proved that for a complexification of a real simplicial arrangement, the complement M is aspherical (also expressed by saying that M is a $K(\pi, 1)$ space.) That is, the universal cover of M is contractible. Since all real reflection arrangements are simplicial, this solved a question raised and partially answered by Brieskorn in [9]. The original study of this sort of problem was the work of Fadell and Neuwirth [25] on the pure braid group. Following [86], the authors introduced in [36] the notion of fiber-type arrangement and observed that for this class M is aspherical, essentially by the iterated fibration argument of Fadell and Neuwirth. So it is natural to ask: for what arrangements is M aspherical? It is known by work of Hattori [42] that not all are — the arrangement defined by $Q = xyz(x + y + z)$ is the simplest example.

Here we wish to touch upon the algebraic consequences of asphericity. Now if M is aspherical, the (known) cohomology of M is isomorphic to the cohomology of the group. Since M has cohomological dimension $\text{rk}(\mathcal{A}) < \infty, \pi_1(M)$ does also. In addition, $\pi_1(M)$ has no torsion, and there is a $K(\pi, 1)$ space, $\pi = \pi_1(M)$, with the homotopy type of a finite complex (namely, M). So here is another open problem:

Problem 2.3. *Are all arrangement groups torsion-free?*

The answer is of course yes for real reflection arrangements and for fiber-type (or supersolvable) arrangements. One approach to this question is to show that all arrangement groups are orderable. Here we say a group G is *orderable* provided that there is a linear order $<$ on G so that $g < h$ implies $cg < ch$ for all $c \in G$. It follows easily that an orderable group has no torsion. The braid group was shown orderable by Dehornoy in [20]; at the Tokyo meeting L. Paris proved that the group of a fiber-type arrangement is orderable [68]. It is not known whether all arrangement groups are orderable. Note that the group of an arrangement has a finite presentation of a fairly restricted type, as described in Section 4.1, and that the relators all lie in the commutator subgroup.

There are some useful observations concerning these ideas in [78]. For instance, we have the following theorem.

Theorem 2.4. *For $j \geq 2$ the Hurewicz map*

$$\phi : \pi_j(M) \to H_j(M)$$

is trivial.

As a consequence, the second homology of $\pi_1(M)$ is isomorphic to $H_2(M)$. In addition, it is mentioned there that the arrangement defined by

$$Q = xyz(y+z)(x-z)(2x+y)$$

has the property that there is no arrangement with aspherical complement with the same intersection lattice in rank one and two. The following result is also proved in [78].

Theorem 2.5. *The complement of a central arrangement of rank three is aspherical provided that the fundamental group has cohomological dimension three and is of type FL.*

A group π is *type FL* provided that $\mathbb{Z}$ (as a trivial $\mathbb{Z}[\pi]$-module) has a finite resolution by free $\mathbb{Z}[\pi]$-modules. An equivalent statement is that there should exist a *finite* CW complex which is a $K(\pi, 1)$-space. Theorem 2.5 shows that for central rank three arrangements asphericity is determined by the fundamental group.

$\mathcal{S}$3. Arrangements with aspherical complements

Much of the early history of the topology of arrangements revolves around the "$K(\pi, 1)$ problem," the problem of determining which arrangements have aspherical complements. (Such an arrangement is called a $K(\pi, 1)$ *arrangement.*) This history is described in some detail in [37] (see also Section 2.4). In addition, we proved an *ad hoc* necessary condition [37, Thm. 3.1] for asphericity involving "simple triangles," and introduced the notion of formal arrangement, which was shown to be a necessary condition for $K(\pi, 1)$ and rational $K(\pi, 1)$ arrangements. A great deal of progress was made in these areas in the intervening years, which we report on in this section.

3.1. Free arrangements are not aspherical

In our earlier survey, we highlighted the Saito conjecture, that all free arrangements are aspherical. In 1995 Edelman and Reiner [23] provided counterexamples, which we briefly describe.

Let S denote the polynomial ring of V. A linear map $\theta : S \to S$ is a derivation if for $f, g \in S$, we have $\theta(fg) = f\theta(g) + g\theta(f)$. The module of $\mathcal{A}$−derivations is defined by

$$D(\mathcal{A}) = \{\theta \mid \theta(Q) \in QS\}$$

where Q is the defining polynomial of the arrangement. Then the arrangement is *free* provided that $D(\mathcal{A})$ is a free S-module.

It is known [86] that reflection arrangements are free; for their many pleasant properties see [65]. In 1975 K. Saito conjectured that free arrangements should be aspherical. In their study of tilings of centrally symmetric octagons in [23], Edelman and Reiner found the family of arrangements given by

$$Q(\mathcal{A}_\alpha) = xyz(x-y)(x-z)(y-z)(x-\alpha y)(x-\alpha z)(y-\alpha z)$$

with $\alpha \in \mathbb{R}$. They proved that the corresponding arrangements are free for all α, while they are not aspherical for $\alpha \neq -1, 0, 1$. The proof of freeness is direct, using addition-deletion [65, Theorem 4.51] while the non-asphericity follows from the "simple triangle" criterion of [37]. The counter-example $\mathcal{A}_{-2}$ is pictured in Figure 4.

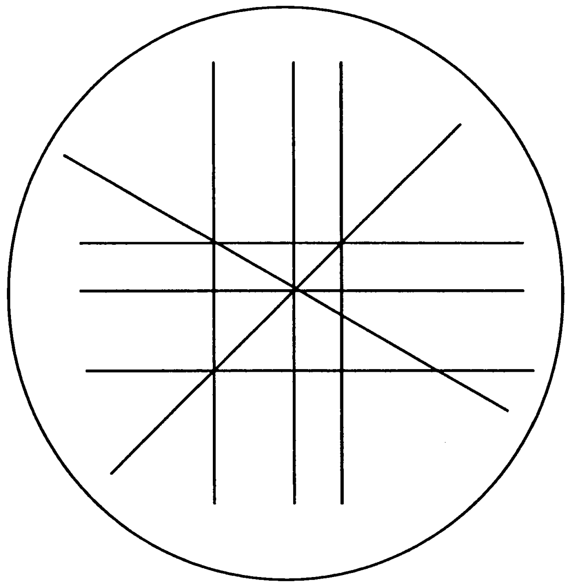


FIGURE 4. Free but not $K(\pi, 1)$

3.2. Formality and related concepts

The fundamental group of arrangement is determined by a generic 3-dimensional section. Based on the idea that $K(\pi, 1)$ arrangements should be extremal in some sense, we developed the notion of formal arrangement in [37]. This has been the subject of several papers since [5, 8, 91, 33], which provide a better understanding of the concept. Here is a "modern" definition, equivalent to the original from [37].

Let $\Phi : \mathbb{C}^n \to V^*$ be given by $\Phi(x) = \sum_{i=1}^n x_i\alpha_i$, where the α_i are the defining forms for $\mathcal{A}$. Let $K = \ker(\Phi)$ and let F be the subspace of K spanned by its elements of weight three (i.e., having three nonzero entries). Then the arrangement $\mathcal{A}$ is *formal* if $F = K$.

The orthogonal complement $K^\perp \subseteq \mathbb{C}^n$ coincides with the point $P_\mathcal{A} \in \mathcal{G}_\ell(\mathbb{C}^n)$ defined in Section 1.2. Thus the arrangement $\mathcal{A}$ is isomorphic to the arrangement in $K^\perp$ formed by the coordinate hyperplanes. In the same way, the orthogonal complement $F^\perp \supseteq K^\perp$ defines an arrangement $\mathcal{A}_F$, called the *formalization* of $\mathcal{A}$. So $\mathcal{A}$ is formal if and only if $\mathcal{A} = \mathcal{A}_F$. If $\mathcal{A}$ is not formal, $\mathcal{A}_F$ has strictly greater rank, and $\mathcal{A}$ is a (not necessarily generic) section of $\mathcal{A}_F$. Also, $\mathcal{A}$ and $\mathcal{A}_F$ have isomorphic generic "planar" (i.e., rank-three) sections.

These properties of formalization were asserted in [37], but the arguments we had in mind were not correct. The clarification described here is due to Yuzvinsky [91]. Examples in [74] show that non-formal arrangements need not be generic sections of their formalizations. The arrangement of Example 2.19 of [74] has the property that the *free* erection of the underlying matroid is not realizable, but (contrary to the assertion in [74]) there is nevertheless a realizable (formal) erection. Matroid "erection" is the reverse of (corank one) truncation; truncation is the matroid-theoretic analogue of generic section. The free erection of an erectible matroid is the unique erection with "the most general position" — see [89].

These observations are enough to establish the following results from [37]. The third assertion follows immediately from the second.

(i) If $\mathcal{A}$ is a $K(\pi,1)$ arrangement, then $\mathcal{A}$ is formal.
(ii) If $\mathcal{A}$ is quadratic, then $\mathcal{A}$ is formal.
(iii) If $\mathcal{A}$ is a rational $K(\pi,1)$ arrangement, then $\mathcal{A}$ is formal.

We asked whether free arrangements are also necessarily formal. This was established by Yuzvinsky.

Theorem 3.1 ([91]). *If $\mathcal{A}$ is a free arrangement, then $\mathcal{A}$ is formal.*

The preceding result was generalized by Brandt and Terao [8]. They define the notion of k-formal arrangement. A formal arrangement has the property that all relations among the defining equations are consequences of relations which are "localized" at rank-two flats, in the sense that an element of K of weight three gives rise to a three-element subset of a rank-two flat. A formal arrangement is *3-formal* if all relations among these local generators of $F = K$ are themselves consequences of relations which are localized at rank-three flats of $\mathcal{A}$. This construction is iterated to define the notion of k-formal arrangement for every $k \geq 2$. See [8] for the precise definition. An arrangement of rank r is automatically k-formal for every $k \geq r$. The original notion of formality coincides with the case $k = 2$.

Theorem 3.2 ([8]). *If $\mathcal{A}$ is a free arrangement of rank r, then $\mathcal{A}$ is k-formal for every $2 \leq k < r$.*

The converse is false [8].

Related work appears in [5], where the authors show that the discriminantal arrangements of Manin and Schechtman [57] (see Section 3.4.2) are formal, and the "very generic" discriminantal arrangements are 3-formal, though none are free.

An arrangement is *locally formal* [91] if, for every flat $X \subseteq [n]$, the arrangement $\mathcal{A}_X = \{H_i \mid i \in X\}$ is formal. Since freeness, quadraticity, and $K(\pi, 1)$-ness are all "hereditary properties," in that they are inherited by the localizations $\mathcal{A}_X$, one has that every free, quadratic, or $K(\pi, 1)$ arrangement is locally formal.

We asked in [37] whether formality is a "combinatorial property", depending only on the underlying matroid. Yuzvinsky constructed counter-examples in [91].

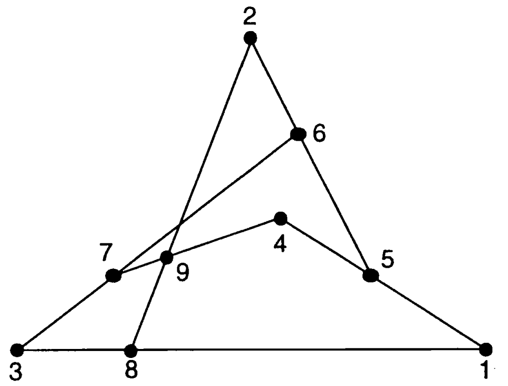
Theorem 3.3 ([91]). *There exist arrangements $\mathcal{A}_1$ and $\mathcal{A}_2$ with the same underlying matroid, such that $\mathcal{A}_1$ is formal and $\mathcal{A}_2$ is not formal.*

In Figure 5 are the dual point configurations of Yuzvinsky's arrangements. The dotted line in Figure 5(b) indicates where to "fold" the configuration to erect it to a rank-four configuration. The nontrivial planes in the erection are

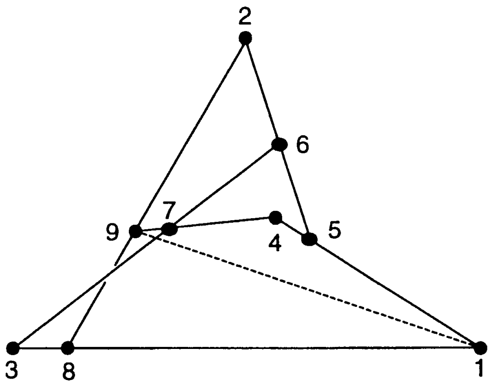
12389, 12456, 13458, 13678, 14579, 23567, 24789, 25689, and 34679.

Note that these two configurations are lattice-isotopic (over $\mathbb{C}$), so neither is free or $K(\pi, 1)$.

If $\mathcal{A}$ is not formal, then the underlying matroid of $\mathcal{A}$ is a *strong map image* (under the identity map) of that of $\mathcal{A}_F$ (see [66] for the general definition), and the two matroids have the same rank-three truncations. These combinatorial properties gave rise to several attempts to replace the notion of formality with some clearly matroidal condition, and strengthen Theorem 3.1 and assertion (i) above. For example one can ask for conditions on a matroid G so that every (complex) realization of G is formal. One is naturally led to the notion of line-closure.



(a) A formal configuration



(b) A non-formal configuration

FIGURE 5. Formality is not matroidal

Let G be a matroid on ground set $[n]$. The *line-closure* of a subset S of $[n]$ is the smallest subset of $[n]$ which contains every line (that is, rank-two flat) spanned by points of S. A set is line-closed if it is equal to its line-closure. The matroid G is *line-closed* if every line-closed subset of $[n]$ is a flat of G. In his current work in progress [33], the first author has established the following result.

Theorem 3.4. *An arrangement $\mathcal{A}$ is quadratic only if the underlying matroid $G(\mathcal{A})$ is line-closed.*

Corollary 3.5. *The underlying matroid of a rational $K(\pi, 1)$ arrangement is necessarily line-closed.*

The converse of Theorem 3.4, that $\mathcal{A}$ is quadratic when $G(\mathcal{A})$ is line-closed, is very likely also true. A crucial step in the proof is yet to be completed, however, so this assertion remains an open problem.

Yuzvinsky [90] defined a *formal matroid* to be a matroid G possessing a basis (of $\mathrm{rk}(G)$ points) whose line-closure is $[n]$. Every line-closed matroid is formal in this sense. In fact a matroid G is line-closed if and only if the line-closure of *every* basis of each flat X is equal to X. Every realization of a formal matroid is formal.

In [33] we define a matroid G to be *taut* if G is not a strong map image of a matroid G' of greater rank with the same points and lines, and *locally taut* if every flat of G is taut. Every line-closed matroid is locally taut, in fact every formal matroid is taut. Every realization of a (locally) taut matroid is (locally) formal. There exist matroids which are taut but not formal [19]. A weak version of the first part of the following problem was suggested by Yuzvinsky in his talk [90].

Problem 3.6. *Prove that the matroid of a free or $K(\pi, 1)$ arrangement is necessarily taut.*

Joseph Kung has pointed out to us that a locally taut matroid is uniquely determined by its points and lines, which suggests the following interesting problem.

Problem 3.7. *Prove that the underlying matroid of a locally formal arrangement (e.g. a free or $K(\pi, 1)$ arrangement) is uniquely determined by its points and lines.*

This last problem is a variant on the following questions from [37], the first of which is Terao's Conjecture, and both of which remain open.

Problem 3.8. *Prove that freeness and $K(\pi, 1)$-ness of arrangements are matroidal properties.*

We will refrain from discussing Terao's Conjecture further, except to pose a weak version which fits the spirit of this paper, and is interesting in its own right.

Problem 3.9. *Prove that freeness is preserved under lattice-isotopy.*

3.3. Tests for asphericity

Some progress was also made on the problem of finding sufficient conditions for an arrangement to be $K(\pi, 1)$. The main results are the weight test of [31] and its application to factored arrangements by Paris [69]. A new technique involving modular flats was recently discovered and presented at the conference [70, 35].

The complement M of a 2-dimensional affine arrangement $\mathcal{A}$ is built up out of $K(\pi, 1)$ spaces, specifically (r, r) torus link complements, in a relatively simple way, as is reflected in the Randell-Salvetti-Arvola presentations (see Section 4.1). In fact this structure mirrors precisely constructions from geometric group theory related to complexes of groups. This observation allows one to construct a relatively well-behaved cell complex which has the homotopy type of the universal cover of M, and to apply the weight test of Gersten and Stallings [83] to derive a test for asphericity of M.

Theorem 3.10 ([31]). *If $\mathcal{A}$ is a complexified affine arrangement in $\mathbb{C}^2$ that admits an $\mathcal{A}$-admissible, aspherical system of weights, then $\mathcal{A}$ is a $K(\pi, 1)$ arrangement.*

The question remains what an $\mathcal{A}$-admissible, aspherical system of weights is. This involves the complex B of bounded faces in the subdivision of $\mathbb{R}^2$ determined by $\mathcal{A}$. A weight system is an assignment of a real number weight to each "corner" of each 2-cell in B. The system is aspherical if the sum of weights around any d-gon at most $d-2$. The system is $\mathcal{A}$-admissible if certain sums of weights at vertices of Γ are at least 2π. See [31] for more detail.

The universal cover complex constructed in [31] may be used in some cases to construct explicit essential spheres showing that M is not aspherical. Radloff [74] used this method to prove some necessary conditions for $K(\pi,1)$-ness, along the lines of the "simple triangle" test of [37], and found several new examples of non-$K(\pi,1)$ arrangements.

Falk and Jambu introduced the notion of *factored* arrangement in [34], originally in an attempt to find a combinatorial criterion for freeness. A *factorization* of an arrangement $\mathcal{A}$ is a partition of $[n]$ such that each flat of $G(\mathcal{A})$ of rank p meets precisely p blocks, and meets one of them in a singleton, for each p. This property is necessary and sufficient for the OS algebra $A(\mathcal{A})$ to have a complete tensor product factorization - see [6, 34, 88, 65]. When $\mathcal{A}$ has a factorization, we say $\mathcal{A}$ is *factored.*

Paris realized that a factorization of a rank-three arrangement provides a template for a very simple $\mathcal{A}$-admissible, aspherical weight system.

Theorem 3.11 ([69]). *If $\mathcal{A}$ is a factored, complexified arrangement in $\mathbb{C}^3$, then $\mathcal{A}$ is a $K(\pi,1)$ arrangement.*

Every supersolvable arrangement is factored, so this result provides a new, wider class of $K(\pi,1)$ arrangements, at least in rank three.

Problem 3.12. *Show that factored arrangements of arbitrary rank are $K(\pi,1)$.*

A flat X of a matroid G is *modular* if $\mathrm{rk}(X \vee Y) + \mathrm{rk}(X \wedge Y) = \mathrm{rk}(X) + \mathrm{rk}(Y)$ for every flat Y. The following result was discovered independently by Paris and Falk-Proudfoot

Theorem 3.13 ([70, 73, 35]). *If X is a modular flat of arbitrary rank in $G(\mathcal{A})$, then there is a topological fibration $M(\mathcal{A}) \to M(\mathcal{A}_X)$ whose fiber is the complement of a projective arrangement.*

This generalizes the corank-one case, which gives rise to fiber-type arrangements, established in [87]. The new result can be used to construct or recognize $K(\pi,1)$ arrangements if the base (whose matroid is the modular flat X) and fiber (whose matroid is the complete principal truncation of $G(\mathcal{A})$ along X) are known to be $K(\pi,1)$. This method is used to construct some interesting new examples in [35]. Refer to Paris' paper [68] in this volume for more details.

3.4. Some crucial examples

In this section we want to briefly discuss some specific and interesting types of arrangements for which the $K(\pi,1)$ problem is unsolved. These might be regarded as test subjects for new techniques; they qualify as "the first unknown cases."

First we cite another improvement to the table of implications in [37]. Recall the definition of parallel arrangement from Section 2.1. In [37] we had listed the implication "parallel $\Longrightarrow$ $K(\pi,1)$" as "not known, of significant interest." In unpublished work, Luis Paris has shown this implication to be false. Specifically, he showed that the Kohno arrangement X_2 (defined in Section 2.1) is not $K(\pi,1)$. The proof establishes that the fundamental group contains a subgroup isomorphic to $\mathbb{Z}^4$; the result then follows from [37, Thm. 3.2]. The copy of $\mathbb{Z}^4$ is generated by a, b, c, and the commutator $[d,e]$, where $a,b,c,d,$ and e are the canonical generators corresponding to the hyperplanes $x \pm z = 0$, $z=0$, and $x+y\pm 2z=0$ respectively.

3.4.1. *Complex reflection arrangements* Fadell and Neuwirth showed in 1962 that the complement of the A_ℓ reflection (or braid) arrangement is $K(\pi,1)$. In 1973 Brieskorn proved this for many real reflection arrangements, followed soon thereafter by Deligne's proof of the general case. Orlik and Solomon extensively studied arrangements of hyperplanes invariant under finite groups generated by complex reflections (see [65, Chapter 6]). It is natural to ask if all such arrangements are aspherical. We believe the conjecture that they are is due to Orlik, though it was proposed long before it ever appeared in print. It is known [65] that the answer is affirmative in all cases except six exceptional, non-complexified arrangements, some of which have rank three. The proofs for the known cases use a variety of techniques, and essentially proceed from the Shepard-Todd classification of irreducible unitary reflection groups (see, e.g., [65]). What seems to be missing is a unifying property, similar to the simplicial property for real reflection arrangements exploited by Deligne. The closest approach to this goal is the work reported in [65, p. 265] which proves the asphericity of arrangements associated to Shephard groups (symmetry group of a regular convex polytope.) Here the problem is reduced to the (already solved) problem for an associated real reflection arrangement.

Problem 3.14. *Give a uniform proof that all unitary reflection arrangements are $K(\pi,1)$.*

3.4.2. *Discriminantal arrangements* Experience seems to show us that questions involving asphericity are quite complex for all arrangements but tractible for restricted classes (reflection, fiber-type, generic). One interesting class is that of the discriminantal arrangements introduced by Manin and Schechtmann [57]. Rather than give the full definition here we will describe the rank three examples, where the problem is already interesting.

Consider a real affine arrangement of lines in the plane, obtained by taking a collection of n points, no three of which are collinear, and drawing all $\binom{n}{2}$ lines through pairs of these points. Then embed this configuration in the plane $z = 1$ in three-space and cone over the origin to obtain a central real three-arrangement. Then complexify.

This process can result in arrangements with distinct matroidal and topological structure, even for fixed n [30, 5]. The discriminantal arrangements are obtained from "very generic" collections of points, for which no three of the $\binom{n}{2}$ lines are concurrent except at the original n points.

The arrangement $C(4)$ is linearly equivalent to the braid arrangement of rank three. An easy calculation shows that the Poincaré polynomial associated to the cohomology of $C(n)$ does not factor over $\mathbb{Z}$ for $n \geq 5$, so that these arrangements are not free and are not of fiber-type. Also $C(n)$ is not simplicial for $n \geq 5$. The arrangements $C(n)$ for $n \geq 6$ are not aspherical, by [37, Thm. 3.1].

For $n = 5$, one obtains a complexified central three-arrangement of 10 planes. This arrangement is not factored. More generally $C(5)$ does not support an admissible, aspherical system of weights, so the weight test fails. On the other hand, all of the standard necessary conditions for asphericity hold.

Problem 3.15. *Determine whether the discriminantal arrangement $C(5)$ is $K(\pi, 1)$.*

A solution to this problem would also determine whether the space of configurations of six points in general position in $\mathbb{C}P^2$ is aspherical [30], a result which would be of significant interest.

3.4.3. *Deformations of reflection arrangements* A "deformation" of a reflection arrangement is an affine arrangement with defining equations of the form

$$\alpha_i(x_1, \dots, x_\ell) = c_{ij},$$

where the α_i are the positive roots in some root system, and $c_{ij} \in \mathbb{R}$. This class of arrangements is of great interest to combinatorialists, and is the subject of the paper of Athanasiadis in this volume [4].

As is our custom, we "cone" to obtain a central arrangement. For instance, based on the root system of type B_2, we obtain the B_2 *Shi arrangement*, defined by the polynomial

$$Q = xyz(x+y)(x-y)(x-z)(y-z)(x+y-z)(x-y-z).$$

(Shi arrangements are obtained by setting $c_{i1} = 0$ and $c_{i2} = 1$ for all i.) This nine-line complexified arrangement has a factorization, given by the partition

$$\{\{4\}, \{1,2,5,7\}, \{3,6,8,9\}\},$$

and is therefore a $K(\pi, 1)$ arrangement. On the other hand, the Shi arrangement constructed in a similar way from the root system of type G_2 is not factored or simplicial, and has no simple triangle.

Problem 3.16. *Decide whether the G_2 Shi arrangement is $K(\pi,1)$.*

More generally, we propose the following.

Problem 3.17. *Decide which Shi arrangements are $K(\pi,1)$.*

$\mathcal{S}$4. Topological properties of the group of an arrangement

At the time of the publication of [37], a presentation of the fundamental group of the complement of a complexified arrangement had been derived [76]. In the meantime, a similar presentation was found for arbitrary complex arrangements [3], and several different "spines" for the complement, some of them modelled on group presentations, were constructed [81, 29, 13, 50]. These group presentations have been used to study the Milnor fibration and Alexander invariants of the complement. We report briefly on these ideas here.

4.1. Presentations of π_1

We have seen earlier in the discussion of the lower central series, Chen groups and group cohomology that certain classes of arrangements (fiber-type, simplicial) have well-behaved fundamental groups. Due to work of Arvola [3], Randell [76] and Salvetti [81] an explicit presentation of $\pi_1(M)$ can be written. See [65, Section 5.3] for a clear exposition of Arvola's presentation for any complex arrangement, and [29] for the explicit presentation and some applications of Randell's presentation, which holds for complexified arrangements and is naturally simpler than the general case. A different approach, using the notion of "labyrinth," is adopted by Dung and Vui in [22] to arrive at similar presentations for arbitrary arrangement groups.

In these presentations one first takes a planar section (or, more precisely, the projective image), so that one is working with an affine arrangement in $\mathbb{C}^2$. Then there is one generator for each line of the arrangement, and one set of relations for each intersection. In all cases the relations consist entirely of commutators, but to date this has not shed much light on the questions of group cohomology, torsion in the fundamental group, or other properties (such as orderability) of the fundamental group. A general theme for questions is: to what extent do arrangement groups mimic the properties of the pure braid groups.

The concept of braid monodromy was introduced by B. Moishezon [60]. Libgober showed in [50] that the braid monodromy presentation of the fundamental group yields a two-complex with the homotopy type of the complement of an algebraic curve (e.g., a line arrangement) transverse to the line at infinity.

Motivated in part by [50], the first author showed in [29] that for arbitrary line arrangements the 2-complex modelled on the presentation of [76] serves as an efficient model for constructing the homotopy type of the complement (in the case of 3-arrangements). This construction was then used to construct a number of examples with different intersection lattice but same homotopy type (see also Section 1.3).

In related work Cohen and Suciu [13] have given an explicit description of the braid monodromy of a complex arrangement, using Hansen's theory of polynomial covering maps. They show that the resulting presentation of the fundamental group is equivalent to the Randell-Arvola presentation via Tietze transformations that do not affect the homotopy type of the associated 2-complex. It follows that the complement is homotopy equivalent to the 2-complex modelled on either of these presentations, generalizing the result of [29]. For this work Cohen and Suciu used extensively the concept of *braided wiring diagram*, which we briefly describe below. The notion of braided wiring diagram generalizes Goodman's concept of wiring diagram [40], and was earlier considered for arrangements in [17]. (Wiring diagrams appear in combinatorics as geometric models for rank-three oriented matroids.) The presentations of [76] and [3] use versions of this idea. In brief, the braided wiring diagram can be thought of as a template for the fundamental group (or, for line arrangements, the homotopy type.)

Here is a sketch of the construction. For examples and further details, in particular, a beautiful derivation using polynomial covering space theory, see [13]. Since we are interested in the fundamental group, consider an affine arrangement $\mathcal{A}$ in $\mathbb{C}^2$. Choose coordinates in $\mathbb{C}^2$ so that the projection to the first coordinate is generic. Suppose that the images $y_1, \ldots, y_n$ of the intersections of the lines have distinct real parts.

Choose a basepoint $y_0 \in \mathbb{C} \setminus \{y_1, \dots, y_n\}$, and assume the real parts of y_i are decreasing with i. Let ξ be a smooth path which begins with y_0 and passes in order through the y_i, horizontal near each y_i. Then the braided wiring diagram is $\mathcal{W} = \{(x, z) \in \xi \times \mathbb{C} \mid Q(x, z) = 0\}$. (Recall that Q is the defining polynomial of the arrangement.)

This braided wiring diagram should be viewed as a picture of the braid monodromy of the fundamental group of the arrangement (or as a picture of the fundamental group itself). In a sense, it carries the attaching (or amalgamating) information as one computes the fundamental group using the Seifert-Van Kampen theorem. Each actual node in the wiring diagram gives a set of relators, as does each crossing. In particular, it is shown in [13] that the braided wiring diagram recovers the Arvola or Randell presentation of $\pi_1(M)$. Indeed, in the real case, the braided wiring diagram can be identified with the usual drawing of the arrangement in $\mathbb{R}^2$.

As is the case with ordinary braids, there are "Markov moves" with which one can modify such a wiring diagram to realize any braid-equivalence of the underlying braid monodromies. These are given explicitly in [13]. Rudimentary moves of this type, called "flips," first appeared in [29]. Among the consequences we note the following results which relate braid monodromy and braided wiring diagrams to lattice isotopy of line arrangements (that is, arrangements in $\mathbb{C}^2$).

Theorem 4.1 ([13]). *Lattice-isotopic arrangements in $\mathbb{C}^2$ have braided wiring diagrams which are related by a finite sequence of Markov moves and their inverses.*

Theorem 4.2 ([13]). *Line arrangements with braid-equivalent monodromies have isomorphic underlying matroids.*

4.2. The Milnor fiber

The defining polynomial $Q = \prod_{i=1}^{n} \alpha_i$ is homogeneous of degree n and can be considered as a map

$$Q : M \to \mathbb{C}^*$$

It is well-known that this map is the projection of a fiber bundle, called the Milnor fibration, and that the Milnor fiber $F = Q^{-1}(1)$ should be of interest. In [79] it was shown that this Milnor fibration is constant in a lattice-isotopic family, so that the Milnor fiber is indeed an invariant of lattice-isotopy. Because of this we propose the following definition, analogous to the definition made in the theory of knots.

Definition 4.3. Two arrangements are called (*topologically*) *equivalent* if they are lattice-isotopic. We say the arrangements have the same (*topological*) *type*.

Thus, arrangements are topologically equivalent if and only if they lie in the same path component of some matroid stratum in the Grassmannian. With this terminology, we have the following result.

Theorem 4.4 ([79]). *The Milnor fiber and fibration are invariants of topological type.*

Now, F is simply an $n-$fold cover of the complement of the projectivized arrangement in $\mathbb{C}P^{\ell-1}$. Since the algorithms of the previous section work to compute the fundamental group of this latter space, questions involving the fundamental group and cohomology of F are also questions involving the group of the arrangement. In particular, while the cohomology of M is determined by the intersection lattice, that of F may not be. The situation is analogous to that of plane curves, where work going back to Zariski [92] shows that not only the type but the position of the singularities affects the irregularity. (The irregularity here is simply half the "excess" in the first betti number of F.)

Early results concerning the Milnor fiber of an arrangement (often in the general context of plane curves) appear in work of Libgober [50, 51, 52, 53] and Randell [75], particularly with respect to Alexander invariants. Libgober's work gave considerable information about the homology of the Milnor fiber in relation to the number and type of singularities of the arrangement, their position and the number of lines. The paper [75] observed that the Alexander polynomial was equal to the characteristic polynomial of the monodromy on the Milnor fiber.

The paper of Artal-Bartolo [2] included an interesting example: for the rank three braid arrangement A_3 the first betti number of the Milnor fiber is seven, an excess of two over the five "predicted" by the number of lines. (This result can be obtained as an interesting exercise by applying the Reidemeister-Schreier rewriting algorithm to the presentations of the fundamental group.) Orlik and Randell [54] showed that in the generic case the cohomology of the Milnor fiber is minimal, given the number of lines, below the middle dimension.

Cohen and Suciu carry forward the study of the Milnor fiber in [11]. Using the group presentation and methods of Fox calculus they give twisted chain complexes whose homology gives that of the Milnor fiber. Their methods are effective, and several explicit examples are given. The monodromy action on the Milnor fiber is of course crucial, and this monodromy is determined as well.

Finally, we note the following problem, which remains open after many years.

Problem 4.5. *Prove that the homology of the Milnor fiber of $\mathcal{A}$ depends only on the underlying matroid.*

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Plumbing Graphs for Normal Surface-Curve Pairs

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Dedicated to Peter Orlik on his 60th birthyear

Abstract.

Consider the set of surface-curve pairs $(X, \mathcal{C})$, where X is a normal surface and $\mathcal{C}$ is an algebraic curve. In this paper, we define a family $\mathcal{F}$ of *normal* surface-curve pairs, which is closed under coverings, and which contains all *smooth* surface-curve pairs $(X, \mathcal{C})$, where X is smooth and $\mathcal{C}$ has smooth irreducible components with normal crossings. We give a modification of W. Neumann's definition of plumbing graphs, their associated 3-dimensional graph manifolds, and intersection matrices, and use this construction to describe rational intersection matrices and boundary manifolds for regular branched coverings.

§1. Introduction

Let $(X, \mathcal{C})$ be a surface-curve pair, consisting of a normal surface X and an algebraic curve $\mathcal{C} \subset X$. The boundary manifold of a regular neighborhood $M(X, \mathcal{C})$ of $\mathcal{C}$ in X can be simply described by taking any smooth model $(\widetilde{X}, \widetilde{\mathcal{C}})$ of $(X, \mathcal{C})$, and using W. Neumann's associated plumbing graphs $\Gamma_{\mathrm{plumb}}(X, \mathcal{C})$ (see [Neu]). The *intersection matrix* $S(X, \mathcal{C})$ of a surface-curve pair $(X, \mathcal{C})$ is the matrix with entries the pairwise rational intersections of irreducible components of $\mathcal{C}$ with respect to some ordering. When $(X, \mathcal{C})$ is a *smooth* surface-curve pair, where X is smooth and $\mathcal{C}$ has smooth irreducible components with normal crossings, the intersection matrix $S(X, \mathcal{C})$ only depends on the combinatorics of $\mathcal{C}$, and thus is also determined by $\Gamma_{\mathrm{plumb}}(X, \mathcal{C})$. Neumann defines the intersection matrix $S(\Gamma_{\mathrm{plumb}})$ for the plumbing graph of a smooth surface-curve pair $(X, \mathcal{C})$, so that $S(X, \mathcal{C}) = S(\Gamma_{\mathrm{plumb}}(X, \mathcal{C}))$.

A modified definition of plumbing graphs is useful for dealing with branched coverings. A (*regular*) *covering of surface-curve pairs*

$$\rho : (Y, \mathcal{D}) \to (X, \mathcal{C})$$

is a finite surjective morphism

$$\rho : Y \to X$$

so that $\mathcal{D} = \rho^{-1}(\mathcal{C})$ and the restriction

$$\rho : Y \setminus \mathcal{D} \to X \setminus \mathcal{C}$$

is a (regular) unbranched covering. Even if $(X, \mathcal{C})$ is a smooth surface-curve pair, the covering $(Y, \mathcal{D})$ of $(X, \mathcal{C})$ need not be smooth.

Let $\mathcal{S}$ be the collection of smooth surface-curve pairs. We will define a family $\mathcal{F}$ of *normal* surface-curve pairs, which contains $\mathcal{S}$ and is closed under coverings, in the sense that: if $(X, \mathcal{C}) \in \mathcal{F}$, and $\rho : (Y, \mathcal{D}) \to (X, \mathcal{C})$ is a covering of surface-curve pairs, then $(Y, \mathcal{D}) \in \mathcal{F}$. We modify Neumann's definition of plumbing graphs and their intersection matrices to describe the local topology of surface-curve pairs in $\mathcal{F}$ and their intersection matrices. This gives a method for studying coverings and computing intersection matrices without having to pass to smooth models, and generalizes the results of [Hir1] and [Hir2], where formulas for intersection matrices of abelian coverings are given.

The reader is reminded of basic definitions and properties of graphs of groups and complexes in Section 2. The modified definition of plumbing graphs, and their associated 3-manifolds and coverings are given in Sections 3. Section 4 contains a definition of normal surface-curve pairs, their associated plumbing graphs, and associated intersection matrices. Formulas for invariants of the plumbing graph of a covering of a normal surface-curve pair from covering data are given in Section 5.

§2. Graphs of groups and complexes

The concept of plumbing graph comes out of a more general construction by which finite CW-complexes and finitely generated groups are described in terms of information attached to the nodes and vertices of a graph. We give the basics of these definitions in this section.

By a *graph* Γ we mean a collection of vertices $\mathcal{V}(\Gamma)$ and oriented edges $\mathcal{Y}(\Gamma)$. For any $y \in \mathcal{Y}(\Gamma)$, we write $o(y)$ for the initial point and $t(y)$ for the terminal point. We will always assume that graphs are *finite* and *connected*. Furthermore, given $y \in \mathcal{Y}(\Gamma)$, we will assume $\overline{y} \in \mathcal{Y}(\Gamma)$, where

$$\begin{aligned} o(\overline{y}) &= t(y), \qquad \text{and} \\ t(\overline{y}) &= o(y). \end{aligned}$$

For any vertex $v \in \mathcal{V}(\Gamma)$, denote by $d(v)$ the degree of the graph Γ at v.

A *graph* of groups $G(\Gamma)$ over Γ is a collection of groups

$$\begin{aligned} &G_v, \quad v \in \mathcal{V}(\Gamma), \\ &G_y, \quad y \in \mathcal{Y}(\Gamma), \end{aligned}$$

so that $G_y = G_{\overline{y}}$; and monomorphisms

$$h : G_y \to G_{t(y)},$$

for each $y \in \mathcal{Y}(\Gamma)$.

A *path* on Γ is an ordered, possibly empty, collection

$$c = (y_1, \ldots, y_k),$$

where

$$y_i \in \mathcal{Y}(\Gamma) \qquad \text{for } i = 1, \ldots, k,$$

and

$$t(y_i) = o(y_{i+1}), \qquad \text{for } i = 1, \ldots, k-1.$$

Given a path $c = (y_1, \ldots, y_k)$ on Γ and collection $r = (r_0, \ldots, r_k)$, where $r_0 \in G_{o(y_1)}$, and $r_i \in G_{t(y_i)}$, for $i = 1, \ldots, k$. Let $|c, r|$ be the word

$$r_0 y_1 r_1 \ldots y_k r_k.$$

Let $F(G(\Gamma))$ be the group of words $|c, r|$ subject to the relations in the vertex and edge groups G_v and G_y, and the relation

$$y r \overline{y} = r_1,$$

if and only if $r = h_y(r_1)$.

The *fundamental group* $\pi_1(G(\Gamma))$ can be defined in two ways. The first is in terms of a basepoint $v_0 \in \mathcal{V}(\Gamma)$. A path $c = (y_1, \ldots, y_k)$ is a *closed circuit based at* v_0, where $v_0 \in \mathcal{V}(\Gamma)$, if

$$v_0 = o(y_1) = t(y_k).$$

The *fundamental group* $\pi_1(G(\Gamma), v_0)$ is defined to be the set of words $|c, r|$, where c is a closed circuit based at v_0.

The second way to describe the fundamental group $\pi_1(G(\Gamma))$ is in terms of a maximal tree inside Γ. A *maximal tree* $\mathcal{T}$ in Γ is a subgraph containing all vertices of Γ, and such that, given any two distinct vertices $v_1, v_2 \in \mathcal{V}(\Gamma)$, there is a unique path $c = (y_1, \ldots, y_k)$ in $\mathcal{T}$ so that

$$y_i \neq \overline{y_{i+1}},$$

for $i = 1, \ldots, k-1$, and $v_1 = o(y_1)$, $v_2 = t(y_k)$. The *fundamental group* $\pi_1(G(\Gamma), \mathcal{T})$ is the group $F(G(\Gamma))$ modulo the normal subgroup generated by the edges in $\mathcal{Y}(\mathcal{T})$ thought of as elements of $F(G(\Gamma))$.

Lemma 0.1 ([Ser], p. 43). *The natural homomorphism*

$$\pi_1(G(\Gamma), v_0) \to \pi_1(G(\Gamma), \mathcal{T}),$$

given by including $\pi_1(G(\Gamma), v_0)$ in $F(G(\Gamma))$ and then taking the quotient by the normal subgroup generated by $\mathcal{Y}(\mathcal{T})$, is an isomorphism.

Given a maximal tree $\mathcal{T}$ of Γ, there are natural maps

$$\psi_v : G_v \to \pi_1(G(\Gamma), \mathcal{T})$$

induced by the natural inclusion of G_v in $F(G(\Gamma))$.

Lemma 0.2 ([Ser], Theorem 11, Corollary 1). *The maps ψ_v are monomorphisms.*

The fundamental group of $G(\Gamma)$ can also be considered as the fundamental group of a naturally associated finite CW-complex. A graph of complexes $\Sigma(\Gamma)$, is a collection of finite CW-complexes

$$X_v, \quad v \in \mathcal{V}(\Gamma),$$

and subcomplexes

$$X_y \subset X_{t(y)}, \quad y \in \mathcal{Y}(\Gamma),$$

such that the induced maps

$$\pi_1(X_y) \to \pi_1(X_{t(y)})$$

are injective, with homeomorphisms

$$h_y : X_{\overline{y}} \to X_y,$$

so that $h_{\overline{y}} = h_y^{-1}$.

Given a graph of complexes $\Sigma(\Gamma)$, the associated *graph complex*, which we will also denote by $\Sigma(\Gamma)$, is the CW-complex obtained by gluing together the X_v along the X_y according to the identifications h_y. Setting $G_v = \pi_1(X_v)$, for $v \in \mathcal{V}(\Gamma)$, and $G_y = \pi_1(X_y)$, for $y \in \mathcal{Y}(\Gamma)$, gives a corresponding graph of groups $G_\Sigma(\Gamma)$.

Theorem 1 ([Hem], Theorem 2.1). *The fundamental group of $G_\Sigma(\Gamma)$ is isomorphic to the fundamental group of $\Sigma(\Gamma)$.*

A *morphism* between graphs of complexes

$$\Psi : \Sigma'(\Gamma') \to \Sigma(\Gamma)$$

is a morphism of graphs

$$\Psi_\Gamma : \Gamma' \to \Gamma$$

and cellular maps

$$\begin{aligned} \Psi_v &: X_v \to X_{\Psi_\Gamma(v)}, \quad v \in \mathcal{V}(\Gamma'), \qquad \text{and} \\ \Psi_y &: X_y \to X_{\Psi_\Gamma(y)}, \quad y \in \mathcal{Y}(\Gamma'), \end{aligned}$$

so that

$$\begin{array}{ccc} X_y & \xrightarrow{\Psi_y} & X_{\Psi_\Gamma(y)} \\ {\scriptstyle h_y}\downarrow & & \downarrow{\scriptstyle h_{\Psi_\Gamma(y)}} \\ X_{t(y)} & \xrightarrow{\Psi_{t(y)}} & X_{\Psi_\Gamma(t(y))} \end{array}$$

commutes, for all $y \in \mathcal{Y}(\Gamma')$.

An *(unbranched) covering*

$$\rho : \Sigma'(\Gamma') \to \Sigma(\Gamma)$$

is a morphism of graph complexes so that

$$\rho_\Gamma : \Gamma' \to \Gamma$$

is onto, and

$$\begin{aligned} \rho_v &: X_v \to X_{\rho_\Gamma(v)}, \quad v \in \mathcal{V}(\Gamma'), \qquad \text{and} \\ \rho_y &: X_y \to X_{\rho_\Gamma(y)}, \quad y \in \mathcal{Y}(\Gamma') \end{aligned}$$

are unbranched coverings. Note that if ρ is an unbranched covering, then the induced map

$$G_{\Sigma'}(\Gamma') \to G_\Sigma(\Gamma)$$

on graphs of groups induces a monomorphism of groups

$$\rho_* : \pi_1(G_{\Sigma'}(\Gamma), v_0) \to \pi_1(G_\Sigma(\Gamma), \rho(v_0)),$$

for any $v_0 \in \Gamma'$.

An unbranched covering

$$\rho : \Sigma'(\Gamma') \to \Sigma(\Gamma)$$

is *regular* if the maps ρ_v and ρ_y are regular coverings, for all $v \in \mathcal{V}(\Gamma')$ and all $y \in \mathcal{Y}(\Gamma')$. Regular coverings $\Sigma'(\Gamma')$ of $\Sigma(\Gamma)$ correspond to epimorphisms

$$\psi : \pi_1(G_\Sigma(\Gamma)) \to F,$$

where F is a finite group.

Fix a maximal tree in Γ. A *lift*

$$\ell : \mathcal{T} \to \Gamma'$$

of $\mathcal{T}$ in the covering graph Γ', is a morphism of graphs so that

$$\begin{aligned} \rho_\Gamma(\ell(v)) &= v, \quad v \in \mathcal{V}(\mathcal{T}), \qquad \text{and} \\ \rho_\Gamma(\ell(y)) &= y, \quad y \in \mathcal{Y}(\mathcal{T}). \end{aligned}$$

Identify $G_v = \pi_1(X_v)$ and $G_y = \pi_1(X_y)$ with the corresponding subgroups of

$$\pi_1(G_\Sigma(\Gamma)) = \pi_1(G_\Sigma(\Gamma), \mathcal{T}).$$

For each $v \in \mathcal{V}(\Gamma)$, let ψ_v be the restriction of ψ to G_v, and, for each $y \in \mathcal{Y}(\Gamma)$, let ψ_y be the restriction of ψ to G_y.

Let

$$\begin{aligned} F_v &= \psi_v(G_v), \quad v \in \mathcal{V}(\Gamma), \qquad \text{and} \\ F_y &= \psi_y(G_y), \quad y \in \mathcal{Y}(\Gamma). \end{aligned}$$

Note that the conjugacy classes of G_v and G_y, and hence F_v and F_y don't depend on the choice of maximal tree $\mathcal{T}$.

For $y \in \mathcal{Y}(\Gamma)$, let $s(y) = \psi(y)$, where we identify $\mathcal{Y}(\Gamma)$ with its natural image in $\pi_1(G_\Sigma(\Gamma), \mathcal{T})$.

The following propositions and corollaries follow from elementary properties of coverings.

Proposition 2. *For $v \in \mathcal{V}(\Gamma)$, the identification*

$$[\alpha F_v] = \alpha\ell(v)$$

gives a one-to-one correspondence between elements in the preimage $\rho^{-1}(v)$ cosets of F_v in F. Furthermore, for $v' \in \rho^{-1}(v)$, the covering

$$\Sigma'_{v'} \to \Sigma_v$$

has defining map

$$\psi_v : \pi_1(\Sigma_v) = G_v \to F_v.$$

Corollary 3. *The number of vertices in $\rho^{-1}(v)$ is*

$$\#|\rho^{-1}(v)| = [F : F_v]$$

where $[F : F_v]$ is the index of F_v in F. For $v' \in \rho^{-1}(v)$, the degree of the covering

$$\Sigma'_{v'} \to \Sigma_v$$

is $\#|F_v|$, the order of F_v.

Similarly, for the edges, we have the following.

Proposition 4. *For $y \in \mathcal{Y}(\Gamma)$, the identification*

$$\alpha\ell(y) = [\alpha F_y]$$

gives a one to one correspondence between the edges in $\rho^{-1}(y)$ and cosets of F_y in F so that

$$t(\alpha\ell(y)) = \alpha(s(y)\ell(t(y)) = [\alpha s(y)F_{t(y)}].$$

Furthermore, the covering

$$\Sigma'_{y'} \to \Sigma_y$$

has defining map

$$\psi_y : \pi_1(\Sigma_y) = G_y \to F_y.$$

Corollary 5. *For $y \in \mathcal{Y}(\Gamma)$,*

$$\#|\rho^{-1}(y)| = [F : F_y];$$

for $y' \in \rho^{-1}(y)$, the covering

$$\Sigma'_{y'} \to \Sigma_y$$

has degree $\#|F_y|$; and, if $t(y) = v$ and $v' \in \rho^{-1}(v)$, we have

$$\#\{y' \in \rho^{-1}(y) \ : \ t(y') = v'\} = \frac{\#F_v}{\#F_y}.$$

§3. Plumbing graphs

In [Wal], F. Waldhausen defines a 3-dimensional *graph manifold* to be a manifold with a torus decomposition into Seifert fibered pieces, noting that this gives the manifold an underlying graph structure. Neumann distills the information using plumbing graphs in [Neu], and develops a calculus for determining the topological equivalence of two graph manifolds. In this section, we review the part of his definition of graph manifold which applies to smooth surface-curve pairs, and then define a modification which we later show applies to normal surface-curve pairs.

A *plumbing graph* $\Gamma_{\text{plumb}} = \langle \Gamma, g, e \rangle$ is a finite connected graph Γ, together with maps

$$\begin{aligned} g &: \mathcal{V}(\Gamma) \to \mathbb{Z}_{\geq 0} \\ e &: \mathcal{V}(\Gamma) \to \mathbb{Z} \end{aligned}$$

Given a plumbing graph Γ_{plumb}, there is an associated graph of complexes $M(\Gamma_{\text{plumb}})$ given as follows. For each vertex $v \in \mathcal{V}(\Gamma)$, let S_v be an oriented surface of genus $g(v)$, with $d(v)$ boundary components, labeled by the edges $y \in \mathcal{Y}(\Gamma)$, where $t(y) = v$; and let $f_v : M_v \to S_v$ be an S^1-bundle map, with trivializations at the boundary components of S_v, so that f_v has Euler number $e(v)$.

Let $h : S^1 \times S^1 \to S^1 \times S^1$ be the automorphism defined by $h(a,b) = (b,a)$. We can think of h as being induced by the action of

$$H = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

on $\pi_1(S^1 \times S^1)$, with respect to the natural identification

$$\pi_1(S^1 \times S^1) = \mathbb{Z} \oplus \mathbb{Z}.$$

Let $T_y \in M_{t(y)}$ be the boundary component of $M_{t(y)}$ associated to the oriented edge y. The local trivialization of f_v at T_y, canonically identifies T_y with $S^1 \times S^1$ so that $f_v|_{T_y}$ is projection onto the second component.

The graph of complexes associated to Γ_{plumb} consists of the manifolds

$$\begin{aligned} X_v &= M_v, \quad v \in \mathcal{V}(\Gamma), \qquad \text{and} \\ X_y &= T_y, \quad y \in \mathcal{Y}(\Gamma). \end{aligned}$$

with gluing maps

$$\begin{tikzcd} T_{\overline{y}} \arrow[r, "h_y"] \arrow[d, equal] & T_y \arrow[d, equal] \\ S^1 \times S^1 \arrow[r, "h"] & S^1 \times S^1 \end{tikzcd}$$

The graph of complexes $M(\Gamma_{\text{plumb}})$ is a graph manifold.

Let $\text{Fin}(S^1 \times S^1)$ be the set of finite unbranched coverings of $S^1 \times S^1$ to itself. A *modified plumbing graph* $\Gamma^{\text{m}}_{\text{plumb}} = \langle \Gamma, g, e, m \rangle$ is a plumbing graph with maps

$$m : \mathcal{Y}(\Gamma) \to \text{Fin}(S^1 \times S^1)$$

so that

(1) the induced maps

$$m(y)_* : \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$$

are non-negative upper triangular matrices in $M_2(\mathbb{Z})$,

$$m(y)_* = \begin{bmatrix} a(y) & b(y) \\ 0 & c(y) \end{bmatrix}$$

where $0 \le b(y) < a(y)$ and $c(y) > 0$; and

(2) the matrices $m(y)_*$ and $Hm(\overline{y})_*$ have the same image in $\mathbb{Z} \oplus \mathbb{Z}$.

Given a modified plumbing graph $\Gamma^{\mathrm{m}}_{\mathrm{plumb}} = \langle \Gamma, g, e, m \rangle$, we define an associated graph manifold $M(\Gamma^{\mathrm{m}}_{\mathrm{plumb}})$ to have vertex and edge manifolds as for $\Gamma_{\mathrm{plumb}} = \langle \Gamma, g, e \rangle$, except that we identify T_y with $S^1 \times S^1$ so that if R is the element of $\mathrm{GL}(2, \mathbb{Z})$ giving

$$m(y)_* R = Hm(\overline{y})_*,$$

then $h_y : T_{\overline{y}} \to T_y$ is the map induced by R. We thus have a commutative diagram

$$\begin{array}{ccc} T_{\overline{y}} & \xrightarrow{h_y} & T_y \\ {\scriptstyle m(\overline{y})}\big\downarrow & & {\scriptstyle m(y)}\big\downarrow \\ S^1 \times S^1 & \xrightarrow{h} & S^1 \times S^1 \end{array} .$$

Since $h = h^{-1}$, it follows that $h_{\overline{y}} = h_y^{-1}$.

Morphisms and coverings of modified plumbing graphs are morphisms and coverings of the associated graph manifolds

$$\Psi : M(\Gamma^{\mathrm{m}}_{\mathrm{plumb}}) \to M(\Gamma^{\mathrm{m}}_{\mathrm{plumb}}{}')$$

such that the following diagram commutes:

$$\begin{array}{ccc} T_y & \xrightarrow{m(y)} & S^1 \times S^1 \\ {\scriptstyle \Psi_y}\big\downarrow & & \big\| \\ T_{\Psi_\Gamma(y)} & \xrightarrow{m(\Psi_\Gamma(y))} & S^1 \times S^1 \end{array} .$$

Given a plumbing graph Γ_{plumb}, one can associate a modified plumbing graph $\Gamma^{\mathrm{m}}_{\mathrm{plumb}}$, by setting all maps $m(y)$ to be the identity. One can easily verify that, in this case, the definitions for the associated graph manifold, and morphisms are the same.

§4. Normal surface-curve pairs

Let X be a normal complex projective surface, and let $\mathcal{C} \subset X$ be an algebraic curve. We will assume for simplicity that $\mathcal{C}$ is connected.

Let $|\mathcal{C}|$ be the set of irreducible curves in $\mathcal{C}$, and let $\mathcal{P} = \mathrm{Sing}(\mathcal{C})$. Let $\mathcal{F}$ be the family of surface-curve pairs $(X, \mathcal{C})$ satisfying the following conditions:

(1) each $C \in |\mathcal{C}|$ is unibranched;
(2) $\mathrm{Sing}(X) \cap \mathcal{C} \subset \mathcal{P}$; and
(3) for each $p \in \mathrm{Sing}(\mathcal{C})$, there is a locally defined finite covering of surface-curve pairs

$$\mu_p : (X, \mathcal{C}) \to (\mathbb{C}^2, \{x = 0\} \cup \{y = 0\})$$

defined near the germ (X, p).

A surface-curve pair $(X, \mathcal{C}) \in \mathcal{F}$ is call a *normal surface-curve pair.*

The following is immediate.

Lemma 5.1. *The family of normal surface-curve pairs is closed under coverings of surface-curve pairs.*

The fundamental group $\pi_1(\mathbb{C}^2 \setminus \{x = 0\} \cup \{y = 0\})$ is canonically isomorphic to the integer lattice $\mathbb{Z} \oplus \mathbb{Z}$, with natural generators given by meridian loops around $\{x = 0\}$ and $\{y = 0\}$. Thus, finite coverings correspond to 2-dimensional lattices of finite index. Given $p \in \mathcal{P}$, and $C, D \in \mathcal{C}$ containing p, let a, b, c be non-negative integers so that $(a, 0)$ and (b, c) generate the sublattice, and $0 \leq b < a$. Note that the numbers a, b, c are uniquely determined given the ordering of C and D. Changing the ordering corresponds to changing the order of the canonical basis for $\mathbb{Z} \oplus \mathbb{Z}$, and hence corresponds to switching columns of the matrix

$$\begin{bmatrix} a & b \\ 0 & c \end{bmatrix},$$

and column-reducing to get

$$\begin{bmatrix} a' & b' \\ 0 & c' \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} R,$$

where $0 \leq b' < a$ and $R \in \mathrm{GL}(2, \mathbb{Z})$.

The matrix R can be obtained from a continued fraction expansion $[m_1, \ldots, m_k]$ for a/b, where

$$\frac{a}{b} = m_1 - \cfrac{1}{m_2 - \cfrac{1}{\cdots \cfrac{1}{m_k}}}.$$

Lemma 5.2. *The matrix R is given by*

$$R = HM_1H \cdots HM_kH,$$

where

$$M_i = (M_i)^{-1} = \begin{bmatrix} 1 & m_i \\ 0 & -1 \end{bmatrix},$$

for $i = 1, \ldots, k$. Furthermore,

$$R^{-1} = HM_kH \cdots HM_1H.$$

One proof of this lemma comes from a study of the singularity (X, p) (see Theorem 7).

Theorem 6 ([Lauf], [Hir2]). *The germ (X,p) is smooth if and only if $b = 0$. In this case, $\mathcal{C}$ must have a normal crossing at p. Otherwise, the germ (X,p) can be desingularized by replacing p by exceptional curves $E_1, \ldots, E_k$, with self-intersections*

$$E_i^2 = -m_i, \qquad \text{for } i = 1, \ldots, k,$$

where $[m_1, \ldots, m_k]$ is the continued fraction expansion for a/b.

Note that reversing the order of the pair of curves C and D passing through to p simply reverses the order of $E_1, \ldots, E_k$. The exceptional curves and the proper transforms of C and D are arranged as in the graph of Figure 1,

Figure 1.

where all edges in the graph correspond to normal crossing intersections.

Given a surface-curve pair $(X, \mathcal{C}) \in \mathcal{F}$, with specified maps μ_p for $p \in \operatorname{Sing}(X)$, there is a canonically associated modified plumbing graph $\Gamma^{\mathrm{m}}_{\mathrm{plumb}} = \Gamma^{\mathrm{m}}_{\mathrm{plumb}}(X, \mathcal{C})$ given as follows. Let $(\widetilde{X}, \widetilde{\mathcal{C}})$ be a minimal desingularization of $(X, \mathcal{C})$ obtained from the μ_p as in [Lauf]. For each $C \in |\mathcal{C}|$, let $\widetilde{C} \in \mathcal{C}'$ be the proper transform of C.

(1) The graph Γ for $\Gamma^{\mathrm{m}}_{\mathrm{plumb}}$ has vertices and edges

$$\begin{aligned} V(\Gamma) &= \{ v_C \ : \ C \in |\mathcal{C}| \ \}, \qquad \text{and} \\ \mathcal{Y}(\Gamma) &= \{ y_{p,C} \ : \ p \in \mathcal{P} \cap \mathcal{C} \ \}, \end{aligned}$$

where $t(y_{p,C}) = v_C$; and for each $p \in \mathcal{P}$, if $C, D \in |\mathcal{C}|$ is the pair of curves so that $p \in C \cap D$, then we have

$$\overline{y_{p,C}} = y_{p,D};$$

(2) for each $C \in |\mathcal{C}|$, let

$$\begin{aligned} g(v_C) &= g(\widetilde{C}) = g(C), \qquad \text{and} \\ e(V_C) &= e(\widetilde{C}) = \widetilde{C}^2; \end{aligned}$$

and

(3) for each $y = y_{p,C} \in \mathcal{Y}(\Gamma)$, let

$$m(y_{p,C}) : S^1 \times S^1 \to S^1 \times S^1$$

be the finite unbranched covering induced by the matrix

$$\begin{bmatrix} a(y) & b(y) \\ 0 & c(y) \end{bmatrix},$$

where $(a(y), 0)$ and $(b(y), c(y))$ generate the image of

$$(\mu_p)_* : \pi_1(X \setminus C \cup D) \to \pi_1(\mathbb{C}^2 \setminus \{x = 0\} \cup \{y = 0\}),$$

and $0 \le b(y) < a(y)$, $0 < c(y)$.

Let $M(X, \mathcal{C})$ be the boundary of a regular neighborhood of $\mathcal{C}$ in X.

Theorem 7. *The graph manifold $M(\Gamma_{\text{plumb}}(X, \mathcal{C}))$ is homeomorphic to $M(X, \mathcal{C})$.*

Proof. For the case when X is smooth see [Neu], p. 333. When X has a singularity at p, since $(X, \mathcal{C})$ is a normal surface-curve pair, there are exactly two curves $C, D \in \mathcal{C}$ so that $p \in C \cap D$. The link S_p of the singularity (X, p) is a lens space, and $X \setminus \mathcal{C}$ looks locally like a cone over $S^3 \setminus L$ near p, where L is an oriented Hopf link. Let T_C and T_D be the torus boundary components of M_C and M_D near p. Then

$$M_p = S_p \setminus U(C_p),$$

where $U(C_p)$ is a regular neighborhood of $\mathcal{C}$ in X, is homeomorphic to a thickened torus with boundary components T_C and T_D. Identifying M_p with the product of a torus and an interval determines a homeomorphism of T_C to T_D, which we will now describe.

Let $y = y_{p,C}$ (so we have $o(y) = D$ and $t(y) = C$), and suppose

$$m(y)_* = \begin{bmatrix} a(y) & b(y) \\ 0 & c(y) \end{bmatrix}.$$

Then (X, p) can be desingularized as in Figure 1.

Give T_C and T_D trivializations so that M_C and M_D have Euler number equal to the self intersections of the proper transforms $\widetilde{C}$ and $\widetilde{D}$ in the minimal desingularization $(\widetilde{X}, \widetilde{\mathcal{C}})$ of $(X, \mathcal{C})$.

Consider the plumbing graph of $(\widetilde{X}, \widetilde{\mathcal{C}})$ over p, which is shown in Figure 1. The vertices corresponding to the E_i have corresponding vertex manifolds which are thickened tori with two boundary components. If we give these boundary components trivializations so that the Euler number of the associated S^1-bundle is $-m_i$, then the boundary components are identified via the product structure by the map

$$S^1 \times S^1 \to S^1 \times S^1$$

corresponding to M_i.

The gluing map

$$h_y : T_C \to T_D$$

can be thought of as a composition of the gluing maps for the plumbing graph of $(\widetilde{X}, \widetilde{\mathcal{C}})$ over p. Thus, h_y is the map corresponding to

$$(h_y)_* = HM_1H \cdots HM_kH$$

as in Lemma 5.2.

By the construction,

$$h \circ m(\overline{y}) = m(y) \circ h_y,$$

and it is also easy to see that $M_i = M_i^{-1}$, for $i = 1, \ldots, k$, and

$$(h_{\overline{y}})_* = HM_kH \cdots HM_1H.$$

Q.E.D.

Given a non-modified plumbing graph Γ_{plumb}, and an ordering of the vertices $v_1, \ldots, v_k \in \mathcal{V}(\Gamma)$, the associated intersection matrix $S(\Gamma_{\text{plumb}})$ is the $k \times k$ matrix with entries $a_{i,j}$, where

$$a_{i,j} = \begin{cases} e(v_i) & \text{if } i = j \\ n(i,j) & \text{otherwise} \end{cases}$$

where $n(i, j)$ is the number of $y \in \mathcal{Y}(\Gamma)$, with $o(y) = v_i$ and $t(y) = v_j$.

When $\Gamma^{\mathrm{m}}_{\mathrm{plumb}}$ is modified, then we define the intersection matrix $S(\Gamma^{\mathrm{m}}_{\mathrm{plumb}})$ to be the matrix with entries $a_{i,j}$ given by

$$a_{i,j} = \begin{cases} e(v_i) + \displaystyle\sum_{\substack{y\in\mathcal{Y}(\Gamma)\\ t(y)=v_i}} \frac{b(y)}{a(y)} & \text{if } i = j \\[2ex] \displaystyle\sum_{\substack{y\in\mathcal{Y}(\Gamma)\\ o(y)=v_i, t(y)=v_j}} \frac{\gcd(a(y), b(y))}{a(y)} & \text{otherwise} \end{cases}$$

Note that the intersection matrices for the modified and non-modified plumbing graphs agree if and only if $b(y) = 0$ for all $y \in \mathcal{Y}(\Gamma)$.

Theorem 8. *If $(X, \mathcal{C})$ is a normal surface-curve pair, then the intersection matrix $S(\Gamma^{\mathrm{m}}_{\mathrm{plumb}}(X, \mathcal{C}))$ equals $S(X, \mathcal{C})$.*

Proof. The formula for intersection numbers of distinct pairs follows directly from [Hir2] (see Lemma 3.5 and Lemma 3.7). For the self intersections, recall that, for any $C \in |\mathcal{C}|$, the pull-back $\overline{C}$ of C in the minimal desingularization is defined to be the divisor equal to the proper transform $\widetilde{C}$ of C plus the unique rational multiples of the exceptional curves, determined by the condition that

$$\overline{C}.E = 0,$$

for any exceptional curve E (see [Mum]). This implies that for each $p \in \mathcal{P} \cap C$, we need only be concerned with the coefficient r_p of the unique exceptional curve E_p over p which intersects $\widetilde{C}$. That is,

$$\begin{aligned} C^2 &= (\overline{C})^2 \\ &= \widetilde{C}.\left(\widetilde{C} + \sum_{p\in\mathcal{P}\cap C} r_p E_p\right) \\ &= (\widetilde{C})^2 + \sum_{p\in\mathcal{P}\cap C} r_p. \end{aligned}$$

The rest follows from the calculations in [Hir2] (see Lemma 3.7). Q.E.D.

§5. Applications to computations on coverings

Let $(X, \mathcal{C})$ be a normal surface-curve pair, and let $\Gamma^{\mathrm{m}}_{\mathrm{plumb}} = \Gamma^{\mathrm{m}}_{\mathrm{plumb}}(X, \mathcal{C})$ be its modified plumbing graph. Let

$$\rho : (Y, \mathcal{D}) \to (X, \mathcal{C})$$

be a regular covering defined by the epimorphism

$$\phi : \pi_1(X, \mathcal{C}) \to F.$$

In this section, we describe the intersection matrix and modified plumbing data for the covering $(Y, \mathcal{D})$ in terms of $\Gamma^{\mathrm{m}}_{\mathrm{plumb}}$, and the induced defining map

$$\psi : \pi_1(G(\Gamma^{\mathrm{m}}_{\mathrm{plumb}}), \mathcal{T}) \to F$$

where $\mathcal{T}$ is a maximal tree in Γ.

Let $F_v = \psi(G_v)$, $F_y = \psi(G_y)$, and let $I_v = \psi(Z_v)$, where Z_v is the subgroup of $G_v = \pi_1(M_v)$ generated by the fiber of the S^1-bundle M_v. For each $y \in \mathcal{Y}(\Gamma)$, let $s(y) = \psi(y)$, where y is considered as an element of $\pi_1(G(\Gamma^{\mathrm{m}}_{\mathrm{plumb}}), \mathcal{T})$. (This $s(y)$ is called the *twisting data* in [Hir1] and [Hir2])

Let Γ' be the graph consisting of vertices

$$\mathcal{V}(\Gamma') = \{[\alpha F_v] \ : \ v \in \mathcal{V}(\Gamma), \ \alpha \in F\}$$

and edges

$$\mathcal{Y}(\Gamma') = \{[\alpha F_y] \ : \ y \in \mathcal{Y}(\Gamma), \ \alpha \in F\};$$

where, if $y' = [\alpha F_y]$, let $\overline{y'} = [\alpha F_{\overline{y}}]$, and let $t(y') = v'$ where $v' = [\alpha s(y) F_v]$.

Lemma 8.1. *The graph Γ' is the underlying graph of the covering, and the map*

$$\rho_\Gamma : \Gamma' \to \Gamma$$

is given by

$$\begin{aligned} \rho_\Gamma([\alpha F_v]) &= v, \quad v \in \mathcal{V}(\Gamma), \qquad \text{and} \\ \rho_\Gamma([\alpha F_y]) &= y, \quad y \in \mathcal{Y}(\Gamma). \end{aligned}$$

Note that this presentation of the graph Γ' contains within it a natural lifting of a maximal tree $\mathcal{T}$ in Γ. Giving an identification of $\mathcal{V}(\Gamma')$ with $|\mathcal{D}|$ requires some extra information. Choose a *section*

$$\tau : \mathcal{T} \to M(X, \mathcal{C}).$$

This amounts to choosing base-points in M_v and M_y, for all $v \in \mathcal{V}(\Gamma)$ and $y \in \mathcal{Y}(\Gamma)$, and connecting paths, for each $y \in \mathcal{Y}(\Gamma)$, connecting the base-point in M_y to the base-point in $M_{t(y)}$. The section τ lifts to the boundary manifold $M(Y, \mathcal{D})$ and gives a natural identification of $|\mathcal{D}|$

with the vertices in $\mathcal{V}(\Gamma')$ so that the lift of $\tau(v)$ lies on $D \in |\mathcal{D}|$ if and only if

$$v_{\alpha D} = [\alpha F_{v_{\rho(D)}}],$$

for all $\alpha \in F$. We will call such an identification a *compatible* identification of $|\mathcal{D}|$ with $\mathcal{V}(\Gamma')$.

The genus associated to vertices in Γ', and hence to the components of $\mathcal{D}$ are given as follows.

Lemma 8.2. *For $v' \in \mathcal{V}(\Gamma')$, and $\rho_\Gamma(v') = v$, the genus $g(v')$ is given by*

$$g(v') = \frac{1}{2}\left(2 - \frac{\#F_v}{\#I_v}(2 - 2g(v) - d(v)) - \sum_{\substack{y\in\mathcal{Y}(\Gamma)\\ t(y)=v}} \frac{\#F_v}{\#F_y}\right).$$

Proof. The formula follows from additive properties of the topological Euler characteristic, Corollary 3, and Corollary 5. Q.E.D.

The map $m : \mathcal{Y}(\Gamma') \to \mathrm{Fin}(S^1 \times S^1)$ can also be written in terms of the covering data and the modified plumbing graph of the base.

Lemma 8.3. *For $y' \in \mathcal{Y}(\Gamma')$, and $y = \rho_\Gamma(y')$, $m(y')$ is the composition*

$$m(y') = m(y) \circ \rho_y,$$

where $\rho_y \in \mathrm{Fin}(S^1 \times S^1)$ is the unique map induced by

$$\psi_y : \mathbb{Z} \oplus \mathbb{Z} = G_y \to F_y,$$

such that

$$m(y')_* = \begin{bmatrix} a'(y) & b'(y) \\ 0 & c'(y) \end{bmatrix},$$

where $0 \leq b'(y) < a'(y)$ and $0 < c'(y)$.

Proof. This lemma is a consequence of the definitions of modified plumbing graphs and Proposition 4, noting that the form of $m(y')$ can be arranged by composing with an automorphism of the domain of ψ_y. Q.E.D.

Lemma 8.3, leads to the following formulas for intersection matrices of coverings, generalizing the results of [Hir2].

Theorem 9. *The intersection matrix $S(Y, \mathcal{D})$, with respect to a compatible identification of $|\mathcal{D}|$ with $\mathcal{V}(\Gamma')$, is given by*

$$[\alpha F_v].[\beta F_w] = \sum_{\substack{y \in \mathcal{Y}(\Gamma) \\ o(y)=v \\ t(y)=w}} \frac{\#\left(\alpha F_v \cap \beta s(y)^{-1} F_w\right)}{\#(I_v + I_w)} \frac{\gcd(a'(y), b'(y))}{a'(y)},$$

when $v, w \in \mathcal{V}(\Gamma)$ are distinct pairs, and

$$[\alpha F_v].[\beta F_v] = \frac{\#\left(\alpha F_v \cap \beta F_v\right)}{(\# I_v)^2}.$$

Proof. The first formula follows from Theorem 8, and Proposition 2, while the second formula follows from [Hir2] (see Lemma 3.3). Q.E.D.

The second formula in Theorem 9 leads to the following formula for the Euler numbers attached to vertices of Γ'.

Lemma 9.1. *Given $v' \in \mathcal{V}(\Gamma')$, and $v = \rho(v')$, the Euler number $e(v')$ is given by*

$$e(v') = \frac{\# F_v}{(\# I_v)^2} - \sum_{\substack{y \in \mathcal{Y}(\Gamma) \\ t(y)=v}} \frac{b'(y)}{a'(y)} \frac{\# F_v}{\# F_y}.$$

Proof. The formula follows from Theorem 8 and Theorem 9. Q.E.D.

This completes the description of the covering modified plumbing graph.

Note that the above formulas depend only on the map ψ restricted to G_v, G_y, Z_v, and $\mathcal{Y}(\Gamma)$. This leads to the question of which defining maps for the boundary manifold ψ are induced by global defining maps on $\pi_1(X \setminus \mathcal{C})$, and thus to the question of the relation between $\pi_1(X \setminus \mathcal{C})$ and $\pi_1(M(X \setminus \mathcal{C}))$.

In general (when $\mathcal{C}$ supports an ample divisor), the fundamental group of the boundary manifold of $\mathcal{C}$ in X surjects onto the fundamental group of $X \setminus \mathcal{C}$ under the map induced by inclusion. To understand the kernel of this map is a harder problem and includes the problem of understanding the effect of locations of singularities on $\mathcal{C}$ on the fundamental group of the complement.

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Polytopes, Invariants and Harmonic Functions

Katsunori Iwasaki

Abstract.

The classical harmonic functions are characterized in terms of the mean value property with respect to the unit ball. Replacing the ball by a polytope, we are led to the notion of polyhedral harmonic functions, i.e., those continuous functions which satisfy the mean value property with respect to a given polytope. The study of polyhedral harmonic functions involves not only analysis but also algebra, including combinatorics of polytopes and invariant theory for finite reflection groups. We give a brief survey on this subject, focusing on some recent results obtained by the author.

§1. Introduction

The harmonic functions are a very important class of functions in mathematics as well as in physics. Let us recall a classical theorem of Gauss and Koebe stating that they are characterized in terms of the mean value property with respect to the unit ball.

Theorem 1.1 (Gauss-Koebe). *Let Ω be a domain in $\mathbb{R}^n$. Any function $f \in C^2(\Omega)$ is harmonic if and only if $f \in C(\Omega)$ satisfies the mean value property with respect to the n-dimensional unit ball B^n with center at the origin:*

$$f(x) = \frac{1}{|B^n|} \int_{B^n} f(x+ry)\, dy \qquad (\forall\, x \in \Omega,\ 0 < \forall\, r < \operatorname{dist}(x, \partial\Omega)),$$

where $|B^n|$ denotes the volume of B^n.

This theorem naturally leads us to the following simple question (see Figure 1).

Problem 1.1. What happens if the ball is replaced by a *polytope* ?

Namely, we are interested in the problem of characterizing those continuous functions which satisfy the mean value property with respect to a given polytope. In this paper we give a brief survey on this subject, focusing on some recent results obtained by the author. See also [22].

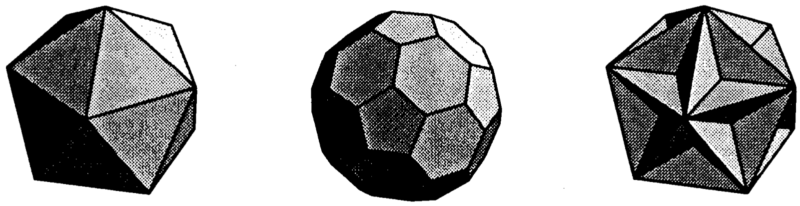

FIG 1. Examples of Polyhedra

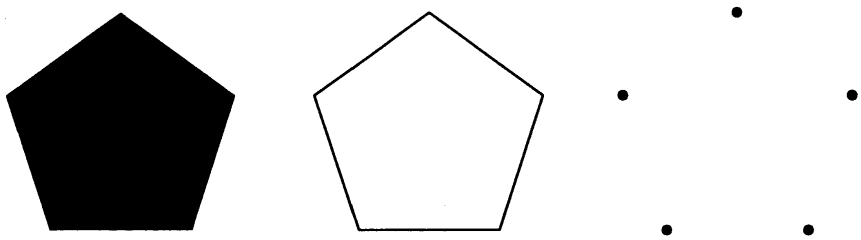

FIG 2. Skeletons of Pentagon

§2. Polyhedral Harmonic Functions

We formulate Problem 1.1 more precisely. Let P be an n-dimensional polytope, and $P(k)$ be the k-skeleton of P for $k = 0, 1, \dots, n$ (see Figure 2). A continuous function $f \in C(\Omega)$ is said to be *$P(k)$-harmonic* if f satisfies the mean value property with respect to $P(k)$, that is, for any $x \in \Omega$, there exists a positive number $r_x > 0$ such that

$$f(x) = \frac{1}{|P(k)|} \int_{P(k)} f(x + ry)\, d\mu_k(y) \qquad (\forall\, x \in \Omega,\ 0 < \forall\, r < r_x),$$

where μ_k is the k-dimensional Euclidean measure and $|P(k)| = \mu_k(P(k))$ is the total measure of $P(k)$. Let $\mathcal{H}_{P(k)}(\Omega)$ denote the linear space of all $P(k)$-harmonic functions on Ω. Then our problem is stated as follows.

Problem 2.1. Characterize the function space $\mathcal{H}_{P(k)}(\Omega)$.

The history of polyhedral harmonics began with the works of Kakutani and Nagumo[24] (1935) and Walsh[28] (1936), who considered the vertex problem ($k = 0$) for a regular convex polygon. Since then several authors have discussed various problems in various settings ([1][2][3][5][6]

[10][11][13][14][25]). See the references in [16] for a more extensive literature. In particular, Friedman and Littman[12] posed a rather surprising question.

Problem 2.2 (Friedman-Littman, 1962). Is $\mathcal{H}_{P(k)}(\Omega)$ finite dimensional ?

This problem had been open until recently when the author was able to settle it in the affirmative.

§3. General Properties

In general the function space $\mathcal{H}_{P(k)}(\Omega)$ satisfies the following properties.

Theorem 3.1 ([16]). *Let P be any n-dimensional polytope and $k \in \{0, 1, \ldots, n\}$. Then,*

(1) *$\mathcal{H}_{P(k)}(\Omega)$ is independent of the domain Ω, namely, the restriction map $\mathcal{H}_{P(k)} := \mathcal{H}_{P(k)}(\mathbb{R}^n) \to \mathcal{H}_{P(k)}(\Omega)$ is an isomorphism;*
(2) *$\mathcal{H}_{P(k)}$ is a finite-dimensional linear space of polynomials;*
(3) *Let $G \subset O(n)$ be the symmetry group of P. Then $\dim \mathcal{H}_{P(k)} \geq |G|$;*
(4) *If G is irreducible, then $\mathcal{H}_{P(k)}$ is a finite-dimensional linear space of harmonic polynomials;*
(5) *$\mathcal{H}_{P(k)}$ is an $\mathbb{R}[\partial]$-module, where $\mathbb{R}[\partial]$ is the ring of linear partial differential operators with constant coefficients.*

This theorem shows that the space $\mathcal{H}_{P(k)}(\Omega)$ of polyhedral harmonic functions is completely different from the space $\mathcal{H}(\Omega)$ of classical harmonic functions. A summary of comparisons between them is given in Table 1. The most remarkable contrast is their dimensionality; $\mathcal{H}_{P(k)}(\Omega)$ is finite dimensional, while $\mathcal{H}(\Omega)$ is infinite dimensional. The finite-dimensionality of $\mathcal{H}_{P(k)}(\Omega)$ gives rise to the problem of computing $\dim \mathcal{H}_{P(k)}(\Omega)$ and, moreover, that of constructing a natural basis of it. In view of (5) of Theorem 3.1, investigating the structure of $\mathcal{H}_{P(k)}(\Omega)$ as an $\mathbb{R}[\partial]$-module is also an interesting problem. Some results in these directions will be presented in Sections 4 and 5. But these problems are yet to be considered more extensively. Hereafter we put $\mathcal{H}_{P(k)} = \mathcal{H}_{P(k)}(\Omega)$, since it is independent of the domain Ω.

§4. Regular Convex Polytopes

Our problem is of particular interest when P admits ample symmetry. A typical instance is the case where P is a *regular convex polytope*

	$\mathcal{H}(\Omega)$: classical	$\mathcal{H}_{P(k)}(\Omega)$: polyhedral
domain Ω	depends on Ω (natural boundary)	independent of Ω
dimension	$\dim \mathcal{H}(\Omega) = \infty$	$\dim \mathcal{H}_{P(k)}(\Omega) < \infty$
functions	transcendental in general	only polynomials
PDEs	$\Delta f = 0$ (single equation)	an infinite system (holonomic)

TABLE 1. Classical vs. Polyhedral Harmonic Functions

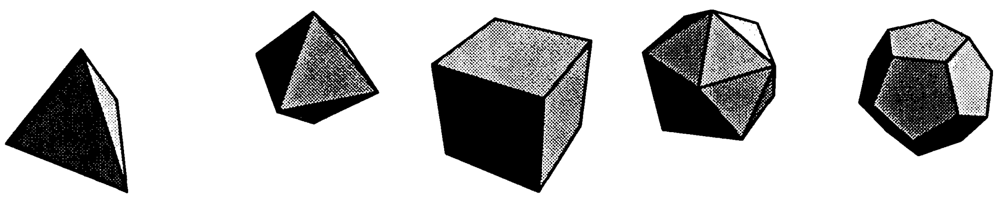

FIG 3. Platonic Solids (Regular Convex Polyhedra)

with center at the origin. We refer to [4] for the classification of regular convex polytopes (see Figure 3 for $n = 3$). In this case it is known that the symmetry group of P is a *finite reflection group*. So we can apply invariant theory for finite reflection groups to characterize the function space $\mathcal{H}_{P(k)}$.

We recall some basic definitions. A finite reflection group is a finite group generated by reflections. Here a *reflection* is an orthogonal transformation $g \in O(n)$ that takes a nonzero vector $v \in \mathbb{R}^n$ to its negative $-v$, while keeping the orthogonal complement H to v pointwise fixed. The hyperplane $H = H_g$ is called the reflecting hyperplane of g. Let $\alpha_g : \mathbb{R}^n \to \mathbb{R}$ be a linear form such that $\operatorname{Ker} \alpha_g = H_g$, (such an α_g is unique up to a nonzero constant multiple). Given a finite reflection group G, let R be the set of all reflections in G. Then the *fundamental* *alternating polynomial* for G is defined by

$$\Delta_G(x) = \prod_{g \in R} \alpha_g(x).$$

It is uniquely determined up to a nonzero constant multiple. We give an example.

Example 4.1. If P is a regular n-simplex with center at the origin, then G is the symmetric group $\mathfrak{S}_n$ acting on $\mathbb{R}^n$ by permuting the coordinates $x_1, \ldots, x_n$. In this case,

$$\Delta_G(x) = \prod_{i<j} \langle p_i - p_j, x \rangle,$$

where $p_0, p_1, \ldots, p_n$ are the vertices of P and $\langle \cdot, \cdot \rangle$ is the Euclidean inner product on $\mathbb{R}^n$.

Theorem 4.1 ([6][10][17][20][23]). *Let P be any n-dimensional regular convex polytope that is not a measure polytope. Let $G \subset O(n)$ be the symmetry group of P, and $\Delta_G(x)$ be the fundamental alternating polynomial for the finite reflection group G. Then,*

(1) *$\mathcal{H}_{P(k)}$ is independent of $k = \dim P(k)$;*
(2) *The dimension of $\mathcal{H}_{P(k)}$ is equal to the order of G:* $\dim \mathcal{H}_{P(k)} = |G|$;
(3) *$\mathcal{H}_{P(k)}$ is generated by $\Delta_G(x)$ as an $\mathbb{R}[\partial]$-module:* $\mathcal{H}_{P(k)} = \mathbb{R}[\partial]\Delta_G(x)$.

The author believes that the same result holds for the measure polytope, although he does not have a complete proof as yet. (This was proved in [13] only for $k = 0$.) The dimension of $\mathcal{H}_{P(k)}$ for each regular convex polytope P is given in Table 2, (the value for the measure polytope is still conjectural).

§5. Triangle Mean Value Property

We explicitly determine $\mathcal{H}_{\Delta(k)}$ for any *triangle* Δ in $\mathbb{R}^2$ and $k = 0, 1, 2$. To state the result we introduce some notations. Let A_1, A_2, A_3 be the vertices of the triangle Δ. (A point A in $\mathbb{R}^2$ is identified with the vector $\overrightarrow{OA}$, where O is the origin in $\mathbb{R}^2$.) The indices i, j, k stand for any permutation of $1, 2, 3$. Let A'_i be the mid-point of the side $\overline{A_j A_k}$:

$$A'_i = \frac{A_j + A_k}{2}.$$

$\dim P$	P : regular solids	$\dim \mathcal{H}_{P(k)}$
2	regular m-gon	$2m$
n	regular n-simplex (tetrahedron)	$(n+1)!$
n	cross polytope (octahedron)	$2^n n!$
n	measure polytope (cube)	$2^n n!$
3	icosahedron	120
3	dodecahedron	120
4	24-cell	1152
4	600-cell	14400
4	120-cell	14400

TABLE 2. Dimension of $\mathcal{H}_{P(k)}$ for Regular Solids

The *reciprocal* triangle Δ' of Δ is defined to be the triangle having A'_1, A'_2, A'_3 as its vertices. Let $B := (1/3)(A_1 + A_2 + A_3)$ be the *barycenter* of Δ, and I' be the *incenter* of Δ' (see Figure 4). Then the center of gravity C_k for $\Delta(k)$ is defined by

$$C_k = \begin{cases} B & (k = 0, 2), \\ I' & (k = 1). \end{cases}$$

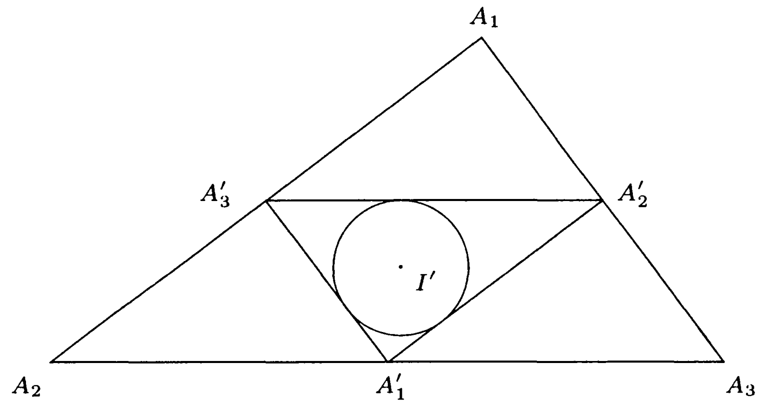

FIG 4. Reciprocal Triangle and Its Incenter

Theorem 5.1 ([21]). *The dimension of the linear space $\mathcal{H}_{\Delta(k)}$ is given by*

$$\dim \mathcal{H}_{\Delta(k)} = \begin{cases} 6 & (C_k = O), \\ 2 & (C_k \neq O). \end{cases}$$

As an $\mathbb{R}[\partial]$-module, $\mathcal{H}_{\Delta(k)}$ is generated by a single homogeneous polynomial $F_k(x)$:

$$\mathcal{H}_{\Delta(k)} = \mathbb{R}[\partial]F_k(x).$$

Explicitly, $F_k(x)$ is given as follows: If $C_k = O$, then

$$F_k(x) = \begin{cases} \prod_{i=1}^{3} \langle A_i'', x\rangle & (k = 0, 2), \\ \sum_{i=1}^{3} \dfrac{\langle A_i'', x\rangle^3}{[a_i(a_j + a_k)]^2} & (k = 1), \end{cases}$$

where a_i is the side-length of $\overline{A_jA_k}$, and A_1'', A_2'', A_3'' are the (unique) vectors satisfying

$$\langle A_i'', A_i'\rangle = 0, \quad \langle A_i'', A_j'\rangle = \frac{1}{a_j} \quad \textit{for} \quad (i,j) = (1,2),\ (2,3),\ (3,1).$$

If $C_k \neq O$, then $F_k(x) = \langle C_k', x\rangle$, where C_k' is a nonzero vector perpendicular to C_k.

§6. Differential Equations

The classical harmonic functions are characterized as the solutions of the Laplace equation $\Delta f = 0$. Note that the Laplace equation is a *single* equation. The $P(k)$-harmonic functions can also be characterized in terms of partial differential equations, though, not by a single equation, but by an *infinite system.* This system is described in terms of some combinatorial data on $P(k)$.

To describe the system we introduce some notations (see also Figure 5). For $j = 0, 1, \dots, n$, let $\{P_{i_j}\}_{i_j \in I_j}$ be the set of all j-dimensional faces of P, where I_j is an index set; H_{i_j} be the j-dimensional affine subspace of $\mathbb{R}^n$ containing P_{i_j}; and $\pi_{i_j} : \mathbb{R}^n \to H_{i_j}$ be the orthogonal projection from $\mathbb{R}^n$ down to the subspace H_{i_j}. Let $p_{i_j} \in \mathbb{R}^n$ be the vector (or point) in $\mathbb{R}^n$ defined by

$$p_{i_j} = \pi_{i_j}(0) \in H_{i_j}.$$

For $i_j \in I_j$ and $i_{j+1} \in I_{j+1}$, we write $i_j \prec i_{j+1}$ if P_{i_j} is a face of $P_{i_{j+1}}$. For $i_j \prec i_{j+1}$, let $\mathbf{n}_{i_j i_{j+1}}$ be the outer unit normal vector of $\partial P_{i_{j+1}}$ in $H_{i_{j+1}}$ at the face P_{i_j}. It is easy to see that the vector $p_{i_j} - p_{i_{j+1}}$ is parallel to $\mathbf{n}_{i_j i_{j+1}}$, so that one can define the incidence number $[i_j : i_{j+1}] \in \mathbb{R}$ by the relation

$$p_{i_j} - p_{i_{j+1}} = [i_j : i_{j+1}]\mathbf{n}_{i_j, i_{j+1}}.$$

For each $k = 0, 1, \dots, n$, let $I(k)$ be the set of *k-flags* defined by

$$I(k) = \{i = (i_0, i_1, \dots, i_k)\,;\, i_j \in I_j, i_0 \prec i_1 \prec \cdots \prec i_k\}.$$

For each k-flag $i = (i_0, i_1, \dots, i_k) \in I(k)$, we set

$$[i] = [i_0 : i_1][i_1 : i_2] \cdots [i_{k-1} : i_k] \qquad (k = 1, \dots, n),$$

with the convention $[i] = 1$ for $k = 0$. Note that $[i]$ is the *signed* volume of the k-simplex having $p_{i_0}, p_{i_1}, \dots, p_{i_k}$ as its vertices. Let $h_m^{(j)}(\xi)$ denote the complete symmetric polynomial of degree m in j-variables:

$$h_m^{(j)}(\xi_1, \dots, \xi_j) = \sum_{m_1 + \cdots + m_j = m} \xi_1^{m_1} \xi_2^{m_2} \cdots \xi_j^{m_j},$$

where the summation is taken over all j-tuples $(m_1, \dots, m_j)$ of nonnegative integers satisfying the indicated condition. Finally we set $\langle \xi, \eta \rangle = \xi_1\eta_1 + \cdots + \xi_n\eta_n$ for two vectors $\xi = (\xi_1, \dots, \xi_n), \eta = (\eta_1, \dots, \eta_n) \in \mathbb{C}^n$. The following theorem gives a characterization of the $P(k)$-harmonic functions in terms of a system of partial differential equations.

Theorem 6.1 ([16]). *Any $f \in \mathcal{H}_{P(k)}(\Omega)$ is real analytic and satisfies the system of partial differential equations*:

$$\tau_m^{(k)}(\partial) f = 0 \qquad (m = 1, 2, 3, \dots), \tag{6.1}$$

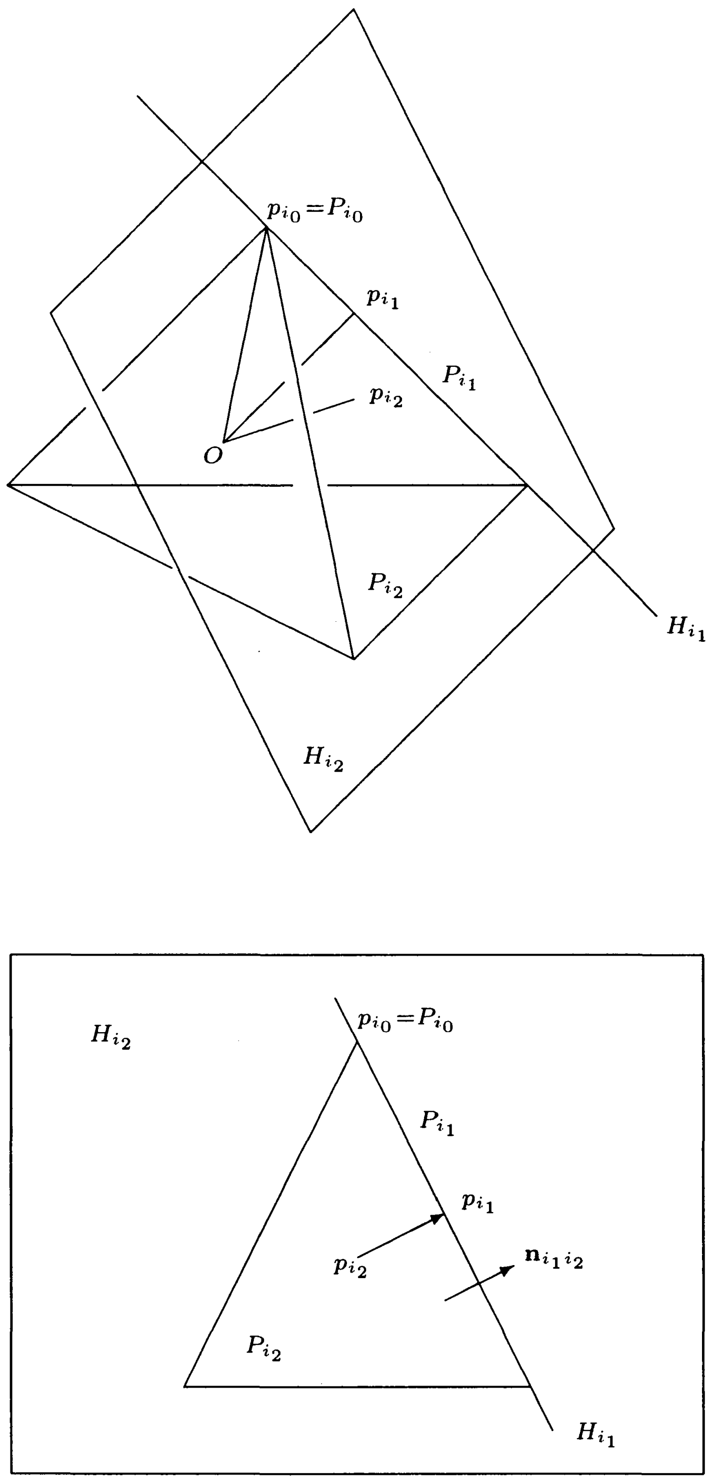

FIG 5. Combinatorial Structure of P

where $\tau_m^{(k)}(\xi)$ *is the homogeneous polynomial of degree* m *defined by*

$$\tau_m^{(k)}(\xi) = \sum_{i \in I(k)} [i] h_m^{(k+1)}(\langle p_{i_0}, \xi \rangle, \langle p_{i_1}, \xi \rangle, \dots, \langle p_{i_k}, \xi \rangle).$$

Conversely, any weak solution of (6.1) *is real analytic, and belongs to* $\mathcal{H}_{P(k)}(\Omega)$.

The system (6.1) enjoys the following remarkable property.

Theorem 6.2 ([16]). *The system* (6.1) *is holonomic. In particular, the solution space of* (6.1) *is finite dimensional.*

The proof of Theorems 6.1 and 6.2 is based on geometry and combinatorics of the polytope P. These theorems play a central role in establishing Theorem 3.1.

§7. Open Problem

Let $\mathcal{H}_n$ be the linear space of all *harmonic polynomials* in n-variables. Note that $\mathcal{H}_n$ is infinite dimensional. By Theorem 3.1, if P is an n-dimensional polytope with ample symmetry (this means that the symmetry group of P is irreducible), then $\mathcal{H}_{P(k)}$ is a finite-dimensional linear subspace of $\mathcal{H}_n$. Now a natural question arises: *As the polytope P approximates the unit ball B^n, does the function space $\mathcal{H}_{P(k)}$ approximate $\mathcal{H}_n$?* More precisely this problem is formulated as follows (see also Figure 6).

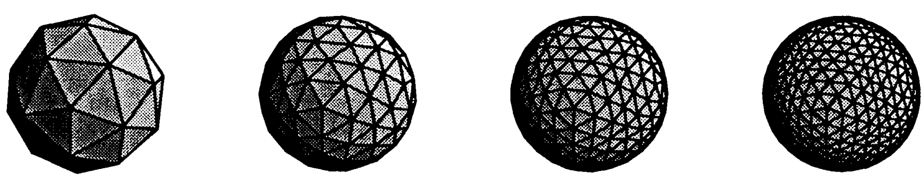

FIG 6. Geodesic Domes

Problem 7.1. Is there an infinite sequence $\{P_m\}_{m \in \mathbb{N}}$ of n-dimensional polytopes with ample symmetry such that the following conditions are satisfied ?

(1) $P_m \to B^n$ as $m \to \infty$ (Hausdorff convergence),

(2) $\cdots \subset \mathcal{H}_{P_{m-1}(k)} \subset \mathcal{H}_{P_m(k)} \subset \mathcal{H}_{P_{m+1}(k)} \subset \cdots$,
(3) $\bigcup_{m\in\mathbb{N}} \mathcal{H}_{P_m(k)} = \mathcal{H}_n$.

If $n = 2$, the answer to this question is *yes* for $k = 0, 1, 2$. Indeed, we can take P_m to be a regular convex m-gon with center at the origin. However, if $n \geq 3$, the problem becomes quite difficult. For the vertex problem ($k = 0$), we can also say that the answer is *yes*, but the proof of it is based on a very deep result from *spherical designs* (see [26]). For the remaining cases $n \geq 3$ and $k = 1, 2, \ldots, n$, the problem is completely open.

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Vassiliev Invariants of Braids and Iterated Integrals

Toshitake Kohno

§ Introduction

The notion of finite type invariants of knots was introduced by Vassiliev in his study of the discriminats of function spaces (see [13]). It was shown by Kontsevich [9] that such invariants, which we shall call the Vassiliev invariants, can be expressed universally by iterated integrals of logarithmic forms on the configuration space of distinct points in the complex plane.

In the present paper we focus on the Vassiliev invariants of braids. Our main object is to clarify the relation between the Vassiliev invariants of braids and the iterated integrals of logarithmic forms on the configuration space which are homotopy invariant. A version of such description for pure braids is given in [6]. We denote by B_n the braid group on n strings. Let J be the ideal of the group ring $\mathbf{C}[B_n]$ generated by $\sigma_i - \sigma_i^{-1}$, where $\{\sigma_i\}_{1\leq i\leq n-1}$ is the set of standard generators of B_n. The vector space of the Vassiliev invariants of B_n of order k with values in $\mathbf{C}$ can be identified with $\mathrm{Hom}(\mathbf{C}[B_n]/J^{k+1}, \mathbf{C})$. Let us stress that such vector space had been studied in terms of the iterated integrals due to K. T. Chen before the work of Vassiliev. We introduce a graded algebra $\tilde{\mathcal{A}}_n$, which is a semi-direct product of the completed universal enveloping algebra of the holonomy Lie algebra of the configuration space and the group algebra of the symmetric group. We construct a homomorphism $\theta : B_n \to \tilde{\mathcal{A}}_n$ expressed as an infinite sum of Chen's iterated integrals, which gives a universal expression of the holonomy of logarithmic connections. This homomorphism may be considered as a prototype of the Kontsevich integral for knots. Using this homomorpshim we shall determine all iterated integrals of logarithmic forms which provide invariants of braids (see Theorem 3.1). As a Corollary we recover the isomorphism

$$\tilde{\mathcal{A}}_n \cong \varprojlim \mathbf{C}[B_n]/J^j$$

which also follows from Chen's theory of iterated integrals (see also [11]). Here we notice that the above isomorphism can be shown over $\mathbf{Q}$ by means of the expression of the Vassiliev invariants based on the Drinfel'd associator defined over $\mathbf{Q}$.

The paper is organized in the following way. In Section 1 we discuss in general the situation of the complement of an arrangement of hyperplanes in a complex vector space and recall basic facts on the itegrability of logarithmic connections. In Section 2 we give a brief summary of fundamental results in Chen's theory of iterated integrals. Section 3 is the main part of the present paper. We describe the Vassiliev invariants of braids and their relation with Chen's iterated integrals of logarithmic forms.

§1. Arrangements and integrable connections

Let H_j, $1 \leq j \leq m$, be affine hyperplanes in the complex vector space $\mathbf{C}^n$ and we denote by f_j a linear form defining H_j. We define the logarithmic differential form ω_j by

$$\omega_j = \frac{1}{2\pi\sqrt{-1}}\, d\log f_j = \frac{1}{2\pi\sqrt{-1}} \frac{df_j}{f_j}.$$

We put $X = \mathbf{C}^n \setminus \cup_{j=1}^m H_j$. Let V be a complex vector space and we consider the trivial vector bundle $E = X \times V$ over X. For $A_j \in \mathrm{End}(V)$, $1 \leq j \leq m$, the 1-form $\omega = \sum_{j=1}^m A_j \omega_j$ defines a connection on the vector bundle E. We have the following Lemma.

Lemma 1.1. *The 1-form $\omega = \sum_{j=1}^m A_j \omega_j$ defines an integrable connection if the condition*

$$[A_{j_p}, A_{j_1} + \cdots + A_{j_s}] = 0, \ 1 \leq p \leq s$$

is satisfied for any maximal family of hyperplanes $H_{j_1}, \cdots, H_{j_s}$ such that

$$\mathrm{codim}_{\mathbf{C}}(H_{j_1} \cap \cdots \cap H_{j_s}) = 2.$$

Proof. For each triplet of hyperplanes H_i, H_j, H_k contained in the set of hyperplanes $\{H_{j_p}\}_{1 \leq p \leq s}$ we have the relation

$$\omega_i \wedge \omega_j + \omega_j \wedge \omega_k + \omega_k \wedge \omega_i = 0.$$

To show the integrability of the connection defined by ω it is sufficient to prove $\omega \wedge \omega = 0$ since $d\omega = 0$. This follows from the above quadratic relations among logarithmic forms. Q.E.D.

The relation among the logarithmic forms used in the proof of Lemma 1.1 is a special case of the relations for describing the structure of the cohomology ring of X given by Orlik and Solomon [12]. We denote by X_n the configuration space of ordered distinct n points in the complex plane. Namely X_n is defined by

$$X_n = \{(z_1, \cdots, z_n) \in \mathbf{C}^n \ ; \ z_i \neq z_j \text{ if } i \neq j\}.$$

Let us consider the action of the symmetric group $\mathcal{S}_n$ on X_n by the permutation of the coordinates. The quotient space $\mathcal{S}_n \setminus X_n$ is denoted by Y_n. We have an unramified covering $\pi : X_n \to Y_n$. We fix basepoints $x \in X_n$ and $y \in Y_n$ satisfying $\pi(x) = y$. The fundamental group of X_n is by definition the braid group on n strings and is denoted by B_n. The fundamental group of Y_n is the pure braid group on n strings and is denoted by P_n. We denote by σ_j, $1 \leq j \leq n-1$, the standard generators of B_n, where σ_j is represented by the braid interchanging the j-th and $(j+1)$-st strings in the positive direction. We put

$$\gamma_{ij} = \sigma_i \cdots \sigma_{j-1} \sigma_j^2 \sigma_{j-1}^{-1} \cdots \sigma_i^{-1}$$

for $1 \leq i < j \leq n$. It is known that P_n is generated by $\gamma_{ij}, 1 \leq i < j \leq n$ (see [1]).

We consider the logarithmic differential forms

$$\omega_{ij} = \frac{1}{2\pi\sqrt{-1}} d\log(z_i - z_j), \quad 1 \leq i < j \leq n$$

defined on X_n. It was shown by Arnold that the cohomology ring of X_n is generated by the de Rham cohomology classes of the above logarithmic forms with the relations

$$\omega_{ij} \wedge \omega_{jk} + \omega_{jk} \wedge \omega_{ki} + \omega_{ki} \wedge \omega_{ij} = 0, \quad i < j < k.$$

Let V be a complex vector space and let A_{ij}, $1 \leq i \neq j \leq n$, be linear transformations of V satisfying $A_{ij} = A_{ji}$. We consider the 1-form $\omega = \sum_{i<j} A_{ij}\omega_{ij}$. As a special case of Lemma 1.1, we see that ω defines an integrable connection if the condition

$$\begin{aligned} &[A_{ik}, A_{ij} + A_{jk}] = 0 \quad i, j, k \text{ distinct} \\ &[A_{ij}, A_{kl}] = 0 \quad i, j, k, l \text{ distinct} \end{aligned}$$

is satisfied. The above relation among A_{ij} is called the infinitesimal pure braid relation.

Now we explain the Knizhnik-Zamolodchikov connection as a typical example. Let $\mathbf{g}$ be a finite dimensional complex semi-simple Lie algebra and we denote by I_μ, $1 \le \mu \le \dim \mathbf{g}$, an orthonormal basis of $\mathbf{g}$ with respect to the Cartan-Killing form. Let $\rho_j : \mathbf{g} \to \mathrm{End}(V_j)$, $1 \le j \le n$, be representations of the Lie algebra $\mathbf{g}$. We put

$$\Omega_{ij} = \sum_\mu 1 \otimes \cdots \otimes 1 \otimes \rho_i(I_\mu) \otimes 1 \otimes \cdots \otimes 1 \otimes \rho_j(I_\mu) \otimes 1 \otimes \cdots \otimes 1$$

for $1 \le i, j \le n$, where $\rho_i(I_\mu)$ stands for the action on the i-th component of the tensor product $V_1 \otimes \cdots \otimes V_n$. By using the fact that the Casimir element $\sum_\mu I_\mu \cdot I_\mu$ lies in the center of the universal enveloping algebra of $\mathbf{g}$ we can show that the above Ω_{ij}, $1 \le i, j \le n$, satisfy the infinitesimal pure braid relation. Therefore the 1-form

$$\omega = \lambda \sum_{i<j} \Omega_{ij} \omega_{ij}$$

defines an integrable connection for any complex parameter λ, which we shall call the Knizhnik-Zamolodchikov connection. As the holonomy of this connecion we obtain linear representations of the pure braid group. We refer the readers to [4] and [7] for a detailed description of these representations.

§2. Review of Chen's iterated integrals

We recall the definition and basic properties of Chen's iterated integrals. Let M be a smooth manifold and we fix two points a and b in M. We denote by $\mathcal{P}_{a,b}(M)$ the set of smooth paths $\gamma : [0,1] \to M$. Let Δ_q denote the simplex defined by

$$\Delta_q = \{(t_1, \cdots, t_q) \in \mathbf{R}^q \ ; \ 0 \le t_1 \le \cdots \le t_q \le 1\}.$$

Let us consider the map

$$\phi : \mathcal{P}_{a,b}(M) \times \Delta_q \to M^q$$

defined by

$$\phi(\gamma, (t_1, \cdots, t_q)) = (\gamma(t_1), \cdots, \gamma(t_q))$$

where M^q stands for the q-fold Cartesian product of the manifold M. Let ω be a differential form of degree p on M. Then integrating the pull-back $\phi^*\omega$ along the fiber of the projection map $\pi : \mathcal{P}_{a,b}(M) \times \Delta_q \to \mathcal{P}_{a,b}(M)$, we obtain

$$\pi_* \phi^* \omega = \int_{\Delta_q} \phi^* \omega,$$

which is considered to be a differential form of degree $p-q$ on the path space $\mathcal{P}_{a,b}(M)$. For differential forms ω_j, $1 \le j \le q$, on M we denote by $\omega_1 \times \cdots \times \omega_q$ the differential form on M^q given by $\pi_1^*\omega_1 \wedge \cdots \wedge \pi_q^*\omega_q$ where $\pi_j : M^q \to M$ denotes the projection on the j-th factor. By applying the above construction we obtain the differential form $\pi_*\phi^*(\omega_1 \times \cdots \times \omega_q)$ on the path space $\mathcal{P}_{a,b}(M)$. The value of $\pi_*\phi^*(\omega_1 \times \cdots \times \omega_q)$ at $\gamma \in \mathcal{P}_{a,b}(M)$ is also denoted by

$$\int_\gamma \omega_1 \cdots \omega_q.$$

which is by definition Chen's iterated integral of $\omega_1, \cdots, \omega_q$ along the path γ. We can show that the differential form $d(\pi_*\phi^*\omega)$ on the path space $\mathcal{P}_{a,b}(M)$ is written as the sum of $\pi_*\phi^*(d\omega)$ and

$$\int_{\partial \Delta_q} \phi^*\omega$$

with a suitable sign convention. This leads us to define the following double complex.

We denote by $A^p(M)$ the vector space of smooth differential forms of deree p on M. We define $\mathcal{C}^{p,-q}(M)$ to be the direct sum

$$\oplus_{p_1+\cdots+p_q=p,\ p_1>0,\cdots,p_q>0}\ [A^{p_1}(M) \otimes \cdots \otimes A^{p_q}(M)].$$

Let us introduce the differentials

$$d_1 : \mathcal{C}^{p,-q} \to \mathcal{C}^{p+1,-q}, \quad d_2 : \mathcal{C}^{p,-q} \to \mathcal{C}^{p,-q+1}$$

by

$$\begin{aligned} d_1(\omega_1 \otimes \cdots \otimes \omega_q) &= \sum_{i=1}^{q} (-1)^i (J\omega_1 \otimes \cdots \otimes J\omega_{i-1} \otimes d\omega_i \otimes \cdots \otimes \omega_q) \\ d_2(\omega_1 \otimes \cdots \otimes \omega_q) &= \sum_{i=1}^{q-1} (-1)^{i-1} (J\omega_1 \otimes \cdots \otimes J\omega_{i-1} \otimes (J\omega_i \wedge \omega_{i+1}) \\ &\qquad \otimes\ \omega_{i+2} \otimes \cdots \otimes \omega_q) \end{aligned}$$

where $J\omega$ stands for $(-1)^{\deg \omega}\omega$. Putting $\mathcal{C}^n = \oplus_{p-q=n}\mathcal{C}^{p,-q}$ and $d = d_1 + d_2$, we obtain the associated total complex $\mathcal{C}^\bullet = \oplus_{n \in \mathbf{Z}}\ \mathcal{C}^n$, which we shall call the bar complex.

We define a linear map $\mu : \mathcal{C}^\bullet \to A^\bullet(\mathcal{P}_{a,b}(M))$ by

$$\mu(\omega_1 \otimes \cdots \otimes \omega_q) = \pi_*\phi^*(\omega_1 \times \cdots \times \omega_q).$$

We have the following Proposition.

Proposition 2.1 (K. T. Chen [2]). *The map*

$$\mu : \mathcal{C}^\bullet \to A^\bullet(\mathcal{P}_{a,b}(M))$$

defines a homomorphism of graded differential algebras.

We fix a basepoint $x \in M$ and we denote by $\Omega_x(M)$ the loop space of M based at x. Namely $\Omega_x(M)$ is by definition the space of paths $\mathcal{P}_{x,x}(M)$. The following is a fundamental result due to K. T. Chen in the case M is simply connected.

Theorem 2.2 (K. T. Chen [2]). *Let M be a simply connected manifold. The above map μ induces an isomorphism of cohomology*

$$H^j(\mathcal{C}^\bullet) \cong H^j_{DR}(\Omega_x(M))$$

where $H^j_{DR}(\Omega_x(M))$ denotes the de Rham cohomology of the loop space $\Omega_x(M)$.

Let us describe the relation between the fundamental group of M and the 0-th cohomology $H^0(\mathcal{C}^\bullet)$ of the bar complex $\mathcal{C}^\bullet$. The iterated integration map

$$\iota : \mathcal{C}^0 \times \Omega_x(M) \to \mathbf{R}$$

defined by $\iota(\omega_1 \otimes \cdots \otimes \omega_q, \gamma) = \int_\gamma \omega_1 \cdots \omega_q$ induces a natural paring map

$$H^0(\mathcal{C}^\bullet) \times \pi_1(M, x) \to \mathbf{R}$$

which gives a bilinear map

$$H^0(\mathcal{C}^\bullet) \times \mathbf{R}[\pi_1(M, x)] \to \mathbf{R}.$$

Here $\mathbf{R}[\pi_1(M, x)]$ stands for the group algebra of $\pi_1(M, x)$ over $\mathbf{R}$. We denote by I the kernel of the augmentation homomorphism $\varepsilon : \mathbf{R}[\pi_1(M, x)] \to \mathbf{R}$. Let us introduce the increasing filtration $\mathcal{F}_k\mathcal{C}^n$, $k \geq 0$, on the bar complex $\mathcal{C}^\bullet$ defined by

$$\mathcal{F}_k\mathcal{C}^n = \oplus_{p-q=n,\ q \leq k} \ \mathcal{C}^{p,-q}.$$

The above filtration is preserved by the differential and induces a filtration on the cohomology of the bar complex $\mathcal{C}^\bullet$. The following Theorem is due to K. T. Chen.

Theorem 2.3 (K. T. Chen [3]). *The iterated integration map induces an isomorphism*

$$\mathcal{F}_k H^0(\mathcal{C}^\bullet) \cong \mathrm{Hom}_{\mathbf{R}}(\mathbf{R}[\pi_1(M, x)]/I^{k+1}, \mathbf{R}).$$

Let us denote by $\mathrm{Lib}(H_1(M,\mathbf{Q}))$ the free Lie algebra over $\mathbf{Q}$ generated by the first homology $H_1(M,\mathbf{Q})$. We consider the dual of the cup product homomorphism

$$\alpha : H_2(M,\mathbf{Q}) \to H_1(M,\mathbf{Q}) \wedge H_1(M,\mathbf{Q})$$

and the ideal in $\mathrm{Lib}(H_1(M,\mathbf{Q}))$ generated by Im α is denoted by $\mathcal{I}$. Here we identify the wedge product with the Lie bracket. The holonomy Lie algebra of M over $\mathbf{Q}$ is defined to be

$$\mathbf{g}(M)_{\mathbf{Q}} = \mathrm{Lib}(H_1(M,\mathbf{Q}))/\mathcal{I}.$$

We have the filtration

$$\mathbf{g}(M)_{\mathbf{Q}} = \Gamma_0 \supset \Gamma_1 \cdots \supset \Gamma_j \supset \cdots$$

defined inductively by $\Gamma_{j+1} = [\Gamma_0, \Gamma_j]$ for $j \geq 0$. As the quotient $\mathbf{g}(M)_{\mathbf{Q}}/\Gamma_j$ we obtain a nilpotent Lie algebra whose universal enveloping algebra is denoted by $U(\mathbf{g}(M)_{\mathbf{Q}}/\Gamma_j)$. We consider the $\mathbf{Q}$-algebra $\mathcal{A}(M)_{\mathbf{Q}}$ defined by the inverse limit

$$\mathcal{A}(M)_{\mathbf{Q}} = \varprojlim U(\mathbf{g}(M)_{\mathbf{Q}}/\Gamma_j).$$

In the case when the manifold M is the complement of hyperlpanes $X = \mathbf{C}^n \setminus \cup_{j=1}^m H_j$ we have an isomorphism

$$U(\mathbf{g}(X)_{\mathbf{Q}}/\Gamma_j) \cong \mathbf{Q}[\pi_1(X,x)]/I^{j+1}$$

which induces an isomorphism of complete Hopf algebras

$$\mathcal{A}(X)_{\mathbf{Q}} \cong \varprojlim \mathbf{Q}[\pi_1(X,x)]/I^j.$$

We refer the readers to [6] for the above isomorphisms. In this case the algebra $\mathcal{A}(X)_{\mathbf{Q}}$ has the following explicit description. We take basis $X_j, 1 \leq j \leq m$, whose dual basis consists of the logarithmic forms $\omega_j, 1 \leq j \leq m$. Let us denote by $\mathbf{Q}\langle\langle X_1, \cdots, X_m\rangle\rangle$ the algebra of formal power series in the non-commutative indeterminates $X_j, 1 \leq j \leq m$, and let $\mathcal{J}$ be its ideal generated by

$$[X_{j_p}, X_{j_1} + \cdots + X_{j_s}], \quad 1 \leq p \leq s$$

for any maximal family of hyperplanes $H_{j_1}, \cdots, H_{j_s}$ such that

$$\mathrm{codim}_{\mathbf{C}}(H_{j_1} \cap \cdots \cap H_{j_s}) = 2.$$

Then we have an isomorphism

$$\mathcal{A}(X)_{\mathbf{Q}} \cong \mathbf{Q}\langle\langle X_1, \cdots, X_m \rangle\rangle / \mathcal{J}$$

as complete Hopf algebras.

For a field k containing $\mathbf{Q}$ we put $\mathcal{A}(X)_k = \mathcal{A}(X)_{\mathbf{Q}} \otimes k$. We are going to construct a homomorphism

$$\theta : \pi_1(X, x) \to \mathcal{A}(X)_{\mathbf{C}}$$

which gives a universal expression of the holonomy of the connection $\omega = \sum_{j=1}^m A_j \omega_j$ in Section 1. We put $\tilde{\omega} = \sum_{j=1}^m X_j \otimes \omega_j$ and we define the map θ as the infinite sum of iterated integrals given by

$$\theta(\gamma) = 1 + \int_\gamma \tilde{\omega} + \cdots + \int_\gamma \underbrace{\tilde{\omega} \cdots \tilde{\omega}}_{k} + \cdots$$

for $\gamma \in \pi_1(X, x)$. Here by definition

$$\int_\gamma \underbrace{\tilde{\omega} \cdots \tilde{\omega}}_{k} = \sum_{j_1, \cdots, j_k} \left(\int_\gamma \omega_{j_1} \cdots \omega_{j_k} \right) X_{j_1} \cdots X_{j_k}.$$

We see that θ induces a homomorphism of algebras

$$\tilde{\theta} : \mathbf{C}[\pi_1(X, x)] \to \mathcal{A}(X)_{\mathbf{C}}.$$

§3. Iterated integrals and invariants of braids

First we describe the notion of Vassiliev invariants of braids by means of the group algebra of the braid group. Let J be the kernel of the natural homomorphism $\pi : \mathbf{C}[B_n] \to \mathbf{C}\mathcal{S}_n$. It turns out that J is the ideal generated by $\sigma_i - \sigma_i^{-1}$, where $\{\sigma_i\}_{1 \le i \le n-1}$ is the set of standard generators of B_n. Let $v : B_n \to \mathbf{C}$ be an invariant of braids with values in $\mathbf{C}$. Extending v linearly we obtain a linear map $\tilde{v} : \mathbf{C}[B_n] \to \mathbf{C}$. We shall say that v is a Vassiliev invariant of order k if $\tilde{v}$ factors through $\mathbf{C}[B_n]/J^{k+1}$. We denote by $\mathcal{V}_k(B_n)$ the complex vector space of the Vassiliev invariants of order k for B_n. We have an identification

$$\mathcal{V}_k(B_n) \cong \mathrm{Hom}_{\mathbf{C}}(\mathbf{C}[B_n]/J^{k+1}, \mathbf{C})$$

as complex vector spaces. We have an increasing sequence of vector spaces

$$\mathcal{V}_0(B_n) \subset \cdots \subset \mathcal{V}_k(B_n) \subset \mathcal{V}_{k+1}(B_n) \subset \cdots$$

whose union $\mathcal{V}(B_n) = \cup_{k\geq 0}\mathcal{V}_k(B_n)$ is called the vector space of Vassiliev invariants of braids.

For the configuration space X_n defined in Section 1, we set $\mathcal{A}_n = \mathcal{A}(X_n)_{\mathbf{C}}$. The algebra $\mathcal{A}_n$ is described as the quotient $\mathbf{C}\langle\langle X_{ij}\rangle\rangle/\mathcal{J}$ where $\mathbf{C}\langle\langle X_{ij}\rangle\rangle$ is the algebra of formal non-commutative power series with indeterminates X_{ij}, $1 \leq i \neq j \leq n$, and $\mathcal{J}$ is the ideal generated by $X_{ij} - X_{ji}$ and the infinitesimal pure braid relations

$$
\begin{aligned}
&[X_{ik}, X_{ij} + X_{jk}] \quad i, j, k \text{ distinct} \\
&[X_{ij}, X_{kl}] \quad i, j, k, l \text{ distinct.}
\end{aligned}
$$

Let us notice that $\mathcal{A}_n$ has a structure of a graded algebra with $\deg X_{ij} = 1$. We denote by $\mathcal{A}_n^p$ the degree p part of $\mathcal{A}_n$. We put $\mathcal{A}_n^{\leq k} = \oplus_{p\leq k}\mathcal{A}_n^p$. We introduce an extension $\tilde{\mathcal{A}}_n$ of the algebra $\mathcal{A}_n$. As a vector space $\tilde{\mathcal{A}}_n$ is defined to be the tensor product $\mathcal{A}_n \otimes \mathbf{C}[\mathcal{S}_n]$. We introduce a structure of an algebra for $\tilde{\mathcal{A}}_n$ by

$$
X_{ij}s = sX_{s(i)s(j)}.
$$

for $X_{ij} \in \mathcal{A}_n$ and $s \in \mathcal{S}_n$. The algebra $\tilde{\mathcal{A}}_n$ is the semi-direct product of $\mathcal{A}_n$ and $\mathbf{C}[\mathcal{S}_n]$ defined by the above relation. This algebra has a structure of a graded algebra with $\deg X_{ij} = 1$ and $\deg s = 0$ for $s \in \mathcal{S}_n$. As in the case of $\mathcal{A}_n$, we denote by $\tilde{\mathcal{A}}_n^p$ the degree p part of $\tilde{\mathcal{A}}_n$. We put $\tilde{\mathcal{A}}_n^{\leq k} = \oplus_{p\leq k}\tilde{\mathcal{A}}_n^p$.

Our next object is to define a linear map

$$
w : \mathcal{V}_k(B_n) \to \mathrm{Hom}_{\mathbf{C}}(\tilde{\mathcal{A}}_n^{\leq k}, \mathbf{C}).
$$

First we observe that an element of $\tilde{\mathcal{A}}_n^{\leq k}$ is written as a linear combination of elements of the form

$$
X_{i_1j_1} \cdots X_{i_pj_p}s, \quad s \in \mathcal{S}_n,\ p \leq k.
$$

We choose $\sigma \in B_n$ such that $p(\sigma) = s$. The map w is defined by

$$
\begin{aligned}
&w(v)(X_{i_1j_1} \cdots X_{i_pj_p}s) \\
&= \tilde{v}((\gamma_{i_1j_1} - 1) \cdots (\gamma_{i_pj_p} - 1)\ \sigma).
\end{aligned}
$$

We see that since v is of order k the map w is well-defined. The above $w(v)$ is called the weight system for v.

Let us consider the loop space $\Omega_y(Y_n)$ with basepoint $y \in Y_n$. An element of $\Omega_y(Y_n)$ is called a geometric braid. We define $\mathcal{C}^0(\Omega_y(Y_n))$ to be the subspace of $\mathcal{A}^0(\Omega_y(Y_n))$ spanned by the functions whose values at $\gamma \in \Omega_y(Y_n)$ are given by the iterated integrals of the form

$$\int_{\tilde{\gamma}} \omega_{i_1 j_1} \cdots \omega_{i_k j_k}$$

with some k, where $\tilde{\gamma}$ is the lift of γ in X_n starting at the basepoint $x \in X_n$. The 0-th cohomology $H^0(\mathcal{C}^\bullet(\Omega_y(Y_n)))$ consists of the iterated integrals of logarithmic forms depending only on the homotopy classes of loops $\gamma \in \Omega_y(Y_n)$. This cohomology group has the increasing filtration $\mathcal{F}_k H^0(\mathcal{C}^\bullet(\Omega_y(Y_n)), k \geq 0$, defined by the length of the iterated integrals. The following Theorem permits us to determine all such iterated integrals combinatorially in terms of the algebra $\tilde{\mathcal{A}}_n$.

Theorem 3.1. *We have isomorphisms*

$$\mathcal{F}_k H^0(\mathcal{C}^\bullet(\Omega_y(Y_n)) \cong \mathcal{V}_k(B_n) \cong \mathrm{Hom}_{\mathbf{C}}(\tilde{\mathcal{A}}_n^{\leq k}, \mathbf{C}).$$

Proof. We defined the map

$$w : \mathcal{V}_k(B_n) \to \mathrm{Hom}_{\mathbf{C}}(\tilde{\mathcal{A}}_n^{\leq k}, \mathbf{C})$$

by taking the associated weight system. To construct the inverse map we consider the universal holonomy homomorphism

$$\theta : B_n \to \tilde{\mathcal{A}}_n$$

defined in the following way. We put $\tilde{\omega} = \sum_{i<j} X_{ij} \otimes \omega_{ij}$ and we define the map θ as the infinite sum of iterated integrals given by

$$\theta(\gamma) = \left(1 + \int_{\tilde{\gamma}} \tilde{\omega} + \cdots + \int_{\tilde{\gamma}} \underbrace{\tilde{\omega} \cdots \tilde{\omega}}_{k} + \cdots \right) p(\gamma)$$

for B_n, where $\tilde{\gamma}$ is the lift of γ in X_n starting at the basepoint $x \in X_n$ as above and $p : B_n \to \mathcal{S}_n$ is the narutal homomorphism. We denote by $\tau : \tilde{\mathcal{A}}_n \to \tilde{\mathcal{A}}_n^{\leq k}$ the truncation map. For $\beta \in \mathrm{Hom}_{\mathbf{C}}(\tilde{\mathcal{A}}_n^{\leq k}, \mathbf{C})$ we obtain a Vassiliev invariant of order k of B_n as the composition $\beta \circ \tau \circ \theta$. One can check that this construction gives the inverse of the map w. We observe that each element in $\mathcal{F}_k H^0(\mathcal{C}^\bullet(\Omega_y(Y_n))$ defines a Vassiliev invariant of braids of order k. Conversely, given a Vassiliev invariant v of braids of order k, we consider the associated weight system $w(v)$. Then as the composition $w(v) \circ \tau \circ \theta$ we recover the iterated integral expression of the Vassiliev invariant v. This shows the first isomorphism. Q.E.D.

Corollary 3.2. *We have an isomorphism*

$$\tilde{\mathcal{A}}_n^{\leq k} \cong \mathbf{C}[B_n]/J^{k+1}$$

which induces an isomorphism of complete Hopf algebras

$$\tilde{\mathcal{A}}_n \cong \varprojlim \mathbf{C}[B_n]/J^j$$

Remark 3.3. It can been shown that the universal Vassiliev invariants with values in $\mathbf{Q}$ can be defined by means of the Drinfel'd associator defined over $\mathbf{Q}$ (see [4], [5] and [10]). Using this expression we can establish the isomorphism in the above Corollary over $\mathbf{Q}$.

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Cohomology of Local systems

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§1. Introduction

This survey is intended to provide a background for the authors paper [23]. The latter was the subject of the talk given by the second author at the Arrangement Workshop. The central theme of this survey is the cohomology of local systems on quasi-projective varieties, especially on the complements to algebraic curves and arrangements of lines in $\mathbf{P}^2$. A few of the results of [23] are discussed in section 4 while the first part of this paper contains some of highlights of Deligne's theory [7] and several examples from the theory of Alexander invariants developed mostly by the first author in the series of papers [17] - [22]. We also included several problems indicating possible further development. The second author uses the opportunity to thank M. Oka and H. Terao for the hard labor of organizing the Arrangement Workshop.

§2. Background on cohomology of local systems

Local systems. A local system of rank n on a topological space X is a homomorphism $\pi_1(X) \to GL(n, \mathbf{C})$. Such a homomorphism defines a vector bundle on X with discrete structure group or a locally constant bundle (cf. [7], I.1). Indeed, if $\tilde{X}_u$ is the universal cover of X then $\tilde{X}_u \times_{\pi_1(X)} \mathbf{C}^n$ is such a bundle (this product is the quotient of $\tilde{X}_u \times \mathbf{C}^n$ by the equivalence relation $(x, v) \sim (x', v')$ if and only if there is $g \in \pi_1(X)$ such that $x' = gx, v' = gv$; this quotient has the projection onto $\tilde{X}_u/\pi_1(X) = X$ with the fiber $\mathbf{C}^n$). Vice versa, any locally constant bundle defines a representation of the fundamental group of the base.

If X is a *complex* manifold, then there is a one-to-one correspondence between the local systems and pairs consisting of a *holomorphic* vector bundle on X and an *integrable* connection on the latter (cf. [7] I.2, Theorem 2.17). If V is a vector bundle then a connection can be viewed as a $\mathbf{C}$-linear map defined for each open set U of X and acting as follows:

[1]partially supported by NSF

$\nabla : V(U) \to \Omega^1(X)(U) \otimes_{\mathcal{O}(U)} V(U))$. Here $V(U)$ (resp. $\mathcal{O}(U)$, resp. $\Omega(X)(U))$ is the space of sections of V (resp. the space of functions and the space of 1-forms) holomorphic on U. It is required from ∇ to satisfy the Leibniz rule $\nabla(f \cdot s) = df \otimes s + f \cdot \nabla(s)$. The integrability requirement is that if one extends ∇ to the maps $\nabla_1 : \Omega^1(X)(U) \otimes_{\mathcal{O}(U)} V(U) \to \Omega^2(X)(U) \otimes_{\mathcal{O}(U)} V(U)$ using the rule $\nabla_1(\omega \otimes v) = d\omega \otimes v - \omega \wedge \nabla v$ then $\nabla_1 \circ \nabla = 0$.

The above correspondence can be described as follows. If V is a locally constant bundle, then on the holomorphic bundle $V \otimes_{\mathbf{C}} \mathcal{O}$, where $\mathcal{O}$ is the trivial bundle, one can define the connection by $\nabla(f \otimes v) = df \otimes v$ where f (resp. v) is a holomorphic function (resp. a section of a locally constant bundle V) on U. The sections v of $V \otimes_{\mathbf{C}} \mathcal{O}$ which are flat with respect to this connection, i.e., such that $\nabla(v) = 0$, coincide with the sections of V. Vice versa, the sections of any holomorphic bundle with integrable connection form a locally constant bundle, i.e., a local system.

Cohomology. The homology of a local system can be defined as the homology of chain complex:

$$\ldots \to C_i(\tilde{X}_u) \otimes_{\pi_1(X)} \mathbf{C}^n \to \ldots$$

Here the chain complex for $\tilde{X}_u$ can be the complex of singular chains, or chains corresponding to a triangulation, or chains with a support, etc.

It is well known that the cohomology of X with constant coefficients can be calculated using the de Rham complex $\mathcal{A}^*(X)$ of C^∞-differential forms(the de Rham theorem). In the case where X is a non singular algebraic variety which is the complement to the union Y of smooth divisors on a projective variety $\bar{X}$ one can define a subcomplex $\mathcal{A}\langle Y \rangle$ of de Rham complex $\mathcal{A}^*(X)$ called log-complex. It consists of C^∞ forms ω on X with the property that near a point of $\bar{X}$ at which Y has local equation $Q = 0$ both $Q\omega$ and $Qd\omega$ admit extension to $\bar{X}$. If the components of Y intersect transversally then the cohomology of the complex $\mathcal{A}\langle Y \rangle$ is also isomorphic to $H^*(X)$. Otherwise it is not valid in general, though under some conditions on the singularities of Y (e.g. if Y is *free*) it is still true (see [4, 34]).

The cohomology of local systems also can be described using differential forms. Before stating this result let us recall that, though the *holomorphic* log-de Rham complex is too small to give full cohomology groups, there is, nevertheless, a way to reconstruct cohomology of X using holomorphic log-forms. Namely, one can consider a double complex $\mathcal{F}^{i,j}$ of sheaves such that all $\mathcal{F}^{i,j}$ are acyclic and $\mathcal{F}^{i,*}$ form a resolution of the sheaf of holomorphic log-forms $\Omega^i(X)\langle Y \rangle$. The cohomology of the double complex $\Gamma(\mathcal{F}^{i,j})$, i.e., the cohomology of the complex $\oplus_{i+j=k}\Gamma(\mathcal{F}^{i,j})$, is called the hypercohomolgy of $\Omega^*(X)\langle Y\rangle$. This construction of hypercohomology, applied verbatim to any complex of sheaves F^* instead of log-complex, yields hypercohomology groups $\mathbf{H}^*(F^*)$. A theorem of Deligne states that the hypercohomolgy $\mathbf{H}^i(\Omega^*(X)\langle Y\rangle)$ is isomorphic to $H^i(X,\mathbf{C})$ (cf. [8]). On the other hand, if X is affine, e.g. a complement to a hypersurface in projective space (cf. section 4.), then the cohomology $H^i(X,\mathbf{C})$ can be found using the complex of *rational* forms (the algebraic de Rham theorem, cf. [12]).

Hypercohomology also yields the cohomology of local systems in terms of differential forms, i.e., give a version of the de Rham theorem for local systems. The (holomorphic) de Rham complex in this case is formed by the sheaves of holomorphic forms with values in the holomorphic bundle $\mathcal{V}$ corresponding to the local system V, i.e., $\Omega^p(\mathcal{V}) = \Omega^p \otimes_{\mathcal{O}} \mathcal{V}$ with the differential given by $\nabla_p(\omega \otimes v) = d\omega \otimes v + (-1)^{\deg(v)}\omega \wedge \nabla v$ (∇_1 above is a special case of the differential in this de Rham complex). Note that integrability $\nabla_1 \circ \nabla = 0$ yields that $\nabla_{p+1} \circ \nabla_p = 0$, i.e., $\Omega^p(\mathcal{V})$ form indeed a complex of sheaves. This de Rham complex is a resolution of the holomorphic bundle $\mathcal{V}$ and it yields "de Rham theorem" $\mathbf{H}^p(\Omega(X)(\mathcal{V})) = H^p(X,V)$ (if V is a trivial local system one obtains the standard de Rham theorem). Moreover, if X is affine, the de Rham theorem with twisted coefficients still holds, i.e., the cohomology of the complex of rational forms with values in $\mathcal{V}$ is isomorphic to $H^i(X,V)$ (cf. [7], II, cor.6.3)

Calculation of cohomology of local systems using logarithmic complex is more subtle (even in the case of normal crossing), i.e., hypercohomology of log-complex yield the cohomology of the local system only if certain conditions are met. Deligne describes such sufficient conditions. The conditions are stated in the case where the connection ∇ has logarithmic poles along Y in the following sense. One assumes that the bundle $\mathcal{V}$ on X is a restriction of a holomorphic bundle $\bar{\mathcal{V}}$ on compactification $\bar{X}$ of X where $Y = \bar{X} - X$ is a divisor with normal crossings. The logarithmic property of ∇ means that in a sufficiently small neighborhood U_p of any point $p \in Y$, such that there exists a choice of sections $e_i \in \Gamma(U_p, \bar{\mathcal{V}})$ forming a basis of any fiber of $\bar{V}$ in U_p, the matrix of ∇ consists of 1-forms having logarithmic poles along Y. The entries of this matrix are $a_{i,j} \in \Omega^1(U_p \cap X \otimes \mathcal{V})$ such that $\nabla(e_i) = \Sigma a_{i,j} \otimes e_j$. The matrix of ∇ can be described in invariant terms as an element of $\Omega^1(X)(\operatorname{End}\mathcal{V})$. On the other hand, near $p \in Y$ where Y is given by $z_1 \cdots z_k = 0$, a log-1-form ω on X can be written as $\Sigma\alpha_i dz_i/z_i$ where α_i are holomorphic in U_p and hence defines a holomorphic section $\operatorname{Res}_Y(\omega) = \Sigma_i \alpha_i|Y$ on Y called the residue of ω. If ∇ is a connection with logarithmic poles then one can define $\operatorname{Res}_Y(\nabla)$ as a matrix formed by the residues of log-1-forms $a_{i,j}$. This matrix can be viewed as an element of $\mathrm{End}(\bar{\mathcal{V}}|_Y)$. Deligne's fundamental theorem ([7], 3.15) states that if none of the eigenvalues of matrices $\mathrm{Res}_Y(\nabla)(p)$ ($p \in Y$) is a positive integer then one has the isomorphism of hypercohomology:

$$\mathbf{H}(X, \Omega^*_{\bar{X}}\langle Y\rangle(\mathcal{V})) = \mathbf{H}(X, \Omega^*_X(\mathcal{V}))$$

Rank one local systems. Rank one local systems on X are just the characters of fundamental group or equivalently of $H_1(X, \mathbb{Z})$. We will assume for simplicity that the latter group is torsion free. In this case the "moduli space" of local systems of rank one is just the torus $\mathrm{Char}(X) = \mathbf{C}^{*b_1}$ where $b_1 = \dim H_1(X, \mathbf{R})$ (presence of torsion in $H_1(X, \mathbf{Z})$ will yield $\mathrm{Char}(X)$ with several connected components, each being a translation of a torus). For higher rank the construction of moduli spaces in considerably more complicated (cf. [31]).

The torus $\mathrm{Char}(X)$ contains subvarieties Σ_i^k that consist of local systems V such that $rkH^k(X, V) \geq i$. Σ_i^k are important invariants of X. They play a crucial role in several problems.

First, these subvarieties of $\mathrm{Char}(X)$ are closely related to the structure of the fundamental group of X or more precisely to the Alexander invariants of the latter. Those can be defined as follows (cf. [22]). Let $\tilde{X}_A$ be an abelian cover of X corresponding to the kernel of a surjection $\phi_A : \pi_1(X) \to A$. The group A is an abelian group of deck transformations. Though $\tilde{X}_A$ of course depends on ϕ_A we shall not specify this dependence since in this paper this wouldn't cause a confusion. The group $H_1(\tilde{X}_A, \mathbf{C})$ can be considered as a module over the group ring of the group of deck transformations of $\tilde{X}_A$, i.e., over $R = \mathbf{C}[A]$. The latter, after a choice of independent generators in A, can be identified with the ring of Laurent polynomials of $rk(A)$ variables. This module is the Alexander invariant of X corresponding to A and is denoted below by $\mathcal{A}(X, A)$. A particularly important case is where $A = H_1(X, \mathbb{Z})$, i.e., the case of universal abelian cover, since in this case $\mathcal{A}(X, A)$ is a homotopy invariant of X.

Definition 2.1. *Let $R^m \to R^n \to \mathcal{A}(X, A) \to 0$ be a presentation of the Alexander invariant. The i-th characteristic variety is the set of zeros in $(\mathbf{C}^*)^{rk(A)}$ of the polynomials in the ideal in R generated by minors of order $n - i + 1$ (i-th Fitting ideal of $\mathcal{A}(X, A)$).*

If $A = H_1(X, \mathbb{Z})$ then $H_1(\tilde{X}_A) = \pi_1'(X)/\pi_1''(X) \otimes \mathbf{C}$, i.e., depends only on the fundamental group of X. For any group G, the Alexander invariant of X such that $\pi_1(X) = G$ provides an invariant of a pair

(G, A) where A is a (free) abelian quotient of G. For any A the module $\mathcal{A}(X, A)$ can be computed directly using Fox calculus.

It turns out (cf. [14], [22]) that the i-th characteristic variety coincides with Σ_i^1 (considered as subvariety of $\mathrm{Char}(A)$ of the space of rank one local systems which factor through A). In the case where $\pi_1(X)$ is abelian and $\pi_j(X) = 0$ for $j < k$ one can similarly interpret Σ_i^k as the set of zeros of polynomials in the i-th Fitting ideal of the module $H_k(\tilde{X}_{H_1(X,\mathbb{Z})})$ (cf. [20] where typical case of such situation, i.e., when X is a complement to a hypersurface in $\mathbf{C}^{k+1}$, is considered).

Second, the characteristic varieties Σ_i^1 determine the one dimensional cohomology of branched and unbranched covers of X (cf. [18], [29] and the next section)

Third, the varieties Σ_i^1 allow one to detect dominant maps of X on curves. These results are going back to classical works of deFranchis on the existence of irrational pencils on algebraic surfaces and to more recent work of Green-Lazarsfeld, Beauville, Simpson, Deligne (when X is projective) and D. Arapura (when X is quasiprojective).

Theorem 2.2 ((D.Arapura) [2]). *Let X be a quasiprojective variety then any irreducible component of characteristic variety is a coset of a subgroup of* $\mathrm{Char}\,\pi_1(X)$. *Moreover each component having a positive dimension corresponds to a holomorphic map $f : X \to C$ on a curve C such that local systems in this component have the form $E \otimes f^*(L)$ where L runs through the local systems on C.*

§3. Local systems on complements to algebraic curves

Now we will restrict our attention to the case where X is a complement to an algebraic curve $\mathcal{C}$ in affine plane $\mathbf{C}^2 = \mathbf{P}^2 - L$. The case where L is in general position relative to the projective closure $\bar{\mathcal{C}}$ of $\mathcal{C}$ is of particular interest since in this case $\pi_1(\mathbf{P}^2 - \bar{\mathcal{C}})$ is just a quotient of $\pi_1(\mathbf{C}^2 - \mathcal{C})$ by an element of its center (cf. [19]) which we will assume here. A closely related case of local systems on complements to hypersurfaces with isolated singularities is considered in [21].

With curve $\mathcal{C}$ and surjection $H_1(\mathbf{C}^2 - \mathcal{C}, \mathbb{Z}) \to A$ on a group A (cf. section 2) one can associate unbranched cover $\widetilde{(\mathbf{C}^2 - \mathcal{C})}_A$ of $\mathbf{C}^2 - \mathcal{C}$ and branched cover of $\mathbf{P}^2$ branched over the projective closure of $\mathcal{C}$ with A as the group of the cover. Since the first Betti number of an algebraic surface is a birational invariant, the first Betti number of a resolution of singularities of the latter cover depends only on $\mathcal{C}$ and the group A. We shall denote a resolution of singularities of a cover of $\mathbf{P}^2$ branched along $\mathcal{C}$ by $Z_{\mathcal{C},A}$ (though it depends, of course, on the surjection on

$H_1(\mathbf{C}^2 - \mathcal{C}) \to A)$. Moreover, the first Betti numbers of $(\widetilde{\mathbf{C}^2 - \mathcal{C}})_A$ and $Z_{\mathcal{C},A}$ depend only on the characteristic varieties of $\mathbf{C}^2 - \mathcal{C}$. More precisely we have (cf. [18])

$$rkH_1((\widetilde{\mathbf{C}^2 - \mathcal{C}})_A) = \Sigma_{\chi \in \mathrm{Char}(A)} \max\{i | \chi \in \Sigma_i^1(\mathbf{C}^2 - \mathcal{C})\}.$$

For the branched case, for $\chi \in \mathrm{Char}\, \pi_1(\mathbf{C}^2 - \mathcal{C})$ denote by $\mathcal{C}_\chi$ the curve formed by the components C_i of $\mathcal{C}$ such that if γ is the boundary of a small 2-disk transversal to C_i then $\chi(\gamma_i) \neq 1$. Then (cf. [29])

$$rkH_1(Z_{\mathcal{C},A}) = \Sigma_{\chi \in \mathrm{Char}(A)} \max\{i | \chi \in \Sigma_i^1(\mathbf{C}^2 - \mathcal{C}_\chi)\}.$$

If (G, A) is a pair as in section 2 where $A = \mathbb{Z}$ then all ideals in $R = \mathbf{C}[A]$ are principal. A generator of the i-th Fitting ideal for the module $\mathcal{A}(X, A)$, (defined up to a unit of R) where X is a space with $\pi_1(X) = G$ is a polynomial called i-th Alexander polynomial $\Delta_i(\mathcal{C})$ of $\mathcal{C}$. Its set of zeros is Σ_i^1. In the case when $rkA > 1$, Fitting ideals for $\mathrm{Ker}(G \to A)/\mathrm{Ker}(G \to A)' \otimes \mathbf{C}$ are not principal in general, though in special case where G is the fundamental group of a complement to a link in a 3-sphere the first Fitting ideal *is* a product of a power of the maximal ideal of the identity of $\mathrm{Char}\, G$ and a principal ideal (whose generator is the multivariable Alexander polynomial). A special feature of the case where $G = \pi_1(\mathbf{C}^2 - \mathcal{C})$ is that one can determine the characteristic varieties in terms of local type of singularities of $\mathcal{C}$ and the geometry of the set of singular points of $\mathcal{C}$ in the projective plane containing $\bar{\mathcal{C}}$. In fact, in the cyclic case, one obtains an expression for the whole Alexander polynomial (cf. [33]). In the rest of this section we will describe this calculation and give some examples only briefly indicating how this can be generalized to the abelian case (i.e., when $rk(A) > 1$) referring the reader to [22] for complete details.

Let us first describe the local data which comes into the description of the Alexander polynomials of algebraic curves (cf. [17]). We want to associate with each germ of a plane curve singularity, say $f(x, y) = 0$ at the origin, a sequence of rational numbers $\kappa_1, ..., \kappa_l$ and corresponding ideals $\mathcal{A}_{\kappa_1}, ..\mathcal{A}_{\kappa_l}$ in the local ring $\mathcal{O}_{(0,0)}$.

Recall that the adjoint ideal of an isolated singularity of a hypersurface V at the origin near which V is given by the equation $g(x_1, ..., x_r) = 0$ consists of germs ϕ in the local ring of the origin such that $\phi \cdot (dx_1 \wedge ... \wedge \hat{dx_i} \wedge ... dx_r)/g_{x_i}$ admits a holomorphic extension over the exceptional set of some resolution of the singularity of V. The adjoint ideal will be denoted as $Adj(g = 0)$. In the case where $g(x_1, ..., x_r)$ is generic for its Newton polytope, a monomial $x_1^{i_1} \cdots x_r^{i_r}$ belongs to the adjoint ideal if and only if $(i_1+1, ..., i_r+1)$ is strictly above the Newton polytope of g (cf. [25]).

In order to define the constants $\kappa_1, ..., \kappa_l$ (constants of quasiadjunction of the singularity of a germ of plane curve $f(x,y)$) let us consider for each element ϕ in the local ring of the origin, the function

$$\Psi_\phi(p) = \min\{k | z^k \cdot \phi \in Adj(z^p = f(x,y))\}.$$

One can show that this function can be written for an appropriate rational number κ_ϕ as $\Psi_\phi(p) = [\kappa_\phi \cdot p]$ where $[\cdot]$ denotes the integer part (this is immediate, in the case where $f(x,y)$ is generic for its Newton polytope, from the description of the adjoint ideal mentioned in the previous paragraph, since in this case the germ of $z^p = f(x,y)$ is generic for its Newton polytope). Moreover, the set of rational numbers $\kappa_\phi, \phi \in \mathcal{O}_{0,0}$, is finite. In fact the set of numbers $-\kappa_\phi$ forms a subset of Arnold-Steenbrink spectrum of $f(x,y) = 0$ belonging to the interval $(0,1)$ (cf. [24]). In particular $\exp(2\pi i \kappa)$ is a root of the local Alexander polynomial of the link of the singularity $f(x,y) = 0$.

It follows from the definition that for each κ the germs ϕ such that $\kappa_\phi < \kappa$ form an ideal called *an ideal of quasiadjunction* and denoted $\mathcal{A}_\kappa$. Now we are ready to describe the Alexander polynomial of $\mathcal{C} \subset \mathbf{C}^2$ (cf. [17]). For each rational κ let us consider the ideal sheaf $\mathcal{I}_\kappa \subset \mathcal{O}_{\mathbf{P}^2}$ such that $\mathcal{O}_{\mathbf{P}^2}/\mathcal{I}_\kappa$ is supported at the singular locus of $\bar{\mathcal{C}}$ and such that the stalk of $\mathcal{I}_\kappa$ at a singular point p of $\mathcal{C}$ consists of germs belonging to the ideal $\mathcal{A}_\kappa$ of the singularity of $\mathcal{C}$ at p.

Theorem 3.1. *The Alexander polynomial of $\mathcal{C}$ having degree d is $\Pi_\kappa[(t-\exp(2\pi i\kappa))(t-\exp(-2\pi i\kappa)]^{\dim H^1(\mathcal{I}_\kappa(d-3-d\cdot\kappa))}$ where the product is over all constants of quasiadjunction of all singular points of $\mathcal{C}$ such that $d \cdot \kappa \in \mathbb{Z}$.*

Examples 1. Let $\bar{\mathcal{C}}$ be given by the equation $f_{3k}^2 + f_{2k}^3 = 0$ where f_l is a generic form of degree l. Then $\bar{\mathcal{C}}$ is a curve of degree $6k$ having $6k^2$ ordinary cusps (i.e., locally given by $x^2+y^3=0$) located at the set of solutions of $f_{2k} = f_{3k} = 0$. Ordinary cusp has only one constant of quasiadjunction $\kappa = 1/6$ and the corresponding ideal of quasiadjunction is just the maximal ideal (this follows directly from the above since the ordinary cusp is weighted homogeneous and hence generic for its Newton polytope). The corresponding sheaf $\mathcal{I}_{1/6}$ admits Koszul resolution $0 \to \mathcal{O}_{\mathbf{P}^2}(-5k) \to \mathcal{O}_{\mathbf{P}^2}(-3k) \oplus \mathcal{O}_{\mathbf{P}^2}(-2k) \to \mathcal{I}_{1/6} \to 0$ which yields $\dim H^1(\mathbf{P}^2, \mathcal{I}_{1/6}(6k-3-6k/6)) = 1$. Therefore the Alexander polynomial is equal to t^2-t+1. This, of course, provides complete description of the cohomology of local systems on the complement to this curve.

2. Let $\bar{C}$ be a sextic with 3 cusps and one singularity of type $x^4 = y^5$ (cf. [26]). We start by describing the constants of quasiadjunction of singularities of this curve which may contribute to the Alexander polynomial. First, the constant of quasiadjunction of singularity $x^4 = y^5$ corresponding to $\phi = x^i \cdot y^j$ is equal to $\kappa_\phi = \max\{(11 - 5i - 4j)/20, 0\}$ as follows from the description of adjoint ideals for polynomials generic for their Newton polytopes mentioned earlier. We noted already that $x^2 = y^3$ has only one constant of quasiadjunction, i.e., $1/6$. Second, since the degree of the curve is 6, the contributing into Alexander polynomial constants of quasiadjunction κ should satisfy $6 \cdot \kappa \in \mathbb{Z}$. Third, it follows again from the description of adjoint ideals that the monomial $x^i y^j$ belongs to the ideal of quasiadjunction corresponding to the constant of quasiadjunction $1/6$ in the local ring of the singularity $x^4 = y^5$ if and only if $x^i y^j z^{[p/6]}$ belongs to the adjoint ideal of $z^p = x^4 - y^5$. This happens if an only if $5p(i+1) + 4p(j+1) + 20((p/6)+1) \geq 20p$ for any positive p. This is equivalent to $5i + 4j > 7(2/3)$, i.e., either $i \geq 2$ or $j \geq 2$ or both $i, j \geq 1$. Hence ϕ, which is a combination of $x^i y^j$, is in the ideal of quasiadjunction of $1/6$ of singularity $x^4 = y^5$ if and only if it is in the square of the maximal ideal. Therefore the intersection index of the $\phi = 0$ with $x^4 = y^5$ is at least 8. The ideal of quasiadjunction corresponding to the constant $1/6$ for the ordinary cusp is the maximal ideal. It follows from the Bezout theorem that $H^0(\mathcal{I}_{1/6}(2)) = 0$. Now $\chi(I_{1/6}(2)) = 0$, because the sum of dimensions of stalks of $\mathcal{O}_{\mathbf{P}^2}(2)/\mathcal{I}_{1/6} = \dim H^0(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(2)) = 6$, whence $H^1(\mathcal{I}_{1/6}(2)) = 0$ and the Alexander polynomial of this curve is 1.

§4. Local systems on the complements to arrangements of hyperplanes

An interesting class of examples where cohomology of local systems and characteristic varieties can be often explicitly computed is formed by complements to hyperplane arrangements. Tools for computations are given by combinatorial invariants of arrangements: the intersection lattice and its Orlik-Solomon algebra.

Let $\mathcal{B}$ be an arrangement $\{H_1, \dots, H_n\}$ of hyperplanes in a complex projective space $\mathbf{P}$ and L its intersection lattice (i.e., the set of all intersections of the hyperplanes ordered opposite to inclusion and augmented by the maximum element 1). Fix some homogeneous linear forms $\alpha_1, \dots, \alpha_n$ such that the zero locus of α_i is H_i . Recall that the Orlik-Solomon algebra S of $\mathcal{B}$ (or of L) is the factor of the exterior algebra over $\mathbb{C}$ on generators $e_1, \dots, e_n$ by the ideal generated by

$$\sum_{j=1}^{p}(-1)^j e_{i_1}\cdots\hat{e}_{i_j}\cdots e_{i_p}$$

for all linearly dependent sets $\{\alpha_{i_1}, \dots, \alpha_{i_p}\}$. Algebra S is graded and generated in degree one. Denote by $\bar{S}$ the subalgebra of S generated by the elements $\sum_{i=1}^{n} a_i e_i$ ($a_i \in \mathbb{C}$) with $\sum_{i=1}^{n} a_i = 0$. According to the projective version of the Brieskorn-Orlik-Solomon theorem ([3, 27]), $\bar{S}$ is isomorphic to the algebra $H^*(M, \mathbb{C})$ where M is the complement of the divisor $D = \bigcup_i H_i$. The isomorphism is given by sending each e_i to the closed 1-form $\omega_i = d\alpha_i/\alpha_i$ and taking the cohomology class of the latter.

The forms ω_i can be used to produce matrix-valued logarithmic forms and local systems of higher rank on M. For a positive integer r, let P_i ($i = 1, \dots, n$) be $r \times r$-matrices over $\mathbb{C}$ such that $\sum_i P_i = 0$ and $\omega \in H^0(\mathbf{P}, \Omega^1\langle\mathcal{B}\rangle \otimes \mathcal{O}^r)$ be defined as

$$\omega = \sum_{i=1}^{n} \omega_i \otimes P_i.$$

Via the construction mentioned in section 2, the form ω defines a connection on $\mathcal{O}^r_M$ which is integrable if an only if $\omega \wedge \omega = 0$. This connection defines the local system of rank r on M and since M is affine the cohomology of this system is the cohomology of the complex of rational forms:

$$\Gamma = \Gamma(M, \Omega^{\star}_M \otimes \mathbb{C}^r)$$

with differential $d + \omega\wedge$ (cf. section 2). The correspondence $e_i \mapsto \omega_i$ defines also an embedding

$$\phi : \bar{S}^* \otimes \mathbb{C}^r \subset \Gamma$$

where $\bar{S}^* \otimes \mathbb{C}^r$ is the complex on $\bar{S} \otimes \mathbb{C}^r$ whose differential is the (left) multiplication by the element $a \in \bar{S}_1$ corresponding to the form ω. In the rank one case which is of the main interest in this note we denote the cohomology of that complex by $H^*(\bar{S}, a)$.

For the arrangement of hyperplanes in general position or for a general position a the embedding ϕ is a quasi-isomorphism. More precise sufficient conditions were obtained in [9, 30] by blowing up at non-normal crossings and applying Deligne's theorem (see section 2). To state the stronger version from [30] note that each $X \in L$ defines the subarrangement $\mathcal{B}_X = \{H \in \mathcal{B} | X \subset H\}$ of $\mathcal{B}$. We put $P_X = \sum_{H_i \in \mathcal{B}_X} P_i$ and call the subspace X *dense* if $\mathcal{B}_X$ is not the product of two non-empty arrangements.

Theorem 4.1 (Schechtman-Terao-Varchenko). *If for any dense subspace $X \in L$ none of the eigenvalues of P_X is a positive integer then ϕ is a quasi-isomorphism.*

Theorem 4.1 brings up the combinatorial problem of computing $H^*(\bar{S}, a)$ for various $a \in \bar{S}_1$. In particular an important question for applications (e.g. to hypergeometric functions, see [1]) is when this cohomology vanishes in all but the maximum dimensions. This question was studied in [35] (cf. also [16]) using sheaves on posets. In particular it was proved there that a sufficient condition for the vanishing is

$$\sum_{H_i \in \mathcal{B}_X} a_i \neq 0$$

for every dense $X \in L$. This work was continued in [11] where a basis of cohomology of the maximum dimension was found that is independent of ω.

Another interesting problem is to investigate connections between the two types of cohomology (of rank 1 local system and of the complex $\bar{S}^*$) when the conditions of Theorem 4.1 cease to hold (so called *resonance* case). There are at least two related but different ways to do that. One way is to relate the characteristic varieties of an arrangement with the respective subvarieties of $\bar{S}_1$. Let us define the latter. The first relevant definition was given by Falk [10] who studied invariants of S.

For an arrangement $\mathcal{B}$ define the *resonance variety*

$$R_k^\ell = R_k^\ell(\mathcal{B}) = \{a \in \bar{S}_1 \,|\, \dim H^\ell(\bar{S}, a) \geq k\}.$$

Clearly each R_k^ℓ is an algebraic subvariety of the linear space $\bar{S}_1$ and the easiest one to study is R_k^1. The studies of these varieties were started in [10]. Their relations with the characteristic varieties were first investigated in [22] and then in [5].

Since we focus on the cohomology of dimension 1 it suffices to consider arrangements of lines in the projective plane since by twisted version of Lefschetz theorem [32] the fundamental group of the complement to an arrangement is the same as the one for the intersection of this arrangement with a generic plane. For this case, the irreducible components of R_k^1 are linear and there is a bijection $\mathcal{W} \mapsto W$ between the set of these components and the set of all the positive dimension components of Σ_k^1 passing through 1. The exponentiation defines the universal covering of W by $\mathcal{W}$ [22, 23] and R_k^1 is the tangent cone of Σ_k^1 at the point 1 [5]. Characteristic varieties also yield a different sufficient condition for the conclusion of Theorem 4.1 [22] to be true. Namely ϕ

is a quasi-isomorphism if

$$(\exp(2\pi i a_1), \dots, \exp(2\pi i a_n)) \notin \mathrm{Char}^1 .$$

The other way to relate the two kinds of cohomology is for individual elements $a \in \bar{S}_1$. This is the main theme of [23]. It starts with the following inequality

$$\dim H^p(M, \mathcal{V}(a)) \geq \dim H^p(\bar{S}, a + N) \qquad (*)$$

for every p and $a \in \bar{S}_1$ where $\mathcal{V}(a)$ is the rank 1 local system defined by the 1-form corresponding to a and N is an arbitrary element of $\bar{S}_1$ with integer coordinates in the canonical basis $(e_1, \dots, e_n)$. This inequality follows immediately from the two observations. First, by multiplying a by $1 + \epsilon$ with $|\epsilon|$ small one makes it satisfy the conditions of Theorem 4.1 and does not change $H^p(\bar{S}, a)$. Now the upper semicontinuity of the dimension of cohomology gives $(*)$ with $N = 0$. Second, adding N to the right hand side of $(*)$ does not change its left hand side since the local system $\mathcal{V}(a)$ is defined by the character of S_1 given by $e_k \mapsto \exp(2\pi i a_k)$. (Note that the differentials $d + \omega_a \wedge$ and $d + \omega_{a+N} \wedge$ are different though isomorphic via multiplication by a rational function.)

The main result of [23] is the following theorem.

Theorem 4.2. *The left-hand side of* $(*)$ *is the supremum of its right-hand side while* N *is running through* $\mathbb{Z}^n$ *for all but finitely many cosets mod* $\mathbb{Z}^n$ *of elements of* R^1.

The proof of this theorem required certain further resutls about irreducible components of both the characteristic and resonance varieties. With every $a \in R^1_k$ one can associate the set $\mathcal{X}(a)$ of multiple points of intersection of lines such that the vector $(a_i | H_i \supset X)$ is not zero but $\sum_{H_i \supset X} a_i = 0$. The set $\mathcal{X}(a)$ defines the collection of subsets of $\mathcal{B}$ of lines passing through a point $X \in \mathcal{X}$. The incidence matrix $J(a)$ of the collection defines the symmetric matrix $Q(a) = J^t J - E$, where E is the matrix whose every entry is 1, that satisfies the conditions of a theorem of Vinberg's ([15], p.48) except that it is decomposable in general. An application of this theorem to the indecomposable components of Q shows that they should be either affine or finite with at least three affine ones. In particular this implies that if $\mathcal{W}$ is an irreducible component of R^1_k and a is an arbitrary nonzero vector form $\mathcal{W}$ then $\mathcal{W}$ is the $k+1$-dimensional subspace of $\bar{S}_1$ given by the linear system $\sum_{H_i \supset X} x_i = 0$ for all $X \in \mathcal{X}(a)$. In particular $\mathcal{W}$ is defined by $\mathcal{X}(a)$. In the ring $\bar{S}$, $\mathcal{W}$ is the annihilator of a in degree one whence any two irreducible components intersect at 0.

On the other hand, let W be a positive dimension component of Σ_k^1 containing 1 with the universal cover $\mathcal{W}$. Suppose W is essential, i.e., it is not the image of a component for a proper subarrangement $\mathcal{B}'$ of $\mathcal{B}$ under the inclusion map $H^1(M', \mathbb{C}^*) \to H^1(M, \mathbb{C}^*)$. Then one can associate with W a pencil of curves whose fibers do not have a common component. Some degenerate fibers are the unions of lines from $\mathcal{B}$ passing through a point from $\mathcal{X}(a)$ where $a \in \mathcal{W}$. If P' is the blow up of the projective plane at $\mathcal{X}(a)$ then matrix Q can be recovered as the minus intersection form on P'. Using the pencil of curves, the Euler characteristic of P' can be computed in two different ways that gives strong restrictions on the size and amount of indecomposable blocks of Q. Combining this with the condition on the blocks of being affine one obtains strong restrictions on arrangements of lines with characteristic varieties of positive dimension. For instance, if each line has precisely three multiple points then the arrangement can be embedded into the Hesse arrangement consisting of 12 lines passing each through 3 of 9 inflection points of a smooth cubic.

The following example from [5] can be used to show that the exceptional finitely many cosets of elements of R^1 from Theorem 4.2 can indeed exist.

Example [5]. The arrangement consists of 7 lines that are the zero loci of the following forms α_i (ordered from left to right): x, $x+y+z$, $x+y-z$, y, $x-y-z$, $x-y+z$, z. These lines define 3 double and 6 triple points of intersection with the latter (viewed as the sets of indices of lines passing through them) being

$$X_1 = \{1,2,5\},\ X_2 = \{1,3,6\},\ X_3 = \{2,3,7\},$$
$$X_4 = \{2,4,6\},\ X_5 = \{3,4,5\},\ X_6 = \{5,6,7\}.$$

The resonance variety $R^1 = R_1^1$ has 3 irreducible components W_1, W_2, and W_3 of dimension 2 defined by the collections

$$\mathcal{X}_1 = \{X_3, X_4, X_5, X_6\},\ \mathcal{X}_2 = \{X_1, X_2, X_3, X_6\},\ \mathcal{X}_3 = \{X_1, X_2, X_4, X_5\}$$

respectively. The pencil of quadrics corresponding to, say $\mathcal{W}_1$, is generated by $x^2 - y^2 - z^2$ and yz.

Consider $a = 1/2(-e_2 + e_3 - e_5 + e_6) \in W_1$. Then $a + N_1 \in W_2$ and $a + N_2 \in W_3$ with $N_1 = e_2 - e_6$ and $N_2 = e_3 - e_6$. Thus $\mathcal{V}(a) = \mathcal{V}(a + N_i)$. It is not hard to see that $\dim H^1(M, \mathcal{V}(a)) = 2$, i.e., $\mathcal{V}(a) \in \Sigma_2^1$. In fact, $\mathcal{V}(a)$ together with the constant system forms a discrete component of Σ_2^1 that is a group of order 2.

§5. Problems

The emerging picture of the cohomology of rank one local systems is far from being complete. We suggest several problems as an attempt to clarify it.

The case of arrangements of lines in the projective plane seems to be the most promising and a majority of our problems is devoted to this case. In them, M is the complement of the union of lines (cf. section 4).

Problem 5.1. *Is it true that every positive dimensional irreducible component of* $\mathrm{Char}^1(M)$ *contains* 1 (*whence is covered by a component of the resonance variety* R^1)*? The above example shows that it is not true for discrete components.*

Problem 5.2. *For* $a \in \bar{S}_1$*, is it possible to compute* $H^1(M, \mathcal{V}(a))$ *knowing* $H^1(\bar{S}, a+N)$ *for all vectors* $N \in \mathbb{Z}^n$*?*

Theorem 4.2 gives the positive answer for almost all $a \in R^1$. Even for the exceptional a in the above example the cocycles with coefficients in $\mathcal{V}(a)$ are generated by differential forms corresponding to cocycles for $a+N$ in $\bar{S}_1$.

This problem may split depending on which a one considers, from R^1 or not. In particular the following particular case of the problem might be easier.

Problem 5.3. *Can there exist* $a \in \bar{S}_1$ *such that* $H^1(\bar{S}, a+N) = 0$ *for all* $N \in \mathbb{Z}^n$ *but* $H^1(X, \mathcal{V}(a)) \neq 0$*?*

Problem 5.4 (Combinatorial invariance of characteristic varieties). *Are the characteristic varieties combinatorial invariants of arrangements, i.e., can one reconstruct them from the lattice of an arrangement?*

It is known that the fundamental groups of the complements to arrangements are *not* invariants of the lattice ([28]). On the other hand, the results of [5], [22], and [23] show that components of positive dimension containing the identity character are. For algebraic curves, Alexander polynomials can not be determined just from the degrees of the curve and the local type of singularities. This follows from seminal example of two sextics with six cusps on and not on a conic. Recent results on Zariski's pairs and triples are discussed in [26].

Problem 5.5 (Realization and classification). *How many components can the characteristic variety have? Can this number be arbitrary large? Can one bound dimensions and the number of components in terms of lattice of arrangement? Can one classify arrangements for* *which the characteristic varieties have components with positive dimension or sufficiently large dimension?*

Results of [23] show that the dimension of the characteristic variety imposes a bound on the number of lines in the arrangement. Similar realization problem for characteristic varieties and Alexander polynomials of algebraic curves (e.g. which polynomials can appear as the Alexander polynomials of algebraic curves or algebraic curves of given degree) seems to be also open. More concretly: how large can the degree of the Alexander polynomial be for a curve with nodes and cusps? For the sextic dual to a non singular plane cubic the Alexander polynomial is equal to $(t^2 - t + 1)^3$. Are there the curves, with nodes and cusps only, for which the degree of Alexander polynomial is bigger than 6? The Alexander polynomial of the complement to an algebraic curve divides the product of local Alexander polynomials and the Alexander polynomial at infinity (cf. [17] and references there). This gives a bound for the degree of the Alexander polynomial in terms of the degree of the curve. For example for a curve with singularities not worse than ordinary cusps we obtain $2(d-2)$. For calculations of Alexander polynomials for curves with more complicated singularites we refer to [6].

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Cohomology rings and nilpotent quotients of real and complex arrangements

Daniel Matei and Alexander I. Suciu

Dedicated to Peter Orlik on his 60^{th} birthday

Abstract.

For an arrangement with complement X and fundamental group G, we relate the truncated cohomology ring, $H^{\leq 2}(X)$, to the second nilpotent quotient, G/G_3. We define invariants of G/G_3 by counting normal subgroups of a fixed prime index p, according to their abelianization. We show how to compute this distribution from the resonance varieties of the Orlik-Solomon algebra mod p. As an application, we establish the cohomology classification of 2-arrangements of $n \leq 6$ planes in $\mathbb{R}^4$.

§0. Introduction

1. Two basic homotopy-type invariants of a path-connected space X are: the cohomology ring, $H^*(X)$, and the fundamental group, $G = \pi_1(X)$. Given X and X', one would like to know:

(I) Is there an isomorphism $H^{\leq q}(X) \cong H^{\leq q}(X')$ between the cohomology rings, up to degree q?

(II) Is there an isomorphism $G/G_{q+1} \cong G'/G'_{q+1}$ between the q^{th} nilpotent quotients?

We single out a class of spaces—including complements of complex hyperplane arrangements, complements of 'rigid' links, and complements of arrangements of transverse planes in $\mathbb{R}^4$—for which the above questions are amenable to a detailed study, capable of yielding classification results. The invariants that we use have a dual nature, being able to capture both the ring-theoretic properties of the cohomology of X, and the group-theoretic properties of the nilpotent quotients of G. Our main

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result is an explicit correspondence between two sets of invariants—one determined by the vanishing cup products in $H^{\leq 2}(X)$, the other by the finite-index subgroups of G/G_3.

2. For $q = 1$, questions (I) and (II) are equivalent, provided H_1 is torsion free. Indeed, $H^1(X) = G/G_2$ under that assumption. For $q = 2$, the two questions are still equivalent, under some additional conditions: If H_2 is also torsion-free, and the cup-product map $\mu : H^1 \wedge H^1 \to H^2$ is surjective, then:

$$H^{\leq 2}(X) \cong H^{\leq 2}(X') \text{ if and only if } G/G_3 \cong G'/G'_3.$$

Section 1 is devoted to a proof of this fact. A key ingredient is the vanishing of the Hurewicz map $\pi_2(X) \to H_2(X)$, which permits us to identify $H^{\leq 2}(X)$ with $H^{\leq 2}(G)$. The other ingredient is the interpretation of the k-invariant of the extension $0 \to G_2/G_3 \to G/G_3 \to G/G_2 \to 0$, in terms of the cup-products of G.

In Section 2, we use commutator calculus to describe the nilpotent quotients of G. We restrict our attention to spaces X, for which $G = \pi_1(X)$ has a finite presentation $G = \mathbb{F}/R$, with $R \subset [\mathbb{F}, \mathbb{F}]$. The cup products in $H^{\leq 2}(G)$ can then be computed from the second order Fox derivatives of the relators.

3. The invariants of the cohomology ring that we use are the resonance varieties, first introduced by Falk [11] in the context of complex hyperplane arrangements. The d^{th} resonance variety of X, with coefficients in a field $\mathbb{K}$, is the set $\mathcal{R}_d(X, \mathbb{K})$ of cohomology classes $\lambda \in H^1(X, \mathbb{K})$ for which there is a subspace $W \subset H^1(X, \mathbb{K})$, of dimension $d + 1$, such that $\mu(\lambda \wedge W) = 0$.

In Section 3, we prove that $\mathcal{R}_d(X, \mathbb{K})$ equals $\mathcal{R}_d(G, \mathbb{K})$, the resonance variety of the Eilenberg-MacLane space $K(G, 1)$. Moreover, we exploit the Fox calculus interpretation of cup products to show that the varieties $\mathcal{R}_d(G, \mathbb{K})$ are the determinantal varieties of the 'linearized' Alexander matrix of G.

4. A well-known invariant of a group G is the number of normal subgroups of fixed prime index. For a commutator-relators group, that number depends only on $n = \operatorname{rank} G/G_2$, and the prime p. In order to get a finer invariant, we consider the distribution of index p subgroups, according to their abelianization. The ν-invariants of the nilpotent quotients G/G_{q+1} are defined as follows:

$$\nu_{p,d}(G/G_{q+1}) = \# \left\{ K \lhd G/G_{q+1} \;\middle|\; \begin{array}{l} [G/G_{q+1} : K] = p \text{ and} \\ \dim_{\mathbb{Z}_p}(\operatorname{Tors} H_1(K)) \otimes \mathbb{Z}_p = d \end{array} \right\}.$$

In Section 4, we show how to compute the ν-invariants of G/G_3 from the stratification of $\mathbb{P}(\mathbb{Z}_p^n)$ by the projectivized $\mathbb{Z}_p$-resonance varieties of X:

$$\nu_{p,d}(G/G_3) = \#(\mathcal{P}_d(X, \mathbb{Z}_p) \setminus \mathcal{P}_{d+1}(X, \mathbb{Z}_p)).$$

This formula makes the computation of the ν-invariants practical. It also makes clear that the mod p resonance varieties of X, which are defined solely in terms of $H^{\leq 2}(X)$, do capture significant group-theoretic information about G/G_3.

5. In the case where X is the complement of a complex hyperplane arrangement, the varieties $\mathcal{R}_d(X, \mathbb{C})$ have been extensively studied by Falk, Yuzvinsky, Libgober, Cohen, and Suciu [11, 33, 19, 20, 7]. The top variety, $\mathcal{R}_1(X, \mathbb{C})$, is a complete invariant of the cohomology ring $H^{\leq 2}(X)$. Moreover, $\mathcal{R}_1(X, \mathbb{C})$ is a union of linear subspaces intersecting only at the origin, and $\mathcal{R}_d(X, \mathbb{C})$ is the union of those subspaces of dimension at least $d + 1$.

In Section 5, we use these results to derive a simple consequence. Since $\mathcal{R}_d(X, \mathbb{C})$ has integral equations, we may consider its reduction mod p. If that variety coincides with $\mathcal{R}_d(X, \mathbb{Z}_p)$, we have:

$$\nu_{p,d-1}(G/G_3) = \frac{p^d - 1}{p - 1} m_d,$$

where m_d is the number of components of $\mathcal{R}_1(X, \mathbb{C})$ of dimension d.

In general, though, this formula fails, due to a different resonance at 'exceptional' primes. For such primes p, the variety $\mathcal{R}_d(X, \mathbb{Z}_p)$ is not necessarily the union of the components of $\mathcal{R}_1(X, \mathbb{Z}_p)$ of dimension $\geq d + 1$. Furthermore, $\mathcal{R}_1(X, \mathbb{C})$ mod p and $\mathcal{R}_1(X, \mathbb{Z}_p)$ may differ in the number of non-local components, as well as in the dimensions of those components. Most strikingly, $\mathcal{R}_1(X, \mathbb{Z}_p)$ may have non-local components, even though $\mathcal{R}_1(X, \mathbb{C})$ mod p has none.

6. Much of the original motivation for this paper came from an effort to understand Ziegler's pair of arrangements of 4 transverse planes in $\mathbb{R}^4$. Those arrangements have isomorphic lattices, but their complements have non-isomorphic cohomology rings, see [34]. In an earlier work [23], we investigated the homotopy types of complements of 2-arrangements, obtaining a complete classification for $n \leq 6$ planes. This left open the problem of classifying cohomology rings for $n > 4$.

In Section 6, we start a study of the varieties $\mathcal{R}_d(X, \mathbb{C})$, where X is the complement of a 2-arrangement. The resonance varieties of real arrangements exhibit a much richer geometry than those of complex arrangements. Most strikingly, $\mathcal{R}_1(X,\mathbb{C})$ may not be a union of linear subspaces, and it may not determine $H^*(X)$.

Using the ν-invariants of G/G_3, we establish the cohomology classification of complements of 2-arrangements of $n \leq 6$ planes in $\mathbb{R}^4$. With one exception, this classification coincides with the homotopy-type classification from [23]. The exception is Mazurovskiĭ's pair [24]. The two complements, X and X', have isomorphic cohomology rings, and thus $G/G_3 \cong G'/G'_3$. On the other hand, $\nu_{3,2}(G/G_4) = 162$ and $\nu_{3,2}(G'/G'_4) = 172$.

As this example shows, the ν-invariants of the third nilpotent quotient *cannot* be computed from the resonance varieties of the cohomology ring. To arrive at a more conceptual understanding of these invariants, one needs to look beyond cup-products, and on to the Massey products. This will be pursued elsewhere.

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CONTENTS

§1. Cohomology ring and second nilpotent quotient

In this section, we introduce a class of spaces that abstract the cohomological essence of hyperplane arrangements. We then relate the cohomology ring of such a space X to the second nilpotent quotient of the fundamental group of X.

1.1. Cohomology ring

All the spaces considered in this paper have the homotopy type of a connected CW-complex with finite 2-skeleton. Let X be such a space. Consider the following conditions on the cohomology ring of X:

(A) The homology groups $H_*(X)$ are free abelian.
(B) The cup-product map $\mu_X : \bigwedge^* H^1(X) \to H^*(X)$ is surjective.

If conditions (A) and (B) only hold for $1 \le * \le n$, we will refer to them as (A_n) and (B_n).

Example 1.2. The basic example we have in mind is that of the complement, $X = \mathbb{C}^\ell \setminus \bigcup_{H \in \mathcal{A}} H$, of a central hyperplane arrangement $\mathcal{A}$ in $\mathbb{C}^\ell$. As shown by Brieskorn [3] (solving a conjecture of Arnol'd), such a space X satisfies conditions (A) and (B). Moreover, as shown by Orlik and Solomon [25], the *intersection lattice* of the arrangement, $L(\mathcal{A}) = \{\bigcap_{H \in \mathcal{B}} H \,|\, \mathcal{B} \subseteq \mathcal{A}\}$, determines the cohomology ring of X, as follows:

$$H^*(X) = \bigwedge^* \mathbb{Z}^n \Big/ \Big(\partial e_{\mathcal{B}} \mid \operatorname{codim} \bigcap_{H \in \mathcal{B}} H < |\mathcal{B}|\Big).$$

Here $\bigwedge^* \mathbb{Z}^n$ is the exterior algebra on generators $e_1, \ldots, e_n$ dual to the meridians of the hyperplanes; and, if $\mathcal{B} = \{H_{i_1}, \ldots, H_{i_r}\}$ is a sub-arrangement, then $e_{\mathcal{B}} = e_{i_1} \cdots e_{i_r}$, and $\partial e_{\mathcal{B}} = \sum_q (-1)^q e_{i_1} \cdots \widehat{e_{i_q}} \cdots e_{i_r}$. See [26] for a thorough treatment of hyperplane arrangements.

Let X be a space satisfying conditions (A_n) and (B_n). The first condition and the Universal Coefficient Theorem (see [2], Theorem 7.1, p. 281) imply that $H_*(X) = H^*(X)$, for $* \le n$. Write $H = H_1(X) = H^1(X)$. Denote by I^* the kernel of the cup-product map. Condition (B_n) can be restated as saying that the following sequence is exact:

$$(1) \qquad 0 \to I^* \xrightarrow{\iota} \bigwedge^* H \xrightarrow{\mu_X} H^*(X) \to 0, \qquad \text{for } * \le n.$$

By condition (A_n), this is in fact a split-exact sequence.

1.3. Hurewicz homomorphism

The following lemma was proved by Randell [27] in the case where X is the complement of a complex hyperplane arrangement.

Lemma 1.4. *If X satisfies conditions* (A_n) *and* (B_n)*, then the Hurewicz homomorphism, $h : \pi_i(X) \to H_i(X)$, is the zero map, for $2 \le i \le n$.*

Proof. The proof is exactly as in [27]. Let $p : \widetilde{X} \to X$ be the universal covering map. Recall that $p_* : \pi_i(\widetilde{X}) \to \pi_i(X)$ is an isomorphism, for $i \ge 2$. By naturality of the Hurewicz map, universal coefficients, and condition (A_n), it is enough to show that $p^* : H^i(X) \to H^i(\widetilde{X})$ is the zero map. This follows from $H^1(\widetilde{X}) = 0$, condition (B_n), and the naturality of cup products: $p^* \circ \mu_X = \mu_{\widetilde{X}} \circ \wedge^i p^*$. Q.E.D.

1.5. Group cohomology

Let G be a group. The (co)homology groups of G are by definition those of the corresponding Eilenberg-MacLane space: $H_*(G) = H_*(K(G,1))$ and $H^*(G) = H^*(K(G,1))$. Consider the following homological conditions on G:

(A′) The homology groups $H_1(G)$ and $H_2(G)$ are finitely generated free abelian.

(B′) The cup-product map $\mu_G : H^1(G) \wedge H^1(G) \to H^2(G)$ is surjective.

Proposition 1.6. *Let X be a space satisfying conditions* (A_2) *and* (B_2), *and let $G = \pi_1(X)$ be its fundamental group. Then the following hold:*

(a) *$H_1(G) \cong H_1(X)$ and $H_2(G) \cong H_2(X)$.*

(b) *The rings $H^{\leq 2}(G)$ and $H^{\leq 2}(X)$ are isomorphic.*

Therefore, G satisfies conditions (A′) *and* (B′).

Proof. Recall X has the homotopy type of a connected CW-complex Y with finite 2-skeleton. A $K(G,1)$ space may be obtained from Y by attaching suitable cells of dimension ≥ 3. The resulting map, $j : X \to K(G,1)$, induces an isomorphism $H_1(X) \cong H_1(G)$. From the Hopf exact sequence $\pi_2(X) \xrightarrow{h} H_2(X) \to H_2(G) \to 0$ and Lemma 1.4, we get $H_2(X) \cong H_2(G)$. This finishes the proof of (a).

By universal coefficients, the map $j^* : H^i(G) \to H^i(X)$ is a group isomorphism, for $i \leq 2$. By naturality of cup products, we have $j^*\mu_G(a \wedge b) = \mu_X(j^*a \wedge j^*b)$. This proves (b). Q.E.D.

Remark 1.7. The above conditions on X also appear in [1, 29]. The surjectivity of $\mu : H^1(X) \wedge H^1(X) \to H^2(X)$ is stated there dually, as the injectivity of the holonomy map, $\mu^\top : H_2(X) \to \bigwedge^2 H_1(X)$.

1.8. Nilpotent quotients

Let G be a finitely generated group. The lower central series of G is defined inductively by $G_1 = G$, $G_{q+1} = [G, G_q]$, where $[G, G_q]$ denotes the subgroup of G generated by the commutators $[x,y] = xyx^{-1}y^{-1}$ with $x \in G$ and $y \in G_q$. The quotient G_q/G_{q+1} is a finitely generated abelian group, called the q^{th} *lower central series quotient* of G. The quotient G/G_{q+1} is a nilpotent group, called the q^{th} *nilpotent quotient* of G. See [21] for details.

We will be mainly interested in the second nilpotent quotient, G/G_3. This group is a central extension of finitely generated abelian groups,

$$0 \to G_2/G_3 \to G/G_3 \to G/G_2 \to 0. \tag{2}$$

The extension is classified by the k-invariant, $\bar{\chi} \in H^2(G/G_2; G_2/G_3)$. The isomorphism type of G/G_3 is determined by G/G_2, G_2/G_3, and $\bar{\chi}$, as follows.

Let G and G' be two groups. Then $G/G_3 \cong G'/G'_3$ if and only if there exist isomorphisms $\phi : G/G_2 \to G'/G'_2$ and $\psi : G_2/G_3 \to G'_2/G'_3$ under which the k-invariants correspond: $\psi_*(\bar{\chi}) = \phi^*(\bar{\chi}') \in H^2(G/G_2; G'_2/G'_3)$.

Now suppose $H = G/G_2$ is torsion-free. As is well-known, $H_*(H) \cong \bigwedge^* H$. The *classifying map* for the extension (2),

$$\chi : \bigwedge^2 H \to G_2/G_3$$

is the image of $\bar{\chi}$ under the epimorphism

$$H^2(H; G_2/G_3) \to \operatorname{Hom}(\bigwedge^2 H, G_2/G_3)$$

provided by the Universal Coefficient Theorem (see [9]). It is given by $\chi(x \wedge y) = [x, y]$ (see [4], Exercise 8, p. 97). The condition that the k-invariants of G/G_3 and G'/G'_3 correspond translates to $\psi \circ \chi = \chi' \circ \wedge^2 \phi$. We shall write this equivalence relation between classifying maps as $\chi \sim \chi'$.

Suppose now that G_2/G_3 is also torsion-free. Then, the universal coefficient map is an isomorphism, and so $\bar{\chi}$ and χ determine each other. Thus, for a group G with G/G_2 and G_2/G_3 torsion-free, the isomorphism type of G/G_3 is completely determined by the equivalence class of the classifying map χ.

1.9. Cup product and commutators

The 5-term exact sequence for the extension $0 \to G_2 \to G \xrightarrow{\alpha} G/G_2 \to 0$ yields:

$$H_2(G) \xrightarrow{\alpha_*} H_2(G/G_2) \xrightarrow{\delta} G_2/G_3 \to 0. \tag{3}$$

Under the identification $H_2(G/G_2) \cong \bigwedge^2 H$, the boundary map δ corresponds to the classifying map χ (see [4], Exercise 6, p. 47). The next lemma interprets the map α_* in terms of the ring structure of $H^*(G)$.

Lemma 1.10. *The map $\alpha_* : H_2(G) \to \bigwedge^2 H$ is the dual of the cup-product map $\mu_G : H^1(G) \wedge H^1(G) \to H^2(G)$.*

Proof. Follows from the commutativity of the diagram

$$\begin{array}{ccc} H^1(H) \wedge H^1(H) & \xrightarrow{\alpha^* \wedge \alpha^*} & H^1(G) \wedge H^1(G) \\ \Big\downarrow \mu_H & & \Big\downarrow \mu_G \\ H^2(H) & \xrightarrow{\alpha^*} & H^2(G) \end{array}$$

and the fact that the top and left arrows are isomorphisms. Q.E.D.

The following proposition generalizes a result proved by Massey and Traldi [22] in the case where G is a link group.

Proposition 1.11. *Let G be a group satisfying conditions* (A′) *and* (B′). *Then G_2/G_3 is torsion-free, and the following is a split exact sequence*:

$$0 \to H_2(G) \xrightarrow{\mu^\top} \bigwedge^2 H \xrightarrow{\chi} G_2/G_3 \to 0. \tag{4}$$

Proof. The proof follows closely that in [22]. By Lemma 1.10, sequence (3) can be written as $H_2(G) \xrightarrow{\mu^\top} \bigwedge^2 H \xrightarrow{\chi} G_2/G_3 \to 0$. By condition (B′), the map $\mu^\top$ is a monomorphism, whence the exactness of (4).

Since $\mu : \bigwedge^2 H \to H^2(G)$ is an epimorphism between finitely generated free abelian groups, it admits a splitting. Hence $\mu^\top$ is a split injection, and so $\chi^\top$ is a split surjection. Since $\bigwedge^2 H$ is torsion-free, G_2/G_3 is also torsion-free. Q.E.D.

Remark 1.12. The injectivity of $\mu^\top : H_2(G) \to \bigwedge^2 H$ is equivalent to the vanishing of $\Phi_3(G)$, where $H_2(G) = \Phi_2(G) \supset \Phi_3(G) \supset \cdots$ is the Dwyer filtration, $\Phi_k(G) = \ker(H_2(G) \to H_2(G/G_{k-1}))$, see [9].

1.13. Isomorphisms

The next result is an immediate consequence of Proposition 1.11:

Proposition 1.14. *Let X be a space satisfying conditions* (A_2) *and* (B_2), *and let $G = \pi_1(X)$. Then $I^2 = G_2/G_3$, and the exact sequence*

$$0 \to I^2 \xrightarrow{\iota} \bigwedge^2 H^1(X) \xrightarrow{\mu} H^2(X) \to 0 \tag{5}$$

is the dual of sequence (4).

We are now ready to establish the correspondence between the truncated cohomology ring of X and the second nilpotent quotient of

$G = \pi_1(X)$. A version of the equivalence $(b) \Leftrightarrow (c)$ below, with χ replaced by $\mu^{\top}$, was first established by Traldi and Sakuma [32], in the case where X is a link complement.

Theorem 1.15. *Let X and X' be two spaces satisfying conditions* (A_2) *and* (B_2)*, and let G and G' be their fundamental groups. The following are equivalent:*

(a) $H^*(X) \cong H^*(X')$ *for* $* \leq 2$;
(b) $G/G_3 \cong G'/G'_3$;
(c) $\chi \sim \chi'$.

Proof. $(a) \Leftrightarrow (c)$. By Proposition 1.14, sequence (5) is exact, and $\chi = \iota^{\top}$. The equivalence follows from the definitions.

$(b) \Leftrightarrow (c)$. By Propositions 1.6 and 1.14, the first two lower central series quotients of G and G' are torsion-free. The equivalence follows from the discussion in 1.8. Q.E.D.

1.16. Invariants of $H^{\leq 2}(X)$ and G/G_3

In view of Theorem 1.15, an invariant of either the truncated cohomology ring $H^{\leq 2}(X)$, or the second nilpotent quotient G/G_3, or the classifying map χ, is an invariant of the other two. We will define in subsequent sections a series of invariants of both $H^{\leq 2}(X)$ and G/G_3, and relate them one to another. For now, let us define invariants of χ, following an idea of Ziegler [34], that originated from Falk's work on minimal models of arrangements [10].

Let $\mu_H : \bigwedge^i H \otimes \bigwedge^j H \to \bigwedge^{i+j} H$ be the multiplication in the exterior algebra $\bigwedge^* H$. Consider the following finitely generated abelian group:

$$Z_{i,j}(\chi) = \operatorname{coker}\left(\bigwedge^i H \otimes \bigwedge^j G_2/G_3 \xrightarrow{\mathrm{id} \otimes \bigwedge^j \chi^{\top}} \bigwedge^i H \otimes \bigwedge^{2j} H \xrightarrow{\mu_H} \bigwedge^{i+2j} H\right).$$

Clearly, if $\chi \sim \chi'$ then $Z_{i,j}(\chi) \cong Z_{i,j}(\chi')$. Thus, the rank and elementary divisors of $Z_{i,j}(\chi)$ provide invariants of both $H^{\leq 2}(X)$ and G/G_3.

§2. Generators and relators

In this section, we write down explicitly some of the maps introduced in the previous section. We start with a review of some basic facts about Hall commutators and the Fox calculus.

2.1. Basic commutators

Let $\mathbb{F}(n)$ be the free group on generators $x_1, \dots, x_n$. A *basic commutator* in $\mathbb{F} = \mathbb{F}(n)$ is defined inductively as follows (see [12, 21]):

(a) Each basic commutator c has length $\ell(c) \in \mathbb{N}$.

(b) The basic commutators of length 1 are the generators $x_1, \dots, x_n$; those of length > 1 are of the form $c = [c_1, c_2]$, where c_1, c_2 are previously defined commutators and $\ell(c) = \ell(c_1) + \ell(c_2)$.

(c) Basic commutators of the same length are ordered arbitrarily; if $\ell(c) > \ell(c')$, then $c > c'$.

(d) If $\ell(c) > 1$ and $c = [c_1, c_2]$, then $c_1 < c_2$; if $\ell(c) > 2$ and $c = [c_1, [c_2, c_3]]$, then $c_1 \geq c_2$.

The basic commutators of the form $c = [x_{i_1}, [x_{i_2}, [\dots [x_{i_{q-1}}, x_{i_q}] \dots]]]$ are called *simple*. We shall write them as $c = [x_{i_1}, x_{i_2}, \dots, x_{i_q}]$. For $q \leq 3$, all basic commutators are simple.

The following theorem of Hall is well-known (see *loc. cit.*):

Theorem 2.2. *The group $\mathbb{F}_q/\mathbb{F}_{q+1}$ is free abelian, and has a basis consisting of the basic commutators of length q.*

In particular, if $w \in \mathbb{F}$ and $c_1, \dots, c_r$ are the basic commutators of length $< q$, then $w^{(q)} := w \bmod \mathbb{F}_q$ may be written uniquely as $w^{(q)} = c_1^{e_1} c_2^{e_2} \cdots c_r^{e_r}$, for some integers $e_1, \dots, e_r$.

The Hall commutators may be used to write down presentations for the nilpotent quotients of a finitely presented group $G = \mathbb{F}/R$. Indeed, if $G = \langle x_1, \dots, x_n \mid r_1, \dots, r_m \rangle$, we have the following presentation for $G/G_q = \mathbb{F}/R\mathbb{F}_q$:

$$G/G_q = \langle x_1, \dots, x_n \mid r_1^{(q)}, \dots, r_m^{(q)}, c_1, \dots, c_l \rangle, \tag{6}$$

where $r_k^{(q)} = r_k \mod \mathbb{F}_q$, and $\{c_h\}_{1 \leq h \leq l}$ are the basic commutators of length q.

2.3. Fox calculus

Let $\mathbb{Z}\mathbb{F}$ be the group ring of $\mathbb{F}$, with augmentation map $\epsilon : \mathbb{Z}\mathbb{F} \to \mathbb{Z}$ given by $\epsilon(x_i) = 1$. To each x_i there corresponds a *Fox derivative*, $\partial_i : \mathbb{Z}\mathbb{F} \to \mathbb{Z}\mathbb{F}$, given by $\partial_i(1) = 0$, $\partial_i(x_j) = \delta_{ij}$ and $\partial_i(uv) = \partial_i(u)\epsilon(v) + u\partial_i(v)$. The higher Fox derivatives, $\partial_{i_1,\dots,i_k}$, are defined inductively in the obvious manner. The composition of the augmentation map with the higher derivatives yields operators $\epsilon_{i_1,\dots,i_k} := \epsilon \circ \partial_{i_1,\dots,i_k} : \mathbb{Z}\mathbb{F} \to \mathbb{Z}$.

Let $\alpha : \mathbb{F}(n) \to \mathbb{Z}^n$ be the abelianization map, given by $\alpha(x_i) = t_i$. The following lemma is left as an exercise in the definitions.

Lemma 2.4. *We have:*

(a) $\partial_i[u,v] = (1 - uvu^{-1})\partial_i u + (u - [u,v])\partial_i v$.
(b) $\alpha(\partial_i[x_{i_1}, x_{i_2}, \dots, x_{i_q}]) = (t_{i_1} - 1)\cdots(t_{i_{q-2}} - 1)\big((t_{i_{q-1}} - 1)\delta_{i,i_q} - (t_{i_q} - 1)\delta_{i,i_{q-1}}\big)$.
(c) $\epsilon_I(w) = 0$, *if* $w \in \mathbb{F}_q$ *and* $|I| < q$.
(d) $\epsilon_I(uv) = \epsilon_I(u) + \epsilon_I(v)$, *if* $u, v \in \mathbb{F}_q$ *and* $|I| = q$.

2.5. Commutator relations

We now make more explicit some of the constructions from section 1, for the following class of groups.

Definition 2.6. A group G is called a *commutator-relators group* if it admits a presentation $G = \mathbb{F}(n)/R$, where R is the normal closure of a finite subset of $[\mathbb{F}, \mathbb{F}]$.

In other words, G has a finite presentation $G = \langle x_1, \dots, x_n | r_1, \dots, r_m \rangle$, and $G/G_2 = \mathbb{Z}^n$. Commutator-relators groups appear as fundamental groups of certain spaces that we shall encounter later on. The following proposition gives sufficient conditions for this to happen.

Proposition 2.7. *Let X be a space that is homotopy equivalent to a finite CW-complex Y, with 1-skeleton $Y^{(1)} = \bigvee_{i=1}^{n} S_i^1$. If $H_1(X) = \mathbb{Z}^n$, then $G = \pi_1(X)$ is a commutator-relators group.*

Proof. The 2-skeleton $Y^{(2)} = \bigvee_{i=1}^{n} S_i^1 \cup \bigcup_{k=1}^{m} e_k^2$ determines a presentation $G = \langle x_1, \dots, x_n \mid r_1, \dots, r_m \rangle$. A presentation matrix for the abelianization of G is $E = (\epsilon_i(r_k))$. Since $H_1(X) = \mathbb{Z}^n$, we have $H_1(G) = \mathbb{Z}^n$. Thus, E is equivalent to the zero matrix, and hence E *is* the zero matrix. Thus, all relators r_k are commutators. Q.E.D.

Now let $\phi : \mathbb{F} \to G$ be the quotient map, and let $\alpha : G \to G/G_2$ be the abelianization map. Set $t_i = \alpha(\phi(x_i))$. Then $\{t_1, \dots, t_n\}$ form a basis for $H_1(G)$, and their Kronecker duals, $\{e_1, \dots, e_n\}$, form a basis for $H^1(G)$.

By the Hopf formula, we have $H_2(G) = R/[R, \mathbb{F}]$. Assume that $H_2(G)$ is free abelian, and let $\theta_k = r_k \bmod [R, \mathbb{F}]$. Then $\{\theta_1, \dots, \theta_m\}$ form a basis for $H_2(G)$, and their duals, $\{\gamma_1, \dots, \gamma_m\}$, form a basis for $H^2(G)$.

Proposition 2.8. *Let G be a commutator-relators group, such that $H_2(G)$ is free abelian. In the basis specified above, the cup-product map $\mu : H^1(G) \wedge H^1(G) \to H^2(G)$ is given by*

$$\mu(e_i \wedge e_j) = \sum_{k=1}^{m} \epsilon_{i,j}(r_k)\gamma_k.$$

Proof. This follows immediately from [13], Theorem 2.3. Q.E.D.

2.9. Links in $\mathbb{S}^3$

We conclude this section with a classical example. Let L be an oriented link in $\mathbb{S}^3$, with components $L_1, \dots L_n$. Its complement, $X = \mathbb{S}^3 \setminus \bigcup_i L_i$, has the homotopy type of a connected, 2-dimensional finite CW-complex. The homology groups of X are computed by Alexander duality: $H_1(X) = \mathbb{Z}^n$, $H_2(X) = \mathbb{Z}^{n-1}$. It follows that Condition (A) is always satisfied for a link complement. If $L = \hat{\beta}$ is the closure of a pure braid $\beta \in P_n$, then X satisfies the assumption of Proposition 2.7, and so $G = \pi_1(X)$ is a commutator-relators group, with presentation $G = \langle x_1, \dots, x_n \mid \beta(x_i)x_i^{-1} = 1,\ 1 \le i < n\rangle$.

For an arbitrary link L, let $\{e_1, \dots, e_n\}$ be the basis for $H^1(X)$ dual to the meridians of L. Choose arcs $c_{i,j}$ in X connecting L_i to L_j, and let $\gamma_{i,j} \in H^2(X)$ be their duals. Then $\{\gamma_{1,n}, \dots, \gamma_{n-1,n}\}$ forms a basis for $H_2(X)$. Let $l_{i,j} = \mathrm{lk}(L_i, L_j)$ be the linking numbers of L. A presentation for the cohomology ring of X is given by:

$$H^*(X) = \left(e_i, \gamma_{i,j} \;\middle|\; \begin{array}{l} e_ie_j = l_{i,j}\gamma_{i,j},\ \gamma_{i,j} + \gamma_{j,k} + \gamma_{k,i} = 0 \\ \gamma_{i,i} = \gamma_{i,j}e_k = e_k\gamma_{i,j} = \gamma_{i,j}\gamma_{k,l} = 0 \end{array} \right). \tag{7}$$

Let $\mathcal{G}$ be the "linking graph" associated to L: It is the complete graph on n vertices, with edges labelled by the linking numbers. If $\mathcal{G}$ possesses a spanning tree T with n vertices, and all edges labelled ± 1, we say that L is *(cohomologically) rigid.* The complement of such a link satisfies condition (B), see [22, 17, 1]. Moreover, G_2/G_3 is free abelian of rank $\binom{n-1}{2}$, with basis $\{x_{ij} \mid ij \notin T \text{ and } i < j\}$. The classifying map $\chi\colon \bigwedge^2 H \to G_2/G_3$ is given by

$$\chi(e_i \wedge e_j) = \begin{cases} x_{ij} & \text{if } ij \notin T, \\ \sum_{\{k \mid ik \notin T\}} l_{i,k}x_{ik} & \text{if } ij \in T, \end{cases}$$

where $x_{ik} = -x_{ki}$, for $i > k$.

We will be mainly interested in those rigid links for which $l_{i,j} = \pm 1$. Examples include the Hopf links, and, more generally, the singularity links of 2-arrangements in $\mathbb{R}^4$ (see 6.1). For such links, the presentation (7) simplifies to:

$$H^*(X) = \left(e_i \;\middle|\; \begin{array}{l} e_i^2 = 0,\ e_ie_j = -e_je_i, \\ l_{i,j}e_ie_j + l_{j,k}e_je_k + l_{k,i}e_ke_i = 0 \end{array} \right). \tag{8}$$

Moreover, the transpose of the classifying map, $\chi^{\top} : G_2/G_3 \to \bigwedge^2 H$, is given by the simple formula

$$\chi^{\top}(x_{ij}) = (e_i - l_{i,j}e_n) \wedge (e_j - l_{i,j}e_n). \tag{9}$$

§3. Resonance varieties

In this section, we define the 'resonance' varieties of the cohomology ring of a space X. We then show that, under certain conditions on X, these varieties are the determinantal varieties of the linearized Alexander matrix of the group $G = \pi_1(X)$.

3.1. Filtration of first cohomology

Let X be a space that satisfies conditions (A_2) and (B_2) of Section 1, and the hypothesis of Proposition 2.7. We thus have: $H^1(X) = \mathbb{Z}^n$, $H^2(X) = \mathbb{Z}^m$, the cup-product map $\mu : H^1(X) \wedge H^1(X) \to H^2(X)$ is surjective, and $G = \pi_1(X)$ is a commutator-relators group.

Lemma 3.2. *Let X be as above, and let $\mathbb{K}$ be a commutative field.*

(a) *The $\mathbb{K}$-cup products may be computed from the integral ones:* $\mu_{\mathbb{K}} = \mu \otimes \mathrm{id}_{\mathbb{K}}$.

(b) *If $H^{\leq 2}(X) \cong H^{\leq 2}(X')$ then $H^{\leq 2}(X, \mathbb{K}) \cong H^{\leq 2}(X', \mathbb{K})$.*

Proof. Let $\kappa : \mathbb{Z} \to \mathbb{K}$ be the homomorphism given by $\kappa(1) = 1$. From the definitions, the coefficient map $\kappa_* : H^*(X, \mathbb{Z}) \to H^*(X, \mathbb{K})$, and the map $\mathrm{id} \otimes \kappa : H^*(X) \otimes \mathbb{Z} \to H^*(X) \otimes \mathbb{K}$ commute with cup products. By the Universal Coefficient Theorem (see [2], Theorem 7.4, p. 282), the map $v : H^*(X) \otimes \mathbb{K} \to H^*(X, \mathbb{K})$, $v([z] \otimes k) = [z \otimes k]$ is an isomorphism for $* \leq 2$. Since $v \circ (\mathrm{id} \otimes \kappa) = \kappa_*$, the map v also commutes with cup products. The conclusions follow. Q.E.D.

Definition 3.3. Let d be an integer, $0 \leq d \leq n$. The d^{th} *resonance variety* of X (with coefficients in $\mathbb{K}$) is the subvariety of $H^1(X, \mathbb{K}) = \mathbb{K}^n$, defined as follows:

$$\mathcal{R}_d(X, \mathbb{K}) = \left\{ \lambda \in H^1(X, \mathbb{K}) \;\middle|\; \begin{array}{l} \exists \text{ subspace } W \subset H^1(X, \mathbb{K}) \text{ such that} \\ \dim W = d + 1 \text{ and } \mu(\lambda \wedge W) = 0 \end{array} \right\}.$$

The resonance varieties form a descending filtration $\mathbb{K}^n = \mathcal{R}_0 \supset \mathcal{R}_1 \supset \cdots \supset \mathcal{R}_{n-1} \supset \mathcal{R}_n = \emptyset$. The *ambient type* of the $\mathbb{K}$-resonance varieties depends only on the truncated cohomology ring $H^{\leq 2}(X, \mathbb{K})$, and thus, by Lemma 3.2 (b), only on $H^{\leq 2}(X)$. More precisely, if $H^{\leq 2}(X) \cong H^{\leq 2}(X')$, there exists a linear automorphism of $\mathbb{K}^n$ taking $\mathcal{R}_d(X, \mathbb{K})$ to $\mathcal{R}_d(X', \mathbb{K})$.

For a group G, define the resonance varieties to be those of the corresponding Eilenberg-MacLane space: $\mathcal{R}_d(G, \mathbb{K}) := \mathcal{R}_d(K(G, 1), \mathbb{K})$.

Proposition 3.4. *Let X be a space satisfying conditions* (A_2) *and* (B_2). *Let $G = \pi_1(X)$. Then $\mathcal{R}_d(X, \mathbb{K}) = \mathcal{R}_d(G, \mathbb{K})$.*

Proof. By Proposition 1.6, the inclusion $j : X \to K(G,1)$ induces an isomorphism $j^* : H^{\leq 2}(G) \to H^{\leq 2}(X)$. The conclusion follows from Lemma 3.2 (b) above. Q.E.D.

3.5. Alexander matrices

Let $G = \langle x_1, \dots, x_n \mid r_1, \dots, r_m \rangle$ be a commutator-relators group. Recall the projection map $\phi : \mathbb{F}(n) \to G$, and the abelianization map, $\alpha : G \to \mathbb{Z}^n$, given by $\alpha(x_i) = t_i$.

Definition 3.6. The *Alexander matrix* of G is the $m \times n$ matrix $A = (\alpha\phi\partial_i(r_k))$ with entries in the Laurent polynomials ring $\mathbb{Z}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$.

Now let $\psi : \mathbb{Z}[t_1^{\pm 1}, \dots, t_n^{\pm 1}] \to \mathbb{Z}[[s_1, \dots, s_n]]$ be the ring homomorphism given by $\psi(t_i) = 1 + s_i$ and $\psi(t_i^{-1}) = \sum_{q \geq 0} (-1)^q s_i^q$. Also, let $\psi^{(q)}$ be the graded q^{th} piece of ψ. Since all the relators of G are commutators, the entries of A are in the ideal $(t_1 - 1, \dots, t_n - 1)$, and so $\psi^{(0)} A$ is the zero matrix.

Definition 3.7. The *linearized Alexander matrix* of G is the $m \times n$ matrix

$$M = \psi^{(1)} A.$$

Note that the entries of M are integral linear forms in $s_1, \dots, s_n$. By Lemma 2.4 (a), (b) we have $\psi^{(1)}\alpha\phi\partial_i(r_k) = \psi^{(1)}\alpha\phi\partial_i(r_k^{(3)})$. Thus, M depends only on the relators of G, modulo length 3 commutators. By Lemma 2.4 (c), (d) those truncated relators are given by $r_k^{(3)} = \prod_{i<j} [x_i, x_j]^{\epsilon_{i,j}(r_k)}$. Thus, the entries of M are:

$$M_{k,j} = \sum_{i=1}^{n} \epsilon_{i,j}(r_k) s_i. \tag{10}$$

The linearized Alexander matrix of a link was first considered by Traldi [31]. If the link L has n components, then M has size $n \times (n-1)$, and its entries are $M_{k,j} = l_{k,j} s_k - \delta_{k,j}(\sum_i l_{k,i} s_i)$.

3.8. Equations for resonance varieties

We now find explicit equations for the varieties $\mathcal{R}_d(X, \mathbb{K})$. In view of Proposition 3.4, that is the same as finding equations for $\mathcal{R}_d(G, \mathbb{K})$, with $G = \pi_1(X)$. Moreover, in view of Lemma 3.2 (a), the formula for the integral cup products from Proposition 2.8 may be used to compute the $\mathbb{K}$-cup products. We will use the notations of that proposition for the rest of this section.

Let M be the linearized Alexander matrix of G. Let $M_{\mathbb{K}}$ be the corresponding matrix of linear forms over $\mathbb{K}$, and let $M(\lambda)$ be the matrix $M_{\mathbb{K}}$ evaluated at $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{K}^n$.

Theorem 3.9. *For G a commutator-relators group with $H_2(G)$ torsion free,*

$$\mathcal{R}_d(G, \mathbb{K}) = \{\lambda \in \mathbb{K}^n \mid \operatorname{rank}_{\mathbb{K}} M(\lambda) < n - d\}.$$

Proof. Let $\lambda = \sum_{i=1}^n \lambda_i e_i \in H^1(G, \mathbb{K}) = \mathbb{K}^n$. We are looking for $v = \sum_{i=1}^n v_i e_i$ such that $\mu(\lambda \wedge v) = 0$ in $H^2(G, \mathbb{K}) = \mathbb{K}^m$. Recall from Proposition 2.8 that $\mu(e_i \wedge e_j) = \sum_{k=1}^m \epsilon_{i,j}(r_k)\gamma_k$. It follows that

$$\mu(\lambda \wedge v) = \sum_{k=1}^{m} \Big(\sum_{1 \le i,j \le n} \lambda_i v_j \epsilon_{i,j}(r_k) \Big) \gamma_k.$$

We thus obtain a linear system of m equations in $v_1, \dots, v_n$:

$$\sum_{j=1}^{n} \Big(\sum_{i=1}^{n} \lambda_i \epsilon_{i,j}(r_k) \Big) v_j = 0,$$

with coefficient matrix $M(\lambda)$.

Now λ belongs to $\mathcal{R}_d(G, \mathbb{K})$ if and only if the space W of solutions of the linear system $M(\lambda) \cdot v = 0$ is at least $(d+1)$-dimensional. That translates into the condition $\operatorname{rank}_{\mathbb{K}} M(\lambda) < n - d$ of the statement, and we are done. Q.E.D.

We will be mainly interested in the coefficient fields $\mathbb{K} = \mathbb{C}$ and $\mathbb{K} = \mathbb{Z}_p$, for some prime p. By the above theorem, the $\mathbb{C}$-resonance varieties have *integral* equations. As we shall see in Section 5, although $\mu_{\mathbb{Z}_p} : H^1(X, \mathbb{Z}_p) \wedge H^1(X, \mathbb{Z}_p) \to H^2(X, \mathbb{Z}_p)$ is the reduction $\operatorname{mod} p$ of $\mu : H^1(X) \wedge H^1(X) \to H^2(X)$, the variety $\mathcal{R}_d(X, \mathbb{Z}_p)$ is *not* necessarily the reduction $\operatorname{mod} p$ of $\mathcal{R}_d(X, \mathbb{C})$.

Example 3.10. Let X be the complement of an n-component rigid link. The matrix $M(\lambda)$ has entries $M(\lambda)_{k,j} = l_{k,j}\lambda_k - \delta_{k,j}(\sum_i l_{k,i}\lambda_i)$. The variety $\mathcal{R}_1(X, \mathbb{K})$ is the zero-locus of a degree $n-2$ homogeneous polynomial obtained by taking the greatest common divisor of the $(n-1) \times (n-1)$ minors of the matrix $M(\lambda)$. At the other extreme, we have $\mathcal{R}_{n-1}(X, \mathbb{K}) = \{\mathbf{0}\}$. Indeed, the off-diagonal entries of $M(\lambda)$ corresponding to the edges of the maximal spanning tree generate the maximal ideal $(\lambda_1, \dots, \lambda_n)$ of $\mathbb{K}[\lambda_1, \dots, \lambda_n]$.

3.11. Projectivized resonance varieties

The affine variety $\mathcal{R}_d(X, \mathbb{K}) \subset \mathbb{K}^n$ is homogeneous, and so defines a projective variety $\mathcal{P}_d(X, \mathbb{K}) \subset \mathbb{P}(\mathbb{K}^n)$. If $H^{\leq 2}(X)$ is isomorphic to $H^{\leq 2}(X')$, there is a projective automorphism $\mathbb{P}(\mathbb{K}^n) \to \mathbb{P}(\mathbb{K}^n)$ taking $\mathcal{P}_d(X, \mathbb{K})$ to $\mathcal{P}_d(X', \mathbb{K})$. The rest of the above discussion applies to the projective resonance varieties in an obvious manner. In particular, we have:

Corollary 3.12. $\mathcal{P}_d(G, \mathbb{K}) = \{\lambda \in \mathbb{P}(\mathbb{K}^n) \mid \operatorname{rank}_{\mathbb{K}} M(\lambda) < n-d-1\}$.

§4. Prime index normal subgroups

In this section, we consider nilpotent quotients of commutator-relators groups. We show how to count the normal subgroups of prime index, according to their abelianization.

4.1. Counting subgroups

Let G be a group. For a prime number p, let $\Sigma_p(G)$ be the set of index p normal subgroups of G, and let $N_p(G) = |\Sigma_p(G)|$ be its cardinality.

Proposition 4.2. *For the free group $\mathbb{F}(n)$, the set $\Sigma_p(\mathbb{F}(n))$ is in bijective correspondence with the projective space $\mathbb{P}(\mathbb{Z}_p^n)$.*

Proof. Every index p normal subgroup of $\mathbb{F}(n)$ is the kernel of an epimorphism $\lambda : \mathbb{F}(n) \to \mathbb{Z}_p$. Such homomorphisms are parametrized by $\mathbb{Z}_p^n \setminus \{\mathbf{0}\}$. Two epimorphisms λ and λ' have the same kernel if and only if $\lambda = q \cdot \lambda'$, for some $q \in \mathbb{Z}_p^*$. Q.E.D.

Corollary 4.3. *Let $G = \mathbb{F}(n)/R$ be a commutator-relators group. For all primes p,*

$$N_p(G) = \frac{p^n - 1}{p - 1}.$$

Proof. Since R consists of commutators,

$$\operatorname{Hom}(G, \mathbb{Z}_p) \cong \operatorname{Hom}(\mathbb{F}(n), \mathbb{Z}_p).$$

Thus, $\Sigma_p(G)$ is in one-to-one correspondence with $\Sigma_p(\mathbb{F}(n)) = \mathbb{P}(\mathbb{Z}_p^n)$. Q.E.D.

4.4. Abelianizing normal subgroups

Let $G = \langle x_1, \ldots, x_n \mid r_1, \ldots, r_m \rangle$ be a commutator-relators group. Let $K \lhd G$ be a normal subgroup of index p, defined by a representation $\lambda : G \to \mathbb{Z}_p$, $\lambda(x_i) = \lambda_i$. Let $\bar{\lambda} : \mathbb{Z}G \to \mathbb{Z}\mathbb{Z}_p$ be the linear extension of λ to group rings. More precisely, we view here $\mathbb{Z}_p$ as a multiplicative group, with generator ζ. Then $\bar{\lambda}(x_i) = \zeta^{\lambda_i}$. Finally, let $\beta : \mathbb{Z}\mathbb{Z}_p \to \mathrm{Mat}(p, \mathbb{Z})$ be the ring homomorphism defined by the (left) regular representation of $\mathbb{Z}_p$.

Definition 4.5. For a given representation $\lambda : G \to \mathbb{Z}_p$, the *twisted Alexander matrix of G* is the $pm \times pn$ matrix

$$A_\lambda = \left(\bar{\lambda}\phi\partial_i(r_k)\right)^\beta$$

obtained from $\left(\bar{\lambda}\phi\partial_i(r_k)\right)$ by replacing each entry e with $\beta(e)$.

Proposition 4.6. *Let G be a commutator-relators group, and let $K = \ker(\lambda : G \to \mathbb{Z}_p)$. The matrix A_λ is a relation matrix for the group $H_1(K) \oplus \mathbb{Z}^{p-1}$.*

A proof can be found in [16]. The matrix A_λ is equivalent (via row-and-column operations) to a diagonal matrix, from which the rank and elementary divisors of $H_1(K)$ can be read off.

4.7. Nilpotent quotients

We now apply the above procedure to a particular class of groups: the nilpotent quotients G/G_q, $q \geq 3$, of a commutator-relators group $G = \mathbb{F}(n)/R$.

Let $\lambda : G/G_q \to \mathbb{Z}_p$ be a non-trivial representation. To describe explicitly the presentation matrix A_λ of Proposition 4.6, we need to examine more closely the Fox derivatives of the relators c_h and $r_k^{(q)}$ in the presentation (6) for G/G_q.

If c is a non-simple basic commutator, then Lemma 2.4 (a), (b) gives $\bar{\lambda}\phi(\partial_i c) = 0$. If $c = [x_{i_1}, x_{i_2}, \ldots, x_{i_q}]$ is a simple commutator, then it follows from Lemma 2.4 (b) that $\bar{\lambda}\phi\,(\partial_i c)$ is either zero or of the form $e = \pm(\zeta^{a_1} - 1)\cdots(\zeta^{a_{q-2}} - 1) \in \mathbb{Z}\mathbb{Z}_p$, for some integers $1 \leq a_j \leq p-1$.

Recall that the truncation $r_k^{(q)}$ is a product of basic commutators of length $< q$. The same argument shows that $\bar{\lambda}\phi\big(\partial_i r_k^{(q)}\big)$ is a linear combination of elements in $\mathbb{Z}\mathbb{Z}_p$ of the form $(\zeta^{a_{i_1}} - 1)\cdots(\zeta^{a_{i_j}} - 1)$, for $j < q-2$.

The following lemma shows the typical simplifications that we will perform on $\left(\bar{\lambda}\phi(\partial c_h)\right)^\beta$ and $\left(\bar{\lambda}\phi(\partial r_k^{(q)})\right)^\beta$.

Lemma 4.8. *The integral $p \times p$ matrix e^{β} corresponding to $e = (\zeta^{a_1} - 1)\cdots(\zeta^{a_k} - 1) \in \mathbb{Z}\mathbb{Z}_p$ has diagonal form*

$$(\underbrace{p^{r-1},\dots,p^{r-1}}_{p-l-1},\underbrace{p^{r},\dots,p^{r}}_{l},0),$$

where $r = \lceil \frac{k-1}{p-1} \rceil$, and $l = k-1-(r-1)(p-1)$. Moreover, there is a sequence of row and column operations, independent of the particular e, that brings e^{β} to that diagonal form.

Proposition 4.9. *Let K be an index p normal subgroup of the free nilpotent quotient $\mathbb{F}(n)/\mathbb{F}(n)_q$. Then:*

$$H_1(K) = \mathbb{Z}^n \oplus (\mathbb{Z}/p^{r-1}\mathbb{Z})^{(n-1)(p-l-1)} \oplus (\mathbb{Z}/p^{r}\mathbb{Z})^{(n-1)l},$$

where $r = \left\lceil \frac{q-2}{p-1} \right\rceil$, and $l = q-2-(r-1)(p-1)$.

Proof. In this case, only commutator relators are present, so Lemma 4.8, applied to each entry $\bar{\lambda}\phi(\partial c_h)$, shows that the matrix A_λ is equivalent the following diagonal matrix:

$$D = (\underbrace{p^{r-1},\dots,p^{r-1}}_{(n-1)(p-l-1)},\underbrace{p^{r},\dots,p^{r}}_{(n-1)l},\underbrace{0,\dots,0}_{n+p-1}). \tag{11}$$

Q.E.D.

Theorem 4.10. *Let $G = \mathbb{F}(n)/R$ be a commutator-relators group. Let K be an index p normal subgroup of G/G_q. Set $r = \left\lceil \frac{q-2}{p-1} \right\rceil$. Then:*

$$H_1(K) = \mathbb{Z}^n \oplus \bigoplus_{i=0}^{r} (\mathbb{Z}/p^i\mathbb{Z})^{d_i},$$

for some positive integers $d_0,\dots,d_r$ such that $d_0+\cdots+d_r = (n-1)(p-1)$ and $d_r \le l(n-1)$.

Proof. Let $K = \ker(\lambda : G/G_q \to \mathbb{Z}_p)$. Consider the relation matrix A_λ, corresponding to the presentation $G/G_q = \mathbb{F}/R\mathbb{F}_q$ from (6). Partition A_λ into two blocks, $A_\lambda = \left(\begin{smallmatrix} B_\lambda \\ C_\lambda \end{smallmatrix}\right)$, where B_λ corresponds to the relators R, and C_λ corresponds to the basic commutators.

Assume that the row and column operations of Lemma 4.8 have already been performed. Then, after moving all the zero columns to the right, A_λ is equivalent to $\left(\begin{smallmatrix} B'_\lambda & 0 \\ D' & 0 \end{smallmatrix}\right)$, where $D = (D'\ 0)$ is the diagonal matrix (11). Since the number of zero diagonal elements of D is $n+p-1$, the rank of K is n. Since the non-zero diagonal elements of D are either p^{r-1} or p^r, the elementary divisors of K are among $p, p^2, \dots, p^r$. The conclusion readily follows. Q.E.D.

4.11. ν-Invariants

In view of Theorem 4.10, we define the following numerical invariants of isomorphism type for the nilpotent quotients of a group.

Definition 4.12. Let G be a commutator-relators group, and let G/G_q be the $(q-1)^{st}$ nilpotent quotient of G. Given a prime p, and a positive integer d, define

$$\nu_{p,d}(G/G_q) = \#\left\{ K \lhd G/G_q \;\middle|\; \begin{array}{l} [G/G_q : K] = p \text{ and} \\ \dim_{\mathbb{Z}_p}(\operatorname{Tors} H_1(K)) \otimes \mathbb{Z}_p = d \end{array} \right\}.$$

Example 4.13. If $q = 3$, then $H_1(K) = \mathbb{Z}^n \oplus \mathbb{Z}_p^d$, for some $0 \le d \le n-1$. So we have invariants $\nu_{p,0}(G/G_3), \dots, \nu_{p,n-1}(G/G_3)$ for the second nilpotent quotient of G. Since $\sum_{d=0}^{n-1} \nu_{p,d} = \frac{p^n-1}{p-1}$, it is enough to compute $\nu_{p,1}, \dots, \nu_{p,n-1}$.

Example 4.14. If $q = 4$, and $p \ge 3$, then $H_1(K) = \mathbb{Z}^n \oplus \mathbb{Z}_p^d$, for some $0 \le d \le 2n-2$. If $p = 2$, then $H_1(K) = \mathbb{Z}^n \oplus \mathbb{Z}_2^{d_1} \oplus \mathbb{Z}_4^{d_2}$, for some $0 \le d = d_1 + d_2 \le n-1$.

4.15. Second nilpotent quotient

We now restrict our attention to G/G_3. From (6), for $q = 3$ we obtain the presentation:

$$G/G_3 = \langle x_1, \dots, x_n \mid r_1^{(3)}, \dots, r_m^{(3)}, c_1, \dots, c_l \rangle, \tag{12}$$

where $l = 2\binom{n+1}{3}$, and $c_1, \dots, c_l$ are the basic commutators $[x_i, [x_j, x_k]]$, with $j < k$ and $i \ge j$.

Theorem 4.16. *Given an epimorphism $\lambda : G/G_3 \to \mathbb{Z}_p$, with kernel K_λ, we have*

$$\dim_{\mathbb{Z}_p}(\operatorname{Tors} H_1(K_\lambda)) \otimes \mathbb{Z}_p = n - 1 - \operatorname{rank}_{\mathbb{Z}_p} M(\lambda).$$

Proof. Recall from the proof of Theorem 4.10 that the relation matrix of the abelian group $H_1(K_\lambda)$ has the following form: $A_\lambda = \left(\begin{smallmatrix} B'_\lambda & 0 \\ C'_\lambda & 0 \end{smallmatrix}\right)$. We have already seen in Proposition 4.9 that C'_λ is equivalent to a diagonal matrix $D' = \left(\begin{smallmatrix} I_{(n-1)(p-2)} & 0 \\ 0 & p \cdot I_{n-1} \end{smallmatrix}\right)$.

Recall also that $r_k^{(3)} = \prod_{i<j} [x_i, x_j]^{\epsilon_{i,j}(r_k)}$. A computation using formula (a) in Lemma 2.4 shows:

$$\bar{\lambda}\phi\left(\frac{\partial r_k^{(3)}}{\partial x_l}\right) = \sum_{i=1}^{n} \epsilon_{i,l}(r_k)(\zeta^{\lambda_i} - 1), \tag{13}$$

for $1 \le l \le n$ and $1 \le k \le m$.

Consider $e = \sum_{\sigma=1}^{p-1} a_\sigma(\zeta^\sigma - 1) \in \mathbb{Z}\mathbb{Z}_p$. Set $a = \sum_{\sigma=1}^{p-1} a_\sigma$. It is readily seen that the matrix e^β is equivalent to:

$$\begin{pmatrix} * & \cdots & * & p\cdot a & 0 \\ \vdots & & \vdots & \vdots & \vdots \\ * & \cdots & * & p\cdot a & 0 \\ * & \cdots & * & \sum_{\sigma=1}^{p-1} a_\sigma \cdot \sigma & 0 \end{pmatrix}. \tag{14}$$

Now (10), together with (13) and (14), imply that B'_λ is equivalent to $\left(\begin{smallmatrix} * & 0 \\ * & M(\lambda)' \end{smallmatrix}\right)$, where $M(\lambda)'$ is some codimension 1 minor of $M(\lambda)$. Hence, A_λ is equivalent to:

$$\begin{pmatrix} * & 0 & 0 \\ * & M(\lambda)' & 0 \\ I_{(n-1)(p-2)} & 0 & 0 \\ 0 & p\cdot I_{n-1} & 0 \end{pmatrix}.$$

The theorem now follows from the following fact: An integral matrix of the form $\left(\begin{smallmatrix} Q \\ p\cdot I_n \end{smallmatrix}\right)$ is equivalent to $\left(\begin{smallmatrix} I_r & 0 \\ 0 & p\cdot I_d \end{smallmatrix}\right)$, where $r = \operatorname{rank}_{\mathbb{Z}_p} Q \otimes \mathrm{id}_{\mathbb{Z}_p}$, and $d = n - r$. Q.E.D.

Corollary 4.17. $\nu_{p,d}(G/G_3) = \#\{K_\lambda \in \Sigma_p(G/G_3) \mid \operatorname{rank}_{\mathbb{Z}_p} M(\lambda) = n - d - 1\}$.

4.18. Resonance varieties and subgroups of G/G_3

The following theorem relates the distribution of index p normal subgroups of G/G_3, according to their abelianization, to the number of points on the n-dimensional projective space over $\mathbb{Z}_p$, according to the stratification by the resonance varieties.

Theorem 4.19. *For G a commutator-relators group with $H_2(G)$ torsion free,*

$$\nu_{p,d}(G/G_3) = \#(\mathcal{P}_d(G,\mathbb{Z}_p) \setminus \mathcal{P}_{d+1}(G,\mathbb{Z}_p)).$$

Proof. Follows from Corollary 3.12 and Corollary 4.17. Q.E.D.

§5. Complex arrangements

We illustrate the techniques developed in the previous sections with the main example of spaces satisfying conditions (A) and (B): complements of complex hyperplane arrangements.

5.1. Cohomology and fundamental group

Let $\mathcal{A}'$ be a complex hyperplane arrangement, with complement X'. Let $\mathcal{A}$ be a generic two-dimensional section of $\mathcal{A}'$, with complement X. Then, by the Lefschetz-type theorem of Hamm and Lê [15], the inclusion $i : X \to X'$ induces an isomorphism $i_* : \pi_1(X) \to \pi_1(X')$ and a monomorphism $i^* : H^2(X') \to H^2(X)$. By the Brieskorn-Orlik-Solomon theorem, the map i^* is, in fact, an isomorphism. So, for our purposes here, we may restrict our attention to $\mathcal{A}$.

Let $\mathcal{A} = \{H_1, \dots, H_n\}$ be an arrangement of n affine lines in $\mathbb{C}^2$, in general position at infinity. Let $v_1, \dots, v_s$ be the intersection points of the lines. If $v_q = H_{i_1} \cap \cdots \cap H_{i_m}$, set $V_q = \{i_1, \dots, i_m\}$ and $\overline{V}_q = V_q \setminus \{\max V_q\}$. The level 2 of the lattice of $\mathcal{A}$ is encoded in the list $L_2(\mathcal{A}) = \{V_1, \dots, V_s\}$, which keeps track of the incidence relations between the points and the lines of the arrangement.

The following properties hold:

(i) The homology groups of $X = \mathbb{C}^2 \setminus \bigcup_i H_i$ are free abelian, of ranks $b_1 = n$, $b_2 = \sum_{q=1}^{s} |\overline{V}_q|$, and $b_i = 0$ for $i > 2$. The cohomology ring is determined by $L_2(\mathcal{A})$ (see [26]):

$$H^*(X) = \left\langle e_1, \dots, e_n \;\middle|\; \begin{array}{l} e_i^2 = 0, \; e_ie_j = -e_je_i \\ e_ie_j + e_je_k + e_ke_i = 0 \;\; \text{for } i,j,k \in V_q, \; 1 \le q \le s \end{array} \right\rangle.$$

(ii) The fundamental group $G = \pi_1(X)$ is a commutator-relators group:

$$G = \langle x_1, \dots, x_n \mid \beta_q(x_i)x_i^{-1} = 1 \quad \text{for } i \in \overline{V}_q \text{ and } q = 1, \dots, s \rangle.$$

The pure braid monodromy generators $\beta_1, \dots, \beta_s$ can be read off from a 'braided wiring diagram' associated to $\mathcal{A}$ (see [5]). Moreover, the space X is homotopy equivalent to the 2-complex given by this presentation (see [18]).

(iii) The second nilpotent quotient is determined by $L_2(\mathcal{A})$:

$$G/G_3 = \left\langle x_1, \dots, x_n \;\middle|\; \begin{array}{ll} [x_i, \prod_{j \in V_q} x_j] & \text{for } i \in \overline{V}_q, \quad 1 \le q \le s \\ [x_i, [x_j, x_k]] & \text{for } 1 \le j < k \le n, \quad j \le i \le n \end{array} \right\rangle.$$

This follows from the presentation in (ii), together with (12) (see also [28]).

(iv) The linearized Alexander matrix is determined by $L_2(\mathcal{A})$. It is obtained by stacking $M_{V_1}(\lambda), \dots, M_{V_s}(\lambda)$, where $M_V(\lambda)$ is the

$|\overline{V}| \times n$ matrix with entries

$$M_V(\lambda)_{i,j} = \delta_{j,V}\Big(\lambda_i - \delta_{i,j}\sum_{k\in V}\lambda_k\Big), \qquad \text{for } i \in \overline{V} \text{ and } 1 \leq j \leq n.$$

For a detailed discussion of the Alexander matrix and the Alexander invariant of $\mathcal{A}$, see [6].

From properties (i) and (ii), we deduce that X satisfies the conditions from Proposition 2.7.

5.2. Resonance varieties over $\mathbb{C}$

The resonance varieties of a complex hyperplane arrangement were introduced by Falk in [11]. Let $\mathcal{A}$ be an arrangement of n affine lines in $\mathbb{C}^2$, in general position at infinity. Set $\mathcal{R}_d(\mathcal{A}) := \mathcal{R}_d(X, \mathbb{C})$. By Theorem 3.1 in [11], this definition agrees with Falk's definition.

Qualitative results as to the nature of the resonance varieties of complex arrangements were obtained by a number of authors, [33, 11, 7, 19, 20]. We summarize some of those results, as follows.

Theorem 5.3. *Let $\mathcal{R}_1(\mathcal{A}) \subset \mathbb{C}^n$ be the resonance variety of an arrangement of n complex hyperplanes. Then:*

(a) *The ambient type of $\mathcal{R}_1(\mathcal{A})$ determines the isomorphism type of $H^{\leq 2}(X)$.*
(b) *$\mathcal{R}_1(\mathcal{A})$ is contained in the hyperplane $\Delta_n := \{\sum_{i=1}^n \lambda_i = 0\}$.*
(c) *Each component C_i of $\mathcal{R}_1(\mathcal{A})$ is a linear subspace.*
(d) *$C_i \cap C_j = \{\mathbf{0}\}$ for $i \neq j$.*
(e) *$\mathcal{R}_d(\mathcal{A}) = \{\mathbf{0}\} \cup \bigcup_{\dim C_i \geq d+1} C_i$.*

Proof. Part (a) was proved in [11]. Part (b) was proved in [33] and [11]. Part (c) was conjectured in [11], and proved in [7] and [19]. Part (d) is proved in [20]. Part (e) follows from [20], Theorem 3.4, as was pointed out to us by S. Yuzvinsky. Q.E.D.

By Theorem 3.9, the resonance varieties $\mathcal{R}_d(\mathcal{A})$ are the determinantal varieties associated to the linearized Alexander matrix, M. For another set of explicit equations, obtained from a presentation of the linearized Alexander invariant, see [7].

All the components of $\mathcal{R}_1(\mathcal{A})$ arise from *neighborly partitions* of subarrangements of $\mathcal{A}$, see [11], [20]. To a partition Π of $\mathcal{A}' \subset \mathcal{A}$, such that a certain bilinear form associated to Π is degenerate, there corresponds a component C_Π of $\mathcal{R}_1(\mathcal{A})$. For each $V \in L_2(\mathcal{A})$ with $|V| \geq 3$, there is a *local* component, $C_V = \Delta_n \cap \{\lambda_i = 0 \mid i \notin V\}$, of dimension $|V| - 1$, corresponding to the partition (V) of $\mathcal{A}_V = \{H_i \mid i \in V\}$. The other components of $\mathcal{R}_1(\mathcal{A})$ are called *non-local.* For more details and examples, see [11, 7, 19, 20].

5.4. Resonance varieties over $\mathbb{Z}_p$

We now turn to the characteristic p resonance varieties, $\mathcal{R}_d(\mathcal{A};\mathbb{Z}_p)$. Recall that the variety $\mathcal{R}_d(\mathcal{A})$ has integral equations, so we may consider its reduction mod p. As we shall see, there are arrangements $\mathcal{A}$ such that $\mathcal{R}_d(\mathcal{A};\mathbb{Z}_p)$ does not coincide with $\mathcal{R}_d(\mathcal{A})$ mod p, for certain primes p. Indeed:

- The number of irreducible components, or the dimensions of the components may be different, as illustrated in Examples 5.9 and 5.10 below.
- The analogues of Theorem 5.3 (a) and (e) fail in general, as seen in Examples 5.8 and 5.10 below.

On the other hand, it seems likely that the analogues of Theorem 5.3 (b), (c) and (d) hold for every prime p.

Now let $\nu_{p,d}(\mathcal{A}) = \nu_{p,d}(G/G_3)$ be the number of normal subgroups of G/G_3 with abelianization $\mathbb{Z}^n \oplus \mathbb{Z}_p^d$, for $0 \leq d \leq n-1$. By properties (i) and (ii) above, Theorem 4.19 applies, and so $\nu_{p,d}(\mathcal{A})$ can be computed from the $\mathbb{Z}_p$-resonance varieties.

Corollary 5.5. *If $\mathcal{R}_d(\mathcal{A},\mathbb{Z}_p) = \mathcal{R}_d(\mathcal{A})$ mod p, for all $d \geq 1$, then*

$$\nu_{p,d-1}(\mathcal{A}) = \frac{p^d - 1}{p-1} m_d,$$

where m_d is the number of components of $\mathcal{R}_1(\mathcal{A})$ of dimension d.

Proof. From the assumption, properties (c)–(e) hold for $\mathcal{R}_d(\mathcal{A},\mathbb{Z}_p)$. Therefore, $\mathcal{P}_d(X,\mathbb{Z}_p)\backslash\mathcal{P}_{d+1}(X,\mathbb{Z}_p)$ consists of m_d disjoint, d-dimensional projective subspaces in $\mathbb{P}(\mathbb{Z}_p^n)$. The formula follows from Theorem 4.19. Q.E.D.

If all the components of $\mathcal{R}_1(\mathcal{A})$ are local, then $m_d = \#\{V \in L_2(\mathcal{A}) \mid |V| = d+1\}$, but the Corollary may not apply, see Example 5.9.

5.6. Examples

We conclude this section with a few examples that illustrate the phenomena mentioned above. The motivation to look at Examples 5.8 and 5.10 came from S. Yuzvinsky, who was the first to realize that there are exceptional primes for these arrangements. His method of computing the corresponding non-local components is different from ours, though.

Example 5.7. Let $\mathcal{A}$ be the reflection arrangement of type A_3, with lattice

$$L_2(\mathcal{A}) = \{123, 145, 246, 356, 16, 25, 34\}.$$

The variety $\mathcal{R}_1(\mathcal{A})$ has 5 components of dimension 2. The non-local component, $C_\Pi = \{\lambda_1 - \lambda_6 = \lambda_2 - \lambda_5 = \lambda_3 - \lambda_4 = 0\} \cap \Delta_6$, corresponds to the partition $\Pi = (16 \mid 25 \mid 34)$, see [11, 7, 19].

For all primes p, Corollary 5.5 applies, giving $\nu_{p,1} = 5(p+1)$.

Example 5.8. Let $\mathcal{A}$ be the realization of the non-Fano plane, with lattice

$$L_2(\mathcal{A}) = \{123, 147, 156, 257, 345, 367, 24, 26, 46\}.$$

The variety $\mathcal{R}_1(\mathcal{A})$ has 9 components of dimension 2. The non-local components are given by the partitions $\Pi_1 = (13 \mid 46 \mid 57)$, $\Pi_2 = (15 \mid 24 \mid 37)$, $\Pi_3 = (17 \mid 26 \mid 35)$ of the corresponding type A_3 sub-arrangements, see [7].

For $p > 2$, Corollary 5.5 applies, and so $\nu_{p,1} = 9(p+1)$.

For $p = 2$, though, $\mathcal{R}_1(\mathcal{A}, \mathbb{Z}_2)$ has a single, 3-dimensional non-local component, $C_\Pi = \{\lambda_1 + \lambda_4 + \lambda_7 = \lambda_2 + \lambda_5 + \lambda_7 = \lambda_3 + \lambda_6 + \lambda_7 = 0\} \cap \Delta_7$, corresponding to $\Pi = (1 \mid 3 \mid 5 \mid 7 \mid 246)$. Furthermore, $\mathcal{R}_2(\mathcal{A}, \mathbb{Z}_2)$ has a single, 1-dimensional component, $C_{\Pi'} = \{\lambda_1 + \lambda_7 = \lambda_3 + \lambda_7 = \lambda_5 + \lambda_7 = \lambda_2 = \lambda_4 = \lambda_6 = 0\}$, corresponding to $\Pi' = (1 \mid 3 \mid 5 \mid 7)$, and $\mathcal{R}_3(\mathcal{A}, \mathbb{Z}_2) = \{\mathbf{0}\}$. Thus, $\nu_{2,1} = 24$ and $\nu_{2,2} = 1$.

Example 5.9. Let $\mathcal{A}$ be one of the realizations of the MacLane matroid, with

$$L_2(\mathcal{A}) = \{123, 456, 147, 267, 258, 348, 357, 168, 15, 24, 36, 78\}.$$

The variety $\mathcal{R}_1(\mathcal{A})$ has 8 local components. Despite the fact that $\mathcal{A}$ supports many neighborly partitions, $\mathcal{R}_1(\mathcal{A})$ has no non-local components, since Falk's degeneracy condition is not satisfied, see [11].

For $p \neq 3$, Corollary 5.5 applies, and so $\nu_{p,1} = 8(p+1)$.

For $p = 3$, though, the degeneracy condition is satisfied, and the variety $\mathcal{R}_1(\mathcal{A}, \mathbb{Z}_3)$ has a non-local, 2-dimensional component,

$$\begin{aligned} C_\Pi = \{\lambda_2 + \lambda_5 + \lambda_8 = \lambda_3 + \lambda_5 - \lambda_8 = \lambda_4 - \lambda_5 - \lambda_8 \\ = \lambda_5 - \lambda_6 - \lambda_8 = \lambda_1 + \lambda_5 = \lambda_7 + \lambda_8 = 0\}, \end{aligned}$$

corresponding to $\Pi = (15 \mid 24 \mid 36 \mid 78)$. Moreover, $\mathcal{R}_2(\mathcal{A}, \mathbb{Z}_3) = \{\mathbf{0}\}$. Hence, $\nu_{3,1} = 36$.

Example 5.10. Let $\mathcal{A}$ be the realization of the affine plane over $\mathbb{Z}_3$, with lattice

$$L_2(\mathcal{A}) = \{123, 456, 789, 147, 258, 369, 159, 357, 168, 249, 267, 348\}.$$

The variety $\mathcal{R}_1(\mathcal{A})$ has 12 local components, and 4 non-local components of dimension 2, see [11, 7, 19, 20].

For $p \neq 3$, Corollary 5.5 applies, and so $\nu_{p,1} = 16(p+1)$.

On the other hand, $\mathcal{R}_1(\mathcal{A}, \mathbb{Z}_3)$ has a single, 3-dimensional non-local component, $C_\Pi = \{\lambda_1 + \lambda_6 + \lambda_8 = \lambda_2 + \lambda_4 + \lambda_9 = \lambda_3 + \lambda_5 + \lambda_7 = \lambda_3 + \lambda_4 + \lambda_8 = \lambda_3 + \lambda_6 + \lambda_9 = \lambda_7 + \lambda_8 + \lambda_9 = 0\}$, corresponding to $\Pi = (123 \mid 456 \mid 789)$, or any other of the partitions that give rise to the 4 non-local components of $\mathcal{R}_1(\mathcal{A})$. Moreover, $\mathcal{R}_2(\mathcal{A}, \mathbb{Z}_3) = C_\Pi$, and $\mathcal{R}_3(\mathcal{A}, \mathbb{Z}_3) = \{\mathbf{0}\}$. Thus, $\nu_{3,1} = 48$ and $\nu_{3,2} = 13$.

Example 5.11. Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be generic plane sections of the two arrangements from [11], Example 4.10. Each arrangement consists of 7 affine lines in $\mathbb{C}^2$, and each resonance variety has only local components. Thus, the ν-invariants of $\mathcal{A}_1$ and $\mathcal{A}_2$ coincide. On the other hand, as shown by Falk, there is no linear automorphism $\mathbb{C}^7 \to \mathbb{C}^7$ restricting to an isomorphism $\mathcal{R}_1(\mathcal{A}_1) \to \mathcal{R}_1(\mathcal{A}_2)$. The same 'polymatroid' argument shows that there is no automorphism $\mathbb{P}(\mathbb{Z}_p^7) \to \mathbb{P}(\mathbb{Z}_p^7)$ restricting to $\mathcal{P}_1(\mathcal{A}_1, \mathbb{Z}_p) \to \mathcal{P}_1(\mathcal{A}_2, \mathbb{Z}_p)$. Thus, the ambient type of the (projective) resonance varieties carries more information than the count of their points.

§6. Real arrangements

We conclude with an application to the classification of arrangements of transverse planes in $\mathbb{R}^4$. Though similar in some respects to central line arrangements in $\mathbb{C}^2$, such arrangements lack a complex structure. That difference manifests itself in the nature of the resonance varieties.

6.1. Arrangements of real planes

A *2-arrangement* in $\mathbb{R}^4$ is a finite collection $\mathcal{A} = \{H_1, \dots, H_n\}$ of transverse planes through the origin of $\mathbb{R}^4$. Such an arrangement $\mathcal{A}$ is a realization of the uniform matroid $U_{2,n}$; thus, its intersection lattice is solely determined by n. Let $X = \mathbb{R}^4 \setminus \bigcup_i H_i$ be the complement of the arrangement. The *link* of the arrangement is $L = \mathbb{S}^3 \cap \bigcup_i H_i$. Clearly, the complement of $\mathcal{A}$ deform-retracts onto the complement of L. The link L is the closure of a pure braid in P_n, see [24], [23]. Hence, $G = \pi_1(X)$ is a commutator-relators group.

The *linking numbers* of $\mathcal{A}$ are by definition those of the link L. They can be computed from the defining equations of $\mathcal{A}$: If $H_i = \{\alpha_i = \alpha'_i = 0\}$, for some linear forms $\alpha_i, \alpha'_i : \mathbb{R}^4 \to \mathbb{R}$, then $l_{i,j} = \operatorname{sgn}(\det(\alpha_i, \alpha'_i, \alpha_j, \alpha'_j))$, see [34]. A presentation for the cohomology ring of X in terms of the linking numbers is given in (8), see also [34].

Arrangements of transverse planes in $\mathbb{R}^4$ fall, in the terminology of [8], into several types: horizontal and non-horizontal, decomposable and indecomposable. A 2-arrangement $\mathcal{A}$ is *horizontal* if it admits a defining polynomial of the form $f(z,w) = \prod_{i=1}^{n}(z + a_i w + b_i \bar{w})$, with a_i, b_i real. From the coefficients of f, one reads off a permutation $\tau \in S_n$. Conversely, given τ, there is a horizontal arrangement, $\mathcal{A}(\tau)$, whose associated permutation is τ. A 2-arrangement is *decomposable* if its link is the $(1, \pm 1)$-cable of the link of another 2-arrangement, and it is *completely decomposable* if its link can be obtained from the unknot by successive $(1, \pm 1)$-cablings. See [23] for details.

6.2. Resonance varieties

Let $\mathcal{R}_d(\mathcal{A}) := \mathcal{R}_d(X, \mathbb{C})$ be the d^{th} resonance variety of $\mathcal{A}$. Recall that the resonance varieties form a tower $\mathbb{C}^n = \mathcal{R}_0 \supset \mathcal{R}_1 \supset \cdots \supset \mathcal{R}_{n-1} = \{\mathbf{0}\}$. Moreover, they are the determinantal varieties of the $n \times (n-1)$ matrix $M(\lambda)$, whose entries are given by $M(\lambda)_{k,j} = l_{k,j}\lambda_k - \delta_{k,j}(\sum_i l_{k,i}\lambda_i)$.

If $\mathcal{A}$ is decomposable, the top resonance variety, $\mathcal{R}_1(\mathcal{A})$, contains as a component the hyperplane $\Delta_n = \{\lambda_1 + \cdots + \lambda_n = 0\}$. Moreover, if $\mathcal{A}$ is completely decomposable, $\mathcal{R}_1(\mathcal{A})$ is the union of a central arrangement of $n-2$ hyperplanes in $\mathbb{C}^n$ (counting multiplicities), with defining equations of the form $\epsilon_1\lambda_1 + \cdots + \epsilon_n\lambda_n = 0$, where $\epsilon_i = \pm 1$. If $\mathcal{A}$ is indecomposable, though, $\mathcal{R}_1(\mathcal{A})$ may contain non-linear components (see Example 6.5).

At the other extreme, all the components of the variety $\mathcal{R}_{n-2}(\mathcal{A})$ are linear. It can be shown that a horizontal arrangement $\mathcal{A}$ is indecomposable if and only if $\mathcal{R}_{n-2}(\mathcal{A}) = \{\mathbf{0}\}$.

Example 6.3. In [34], Ziegler provided the first examples of 2-arrangements with isomorphic intersection lattices, but non-isomorphic cohomology rings. Those arrangements are: $\mathcal{A} = \mathcal{A}(1234)$ and $\mathcal{A}' = \mathcal{A}(2134)$. We can distinguish their cohomology rings by counting the components of their resonance varieties:

$$\mathcal{R}_1(\mathcal{A}) = \Delta_4\,, \quad \mathcal{R}_1(\mathcal{A}') = \Delta_4 \cup \{\lambda_1 + \lambda_2 - \lambda_3 - \lambda_4 = 0\}.$$

The example $\mathcal{A}'$ shows that the analogues of Theorem 5.3 (b), (d), (e) do not hold for 2-arrangements:

- The second component of $\mathcal{R}_1(\mathcal{A}')$ does *not* lie in the hyperplane Δ_4.
- The two components of $\mathcal{R}_1(\mathcal{A}')$ do *not* intersect only at the origin, but rather, in the 2-dimensional subspace $\{\lambda_1+\lambda_2 = \lambda_3+\lambda_4 = 0\}$.
- We have $\mathcal{R}_2(\mathcal{A}') = \{\lambda_1 + \lambda_2 = \lambda_3 = \lambda_4 = 0\} \cup \{\lambda_1 = \lambda_2 = \lambda_3 + \lambda_4 = 0\}$, and thus the stratification of $\mathcal{R}_1$ by $\mathcal{R}_d$'s is *not* by dimension of components.

Example 6.4. Let $\mathcal{A} = \mathcal{A}(321456)$ and $\mathcal{A}' = \mathcal{A}(213456)$. Then:

$$\begin{aligned}
&\mathcal{R}_1(\mathcal{A}) = \mathcal{R}_1(\mathcal{A}') = \Delta_6 \cup \{\lambda_1 + \lambda_2 - \lambda_3 - \lambda_4 - \lambda_5 - \lambda_6 = 0\},\\
&\mathcal{R}_2(\mathcal{A}) = \mathcal{R}_1(\mathcal{A}), \qquad \mathcal{R}_2(\mathcal{A}') = \Delta_6.
\end{aligned}$$

This example shows that the analogue of Theorem 5.3 (a) does not hold for 2-arrangements: The variety $\mathcal{R}_1$ fails to determine $\mathcal{R}_2$, and thereby fails to determine the cohomology ring of the complement.

Example 6.5. The horizontal arrangement $\mathcal{A}(31425)$ is indecomposable. Its resonance varieties are:

$$\begin{aligned}
\mathcal{R}_1 = \{&\lambda_1^3 - \lambda_2^3 - \lambda_3^3 + \lambda_4^3 - \lambda_5^3 + \lambda_1^2\lambda_2 - \lambda_1\lambda_2^2 + \lambda_1^2\lambda_3 - \lambda_1\lambda_3^2 - \lambda_1^2\lambda_4 -\\
&\lambda_1\lambda_4^2 + \lambda_1^2\lambda_5 - \lambda_1\lambda_5^2 + \lambda_2^2\lambda_3 + \lambda_2\lambda_3^2 - \lambda_2^2\lambda_4 + \lambda_2\lambda_4^2 + \lambda_2^2\lambda_5 + \lambda_2\lambda_5^2 -\\
&\lambda_3^2\lambda_4 + \lambda_3\lambda_4^2 + \lambda_3^2\lambda_5 + \lambda_3\lambda_5^2 + \lambda_4^2\lambda_5 - \lambda_4\lambda_5^2 + 2\lambda_1\lambda_2\lambda_3 - 2\lambda_1\lambda_2\lambda_4 +\\
&2\lambda_1\lambda_2\lambda_5 - 2\lambda_1\lambda_3\lambda_4 + 2\lambda_1\lambda_3\lambda_5 - 2\lambda_1\lambda_4\lambda_5 + 2\lambda_2\lambda_3\lambda_4 - 2\lambda_2\lambda_3\lambda_5 +\\
&2\lambda_2\lambda_4\lambda_5 + 2\lambda_3\lambda_4\lambda_5 = 0\}\\
\mathcal{R}_2 = \{&\lambda_1 + \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = 0\} \cup \{\lambda_1 + \lambda_3 = \lambda_2 = \lambda_4 = \lambda_5 = 0\} \cup\\
&\{\lambda_2 + \lambda_4 = \lambda_1 = \lambda_3 = \lambda_5 = 0\} \cup \{\lambda_3 + \lambda_4 = \lambda_1 = \lambda_2 = \lambda_5 = 0\} \cup\\
&\{\lambda_1 + \lambda_5 = \lambda_2 = \lambda_3 = \lambda_4 = 0\} \cup \{\lambda_4 + \lambda_5 = \lambda_1 = \lambda_2 = \lambda_3 = 0\} \cup\\
&\{\lambda_1 - \lambda_4 = \lambda_2 = \lambda_3 = \lambda_5 = 0\} \cup \{\lambda_2 - \lambda_3 = \lambda_1 = \lambda_4 = \lambda_5 = 0\} \cup\\
&\{\lambda_2 - \lambda_5 = \lambda_1 = \lambda_3 = \lambda_4 = 0\} \cup \{\lambda_3 - \lambda_5 = \lambda_1 = \lambda_2 = \lambda_4 = 0\}\\
\mathcal{R}_3 = \{&\mathbf{0}\}
\end{aligned}$$

This example shows that the analogue of Theorem 5.3 (c) does not hold for 2-arrangements: The variety $\mathcal{R}_1$ is not linear.

6.6. Ziegler invariant

The cohomology rings of the arrangements in Example 6.3 were distinguished by Ziegler by means of an invariant closely related to one of the Z-invariants introduced in 1.16.

Recall the sequence $0 \to G_2/G_3 \to G/G_3 \to H \to 0$. This central extension is determined by the map $\chi^\top : G_2/G_3 \to \bigwedge^2 H$, given explicitly by (9). The invariant $Z_{0,1}(\chi) = \operatorname{coker} \chi^{\top}$ equals $H^2(G) = \mathbb{Z}^{n-1}$. More information is carried by the next invariant,

$$Z_{0,2}(\chi) = \operatorname{coker}\Big(\mu_H \circ \bigwedge^2 \chi^{\top} : \bigwedge^2 G_2/G_3 \to \bigwedge^4 H\Big).$$

Set $Z(\mathcal{A}) := Z_{0,2}(\chi)$. It can be shown that $Z(\mathcal{A}) = \mathbb{Z}^{\binom{n-1}{3}-r} \oplus \mathbb{Z}_2^r$, where r is some integer that can be read off from the linking graph $\mathcal{G}$ of the link of $\mathcal{A}$.

For example, $Z(\mathcal{A}(1234)) = \mathbb{Z}$ and $Z(\mathcal{A}(2134)) = \mathbb{Z}_2$, showing again that the two arrangements have different cohomology rings. But $Z(\mathcal{A})$ is not a complete invariant of the cohomology ring. For example, $Z(\mathcal{A}(21435)) = Z(\mathcal{A}(31425)) = \mathbb{Z}_2^4$, although the two arrangements are distinguished by the ν-invariants (see below).

n	$\mathcal{A}$	$\nu_{3,0}$	$\nu_{3,1}$	$\nu_{3,2}$	$\nu_{3,3}$	$\nu_{3,4}$
3	$\mathcal{A}(123)$	9	4			
4	$\mathcal{A}(1234)$	27	0	13		
	$\mathcal{A}(2134)$	18	20	2		
5	$\mathcal{A}(12345)$	81	0	0	40	
	$\mathcal{A}(21345)$	54	27	35	5	
	$\mathcal{A}(21435)$	36	66	17	2	
	$\mathcal{A}(31425)$	51	60	10	0	
6	$\mathcal{A}(123456)$	243	0	0	0	121
	$\mathcal{A}(213456)$	162	81	0	107	14
	$\mathcal{A}(321456)$	162	0	162	32	8
	$\mathcal{A}(215436)$	108	126	87	38	5
	$\mathcal{A}(214356)$	108	108	121	24	3
	$\mathcal{A}(312546)$	72	186	90	14	2
	$\mathcal{A}(341256)$	81	162	112	6	3
	$\mathcal{A}(314256)$	117	162	74	10	1
	$\mathcal{A}(241536)$	108	200	48	8	0
	$\mathcal{L}$	81	162	112	6	3
	$\mathcal{M}$	144	160	60	0	0

TABLE 1. Arrangements of $n \leq 6$ planes in $\mathbb{R}^4$: Number $\nu_{3,d}$ of index 3 subgroups, according to their abelianization, $\mathbb{Z}^n \oplus \mathbb{Z}_3^d$.

6.7. Classification for $n \leq 6$

Let G the group of an arrangement of n transverse planes in $\mathbb{R}^4$, and G/G_3 its second nilpotent quotient. As can be seen in Table 1, the $\nu_{3,d}$-invariants completely classify the second nilpotent quotients (and, thereby the cohomology rings) of 2-arrangement groups, for $n \leq 6$, with a lone exception.

The exception is Mazurovskiĭ's pair, $\mathcal{K} = \mathcal{A}(341256)$ and $\mathcal{L}$. The corresponding configurations of skew lines in $\mathbb{R}^3$ were introduced in [24]. Explicit equations for $\mathcal{K}$ and $\mathcal{L}$ can be found in [23]. As noted in [24], the links of $\mathcal{K}$ and $\mathcal{L}$ have the same linking numbers. Thus, $H^*(X_\mathcal{K};\mathbb{Z}) \cong H^*(X_\mathcal{L};\mathbb{Z})$, and $G_\mathcal{K}/(G_\mathcal{K})_3 \cong G_\mathcal{L}/(G_\mathcal{L})_3$. On the other hand, $G_\mathcal{K}/(G_\mathcal{K})_4 \not\cong G_\mathcal{L}/(G_\mathcal{L})_4$, as can be seen from the distribution of the abelianization of their index 3 subgroups, shown in Table 2.

d	0	1	2	3	4	5	6	7	8
$G_\mathcal{K}/(G_\mathcal{K})_4$	81	0	162	0	112	0	6	0	3
$G_\mathcal{L}/(G_\mathcal{L})_4$	81	0	172	24	78	6	0	0	3

TABLE 2. The groups $G_\mathcal{K}/(G_\mathcal{K})_4$ and $G_\mathcal{L}/(G_\mathcal{L})_4$: Number of index 3 subgroups, according to their abelianization, $\mathbb{Z}^6 \oplus \mathbb{Z}_3^d$.

We summarize the above discussion, as follows:

Theorem 6.8. *Let $(\mathcal{A},\mathcal{A}') \neq (\mathcal{K},\mathcal{L})$ be a pair of 2-arrangements of $n \leq 6$ planes in $\mathbb{R}^4$. Then $H^*(X) \cong H^*(X')$ if and only if $X \simeq X'$.*

In other words, up to 6 planes, and with the exception of Mazurovskiĭ's pair, the classification of complements of 2-arrangements up to cohomology-ring isomorphism coincides with the homotopy-type classification. As shown in [23], the latter coincides with the isotopy-type classification, modulo mirror images.

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Recent progress of intersection theory for twisted (co)homology groups

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§1. Introduction

Maybe you have ever seen at least one of the following formulae:

$$B(p,q)B(-p,-q) = \frac{2\pi i(p+q)}{pq}\frac{1-e^{2\pi i(p+q)}}{(1-e^{2\pi ip})(1-e^{2\pi iq})},$$
$$\Gamma(p)\Gamma(1-p) = \frac{\pi}{\sin \pi p}, \quad \left(\int_{-\infty}^{\infty} e^{-t^2/2}dt\right)^2 = 2\pi,$$

where

$$B(p,q) := \int_0^1 t^p(1-t)^q \frac{dt}{t(1-t)}, \quad \Gamma(p) := \int_0^\infty t^p e^{-t}\frac{dt}{t}$$

are the Gamma and the Beta functions.

In this paper, we give a geometric meaning for these formulae: If one regards such an integral as the dual pairing between a (kind of) cycle and a (kind of) differential form, then the value given in the right hand side of each formula is the product of the *intersection* numbers of the two cycles and that of the two forms appeared in the left-hand side.

Of course the intersection theory is not made only to *explain* these well known formulae; for applications, see [CM], [KM], [Y1].

§2. Twisted (co)homology groups

Let $l_1, \dots, l_{n+1}$ be polynomials of degree 1 in $t_1, \dots, t_k$, $(n \geq k \geq 1)$ and $\alpha_1, \dots, \alpha_{n+1}$ be complex numbers satisfying

Assumption 1. $\alpha_j \notin \mathbb{Z}, \quad \alpha_0 := -\alpha_1 - \cdots - \alpha_{n+1} \notin \mathbb{Z}.$

Put

$$
\begin{aligned}
L_j &= \text{ hyperplane defined by } l_j, \quad j = 1, \dots, n+1, \\
T &= \mathbb{C}^k - \cup_{j=1}^{n+1} L_j \\
&= \mathbb{P}^k - \cup_{j=0}^{n+1} L_j, \quad L_0 : \text{ hyperplane at infinity}, \\
u &= \prod_{j=1}^{n+1} l_j^{\alpha_j} : \quad \text{multi-valued function on } T, \\
\mathcal{L},\ \check{\mathcal{L}} &: \text{local systems caused by } u^{-1} \text{ and } u, \text{ respectively}, \\
\omega &= \sum_{j=1}^{n} \alpha_j \frac{dl_j}{l_j} : \quad \text{single-valued 1-form on } T, \\
\nabla &= d + \omega\wedge, \quad \check{\nabla} = d - \omega\wedge : \quad \text{derivations.}
\end{aligned}
$$

Assumption 2. No $k+1$ hyperplanes in $\{L_j\}_{j=0}^{n+1}$ intersect in $\mathbb{P}^k$.

Denoting the k-dimensional cohomology groups (with compact support) and the (locally finite) homology groups by the usual symbols, we have the three natural dual parings (explained below):

$$
\begin{array}{ccc}
H_c^k(T, \mathcal{L}) & \longleftrightarrow & H^k(T, \check{\mathcal{L}}) \\
\updownarrow & & \updownarrow \\
H_k^{lf}(T, \check{\mathcal{L}}) & & H_k(T, \mathcal{L}).
\end{array}
$$

All other dimensional (co)homology groups vanish. By de Rham's theorem, cohomology classes can be represented by smooth global forms:

$$
H_c^k(T, \mathcal{L}) \cong H^k(\mathcal{E}_c^\bullet, \nabla), \quad H^k(T, \check{\mathcal{L}}) \cong H^k(\mathcal{E}^\bullet, \check{\nabla}),
$$

where $\mathcal{E}^p$ and $\mathcal{E}_c^p$ are spaces of smooth p-forms on T and those with compact support. Through these isomorphisms, the columns in the above diagram can be realized by the integration

$$
\langle \varphi, \delta \rangle := \int_\delta \varphi u, \quad \text{or} \quad \langle \psi, \gamma \rangle := \int_\gamma \psi u^{-1}
$$

of k-forms along k-cycles, where

$$
\varphi \in H_c^k(T, \mathcal{L}), \quad \delta \in H_k^{lf}(T, \check{\mathcal{L}}), \quad \text{or} \quad \psi \in H^k(T, \check{\mathcal{L}}), \quad \gamma \in H_k(T, \mathcal{L}),
$$

respectively. Such an integration is often called a *hypergeometric integral* (HG integral for short) because if one let the hyperplanes L_j move then the integral defines a hypergeometric function of type $(k+1, n+2)$. When $k = 1, n = 2$ this is indeed the Gauss hypergeometric function.

The first row is the *intersection form for cohomology groups*, and can be represented by the integral

$$\varphi \cdot \psi := \int_T (\varphi u) \wedge (\psi u^{-1}) = \int_T \varphi \wedge \psi$$

of $2k$-forms over T, where $\varphi \in H^k(\mathcal{E}_c^\bullet, \nabla)$, $\psi \in H^k(\mathcal{E}^\bullet, \check{\nabla})$. (N.B. In [KY1], $\psi \wedge \varphi$ is used in place of $\varphi \wedge \psi$.)

Now these three pairings induce the Poincaré isomorphisms:

$$H_c^k(T, \mathcal{L}) \cong H_k(T, \mathcal{L}), \qquad H^k(T, \check{\mathcal{L}}) \cong H_k^{lf}(T, \check{\mathcal{L}}).$$

Thus through these two isomorphisms the intersection form for cohomolog groups induces the dual pairing, called the *intersection form for homology groups*, of the two homology groups. In this way we have the four *compatible* pairings:

$$\begin{array}{ccccl} H_c^k(T,\mathcal{L}) & \longleftrightarrow & H^k(T,\check{\mathcal{L}}) & & \text{intersection form} \cdot \text{for coh.} \\ \updownarrow & & \updownarrow & & \\ H_k^{lf}(T,\check{\mathcal{L}}) & \longleftrightarrow & H_k(T,\mathcal{L}) & & \text{intersection form} \cdot \text{for hom.} \\ & & & & \\ \text{HG integral} & & \text{HG integral} & & \end{array}$$

Let us take bases as

$$\begin{array}{ll} \varphi^i \in H_c^k(T, \mathcal{L}), & \psi^i \in H^k(T, \check{\mathcal{L}}), \\ \delta_i \in H_k^{lf}(T, \check{\mathcal{L}}), & \gamma_i \in H_k(T, \mathcal{L}). \end{array}$$

Denoting the matrix $(\langle \varphi^i, \delta_j \rangle)_{ij}$ by $(\langle \varphi, \delta \rangle)$ and $(\delta_i \cdot \gamma_j)_{i,j}$ by $(\delta \cdot \gamma)$, we have

$$(\varphi \cdot \psi) = (\langle \varphi, \delta \rangle)(\gamma \cdot \delta)^{-1}\, {}^t(\langle \psi, \gamma \rangle),$$

which gives quadratic relations among the HG integrals.

Note that up to now we presented abstract nonsense which is valid for any complex manifold and for any local system. Our task is, for the special T and $\mathcal{L}$ given above, to pick a suitable basis of each (co)homology group and evaluate the intersection numbers.

§3. Intersection form for cohomology groups

To pick an explicit basis of the cohomology groups, holomorphic forms or possibly algebraic forms are better. Recall the isomorphisms, due to comparison theorems,

$$\begin{aligned} H^k(T, \mathcal{L}) &\cong H^k(\mathcal{E}^\bullet, \nabla) \cong H^k(\Omega^\bullet, \nabla) \\ &\cong H^k(\Omega^\bullet(*L), \nabla) \cong H^k(\Omega^\bullet(\log L), \nabla), \end{aligned}$$

where Ω^p, $\Omega^p(*L)$ and $\Omega^p(\log L)$ are spaces of holomorphic forms on T, algebraic forms and logarithmic forms with poles only along $\cup_{j=0}^{n+1} L_j$, respectively.

For a multi-index $I = (i_0, \dots, i_k)$, $0 \leq i_0 < \cdots < i_k \leq n+1$, we define a logarithmic k-form

$$\varphi_I = \frac{dl_{i_0}}{l_{i_1}} \wedge \cdots \wedge \frac{dl_{i_{k-1}}}{l_{i_k}}.$$

For example, the following $\binom{n}{k}$ forms give a basis of $H^k(\Omega^\bullet(\log L), \nabla)$:

$$\varphi_I, \qquad i_0 = 0 < i_1 < \cdots < i_k \leq n.$$

It is known (e.g. [DM]) and easy to prove, under Assumption 1, the isomorphism

$$H_c^k(T, \mathcal{L}) \cong H^k(T, \mathcal{L}).$$

Thus together with the isomorphism $H^k(T, \mathcal{L}) \cong H^k(\Omega^\bullet(\log L), \nabla)$ above, we can let φ_I represent also an element of $H_c^k(T, \mathcal{L})$. We wish to evaluate the intersection numbers of these forms. The key point is to represent the isomorphism

$$\iota : H^k(\Omega^\bullet(\log L), \nabla) \xrightarrow{\cong} H^k(\mathcal{E}_c^\bullet, \nabla) \quad (\cong H_c^k(T, \mathcal{L})),$$

explicit enough so that the $2k$-dimensional integral

$$\int \iota(\varphi_I) \wedge \varphi_J$$

is computable. This can be done and we get

Theorem 1. *The intersection number $\varphi_I \cdot \varphi_J$ of*

$$\varphi_I \in H_c^k(T, \mathcal{L}) \quad \textit{and} \quad \varphi_J \in H^k(T, \check{\mathcal{L}}),$$

where $I = \{i_0, \dots, i_k\}$, $0 \leq i_0 < \cdots < i_k \leq n+1$, $J = \{j_0, \dots, j_k\}$, $0 \leq j_0 < \cdots < j_k \leq n+1$, is equal to the (I, J)-minor of the tri-diagonal symmetric matrix

$$Int_{coh}(\alpha) = 2\pi\sqrt{-1} \begin{pmatrix} 1/\alpha_0 + 1/\alpha_1 & 1/\alpha_1 & 0 & \cdots \\ 1/\alpha_1 & 1/\alpha_1 + 1/\alpha_2 & 1/\alpha_2 & \\ 0 & 1/\alpha_2 & 1/\alpha_2 + 1/\alpha_3 & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix}.$$

Actual value of $\varphi_I \cdot \varphi_J$ *is given as follows*:

$$(2\pi\sqrt{-1})^k \frac{\sum_{i\in I}\alpha_i}{\prod_{i\in I}\alpha_i} \quad \text{if} \quad I = J,$$

$$(2\pi\sqrt{-1})^k \frac{(-1)^{\mu+\nu}}{\prod_{i\in I\cap J}\alpha_i} \quad \text{if} \quad \#(I\cap J) = k,$$

0, *otherwise, where* μ *and* ν *are determined by* $\{i_\mu\} = I - J$ *and* $\{j_\nu\} = J - I$.

Though there are technical difficulties for general k, the essential idea of the proof can be seen from that of the case $k = 1$. So we prove this theorem only when $k = 1$, and when $k \geq 2$ we describe where the difficulty lies and how we can manage.

3.1. Proof of Theorem 1 when $k = 1$. We express the image $\iota(\varphi_I)$ explicitly. We find a smooth function f on T such that $\varphi_I - \nabla f$ is compactly supported. This means that $\varphi_I - \nabla f$ represents the class $\iota(\varphi_I)$ of $H_c^1(T, \mathcal{L})$.

We can find a convergent power series f_p centered at the point L_p satisfying $\nabla f_p = \varphi_I$. Let h_p be a smooth real function on $\mathbb{P}^1$ such that $h_p(t) = 0$ $(t \notin U_p)$, $0 < h_p(t) < 1$ $(t \in U_p \setminus V_p)$, $h_p(t) = 1$ $(t \in V_p)$, where $L_p \in V_p \subset U_p$, and U_p is a small neighborhood of L_p. Regarding $f := \sum_{p=0}^{n+1} h_p f_p$ as defined on T, we have

$$\varphi_I - \nabla f = \varphi_I - \sum_{p=0}^{n+1}[h_p\nabla(f_p) + f_p dh_p] = \sum_{p=0}^{n+1}[(1-h_p)\varphi_I - f_p dh_p],$$

which is of compact support on T. The Stokes theorem and the residue theorem yields

$$\begin{aligned}\int_T \iota(\varphi_I)\wedge\varphi_J &= \sum_{p=0}^{n+1}\int_T [(1-h_p)\varphi_I - f_p dh_p]\wedge\varphi_J \\ &= \sum_{p=0}^{n+1}\int_{U_p\setminus V_p} -f_p dh_p\wedge\varphi_J = \sum_{p=0}^{n+1}\int_{\partial(U_p\setminus V_p)} -h_p f_p\varphi_J \\ &= \sum_{p=0}^{n+1}\int_{\partial V_p} f_p\varphi_J = 2\pi\sqrt{-1}\sum_{p=0}^{n+1}\mathrm{Res}_{L_p}(f_p\varphi_J).\end{aligned}$$

Completion of the proof is now immediate.

3.2. Strategy for $k \geq 2$. We prepare some notation. Let L_{P^q} be the intersection of $L_{p_1}, L_{p_2}, \dots, L_{p_q}$, and let U_{P^q} be a small tubular neighborhood of L_{P^q} in $\mathbb{P}^k$, where P^q is a multi-index with cardinality q, say,

$$P^q = \{p_1, p_2 \dots, p_q\}, \quad 0 \leq p_1 < p_2 < \cdots < p_q \leq n+1.$$

For multi-indices P^{q-1} and P^q, if $P^{q-1} \subset P^q$, then we put

$$\delta(P^{q-1}; P^q) = (-1)^r, \quad \text{where} \quad \{p_r\} := P^q \setminus P^{q-1}.$$

Step 1. Construct a system of holomorphic $(k-q)$-forms f_{P^q} on $U_{P^q} \cap T$ such that

$$\begin{aligned} \nabla(f_{P^1}) &= \varphi_I, \\ \nabla(f_{P^q}) &= \sum_{P^{q-1} \subset P^q} \delta(P^{q-1}; P^q) f_{P^{q-1}} \quad (2 \leq q \leq k); \end{aligned}$$

these can be obtained as convergent power series. Complexity lies on the fact that the singularities $\cup L_{p=j}$ are not isolated.
Step 2. By patching f_{P^q} inductively by the help of partition of the unity on $\cup_{j=0}^{n+1} U_j$, we get a smooth $(k-1)$-form f on T such that

$$\nabla f = \varphi_I \quad \text{in} \quad \cup_{j=0}^{n+1} U_j.$$

Since $\varphi_I - \nabla f$ is of compact support on T and is cohomologous to φ_I in $H^k(\mathcal{E}, \nabla)$, it represents $\iota(\varphi_I)$.
Step 3. Repeated use of the Stokes theorem and the residue theorem leads to

$$\begin{aligned} \int_T \iota(\varphi_I) \wedge \varphi_J &= \int_T -df \wedge \varphi_J \\ &= (2\pi\sqrt{-1})^k \sum_{P^k} \mathrm{Res}_{L_{P^k}}(f_{P^k} \varphi_J), \end{aligned}$$

which will imply the theorem.

§4. Intersection form for homology groups

Since we assumed that our hyperplane arrangement is in general position (Assumption 2), we can continuously deform the arrangement, keeping its intersection pattern, into a real arrangement, by which we mean all the linear forms l_j are defined over the real numbers. So we assume that our arrangement is real.

Note that there are many arrangements not in general position that one can not deform into a real one.

Let $T_{\mathbb{R}}$ be the real locus of T. $\binom{n}{k}$ bounded chambers support cycles forming a basis of $H_k^{lf}(T,\check{\mathcal{L}})$. One can load any branch of u on the chambers; too much freedom annoys us. In order to make it in a systematic way, we further deform the arrangement and put the hypersurfaces in a specially nice way. Then the k-dimensional cases can be reduced to the simplest case $k = 1$.

Loaded cycles: We represent elements of $H_p(T,\check{\mathcal{L}})$ by *loaded p-cycles*, which is convenient here and will be indispensable in §8.2. A loaded p-chain is a formal sum of loaded p-simplexes. A loaded p-simplex is a topological simplex on which a branch of u is assigned. The boundary operator is naturally defined. For example, the boundary of a loaded path (1-chain) is given by

$$\begin{aligned}&(\text{ending point loaded with the value of the function there})\\ -&(\text{starting point loaded with the value of the function there}).\end{aligned}$$

The boundary of a higher dimensional loaded chain is defined in an obvious way. A loaded p-chain is called a *loaded p-cycle* if its boundary vanishes.

4.1. Case $k = 1$. Let $x_1,\dots,x_{n+1}$ be distinct real points on $\mathbb{P}^1$ satisfying $x_1 < \cdots < x_{n+1}$. Then the multi-valued function

$$u = \prod_{j=1}^{n} l_j^{\alpha_j}, \qquad l_j = t - x_j$$

is defined on $T = \mathbb{P}^1 - \{x_1,\dots,x_{n+1}, x_0 = \infty\}$. On each oriented interval $\overrightarrow{(x_p, x_{p+1})}$, we load a branch of the function u determined by

$$\arg(t - x_j) = \begin{cases} 0 & j \le p, \\ -\pi & p+1 \le j, \end{cases}$$

and call this loaded path $\check{I}_p$. Note that if you analytically continue the branch of u corresponding to some loaded path $\check{I}_j$ through the *lower* half part of the t-plane T, then you get the branches of u corresponding to other loaded paths $\check{I}_i$. But if you do the same starting from a point in (x_j, x_{j+1}), passing through the *upper* part and ending at a point in (x_{j-1}, x_j), you get

$$c_j := e^{2\pi i \alpha_j}$$

times the branch u corresponding to the loaded paths $\check{I}_{j-1}$.

Anyway, $\check{I}_j$ represent elements of $H_1^{lf}(T,\check{\mathcal{L}})$. For example, n non-compact loaded cycles $\check{I}_1,\ldots,\check{I}_n$ form a basis. Loading u^{-1} in place of u, we get non-compact loaded cycles I_j; for example, $I_1,\ldots,I_n$ form a basis of $H_1^{lf}(T,\mathcal{L})$.

As we did in §3, to define intersection numbers, we must make a compact counterpart regI_j, regularization of I_j. This can be done by attaching two circles at the ends:

$$-\frac{c_j}{d_j}\,C_\epsilon^j+\overrightarrow{(x_j+\varepsilon,x_{j+1}-\varepsilon)}+\frac{c_{j+1}}{d_{j+1}}\,C_{-\epsilon}^{j+1},\quad d_j:=c_j-1,$$

where $C_{\pm\epsilon}^j$ is the positively oriented circle of radius $\epsilon>0$ center at x_j starting at $x_j\pm\epsilon$ (see Figure 1), and by loading u^{-1} along the three paths, where the branch of u^{-1} at each starting point is that of I_j. Note that regI_j is homologous to I_j in $H_1^{lf}(T,\mathcal{L})$. reg$I_1,\ldots,$regI_n form a basis of $H_1(T,\mathcal{L})$.

Let us evaluate the intersection number reg$I_i\cdot\check{I}_j$. As is explained in §2, the definition is made through the intersection number of cohomology groups; it is a, so to speak, indirect *analytic* definition. In the following, we give a direct it topological definition, by which one can evaluate intersection numbers explicitly. These two definitions agree (see [KY1]); this fact will be referred to the *compatibility of intersection forms for homology and cohomology groups.*

Deform the support of $\check{I}_j$ so that it intersects transversally with that of regI_i; any deformation will do. At each intersection point of the two supports, multiply the values of the two functions loaded to make the local intersection number at this point. Then sum up all the local intersection numbers, and finally change the sign to get reg$I_i\cdot\check{I}_j$ (see Figure 1). Here is an actual computation:

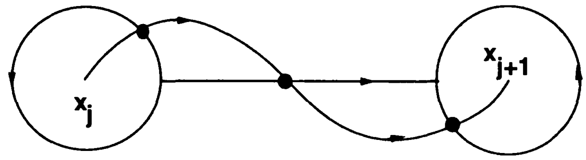

FIG 1. Intersection of $regI_j$ and I_j

$$(regI_j)\cdot\check{I}_j=-\left(\frac{c_j}{d_j}-1+\frac{c_{j+1}}{d_{j+1}}\right)=-\left(\frac{d_{j,j+1}}{d_jd_{j+1}}\right),$$

$$(regI_j)\cdot\check{I}_{j-1}=\frac{1}{d_j},\qquad(regI_{j-1})\cdot\check{I}_j=\frac{c_j}{d_j},$$

0, otherwise, where $d_{ij} = c_i c_j - 1$. Therefore the *intersection matrix* $Int_{hom}(\alpha) = (regI_i \cdot \check{I}_j)_{ij}$ is given by the following tri-diagonal matrix

$$Int_{hom}(\alpha) = -\begin{pmatrix} d_{12}/d_1 d_2 & -c_2/d_2 & 0 & \cdots \\ -1/d_2 & d_{23}/d_2 d_3 & -c_3/d_3 & \ddots \\ 0 & -1/d_3 & d_{34}/d_3 d_4 & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix}.$$

(N.B. The intersection matrix in [KY1] is given by $-{}^t Int_{hom}(\alpha) = Int_{hom}(-\alpha)$ according to the definition of the intersection form for cohomology groups made there (cf. §2).)

4.2. Case $k \geq 2$. For given $n+1$ real points on $\mathbb{C}$

$$x_1 < \cdots < x_j < \cdots < x_{n+1}, \quad x_0 = \infty,$$

we define $n+1$ real hyperplanes $L_1, \ldots, L_n$ in $t = (t_1, \ldots, t_k)$-space by

$$l_j := t_r + (-x_j)t_{r-1} + \cdots + (-x_j)^{r-1} t_1 + (-x_j)^r, \quad 1 \leq j \leq n,$$

and L_0 the hyperplane at infinity. This arrangent $\{L_0, \ldots, L_n\}$ is called a Veronese arrangement, since an embedding of $\mathbb{P}^1$ into $\mathbb{P}^k$ by

$$t_0 = s^k,\ t_1 = s^{k-1}, \ldots, t_{k-1} = s,\ t_k = 1$$

is called the Veronese embedding. When $k = 2$ and $n = 4$, the arrangement is illustrated in Figure 2. Set

$$U = \prod_{j=1}^{n} l_j(t)^{\alpha_j},$$

where $l_j(t)$ is the linear form in t just defined above. For a multi-index,

$$I = (i_1, \ldots, i_k), \quad 1 \leq i_1 < \cdots < i_k \leq n,$$

we define loaded cycles $D_I \in H_k^{lf}(T, \mathcal{L})$ and $\check{D}_I \in H_k^{lf}(T, \check{\mathcal{L}})$ with support on the chamber (see Figure 2)

$$|D_I| = \{t \in T_{\mathbb{R}} \mid (-1)^{P(j)} l_j(t) > 0, \quad 1 \leq j \leq n\},$$

loaded with U^{-1} and U, respectively, with

$$\arg l_j = -P(j)\pi, \quad 1 \leq j \leq n,$$

where $P(j)$ denotes the cardinality of $\{p \mid i_p < j\}$. Since each loaded

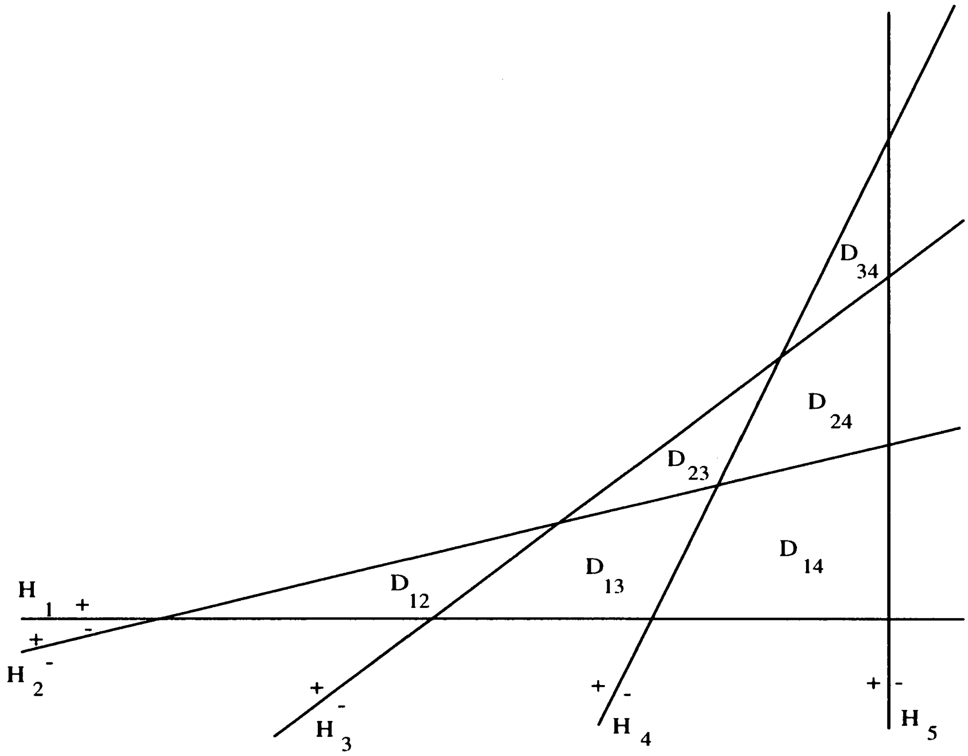

FIG 2. A Veronese arrangement ($k = 2$, $n = 4$) and the chambers

cycle is locally a direct product of 1-dimensional cycles, the regularizations $\text{reg}D_I \in H_k(T, \mathcal{L})$ are naturally defined. We now state the result, which is very similar to Theorem 1.

Theorem 2. *For multi-indices $I = (i_1 \ldots i_k)$, $1 \le i_1 < \cdots < i_k \le n$, $J = (j_1 \ldots j_k)$, $1 \le j_1 < \cdots < j_k \le n$, the intersection number $\text{reg}D_I \cdot \check{D}_J$ is equal to the (I, J)-minor of the matrix $Int_{hom}(\alpha)$.*

For rigorous proofs, see [KY2]. This theorem can be naturally understood if you write

$$D_J = I_{j_1} \wedge \cdots \wedge I_{j_k}, \quad J = (j_1, \ldots, j_k)$$

which is justified in [IK2].

§5. Quadratic relations

As we pointed out at the end of §2 (see also the middle of §4.1), the compatibility of the intersection forms for homology groups and cohomology groups, which is a general, universal and abstract equality, produces explicit quadratic relations among hypergeometric integrals — twisted analogues of the Riemann equality for periods.

The simplest example is the one in §1

$$B(p,q)B(-p,-q)=\frac{2\pi i(p+q)}{pq}\cdot\frac{1-e^{2\pi i(p+q)}}{(1-e^{2\pi ip})(1-e^{2\pi iq})}.$$

Now we know the meaning of the right-hand side: It is the product of the intersection number of the forms

$$\frac{dt}{t(1-t)}\in H^1(\Omega^\bullet(\log L),\nabla)\quad\text{and}$$
$$\frac{dt}{t(1-t)}\in H^1(\Omega^\bullet(\log L),\check{\nabla}),\quad L=\{0,1,\infty\}$$

and that of the cycles

$$(0,1)\otimes u^{-1}\in H_1(T,\mathcal{L})\quad\text{and}\quad(0,1)\otimes u\in H_1(T,\check{\mathcal{L}}),\quad u:=t^p(1-t)^q.$$

Here is another example due to Gauss:

$$\begin{aligned}&F(a,b,c;x)F(1-a,1-b,2-c;x)\\&\quad=\;F(a+1-c,b+1-c,2-c;x)F(c-a,c-b,c;x),\end{aligned}$$

where F is the hypergeometric function (cf. [CM], [Mat1]).

Twisted analogues of Riemann inequality. When $\alpha_j\in\mathbb{R}$, we can speak about the Hodge structure on the cohomology groups, and get twisted analogues of Riemann inequality. [HY] studies these when $k=1$.

§6. Further study

So far, we worked on the projective spaces $\mathbb{P}^k$, linear forms l_j, function $u=\prod l_j^{\alpha_j}$, 1-form $\omega=du/u$, etc, under Assumption 1: $\alpha_j\notin\mathbb{Z}$, and Assumption 2: no $k+1$ hyperplanes in $\{L_j\}$ intersect.

For a general arrangement, without Assumption 2 but with a genericity for α_j corresponding to Assumption 1, the structure of the cohomology group can be described in terms of the so-called Orlik-Solomon algebra, and an explicit basis of the homology group is known, if the arrangement is *real.* By successive blowing-up one can make the proper transform of the arrangement normally crossing — there is a systematic way to do this — then one can, in principle, evaluate the intersection numbers (cf. [KY2], [Yos2]). We expect that these intersection numbers can be expressed combinatorially in a closed form.

For imaginary arrangements, $k \geq 2$, or non-linear arrangements (cf. [KY2]), little is known about explicit cycles.

Motivated by an integral whose integrand involves hypergeometric functions, Hanamura, Ohara and Takayama study intersection theory when the rank of the local system $\mathcal{L}$ is larger than 1 (cf. [Oha1,2], [OT]). They use hyperplane-section method, which is expected to be effective also to the previous problem.

Recall the famous limit formula:

$$(1+\lambda t)^{1/\lambda} \longrightarrow e^t, \quad \text{as} \quad \lambda \longrightarrow 0$$

and a less famous one

$$(1+\lambda t)^{1/\lambda(\mu-\lambda)}(1+\mu t)^{1/\mu(\lambda-\mu)} \longrightarrow e^{t^2/2}, \quad \text{as} \quad \lambda,\ \mu \longrightarrow 0.$$

In §1, starting from the Beta integral you find two 'limit' integrals, one of them is the Gamma function. These formulae suggest another direction of generalization of the theories stated above, that is, to consider for example

$$u = \prod_{j=1}^{m}(t-x_j)^{\alpha_j}\exp f, \quad \omega = d\log u = \sum_{j=1}^{m}\alpha_j\frac{dt}{t-x_j}+df, \quad \nabla = d+\omega\wedge,$$

where f is a polynomial in t. The corresponding hypergeometric integrals represent various *confluent* hypergeometric functions; the extreme ones are those without l_j; such integrals are called *generalized Airy integrals*, because

$$\int \exp(-t^3/3+xt)dt$$

represents the Airy function.

In the following sections we study the confluent cases. Since the above limit formulae are delicate, if you know what I mean, the above theories in §§2 – 5 do *not* directly imply those for confluent cases; we must establish it independently. Of course you can expect some limit relations among them (see [KHT2], [Ha2]).

§7. Confluent cases, general frame

Let $n_1 \geq \cdots \geq n_m$ be natural numbers and L_j $(1 \leq j \leq m)$ be hyperplanes in $\mathbb{P}^k$ defined by linear forms l_j of $t_1, \dots, t_k$; put $T = \mathbb{P}^k \setminus \cup_{j=1}^m L_j$. We define a rational exact 1-form ω'_j with n_j-fold poles along L_j; this is explicitly given in §9. Put

$$\omega = \sum_{j=1}^m \left(\alpha_j \frac{dl_j}{l_j} + \omega'_j \right), \quad \nabla = d + \omega \wedge$$

and consider the following complex

$$0 \longrightarrow \Omega^0(*L) \xrightarrow{\nabla} \Omega^1(*L) \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} \Omega^k(*L) \longrightarrow 0.$$

We want to define the intersection pairing between $H^k(\Omega^\bullet(*L), \nabla)$ and $H^k(\Omega^\bullet(*L), \check{\nabla})$ as we did in non-confluent cases. However, we can easily see that

$$H^k(\mathcal{E}^\bullet, \nabla) \not\simeq H^k(\Omega^\bullet(*L), \nabla) \not\simeq H^k(\mathcal{E}_c^\bullet, \nabla)$$

in general. So we need to introduce a reasonable cohomology theory on which a perfect pairing can be naturally defined. We also want to have a suitable homology theory and Poincaré isomorphisms to get intersection numbers for homology groups. Up to now only two extreme cases are studied:

Case $k = 1$,

Case $T = \mathbb{C}^k$, i.e. ω admites poles only along the hyperplane at infinity.

§8. Confluent cases $k = 1$

8.1. Twisted de Rham cohomology groups.

A smooth function f defined in a neighborhood U of the point x is said to be rapidly decreasing at x if f satisfies

$$\frac{\partial^{p+q}}{\partial t^p \partial \bar{t}^q} f(x) = 0, \quad p, q = 0, 1, 2, \dots.$$

Let $\mathcal{S}^p$ be the vector space of smooth p-forms on $\mathbb{P}^1$ which are rapidly decreasing at $x_i (= L_i)$ for every i. A smooth function f defined in $U \setminus \{x\}$ is said to be polynomially growing at x if there exists $r \in \mathbb{N}$ such that $(t - x)^r f$ is smooth on U. Let $\mathcal{P}^p$ be the vector space of smooth p-forms f on T which are polynomially growing at x_i for every i.

We consider two complexes with differential ∇ :

$$\begin{aligned}(\mathcal{S}^\bullet,\nabla) &: \mathcal{S}^0 \xrightarrow{\nabla} \mathcal{S}^1 \xrightarrow{\nabla} \mathcal{S}^2 \xrightarrow{\nabla} 0,\\ (\mathcal{P}^\bullet,\nabla) &: \mathcal{P}^0 \xrightarrow{\nabla} \mathcal{P}^1 \xrightarrow{\nabla} \mathcal{P}^2 \xrightarrow{\nabla} 0.\end{aligned}$$

The cohomology groups $H^k(\mathcal{S}^\bullet,\nabla)$ and $H^k(\mathcal{P}^\bullet,\nabla)$ are called rapidly decreasing and polynomially growing twisted de Rham cohomology groups with respect to ∇, respectively. The inclusions

$$(\Omega^\bullet(*L),\nabla) \subset (\mathcal{P}^\bullet,\nabla), \quad (\mathcal{S}^\bullet,\nabla) \subset (\mathcal{P}^\bullet,\nabla)$$

of complexes induce the following isomorphisms among twisted de Rham cohomology groups.

Theorem 3. $H^p(\Omega^\bullet(*L),\nabla) \simeq H^p(\mathcal{P}^\bullet,\nabla) \simeq H^p(\mathcal{S}^\bullet,\nabla), \quad p = 0,1,2.$

The first isomorphism can be proved by the help of $\bar\partial$-calculus. Since the injectivity of the natural map $H^p(\mathcal{S}^\bullet,\nabla) \to H^p(\mathcal{P}^\bullet,\nabla)$ is easy, we mention briefly its surjectivity when $p = 1$. For a ∇-closed form $\varphi \in \Omega^1(*L)$, there exists a unique formal meromorphic Laurent series F_i around x_i satisfying $\nabla F_i = \varphi$. If $n_i \geq 2$, F_i is divergent in general, however, there exists a polynomially growing smooth function f_i with the same expansion as F_i. Thus the form

$$\varphi - \sum_{i=0}^{m} \nabla(h_i f_i)$$

is in $\mathcal{S}^1$, where h_i is a smooth function defined in §3.1. This implies the surjectivity.

8.2. Twisted homology groups. Let $\triangle$ be a singular p-simplex in T, define a function $u_\triangle$ on $\triangle$ by

$$u_\triangle(t) = \exp\left(\int^t \omega\right),$$

where the path of the integration is in $\triangle$. We consider only chains ρ such that if x_i belongs to the closure of $\rho = \sum_j b_j \triangle_j$ in $\mathbb{P}^1$ then

$$\lim_{t\to x_i,\ t\in\rho} (t-x_i)^r u_\rho(t) = 0, \quad r = 0,1,2,\ldots,$$

where $u_\rho(t) = u_{\triangle_j}(t)$ $(t \in \triangle_j)$. Let $C_p(T,\omega)$ be the space of loaded p-chains $\sum_j b_j \triangle_j \otimes u_{\triangle_j}$ for all such p-chains $\rho = \sum_j b_j \triangle_j$. The boundary operator ∂_ω on $C_\bullet(T,\omega)$ is naturally defined, and we get the p-th homology group $H_p(C_\bullet(T,\omega),\partial_\omega)$ as we did in §4. There is a natural pairing between $H^1(\mathcal{S}^\bullet,\nabla)$ and $H_1(C_\bullet(T,\omega),\partial_\omega)$ through the (confluent) hypergeometric integral

$$\langle\varphi,\gamma\rangle = \sum_j b_j \int_{\Delta_j} u_{\Delta_j}(t)\varphi,$$

where $\varphi \in \mathcal{S}^1, \gamma = \sum_j b_j\ \Delta_j \otimes u_{\Delta_j}(t) \in C_1(T,\omega)$.

Theorem 4. *The pairing between $H^1(\mathcal{S}^\bullet,\nabla)$ and $H_1(C_\bullet(T,\omega),\partial_\omega)$ is perfect.*

8.3. Intersection pairings.

There is a natural pairing between $\mathcal{S}^1$ and $\mathcal{P}^1$ by

$$\int_{\mathbb{P}^1} \varphi\wedge\psi, \quad \varphi\in\mathcal{S}^1,\ \psi\in\mathcal{P}^1.$$

This pairing descends to the perfect pairing $\cdot$ between $H^1(\mathcal{S}^\bullet,\nabla)$ and $H^1(\mathcal{P}^\bullet,\check{\nabla})$. Theorem 3 yields the isomorphism $\iota : H^1(\Omega^\bullet(*L),\nabla) \to H^1(\mathcal{S}^\bullet(*L),\nabla)$, which induces the intersection pairing of $H^1(\Omega^\bullet(*L),\nabla)$ and $H^1(\Omega^\bullet(*L),\check{\nabla})$ by

$$\varphi\cdot\psi = \int_{\mathbb{P}^1} \iota(\varphi)\wedge\psi.$$

Theorem 5. *The intersection number $\varphi\cdot\psi$ of $\varphi \in H^1(\Omega^\bullet(*L),\nabla)$ and $\psi\in H^1(\Omega^\bullet(*L),\check{\nabla})$ is given by*

$$\varphi\cdot\psi = 2\pi i \sum_{j=0}^{m} \mathrm{Res}_{t=x_j}(F_j\psi),$$

where F_j is the meromorphic formal Laurent series around x_j satisfying $\nabla F_j = \varphi$.

Note that we can evaluate the intersection number $\varphi\cdot\psi$ by this theorem; see examples in the next subsection.

So far in this section, we defined three pairings:

$$\begin{array}{ccc} H^1(\Omega^\bullet(*L),\nabla)\cong H^1(\mathcal{S}^\bullet,\nabla) & \longleftrightarrow & H^1(\mathcal{P}^\bullet,\check{\nabla})\cong H^1(\Omega^\bullet(*L),\check{\nabla}) \\ \updownarrow & & \updownarrow \\ H_1(C_\bullet(T,\omega),\partial_\omega) & & H_1(C_\bullet(T,-\omega),\partial_{-\omega}). \end{array}$$

These pairings define a pairing between the two homology groups.

Theorem 6. *Suppose for two loaded cycles*

$$\rho^+ = \sum_i b_i \, \Delta_i^+ \otimes u_{\Delta_i^+}(t) \in H_1(C_\bullet(T,\omega), \partial_\omega)$$

and

$$\rho^- = \sum_j b_j \, \Delta_j^- \otimes u_{\Delta_j^-}^{-1}(t) \in H_1(C_\bullet(T,-\omega), \partial_{-\omega}),$$

Δ_i^+ and Δ_j^- meet transversally at finitely many points. Then the intersection number $\rho^+ \cdot \rho^-$ is equal to

$$\rho^+ \cdot \rho^- = \sum_{i,j} \sum_{v \in \Delta_i^+ \cap \Delta_j^-} b_i b_j [u_{\Delta_i^+}(t)]_{t=v} [u_{\Delta_j^-}^{-1}(t)]_{t=v} I_v(\Delta_i^+, \Delta_j^-),$$

where $I_v(\Delta_i^+, \Delta_j^-)$ is the topological intersection number of Δ_i^+ and Δ_j^- at $v \in T$.

8.4. Examples. The compatibility of the parings yields quadratic relations among confluent hypergeometric functions.

Let $\omega = -tdt$, so $u(t) = e^{-t^2/2}$. The (co)homology groups in question are 1-dimensional. Put

$$\rho^+ = [-\infty, \infty] \otimes e^{-t^2/2}, \quad \rho^- = [-i\infty, i\infty] \otimes e^{t^2/2}.$$

Let us compute the intersection number $dt \cdot dt$ applying Theorem 5. Since the pole of ω is at ∞ only, we solve the equation $\nabla F = dt$ at ∞. By a straightforward calculation, we have

$$F = -s + s^3 - 2s^5 + 2 \cdot 4s^7 - 2 \cdot 4 \cdot 6s^9 + \cdots, \quad s = 1/t.$$

Since $\mathrm{Res}_{s=0}(F(s)(-ds/s^2)) = 1$, $dt \cdot dt$ equals $2\pi i$. One can easily see that Theorem 6 implies $\rho^+ \cdot \rho^- = 1$. Since

$$\langle dt, \gamma^+ \rangle = \int_{-\infty}^{+\infty} e^{-t^2/2} dt, \quad \langle dt, \gamma^- \rangle = \int_{-i\infty}^{+i\infty} e^{t^2/2} dt = i \int_{-\infty}^{+\infty} e^{-t^2/2} dt,$$

we have the formula announced in §1:

$$\left(\int_{-\infty}^{\infty} e^{-t^2/2} dt \right) \cdot 1 \cdot \left(i \int_{-\infty}^{\infty} e^{-t^2/2} dt \right) = 2\pi i.$$

We present two more examples: the inversion formula for the gamma function

$$\Gamma(\alpha)\Gamma(1-\alpha) = \frac{\pi}{\sin \pi\alpha}$$

and Lommel's formula

$$J_a(z)J_{-a+1}(z) + J_{a-1}(z)J_{-a}(z) = \frac{2\sin(\pi a)}{\pi z},$$

which holds for the Bessel function with parameter $a \in \mathbb{C} \setminus \mathbb{Z}$

$$J_a(z) = \left(\frac{z}{2}\right)^a \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(a+k+1)} \left(\frac{z}{2}\right)^k,$$

where $z \in \{z \in \mathbb{C} \mid \mathrm{Re}(z) > 0\}$ and the argument of z is in $(-\pi/2, \pi/2)$. For details including proofs, refer to [MMT].

[Ha2] shows that such quadratic relations are indeed obtained from these in §5 by confluence process.

§9. Confluent cases, generalized Airy $k \geq 2$

Let ω be an exact 1-form on $T = \mathbb{C}^k$, with parameters $\alpha_1, \ldots, \alpha_n$, defined as

$$\omega = d\theta_{n+1}(t) + \sum_{j=1}^{n} \alpha_j d\theta_j(t),$$

where θ_j are polynomials in $t = (t_1, \ldots, t_k)$ of degree j defined by

$$\log(1 + t_1 X + t_2 X^2 + \cdots + t_k X^k) = \sum_{j \geq 1} \theta_j(t) X^j;$$

for example, $\theta_1(t) = t_1$, $\theta_2(t) = t_2 - t_1^2/2$, $\theta_3(t) = t_3 - t_1 t_2 + t_1^3/3$. Note that the form ω has poles of order $n+2$ along the hyperplane L at infinity. Let $H^p(\Omega^\bullet, \nabla)$ be the p-th cohomology group of the complex

$$(\Omega^\bullet, \nabla) : 0 \longrightarrow \Omega^0 \xrightarrow{\nabla} \Omega^1 \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} \Omega^k \longrightarrow 0,$$

where Ω^p the vector space of polynomial p-forms. In [Kim2], it is shown that only $H^k(\Omega^\bullet, \nabla)$ survives and is $\binom{n}{k}$-dimensional, further it is conjectured that there exists a basis expressed in terms of Schur polynomials. This conjecture is established in [IM]. In order to state this, we consider the map

$$\phi : \mathbb{C}^k \ni s = (s_1, \ldots, s_k) \mapsto t = (t_1, \ldots, t_k) = (e_1(s), \ldots, e_k(s)) \in \mathbb{C}^k,$$

where $e_j(s)$ is the elementary symmetric polynomial of degree j.

Theorem 7. *$H^k(\Omega^\bullet, \nabla)$ can be spanned by*

$$\Theta_I = d\theta_{i_1} \wedge \cdots \wedge d\theta_{i_k}, \quad I = (i_1, \ldots, i_k),\ 1 \le i_1 < \cdots < i_k \le n.$$

The pull buck $\phi^*(\Theta_I)$ of Θ_I by ϕ is given by

$$\phi^*(\Theta_I) = Sc_\lambda(s)\Delta(s) ds_1 \wedge \cdots \wedge ds_k,$$

where $Sc_\lambda(s)$ is the Schur polynomial attached to the Young digram $\lambda = (i_k - k, \ldots, i_1 - 1)$ and $\Delta(s)$ is the difference product of $s_1, \ldots, s_k$.

Let us define the intersection pairing $H^k(\Omega^\bullet, \nabla)$ and $H^k(\Omega^\bullet, \check{\nabla})$. Note that the map ϕ induces the biholomorphic map from the quotient variety $(\mathbb{P}^1)^k / S_k$ to $\mathbb{P}^k$. We can easily see that

$$\phi^*(d\theta_i(t_1, \ldots, t_k)) = \sum_{j=1}^{k} d\theta_i(s_j, 0, \ldots, 0).$$

We regard $\phi^*(\omega)$ as a meromorphic 1-form on $(\mathbb{P}^1)^k / S_k$. We can deform $\phi^*(\Theta_I)$ into a S_k-invariant rapidly decreasing k-form $\iota(\phi^*(\Theta_I))$ on $\mathbb{C}^k$ by adding $dF + \phi^*(\omega) \wedge F$, where F is a S_k-invariant polynomially growing $(k-1)$-form on $\mathbb{C}^k$. Since $(\mathbb{P}^1)^k$ is the $k!$-fold covering of $(\mathbb{P}^1)^k / S_k$, we define the intersection number $\Theta_I \cdot \Theta_J$ for $\Theta_I \in H^k(\Omega^\bullet, \nabla)$ and $\Theta_J \in H^k(\Omega^\bullet, \check{\nabla})$ as

$$\langle \Theta_I, \Theta_J \rangle = \frac{1}{k!} \int_{\mathbb{C}^k} \iota(\phi^*(\Theta_I)) \wedge \phi^*(\Theta_J).$$

Theorem 8. *The intersection number $\langle \Theta_I, \Theta_J \rangle$ is equal to the skew Schur polynomial $Sc_{\lambda/\tilde{\mu}}(\alpha)$ with elementary symmetric polynomials as variables, where $\tilde{\mu}$ is the complement of the Young diagram $\mu = (j_k - k, \ldots, j_1 - 1)$ in the $k \times (n-k)$ rectangle, and $\lambda/\tilde{\mu}$ is the skew Young diagram of $\lambda = (i_k - k, \ldots, i_1 - 1)$ and $\tilde{\mu}$.*

The cohomology theory introduced in this section will be presented in full in [IM]. The homological counter part is still unsettled.

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Remarks on critical points of phase functions and norms of Bethe vectors

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Dedicated to Peter Orlik on his sixtieth birthday

Abstract.

We consider a tensor product of a Verma module and the basic linear representation of $sl(n+1)$. We prove that the corresponding phase function, which is used in the solutions of the KZ equation with values in the tensor product, has a unique critical point and show that the Hessian of the logarithm of the phase function at this critical point equals the Shapovalov norm of the corresponding Bethe vector in the tensor product.

§1. Introduction

Let $\mathfrak{g}$ be a simple Lie algebra with simple roots α_i and Chevalley generators e_i, f_i, h_i, $i = 1, \dots, n$. Let V_1, V_2 be representations of $\mathfrak{g}$ with highest weights λ_1, λ_2. The Knizhnik-Zamolodchikov (KZ) equation on a function u with values in $V_1 \otimes V_2$ has the form

$$\kappa \frac{\partial}{\partial z_1} u = \frac{\Omega}{z_1 - z_2} u, \qquad \kappa \frac{\partial}{\partial z_2} u = \frac{\Omega}{z_2 - z_1} u,$$

where $\Omega \in \mathrm{End}(V_1 \otimes V_2)$ is the Casimir operator. Solutions with values in the space of singular vectors of weight $\lambda_1 + \lambda_2 - \sum_{i=1}^n l_j \alpha_j$ are given by hypergeometric integrals with $l = \sum_{i=1}^n l_j$ integrations, see [SV].

For an ordered set of numbers $I = \{i_1, \dots, i_m\}$, $i_k \in \{1, \dots, n\}$, and a vector v in a representation of $\mathfrak{g}$, denote $f^I v = f_{i_1} \dots f_{i_m} v$. The hypergeometric solutions of the KZ equation have the form

$$u = \sum u_{I,J} f^I v_1 \otimes f^J v_2, \qquad u_{I,J} = \int_\gamma \tilde{\Phi} \tilde{\omega}_{I,J} dt_1 \wedge \cdots \wedge dt_l,$$

where v_1, v_2 are highest weight vectors of V_1, V_2; the summation is over all pairs of ordered sets I, J, such that their union $\{i_k, j_s\}$ contains a

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number i exactly l_i times, $i = 1, \ldots, n$; γ is a suitable cycle; $\tilde{\omega}_{I,J} = \tilde{\omega}_{I,J}(z_1, z_2, t_1, ..., t_l)$ are suitable rational functions, the function $\tilde{\Phi} = \tilde{\Phi}(z_1, z_2, t_1, ..., t_l)$, called the phase function, is given by

$$\tilde{\Phi} = (z_1 - z_2)^{(\lambda_1,\lambda_2)/\kappa} \prod_{j=1}^{l} (t_j - z_1)^{-(\lambda_1,\alpha_{t_j})/\kappa} (t_j - z_2)^{-(\lambda_2,\alpha_{t_j})/\kappa} \times \prod_{1 \le i < j \le l} (t_i - t_j)^{(\alpha_{t_i},\alpha_{t_j})/\kappa}.$$

Here (,) is the Killing form and α_{t_i} denotes the simple root assigned a the variable t_i by the following rule. The first l_1 variables $t_1, \ldots, t_{l_1}$ are assigned to the simple root α_1, the next l_2 variables $t_{l_1+1}, \ldots, t_{l_1+l_2}$ to the second simple root α_2, and so on.

Define the normalized phase function Φ by the formula

$$\Phi(\lambda_1, \lambda_2, \kappa) = \prod_{j=1}^{l} t_j^{-(\lambda_1,\alpha_{t_j})/\kappa} (1 - t_j)^{-(\lambda_2,\alpha_{t_j})/\kappa} \times \prod_{1 \le i < j \le l} (t_i - t_j)^{(\alpha_{t_i},\alpha_{t_j})/\kappa}. \tag{1}$$

We also substitute $z_1 = 0$, $z_2 = 1$ in the rational functions $\tilde{\omega}_{I,J}$ and denote the result $\omega_{I,J}$.

Conjecture 1. *If the space of singular vectors of weight $\lambda_1 + \lambda_2 - \sum_{i=1}^{n} l_j \alpha_j$ is one-dimensional, then there is a region Δ of the form $\Delta = \{t \in \mathbb{R}^l \mid 0 < t_{\sigma_l} < \cdots < t_{\sigma_1} < 1\}$ for some permutation σ, such that the integral $\int_\Delta \Phi dt$ can be computed explicitly. Moreover, up to a rational number independent on $\lambda_1, \lambda_2, \kappa$, it is equal to an alternating product of Euler Γ-functions whose arguments are linear functions of weights λ_1, λ_2.*

Example. *The Selberg integral.* Let $\mathfrak{g} = sl(2)$. Let V_1 and V_2 be $sl(2)$ modules with highest weights $\lambda_1, \lambda_2 \in \mathbb{C}$. Then the normalized phase function (1) has the form

$$\Phi(\lambda_1, \lambda_2, \kappa) = \prod_{j=1}^{l} t_j^{-\lambda_1/\kappa} (1 - t_j)^{-\lambda_2/\kappa} \prod_{1 \le i < j \le l} (t_i - t_j)^{2/\kappa}. \tag{2}$$

Conjecture 1 holds for $\mathfrak{g} = sl(2)$ according to the Selberg formula

$$l! \int_\Delta \Phi(\lambda_1, \lambda_2, \kappa) dt_1 \dots dt_l$$
$$= \prod_{j=0}^{l-1} \frac{\Gamma((-\lambda_1 + j)/\kappa + 1)\Gamma((-\lambda_2 + j)/\kappa + 1)\Gamma((j+1)/\kappa + 1)}{\Gamma((-\lambda_1 - \lambda_2 + (2l - j - 2))/\kappa + 2)\Gamma(1/\kappa + 1)},$$

where $\Delta = \{t \in \mathbb{R}^l \mid 0 < t_1 < \dots < t_l < 1\}$. □

Using the phase function Φ and the rational functions $\omega_{I,J}$, one can construct singular vectors in $V_1 \otimes V_2$. Namely, if t^0 is a critical point of the function Φ, then the vector $\sum \omega_{I,J}(t^0) f^I v_1 \otimes f^J v_2$ is singular, see [RV]. The equation for critical points, $d\Phi = 0$, is called the *Bethe equation* and the corresponding singular vectors are called the *Bethe vectors.*

Conjecture 2. *If the space of singular vectors of a given weight in $V_1 \otimes V_2$ is one-dimensional, then the corresponding phase function has exactly one critical point modulo permutations of variables t_i assigned to the same simple root.*

Example. The conjecture holds for $\mathfrak{g} = sl(2)$. If $(t_1, \dots, t_l)$ is a critical point of the function $\Phi(\lambda_1, \lambda_2, \kappa)$ given by (2), then

$$\sigma_k(t) = \binom{l}{k} \prod_{j=1}^{k} \frac{\lambda_1 - l + j}{\lambda_1 + \lambda_2 - 2l + j + 1},$$

where $\sigma_1(t) = \sum t_j$, $\sigma_2(t) = \sum t_i t_j$, etc, are the standard symmetric functions, see [V], so there is a unique critical point up to permutations of coordinates. □

The rational functions $\omega_{I,J}(t)$ are invariant with respect to permutation of variables assigned to the same simple root. Thus, Conjecture 2 implies that there is a unique Bethe vector X.

The space $V_1 \otimes V_2$ has a natural bilinear form B, called the Shapovalov form, which is the tensor product of Shapovalov forms of factors.

Conjecture 3. *The length of a Bethe vector X equals the Hessian of the logarithm of the phase function Φ with $\kappa = 1$ at a critical point t^0,*

$$B(X, X) = det\left(\frac{\partial^2}{\partial t_i \partial t_j} \ln \Phi(t^0)\right).$$

Example. The conjecture holds for $\mathfrak{g} = sl(2)$, see [V]. $\square$

In this paper we prove Conjectures 1, 2 and 3 for the case when $\mathfrak{g} = sl(n+1)$, V_1 is a Verma module and V_2 is the basic linear representation.

§2. The integral

Let

$$\bar{\Phi}_n(\alpha,\beta) = t_1^{\alpha_1}(1-t_1)^{\beta_1}\prod_{j=2}^{n} t_j^{\alpha_j}(t_j-t_{j-1})^{\beta_j}. \tag{3}$$

Theorem 1. *Let $\alpha_i > 0$, $\beta_i > 0$, $i = 1,\dots n$. Then*

$$\int_{\Delta_n}\bar{\Phi}_n(\alpha,\beta)dt_1\dots dt_n$$
$$= \prod_{j=1}^{n}\frac{\Gamma(\beta_j+1)\Gamma(\alpha_j+\dots+\alpha_n+\beta_{j+1}+\dots+\beta_n+n-j+1)}{\Gamma(\alpha_j+\dots+\alpha_n+\beta_j+\dots+\beta_n+n-j+2)},$$

where $\Delta_n = \{t\in\mathbb{R}^n \mid 0 < t_n < \dots < t_1 < 1\}$.

Proof. The formula is clearly true for $n = 1$.

Fix $t_1,\dots,t_{n-1}$ and integrate with respect to t_n. We obtain the recurrent relation

$$\int_{\Delta_n}\bar{\Phi}_n(\alpha,\beta)dt_1\dots dt_n = \frac{\Gamma(\alpha_n+1)\Gamma(\beta_n+1)}{\Gamma(\alpha_n+\beta_n+2)}\times$$
$$\times\int_{\Delta_{n-1}}\bar{\Phi}_{n-1}(\alpha_1,\dots,\alpha_{n-1},\beta_1,\dots,\beta_{n-2},\beta_{n-1}+\beta_n+\alpha_n+1)dt_1\dots dt_{n-1},$$

which implies the Theorem. Q.E.D.

§3. The critical point

Let $\mathfrak{g} = sl(n+1)$. Let V_1 be a Verma module of highest weight λ, $(\lambda,\alpha_i) = \lambda_i$. Let V_2 be the basic linear representation, that is the irreducible representation with highest weight ω, $(\omega,\alpha_i) = \delta_{i,1}$.

The nontrivial subspaces of singular vectors of a given weight in the tensor product $V_1\otimes V_2$ are one dimensional and have weights $\lambda+\omega-\sum_{i=1}^{k}\alpha_i$, $k = 0,\dots,n$. The computations for weights $\lambda+\omega-\sum_{i=1}^{k}\alpha_i$, $k < n$, are reduced to the case $\mathfrak{g} = sl(k+1)$. Consider the normalized phase function $\Phi_n(\lambda,\kappa)$ corresponding to the weight $\lambda+\omega-\sum_{i=1}^{n}\alpha_i$.

We have $\Phi_n(\lambda,\kappa) = \Phi(\lambda,\omega,\kappa)$, where $\Phi(\lambda,\omega,\kappa)$ is given by (1). Note that

$$\Phi_n(\lambda,\kappa) = \bar{\Phi}_n(-\lambda_1/\kappa,\dots,-\lambda_n/\kappa,-1/\kappa,\dots,-1/\kappa),$$

where $\bar{\Phi}_n$ is given by (3).

Theorem 2. *The function $\Phi_n(\lambda,\kappa)$ has exactly one critical point $t^n = (t_1^n,\dots,t_n^n)$ given by*

$$t_j^n(\lambda_1,\dots,\lambda_n) = \prod_{i=1}^{j} \frac{\lambda_i+\dots+\lambda_n+n-i}{\lambda_i+\dots+\lambda_n+n-i+1}.$$

Proof. The computation is obvious if $n=1$.
The equation $\partial\Phi_n/\partial t_n = 0$ has the form

$$t_n^n = \frac{\lambda_n}{\lambda_n+1} t_{n-1}^n.$$

Substituting for t_n^n in the equations $\partial\Phi_n/\partial t_i = 0$, $i=1,\dots,n-1$ and comparing the result with the equation $d\Phi_{n-1}=0$, we obtain

$$t_k^n(\lambda_1,\dots,\lambda_n) = t_k^{n-1}(\lambda_1\dots,\lambda_{n-2},\lambda_{n-1}+\lambda_n+1), \qquad k=1,\dots,n-1.$$

This recurrent relation implies the Theorem. Q.E.D.

§4. The norm of the Bethe vector

Let V be a $\mathfrak{g}$ module with highest weight vector v. The Shapovalov form $B(\ ,\ ) : V\otimes V \to \mathbb{C}$ is the unique symmetric bilinear form with the properties

$$B(e_i x, y) = B(x, f_i y), \qquad B(v,v) = 1,$$

for any $x,y\in V$. The Shapovalov form on a tensor product of modules is the tensor product of Shapovalov forms of factors.

Let $\mathfrak{g} = sl(n+1)$. Let $V_1 = V_\lambda$ be a Verma module of highest weight λ. Let $V_2 = V_\omega$ be the basic linear representation. Then the space of singular vecors in $V_\lambda\otimes V_\omega$ of weight $\lambda+\omega-\sum_{i=1}^n \alpha_i$ is one-dimensional and is spanned by the Bethe vector $X^n(\lambda)$ corresponding to the critical point of the function $\Phi_n(\lambda,\kappa)$. The Bethe vector has the form

$$X^n(\lambda) = x_0^n\otimes f_n\dots f_1 v_0 + x_1^n\otimes f_{n-1}\dots f_1 v_0 + \dots + x_n^n \otimes v_0,$$

where $x_i^n \in V_\lambda$ and v_0 is the highest weight vector in V_ω. Here, $x_0^n = a^n v_\lambda$, where v_λ is the highest weight vector in V_λ and a^n is the value of the corresponding rational function

$$\omega_{\emptyset,(n,n-1,\ldots,1)}(t) = \frac{1}{t_1 - 1} \prod_{i=1}^{n-1} \frac{1}{t_{i+1} - t_i}$$

at the critical point t^n of function $\Phi_n(\lambda, \kappa)$, given by Theorem 2. For a description of all other rational functions whose values at t^n determine $x_1^n, \ldots, x_n^n$, see [SV]. We have

$$a^n = (-1)^n \prod_{k=1}^{n} \frac{(\lambda_k + \cdots + \lambda_n + n - k + 1)^{n-k+1}}{(\lambda_k + \cdots + \lambda_n + n - k)^{n-k}}.$$

Theorem 3.

$$(4) B(X^n(\lambda), X^n(\lambda)) = \prod_{k=1}^{n} \frac{(\lambda_k + \cdots + \lambda_n + n - k + 1)^{2(n-k)+3}}{(\lambda_k + \cdots + \lambda_n + n - k)^{2(n-k)+1}}.$$

Proof. We also claim

$$(5) \qquad B(x_n^n, x_n^n) = \frac{B(X^n(\lambda), X^n(\lambda))}{\lambda_1 + \cdots + \lambda_n + n}.$$

Formulas (4), (5) are readily checked for $n = 1$.

The vectors $\{v_0, f_1 v_0, f_2 f_1 v_0, \ldots, f_n \cdots f_1 v_0\}$ form an orthonormal basis of V_ω with respect to its Shapovalov form. Clearly, we have

$$B(X^n(\lambda), X^n(\lambda)) = \left(\frac{a^n(\lambda)}{a^{n-1}(\lambda')}\right)^2 B(X^{n-1}(\lambda'), X^{n-1}(\lambda')) + B(x_n^n, x_n^n),$$

where λ' is the $sl(n)$ weight, such that $(\lambda', \alpha_i) = \lambda_{i+1}$, $i = 1, \ldots, n-1$.

The vector X^n is singular. In particular it means that $e_i x_n^n = 0$ for $i > 1$ and $e_1 x_n^n = -x_{n-1}^n$. The vector x_n^n has the form $x_n^n = \sum_\sigma b_\sigma^n f_{\sigma(1)} \cdots f_{\sigma(n)} v_\lambda^n$, where the coefficients b_σ^n are the values of the corresponding rational functions at the critical point given by Theorem 2.

Let $b^n = b_{\sigma=\mathrm{id}}^n$. Then we have

$$B(x_n^n, x_n^n) = B(x_n^n, b^n f_1 \ldots f_n v_\lambda^n) = -b^n B(x_{n-1}^n, f_2 \ldots f_n v_\lambda^n) =$$
$$= -b^n \frac{a_n}{a_{n-1}} B(x_{n-1}^{n-1}, f_1, \ldots f_{n-1} v_{\lambda'}^{n-1}) = -\frac{b^n}{b^{n-1}} \frac{a_n}{a_{n-1}} B(x_{n-1}^{n-1}, x_{n-1}^{n-1}),$$

where x_{n-1}^{n-1} is a component of the singular vector in $V_{\lambda'} \otimes V_\omega$.

The coefficient b^n is the value of the function

$$\omega_{(n,n-1,\dots,1),\emptyset}(t) = \frac{1}{t_n}\prod_{i=1}^{n-1}\frac{1}{t_i - t_{i+1}}$$

at the critical point t^n, given by Theorem 2. We have

$$b^n = (-1)^{n-1}\frac{a_n}{\lambda_1 + \cdots + \lambda_n + n}\prod_{k=1}^{n}\frac{\lambda_k + \cdots + \lambda_n + n - k + 1}{\lambda_k + \cdots + \lambda_n + n - k}$$

Now, formulas (4), (5) are proved by induction on n. Q.E.D.

Theorem 4.

$$B(X^n(\lambda), X^n(\lambda)) = det\left(\frac{\partial^2}{\partial t_i \partial t_j}\ln\Phi_n(\lambda, \kappa = 1)(t^n)\right),$$

where t^n is the critical point of the phase function $\Phi_n(\lambda,\kappa)$ given by Theorem 2.

Proof. It is sufficient to prove the Theorem for $\lambda_i > 0, \kappa < 0$. We tend κ to zero and compute the asymptotics of the integral $\int_{\Delta_n}\Phi_n dt$.

On the one hand, the integral is evaluated by Theorem 1. We compute the asymptotics using the Stirling formula for Γ-functions.

On the other hand, the asymptotics of the same integral can be computed by the method of stationary phase, since the critical point t^n of the function Φ_n is non-degenerate by Theorem 1.2.1 in [V]. Then the asymptotics of the integral is

$$(2\pi\kappa)^{l/2}\Phi_n(\lambda,\kappa)(t^n)\left(\mathrm{Hess}(\kappa\ln\Phi_n(\lambda,\kappa)(t^n)\right)^{-1/2}.$$

Note that $\kappa\ln\Phi_n(\lambda,\kappa) = \ln\Phi_n(\lambda,1)$, and

$$\Phi_n(\lambda,\kappa)(t^n) = \prod_{k=1}^{n}\frac{(\lambda_k + \cdots + \lambda_n + n - k + 1)^{(\lambda_k+\cdots+\lambda_n+n-k+1)/\kappa}}{(\lambda_k + \cdots + \lambda_n + n - k)^{(\lambda_k+\cdots+\lambda_n+n-k)/\kappa}}.$$

Comparing the results we compute the Hessian explicitly and prove the Theorem. Q.E.D.

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Local system homology of arrangement complements

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Abstract.

We use the critical points of a multivalued holomorphic function and Morse theory to find a basis for a local system homology group defined on the complement of an arrangement of hyperplanes. This generalizes results of Kohno [4] and Douai-Terao [2] from complexified real arrangements to all arrangements. We also show that the set of critical points satisfies the same recursion with respect to deletion and restriction as the βnbc *set of the arrangement.*

§1. Introduction

A finite set of hyperplanes in $\mathbb{C}^\ell$ is called an affine arrangement, $\mathcal{A}$. Let

$$N(\mathcal{A}) = \cup_{H\in\mathcal{A}} H, \quad M(\mathcal{A}) = \mathbb{C}^\ell - N(\mathcal{A})$$

be the divisor and the complement of $\mathcal{A}$. We assume that $\mathcal{A}$ is essential: $\mathcal{A}$ contains ℓ linearly independent hyperplanes. For each $H \in \mathcal{A}$ choose an exponent $\lambda_H \in \mathbb{C}$ and let $\lambda = \{\lambda_H \mid H \in \mathcal{A}\}$. Let γ_H be the standard generator of $\pi_1(M)$ linking H. Define a rank one local system $\mathcal{L}_\lambda$ on M by $\gamma_H \mapsto \exp(-2\pi i \lambda_H)$ and let $\mathcal{L}_\lambda^\vee$ denote its dual. Let $z_1, \ldots, z_\ell$ be coordinates in $\mathbb{C}^\ell$. For each hyperplane $H \in \mathcal{A}$, choose a polynomial of degree one, α_H, with $H = \ker \alpha_H$. Call $Q(\mathcal{A}) = \prod_{H\in\mathcal{A}} \alpha_H$ a defining polynomial for $\mathcal{A}$. Define a multivalued holomorphic function on M by

$$\Phi_\lambda = \prod_{H\in\mathcal{A}} \alpha_H^{\lambda_H}.$$

The hypergeometric pairing

$$H_\ell(M;\mathcal{L}_\lambda^\vee) \times H^\ell(M;\mathcal{L}_\lambda) \longrightarrow \mathbb{C}$$

[1,2] Partially supported by NSF

is defined by $(\delta, \phi) \mapsto \int_\delta \Phi_\lambda \phi$. Under suitable nonresonance conditions on the exponents λ, the groups in the hypergeometric pairing have rank $\beta = (-1)^\ell \chi(M)$, where χ denotes Euler characteristic. Falk and Terao [3] constructed a basis for $H^\ell(M; \mathcal{L}_\lambda)$, but a basis for $H_\ell(M; \mathcal{L}_\lambda^\vee)$ is known only for complexified real arrangements [4], [2], defined in terms of the bounded chambers of the underlying real arrangement. Our main objective is to use Morse theory to construct a geometric homology basis for all arrangements.

Deletion and restriction is a basic tool for various induction arguments concerning arrangements [5]. Given a linear order on the hyperplanes of $\mathcal{A}$, Ziegler [8] associated with it a combinatorially defined set of cardinality β, called βnbc, and proved that βnbc satisfies a simple recursion with respect to deletion and restriction. We call a basis for either group in the hypergeometric pairing a βnbc-basis if its elements are labeled by βnbc and they satisfy the βnbc recursion of Theorem 4.2. The cohomology basis constructed by Falk and Terao [3] is a βnbc-basis. The homology basis constructed by Douai and Terao [2] for complexified real arrangements is also a βnbc-basis. We show that our homology basis is also a βnbc-basis.

§2. A Morse theoretic argument

A point $p \in M$ is a critical point of Φ_λ if and only if $d(\log \Phi_\lambda)(p) = 0$. Properties of this critical set were established in [6], [7]:

Proposition 2.1. *For a dense open set of exponents λ, every critical point of Φ_λ in M is nondegenerate and has index ℓ. The number of these critical points is $\beta = (-1)^\ell \chi(M)$. Denote a critical set satisfying these conditions*

$$C(\mathcal{A}, \lambda) = \{p \in M \mid d(\log \Phi_\lambda)(p) = 0\}. \quad Q.E.D.$$

We would like to apply a Morse theoretic argument analogous to [7, Sect. 5]. This argument relies on separating the divisors of zeroes and of poles of a certain generalized meromorphic function whose orders along its divisors generally are complex numbers, the exponents in our case. However, this procedure is only possible when the exponents are rational numbers or rational multiples of the same real number. Our strategy is to prove first the required homotopy result working with strictly positive rational exponents and subsequently extend its validity by perturbing the relevant Morse function slightly around the rational point. Since any open set in the space of exponents contains a rational point, the result will thus have been proven for all exponents with strictly positive real part and small imaginary part.

Consider $\mathbb{C}^\ell \subset \mathbb{P}^\ell$ as the complement of the infinite hyperplane, P, defined by $z_0 = 0$. Assign it the exponent $\lambda_P = -\sum_{H\in\mathcal{A}} \lambda_H$. Let $\hat{H}$ denote the projective closure of H and define $\mathcal{A}_\infty = \{\hat{H} \mid H \in \mathcal{A}\} \cup \{P\}$. Then $A = \hat{N} \cup P$ is the divisor of the projective arrangement $\mathcal{A}_\infty$ and we have $\mathbb{P}^\ell - A = M = \mathbb{C}^\ell - N$. Let $\hat{\alpha}_H$ be the homogenized α_H and define

$$\hat{\Phi}_\lambda = z_0^{\lambda_P} \prod (\hat{\alpha}_H)^{\lambda_H}.$$

Assume that the exponents associated with the hyperplanes are rational and strictly positive, $\lambda_H \in \mathbb{Q}_{>0}$ for all $H \in \mathcal{A}$. The exponent λ_P at infinity is thus rational and strictly negative. There is an integer $m > 0$ so that $m\lambda_H$ is an integer for all $H \in \mathcal{A}$. The function $(\hat{\Phi}_\lambda)^m$ is a meromorphic function on $\mathbb{P}^\ell$. Its set of critical points in $\mathbb{P}^\ell - A = M$ is $C(\mathcal{A}, \lambda)$ and the same holds for $|\hat{\Phi}_\lambda^m|^2$. We apply exactly the same blow–up procedure as in [7]. We form a birational map $\sigma : X \to \mathbb{P}^\ell$ by recursively blowing up points on A until $\sigma^{-1}(A) = D$ has normal crossings. Then $D = D_0 \cup D_\infty \cup D'$ where the divisor D_0 of zeroes of the meromorphic function $\sigma^*\hat{\Phi}_\lambda^m$ is disjoint from the divisor D_∞ of poles and D' consists of exceptional divisors connecting D_0 with D_∞ along whose generic points $\sigma^*\hat{\Phi}_\lambda^m$ neither vanishes nor diverges. Note that $\sigma^{-1}(N) \subset D_0$ and that in fact there are generally more irreducible components in D_0 than in $\sigma^{-1}(N)$.

Let T_0 and T_∞ respectively be sufficiently small open regular neighborhoods of D_0 and D_∞ in X and let $\overline{T}_0$, $\overline{T}_\infty$ be their closures. By the same argument as that leading to Theorem 5.3 of [7] , the C^∞ extension F of $\sigma^*|\hat{\Phi}_\lambda^m|^2$ to $X - D_\infty$ defines a Morse function on the compact manifold $X - (T_0 \cup T_\infty)$ with boundary $\partial\overline{T}_0 \cup \partial\overline{T}_\infty$ none of whose critical points lies on $(X - (T_0 \cup T_\infty)) \cap D'$. There is a homotopy equivalence

$$X - (T_0 \cup T_\infty) \cong \partial\overline{T}_0 \cup e_1 \cup \cdots \cup e_\beta,$$

with the following properties:

(1) Each e_i is an ℓ–cell constructed in terms of the level sets of F and it contains exactly one critical point of F so the number of cells is β.

(2) The right hand side is in fact a deformation retract of $X - (T_0 \cup T_\infty)$, hence of $X - (D_0 \cup D_\infty)$ as well.

The next step consists of deleting D', or actually an arbitrarily small neighborhood T' of D'. It is fairly obvious that deletion of T' from

$\partial\overline{T}_0 \cup e_1 \cup \cdots \cup e_\beta$ yields a deformation retract of $X - (T_0 \cup T_\infty \cup T')$. Note that the critical points of F are not on D'. In view of the manner the cells e_i are constructed, one may choose the e_i so that $e_i \cap T' = \emptyset$ for all $i = 1, \dots, \beta$. This granted, we have the homotopy equivalence

$$X - (T_0 \cup T_\infty \cup T') \cong (\partial\overline{T}_0 - \partial\overline{T}_0 \cap T') \cup e_1 \cup \cdots \cup e_\beta.$$

Note that $T = \sigma(T_0 \cup T_\infty \cup T')$ is an open tubular neighborhood of A. Since σ is an isomorphism outside D, the left hand side is isomorphic to the complement $\mathbb{P}^\ell - T$. It follows that its affine part, $T_a = T \cap \mathbb{C}^\ell$, is an open tubular neighborhood of N. Let $\overline{T}_a$ be its closure in $\mathbb{C}^\ell$. Then $\partial\overline{T}_0 - \partial\overline{T}_0 \cap T' \cong \partial\overline{T}_a$.

Theorem 2.2. *Let $\mathcal{A}$ be an essential ℓ-arrangement. Suppose the exponents λ_H have strictly positive real parts and small imaginary parts. Then there is a homotopy equivalence*

$$\mathbb{C}^\ell - T_a = \mathbb{P}^\ell - T \cong \partial\overline{T}_a \cup \bigcup_{p \in C(\mathcal{A},\lambda)} e_p,$$

where e_p is an ℓ-cell attached to $\partial\overline{T}_a$ and "centered" at the critical point $p \in C(\mathcal{A}, \lambda)$.

Proof. We proved the result for any rational point λ in the construction preceding the theorem. It remains to prove it for nearby points λ'. The location of the critical points varies analytically with λ'. If λ' is close to λ, then the cell attached to a critical point p' of $\Phi_{\lambda'}$ is at most a slight deformation of the cell centered at the nearby critical point p of Φ_λ. Q.E.D.

§3. Locally finite homology

Let $\mathcal{L}$ be any rank one complex local system on $\mathbb{P}^\ell - A = M = \mathbb{C}^\ell - N$. We consider the locally finite homology $H_*^{lf}(M; \mathcal{L})$ of M with coefficients in $\mathcal{L}$, defined as the dual to the compactly supported cohomology of $\mathbb{P}^\ell - A$ with coefficients in the dual local system $\mathcal{L}^\vee$:

$$H_i^{lf}(M; \mathcal{L}) = H_c^i(M; \mathcal{L}^\vee)^*.$$

It may be thought of as the homology of the complex of chains of the form $\sum_\sigma a_\sigma \sigma$. These are possibly infinite linear combinations of oriented simplices σ of a given triangulation of M whose coefficients are local sections $a_\sigma \in \mathcal{L}(\sigma) = H^0(\sigma, \mathcal{L})$.

In order to compute the locally finite homology we first identify it with a relative homology group in Proposition 3.1. Its proof is related to ideas in [1, pp. 10–12]. The Morse theoretic result of Theorem 2.2 on the homotopy of M permits explicit determination which part of the homology comes from the vicinity of A and which part from the attached ℓ–cells.

Proposition 3.1. *Let T be a small open neighborhood of A in $\mathbb{P}^\ell$, let $\overline{T}$ denote its closure, and let $\partial\overline{T}$ denote the boundary of $\overline{T}$. There is an isomorphism*

$$\imath_*\alpha : H_i\big(\mathbb{P}^\ell - T,\ \partial(\mathbb{P}^\ell - T); \mathcal{L}\big) \to H_i^{lf}(\mathbb{P}^\ell - A; \mathcal{L}).$$

Proof. Consider the pair $\big(\mathbb{P}^\ell - T, \partial(\mathbb{P}^\ell - T)\big)$ consisting of the (oriented) topological manifold $\mathbb{P}^\ell - T$ together with its boundary. Notice that $\partial(\mathbb{P}^\ell - T)$ is $\partial\overline{T}$ up to orientation and that removing the boundary from $\mathbb{P}^\ell - T$ gives the open topological manifold $\mathbb{P}^\ell - T \cup \partial(\mathbb{P}^\ell - T) = \mathbb{P}^\ell - \overline{T}$. The Lefschetz duality isomorphism $H_i\big(\mathbb{P}^\ell - T,\ \partial(\mathbb{P}^\ell - T); G\big)^* \cong H^{2\ell-i}(\mathbb{P}^\ell - \overline{T}; G)$ is well known for any constant abelian group of coefficients G. It generalizes to local coefficients in the form

$$H_i\big(\mathbb{P}^\ell - T, \partial(\mathbb{P}^\ell - T); \mathcal{L}\big) \cong H^{2\ell-i}(\mathbb{P}^\ell - \overline{T}; \mathcal{L})^*.$$

Combined with Poincaré duality,

$$\begin{array}{ccc} H^{2\ell-i}(\mathbb{P}^\ell - \overline{T}; \mathcal{L}^\vee) & \xrightarrow{\sim} & H_c^i(\mathbb{P}^\ell - \overline{T}; \mathcal{L}^\vee) \\ & & \| \\ & & H_i^{lf}(\mathbb{P}^\ell - \overline{T}; \mathcal{L})^* \end{array}$$

we obtain an isomorphism

$$\alpha : H_i\big(\mathbb{P}^\ell - T,\ \partial(\mathbb{P}^\ell - T); \mathcal{L}\big) \xrightarrow{\sim} H_i^{lf}(\mathbb{P}^\ell - \overline{T}; \mathcal{L}).$$

The inclusion map $\imath : \mathbb{P}^\ell - \overline{T} \hookrightarrow \mathbb{P}^\ell - A$ is a homotopy equivalence. It induces the isomorphism

$$\imath_* : H_i^{lf}(\mathbb{P}^\ell - \overline{T}; \mathcal{L}) \xrightarrow{\sim} H_i^{lf}(\mathbb{P}^\ell - A; \mathcal{L}).$$

Thus we have

$$\begin{array}{ccc} \imath^* : H_c^i(\mathbb{P}^\ell - A; \mathcal{L}^\vee) & \xrightarrow{\sim} & H_c^i(\mathbb{P}^\ell - \overline{T}; \mathcal{L}^\vee) \\ \| & & \| \\ H_i^{lf}(\mathbb{P}^\ell - A; \mathcal{L})^* & & H_i^{lf}(\mathbb{P}^\ell - \overline{T}; \mathcal{L})^*, \end{array}$$

and the conclusion follows. Q.E.D.

The isomorphism in Proposition 3.1 can be described concretely as follows. A class in $H_i(\mathbb{P}^\ell - T,\ \partial(\mathbb{P}^\ell - T); \mathcal{L})$ is represented by a relative cycle of the form $c = c' + c''$, where $c' \in C_i(\mathbb{P}^\ell - T; \mathcal{L})$ with boundary $\partial c' \in C_{i-1}(\partial\overline{T}; \mathcal{L})$ and $c'' \in C_i(\partial\overline{T}; \mathcal{L})$. Then $\alpha(c) = c' - \partial c'$ is the interior of c'. Finally, $\imath_*\alpha(c) = \imath_*(c' - \partial c')$ is obtained by retracting $\mathbb{P}^\ell - \overline{T}$ to $\mathbb{P}^\ell - A$ by letting $\partial\overline{T}$ "collapse" onto A.

Definition 3.2. *Let $\delta_p = \imath_*\alpha(e_p)$ be the locally finite cycle obtained from the interior of the cell e_p of Theorem 2.2 by letting $\partial\overline{T}$ "collapse" onto A and define*

$$\Delta(\mathcal{A}, \lambda) = \{\delta_p \mid p \in C(\mathcal{A}, \lambda)\}.$$

Lemma 3.3. *Let $\mathcal{L}$ be a rank one complex local system on M. Then $\dim H_\ell^{lf}(M; \mathcal{L}) \geq \beta$ and $\Delta(\mathcal{A}, \lambda)$ provides β linearly independent cycles.*

Proof. The boundaries of the the ℓ–cells e_p of Theorem 2.2. lie on $\partial\overline{T}$. Since e_p is simply connected, the sections of any local system $\mathcal{L}$ on $\mathbb{P}^\ell - A$ over the cell e_p are $H^0(e_i, \mathcal{L}) = \mathbb{C}$, the constant functions on e_p. It follows that any linear combination $\sum a_p e_p$ with constant coefficients $a_p \in \mathbb{C} = H^0(e_p, \mathcal{L})$ is a relative cycle in $H_\ell(\mathbb{P}^\ell - T,\ \partial(\mathbb{P}^\ell - T); \mathcal{L})$. Since the e_p are mutually disjoint, the space of such cycles has dimension β. Next we argue that the cells e_p are homologically independent. Recall from Theorem 2.2 that the space $\mathbb{P}^\ell - T$ is homotopically retractible to $\partial\overline{T}_a \cup_{p \in C(\mathcal{A},\lambda)} e_p$. Thus every $(\ell+1)$–chain in $C_*(\mathbb{P}^\ell - T)$ may be retracted to lie within $\partial\overline{T}$. Hence the boundaries $\partial C_{\ell+1}(\mathbb{P}^\ell - T,\ \partial(\mathbb{P}^\ell - T); \mathcal{L})$ form a subspace of $C_\ell(\partial(\mathbb{P}^\ell - T); \mathcal{L})$.

Here we should note that there does not have to be any relationship between the local system $\mathcal{L}$ and the function Φ_λ in terms of which the cells e_p are defined. Q.E.D.

Theorem 3.4. *If $\mathcal{L}$ is generic, then $\Delta(\mathcal{A}, \lambda)$ is a basis for $H_\ell^{lf}(M; \mathcal{L}) = H_\ell(M; \mathcal{L})$. Thus $\Delta(\mathcal{A}, \lambda)$ is a basis for $H_\ell(M; \mathcal{L}_\lambda^\vee)$ for generic λ.*

Proof. It is known [4] that for a sufficiently generic local system $\mathcal{L}$, $H_i(M; \mathcal{L}) = 0 = H_{2\ell-i}^{lf}(M; \mathcal{L})$ for $i \neq \ell$. In this case the dimension of the only nonvanishing homology group is $\dim H_\ell^{lf}(M; \mathcal{L}) = \beta$. Q.E.D.

§4. The βnbc homology basis

We review first some combinatorial constructions for arrangements. A set $S \subseteq \mathcal{A}$ is dependent if $\bigcap_{H \in S} H \neq \emptyset$ and $\operatorname{codim}(\cap_{H \in S} H) < |S|$.

A subset of $\mathcal{A}$ which has nonempty intersection and is not dependent is called independent. Maximal independent sets are called frames. Since $\mathcal{A}$ is essential, every frame has cardinality ℓ and intersection a point. Introduce a linear order in the arrangement by writing $\mathcal{A} = \{H_1, \dots, H_n\}$ and setting $H_i < H_j$ if $i < j$. An inclusion-minimal dependent set is called a circuit. A broken circuit is a subset S of $\mathcal{A}$ for which there exists $H < \min(S)$ such that $H \cup S$ is a circuit. The non-broken circuit complex of $\mathcal{A}$ is the collection of subsets of $\mathcal{A}$ which have nonempty intersection and contain no broken circuits. Maximal sets of this complex are frames of $\mathcal{A}$ called nbc-frames. We call an ordered frame $(H_{i_1}, \dots, H_{i_\ell})$ standard if $i_1 < \dots < i_\ell$.

Definition 4.1. *An nbc-frame B is called a* βnbc-frame *if for every $H \in B$ there exists $H' < H$ in $\mathcal{A}$ such that $(B - \{H\}) \cup \{H'\}$ is a frame. The set of standard ordered* βnbc*-frames of $\mathcal{A}$ is denoted* $\beta\mathrm{nbc}(\mathcal{A})$.

The notation and terminology reflect the fact that $|\beta\mathrm{nbc}(\mathcal{A})| = \beta$. The set $\beta\mathrm{nbc}(\mathcal{A})$ satisfies a recursion with respect to deletion and restriction [8, Thm. 1.5]. Given $H \in \mathcal{A}$, let $\mathcal{A}' = \mathcal{A} - \{H\}$ be the arrangement with H deleted and let $\mathcal{A}'' = \{H \cap K \mid K \in \mathcal{A}', H \cap K \neq \emptyset\}$ be the arrangement restricted to H. We will always choose the last hyperplane $H = H_n$ for the triple $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$. The linear order on $\mathcal{A}'$ is inherited from $\mathcal{A}$. The linear order on $\mathcal{A}''$ is determined by labeling each hyperplane K of $\mathcal{A}''$ by the smallest hyperplane $\nu(K)$ of $\mathcal{A}$ containing it. Clearly $\nu(K) < H_n$ for all $K \in \mathcal{A}''$. If $S'' = (K_{i_1}, \dots, K_{i_p})$ is a set of hyperplanes in $\mathcal{A}''$, define $\nu(S'') = (\nu(K_{i_1}), \dots, \nu(K_{i_p}))$.

Theorem 4.2 (Ziegler). *Let $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$ be a triple and assume that $\mathcal{A}'$ is essential. Write $\overline{\beta\mathrm{nbc}}(\mathcal{A}'') = \{(\nu(B''), H_n) \mid B'' \in \beta\mathrm{nbc}(\mathcal{A}'')\}$. There is a disjoint union*

$$\beta\mathrm{nbc}(\mathcal{A}) = \beta\mathrm{nbc}(\mathcal{A}') \cup \overline{\beta\mathrm{nbc}}(\mathcal{A}''). \quad \textit{Q.E.D.}$$

We refer to this formula as the βnbc recursion. When $\ell = 1$, $\mathcal{A} = \{H_1, \dots, H_n\}$ consists of $n \geq 1$ points in $\mathbb{C}$. Here $\beta\mathrm{nbc}(\mathcal{A}) = \{H_2, \dots, H_n\}$. In order to interpret the βnbc recursion in this case, we agree that $\beta\mathrm{nbc}(\mathcal{A}'')$ has one element, the empty set, so $\overline{\beta\mathrm{nbc}}(\mathcal{A}'') = \{H_n\}$. Introduce a linear order in $\beta\mathrm{nbc}(\mathcal{A})$ using the lexicographic order on the hyperplanes read from right to left. Note that in this order elements of $\beta\mathrm{nbc}(\mathcal{A}')$ are always smaller than elements of $\overline{\beta\mathrm{nbc}}(\mathcal{A}'')$.

Next we show that the homology basis constructed in Theorem 3.4 is a βnbc basis. Since $\Delta(\mathcal{A}, \lambda)$ is labeled by $C(\mathcal{A}, \lambda)$, it suffices to show that the critical points may be labeled by βnbc and satisfy the βnbc recursion. The weights λ' of $\mathcal{A}'$ are inherited from $\mathcal{A}$: if $H \in \mathcal{A}'$, then $\lambda'_H = \lambda_H$. The weights λ'' of $\mathcal{A}''$ are defined by $\lambda''_K = \sum \lambda_H$ for $K \subset H$ and $H \in \mathcal{A}'$. We may assume that λ' and λ'' are generic.

Lemma 4.3. *If $\mathcal{A}'$ is essential, then the map $\lambda_n \mapsto 0$ induces a bijection onto the disjoint union:*

$$\tau : C(\mathcal{A}, \lambda) \longrightarrow C(\mathcal{A}', \lambda') \cup C(\mathcal{A}'', \lambda'').$$

Proof. Choose coordinates so that $H_n = \ker z_1$ and write $\lambda_n = t$. Then $Q = z_1 Q'$ where Q' defines the essential arrangement $\mathcal{A}'$. Similarly $\Phi = z_1^t \Phi'$ where Φ' is the corresponding multivalued holomorphic function for $\mathcal{A}'$ whose critical points are $C(\mathcal{A}', \lambda')$. Let Φ'' denote the restriction of Φ' to H_n. The partial derivatives of Φ are

$$\begin{aligned}
\partial_{z_1} \log \Phi &= g_1/(z_1 Q') \\
\partial_{z_i} \log \Phi &= g_i/Q' \quad \text{for } 2 \le i \le \ell \text{ where} \\
g_1 &= tQ' + z_1 Q' \partial_{z_1} \log \Phi' \\
g_i &= Q' \partial_{z_i} \log \Phi' \quad \text{for } 2 \le i \le \ell
\end{aligned}$$

Note that only g_1 depends on t. The zero set of the equations $\{g_i = 0 \mid 1 \le i \le \ell\}$ in M is $C(\mathcal{A}, \lambda)$ when $t \neq 0$. Setting $t = 0$ gives the solutions $C(\mathcal{A}', \lambda')$ of critical points of Φ' in M', the complement of $\mathcal{A}'$. Next consider the points on H_n defined by $\{z_1 = 0, g_i = 0 \mid 1 \le i \le \ell\}$. For $t \neq 0$, these equations have no solution on $M'' = H_n \cap M'$, the complement of $\mathcal{A}''$. Adding $t = 0$ to these equations gives exactly the critical points of Φ'' on M''. Q.E.D.

Theorem 4.4. *There is a labeling of the critical points*

$$\rho : C(\mathcal{A}, \lambda) \longrightarrow \beta\mathrm{nbc}(\mathcal{A})$$

which is a bijection and respects the βnbc recursion.

Proof. We argue by double induction on ℓ and n. We may assume that in each arrangement the first ℓ hyperplanes are linearly independent, so the induction starts with this essential arrangement and when we add the next hyperplane all labels come from the restriction where the induction hypothesis holds since it has lower dimension. If $\ell = 1$ and $n = 2$, then $\beta\mathrm{nbc}(\mathcal{A}) = \{(H_2)\}$ and $C(\mathcal{A}, \lambda) = \{p\}$ is a singleton. Define $\rho(p) = (H_2)$. For the induction step we use the recursions in Theorem 4.2 and Lemma 4.4. We may assume that $\rho' : C(\mathcal{A}', \lambda') \to \beta\mathrm{nbc}(\mathcal{A}')$

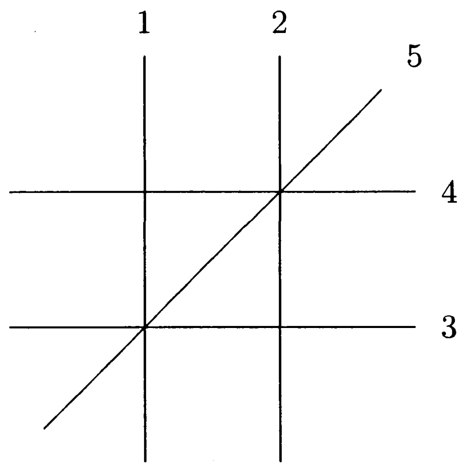

FIG 1. The Selberg arrangement

and $\rho'' : C(\mathcal{A}'', \lambda'') \to \beta\text{nbc}(\mathcal{A}'')$ satisfy the theorem. For $p \in C(\mathcal{A}, \lambda)$ define

$$\rho(p) = \begin{cases} \rho'(\tau(p)) & \text{if } \tau(p) \in C(\mathcal{A}', \lambda') \\ (\nu\rho''(\tau(p)), H_n) & \text{if } \tau(p) \in C(\mathcal{A}'', \lambda''). \end{cases}$$

This labeling respects the recursion by construction. Q.E.D.

Corollary 4.5. *The critical point labeling of the open cells*

$$\Delta(\mathcal{A}, \lambda) = \{\delta_{\rho(p)} \mid \rho(p) \in \beta\text{nbc}(\mathcal{A})\}$$

provides a βnbc *basis for* $H_\ell(M; \mathcal{L}_\lambda^\vee)$. *Q.E.D.*

It is interesting to note that if $\mathcal{A}$ is a complexified real arrangement and the critical points lie in the bounded chambers, then the critical point labeling may differ from the bounded chamber labeling of [2].

Example 4.6. *Let* $Q = (z_1 + 1)(z_1 - 1)(z_2 + 1)(z_2 - 1)(z_1 - z_2)$ *be the Selberg arrangement with the given linear order, see Figure 1. Then* $\beta\text{nbc}(\mathcal{A}) = \{(2,4), (2,5)\}$. *The bounded chamber labeling of* [2] *is independent of* λ. *It assigns* $(2,4)$ *to the upper left chamber and* $(2,5)$ *to the lower right. The critical point labeling depends on* λ. *Let* $\lambda_5 = t$ *be a small positive real number. With* $\lambda = (1.5, 0.5, 0.9, 0.3, t)$ *the critical point labels are the same, but with* $\lambda = (1.5, 0.5, 0.3, 0.9, t)$ *the labels are reversed.*

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On the fundamental group of the complement of a complex hyperplane arrangement

Luis Paris

Dedicated to Professor Peter Orlik on his 60th birthday

§1. Introduction

Let $\mathbf{K}$ be a field, and let $V = \mathbf{K}^l$ be a finite dimensional vector space over $\mathbf{K}$. An *arrangement of hyperplanes* in V is a finite family $\mathcal{A}$ of affine hyperplanes of V. The *complement* of $\mathcal{A}$ is defined by

$$M(\mathcal{A}) = V \setminus \bigcup_{H \in \mathcal{A}} H .$$

If $\mathbf{K}$ is $\mathbf{C}$, then the complement $M(\mathcal{A})$ is an open and connected subset of V.

The present paper is concerned with fundamental groups of complements of complex arrangements of hyperplanes.

The most popular such a group is certainly the pure braid group; it appears as the fundamental group of the complement of the "braid arrangement" (see [OT]). So, $\pi_1(M(\mathcal{A}))$ can be considered as a generalization of the pure braid group, and one can expect to show that many properties of the pure braid group also hold for $\pi_1(M(\mathcal{A}))$. However, the only general known results on this group are presentations [Ar], [CS1], [Ra], [Sa1]. Many interesting questions remain, for example, to know whether such a group is torsion free.

We focus in this paper on two families of arrangements of hyperplanes, to the fundamental group of which many well-known results on the pure braid group can be extended. Both of them, of course, contain the braid arrangement. These families are the "simplicial arrangements" and the "supersolvable arrangements". Note that there is another well-understood family of arrangements, the "reflection arrangements" (see

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[OT, Ch. 6] and [BMR]), which contains the braid arrangement, and which is not treated in the present paper.

The methods to approach each of these two families are completely different. The first method, which applies to simplicial arrangements, consists on associating with a real arrangement $\mathcal{A}$ a groupoid $G(\mathcal{A})$ that we call Deligne groupoid. Any vertex group of $G(\mathcal{A})$ is isomorphic to $\pi_1(M(\mathcal{A}_{\mathbf{C}}))$, where $\mathcal{A}_{\mathbf{C}}$ is the complexification of $\mathcal{A}$. If $\mathcal{A}$ is a simplicial arrangement, then it is shown that there exists an "automatic structure" on $G(\mathcal{A})$. Then, follow many properties of $\pi_1(M(\mathcal{A}))$. This is the object of Section 2. The second method, which applies to supersolvable arrangements, consists on proving the existence of certain fibrations. In the case of supersolvable arrangements, these fibrations give rise to a presentation of $\pi_1(M(\mathcal{A}))$ as an "iterated semidirect product" of free groups. This is the object of Section 3.

§2. The Deligne groupoid

Throughout this section, $\mathbf{K}$ is $\mathbf{R}$, and $\mathcal{A}$ is a (real) arrangement of hyperplanes in V. The *complexification* of V is $V_{\mathbf{C}} = \mathbf{C}^l$. The *complexification* of a hyperplane H is the hyperplane $H_{\mathbf{C}}$ of $V_{\mathbf{C}}$ having the same equation as H. The *complexification* of $\mathcal{A}$ is the arrangement $\mathcal{A}_{\mathbf{C}} = \{H_{\mathbf{C}}; H \in \mathcal{A}\}$ in $V_{\mathbf{C}}$.

DEFINITION. A *groupoid* is a category such that there is a morphism between any two objects, and such that each morphism is invertible.

A group is a groupoid with exactly one object. An object of a groupoid G is called *vertex* of G. For any vertex x, the set of morphisms from x to itself forms a group called *vertex group* and denoted by G_x.

Now, in order to define the Deligne groupoid $G(\mathcal{A})$ associated with a real arrangement of hyperplanes $\mathcal{A}$, we first give some terminology on oriented graphs.

DEFINITION. An *oriented graph* Γ is the following data:

1) a set $V(\Gamma)$ of *vertices*,

2) a set $A(\Gamma)$ of *arrows*,

3) a mapping $s : A(\Gamma) \to V(\Gamma)$ called *source*, and a mapping $t : A(\Gamma) \to V(\Gamma)$ called *target*.

Consider the abstract set $A(\Gamma)^{-1} = \{a^{-1}; a \in A(\Gamma)\}$ in one-to-one correspondence with $A(\Gamma)$, and set $s(a^{-1}) = t(a)$ and $t(a^{-1}) = s(a)$, for a in $A(\Gamma)$. A *path* of Γ is an expression

$$g = a_1^{\varepsilon_1} \dots a_d^{\varepsilon_d} ,$$

where $a_i \in A(\Gamma)$, $\varepsilon_i \in \{\pm 1\}$, and $t(a_i^{\varepsilon_i}) = s(a_{i+1}^{\varepsilon_{i+1}})$ for all $i = 1, \dots, d-1$. The vertex $s(a_1^{\varepsilon_1})$ is called *source* of g and is denoted by $s(g)$, and the vertex $t(a_d^{\varepsilon_d})$ is called *target* of g and is denoted by $t(g)$. The integer d is the *length* of g. Any vertex is assumed to be a path of length 0. For a path $f = a_1^{\varepsilon_1} \dots a_d^{\varepsilon_d}$, we write $f^{-1} = a_d^{-\varepsilon_d} \dots a_1^{-\varepsilon_1}$. For two paths $f = a_1^{\varepsilon_1} \dots a_d^{\varepsilon_d}$ and $g = b_1^{\mu_1} \dots b_k^{\mu_k}$ with $t(f) = s(g)$, we write $fg = a_1^{\varepsilon_1} \dots a_d^{\varepsilon_d} b_1^{\mu_1} \dots b_k^{\mu_k}$. A *positive path* is a path $f = a_1^{\varepsilon_1} \dots a_d^{\varepsilon_d}$ with $\varepsilon_1 = \varepsilon_2 = \dots = \varepsilon_d = 1$. The *distance* between two vertices x and y is defined to be the minimal length of a path connecting x and y. Any path which achieves this minimum is called *minimal path.*

Call an oriented graph *connected* if there is a path connecting any two vertices.

DEFINITION. Let Γ be an oriented connected graph. A *congruence* on Γ is an equivalence relation $\sim$ on the set of paths of Γ, which satisfies the following conditions:

1) if $f \sim g$, then $s(f) = s(g)$ and $t(f) = t(g)$,
2) $ff^{-1} \sim s(f)$ for any path f,
3) if $f \sim g$, then $f^{-1} \sim g^{-1}$,
4) if $f \sim g$, h_1 is a path with $t(h_1) = s(f) = s(g)$, and h_2 is a path with $s(h_2) = t(f) = t(g)$, then $h_1 f h_2 \sim h_1 g h_2$.

A congruence $\sim$ on a connected oriented graph Γ determines a groupoid $G(\Gamma, \sim)$: the objects of $G(\Gamma, \sim)$ are the vertices, and the morphisms of $G(\Gamma, \sim)$ are the equivalence classes of paths.

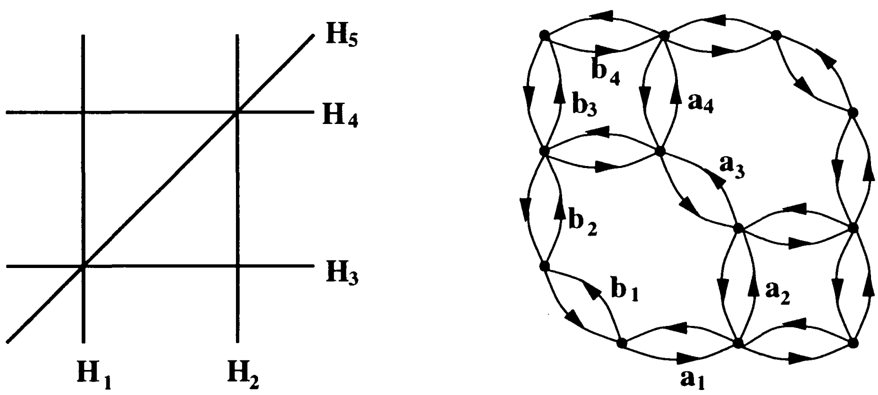
Let $\mathcal{A}$ be a (real) arrangement of hyperplanes. Now, we associate with $\mathcal{A}$ a connected oriented graph $\Gamma(\mathcal{A})$ and a congruence $\sim$ on $\Gamma(\mathcal{A})$, and we define the Deligne groupoid $G(\mathcal{A})$ associated with $\mathcal{A}$ to be $G(\Gamma(A), \sim)$.

DEFINITION. A *chamber* of $\mathcal{A}$ is a connected component of $M(\mathcal{A}) = V \backslash (\cup_{H \in \mathcal{A}} H)$. Call two chambers C and D *adjacent* if there exists exactly one hyperplane in $\mathcal{A}$ which separates C and D. Let $\Gamma(\mathcal{A})$ be the oriented graph whose vertices are the chambers, and whose arrows are the pairs (C, D) of adjacent chambers. Note that (C, D) and (D, C) are distinct arrows of $\Gamma(\mathcal{A})$, if C, D are adjacent chambers. Let $\sim$ be the smallest congruence on $\Gamma(\mathcal{A})$ satisfying: if α and β are both positive minimal paths with the same source and the same target, then $\alpha \sim \beta$. The *Deligne groupoid* of $\mathcal{A}$ is defined to be the groupoid $G(\mathcal{A}) = G(\Gamma(\mathcal{A}), \sim)$ associated with $\Gamma(\mathcal{A})$ and $\sim$. Note that, for two chamber C, D, there is a unique equivalence class of positive minimal paths with source C and target D. This class will be denoted by $\delta(C, D)$.

EXAMPLE. Consider the arrangement of lines $\mathcal{A} = \{H_1, \ldots, H_5\}$ drawn in Figure 1. Then $\Gamma(\mathcal{A})$ is the oriented graph also drawn in Figure 1. Let

$$\alpha = a_1 a_2 a_3 a_4 \ , \quad \beta = b_1 b_2 b_3 b_4 \ .$$

Then α and β are both positive minimal paths with the same source and the same target, thus $\alpha \sim \beta$.



A $\qquad$ Γ(A)

FIGURE 1

THEOREM 2.1 (PARIS [Pa1], SALVETTI [Sa1]). *Let $\mathcal{A}$ be a (real) arrangement of hyperplanes. Then any vertex group of $G(\mathcal{A})$ is isomorphic to the fundamental group of $M(\mathcal{A}_{\mathbf{C}})$.*

The Deligne groupoid was first introduced, and Theorem 2.1 was proved, in [De] for simplicial arrangements.

DEFINITION. Let $\mathcal{A}$ be a (real) arrangement of hyperplanes. We say that $\mathcal{A}$ is *central* if all the hyperplanes of $\mathcal{A}$ contain the origin. We say further that $\mathcal{A}$ is *essential* if the intersection of all the elements of $\mathcal{A}$ is $\{0\}$. Call $\mathcal{A}$ *simplicial* if it is central and essential, and if all the chambers of $\mathcal{A}$ are cones over simplices.

Two results on simplicial arrangements are particularly interesting. The first one (Theorem 2.3) is due to Deligne [De], and the second one (Theorem 2.5) is due to Charney [Ch]. Many properties of the fundamental group of $M(\mathcal{A}_{\mathbf{C}})$ are derived from these theorems. The proofs of both are very close from the work of Garside [Ga] and Thurston [Th] on braid groups. They are both strongly based on the following lemma 2.2. Note that, by [Pa2], the conclusion of Lemma 2.2 is true if and only if $\mathcal{A}$ is a simplicial arrangement.

Let $\mathcal{A}$ be a (real) arrangement of hyperplanes. Let f, g be two positive paths of $\Gamma(\mathcal{A})$ with $s(f) = s(g)$. We say that f *begins* with g

if there exists a positive path h such that $s(h) = t(g)$, $t(h) = t(f)$, and $f \sim gh$. For a positive path f, let $\mathrm{Begin}(f)$ denote the set of positive minimal paths α such that f begins with α.

LEMMA 2.2 (DELIGNE [De]). *Let $\mathcal{A}$ be a simplicial arrangement of hyperplanes. For every positive path f of $\Gamma(\mathcal{A})$, there exists a (unique up to equivalence) positive minimal path α such that* $\mathrm{Begin}(f) = \mathrm{Begin}(\alpha)$. *In particular, f begins with α.*

A space M is called an *Eilenberg-MacLane space* if its universal cover is contractible. Such a space is specially interesting to study its fundamental group because the homologies of M and $\pi_1(M)$ are equal and, consequently, many topological properties of M reflect on $\pi_1(M)$. We refer to [Br] for more details on the subject.

THEOREM 2.3 (DELIGNE [De]). *Let $\mathcal{A}$ be a simplicial arrangement of hyperplanes. Then $M(\mathcal{A}_{\mathbf{C}})$ is an Eilenberg-MacLane space.*

COROLLARY 2.4. *Let $\mathcal{A}$ be a simplicial arrangement of hyperplanes.*
i) $\pi_1(M(\mathcal{A}_{\mathbf{C}}))$ *is torsion free.*
ii) $\pi_1(M(\mathcal{A}_{\mathbf{C}}))$ *has finite cohomological dimension.*
iii) $H_*(\pi_1(M(\mathcal{A}_{\mathbf{C}})), \mathbf{Z})$ *is torsion free (by* [OS]).

Automatic groups form a large class of groups which contains all finite groups, abelian groups, free groups, fundamental groups of compact hyperbolic manifolds, and, more generally, hyperbolic groups in Gromov's sense [GH]. On the other hand, if an automatic group is nilpotent, then it is virtually abelian. More generally, if a subgroup of a biautomatic group is nilpotent, then it is virtually abelian [GS]. Briefly, an automatic group is a group provided with an extra combinatorial structure which "controls" the words and their lengths in the group. Such a structure allows to compute the growth function of the group, gives isoperimetric inequalities, and furnishes algorithms to solve the word problem and, if the structure is biautomatic, to solve the conjugacy problem. A finite index subgroup of an automatic group "inherits" the automatic structure from the group. Conversely, if a finite index subgroup is automatic, then the automatic structure of the subgroup can be extended to the whole group. The theory of (bi)automatic groupoids is identical to the theory of (bi)automatic groups. In particular, an automatic groupoid has finitely many vertices, and every vertex group inherits the automatic structure from the groupoid. We refer to [ECH] for a general exposition on the subject.

A natural question is whether the Deligne groupoid $G(\mathcal{A})$ of a real arrangement $\mathcal{A}$ admits an automatic structure. This question has been solved by Charney [Ch] in the case of simplicial arrangements. This is the subject of the remainder of the section.

Now, we give a precise definition of a (bi)automatic groupoid and, after stating Charney's theorem, we show the automatic structure on $G(\mathcal{A})$ when $\mathcal{A}$ is a simplicial arrangement. We will notice that the definition of this automatic structure highly depends on Lemma 2.2 above.

Let A be a finite set (of letters). We write A^* for the free monoid generated by A. The elements of A will be called *words*.

DEFINITION. A *finite state automaton* is a quintuple $\mathcal{F} = (V, A, \mu, Y, v_0)$, where V is a finite set called *state set*, A is a finite set called the *alphabet*, $\mu : V \times A \to V$ is a function called the *transition function*, Y is a subset of V called the *accept state set*, and v_0 is an element of V called *start state*. For $v \in V$ and $f = x_1 \dots x_n \in A^*$ we define the state $\mu(v, f)$ inductively on n by:

$$\mu(v,f) = \begin{cases} v & \text{if } n = 0 \\ \mu(\mu(v, x_1 \dots x_{n-1}), x_n) & \text{if } n \geq 1 \end{cases}$$

Then

$$L_{\mathcal{F}} = \{f \in A^* \ ; \ \mu(v_0, f) \in Y\}$$

is called the *language recognized* by $\mathcal{F}$. A *regular language* is a language recognized by a finite state automaton.

DEFINITION. Let G be a groupoid. A set S of morphisms is called a *generating set* if every morphism of G is the composition of finitely many elements of S. The *length* of a morphism f (with repect to S), denoted by $\lg_S(f)$, is the shortest length of a word in S^* which represents f. Let f, g be two morphisms with the same source. The *distance* between f and g, denoted by $d_S(f, g)$, is the length of $f^{-1}g$.

REMARK. Let $\mathcal{A}$ be a real arrangement of hyperplanes and $\Gamma = \Gamma(\mathcal{A})$. Then $G(\mathcal{A})$ has a natural generating set: $A(\Gamma) \cup A^{-1}(\Gamma)$. However, we will see later that this is not the generating set used to define the automatic structure on $G(\mathcal{A})$ when $\mathcal{A}$ is a simplicial arrangement.

DEFINITION. Let G be a groupoid and S a generating set of G. For $f \in S^*$, we denote by $\bar{f}$ the morphism of G represented by f if it exists. A language L in S^* *represents* G if every element of L represents a morphism and every morphism is represented by an element of L. For $f = x_1 \dots x_n \in L$ and a positive integer t, we write $\bar{f}(t) = \overline{x_1 \dots x_t}$ if $1 \leq t \leq n$ and $\bar{f}(t) = \bar{f}$ if $t \geq n$. Let κ be a positive integer. We say that L has the *κ-fellow traveller property* if, for all $f, g \in L$ such that $\bar{f}$ and $\bar{g}$ have the same source, we have:

$$d_S(\bar{f}(t), \bar{g}(t)) \leq \kappa \cdot d_S(\bar{f}, \bar{g})$$

for all integer $t \geq 1$.

DEFINITION. A groupoid G is *automatic* if there exist a finite generating set S of G, a constant $\kappa \geq 1$, and a regular language L in S^*, such that L represents G and has the κ-fellow traveller property. If, in addition, the language L^{-1} in $(S^{-1})^*$, obtained by formally inverting the elements of L, also has the κ-fellow traveller property, then G is called *biautomatic.*

THEOREM 2.5 (CHARNEY [Ch]). *Let $\mathcal{A}$ be a simplicial arrangement of hyperplanes, and let $G(\mathcal{A})$ be the Deligne groupoid of $\mathcal{A}$. Then $G(\mathcal{A})$ is biautomatic.*

COROLLARY 2.6. *Let $\mathcal{A}$ be a simplicial arrangement of hyperplanes.*
i) $\pi_1(M(\mathcal{A}_{\mathbf{C}}))$ *is biautomatic.*
ii) $\pi_1(M(\mathcal{A}_{\mathbf{C}}))$ *has the conjugacy problem solvable.*
iii) $\pi_1(M(\mathcal{A}_{\mathbf{C}}))$ *has a quadratic isoperimetric inequality.*

Let $\mathcal{A}$ be a simplicial arrangement. We turn now to give the definition of the finite state automaton $\mathcal{F} = (V, S, \mu, Y, v_0)$ which determines the automatic structure on $G(\mathcal{A})$. We refer to [Ch] for the proof that both, the language L recognized by $\mathcal{F}$ and its inverse L^{-1}, have the 6 -fellow traveller property.

Let $\Gamma = \Gamma(\mathcal{A})$. Recall that the vertex set $V(\Gamma)$ is the set of chambers of $\mathcal{A}$. For $C, D \in V(\Gamma)$, we denote by $\delta(C, D)$ the (unique) equivalence class of positive minimal paths with source C and target D. We write $\Delta_C = \delta(C, -C)$ for all $C \in V(\Gamma)$. Then

$$S = \{\delta(C, D) \ ; \ C, D \in V(\Gamma)\} \cup \{\Delta_C^{-1} \ ; \ C \in V(\Gamma)\} \ .$$

It is true but non trivial that S generates $G(\mathcal{A})$. Note also that $\delta(C, C)$ is the identity morphism on C and lies in S for all $C \in V(\Gamma)$. We set

$$V = S \cup \{v_0, v_1\} \ , \quad Y = S \ ,$$

where v_0, v_1 are "abstract" states. v_0 is the start state and v_0, v_1 are the only non accept states. So, it remains to define the transition function $\mu : V \times S \to V$. We say that $\delta(C_0, C_1)\delta(C_1, C_2)$ is *normal* if, according to Lemma 2.2, $\delta(C_0, C_1)$ is the (unique) class of positive minimal paths such that

$$\mathrm{Begin}(\delta(C_0, C_1)\delta(C_1, C_2)) = \mathrm{Begin}(\delta(C_0, C_1)).$$

Now:

$$\begin{array}{lll}
\mu(v_0, x) & = x & \text{for all } x \in S \\
\mu(v_1, x) & = v_1 & \text{for all } x \in S \\
\mu(\Delta_C^{-1}, \Delta_D^{-1}) & = \begin{cases} \Delta_D^{-1} \\ v_1 \end{cases} & \begin{array}{l} \text{if } D = -C \\ \text{otherwise} \end{array} \\
\mu(\Delta_C^{-1}, \delta(D_0, D_1)) & = \begin{cases} \delta(D_0, D_1) \\ v_1 \end{cases} & \begin{array}{l} \text{if } D_0 = C \text{ and } D_1 \neq -C \\ \text{otherwise} \end{array} \\
\mu(\delta(C_0, C_1), \Delta_D^{-1}) & = v_1 & \\
\mu(\delta(C_0, C_1), \delta(D_0, D_1)) & = \begin{cases} \delta(D_0, D_1) \\ \\ v_1 \end{cases} & \begin{array}{l} \text{if } C_1 = D_0 \text{ and } \delta(C_0, C_1)\delta(D_0, D_1) \\ \quad \text{is normal} \\ \text{otherwise} \end{array}
\end{array}$$

§3. Fibration theorem

Let $\mathcal{A}$ be an arrangement of hyperplanes. The *intersection poset* of $\mathcal{A}$ is the set $L(\mathcal{A})$ of nonempty intersections of elements of $\mathcal{A}$, partially ordered by reverse inclusion. It admits a *rank* function defined by $r(X) = \text{Codim}X$, for X in $L(\mathcal{A})$. The space V is the unique minimal element of $L(\mathcal{A})$, and, by [Sa1], all the maximal elements have the same rank. Call the arrangement $\mathcal{A}$ *essential* if the maximal elements are points. Let X, Y in $L(\mathcal{A})$. Their *meet* is defined to be $X \wedge Y = \cap\{H \in \mathcal{A}; X \cup Y \subseteq H\}$. If $X \cap Y \neq \emptyset$, their *join* is defined to be $X \vee Y = X \cap Y$.

Throughout this section, $\mathbf{K}$ is $\mathbf{C}$, $\mathcal{A}$ is a (complex) arrangement of hyperplanes in V, and X is a linear subspace which is not necessarily in $L(\mathcal{A})$.

We say that a hyperplane H of $\mathcal{A}$ is *parallel* to X if either $H \cap X = \emptyset$ or $X \subseteq H$. Consider the projection $p_X : V \to V/X$. If H is parallel to X, then $p_X(H)$ is a hyperplane of V/X. Let $\mathcal{A}/X = \{p_X(H); H \in \mathcal{A}$ and H parallel to $X\}$. Then the projection p_X induces a projection $p_X : M(\mathcal{A}) \to M(\mathcal{A}/X)$.

PROPOSITION 3.1 (PARIS [Pa3]). *The projection $p_X : M(\mathcal{A}) \to M(\mathcal{A}/X)$ admits a cross-section $s_X : M(\mathcal{A}/X) \to M(\mathcal{A})$.*

DEFINITION. Call $Y \in L(\mathcal{A})$ *horizontal* with respect to X if $p_X(Y) = V/X$. Let Hor_X denote the set of horizontal elements of $L(\mathcal{A})$. The *bad set* of $M(\mathcal{A}/X)$ is

$$B_X = \cup\{p_X(Y) \cap M(\mathcal{A}/X)\ ; Y \in L(\mathcal{A}) \setminus \text{Hor}_X\}\ .$$

THEOREM 3.2 (PARIS [Pa3]). *Let*

$$N_X = M(\mathcal{A}/X) \setminus B_X\ , \quad M_X = p_X^{-1}(N_X) \cap M(\mathcal{A})\ .$$

Then the restriction $p_X : M_X \to N_X$ of p_X to M_X is a locally trivial C^∞ fibration.

REMARK. i) The restriction of s_X to N_X determines a cross-section $s_X : N_X \to M_X$ of the fibration.

ii) Let $y_0 \in N_X$, and let $z_0 = s_X(y_0) \in M_X$. The fiber of p_X containing z_0 is the complement of the arrangement $\mathcal{A}^X_{z_0}$ in $(z_0 + X)$ defined by

$$\mathcal{A}^X_{z_0} = \{(z_0 + X) \cap H \ ; \ H \in \mathcal{A} \text{ and } H \text{ not parallel to } X\} \ .$$

So, by [Hu, Ch. V, Prop. 6.2]:

COROLLARY 3.3. *The following sequence is exact and splits.*

$$1 \to \pi_1(M(\mathcal{A}^X_{z_0}), z_0) \to \pi_1(M_X, z_0) \to \pi_1(N_X, y_0) \to 1$$

Another direct consequence of Theorem 3.2 is, by [Pa3]:

COROLLARY 3.4. *The following sequence is exact and splits.*

$$\pi_1(M(\mathcal{A}^X_{z_0}), z_0) \to \pi_1(M(\mathcal{A}), z_0) \to \pi_1(M(\mathcal{A}/X), y_0) \to 1$$

Note that the morphism $\pi_1(M(\mathcal{A}^X_{z_0}), z_0) \to \pi_1(M(\mathcal{A}), z_0)$ is not injective in general.

DEFINITION. Assume that $\mathcal{A}$ is a (complex) central arrangement of hyperplanes. Call X in $L(\mathcal{A})$ *modular* if $X \wedge Y = X + Y$ for all Y in $L(\mathcal{A})$. Call $\mathcal{A}$ *supersolvable* if it is essential and there exists a chain $0 > X_1 > \cdots > X_l = V$ in $L(\mathcal{A})$ such that X_μ is modular and $\dim X_\mu = \mu$ for all $\mu = 1, \ldots, l$.

Theorem 3.2 is particularly interesting if X is modular, because of the following theorem.

THEOREM 3.5 (TERAO [Te]). *Let $\mathcal{A}$ be a central arrangement of hyperplanes, and let X be a modular element of $L(\mathcal{A})$. Then the bad set B_X is empty.*

COROLLARY 3.6. *Let $\mathcal{A}$ be a central arrangement of hyperplanes, and let X be a modular element of $L(\mathcal{A})$. Then $p_X : M(\mathcal{A}) \to M(\mathcal{A}/X)$ is a locally trivial C^∞ fibration.*

COROLLARY 3.7. *Let $\mathcal{A}$ be a central arrangement of hyperplanes, and let X be a modular element of $L(\mathcal{A})$. Then the following sequence is exact and splits.*

$$1 \to \pi_1(M(\mathcal{A}^X_{z_0}), z_0) \to \pi_1(M(\mathcal{A}), z_0) \to \pi_1(M(\mathcal{A}/X), y_0) \to 1$$

Falk and Proudfoot [FP] have independently proved Corollary 3.6 using the same argument as sketched below. Corollary 3.6 is a classical and well-known result in the case of a modular element of dimension 1 [Te]. One can easily verify in this particular case that each fiber is diffeomorphic to $\mathbf{C}$ minus $|\mathcal{A} \setminus \mathcal{A}_X|$ points; however, the existence of trivializing neighborhoods is not explicitly proved in [Te]. To prove the existence of trivializing neighborhoods, one can apply the techniques shown below or, maybe, use simpler arguments like those given by Fadell and Neuwirth in [FN].

The proof of Theorem 3.2 is an application of Thom's first isotopy lemma that we state now.

Let M be a C^∞ manifold, and let A be a subset of M. A C^∞ *Whitney prestratification* of A is a partition $\mathcal{P}$ of $\mathcal{A}$ into subsets, that are called *strata*, satisfying the following conditions:

1) each stratum is a C^∞ submanifold of M;

2) $\mathcal{P}$ is locally finite;

3) if $U, V \in \mathcal{P}$ are such that $\bar{U} \cap V \neq \emptyset$, then $V \subseteq \bar{U}$ (in that case we write $V < U$);

4) if $U, V \in \mathcal{P}$ are such that $V < U$, then (U, V) satisfies the *Whitney Condition (b)* defined in [Ma].

THEOREM 3.8 (MATHER [Ma]). *Let M, N be two C^∞ manifolds, let $f : M \to N$ be a C^∞ function, let A be a subset of M, and let $\mathcal{P}$ be a Whitney prestratification of A. Assume that the restriction $f|_A : A \to N$ is a proper map, and that the restriction $f|_U : U \to N$ is a submersion for every $U \in \mathcal{P}$. Then $f|_A : A \to N$ is a locally trivial C^0 fibration, and $f|_U : U \to N$ is a locally trivial C^∞ fibration for all $U \in \mathcal{P}$.*

We turn now to the proof of Theorem 3.2. Let X be a linear subspace. We assume that $X = \mathbf{C}^d$, $V/X = \mathbf{C}^{l-d}$, and $p_X : \mathbf{C}^d \times \mathbf{C}^{l-d} \to \mathbf{C}^{l-d}$ is the projection on the second coordinate. Let $\mathbf{P}^d$ denote the complex projective space of dimension d. Consider the embedding of $\mathbf{C}^d$ in $\mathbf{P}^d$, and still denote by $p_X : \mathbf{P}^d \times \mathbf{C}^{l-d} \to \mathbf{C}^{l-d}$ the projection on the second coordinate. The proof of Theorem 3.2 given in [FP] and [Pa3] consists on defining a Whitney prestratification on $\mathbf{P}^d \times N_X$ so that M_X is a stratum and the restriction of the projection $p_X : \mathbf{P}^d \times N_X \to N_X$ on each stratum is a submersion. $p_X : \mathbf{P}^d \times N_X \to N_X$ being obviously a proper map, by Theorem 3.8, it follows that $p_X : M_X \to N_X$ is a locally trivial C^∞ fibration.

We focus now on the family of supersolvable arrangements. So, from now on, $\mathcal{A}$ is supposed to be a (complex) supersolvable arrangement of hyperplanes.

First, notice that, iterating Corollaries 3.6 and 3.7, one obtains the following two theorems.

THEOREM 3.9 (TERAO [Te]). *$M(\mathcal{A})$ is an Eilenberg-MacLane space.*

THEOREM 3.10 (FALK and RANDELL [FR1]). *$\pi_1(M(\mathcal{A}))$ can be presented as*

$$\pi_1(M(\mathcal{A})) = F_1 \rtimes (F_2 \rtimes (\dots (F_{l-1} \rtimes \mathbf{Z}) \dots)) ,$$

where $F_1, \dots, F_{l-1}$ are free groups.

Like for complexifications of simplicial arrangements, Theorem 3.9 implies:

COROLLARY 3.11. i) *$\pi_1(M(\mathcal{A}))$ is torsion free.*
ii) *$\pi_1(M(\mathcal{A}))$ has finite cohomological dimension.*
iii) *$H_*(\pi_1(M(\mathcal{A})), \mathbf{Z})$ is torsion free (by [OS]).*

It is known not only that $\pi_1(M(\mathcal{A}))$ can be written as an iterated semidirect product of free groups, but also that the succesive actions on the free groups are trivial at the homology level:

LEMMA 3.12. *Let $\pi_1(M(\mathcal{A})) = F_1 \rtimes (F_2 \rtimes (\dots (F_{l-1} \rtimes \mathbf{Z}) \dots))$ be the decomposition of Theorem 3.10. Then $F_{\mu+1} \rtimes (\dots (F_{l-1} \rtimes \mathbf{Z}) \dots)$ acts trivially on the homology of F_μ for all $\mu = 1, \dots, l-1$.*

This last lemma is the key of the proof of many properties of $\pi_1(M(\mathcal{A}))$. Cohen and Suciu [CS2] have recently proved that the action of $F_{\mu+1} \rtimes (\dots (F_{l-1} \rtimes \mathbf{Z}) \dots)$ on F_μ is actually a "pure braid action", which is a stronger statement.

We focus now on one of these properties, the biordering, which is not so well-known, and refer to [FR2] and [FR3] for an exposition on the other properties of $\pi_1(M(\mathcal{A}))$ (LCS formula, rational $K(\pi, 1)$ property, Koszul property, etc...).

DEFINITION. Call a group G *biorderable* if there exists a total ordering $<$ on G such that $f < g$ implies $h_1 f h_2 < h_1 g h_2$ for all $h_1, h_2, f, g \in G$.

Recall that $\mathcal{A}$ denotes a complex supersolvable arrangement of hyperplanes. We turn now to show that $\pi_1(M(\mathcal{A}))$ is biorderable and explain some consequences of this fact. We refer to [MR] for a general exposition on biorderable groups.

Let G be a biorderable group. Say that $g \in G$ is *positive* if $g > 1$, and denote by P the set of positive elements. Then one has the disjoint union $G = P \sqcup P^{-1} \sqcup \{1\}$. Moreover, $P \cdot P \subseteq P$, and $gPg^{-1} = P$ for all $g \in G$. Conversely, these conditions imply that G is biorderable, namely:

PROPOSITION 3.13. *Let G be a group, and let $P \subseteq G$ be a subset such that $G = P \sqcup P^{-1} \sqcup \{1\}$ is a disjoint union, $P \cdot P \subseteq P$, and $gPg^{-1} = P$ for all $g \in G$. Then G is biorderable, the ordering being given by $g > f$ if $gf^{-1} \in P$.*

Now, consider an exact sequence

$$1 \to K \to G \xrightarrow{\phi} H \to 1 ,$$

and assume that K and H are both biorderable. Let P_K and P_H denote the sets of positive elements of K and H, respectively. Say that $g \in G$ is positive if either $\phi(g) \in P_H$, or $\phi(g) = 1$ (namely, $g \in K$) and $g \in P_K$. Let P denote the set of positive elements. We clearly have the disjoint union $G = P \sqcup P^{-1} \sqcup \{1\}$ and the inclusion $P \cdot P \subseteq P$. Moreover, we have $gPg^{-1} = P$ for all $g \in G$ if and only if we have $gP_Kg^{-1} = P_K$ for all $g \in G$. This last condition holds if K is central, so:

PROPOSITION 3.14. *Let*

$$1 \to K \to G \to H \to 1$$

be an exact sequence such that K and H are both biorderable and K is central in G. Then G is also biorderable.

DEFINITION. Let G be a group. For two subgroups A, B of G, let $[A, B]$ denote the subgroup generated by $\{aba^{-1}b^{-1}; a \in A \text{ and } b \in B\}$. The subgroups G_n of the *lower central series* of G are defined recursively by

$$G_1 = G, \quad G_{n+1} = [G_n, G] \quad n = 1, 2, \dots$$

A group G for which $\cap_{n=1}^{\infty} G_n = \{1\}$ and G_n/G_{n+1} is torsion free for all n, is called *residually nilpotent without torsion.*

An important result obtained with Lemma 3.12 is:

THEOREM 3.15 (FALK and RANDELL [FR1]). *$\pi_1(M(\mathcal{A}))$ is residually nilpotent without torsion.*

Now, $\pi_1(M(\mathcal{A}))$ is biorderable because of the following.

PROPOSITION 3.16. *Let G be a residually nilpotent without torsion group. Then G is biorderable.*

Proof. We first prove that G/G_n is biorderable by induction on n. Consider the exact sequence

$$1 \to G_n/G_{n+1} \to G/G_{n+1} \to G/G_n \to 1 .$$

The group G_n/G_{n+1} is a free abelian group thus is biorderable (take, for example, the lexicographic order), G/G_n is biorderable by induction hypothesis, and G_n/G_{n+1} is central in G/G_{n+1}, thus G/G_{n+1} is biorderable by Proposition 3.14.

Now, call $g \in G$ positive if there exists some $n \geq 1$ such that the class $[g] \in G/G_n$ of g is not the identity and is positive. By the definition of the ordering on G/G_n, this definition does not depend on the choice of n. Let P denote the set of positive elements. The condition $\cap_{n=1}^{\infty} G_n = \{1\}$ implies that $G = P \sqcup P^{-1} \sqcup \{1\}$ is a disjoint union. Moreover, one can easily verify that $P \cdot P \subseteq P$ and $gPg^{-1} = P$ for all $g \in G$. Q.E.D.

An alternative proof of the fact that $\pi_1(M(\mathcal{A}))$ is biorderable can be found in [KR]. There exists a "natural" ordering on any finitely generated free group called *Magnus ordering*. The key of the proof of Kim and Rolfsen is the following lemma.

LEMMA 3.17 (KIM and ROLFSEN [KR]). *Let F be a finitely generated free group, let P be the set of positive elements of F with respect to the Magnus ordering, and let $\alpha \in \mathrm{Aut}(F)$ which acts trivially on the homology of F. Then $\alpha(P) = P$.*

We turn now to give this alternative proof. Let $\pi_1(M(\mathcal{A})) = F_1 \rtimes (F_2 \rtimes (\dots(F_{l-1} \rtimes \mathbf{Z})\dots))$ be the decomposition of $\pi_1(M(\mathcal{A}))$ of Theorem 3.10. Write $H_\mu = F_\mu \rtimes (\dots(F_{l-1} \rtimes \mathbf{Z})\dots)$ for $\mu = 1, \dots, l$. It clearly suffices to show that $H_{\mu-1}$ is biorderable if H_μ is biorderable. Let $g \in H_{\mu-1}$. Since $H_{\mu-1} = F_{\mu-1} \rtimes H_\mu$, g can be uniquely written in the form $g = g_1 g_2$, where $g_1 \in F_{\mu-1}$ and $g_2 \in H_\mu$. Say that g is positive if either g_2 is positive, or $g_2 = 1$ and g_1 is positive with repect to the Magnus ordering. Let P denote the set of positive elements of $H_{\mu-1}$. The fact that H_μ acts trivially on the homology of $F_{\mu-1}$ (Lemma 3.12) and Lemma 3.17 imply that $gPg^{-1} = P$ for all $g \in H_{\mu-1}$. The disjoint union $H_{\mu-1} = P \sqcup P^{-1} \sqcup \{1\}$ and the inclusion $P \cdot P \subseteq P$ are obvious.

We turn now to investigate two properties of biorderable groups. The first one says that a biorderable group has no generalized torsion, and the second one says that the group ring of a biorderable group has no zero divisors.

DEFINITION. Let G be a group. An element $g \in G$ is said to be a *generalized torsion element* if there exist $h_1 \dots, h_r \in G$ such that

$$(h_1 g h_1^{-1})(h_2 g h_2^{-1}) \dots (h_r g h_r^{-1}) = 1 .$$

PROPOSITION 3.18. *A biorderable group contains no generalized torsion elements.*

Proof. Let $g \in G$, $g \neq 1$. Let $h_1, \dots, h_r \in G$. Either $g > 1$ or $g < 1$. Assume $g > 1$. Then $(h_i g h_i^{-1}) > 1$ for all $i = 1, \dots, r$, thus

$$1 < (h_1 g h_1^{-1})(h_2 g h_2^{-1}) \dots (h_r g h_r^{-1}) \neq 1 \ . \qquad \text{Q.E.D.}$$

Proposition 3.18 says in particular that biorderable groups are torsion free.

PROPOSITION 3.19. *Let G be a biorderable group. Then $\mathbf{Z}G$ contains no zero divisors.*

Proof. Let α, β be non zero elements of $\mathbf{Z}G$. We write

$$\alpha = a_1 g_1 + \dots + a_p g_p \ , \quad \beta = b_1 f_1 + \dots + b_q f_q \ ,$$

where $a_1, \dots, a_p, b_1, \dots, b_q \in \mathbf{Z} \setminus \{0\}$, $g_1, \dots, g_p, f_1, \dots, f_q \in G$, $g_1 < g_2 < \dots < g_p$, and $f_1 < f_2 < \dots < f_q$. Then

$$\begin{aligned} &\alpha\beta = \sum_{i,j} (a_i b_j)(g_i f_j) \ , \\ &g_1 f_1 < g_i f_j \quad \text{if } i \neq 1 \text{ or } j \neq 1 \ , \\ &a_1 b_1 \neq 0 \ , \end{aligned}$$

thus $\alpha\beta \neq 0$. Q.E.D.

Let G be a biorderable group. We pointed out before that G is then torsion free because it has no generalized torsion elements. The fact that $\mathbf{Z}G$ has no zero divisors also implies that G is torsion free. Indeed, if g is a torsion element of a group G (say of order k), then

$$(1 - g)(1 + g + g^2 + \dots + g^{k-1}) = 1 - g^k = 0 \ ,$$

thus $(1 - g)$ is a zero divisor. It is not known whether $\mathbf{Z}G$ has no zero divisor in general if G is torsion free.

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Logarithmic forms and anti-invariant forms of reflection groups

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Dedicated to Peter Orlik on his sixtieth birthday

Abstract.

Let W be a finite group generated by unitary reflections and $\mathcal{A}$ be the set of reflecting hyperplanes. We will give a characterization of the logarithmic differential forms with poles along $\mathcal{A}$ in terms of anti-invariant differential forms. If W is a Coxeter group defined over $\mathbf{R}$, then the characterization provides a new method to find a basis for the module of logarithmic differential forms out of basic invariants.

Basic definitions. Let V be an ℓ-dimensional unitary space. Let $W \subset \mathbf{GL}(V)$ be a finite group generated by unitary reflections and $\mathcal{A}$ be the set of reflecting hyperplanes. We say that W is a *unitary reflection group* and $\mathcal{A}$ is the corresponding *unitary reflection arrangement.* Let S be the algebra of polynomial functions on V. The algebra S is naturally graded by $S = \bigoplus_{q \geq 0} S_q$ where S_q is the space of homogeneous polynomials of degree q. Thus $S_1 = V^*$ is the dual space of V. Let Der_S be the S-module of $\mathbf{C}$-derivations of S. We say that $\theta \in \mathrm{Der}_S$ is homogeneous of degree q if $\theta(S_1) \subseteq S_q$. Choose for each hyperplane $H \in \mathcal{A}$ a linear form $\alpha_H \in V^*$ such that $H = \ker(\alpha_H)$. Define $Q \in S$ by

$$Q = \prod_{H \in \mathcal{A}} \alpha_H \,.$$

The polynomial Q is uniquely determined, up to a constant multiple, by the group W. When convenient we choose a basis $e_1, \dots, e_l$ for V and let $x_1, \dots, x_l$ denote the dual basis for V^*. Let $\langle \ , \ \rangle : V^* \times V \to \mathbf{C}$ denote the natural pairing. Thus $\langle x_i, e_j \rangle = \delta_{ij}$. For each $v \in V$ let $\partial_v \in \mathrm{Der}_S$ be the unique derivation such that $\partial_v x = \langle x, v \rangle$ for $x \in V^*$. Define $\partial_i \in \mathrm{Der}_S$ by $\partial_i = \partial_{e_i}$. Then $\partial_i x_j = \delta_{ij}$ and Der_S is a free S-module

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with basis $\partial_1, \ldots, \partial_l$. There is a natural isomorphism $S \otimes V \to \mathrm{Der}_S$ of S-modules given by

$$f \otimes v \mapsto f\partial_v$$

for $f \in S$ and $v \in V$. Let $\Omega^1 = \mathrm{Hom}_S(\mathrm{Der}_S, S)$ be the S-module dual to Der_S. Define $d : S \to \Omega^1$ by $df(\theta) = \theta(f)$ for $f \in S$ and $\theta \in \mathrm{Der}_S$. Then $d(ff') = (df)f' + f(df')$ for $f, f' \in S$. Furthermore, Ω^1 is a free S-module with basis $dx_1, \ldots, dx_l$ and $df = \sum_{i=1}^{l} (\partial_i f)dx_i$. There is a natural isomorphism $S \otimes V^* \to \Omega^1$ of S-modules given by

$$f \otimes x \mapsto f dx$$

for $f \in S$ and $x \in V^*$. The modules Der_S and Ω^1 inherit gradings from S which are defined by $\deg(f\partial_v) = \deg(f)$ and $\deg(fdx) = \deg(f)$ if $f \in S$ is homogeneous. Define $\Omega^p = \bigwedge_S^p \Omega^1$ $(p = 1, \ldots, \ell)$. Let $\Omega^0 = S$. The S-module Ω^p is free with a basis $\{dx_{i_1} \wedge \cdots \wedge dx_{i_p} \mid 1 \le i_1 < \cdots < i_p \le \ell\}$. It is naturally isomorphic to $S \otimes_{\mathbf{C}} \bigwedge^p V^*$. Let $\Omega^p(\mathcal{A})$ be the S-module of *logarithmic p-forms* with poles along $\mathcal{A}$ [Sai3][OrT]:

$$\Omega^p(\mathcal{A}) = \left\{ \frac{\eta}{Q} \,\middle|\, \eta \in \Omega^p, d\left(\frac{\eta}{Q}\right) \in \frac{1}{Q}\Omega^{p+1} \right\}$$

where d is the exterior differentiation.

The unitary reflection group W acts contragradiently on V^* and thus on S. The modules Der_S and Ω^p $(p = 0, \ldots, \ell)$ also have W-module structures so that the above isomorphisms are W-module isomorphisms. If M is an $\mathbf{C}[W]$-module let $M^W = \{x \in M \mid wx = x \text{ for all } w \in W\}$ denote the space of invariant elements in M. Let $M^{\det^{-1}} = \{x \in M \mid wx = \det(w)^{-1}x \text{ for all } w \in W\}$ denote the space of anti-invariant elements in M. Let $R = S^W$ be the invariant subring of S under W. By a theorem of Shephard, Todd, and Chevalley [Bou, V.5.3, Theorem 3] there exist algebraically independent homogeneous polynomials $f_1, \ldots, f_l \in R$ such that $R = \mathbf{C}[f_1, \ldots, f_l]$. They are called *basic invariants.* Elements of $S^{\det^{-1}}$ and $(\Omega^p)^{\det^{-1}}$ are called *anti-invariants* and *anti-invariant p-forms* respectively. It is well-known that $S^{\det^{-1}} = RQ$.

The main theorem. The following theorem gives the relationship between logarithmic forms and anti-invariant forms.

Theorem 1. *For* $0 \le p \le \ell$,

$$\Omega^p(\mathcal{A}) = \frac{1}{Q}(\Omega^p)^{\det^{-1}} \otimes_R S.$$

Proof. When p=0, the result follows from the formula $S^{\det^{-1}}$=RQ. Let $p>0$. Let $x_1, \dots, x_\ell$ be an orthonormal basis for V^*. Let $\theta_1, \dots, \theta_\ell$ be an R-basis for Der_S^W. Then, by [OrT, Theorem 6.59], $\theta_1, \dots, \theta_\ell$ is known to be an S-basis for the module $D(\mathcal{A})$ of $\mathcal{A}$-derivations, where

$$D(\mathcal{A}) = \{\theta \in \mathrm{Der}_S \mid \theta(Q) \in QS\}.$$

By the contraction $\langle\ ,\ \rangle$ of a 1-form and a derivation, the S-modules $D(\mathcal{A})$ and $\Omega^1(\mathcal{A})$ are S-dual to each other [Sai3, p.268] [OrT, Theorem 4.75] . Let $\{\omega_1, \dots, \omega_\ell\} \subset \Omega^1(\mathcal{A})$ be dual to $\{\theta_1, \dots, \theta_\ell\}$. In other words, $\langle\theta_i, \omega_j\rangle = \delta_{ij}$ (Kronecker's delta). Then $\{\omega_1, \dots, \omega_\ell\}$ is an S-basis for $\Omega^1(\mathcal{A})$. Then each ω_i is obviously W-invariant and

$$\omega_i \in (\frac{1}{Q}\Omega^1)^W = \frac{1}{Q}(\Omega^1)^{\det^{-1}}.$$

Therefore we have

$$\Omega^1(\mathcal{A}) \subseteq \frac{1}{Q}(\Omega^1)^{\det^{-1}} \otimes_R S.$$

By [OrT, Proposition 4.81], the set $\{\omega_{i_1}\wedge\cdots\wedge\omega_{i_p} \mid 1 \le i_1 < \cdots < i_p \le \ell\}$ is a basis for $\Omega^p(\mathcal{A})$. In particular, $\omega_{i_1} \wedge \cdots \wedge \omega_{i_p} \in (1/Q)\Omega^p$. Since $\omega_{i_1} \wedge \cdots \wedge \omega_{i_p}$ is W-invariant, $Q(\omega_{i_1} \wedge \cdots \wedge \omega_{i_p}) \in (\Omega^p)^{\det^{-1}}$. This shows that

$$\Omega^p(\mathcal{A}) \subseteq \frac{1}{Q}(\Omega^p)^{\det^{-1}} \otimes_R S.$$

Conversely let $\omega \in (1/Q)(\Omega^p)^{\det^{-1}}$. Then $Q\omega \in \Omega^p \subseteq \Omega^p(\mathcal{A})$. Thus $Q\omega$ can be uniquely expressed as

$$Q\omega = \sum_{i_1<\cdots<i_p} f_{i_1\cdots i_p}\omega_{i_1} \wedge \cdots \wedge \omega_{i_p}\ (f_{i_1\cdots i_p} \in S).$$

Act $w \in W$ on both sides, and we get

$$\det(w)^{-1}Q\omega = w(Q)\omega = \sum_{i_1<\cdots<i_p} w(f_{i_1\cdots i_p})\omega_{i_1} \wedge \cdots \wedge \omega_{i_p}.$$

Therefore, by the uniqueness of the expression, we have

$$\det(w)^{-1}f_{i_1\cdots i_p} = w(f_{i_1\cdots i_p})\ (w \in W)$$

and $f_{i_1\cdots i_p} \in S^{\det^{-1}} = RQ$. This implies that each $f_{i_1\cdots i_p}/Q$ lies in S and that

$$\omega = \sum_{i_1<\cdots<i_p} \left(\frac{f_{i_1\cdots i_p}}{Q}\right)\omega_{i_1} \wedge \cdots \wedge \omega_{i_p} \in \Omega^p(\mathcal{A}).$$

Thus we have shown the inclusion

$$\frac{1}{Q}(\Omega^p)^{\det^{-1}} \otimes_R S \subseteq \Omega^p(\mathcal{A}).$$

Q.E.D.

Taking the W-invariant parts of the both sides in Theorem 1, we have

Corollary 2. *For* $0 \leq p \leq \ell$,

$$(\Omega^p(\mathcal{A}))^W = \frac{1}{Q}(\Omega^p)^{\det^{-1}}.$$

The following theorem is a special case of a theorem obtained by Shepler [She1].

Theorem 3 (Shepler). *For* $0 \leq p \leq \ell$,

$$(\Omega^p)^{\det^{-1}} = Q^{1-p} \bigwedge_R^p (\Omega^1)^{\det^{-1}}.$$

Proof. Let $p = 0$. We naturally interpret the "empty exterior product" to be equal to the coefficient ring. Thus the result follows from the formula $S^{\det^{-1}} = RQ$. Let $p > 0$. In the proof of Theorem 1, we have already shown that the both sides have the same R-basis

$$\{Q(\omega_{i_1} \wedge \cdots \wedge \omega_{i_p}) \mid 1 \leq i_1 < \cdots < i_p \leq \ell\}.$$

Q.E.D.

The Coxeter case. From now on we assume that W is a *Coxeter group*. In other words, for an ℓ-dimensional Euclidean space V, $W \subset \mathbf{GL}(V)$ is a finite group generated by orthogonal reflections and W acts irreducibly on V. The objects like S, R, and Ω^p are defined over $\mathbf{R}$. Note that $\det(w)$ is either $+1$ or -1 for any $w \in W$ and thus $\det = \det^{-1}$.

Recall the definition of the $\mathbf{R}$-linear map $\hat{d} : S \longrightarrow \Omega^1$ in [SoT, Proposition 6.1]:

$$\hat{d}f = \sum_{i=1}^{\ell}(\partial_i f)d(Q(Dx_i)).$$

Here D is a Saito derivation introduced in [Sai2][SYS]. The following proposition is Proposition 6.1 in [SoT]:

Proposition 4 (Solomon-Terao). *Let $f_1, \dots, f_\ell$ be basic invariants. Then*

$$(\Omega^1)^{\det} = R\hat{d}f_1 \oplus \cdots \oplus R\hat{d}f_\ell.$$

From Theorem 3 and Proposition 4 we get

Corollary 5. *For $0 \leq p \leq \ell$,*

$$(\Omega^p)^{\det} = \bigoplus_{1 \leq i_1 < \cdots < i_p \leq \ell} RQ^{1-p}(\hat{d}f_{i_1} \wedge \cdots \wedge \hat{d}f_{i_p}).$$

Using Theorem 1, we have

Corollary 6. *For $0 \leq p \leq \ell$,*

$$\Omega^p(\mathcal{A}) = \bigoplus_{1 \leq i_1 < \cdots < i_p \leq \ell} SQ^{-p}(\hat{d}f_{i_1} \wedge \cdots \wedge \hat{d}f_{i_p}).$$

This corollary gives a new method using the new differential operator $\hat{d}$ to calculate a basis for the module of logarithmic forms.

Taking the W-invariant parts of the both sides in Corollary 6, we also have

Corollary 7. *For $0 \leq p \leq \ell$,*

$$(\Omega^p(\mathcal{A}))^W = \bigoplus_{1 \leq i_1 < \cdots < i_p \leq \ell} RQ^{-p}(\hat{d}f_{i_1} \wedge \cdots \wedge \hat{d}f_{i_p}).$$

Example 8 (B_2). When W is the Coxeter group of type B_2, we can choose

$$f_1 = \frac{1}{2}(x_1^2 + x_2^2),\ f_2 = \frac{1}{4}(x_1^4 + x_2^4).$$

Then, as seen in [SoT, §5.2], the operator $\hat{d}$ in Proposition 4 satisfies

$$\hat{d}x_1 = -dx_2,\ \hat{d}x_2 = dx_1.$$

Thus

$$\hat{d}f_1 = -x_1dx_2 + x_2dx_1,\ \hat{d}f_2 = -x_1^3dx_2 + x_2^3dx_1.$$

Then $\hat{d}f_1$ and $\hat{d}f_2$ form an R-basis for $(\Omega^1)^{\det}$ and $\hat{d}f_1/Q$ and $\hat{d}f_2/Q$ form an S-basis for $\Omega^1(\mathcal{A})$ as Corollaries 5 and 6 assert.

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