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Robinson-Schensted Correspondence and Left Cells

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§1. Introduction

This is based on [A]. In [A], I explained several theorems which focused on a famous theorem of [KL] that two elements of the symmetric group belong to a same left cell if and only if they share a common Q-symbol. The first half of [A] was about a direct proof of this theorem (Theorem A), and the second half was about relation between primitive ideals and left cells, and I explained another proof of this theorem.

The reason why I gave a direct proof which was different from the proof in [KL] was that the proof in [KL] was hard to read: It relied on [V1, 6], which in turn relied on [Jo1], the full paper of which is not yet available even today. Note also that the theorem itself is not stated in [KL]. But the beginning part of the proof of [KL, Theorem 1.4] gives some explanation on the relation between left cells in the sense of Kazhdan and Lusztig and Vogan's generalized τ-invariants in the theory of primitive ideals. In this picture, Theorem A is derived from Joseph's theorem.

Lack of a clear proof in the literature lead Garsia and McLarnan to the publication of [GM]. [1] The proof given in [GM] is close to [A], but the line of the proof in [GM] is interrupted with combinatorics of tableaux, which is not necessary. In fact, after we read to the fourth section of [KL], which is the section for some preliminaries to the proof of [KL, Theorem 1.4], we can give a short and elementary proof of Theorem A in a direct way, as I will show below.

I rush to say that my proof was not so original: It copied argument in [Ja1, Satz 5.25] for Joseph's theorem. This was the reason why I

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[1]It is worth mentioning that Garsia and McLarnan wrote in [GM] that they were benefitted by A.Björner's lecture notes and R.King's lecture notes. Both of these notes are still not available, and it seems that preliminary version of them were circulated in a very restricted group of people around the time.

did not publish it in English. But after a decade has passed, we still have no suitable literature which includes the direct proof. Further, we have new development in the last decade. For example, we have better understanding of this theorem in *jeu de taquin* context (see [H],[BSS]); the study of solvable lattice models in Kyoto school lead to the theory of crystal bases and canonical bases, by which we can understand Theorem A in the crystal base theory context.

I have therefore decided to add this short note to this volume in order to explain this proof and new development. I also prepare enough papers in the references for reader's convenience. I note that there is a sketch of proof of Theorem A in [BV, p.172]. It involves the notion of wave front sets. I do not recommend [BV] for knowing the proof of Theorem A. One reason is that we do not need wave front sets for the proof of Theorem A itself.

The non-direct proof I explained in the second half of [A] is the proof which is indicated in [KL], and the proof was taken from Jantzen's lecture notes [Ja1]. Thus I do not reproduce it, and I only give statements of several theorems (Theorem B,C,D) which concludes Theorem A.

I give some bibliographical comments on the second proof. It is obtained by combining two theorems; one known as Joseph's theorem, which states that two primitive ideals with a same integral regular central character coincide if and only if their Q-symbol coincide (Theorem B), and another theorem due to Joseph and Vogan, which relates the inclusion relation of primitive ideals to non-vanishing condition of certain multiplicities (Theorem C), and thus to order relation of left cells of the symmetric group (Theorem D).

In a survey [Bo], it is stated that Theorem A was proved in [Jo1], and simple proof could be found in [V1] and [Ja1]. But as I stated, [Jo1] is not published, and [V1] is based on [Jo1]. Thus to read [V1], one has to reproduce arguments by oneself. Nevertheless, it is Joseph's theorem, and his idea came from the explicit form of Goldie rank representations in type A case [Jo2, Proposition 8.4], which makes the number of primitive ideals with a common regular central character equal to the number of involutions. That is, Duflo's map is bijective, which proves Theorem B. The proof of Theorem B is given in [Jo3, Corollary 5.3]. We do not follow his line and I refer to [Ja1, Satz 5.25].

Theorem C is proved in [V2, Theorem 3.2]. One implication is due to [Jo5, Theorem 5.3], which is reformulated in [V2, Proposition 3.1]. It is not difficult to derive Theorem D: That Theorem C implies Theorem D (see also [Jo5, Conjecture C]) is stated in the introduction of [V2]. The proof here is based on [Ja1, Corollar 7.13] and [Ja1, Lemma 14.9]. Since the Kazhdan-Lusztig conjecture proved by Brylinski-Kashiwara and Beilinson-Berstein is very well known, I take it for granted when I explain Theorem D. But it is of cource a very deep result.

Joseph's Goldie rank representation was related to Springer representation and the theory evolved into a beautiful geometric representation theory. Since this part is not at all combinatorial, I do not mention it. I only refer to [BB3] for this development. There is also new direction for the generalization of the Robinson-Schensted correspondence related to the primitive ideal theory. I refer [Ga] and [Tr]. [2]

§2. Preliminaries

2.1. P-symbols and Q-symbols

Let S_n be the symmetric group of degree n. Namely, its underlying set is the set of bijective maps from $\{1,\dots,n\}$ to itself, and the group structure is given by composition of maps. Let w be an element in S_n, and denote the image of $i \in \{1,\dots,n\}$ under the map w by w_i. Throughout the paper, we identify w with the sequence $w_1 \cdots w_n$, which is a permutation of $1,\dots,n$.

Let $\mathbb{N} = \{1,2,\dots\}$ be the set of natural numbers. A **Young diagram** λ is a finite subset of $\mathbb{N}\times\mathbb{N}$ which satisfies the condition that if $(x,y)\in\lambda$, then $\{(x-1,y),(x,y-1)\}\cap\mathbb{N}\times\mathbb{N}\in\lambda$. $(x,y)\in\lambda$ is called a **node** of λ. x is called the row number of the node, and y is called the column number of the node.

A **tableau** T of shape λ is a map from λ to $\mathbb{N}$. The image of a node (x,y) of λ under the map is called the entry of the node, and is denoted by $T(x,y)$. We only consider the tableaux satisfying

$$T(x,y) \le T(x',y') \quad (x \le x', y \le y').$$

If it also satisfies $T(x,y) < T(x',y)$ $(x < x')$ (resp. $T(x,y) < T(x,y')$ $(y < y')$), T is called a **column strict** (resp. **row strict**) **semi-standard tableau.** If the entries of T are precisely $\{1,\dots,n\}$, we call T a **standard tableau.**

Let T be a column strict semi-standard tableau, k be a natural number. We denote the set of nodes in the i th row by $R_i(T)$, and denote the maximal column number of the nodes in $R_i(T)$ by c_i. Assume that we are given a natural number k_i. If k_i is equal or greater than all entries of $R_i(T)$, we add the node (i,c_i+1) to $R_i(T)$ and make its entry be k_i.

[2]There is one more dirction: generalization of the Steinberg's theorem [St] is given by M. van Leeuwen. This is the direction to the geometry of flag varieties.

If it is not the case, we consider the nodes of $R_i(T)$ whose entries are greater than k_i, and pick up the node of minimal column number among them. We then change its entry to k_i, and we make k_{i+1} be the original entry of the node. This latter procedure is called **bumping procedure**. We set $k_1 = k$, and continue the bumping procedure until no bumping occurs. This is called a **row insertion algorithm**, and it results in a new column strict semi-standard tableau, which we denote by $T \leftarrow k$. In the similar way, we can define a column insertion algorithm $k \rightarrow T$.

Definition 2.1. *Let $w = w_1 \cdots w_n$ be a permutation. Two standard tableaux $P(w)$ and $Q(w)$ defined by*

$$\begin{aligned} P(w) &= \emptyset \leftarrow w_1 \leftarrow \cdots \leftarrow w_n, \\ Q(w) &= P(w^{-1}) \end{aligned}$$

are called the **P-symbol** *of w and the* **Q-symbol** *of w respectively. The correspondence between w and the pair $(P(w), Q(w))$ is called the* **Robinson-Schensted correspondence**. *We often write $P(w) = \emptyset \leftarrow w_1 \cdots w_n$ for short.*

It is known that $P(w) = w_1 \rightarrow \cdots \rightarrow w_k \rightarrow \emptyset \leftarrow w_{k+1} \leftarrow \cdots \leftarrow w_n$ holds for any k.

Example 2.2. *If $w = 31524$, then we have*

$$P(w) = \begin{matrix} 1 & 2 & 4 \\ 3 & 5 & \end{matrix} \qquad Q(w) = \begin{matrix} 1 & 3 & 5 \\ 2 & 4 & \end{matrix} .$$

More familiar definition of the Q-symbol is by the "recording" tableau, which records the node added by each insertion procedure. It is a well known theorem that it coincides with $P(w^{-1})$.

Remark We have a two dimensional pictorial algorithm to compute P-symbols and Q-symbols due to S.V.Fomin [Fo3, 4.2.4]. In his picture, we know at a glance that $Q(w)$ equals $P(w^{-1})$.

Remark If two elements in S_n have a common P-symbol, we say that these belong to a same left Knuth class. Similarly, if these have a common Q-symbol, we say that these belong to a same right Knuth class.

Although we do not go into the combinatorial structures of the Robinson-Schensted correspondence, it is worth referring to the relation of the Robinson-Schensted correspondence to the *jeu de taquin* sliding algorithm. Namely, the insertion algorithm may be viewed as *jeu de taquin* moves, and *jeu de taquin* equivalence classes are the same as left

Knuth classes. On the other hand, right Knuth classes are the same as Haiman's dual equivalence classes.

To be more precise, let λ_n be the staircase Young diagram of size $n(n-1)/2$. Then λ_{n+1}/λ_n consists of n one-node components. The tableaux of shape λ_{n+1}/λ_n are called **permutation tableaux**. By reading entries from left to right, we identify permutation tableaux with permutations of $1,\ldots,n$. This rule extends to more general tableaux. We read entries of such a tableau row by row from left to right starting with the node on the south-west end and ending up with the node on the north-east end. Then the corresponding permutation tableau is *jeu de taquin* equivalent to the original tableau.

We take a tableau T of shape λ_n, and compute **switching** of a permutation tableau w and T [BSS]. Since all tableaux of a same non skew shape are dual equivalent [BSS, Proposition 4.2], the non skew tableau produced by the switching is independent of the choice of T [BSS, Theorem 4.3], and it is *jeu de taquin* equivalent to the permutation tableau w [BSS, Theorem 3.1]. This is the P-symbol of w. Further, two permutation tableaux are in a same dual equivalence class if and only if they have a common Q-symbol [H, Theorem 2.12]. These give the Robinson-Schensted corespondence in the *jeu de taquin* context. In fact, this view point also appears in the crystal base theory [BKK].

2.2. KL polynomials

Let q be a variable, and let $\mathcal{H}_n$ be the Hecke algebra of the symmetric group S_n. Namely, $\mathcal{H}_n$ is the algebra over $\mathbb{Q}(q)$ defined by generators $T_1,\ldots,T_{n-1}$ and relations

$$(T_i-q)(T_i+1)=0, \quad T_iT_{i+1}T_i=T_{i+1}T_iT_{i+1}, \quad T_iT_j=T_jT_i \ (j\geq i+2).$$

Let $s_i=(i,i+1)$ be the transposition of i and $i+1$. We set $T_{s_i}=T_i$. For general $w\in S_n$, we find a reduced expression $w=s_{i_1}\cdots s_{i_r}$ and set $T_w=T_{i_1}\cdots T_{i_r}$. The length r of reduced expressions is constant on the set of reduced expressions of w, which is denoted by $l(w)$. It is also well known that T_w does not depend on the choice of the reduced expression, and $\{T_w|w\in S_n\}$ is a basis of $\mathcal{H}_n$. If y is obtained by the product of a subword of a reduced expression of w, we write $y\leq w$. This order is called **Bruhat order**.

Definition 2.3. *The following two conditions uniquely define the polynomials $P_{y,w}(q)\in\mathbb{Z}[q]$ $(y\leq w)$, which are called* **Kazhdan-Lusztig polynomials***:*

$$\begin{aligned} C_w &= \textstyle\sum_{y\leq w}(-1)^{l(w)-l(y)}q^{\frac{l(w)}{2}-l(y)}P_{y,w}(q^{-1})T_y \\ &= \textstyle\sum_{y\leq w}(-1)^{l(w)-l(y)}q^{-\frac{l(w)}{2}+l(y)}P_{y,w}(q)T_{y^{-1}}^{-1}, \end{aligned}$$

and $P_{w,w}(q) = 1, \quad \deg P_{y,w}(q) \leq \frac{l(w)-l(y)-1}{2} \quad (y < w).$

We call the first property the bar invariance property, and the second property the degree property. For the definition, we can use the following element instead of C_w.

$$C'_w = q^{-\frac{l(w)}{2}} \sum_{y \leq w} P_{y,w}(q) T_y$$

If $\frac{l(w)-l(y)-1}{2}$ is an integer, we denote the coefficient of $q^{\frac{l(w)-l(y)-1}{2}}$ in $P_{y,w}(q)$ by $\mu(y,w)$. If $\mu(y,w) \neq 0$ for $y < w$ or $\mu(w,y) \neq 0$ for $y > w$ occurs, we write $\mu(y|w) \neq 0$. Note that if we write $\mu(y|w) \neq 0$, it particularly implies that $y > w$ or $y < w$ holds.

The welldefinedness of $P_{y,w}(q)$ is non trivial, and in fact is one of the main theorems [KL, Theorem 1.1]. The uniqueness of $P_{y,w}(q)$ is easy to prove, but for the existence of these polynomials, we need to construct C_w $(w \in S_n)$. In [KL, 2.2], these C_w are inductively constructed by setting $C_e = 1$ and

$$C_w = C_{s_i} C_{s_i w} - \sum_{\substack{z < w, s_i z < z \\ \mu(z,w) \neq 0}} \mu(z, s_i w) C_z$$

for $s_i \in \mathcal{L}(w) = \{s_j | \ s_j w < w\}$. Since $C_{s_i} = q^{-\frac{1}{2}} T_i - q^{\frac{1}{2}}$, we can give an inductive definition of Kazhdan-Lusztig polynoimals as follows.

Definition 2.4.

$$\begin{aligned} P_{y,w}(q) &= q^{1-c} P_{s_i y, s_i w}(q) + q^c P_{y, s_i w}(q) \\ &\quad - \sum_{\substack{y \leq z \leq s_i w, s_i z < z \\ \mu(z, s_i w) \neq 0}} \mu(z, s_i w) q^{\frac{l(w)-l(z)}{2}} P_{y,z}(q) \end{aligned}$$

where $c = 1$ *if* $s_i y < y$ *and* $c = 0$ *if* $s_i y > y$.

That the right hand side does not depend on the choice of s_i comes from the welldefinedness result. We can also construct C_w by $C_e = 1$ and

$$C_w = C_{w s_i} C_{s_i} - \sum_{\substack{z < w, z s_i < z \\ \mu(z,w) \neq 0}} \mu(z, s_i w) C_z$$

for $s_i \in \mathcal{R}(w) = \{s_j | \ w s_j < w\}$, which leads to a similar inductive definition of Kazhdan-Lusztig polynomials. Note that $\mathcal{R}(w) = \mathcal{L}(w^{-1})$.

Lemma 2.5. *(1)* $P_{y,w}(0) = 1$.
(2) $y < w, s_i y > y, s_i w < w$ *imply* $P_{y,w}(q) = P_{s_i y,w}(q)$.
(3) $P_{y^{-1},w^{-1}}(q) = P_{y,w}(q)$.

(1) follows from the inductive definition 2.4. (2) is proved by induction on $l(y)$, which also uses 2.4. (3) follows from the first definition of Kazhdan-Lusztig polynomials: If we replace $P_{y,w}(q)$ by $P_{y^{-1},w^{-1}}(q)$ in the definition of C_w and apply the anti-involution defined by $T_i \mapsto T_i$, we have $C_{w^{-1}}$. Thus we have the bar invariance property. The degree property is obvious. Hence $P_{y,w}(q)$ and $P_{y^{-1},w^{-1}}(q)$ must coincide.

Another corollary of this construction of C_w is the **Kazhdan-Lusztig representation** of the regular representation, which is the matrix representation with respect to the basis $\{C_w\}$. For the left regular representation, we have

$$T_i C_w = \begin{cases} -C_w & (\text{if } s_i w < w) \\ qC_w + q^{\frac{1}{2}} C_{s_i w} + \sum\limits_{\substack{z<w, s_i z<z \\ \mu(z,w)\neq 0}} \mu(z,w) q^{\frac{1}{2}} C_z & (\text{if } s_i w > w). \end{cases}$$

We have the same formula for the right regular representation. We can now introduce the notion of left cells and right cells.

Definition 2.6. *If there exists a sequence* $y = x_1, x_2, \ldots, x_r = w$ *such that* $\mathcal{L}(x_i) \not\subset \mathcal{L}(x_{i+1})$, $\mu(x_i|x_{i+1}) \neq 0$ *for* $1 \le i < r$, *we write* $y \underset{L}{\le} w$.

If there exists a sequence $y = x_1, x_2, \ldots, x_r = w$ *such that* $\mathcal{R}(x_i) \not\subset \mathcal{R}(x_{i+1})$, $\mu(x_i|x_{i+1}) \neq 0$ *for* $1 \le i < r$, *we write* $y \underset{R}{\le} w$.

Note that $y \underset{R}{\le} w$ *if and only if* $y^{-1} \underset{L}{\le} w^{-1}$. *If both* $y \underset{L}{\le} w$ *and* $w \underset{L}{\le} y$ *hold, we write* $y \underset{L}{\sim} w$. *Similarly, if both* $y \underset{R}{\le} w$ *and* $w \underset{R}{\le} y$ *hold, we write* $y \underset{R}{\sim} w$.

These relations partition S_n *into equivalence classes, which are called* **left cells** *and* **right cells** *respectively.*

At a first look, the definition of the relation $y \underset{L}{\le} w$ seems to be very artificial. To understand it in a more natural way, we set $q = 1$ and denote $C_w|_{q=1}$ by $a(w)$. (The specialization to $q = 1$ is only for simplifying the situation to more familiar setting of the symmetric group, and is not at all essential.) Then we have the following lemma by using Lemma 2.5 and the Kazhdan-Lusztig representation specialized to $q = 1$.

Lemma 2.7. *Let $y \neq w$ be two elements of S_n. Then we have (1)$\Leftrightarrow$(2) where*

(1) $s_i \in \mathcal{L}(y) \setminus \mathcal{L}(w)$ and $\mu(y|w) \neq 0$.

(2) $a(y)$ appears in $s_i a(w)$.

Hence, the left regular representation of the symmetric group with the specific basis $\{a(w)\}$ gives a natural meaning of the relation $y \underset{L}{\leq} w$ as follows.

Let $\overline{V_w}^L$ be the left ideal uniquely defined by the following three conditions.

(1) $a(w) \in \overline{V_w}^L$.

(2) $\overline{V_w}^L$ is spanned by a subset of $\{a(x)\}$.

(3) If a left ideal satisfies (1) and (2), it contains $\overline{V_w}^L$.

Then we have $y \underset{L}{\leq} w \Leftrightarrow \overline{V_y}^L \subset \overline{V_w}^L$. Similar formula exists for $y \underset{R}{\leq} w$.

§3. The RS correspondence and the left cell

3.1. The Kazhdan-Lusztig theorem

The following theorem is the theorem of Kazhdan and Lusztig which we are going to prove.

Theorem A For $y, w \in S_n$, we have $y \underset{L}{\sim} w \Leftrightarrow Q(y) = Q(w)$.

Example 3.1 (The S_3 case). *Left cells are* $\{123\}$, $\{213, 312\}$, $\{132, 231\}$ *and* $\{321\}$. *Their Q-symbols are*

$$\begin{matrix} 1 & 2 & 3 \end{matrix}\,, \quad \begin{matrix} 1 & 3 \\ 2 & \end{matrix}\,, \quad \begin{matrix} 1 & 2 \\ 3 & \end{matrix}\,, \quad \begin{matrix} 1 \\ 2 \\ 3 \end{matrix}\,.$$

For the S_4 case, see [Shi, p.20].

3.2. A theorem of Knuth

We write $y \equiv w$ if $P(y) = P(w)$. To describe this equivalence relation, we introduce Knuth relations.

Definition 3.2. *Let $y_1 \cdots y_n$ be a permutation of $1, \dots, n$. We set w as follows.*

If $y_{i+1} < y_i < y_{i+2}$, we set $w = y_1 \cdots y_i y_{i+2} y_{i+1} \cdots y_n$.

If $y_{i+1} < y_{i+2} < y_i$, we set $w = y_1 \cdots y_{i+1} y_i y_{i+2} \cdots y_n$.

We have $y \equiv w$, and we say that y and w are in Knuth relation.

The following theorem is due to Knuth [K, Theorem 6].

Theorem 3.3. *Let $y, w \in S_n$. Then $y \equiv w$ if and only if these permutations are connected by a chain of Knuth relations.*

Let $D_{ij} := \{ w \mid ws_i < w, ws_j > w \}$ where $j = i \pm 1$. If $y \in D_{ij}$, we consider the right coset $y < s_i, s_j >$ and take the distinguished coset representative y^0. Then we have either $y = y^0 s_i$ or $y^0 s_j s_i$. We set $K_{ij}(y) = y^0 s_i s_j$ in the former case, and $K_{ij}(y) = y^0 s_j$ in the latter case. Note that K_{ij} is a bijective map from D_{ij} to D_{ji}. If $j = i+1$, this is the rule to obtain w from y in the Knuth relation, and if $j = i-1$, this is the rule to obtain y from w in the Knuth relation. This description of Knuth relations is convenient for our purpose. The following lemma shows that two elements in Knuth relation are in a right cell.

Lemma 3.4. *If $w \in D_{ij}$, we have $K_{ij}(w) \underset{R}{\sim} w$.*

(Proof) Since $w \in D_{ij}$, we have $w = w^0 s_i$ or $w = w^0 s_j s_i$ where w^0 is the distinguished coset representative of $w\langle s_i, s_j\rangle$. By the same proof as in Lemma 2.7 we have $\mu(w^0 s_i, w^0 s_i s_j) = 1$, and $\mu(w^0 s_j, w^0 s_j s_i) = 1$. In either cases, we have $\mu(w|K_{ij}(w)) \neq 0$. Since $s_i \in \mathcal{R}(w) \setminus \mathcal{R}(K_{ij}(w))$ and $s_j \in \mathcal{R}(K_{ij}(w)) \setminus \mathcal{R}(w)$, we have $\mathcal{R}(w) \not\subset \mathcal{R}(K_{ij}(w))$ and $\mathcal{R}(w) \not\supset \mathcal{R}(K_{ij}(w))$. We have the result. Q.E.D

Remark We have another way to describe the Knuth relation as follows.

If $w < s_i w$ and $\mathcal{L}(w) \not\subset \mathcal{L}(s_i w)$, then we have $w^{-1} \equiv w^{-1} s_i$.

In fact, if we take $s_j \in \mathcal{L}(w) \setminus \mathcal{L}(s_i w)$, the choice of s_i, s_j leads to $s_j w < w$ and $s_j s_i w > s_i w$. These s_i and s_j can not be commutable elements. Thus $w^{-1} \in D_{ji}$ and we are in the latter case in the definition of K_{ij}. If we consider the case that $y < s_j y$ and $\mathcal{L}(y) \not\subset \mathcal{L}(s_j y)$, where we take $s_i \in \mathcal{L}(y) \setminus \mathcal{L}(s_j y)$ such that $y^{-1} \in D_{ij}$, we meet the former case in the definition of K_{ij}, and we have $y^{-1} \equiv y^{-1} s_j$. But this statement is the same as the previous case.

3.3. Preparatory results for the proof of Theorem A

The following three propositions are proved in [KL]. I avoid repetition as long as the readability of the proof is guaranteed.

Proposition 3.5 ([KL, Proposition 2.4]). *If $y \underset{L}{\leq} w$, then we have* $\mathcal{R}(y) \supset \mathcal{R}(w)$.

(Proof) It is enough to prove it for the case that $\mathcal{L}(y) \not\subset \mathcal{L}(w)$ and $\mu(y|w) \neq 0$. By Lemma 2.7, we have $y = s_i w > w$ for some i or $y < w$ and $\mu(y,w) \neq 0$. In the former case, we consider the double coset $\langle s_i \rangle w \langle s_j \rangle$ for each $s_j \in \mathcal{R}(w)$. Then we can easily conclude that $s_j \in \mathcal{R}(s_i w)$. Thus we have $\mathcal{R}(y) \supset \mathcal{R}(w)$. In the latter case, we assume to the contrary that there is $s_j \in \mathcal{R}(w) \setminus \mathcal{R}(y)$. By Lemma 2.5(3), our assumption $\mu(y|w) \neq 0$ is equal to $\mu(y^{-1}|w^{-1}) \neq 0$. We also have $s_j \in \mathcal{L}(w^{-1}) \setminus \mathcal{L}(y^{-1})$. By Lemma 2.7, we know that $a(w^{-1})$ appears in $s_j a(y^{-1})$. Since $y < w$, we have $w^{-1} > y^{-1}$ and thus we have $w^{-1} = s_j y^{-1}$. By the same argument in the former case, $w = y s_j > y$ implies $\mathcal{L}(y) \subset \mathcal{L}(w)$. It contradicts our assumption that $\mathcal{L}(y) \not\subset \mathcal{L}(w)$. Q.E.D

Proposition 3.6 ([KL, Theorem 4.2]). *If $y \neq w \in D_{ij}$ and $\mu(y|w) \neq 0$, then we have $\mu(K_{ij}(y)|K_{ij}(w)) \neq 0$.*

Remark By the definition of D_{ij}, there are two possibilities for y and w respectively. Namely,

$$\begin{aligned} ys_i < y &= K_{ij}(y)s_j < ys_j = K_{ij}(y) < K_{ij}(y)s_i, \\ ys_j > y &= K_{ij}(y)s_i > ys_i = K_{ij}(y) > K_{ij}(y)s_j, \end{aligned}$$

$$\begin{aligned} ws_i < w &= K_{ij}(w)s_j < ws_j = K_{ij}(w) < K_{ij}(w)s_i, \\ ws_j > w &= K_{ij}(w)s_i > ws_i = K_{ij}(w) > K_{ij}(w)s_j. \end{aligned}$$

Let y_i, w_i, $(i = 1, 2)$ and s, t be as follows.
(a) If both y and w are in the former case, we set

$$y_1 = K_{ij}(y),\ y_2 = y,\ s = s_j,\ t = s_i,\ w_1 = K_{ij}(w),\ w_2 = w.$$

(b) If both y and w are in the latter case, we set

$$y_1 = y,\ y_2 = K_{ij}(y),\ s = s_i,\ t = s_j,\ w_1 = w,\ w_2 = K_{ij}(w).$$

(c) If y is in the latter case and w is in the former case, we set

$$w_1 = K_{ij}(y),\ w_2 = y,\ s = s_j,\ t = s_i,\ y_1 = K_{ij}(w),\ y_2 = w.$$

Then we are reduced to the following two cases.

$$\begin{array}{ll} (1)\ y_2t < y_2 = y_1s < y_1 < y_1t, & w_2t < w_2 = w_1s < w_1 < w_1t, \\ (2)\ y_2t < y_2 = y_1s < y_1 < y_1t, & w_1s < w_1 < w_1t = w_2 < w_2s. \end{array}$$

We have to show $\mu(y_1|w_1) = \mu(y_2|w_2)$ for these two cases. Then we have come to the beginning of the proof in [KL, Theorem 4.2(ii)].

Proposition 3.7 ([KL, Corollary 4.3]). *Let $y, w \in D_{ij}$. Then $y \underset{L}{\sim} w$ implies $K_{ij}(y) \underset{L}{\sim} K_{ij}(w)$.*

(Proof) We can assume that $\mathcal{L}(y) \not\subset \mathcal{L}(w)$ ($\mathcal{L}(y) \not\supset \mathcal{L}(w)$), and $\mu(y|w) \neq 0$. By Lemma 3.4 and Proposition 3.5, we have $\mathcal{L}(K_{ij}(y)) = \mathcal{L}(y)$, $\mathcal{L}(K_{ij}(w)) = \mathcal{L}(w)$. Hence we have $\mathcal{L}(K_{ij}(y)) \not\subset \mathcal{L}(K_{ij}(w))$ $((\mathcal{L}(K_{ij}(y)) \not\supset \mathcal{L}(K_{ij}(w)))$. We also have $\mu(K_{ij}(y)|K_{ij}(w)) \neq 0$ by Proposition 3.6. We are through. Q.E.D

3.4. Proof of Theorem A

One implication is easy.

Proposition 3.8. *If $Q(y) = Q(w)$, then $y \underset{L}{\sim} w$.*

(Proof) Since $Q(y) = P(y^{-1})$ and $Q(w) = P(w^{-1})$, y^{-1} is connected to w^{-1} by a chain of Knuth relations. Thus it is enough to prove that $w^{-1} = K_{ij}(y^{-1})$ ($y^{-1} \in D_{ij}$) implies $y \underset{L}{\sim} w$. But Lemma 3.4 shows that $y^{-1} \underset{R}{\sim} w^{-1}$, which is $y \underset{L}{\sim} w$. Q.E.D

It remains to prove that $y \underset{L}{\sim} w$ implies $Q(y) = Q(w)$. For each partition π, we define a standard tableau P_π by setting the entries of the i th column of P_π to be $\sum_{j=1}^{i-1} l_j + 1, \ldots, \sum_{j=1}^{i} l_j$ from top to bottom, where $l_1, l_2, \ldots$ are column lengths of π. We denote the shapes of $Q(y)$ and $Q(w)$ by π_1 and π_2 respectively. We define $\hat{y}$, $\hat{w}$ by $(P(\hat{y}), Q(\hat{y})) = (P_{\pi_1}, Q(y))$ and $(P(\hat{w}), Q(\hat{w})) = (P_{\pi_2}, Q(w))$. By Proposition 3.8, we have $y \underset{L}{\sim} \hat{y}$ and $w \underset{L}{\sim} \hat{w}$. Thus we have $\hat{y} \underset{L}{\sim} \hat{w}$. To prove that $Q(\hat{y}) = Q(\hat{w})$, we define y' and w" by

$$(P(y'), Q(y')) = (P_{\pi_1}, P_{\pi_1}) \qquad (P(w"), Q(w")) = (P_{\pi_2}, P_{\pi_2}).$$

By the theorem of Knuth, we can write

$$\begin{aligned} y' &= K_{i_1 j_1} \circ \cdots \circ K_{i_r j_r}(\hat{y}) \\ w" &= K_{i'_1 j'_1} \circ \cdots \circ K_{i'_s j'_s}(\hat{w}) \end{aligned}$$

We shall define w' and y" by

$$\begin{aligned} w' &= K_{i_1j_1}\circ\cdots\circ K_{i_rj_r}(\hat{w}) \\ y" &= K_{i'_1j'_1}\circ\cdots\circ K_{i'_sj'_s}(\hat{y}) \end{aligned}$$

Recall that Proposition 3.5 tells that $\mathcal{R}(\hat{y}) = \mathcal{R}(\hat{w})$. Hence $\hat{y}\in D_{i_rj_r}$ implies $\hat{w}\in D_{i_rj_r}$. We then have welldefined $K_{i_rj_r}(\hat{w})$, which satisfies $K_{i_rj_r}(\hat{y}) \underset{L}{\sim} K_{i_rj_r}(\hat{w})$ by Proposition 3.7. We continue the argument and conclude that these y" and w' are welldefined. y' and w' satisfy $\mathcal{R}(y') = \mathcal{R}(w')$, $P(y') = Q(y') = P_{\pi_1}$ and $P(w') = P_{\pi_2}$. Similarly, y" and w" satisfy $\mathcal{R}(y") = \mathcal{R}(w")$, $P(y") = P_{\pi_1}$ and $P(w") = Q(w") = P_{\pi_2}$. Note that y' is the permutation

$$l_1, l_1-1, \cdots, 1, l_1+l_2, \cdots, l_1+1, \cdots.$$

Similarly, w" is the permutation

$$l'_1, l'_1-1, \cdots, 1, l'_1+l'_2, \cdots, l'_1+1, \cdots.$$

where we denote column lengths of π_2 by $l'_1, l'_2, \ldots$. Since $\mathcal{R}(y') = \mathcal{R}(w')$, the first l_1 letters of w' are in the decreasing order, the next l_2 letters are in the decreasing order, etc. Similarly, the first l'_1 letters of y" are in the decreasing order, the next l'_2 letters are in the decreasing order, etc.

By inserting the first l_1 letters of w' to $\emptyset$, we know that the first column of π_2 must have the length equal or greater than l_1. By using y", we have the opposite inequality. We have $l_1 = l'_1$. It also implies that the next l_2 decreasing letters of w' do not produce bumping, since if otherwise we have $l'_1 > l_1$. Thus we have that $l'_2 \geq l_2$. We use y" to have the opposite inequality. Continuing the same argument, we conclude that $\pi_1 = \pi_2$ and $Q(w') = P_{\pi_2}$. (We also have $Q(y") = P_{\pi_1}$.) Therefore, we have $y' = w'$, which implies $\hat{y} = \hat{w}$. We have proved $Q(y) = Q(w)$. Q.E.D

3.5. Theorem A in the crystal base theory context

An occurence of the Robinson-Schensted algorithm in the tensor product representation of the vector representation of $U_q(\mathfrak{gl}_n)$ was first observed in [DJM]. The tensor product representation itself can be viewed as an example of Demazure modules [KMOTU, Theorem 3.1], and we may consider generalization into this direction, but we restrict ourselves to the original case. Then the crystal base is induced by the canonical base and we now have a good understanding of the base (see [SV]) and of the Robinson-Schensted algorithm in the crystal base theory context.

Let U_q be the quantum algebra of $\mathfrak{gl}_r$, and Δ be Lusztig's coproduct:

$$\begin{aligned}
\Delta(e_i) &= \; e_i \otimes 1 + q^{\epsilon_i - \epsilon_{i+1}} \otimes e_i \quad (i = 1, \ldots, r{-}1), \\
\Delta(f_i) &= \; 1 \otimes f_i + f_i \otimes q^{-\epsilon_i + \epsilon_{i+1}} \quad (i = 1, \ldots, r{-}1), \\
\Delta(q^h) &= \; q^h \otimes q^h \quad (h \in \mathbb{Z}\epsilon_1 + \cdots + \mathbb{Z}\epsilon_n).
\end{aligned}$$

Let $V = \mathbb{Q}(q)^r$ be its vector representation given by

$$e_i = E_{i,i+1}, \quad f_i = E_{i+1,i}, \quad q^{\epsilon_i} = qE_{ii} + \sum_{j \neq i} E_{jj}$$

where E_{ij} are matrix units. Natural base elements $v_1 = (1, 0, \ldots, 0)^{\mathrm{T}}$, $v_2 = (0, 1, \ldots, 0)^{\mathrm{T}}, \ldots$ induce a base at $q = \infty$ in the sense of Kashiwara-Lusztig. In the following, we exclusively work with bases at infinity, and call them crystal bases instead of bases at $q = \infty$. We set $L = \oplus_{i=1}^{r} \mathbb{Q}[q^{-1}]_{(q^{-1})} v_i$, $B = \{ v_i \bmod q^{-1}L\} \subset L/q^{-1}L$. Then (L, B) is the crystal base of V stated above, and $V^{\otimes n}$ has $(L^{\otimes n}, B^{\otimes n})$ as its crystal base. To describe the tensor structure on $B^{\otimes n}$, we introduce $\varphi_i(b), \epsilon_i(b)$ by

$$\varphi_i(b) = \max\{ k |\ \tilde{f}_i^k(b) \neq 0\}, \quad \epsilon_i(b) = \max\{ k |\ \tilde{e}_i^k(b) \neq 0\}$$

where $\tilde{e}_i$ and $\tilde{f}_i$ are Kashiwara operators. Then

$$\begin{aligned}
\tilde{e}_i(b_1 \otimes b_2) &= \begin{cases} b_1 \otimes \tilde{e}_i(b_2) & (\epsilon_i(b_1) \leq \varphi_i(b_2)), \\ \tilde{e}_i(b_1) \otimes b_2 & (\epsilon_i(b_1) > \varphi_i(b_2)), \end{cases} \\
\tilde{f}_i(b_1 \otimes b_2) &= \begin{cases} b_1 \otimes \tilde{f}_i(b_2) & (\epsilon_i(b_1) < \varphi_i(b_2)), \\ \tilde{f}_i(b_1) \otimes b_2 & (\epsilon_i(b_1) \geq \varphi_i(b_2)). \end{cases}
\end{aligned}$$

Let $V_q(\lambda)$ be the irreducible highest weight module of U_q associated with $\lambda = \sum \lambda_i \epsilon_i$. We identify λ with the corresponding Young diagram. Then it is well known that $V_q(\lambda) \otimes V$ is multiplicity free. Hence, we can uniquely define the submodule of $V^{\otimes n}$ for each increasing sequence of Young diagrams. We identify the increasing sequence with a standard tableau Q, which we call the recording tableau. We denote the submodule by $V_q(Q)$. If the shape of Q is λ, we have $V_q(Q) \simeq V_q(\lambda)$.

Proposition 3.9. *(1) Let Q be a standard tableau of size n–1, and $\mathcal{T}_Q$ be the set of tableaux obtained from Q by adding $\boxed{n}$. Let $(L(Q), B(Q))$ be a crystal base of $V_q(Q)$. We set $L(T) = (L(Q) \otimes L) \cap V_q(T)$. Then we have*

$$L(Q) \otimes L = \bigoplus_{T \in \mathcal{T}_Q} L(T).$$

We nextly set $B(T) = (B(Q) \otimes B) \cap (L(T)/q^{-1}L(T))$. *Then we have*

$$(L(Q) \otimes L, B(Q) \otimes B) = \bigoplus_{T \in \mathcal{T}_Q} (L(T), B(T)).$$

(2) Let $L(Q) = L^{\otimes n} \cap V_q(Q)$. *Then we have* $L^{\otimes n} = \oplus L(Q)$. *If we further set* $B(Q) = B^{\otimes n} \cap (L(Q)/q^{-1}L(Q))$, *we have* $(L^{\otimes n}, B^{\otimes n}) = \oplus(L(Q), B(Q))$.

(Proof) (1) Let v_T be the highest weight vector which generates the highest weight space of $L(T)$. Since $L(T)$ and the lattice generated by $\tilde{f}_{i_1} \cdots \tilde{f}_{i_*} v_T$ are crystal lattices of $V_q(T)$, the uniqueness theorem of crystal bases concludes that they coincide. The uniqueness theorem also guarantees that there exists an automorphism of $V_q(Q) \otimes V$ such that it maps $L(Q) \otimes L$ to $\oplus L(T)$. Since $V_q(Q) \otimes V$ is multiplicity free, the automorphism is scalar multiplication on each $V_q(T)$. Thus by looking at highest weight spaces, we have that the automorphism is the identity. By descending induction on weights, we can prove $B(Q) \otimes B = \sqcup B(T)$.
(2) We prove it by induction on n. Assume that it holds for n. Then we have $(L(Q) \otimes L, B(Q) \otimes B) = \oplus(L(T), B(T))$ by (1) where

$$\begin{aligned} L(T) &= (L(Q) \otimes L) \cap V_q(T) = ((L^{\otimes n} \cap V_q(Q)) \otimes L) \cap V_q(T) \\ &= \left(L^{\otimes n+1} \cap V_q(Q) \otimes L\right) \cap V_q(T) \\ &= \left(L^{\otimes n+1} \cap V_q(Q) \otimes V\right) \cap V_q(T) \\ &= L^{\otimes n+1} \cap V_q(T). \end{aligned}$$

Q.E.D

The crystal graph of $V_q(\lambda)$ has description in terms of semistandard tableaux as follows [KN].

We write $\boxed{i}$ for $v_i \bmod q^{-1} \in B$. Let $B(\lambda)$ be the set of column strict semi-standard tableaux of shape λ. For each $T \in B(\lambda)$, we read its entries row by row, starting from the bottom row. This reading gives an injection from $B(\lambda)$ to $B^{\otimes n}$. For example, we have

$$\begin{array}{llll} 1 & 1 & 2 & 4 \\ 2 & 3 & & \\ 4 & & & \end{array} \qquad \Longrightarrow \boxed{4} \otimes \boxed{2} \otimes \boxed{3} \otimes \boxed{1} \otimes \boxed{1} \otimes \boxed{2} \otimes \boxed{4}.$$

We induce the crystal structure on $B(\lambda)$ through this inclusion: the Kashiwara operators $\tilde{e}_i, \tilde{f}_i$ act on these monomial tensors by changing the leftmost $\boxed{i+1}$ or the rightmost $\boxed{i}$ of the sequence which is obtained by removing consecutive $\boxed{i+1} \otimes \boxed{i}$ as many as possible. Thus if $\tilde{e}_i T \neq$

$0(\tilde{f}_i T \neq 0)$, then $\tilde{e}_i T \in B(\lambda)(\tilde{f}_i T \in B(\lambda))$. This embedding is in fact the embedding of $B(\lambda)$ into the set of permutation tableaux by *jeu de taquin* moves, and the inverse is given by taking P-symbols, namely by the insertion algorithm. See [Fo3] for example.

Let Q be a standard tableau of shape λ. We identify $B(Q)$ with $B(\lambda)$. Note that there exists a unique isomorphism of the crystals $(L(Q), B(Q))$ and $(L(\lambda), B(\lambda))$. The following is the modern version of the Date-Jimbo-Miwa theorem. We refer [BKK] for its generalization to superalgebras.

Theorem 3.10. *We identify $B(Q)$ with $B(\lambda)$ as above. Then the following hold.*

(1) If $b = \boxed{i_1} \otimes \cdots \otimes \boxed{i_n} \in B(Q)$, then the Q-symbol of $\emptyset \leftarrow i_1 i_2 \cdots i_n$ is Q.
(2) Let $P(b)$ be the P-symbol of $\emptyset \leftarrow i_1 i_2 \cdots i_n$. Then the identification of $B(Q)$ with $B(\lambda)$ is given by the map $b \mapsto P(b)$.

(Proof) We first recall that the bumping procedure $(T, i) \mapsto (T \leftarrow i)$ gives the isomorphism of crystals between $B(\lambda) \otimes B$ and $\sqcup_{|\mu/\lambda|=1} B(\mu)$. (As I have explained, we can think of the insertion via *jeu de taquin* moves. Hence it is enough to establish the isomorphism for a *jeu de taquin* move, which is easy.)

This isomorphism leads to a crystal automorphism on $B(Q) \otimes B$ as follows.

$$B(Q) \otimes B \xrightarrow{\sim} B(\lambda) \otimes B \xrightarrow{\sim} \bigsqcup_{|\mu/\lambda|=1} B(\mu) \xleftarrow{\sim} \bigsqcup_{T \in \mathcal{T}_Q} B(T) = B(Q) \otimes B$$

where the second isomorphism is given by the insertion algorithm. Since $V_q(Q) \otimes V$ is multiplicity free, the automorphism must be the identity. Hence the isomorphism $B(T) \xrightarrow{\sim} B(\mu)$ for $T \in \mathcal{T}_Q$ of shape μ is given by restricting the following isomorphism to $B(T)$. Note again that the second isomorphism is given by the insertion algorithm.

$$B(Q) \otimes B \xrightarrow{\sim} B(\lambda) \otimes B \xrightarrow{\sim} \bigsqcup_{|\mu/\lambda|=1} B(\mu)$$

Thus if the Robinson-Schensted algorithm gives the isomorphism $B(Q) \xrightarrow{\sim} B(\lambda)$ such that the Q-symbols of the elements in its image are constant Q, then the Robinson-Schensted algorithm gives the isomorphism $B(T) \xrightarrow{\sim} B(\mu)$, and the Q-symbols of the elements in its image are constant T. Therefore the induction proceeds. Q.E.D

We now turn to the q^2-Schur algebra. We refer [Du] for the details. We consider the Hecke algebra whose deformation parameter q is replaced by q^2. We also denote it by $\mathcal{H}_n$ by abuse of notion. $V^{\otimes n}$ has $\mathcal{H}_n$ action given by

$$v_{i_1}\otimes\cdots\otimes v_{i_n}T_k=\begin{cases} qv_{i_1}\otimes\cdots\otimes v_{i_{k+1}}\otimes v_{i_k}\otimes\cdots\otimes v_{i_n} & (i_k>i_{k+1})\\ q^2v_{i_1}\otimes\cdots\otimes v_{i_n} & (i_k=i_{k+1})\\ qv_{i_1}\otimes\cdots\otimes v_{i_{k+1}}\otimes v_{i_k}\otimes\cdots\otimes v_{i_n} & \\ \quad +(q^2-1)v_{i_1}\otimes\cdots\otimes v_{i_n} & (i_k<i_{k+1})\end{cases}$$

It commutes with U_q action. The endomorphism ring $\mathrm{End}_{\mathcal{H}_n}(V^{\otimes n})$ is called the **q-Schur algebra**, which is denoted by $\mathcal{S}_{r,n}$. It is well known that it is a quotient algebra of U_q. If we denote the μ-weight space of $V^{\otimes n}$ by V_μ, then we obviously have $\mathcal{S}_{r,n}=\oplus_{\mu,\nu}\mathrm{End}_{\mathcal{H}_n}(V_\nu,V_\mu)$.

We now assume $r=n$ and set $\omega=\epsilon_1+\cdots+\epsilon_n$. Then $\mathcal{H}_n\simeq\mathrm{End}_{\mathcal{H}_n}(V_\omega,V_\omega)$, and we can identify $\mathcal{H}_n$ with the subalgebra of $\mathcal{S}_{n,n}$.

On the other hand, if we set $x_\mu=\sum_{w\in S_\mu}T_w$ where S_μ is the Young subgroup associated with μ, the weight space V_μ is isomorphic to $x_\mu\mathcal{H}_n$. Hence we can also identify V_ω with $\mathcal{H}_n$. This identification is given by

$$v_{w_1}\otimes\cdots\otimes v_{w_n}\mapsto(q^2)^{-l(ww_0)/2}T_{ww_0}.$$

In particular, the Kazhdan-Lusztig basis element C'_w is identified with

$$\sum_{y\le w}P_{y,w}(q^2)q^{l(y)-l(w)}v_{y_n}\otimes\cdots\otimes v_{y_1}.$$

We have $P_{y,w}(q^2)q^{l(y)-l(w)}\in\mathbb{Z}[q^{-1}]$, and $C'_w\equiv v_{w_n}\otimes\cdots\otimes v_{w_1}\mod q^{-1}$.

The tensor space and the q^2-Schur algebra have bar operations, which satisfy $\bar{x}\bar{v}=\overline{xv}$ ($x\in\mathcal{S}_{n,n},v\in V^{\otimes n}$), and $\overline{v_n\otimes\cdots\otimes v_1}=v_n\otimes\cdots\otimes v_1$. The bar operation on the tensor space coincides with the bar operation introduced in 2.2 if restricted to $\mathcal{H}=V_\omega\subset V^{\otimes n}$.

By these reasons, we conclude that these are canonical basis elements arising from the crystal base we have considered above. We also remark that the canonical basis of the q^2-Schur algebra is the image of the canonical basis of the modified quantized enveloping algebra by the work [SV]. In fact, because of $(V^{\otimes n})_\omega=\oplus S_q(Q)$ where $S_q(Q)=V_q(Q)_\omega$, these Kazhdan-Lusztig basis elements are partitioned into the disjoint union $\sqcup B(Q)_\omega$ at $q=\infty$.

Recall that these C'_w are obtained from C_w by applying a $\mathbb{Q}$-algebra automorphism of $\mathcal{H}_n$. Thus the vector spaces $S_{\underset{L}{\le} w}$, $S_{\underset{L}{<} w}$ generated by

$\{C'_y | y \underset{L}{\leq} w\}$, $\{C'_y | y \underset{L}{<} w\}$ respectively are $\mathcal{H}_n$-modules. It is known that the factor module $S_{\underset{L}{\leq} w}/S_{\underset{L}{<} w}$ is irreducible. We now take the U_q-submodules $V_{\underset{L}{\leq} w}$, $V_{\underset{L}{<} w}$ of $V^{\otimes n}$ generated by $S_{\underset{L}{\leq} w}$, $S_{\underset{L}{<} w}$ respectively. By applying compositions of $\tilde{e}_i, \tilde{f}_i$ to $\{C'_y | y \underset{L}{\leq} w\}$, $\{C'_y | y \underset{L}{<} w\}$, we also have crystal bases of $V_{\underset{L}{\leq} w}$ and $V_{\underset{L}{<} w}$, which we denote by $(L_{\underset{L}{\leq} w}, B_{\underset{L}{\leq} w})$, $(L_{\underset{L}{<} w}, B_{\underset{L}{<} w})$. $B_{\underset{L}{<} w}$ is a union of connected components of $B_{\underset{L}{\leq} w}$. Since $V_{\underset{L}{\leq} w}/V_{\underset{L}{<} w}$ is irreducible, $B_{\underset{L}{\leq} w} \backslash B_{\underset{L}{<} w}$ coincides with one of $B(Q)_\omega$. Therefore, we have Theorem A again.

3.6. Theorem A derived from the primitive ideal theory

Definition 3.11. *The annihilator ideal of $L(\lambda)$ in $U(\mathfrak{g})$ is denoted by $I(\lambda) := Ann(L(\lambda))$, and is called a* **primitive ideal**.

The following is a theorem of Joseph.

Theorem B $Q(y) = Q(w) \Leftrightarrow I(y \cdot 0) = I(w \cdot 0)$.

By the translation principle, 0 can be replaced by any dominant integral weight.

The proof of this theorem depends on the following proposition.

Proposition 3.12. **(1)** *Let $y, w \in D_{ij}$ and assume $I(y \cdot 0) \subset I(w \cdot 0)$. Then we have $I(K_{ij}(y) \cdot 0) \subset I(K_{ij}(w) \cdot 0)$.*

(2) *If $Q(y) = Q(w)$, we have $I(y \cdot 0) = I(w \cdot 0)$.*

(1) is proved in [Ja1, Satz 5.9]. (2) is proved in [Ja1, Satz 5.18]. Once this proposition is established, the proof of Theorem B goes precisely the same as the proof of Theorem A.

By [Ja1, Corollar 6.26], [Ja1, Satz 7.9], [Ja1, Satz 7.12], we have the following theorem of Vogan. We state it in weaker form since it is enough for our purpose.

Theorem C Let λ, μ_1, μ_2 be dominant integral weights. Then we have that $I(y \cdot \lambda) \subset I(w \cdot \lambda)$ holds if and only if there exists a finite dimensional module E such that

$$\left[L(y^{-1} \cdot \mu_1) \otimes E : L(w^{-1} \cdot \mu_2)\right] \neq 0.$$

This theorem leads to Theorem D below. Recall that Kazhdan-Lusztig conjecture states that if we define $a(y,w)$ by

$$L(y\cdot 0)=\sum_{y\leq w} a(y,w)M(w\cdot 0),$$

we have $a(y,w)=(-1)^{l(w)-l(y)}P_{w_0w,w_0y}(1)$. This is proved by Brylinski and Kashiwara, Beilinson and Bernstein. Thus, there is a linear isomorphism between $K_0(\mathcal{O}_0)$ and $\mathbb{Z}W$ which sends $M(w_0w^{-1}\cdot 0)$ to w and $L(w_0w^{-1}\cdot 0)$ to $a(w)$. By introducing W-action on $K_0(\mathcal{O}_0)$ by $\tau M(w_0w^{-1}\cdot 0)=M(w_0w^{-1}\tau^{-1}\cdot 0)$, we can make it into a W-module isomorphism. Hence it is possible to translate statements for $K_0(\mathcal{O}_0)$ to those for the Weyl group. The following theorem is due to Joseph and Vogan. The formulation is due to Joseph [Jo5], and Vogan gives the proof in proving Theorem C. See [Ja1, Lemma 14.9] for the proof.

Theorem D $I(yw_0\cdot 0)\subset I(ww_0\cdot 0)\Leftrightarrow a(w)\in \overline{V_y}^L$.

This theorem shows that $y \underset{L}{\sim} w \Leftrightarrow I(yw_0\cdot 0)=I(ww_0\cdot 0)$. We then use Theorem B to conclude that $y \underset{L}{\sim} w \Leftrightarrow Q(yw_0)=Q(ww_0)$. Schützenberger's theorem [Sch1] tells that if we apply evacuation procedure to $Q(w)$, we obtain the transpose of $Q(ww_0)$ [S, Theorem 3.114]. Thus we have established Theorem A again.

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Crystal Bases for Quantum Superalgebras

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§1. Introduction

Associated with each integrable module M for the quantized enveloping algebra $U_q(\mathfrak{g})$ of a symmetrizable Kac-Moody Lie algebra $\mathfrak{g}$, there is a remarkable basis at $q = 0$, the *crystal base*, which was introduced by Kashiwara [Ka1]. If $\mathbf{A}$ denotes the local ring of all rational functions $f/g \in \mathbf{Q}(q)$ with $g(0) \neq 0$, then M contains an $\mathbf{A}$-lattice L, called the *crystal lattice*. The crystal base is a certain basis B for the $\mathbf{Q}$-vector space L/qL, which possesses many noteworthy features. It is well-behaved with respect to tensor products; it is preserved under the action of the modified root vector operators $\tilde{e}_i$ and $\tilde{f}_i$ (what are often called *Kashiwara operators*); and it has important connections with combinatorial bases of tableaux (see [MM], [KN], [KM], and [L]). Crystal bases play a prominent role in two-dimensional solvable lattice models, where the parameter q corresponds to the temperature in the lattice model. Since $q = 0$ corresponds to absolute zero temperature, one expects special behavior at this particular value, and the crystal base reflects this exceptional behavior.

In this work we describe a crystal base theory for quantum superalgebras. Basic definitions and general results on crystal bases for Kac-Moody superalgebras are presented in Sections 2, 3, and 4. Section 5 describes crystal bases for the orthosymplectic Lie superalgebra $\mathfrak{osp}(1, 2n)$, and Section 6, for affine Kac-Moody superalgebras. Sections 7, 8, and

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9 discuss crystal bases for the general linear Lie superalgebra $\mathfrak{gl}(m,n)$. More details on crystal bases for $\mathfrak{gl}(m,n)$ can be found in [BKK] where the theory is developed, in [MZ] and [C] for $\mathfrak{osp}(1,2n)$, and in [Je] for Kac-Moody Lie superalgebras.

§2. Quantized Universal Enveloping Algebras for Kac-Moody Superalgebras

We first recall the definition of the *quantized universal enveloping algebra* for a Kac-Moody superalgebra (cf. [BKM]). Let I be a finite index set, and assume $A = (a_{i,j})_{i,j \in I}$ is a generalized Cartan matrix. Thus A satisfies: (i) $a_{i,i} = 2$ for all $i \in I$, (ii) $a_{i,j} \in \mathbb{Z}_{\leq 0}$ for $i \neq j$, (iii) $a_{i,j} = 0$ if and only if $a_{j,i} = 0$ $(i, j \in I)$. In this paper, we assume that A is symmetrizable; i.e., there exists an invertible diagonal matrix $D = \mathrm{diag}(\ell_i |\ i \in I)$ with $\ell_i \in \mathbb{Z}_{>0}$ such that DA is symmetric.

Let I^{odd} be a subset of I and set $I^{\mathrm{even}} = I \setminus I^{\mathrm{odd}}$. The elements of I^{odd} (resp. I^{even}) are called the *odd* (resp. *even*) indices. The *parity function* p is defined by $p(i) = 1$ if $i \in I^{\mathrm{odd}}$ and $p(i) = 0$ if $i \in I^{\mathrm{even}}$. We say that the generalized Cartan matrix A is *colored by* I^{odd} if $a_{i,j} \in 2\mathbb{Z}$ for all $i \in I^{\mathrm{odd}}$, $j \in I$.

Let $\mathfrak{h}$ be a vector space of dimension $2|I| - \mathrm{rank} A$. Let $\Pi = \{\alpha_i |\ i \in I\} \subset \mathfrak{h}^*$ and $\Pi^\vee = \{h_i |\ i \in I\} \subset \mathfrak{h}$ be linearly independent sets such that $\alpha_j(h_i) = a_{i,j}$ for all $i, j \in I$. Then the triple $(\mathfrak{h}, \Pi, \Pi^\vee)$ forms a realization of A in the sense of [K4, Chap. 1].

Definition 2.1. *Assume $A = (a_{i,j})_{i,j \in I}$ is a generalized Cartan matrix colored by $I^{\mathrm{odd}} \subset I$, and let $(\mathfrak{h}, \Pi, \Pi^\vee)$ be a realization of A. The* **Kac-Moody superalgebra** $\mathfrak{g} = \mathfrak{g}(A, I^{\mathrm{odd}})$ *is the Lie superalgebra generated by e_i, f_i $(i \in I)$ and $\mathfrak{h}$ with defining relations*

$$
\begin{aligned}
&[h, h'] = 0 \quad \textit{for all} \quad h, h' \in \mathfrak{h}, \\
&[h, e_i] = \alpha_i(h) e_i, \quad [h, f_i] = -\alpha_i(h) f_i, \\
&[e_i, f_j] = \delta_{i,j} h_i, \\
&(ad\, e_i)^{1-a_{i,j}}(e_j) = (ad\, f_i)^{1-a_{i,j}}(f_j) = 0 \quad (i \neq j).
\end{aligned}
\tag{2.2}
$$

The free abelian group $Q = \bigoplus_{i \in I} \mathbb{Z}\alpha_i$ is the *root lattice* associated with the data $(A, \Pi, \Pi^\vee)$. For an element $\alpha = \sum_i k_i \alpha_i \in Q$, the parity of α is defined to be $p(\alpha) = \sum_i k_i p(i) \in \mathbb{Z}_2$. We say that α is *even* (resp. odd) if $p(\alpha) = 0$ (resp. $p(\alpha) = 1$). Let $Q_+ = \bigoplus_{i \in I} \mathbb{Z}_{\geq 0}\alpha_i$ and $Q_- = -Q_+$. There is a partial ordering $\geq$ on $\mathfrak{h}^*$ defined by $\lambda \geq \mu$ if and only if $\lambda - \mu \in Q_+$ $(\lambda, \mu \in \mathfrak{h}^*)$. Since A is symmetrizable there is a nondegenerate symmetric bilinear form $(\ |\ )$ on $\mathfrak{h}^*$ satisfying $(\alpha_i|\alpha_j) = \ell_i a_{i,j}$ for $i, j \in I$. For each $i \in I$, let $r_i \in GL(\mathfrak{h}^*)$ be the simple reflection defined by $r_i(\lambda) = \lambda - \lambda(h_i)\alpha_i$ $(\lambda \in \mathfrak{h}^*)$. The subgroup of $GL(\mathfrak{h}^*)$ generated by r_i's is the *Weyl group* of the data $(A, \Pi, \Pi^\vee)$.

Let q be an indeterminate and set $q_i = q^{\ell_i}$ $(i \in I)$. For nonnegative integers n and N, we define the q-binomial coefficients as follows:

$$\begin{aligned} &[n]_i = \frac{(-1)^{np(i)}q_i^n - q_i^{-n}}{(-1)^{p(i)}q_i - q_i^{-1}}, \\ &[n]_i! = [n]_i[n-1]_i \cdots [2]_i[1]_i, \qquad [0]_i! = 1, \\ &\begin{bmatrix} N \\ n \end{bmatrix}_i = \frac{[N]_i!}{[n]_i![N-n]_i!}. \end{aligned} \tag{2.3}$$

Let $P^\vee$ be a $\mathbb{Z}$-lattice of $\mathfrak{h}$ containing all h_i's and satisfying $\alpha_i(h) \in \mathbb{Z}$ for all $i \in I$ and $h \in P^\vee$. The lattice $P^\vee$ is referred to as the *dual weight lattice*, and $P = \{\lambda \in \mathfrak{h}^* |\ \lambda(P^\vee) \subset \mathbb{Z}\}$ is the *weight lattice.*

Definition 2.4. *Let $A = (a_{i,j})_{i,j \in I}$ be a generalized Cartan matrix colored by $I^{\mathrm{odd}} \subset I$, and let $\mathfrak{g} = \mathfrak{g}(A, I^{\mathrm{odd}})$ be the corresponding Kac-Moody superalgebra. The* **quantized universal enveloping algebra** *$U_q(\mathfrak{g})$ is the associative algebra over $\mathbf{Q}(q)$ with 1 generated by the elements q^h $(h \in P^\vee)$, e_i, f_i $(i \in I)$ with defining relations*

$$\begin{aligned} &q^0 = 1, \quad q^h q^{h'} = q^{h+h'} \text{ for all } h, h' \in P^\vee, \\ &q^h e_i q^{-h} = q^{\alpha_i(h)} e_i, \quad q^h f_i q^{-h} = q^{-\alpha_i(h)} f_i, \\ &e_i f_j - (-1)^{p(i)p(j)} f_j e_i = \delta_{i,j} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}, \\ &(ad_q e_i)^{1-a_{i,j}}(e_j) = (ad_q f_i)^{1-a_{i,j}}(f_j) = 0 \quad (i \neq j), \end{aligned} \tag{2.5}$$

where $K_i = q^{\ell_i h_i}$ for $i \in I$ and $ad_q\, x(y) = xy - q^{(\alpha|\beta)}(-1)^{p(\alpha)p(\beta)} yx$ for $x \in \mathfrak{g}_\alpha$, $y \in \mathfrak{g}_\beta$.

Proposition 2.6. ([BKM]) *The algebra $U_q(\mathfrak{g})$ has a Hopf superalgebra structure with comultiplication Δ, counit ε, and antipode S*

defined by

$$\tag{2.7} \begin{aligned} &\Delta(q^h) = q^h \otimes q^h \quad \textit{for} \quad h \in P^\vee, \\ &\Delta(e_i) = e_i \otimes K_i^{-1} + 1 \otimes e_i, \\ &\Delta(f_i) = f_i \otimes 1 + K_i \otimes f_i \quad \textit{for} \quad i \in I, \end{aligned}$$

$$\tag{2.8} \begin{aligned} &\varepsilon(q^h) = 1 \quad \textit{for } h \in P^\vee, \\ &\varepsilon(e_i) = \varepsilon(f_i) = 0 \quad \textit{for } i \in I, \end{aligned}$$

$$\tag{2.9} \begin{aligned} &S(q^h) = q^{-h} \quad \textit{for } h \in P^\vee, \\ &S(e_i) = -e_i K_i, \quad S(f_i) = -K_i^{-1} f_i \quad \textit{for } i \in I. \end{aligned}$$

The $\mathbf{Q}(q)$-subalgebra U_q^0 of $U_q(\mathfrak{g})$ generated by the elements q^h ($h \in P^\vee$) is isomorphic to the group algebra $\mathbf{Q}(q)[P^\vee]$. Let U_q^+ (resp. U_q^-) be the $\mathbf{Q}(q)$-subalgebra of $U_q(\mathfrak{g})$ with 1 generated by the elements e_i (resp. f_i) for $i \in I$. Then we have the following *triangular decomposition* which can be regarded as a quantum analogue of Poincaré-Birkhoff-Witt Theorem.

Proposition 2.10. ([BKM]) *There is a* $\mathbf{Q}(q)$*-linear isomorphism*

$$\tag{2.11} U_q(\mathfrak{g}) \cong U_q^- \otimes U_q^0 \otimes U_q^+.$$

Example 2.12. The simplest example of a quantized universal enveloping algebra associated with a Kac-Moody superalgebra is the *orthosymplectic quantum superalgebra* $U_q(\mathfrak{osp}(1,2))$, which is generated by the elements e, f, and $K^{\pm 1}$ and has defining relations

$$\tag{2.13} KeK^{-1} = q^2 e, \quad KfK^{-1} = q^{-2} f, \quad ef + fe = \frac{K - K^{-1}}{q - q^{-1}}.$$

§3. Integrable Representations

In this section, we introduce the notion of integrable $U_q(\mathfrak{g})$-modules in category $\mathcal{O}$. Let $\mathfrak{g} = \mathfrak{g}(A, I^{\mathrm{odd}})$ be a Kac-Moody superalgebra associated with a generalized Cartan matrix $A = (a_{i,j})_{i,j \in I}$ colored by the set of odd indices I^{odd}. Let $U_q(\mathfrak{g})$ be the corresponding quantized universal enveloping algebra.

A $U_q(\mathfrak{g})$-module M is a *weight module* if it admits a *weight space decomposition*

$$M = \bigoplus_{\lambda \in P} M_\lambda, \quad \text{where} \quad M_\lambda = \{v \in M \mid q^h v = q^{\lambda(h)} v \ \text{ for all } \ h \in P^\vee\}.$$

The category $\mathcal{O}$ consists of the weight modules with finite dimensional weight spaces such that there exist $\mu_1, \cdots, \mu_s$ in P satisfying

$$\mathrm{wt}(M) \subset D(\mu_1) \cup \cdots \cup D(\mu_s),$$

where $\mathrm{wt}(M) = \{\lambda \in P \mid M_\lambda \neq 0\}$ and $D(\mu) = \{\tau \in P \mid \tau \leq \mu\}$. The morphisms are $U_q(\mathfrak{g})$-module homomorphisms.

Among the $U_q(\mathfrak{g})$-modules in category $\mathcal{O}$, the most interesting ones are highest weight modules. A $U_q(\mathfrak{g})$-module M is a *highest weight module with highest weight* $\lambda \in P$ if there exists a nonzero vector $v_\lambda \in M$ such that (i) $q^h v_\lambda = q^{\lambda(h)} v_\lambda$ for all $h \in P^\vee$, (ii) $e_i v_\lambda = 0$ for all $i \in I$, (iii) $M = U_q(\mathfrak{g}) v_\lambda$. The vector v_λ, which is called a *highest weight vector* of M, is unique up to a constant multiple.

Let $\lambda \in P$ and let $J(\lambda)$ be the left ideal of $U_q(\mathfrak{g})$ generated by e_i ($i \in I$) and $q^h - q^{\lambda(h)} 1$ ($h \in P^\vee$). Then the quotient $M(\lambda) = U_q(\mathfrak{g})/J(\lambda)$ is given a $U_q(\mathfrak{g})$-module structure by left multiplication. It is easy to see that $M(\lambda)$ is a highest weight module with highest weight λ and highest weight vector $v_\lambda = 1 + J(\lambda)$. The $U_q(\mathfrak{g})$-module $M(\lambda)$ is the *Verma module* with highest weight λ. As a U_q^--module, $M(\lambda)$ is free of rank 1 generated by the highest weight vector $v_\lambda = 1 + J(\lambda)$, and every highest weight $U_q(\mathfrak{g})$-module with highest weight λ is a homomorphic image of $M(\lambda)$. The module $M(\lambda)$ has a unique maximal submodule $N(\lambda)$, and its unique irreducible quotient $V(\lambda) = M(\lambda)/N(\lambda)$ is again a highest weight $U_q(\mathfrak{g})$-module with highest weight λ.

A $U_q(\mathfrak{g})$-module M is said to be *integrable* if all e_i and f_i ($i \in I$) are locally nilpotent on M. We denote by $\mathcal{O}_{int}$ the subcategory of $\mathcal{O}$ consisting of integrable $U_q(\mathfrak{g})$-modules in category $\mathcal{O}$. For each $i \in I$, let $U_q(\mathfrak{g})_i$ be the subalgebra of $U_q(\mathfrak{g})$ generated by e_i, f_i, $K_i^{\pm 1}$. Then it is easy to verify that

$$U_q(\mathfrak{g})_i \cong \begin{cases} U_q(\mathfrak{sl}_2) & \text{if } \ i \in I^{\text{even}}, \\ U_q(\mathfrak{osp}(1,2)) & \text{if } \ i \in I^{\text{odd}}. \end{cases} \tag{3.1}$$

A $U_q(\mathfrak{g})$-module M is in category $\mathcal{O}_{int}$ if and only if it has a weight space decomposition with finite dimensional weight spaces, and for each $i \in I$, M is locally $U_q(\mathfrak{g})_i$-finite, i.e., for each $v \in M$, $\dim U_q(\mathfrak{g})_i v < \infty$. In particular, a $U_q(\mathfrak{g})$-module in category $\mathcal{O}_{int}$ is a direct sum of finite dimensional irreducible $U_q(\mathfrak{g})_i$-modules for all $i \in I$.

The category $\mathcal{O}_{int}$ is semisimple, and its irreducible objects can be characterized as follows.

Proposition 3.2. ([Je]) *The irreducible highest weight $U_q(\mathfrak{g})$-module $V(\lambda)$ is integrable if and only if $\lambda \in P^+$, where*

$$P^+ = \{\lambda \in P | \ \lambda(h_i) \geq 0 \ \ \textit{for all } i \in I, \ \ \lambda(h_i) \in 2\mathbf{Z} \ \ \textit{for all } i \in I^{\text{odd}}\}.$$

The irreducible integrable highest weight modules over $U_q(\mathfrak{g})$ are quantum deformations of those over the Kac-Moody superalgebra $\mathfrak{g}$.

Proposition 3.3. ([BKM]) *If $\lambda \in P^+$, then the irreducible highest weight module $V(\lambda)$ over the quantized universal enveloping algebra $U_q(\mathfrak{g})$ is a quantum deformation of the irreducible highest weight module $\overline{V}(\lambda)$ over the Kac-Moody superalgebra $\mathfrak{g}$. The character of $V(\lambda)$ is given by the Weyl-Kac character formula:*

$$chV(\lambda) = \frac{\sum_{w \in W} (-1)^{l(w)} e^{w(\lambda+\rho)-\rho}}{\prod_{\alpha \in Q_+} (1-(-1)^{p(\alpha)} e^{-\alpha})^{1-2p(\alpha) \dim \mathfrak{g}_\alpha}}, \tag{3.4}$$

where $\rho \in P$ is a linear functional satisfying $\rho(h_i) = 1$ for all $i \in I$, the parity function p is as in Section 2, and W is the Weyl group.

§4. Crystal Bases for Kac-Moody Superalgebras

Let $M = \bigoplus_{\lambda \in P} M_\lambda$ be a $U_q(\mathfrak{g})$-module in category $\mathcal{O}_{int}$. Fix $i \in I$ and for any $k \in \mathbf{Z}_{\geq 0}$ define

$$e_i^{(k)} = \frac{1}{[k]_i!} e_i^k, \quad f_i^{(k)} = \frac{1}{[k]_i!} f_i^k.$$

Then every element $u \in M_\lambda$ can be uniquely expressed as

$$u = \sum_{k=0}^{N} f_i^{(k)} u_k, \quad \text{where } \ N \in \mathbf{Z}_{\geq 0} \ \text{ and } \ u_k \in M_{\lambda + k\alpha_i} \cap \ker e_i. \tag{4.1}$$

We define the *Kashiwara operators* $\tilde{e}_i$ and $\tilde{f}_i$ by

$$(4.2) \qquad \tilde{e}_i u = \sum_{k=1}^{N} f_i^{(k-1)} u_k, \quad \tilde{f}_i u = \sum_{k=0}^{N} f_i^{(k+1)} u_k.$$

Let $\mathbf{A} = \{f(q)/g(q) \in \mathbf{Q}(q) | \ f, g \in \mathbf{Q}[q], \ g(0) \neq 0\}$ be the localization of $\mathbf{Q}[q]$ at $q = 0$. Thus $\mathbf{A}$ consists of all rational functions that are regular at $q = 0$.

Definition 4.3. *Let M be a $U_q(\mathfrak{g})$-module in the category $\mathcal{O}_{\text{int}}$. A free $\mathbf{A}$-submodule L of M is a* **crystal lattice** *if*

(i) *L generates M as a vector space over $\mathbf{Q}(q)$.*
(ii) *L has a weight decomposition $L = \bigoplus_{\lambda \in P} L_\lambda$, where $L_\lambda = L \cap M_\lambda$.*
(iii) *$\tilde{e}_i L \subset L$ and $\tilde{f}_i L \subset L$ for any $i \in I$.*

Definition 4.4. *Let M be a $U_q(\mathfrak{g})$-module in the category $\mathcal{O}_{\text{int}}$. A* **crystal base** *of M is a pair (L, B) such that*

(i) *L is a crystal lattice of M,*
(ii) *B is a pseudo-basis of L/qL, that is, $B = B' \cup (-B')$ for some $\mathbf{Q}$-basis B' of L/qL,*
(iii) *B has a weight decomposition $B = \bigsqcup_{\lambda \in P} B_\lambda$, where $B_\lambda = B \cap (L_\lambda/qL_\lambda)$,*
(iv) *$\tilde{e}_i B \subset B \sqcup \{0\}$ and $\tilde{f}_i B \subset B \sqcup \{0\}$ for all $i \in I$,*
(v) *for any $b, b' \in B$ and $i \in I$, we have $b = \tilde{f}_i b'$ if and only if $b' = \tilde{e}_i b$.*

The set $B/\{\pm 1\}$ is given a colored oriented graph structure with the i-arrow defined by $b \xrightarrow{i} b'$ if and only if $\tilde{f}_i b = b'$ for $b, b' \in B/\{\pm 1\}$. We call $B/\{\pm 1\}$ the *crystal graph* of M.

For $b \in B/\{\pm 1\}$ and $i \in I$, let

$$(4.5) \qquad \begin{aligned} \varepsilon_i(b) &= \max\{n \in \mathbf{Z}_{\geq 0} \mid \tilde{e}_i^n b \neq 0\}, \\ \varphi_i(b) &= \max\{n \in \mathbf{Z}_{\geq 0} \mid \tilde{f}_i^n b \neq 0\}. \end{aligned}$$

Then it follows from the representation theory of $U_{q_i}(\mathfrak{sl}_2)$ and $U_{q_i}(\mathfrak{osp}(1,2))$ that

$$\langle h_i, \mathrm{wt}(b) \rangle = \varphi_i(b) - \varepsilon_i(b) \quad \text{for } i \in I,$$

where $\mathrm{wt}(b)$ is the weight of b.

As in the case of Kac-Moody algebras, the crystal graphs for Kac-Moody superalgebras exhibit nice behavior with respect to tensor products.

Theorem 4.6. *Let M_ν be a $U_q(\mathfrak{g})$-module in category $\mathcal{O}_{\mathrm{int}}$, and let (L_ν, B_ν) be a crystal base of M_ν for $\nu = 1, 2$. Set $L = L_1 \otimes_{\mathbf{A}} L_2$ and $B = B_1 \otimes B_2 = B_1 \times B_2$. Then (L, B) is a crystal base of $M_1 \otimes M_2$, and the crystal graph structure of $B_1 \otimes B_2$ is given by*

$$\tilde{e}_i(b_1 \otimes b_2) = \begin{cases} \tilde{e}_i(b_1) \otimes b_2 & \text{if } \varphi_i(b_1) \geq \varepsilon_i(b_2), \\ b_1 \otimes \tilde{e}_i(b_2) & \text{if } \varphi_i(b_1) < \varepsilon_i(b_2), \end{cases}$$

$$\tilde{f}_i(b_1 \otimes b_2) = \begin{cases} \tilde{f}_i(b_1) \otimes b_2 & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2) \\ b_1 \otimes \tilde{f}_i(b_2) & \text{if } \varphi_i(b_1) \leq \varepsilon_i(b_2). \end{cases} \tag{4.7}$$

Hence the tensor product rule for crystal graphs of Kac-Moody superalgebras is the same as the one for Kac-Moody algebras (cf. [Ka2]).

For a dominant integral weight $\lambda \in P^+$, let $V(\lambda)$ be the irreducible highest weight $U_q(\mathfrak{g})$-module with highest weight λ, and let $L(\lambda)$ be the free $\mathbf{A}$-submodule of $V(\lambda)$ spanned by the vectors of the form $\tilde{f}_{i_1}\tilde{f}_{i_2}\cdots\tilde{f}_{i_r}v_\lambda$, where v_λ is the highest weight vector of $V(\lambda)$. Set

$$B(\lambda) = \{\tilde{f}_{i_1}\tilde{f}_{i_2}\cdots\tilde{f}_{i_r}v_\lambda \in L(\lambda)/qL(\lambda) \mid i_k \in I\} \setminus \{0\}. \tag{4.8}$$

Then, just as in the case of Kac-Moody algebras, we have the existence theorem for the crystal base for Kac-Moody superalgebras.

Theorem 4.9. ([Je]) *The pair $(L(\lambda), B(\lambda))$ is a crystal base of $V(\lambda)$.*

Although the proof of Theorem 4.9 is rather long and complicated, it is still a straightforward generalization of Kashiwara's *grand loop argument* ([Ka2]) once we define the super-version of the q-analogue of bosons. See [Je] for more details. The uniqueness theorem for the crystal base for Kac-Moody superalgebras can be proved in the same manner as for Kac-Moody algebras.

Theorem 4.10. (cf. [Je], [Ka2]) *Let M be a $U_q(\mathfrak{g})$-module in category $\mathcal{O}_{\text{int}}$ and let $M \cong \bigoplus_{\lambda \in P^+} V(\lambda)$ be the decomposition of M into a direct sum of irreducible highest weight modules. Then for any crystal base (L, B) of M, there exists an isomorphism $\Phi : M \to \bigoplus_{\lambda \in P^+} V(\lambda)$ such that $\Phi(L) \cong \bigoplus_{\lambda \in P^+} L(\lambda)$ and $\overline{\Phi}(B) \cong \bigsqcup_{\lambda \in P^+} B(\lambda)$, where $\overline{\Phi} : L \to \bigoplus_{\lambda \in P^+} L(\lambda)/qL(\lambda)$ is the $\mathbf{Q}$-linear isomorphism induced by Φ. In particular, if $\lambda \in P^+$, any crystal base (L, B) of $V(\lambda)$ is isomorphic to $(L(\lambda), B(\lambda))$.*

§5. Crystal Graphs for the Quantum Superalgebra $U_q(\mathfrak{osp}(1, 2n))$

In this section, we present an explicit description of the crystal graph of a finite dimensional irreducible module for the quantum superalgebra $U_q(\mathfrak{osp}(1, 2n))$. Among the finite dimensional simple Lie superalgebras, the superalgebras $\mathfrak{osp}(1, 2n)$ are distinguished as the only ones that are Kac-Moody superalgebras as defined in Section 2. We apply the crystal base theory for Kac-Moody superalgebras described in Section 4 to the superalgebras $\mathfrak{osp}(1, 2n)$ and obtain a realization of the crystal graphs in terms of certain semistandard Young tableaux. The basic technique used here originated in [KN], where the crystal graphs for finite dimensional irreducible modules over classical Lie algebras were realized in terms of certain semistandard Young tableaux. Our presentation follows that in [C], which was based on the approach of [KN]. Musson and Zou [MZ] have also developed a crystal base theory for $\mathfrak{osp}(1, 2n)$. Their methods derive from those in [J] and [Ka1], and they work only for $\mathfrak{osp}(1, 2n)$. Our method is based on the general crystal base theory for arbitrary Kac-Moody superalgebras, which includes some affine cases. Moreover, it yields an explicit tableau description of crystal graphs.

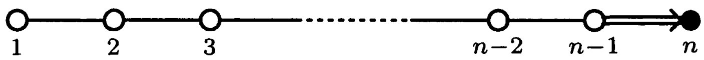
The representation theory of the Lie superalgebra $\mathfrak{osp}(1, 2n)$ is known to closely resemble the representation theory of the Lie algebra $\mathfrak{so}(2n+1)$ (see for example, [RS]). As a result, the crystal graphs of finite dimensional irreducible representations of $U_q(\mathfrak{osp}(1, 2n))$ have virtually the same description as the crystal graphs for $U_q(\mathfrak{so}(2n+1))$-modules. There exist finite dimensional irreducible representations of $U_q(\mathfrak{osp}(1, 2n))$ that are not quantum deformations of $\mathfrak{osp}(1, 2n)$-modules (these can be found in [Z1]). Since in this paper we focus only on the $U_q(\mathfrak{osp}(1, 2n))$-modules that are quantum deformations of $\mathfrak{osp}(1, 2n)$-modules, we don't have to consider ones that correspond to the so-called *spinor representations* of $\mathfrak{so}(2n + 1)$.

The index set for $\mathfrak{osp}(1, 2n)$ is $I = \{1, 2, \cdots, n\}$, and there is just one odd index, $I^{odd} = \{n\}$. The associated Cartan matrix is a generalized

Cartan matrix of type $B(0,n)$:

$$A=(a_{ij})_{i,j\in I}=\begin{pmatrix} 2 & -1 & \cdots & & & \\ -1 & 2 & \cdots & & & \\ & & \ddots & & & \\ & & \cdots & 2 & -1 & \\ & & \cdots & -1 & 2 & -1 \\ & & \cdots & & -2 & 2 \end{pmatrix} \tag{5.1}$$

which corresponds to the Dynkin diagram



Definition 5.2. *The* **quantum superalgebra** $U_q(\mathfrak{osp}(1,2n))$ *is the quantized universal enveloping algebra of the Kac-Moody superalgebra associated with the data* (A, I^{odd})*, where* $A=(a_{ij})_{i,j\in I}$ *is given in (5.1) and* $I^{odd}=\{n\}$.

Thus, $U_q(\mathfrak{osp}(1,2n))$ is the associative algebra over $\mathbf{Q}(q)$ generated by the elements e_i, f_i, $K_i^{\pm 1}$ $(i=1,\cdots,n)$ with defining relations given by (2.5). Let $(\mathfrak{h},\Pi,\Pi^\vee)$ denote a realization of the generalized Cartan matrix of type $B(0,n)$, and assume ϵ_i $(i=1,2,\cdots,n)$ is an orthonormal basis of $\mathfrak{h}^*$. Then the simple roots of the Lie superalgebra $\mathfrak{osp}(1,2n)$ are given by

$$\begin{aligned} \alpha_i &= \epsilon_i-\epsilon_{i+1} \quad (1\le i\le n-1), \\ \alpha_n &= \epsilon_n. \end{aligned}$$

The basic representation of $U_q(\mathfrak{osp}(1,2n))$ is its *vector representation*, which is the $(2n+1)$-dimensional space

$$\mathbf{V}=\left(\bigoplus_{i=1}^{n}\mathbf{Q}(q)v_i\right)\oplus\mathbf{Q}(q)v_0\oplus\left(\bigoplus_{i=1}^{n}\mathbf{Q}(q)v_{\bar{i}}\right)$$

over $\mathbf{Q}(q)$ with basis $\{v_i, v_{\bar{i}} \mid i=1,2,\cdots,n\}\cup\{v_0\}$ and with $U_q(\mathfrak{osp}(1,2n))$-

action defined by

$$
(5.3)\qquad
\begin{aligned}
K_i v_j &= \begin{cases} q v_j & \text{if } j = i, \overline{i+1}, \\ q^{-1} v_j & \text{if } j = i+1, \overline{i}, \\ v_j & \text{otherwise}, \end{cases} \\
e_i v_j &= \begin{cases} v_{j-1} & \text{if } j = i+1, \overline{i}, \\ 0 & \text{otherwise}, \end{cases} \\
f_i v_j &= \begin{cases} v_{j+1} & \text{if } j = i, \overline{i+1}, \\ 0 & \text{otherwise} \end{cases}
\end{aligned}
$$

for $i = 1, 2, \cdots, n-1$, and

$$
(5.4)\qquad
\begin{aligned}
K_n v_j &= \begin{cases} q^2 v_j & \text{if } j = n, \\ q^{-2} v_j & \text{if } j = \bar{n}, \\ v_j & \text{otherwise}, \end{cases} \\
e_n v_j &= \begin{cases} v_0 & \text{if } j = \bar{n}, \\ [2]_n v_n & \text{if } j = 0, \\ 0 & \text{otherwise}, \end{cases} \\
f_n v_j &= \begin{cases} v_0 & \text{if } j = n, \\ [2]_n v_{\bar{n}} & \text{if } j = 0, \\ 0 & \text{otherwise}. \end{cases}
\end{aligned}
$$

In these expressions it is understood that $\bar{i} \pm 1 = \overline{i \mp 1}$ for $i = 1, 2, \cdots, n-1$.

Let

$$
(5.5)\qquad
\begin{aligned}
&\mathbf{L} = \left(\bigoplus_{i=1}^{n} \mathbf{A} v_i \right) \oplus \mathbf{A} v_0 \oplus \left(\bigoplus_{i=1}^{n} \mathbf{A} v_{\bar{i}} \right) \\
&\text{and} \quad \mathbf{B} = \{ \overline{v_j}, \overline{v_{\bar{j}}} \mid j = 1, \cdots, n \} \cup \{ \overline{v_0} \}.
\end{aligned}
$$

Then $(\mathbf{L}, \mathbf{B})$ is a crystal base of $\mathbf{V}$ with crystal graph given by

$$
\boxed{1} \xrightarrow{1} \boxed{2} \xrightarrow{2} \cdots \xrightarrow{n-1} \boxed{n} \xrightarrow{n} \boxed{0} \xrightarrow{n} \boxed{\bar{n}} \xrightarrow{n-1} \cdots \xrightarrow{2} \boxed{\bar{2}} \xrightarrow{1} \boxed{\bar{1}}
$$

Here we identify $\overline{v_j} = \boxed{j}$ for $j = 1, \cdots, n, 0, \bar{n}, \overline{n-1}, \cdots, \bar{1}$.

The *fundamental weights* of $\mathfrak{osp}(1,2n)$ are defined by $\omega_i(h_j) = \delta_{i,j}$ $(i,j = 1,2,\cdots,n)$. Alternately, $\omega_i = \epsilon_1 + \cdots + \epsilon_i$, $(1 \le i \le n-1)$, and $\omega_n = \frac{1}{2}(\epsilon_1 + \cdots + \epsilon_n)$. The finite dimensional irreducible $\mathfrak{osp}(1,2n)$-modules are parametrized by their highest weights λ, which have the form

$$\begin{aligned}\lambda &= a_1\omega_1 + \cdots + a_{n-1}\omega_{n-1} + 2a_n\omega_n \quad \text{with} \quad a_i \in \mathbf{Z}_{\ge 0},\\ &= \lambda_1\epsilon_1 + \cdots + \lambda_{n-1}\epsilon_{n-1} + \lambda_n\epsilon_n,\end{aligned}$$

where $\lambda_i = a_1 + \cdots + a_i$ $(i = 1,2,\cdots,n)$ ([K3]). Hence λ corresponds to a partition $(\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n \ge 0)$ of $N = a_1 + 2a_2 + \cdots + na_n$ having at most n parts, and the finite dimensional irreducible $U_q(\mathfrak{osp}(1,2n))$-module $V(\lambda)$ can be embedded into $\mathbf{V}^{\otimes N}$. Therefore the crystal graph $B(\lambda)$ of $V(\lambda)$ is isomorphic to the connected component of $\mathbf{B}^{\otimes N}$ containing a highest weight vector u_λ of weight λ (i.e., $\mathrm{wt} u_\lambda = \lambda$ and $\tilde{e}_i u_\lambda = 0$ for all $i = 1,2,\cdots,n$).

The methods in [KN] allow us to identify the crystal graph $B(\lambda)$ with a certain set of Young tableaux which are semistandard relative to the ordering

$$1 < 2 < \cdots < n < 0 < \overline{n} < \cdots < \overline{1}$$

on the elements of $\mathbf{B}$ in the following way.

Suppose first that $\lambda = \epsilon_1 + \cdots + \epsilon_k$ $(1 \le k \le n)$. Then $\lambda = \omega_k$ if $k = 1,\cdots,n-1$ and $\lambda = 2\omega_n$ if $k = n$. Let $u_\lambda = \boxed{1}\otimes\boxed{2}\otimes\cdots\otimes\boxed{k} \in \mathbf{B}^{\otimes k}$. Then it is easy to see that $\mathrm{wt} u_\lambda = \lambda = \epsilon_1 + \cdots + \epsilon_k$ and $\tilde{e}_i u_\lambda = 0$ for all $i = 1,\cdots,n$. The explicit description of the crystal graph $B(\epsilon_1 + \cdots + \epsilon_k)$ is given in the next result.

Proposition 5.6. ([KN]) *Let $B(Y_k)$ be the set of all vectors in $\mathbf{B}^{\otimes k}$ of the form* $\begin{array}{|c|}\hline j_1 \\ \hline \vdots \\ \hline j_k \\ \hline\end{array} = \boxed{j_1}\otimes\cdots\otimes\boxed{j_k} \in \mathbf{B}^{\otimes k}$ *satisfying the following conditions:*

(a) $1 \le j_1 \le \cdots \le j_k \le \overline{1}$, *but 0 is the only entry that can be repeated.*

(b) *if $j_r = p$ and $j_s = \overline{p}$ $(1 \le p \le n)$, then $r - s + k + 1 \le p$.*

Then $B(\epsilon_1 + \cdots + \epsilon_k) \cong B(Y_k)$.

Next suppose $\lambda = (\epsilon_1 + \cdots + \epsilon_k) + (\epsilon_1 + \cdots + \epsilon_l)$ with $1 \le k \le l \le n$. For $u_\lambda = \begin{array}{|c|c|}\hline 1 & 1 \\ \vdots & \vdots \\ \hline k & k \\ \hline \\ \cline{1-1} l \\ \cline{1-1}\end{array} = \begin{array}{|c|}\hline 1 \\ \\ k \\ \hline\end{array} \otimes \begin{array}{|c|}\hline 1 \\ \\ k \\ \hline \\ l \\ \hline\end{array} = \boxed{1} \otimes \cdot\cdot \otimes \boxed{k} \otimes \boxed{1} \otimes \cdots \otimes \boxed{l} \in B(Y_k) \otimes B(Y_l) \subset \mathbf{B}^{\otimes(k+l)}$, it is easy to see that $\mathrm{wt} u_\lambda = \lambda = (\epsilon_1 + \cdots + \epsilon_k) + (\epsilon_1 + \cdots + \epsilon_l)$ and $\tilde{e}_i u_\lambda = 0$ for all $i = 1, 2, \cdots, n$.

To describe the connected component of $\mathbf{B}^{\otimes(k+l)}$ containing u_λ, we need to introduce some terminology.

Definition 5.7. *For*

$$w = u \otimes v = \begin{array}{|c|}\hline j_1 \\ \vdots \\ j_k \\ \hline\end{array} \otimes \begin{array}{|c|}\hline i_1 \\ \\ \vdots \\ \\ i_l \\ \hline\end{array} = \begin{array}{|c|c|}\hline i_1 & j_1 \\ \vdots & \vdots \\ & j_k \\ \hline \\ \cline{1-1} i_l \\ \cline{1-1}\end{array} \in B(Y_k) \otimes B(Y_l) \subset \mathbf{B}^{\otimes(k+l)},$$

(a) *w is in the* (a,b)**-configuration** *for $1 \le a \le b < n$, if there exist positive integers $1 \le p \le q < r \le s \le k$ such that $i_p = a$, $j_q = b$, $j_r = \overline{b}$, $j_s = \overline{a}$ or $i_p = a$, $i_q = b$, $i_r = \overline{b}$, $j_s = \overline{a}$.*

(b) *w is in the* (a,n)**-configuration** *for $1 \le a < n$ if there exist positive integers $1 \le p \le q < r = q+1 \le k$ such that $i_p = a$, $j_s = \overline{a}$, and $i_q, i_{q+1} \in \{n, 0, \overline{n}\}$ or $i_p = a$, $j_s = \overline{a}$, and $j_q, j_{q+1} \in \{n, 0, \overline{n}\}$.*

(c) *w is in the* (n,n)**-configuration** *if there exist positive integers $1 \le p < q \le r = s = k$ such that $i_p = n$ or 0, and $j_q = 0$ or $\overline{n}$.*

Then the crystal graph $B(\lambda)$ with $\lambda = (\epsilon_1 + \cdots + \epsilon_k) + (\epsilon_1 + \cdots + \epsilon_l)$ for the finite dimensional irreducible $U_q(\mathfrak{osp}(1, 2n))$-module $V(\lambda)$ can be described as follows:

Proposition 5.8. *Let $B(Y_{k,l})$ $(k \le l)$ be the set of vectors in $B(Y_k) \otimes B(Y_l)$ of the form* $w = u \otimes v = \begin{array}{|c|}\hline j_1 \\ \vdots \\ j_k \\ \hline\end{array} \otimes \begin{array}{|c|}\hline i_1 \\ \\ \vdots \\ \\ i_l \\ \hline\end{array} = \begin{array}{|c|c|}\hline i_1 & j_1 \\ \vdots & \vdots \\ & j_k \\ \hline \\ \cline{1-1} i_l \\ \cline{1-1}\end{array}$ *satisfying the conditions:*

(i) *$i_r \le j_r$ for $1 \le r \le k$ and i_r and j_r cannot be 0 simultaneously.*

(ii) *whenever w is in the (a,b)-configuration for some a,b with $1 \leq a \leq b \leq n$, then*

$$(q-p)+(s-r)<b-a,$$

where p,q,r,s correspond to a,b as in Definition 5.7.

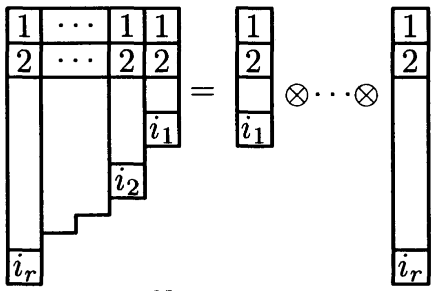
Then for $\lambda = (\epsilon_1 + \cdots + \epsilon_k) + (\epsilon_1 + \cdots + \epsilon_l)$,

$$B(\lambda) \cong B(Y_{k,l}).$$

Now for an arbitrary dominant integral weight λ of $U_q(\mathfrak{osp}(1,2n))$ with

$$\begin{aligned}\lambda &= a_1\omega_1 + \cdots + a_{n-1}\omega_{n-1} + 2a_n\omega_n \\ &= (\epsilon_1 + \cdots + \epsilon_{i_1}) + \cdots + (\epsilon_1 + \cdots + \epsilon_{i_r}),\end{aligned}$$

where $1 \leq i_1 \leq i_2 \leq \cdots \leq i_r \leq n$, let $u_\lambda =$

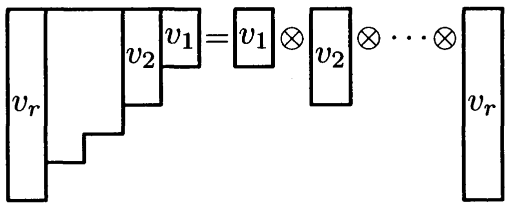


.

Then u_λ is a highest weight vector of weight λ in $\mathbf{B}^{\otimes N}$ whose connected component of $\mathbf{B}^{\otimes N}$ is described in the following theorem.

Theorem 5.9. *Let $B(Y)$ be the set of vectors of the form*

$v = v_1 \otimes \cdots \otimes v_r =$



$\in B(Y_{i_1}) \otimes \cdots \otimes B(Y_{i_r})$

such that $v_k \otimes v_{k+1} \in B(Y_{i_k,i_{k+1}})$ for all $k = 1, \cdots, r-1$. Then

$$B(\lambda) \cong B(Y).$$

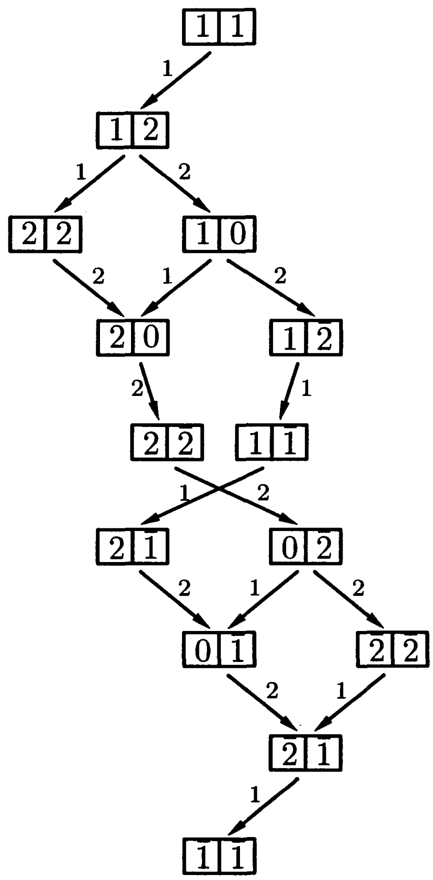
It is helpful to illlustrate these results with some examples.

Example 5.10.

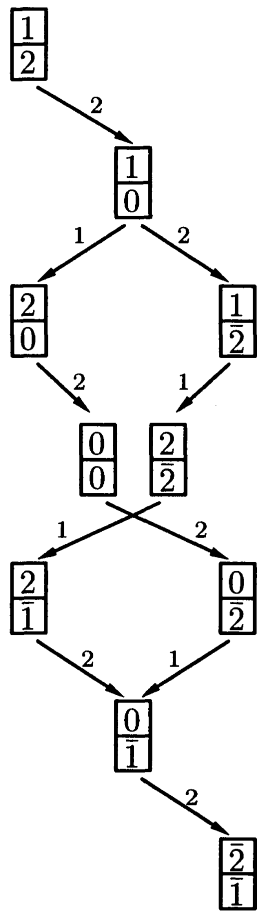
(a) The crystal graph $B(2\epsilon_1)$ over $U_q(\mathfrak{osp}(1,4))$.

(b) The crystal graph $B(\epsilon_1 + \epsilon_2)$ over $U_q(\mathfrak{osp}(1,4))$.

a)



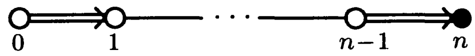
b)



§6. Quantum Affine Superalgebras

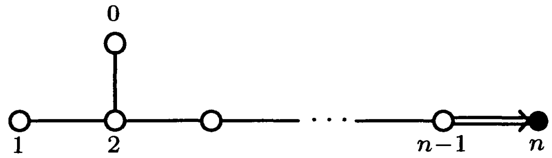
The only families of affine Lie superalgebras that belong to the class of Kac-Moody superalgebras are $B^{(1)}(0,n)$, $A^{(2)}(0,2n-1)$, $C^{(2)}(n+1)$ and $A^{(4)}(0,2n)$. Their Dynkin diagrams are

$B^{(1)}(0,n) \quad (n \geq 2)$



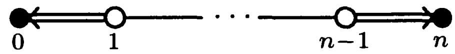
$B^{(1)}(0,1)$

$A^{(2)}(0,2n-1) \quad (n \geq 3)$



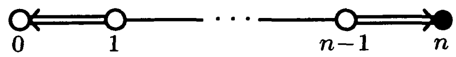
$A^{(2)}(0,3)$

$C^{(2)}(n+1) \quad (n \geq 2)$

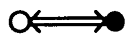


$C^{(2)}(2)$

$A^{(4)}(0,2n) \quad (n \geq 2)$



$A^{(4)}(0,2)$



For simplicity, we restrict our considerations to $B^{(1)}(0,n)$ $(n \geq 2)$, $A^{(2)}(0,2n-1)$ $(n \geq 3)$, and $A^{(4)}(0,2n)$ $(n \geq 2)$ only. For these classes of affine Kac-Moody superalgebras, tableaux bases are not known. However, we can take a different tack and develop the theory of *perfect crystals* as in [KMN1]. To define the notion of perfect crystals, we require some preliminaries.

Let $\mathfrak{g}$ be an affine Kac-Moody superalgebra corresponding to one of these diagrams, and let $I = \{0, 1, \cdots, n\}$ be the index set for the simple roots. We denote by $U'_q(\mathfrak{g})$ the quantum superalgebra corresponding to the derived subalgebra $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$. Let $P' = \bigoplus_{i=0}^{n} \mathbf{Z}\Lambda_i$ be the weight lattice of $\mathfrak{g}'$ with dominant integral weights $(P')^+$, and let $(P')^\vee = \bigoplus_{i=0}^{n} \mathbf{Z}h_i$ be the dual weight lattice of $\mathfrak{g}'$. For an element b in a crystal graph B of a $U'_q(\mathfrak{g})$-module V, set

$$\varepsilon(b) = \sum_{i=0}^{n} \varepsilon_i(b)\Lambda_i, \quad \text{and} \quad \varphi(b) = \sum_{i=0}^{n} \varphi_i(b)\Lambda_i. \tag{6.1}$$

Definition 6.2. *Assume B is a crystal graph of a finite dimen-*

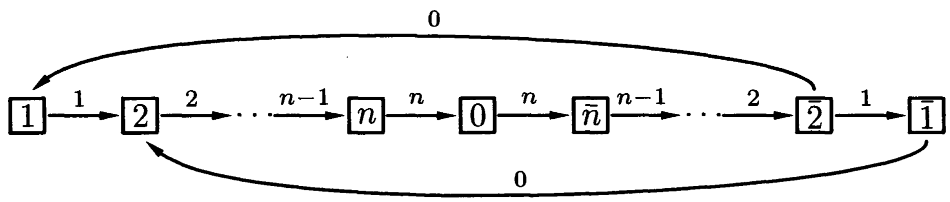
sional $U'_q(\mathfrak{g})$-module and let $l > 0$ be a positive integer. Then B is a **perfect crystal of level** $l > 0$ *if*

(i) *$B \otimes B$ is connected,*
(ii) *there exists a weight $\lambda_0 \in P'$ such that $|B_{\lambda_0}| = 1$ and $wt(B) \subset \lambda_0 + \sum_{i \neq 0} \mathbf{Z}_{\leq 0}\alpha_i$,*
(iii) *$\langle c, \varepsilon(b) \rangle \geq l$ for all $b \in B$, where c denotes the canonical central element for the affine Kac-Moody superalgebra $\mathfrak{g}$,*
(iv) *for each dominant integral weight $\lambda \in (P')^+$ of level l, there exist unique elements b^λ and b_λ in B such that $\varepsilon(b^\lambda) = \lambda$, $\varphi(b_\lambda) = \lambda$.*

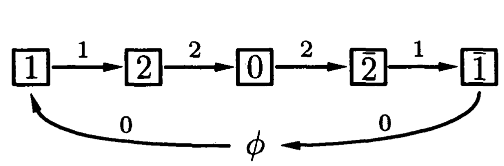
Perfect crystals play a crucial role in realizing the crystal graphs of irreducible highest weight modules over quantum affine superalgebras. We first present some illustrative examples.

Example 6.3.

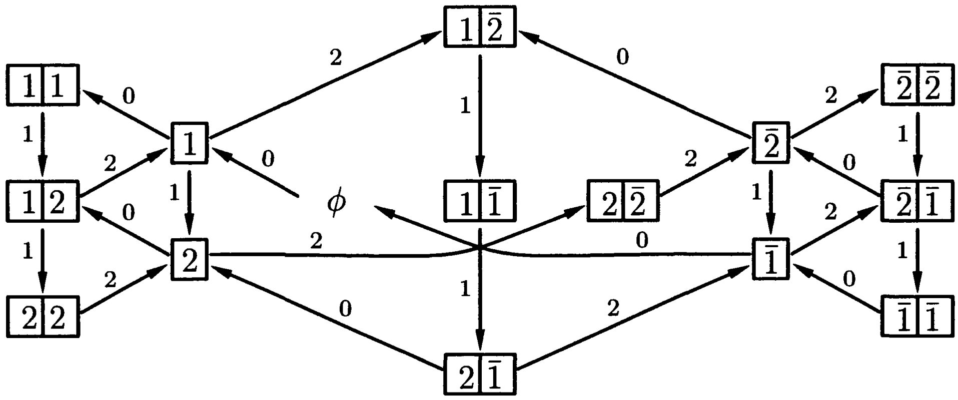
(a) Level 1 perfect crystal for $A^{(2)}(0, 2n-1)$



(b) Level 1 perfect crystal for $A^{(4)}(0, 4)$



(c) Level 2 perfect crystal for $A^{(4)}(0, 4)$



Perfect crystals give rise to the following important crystal isomorphism.

Theorem 6.4. ([KMN2]) *Assume* $\lambda \in (P')^+_l = \{\mu \in (P')^+ \mid \langle c, \mu \rangle = l\}$ *is a dominant integral weight of level* l, *and let* $B(\lambda)$ *be the crystal graph of the irreducible highest weight module* $V(\lambda)$ *over* $U'_q(\mathfrak{g})$. *Then for any perfect crystal* B *of level* l, *there is an isomorphism of crystals*

$$B(\lambda) \otimes B \cong B(\lambda'), \qquad u_\lambda \otimes b^\lambda \longmapsto u_{\lambda'},$$

where b^λ *is the unique element in* B *such that* $\varepsilon(b^\lambda) = \lambda$ *and* $\lambda' = \lambda + wt(b^\lambda)$.

Thanks to Theorem 6.4, the crystal graph $B(\lambda)$ has a *path realization*. Start with $\lambda = \lambda_0$ a dominant integral weight of $U'_q(\mathfrak{g})$ of level l, and let B be a perfect crystal of level l. By repeating the isomorphism of crystal graphs given in Theorem 6.4, we obtain

$$\begin{aligned} B(\lambda) \otimes B &\cong B(\lambda_1), \quad u_\lambda \otimes b_0 \mapsto u_{\lambda_1}, \\ B(\lambda_1) \otimes B &\cong B(\lambda_2), \quad u_{\lambda_1} \otimes b_1 \mapsto u_{\lambda_2}, \\ B(\lambda_2) \otimes B &\cong B(\lambda_3), \quad u_{\lambda_2} \otimes b_2 \mapsto u_{\lambda_3}, \\ &\cdots\cdots \end{aligned}$$

where $b_k = b^{\lambda_k}$ for $k = 0, 1, 2, \cdots$. Since there are only finitely many dominant integral weights of $U'_q(\mathfrak{g})$ of a given level, we must have $\lambda_N = \lambda_0 = \lambda$ for some $N > 0$. Thus, there is a chain of crystal isomorphisms

$$\begin{aligned} B(\lambda) &\cong B(\lambda_{N-1}) \otimes B \cong B(\lambda_{N-2}) \otimes B \otimes B \\ &\cong \cdots \cong B(\lambda_0) \otimes B \otimes \cdots \otimes B \end{aligned}$$

such that

$$\begin{aligned} u_\lambda &\longmapsto u_{\lambda_{N-1}} \otimes b_{N-1} \longmapsto u_{\lambda_{N-2}} \otimes b_{N-2} \otimes b_{N-1} \\ &\longmapsto \cdots \longmapsto u_\lambda \otimes b_0 \otimes \cdots \otimes b_{N-1}. \end{aligned}$$

The sequence

$$\begin{aligned} p_\lambda = (p_\lambda(k))_{k\geq 1} &= \cdots \otimes p_\lambda(k) \otimes \cdots \otimes p_\lambda(2) \otimes p_\lambda(1) \\ &= \cdots \otimes b_0 \otimes \cdots \otimes b_{N-1} \otimes b_0 \otimes \cdots \otimes b_{N-1} \end{aligned}$$

is called the *ground-state path* of weight λ. A λ-*path* is a sequence $p = (p(k))_{k\geq 1} = \cdots \otimes p(k) \otimes \cdots \otimes p(2) \otimes p(1)$ with $p(k) \in B$ such that $p(k) = p_\lambda(k)$ for all k sufficiently large. Let $\mathcal{P}(\lambda, B)$ be the set of all λ-paths in B and define the crystal structure on $\mathcal{P}(\lambda, B)$ using the tensor product rule to obtain the following.

Theorem 6.5.

$$B(\lambda) \cong \mathcal{P}(\lambda, B).$$

Because the structure of perfect crystals for quantum affine Kac-Moody superalgebras is the same as for perfect crystals for quantum affine Kac-Moody algebras, their description can be found in [KMN2] and [KK]. We close this section by giving an example of path realization.

Example 6.6. Let B be a perfect crystal of level 1 given in Example 6.3 (b). By Theorem 6.4, there is an isomorphism of crystal graphs

$$B(\Lambda_0) \otimes B \cong B(\Lambda_0), \qquad u_{\Lambda_0} \otimes \phi \longmapsto u_{\Lambda_0}.$$

Hence the ground-state path p_{Λ_0} is given by

$$p_{\Lambda_0} = \cdots \otimes \phi \otimes \phi \otimes \phi = (\cdots \phi, \phi, \phi),$$

and the path realization of $B(\Lambda_0)$ is

$$\begin{tikzcd}
 & (\cdots,\phi,\phi,\phi) \arrow[d, "0"'] & \\
 & (\cdots,\phi,\phi,1) \arrow[dl, "1"'] & \\
(\cdots,\phi,\phi,2) \arrow[d, "0"'] \arrow[dr, "2"] & & \\
(\cdots,\phi,1,2) \arrow[dr, "2"] & (\cdots,\phi,\phi,0) \arrow[d, "0"'] \arrow[dr, "2"] & \\
 & (\cdots,\phi,1,0) \arrow[dl, "1"'] \arrow[dr, "2"] & (\cdots,\phi,\phi,\bar{2}) \arrow[dl, "1"'] \arrow[d, "0"'] \\
(\cdots,\phi,2,0) & (\cdots,\phi,\phi,\bar{1}) & (\cdots,\phi,1,\bar{2}) \\
\vdots & \vdots & \vdots
\end{tikzcd}$$

§7. The Quantum Superalgebra $U_q(\mathfrak{gl}(m,n))$

In this section we focus on the q-analogue of one of the basic Lie superalgebras – the general linear Lie superalgebra $\mathfrak{gl}(m,n)$.

Suppose $V = V_0 \oplus V_1$ is a $\mathbf{Z}_2$-graded complex vector space such that $\dim V_0 = m$ and $\dim V_1 = n$. For $i = 0, 1$, let $\mathrm{End}(V)_i = \{x \in \mathrm{End}(V) \mid xV_j \subseteq V_{i+j}\}$ (subscripts are read mod 2). Then $\mathfrak{gl}(m,n)$ is $\mathrm{End}(V) = \mathrm{End}(V)_0 \oplus \mathrm{End}(V)_1$ regarded as a Lie superalgebra under the supercommutator product

$$[x,y] = xy - (-1)^{ij} yx, \qquad x \in \mathrm{End}(V)_i, \quad y \in \mathrm{End}(V)_j.$$

Set $\mathbf{B} = \mathbf{B}_+ \sqcup \mathbf{B}_-$, where $\mathbf{B}_+ = \{\overline{m}, \ldots, \overline{1}\}$, and $\mathbf{B}_- = \{1, \ldots, n\}$. We can think of V_0 (resp. V_1) as having a basis indexed by the elements of $\mathbf{B}_+$ (resp. $\mathbf{B}_-$), so that $\mathfrak{gl}(m,n)$ can be viewed as matrices having rows and columns indexed by $\mathbf{B}$. The diagonal matrices in $\mathfrak{gl}(m,n)$ can be taken to be a Cartan subalgebra for $\mathfrak{gl}(m,n)$. Let $P = \bigoplus_{b \in \mathbf{B}} \mathbf{Z}\epsilon_b$ be the lattice of integral weights and $P^\vee = \bigoplus_{b \in \mathbf{B}} \mathbf{Z}E_{b,b}$ the dual weight lattice of $\mathfrak{gl}(m,n)$, where ϵ_b denotes the projection of a matrix onto its (b,b)-entry, and $E_{b,b}$ is the standard matrix unit. Then the symmetric bilinear form on P is given by

$$(\epsilon_a, \epsilon_{a'}) = \begin{cases} 1 & \text{if } a = a' \in \mathbf{B}_+, \\ -1 & \text{if } a = a' \in \mathbf{B}_-, \\ 0 & \text{otherwise.} \end{cases}$$

We assume that the index set for the simple roots of $\mathfrak{g} = \mathfrak{gl}(m,n)$ is $I = I^{\text{even}} \sqcup I^{\text{odd}}$ where

$$I^{\text{even}} = \{\overline{m-1}, \dots, \overline{1}, 1, \dots, n-1\},$$
$$I^{\text{odd}} = \{0\},$$

and the simple roots are given by

$$\alpha_i = \begin{cases} \epsilon_{\overline{a+1}} - \epsilon_{\overline{a}} & \text{if } i = \overline{a} \text{ and } a = m-1, \dots, 1, \\ \epsilon_{\overline{1}} - \epsilon_1 & \text{if } i = 0, \\ \epsilon_i - \epsilon_{i+1} & \text{if } i = 1, \dots, n-1. \end{cases}$$

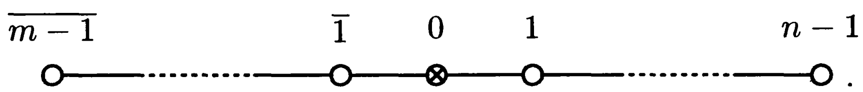
The coroot corresponding to α_i is the unique $h_i \in P^\vee$ satisfying

$$\ell_i \langle h_i, \lambda \rangle = (\alpha_i, \lambda) \quad \text{for any} \lambda \in P,$$

where

$$\ell_i = \begin{cases} 1 & \text{if } i = \overline{m-1}, \dots, \overline{1} \text{ or } 0, \\ -1 & \text{if } i = 1, \dots, n-1. \end{cases}$$

Relative to this indexing of simple roots, the Dynkin diagram is given by



This diagram corresponds to the matrix

(7.1)

$$A = (a_{i,j})_{i,j \in I} = \begin{pmatrix} 2 & -1 & 0 & \cdots &&&&&& \\ -1 & 2 & -1 & \cdots &&&&&& \\ 0 & -1 & \ddots &&&&&&& \\ &&& 2 & -1 &&&&& \\ &&& -1 & 0 & 1 &&&& \\ &&&& -1 & 2 & -1 &&& \\ &&&&& -1 & 2 & -1 && \\ &&&&&&&& \ddots & -1 \\ &&&&&&&& -1 & 2 \end{pmatrix}$$

with rows and columns indexed by the elements of $I = \{\overline{m-1}, \ldots, \overline{1}, 0, 1, \ldots, n-1\}$. Note that $a_{0,0} = 0$ and $a_{0,1} = 1$ so that A is not a generalized Cartan matrix of the type considered in Section 2.

As in [K1], we can construct the contragredient Lie superalgebra $\mathfrak{g} = \mathfrak{g}(A, I^{\text{odd}})$, $I^{\text{odd}} = \{0\}$, associated with the Cartan data (A, I^{odd}). The Lie superalgebra $\mathfrak{g} = \mathfrak{g}(A, I^{\text{odd}})$ is isomorphic to the special linear Lie superalgebra of matrices of supertrace zero:

$$\mathfrak{sl}(m,n) = \{x \in \mathfrak{gl}(m,n) \mid \operatorname{str}(x) = 0\},$$

where for an $(m+n) \times (m+n)$ matrix $x = (x_{b,b'})_{b,b' \in \mathbf{B}}$, its *supertrace* is given by

$$\operatorname{str}(x) = \sum_{b \in \mathbf{B}_+} x_{b,b} - \sum_{b \in \mathbf{B}_-} x_{b,b}.$$

The general linear Lie superalgebra $\mathfrak{gl}(m,n)$ is a 1-dimensional central extension of $\mathfrak{sl}(m,n)$.

Definition 7.2. ([KT], [Y]) *The* **quantum superalgebra** *$U_q(\mathfrak{gl}(m,n))$ is the associative algebra over $\mathbf{Q}(q)$ with 1 generated by the elements e_i, f_i $i \in I = I^{\text{even}} \cup I^{\text{odd}}$ and q^h $(h \in P^\vee)$ with defining relations*

(7.3)
$$\begin{aligned}
&q^0 = 1, \quad q^h q^{h'} = q^{h+h'} \quad \textit{for } h, h' \in P^\vee, \\
&q^h e_i q^{-h} = q^{\alpha_i(h)} e_i, \quad q^h f_i q^{-h} = q^{-\alpha_i(h)} f_i \quad \textit{for } \ h \in P^\vee, i \in I, \\
&e_i f_j - (-1)^{p(i)p(j)} f_j e_i = \delta_{i,j}(K_i - K_i^{-1})/(q_i - q_i^{-1}) \quad \textit{for } i,j \in I, \\
&(ad_q e_i)^{1-a_{i,j}}(e_j) = (ad_q f_i)^{1-a_{i,j}}(f_j) = 0 \\
&\qquad\qquad \textit{if } i \neq j \in I^{\text{even}}, \textit{ or if } i \in I^{\text{even}} \textit{ and } j = 0, \\
&e_0^2 = f_0^2 = 0, \\
&e_0 e_{\overline{1}} e_0 e_1 + e_{\overline{1}} e_0 e_1 e_0 + e_0 e_1 e_0 e_{\overline{1}} \\
&\qquad\qquad + e_1 e_0 e_{\overline{1}} e_0 - (q + q^{-1}) e_0 e_{\overline{1}} e_1 e_0 = 0, \\
&f_0 f_{\overline{1}} f_0 f_1 + f_{\overline{1}} f_0 f_1 f_0 + f_0 f_1 f_0 f_{\overline{1}} \\
&\qquad\qquad + f_1 f_0 f_{\overline{1}} f_0 - (q + q^{-1}) f_0 f_{\overline{1}} f_1 f_0 = 0.
\end{aligned}$$

Here, p denotes the parity map with $p(i) = 0$ if $i \neq 0$ and $p(0) = 1$, $q_i = q^{\ell_i}$, and $K_i = q^{\ell_i h_i}$ for $i \in I$.

The Hopf superalgebra structure on $U_q(\mathfrak{gl}(m,n))$ has comultiplication Δ, counit ϵ, and antipode S specified by the formulas in (2.7)-(2.9).

It follows from (7.3) that the subalgebra $U_q(\mathfrak{gl}(m,n))_i$ generated by e_i, f_i, $K_i^{\pm 1}$ is isomorphic to $U_q(\mathfrak{sl}_2)$ for $i \neq 0$ and to the quantum superalgebra $U_q(\mathfrak{sl}(1,1))$ for $i = 0$.

We now define the category of $U_q(\mathfrak{gl}(m,n))$-modules for which the crystal base theory is developed in [BKK].

Definition 7.4. *The category $\mathcal{O}_{\text{int}}$ is the category of $\mathbf{Z}_2$-graded finite dimensional $U_q(\mathfrak{gl}(m,n))$-modules M and $U_q(\mathfrak{gl}(m,n))$-module homomorphisms which satisfy the following constraints:*

(i) *M has a weight decomposition $M = \bigoplus_{\lambda \in P} M_\lambda$, where*
$M_\lambda = \{u \in M \mid q^h u = q^{\langle h, \lambda \rangle} u \quad \text{for all } h \in P^\vee\}$.
(ii) *if $M_\mu \neq 0$, then $\mu(h_0) \geq 0$.*
(iii) *if $\mu(h_0) = 0$, then $e_0 M_\mu = f_0 M_\mu = 0$. Thus M is a direct sum of 1-dimensional or 2-dimensional irreducible modules over $U_q(\mathfrak{gl}(m,n))_0 \cong U_q(\mathfrak{sl}(1,1))$.*

The category $\mathcal{O}_{\text{int}}$ is stable under taking subquotients and tensor products. In [BKK] it is conjectured that the modules in $\mathcal{O}_{\text{int}}$ are completely reducible.

Proposition 7.5. ([BKK]) *Let $V(\lambda)$ be an irreducible highest weight $U_q(\mathfrak{gl}(m,n))$-module with highest weight $\lambda = \sum_{b \in \mathbf{B}} \lambda_b \epsilon_b \in P$, where $\mathbf{B} = \{\overline{m} < \overline{m-1} < \cdots < \overline{2} < \overline{1} < 1 < 2 < \cdots < n-1 < n\}$. If $V(\lambda)$ belongs to category $\mathcal{O}_{\text{int}}$, then we have*

(i) *$\lambda_b \geq \lambda_{b'}$ for $b < b'$.*
(ii) *if $\lambda_b > 0$ for some $b = 1, \cdots, n$, then $\lambda_0 \geq b$.*

§8. Crystal bases for $U_q(\mathfrak{gl}(m,n))$

Whenever M is in the category $\mathcal{O}_{\text{int}}$ for $U_q(\mathfrak{gl}(m,n))$ and $i \in I^{\text{even}}$, then for any $u \in M$ of weight $\lambda \in P$, there is a unique expression

$$u = \sum_{k \geq 0, -\langle h_i, \lambda \rangle} f_i^{(k)} u_k$$

with $e_i u_k = 0$ for each k. For $U_q(\mathfrak{gl}(m,n))$ (and other contragredient Lie superalgebras) we use the divided powers

$$f_i^{(k)} = \frac{1}{[k]_i!} f_i^k,$$

where

$$(8.1)\qquad \begin{aligned} &[k]_i = (q_i^k - q_i^{-k})/(q_i - q_i^{-1}), \\ &[k]_i! = \prod_{n=1}^{k} [n]_i \quad \text{for } k \geq 1, \quad \text{and} \quad [0]! = 1. \end{aligned}$$

It is convenient to adopt the convention that $f_i^{(k)} = 0$ for $k < 0$.

Then the Kashiwara operators are defined by

Case (1): for $i = \overline{m-1}, \cdots, \overline{1}$,

$$(8.2)\qquad \tilde{e}_i u = \sum_k f_i^{(k-1)} u_k, \qquad \tilde{f}_i u = \sum_k f_i^{(k+1)} u_k.$$

Case (2): for $i = 1, \cdots, n-1$,

$$(8.3)\qquad \tilde{e}_i u = \sum_k q_i^{\lambda(h_i)+1} f_i^{(k-1)} u_k, \quad \tilde{f}_i u = \sum_k q_i^{-\lambda(h_i)+1} f_i^{(k+1)} u_k.$$

Case (3): for $i = 0$,

$$(8.4)\qquad \tilde{e}_i u = q_i^{-1} K_i e_i u, \quad \tilde{f}_i u = f_i u.$$

As before, let $\mathbf{A}$ denote the subring of $\mathbf{Q}(q)$ consisting of all rational functions $f/g \in \mathbf{Q}(q)$ such that $g(0) \neq 0$. Assume M is a $U_q(\mathfrak{gl}(m,n))$-module in the category $\mathcal{O}_{\text{int}}$. A free $\mathbf{A}$-submodule L of M is a **crystal lattice** if it satisfies the conditions in Definition 2.10 (but using the Kashiwara operators in (8.4)). A **crystal base** of M is a pair (L, B), where B is a subset of L/qL for which (i)-(v) of Definition 2.11 hold. The associated crystal of (L, B) consists of $B/\{\pm 1\}$ with the structure of a colored oriented graph where b, $b' \in B/\{\pm 1\}$ are joined by the i-arrow, $b \xrightarrow{\mathrm{i}} b'$, if $\tilde{f}_i b = b'$.

Lemma 8.5. (See Lemma 2.7 of [BKK].) *Let M be a $U_q(\mathfrak{gl}(m,n))$-module in $\mathcal{O}_{\text{int}}$ with a crystal base (L, B). Assume that*

(a) *the associated crystal is connected, and*
(b) *there is a weight λ such that* $\dim M_\lambda = 1$.

Then

(i) L/qL *is an irreducible module over the algebra generated by the* $\tilde{e}_i$*'s and the* $\tilde{f}_i$*'s.*
(ii) M *is an irreducible* $U_q(\mathfrak{g})$*-module.*
(iii) $L'_\lambda = L_\lambda$ *for* L' *a crystal lattice implies* $L' = L$.
(iv) *The crystal base of* M *is unique up to a constant multiple.*

The antiautomorphism η of $U_q(\mathfrak{gl}(m,n))$ defined by

$$\begin{aligned}\eta(q^h) &= q^h\\ \eta(e_i) &= q_i f_i K_i^{-1}\\ \eta(f_i) &= q_i^{-1} K_i e_i\end{aligned}$$

satisfies $\eta^2 = \mathrm{id}$. We say that a symmetric bilinear form $(\cdot,\cdot)$ on a $U_q(\mathfrak{gl}(m,n))$-module M is a *polarization* if $(au, v) = (u, \eta(a)v)$ holds for any $u, v \in M$ and $a \in U_q(\mathfrak{gl}(m,n))$.

It is an easy consequence of the relation $\Delta \circ \eta = (\eta \otimes \eta) \circ \Delta$ that the following holds:

Lemma 8.6. ([BKK]) *Let* M_1 *and* M_2 *be two* $U_q(\mathfrak{gl}(m,n))$*-modules with polarizations. Then the symmetric bilinear form* $(\cdot,\cdot)$ *on* $M_1 \otimes M_2$ *defined by* $(u_1 \otimes u_2, v_1 \otimes v_2) = (u_1, v_1)(u_2, v_2)$ *is a polarization.*

The Kashiwara operators $\tilde{e}_i$ and $\tilde{f}_i$ on the modules M in $\mathcal{O}_{\mathrm{int}}$ are defined so that $\tilde{e}_i$ and $\tilde{f}_i$ are adjoints of each other at $q = 0$ with respect to a polarization. More precisely:

Proposition 8.7. ([BKK]) *Let* M *be a* $U_q(\mathfrak{gl}(m,n))$*-module in* $\mathcal{O}_{\mathrm{int}}$ *with a crystal lattice* L, *and let* $(\,\cdot\,,\,\cdot\,)$ *be a polarization of* M. *Assume* $(L, L) \subset \mathbf{A}$. *Then the induced* $\mathbf{Q}$*-valued symmetric bilinear form* $(\,\cdot\,,\,\cdot\,)_0$ *on* L/qL *satisfies* $(\tilde{e}_i u, v)_0 = (u, \tilde{f}_i v)_0$ *for any* $u,\ v \in L/qL$.

Definition 8.8. ([BKK]) *A crystal base* (L, B) *for a* $U_q(\mathfrak{gl}(m,n))$*-module* M *is said to be polarizable if there is a polarization* $(\cdot,\cdot)$ *of* M *such that* $(L, L) \subset \mathbf{A}$, *and the induced* $\mathbf{Q}$*-valued symmetric bilinear form* $(\cdot,\cdot)_0$ *on* L/qL *satisfies*

$$(b,b')_0 = \begin{cases} \pm 1 & \text{if } b' = \pm b, \\ 0 & \text{otherwise} \end{cases}$$

for all $b, b' \in B$.

Assume M_1 and M_2 are $U_q(\mathfrak{gl}(m,n))$-modules in the category $\mathcal{O}_{\text{int}}$, and let (L_1, B_1) and (L_2, B_2) be their crystal bases. Proposition 8.9 below, which is proved in [BKK], says that $M_1 \otimes M_2$ has a crystal base given by $L = L_1 \otimes_{\mathbf{A}} L_2$ and $B = B_1 \otimes B_2 \subset (L_1/qL_1) \otimes (L_2/qL_2) = L/qL$. To describe the action of the Kashiwara operators on B we require ε_i and φ_i, which are defined exactly as in (4.5).

Proposition 8.9. *Suppose* (L_ν, B_ν) *is a crystal base of* M_ν, $\nu = 1, 2$. *Then*

(i) (L, B) *is a crystal base of* $M_1 \otimes M_2$.

(ii) *The actions of* $\tilde{e}_i$ *and* $\tilde{f}_i$ *on* $b_1 \otimes b_2$ $(b_1 \in B_1$ *and* $b_2 \in B_2)$ *are as follows:*

(a) *If* $i = \overline{m-1}, \cdots, \overline{1}$, *then*

$$\tilde{e}_i(b_1 \otimes b_2) = \begin{cases} \tilde{e}_i(b_1) \otimes b_2 & \text{if } \varphi_i(b_1) \geq \varepsilon_i(b_2), \\ b_1 \otimes \tilde{e}_i(b_2) & \text{if } \varphi_i(b_1) < \varepsilon_i(b_2), \end{cases}$$

$$\tilde{f}_i(b_1 \otimes b_2) = \begin{cases} \tilde{f}_i(b_1) \otimes b_2 & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2), \\ b_1 \otimes \tilde{f}_i(b_2) & \text{if } \varphi_i(b_1) \leq \varepsilon_i(b_2). \end{cases}$$

(b) *If* $i = 1, \cdots, n-1$, *then*

$$\tilde{e}_i(b_1 \otimes b_2) = \begin{cases} b_1 \otimes \tilde{e}_i(b_2) & \text{if } \varphi_i(b_2) \geq \varepsilon_i(b_1), \\ \tilde{e}_i(b_1) \otimes b_2 & \text{if } \varphi_i(b_2) < \varepsilon_i(b_1), \end{cases}$$

$$\tilde{f}_i(b_1 \otimes b_2) = \begin{cases} b_1 \otimes \tilde{f}_i(b_2) & \text{if } \varphi_i(b_2) > \varepsilon_i(b_1), \\ \tilde{f}_i(b_1) \otimes b_2 & \text{if } \varphi_i(b_2) \leq \varepsilon_i(b_1). \end{cases}$$

(c) *If* $i = 0$, *then*

$$\tilde{e}_i(b_1 \otimes b_2) = \begin{cases} \tilde{e}_i(b_1) \otimes b_2 & \text{if } \langle h_i, wt(b_1) \rangle > 0, \\ \pm b_1 \otimes \tilde{e}_i(b_2) & \text{if } \langle h_i, wt(b_1) \rangle = 0, \end{cases}$$

$$\tilde{f}_i(b_1 \otimes b_2) = \begin{cases} \tilde{f}_i(b_1) \otimes b_2 & \text{if } \langle h_i, wt(b_1) \rangle > 0, \\ \pm b_1 \otimes \tilde{f}_i(b_2) & \text{if } \langle h_i, wt(b_1) \rangle = 0. \end{cases}$$

The sign in part (c) depends on the parity of b_1 and i.

Part (i) of the next theorem is an immediate consequence of Definition 8.8 and Proposition 8.9. Part (ii) uses the polarization on a module M to show that the orthogonal subspace $N^\perp$ of any submodule N of M forms a complement. Part (iii) then follows directly from (i) and (ii).

Theorem 8.10.

(i) *Let (L_ν, B_ν) be a polarizable crystal base of $M_\nu \in \mathcal{O}_{\mathrm{int}}$ ($\nu = 1, 2$). Then $(L_1 \otimes_A L_2, B_1 \otimes B_2)$ is a polarizable crystal base of $M_1 \otimes M_2$.*

(ii) *If M is a $U_q(\mathfrak{gl}(m,n))$-module in $\mathcal{O}_{\mathrm{int}}$ with a polarizable crystal base, then M is completely reducible.*

(iii) *If M_ν ($\nu = 1, \dots, k$) is a $U_q(\mathfrak{gl}(m,n))$-module in $\mathcal{O}_{\mathrm{int}}$ with a polarizable crystal base, then $M_1 \otimes \cdots \otimes M_k$ is completely reducible.*

§9. Young Tableaux and Crystal Graphs for $U_q(\mathfrak{gl}(m,n))$

The result in (ii) of Theorem 8.10 is particularly striking because most modules for contragredient Lie superalgebras and their quantized enveloping algebras are not completely reducible. In this section we study the natural $(m+n)$-dimensional module $\mathbf{V}$ of $U_q(\mathfrak{gl}(m,n))$ and its tensor powers. Critical to the discussion will be the fact that $\mathbf{V}$ belongs to $\mathcal{O}_{\mathrm{int}}$ and has a polarizable crystal base. Then its tensor powers $\mathbf{V}^{\otimes k}$ have a polarizable crystal base and are completely reducible by (iii). In the nonquantum setting Berele and Regev [BR] have studied the tensor powers $V^{\otimes k}$ of the $(m+n)$-dimensional module V for $\mathfrak{gl}(m,n)$. They have shown that $V^{\otimes k}$ is completely reducible, and its summands have a combinatorial basis indexed by certain tableaux. Our idea in [BKK] was to exploit that tableau basis to describe a crystal base for the summands of $\mathbf{V}$. This line of attack follows the papers [KN], [KM], [L], and [MM], which construct crystal bases for the finite dimensional simple Lie algebras of types A_n, B_n, C_n, D_n, E_6, and G_2, and for the fundamental representation of the affine Lie algebra $\widehat{\mathfrak{sl}}(n)$ using tableaux. In earlier work (not using tableaux) Zou introduced a crystal base for the Lie superalgebra $\mathfrak{sl}(2,1)$ and studied its properties. However, Zou's notion of a crystal base in [Z2] differs from the one in Definition 2.11 since his base is invariant under some but not all of the Kashiwara operators. Zou's recent paper [Z3] has followed the approach of [BKK] to produce crystal bases for the family of simple Lie superalgebras $\Gamma(\sigma_1, \sigma_2, \sigma_3)$.

The simplest representation of $U_q(\mathfrak{gl}(m,n))$ is its $(m{+}n)$-dimensional vector representation $\mathbf{V} = \mathbf{V}_+ \oplus \mathbf{V}_-$, where $\mathbf{V}_\pm = \bigoplus_{b\in\mathbf{B}_\pm} \mathbf{Q}(q)v_b$, (where $\mathbf{B}_\pm$ is as in Section 6), and the action is specified by

$$q^h v_b = q^{\epsilon_b(h)} v_b,$$

$$e_i v_b = \begin{cases} v_{\overline{k+1}} & \text{if } i = \overline{k} \text{ and } b = \overline{k} \text{ with } k = 1,\dots,m-1 \\ v_{\overline{1}} & \text{if } i = 0 \text{ and } b = 1, \\ v_k & \text{if } i = k \text{ and } b = k+1 \text{ with } k = 1,\dots,n-1, \\ 0 & \text{otherwise,} \end{cases}$$

$$f_i v_b = \begin{cases} v_{\overline{k}} & \text{if } i = \overline{k} \text{ and } b = \overline{k+1} \text{ with } k = 1,\dots,m-1, \\ v_1 & \text{if } i = 0 \text{ and } b = \overline{1}, \\ v_{k+1} & \text{if } i = k \text{ and } b = k \text{ with } k = 1,\dots,n-1, \\ 0 & \text{otherwise.} \end{cases}$$

The $U_q(\mathfrak{gl}(m,n))$-module $\mathbf{V}$ belongs to the category $\mathcal{O}_{\text{int}}$, and $\mathbf{L} = \bigoplus_{b\in\mathbf{B}} \mathbf{A}v_b$ is a crystal lattice. The set $\{\pm v_b \mod q\mathbf{L} \mid b \in \mathbf{B}\}$ determines a crystal base of $\mathbf{V}$ with associated crystal:

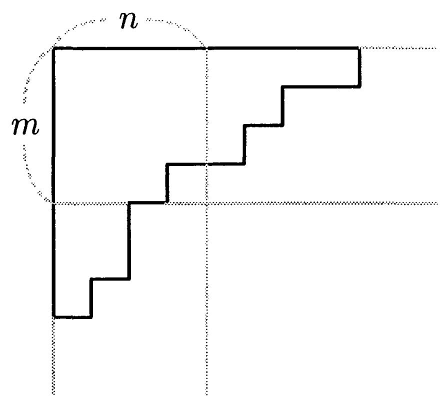
$$\boxed{\overline{m}} \xrightarrow{\overline{m-1}} \boxed{\overline{m-1}} \xrightarrow{\overline{m-2}} \cdots$$

$$\cdots \xrightarrow{\overline{2}} \boxed{\overline{2}} \xrightarrow{\overline{1}} \boxed{\overline{1}} \xrightarrow{0} \boxed{1} \xrightarrow{1} \boxed{2} \xrightarrow{2} \cdots$$

$$\cdots \xrightarrow{n-2} \boxed{n-1} \xrightarrow{n-1} \boxed{n}$$

With respect to the symmetric bilinear form on $\mathbf{V}$ which has $\{v_b \mid b \in \mathbf{B}\}$ as an orthonormal basis, $(\mathbf{L}, \mathbf{B} \sqcup (-\mathbf{B}))$ is a polarizable crystal base. As we have mentioned before, Theorem 8.10 (iii) says that $\mathbf{V}^{\otimes k}$ is completely reducible for all $k \geq 1$. Moreover, $(\mathbf{L}^{\otimes k}, (\mathbf{B} \sqcup (-\mathbf{B}))^{\otimes k})$ is a polarizable crystal base for $\mathbf{V}^{\otimes k}$.

A *Young diagram* is a collection of boxes arranged in left-justified rows with a weakly decreasing number of boxes in each row. A diagram obtained by removing a smaller Young diagram from a larger one containing it is a *skew Young diagram.* We say that a Young diagram is an (m,n)-hook Young diagram if the number of boxes in the $(m+1)$st row is less than or equal to n. Thus, an (m,n)-hook Young diagram fits inside the (m,n)-hook as displayed below.



We assign an order to the elements of $\mathbf{B}$ by saying

$$\overline{m} < \overline{m-1} < \cdots < \overline{2} < \overline{1} < 1 < 2 < \cdots < n-1 < n.$$

Then a *semistandard tableau* is obtained by filling a skew Young diagram with elements of $\mathbf{B}$ such that

(i) the entries in each row are increasing, allowing repetition of the elements of $\mathbf{B}_+ = \{\overline{m}, \overline{m-1}, \ldots, \overline{2}, \overline{1}\}$ but not of the elements of $\mathbf{B}_- = \{1, 2, \ldots, n-1, n\}$.

(ii) the entries in each column are increasing, permitting repetition of the elements of $\mathbf{B}_-$ but not of $\mathbf{B}_+$.

It is not difficult to see that a Young diagram can be made into a semistandard tableau with entries in $\mathbf{B}$ if and only if it is an (m, n)-hook Young diagram. These semistandard tableaux were introduced by Berele and Regev. In [BR] they show that the irreducible summands of the tensor powers $V^{\otimes k}$ of the $(m+n)$-dimensional module V for $\mathfrak{gl}(m, n)$ can be indexed by the (m, n)-hook Young diagrams. A basis for the summand indexed by Y is in one-to-one correspondence with the semistandard tableaux of shape Y. The weight of a tableau is $\sum_{b \in \mathbf{B}} \omega_b \epsilon_b$ where ω_b is the number of its entries which are equal to b.

If N is the number of boxes contained in a Young diagram Y, then we can embed the set $B(Y)$ of semistandard tableaux of shape Y into $\mathbf{B}^{\otimes N}$ by reading the entries $\{b_1, \ldots, b_N\}$ of the tableau and identifying the tableau with $b_1 \otimes \cdots \otimes b_N$. There are many different ways this reading can be done, and we single out certain special ones.

Suppose β and β' are boxes in a skew Young diagram with β lying in position (i, j) (row i and column j) and β' in position (i', j'). Then we say β is strictly higher than β' if $\beta \neq \beta'$ and $i \leq i'$ and $j \geq j'$. This just amounts to saying that β lies northeast of β'. In an *admissible reading*, box β is read before box β' whenever β is strictly higher than β'. For example, if we start with the rightmost column and read the entries from top to bottom, and then read the next column from top to bottom, and continue until the bottom entry in the leftmost column is read, we obtain an admissible reading, which we term a *Japanese (or Chinese) reading.* Similarly, reading the rows from right to left starting with the rightmost entry in the top row and proceeding to the bottommost row gives an admissible reading, which we call an *Arabic (or Hebrew) reading.* These particular admissible readings are illustrated in the following figure.

$$\begin{array}{|c|c|c|}\cline{2-3} \multicolumn{1}{c|}{} & \bar{3} & \bar{2} \\ \hline \bar{4} & \bar{1} & 1 \\ \hline 1 & 3 & \multicolumn{1}{c}{} \\ \cline{1-2} 2 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array} \quad = \quad \boxed{\bar{2}} \otimes \boxed{1} \otimes \boxed{\bar{3}} \otimes \boxed{\bar{1}} \otimes \boxed{3} \otimes \boxed{\bar{4}} \otimes \boxed{1} \otimes \boxed{2}$$

Japanese reading

$$\begin{array}{|c|c|c|}\cline{2-3} \multicolumn{1}{c|}{} & \bar{3} & \bar{2} \\ \hline \bar{4} & \bar{1} & 1 \\ \hline 1 & 3 & \multicolumn{1}{c}{} \\ \cline{1-2} 2 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array} \quad = \quad \boxed{\bar{2}} \otimes \boxed{\bar{3}} \otimes \boxed{1} \otimes \boxed{\bar{1}} \otimes \boxed{\bar{4}} \otimes \boxed{3} \otimes \boxed{1} \otimes \boxed{2}$$

Arabic reading

Theorem 9.1. *(Compare Thm. 4.4 of* [BKK]*.) Let Y be a skew Young diagram and $B(Y)$ be the set of semistandard tableaux of shape Y. Then*

(i) *For any admissible reading $\psi : B(Y) \to \mathbf{B}^{\otimes N}$ of Y, the image $\psi(B(Y))$ is stable under the operators $\tilde{e}_i$ and $\tilde{f}_i$ for all $i \in I$.*
(ii) *The induced crystal structure on $B(Y)$ does not depend on the admissible reading.*

For a crystal base (L, B) of a $U_q(\mathfrak{gl}(m,n))$-module, we say that an element $b \in B_\lambda$ is a *genuine highest weight vector* of B if $B_\lambda = \{b\}$ and $\mathrm{Wt}(B) \subset \lambda - Q_+$, where $\mathrm{Wt}(B)$ is the set of weights of the crystal B and $Q_+ = \sum_{\alpha\in\Delta^+} \mathbf{Z}_{\geq 0}\alpha$. Analogously, $b \in B_\mu$ is a *genuine lowest weight vector* of B if $B_\mu = \{b\}$ and $\mathrm{Wt}(B) \subset \mu + Q_+$. A genuine highest (resp. lowest) weight vector is unique whenever it exists. Moreover a genuine highest (resp. lowest) weight vector satisfies $\tilde{e}_i b = 0$ (resp. $\tilde{f}_i b = 0$) for all $i \in I$. It is possible for an element $b \in B$ to satisfy one of these properties without being a genuine highest or lowest weight vector. Such vectors we term *fake highest (or lowest) weight vectors.* The existence of fake highest (or lowest) weight vectors complicates the question of the connectedness of the crystal graph. However, we have

Proposition 9.2. ([BKK]) *The crystal $B(Y)$ associated with any (m,n)-hook Young diagram Y is connected.*

Suppose Y_0 is an (m,n)−hook Young diagram. In [BKK] we have developed a combinatorial procedure for decomposing the tensor product $B(Y_0) \otimes \mathbf{B}$ into connected components $B(Y)$ corresponding to diagrams Y obtained from Y_0 by adding a box. A box in a diagram is a *corner* if there are no boxes in the diagram to its right or beneath it. A place where a box can be adjoined to a diagram to create a corner of a larger diagram is said to be a *co-corner.*

Theorem 9.3. ([BKK]) *Assume Y_0 is an (m,n)-hook Young diagram, and let $B(Y_0)$ be the set of all semistandard tableaux of shape Y_0 endowed with a crystal structure by an admissible reading. Then the tensor product of crystals $B(Y_0) \otimes \mathbf{B}$ has the following decomposition into connected components:*

$$B(Y_0) \otimes \mathbf{B} \cong \bigoplus_{Y} B(Y),$$

where Y runs over the set of all (m,n)-hook Young diagrams obtained from Y_0 by adding a box to a co-corner of Y_0.

As an immediate consequence we obtain

Corollary 9.4. *Any connected component of $\mathbf{B}^{\otimes k}$ of k copies of $\mathbf{B}$ is isomorphic (as a crystal) to $B(Y)$ for some (m,n)-hook Young diagram Y having k-boxes. Moreover, for any skew Young diagrams Y_1 and Y_2, the connected components of the tensor product of crystals $B(Y_1) \otimes B(Y_2)$ have the form $B(Y)$, where Y is an (m,n)-hook Young diagram.*

Consider the set $\widetilde{P}$ of weights $\lambda \in \bigoplus_{b\in\mathbf{B}} \mathbf{Z}\epsilon_b$ satisfying

(i) $\langle h_i, \lambda\rangle \geq 0$ for all $i \in I$,

(ii) $\langle h_0 - h_1 - \cdots - h_k, \lambda\rangle \geq k$ for all $k \in \{1, \ldots, n-1\}$ such that $\langle h_k, \lambda\rangle > 0$.

These conditions exactly translate to the ones encountered in Proposition 7.5. The weights in $\widetilde{P}$ play a distinguished role because of the following.

Proposition 9.5. ([BKK]) *If $V(\lambda)$ is an irreducible $U_q(\mathfrak{gl}(m,n))$-module in $\mathcal{O}_{\text{int}}$ with highest weight λ, then $\lambda \in \widetilde{P}$.*

Suppose that $\lambda \in \widetilde{P}^+ = \widetilde{P} \bigcap \bigoplus_{b \in \mathbf{B}} \mathbf{Z}_{\geq 0} \epsilon_b$ and write $\lambda = a_1 \epsilon_{\overline{m}} + a_2 \epsilon_{\overline{m-1}} + \cdots + a_m \epsilon_{\overline{1}} + d_1 \epsilon_1 + \cdots + d_n \epsilon_n$. Then we can create an (m,n)-hook tableau H_λ by this procedure:

(1) row i has a_i boxes all filled with the entry $\overline{i}$ for $i = 1, \ldots, m$,
(2) starting below row m, column j has d_j boxes that are filled with j for $j = 1, \ldots, n$.

The weight of the tableau H_λ is λ, and its shape is the Young diagram that we denote by Y_λ. Any semistandard tableau with shape Y_λ has weight in $\lambda - Q_+$. Thus, H_λ is a genuine highest weight vector in $B(Y_\lambda)$.

Theorem 9.6. ([BKK]) *If $\lambda \in \widetilde{P}$, then the irreducible $U_q(\mathfrak{gl}(m,n))$-module $V(\lambda)$ with highest weight λ is in $\mathcal{O}_{\text{int}}$, and it has a polarizable crystal base. If $\lambda \in \widetilde{P}^+$, then the associated crystal is isomorphic to $B(Y_\lambda)$.*

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Approximation Results for Kazhdan-Lusztig Polynomials

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§1. Introduction

In their fundamental paper [7] Kazhdan and Lusztig defined, for every Coxeter group W, a family of polynomials, indexed by pairs of elements of W, which have become known as the Kazhdan-Lusztig polynomials of W (see, e.g., [6], Chap. 7). These polynomials are intimately related to the Bruhat order of W and to the geometry of Schubert varieties, and have proven to be of fundamental importance in representation theory. In order to prove the existence of these polynomials Kazhdan and Lusztig used another family of polynomials (see [7], §2) which are intimately related to the multiplicative structure of the Hecke algebra associated to W. These polynomials are known as the R-polynomials of W (see, e.g., [6], §7.5) and their importance stems mainly from the fact that their knowledge is equivalent to that of the Kazhdan-Lusztig polynomials.

The main idea of this work is to use the theory of P-kernels developed by Stanley in [10] to approximate the Kazhdan-Lusztig polynomials with other "KLS-functions" (see §2 for definitions) that are easier to compute. In particular, we characterize the pairs $u, v \in W$ such that the Kazhdan-Lusztig polynomials of the subintervals of $[u, v]$ satisfy certain vanishing properties or, more generally, coincide with some given function in the incidence algebra of W, up to a given order. Two of our results generalize and refine previous ones that have appeared in [7] and [3].

The theory of P-kernels also naturally leads to define and study certain polynomials, indexed by pairs of elements of W, that are "dual"

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to the R-polynomials of W in a very precise sense. To the best of our knowledge, although their definition is quite natural, these polynomials have never been considered before in the literature. Similarly, we are led to the study of the "dual" of the zeta function of a locally Eulerian poset, which also seems to be a new object.

The organization of the paper is as follows. In the next section we collect notation, definitions, and results, that are used in the sequel. In §3 we prove our main results (Theorems 3.1 and 3.2). These are purely combinatorial results that "compare" two KLS-functions in terms of their kernels. In section 4 we define a natural involution on kernels and KLS-functions and study in some detail the dual of the zeta function of a locally Eulerian poset, and of the R-polynomials of a Coxeter group. To the best of our knowledge, these objects have never been considered before. We also study how a local change in a KLS-function affects the corresponding kernel. In section 5 we apply the results obtained in the two previous ones to the Kazhdan-Lusztig polynomials. In particular, we characterize the intervals of W such that the Kazhdan-Lusztig polynomials of its subintervals (respectively, lower subintervals) are equal to 1 up to a given order. Finally, in section 6, we discuss some conjectures and open problems arising from the present work.

§2. Notation, Definitions, and preliminaries

In this section we collect some definitions, notation and results that will be used in the rest of this work. We let $\mathbf{P} \stackrel{\text{def}}{=} \{1, 2, 3, \dots\}$, $\mathbf{N} \stackrel{\text{def}}{=} \mathbf{P} \cup \{0\}$, $\mathbf{Z}$ be the ring of integers, $\mathbf{Q}$ be the field of rational numbers, and $\mathbf{R}$ be the field of real numbers; for $a \in \mathbf{N}$ we let $[a] \stackrel{\text{def}}{=} \{1, 2, \dots, a\}$ (where $[0] \stackrel{\text{def}}{=} \emptyset$). The cardinality of a set A will be denoted by $|A|$. We write $A \subset B$ to mean that $A \subseteq B$ and $A \neq B$. Given a polynomial $P(q)$, and $i \in \mathbf{Z}$, we denote by $[q^i](P(q))$ the coefficient of q^i in $P(q)$. For $a \in \mathbf{Q}$ we let $\lfloor a \rfloor$ (respectively, $\lceil a \rceil$) denote the largest integer $\leq a$ (respectively, smallest integer $\geq a$). Given $A(q) \in \mathbf{R}[q]$ and $d \in \mathbf{P}$ we say that $A(q)$ is *symmetric* (respectively, *antisymmetric*) with respect to d if $q^d A\left(\frac{1}{q}\right) = A(q)$ (respectively, $q^d A\left(\frac{1}{q}\right) = -A(q)$).

For $j \in \mathbf{Q}$ we define operators U_j, $D_j : \mathbf{R}[q] \to \mathbf{R}[q]$ by letting

$$U_j\left(\sum_{i \geq 0} a_i q^i\right) \stackrel{\text{def}}{=} \sum_{i \geq j} a_i q^i,$$

and

$$D_j\left(\sum_{i\geq 0} a_i q^i\right) \stackrel{\text{def}}{=} \sum_{i=0}^{\lfloor j\rfloor} a_i q^i.$$

Note that U_j and D_j are linear and idempotent, and that $D_j = D_{\lfloor j\rfloor}$ and $U_j = U_{\lceil j\rceil}$, for all $j \in \mathbf{Q}$. The following lemma will be used repeatedly in this paper and its simple verification is omitted.

Lemma 2.1. *Let $A(q), B(q) \in \mathbf{R}[q]$, and $k \in \mathbf{Z}$. Then*

$$D_k(AB) = D_k(D_k(A)D_k(B)).$$

We follow [9], Chap. 3, for notation and terminology concerning partially ordered sets. In particular, given a partially ordered set (or, poset, for short) P we let $\mathrm{Int}(P) \stackrel{\text{def}}{=} \{(x,y) \in P^2 : x \leq y\}$, and given $u, v \in P$ we let $[u,v] \stackrel{\text{def}}{=} \{x \in P : u \leq x \leq v\}$, and define $[u,v)$ and $(u,v]$ similarly. We consider $[u,v]$ as a poset with the partial ordering induced by P. We say that a poset P is *locally finite* if $|[x,y]| < +\infty$ for all $(x,y) \in \mathrm{Int}(P)$, and in this case we denote by ζ_P (respectively, μ_P, δ_P) the *zeta* (respectively, *Möbius, delta*) function of P. We will usually omit the index P if there is no danger of confusion.

Given a finite graded poset P and $S \subseteq \mathbf{N}$ we let $P_S \stackrel{\text{def}}{=} \{x \in P : l(x) \in S\}$, where $l : P \to \mathbf{N}$ is the rank function of P, and $\alpha(P;S)$ be the number of maximal chains of P_S. We also let $P_i \stackrel{\text{def}}{=} P_{\{i\}}$ if $i \in \mathbf{N}$. We call $G(P) \stackrel{\text{def}}{=} \sum_{i\geq 0} |P_i| q^i$ the *rank generating function* of P, $d \stackrel{\text{def}}{=} \deg(G(P))$ the *rank* of P, and the collection of numbers $\{\alpha(P;S)\}_{S\subseteq[d]}$ the *flag f-vector* of P. We say that a finite graded poset P as above is *rank symmetric* if $G(P)$ is symmetric with respect to d, and is *Eulerian* if P has a $\hat{0}$ and $\hat{1}$ and $\mu(x,y) = (-1)^{l(y)-l(x)}$ for all $x, y \in P$, $x \leq y$. Following [10, §7, p. 835] (respectively, [11]) we say that a locally finite poset P is *locally Eulerian* (respectively, *locally rank symmetric*) if $[x,y]$ is Eulerian (respectively, *rank symmetric*) for all $(x,y) \in \mathrm{Int}(P)$.

Recall (see, e.g., [9], §3.6) that given a locally finite poset P and a commutative ring R the *incidence algebra* of P with coefficients in R, denoted $I(P;R)$, is the set of all functions $f : \mathrm{Int}(P) \to R$ with sum and product defined by

$$(f+g)(x,y) \stackrel{\text{def}}{=} f(x,y) + g(x,y)$$

and

$$(fg)(x,y) \stackrel{\mathrm{def}}{=} \sum_{x \leq z \leq y} f(x,z)\, g(z,y),$$

for all $f, g \in I(P;R)$ and $(x,y) \in \mathrm{Int}(P)$. It is well known (see, e.g., [9], §3.6, and Proposition 3.6.2) that $I(P;R)$ is an associative algebra having δ as identity element, and that an element $f \in I(P;R)$ is invertible if and only if $f(x,x) \in R$ is invertible for all $x \in P$. If f is invertible then we denote by f^{-1} its (two-sided) inverse. Given $f \in I(P;R)$ we define $f^* \in I(P^*;R)$ (where P^* denotes the order dual of P) by letting

$$f^*(v,u) \stackrel{\mathrm{def}}{=} f(u,v)$$

for all $(v,u) \in \mathrm{Int}(P^*)$. Note that $\zeta_{P^*} = \zeta_P^*$, $\delta_{P^*} = \delta_P^*$, and $\mu_{P^*} = \mu_P^*$. We adopt the convention that $f(u,v) \stackrel{\mathrm{def}}{=} 0$ if $f \in I(P;R)$ and $u, v \in P$, $u \not\leq v$.

Let P be a locally finite poset. We say that a function $\rho : \mathrm{Int}(P) \to \mathbf{N}$ is a *weak rank function* for P if it has the following two properties:

i): if $u < v$ then $\rho(u,v) > 0$;

ii): if $u \leq a \leq v$ then $\rho(u,v) = \rho(u,a) + \rho(a,v)$.

Note that a weak rank function always exists and that if ρ is a weak rank function for P then ρ^* is a weak rank function for P^*. The concept of a weak rank function enables us to extend the main definitions of §6 of [10] from the locally graded case (i.e., posets P such that $[x,y]$ is a finite graded poset for all $(x,y) \in \mathrm{Int}(P)$) to the locally finite case.

Let P and ρ be as above and $I(P) \stackrel{\mathrm{def}}{=} I(P;\mathbf{R}[q])$. Following Stanley (see [10],p. 830, and Proposition 6.11, p. 835) we let

$$\tilde{I}(P) \stackrel{\mathrm{def}}{=} \{f \in I(P) : \ \deg(f(x,y)) \leq \rho(x,y), \quad \text{for all } (x,y) \in \mathrm{Int}(P)\},$$

and

$$\begin{aligned} I_{\frac{1}{2}}(P) \quad &\stackrel{\mathrm{def}}{=} \quad \{f \in \tilde{I}(P) : \ \deg(f(u,v)) \leq \frac{1}{2}(\rho(u,v) - 1) \text{ for } u < v, \\ &\qquad\quad \text{and } \ f(u,u) = 1 \ \text{ for all } \ u \in P\}. \end{aligned}$$

Note that $\tilde{I}(P)$ is a subalgebra of $I(P)$ and that, if $f \in I(P)$ is invertible, then $f \in \tilde{I}(P)$ if and only if $f^{-1} \in \tilde{I}(P)$. Given $f \in \tilde{I}(P)$ and $k \in \mathbf{Q}$ we let

$$\overline{f}(u,v) \stackrel{\mathrm{def}}{=} q^{\rho(u,v)} f(u,v) \left(\frac{1}{q}\right),$$

and

$$D_k(f)(u,v) \stackrel{\text{def}}{=} D_k(f(u,v)),$$

for all $u, v \in P$, $u \leq v$. Notice that $\tilde{I}(P)$, $I_{\frac{1}{2}}(P)$, and the involution $^{-}$ all depend also on ρ. However, throughout this work ρ will always be fixed, so no confusion should arise. Recall (see [10], Definition 6.2, p. 830) that an element $K \in I(P)$ is called a *P-kernel* (or, more simply, a *kernel*) if K is *unitary* (i.e., $K(u,u) = 1$ for all $u \in P$) and there exists an element $f \in I(P)$ such that:

i): f is invertible in $I(P)$;

ii): $fK = \overline{f}$.

An element $f \in I(P)$ satisfying ii) above is called *K-totally acceptable* (see [10], Definition 6.2, p.830). The next result was first proved by Stanley in the locally graded case (see [10], Corollary 6.7), and by the author in the locally finite one (see [5, Theorem 6.2]).

Theorem 2.2. *Let P be a locally finite poset and $K \in I(P)$ a P-kernel. Then there exists a unique K-totally acceptable element $\gamma \in I_{\frac{1}{2}}(P)$.*

We call the element γ whose existence and uniqueness is guaranteed by the preceding theorem the *Kazhdan-Lusztig-Stanley function* (or *KLS-function*, for short) of K. As noted in [10], §§6 and 7, the function γ specializes to many interesting objects depending on the particular choice of the poset P and kernel K.

There is a simple way to decide if a given element $K \in I(P)$ is a P-kernel or not. The following result was first proved by Stanley in [10] (see Theorem 6.5, p. 831) in the case that P is locally graded. However, his proof carries over unchanged to the present more general setting.

Theorem 2.3. *Let P be a locally finite poset and $K \in I(P)$ be such that $K(u,u) = 1$ for all $u \in P$. Then K is a P-kernel if and only if $K\overline{K} = \delta$.*

Note that Theorem 2.2 defines a map from the set of P-kernels to $I_{\frac{1}{2}}(P)$ and that, by Theorem 2.3, the map $f \mapsto f^{-1}\overline{f}$ is its inverse. Thus the correspondence $K \mapsto \gamma$ of Theorem 2.2 is a bijection. We call this bijection the *KLS-correspondence* of P and the elements of $I_{\frac{1}{2}}(P)$ the *KLS-functions* of P.

For a locally finite poset P define an element $\chi_P \in I(P)$ by letting

$$\chi_P(u,v) \stackrel{\text{def}}{=} \sum_{a \in [u,v]} \mu(u,a) q^{\rho(a,v)} \tag{1}$$

for all $(u, v) \in \mathrm{Int}(P)$ ($\chi_P(u, v)$ is often called the *characteristic polynomial* of $[u, v]$, see, e.g., [9, §3.10, p.128]). It is then clear from the definitions (see also [10, Example 6.8, p. 833]) that χ_P is a P-kernel, and that ζ is its KLS-function. We call χ_P the *characteristic kernel* of P. Note that, in general, $\chi_P^* \neq \chi_{P^*}$ even if P^* is weakly graded by ρ^*.

We follow [6] for general Coxeter groups notation and terminology. Given a Coxeter system (W, S) and $\sigma \in W$ we denote by $l(\sigma)$ the length of σ in W, with respect to S, and we let $D(\sigma) \stackrel{\mathrm{def}}{=} \{s \in S : l(s\sigma) < l(\sigma)\}$, and $\varepsilon_\sigma \stackrel{\mathrm{def}}{=} (-1)^{l(\sigma)}$. We denote by e the identity of W, and we let $T \stackrel{\mathrm{def}}{=} \{\sigma s \sigma^{-1} : \sigma \in W,\ s \in S\}$ be the set of reflections of W. We will always assume that W is partially ordered by (strong) *Bruhat order*. Recall (see, e.g., [6], §5.9) that this means that $x \leq y$ if and only if there exist $r \in \mathbf{N}$ and $t_1, \ldots, t_r \in T$ such that $t_r \ldots t_1 x = y$ and $l(t_i \ldots t_1 x) > l(t_{i-1} \ldots t_1 x)$ for $i = 1, \ldots, r$. It is well known (see, e.g., [6, §8.5], Proposition 1, iv)) that intervals of W are finite Eulerian posets, and it is clear that $\rho(x, y) \stackrel{\mathrm{def}}{=} l(y) - l(x)$ for $(x, y) \in \mathrm{Int}(W)$ is a weak rank function for W. The following two results are well known and we refer the reader to [6, §7.5] and to [6, §§7.9-11] for their proofs.

Theorem 2.4. *Let (W, S) be a Coxeter system. Then there is a unique family of polynomials $\{R_{u,v}(q)\}_{u,v \in W} \subseteq \mathbf{Z}[q]$ such that, for all $u, v \in W$:*

i): $R_{u,v}(q) = 0$ *if* $u \not\leq v$;

ii): $R_{u,u}(q) = 1$;

iii): *if* $u < v$ *and* $s \in D(v)$ *then*

$$R_{u,v}(q) = \begin{cases} R_{su,sv}(q), & \text{if } s \in D(u), \\ (q-1) R_{u,sv}(q) + q R_{su,sv}(q), & \text{if } s \notin D(u). \end{cases}$$

Theorem 2.5. *Let (W, S) be a Coxeter system. Then there is a unique family of polynomials $\{P_{u,v}(q)\}_{u,v \in W} \subseteq \mathbf{Z}[q]$, such that, for all $u, v \in W$:*

i): $P_{u,v}(q) = 0$ *if* $u \not\leq v$;

ii): $P_{u,u}(q) = 1$;

iii): $deg(P_{u,v}(q)) \leq \lfloor \frac{1}{2} (l(v) - l(u) - 1) \rfloor$, *if* $u < v$;

iv):

$$q^{l(v)-l(u)} P_{u,v}\left(\frac{1}{q}\right) = \sum_{u \leq z \leq v} R_{u,z}(q) P_{z,v}(q),$$

if $u \leq v$.

The polynomials $R_{u,v}(q)$ and $P_{u,v}(q)$, whose existence is guaranteed by the two previous theorems, are called the *R-polynomials* and

Kazhdan-Lusztig polynomials of W. There is one more property of the polynomials $R_{u,v}(q)$ that we will use, and that we recall here for the reader's convenience. A proof of it can be found in [6, §7.8].

Proposition 2.6. *Let (W, S) be a Coxeter system. Then*

$$q^{l(v)-l(u)} R_{u,v}\left(\frac{1}{q}\right) = (-1)^{l(v)-l(u)} R_{u,v}(q)$$

for all $u, v \in W$.

We define two elements $\Re, \wp \in \tilde{I}(W)$ by letting

$$\Re(u,v) \stackrel{\text{def}}{=} (-1)^{l(v)-l(u)} R_{u,v}(q), \tag{2}$$

and

$$\wp(u,v) \stackrel{\text{def}}{=} P_{u,v}(q), \tag{3}$$

for all $u, v \in W$, $u \leq v$. It then follows immediately from Proposition 2.6 and Theorem 2.5 that $\overline{\Re}\wp = \overline{\wp}$ and that $\wp \in I_{\frac{1}{2}}(W)$. Therefore $(\overline{\Re})^*$ is a W^*-kernel and $\wp^*$ is its KLS-function.

Given $u, v \in W$, $u \leq v$, we define a polynomial $Q_{u,v}(q) \in \mathbf{Z}[q]$ by letting

$$Q_{u,v}(q) \stackrel{\text{def}}{=} (-1)^{l(v)-l(u)} \wp^{-1}(u,v). \tag{4}$$

(note that $\wp$ is invertible in $I(W)$ by part ii) of Theorem 2.5). It then follows immediately from well known results (see, e.g., [8], p. 190) that $Q_{u,v}(q)$ is the inverse Kazhdan-Lusztig polynomial of u, v.

The R-polynomials are much better understood than the Kazhdan-Lusztig polynomials (see, e.g., [6, p. 159]). For example, it is well known (see, e.g., [6, §§7.4-5]) and easy to see, that $R_{x,y}(q)$ is always a monic polynomial of degree $l(y) - l(x)$, while neither the degree nor the leading term of $P_{x,y}(q)$ can be easily predicted. Therefore, some of the recent research on Kazhdan-Lusztig polynomials (see, e.g., [3]) has focused on using the R-polynomials to gain information on the Kazhdan-Lusztig polynomials. This is the case for the present work also.

Throughout this paper, unless otherwise explicitly stated, (W, S) denotes a Coxeter system, P a locally finite poset, and $\rho : \text{Int}(P) \to \mathbf{N}$ a weak rank function for P.

§3. Comparison Results

In this section we derive the main results on which our applications to Kazhdan-Lusztig polynomials are based. These are purely combinatorial results which "compare" two KLS-functions in terms of their kernels. They can also be seen as giving some fundamental properties of the KLS-correspondence of a locally finite poset.

Theorem 3.1. *Let $k \in \mathbf{Z}$, $u, v \in P$, K_1, K_2 be P-kernels, and γ_1, γ_2 be their KLS-functions. Then the following are equivalent:*

i): $D_k(\gamma_1(x,y)) = D_k(\gamma_2(x,y))$ *for all* $x, y \in [u,v]$;
ii): $D_k(K_1(x,y)) = D_k(K_2(x,y))$ *for all* $x, y \in [u,v]$.

Proof. Assume that ii) holds. We proceed by induction on $\rho(u,v)$. If $\rho(u,v) = 0$ then $u = v$ and i) coincides with ii). So assume $\rho(u,v) > 0$, and let $x, y \in [u,v]$. Then from ii) and our induction hypothesis we conclude that

$$\begin{aligned} D_k(\overline{\gamma_1}(x,y) - \gamma_1(x,y)) &= D_k\left(\sum_{x \le a < y} \gamma_1(x,a) K_1(a,y)\right) \\ &= D_k\left(\sum_{x \le a < y} D_k(\gamma_1(x,a)) D_k(K_1(a,y))\right) \\ &= D_k\left(\sum_{x \le a < y} D_k(\gamma_2(x,a)) D_k(K_2(a,y))\right) \\ &= D_k\left(\sum_{x \le a < y} \gamma_2(x,a) K_2(a,y)\right) \\ &= D_k(\overline{\gamma_2}(x,y) - \gamma_2(x,y)). \end{aligned}$$

Since $\gamma_1, \gamma_2 \in I_{\frac{1}{2}}(P)$ this implies that $D_k(\gamma_1(x,y)) = D_k(\gamma_2(x,y))$ and this proves i).

Assume now that i) holds. We proceed again by induction on $\rho(u,v)$, ii) being clearly true if $\rho(u,v) = 0$. So assume that $\rho(u,v) > 0$ and let $x, y \in [u,v]$, $x < y$. Then from i) and the induction hypothesis we conclude that

$$\begin{aligned}
D_k(\overline{\gamma_1}(x,y) - K_1(x,y)) &= D_k\left(\sum_{x<a\leq y} \gamma_1(x,a)K_1(a,y)\right) \\
&= D_k\left(\sum_{x<a\leq y} D_k(\gamma_1(x,a))D_k(K_1(a,y))\right) \\
&= D_k\left(\sum_{x<a\leq y} D_k(\gamma_2(x,a))D_k(K_2(a,y))\right) \\
&= D_k\left(\sum_{x<a\leq y} \gamma_2(x,a)K_2(a,y)\right) \\
&= D_k(\overline{\gamma_2}(x,y) - K_2(x,y)). \qquad (5)
\end{aligned}$$

Now if $k < \left\lceil \frac{\rho(x,y)+1}{2} \right\rceil$ then since $\gamma_1, \gamma_2 \in I_{\frac{1}{2}}(P)$ we have from (5) that

$$D_k(K_1(x,y)) = D_k(K_2(x,y)),$$

as desired. If $k \geq \frac{\rho(x,y)+1}{2}$ then we conclude from our hypothesis i) that

$$\gamma_1(x,y) = D_k(\gamma_1(x,y)) = D_k(\gamma_2(x,y)) = \gamma_2(x,y).$$

Hence

$$\begin{aligned}
D_k(K_1(x,y)) &= D_k(K_1(x,y) - \overline{\gamma_1}(x,y)) + D_k(\overline{\gamma_1}(x,y)) \\
&= D_k(K_2(x,y) - \overline{\gamma_2}(x,y)) + D_k(\overline{\gamma_2}(x,y)) \\
&= D_k(K_2(x,y))
\end{aligned}$$

and ii) holds also in this case.

Note that it is not true, in general, that if K is a P-kernel and $k \in \mathbf{N}$, then $D_k(K)$ is also a P-kernel. For example, if P is the Boolean algebra of rank 2 and $K = \chi_P$ (the characteristic kernel of P) then $D_1(K)$ is not a P-kernel since

$$\sum_{\hat{0}\leq a\leq \hat{1}} D_1(\chi_P)(\hat{0},a)\,\overline{D_1(\chi_P)}(a,\hat{1}) = (q^2-2q) - 2(q-1)^2 + (1-2q) \neq 0,$$

which would contradict Theorem 2.3. Thus Theorem 3.1 is not a special case of Theorem 2.2.

The next result is also a "comparison result" except that it does not require any knowledge of the P-kernel corresponding to one of the KLS-functions involved.

Theorem 3.2. *Let $k \in \mathbf{Z}$, $u, v \in P$, $f \in I_{\frac{1}{2}}(P)$, K be a P-kernel, and γ be its KLS-function. Then the following are equivalent:*

i): $D_k(\gamma(u,x)) = D_k(f(u,x))$ *for all* $x \in [u,v]$*;*

ii): $D_k\left(\sum_{a \in [u,x)} f(u,a)K(a,x)\right) = D_k\left(\overline{f}(u,x) - f(u,x)\right)$ *for all* $x \in [u,v]$.

Proof. Assume that i) holds. Then we have from our hypotheses that

$$\begin{aligned} D_k\left(\sum_{a\in[u,x)} f(u,a)K(a,x)\right) &= D_k\left(\sum_{a\in[u,x)} D_k(f(u,a))D_k(K(a,x))\right) \\ &= D_k\left(\sum_{a\in[u,x)} D_k(\gamma(u,a))D_k(K(a,x))\right) \\ &= D_k\left(\sum_{a\in[u,x)} \gamma(u,a)K(a,x)\right) \\ &= D_k(\overline{\gamma}(u,x) - \gamma(u,x)) \\ &= D_k(\overline{f}(u,x) - f(u,x)) \end{aligned}$$

for all $x \in [u,v]$, as desired.

Conversely, assume that ii) holds. We proceed by induction on $\rho(u,v)$, i) being clear if $u = v$. So assume that $\rho(u,v) \geq 1$. Then by our induction hypothesis we have that $D_k(\gamma(u,x)) = D_k(f(u,x))$ for all $x \in [u,v)$. Hence we have from our hypothesis ii) that

$$\begin{aligned} D_k(\overline{f}(u,v) - f(u,v)) &= D_k\left(\sum_{a\in[u,v)} f(u,a)K(a,v)\right) \\ &= D_k\left(\sum_{a\in[u,v)} D_k(f(u,a))D_k(K(a,v))\right) \\ &= D_k\left(\sum_{a\in[u,v)} D_k(\gamma(u,a))D_k(K(a,v))\right) \\ &= D_k\left(\sum_{a\in[u,v)} \gamma(u,a)K(a,v)\right) \\ &= D_k(\overline{\gamma}(u,v) - \gamma(u,v)). \end{aligned}$$

Since $\gamma, f \in I_{\frac{1}{2}}(P)$ this implies that $D_k(f(u,v)) = D_k(\gamma(u,v))$, and i) follows.

§4. New kernels from old

The applicability of the results obtained in the previous section depends to some extent on the explicit knowledge of P-kernels and their corresponding KLS-functions. Although on almost all posets there are infinitely many P-kernels it is difficult to find pairs of a P-kernel and its KLS-function that can *both* be described explicitly. For example, it follows easily from Theorem 2.3 that if K is a P-kernel then $\overline{K}$ is also a P-kernel and K^* is a P^*-kernel. Thus, to each kernel K there are naturally associated three other kernels, namely $\overline{K}$, K^*, and $\overline{K^*}$ (note that this process does not go on indefinitely, since $(\overline{K})^* = \overline{K^*}$ if P^* is weakly graded by ρ^*, as is usually the case). However, while it is known (see [10, Proposition 8.1]) that the KLS-function of $\overline{K^*}$ is $(\gamma^{-1})^*$ if γ is the KLS-function of K, no simple expression is known for the KLS-functions of K^* or of $\overline{K}$ in terms of the KLS-function of K. Similarly, it is obvious that if $\gamma \in I_{\frac{1}{2}}(P)$ then $\gamma^* \in I_{\frac{1}{2}}(P^*)$, and $\gamma^{-1}, D_k(\gamma) \in I_{\frac{1}{2}}(P)$, but no simple expression is known for the corresponding kernels in terms of the kernel of γ.

In this section we examine in some detail two particularly interesting such pairs, and we introduce a process that, given a pair (K, γ) of a P-kernel and its corresponding KLS-function, produces explicitly another such pair. The results in this section are applied in the next one to the study of Kazhdan-Lusztig polynomials.

We begin by studying a process that could be called "deformation" of a KLS-function. For $g \in \mathbf{R}[q]$ and $x, y \in P$, $x < y$, we define an element $g_{x,y} \in I(P)$ by letting

$$g_{x,y}(u,v) \stackrel{\text{def}}{=} \begin{cases} 0, & \text{if } (u,v) \neq (x,y), \\ g(q), & \text{if } (u,v) = (x,y), \end{cases}$$

for all $(u,v) \in \mathrm{Int}(P)$.

Proposition 4.1. *Let $f \in I(P)$ be unitary, $g \in \mathbf{R}[q]$, and $x, y \in P$, $x < y$. Then*

$$(f + g_{x,y})^{-1}(u,v) = f^{-1}(u,v) - g(q) f^{-1}(u,x) f^{-1}(y,v)$$

for all $(u,v) \in Int(P)$.

Proof. We proceed by induction on $\rho(u,v)$, the result being clear if $\rho(u,v) = 0$. So let $\rho(u,v) \geq 1$. We may clearly assume that $[x,y] \subseteq$

$[u, v]$. Then we have that, if $u < x$,

$$\begin{aligned}
(f+g_{x,y})^{-1}(u,v) &= -\sum_{u<a\le v} (f+g_{x,y})(u,a)(f^{-1}(a,v) - g(q)f^{-1}(a,x)f^{-1}(y,v)) \\
&= -\sum_{u<a\le v} f(u,a)(f^{-1}(a,v) - g(q)f^{-1}(a,x)f^{-1}(y,v)) \\
&= f^{-1}(u,v) + g(q)\sum_{u<a\le v} f(u,a)f^{-1}(a,x)f^{-1}(y,v) \\
&= f^{-1}(u,v) + g(q)f^{-1}(y,v)\sum_{u<a\le x} f(u,a)f^{-1}(a,x) \\
&= f^{-1}(u,v) - g(q)f^{-1}(y,v)f^{-1}(u,x),
\end{aligned}$$

as desired. On the other hand, if $u = x$ then

$$\begin{aligned}
(f+g_{x,y})^{-1}(x,v) &= -\sum_{x<a\le v} (f+g_{x,y})(x,a)(f^{-1}(a,v) - g(q)f^{-1}(a,x)f^{-1}(y,v)) \\
&= -\sum_{x<a\le v} (f+g_{x,y})(x,a)f^{-1}(a,v) \\
&= -\sum_{x<a\le v} f(x,a)f^{-1}(a,v) - g(q)f^{-1}(y,v) \\
&= f^{-1}(x,v) - g(q)f^{-1}(y,v),
\end{aligned}$$

and the result again follows.

Suppose now that $f \in I_{\frac{1}{2}}(P)$, K is the P-kernel of f, and $g \in \mathbf{R}[q]$, $x, y \in P$, $x < y$, are such that $\deg(g) \le \frac{1}{2}(\rho(x,y) - 1)$. Then $f + g_{x,y} \in I_{\frac{1}{2}}(P)$, and we denote by $K_{x,y}(g)$ the P-kernel corresponding to $f + g_{x,y}$. The next result gives an explicit expression for $K_{x,y}(g)$ in terms of K, g, f, and x, y.

Theorem 4.2. *Let $f \in I_{\frac{1}{2}}(P)$, $g \in \mathbf{R}[q]$, and $x, y \in P$, $x < y$, be such that $f + g_{x,y} \in I_{\frac{1}{2}}(P)$. Then*

$$K_{x,y}(g)(u,v) = \begin{cases} K(u,v) + f^{-1}(u,x)\left(q^{\rho(x,y)}g\left(\frac{1}{q}\right) - g(q)\right), & \textit{if } y = v, \\ K(u,v) - g(q)f^{-1}(u,x)K(y,v), & \textit{otherwise}, \end{cases}$$

for all $(u, v) \in Int(P)$.

Proof. Since $K_{x,y}(g)$ is the P-kernel corresponding to $f + g_{x,y}$ and K is the P-kernel of f there follows from the definitions that $K_{x,y}(g) =$

$(f + g_{x,y})^{-1}\overline{(f + g_{x,y})}$, and $K = f^{-1}\overline{f}$. We may clearly assume that $[x, y] \subseteq [u, v]$. Then we have from Proposition 4.1 that, if $y < v$,

$$\begin{aligned}
K_{x,y}(g)(u,v) &= \sum_{u\leq a\leq v} (f + g_{x,y})^{-1}(u,a)\overline{(f + g_{x,y})}(a,v) \\
&= \sum_{u\leq a\leq v} (f + g_{x,y})^{-1}(u,a)\overline{f}(a,v) \\
&= \sum_{u\leq a\leq v} (f^{-1}(u,a) - g(q)f^{-1}(u,x)f^{-1}(y,a))\overline{f}(a,v) \\
&= K(u,v) - g(q)f^{-1}(u,x) \sum_{y\leq a\leq v} f^{-1}(y,a)\overline{f}(a,v) \\
&= K(u,v) - g(q)f^{-1}(u,x)K(y,v),
\end{aligned}$$

as desired. On the other hand, if $y = v$ then

$$\begin{aligned}
K_{x,y}(g)(u,y) &= \sum_{u\leq a\leq y} (f + g_{x,y})^{-1}(u,a)\overline{(f + g_{x,y})}(a,y) \\
&= \sum_{u\leq a\leq y} (f^{-1}(u,a) - g(q)f^{-1}(u,x)f^{-1}(y,a))\overline{(f + g_{x,y})}(a,y) \\
&= \sum_{u\leq a\leq y} f^{-1}(u,a)\overline{(f + g_{x,y})}(a,y) - g(q)f^{-1}(u,x) \\
&= \sum_{u\leq a\leq y} f^{-1}(u,a)\overline{f}(a,y) + f^{-1}(u,x)q^{\rho(x,y)}g\left(\frac{1}{q}\right) - g(q)f^{-1}(u,x) \\
&= K(u,y) + f^{-1}(u,x)\left(q^{\rho(x,y)}g\left(\frac{1}{q}\right) - g(q)\right),
\end{aligned}$$

and the result again follows.

As noted at the beginning of this section, given a P-kernel K and its KLS-function γ, no simple formula is known for the KLS-function of $\overline{K}$ nor for the P-kernel of γ^{-1}. We believe that these objects are interesting and worthy of investigation. For this reason, and for convenience, we introduce here a notation for them. Namely, given $\gamma \in I_{\frac{1}{2}}(P)$ we let γ' be the KLS-function of $\overline{K}$ (where K is the P-kernel of γ). Similarly, given a P-kernel K we let K' be the P-kernel of γ^{-1} (where γ is the KLS-function of K). Note that these definitions don't overlap since no element of $I_{\frac{1}{2}}(P) \setminus \{\delta\}$ can be a P-kernel, by Theorem 2.3. Also, note that $(\gamma')' = \gamma$ and $(K')' = K$ for any $\gamma \in I_{\frac{1}{2}}(P)$ and P-kernel K. In the rest of this section we look in some detail at two particularly interesting cases of this operation. Namely, we look at ζ' and $\Re'$.

Let $\chi \in I(P)$ be the characteristic kernel of P. Then from (1) we have that

$$\overline{\chi}(u,v) = \sum_{a\in[u,v]} \mu(u,a) q^{\rho(u,a)}, \tag{6}$$

for all $(u,v) \in \mathrm{Int}(P)$. Since the KLS-function of χ is the zeta function of P, we expect ζ' to be a fundamental enumerative invariant of P. For simplicity, and because of the applications that we are interested in, we limit ourselves to the case that P is locally Eulerian. As a weak rank function for P we take, for $(x,y) \in \mathrm{Int}(P)$, $\rho(x,y)$ to be the common length of all the maximal chains in $[x,y]$ (see also [10, p. 829]). Note that in this case we have from (6) that

$$\overline{\chi}(u,v) = \sum_{a\in[u,v]} (-q)^{\rho(u,a)} \tag{7}$$

for all $(u,v) \in \mathrm{Int}(P)$.

We begin by showing that there is one case in which ζ' is extremely easy to compute.

Proposition 4.3. *Let P be a locally Eulerian poset and $u,v \in P$, $u < v$. Then $[u,v]$ is locally rank symmetric if and only if $\zeta'(x,y) = (-1)^{\rho(x,y)}$ for all $(x,y) \in$ Int$([u,v])$.*

Proof. It is clear from our definition (1) and (7) that $[u,v]$ is locally rank symmetric if and only if $\overline{\chi}(x,y) = (-1)^{\rho(x,y)}\chi(x,y)$ for all $(x,y) \in \mathrm{Int}([u,v])$. But it is easy to see that $(-1)^{\rho(x,y)}\chi(x,y)$ is a P-kernel and $(-1)^{\rho(x,y)}$ is its KLS-function, so the result follows from Theorem 3.1.

If Proposition 4.3 does not apply, however, things are considerably more subtle. Given three integers s,k,d with $0 \le k \le d$ and $0 \le s$ we let $\mathbf{S}_{s,k}(d)$ be the set of all sequences $(a_1,\dots,a_{2s+1}) \in [d]^{2s+1}$ such that:

i): $a_1 \le a_2 \le \dots \le a_{2s+1}$;
ii): $\sum_{j=1}^{i}(-1)^{j+i}a_{2j-1} > \frac{1}{2}a_{2i}$ for $i = 1,\dots,s$;
iii): $\sum_{j=1}^{s+1}(-1)^{s+1-j}a_{2j-1} = d-k$.

We then let

$$\mathbf{S}_k(d) = \bigcup_{s\ge 0} \mathbf{S}_{s,k}(d).$$

For example, $\mathbf{S}_{0,k}(d) = \{(d-k)\}$, $\mathbf{S}_{1,1}(d) = \{(1,1,d)\}$, $\mathbf{S}_{1,2}(d) = \{(1,1,d-1),(2,2,d),\ (2,3,d)\}$, $\mathbf{S}_{2,2}(d) = \{(1,1,3,3,d)\}$, and $\mathbf{S}_{2,3}(6) = \{(1,1,3,3,5),(1,1,4,4,6),\ (1,1,4,5,6),(2,2,5,5,6),(2,3,5,5,6)\}$.

Notice that if $(a_1, \dots, a_{2s+1}) \in \mathbf{S}_{s,k}(d)$, with $s \geq 1$, then from ii) (for $i = s$) and iii) we conclude that $a_{2s+1} - d + k > \frac{a_{2s}}{2}$. In particular, this shows that if $(a_1, \dots, a_{2s+1}) \in \mathbf{S}_k(d) \setminus \mathbf{S}_{0,k}(d)$ and $k < \frac{d}{2}$ then $a_{2s} \leq d - 1$.

It is not apparent from our definitions that $\mathbf{S}_{s,k}(d) = \emptyset$ for $s > k$, but this is indeed the case.

Lemma 4.4. *Let $s, k, d \in \mathbf{N}$, $k \leq d$, and $(a_1, \dots, a_{2s+1}) \in \mathbf{S}_{s,k}(d)$. Then*

$$(8) \qquad a_1 < a_3 - a_1 < a_5 - a_3 + a_1 < \dots < a_{2s-1} - a_{2s-3} + \dots$$

In particular, $s \leq k$.

Proof. From part ii) of the definition of $\mathbf{S}_{s,k}(d)$ we deduce that

$$\sum_{j=1}^{i} (-1)^{j+i} a_{2j-1} > \frac{a_{2i-1}}{2}$$

which can be written as

$$\sum_{j=1}^{i-1} (-1)^{j+i-1} a_{2j-1} < \sum_{j=1}^{i} (-1)^{j+i} a_{2j-1}$$

for $i = 1, \dots, s$, and this proves (8). In particular, (8) implies that $a_{2s-1} - a_{2s-3} + \dots \geq s$. Therefore, using iii),

$$d \geq a_{2s+1} = \sum_{j=1}^{s+1} (-1)^{s+1-j} a_{2j-1} + \sum_{j=1}^{s} (-1)^{s-j} a_{2j-1} \geq d - k + s,$$

and the second statement also follows.

We are now ready to state and prove the second main result of this section. This gives an explicit formula for ζ' in terms of the flag f-vector of the intervals of P. For a sequence $A \stackrel{\text{def}}{=} (a_1, \dots, a_r) \in \mathbf{N}^r$ with $a_1 \leq \dots \leq a_r$ and $u, v \in P$ we let $\alpha([u,v]; (a_1, \dots, a_r)) \stackrel{\text{def}}{=} \alpha([u,v];\ \{x \in \mathbf{N} :\ x = a_i \text{ for some } i \in [r]\})$, and $\sum_{a \in A} a \stackrel{\text{def}}{=} a_1 + \dots + a_r$. So, for example, $\alpha([u,v]; (1,1,3,3,5)) = \alpha([u,v]; \{1,3,5\})$.

Theorem 4.5. *Let P be a locally Eulerian poset, $u, v \in P$, $u < v$, and $k \leq \frac{1}{2}(\rho(u,v) - 1)$. Then*

$$[q^k](\zeta'(u,v)) = \sum_{A \in \mathbf{S}_k(d)} (-1)^{\sum_{a \in A} a}\, \alpha([u,v]; A),$$

where $d \stackrel{\text{def}}{=} \rho(u,v)$.

Proof. We proceed by induction on $\rho(u,v)$, the result being clear if $\rho(u,v) = 1$. From the definition of ζ', (7), and the fact that $k \leq \frac{1}{2}(d-1)$ we conclude that

$$[q^{d-k}](\overline{\zeta'}(u,v) - \overline{\chi}(u,v)) = [q^{d-k}]\left(\sum_{u<a\leq v} \zeta'(u,a)\overline{\chi}(a,v)\right)$$

$$= \sum_{u<a<v} \sum_{i=0}^{\lfloor \frac{\rho(u,a)-1}{2} \rfloor} [q^i](\zeta'(u,a))[q^{d-k-i}](\overline{\chi}(a,v))$$

$$= \sum_{u<a<v} \sum_{i=0}^{\lfloor \frac{\rho(u,a)-1}{2} \rfloor} \sum_{A \in \mathbf{S}_i(\rho(u,a))} (-1)^{\sum_{x\in A} x} \alpha([u,a];A)(-1)^{d-k-i}|[a,v]_{d-k-i}|$$

$$= \sum_{j=1}^{d-1} \sum_{i=0}^{\lfloor \frac{j-1}{2} \rfloor} \sum_{A \in \mathbf{S}_i(j)} (-1)^{\sum_{x\in A} x+d-k-i} \sum_{a\in[u,v]_j} \alpha([u,a];A)|[a,v]_{d-k-i}|$$

(9)

$$= \sum_{j=1}^{d-1} \sum_{i=max(0,j-k)}^{\lfloor \frac{j-1}{2} \rfloor} \sum_{A \in \mathbf{S}_i(j)} (-1)^{\sum_{x\in A} x+d-k-i} \alpha([u,v];A,j,j+d-k-i).$$

Now notice that if $A \in \mathbf{S}_i(j)$, $0 \leq i \leq \lfloor \frac{j-1}{2} \rfloor$ and $j \in [d-1]$ (with $i \geq j-k$) then $(A,j,j+d-k-i) \in \mathbf{S}_k(d) \setminus \mathbf{S}_{0,k}(d)$. Conversely, if $B = (b_1, \ldots, b_{2s+3}) \in \mathbf{S}_k(d) \setminus \mathbf{S}_{0,k}(d)$ then $(b_1, \ldots, b_{2s+1}) \in \mathbf{S}_{b_{2s+2}+d-k-b_{2s+3}}(b_{2s+2})$, $0 \leq b_{2s+2} + d - k - b_{2s+3} < \frac{b_{2s+2}}{2}$, (by Lemma 4.4), $b_{2s+2} \in [d-1]$, by the remarks preceding Lemma 4.4, and $b_{2s+2} + d - k - b_{2s+3} \geq b_{2s+2} - k$. Therefore we conclude from (9) that

$$[q^{d-k}](\overline{\zeta'}(u,v) - \overline{\chi}(u,v)) = \sum_{B\in \mathbf{S}_k(d)\setminus \mathbf{S}_{0,k}(d)} (-1)^{\sum_{x\in B} x}\alpha([u,v];B)$$

and the result follows since

$$[q^{d-k}](\overline{\chi}(u,v)) = (-1)^{d-k}|[u,v]_{d-k}| = (-1)^{d-k}\alpha([u,v];d-k)$$

and $\mathbf{S}_{0,k}(d) = \{(d-k)\}$. This concludes the induction step and hence the proof.

Using the Bayer-Billera relations for flag f-vectors of Eulerian posets (see [1, Theorem 2.1]) it is possible to simplify somewhat the expression given in Theorem 4.5, especially for small values of k.

Corollary 4.6. *Let P be a locally Eulerian poset, and $u, v \in P$, $u < v$. Then*

$$[q^0](\zeta'(u,v)) = (-1)^{\rho(u,v)},$$

$$[q](\zeta'(u,v)) = (-1)^{\rho(u,v)}(|[u,v]_1| - |[u,v]_1^*|),$$

$$[q^2](\zeta'(u,v)) = (-1)^{\rho(u,v)}(|[u,v]_2| - \alpha([u,v];\{1,\rho(u,v)-1\}) + |[u,v]_2^*|).$$

Proof. The first two formulas follow immediately from Theorem 4.5 and the definition of $\mathbf{S}_k(d)$, keeping in mind that $\mathbf{S}_{s,k}(d) = \emptyset$ if $s > k$ by Lemma 4.4. For $k = 2$ we obtain in the same way that

$$\begin{aligned}[q^2](\zeta'(u,v)) &= (-1)^{\rho(u,v)}(|[u,v]_2^*| - \alpha([u,v];\{1,\rho(u,v)-1\}) + |[u,v]_2| \\ &\quad -\alpha([u,v];\{2,3\}) + \alpha([u,v];\{1,3\})).\end{aligned}$$

But since $[u,v]$ is Eulerian we have that

$$2\alpha([u,v];\{2,3\}) = \alpha([u,v];\{1,2,3\}) = 2\alpha([u,v];\{1,3\}),$$

and the result follows.

We conclude this section by looking at the W-kernel $\Re'$. Recall from section 2 that $\overline{\Re}^*$ is a W^*-kernel and $\wp^*$ is its KLS-function. It therefore follows from Proposition 8.1 of [10] that $\Re$ is a W-kernel and $\wp^{-1}$ is its KLS-function. Therefore, by our definition, $\Re'$ is the W-kernel of $\wp$. Note that, from this point of view, $\Re'$ is an even more natural object to consider than $\Re$ itself. We let

$$S_{x,y}(q) \stackrel{\text{def}}{=} \Re'(x,y)$$

for all $(x,y) \in \text{Int}(W)$.

Our aim is to obtain some information about the polynomials $S_{x,y}(q)$. Despite the naturality of their definition these polynomials seem to have never been considered before. As the following results show, they have properties that are very similar to those of the R-polynomials.

Proposition 4.7. *Let $x, y \in W$, $x \leq y$. Then*

$$S_{x,y}(q) = \sum_{x \leq a \leq y} \varepsilon_x \varepsilon_a Q_{x,a}(q)\, q^{l(y)-l(a)} P_{a,y}\left(\frac{1}{q}\right). \tag{10}$$

In particular, $S_{x,y}(q)$ is a monic polynomial of degree $l(y) - l(x)$, and $S_{x,y}(0) = \varepsilon_x \varepsilon_y$.

Proof. The first assertion is essentially just a restatement of our definitions. In fact, it follows from them that $\wp \Re' = \overline{\wp}$ in $I(W)$ and hence that $\Re' = \wp^{-1}\overline{\wp}$, which, by (3) and (4), implies (10). The second statement follows from the first one and the facts that $\wp$, $\wp^{-1} \in I_{\frac{1}{2}}(W)$ and $Q_{x,y}(0) = P_{x,y}(0) = 1$ for $(x, y) \in \mathrm{Int}(W)$.

Note the similarity of (10) with the formula for the R-polynomials

$$R_{x,y}(q) = \sum_{x \le a \le y} \varepsilon_x \varepsilon_a P_{x,a}(q) q^{l(y)-l(a)} Q_{a,y}\left(\frac{1}{q}\right) \tag{11}$$

for all $(x, y) \in \mathrm{Int}(W)$. Because of (10) and (11), many other formulas for the R-polynomials have analogues for the polynomials $S_{x,y}(q)$. We give below two as an example (cf. Corollaries 5.3 and 7.7 in [3]).

Corollary 4.8. *Let $x, y \in W$, $x \le y$. Then*

$$[q](S_{x,y}) = \varepsilon_x \varepsilon_y ([q](Q_{x,y}) - |[x, y]_1^*|)$$

and

$$[q^{l(y)-l(x)-1}](S_{x,y}) = [q](P_{x,y}) - |[x, y]_1|.$$

It is of course possible to obtain from (10) similar formulas for all the coefficients of $S_{x,y}(q)$, but we see no reason to do this explicitly here.

Proposition 4.9. *Let (W, S) be a finite Coxeter system, and $x, y \in W$, $x \le y$. Then*

$$q^{l(y)-l(x)} S_{x,y}\left(\frac{1}{q}\right) = \varepsilon_x \varepsilon_y S_{w_0 y, w_0 x}(q),$$

where w_0 denotes the longest element of W.

Proof. It is well known (see, e.g., [6, Proposition 7.13]) that if W is finite then $Q_{x,y}(q) = P_{w_0 y, w_0 x}(q)$ for all $(x, y) \in \mathrm{Int}(W)$. Hence we conclude from (10) that

$$\begin{aligned} q^{l(y)-l(x)} S_{x,y}\left(\frac{1}{q}\right) &= \sum_{x \le a \le y} \varepsilon_x \varepsilon_a q^{l(a)-l(x)} P_{w_0 a, w_0 x}\left(\frac{1}{q}\right) P_{a,y}(q) \\ &= \sum_{w_0 y \le b \le w_0 x} \varepsilon_x \varepsilon_{w_0 b} q^{l(w_0 x)-l(b)} P_{b, w_0 x}\left(\frac{1}{q}\right) Q_{w_0 y, b}(q) \\ &= \varepsilon_x \varepsilon_y S_{w_0 y, w_0 x}(q), \end{aligned}$$

as desired.

Proposition 4.9 also holds for the R-polynomials (see, e.g., [6, Propositions 7.6 and 7.8]). After seeing all these similarities it is natural to suspect that the polynomials $S_{x,y}(q)$ might just be the R-polynomials in disguise. This, however, is not true even for finite Weyl groups. For example, if $W = S_4$ then one can compute that

$$S_{1234,3412}(q) = q^4 - 2q^3 + 4q^2 - 4q + 1,$$

and this is not an R-polynomial by Proposition 2.6.

§5. Applications to Kazhdan-Lusztig polynomials

In this section we apply the results obtained in the two previous ones to Kazhdan-Lusztig polynomials. In particular, we characterize the intervals $[u, v]$ in W such that the Kazhdan-Lusztig polynomials of its subintervals coincide with ζ, ζ', or $\wp^{-1}$ up to a given order, and we obtain refinements of two results that originally appeared in [3] and [7].

We begin by comparing a deformation of the characteristic kernel of W^* with the kernel $(\overline{\mathfrak{R}})^*$, where $\mathfrak{R}$ is defined by (2). For brevity, throughout this section, we write χ instead of χ_{W^*}. Note first that, since W^* is locally Eulerian, we obtain from (1) that

$$\chi(y,x) = \varepsilon_x \varepsilon_y \sum_{i=0}^{l(y)-l(x)} |[x,y]_i|(-q)^i \tag{12}$$

for all $(y, x) \in \mathrm{Int}(W^*)$.

Theorem 5.1. *Let $u, v \in W$, $u < v$, $k \in \mathbf{N}$, and $f \in \mathbf{R}[q]$, $deg(f) \leq \frac{1}{2}(l(v) - l(u) - 1)$. Then the following are equivalent:*

i): *$D_k(P_{y,x}) = 1$ for all $[y, x] \subset [u, v]$, and $D_k(P_{u,v}) = D_k(1 + f)$;*

ii): *$D_k(R_{y,x}) = D_k(\chi(x, y))$ for all $[y, x] \subset [u, v]$, and*

$$D_k\left(f(q) - q^{l(v)-l(u)} f\left(\frac{1}{q}\right)\right) = D_k(\chi(v,u) - R_{u,v}).$$

Proof. Let $\chi_{v,u}(f)$ be the W^*-kernel of $\zeta + f_{v,u} \in I_{\frac{1}{2}}(W^*)$. Then we have from Theorem 4.2 that, if $[y, x] \subseteq [u, v]$

$$\chi_{v,u}(f)(x,y) = \chi(x,y) + \zeta^{-1}(x,v)\left(q^{l(v)-l(u)} f\left(\frac{1}{q}\right) - f(q)\right)$$

if $u = y$, while $\chi_{v,u}(f)(x, y) = \chi(x, y)$ otherwise.

On the other hand, we know that $(\overline{\Re})^*$ is a W^*-kernel and $\wp^*$ is its KLS function. Furthermore, it follows from our definitions that

$$[q^j](\overline{\Re}^*(x,y)) = [q^j](\overline{\Re}(y,x)) = [q^j]\left(q^{l(x)-l(y)}\varepsilon_x\varepsilon_y R_{y,x}\left(\frac{1}{q}\right)\right) = [q^j](R_{y,x}(q)),$$

by Proposition 2.6, and

$$[q^j](\wp^*(x,y)) = [q^j](\wp(y,x)) = [q^j](P_{y,x})$$

for all $(x,y) \in \mathrm{Int}(W^*)$, so the result follows from Theorem 3.1.

If $P_{y,x}(q) = 1$ for all $[y,x] \subset [u,v]$ then much more precise information can be obtained, as the next result shows. Note that if one uses Theorem 2.5 as a recursion for computing Kazhdan-Lusztig polynomials these are the first "non-trivial" (i.e., $\neq 1$) Kazhdan-Lusztig polynomials that one generates.

Proposition 5.2. *Let $u, v \in W$, $u < v$, and $d \stackrel{\text{def}}{=} l(v) - l(u)$. Suppose that $P_{y,x} = 1$ for all $[y,x] \subset [u,v]$. Then:*

i): $P_{u,v} = 1 + D_{\frac{d-1}{2}}(\chi(v,u) - R_{u,v})$;
ii): $R_{u,v} = \chi(v,u) + 1 - P_{u,v}(q) - q^d + q^d P_{u,v}\left(\frac{1}{q}\right)$;
iii): $(1+\varepsilon_u\varepsilon_v)R_{u,v}(q) = \chi(v,u)(q) + q^d\chi(v,u)\left(\frac{1}{q}\right)$;
iv): $P_{u,v}(q) = 1 + \frac{1}{2}\sum_{i=0}^{\lfloor\frac{d-1}{2}\rfloor}(-q)^i(|[u,v]_i| - |[u,v]_{d-i}|)$ *if d is even;*
v): *$[u,v]$ is rank-symmetric if d is odd.*

Proof. Taking $k = l(v) - l(u)$, and $f(q) \stackrel{\text{def}}{=} P_{u,v}(q) - 1$ in Theorem 5.1 yields ii), from which i) follows immediately. From ii) we conclude that

$$\begin{aligned} q^d\left(-1 + \left(\frac{1}{q}\right)^d - \chi(v,u)\left(\frac{1}{q}\right)\right) &= q^d\left(\left(\frac{1}{q}\right)^d P_{u,v}(q) - P_{u,v}\left(\frac{1}{q}\right) - R_{u,v}\left(\frac{1}{q}\right)\right) \\ &= P_{u,v}(q) - q^d P_{u,v}\left(\frac{1}{q}\right) - \varepsilon_u\varepsilon_v R_{u,v}(q) \\ &= 1 - q^d + \chi(v,u)(q) - (1+\varepsilon_u\varepsilon_v)R_{u,v}(q)\,, \end{aligned}$$

where we have used ii) again and Proposition 2.6 and iii) follows.

Now if $\varepsilon_u\varepsilon_v = 1$ we conclude from iii) and i) that

$$P_{u,v}(q) = 1 + \frac{1}{2}D_{\frac{d-1}{2}}\left(\chi(v,u)(q) - q^d\chi(v,u)\left(\frac{1}{q}\right)\right),$$

and iv) follows from (12). If $\varepsilon_u \varepsilon_v = -1$ it follows from iii) that

$$\chi(v,u)(q) = -q^d \chi(v,u)\left(\frac{1}{q}\right),$$

and v) follows from (12).

It should be noted that the preceding result is yet another piece of evidence in favor of the "feeling" mentioned in [3, p. 384], that the Kazhdan-Lusztig polynomials somehow "measure" the difference between the R-polynomials and the rank generating functions.

We note the following interesting reformulation of part v) of Proposition 5.2.

Corollary 5.3. *Let $u, v \in W$, $u < v$, be such that $[u,v]$ is not rank-symmetric and has odd rank. Then there exists $[x,y] \subset [u,v]$ such that $P_{x,y}(q) \neq 1$.*

The following result characterizes the intervals of W for which the zeta function is a "good approximation" of the Kazhdan-Lusztig polynomials.

Proposition 5.4. *Let $u, v \in W$, $u < v$, and $k \in \mathbf{N}$. Then the following are equivalent:*

i): $D_k(P_{x,y}) = 1$ *for all* $[x,y] \subseteq [u,v]$;
ii): $\varepsilon_x \varepsilon_y [q^j](R_{x,y}) = (-1)^j |[x,y]_j|$ *for all* $x, y \in [u,v]$ *and* $j \in [k]$.

Proof. This follows immediately from (12) and Theorem 5.1.

Note that, when $k = l(v) - l(u)$, Proposition 5.4 reduces to Proposition 5.6 of [3].

Most of the results that we have derived so far in this section have analogues that are obtained by considering the W^*-kernel $\overline{\chi}$ instead of χ. We state here one of them as an example. It is a "dual" of Proposition 5.4, and characterizes the intervals of W having the property that the Kazhdan-Lusztig polynomials of its subintervals coincide, up to a given order, with the function ζ' studied in section 4. Note that we write ζ' for $(\zeta_{W^*})'$.

Proposition 5.5. *Let $u, v \in W$, $u < v$, and $k \in \mathbf{N}$. Then the following are equivalent:*

i): $D_k(P_{x,y}) = \varepsilon_x \varepsilon_y D_k(\zeta'(y,x))$ *for all* $x, y \in [u,v]$;
ii): $\varepsilon_x \varepsilon_y [q^j](R_{x,y}) = (-1)^j |[x,y]_j^*|$ *for all* $x, y \in [u,v]$ *and* $j \in [k]$.

The last two propositions have the following curious consequence.

Corollary 5.6. *Let $u, v \in W$, $u < v$. Then the following are equivalent:*

i): $P_{x,y} = 1$ *for all* $u \leq x \leq y \leq v$;
ii): $P_{x,y} = \varepsilon_x \varepsilon_y \zeta'(y,x)$ *for all* $u \leq x \leq y \leq v$.

Proof. This follows immediately from Propositions 5.4, 5.5 and 2.6.

As another application of Theorem 3.1 we obtain the following result which characterizes the intervals $[u,v]$ of W for which the Kazhdan-Lusztig and inverse Kazhdan-Lusztig polynomials coincide on the subintervals of $[u,v]$, up to a given order.

Proposition 5.7. *Let $u, v \in W$, and $k \in \mathbf{N}$. Then the following are equivalent:*

i): $D_k(P_{x,y}) = D_k(Q_{x,y})$ *for all* $x, y \in [u,v]$;
ii): $D_k(R_{x,y}) = D_k(S_{x,y})$ *for all* $x, y \in [u,v]$.

Proof. We know that $\Re$ and $\Re'$ are W-kernels and that $\wp^{-1}$ and $\wp$ are their respective KLS-functions. This easily implies that $(x,y) \mapsto R_{x,y}(q)$ is a W-kernel and that $(x,y) \mapsto Q_{x,y}(q)$ is its KLS-function, so the result follows from Theorem 3.1.

We conclude this section with an application of Theorem 3.2, which gives a refinement of Lemma 2.6 (ii) of [7] (the case $k = l(v) - l(u)$).

Corollary 5.8. *Let $k \in \mathbf{N}$, and $u, v \in W$, $u \leq v$. Then the following are equivalent:*

i): $D_k(P_{x,v}) = 1$ *for all* $x \in [u,v]$;
ii): $D_k\left(\sum_{a \in [x,v]} R_{x,a}\right) = D_k(q^{l(v)-l(x)})$ *for all* $x \in [u,v]$.

Proof. This follows immediately from Theorem 3.2 by taking $P = W^*$, $K = \overline{\Re}^*$, $\gamma = \wp^*$, and $f = \zeta_{W^*}$.

A dual result can be obtained by taking $P = W$, $K = \Re'$, $\gamma = \wp$, and $f = \zeta$, we leave its statement to the interested reader.

§6. Conjectures and open problems

In this section we discuss some conjectures and open problems arising from the present work.

The first one is naturally suggested by Corollary 5.6.

Conjecture 6.1. *Let $u, v \in W$, $u < v$, and $k \in \mathbf{N}$. Then the following are equivalent:*

i): $D_k(P_{x,y}) = 1$ *for all* $u \leq x \leq y \leq v$;

ii): $D_k(P_{x,y}) = \varepsilon_x \varepsilon_y D_k(\zeta'(y,x))$ *for all* $u \le x \le y \le v$.

By Propositions 5.4 and 5.5 Conjecture 6.1 is equivalent to the following one.

Conjecture 6.2. *Let* $u, v \in W$, $u < v$, *and* $k \in \mathbf{N}$. *Then the following are equivalent:*

i): $\varepsilon_x \varepsilon_y [q^j](R_{x,y}) = (-1)^j |[x,y]_j|$ *for all* $x, y \in [u,v]$ *and* $j \in [k]$;

ii): $\varepsilon_x \varepsilon_y [q^j](R_{x,y}) = (-1)^j |[x,y]^*_j|$ *for all* $x, y \in [u,v]$ *and* $j \in [k]$.

A consequence of Conjecture 6.2 is the following one.

Conjecture 6.3. *Let* $u, v \in W$, $u < v$, *and* $k \in \mathbf{N}$. *Suppose that* $D_k(P_{x,y}) = 1$ *for all* $(x,y) \in Int([u,v])$. *Then* $|[x,y]_j| = |[x,y]^*_j|$ *for all* $x, y \in [u,v]$ *and* $j \in [k]$.

Note that this conjecture holds for $k = 1$ by Proposition 5.4 and Corollary 5.3 of [3]. We now show that it also holds for $k = 2$.

Proposition 6.4. *Let* $u, v \in W$, $u < v$, *be such that* $[q](P_{x,y}) = [q^2](P_{x,y}) = 0$ *for all* $x, y \in [u,v]$. *Then* $|[x,y]_i| = |[x,y]^*_i|$ *for all* $x, y \in [u,v]$, $i = 1, 2$.

Proof. We already know that $|[x,y]_1| = |[x,y]^*_1|$ for all $x, y \in [u,v]$. Also, we know from [3, Corollary 5.4] that

$$\begin{aligned}[q^2](P_{x,y}) &= \varepsilon_x \varepsilon_y [q^2](R_{x,y}) + \sum_{a \in [x,y]^*_1} \varepsilon_x \varepsilon_a [q](R_{x,a}) + |[x,y]^*_2| \\ &\quad + \alpha([x,y]^*; \{1,3\}) - \sum_{a \in [x,y]^*_3} [q](R_{a,y}). \end{aligned} \tag{13}$$

On the other hand, from Proposition 5.4 and our hypotheses we deduce that

$$\varepsilon_x \varepsilon_y [q](R_{x,y}) = -|[x,y]_1| = -|[x,y]^*_1|$$

for all $x, y \in [u,v]$. Hence from (13) and our hypotheses we conclude that

$$\begin{aligned} 0 &= \varepsilon_x \varepsilon_y [q^2](R_{x,y}) - \sum_{a \in [x,y]^*_1} |[x,a]^*_1| + |[x,y]^*_2| \\ &\quad + \alpha([x,y]^*; \{1,3\}) - \sum_{a \in [x,y]^*_3} |[a,y]^*_1| \\ &= \varepsilon_x \varepsilon_y [q^2](R_{x,y}) - \alpha([x,y]^*; \{1,2\}) + |[x,y]^*_2| \\ &\quad + \alpha([x,y]^*; \{1,3\}) - \alpha([x,y]^*; \{1,3\}) \\ &= \varepsilon_x \varepsilon_y [q^2](R_{x,y}) - |[x,y]^*_2|, \end{aligned} \tag{14}$$

for all $x, y \in [u, v]$. On the other hand, from Proposition 5.4 we have that

$$\varepsilon_x \varepsilon_y [q^2](R_{x,y}) = |[x, y]_2| \tag{15}$$

for all $x, y \in [u, v]$, and the result follows from (14) and (15).

It is a well known conjecture (see, [6, p. 159]) that the coefficients of Kazhdan-Lusztig polynomials are always nonnegative. Using Theorem 5.1 we can derive from it the following (much weaker) conjecture, that should be more tractable.

Conjecture 6.5. *Let $k \in \mathbf{P}$ and $u, v \in W$, $u < v$, be such that $D_k(P_{x,y}) = 1$ for all $[x, y] \subset [u, v]$. Then*

$$\varepsilon_u \varepsilon_v (-1)^i |[u, v]_i| \geq [q^i](R_{u,v})$$

for $i = 0, \dots, min(k, \lfloor \frac{d}{2} \rfloor)$, and

$$(-1)^i |[u, v]_i^*| \leq \varepsilon_u \varepsilon_v [q^i](R_{u,v}),$$

for $i = d - k, \dots, \lfloor \frac{d}{2} \rfloor$.

There is another related conjecture which we wish to mention. It was observed in [3, p. 384] (see also [4], Problem 5.1), that the polynomial $\varepsilon_u \varepsilon_v (\chi(v, u)(-q) - R_{u,v}(-q))$ seems to have always nonnegative coefficients. If this is true, and the nonnegativity conjecture holds, then part ii) of Proposition 5.2, shows that the following must also hold.

Conjecture 6.6. *Let $u, v \in W$, $u < v$, be such that $P_{x,y} = 1$ for all $[x, y] \subset [u, v]$. Then:*

i): $P_{u,v}(q) = 1$ *if* $\varepsilon_u \varepsilon_v = 1$;

ii): $[q^{2i}](P_{u,v}) = 0$ *if* $\varepsilon_u \varepsilon_v = -1$, *and* $i \geq 1$.

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Plane partitions II: $5\frac{1}{2}$ symmetry classes

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Abstract.

We present new, simple proofs for the enumeration of five of the ten symmetry classes of plane partitions contained in a given box. Four of them are derived from a simple determinant evaluation, using combinatorial arguments. The previous proofs of these four cases were quite complicated. For one more symmetry class we give an elementary proof in the case when two of the sides of the box are equal. Our results include simple evaluations of the determinants $\det\left(\delta_{ij}+\binom{x+i+j}{i}\right)_{0\le i,j\le n-1}$ and $\det\left(\binom{x+i+j}{2j-i}\right)_{0\le i,j\le n-1}$, notorious in plane partition enumeration, whose previous evaluations were quite intricate.

1. Introduction

A plane partition is an array of nonnegative integers with the property that all rows and columns are weakly decreasing. By a well-known bijection (see [**9**] or [**18**]), plane partitions contained in an $a\times b$ rectangle and with entries at most c can be identified with lozenge tilings of a hexagon $H(a,b,c)$ with side-lengths a, b, c, a, b, c (in cyclic order) and angles of 120° (a lozenge tiling of a region on the triangular lattice is a tiling by unit rhombi with angles of 60° and 120°).

In [**19**] Stanley describes ten natural symmetry classes of plane partitions. Strikingly, the number of elements in each symmetry class is

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given by a simple product formula. The available proofs, however, are in many cases quite intricate (see [**19**], [**3**], [**13**] and [**21**]). In this paper we present simple proofs for five symmetry classes, and for one more we give an elementary proof in the case when two of the numbers a, b and c are equal.

Our proofs employ Kuperberg's observation [**13**] that the bijection mentioned in the first paragraph maps symmetry classes of plane partitions to symmetry classes of tilings of $H(a,b,c)$. The three basic symmetries, in the context of tilings T, are:

(1) the reflection $t : T \mapsto T^t$ (called *transposition*) in the diagonal joining the two vertices of $H(a,b,c)$ where sides of lengths a and b meet (this assumes $a = b$),

(2) the *rotation* $r : T \mapsto T^r$ by 120° around the center of $H(a,b,c)$ (assuming $a = b = c$), and

(3) the rotation $k : T \mapsto T^k$ by 180° (called *complementation*) around the center of $H(a,b,c)$.

If a tiling is invariant under one of these symmetries, it is called symmetric, cyclically-symmetric or self-complementary, respectively.

We employ simple combinatorial arguments to deduce four difficult symmetry classes from a determinant evaluation due to Andrews and Burge [**4**], which was later generalized by Krattenthaler [**12**], and then proved in a very simple way by Amdeberhan [**1**]. The main tool used in our proofs is the Factorization Theorem for perfect matchings presented in [**6**].

The first of this group of four symmetry classes is the case of cyclically symmetric plane partitions (i.e., $T^r = T$), first proved by Andrews [**2**] (for another proof and a q-version, see [**15**]). In fact, Andrews' result [**2**, Theorem 8] is a generalization of this case, and it gives a simple product formula for

$$\det\left(\delta_{ij} + \binom{x+i+j}{i}\right)_{0\le i,j\le n-1}. \tag{1.1}$$

Our proof also addresses this more general result, and answers thus the problem suggested by Mills, Robbins and Rumsey [**16**] of finding a simple solution for the evaluation of (1.1).

The next case we treat is that of cyclically symmetric transposed-complementary plane partitions (i.e., $T^r = T$ and $T^t = T^k$), first proved by Mills, Robbins and Rumsey [**16**]. Again, we solve the more general problem of evaluating the determinant

$$\det\left(\binom{x+i+j}{2j-i}\right)_{0\le i,j\le n-1}. \tag{1.2}$$

(It is in fact this more general result that is obtained in [**16**].)

The last two cases in this group of four are those of cyclically symmetric self-complementary (i.e., $T^r = T^k = T$) and totally symmetric self-complementary (i.e., $T^t = T^r = T^k = T$) plane partitions, which were first proved by Kuperberg [**13**] and Andrews [**3**], respectively.

The fifth case we deal with is that of transposed-complementary plane partitions (i.e., $T^t = T^k$), which was first proved by Proctor [**17**] using arguments from representation theory. We deduce it as a simple consequence of results in [**8**] on the tiling generating function of certain regions on the triangular lattice. (We note here that three of the five cases mentioned so far — those of invariance under the groups $\langle r\rangle$, $\langle tk\rangle$, and $\langle r, tk\rangle$ — are among the four that were given uniform solutions by Kuperberg [**14**] using representation theory; the fourth case covered in [**14**] is the base case.)

Finally, the "half" case — which we provide a simple proof for, based on the aforementioned results of [**8**], in case two of the numbers a, b and c are equal — is that of self-complementary plane partitions (i.e., $T^k = T$), which was first proved by Stanley [**19**] using the theory of symmetric functions.

In fact, one more "half-case" could be added to the ones mentioned above: if two of the numbers a, b and c are equal, the base case (i.e., no symmetry requirements) follows directly by specializing $k = 0$ in [**8**, Theorem 1.1(a)].

2. A determinant with two tiling interpretations

The determinant evaluation mentioned in the Introduction from which we will derive the first four symmetry classes is the following.

Theorem 2.1 (Krattenthaler [12]). *Let x, y and n be nonnegative integers with $x + y > 0$, and set*

$$K_n(x,y) = \left(\frac{(x+y+i+j-1)!}{(x+2i-j)!\,(y+2j-i)!}\right)_{0\le i,j\le n-1}. \tag{2.1}$$

Then we have

(2.2)
$$\det(K_n(x,y)) = \prod_{i=0}^{n-1} \frac{i!\,(x+y+i-1)!\,(2x+y+2i)_i\,(x+2y+2i)_i}{(x+2i)!\,(y+2i)!},$$

where $(a)_k := a(a+1)\cdots(a+k-1)$ *is the shifted factorial.*

Proof (Amdeberhan [**1**]). We use the fact that for any matrix $A = (a_{ij})_{0\le i,j\le n-1}$ we have

$$\det A = \frac{(\det A_0^0)(\det A_{n-1}^{n-1}) - (\det A_0^{n-1})(\det A_{n-1}^0)}{\det A_{0,n-1}^{0,n-1}}, \tag{2.3}$$

where $A_{i_1,\dots,i_k}^{j_1,\dots,j_k}$ is the submatrix of A obtained by removing rows indexed by $i_1,\dots,i_k$ and columns indexed by $j_1,\dots,j_k$ (see e.g. [**11**]).

Take $A = K_n(x,y)$ in (2.3). It is readily seen that the five determinants on the right hand side can be written in the form $\det K_m(x',y')$, with $m < n$ and suitable x' and y'. More precisely, we obtain

$$\det K_n(x,y) = (\det K_{n-1}(x+1,y+1)\det K_{n-1}(x,y) - \det K_{n-1}(x+2,y-1)\det K_{n-1}(x-1,y+2))/\det K_{n-2}(x+1,y+1).$$

It is easy to check that the expression on the right hand side of (2.2) also satisfies the above recurrence. Thus, (2.2) follows by induction on n. $\square$

We define a *region* to be any subset of the plane that can be obtained as the union of finitely many unit triangles of the regular triangular lattice. In a lozenge tiling of a region, we allow the tile positions to be weighted. The weight of a tiling is the product of the weights of the positions occupied by lozenges. The tiling generating function $L(R)$ of a region R is the sum of the weights of all its tilings.

We now introduce two types of regions whose tiling generating functions turn out to be very closely related to the determinant in the statement of Theorem 2.1.

Let n and x be nonnegative integers. Consider the pentagonal region illustrated by Figure 2.1, where the top side has length x, the southeastern side has length n, and the western and northeastern sides follow zig-zag paths of length $2n$. Weight the n tile positions fitting in the indentations of the northeastern boundary by $1/2$ (we indicate weightings by $1/2$ in our figures by placing shaded ovals in the corresponding

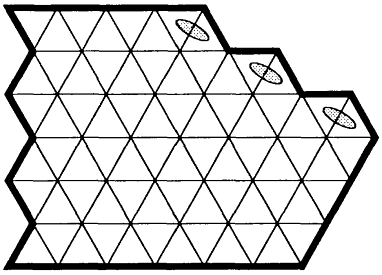

FIGURE 2.1. $A_{3,4}$.

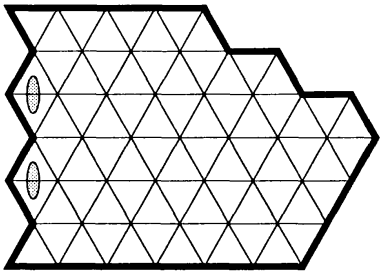

FIGURE 2.2. $B_{3,4}$.

tile positions; see Figure 2.1); weight all the others by 1. Denote the resulting region by $A_{n,x}$.

Let $B_{n,x}$ be the region with the same boundary as $A_{n,x}$, but having the $n-1$ tile positions fitting in the indentations of the western boundary weighted by 1/2, and all other tile positions weighted by 1 (see Figure 2.2).

The close connection between these regions and the determinant of the matrix (2.1) is expressed by the following result.

Proposition 2.2.

$$L(A_{n,x}) = \frac{1}{2^n} \det(K_n(x,0)) \prod_{i=0}^{n-1} (x+3i) \tag{2.4}$$

$$L(B_{n,x}) = \frac{1}{2^n} \det(K_n(x,0)) \prod_{i=0}^{n-1} (2x+3i). \tag{2.5}$$

Proof. We use the well-known procedure of encoding tilings of a region as families of non-intersecting lattice paths (see e.g. [**8**, §4]). By this, every tiling T of $A_{n,x}$ is identified with an n-tuple of paths of rhombi of T, each going from the western boundary of $A_{n,x}$ to its northeastern boundary. It follows that $L(A_{n,x})$ is equal to the generating function of n-tuples of non-intersecting lattice paths on the square lattice taking steps north and east, starting at the points $u_i = (i, 2n-2i-1)$ and ending at $v_i = (x+2i, 2n-i-1)$, $i = 0, \dots, n-1$, where paths with the last step horizontal are weighted by 1/2 (the weight of a family of paths is the product of the weights of its elements).

It is immediate to check that the u_i's and v_j's satisfy the requirements in the hypothesis of the basic theorem of Gessel and Viennot on non-intersecting lattice paths (see e.g. [**20**, Theorem 1.2] or [**10**]). We obtain that the above generating function of non-intersecting lattice paths equals

$$\det\left((a_{ij})_{0\le i,j\le n-1}\right), \tag{2.6}$$

where a_{ij} is the generating function of lattice paths from u_i to v_j. A straightforward calculation yields

$$\begin{aligned} a_{ij} &= \frac{1}{2}\binom{x+i+j-1}{2i-j} + \binom{x+i+j-1}{2i-j-1} \\ &= \frac{x+3i}{2}\frac{(x+i+j-1)!}{(x-i+2j)!\,(2i-j)!}. \end{aligned} \tag{2.7}$$

Therefore, by factoring out $(x+3i)/2$ along row i of the matrix in (2.6), we obtain (2.4).

To prove (2.5) we proceed similarly. Encoding tilings as lattice paths, we obtain that $L(B_{n,x})$ is equal to the generating function of n-tuples of non-intersecting lattice paths starting and ending at the same points as above, but now with paths having the *first* step *vertical* weighted by $1/2$. It is easy to see that in this case we have

$$\begin{aligned} a_{ij} &= \frac{1}{2}\binom{x+i+j-1}{2i-j-1} + \binom{x+i+j-1}{2i-j} \\ &= \frac{2x+3j}{2}\frac{(x+i+j-1)!}{(x-i+2j)!\,(2i-j)!}. \end{aligned} \tag{2.8}$$

By factoring out $(2x+3j)/2$ along column j we obtain (2.5). $\square$

3. Cyclically symmetric plane partitions

By (2) of the Introduction, this case amounts to enumerating r-invariant tilings of $H(n,n,n)$, where n is a positive integer.

We generalize this problem as follows. Consider the hexagonal region having sides of lengths n, $n+x$, n, $n+x$, n, $n+x$ (in cyclic order), where $n\ge 1$ and $x\ge 0$ are integers. Let $H_{n,x}$ be the region obtained from this hexagon by removing a triangular region of side x from its center, so that its vertices point towards the shorter edges of the hexagon (this is illustrated in Figure 3.1 for $n=4$, $x=2$). Denote by $CS(n,x)$ the number of lozenge tilings of $H_{n,x}$ that are invariant under rotation by $120°$. Clearly, $CS(n,0)$ is the number of r-invariant tilings of $H(n,n,n)$.

The result below was inspired by Stembridge's proof of the special case $x=0$ (see [**21**, Lemma 2.4]), first proved by Andrews [**2**, Theorem 4].

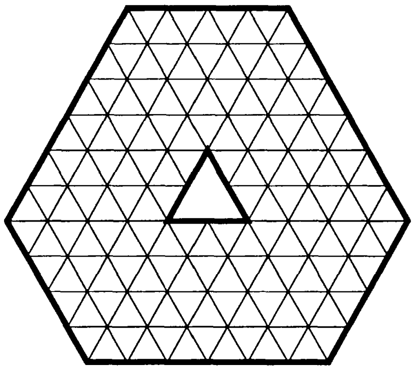

FIGURE 3.1.

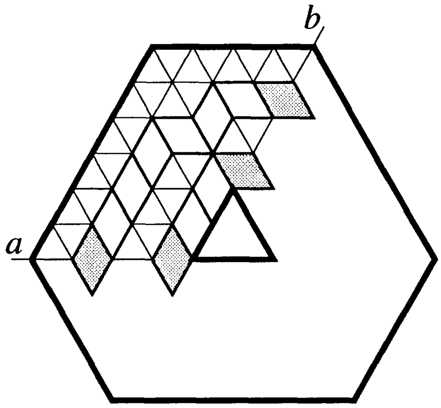

FIGURE 3.2.

Lemma 3.1.

$$CS(n,x) = \det\left(\delta_{ij} + \binom{x+i+j}{i}\right)_{0\le i,j\le n-1}.$$

Proof. Let a and b be two lattice rays originating at two vertices of the removed triangle so that they determine a fundamental region F for the action of r on $H_{n,x}$ (see Figure 3.2). Both a and b dissect n tile positions in $H_{n,x}$. Label these positions, starting with the ones closest to the removed triangle, by $0, 1, \dots, n-1$.

Clearly, for any r-invariant tiling of $H_{n,x}$, the sets of tiles crossed by a and b have the same labels. We claim that the number of tilings for which this set of labels is $0 \le i_1 < \cdots < i_k \le n-1$, $1 \le k \le n$ is equal to the principal minor of the matrix $B = \left(\binom{x+i+j}{i}\right)_{0\le i,j\le n-1}$ corresponding to these labels.

Indeed, r-invariant tilings are determined by their intersection with the fundamental region F, so such tilings with corresponding labels $i_1, \dots, i_k$ can be identified with tilings of the region $F(i_1, \dots, i_k)$ obtained from F by removing unit triangles along a and b in positions $i_1 \dots, i_k$.

In turn, using the standard encoding of lozenge tilings as families of non-intersecting lattice paths, the tilings of $F(i_1, \dots, i_k)$ are easily seen to be in bijection with k-tuples of non-intersecting lattice paths on the square lattice, taking steps north and east, starting at $u_\mu = (n-i_\mu-1, 0)$ and ending at $v_\mu = (n-1, x+i_\mu)$, $\mu = 1, \dots, k$. Apply the Gessel-Viennot theorem [**20**, Theorem 1.2]. The determinant corresponding to (2.6) is easily seen to be in this case precisely the principal minor of B corresponding to row and column indices $i_1, \dots, i_k$, thus proving our claim.

We obtain that $CS(n,x)$ is equal to the sum of all principal minors of B, i.e., to $\det(I+B)$. □

Theorem 3.2. *For $n, x \geq 1$ we have*

$$CS(2n,2x+1) = \frac{n!\,(x-1)!}{(2n)!} \prod_{i=0}^{n} \frac{(x+2i)_{i+1}}{(x+n+i)!} \prod_{i=0}^{n-1} \frac{[i!]^2\,[(2x+2i+2)_{i+1}]^2\,(x+i)!\,(x+2i+1)_i}{[(2i)!]^2} \tag{3.1}$$

$$CS(2n-1,2x+1) = \frac{(x-1)!\,(2x+2n)_n}{(x+n-1)!} \prod_{i=0}^{n-1} \frac{[i!]^2\,[(2x+2i)_i]^2\,(x+i)!\,(x+2i)_{i+1}\,(x+2i+1)_i}{[(2i)!]^2\,(x+n+i)!}. \tag{3.2}$$

Proof. Lozenge tilings of $H_{2n,2x+1}$ can naturally be identified with perfect matchings of the "dual" graph G, i.e., the graph whose vertices are the unit triangles of $H_{2n,2x+1}$ and whose edges connect precisely those unit triangles that share an edge (a perfect matching of a graph is a collection of vertex-disjoint edges collectively incident to all vertices of the graph; we will often refer to a perfect matching simply as a *matching*). Therefore, $CS(2n,2x+1)$ is the number of matchings of G invariant under the rotation r by $120°$ around the center of G.

Consider the action of the group generated by r on G, and let $\tilde{G}$ be the orbit graph. It follows easily that the r-invariant matchings of G can be identified with the matchings of $\tilde{G}$.

As illustrated in Figure 3.3 (for $n=3$, $x=2$), the graph $\tilde{G}$ can be embedded in the plane so that it admits a symmetry axis ℓ. Moreover, it can be readily checked that the Factorization Theorem [**6**, Theorem 1.2] for perfect matchings can be applied to $\tilde{G}$. We obtain that

$$M(\tilde{G}) = 2^{2n} M(\tilde{G}^+) M(\tilde{G}^-), \tag{3.3}$$

where $M(G)$ denotes the matching generating function of G, and $\tilde{G}^+$ and $\tilde{G}^-$ (illustrated in Figure 3.4) are the connected components of the

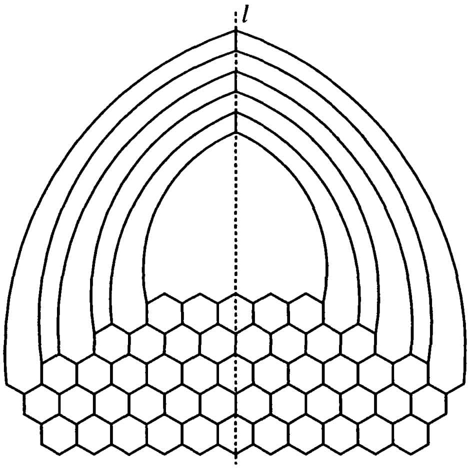

FIGURE 3.3.

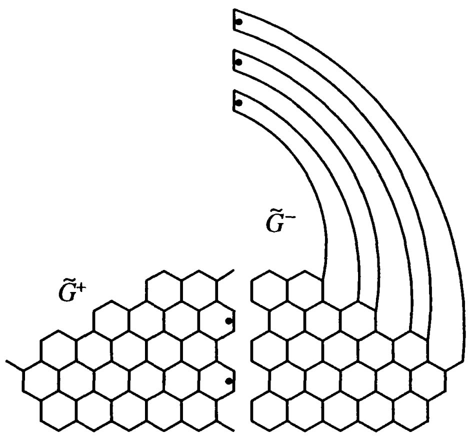

FIGURE 3.4.

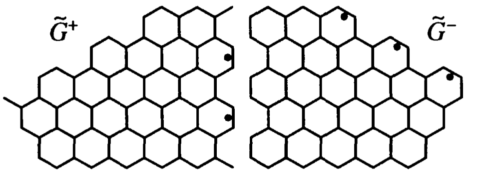

FIGURE 3.5.

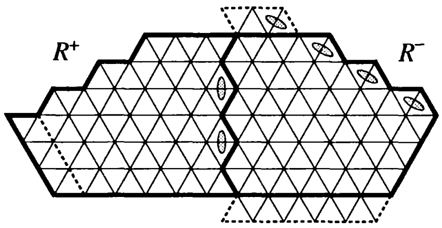

FIGURE 3.6.

subgraph obtained from $\tilde{G}$ by deleting the top $2n$ edges immediately to the left of ℓ, the bottom $2n$ edges immediately to the right of ℓ, and changing the weight of the $2n-1$ edges along ℓ to $1/2$ (the matching generating function of a graph is the sum of the weights of all its perfect matchings, the weight of a matching being the product of weights of its edges).

Clearly, the graphs $\tilde{G}^+$ and $\tilde{G}^-$ can be redrawn as shown in Figure 3.5. Using again the duality between matchings and tilings, we arrive at two regions R^+ and R^- whose tilings can be identified, preserving weights, with the matchings of $\tilde{G}^+$ and $\tilde{G}^-$ (see Figure 3.6; the boundaries of R^+ and R^- are shown in solid lines). However, because of forced tiles, it is readily seen that $L(R^+) = L(B_{n,x+1})$ and $L(R^-) = 2L(A_{n+1,x})$ (compare Figure 3.6 to Figures 2.1 and 2.2; the places where the boundaries of $B_{n,x+1}$ and $A_{n+1,x}$ differ from those of R^+ and R^- are indicated by dashed lines in Figure 3.6). Therefore, (3.3) can be rewritten as

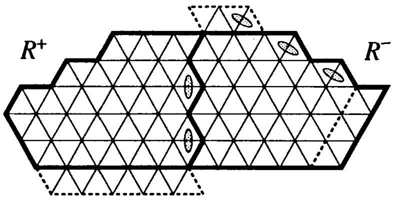

Figure 3.7.

$$CS(2n, 2x+1) = 2^{2n+1} L(A_{n+1,x}) L(B_{n,x+1}). \tag{3.4}$$

By Proposition 2.2 and Theorem 2.1, the above equality yields an explicit product formula for $CS(2n, 2x+1)$. After some manipulation one arrives at (3.1).

To prove (3.2) we proceed similarly. Take G to be the graph dual to $H_{2n-1,2x+1}$, construct the orbit graph $\tilde{G}$ as above and apply the Factorization Theorem to $\tilde{G}$. One obtains

$$M(\tilde{G}) = 2^{2n-1} M(\tilde{G}^+) M(\tilde{G}^-) \tag{3.5}$$

(the change in the exponent of 2 is due the fact that the "width" of $\tilde{G}$ — cf. [**6**], half the number of vertices on ℓ — is now $2n-1$).

The regions R^+ and R^- dual to $\tilde{G}^+$ and $\tilde{G}^-$ satisfy this time $L(R^+) = L(B_{n,x+1})$ and $L(R^-) = 2L(A_{n,x})$ (Figure 3.7 illustrates the case $n = 3$, $x = 2$). Therefore, (3.5) implies

$$CS(2n-1, 2x+1) = 2^{2n} L(A_{n,x}) L(B_{n,x+1}). \tag{3.6}$$

This provides, by Proposition 2.2 and Theorem 2.1, a product formula for $CS(2n-1, 2x+1)$, which one easily brings to the form (3.2). □

By Lemma 3.1, for fixed n, the expressions on the right hand side in (3.1) and (3.2) are polynomials in x. Define $P_{2n}(x)$ and $P_{2n-1}(x)$ to be the polynomials on the right hand side in (3.1) and (3.2), respectively.

Corollary 3.3. *With the above definition of the polynomials P_n, for all $n \geq 1$ we have*

$$CS(n, x) = P_n\left(\frac{x-1}{2}\right) \tag{3.7}$$

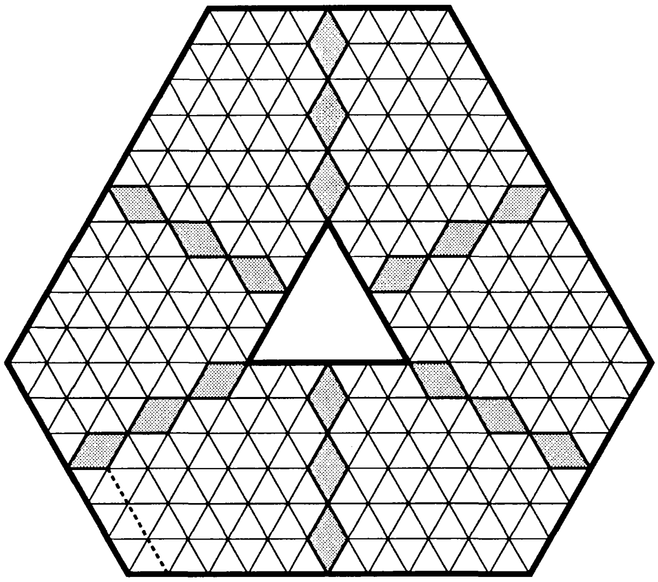

Figure 4.1.

as polynomials in x.

Proof. By Theorem 3.2, (3.7) holds if x is odd and $x \geq 3$. Since the two sides of (3.7) are polynomials (the left hand side by Lemma 3.1), they must be equal. □

REMARK 3.4. By Lemma 3.1 and Corollary 3.3 we obtain an expression for

$$\det\left(\delta_{ij} + \binom{x+i+j}{i}\right)_{0\leq i,j\leq n-1}$$

as a product of linear factors in x. This is equivalent to Theorem 8 of [2].

4. Cyclically symmetric transposed-complementary plane partitions

By (1), (2) and (3) of the Introduction, this case is equivalent to counting tilings of $H(n,n,n)$ that are invariant under the rotation r and the reflection t' across a symmetry axis of $H(n,n,n)$ not containing any of its vertices. More generally, we determine the number $CSTC(n,x)$ of r,t'-invariant tilings of the regions $H_{n,x}$ (defined at the beginning of Section 3). It is easy to see that $H_{n,x}$ has no such tilings unless n and x are both even.

Define the region $C_{n,x}$ to be the region having the same boundary as $A_{n,x}$ (see Figure 2.1), but with all tile positions weighted by 1.

Lemma 4.1. $CSTC(2n, 2x) = L(C_{n,x})$.

Proof. Suppose T is an r, t'-invariant tiling of $H_{2n,2x}$. It follows that T is invariant under reflection in the three symmetry axes of $H_{2n,2x}$. This implies that in T the $6n$ tile positions along these symmetry axes are occupied by lozenges (see Figure 4.1). The set of these $6n$ lozenges disconnects $H_{2n,2x}$ in six congruent pieces. Removing n forced lozenges from one of these pieces one obtains a region congruent to $C_{n,x}$ (this is indicated by the dotted line in Figure 4.1). The group generated by r and t' acts transitively on the set of these pieces. Therefore, the restriction of T to one of the pieces gives a bijection between r, t'-invariant tilings of $H_{2n,2x}$ and tilings of $C_{n,x}$. □

Theorem 4.2.

$$2\frac{CSTC(2n+2, 2x)}{CSTC(2n, 2x)} = \frac{CS(2n+1, 2x)}{CS(2n, 2x)}. \tag{4.1}$$

Proof. We deduce (4.1) by working out the analogs of (3.4) and (3.6) for the case when the second argument on the left hand side is even.

We proceed along the same lines as in the proof of Theorem 3.2. Let G be the graph dual to $H_{2n,2x}$, and let $\tilde{G}$ be the orbit graph of the action of $\langle r \rangle$ on G. As in the proof of Theorem 3.2, we can embed $\tilde{G}$ in the plane so that it admits a symmetry axis, and we can apply the Factorization Theorem of [**6**]. This expresses the number of perfect matchings of $\tilde{G}$ as a product involving the matching generating functions of two subgraphs. These two subgraphs can be redrawn in the plane such that they are the dual graphs of two regions R_1^+ and R_1^- on the triangular lattice. For $n = 2$, $x = 1$, these regions are illustrated in Figure 4.2 (their boundaries are shown in solid lines).

Therefore, since $M(\tilde{G}) = CS(2n, 2x)$, we can phrase the result of applying the Factorization Theorem to $\tilde{G}$ as

$$CS(2n, 2x) = 2^{2n} L(R_1^+) L(R_1^-). \tag{4.2}$$

However, by removing the n forced lozenges along the left boundary of R_1^+, we are left with a region congruent to $C_{n,x}$ (this is indicated by the dotted line in Figure 4.2). Thus, (4.2) implies

$$CS(2n, 2x) = 2^{2n} L(C_{n,x}) L(R_1^-). \tag{4.3}$$

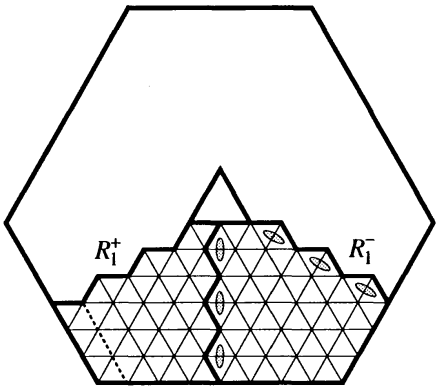

FIGURE 4.2.

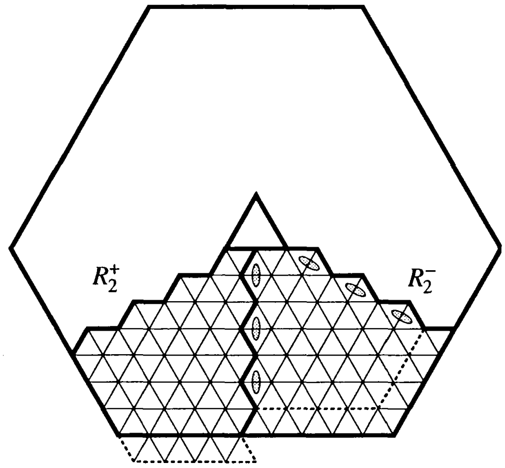

FIGURE 4.3.

Similarly, starting from the graph dual to $H_{2n+1,2x}$, considering its orbit graph under the action of $\langle r \rangle$ and applying the Factorization Theorem to it, we obtain after rephrasing everything in terms of tilings that

$$CS(2n+1,2x) = 2^{2n+1} L(R_2^+) L(R_2^-), \tag{4.4.}$$

for two regions R_2^+ and R_2^- which are illustrated in Figure 4.3 for $n = 2$, $x = 1$ (their boundaries are shown in solid lines). However, R_2^+ is congruent to the region obtained from $C_{n+1,x}$ after removing the $n + x$ forced lozenges along its base. Moreover, the region obtained from R_2^- by removing all forced lozenges is isomorphic to R_1^-. Therefore, (4.4) becomes

$$CS(2n+1,2x) = 2^{2n+1} L(C_{n+1,x}) L(R_1^-). \tag{4.5.}$$

Dividing (4.3) and (4.5) side by side and using Lemma 4.1 we obtain (4.1). □

Corollary 4.3.

$$CSTC(2n,2x) = \frac{1}{2^n} \prod_{k=0}^{n-1} \frac{CS(2k+1,2x)}{CS(2k,2x)}. \tag{4.6}$$

(*By Corollary 3.3, this provides an explicit formula for* $CSTC(2n,2x)$.)

Proof. Take the side by side product of (4.1) for $n = 0, 1, \ldots, n-1$. □

REMARK 4.4. Using the standard encoding of lozenge tilings as families of non-intersecting lattice paths, and then employing the Gessel-Viennot theorem [**20**, Theorem 1.2], it is easy to see that

$$L(C_{n,x}) = \det\left(\binom{x+i+j}{2j-i}\right)_{0\le i,j\le n-1}. \tag{4.7}$$

Therefore, by (4.7), Lemma 4.1 and Corollary 4.3 we obtain an expression for

$$\det\left(\binom{x+i+j}{2j-i}\right)_{0\le i,j\le n-1}$$

as a product of linear factors in x. Such a formula was first proved by Mills, Robbins and Rumsey in [**16**, Theorem 7].

REMARK 4.5. Following the notation of [**5**], let

$$Z_n(x) = \det\left(\delta_{ij} + \binom{x+i+j}{i}\right)_{0\le i,j\le n-1},$$

$$T_n(x) = \det\left(\binom{x+i+j}{2j-i}\right)_{0\le i,j\le n-1},$$

$$R_n(x) = \det\left(\binom{x+i+j}{2i-j} + 2\binom{x+i+j+2}{2i-j+1}\right)_{0\le i,j\le n-1}.$$

Encode the tilings of the region R_1^- in (4.3) as n-tuples of non-intersecting paths of rhombi going from the western boundary to the northeastern boundary of R_1^- (see Figure 4.2). Identify, as usual, these paths of rhombi with lattice paths on $\mathbf{Z}^2$. Apply the Gessel-Viennot theorem on non-intersecting lattice paths [**20**, Theorem 1.2]. It is easy to see that the (i,j)-entry of the Gessel-Viennot matrix M is in this case

$$M_{ij} = \binom{x+i+j}{2i-j} + \frac{1}{2}\binom{x+i+j}{2i-j-1} + \frac{1}{2}\binom{x+i+j}{2i-j+1} + \frac{1}{4}\binom{x+i+j}{2i-j},$$

for $i,j=0,\dots,n-1$. A simple calculation shows that

$$M_{ij} = \frac{1}{4}\left(R_n(x)\right)_{ij}.$$

Therefore, (4.3) and (4.5) can be written as

$$\begin{aligned} Z_{2n}(2x) &= T_n(x)R_n(x), \\ Z_{2n+1}(2x) &= 2T_{n+1}(x)R_n(x). \end{aligned}$$

These are precisely relations (2.5) and (2.6) of [**5**], which were deduced there from [**16**, Theorem 5].

REMARK 4.6. The case $x = 0$ of Theorem 4.2 is the object of Theorem 6.2 of [**6**].

5. Cyclically symmetric self-complementary and totally symmetric self-complementary plane partitions

It is easy to see that in order for the hexagon $H(a, b, c)$ to have tilings in any of the two symmetry classes mentioned in the title of this section one needs to have $a = b = c = 2n$, with n a positive integer. Denote by $CSSC(2n)$ and $TSSC(2n)$ the number of tilings of $H(2n, 2n, 2n)$ in the two symmetry classes, respectively.

The following result was first proved by Kuperberg [**13**].

Theorem 5.1.

$$CSSC(2n) = \left(\prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!}\right)^2. \tag{5.1}$$

Proof. In [**7**] it is shown (see [**7**, (2.3)]) that a simple consequence of the Factorization Theorem for perfect matchings [**6**, Theorem 1.2] is that

$$CSSC(2n) = 2^n L(A_{n,1}). \tag{5.2}$$

(The derivation of this result follows along the lines of the proofs of (3.4), (3.6), (4.3) and (4.5).) Using Proposition 2.2 and Theorem 2.1 we obtain a product formula for $CSSC(2n)$, which is easily seen to be equivalent to (5.1). □

REMARK 5.2. Following the notation of [**5**], let

$$W_n(x) = \left(\binom{x+i+j+1}{2i-j+1} + \binom{x+i+j}{2i-j}\right)_{0\le i,j\le n-1}.$$

Based on the fact that the related determinants $Z_n(x)$ and $T_n(x)$ defined in Remark 4.5 have close connections with plane partition enumeration problems, Andrews and Burge suggest in [**5**] that the same might be true for det $W_n(x)$. Relation (5.2) allows us to give what appears to be the first such connection.

Indeed, let

$$w_n(x) = \left(\binom{x+i+j+1}{2i-j} + \binom{x+i+j}{2i-j-1} \right)_{0 \le i,j \le n-1}.$$

It is readily checked that the matrix obtained from $w_n(x)$ by removing the first row and column is precisely $W_{n-1}(x+2)$. Since the top left entry of $w_n(x)$ is 1, we obtain that

$$\det W_{n-1}(x+2) = \det w_n(x). \tag{5.3}$$

A straightforward calculation reveals that $w_n(x)_{ij} = (x+3i+1)K_n(x+1,0)_{ij}$. Therefore, by (2.4), we deduce that

$$\det w_n(x) = 2^n L(A_{n,x+1}). \tag{5.4}$$

From (5.3) and (5.4), it follows that det $W_{n-1}(x+2) = 2^n L(A_{n,x+1})$. Therefore, by (5.2), we obtain that det $W_{n-1}(2) = CSSC(2n)$.

In [**7**] there is presented a direct proof of the fact that

$$CSSC(2n) = TSSC(2n)^2.$$

(In outline, by combinatorial arguments, an expression is derived for $CSSC(2n)$ as the determinant of a certain matrix, which is then transformed by elementary row and column operations to an antisymmetric matrix whose Pfaffian was previously known to give $TSSC(2n)$).

Therefore, we obtain by Theorem 5.1 the following result, first proved by Andrews [**3**].

Corollary 5.3.

$$TSSC(2n) = \prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!}.$$

REMARK 5.4. By (3.4) and (3.6) we have

$$\frac{CS(2n, 2x+1)}{CS(2n-1, 2x+1)} = 2\frac{L(A_{n+1,x})}{L(A_{n,x})}. \tag{5.5}$$

On the other hand, from (5.2) we deduce

$$\frac{CSSC(2n+2)}{CSSC(2n)} = 2\frac{L(A_{n+1,1})}{L(A_{n,1})}.$$

This relation and (5.5) specialized to $x = 1$ imply

$$\frac{CSSC(2n+2)}{CSSC(2n)} = \frac{CS(2n, 3)}{CS(2n-1, 3)}. \tag{5.6}$$

One may regard (5.6) as giving a proof of the cyclically symmetric, self-complementary case based on the solution of the cyclically symmetric case, which was solved fifteen years earlier (see [**2**] and [**13**]).

6. Transposed-complementary plane partitions

By (1) and (3) of the Introduction, this case amounts to finding the number $TC(a, a, 2b)$ of tilings of the hexagon $H(a, a, 2b)$ that are symmetric with respect to its symmetry axis ℓ perpendicular to the sides of length $2b$ (see Figure 6.1; it is easy to see that the indicated form of the arguments represents the general case).

The following result was first proved (in an equivalent form) by Proctor [**17**].

Theorem 6.1.

$$TC(a, a, 2b) = \prod_{i=1}^{\lceil \frac{a}{2} \rceil} \frac{(b+i)_{a-2i+1}}{(i)_{a-2i+1}} \prod_{i=1}^{\lceil \frac{a-1}{2} \rceil} \frac{(2b+2i+1)_{a-2i;\,2}}{(2i+1)_{a-2i;\,2}},$$

where $(a)_{k;\,s} := a(a+s)(a+2s)\cdots(a+(k-1)s)$ *is the shifted factorial of step* s.

Proof. In any tiling T of $H(a, a, 2b)$ symmetric with respect to ℓ, the a tile positions along ℓ are occupied by lozenges. This set of lozenges divides our hexagon in two congruent pieces, and T is determined by its restriction to the left piece S, say (see Figure 6.1). Therefore, $TC(a, a, 2b)$ is just the number of tilings of S.

However, the region obtained from S by removing the forced lozenges (see Figure 6.1) is readily recognized as being a member of the family of

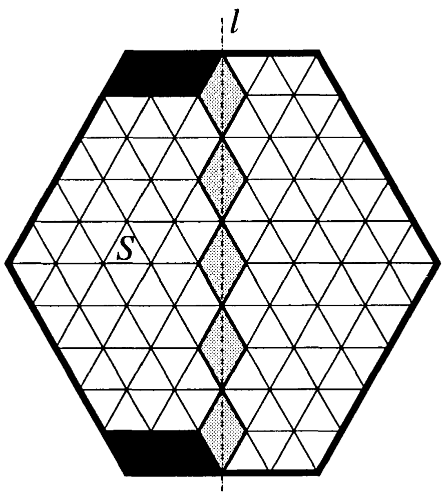

Figure 6.1.

regions $\bar{R}_{\mathbf{l},\mathbf{q}}(x)$ defined in [**8**, §2] (here $\mathbf{l}$ and $\mathbf{q}$ are lists of strictly increasing positive integers, and x is integer). More precisely, S is congruent to the region $\bar{R}_{[a-1],\emptyset}(b)$, where $[n]$ denotes the list $(1,\dots,n)$.

Therefore, Proposition 2.1 of [**8**] and formulas (1.6), (1.2) and (1.4) of [**8**] provide an expression for $L(S)$ (hence, for $TC(a,a,2b)$) as a product of linear polynomials in b. After some manipulation, this expression becomes the right hand side of the equality in the statement of the Theorem. □

7. Self-complementary plane partitions

This case amounts to enumerating tilings of $H(a,b,c)$ that are invariant under the rotation k by 180°, and it was first proved by Stanley [**19**]. In this section we give a simple proof in the case when two of the numbers a, b and c are equal.

Let $SC(a,a,b)$ be the number of k-invariant tilings of $H(a,a,b)$. It is easy to see that this number is 0 unless a or b is even. Let $PP(a,b,c)$ be the total number of tilings of $H(a,b,c)$ (for an explicit product formula, due to Macmahon, see e.g.[8, p.2]).

Theorem 7.1.

$$SC(2x,2x,2y) = PP(x,x,y)^2 \tag{7.1}$$

$$SC(2x,2x,2y+1) = PP(x,x,y)PP(x,x,y+1) \tag{7.2}$$

$$SC(2x+1,2x+1,2y) = PP(x,x+1,y)^2. \tag{7.3}$$

Proof. Following the same reasoning as in proving (3.4), (3.6), (4.3), (4.5) and (5.1), one sees that the Factorization Theorem of [**6**] can be

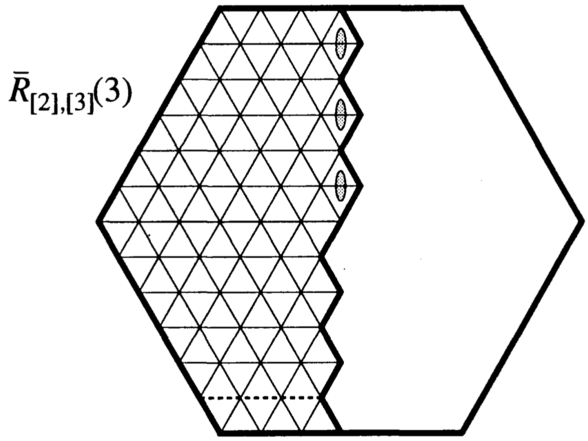

Figure 7.1. $a = 6$, $b = 6$.

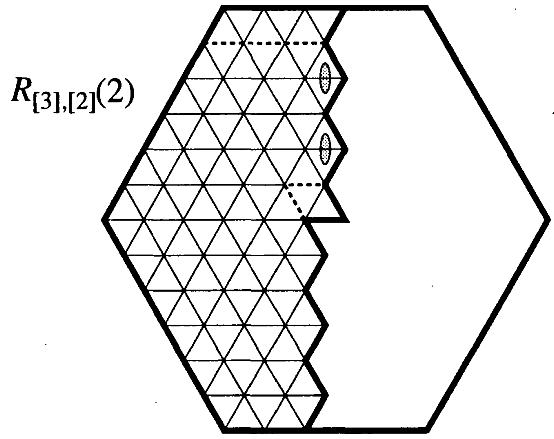

FIGURE 7.2. $a = 6$, $b = 5$.

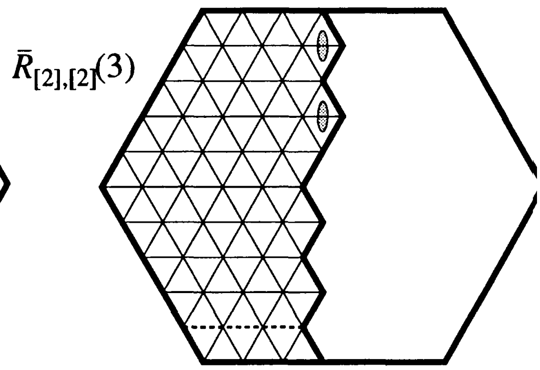

FIGURE 7.3. $a = 5$, $b = 6$.

used to express the number of k-invariant tilings of $H(a, a, b)$ as a power of 2 times the tiling generating function of a certain subregion with some tile positions weighted by $1/2$ (the precise shape of this region depends on the parities of a and b). Furthermore, after removing the forced lozenges from this region, the leftover piece is readily recognized to belong to one of the families $R_{\mathbf{l},\mathbf{q}}(x)$ or $\bar{R}_{\mathbf{l},\mathbf{q}}(x)$ defined in [**8**, §2].

More precisely, for $a = 2x$, $b = 2y$, we obtain that

$$L(H(2x, 2x, 2y)) = 2^x L(\bar{R}_{[x-1],[x]}(y)) \tag{7.4}$$

(see Figure 7.1; as usual, the dotted lines indicate removal of forced lozenges). Similarly, we deduce

$$L(H(2x,2x,2y+1)) = 2^x L(R_{[x],[x-1]}(y)) \tag{7.5}$$

$$L(H(2x+1,2x+1,2y)) = 2^x L(\bar{R}_{[x],[x]}(y)) \tag{7.6}$$

(see Figures 7.2 and 7.3).

By Proposition 2.1 of [8], (7.4)–(7.6) provide product formulas for $SC(a,a,b)$, and these are easily seen to agree with (7.1)–(7.3). □

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Invariants for Representations of Weyl Groups, Two-sided Cells, and Modular Representations of Iwahori-Hecke Algebras

Akihiko Gyoja, Kyo Nishiyama and Kenji Taniguchi

§1. Introduction

1.1. q-Series identity.

Let $s_\lambda(x)$ be the Schur function in infinite variables $x = (x_1, x_2, \dots)$ corresponding to a Young diagram λ. For each node v in the diagram λ, $h(v)$ denotes the hook length of λ at v. Cf. [9] for the Young diagrams and related notions. In a recent work [7], Kawanaka obtained a q-series identity

$$\sum_\lambda I_\lambda(q) s_\lambda(x) = \prod_i \prod_{r=0}^{\infty} \frac{1 + x_i q^{r+1}}{1 - x_i q^r} \prod_{i<j} \frac{1}{1 - x_i x_j}, \tag{1}$$

where

$$I_\lambda(q) = \prod_{v \in \lambda} \frac{1 + q^{h(v)}}{1 - q^{h(v)}}, \tag{2}$$

and the sum on the left hand side of (1) is taken over all Young diagrams λ. If $q = 0$, then (1) reduces to the Schur-Littlewood identity.

Using (1), Kawanaka showed that for a Youndg diagram λ with n nodes, (2) is expressed as

$$I_\lambda(q) = |\mathfrak{S}_n|^{-1} \sum_{s \in \mathfrak{S}_n} \chi_\lambda(s^2) \frac{\det(1 + q\rho(s))}{\det(1 - q\rho(s))}, \tag{3}$$

where χ_λ is the irreducible character of the symmetric group $\mathfrak{S}_n$ corresponding to λ and $\rho : \mathfrak{S}_n \to GL_n(\mathbb{Z})$ is the representation of $\mathfrak{S}_n$ by permutation matrices.

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Since (3) is expressed in terms of the symmetric group and its representation, we can generalize such rational function for characters of other Weyl groups.

Definition 1.1. Let W be a Weyl group acting on a complex vector space $\mathfrak{h}$ faithfully as a reflection group. For a character χ of a finite dimensional representation π of W, we define a rational function of an indeterminate q by

$$I_W(\chi; q) = |W|^{-1} \sum_{w \in W} \chi(w^2) \frac{\det(1 + qw|_{\mathfrak{h}})}{\det(1 - qw|_{\mathfrak{h}})},$$

and we call it the *Kawanaka invariant* of π.

The main object of this paper is the expression for the Kawanaka invariants. We have obtained it in the B_l-case, which is stated in §2 (Theorem 2.1). This is not an immediate corollary of Kawanaka's result; in fact, we need a non-trivial argument. If we proceed to the D_l-case, the situation becomes much more difficult. We succeeded in expressing it by means of the Littlewood-Richardson coefficients (Theorem 2.2) and we obtained a conjectural formula for it (Conjecture 3.2). These are included in §2 and §3.

1.2. Invariants for cells.

The Kawanaka invariant plays a role as an invariant for two-sided cells.

In [4], a polynomial invariant

$$\tau^*(\chi; t) := \chi(e)^{-1} \sum_{w \in W} \chi(w) t^{\dim \mathfrak{h}^w}$$

is defined for a character χ of a finite dimensional representation π of W. Here, $\mathfrak{h}^w$ is the subset of w-fixed vectors in $\mathfrak{h}$. It is observed that, if W is of type A_l or B_l, then τ^* characterizes the two-sided cells. If W is not of these types, some deviation occurs. Trying to save this defect, a modified invariant

$$\tilde{\tau}(\chi; q, y) := |W|^{-1} \sum_{w \in W} \chi(w) \frac{\det(1 + yw|_{\mathfrak{h}})}{\det(1 - qw|_{\mathfrak{h}})},$$

motivated by [1]Chap. V, §5, Ex. 3, is introduced, and the relationship between $\tilde{\tau}$ and the two-sided cells is studied in [4]. Note that

$$|W|^{-1} \tau^*(\chi; t) = \lim_{q \to 1} \tilde{\tau}(\chi; q, -1 + t(1 - q)).$$

Hence, in principle, we can extract information on τ^* from $\tilde{\tau}$. In other words, $\tilde{\tau}$ is a refinement of the invariant τ^*.

Because of the resemblance between the definition of $\tilde{\tau}$ and I_W, we expect that the Kawanaka invariant is also related to the two-sided cells. Detailed discussion on the two-sided cells and invariants τ^*, $\tilde{\tau}$, I_W is contained in §4.

Added on March 23, 1999.

After completing the first draft, the authors learned from Kawanaka his recent result, which incidentally implies our Conjecture 3.2. Thus our conjecture is affirmatively settled.

§2. Expression of the Kawanaka invariant

In this section, we present closed expression for Kawanaka invariants.

2.1. A_l-case.

As is explained in §1, the Kawanaka invariant for representations of symmetric group $\mathfrak{S}_l$ is given by

$$I_{\mathfrak{S}_l}(\chi_\lambda; q) = \prod_{v\in\lambda} \frac{1+q^{h(v)}}{1-q^{h(v)}}.$$

2.2. B_l-case.

In the B_l-case, we have similar expression. The irreducible representation of $W = W(B_l) \simeq \mathfrak{S}_l \ltimes \mathbb{Z}_2^l$ is parametrized by the ordered pair (λ', λ'') of Young diagrams (cf. [8]). Let $\chi_{\lambda',\lambda''}$ be the corresponding irreducible character.

Theorem 2.1 ([5]). *We have*

$$\begin{aligned} I_{W(B_l)}(\chi_{\lambda',\lambda''}; q) &= \prod_{v'\in\lambda'} \frac{1+q^{2h(v')}}{1-q^{2h(v')}} \prod_{v''\in\lambda''} \frac{1+q^{2h(v'')}}{1-q^{2h(v'')}} \\ &= I_{\mathfrak{S}_{l'}}(\chi_{\lambda'}; q^2) I_{\mathfrak{S}_{l''}}(\chi_{\lambda''}; q^2), \end{aligned}$$

where $l' = |\lambda'|$ and $l'' = |\lambda''|$.

2.3. D_l-case.

Let us denote the restriction of $\chi_{\lambda',\lambda''}$ of $W(B_l)$ to $W(D_l) \simeq \mathfrak{S}_l \ltimes \mathbb{Z}_2^{l-1}$ by the same symbol $\chi_{\lambda',\lambda''}$. If $\lambda' \neq \lambda''$, then $\chi_{\lambda',\lambda''}$ is an irreducible character. If $\lambda = \lambda' = \lambda''$, then $\chi_{\lambda,\lambda}$ decomposes into two inequivalent irreducible characters χ^{I}_{λ} and χ^{II}_{λ}, which are interchanged by the outer automorphism induced from the conjugation by the non-unit element of $W(B_l)/W(D_l)$. So we have $I_{W(D_l)}(\chi^{I}_{\lambda};q) = I_{W(D_l)}(\chi^{II}_{\lambda};q) = I_{W(D_l)}(\chi_{\lambda,\lambda};q)/2$. Therefore, it is enough to compute $I_{W(D_l)}(\chi_{\lambda',\lambda''};q)$ for obtaining Kawanaka invariants in the D_l-case.

Denote by ε the one dimensional representation of $W(B_l)$, induced from $W(B_l) \twoheadrightarrow W(B_l)/W(D_l) \simeq \{0,1\} \ni \epsilon \mapsto (-1)^{\epsilon}$. Since $W(D_l) = \operatorname{Ker}\varepsilon$ and $|W(B_l)| = 2|W(D_l)|$, we have

$$I_{W(D_l)}(\chi_{\lambda',\lambda''};q) = I_{W(B_l)}(\chi_{\lambda',\lambda''};q) + I^*(\chi_{\lambda',\lambda''};q),$$

where

$$I^*(\chi_{\lambda',\lambda''};q) = |W(B_l)|^{-1} \sum_{w \in W(B_l)} \chi(w^2)\varepsilon(w)\frac{\det(1+qw|_{\mathfrak{h}})}{\det(1-qw|_{\mathfrak{h}})}.$$

Since the explicit form of $I_{W(B_l)}(\chi_{\lambda',\lambda''};q)$ is known (Theorem 2.1), in order to determine the explicit form of $I_{W(D_l)}(\chi_{\lambda',\lambda''};q)$ it is enough to determine $I^*(\chi_{\lambda',\lambda''};q)$.

Unfortunately, we have not obtained a closed formula of I^*. The next theorem is the expression by means of the Littlewood-Richardson coefficients.

Theorem 2.2 ([5]). *Denote by $c^{\lambda}_{\nu,\mu}$ the Littlewood-Richardson coefficient. Then $I^*(\chi_{\lambda',\lambda''};q)$ is given by*

(4)

$$I^*(\chi_{\lambda',\lambda''};q) = \sum_{N=0}^{\min\{|\lambda'|,|\lambda''|\}} q^{l-2N} \times \sum_{\nu',\nu''}\left(\sum_{|\mu|=N} c^{\lambda'}_{\nu',\mu}c^{\lambda''}_{\nu'',\mu}\right) G(\chi_{\nu'};q^2)G(\chi_{\nu''};q^2),$$

where

$$G(\chi_{\lambda};q) = q^{n(\lambda)}\prod_{v\in\lambda}\frac{1+q^{c(v)}}{1-q^{h(v)}}.$$

2.4. Other cases.

For Weyl groups of exceptional types and for dihedral groups, we have calculated the Kawanaka invariants of all the irreducible representations explicitly.

§3. Conjectures on Kawanaka invariants of type D_l

In this section, we give two conjectures, which are formulated in [5]. The first one follows from the second one. The second one is of purely combinatorial nature, which involves only an identity of polynomial functions.

3.1. Conjectural formula for I^*.

For partitions λ' and λ'' with $l(\lambda') \leq 3$ and $|\lambda''| \leq 3$, we calculated (4) explicitly with the help of *Mathematica*, and we obtained a conjectural formula of $I^*(\chi_{\lambda',\lambda''}; q)$.

Definition 3.1 (The rational function $T_{\lambda',\lambda''}(q)$). If $\lambda' = (\lambda'_1 \geq \lambda'_2 \geq \cdots \geq \lambda'_n \geq 0)$ and $\lambda'' = (\lambda''_1 \geq \lambda''_2 \geq \cdots \geq \lambda''_n \geq 0)$ are a pair of partitions, put $\mu'_i := \lambda'_i + n - i$, $\mu''_i := \lambda''_i + n - i$, and define new partitions by $\mu' := (\mu'_1, \mu'_2, \cdots)$ and $\mu'' := (\mu''_1, \mu''_2, \cdots)$. Put

$$\begin{aligned} T_{\lambda',\lambda''}(q) :=& 2^n q^{|\mu'|+|\mu''|} \prod_{v'\in\lambda'} \frac{1+q^{2h(v')}}{1-q^{2h(v')}} \prod_{v''\in\lambda''} \frac{1+q^{2h(v'')}}{1-q^{2h(v'')}} \\ & \times \frac{\prod_{1\leq i<j\leq n}(q^{2\mu'_j}+q^{2\mu'_i})(q^{2\mu''_j}+q^{2\mu''_i})}{\prod_{1\leq i,j\leq n}(q^{2\mu'_i}+q^{2\mu''_j})}. \end{aligned}$$

Our first conjecture is as follows.

Conjecture 3.2 (A closed formula for $I^*(\chi_{\lambda',\lambda''}; q)$).

$$I^*(\chi_{\lambda',\lambda''}; q) = T_{\lambda',\lambda''}(q).$$

Example 3.3.
1. If $\lambda'' = \emptyset$, we get $I^*(\chi_{\lambda',\emptyset}; q) = T_{\lambda',\emptyset}(q)$ from (4).
2. If λ' and λ'' correspond to trivial representations, i.e. $\lambda' = [l']$, $\lambda'' = [l'']$, we can prove $I^*(\chi_{[l'],[l'']}; q) = T_{[l'],[l'']}(q)$ by induction on $\min\{l', l''\}$.
3. As is written at the beginning of this subsection, if $l(\lambda') \leq 3$ and $|\lambda''| \leq 3$, our conjecture is true. We check it by the aid of *Mathematica*.

Remark 3.4. If $\lambda = \lambda' = \lambda''$, it is not difficult to see

$$T_{\lambda,\lambda}(q) = \left(\prod_{v\in\lambda} \frac{1+q^{2h(v)}}{1-q^{2h(v)}}\right)^2 = I_{\mathfrak{S}_{|\lambda|}}(\chi_\lambda; q^2)^2.$$

3.2. A recursive formula for I^*.

Toward the proof of Conjecture 3.2, we exploited a recursive formula for $I^*(\chi_{\lambda',\lambda''};q)$.

Define an inner product on the space of symmetric functions with n variables $y=(y_1,\cdots,y_n)$ by $\langle s_{\lambda'}(y), s_{\lambda''}(y)\rangle_{GL_n(y)} := \delta_{\lambda',\lambda''}$, where $s_\lambda(y)$'s are the Schur functions. For infinitely many variables $x=(x_1,x_2,\cdots)$, consider $s_\lambda(x,y)$'s as symmetric functions in y, and put $\tilde{I}(\chi_{\lambda';\lambda''},x) := \langle s_{\lambda'}(x,y), s_{\lambda''}(x,y)\rangle_{GL_n(y)}$. Consider the specialization

$$\text{elementary symmetric function } e_r(x) \mapsto q^r \prod_{i=1}^{r} \frac{1+q^{2i-2}}{1-q^{2i}}.$$

By this specialization, $s_\lambda(x)$ becomes $q^{|\lambda|}G(\lambda;q^2)$, and $I^*(\chi_{\lambda',\lambda''};q)$ is the result coming out from $\tilde{I}(\chi_{\lambda',\lambda''};x) = \sum_\mu s_{\lambda'/\mu}(x)s_{\lambda''/\mu}(x)$.

Theorem 3.5 (A recursive formula for I^*). *Fix partitions λ', λ'' and a positive integer r. Denote by $V(r)$ the set of all vertical r-strips, i.e., the skew diagrams which have at most one square in each row. Then*

$$\sum_{\substack{\mu' \\ \mu'-\lambda'\in V(r)}} I^*(\chi_{\mu',\lambda''},q) = \sum_{\substack{i,j\geq 0\\ i+j=r}} e_i \sum_{\substack{\mu'' \\ \lambda''-\mu''\in V(j)}} I^*(\chi_{\lambda',\mu''},q).$$

Thanks to this theorem, our first conjecture reduces to the following second conjecture.

Conjecture 3.6. *$T_{\lambda',\lambda''}$ satisfies the same recursive formula.*

§4. Application – Invariants for two-sided cells

In this section, we discuss the two-sided cells and the invariants τ^*, $\tilde{\tau}$, I_W.

Here we do not reproduce the definition of the two-sided cell [8] §4.2, but we note that this concept is important in the representation theory, e.g., in the work of A. Joseph [6] on the classification of primitive ideals of the enveloping algebras of complex semisimple Lie algebras, and in the work of G.Lusztig [8] on the classification and the description of irreducible characters of finite Chevalley groups.

4.1. Invariant τ^*.

Let us recall the definition of τ^* and $\tilde{\tau}$. We assume the same notation as in Definition 1.1.

Definition 4.1. For a character χ of a finite dimensional representation of a Weyl group W, we define

$$\tau^*(\chi;t) = \chi(e)^{-1}\sum_{w\in W}\chi(w)t^{\dim \mathfrak{h}^w} \quad \text{and}$$
$$\tilde{\tau}(\chi;q,y) = |W|^{-1}\sum_{w\in W}\chi(w)\frac{\det(1+yw|_{\mathfrak{h}})}{\det(1-qw|_{\mathfrak{h}})}.$$

Example 4.2. Let χ_λ be the irreducible character of $\mathfrak{S}_l$, associated to the Young diagram λ. Then we have

$$\tau^*(\chi_\lambda;t) = \prod_{v\in\lambda}(t+c(v)) \quad \text{and}$$
$$\tilde{\tau}(\chi_\lambda;q,y) = q^{n(\lambda)}\prod_{v\in\lambda}\frac{1+yq^{c(v)}}{1-q^{h(v)}},$$

where $c(v)$'s are the contents, and

$$n(\lambda) := \sum_{i>0}(i-1)\lambda_i \text{ where } \lambda = (\lambda_1 \geq \lambda_2 \geq \dots).$$

For the Weyl group of type B_l, we also have a similar formula for τ^* and $\tilde{\tau}$ (Cf. [4]). Especially, they are factorized analogously.

Looking over these results, we can observe a curious phenomenon.

Observation 4.3 ([4]). *Let W be the Weyl group of type A_l or B_l ($l > 2$), then for two irreducible character χ and χ' of W, the two invariants $\tau^*(\chi;t)$ and $\tau^*(\chi';t)$ coincide if and only if χ and χ' belong to the same two-sided cell.*

The arguments used in the theory of two-sided cells is sometimes very deep, based on IC-complexes, D-modules, and so on. Sometimes it is very ad hoc. Therefore it is surprising that such an easy invariant like τ^* characterizes two-sided cells. However such a heavenly simple picture is not true in general. Even if we replace τ^* by $\tilde{\tau}$ in the Observation 4.3, we can not extend the simple picture Observation 4.3 for general W. Therefore we want to understand the deviation itself.

4.2. Refined two-sided cells.

For the above purpose, we introduce a certain refinement of the two-sided cells.

Definition 4.4 (Iwahori-Hecke algebra). For an irreducible Weyl group W, let S be the set of simple reflections. Let $\{q_s\}_{s\in S}$ be a set of indeterminates such that $q_s = q_{s'}$ if and only if s and s' are W-conjugate and such that the different q_s's are algebraically independent. Put $R := \mathbb{Z}[q_s^{1/2}, q_s^{-1/2}]_{s\in S}$. Let K be the fractional field $\mathrm{Frac}(R)$ of R, and $H(W)_R = \bigoplus_{w\in W} RT_w$ the free R-module generated by the formal basis parametrized by W. Then an associative R-algebra structure of $H(W)_R$ is given by

$$\begin{aligned}&T_wT_{w'} = T_{ww'} \text{ if } l(w)+l(w') = l(ww'), \text{ and}\\ &(T_s+1)(T_s-q_s) = 0 \text{ for } s\in S.\end{aligned}$$

Now consider the specialization

$$R \overset{\mathrm{mod}\,p}{\longrightarrow} \mathrm{Frac}(R\otimes \mathbb{Z}/p\mathbb{Z}), \tag{5}$$

and consider the modular representation theory of $H(W)_K := H(W)_R\otimes K$ with respect to this specialization; in particular, consider the blocks of $H(W)_K^\vee$. Here $H(W)_K^\vee$ is the set of irreducible characters of $H(W)_K$, or equivalently, the set of irreducible representations modulo isomorphism.

Recall that $H(W)_K^\vee$ can be identified with $W^\vee$:

$$H(W)_K^\vee = W^\vee.$$

Definition 4.5 (The equivalence relation $\underset{*}{\sim}$). For two characters $\chi, \chi' \in H(W)_K^\vee = W^\vee$, and for a prime number p, define equivalence relations $\underset{p}{\sim}$ and $\underset{*}{\sim}$ by

1. $\chi \underset{p}{\sim} \chi'$ if and only if χ and χ' belong to the same block of $H(W)_K^\vee$ with respect to the specialization (5).
2. $\chi \underset{*}{\sim} \chi'$ if and only if there exist prime numbers $p_1,\dots,p_n$ and irreducible characters $\chi_1,\dots,\chi_{n-1}$ such that

$$\chi \underset{p_1}{\sim} \chi_1 \underset{p_2}{\sim} \cdots \underset{p_{n-1}}{\sim} \chi_{n-1} \underset{p_n}{\sim} \chi'.$$

Theorem 4.6 ([3], [4] § 4.2). *Assume that W is of type A_l, D_l or E_l. Then $\chi \underset{*}{\sim} \chi'$ if and only if χ and χ' belong to the same two-sided cell. In general, the implication 'only if' holds.*

In the sequel, let us call *refined two-sided cells*, the equivalence classes in $W^\vee$ with respect to the equivalence relation $\underset{*}{\sim}$.

4.3. Invariants $\tilde{\tau}$ and I_W.

We have calculated $\tilde{\tau}$'s and the Kawanaka invariants systematically using *Mathematica* and MAPLE in [4] and [5]. Looking over the results of the calculation, we have made some observations. For the statement of our observation, we need the following definition.

Definition 4.7 (Modified exceptional representations). Put

$$W^{\vee}_{\text{ex.m}} = \begin{cases} \{\chi \in W^{\vee} \mid \dim \chi = 2\}, & \text{if } W = W(G_2), \\ \{\chi \in W^{\vee} \mid \dim \chi = 512\}, & \text{if } W = W(E_7), \\ \{\chi \in W^{\vee} \mid \dim \chi = 4096\}, & \text{if } W = W(E_8), \\ \phi, & \text{otherwise.} \end{cases}$$

Observation 4.8. 1. *An irreducible character $\chi \in W^{\vee} \setminus W^{\vee}_{\text{ex.m}}$ forms a refined two-sided cell by itself if and only if*

$$\tilde{\tau}(\chi; q, y) = q^n \prod_{i=1}^{l} \frac{1 + yq^{c_i}}{1 - q^{h_i}}, \quad l = \dim \mathfrak{h}$$

with some integers n, $\{c_i\}_{1 \le i \le l}$ and $\{h_i\}_{1 \le i \le l}$, which are uniquely determined by χ.

2. *If $\chi \in W^{\vee}$ forms a refined two-sided cell by itself, then*

$$I_W(\chi; q) = \prod_{i=1}^{l} \frac{1 + q^{h_i}}{1 - q^{h_i}}, \quad l = \dim \mathfrak{h}$$

with the same integers $\{h_i\}_i$ as above.

Note that, in the A_l or B_l-case, every irreducible character $\chi \in W^{\vee}$ forms a refined two-sided cell by itself and $\tilde{\tau}$ is factorized as above. See Example 4.2.

In this way, we observed that the invariants $\tilde{\tau}$ and the Kawanaka invariants I are related to the two-sided cells and the refined two-sided cells.

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Finite Crystals and Paths

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Dedicated to Professor Tetsuji Miwa on his fiftieth birthday

Abstract.

We consider a category of finite crystals of a quantum affine algebra whose objects are not necessarily perfect, and set of paths, semi-infinite tensor product of an object of this category with a certain boundary condition. It is shown that the set of paths is isomorphic to a direct sum of infinitely many, in general, crystals of integrable highest weight modules. We present examples from $C_n^{(1)}$ and $A_{n-1}^{(1)}$, in which the direct sum becomes a tensor product as suggested from the Bethe Ansatz.

§1. Introduction

The main object of this note is to define a set of paths from a *finite* crystal B, which is not necessarily perfect, and investigate its crystal structure. The set of paths $\mathcal{P}(\boldsymbol{p}, B)$ is, roughly speaking, a subset of the semi-infinite tensor product $\cdots \otimes B \otimes \cdots \otimes B \otimes B$ with a certain boundary condition related to $\boldsymbol{p}$. If B is perfect, it is known [KMN1] that as crystals, $\mathcal{P}(\boldsymbol{p}, B)$ is isomorphic to the crystal base $B(\lambda)$ of an integrable highest weight module with highest weight λ of the quantum affine algebra $U_q(\mathfrak{g})$. While trying to generalize this notion, we had two examples in mind: (a) $\mathfrak{g} = C_n^{(1)}, B = B^{1,l}$ (l : odd); (b) $\mathfrak{g} = A_{n-1}^{(1)}, B = B^{1,l} \otimes B^{1,m}$ ($l \geq m$). For this parametrization of finite crystals, we refer to [HKOTY]. $B^{1,l}$ stands for the crystal base of an irreducible finite-dimensional $U'_q(\mathfrak{g})$-module. In case (a) (resp. (b)) this finite-dimensional module is isomorphic to $V_{l\overline{\Lambda}_1} \oplus V_{(l-2)\overline{\Lambda}_1} \oplus \cdots \oplus V_{\overline{\Lambda}_1}$ (resp. $V_{l\overline{\Lambda}_1}$) as $U_q(\overline{\mathfrak{g}})$-module, where V_λ is the irreducible finite-dimensional module with highest weight λ. In both cases B is not perfect except when $l = m$ in (b). For precise treatment see section 4.1 for (a) and 4.2 for (b).

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Let us consider case (a) first. When $l = 1$ it has already been known [DJKMO] that the formal character of $\mathcal{P}(\boldsymbol{p}, B^{1,1})$ for suitable $\boldsymbol{p}$ agrees with that of the irreducible highest weight $A^{(1)}_{2n-1}$-module with fundamental highest weight Λ_i regarded as $C^{(1)}_n$-module via the natural embedding $C^{(1)}_n \hookrightarrow A^{(1)}_{2n-1}$. On the other hand, the Bethe Ansatz suggests [Ku] that $\mathcal{P}(\boldsymbol{p}, B^{1,l})$ is equal to $B(\lambda) \otimes \mathcal{P}(\boldsymbol{p}^\dagger, B^{1,1})$ for suitable $\boldsymbol{p}, \boldsymbol{p}^\dagger$ and a level $\frac{l-1}{2}$ dominant integral weight λ at the level of the Virasoro central charge.

Let us turn to case (b). In [HKMW] the $U'_q(\widehat{sl}_2)$-invariant integrable vertex model with alternating spins is considered. To translate the physical states and operators of this model into the language of representation theory of the quantum affine algebra $U_q(\widehat{sl}_2)$, they considered a set of paths with alternating spins and showed that it is isomorphic to the tensor product of crystals with highest weights. Another appearance of example (b) can be found in [HKKOTY]. They considered the inductive limit of $(B^{1,l})^{\otimes L_1} \otimes (B^{1,m})^{\otimes L_2}$ when $L_1, L_2 \to \infty, L_1 \equiv r_1, L_1 + L_2 \equiv r_2$ (mod n), and showed that there is a weight preserving bijection between the limit and $B((l-m)\Lambda_{r_1}) \otimes B(m\Lambda_{r_2})$. Since there is a natural isomorphism $B^{1,l} \otimes B^{1,m} \simeq B^{1,m} \otimes B^{1,l}$, the above result claims that $\mathcal{P}(\boldsymbol{p}, B^{1,l} \otimes B^{1,m})$ for suitable $\boldsymbol{p}$ is bijective to $B((l-m)\Lambda_{r_1}) \otimes B(m\Lambda_{r_2})$ with weight preserved. These results are consistent with the earlier Bethe ansatz calculations on "mixed spin" models [AM, DMN].

If we forget about the degree of the null root δ from weight, this phenomenon is explained using the theory of crystals with core [KK]. (See also [HKMW] section 3.2.) Let $\{B_k\}_{k \geq 1}$ be a coherent family of perfect crystals and B'_m be a perfect crystal of level m. Fix l such that $l \geq m$ and take dominant integral weights λ and μ of level $l-m$ and m. Then there exists an isomorphism of crystals:

$$\begin{aligned} B(\lambda) \otimes B(\mu) &\simeq B(\sigma\lambda) \otimes B_{l-m} \otimes B(\sigma'\mu) \otimes B'_m \\ &\simeq B(\sigma\lambda) \otimes B(\sigma\sigma'\mu) \otimes (B_l \otimes B'_m), \end{aligned}$$

where σ and σ' are automorphisms on the weight lattice P related to $\{B_k\}_{k\geq 1}$ and B'_m. Iterating this isomorphism infinitely many times, we can expect

$$\mathcal{P}(\boldsymbol{p}^{(\lambda,\mu)}, B_l \otimes B'_m) \simeq B(\lambda) \otimes B(\mu)$$

as $P/\mathbf{Z}\delta$-weighted crystals with suitable $\boldsymbol{p}^{(\lambda,\mu)}$.

In both cases (a),(b) we have illustrated above, what we expect is an isomorphism of P-weighted crystals of the following type:

$$\mathcal{P}(\boldsymbol{p}, B) \simeq B(\lambda) \otimes \mathcal{P}(\boldsymbol{p}^\dagger, B^\dagger) \tag{1.1}$$

and we shall prove it in this paper. First we examine the crystal structure of $\mathcal{P}(\boldsymbol{p}, B)$ and show it is isomorphic to a direct sum of $B(\lambda)$'s. Therefore, the structure of $\mathcal{P}(\boldsymbol{p}, B)$ is completely determined by the set of highest weight elements. In the LHS of (1.1), such set $\mathcal{P}(\boldsymbol{p}, B)_0$ is easy to describe, and in the RHS, this set turns out to be the set of restricted paths $\mathcal{P}^{(\lambda)}(\boldsymbol{p}^\dagger, B^\dagger)$, which is familiar to the people in solvable lattice models. Thus establishing a weight preserving bijection between $\mathcal{P}(\boldsymbol{p}, B)_0$ and $\mathcal{P}^{(\lambda)}(\boldsymbol{p}^\dagger, B^\dagger)$ directly, we can show (1.1).

§2. Crystals

2.1. Notation

Let $\mathfrak{g}$ be an affine Lie algebra. We denote by I the index set of its Dynkin diagram. Note that 0 is included in I. Let α_i, h_i, Λ_i $(i \in I)$ be the simple roots, simple coroots, fundamental weights for $\mathfrak{g}$. Let $\delta = \sum_{i\in I} a_i\alpha_i$ denote the standard null root, and $c = \sum_{i\in I} a_i^\vee h_i$ the canonical central element, where $a_i, a_i^\vee$ are positive integers as in [Kac]. We assume $a_0 = 1$. Let $P = \bigoplus_{i\in I} \mathbf{Z}\Lambda_i \oplus \mathbf{Z}\delta$ be the weight lattice, and set $P^+ = \sum_{i\in I} \mathbf{Z}_{\geq 0}\Lambda_i \oplus \mathbf{Z}\delta$.

Let $U_q(\mathfrak{g})$ be the quantum affine algebra associated to $\mathfrak{g}$. For the definition of $U_q(\mathfrak{g})$ and its Hopf algebra structure, see e.g. section 2.1 of [KMN1]. For $J \subset I$ we denote by $U_q(\mathfrak{g}_J)$ the subalgebra of $U_q(\mathfrak{g})$ generated by e_i, f_i, t_i $(i \in J)$. In particular, $U_q(\mathfrak{g}_{I\setminus\{0\}})$ is identified with the quantized enveloping algebra for the simple Lie algebra whose Dynkin diagram is obtained by deleting the 0 vertex from that of $\mathfrak{g}$. We also consider the quantum affine algebra without derivation $U'_q(\mathfrak{g})$. As its weight lattice, the classical weight lattice $P_{cl} = P/\mathbf{Z}\delta$ is needed. We canonically identify P_{cl} with $\bigoplus_{i\in I} \mathbf{Z}\Lambda_i \subset P$. For the precise treatment, see section 3.1 of [KMN1]. We further define the following subsets of P_{cl}: $P_{cl}^0 = \{\lambda \in P_{cl} \mid \langle\lambda, c\rangle = 0\}$, $P_{cl}^+ = \{\lambda \in P_{cl} \mid \langle\lambda, h_i\rangle \geq 0 \text{ for any } i\}$, $(P_{cl}^+)_l = \{\lambda \in P_{cl}^+ \mid \langle\lambda, c\rangle = l\}$. For $\lambda, \mu \in P_{cl}$, we write $\lambda \geq \mu$ to mean $\lambda - \mu \in P_{cl}^+$.

2.2. Crystals and crystal bases

We summarize necessary facts in crystal theory. Our basic references are [K1], [KMN1] and [AK].

A crystal B is a set B with the maps

$$\tilde{e}_i, \tilde{f}_i : B \sqcup \{0\} \longrightarrow B \sqcup \{0\}$$

satisfying the following properties:

$\tilde{e}_i 0 = \tilde{f}_i 0 = 0,$

for any b and i, there exists $n > 0$ such that $\tilde{e}_i^n b = \tilde{f}_i^n b = 0$,
for $b, b' \in B$ and $i \in I$, $\tilde{f}_i b = b'$ if and only if $b = \tilde{e}_i b'$.

If we want to emphasize I, B is called an I-crystal. A crystal can be regarded as a colored oriented graph by defining

$$b \xrightarrow{i} b' \quad \Longleftrightarrow \quad \tilde{f}_i b = b'.$$

For an element b of B we set

$$\varepsilon_i(b) = \max\{n \in \mathbf{Z}_{\geq 0} \mid \tilde{e}_i^n b \neq 0\}, \quad \varphi_i(b) = \max\{n \in \mathbf{Z}_{\geq 0} \mid \tilde{f}_i^n b \neq 0\}.$$

We also define a P-weighted crystal. It is a crystal with the weight decomposition $B = \sqcup_{\lambda \in P} B_\lambda$ such that

$$\tilde{e}_i B_\lambda \subset B_{\lambda+\alpha_i} \sqcup \{0\}, \quad \tilde{f}_i B_\lambda \subset B_{\lambda-\alpha_i} \sqcup \{0\}, \tag{2.1}$$

$$\langle h_i, wt\, b\rangle = \varphi_i(b) - \varepsilon_i(b). \tag{2.2}$$

Set

$$\varepsilon(b) = \sum_{i \in I} \varepsilon_i(b)\Lambda_i, \quad \varphi(b) = \sum_{i \in I} \varphi_i(b)\Lambda_i.$$

Then (2.2) is equivalent to $\varphi(b) - \varepsilon(b) = wt\, b$. P_{cl}-weighted crystal is defined similarly.

For two weighted crystals B_1 and B_2, the tensor product $B_1 \otimes B_2$ is defined.

$$B_1 \otimes B_2 = \{b_1 \otimes b_2 \mid b_1 \in B_1, b_2 \in B_2\}.$$

The actions of $\tilde{e}_i$ and $\tilde{f}_i$ are defined by

$$\tilde{e}_i(b_1 \otimes b_2) = \begin{cases} \tilde{e}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) \geq \varepsilon_i(b_2) \\ b_1 \otimes \tilde{e}_i b_2 & \text{if } \varphi_i(b_1) < \varepsilon_i(b_2), \end{cases} \tag{2.3}$$

$$\tilde{f}_i(b_1 \otimes b_2) = \begin{cases} \tilde{f}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2) \\ b_1 \otimes \tilde{f}_i b_2 & \text{if } \varphi_i(b_1) \leq \varepsilon_i(b_2). \end{cases} \tag{2.4}$$

Here $0 \otimes b$ and $b \otimes 0$ are understood to be 0. ε_i, φ_i and wt are given by

$$\varepsilon_i(b_1 \otimes b_2) = \max(\varepsilon_i(b_1), \varepsilon_i(b_1) + \varepsilon_i(b_2) - \varphi_i(b_1)), \tag{2.5}$$

$$\varphi_i(b_1 \otimes b_2) = \max(\varphi_i(b_2), \varphi_i(b_1) + \varphi_i(b_2) - \varepsilon_i(b_2)), \tag{2.6}$$

$$wt\,(b_1 \otimes b_2) = wt\, b_1 + wt\, b_2. \tag{2.7}$$

Definition 2.1 ([AK]). *We say a P (or P_{cl})-weighted crystal is regular, if for any $i, j \in I$ ($i \neq j$), B regarded as $\{i,j\}$-crystal is a disjoint union of crystals of integrable highest weight modules over $U_q(\mathfrak{g}_{\{i,j\}})$.*

Crystal is a notion obtained by abstracting the properties of crystal bases [K1]. Let $V(\lambda)$ be the integrable highest weight $U_q(\mathfrak{g})$-module with highest weight $\lambda \in P^+$ and highest weight vector u_λ. It is shown in [K1] that $V(\lambda)$ has a crystal base $(L(\lambda), B(\lambda))$. We regard u_λ as an element of $B(\lambda)$ as well. $B(\lambda)$ is a regular P-weighted crystal. A finite-dimensional integrable $U'_q(\mathfrak{g})$-module V does not necessarily have a crystal base. If V has a crystal base (L, B), then B is a regular P^0_{cl}-weighted crystal with finitely many elements.

Let W be the affine Weyl group associated to $\mathfrak{g}$, and s_i be the simple reflection corresponding to α_i. W acts on any regular crystal B [K2]. The action is given by

$$S_{s_i} b = \begin{cases} \tilde{f}_i^{\langle h_i, wt\, b\rangle} b & \text{if } \langle h_i, wt\, b\rangle \geq 0 \\ \tilde{e}_i^{-\langle h_i, wt\, b\rangle} b & \text{if } \langle h_i, wt\, b\rangle \leq 0. \end{cases}$$

An element b of B is called *i-extremal* if $\tilde{e}_i b = 0$ or $\tilde{f}_i b = 0$. b is called *extremal* if $S_w b$ is i-extremal for any $w \in W$ and $i \in I$.

Definition 2.2 ([AK] Definition 1.7). *Let B be a regular P^0_{cl}-weighted crystal with finitely many elements. We say B is simple if it satisfies*

(1) *There exists $\lambda \in P^0_{cl}$ such that the weights of B are in the convex hull of $W\lambda$.*
(2) $\sharp B_\lambda = 1$.
(3) *The weight of any extremal element is in $W\lambda$.*

Remark 2.3. *Let B be a regular P^0_{cl}-weighted crystal with finitely many elements. We have the following criterion for simplicity. Let $B(\lambda)$ denote the crystal base of the irreducible highest weight $U_q(\mathfrak{g}_{I\setminus\{0\}})$-module with highest weight λ. If B decomposes into $B \simeq \bigoplus_{j=0}^m B(\lambda_j)$ as $U_q(\mathfrak{g}_{I\setminus\{0\}})$-crystal and λ_j satisfies*

(1) $\lambda_j \in \lambda_0 + \sum_{i\neq 0} \mathbf{Z}_{\leq 0}\alpha_i$ *and* $\lambda_j \neq \lambda_0$ *for any* $j \neq 0$,
(2) *The highest weight element of $B(\lambda_j)$ is not 0-extremal for any $j \neq 0$,*

then B is simple.

Proposition 2.4 ([AK] Lemma 1.9 & 1.10). *Simple crystals have the following properties.*

(1) *A simple crystal is connected.*
(2) *The tensor product of simple crystals is also simple.*

2.3. Category $\mathcal{C}^{fin}$

Let B be a regular P_{cl}^0-weighted crystal with finitely many elements. For B we introduce the *level* of B by

$$\mathit{lev}\, B = \min\{\langle c, \varepsilon(b)\rangle \mid b \in B\} \in \mathbf{Z}_{\geq 0}.$$

Note that $\langle c, \varepsilon(b)\rangle = \langle c, \varphi(b)\rangle$ for any $b \in B$. We also set $B_{\min} = \{b \in B \mid \langle c, \varepsilon(b)\rangle = \mathit{lev}\, B\}$ and call an element of $B_{\min}$ *minimal.*

Definition 2.5. *We denote by $\mathcal{C}^{fin}(\mathfrak{g})$ (or simply $\mathcal{C}^{fin}$) the category of crystal B satisfying the following conditions:*

(1) *B is a crystal base of a finite-dimensional $U_q'(\mathfrak{g})$-module.*
(2) *B is simple.*
(3) *For any $\lambda \in P_{cl}^+$ such that $\langle c, \lambda\rangle \geq \mathit{lev}\, B$, there exists $b \in B$ satisfying $\varepsilon(b) \leq \lambda$. It is also true for φ.*

We call an object of $\mathcal{C}^{fin}(\mathfrak{g})$ *finite crystal.*

Remark 2.6. (i) *Condition (1) implies B is a regular P_{cl}^0-weighted crystal with finitely many elements.*

(ii) *Set $l = \mathit{lev}\, B$. Condition (3) implies that the maps ε and φ from $B_{\min}$ to $(P_{cl}^+)_l$ are surjective. (cf. (4.6.5) in* [KMN1]*)*
(iii) *Practically, one has to check condition (3) only for $\lambda \in P_{cl}^+$ such that there is no $i \in I$ satisfying $\lambda - \Lambda_i \geq 0$ and $\langle c, \lambda - \Lambda_i\rangle \geq \mathit{lev}\, B$. In particular, if $a_i^\vee = 1$ for any $i \in I$ ($\mathfrak{g} = A_n^{(1)}, C_n^{(1)}$), the surjectivity of ε and φ assures (3).*
(iv) *The authors do not know a crystal satisfying (1) and (2), but not satisfying (3).*

Let B_1 and B_2 be two finite crystals. Definition 2.5 (1) and the existence of the universal R-matrix assures that we have a natural isomorphism of crystals.

$$B_1 \otimes B_2 \simeq B_2 \otimes B_1. \tag{2.8}$$

The following lemma is immediate.

Lemma 2.7. *Let B_1, B_2 be finite crystals.*

(1) *$\mathit{lev}(B_1 \otimes B_2) = \max(\mathit{lev}\, B_1, \mathit{lev}\, B_2)$.*
(2) *If $\mathit{lev}\, B_1 \geq \mathit{lev}\, B_2$, then $(B_1 \otimes B_2)_{\min} = \{b_1 \otimes b_2 \mid b_1 \in (B_1)_{\min}, \varphi_i(b_1) \geq \varepsilon_i(b_2)$ for any $i\}$.*
(3) *If $\mathit{lev}\, B_1 \leq \mathit{lev}\, B_2$, then $(B_1 \otimes B_2)_{\min} = \{b_1 \otimes b_2 \mid b_2 \in (B_2)_{\min}, \varphi_i(b_1) \leq \varepsilon_i(b_2)$ for any $i\}$.*

$\mathcal{C}^{fin}(\mathfrak{g})$ forms a tensor category.

Proposition 2.8. *If B_1 and B_2 are objects of $\mathcal{C}^{fin}(\mathfrak{g})$, then $B_1 \otimes B_2$ is also an object of $\mathcal{C}^{fin}(\mathfrak{g})$.*

Proof. We need to check the conditions in Definition 2.5 for $B_1 \otimes B_2$. (1) is obvious and (2) follows from Proposition 2.4 (2).

Let us prove condition (3) for ε. Set $l_1 = \mathit{lev}\, B_1, l_2 = \mathit{lev}\, B_2$. Using (2.8) if necessary, we can assume $l_1 \geq l_2$. Thus we have $\mathit{lev}\, B_1 \otimes B_2 = l_1$. For any $\lambda \in P_{cl}^+$ such that $\langle c, \lambda \rangle \geq l_1$, one can take $b_1 \in B_1$ satisfying $\varepsilon(b_1) \leq \lambda$. Since $\langle c, \varphi(b_1) \rangle \geq l_1 \geq l_2$, one can take $b_2 \in B_2$ satisfying $\varepsilon(b_2) \leq \varphi(b_1)$. In view of (2.5) one has $\varepsilon(b_1 \otimes b_2) = \varepsilon(b_1) \leq \lambda$.

For the proof of φ, repeat a similar exercise for $B_2 \otimes B_1 (\simeq B_1 \otimes B_2)$ using (2.6). ∎

2.4. Category $\mathcal{C}^h$

If an element b of a crystal B satisfies $\tilde{e}_i b = 0$ for any i, we call it a *highest weight* element.

Definition 2.9. *We denote by $\mathcal{C}^h(I, P)$ (or simply $\mathcal{C}^h$) the category of regular P-weighted crystal B satisfying the following condition:*

> *For any $b \in B$, there exist $l \geq 0, i_1, \cdots, i_l \in I$ such that $b' = \tilde{e}_{i_1} \cdots \tilde{e}_{i_l} b \in B$ is a highest weight element.*

Clearly, $\mathcal{C}^h(I, P)$ forms a tensor category.

Proposition 2.10 ([KMN1] Proposition 2.4.4). *An object of $\mathcal{C}^h(I, P)$ is isomorphic to a direct sum (disjoint union) of crystals $B(\lambda)$ ($\lambda \in P^+$) of integrable highest weight $U_q(\mathfrak{g})$-modules.*

Let O be an object of $\mathcal{C}^h(I, P)$. By O_0 we mean the set of highest weight elements in O. Suppose that $O_0 = \{b_j \mid j \in J\}$ and $\mathit{wt}\, b_j = \lambda_j \in P^+$, then from the above proposition we have an isomorphism

$$O \simeq \bigoplus_{j \in J} B(\lambda_j) \quad \text{as } P\text{-weighted crystals.}$$

J can be an infinite set.

The following lemma is standard.

Lemma 2.11. *Let B_1, B_2 be weighted crystals. Then $b_1 \otimes b_2 \in B_1 \otimes B_2$ is a highest weight element, if and only if b_1 is a highest weight element and $\tilde{e}_i^{\langle h_i, \mathit{wt}\, b_1 \rangle + 1} b_2 = 0$ for any i.*

Let O be an object of $\mathcal{C}^h(I,P)$. From this lemma we have the following bijection.

$$\begin{array}{ccc} (B(\lambda)\otimes O)_0 & \longrightarrow & O^{\leq\lambda} := \{b\in O \mid \tilde{e}_i^{\langle h_i,\lambda\rangle+1}b = 0 \text{ for any } i\} \\ u_\lambda\otimes b & \mapsto & b. \end{array}$$

Note that $O^{\leq 0} = O_0$.

§3. Paths

In this section we construct a set of paths from a finite crystal and consider its structure.

3.1. Energy function

Let us recall the energy function used in [NY] to identify the Kostka-Foulkes polynomial with a generating function over classically restricted paths.

Let B_1 and B_2 be two finite crystals. Suppose $b_1\otimes b_2\in B_1\otimes B_2$ is mapped to $\tilde{b}_2\otimes\tilde{b}_1\in B_2\otimes B_1$ under the isomorphism (2.8). A $\mathbf{Z}$-valued function H on $B_1\otimes B_2$ is called an *energy function* if for any i and $b_1\otimes b_2\in B_1\otimes B_2$ such that $\tilde{e}_i(b_1\otimes b_2)\neq 0$, it satisfies

$$\begin{aligned} H(\tilde{e}_i(b_1\otimes b_2)) &= H(b_1\otimes b_2)+1 && \text{if } i=0, \varphi_0(b_1)\geq\varepsilon_0(b_2),\\ &&& \quad\varphi_0(\tilde{b}_2)\geq\varepsilon_0(\tilde{b}_1),\\ &= H(b_1\otimes b_2)-1 && \text{if } i=0, \varphi_0(b_1)<\varepsilon_0(b_2),\\ &&& \quad\varphi_0(\tilde{b}_2)<\varepsilon_0(\tilde{b}_1),\\ &= H(b_1\otimes b_2) && \text{otherwise.} \end{aligned} \tag{3.1}$$

When we want to emphasize $B_1\otimes B_2$, we write $H_{B_1B_2}$ for H. The existence of such function can be shown in a similar manner to section 4 of [KMN1] based on the existence of *combinatorial R-matrix.* The energy function is unique up to additive constant, since $B_1\otimes B_2$ is connected. By definition, $H_{B_1B_2}(b_1\otimes b_2) = H_{B_2B_1}(\tilde{b}_2\otimes\tilde{b}_1)$.

If the tensor product $B_1\otimes B_2$ is homogeneous, i.e., $B_1 = B_2$, we have $\tilde{b}_2 = b_1, \tilde{b}_1 = b_2$. Thus (3.1) is rewritten as

$$\begin{aligned} H(\tilde{e}_i(b_1\otimes b_2)) &= H(b_1\otimes b_2)+1 && \text{if } i=0, \varphi_0(b_1)\geq\varepsilon_0(b_2),\\ &= H(b_1\otimes b_2)-1 && \text{if } i=0, \varphi_0(b_1)<\varepsilon_0(b_2),\\ &= H(b_1\otimes b_2) && \text{if } i\neq 0. \end{aligned} \tag{3.2}$$

The following proposition, which is shown by case-by-case checking, reduces the energy function of a tensor product to that of each component.

Proposition 3.1. *Set* $B = B_1 \otimes B_2$, *then*

$$\begin{aligned} H_{BB}((b_1 \otimes b_2) \otimes (b_1' \otimes b_2')) &= H_{B_1B_2}(b_1 \otimes b_2) + H_{B_1B_1}(\tilde{b}_1 \otimes b_1') \\ &\quad + H_{B_2B_2}(b_2 \otimes \tilde{b}_2') + H_{B_1B_2}(b_1' \otimes b_2'). \end{aligned}$$

Here $\tilde{b}_1, \tilde{b}_2'$ *are defined as*

$$\begin{array}{rcl} B_1 \otimes B_2 & \simeq & B_2 \otimes B_1 \\ b_1 \otimes b_2 & \mapsto & \tilde{b}_2 \otimes \tilde{b}_1 \\ b_1' \otimes b_2' & \mapsto & \tilde{b}_2' \otimes \tilde{b}_1'. \end{array}$$

Remark 3.2. *Decomposition of the energy function is not unique. For instance, the following also gives such decomposition.*

$$\begin{aligned} H_{BB}((b_1 \otimes b_2) \otimes (b_1' \otimes b_2')) &= H_{B_2B_1}(b_2 \otimes b_1') + H_{B_1B_1}(b_1 \otimes \check{b}_1') \\ &\quad + H_{B_2B_2}(\check{b}_2 \otimes b_2') + H_{B_1B_2}(\check{b}_1' \otimes \check{b}_2), \end{aligned}$$

where

$$\begin{array}{rcl} B_2 \otimes B_1 & \simeq & B_1 \otimes B_2 \\ b_2 \otimes b_1' & \mapsto & \check{b}_1' \otimes \check{b}_2. \end{array}$$

3.2. Set of paths $\mathcal{P}(\mathbf{p}, B)$

We shall define a set of paths from any finite crystal in $\mathcal{C}^{fin}$ imitating the construction in section 4 of [KMN1] from a perfect crystal.

Definition 3.3. *An element* $\mathbf{p} = \cdots \otimes \mathbf{b}_j \otimes \cdots \otimes \mathbf{b}_2 \otimes \mathbf{b}_1$ *of the semi-infinite tensor product of* B *is called a reference path if it satisfies* $\mathbf{b}_j \in B_{\min}$ *and* $\varphi(\mathbf{b}_{j+1}) = \varepsilon(\mathbf{b}_j)$ *for any* $j \geq 1$.

Definition 3.4. *Fix a reference path* $\mathbf{p} = \cdots \otimes \mathbf{b}_j \otimes \cdots \otimes \mathbf{b}_2 \otimes \mathbf{b}_1$. *We define a set of paths* $\mathcal{P}(\mathbf{p}, B)$ *by*

$$\mathcal{P}(\mathbf{p}, B) = \{p = \cdots \otimes b_j \otimes \cdots \otimes b_2 \otimes b_1 \mid b_j \in B, b_k = \mathbf{b}_k \text{ for } k \gg 1\}.$$

An element of $\mathcal{P}(\mathbf{p}, B)$ is called a *path*. For convenience we denote b_k by $p(k)$ and $\cdots \otimes b_{k+2} \otimes b_{k+1}$ by $p[k]$ for $p = \cdots \otimes b_j \otimes \cdots \otimes b_2 \otimes b_1$.

Definition 3.5. *For a path* $p \in \mathcal{P}(\mathbf{p}, B)$, *set*

$$\begin{aligned} E(p) &= \sum_{j=1}^{\infty} j(H(p(j+1) \otimes p(j)) - H(\mathbf{p}(j+1) \otimes \mathbf{p}(j))), \\ W(p) &= \varphi(\mathbf{p}(1)) + \sum_{j=1}^{\infty} (wt\, p(j) - wt\, \mathbf{p}(j)) - E(p)\delta. \end{aligned}$$

$E(p)$ *and* $W(p)$ *are called the energy and weight of* p.

We distinguish $W(p) \in P$ from $wt\, p = \varphi(\mathbf{p}(1)) + \sum_{j=1}^{\infty}(wt\, p(j) - wt\, \mathbf{p}(j)) \in P_{cl}$.

Remark 3.6. (i) *If B is perfect, the set of reference paths is bijective to $(P_{cl}^{+})_l$, where $l = \mathrm{lev} B$. For $\lambda \in (P_{cl}^{+})_l$ take a unique $\mathbf{b}_1 \in B_{\min}$ such that $\varphi(\mathbf{b}_1) = \lambda$. The condition $\varphi(\mathbf{b}_{j+1}) = \varepsilon(\mathbf{b}_j)$ fixes $\mathbf{p} = \cdots \otimes \mathbf{b}_j \otimes \cdots \otimes \mathbf{b}_1$ uniquely.*

(ii) *In* [KMN1] *$\mathbf{p}$ is called a ground state path, since $E(p) \geq E(\mathbf{p})$ for any $p \in \mathcal{P}(\mathbf{p}, B)$. But if B is not perfect, it is no longer true in general.*

The following theorem is essential for our consideration below.

Theorem 3.7. *Assume rank $\mathfrak{g} > 2$. Then $\mathcal{P}(\mathbf{p}, B)$ is an object of $\mathcal{C}^h$.*

Proof. Assume $\tilde{e}_i p = \cdots \otimes \tilde{e}_i b_j \otimes \cdots \otimes b_1 \neq 0$. Note that $E(\tilde{e}_i p) = E(p) - \delta_{i0}$ and $wt\, \tilde{e}_i b_j = wt\, b_j + \alpha_i - \delta_{i0}\delta \in P_{cl}$. By Definition 3.5 it is immediate to see $\mathcal{P}(\mathbf{p}, B)$ is a P-weighted crystal. Thus one has to check the following:

(i) If for any $i, j \in I$ $(i \neq j)$, $\mathcal{P}(\mathbf{p}, B)$ regarded as $\{i, j\}$-crystal is a disjoint union of crystals of integrable highest weight modules over $U_q(\mathfrak{g}_{\{i,j\}})$.

(ii) For any $p \in \mathcal{P}(\mathbf{p}, B)$, there exist $l \geq 0, i_1, \cdots, i_l \in I$ such that $p' = \tilde{e}_{i_1} \cdots \tilde{e}_{i_l} p \in \mathcal{P}(\mathbf{p}, B)$ is a highest weight element.

We prove (i) first. For $p \in \mathcal{P}(\mathbf{p}, B)$ take m, m' such that $p(k) = \mathbf{p}(k)$ for $k > m$ and $m' \gg m$. Note that if $\tilde{f}_{i_N} \cdots \tilde{f}_{i_1} p[m] = p[m'] \otimes b'_{m'} \otimes \cdots \otimes b'_{m+1}$, then $b'_k = \mathbf{p}(k)$ for $k > m+N$. From the assumption, $U_q(\mathfrak{g}_{\{i,j\}})$ is the quantized enveloping algebra associated to a finite-dimensional Lie algebra. Since B is regular, the connected component containing $p[m]$, as $\{i, j\}$-crystal, can be considered to be in $B(\varphi(p[m'])) \otimes B^{\otimes(m'-m)}$. Since $\varepsilon(p[m]) = 0$, we can regard $p[m]$ as highest weight element of some $\{i, j\}$-crystal B_0 which is isomorphic to the crystal of an integrable highest weight $U_q(\mathfrak{g}_{\{i,j\}})$-module. Hence p is contained in a component of the $\{i, j\}$-crystal $B_0 \otimes B^{\otimes m}$, which is a disjoint union of crystals of integrable highest weight $U_q(\mathfrak{g}_{\{i,j\}})$-modules.

To prove (ii) for $p = \cdots \otimes b_k \otimes \cdots \otimes b_1 \in \mathcal{P}(\mathbf{p}, B)$, we take the minimum integer m such that $p' = p[m]$ is a highest weight element. We prove by induction on m.

First let us show that there exist $l \geq 0, i_1, \cdots, i_l \in I$ such that $\tilde{e}_{i_1} \cdots \tilde{e}_{i_l}(p' \otimes b_m)$ is a highest weight element. The proof is essentially the same as a part of that of Theorem 4.4.1 in [KMN1]. Nevertheless we repeat it for the sake of self-containedness. Suppose that there does not exist such $i_1, \cdots, i_l$. Then there exists an infinite sequence $\{i_\nu\}$ in I such that

$$\tilde{e}_{i_k} \cdots \tilde{e}_{i_1}(p' \otimes b_m) \neq 0.$$

Since $\tilde{e}_{i_k} \cdots \tilde{e}_{i_1}(p' \otimes b_m) = p' \otimes \tilde{e}_{i_k} \cdots \tilde{e}_{i_1} b_m$ and B is a finite set, there exists $b^{(1)} \in B$ and $j_1, \cdots, j_l$ such that

$$p' \otimes b^{(1)} = \tilde{e}_{j_l} \cdots \tilde{e}_{j_1}(p' \otimes b^{(1)}).$$

Hence setting $b^{(\nu+1)} = \tilde{e}_{j_\nu} b^{(\nu)}$, we have

$$\tilde{e}_{j_\nu}(p' \otimes b^{(\nu)}) = p' \otimes b^{(\nu+1)} \text{ and } b^{(l+1)} = b^{(1)}.$$

In view of (2.6) we have $\varphi_i(p') \geq \varphi_i(b_{m+1})$ for any i. Thus by (2.3) we have $\varepsilon_{j_\nu}(b^{(\nu)}) > \varphi_{j_\nu}(p') \geq \varphi_{j_\nu}(b')$ for some $b' \in B$. Hence we have

$$\tilde{e}_{j_\nu}(b' \otimes b^{(\nu)}) = b' \otimes b^{(\nu+1)}.$$

Therefore, from (3.2), we have

$$H(b' \otimes b^{(\nu+1)}) = H(b' \otimes b^{(\nu)}) - \delta_{i_\nu 0}.$$

Hence $H(b' \otimes b^{(l+1)}) = H(b' \otimes b^{(1)}) - \sharp\{\nu \mid j_\nu = 0\}$, which implies there is no ν such that $j_\nu = 0$. On the other hand, $\sum_\nu \alpha_{j_\nu} = 0 \bmod \mathbf{Z}\delta$ and hence $\sum_\nu \alpha_{j_\nu}$ is a positive multiple of δ, which contradicts $0 \notin \{j_1, \cdots, j_l\}$.

Now set $p'' = p' \otimes b_m (= p[m-1]), b'' = b_{m-1} \otimes \cdots \otimes b_1$. Notice that for any $i \in I$ satisfying $\tilde{e}_i p'' \neq 0$, there exists $k \geq 1$ such that

$$\tilde{e}_i^k (p'' \otimes b'') = \tilde{e}_i p'' \otimes \tilde{e}_i^{k-1} b''.$$

Therefore there exist $l \geq 0, (i_1, k_1), \cdots, (i_l, k_l) \in I \times \mathbf{Z}_{>0}$ such that

$$\tilde{e}_{i_1}^{k_1} \cdots \tilde{e}_{i_l}^{k_l} p = \tilde{e}_{i_1} \cdots \tilde{e}_{i_l} p'' \otimes \tilde{e}_{i_1}^{k_1 - 1} \cdots \tilde{e}_{i_l}^{k_l - 1} b''$$

and $\tilde{e}_{i_1} \cdots \tilde{e}_{i_l} p''$ is a highest weight element. Now we can use the induction assumption and complete the proof. ∎

Remark 3.8. *As seen in the proof, the theorem does not require the condition $\mathbf{b}_j \in B_{\min}$ for the reference path $\boldsymbol{p} = \cdots \otimes \mathbf{b}_j \otimes \cdots \otimes \mathbf{b}_1$.*

The following proposition describes the set of highest weight elements in $\mathcal{P}(\boldsymbol{p}, B)$.

Proposition 3.9.

$$\mathcal{P}(\mathbf{p},B)_0=\{p\in\mathcal{P}(\mathbf{p},B)\mid p(j)\in B_{\min},\varphi(p(j+1))=\varepsilon(p(j))\ \mathit{for}\ \forall j\}.$$

Proof. Assume $p=\cdots\otimes b_j\otimes\cdots\otimes b_1$ is a highest weight element. We prove the following by induction on m in decreasing order.

(i) $b_m\in B_{\min},\varphi(b_{m+1})=\varepsilon(b_m)$
(ii) $\varphi(p[m-1])=\varphi(b_m)$

These conditions are satisfied for sufficiently large m. From (ii) for $m+1$ we have $\varphi(p[m])=\varphi(b_{m+1})$. From Lemma 2.11 we see that $p[m]$ is a highest weight element and $\varepsilon(b_m)\le wt\,p[m]=\varphi(p[m])=\varphi(b_{m+1})$. Combining this with (i) for $m+1$, we can conclude (i) for m. For (ii) use (2.6). ∎

As seen in the proof, we obtain

Corollary 3.10. *If $p\in\mathcal{P}(\mathbf{p},B)_0$, then $wt\,p[j]=\varphi(p(j+1))$.*

3.3. Restricted paths

When B is perfect the set of *restricted* paths was defined in [DJO] and shown to be bijective to $(B(\lambda)\otimes B(\mu))_0$ for some $\lambda,\mu\in P_{cl}^+$. Here we shall consider restricted paths for any finite crystal B.

For $\lambda\in P_{cl}^+$ and $p\in\mathcal{P}(\mathbf{p},B)$, we introduce a sequence of weights $\{\lambda_j(p)\}_{j\ge0}$ by

$$\lambda_j(p)=\lambda+\varphi(p(j+1))\ \text{for}\ j\gg1,$$
$$\lambda_{j-1}(p)=\lambda_j(p)+wt\,p(j).$$

Notice that this definition is well-defined by virtue of the property of the reference path. In fact, $\lambda_j(p)=\lambda+wt\,p[j]$.

Definition 3.11. *For $\lambda\in P_{cl}^+$ we define a subset $\mathcal{P}^{(\lambda)}(\mathbf{p},B)$ of $\mathcal{P}(\mathbf{p},B)$ by*

$$\mathcal{P}^{(\lambda)}(\mathbf{p},B)=\{p\in\mathcal{P}(\mathbf{p},B)\mid \tilde{e}_i^{\langle h_i,\lambda_j(p)\rangle+1}p(j)=0\ \mathit{for}\ \forall i,j\}.$$

An element of $\mathcal{P}^{(\lambda)}(\mathbf{p},B)$ is called a *restricted* path.

Proposition 3.12. *For $\lambda\in P_{cl}^+$ we have*

$$\mathcal{P}(\mathbf{p},B)^{\le\lambda}=\mathcal{P}^{(\lambda)}(\mathbf{p},B).$$

Proof. Assume $p=\cdots\otimes b_j\otimes\cdots\otimes b_1\in\mathcal{P}(\mathbf{p},B)^{\le\lambda}$, which is equivalent to saying $u_\lambda\otimes p$ is a highest weight element. So is $u_\lambda\otimes p[j]\otimes b_j$ by Lemma 2.11. Using this lemma again we get $\varepsilon(b_j)\le wt\,(u_\lambda\otimes p[j])=\lambda_j(p)$.

To show the inverse inclusion, assume $p = \cdots \otimes b_j \otimes \cdots \otimes b_1 \in \mathcal{P}^{(\lambda)}(\mathbf{p}, B)$. We prove $\varepsilon(p[j]) \leq \lambda$ by induction on j in decreasing order. We know $\varepsilon(p[j]) = 0$ for sufficiently large j. Supposing $\varepsilon(p[j]) \leq \lambda$ we immediately obtain $\varepsilon(p[j] \otimes b_j) \leq \lambda$ from (2.5) and the condition $\varepsilon(b_j) \leq \lambda_j(p)$. ∎

As seen in the proof we have $\lambda_j(p) \in P_{cl}^+$ and its level is $\langle c, \lambda \rangle + \mathit{lev}\, B$.

Combining the results in section 2.4, Theorem 3.7 and Proposition 3.12, we obtain

Theorem 3.13. *Let $\mathcal{P}(\mathbf{p}, B)$ and $\mathcal{P}(\mathbf{p}^\dagger, B^\dagger)$ be two sets of paths. If for certain $\lambda \in P_{cl}^+$, there exists a bijection*

$$\begin{array}{lccc} (3.3) & \mathcal{P}(\mathbf{p}, B)_0 & \longrightarrow & \mathcal{P}^{(\lambda)}(\mathbf{p}^\dagger, B^\dagger) \\ & p & \mapsto & p^\dagger \end{array}$$

such that $W(p) = \lambda + W(p^\dagger)$, then we have an isomorphism of P-weighted crystals

$$\mathcal{P}(\mathbf{p}, B) \simeq B(\lambda) \otimes \mathcal{P}(\mathbf{p}^\dagger, B^\dagger).$$

They are isomorphic to a direct sum of crystals of integrable highest weight $U_q(\mathfrak{g})$-modules, and their highest weight elements are parametrized by (3.3).

§4. Examples

We shall give two examples to which we can apply Theorem 3.13 efficiently.

4.1. Example 1

We present a useful proposition first. Similar to $O^{\leq \lambda}$ we define $B^{\leq \lambda}$ for a finite crystal B and $\lambda \in P_{cl}^+$ by

$$B^{\leq \lambda} = \{b \in B \mid \tilde{e}_i^{\langle h_i, \lambda \rangle + 1} b = 0 \text{ for any } i\}.$$

Note that if $\mathit{lev}\, B = l$, then $B_{\min} = \bigsqcup_{\lambda \in (P_{cl}^+)_l} B^{\leq \lambda}$.

Proposition 4.1. *Let B and $B^\dagger$ be finite crystals such that $\mathit{lev}\, B \geq \mathit{lev}\, B^\dagger$, and $\mathbf{p} = \cdots \otimes \mathbf{b}_j \otimes \cdots \otimes \mathbf{b}_1$ be a reference path for B. Suppose there exists a map $t : B_{\min} \longrightarrow B^\dagger$ satisfying the following conditions:*

1. *For any $\mu \in (P_{cl}^+)_l$ ($l = \mathit{lev}\, B$), $t|_{B^{\leq \mu}}$ is a bijection onto $(B^\dagger)^{\leq \mu}$.*
2. *$\mathit{wt}\, t(b) = \mathit{wt}\, b$ for any $b \in B_{\min}$.*
3. *$H_{B^\dagger B^\dagger}(t(b_1) \otimes t(b_2)) = H_{BB}(b_1 \otimes b_2)$ up to global additive constant for any $(b_1, b_2) \in B_{\min}^2$ such that $\varphi(b_1) = \varepsilon(b_2)$.*

(4) $\mathbf{p}^\dagger = \cdots \otimes t(\mathbf{b}_j) \otimes \cdots \otimes t(\mathbf{b}_1)$ *is a reference path for* $B^\dagger$.
Then setting $\lambda = \varphi(\mathbf{b}_1) - \varphi(t(\mathbf{b}_1))$, *we have*

$$\mathcal{P}(\mathbf{p}, B) \simeq B(\lambda) \otimes \mathcal{P}(\mathbf{p}^\dagger, B^\dagger).$$

Proof. Consider the following map.

$$\begin{array}{rcl} \mathcal{P}(\mathbf{p}, B)_0 & \longrightarrow & \mathcal{P}(\mathbf{p}^\dagger, B^\dagger) \\ p = \cdots \otimes b_j \otimes \cdots \otimes b_1 & \mapsto & p^\dagger = \cdots \otimes t(b_j) \otimes \cdots \otimes t(b_1) \end{array}$$

From Theorem 3.13 it suffices to show that this map is a bijection onto $\mathcal{P}^{(\lambda)}(\mathbf{p}^\dagger, B^\dagger)$ such that $W(p) = \lambda + W(p^\dagger)$. Preservation of weight is immediate. To show the bijectivity one has to notice that $wt\, p^\dagger[j] - wt\, p[j]$ does not depend on j. Thus one has $wt\, p^\dagger[j] - wt\, p[j] = wt\, p^\dagger - wt\, p = -\lambda$, and hence

$$\lambda_j(p^\dagger) = \lambda + wt\, p^\dagger[j] = wt\, p[j] = \varphi(b_{j+1}) = \varepsilon(b_j).$$

Note that $p \in \mathcal{P}(\mathbf{p}, B)_0$ (cf. Proposition 3.9 & Corollary 3.10). In view of (1) this equality concludes the bijectivity. ∎

We now consider the $C_n^{(1)}$ case. For an odd positive integer l, consider a finite crystal $B^{1,l}$ given by

$$B^{1,l} = \left\{ (x_1, \dots, x_n, \overline{x}_n, \dots, \overline{x}_1) \left| \begin{array}{l} x_i, \overline{x}_i \in \mathbf{Z}_{\geq 0} \,\forall\, i = 1, \cdots, n \\ \sum_{i=1}^n (x_i + \overline{x}_i) \in \{l, l-2, \dots, 1\} \end{array} \right. \right\}.$$

The crystal structure of $B^{1,l}$ is given by

$$\begin{aligned}
\tilde{e}_0 b &= \begin{cases} (x_1 - 2, x_2, \dots, \overline{x}_2, \overline{x}_1) & \text{if } x_1 \geq \overline{x}_1 + 2, \\ (x_1 - 1, x_2, \dots, \overline{x}_2, \overline{x}_1 + 1) & \text{if } x_1 = \overline{x}_1 + 1, \\ (x_1, x_2, \dots, \overline{x}_2, \overline{x}_1 + 2) & \text{if } x_1 \leq \overline{x}_1, \end{cases} \\
\tilde{e}_i b &= \begin{cases} (x_1, \dots, x_i + 1, x_{i+1} - 1, \dots, \overline{x}_1) & \text{if } x_{i+1} > \overline{x}_{i+1}, \\ (x_1, \dots, \overline{x}_{i+1} + 1, \overline{x}_i - 1, \dots, \overline{x}_1) & \text{if } x_{i+1} \leq \overline{x}_{i+1}, \end{cases} \\
\tilde{e}_n b &= (x_1, \dots, x_n + 1, \overline{x}_n - 1, \dots, \overline{x}_1), \\
\tilde{f}_0 b &= \begin{cases} (x_1 + 2, x_2, \dots, \overline{x}_2, \overline{x}_1) & \text{if } x_1 \geq \overline{x}_1, \\ (x_1 + 1, x_2, \dots, \overline{x}_2, \overline{x}_1 - 1) & \text{if } x_1 = \overline{x}_1 - 1 \\ (x_1, x_2, \dots, \overline{x}_2, \overline{x}_1 - 2) & \text{if } x_1 \leq \overline{x}_1 - 2, \end{cases} \\
\tilde{f}_i b &= \begin{cases} (x_1, \dots, x_i - 1, x_{i+1} + 1, \dots, \overline{x}_1) & \text{if } x_{i+1} \geq \overline{x}_{i+1}, \\ (x_1, \dots, \overline{x}_{i+1} - 1, \overline{x}_i + 1, \dots, \overline{x}_1) & \text{if } x_{i+1} < \overline{x}_{i+1}, \end{cases} \\
\tilde{f}_n b &= (x_1, \dots, x_n - 1, \overline{x}_n + 1, \dots, \overline{x}_1),
\end{aligned}$$

where $b = (x_1, \ldots, x_n, \overline{x}_n, \ldots, \overline{x}_1)$ and $i = 1, \ldots, n-1$. If some component becomes negative upon application, it should be understood as 0. The values of ε_i, φ_i read

$$\begin{array}{ll} \varepsilon_0(b) = \frac{l-s(b)}{2} + (x_1 - \overline{x}_1)_+, & \varphi_0(b) = \frac{l-s(b)}{2} + (\overline{x}_1 - x_1)_+, \\ \varepsilon_i(b) = \overline{x}_i + (x_{i+1} - \overline{x}_{i+1})_+, & \varphi_i(b) = x_i + (\overline{x}_{i+1} - x_{i+1})_+, \\ \varepsilon_n(b) = \overline{x}_n, & \varphi_n(b) = x_n. \end{array}$$

Here $s(b) = \sum_{i=1}^n (x_i + \overline{x}_i), (x)_+ = \max(x, 0)$ and $i = 1, \cdots, n-1$. $B^{1,l}$ is a level $\frac{l+1}{2}$ non-perfect crystal. Now for a fixed l set $B = B^{1,l}$. The minimal elements of B are grouped as $B_{\min} = \bigsqcup_{\mu \in (P_{cl}^+)_{\frac{l+1}{2}}} B^{\leq \mu}$, where for $\mu = \mu_0 \Lambda_0 + \cdots + \mu_n \Lambda_n$. The set $B^{\leq \mu}$ is given by

$$\begin{aligned} B^{\leq \mu} &= \{b_k^\mu \mid \mu_{k-1} > 0, 1 \leq k \leq n\} \cup \{b_{\overline{k}}^\mu \mid \mu_k > 0, 1 \leq k \leq n\}, \\ b_k^\mu &= (\mu_1, \ldots, \mu_{k-1} - 1, \mu_k + 1, \ldots, \mu_n, \mu_n, \ldots, \mu_{k-1} - 1, \ldots, \mu_1), \\ b_{\overline{k}}^\mu &= (\mu_1, \ldots, \mu_k - 1, \ldots, \mu_n, \mu_n, \ldots, \mu_1). \end{aligned}$$

Next consider $B^\dagger = B^{1,1}$ by taking l to be 1. Setting

$$b_k^\dagger = (x_i = \delta_{ik}, \overline{x}_i = 0), \quad b_{\overline{k}}^\dagger = (x_i = 0, \overline{x}_i = \delta_{ik})$$

for $1 \leq k \leq n$, one has

$$(B^\dagger)^{\leq \mu} = \{b_k^\dagger \mid \mu_{k-1} > 0, 1 \leq k \leq n\} \cup \{b_{\overline{k}}^\dagger \mid \mu_k > 0, 1 \leq k \leq n\}$$

for μ as above. Define the map $t : B_{\min} \to B^\dagger$ by

$$t|_{B^{\leq \mu}} : b_k^\mu \mapsto b_k^\dagger \qquad \text{for } k \in \{1, \ldots, n, \overline{n}, \ldots, \overline{1}\}.$$

We are to show that this t satisfies the conditions (1) – (4) in Proposition 4.1. For our purpose fix a dominant integral weight $\lambda \in (P_{cl}^+)_{\frac{l-1}{2}}$ and define $\boldsymbol{p} = \cdots \otimes \boldsymbol{b}_j \otimes \cdots \otimes \boldsymbol{b}_1$ by

$$\boldsymbol{b}_j = \begin{cases} b_{\overline{i}}^{\lambda + \Lambda_i} & \text{if } j \equiv i \,(\mathrm{mod}\ 2n) \text{ for some } i \,(1 \leq i \leq n), \\ b_i^{\lambda + \Lambda_{i-1}} & \text{if } j \equiv 1 - i \,(\mathrm{mod}\ 2n) \text{ for some } i \,(1 \leq i \leq n). \end{cases}$$

Note that $\varepsilon(b_{\overline{i}}^{\lambda+\Lambda_i}) = \varphi(b_i^{\lambda+\Lambda_{i-1}}) = \lambda + \Lambda_i, \varepsilon(b_i^{\lambda+\Lambda_{i-1}}) = \varphi(b_{\overline{i}}^{\lambda+\Lambda_i}) = \lambda + \Lambda_{i-1}$. $\boldsymbol{p}$ becomes a reference path. Let us check (1) – (4) in Proposition 4.1. (1),(2) and (4) are straightforward. To check (3) one can use the formula for H_{BB} in [KKM] section 5.7. (In [KKM] our non-perfect case is not considered. However, the formula itself is valid. Since the formula in [KKM] contains some misprints, we rewrite it below.)

$$H_{B^{1,l}B^{1,l}}(b \otimes b') = \max_{1 \leq j \leq n} (\theta_j(b \otimes b'), \theta'_j(b \otimes b'), \eta_j(b \otimes b'), \eta'_j(b \otimes b')),$$

$$\begin{aligned}
\theta_j(b \otimes b') &= \sum_{k=1}^{j-1}(\overline{x}_k - \overline{x}'_k) + \frac{1}{2}(s(b') - s(b)),\\
\theta'_j(b \otimes b') &= \sum_{k=1}^{j-1}(x'_k - x_k) + \frac{1}{2}(s(b) - s(b')),\\
\eta_j(b \otimes b') &= \sum_{k=1}^{j-1}(\overline{x}_k - \overline{x}'_k) + (\overline{x}_j - x_j) + \frac{1}{2}(s(b') - s(b)),\\
\eta'_j(b \otimes b') &= \sum_{k=1}^{j-1}(x'_k - x_k) + (x'_j - \overline{x}'_j) + \frac{1}{2}(s(b) - s(b')),
\end{aligned}$$

where $b = (x_1, \dots, x_n, \overline{x}_n, \dots, \overline{x}_1), b' = (x'_1, \dots, x'_n, \overline{x}'_n, \dots, \overline{x}'_1)$.

Therefore, the isomorphism in Proposition 4.1 holds with notations above.

4.2. Example 2

We consider the $A^{(1)}_{n-1}$ case. Let $B^{1,l}$ be the crystal base of the symmetric tensor representation of $U'_q(A^{(1)}_{n-1})$ of degree l. As a set it reads

$$B^{1,l} = \{(a_0, a_1, \cdots, a_{n-1}) \mid a_i \in \mathbf{Z}_{\geq 0}, \sum_{i=0}^{n-1} a_i = l\}.$$

For convenience we extend the definition of a_i to $i \in \mathbf{Z}$ by setting $a_{i+n} = a_i$ and use a simpler notation (a_i) for $(a_0, a_1, \cdots, a_{n-1})$. For instance, (a_{i-1}) means $(a_{n-1}, a_0, \cdots, a_{n-2})$. The actions of $\tilde{e}_r, \tilde{f}_r$ $(r = 0, \cdots, n-1)$ are given by

$$\tilde{e}_r(a_i) = (a_i - \delta^{(n)}_{i,r} + \delta^{(n)}_{i,r-1}), \quad \tilde{f}_r(a_i) = (a_i + \delta^{(n)}_{i,r} - \delta^{(n)}_{i,r-1}).$$

Here $\delta^{(n)}_{ij} = 1$ $(i \equiv j \bmod n)$, $= 0$ (otherwise). If some component becomes negative upon application, it should be understood as 0. The values of ε, φ read as follows.

$$\varepsilon((a_i)) = \sum_{i=0}^{n-1} a_i \Lambda_i, \quad \varphi((a_i)) = \sum_{i=0}^{n-1} a_{i-1} \Lambda_i.$$

Thus $lev B^{1,l} = l$ and all elements are minimal. We introduce a $\mathbf{Z}$-linear automorphism σ on P_{cl} by $\sigma \Lambda_i = \Lambda_{i-1}$ $(\Lambda_{-1} = \Lambda_{n-1})$.

Now consider the finite crystal $B = B^{1,l} \otimes B^{1,m}$ $(l \geq m)$ and set $B^\dagger = B^{1,m}$. From Lemma 2.7 (1) the level of B is l. Fix two dominant integral weights $\lambda = \sum_{i=0}^{n-1} \lambda_i \Lambda_i \in (P_{cl}^+)_{l-m}, \mu = \sum_{i=0}^{n-1} \mu_i \Lambda_i \in (P_{cl}^+)_m$. From (λ, μ) we define a path

$$\boldsymbol{p}^{(\lambda,\mu)}(j) = (\lambda_{i+j} + \mu_{i+2j}) \otimes (\mu_{i+2j-1}) \in B.$$

From Lemma 2.7 (2) we see $\boldsymbol{p}^{(\lambda,\mu)}(j) \in B_{\min}$ and by (2.5),(2.6) we obtain $\varepsilon(\boldsymbol{p}^{(\lambda,\mu)}(j)) = \sigma^j \lambda + \sigma^{2j} \mu = \varphi(\boldsymbol{p}^{(\lambda,\mu)}(j+1))$. Therefore $\boldsymbol{p}^{(\lambda,\mu)}$ is a reference path.

We would like to show

$$\text{(4.1)} \qquad \mathcal{P}(\boldsymbol{p}^{(\lambda,\mu)}, B) \simeq B(\lambda) \otimes \mathcal{P}(\boldsymbol{p}^{(\mu)}, B^\dagger) \quad \text{as } P\text{-weighted crystals}$$

with $\boldsymbol{p}^{(\mu)}(j) = (\mu_{i+j})$. To do this, consider the following map

$$\text{(4.2)} \qquad \begin{array}{rcl} \mathcal{P}(\boldsymbol{p}^{(\lambda,\mu)}, B)_0 & \longrightarrow & \mathcal{P}(\boldsymbol{p}^{(\mu)}, B^\dagger) \\ p & \mapsto & p^\dagger \end{array}$$

given by $p^\dagger(j) = (b_{i-j+1}^{(j)})$ for $p(j) = (a_i^{(j)}) \otimes (b_i^{(j)})$. Note that $\boldsymbol{p}^{(\lambda,\mu)}$ is sent to $\boldsymbol{p}^{(\mu)}$ under this map. By Theorem 3.13 it suffices to check the following items:

(i) The map (4.2) is a bijection onto $\mathcal{P}^{(\lambda)}(\boldsymbol{p}^{(\mu)}, B^\dagger)$.
(ii) $wt\, p - wt\, p^\dagger = \lambda$.
(iii) $E(p) = E(p^\dagger)$.

Since $p \in \mathcal{P}(\boldsymbol{p}^{(\lambda,\mu)}, B)_0$, one obtains (cf. Lemma 2.7 (2), Proposition 3.9)

$$\text{(4.3)} \qquad \varphi_i((a_i^{(j)})) = a_{i-1}^{(j)} \geq b_i^{(j)} = \varepsilon_i((b_i^{(j)}))$$

$$\text{(4.4)} \qquad \varphi_i(p(j)) = a_{i-1}^{(j)} + b_{i-1}^{(j)} - b_i^{(j)} = a_i^{(j-1)} = \varepsilon_i(p(j-1))$$

for any i, j. Taking sufficiently large J and using (4.4), one has

$$\begin{aligned} wt\, p^\dagger[j] &= \sum_i b_{i-J+1}^{(J)} \Lambda_i + \sum_{k=j+1}^{J} \sum_i (b_{i-k}^{(k)} - b_{i-k+1}^{(k)}) \Lambda_i \\ &= \sum_i (b_{i-J+1}^{(J)} - a_{i-J}^{(J)} + a_{i-j}^{(j)}) \Lambda_i \\ &= \sum_i a_{i-j}^{(j)} \Lambda_i - \lambda. \end{aligned}$$

Thus the condition $\varepsilon(p^\dagger(j)) \leq \lambda_j(p^\dagger)$ is equivalent to saying $b_{i-j+1}^{(j)} \leq a_{i-j}^{(j)}$ for any i, which is guaranteed by (4.3). This proves (i). For (ii) one only has to notice that $wt\, p[j] = \varphi(p(j+1)) = \sum_i a_i^{(j)} \Lambda_i$.

In order to prove (iii), we set

$$E_L^{diff} = \sum_{j=1}^{L} j\{H_{BB}(((a_i^{(j+1)})\otimes(b_i^{(j+1)}))\otimes((a_i^{(j)})\otimes(b_i^{(j)}))) \\ -H_{B^\dagger B^\dagger}((b_{i-(j+1)+1}^{(j+1)})\otimes(b_{i-j+1}^{(j)}))\}.$$

We can assume $(a_i^{(j)})\otimes(b_i^{(j)})\in B_{\min}$ for $1\le j\le L+1$. Under such assumption the isomorphism $B^{1,l}\otimes B^{1,m}\simeq B^{1,m}\otimes B^{1,l}$ sends $(a_i)\otimes(b_i)$ to $(b_{i+1})\otimes(a_i-b_{i+1}+b_i)$ [NY]. Thus, from Proposition 3.1 we have

$$H_{BB}(((a_i)\otimes(b_i))\otimes((a'_i)\otimes(b'_i))) = b_0+a'_0+b'_0+H_{B^\dagger B^\dagger}((b_i)\otimes(b'_{i+1})).$$

Let us recall the following formula for $H_{B^{1,m}B^{1,m}}$ (cf. [KKM] section 5.1).

$$H_{B^{1,m}B^{1,m}}((b_i)\otimes(b'_i)) = \max_{0\le j\le n-1}\left(\sum_{k=0}^{j-1}(b'_k-b_k)+b'_j\right)$$

From this one gets

$$H_{B^\dagger B^\dagger}((b_i^{(j+1)})\otimes(b_{i+1}^{(j)})) - H_{B^\dagger B^\dagger}((b_{i-j}^{(j+1)})\otimes(b_{i-j+1}^{(j)})) \\ = \sum_{k=1}^{j}(b_{k-j-1}^{(j+1)}-b_{k-j}^{(j)}).$$

Using above facts and (4.4) one obtains

$$E_L^{diff} = \sum_{j=1}^{L}\sum_{k=0}^{j-1} a_{-k}^{(L)} + L\sum_{k=0}^{L} b_{-k}^{(L+1)}.$$

This completes (iii). We have finished proving (4.1). It is also known [KMN2] that $\mathcal{P}(\boldsymbol{p}^{(\mu)},B^{1,m})\simeq B(\mu)$. Therefore we have

$$\mathcal{P}(\boldsymbol{p}^{(\lambda,\mu)},B^{1,l}\otimes B^{1,m})\simeq B(\lambda)\otimes B(\mu)\quad\text{as } P\text{-weighted crystals.}$$

The multi-component version is straightforward. Consider the finite crystal $B^{1,l_1}\otimes\cdots\otimes B^{1,l_s}$ ($l_1\ge\cdots\ge l_s\ge l_{s+1}=0$). For $\lambda^{(i)}\in(P_{cl}^+)_{l_i-l_{i+1}}$ ($1\le i\le s$) we define a reference path $\boldsymbol{p}^{(\lambda_1,\cdots,\lambda_s)}$ by

$$\text{the } k\text{-th tensor component of } \boldsymbol{p}^{(\lambda_1,\cdots,\lambda_s)}(j) \\ = (\lambda_{i+kj-k+1}^{(k)}+\lambda_{i+(k+1)j-k+1}^{(k+1)}+\cdots+\lambda_{i+sj-k+1}^{(s)}).$$

Then we have

$$\mathcal{P}(p^{(\lambda_1,\cdots,\lambda_s)}, B^{1,l_1} \otimes \cdots \otimes B^{1,l_s}) \simeq B(\lambda_1) \otimes \cdots \otimes B(\lambda_s).$$

The proof will be given elsewhere.

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Minor Summation Formulas of Pfaffians, Survey and A New Identity

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Abstract.

In this paper we treat the minor summation formulas of pfaffians presented in [IW1] and derive several basic formulas concerning pfaffians from it. We also present a pfaffian version of the Plücker relation and give a new pfaffian identity as its application.

Chapter I. Introduction

In this short note we treat the minor summation formulas of pfaffians presented in [IW1] and derive several basic formulas concerning pfaffians. We also present a pfaffian version of the Plücker relations and give a new pfaffian identity as its application in Chapter III.

The minor summation formula we call here is an identity which involves pfaffians for a weighted sum of minors of a given matrix. The first appearance of this kind of minor sum is when one tries to count the number of the totally symmetric plane partitions (see [O1]). Once we establish the minor summation formula full in general, one gets various applications (see, e.g., [IOW], [KO], [O2]). Indeed, for example, using the minor summation formula we obtained quite a number of generalizations of the Littlewood formulas concerning various generating functions of the Schur polynomials (see [IW2,3,4]).

Though the notion of pfaffians is less familiar than that of determinants it is also known by a square root of the determinant of a skew

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symmetric matrix. We recall now a more combinatorial definition of pfaffians. Let $\mathfrak{S}_n$ be the symmetric group on the set of the letters $1, 2, \dots, n$ and, for each permutation $\sigma \in \mathfrak{S}_n$, let $\operatorname{sgn}\sigma$ stand for $(-1)^{\ell(\sigma)}$, the sign of σ, where $\ell(\sigma)$ is the number of inversions of σ.

Let $n = 2r$ be even. Let H be the subgroup of $\mathfrak{S}_n$ generated by the elements $(2i-1, 2i)$ for $1 \le i \le r$ and $(2i-1, 2i+1)(2i, 2i+2)$ for $1 \le i < r$. We set a subset $\mathfrak{F}_n$ of $\mathfrak{S}_n$ to be

$$\mathfrak{F}_n = \left\{ \sigma = (\sigma_1, \dots, \sigma_n) \in \mathfrak{S}_n \,\middle|\, \begin{array}{l} \sigma_{2i-1} < \sigma_{2i} \ (1 \le i \le r) \\ \sigma_{2i-1} < \sigma_{2i+1} \ (1 \le i \le r-1) \end{array} \right\}.$$

An element of $\mathfrak{F}_n$ is called a *perfect* matching or a 1-*factor*. For each $\pi \in \mathfrak{S}_n$, $H\pi \cap \mathfrak{F}_n$ has a unique element σ. Let $B = (b_{ij})_{1 \le i,j \le n}$ be an n by n skew-symmetric matrix with entries b_{ij} in a commutative ring. The *pfaffian* of B is then defined as follows:

$$\operatorname{pf}(B) = \sum_{\sigma \in \mathfrak{F}_n} \operatorname{sgn}\sigma \, b_{\sigma(1)\sigma(2)} \cdots b_{\sigma(n-1)\sigma(n)}. \tag{1.1}$$

Chapter II. Pfaffian Identities

Let us denote by $\mathbb{N}$ the set of nonnegative integers, and by $\mathbb{Z}$ the set of integers. Let $[n]$ denote the subset $\{1, 2, \dots, n\}$ of $\mathbb{N}$ for a positive integer n.

Let n, M and N be positive integers such that $n \le M, N$ and let T be any M by N matrix. For n-element subsets $I = \{i_1 < \cdots < i_n\} \subseteq [M]$ and $J = \{j_1 < \cdots < j_n\} \subseteq [N]$ of row and column indices, let $T^I_J = T^{i_1 \cdots i_n}_{j_1 \cdots j_n}$ denote the sub-matrix of T obtained by picking up the rows and columns indexed by I and J. In the case that $n = M$ and I contains all row indices, we omit $I = [M]$ from the above expression and simply write $T_J = T^I_J$. Similarly we write T^I for T^I_J if $n = N$ and $J = [N]$.

Let B be an arbitrary N by N skew symmetric matrix; that is, $B = (b_{ij})$ satisfies $b_{ij} = -b_{ji}$. In Theorem 1 of the paper [IW1], we obtained a formula concerning a certain summation of minors which we call the minor summation formula of pfaffians:

Theorem 2.1. *Let $n \le N$ and assume n is even. Let $T = (t_{ij})_{1 \le i \le n, 1 \le j \le N}$ be any n by N matrix, and let $B = (b_{ij})_{1 \le i,j \le N}$ be any N by N skew symmetric matrix. Then*

$$\sum_{\substack{I \subset [N] \\ \sharp I = n}} \operatorname{pf}(B^I_I) \det(T_I) = \operatorname{pf}(Q), \tag{2.1}$$

where Q is the n by n skew-symmetric matrix defined by $Q = TB\,{}^tT$, i.e.

$$Q_{ij} = \sum_{1 \le k < l \le N} b_{kl} \det(T^{ij}_{kl}), \qquad (1 \le i, j \le n). \tag{2.2}$$

We note that another proof of this minor summation formula and some other extensions using the so-called lattice path methods will be given in the forthcoming paper [IW5].

We now add on one useful formula which relates to the skew symmetric part of a general square matrix. Actually the following type of pfaffians may arise naturally when we consider the imaginary part of a Hermitian form.

Corollary 2.1. *Fix positive integers m, n such that $m \le 2n$. Let A and B be arbitrary $n \times m$ matrices, and X be an $n \times n$ symmetric matrix. (i.e. ${}^tX = X$). Let P be the skew symmetric matrix defined by $P = {}^tAXB - {}^tBXA$. Then we have*

$$\operatorname{pf}(P) = \sum_{\substack{K \subseteq [2n] \\ \sharp K = m}} \operatorname{pf}\left(\begin{pmatrix} O_n & X \\ -X & O_n \end{pmatrix}^K_K \right) \det\left(\begin{pmatrix} A \\ B \end{pmatrix}^K \right).$$

In particular, when $m = 2n$ we have

$$\operatorname{pf}(P) = \det(X) \det\left(\begin{pmatrix} A \\ B \end{pmatrix} \right).$$

Proof. Apply the above theorem to the $2n \times 2n$ skew symmetric matrix $\begin{pmatrix} O_n & X \\ -X & O_n \end{pmatrix}$ and the $2n \times m$ matrix $\begin{pmatrix} A \\ B \end{pmatrix}$. Then the elementary identity

$${}^t\begin{pmatrix} A \\ B \end{pmatrix} \begin{pmatrix} O_n & X \\ -X & O_n \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = {}^tAXB - {}^tBXA$$

immediately asserts the corollary.

As a corollary of the theorem above we have the following expansion formula (cf. [Ste], [IW1]):

Corollary 2.2. *Let A and B be m by m skew symmetric matrices. Put $n = [\frac{m}{2}]$, the integer part of $\frac{m}{2}$. Then*

$$\operatorname{pf}(A + B) = \sum_{r=0}^{n} \sum_{\substack{I \subseteq [m] \\ \sharp I = 2r}} (-1)^{|I| - r} \operatorname{pf}(A^I_I) \operatorname{pf}(B^{\overline{I}}_{\overline{I}}), \tag{2.3}$$

where we denote by $\overline{I}$ the complement of I in $[m]$ and $|I|$ is the sum of the elements of I (i.e. $|I| = \sum_{i\in I} i$).

In particular, we have the expansion formula of pfaffian with respect to any column (row): For any i, j we have

$$\delta_{ij}\,\mathrm{pf}(A) = \sum_{k=1}^{m} a_{ki}\gamma(k,j), \tag{2.4}$$

$$\delta_{ij}\,\mathrm{pf}(A) = \sum_{k=1}^{m} a_{ik}\gamma(j,k), \tag{2.5}$$

where

$$\gamma(i,j) = \begin{cases} (-1)^{i+j-1}\,\mathrm{pf}(A^{[ij]}) & \text{if } i < j, \\ 0 & \text{if } i = j, \\ (-1)^{i+j}\,\mathrm{pf}(A^{[ij]}) & \text{if } j < i. \end{cases} \tag{2.6}$$

and $A^{[ij]}$ stands for the $(m-2)$ by $(m-2)$ skew symmetric matrix which is obtained from A by removing both the i,j-th rows and i,j-th columns for $1 \le i \ne j \le m$.

We close this chapter by noting the fact that one may give a proof of the fundamental relation; $\mathrm{pf}(A)^2 = \det(A)$, for a skew symmetric matrix A without any use of a process of the "diagonalization" by employing the expansion formula above and the Lewis-Carroll formula for determinants discussing below.

Chapter III. The Lewis-Carroll formula, etc.

In this chapter we provide a Pfaffian version of Lewis-Carroll's formula and Plücker's relations. The latter relations are also treated in [DW], and in [Kn] it is called the (generalized) basic identity. First of all we recall the so-called Lewis-Carroll formula, or known as the Jacobi formula among minor determinants.

Proposition 3.1. *Let A be an n by n matrix and $\widetilde{A}$ be the matrix of its cofactors. Let $r \le n$ and $I, J \subseteq [n]$, $\sharp I = \sharp J = r$. Then*

$$\det \widetilde{A}_I^J = (-1)^{r(|I|+|J|)}(\det A)^{r-1}\,\det A_{\overline{J}}^{\overline{I}}, \tag{3.1}$$

where $\overline{I}, \overline{J} \subseteq [n]$ stand for the complementary of I, J, respectively.

Example 1. We give here a few examples of Lewis-Carroll's formula for matrices of small degree.

$$\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} - \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} = a_{11}\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}. \tag{3.2}$$

We give one more;

$$\begin{aligned} &\begin{vmatrix} a_{11} & a_{14} \\ a_{21} & a_{24} \end{vmatrix}\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{vmatrix} - \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}\begin{vmatrix} a_{11} & a_{12} & a_{14} \\ a_{31} & a_{32} & a_{34} \\ a_{41} & a_{42} & a_{44} \end{vmatrix} \\ &+ \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}\begin{vmatrix} a_{11} & a_{13} & a_{14} \\ a_{31} & a_{33} & a_{34} \\ a_{41} & a_{43} & a_{44} \end{vmatrix} = a_{11}\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix}. \end{aligned} \tag{3.3}$$

Hereafter we write A_I for A_I^I for short. We hope that it doesn't cause the reader any confusion since we only treat square matrices. Let m be an even integer and A be an m by m skew symmetric matrix. Assume that $\operatorname{pf}(A)$ is nonzero, that is, A is non-singular.

Let $\Delta(i,j) = (-1)^{i+j} \det A^{ij}$ denote the (i,j)-cofactor of A. If we multiply the both sides of (2.6) by $\operatorname{pf}(A)$ and use the fundamental relation between determinants and pfaffians: $\det A = [\operatorname{pf}(A)]^2$, we obtain

$$\sum_{i=1}^{m} a_{ij}\gamma(i,k) \operatorname{pf}(A) = \delta_{jk} \, [\operatorname{pf}(A)]^2 = \delta_{jk} \det A. \tag{3.4}$$

Comparing this with the cofactor expansion of $\det A$, we obtain the following relation between $\Delta(i,j)$ and $\gamma(i,j)$:

$$\Delta(i,j) = \gamma(i,j) \operatorname{pf}(A). \tag{3.5}$$

The following relation is considered as a pfaffian version of the Lewis-Carroll formula.

Theorem 3.1. *Let m be an even integer and A be an m by m skew symmetric matrix. Let $\widehat{A} = (\gamma(j,i))$. Then, for any $I \subseteq [m]$ such that $\sharp I = 2r$, we have*

$$\operatorname{pf}\left[(\widehat{A})_I\right] = (-1)^{|I|} \, [\operatorname{pf}(A)]^{r-1} \operatorname{pf}(A_{\overline{I}}). \tag{3.6}$$

Example 2. Taking $m = 6$, $t = 1$ and $I = \{1, 2, 3, 4\}$ in the above theorem, we see

$$\gamma(1,2)\gamma(3,4) - \gamma(1,3)\gamma(2,4) + \gamma(1,4)\gamma(2,3) = \mathrm{pf}(A)\,\mathrm{pf}(A_{\{5,6\}}).$$

Hence by definition, we see that this turns out to be

$$\begin{aligned}\mathrm{pf}(A_{\{3,4,5,6\}})\,\mathrm{pf}(A_{\{1,2,5,6\}}) - \mathrm{pf}(A_{\{2,4,5,6\}})\,\mathrm{pf}(A_{\{1,3,5,6\}})& \\ + \mathrm{pf}(A_{\{2,3,5,6\}})\,\mathrm{pf}(A_{\{1,4,5,6\}}) = \mathrm{pf}(A)\,\mathrm{pf}(A_{\{5,6\}}),&\end{aligned} \tag{3.7}$$

that is, in more familiar form we see

$$\begin{aligned}
&\mathrm{pf}\begin{pmatrix} 0 & a_{34} & a_{35} & a_{36} \\ -a_{34} & 0 & a_{45} & a_{46} \\ -a_{35} & -a_{45} & 0 & a_{56} \\ -a_{36} & -a_{46} & -a_{56} & 0 \end{pmatrix} \mathrm{pf}\begin{pmatrix} 0 & a_{12} & a_{15} & a_{16} \\ -a_{12} & 0 & a_{25} & a_{26} \\ -a_{15} & -a_{25} & 0 & a_{56} \\ -a_{16} & -a_{26} & -a_{56} & 0 \end{pmatrix} \\
-&\mathrm{pf}\begin{pmatrix} 0 & a_{24} & a_{25} & a_{26} \\ -a_{24} & 0 & a_{45} & a_{46} \\ -a_{25} & -a_{45} & 0 & a_{56} \\ -a_{26} & -a_{46} & -a_{56} & 0 \end{pmatrix} \mathrm{pf}\begin{pmatrix} 0 & a_{13} & a_{15} & a_{16} \\ -a_{13} & 0 & a_{35} & a_{36} \\ -a_{15} & -a_{35} & 0 & a_{56} \\ -a_{16} & -a_{36} & -a_{56} & 0 \end{pmatrix} \\
+&\mathrm{pf}\begin{pmatrix} 0 & a_{23} & a_{25} & a_{26} \\ -a_{23} & 0 & a_{35} & a_{36} \\ -a_{25} & -a_{35} & 0 & a_{56} \\ -a_{26} & -a_{36} & -a_{56} & 0 \end{pmatrix} \mathrm{pf}\begin{pmatrix} 0 & a_{14} & a_{15} & a_{16} \\ -a_{14} & 0 & a_{45} & a_{46} \\ -a_{15} & -a_{45} & 0 & a_{56} \\ -a_{16} & -a_{46} & -a_{56} & 0 \end{pmatrix} \\
=&\mathrm{pf}\begin{pmatrix} 0 & a_{56} \\ -a_{56} & 0 \end{pmatrix} \mathrm{pf}\begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ -a_{12} & 0 & a_{23} & a_{24} & a_{25} & a_{26} \\ -a_{13} & -a_{23} & 0 & a_{34} & a_{35} & a_{36} \\ -a_{14} & -a_{24} & -a_{34} & 0 & a_{45} & a_{46} \\ -a_{15} & -a_{25} & -a_{35} & -a_{45} & 0 & a_{56} \\ -a_{16} & -a_{26} & -a_{36} & -a_{46} & -a_{56} & 0 \end{pmatrix}.
\end{aligned}$$

We next state a pfaffian version of the Plücker relations (or known as the Grassmann-Plücker relations) for determinants which is a quadratic relations among several subpfaffians. This identity is also proved in the book [Hi] and a recent paper [DW] in the framework of an exterior algebra.

Theorem 3.2. *Suppose m, n are odd integers. Let A be an $(m+n) \times (m+n)$ skew symmetric matrices of odd degrees. Fix a sequence of integers $I = \{i_1 < i_2 < \cdots < i_m\} \subseteq [m+n]$ such that $\sharp I = m$. Denote the complement of I by $\overline{I} = \{k_1, k_2, \ldots, k_n\} \subseteq [m+n]$ which has the* *cardinality n. Then the following relation holds.*

$$\sum_{j=1}^{m}(-1)^{j-1}\operatorname{pf}(A_{I\setminus\{i_j\}})\operatorname{pf}(A_{\{i_j\}\cup\overline{I}}) = \sum_{j=1}^{n}(-1)^{j-1}\operatorname{pf}(A_{I\cup\{k_j\}})\operatorname{pf}(A_{\overline{I}\setminus k_j}). \tag{3.8}$$

The following assertion, which is called by the basic identity in [Kn] is a special consequence of the formula above.

Corollary 3.1. *Let A be a skew symmetric matrix of degree N. Fix a subset $I = \{i_1, i_2, \dots, i_{2k}\} \subseteq [N]$ such that $\sharp I = 2k$. Take an integer l which satisfies $2k + 2l \le N$. Then*

$$\begin{aligned}&\operatorname{pf}(A_{1,2,\dots,2l})\operatorname{pf}(A_{i_1,i_2,\dots,i_{2k},1,\dots,2l})\\ &= \sum_{j=1}^{2k-1}(-1)^{j-1}\operatorname{pf}(A_{i_1,1,2,\dots,2l,i_{j+1}})\operatorname{pf}(A_{i_2,\dots,\widehat{i_{j+1}},\dots,i_{2k},1,\dots,2l}).\end{aligned} \tag{3.9}$$

The theorem stated below is proved by induction using this basic identity. Its proof will be given in the forthcoming paper [IW5].

Theorem 3.3.

$$\begin{aligned}&\operatorname{pf}\left(\frac{y_i - y_j}{a + b(x_i + x_j) + cx_ix_j}\right)_{1\le i,j\le 2n} \times \prod_{1\le i<j\le 2n}\{a + b(x_i + x_j) + cx_ix_j\}\\ &= (ac - b^2)^{\frac{n(n-1)}{2}} \sum_{\substack{I\subseteq[2n]\\ \sharp I=n}} (-1)^{|I|-\frac{n(n+1)}{2}} y_I\Delta_I(x)\Delta_{\overline{I}}(x)J_I(x)J_{\overline{I}}(x),\end{aligned}$$

where the sum runs over all n-element subset $I = \{i_1 < \cdots < i_n\}$ of $[2n]$ and $\overline{I} = \{j_1 < \cdots < j_n\}$ is the complementary subset of I in $[2n]$. Further we write

$$\Delta_I(x) = \prod_{\substack{i,j\in I\\ i<j}}(x_i - x_j),$$

$$J_I(x) = \prod_{\substack{i,j\in I\\ i<j}}\{a + b(x_i + x_j) + cx_ix_j\},$$

$$y_I = \prod_{i\in I} y_i.$$

As a corollary of this theorem we obtain the following identity in [Su2]. Indeed, if we put $a = c = 1, b = 0$ in the theorem, then we have the

Corollary 3.2.

$$\mathrm{pf}\left(\frac{y_i - y_j}{1 + x_i x_j}\right)_{1\le i,j\le 2n} \times \prod_{1\le i<j\le 2n} (1 + x_i x_j) = \sum_{\lambda,\mu} a_{\lambda+\delta_n,\mu+\delta_n}(x, y),$$

where the sums runs over pairs of partitions

$$\lambda = (\alpha_1 - 1, \cdots, \alpha_p - 1|\alpha_1, \cdots, \alpha_p), \mu = (\beta_1 - 1, \cdots, \beta_p - 1|\beta_1, \cdots, \beta_p)$$

in Frobenius notation with $\alpha_1, \beta_1 < n - 1$. Also, for α and β partitions (compositions, in general) of length n, we put

$$a_{\alpha,\beta}(x, y) = \sum_{\sigma\in\mathfrak{S}_{2n}} \epsilon(\sigma)\sigma(x_1^{\alpha_1} y_1 \cdots x_n^{\alpha_n} y_n x_{n+1}^{\beta_1} \cdots x_{2n}^{\beta_n}),$$

where $\sigma \in \mathfrak{S}_{2n}$ acts on each of two sets of variables $\{x_1, \cdots, x_n\}$ and $\{y_1, \cdots, y_n\}$ by permuting indices, and $\delta_n = (n - 1, n - 2, \cdots, 0)$.

Proof. Recall that

$$\sum_{\lambda=(\alpha_1-1,\cdots,\alpha_p-1|\alpha_1,\cdots,\alpha_p)} s_\lambda(x_1, \cdots, x_n) = \prod_{1\le i<j\le n} (1 + x_i x_j), \tag{3.10}$$

where $s_\lambda = s_\lambda(x_1, \cdots, x_n) = a_{\lambda+\delta_n}/a_{\delta_n}$ and $a_\alpha = \det(x_i^{\alpha_j})_{1\le i,j\le n}$ for a composition α. We write $a_\alpha(I) = a_\alpha(x_{i_1}, \cdots, x_{i_n})$ for $I = \{i_1 < \cdots < i_n\} \subseteq [2n]$. By the theorem and (3.10) we see

$$\begin{aligned}
&\mathrm{pf}\left(\frac{y_i - y_j}{1 + x_i x_j}\right)_{1\le i,j\le 2n} \times \prod_{1\le i<j\le 2n} (1 + x_i x_j) \\
&= \sum_{\substack{I\subseteq[2n] \\ \sharp I=n}} \sum_{\lambda,\mu} (-1)^{|I|-\frac{n(n+1)}{2}} y_I a_{\lambda+\delta_n}(I) a_{\mu+\delta_n}(\overline{I}) \\
&= \sum_{\lambda,\mu} \sum_{i_1<\cdots<i_n} \sum_{\sigma,\tau\in\mathfrak{S}_n} (-1)^{|I|-\frac{n(n+1)}{2}} \epsilon(\sigma)\epsilon(\tau) \\
&\qquad \times \sigma(x_{i_1}^{\lambda_1+n-1} y_{i_1} \cdots x_{i_n}^{\lambda_n} y_{i_n}) \tau(x_{j_1}^{\mu_1+n-1} \cdots x_{j_n}^{\mu_n}),
\end{aligned}$$

where $\overline{I} = \{j_1, \cdots, j_n\}$. Thus, the last sum is turned to be

$$= \sum_{\lambda,\mu} \sum_{\sigma,\tau \in \mathfrak{S}_{2n}} \epsilon(\sigma)\sigma(x_1^{\lambda_1+n-1} y_1 \cdots x_n^{\lambda_n} y_n x_{n+1}^{\mu_1+n-1} \cdots x_{2nn}^{\mu_n})$$

$$= \sum_{\lambda,\mu} a_{\lambda+\delta_n,\mu+\delta_n}(x,y).$$

This completes the proof of the corollary.

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Factorization of Kazhdan–Lusztig Elements for Grassmanians

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Abstract.

We show that the Kazhdan-Lusztig basis elements C_w of the Hecke algebra of the symmetric group, when $w \in S_n$ corresponds to a Schubert subvariety of a Grassmann variety, can be written as a product of factors of the form $T_i + f_j(v)$, where f_j are rational functions.

§1. Notation

In this section, we briefly list the main facts and notations related to Kazhdan–Lusztig polynomials and their parabolic analogues (see [D], [S]). We use the following notations:

$\mathcal{H}$—the Hecke algebra of the symmetric group S_n; we consider it as an algebra over the field $\mathbf{Q}(v)$ (the variable v is related to the variable q used by Kazhdan and Lusztig via $v = q^{1/2}$), and we write the quadratic relation in the form

$$(T_i - v)(T_i + v^{-1}) = 0.$$

C_w—KL basis in $\mathcal{H}$, which we define by the conditions $\overline{C_w} = C_w$, $C_w - T_w \in \oplus v\mathbf{Z}[v]T_y$.

For any subset $J \subset \{1, \dots, n-1\}$, we denote by $W_J \subset S_n$ the corresponding parabolic subgroup, and by W^J the set of minimal length representatives of cosets S_n/W_J. We also denote by M^J the $\mathcal{H}$-module induced from the one-dimensional representation of $\mathcal{H}(W_J)$, given by $T_j m_1 = -v^{-1} m_1, j \in J$. We denote $m_y = T_y m_1, y \in W^J$ the usual basis in M^J.

We define the parabolic KL basis $C^J_y, y \in W^J$ in M^J by $\overline{C^J_y} = C^J_y, C^J_y - m_y \in \oplus_{z \in W^J} v\mathbf{Z}[v] m_z$.

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Denote for brevity $C_J = C_{w_0^J}$ the element of KL basis in $\mathcal{H}$ corresponding to the element of w_0^J of maximal length in W_J. The following result is well-known (see, e.g., [S]).

Lemma 1. (i)

$$C_J = \sum_{w \in W^J} (-v)^{l(w_0^J) - l(w)} T_w.$$

(ii) *Let $w \in W$ be such that it is an element of maximal length in the coset wW_J (which is equivalent to $w = \tau w_0^J$ for some $\tau \in W^J$). Then $C_w = XC_J$ for some $X \in \oplus_{y \in W^J} \mathbf{Z}[v^{\pm 1}]T_y$.*

(iii) *Let $X \in \oplus_{y \in W^J} \mathbf{Z}[v^{\pm 1}]T_y$. Then*

$$Xm_1 = C_\tau^J \iff XC_J = C_{\tau w_0^J}.$$

Let us now consider the special case of the above situation. From now on, fix $k \le n-1$, and let $J = \{1, \dots, k-1, k+1, \dots, n-1\}$ so that $W_J = S_k \times S_{n-k}$ is a maximal parabolic subgroup in S_n. In this case, the module M^J can be described as follows:

$$M = \bigoplus_{\varepsilon \in E} \mathbf{Q}(v)\varepsilon,$$

$$T_i\varepsilon = \begin{cases} s_i\varepsilon, & (\varepsilon_i, \varepsilon_{i+1}) = (+-), \\ -v^{-1}\varepsilon, & (\varepsilon_i, \varepsilon_{i+1}) = (--) \text{ or } (++), \\ s_i\varepsilon + (v - v^{-1})\varepsilon, & (\varepsilon_i, \varepsilon_{i+1}) = (-+), \end{cases} \tag{1}$$

where E is the set of all length n sequences of pluses and minuses which contain exactly k pluses. The relation of this with the previous notation is given by $m_y \leftrightarrow y(\mathbf{1}) = T_y(\mathbf{1})$, where

$$\mathbf{1} = (\underbrace{+\cdots+}_{k}\underbrace{-\cdots-}_{n-k}). \tag{2}$$

In particular, $m_1 \leftrightarrow \mathbf{1}$.

The set of minimal length representatives W^J also admits a description in terms of Young diagrams. Namely, let λ be a Young diagram which fits inside the $k \times (n-k)$ rectangle. Define $w_\lambda \in S_n$ by

$$w_\lambda = \prod_{(i,j) \in \lambda} s_{k+j-i}, \tag{3}$$

where (i, j) stands for the box in the i-th row and j-th column, and the product is taken in the following order: we start with the lower right corner and continue along the row, until we get to the first column; then we repeat the same with the next row, and so on until we reach the upper left corner.

Example 1. Let λ be the diagram shown below, and $k = 7$ (to assist the reader, we put the numbers $k + j - i$ in the diagram).

7	8	9	10	11	12
6	7	8			
5	6	7			
4					
3					

Then $w_\lambda = s_3 \cdot s_4 \cdot s_7 s_6 s_5 \cdot s_8 s_7 s_6 \cdot s_{12} s_{11} s_{10} s_9 s_8 s_7$ (for easier reading, we separated products corresponding to different rows by $\cdot$).

The proof of the following proposition is straightforward.

Proposition 2. *The corespondence $\lambda \mapsto w_\lambda$, where w_λ is defined by (3), is a bijection between the set of all Young diagrams which fit inside the $k \times (n-k)$ rectangle and W^J.*

§2. The main theorem

As before, we fix $k \leq n-1$ and let $J = \{1, \ldots, k-1, k+1, \ldots, n-1\}$. Unless otherwise specified, we only use Young diagrams which fit inside the $k \times (n-k)$ rectangle.

For a Young diagram λ, we define the shifts $r_{i,j} \in \mathbf{Z}_{>0}, (i,j) \in \lambda$ by the following relation

$$r_{ij} = \max(r_{i,j+1}, r_{i+1,j}) + 1, \tag{4}$$

where we let $r_{ij} = 0$ if $(i,j) \notin \lambda$.

Example 2. For the diagram λ from Example 1, the shifts r_{ij} are shown below.

6	5	4	3	2	1
4	3	2			
3	2	1			
2					
1					

Next, let us define for each diagram λ an element $X_\lambda \in \mathcal{H}$ by

$$X_\lambda = \prod_{(i,j)\in\lambda} \left(T_{k+j-i} - \frac{v^{r_{ij}}}{[r_{ij}]} \right) \tag{5}$$

where, as usual, $[r] = (v^r - v^{-r})/(v - v^{-1})$, and the product is taken in the same order as in (3).

The main result of this paper is the following theorem.

Theorem 3. *Let λ be a Young diagram. Then*

$$X_\lambda \mathbf{1} = C^J_{w_\lambda}.$$

Note that by Lemma 1, this is equivalent to

$$X_\lambda C_J = C_{w_\lambda w_0^J}. \tag{6}$$

We remind the reader that the Kazhdan-Lusztig elements $C_{ww_0^J}$, where $w \in W^J$, and W_J is a maximal parabolic in S_n (they are also known as KL elements for Grassmanians), have been studied in a number of papers. A combinatorial description was given in [LS1]; it was interpreted geometrically in [Z], and in terms of representations of quantum $\mathfrak{gl}_m$ in [FKK]. However, it is unclear how these results are related with the factorization given by the theorem above. A similar factorization was given in [L] for those permutations which correspond to non-singular Schubert varieties—i.e., for those w such that, for any $v \in S_n$, the Kazhdan-Lusztig polynomial $P_{v,w}$ is either 1 or 0.

Note that one can easily check that the elements X_λ are invariant under the Kazhdan–Lusztig involution: $\overline{X_\lambda} = X_\lambda$; thus, all the difficulty is in proving that they are integral and have the right specialization at $v = 0$.

A crucial step in proving this theorem is the following proposition.

Proposition 4. *Theorem 3 holds when λ is the $k \times (n-k)$ rectangle.*

Proof. For any $w \in S_n$, choose a reduced expression $w = s_{i_\ell} \dots s_{i_1}$. Define the element $\nabla_w \in \mathcal{H}$ by

$$\nabla_w = \left(T_{i_\ell} - \frac{v^{r_\ell}}{[r_\ell]} \right) \dots (T_{i_1} - v), \tag{7}$$

where $r_1, \dots, r_\ell \in \mathbf{Z}_+$ are defined as follows: if $s_{i_{m-1}} \dots s_{i_1}(1, \dots, n) = (\dots, a, b, \dots)$ (in i_m-th, $(i_m + 1)$-st places), then $r_m = b - a$. Then $\{\nabla_w, w \in S_n\}$ is a Yang-Baxter basis of the Hecke algebra, and we have (see [DKLLST, §3]):

Lemma 5. (i) *The element ∇_w does not depend on the choice of reduced expression.*

(ii) *If w_0^J is the longest element in some parabolic subgroup $W_J \subset S_n$, then $\nabla_{w_0^J} = C_J$.*

Now, let us prove our proposition, i.e. that $X_\lambda C_J$ is a KL element for rectangular λ. In this case, w_λ is the longest element in W^J:

$$w_\lambda(\mathbf{1}) = (\underbrace{-\cdots-}_{n-k}\underbrace{+\cdots+}_{k}).$$

Let us choose the following reduced expression for the longest element w_0 in S_n: $w_0 = w_\lambda w_0^J$, where we take for w_λ the reduced expression given by (3). Then one easily sees that definition (7) in this case gives

$$\nabla_{w_0} = X_\lambda \nabla_{w_0^J}.$$

By Lemma 5, we get $C_{w_0} = X_\lambda C_J$, which is exactly the statement of the proposition. Q.E.D.

The proof in the general case is based on the following proposition. Denote

(8) $\quad O(v^m) = \{f \in \mathbf{Q}(v) | f \text{ has zero of order} \geq m \text{ at } v = 0\}.$

Proposition 6.

$$X_\lambda \mathbf{1} = w_\lambda(\mathbf{1}) + \sum_{\varepsilon \in E} O(v)\varepsilon.$$

A proof of this proposition is given in Section 3.

Now we can give a proof of the main theorem. First, one easily checks the invariance under the bar involution, since

$$\overline{T_i - \frac{v^r}{[r]}} = T_i - \frac{v^r}{[r]}.$$

Combining this with Proposition 6, we see that it remains to show that $X_\lambda C_J$ are integral, i.e. $X_\lambda C_J \in \oplus \mathbf{Z}[v^{\pm 1}]T_w$ (note that it is not true that X_λ itself is integral.) This will be done by induction.

Let λ be a Young diagram. Then we claim that any such diagram can be presented as a union $\lambda = \lambda' \sqcup \mu$, where μ is a rectangle, and λ' is again a Young diagram such that for $(i,j) \in \lambda'$, the shifts $r^{\lambda'}_{(i,j)} = r^{\lambda}_{(i,j)}$. It can be formally proved as follows: if one writes the successive widths and heights of the stairs of the diagram

$$\infty, (a_1, b_1), (a_2, b_2), \ldots (a_k, b_k), \infty$$

then there is at least one index i for which $a_i \le b_{i-1}$ and $b_i \le a_{i+1}$. In that case, the rectangle μ has the lower right corner i.

Example 3. For the diagram λ from Example 1, the sequence (a_k, b_k) is given by $\infty, (1,2), (2,2), (3,1), \infty$, and the subdiagram μ is the shaded 2×2 square, as shown below. As before, we also included the shifts r_{ij} in this diagram. The subsets I^μ, J^μ in this case are given by $I^\mu = \{6,7,8\}, J^\mu = \{6,8\}$.

6	5	4	3	2	1
4	3	2			
3	2	1			
2					
1					

Let us choose for λ the presentation $\lambda = \lambda' \sqcup \mu$, where μ is a rectangle, as above. Then $X_\lambda = X_\mu X_{\lambda'}$.

Define the subsets $I^\mu, J^\mu \subset \{1, \dots, n-1\}$ by $I^\mu = \{k'-a+1, \dots, k'+b-1\}$, $J^\mu = I^\mu \setminus \{k'\}$, where $k' = k - i + j$, (i,j)—coordinates of the UL corner of μ, a and b are numbers of rows and columns in μ respectively.

We need to show that $X_\mu X_{\lambda'} C_J \in \sum \mathbf{Z}[v^{\pm 1}] T_y$. By induction assumption, we may assume that $X_{\lambda'} C_J = C_\sigma$, where we denoted for brevity $\sigma = w_{\lambda'} w_0^J$. It is easy to show that if μ is chosen as before, then σ is the maximal length element in the coset $W_{J^\mu}\sigma$. Thus, by Lemma 1, we can write $C_\sigma = C_{J^\mu} Y$ for some integral $Y \in \mathcal{H}$. Therefore, $X_\mu X_{\lambda'} C_J = X_\mu C_{J^\mu} Y$. Since W_{I^μ} is itself a symmetric group, and W_{J_μ} is a maximal parabolic subgroup in it, we can use Proposition 4, which gives $X_\mu C_{J^\mu} = C_{I^\mu}$, and therefore, $X_\mu X_{\lambda'} C_J = C_{I^\mu} Y \in \sum \mathbf{Z}[v^{\pm 1}] T_w$. Q.E.D.

§3. Proof of regularity at $v = 0$

In this section we give the proof of Proposition 6. Before doing so, let us introduce some notation.

As before, assume that we are given n, k, λ and a collection of positive integers $r_{ij}, (i,j) \in \lambda$ (not necessarily defined as in (4)). Let $\varepsilon \in E$ be a sequence of pluses and minuses. We define the *weight* $r_\lambda(\varepsilon)$ as folllows.

Define $a(i), i = 1 \dots k$ by $a(i) = k + \lambda_i - i + 1$. Equivalently, these numbers can be characterized by saying that $w_\lambda(\mathbf{1})$ has pluses exactly at positions $a(k), \dots, a(1)$.

Define $r_\lambda(\varepsilon) = \sum_{t=1}^n r_t(\varepsilon)$, where $r_t(\varepsilon)$ is defined as follows:

(i) if $t = a(i), \varepsilon_t = -$ then $r_t(\varepsilon) = r_{i,\lambda_i} - 1$

(ii) if $a(i) > t > a(i+1), \varepsilon_t = +$ then $r_t(\varepsilon) = r_{i,j}, k + j - i = t$

(iii) otherwise, $r_t(\varepsilon) = 0$

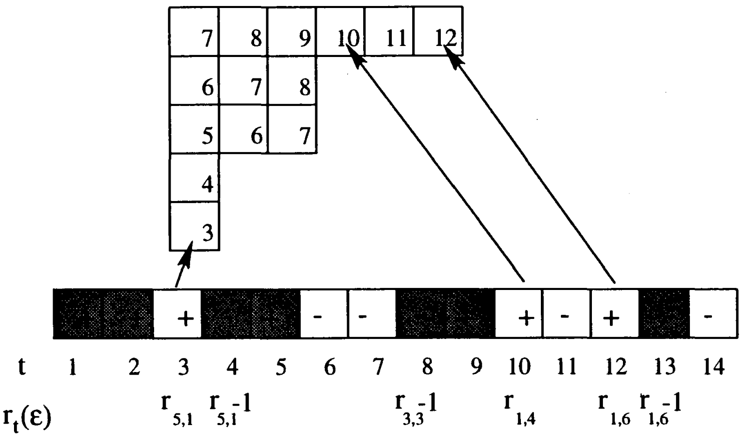
In a sense, $r_\lambda(\varepsilon)$ measures the discrepancy between ε and $w_\lambda(\mathbf{1})$. Indeed, let us denote the numbers of rows and columns in λ by i, j respectively, and let ε be such that

$$\text{(9)} \qquad \begin{aligned} \varepsilon_t &= + \text{ for } t \leq k - i, \\ \varepsilon_t &= - \text{ for } t > k + j. \end{aligned}$$

Then one easily sees that

$$\text{(10)} \qquad r_\lambda(\varepsilon) \geq 0, \qquad r_\lambda(\varepsilon) = 0 \iff \varepsilon = w_\lambda(\mathbf{1})$$

Example 4. Below we illustrate the calculation of $r_\lambda(\varepsilon)$, where λ is the diagram used in Example 1. The positions $a(i)$ are shaded (thus, the sequence of colors encodes $w_\lambda(\mathbf{1})$, with "shaded"$\leftrightarrow +$, "unshaded"$\leftrightarrow -$), and we connected unshaded pluses with the corresponding box (i,j), defined in (ii) above. For convenience of the reader, we also put the numbers $k + j - i$ (not the shifts r_{ij}!) in the diagram.



Lemma 7. *Let λ be any Young diagram inside the $k \times (n-k)$ rectangle, and let $r_{ij}, (i,j) \in \lambda$, be positive integers satisfying $r_{ij} > r_{i,j+1}, r_{ij} > r_{i+1,j}$. Define $\mathcal{L}_\lambda \subset M^J$ by*

$$\mathcal{L}_\lambda = \sum_{\varepsilon \in E} O(v^{r_\lambda(\varepsilon)})\varepsilon.$$

Then

$$X_\lambda \mathbf{1} \in \mathcal{L}_\lambda.$$

Before proving this lemma note that due to (10), this lemma immediately implies Proposition 6.

Proof. The proof is by induction. Let (i,j) be a corner of λ, and $\lambda' = \lambda - (i,j)$, so that $X_\lambda = \left(T_{k-i+j} - \frac{v^{r_{ij}}}{[r_{ij}]}\right) X_{\lambda'}$. Since $\frac{v^r}{[r]} \in O(v^{2r-1})$, it suffices to prove that $\left(T_{k-i+j} + O(v^{2r_{ij}-1})\right)\mathcal{L}_{\lambda'} \subset \mathcal{L}_\lambda$. Since this operation only changes $\varepsilon_a, \varepsilon_{a+1}$ $(a = k-i+j)$, we need to consider 4 cases: $(++), (+-), (-+), (--)$. This is done explicitly. For example, for the $(+-)$ case, we have

$$(T_a + O(v^{2r_{ij}-1}))(\dots + - \dots) = (\dots - + \dots) + O(v^{2r_{ij}-1})(\dots + - \dots)$$

In this case, the first summand has the same weight and comes with the same power of v as the original ε (note that in the original ε, this $(+-)$ didn't contribute to the weight), so it is in $\mathcal{L}_\lambda$. As for the second summand, its weight is increased by $2r_{ij} - 1$ (the plus contributes r and the minus, $r-1$), but it comes with the factor $O(v^{2r_{ij}-1})$, so again, it is in $\mathcal{L}_\lambda$. The other cases are treated similarly.

Q.E.D.

§4. Divided differences and parabolic Kazhdan-Lusztig bases

In this section, we give a factorization for the dual Kazhdan–Lusztig basis for Grassmanians.

To induce a parabolic module, one can start from the 1-dimensional representation $T_j \mapsto v$ instead of $T_j \mapsto -1/v$ which was used in §1. We now denote the corresponding module by M' and its Kazhdan-Lusztig basis by C'^J_y to distinguish from previous case. Note that there exists a natural pairing between M and M', and C^J_y and C'^J_y are dual bases with respect to this pairing (see, e.g., [S], [FKK]). However, we will not use this pairing.

A simple element $T_i - v$ acts now by

$$M' = \bigoplus_{\varepsilon \in E} \mathbf{Q}(v)\varepsilon,$$

(11)
$$(T_i - v)\varepsilon = \begin{cases} s_i\varepsilon - v\varepsilon, & (\varepsilon_i, \varepsilon_{i+1}) = (+-), \\ 0, & (\varepsilon_i, \varepsilon_{i+1}) = (--) \text{ or } (++), \\ s_i\varepsilon - v^{-1}\varepsilon, & (\varepsilon_i, \varepsilon_{i+1}) = (-+). \end{cases}$$

Consider the space $\mathcal{P}(k,n)$ of polynomials in $x_1, \dots, x_n$ of total degree $n-k$, and of degree at most 1 in each x_i. For any partition λ, denote by $x^{[\lambda]}$ the monomial $w_\lambda(x_{k+1} \cdots x_n)$, the symmetric group acting now by permutation of the x_i. In other words, if $w_\lambda(\mathbf{1}) = (\varepsilon_1, \dots, \varepsilon_n)$, then $x^{[\lambda]}$ is the product of the x_i's for those i such that $\varepsilon_i = -$.

Consider the isomorphism of vector spaces

$$M' \simeq \mathcal{P}(k,n) \\ w_\lambda(\mathbf{1}) \mapsto v^{-|\lambda|} x^{[\lambda]}. \tag{12}$$

Then $T_i - v$ induces the operator ∇_i, acting only on x_i, x_{i+1} as follows:

$$\begin{cases} \nabla_i(x_i) = v x_{i+1} - v^{-1} x_i, \\ \nabla_i(1) = \nabla_i(x_i x_{i+1}) = 0, \\ \nabla_i(x_{i+1}) = -v x_{i+1} + v^{-1} x_i, \end{cases} \tag{13}$$

Therefore ∇_i is the operator

$$f \mapsto (v x_{i+1} - v^{-1} x_i)\, \partial_i(f)$$

denoting by ∂_i the divided difference

$$f \mapsto \frac{f - f^{s_i}}{x_i - x_{i+1}}$$

(for a more general action of the Hecke algebra on the ring of polynomials, see [LS2], [DKLLST]).

We intend to show that divided differences easily furnish the Kazhdan-Lusztig basis of $\mathcal{P}(k,n)$ (i.e. the image of the Kazhdan-Lusztig basis $C'_y, y \in W^J$ of M').

To any element $\varepsilon := w_\lambda(\mathbf{1})$ of E one associates a polynomial Q_ε as follows

1) pair recursively $-, +$ (as one pairs opening and closing parentheses)

2) replace each pair $(-,+)$, where $-$ is in position i and $+$ in position j, with a $x_i - v^{j+1-i} x_j$

3) replace each single $-$, in position i, by x_i

The product of all these factors by $v^{-|\lambda|}$, where $|\lambda| = \lambda_1 + \lambda_2 + \cdots$, is by definition Q_ε.

Theorem 8. *Let E be the set of sequences of $(+,-)$ of length n with k pluses. Then the collection of polynomials Q_ε, $\varepsilon \in E$, is the Kazhdan-Lusztig basis of the space $\mathcal{P}(k,n)$.*

Proof. We shall show that

$$Q_\varepsilon = \nabla_j \cdots \nabla_h (x_1 \cdots x_k)$$

when $\varepsilon = w_\lambda(\mathbf{1})$, and when $s_j \cdots s_h$ is a reduced decomposition of w_λ. Now, it is clear that the inverse image of Q_ε in M' is invariant under involution, and it is easy to check the powers of v to get that for $v = 0$, it specializes to ε.

Assume by induction that we already know Q_ε. Let us add on the right of ε sufficiently many pluses, so that all minuses are now paired (the original polynomial is recovered from the new one by specializing $x_{n+1}, x_{n+2}, \ldots$ to 0). Take now any simple transposition s_i such that $\varepsilon_i = +, \varepsilon_{i+1} = -$. The variables x_i, x_{i+1} involve two or one factor in Q_ε, depending whether ε_i is paired or not. The only possible cases for those factors and their images under ∇_i are

$$(x_{i-a} - v^{a+1}x_i)(x_{i+1} - v^{b+1}x_{i+b+1}) \mapsto (x_{i-a} - v^{a+b+2}x_{i+b+1})(v^{-1}x_i - vx_{i+1})$$
$$(x_{i+1} - v^{b+1}x_{i+b+1}) \mapsto (v^{-1}x_i - vx_{i+1})$$

but now the new pairing of $-, +$ differs from the previous one exactly in the places described by the factors on the right. Q.E.D.

Corollary 9. *Let $\sigma_j \cdots \sigma_h$ be a reduced decomposition of $w \in W^J$. Then the corresponding Kazhdan-Lusztig element $C'^J_w \in M'$ is equal to $(T_j - v) \cdots (T_h - v)(\mathbf{1})$.*

This factorization is equivalent to the one given in [FKK, Theorem 3.1]. One can check on examples that this factorization is compatible, via the duality between the two modules M and M', with the factorization given by Theorem 3. However, deducing Theorem 3 from Theorem 8 seems more intricate than proving the two factorization properties directly.

Example 5. Let $\lambda = [5, 3, 2]$ and $\mu = [5, 3, 3]$. Then one has

places	1	2	3	4	5	6	7	8	9
$w_\lambda(\mathbf{1})$	$+$	$-$	$-$	$+$	$-$	$+$	$-$	$-$	$+$
	$+$	$-$					$-$		
pairing			$-$	$+$	$-$	$+$		$-$	$+$
polynomial		x_2					x_7		
			$(x_3$	$-v^2x_4)$	$(x_5$	$-v^2x_6)$		$(x_8$	$-v^2x_9)$
$w_\mu(\mathbf{1})$	$+$	$-$	$-$	$-$	$+$	$+$	$-$	$-$	$+$
	$+$	$-$					$-$		
pairing			$-$			$+$			
				$-$	$+$			$-$	$+$
polynomial		x_2					x_7		
			x_3			$-v^4x_6$			
				$(x_4$	$-v^2x_5)$			$(x_8$	$-v^2x_9)$

and thus

$$
\begin{aligned}
Q_{w_\lambda(\mathbf{1})} &= v^{-10}\, x_2x_7\,(x_3 - v^2x_4)\,(x_5 - v^2x_6)\,(x_8 - v^2x_9) \\
Q_{w_\mu(\mathbf{1})} &= v^{-11}\, x_2x_7\,(x_3 - v^4x_6)\,(x_4 - v^2x_5)\,(x_8 - v^2x_9).
\end{aligned}
\tag{14}
$$

Note that the pairing between $-, +$, which was a key point in the description of Kazhdan-Lusztig polynomials for Grassmannians in [LS1], is provided by divided differences, starting from the monomial $x_{k+1}\cdots x_n$.

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Littlewood-Richardson Coefficients and Kazhdan-Lusztig Polynomials

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Abstract.

We show that the Littlewood-Richardson coefficients are values at 1 of certain parabolic Kazhdan-Lusztig polynomials for affine symmetric groups. These q-analogues of Littlewood-Richardson multiplicities coincide with those previously introduced in [21] in terms of ribbon tableaux.

§1. Introduction

Let $\lambda = (\lambda_1 \geq \ldots \geq \lambda_r \geq 0)$ and $\mu = (\mu_1 \geq \ldots \geq \mu_r \geq 0)$ denote two partitions of length $\leq r$, identified in the usual way with dominant integral weights of the complex Lie algebra $\mathfrak{gl}_r$. It was shown by Lusztig [28] that the multiplicity $K_{\lambda,\mu}$ of the weight μ in the finite-dimensional irreducible representation $W(\lambda)$ of $\mathfrak{gl}_r$ with highest weight λ is the value at 1 of a certain Kazhdan-Lusztig polynomial P_{n_μ,n_λ} for the affine symmetric group $\widehat{\mathfrak{S}}_r$. (For the definition of $\widehat{\mathfrak{S}}_r$ and n_λ, see below Section 2.1). Moreover, Lusztig proved [27] that the polynomial $P_{n_\mu,n_\lambda}(q)$ is equal to the Kostka-Foulkes polynomial $K_{\lambda,\mu}(q)$ defined as the coefficient of the Schur function s_λ on the basis of Hall-Littlewood function $P_\mu(q)$ [33]. A combinatorial expression of $K_{\lambda,\mu}(q)$ had previously been given by Lascoux and Schützenberger in terms of semi-standard Young tableaux [35, 33].

It is well known that $K_{\lambda,\mu}$ is also equal to the multiplicity of $W(\lambda)$ as an irreducible component of the tensor product

$$W(\mu_1) \otimes \cdots \otimes W(\mu_r)$$

of symmetric powers of the vector representation of $\mathfrak{gl}_r$. Let now $\nu^{(1)}$, … , $\nu^{(s)}$ be arbitrary dominant weights and let $c^{\lambda}_{\nu^{(1)},\ldots,\nu^{(s)}}$ denote the

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multiplicity of $W(\lambda)$ in

$$W(\nu^{(1)}) \otimes \cdots \otimes W(\nu^{(s)}) .$$

A q-analogue $c^{\lambda}_{\nu^{(1)},\ldots,\nu^{(s)}}(q)$ of this multiplicity has been introduced in [21] by means of certain generalizations of semi-standard Young tableaux called ribbon tableaux, and it has been proved that when the partitions $\nu^{(j)}$ have only one part μ_j

$$c^{\lambda}_{\mu_1,\ldots,\mu_r}(q) = K_{\lambda,\mu}(q).$$

The purpose of this paper is to establish that for all $\nu^{(1)},\ldots,\nu^{(s)},\lambda$ the $c^{\lambda}_{\nu^{(1)},\ldots,\nu^{(s)}}(q)$ are Kazhdan-Lusztig polynomials for the group $\widehat{\mathfrak{S}}_r$.

Let us outline how this result is obtained. As mentioned by Lusztig in [28], the expression of the weight multiplicity $K_{\lambda,\mu}$ as a value at 1 of a Kazhdan-Lusztig polynomial might be deduced from the conjecture of [26] for the characters of irreducible representations of GL_r over an algebraically closed field of characteristic $n \geq r$ together with the Steinberg tensor product theorem. In [30, 31] a similar conjecture was formulated for the characters of irreducible representations of $U_q(\mathfrak{gl}_r)$ when q^2 is a primitive nth root of 1. A remarkable feature of the quantum conjecture is that the restriction $n \geq r$ is no longer necessary. This conjecture is now proved due to work of Kazhdan-Lusztig and Kashiwara-Tanisaki. On the other hand Lusztig has derived in [30] an analogue of the Steinberg tensor product theorem for the quantum case. From these two facts, it is easy to deduce that the Littlewood-Richardson multiplicities are value at 1 of Kazhdan-Lusztig polynomials (see below, Section 3).

However this would not provide the link with the q-analogues defined by means of ribbon tableaux. We shall therefore follow a different approach and rely on the construction given in [22] of a canonical basis in the level 1 Fock space representation of the quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_n)$. This canonical basis satisfies a formal q-analogue of Steinberg's tensor product theorem which may be formulated in terms of the combinatorics of ribbon tableaux. On the other hand, Varagnolo and Vasserot [42] have recently verified a conjecture of [22]. They proved that the coefficients of the expansion of this canonical basis on the standard basis of q-wedge products coincide with the Kazhdan-Lusztig polynomials occuring in Lusztig's conjecture. Using these two results, we are able to express the $c^{\lambda}_{\nu^{(1)},\ldots,\nu^{(s)}}(q)$ as Kazhdan-Lusztig polynomials.

More precisely, they belong to a family of parabolic analogues of Kazhdan-Lusztig polynomials introduced by Deodhar [4, 5]. There are two types of such polynomials associated with the Hecke algebra modules obtained by inducing respectively the characters $T_i \mapsto -q$ and $T_i \mapsto q^{-1}$ of a parabolic subalgebra. The $c^{\lambda}_{\nu^{(1)},\ldots,\nu^{(s)}}(q)$ turn out to belong to the family denoted by $\widetilde{P}^J_{x,y}$ in [5] and by $n_{x,y}$ in [39], which is less well understood. In particular $\widetilde{P}^J_{x,y}$ may be 0 even if $x < y$ in the Bruhat ordering. Also, since the $\widetilde{P}^J_{x,y}$ are equal to alternating sums of ordinary Kazhdan-Lusztig polynomials, it is not a priori clear whether these polynomials have non-negative coefficients. However, according to experts, it seems probable that they admit a geometrical interpretation in terms of Schubert varieties of finite codimension in an affine flag manifold [1]. This would settle the positivity conjecture VI.3 of [21]. Note that in the case of two factors the polynomials $c^{\lambda}_{\nu^{(1)},\nu^{(2)}}(q)$ are known to have non-negative coefficients because of their combinatorial interpretation in terms of Yamanouchi domino tableaux given in [2].

That the non-vanishing of the polynomials $\widetilde{P}^J_{x,y}$ is a difficult problem should not be too surprising. Indeed, our result shows that this contains as a special case the non-vanishing of the Littlewood-Richardson coefficients. There has been some important recent progress by Klyachko on this classical subject [19] using toric vector bundles on the projective plane (see the reviews of Zelevinsky [44] and Fulton [8]). Maybe some new understanding will arise from the connection with affine Schubert varieties.

A few comments concerning the growing literature on q-analogues of Littlewood-Richardson coefficients are in order. In [36] Shimozono and Weyman have studied the Poincaré polynomials of isotypic components of some virtual graded GL_r-modules supported in the closure of a nilpotent conjugacy class. These are q-analogues of Littlewood-Richardson multiplicities $c^{\lambda}_{\nu^{(1)},\ldots,\nu^{(s)}}$ satisfying a q-Kostant formula and a Morris-like recurrence. In the case where all partitions $\nu^{(j)}$ are rectangular (*i.e.* the corresponding weights are multiples of a single fundamental weight) and are arranged in non-increasing order of width, these polynomials have non-negative coefficients. (This is not true in general, but see [36], Conjecture 4.) In this case, a combinatorial interpretation in terms of semi-standard Young tableaux was given by Shimozono [37, 38], which shows that they coincide with the generalized Kostka-Foulkes polynomials studied by Schilling and Warnaar [34] in relation with exactly solvable lattice models and Rogers-Ramanujan type identities. A different combinatorial interpretation using rigged configurations has been conjectured by Kirillov and Shimozono [18] and recently verified [17].

[1]Added 09/1999. This has eventually been proved by Kashiwara and Tanisaki (preprint math.RT/9908153).

It is believed that for rectangular shapes in non-increasing order these Poincaré polynomials are equal to the corresponding $c^{\lambda}_{\nu^{(1)},\ldots,\nu^{(s)}}(q)$ but the reason for that is still unclear.

Let us describe more precisely the contents of this paper. The results rely mainly on four sources, namely the parabolic analogue of Kazhdan-Lusztig polynomials developed by Deodhar in [4, 5], our joint paper with Lascoux on ribbon tableaux and generalizations of Kostka-Foulkes polynomials [21], our previous note [22], and the paper of Varagnolo and Vasserot [42]. Since [22] contains no proofs, and since only a small part of [21] and [42] is needed to obtain our results, we thought it would be appropriate to provide a self-contained exposition of this material. Thus the style of the paper is openly expository and we hope it can be read without a previous knowledge of these four sources. However for what concerns parabolic Kazhdan-Lusztig polynomials, we decided to omit the proofs because they can be found in the optimum exposition by Soergel of Kazhdan-Lusztig theory from scratch [39].

So in Section 2 we explain all the necessary background on (extended) affine symmetric groups $\widehat{\mathfrak{S}}_r$ and their Hecke algebras $\widehat{H}_r$. In particular we introduce the two presentations (Coxeter-type and Bernstein-type) and give the relations between them. Following [42] we construct a representation of $\widehat{H}_r$ on the weight lattice $\mathcal{P}_r$ of $\mathfrak{gl}_r$ and introduce its two Kazhdan-Lusztig bases. The coefficients of these bases on the basis of weights are the parabolic Kazhdan-Lusztig polynomials (for various parabolic subgroups).

In Section 3, we recall the Lusztig conjecture for quantum $\mathfrak{gl}_r$ at an nth root of 1, the tensor product theorem, and using a formula of Littlewood we deduce from this that the Littlewood-Richardson coefficients are value at 1 of parabolic Kazhdan-Lusztig polynomials (Theorem 3.3).

In Section 4 we recall following [21] the definitions of ribbon tableaux and their spin, we introduce the q-analogues $c^{\lambda}_{\nu^{(1)},\ldots,\nu^{(s)}}(q)$, and we state our main result (Theorem 4.1).

In Section 5 we explain the construction of [42] and consider a quotient $\mathcal{F}_r$ of $\mathcal{P}_r$ whose bases are naturally labelled by dominant integral $\mathfrak{gl}_r$-weights. This space can be identified in a natural way with the (finitized) q-deformed Fock space of Kashiwara, Miwa and Stern [15] considered in [22]. Projecting on $\mathcal{F}_r$ the Kazhdan-Lusztig involution of $\mathcal{P}_r$ one gets the involution defined in [22] in terms of q-wedge products. This implies that the canonical bases of [22] have coefficients given by some parabolic Kazhdan-Lusztig polynomials (Theorem 5.12).

In Section 6 we study the action of the center $Z(\widehat{H}_r)$ of $\widehat{H}_r$ on $\mathcal{F}_r$ and show that it can be expressed via the combinatorics of ribbon tableaux.

We then prove that the vectors $G^-_{\lambda+\rho}$ of the canonical basis indexed by non-restricted weights λ are obtained from the restricted ones by acting with an element of $Z(\widehat{H}_r)$. This should be regarded as an analogue in this setting of the Steinberg-Lusztig tensor product theorem. Then we give the proof of Theorem 4.1.

In Section 7 we review the construction of Kashiwara, Miwa and Stern of the Fock space $\mathbf{F}_\infty$ obtained by taking the limit $r \to \infty$ in $\mathcal{F}_r$. It affords a level 1 integrable representation of the quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_n)$. We investigate the behaviour of the canonical bases of $\mathbf{F}_\infty$ introduced in [22] with respect to the semi-linear involution induced by the conjugation of partitions, and derive from this a symmetry of the polynomials $c^\lambda_{\nu^{(1)},\dots,\nu^{(s)}}(q)$ (Theorem 7.13) and an inversion formula for parabolic Kazhdan-Lusztig polynomials (Corollary 7.15). This formula, together with a result of Du, Parshall and Scott [7], provides an alternative proof of Soergel's character formula for tilting modules in type A (Remark 7.16).

Finally Section 8 provides some numerical tables of q-Littlewood-Richardson multiplicities and Kazhdan-Lusztig polynomials, which may serve as examples of the results discussed in the text.

§2. Affine symmetric groups and their Hecke algebras

2.1. Affine symmetric groups

Let $\widetilde{\mathfrak{S}}_r$ denote the Coxeter group of type $\widetilde{A}_{r-1}$. For $r = 2$, this is the group generated by s_0, s_1 subject to the relations $s_0^2 = s_1^2 = 1$. For $r > 2$, $\widetilde{\mathfrak{S}}_r$ is generated by $s_0, s_1, \dots, s_{r-1}$ subject to

$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, \tag{1}$$

$$s_i s_j = s_j s_i, \qquad (i - j \neq \pm 1), \tag{2}$$

$$s_i^2 = 1, \tag{3}$$

where the subscripts are understood modulo r. The subgroup generated by $s_1, \dots, s_{r-1}$ is isomorphic to the symmetric group $\mathfrak{S}_r$. The group $\widetilde{\mathfrak{S}}_r$ has a concrete realization as an affine reflection group. Let $(\epsilon_1, \dots, \epsilon_r)$ denote the standard basis of $\mathbf{R}^r$, and define a scalar product by putting $(\epsilon_i, \epsilon_j) = \delta_{ij}$. Set $\alpha_i = \epsilon_i - \epsilon_{i+1}$ $(1 \le i \le r-1)$ and $\alpha_0 = \epsilon_r - \epsilon_1$. Let $\mathfrak{h}_r$ denote a Cartan subalgebra of $\mathfrak{gl}_r$. We identify $\mathbf{R}^r$ with (the real part of) $\mathfrak{h}_r^*$ in the usual way, so that $P = P_r := \bigoplus_{i=1}^r \mathbf{Z}\epsilon_i$ becomes the weight lattice, $Q = Q_r := \bigoplus_{i=1}^{r-1} \mathbf{Z}\alpha_i$ the root lattice, α_i $(1 \le i \le r-1)$ the simple roots, $-\alpha_0$ the highest root, *etc.* For $\alpha \in \mathbf{R}^r$ and $m \in \mathbf{Z}$, denote by $S_{\alpha,m}$ the affine reflection defined by

$$S_{\alpha,m}(\lambda) = \lambda - 2\frac{(\lambda,\alpha)+m}{(\alpha,\alpha)}\,\alpha.$$

Then for any $m \in \mathbf{Z}^*$, the assignment

$$s_0 \mapsto S_{\alpha_0,m}\,, \quad s_i \mapsto S_{\alpha_i,0} \quad (1 \le i \le r-1)$$

defines a faithful representation π_m of $\widetilde{\mathfrak{S}}_r$ as a discrete subgroup of the group of affine transformations of $\mathbf{R}^r$. In coordinates, we have

$$\begin{aligned} &\pi_m(s_i)(\lambda) = (\lambda_1, \dots, \lambda_{i+1}, \lambda_i, \dots, \lambda_r), && (1 \le i \le r-1),\\ &\pi_m(s_0)(\lambda) = (\lambda_r + m, \lambda_2, \dots, \lambda_{r-1}, \lambda_1 - m). \end{aligned}$$

Note that for $s \in \mathfrak{S}_r$, $\pi_m(s)$ does not depend on m. We shall therefore simplify the notation and write $s\lambda$ in place of $\pi_m(s)(\lambda)$.

This realization shows that $\widetilde{\mathfrak{S}}_r$ contains a large commutative subgroup, namely the image under π_m^{-1} of the group of translations by the vectors of the lattice mQ. Write $\mathrm{Tr}(\lambda)$ for the translation by $\lambda \in \mathbf{R}^r$, and let t_i denote the element of $\widetilde{\mathfrak{S}}_r$ corresponding to $\mathrm{Tr}(m\alpha_i)$ under π_m. Then one can check that

$$\begin{aligned} t_1 &= (s_0 s_{r-1} s_{r-2} \cdots s_3 s_2)(s_3 s_4 \cdots s_{r-1} s_0 s_1),\\ t_2 &= (s_1 s_0 s_{r-1} \cdots s_4 s_3)(s_4 s_5 \cdots s_0 s_1 s_2),\\ &\ \vdots \qquad\qquad \vdots\\ t_{r-1} &= (s_{r-2} s_{r-3} s_{r-4} \cdots s_1 s_0)(s_1 s_2 \cdots s_{r-3} s_{r-2} s_{r-1}),\\ t_0 &= (s_{r-1} s_{r-2} s_{r-3} \cdots s_2 s_1)(s_1 s_2 \cdots s_{r-2} s_{r-1} s_0). \end{aligned}$$

It will be convenient to enlarge $\widetilde{\mathfrak{S}}_r$ by adding the translations by vectors of the lattice mP. Abstractly, this extended affine symmetric group that we shall denote by $\widehat{\mathfrak{S}}_r$ may be defined as the group generated by $s_0, s_1, \dots, s_{r-1}, \tau$ subject to relations (1), (2), (3) together with

(4) $$\tau s_i = s_{i+1} \tau,$$

where again subscripts are understood modulo r. It is clear that each $w \in \widehat{\mathfrak{S}}_r$ can be written in a unique way as

(5) $$w = \tau^k \sigma, \qquad (k \in \mathbf{Z},\ \sigma \in \widetilde{\mathfrak{S}}_r).$$

An alternative useful presentation is as follows. The group $\widehat{\mathfrak{S}}_r$ is generated by the elements $s_1, \dots, s_{r-1}, y_1, \dots, y_r$ subject to relations (1), (2), (3) with all indices between 1 and $r-1$ together with

$$y_i y_j = y_j y_i, \tag{6}$$
$$s_i y_j = y_j s_i \text{ for } j \neq i, i+1, \tag{7}$$
$$s_i y_i s_i = y_{i+1}. \tag{8}$$

The homomorphism π_m can then be extended to $\widehat{\mathfrak{S}}_r$ by setting

$$\pi_m(y_i) := \mathrm{Tr}(m\epsilon_i), \qquad \pi_m(\tau) := S_{\alpha_1,0} S_{\alpha_2,0} \cdots S_{\alpha_{r-1},0} \mathrm{Tr}(m\epsilon_r),$$

or in coordinates

$$\begin{aligned} \pi_m(y_i)(\lambda) &= (\lambda_1, \dots, \lambda_i + m, \dots, \lambda_r), && (1 \le i \le r), \\ \pi_m(\tau)(\lambda) &= (\lambda_r + m, \lambda_1, \dots, \lambda_{r-2}, \lambda_{r-1}). \end{aligned}$$

The following equations relate the two above presentations of $\widehat{\mathfrak{S}}_r$:

$$y_i = s_{i-1} s_{i-2} \cdots s_1 s_0 s_{r-1} s_{r-2} \cdots s_{i+1} \tau, \qquad (1 \le i \le r) \tag{9}$$
$$\tau = s_1 s_2 \cdots s_{r-1} y_r, \tag{10}$$
$$s_0 = s_{r-1} s_{r-2} \cdots s_2 s_1 s_2 \cdots s_{r-1} y_1^{-1} y_r. \tag{11}$$

(In (9) the subscripts are understood modulo r.)

Note that $\widehat{\mathfrak{S}}_r$ is not a Coxeter group. However, one can still define a Bruhat order and a length function. Let $w = \tau^k \sigma, w' = \tau^m \sigma'$ with $k, m \in \mathbf{Z}$, $\sigma, \sigma' \in \widetilde{\mathfrak{S}}_r$. We say that $w < w'$ if and only if $k = m$ and $\sigma < \sigma'$, and we put $\ell(w) := \ell(\sigma)$. Define

$$A_{r,m} := \left| \begin{array}{ll} \{\lambda \in \mathbf{R}^r \mid m > \lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_r \ge 0\} & \text{if } \; m > 0, \\ \{\lambda \in \mathbf{R}^r \mid m < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_r \le 0\} & \text{if } \; m < 0, \end{array} \right.$$

and $\mathcal{A}_{r,m} := A_{r,m} \cap P$ (see Figure 1). Then $A_{r,m}$ is a fundamental domain for the action of $\widehat{\mathfrak{S}}_r$ on $\mathbf{R}^r$ via π_m, that is, each orbit intersects it in a unique point. Let $\lambda \in P$, and let ν be the intersection of $\mathcal{A}_{r,m}$ with the orbit of λ. Then there is a unique $w(\lambda, m) \in \widehat{\mathfrak{S}}_r$ of minimal length such that $\pi_m(w(\lambda, m))(\nu) = \lambda$. Let $\mathfrak{S}_{\nu,m}$ be the parabolic subgroup consisting of the w such that $\pi_m(w)(\nu) = \nu$. (Since $|\nu_1 - \nu_r| < m$, $\mathfrak{S}_{\nu,m} \subset \mathfrak{S}_r$.) Then $w(\lambda, m)$ is the minimal length representative of the coset

$$w(\lambda, m)\mathfrak{S}_{\nu,m} = \{w \in \widehat{\mathfrak{S}}_r \mid \pi_m(w)(\nu) = \lambda\}.$$

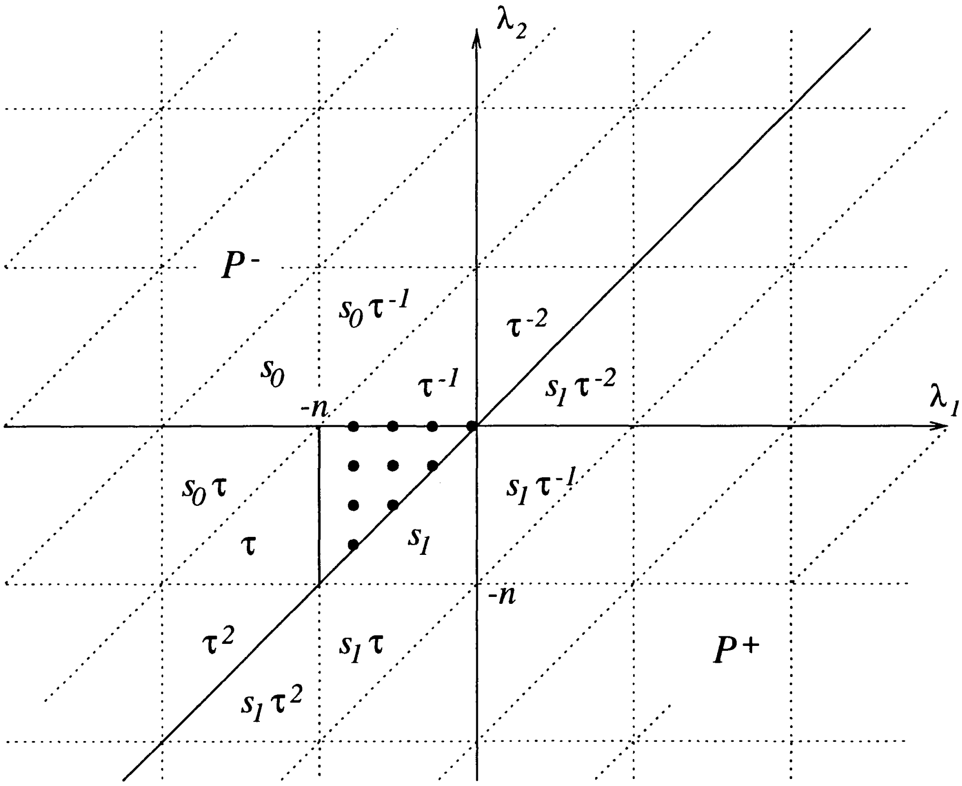

Fig. 1. The action of $\widehat{\mathfrak{S}}_2$ on P_2 via π_{-n}

In this way, we can associate to the data (λ, m) a certain element $w(\lambda, m)$ of $\widehat{\mathfrak{S}}_r$. This will allow us to pass from the indexation by weights of the Littlewood-Richardson coefficients to the indexation by elements of $\widehat{\mathfrak{S}}_r$ of the Kazhdan-Lusztig polynomials.

Example 2.1. Take $r = 3$ and $\lambda = (5, 3, 0)$. Then

$$w(\lambda, 3) = \tau^2 s_1, \quad w(\lambda, 2) = \tau^3 s_0 s_1 s_2, \quad w(\lambda, -2) = \tau^{-5} s_1 s_0 s_2 s_1 s_2 s_0.$$

◇

For $\lambda = (\lambda_1, \ldots, \lambda_r) \in P$ set $\lambda_0 := \lambda_r + m$, $\lambda_{r+1} := \lambda_1 - m$ and define the descent function

$$\operatorname{desc}(\lambda, i, m) := \begin{cases} 1 & \text{if } \lambda_i > \lambda_{i+1}, \\ 0 & \text{if } \lambda_i = \lambda_{i+1}, \\ -1 & \text{if } \lambda_i < \lambda_{i+1}, \end{cases} \qquad (0 \le i \le r).$$

Note that

$$\operatorname{desc}(\lambda, 0, m) = \operatorname{desc}(\lambda, r, m), \quad \operatorname{desc}(\lambda, i, m) = \operatorname{desc}(\pi_m(\tau)(\lambda), i + 1, m).$$

Geometrically, $\mathrm{desc}(\lambda, i, m) = 0$ means that λ lies on the reflecting hyperplane $\mathcal{H}_m$ of $\pi_m(s_i)$, *i.e.* $\pi_m(s_i)(\lambda) = \lambda$, and $\mathrm{desc}(\lambda, i, m) = \mathrm{sgn}\,(m)$ means that λ belongs to the 1/2-space defined by $\mathcal{H}_m$ which contains the fundamental domain $\mathcal{A}_{r,m}$, *i.e.* $s_i w(\lambda, m) > w(\lambda, m)$.

Lemma 2.1. *Let $\lambda \in P$, $i \in \{0, \dots, r-1\}$ and $m \in \mathbf{Z}^*$. Let $\nu = w(\lambda, m)^{-1}(\lambda)$ be the point of $\mathcal{A}_{r,m}$ congruent to λ under π_m. One has the three following alternatives:*

$$\begin{array}{lll}
\text{(i)} \quad \mathrm{desc}(\lambda, i, m) = \mathrm{sgn}\,(m) & \Longleftrightarrow & s_i w(\lambda, m) = w(s_i\lambda, m) > w(\lambda, m), \\
\text{(ii)} \quad \mathrm{desc}(\lambda, i, m) = 0 & \Longleftrightarrow & s_i w(\lambda, m) = w(\lambda, m) s_j \\
 & & \textit{for some } s_j \in \mathfrak{S}_{\nu,m}, \\
\text{(iii)} \quad \mathrm{desc}(\lambda, i, m) = -\mathrm{sgn}\,(m) & \Longleftrightarrow & s_i w(\lambda, m) = w(s_i\lambda, m) < w(\lambda, m).
\end{array}$$

Proof — This is a reformulation of Lemma 2.1 (iii) of [4]. Indeed, $\mathrm{desc}(\lambda, i, m) = -\mathrm{sgn}\,(m)$ if and only if $s_i w(\lambda, m) < w(\lambda, m)$ and in this case $s_i w(\lambda, m) = w(s_i\lambda, m)$ by [4]. Also, $\mathrm{desc}(\lambda, i, m) = 0$ if and only if $s_i w(\lambda, m)\nu = w(\lambda, m)\nu$ which shows that $s_i w(\lambda, m)$ belongs to the same coset as $w(\lambda, m)$ and is not minimal in this coset. In this case, by [4], there exists $s_j \in \mathfrak{S}_{\nu,m}$ such that $s_i w(\lambda, m) = w(\lambda, m)s_j$. Finally, $\mathrm{desc}(\lambda, i, m) = \mathrm{sgn}\,(m)$ if and only if $s_i w(\lambda, m) > w(\lambda, m)$ and $s_i w(\lambda, m)\nu \neq w(\lambda, m)\nu$. In that case, again by [4], $s_i w(\lambda, m)$ is minimal in its coset and thus equal to $w(s_i\lambda, m)$. $\square$

If ν is regular, that is, $\mathfrak{S}_{\nu,m} = \{1\}$, then case (ii) does not occur and we obtain the following criterion

$$(12) \qquad s_i w > w \iff \mathrm{desc}(w\nu, i, m) = \mathrm{sgn}\,(m), \qquad (w \in \widehat{\mathfrak{S}}_r).$$

In particular, taking $m = r$ and $\nu = \rho := (r-1, r-2, \dots, 1, 0)$ we get that

$$(13) \qquad s_i w > w \iff \mathrm{desc}(w\rho, i, r) = 1, \qquad (w \in \widehat{\mathfrak{S}}_r).$$

For $\lambda \in P$, set $y^\lambda := y_1^{\lambda_1} \cdots y_r^{\lambda_r}$. Every $w \in \widehat{\mathfrak{S}}_r$ has a unique decomposition of the form $w = y^\lambda s$, where $\lambda \in P$ and $s \in \mathfrak{S}_r$. Therefore each coset $w\mathfrak{S}_r$ contains a unique element y^λ. It follows from (8) that for $s \in \mathfrak{S}_r$, $sy^\lambda = y^{s\lambda}s$. This implies that each double coset $\mathfrak{S}_r w \mathfrak{S}_r$ in $\widehat{\mathfrak{S}}_r$ contains a unique element y^λ with $\lambda \in P^+ := \{\mu \in P \mid \mu_1 \geq \mu_2 \geq \dots \geq \mu_r\}$, the set of dominant weights. For $\lambda \in P^+$, we denote by n_λ the element of maximal length in $\mathfrak{S}_r y^\lambda \mathfrak{S}_r$.

Lemma 2.2. *Let $\lambda \in P^+$, $\mu \in P^- := -P^+$ and $s \in \mathfrak{S}_r$. We have*

$$\ell(sy^\lambda) = \ell(s) + \ell(y^\lambda), \qquad \ell(y^\mu s) = \ell(y^\mu) + \ell(s).$$

In particular $n_\lambda = w_0 y^\lambda$*, where* w_0 *denotes the longest element of* $\mathfrak{S}_r$.

Proof — If $\lambda \in P^+$ then $\alpha := y^\lambda \rho$ satisfies $\alpha_1 > \alpha_2 > \cdots > \alpha_r$. Let $s = s_{i_1} \cdots s_{i_k}$ be a reduced decomposition of s. By repeated applications of (13) we see that $\ell(sy^\lambda) = \ell(y^\lambda) + k$, which proves the first statement. The case of μ is similar. Finally, $w_0 y^\lambda$ belongs to the double coset of y^λ and for $s \in \mathfrak{S}_r$ ($s \neq 1$), $\ell(sw_0 y^\lambda) = \ell(sw_0) + \ell(y^\lambda) < \ell(w_0) + \ell(y^\lambda)$ so that $sw_0 y^\lambda$ is not maximal. The argument is similar for right multiplication by s, since $w_0 y^\lambda = y^{w_0\lambda} w_0$, and $w_0\lambda \in P^-$. □

Example 2.2. Take $r = 3$. Then,

$$\begin{aligned} n_{(2,1,0)} &= s_2 s_1 s_2 y_1^2 y_2 = s_2 s_1 s_2 s_0 s_2 s_1 s_2 \tau^3, \\ n_{(1,1,1)} &= s_2 s_1 s_2 y_1 y_2 y_3 = s_2 s_1 s_2 \tau^3. \end{aligned}$$

◇

In fact, Lemma 2.2 easily follows from a general formula of Iwahori and Matsumoto ([11], Prop. 1.23) which in our case reads

$$\ell(sy^\lambda) = \sum_{\substack{i<j \\ s(i)<s(j)}} |\lambda_i - \lambda_j| + \sum_{\substack{i<j \\ s(i)>s(j)}} |\lambda_i - \lambda_j + 1|, \tag{14}$$

where $s \in \mathfrak{S}_r$ and $\lambda \in P$. In particular, if $\lambda \in P^+$ then $\ell(y^\lambda) = \sum_{i=1}^r (r+1-2i)\lambda_i$, which shows that

$$\ell(y^\lambda) + \ell(y^\mu) = \ell(y^{\lambda+\mu}), \qquad (\lambda, \mu \in P^+). \tag{15}$$

Lemma 2.3. *Let* $\lambda \in P^+$ *and set* $\lambda^* := w_0(-\lambda)$*. Then, for all* $n \geq r$ *one has*

$$w(n\lambda + \rho, -n) = n_{\lambda^*}\, \tau^{-r+1}.$$

Proof — Since $n \geq r$, the weight

$$\nu := \pi_{-n}(\tau^{r-1} w_0)(\rho) = (1-n, 2-n, \ldots, r-1-n, 0)$$

belongs to $\mathcal{A}_{r,-n}$ and we have

$$n\lambda + \rho = \pi_{-n}(y^{-\lambda})(\rho) = \pi_{-n}(y^{-\lambda} w_0 \tau^{-r+1})(\nu).$$

The stabilizer of ν in $\pi_{-n}(\widehat{\mathfrak{S}}_r)$ is trivial, that is, ν is a regular weight. Therefore we get

$$w(n\lambda + \rho, -n) = y^{-\lambda} w_0 \tau^{-r+1} = w_0 y^{w_0(-\lambda)} \tau^{-r+1} = n_{\lambda^*}\, \tau^{-r+1}.$$

□

2.2. Affine Hecke algebras

The Hecke algebra $\widehat{H}_r := H(\widehat{\mathfrak{S}}_r)$ is the algebra over $\mathbf{Z}[q, q^{-1}]$ with basis T_w ($w \in \widehat{\mathfrak{S}}_r$) and multiplication defined by

$$T_w T_{w'} = T_{ww'} \text{ if } \ell(ww') = \ell(w) + \ell(w'), \tag{16}$$

$$(T_{s_i} - q^{-1})(T_{s_i} + q) = 0. \tag{17}$$

There is a canonical involution $x \mapsto \overline{x}$ of $\widehat{H}_r$ defined as the unique ring homomorphism such that $\overline{q} = q^{-1}$ and $\overline{T_w} = (T_{w^{-1}})^{-1}$.

To simplify notation, we put $T_i := T_{s_i}$ and we write τ instead of T_τ. Then we have the two following presentations of $\widehat{H}_r$ corresponding to the two above presentations of $\widehat{\mathfrak{S}}_r$ (see [27, 29]). First, $\widehat{H}_r$ is the algebra generated by T_i ($0 \le i \le r-1$) and an invertible element τ subject to the relations

$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, \tag{18}$$

$$T_i T_j = T_j T_i, \qquad (i - j \neq \pm 1), \tag{19}$$

$$(T_i - q^{-1})(T_i + q) = 0, \tag{20}$$

$$\tau T_i = T_{i+1} \tau. \tag{21}$$

Alternatively, $\widehat{H}_r$ is the algebra generated by T_i ($1 \le i \le r-1$) and invertible elements Y_i ($1 \le i \le r$) subject to the relations (18), (19), (20) with subscripts between 1 and $r-1$ together with

$$Y_i Y_j = Y_j Y_i, \tag{22}$$

$$T_i Y_j = Y_j T_i \text{ for } j \neq i, i+1, \tag{23}$$

$$T_i Y_i T_i = Y_{i+1}. \tag{24}$$

The following equations relate the two above presentations of $\widehat{H}_r$:

$$Y_i = T_{i-1} T_{i-2} \cdots T_1 T_0^{-1} T_{r-1}^{-1} T_{r-2}^{-1} \cdots T_{i+1}^{-1} \tau, \qquad (1 \le i \le r) \tag{25}$$

$$\tau = T_1^{-1} T_2^{-1} \cdots T_{r-1}^{-1} Y_r, \tag{26}$$

$$T_0 = T_{r-1}^{-1} T_{r-2}^{-1} \cdots T_2^{-1} T_1^{-1} T_2^{-1} \cdots T_{r-1}^{-1} Y_1^{-1} Y_r. \tag{27}$$

(In (18)(19)(21)(25) the subscripts are understood modulo r.)

Note that for $\lambda \in P$, we have two natural elements in $\widehat{H}_r$ corresponding to the translation by λ, namely, $Y^\lambda := Y_1^{\lambda_1} \cdots Y_r^{\lambda_r}$ and $T_\lambda := T_{y^\lambda}$. They do not coincide in general. (For example if $r = 3$, $T_{y_2} = T_1 T_0 \tau$ and $Y_2 = T_1 T_0^{-1} \tau$.) In fact the T_λ do not commute in general. However it follows from (15) that $T_\lambda T_\mu = T_{\lambda+\mu} = T_\mu T_\lambda$ if $\lambda, \mu \in P^+$. Let $\lambda \in P$

be written as $\lambda = \lambda' - \lambda''$ with $\lambda', \lambda'' \in P^+$. Bernstein has introduced an element $\widehat{T}_\lambda \in \widehat{H}_r$ by

$$\widehat{T}_\lambda := T_{\lambda'} T_{\lambda''}^{-1}.$$

This element is well-defined, *i.e.* it does not depend on the choice of λ' and λ'', and $\widehat{T}_\lambda \widehat{T}_\mu = \widehat{T}_{\lambda+\mu} = \widehat{T}_\mu \widehat{T}_\lambda$ for all $\lambda, \mu \in P$. With this notation one can check that

$$Y^\lambda = \overline{\widehat{T}_\lambda} = T_{-\lambda'}^{-1} T_{-\lambda''} = T_{-\lambda''} T_{-\lambda'}^{-1}. \tag{28}$$

In particular, if $\lambda \in P^-$ then

$$Y^\lambda = T_\lambda. \tag{29}$$

2.3. Action of $\widehat{H}_r$ on the weight lattice

Let $\mathcal{P} = \mathcal{P}_r := \mathbf{Z}[q, q^{-1}] \otimes_{\mathbf{Z}} P$. We shall use the descent function to q-deform the representation π_m of $\widehat{\mathfrak{S}}_r$ on P into a representation Π_m of $\widehat{H}_r$ on $\mathcal{P}$. Indeed, it follows from Lemma 2.1 that $\widehat{H}_r$ acts on $\mathcal{P}$ by $\Pi_m(\tau)(\lambda) := \pi_m(\tau)(\lambda)$ and for $0 \le i \le r-1$,

$$\Pi_m(T_i)(\lambda) := \begin{cases} \pi_m(s_i)(\lambda) & \text{if} \quad \mathrm{desc}(\lambda, i, m) = \mathrm{sgn}\,(m), \\ q^{-1}\lambda & \text{if} \quad \mathrm{desc}(\lambda, i, m) = 0, \\ \pi_m(s_i)(\lambda) + (q^{-1} - q)\lambda & \text{if} \quad \mathrm{desc}(\lambda, i, m) = -\mathrm{sgn}\,(m). \end{cases}$$

Warning From now on in order to simplify the notation we shall often omit the dependence on m and write for example $T_i\lambda$ in place of $\Pi_m(T_i)(\lambda)$, or $s_i\lambda$ in place of $\pi_m(s_i)(\lambda)$. We hope that this will not create confusion.

In terms of the Kazhdan-Lusztig elements $C_i' := T_i + q$ and $C_i := T_i - q^{-1}$ we have

$$C_i'\lambda = \begin{cases} s_i\lambda + q\lambda & \text{if} \quad \mathrm{desc}(\lambda, i, m) = \mathrm{sgn}\,(m), \\ (q + q^{-1})\lambda & \text{if} \quad \mathrm{desc}(\lambda, i, m) = 0, \\ s_i\lambda + q^{-1}\lambda & \text{if} \quad \mathrm{desc}(\lambda, i, m) = -\mathrm{sgn}\,(m), \end{cases} \qquad (0 \le i \le r-1),$$

$$C_i\lambda = \begin{cases} s_i\lambda - q^{-1}\lambda & \text{if} \quad \mathrm{desc}(\lambda, i, m) = \mathrm{sgn}\,(m), \\ 0 & \text{if} \quad \mathrm{desc}(\lambda, i, m) = 0, \\ s_i\lambda - q\lambda & \text{if} \quad \mathrm{desc}(\lambda, i, m) = -\mathrm{sgn}\,(m), \end{cases} \qquad (0 \le i \le r-1).$$

These formulas show that the $\widehat{H}_r$-module $\mathcal{P}$ decomposes as

$$\mathcal{P} = \bigoplus_{\nu \in \mathcal{A}_{r,m}} \widehat{H}_r \nu.$$

Moreover, each summand of the right-hand side is isomorphic to an induced module. Indeed, let $\widehat{H}_{\nu,m}$ be the subalgebra of $\widehat{H}_r$ generated by the T_i such that $s_i\nu = \nu$, and let $\mathbf{1}_{q^{-1}}$ denote the 1-dimensional $\widehat{H}_{\nu,m}$-module in which T_i acts by multiplication by q^{-1}. Then

$$\widehat{H}_r\nu \simeq \widehat{H}_r \otimes_{\widehat{H}_{\nu,m}} \mathbf{1}_{q^{-1}},$$

the isomorphism being given by

$$\lambda \in \widehat{\mathfrak{S}}_r\nu \mapsto T_{w(\lambda,m)} \otimes 1. \tag{30}$$

In particular $\lambda = T_{w(\lambda,m)}\nu$.

2.4. Kazhdan-Lusztig polynomials

The module $\mathcal{P}_\nu := \widehat{H}_r\nu$ is a parabolic module of the type considered by Deodhar in [4, 5]. (Note that if ν is a regular weight, then $\mathcal{P}_\nu$ is just the regular representation of $\widehat{H}_r$.) Therefore $\mathcal{P}_\nu$ has two Kazhdan-Lusztig bases constructed as follows (see [39]). Define a semi-linear involution on $\mathcal{P}_\nu$ by

$$\overline{q} := q^{-1}, \qquad \overline{x\nu} := \overline{x}\nu \quad (x \in \widehat{H}_r),$$

and two lattices

$$L_\nu^+ := \bigoplus_{\lambda\in\widehat{\mathfrak{S}}_r\nu} \mathbf{Z}[q]\lambda, \qquad L_\nu^- := \bigoplus_{\lambda\in\widehat{\mathfrak{S}}_r\nu} \mathbf{Z}[q^{-1}]\lambda.$$

Then there are two bases C_λ^+, C_λ^- $(\lambda \in \widehat{\mathfrak{S}}_r\nu)$ characterized by

$$\overline{C_\lambda^+} = C_\lambda^+, \qquad \overline{C_\lambda^-} = C_\lambda^-,$$

and

$$C_\lambda^+ \equiv \lambda \bmod qL_\nu^+, \qquad C_\lambda^- \equiv \lambda \bmod q^{-1}L_\nu^-.$$

When ν is regular these bases coincide with the Kazhdan-Lusztig bases C'_w and C_w respectively under the isomorphism (30).

These bases can be computed recursively as follows [39]. First, by definition, $C_\nu^+ = C_\nu^- = \nu$, and more generally $C_{\tau^k\nu}^+ = C_{\tau^k\nu}^- = \tau^k\nu$ ($k \in \mathbf{Z}$). Let $\lambda \in \widehat{\mathfrak{S}}_r\nu$ and suppose that C_μ^+ (*resp.* C_μ^-) has already been calculated for all $\mu < \lambda$, that is, such that $w(\mu,m) < w(\lambda,m)$. Then compute $v_\lambda^+ = C'_i\,C_\mu^+$ (*resp.* $v_\lambda^- = C_i\,C_\mu^-$) where μ and i satisfy $s_i(\mu) = \lambda$

and $\operatorname{desc}(\mu, i, m) = \operatorname{sgn}(m)$. Then v_λ^+ (*resp.* v_λ^-) is invariant under the bar-involution and belongs to L_ν^+ (*resp.* L_ν^-). Write

$$v_\lambda^+ \equiv \lambda + \sum_\alpha a_\alpha \alpha \bmod qL_\nu^+, \qquad (\textit{resp. } v_\lambda^- \equiv \lambda + \sum_\beta b_\beta \beta \bmod q^{-1}L_\nu^-),$$

where $a_\alpha,\ b_\beta \in \mathbf{Z}$. The weights α, β occuring in the right-hand side are certainly $< \lambda$ and we obtain

$$C_\lambda^+ = v_\lambda^+ - \sum_\alpha a_\alpha C_\alpha^+ \qquad (\textit{resp. } C_\lambda^- = v_\lambda^- - \sum_\beta b_\beta C_\beta^-).$$

Example 2.3. Let us take $r = 3$, $m = -2$ and compute $C^-_{(0,6,1)}$. We have $w((0,6,1),-2) = s_2 s_0 s_1 s_2 s_0 \tau^{-4}$ and

$$\nu := w((0,6,1),-2)^{-1}(0,6,1) = (-1,0,0).$$

Clearly,

$$C^-_{(2,2,3)} = C^-_{\tau^{-4}(-1,0,0)} = (2,2,3).$$

Then we compute successively ($t = q^{-1}$)

$$\begin{aligned}
v^-_{(1,2,4)} = C^-_{(1,2,4)} &= (1,2,4) - t(2,2,3),\\
v^-_{(1,4,2)} = C^-_{(1,4,2)} &= (1,4,2) - t(1,2,4) - t(2,3,2) + t^2(2,2,3),\\
v^-_{(4,1,2)} = C^-_{(4,1,2)} &= (4,1,2) - t(1,4,2) - t(2,1,4) + t^2(1,2,4)\\
&\quad -t(3,2,2) + t^2(2,3,2),\\
v^-_{(0,1,6)} = C^-_{(0,1,6)} &= (0,1,6) - t(4,1,2) - t(0,4,3) + t^2(1,4,2)\\
&\quad +t^2(2,2,3) - t(1,2,4) - t(0,2,5)\\
&\quad +t^2(3,2,2) + t^2(0,3,4) - t^3(2,3,2),\\
v^-_{(0,6,1)} &= (0,6,1) - t(0,1,6) - t(4,2,1) + t^2(4,1,2) - t(0,3,4)\\
&\quad +(0,4,3) + 2t^2(1,2,4) - 2t(1,4,2)\\
&\quad +2t^2(2,3,2) - 2t^3(2,2,3) - t(0,5,2)\\
&\quad +t^2(0,2,5) + t^2(0,4,3) - t^3(0,3,4).
\end{aligned}$$

We see that $v^-_{(0,6,1)} \equiv (0,6,1) + (0,4,3) \bmod tL_\nu^-$. Thus subtracting the previously calculated element

$$C^-_{(0,4,3)} = (0,4,3) - t(0,3,4) - t(1,4,2) + t^2(1,2,4) + t^2(2,3,2) - t^3(2,2,3)$$

we get

$$C^-_{(0,6,1)} = (0,6,1) - t(0,1,6) - t(4,2,1) + t^2(4,1,2) + t^2(1,2,4) \\ -t(1,4,2) + t^2(2,3,2) - t^3(2,2,3) - t(0,5,2) \\ +t^2(0,2,5) + t^2(0,4,3) - t^3(0,3,4).$$

◇

Put

$$C'_w = \sum_{x\in\widehat{\mathfrak{S}}_r} P_{x,w}(q)\, T_x.$$

Then

$$C_w = \sum_{x\in\widehat{\mathfrak{S}}_r} P_{x,w}(-q^{-1})\, T_x.$$

The $P_{x,w}$ are the Kazhdan-Lusztig polynomials (up to a factor $q^{\ell(w)-\ell(x)}$ and the change of variable $q \mapsto q^{-2}$). Similarly for $\lambda \in \widehat{\mathfrak{S}}_r\nu$ write

$$C^+_\lambda = \sum_{\mu\in\widehat{\mathfrak{S}}_r\nu} P^+_{\mu,\lambda}(q)\, \mu, \qquad C^-_\lambda = \sum_{\mu\in\widehat{\mathfrak{S}}_r\nu} P^-_{\mu,\lambda}(-q^{-1})\, \mu.$$

Then $P^+_{\mu,\lambda}$ and $P^-_{\mu,\lambda}$ are respectively equal to Deodhar's polynomials $P^J_{w(\mu,m),w(\lambda,m)}$ and $\widetilde{P}^J_{w(\mu,m),w(\lambda,m)}$ (again up to a factor $q^{\ell(w)-\ell(x)}$ and the change of variable $q \mapsto q^{-2}$), where J is the set of indices i of the Coxeter generators $s_i \in \mathfrak{S}_{\nu,m}$. Their expression in terms of ordinary Kazhdan-Lusztig polynomials is given by

Theorem 2.4 (Deodhar [4, 5]). *Let $w_{0,\nu}$ be the longest element of $\mathfrak{S}_{\nu,m}$. Then*

$$P^+_{\mu,\lambda} = P_{w(\mu,m)w_{0,\nu}\,,\,w(\lambda,m)w_{0,\nu}}, \quad P^-_{\mu,\lambda} = \sum_{z\in\mathfrak{S}_{\nu,m}} (-q)^{\ell(z)} P_{w(\mu,m)z\,,\,w(\lambda,m)}.$$

We shall also need the following simple observation (see [39], Remark 3.2.4). Suppose that $\mathrm{desc}(\lambda,i,m) = \mathrm{desc}(\mu,i,m) = -\mathrm{sgn}\,(m)$. Then

$$P^+_{s_i\mu,\lambda} = qP^+_{\mu,\lambda}, \qquad P^-_{s_i\mu,\lambda} = qP^-_{\mu,\lambda}. \tag{31}$$

This follows from the fact that if $\mathrm{desc}(\lambda,i,m) = -\mathrm{sgn}\,(m)$ then

$$C'_i C^+_\lambda = (q+q^{-1})\, C^+_\lambda, \qquad C_i C^-_\lambda = -(q+q^{-1})\, C^-_\lambda.$$

§3. Littlewood-Richardson coefficients and Kazhdan-Lusztig polynomials

3.1. The Lusztig conjecture

Let $U_q(\mathfrak{gl}_r)$ be the quantum enveloping algebra of $\mathfrak{gl}_r$. This is a $\mathbf{Q}(q)$-algebra with generators E_i, F_i, $q^{\pm\epsilon_j}$ $(1 \le i \le r-1,\ 1 \le j \le r)$. The relations are standard [14] and will be omitted. To avoid confusion when q is specialized to a complex number, we shall write $K_j^{\pm}$ in place of $q^{\pm\epsilon_j}$. Let $U_{q,\mathbf{Z}}(\mathfrak{gl}_r)$ denote the $\mathbf{Z}[q,q^{-1}]$-subalgebra generated by the elements

$$E_i^{(k)} := \frac{E_i^k}{[k]!}, \quad F_i^{(k)} := \frac{F_i^k}{[k]!}, \quad K_j^{\pm}, \qquad (k \in \mathbf{N}),$$

where $[k]! := [k][k-1]\cdots[2][1]$ and $[k] := (q^k - q^{-k})/(q-q^{-1})$. Let $\zeta \in \mathbf{C}$ be such that ζ^2 is a primitive nth root of 1. One defines $U_\zeta(\mathfrak{gl}_r) := U_{q,\mathbf{Z}}(\mathfrak{gl}_r) \otimes_{\mathbf{Z}[q,q^{-1}]} \mathbf{C}$ where $\mathbf{Z}[q,q^{-1}]$ acts on $\mathbf{C}$ by $q \mapsto \zeta$ [30, 31].

Let $\lambda \in P_r^+$. There is a unique finite-dimensional $U_q(\mathfrak{gl}_r)$-module (of type 1) $W_q(\lambda)$ with highest weight λ. Its character is the same as in the classical case and is given by Weyl's character formula

$$\operatorname{ch} W_q(\lambda) = s_\lambda(e^{\epsilon_1}, \ldots, e^{\epsilon_r}), \tag{32}$$

where s_λ denotes the Schur function (see [33]). Fix a highest weight vector $u_\lambda \in W_q(\lambda)$ and denote by $W_{q,\mathbf{Z}}(\lambda)$ the $U_{q,\mathbf{Z}}(\mathfrak{gl}_r)$-submodule of $W_q(\lambda)$ generated by acting on u_λ. Finally, put

$$W_\zeta(\lambda) := W_{q,\mathbf{Z}}(\lambda) \otimes_{\mathbf{Z}[q,q^{-1}]} \mathbf{C}.$$

This is a $U_\zeta(\mathfrak{gl}_r)$-module called a Weyl module [30]. By definition $\operatorname{ch} W_\zeta(\lambda) = \operatorname{ch} W_q(\lambda)$.

There is a unique simple quotient of $W_\zeta(\lambda)$ denoted by $L(\lambda)$. Its character is given in terms of the characters of the Weyl modules by the Lusztig conjecture. Put $m = -n$ (this assumption will be in force for the whole Section 3) and consider the action of $\widehat{\mathfrak{S}}_r$ on P via π_m. For $\lambda \in P^+$ write $\nu := w(\lambda+\rho, m)^{-1}(\lambda+\rho)$. Then

Theorem 3.1 (Kazhdan-Lusztig, Kashiwara-Tanisaki).

$$\operatorname{ch} L(\lambda) = \sum_w (-1)^{\ell(w(\lambda+\rho,m))-\ell(w)} P_{w,w(\lambda+\rho,m)}(1) \operatorname{ch} W_\zeta(w(\nu)-\rho),$$

where the sum runs over the $w \in \widehat{\mathfrak{S}}_r$ such that $w < w(\lambda+\rho,m)$ and $w(\nu)-\rho \in P^+$.

Note that if λ is a singular weight the coefficient of a given Weyl module $W_\zeta(\mu)$ in the right-hand side of Theorem 3.1 is an alternating sum of $P_{w,w(\lambda,m)}(1)$ over the stabilizer $\mathfrak{S}_{\nu,m}$. In fact, using the notation of Section 2.4 one can rewrite Theorem 3.1 as

$$\operatorname{ch} L(\lambda) = \sum_{\mu} P^{-}_{\mu+\rho,\lambda+\rho}(-1)\ \operatorname{ch} W_\zeta(\mu), \tag{33}$$

where the sum is over the $\mu \in P^+$ such that $\mu + \rho \in \widehat{\mathfrak{S}}_r \nu$.

Example 3.1. Take $r = 3$, $n = 2$ and $\lambda = (4,0,0)$. Then $\lambda + \rho = (6,1,0)$ and for $m = -n = -2$, one has

$$\begin{aligned} C^{-}_{(6,1,0)} &= (6,1,0) - q^{-1}(6,0,1) - q^{-1}(1,6,0) + q^{-2}(0,6,1) \\ &+q^{-2}(1,0,6) - q^{-3}(0,1,6) - q^{-1}(5,2,0) + q^{-2}(5,0,2) \\ &+q^{-2}(2,5,0) - q^{-3}(0,5,2) - q^{-3}(2,0,5) + q^{-4}(0,2,5) \\ &+q^{-2}(4,3,0) - q^{-3}(4,0,3) - q^{-3}(3,4,0) + q^{-4}(0,4,3) \\ &+q^{-4}(3,0,4) - q^{-5}(0,3,4). \end{aligned}$$

It follows that the character of $L(4,0,0)$ for $\zeta^2 = -1$ is given by

$$\operatorname{ch} L(4,0,0) = \operatorname{ch} W_\zeta(4,0,0) - \operatorname{ch} W_\zeta(3,1,0) + \operatorname{ch} W_\zeta(2,2,0).$$

◇

3.2. The tensor product theorem

Let Fr denote the Frobenius map from $U_\zeta(\mathfrak{gl}_r)$ to the (classical) enveloping algebra $U(\mathfrak{gl}_r)$ [30, 3]. This is the algebra homomorphism defined by $\mathrm{Fr}(K_j) = 1$ and

$$\mathrm{Fr}(E_i^{(k)}) = \begin{cases} E_i^{(k/n)} & \text{if } n \text{ divides } k, \\ 0 & \text{otherwise,} \end{cases} \quad \mathrm{Fr}(F_i^{(k)}) = \begin{cases} F_i^{(k/n)} & \text{if } n \text{ divides } k, \\ 0 & \text{otherwise.} \end{cases}$$

(Here we slightly abuse notation and denote by the same symbols the Chevalley generators of $U_\zeta(\mathfrak{gl}_r)$ and those of $U(\mathfrak{gl}_r)$.) Given a $U(\mathfrak{gl}_r)$-module M, one can thus define a $U_\zeta(\mathfrak{gl}_r)$-module M^{Fr} by composing the action of $U(\mathfrak{gl}_r)$ with Fr. If M is a finite-dimensional module with character the symmetric Laurent polynomial $\operatorname{ch} M = \varphi(e^{\epsilon_1}, \dots, e^{\epsilon_r})$, then

$$\operatorname{ch} M^{\mathrm{Fr}} = p_n(\varphi)(e^{\epsilon_1}, \dots, e^{\epsilon_r}) := \varphi(e^{n\epsilon_1}, \dots, e^{n\epsilon_r}),$$

the so-called plethysm of φ with the power sum p_n (see [33]). In particular, the character of the pullback $W(\lambda)^{\mathrm{Fr}}$ of the classical Weyl module $W(\lambda)$ is the plethysm $p_n(s_\lambda)$.

Theorem 3.2 (Lusztig [30]). *Let $\lambda \in P^+$. Write $\lambda = \lambda^{(0)} + n\lambda^{(1)}$, where $\lambda^{(0)}$ is n-restricted, that is,*

$$0 \leq \lambda_i^{(0)} - \lambda_{i+1}^{(0)} < n \qquad (1 \leq i \leq r-1).$$

The simple $U_\zeta(\mathfrak{gl}_r)$-module $L(\lambda)$ is isomorphic to the tensor product

$$L(\lambda) \simeq L(\lambda^{(0)}) \otimes W(\lambda^{(1)})^{\mathrm{Fr}}.$$

Consider now the particular case when λ is a partition whose parts are all divisible by n. Then, writing $n\lambda$ in place of λ, we deduce from Theorem 3.2 and Eq. (33) that $p_n(s_\lambda) = \operatorname{ch} L(n\lambda)$ is given by

$$(34) \quad p_n(s_\lambda) = \sum_\mu P^-_{\mu+\rho, n\lambda+\rho}(-1) \operatorname{ch} W_\zeta(\mu) = \sum_\mu P^-_{\mu+\rho, n\lambda+\rho}(-1) s_\mu,$$

where the sum is over the $\mu \in P^+$ such that $\mu + \rho \in \widehat{\mathfrak{S}}_r(n\lambda + \rho) = \widehat{\mathfrak{S}}_r\rho$.

3.3. Expression of the Littlewood-Richardson coefficients

Let $\lambda \in \mathbb{P}_r^+ = \{\lambda \in P \mid \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_r \geq 0\}$, the set of partitions of length $l(\lambda) \leq r$. It is a well-known result of Littlewood [25] that the coefficients in the expansion of $p_n(s_\lambda)$ on the basis of Schur functions are Littlewood-Richardson multiplicities. More precisely, if $\mu \in \mathbb{P}_r^+$ is such that $\mu + \rho \in \widehat{\mathfrak{S}}_r\rho$ then there is a unique expression

$$\mu + \rho = \gamma + n\alpha, \qquad (\gamma = s\rho,\ s \in \mathfrak{S}_r,\ \alpha \in \mathbf{N}^r)$$

such that $i < j$ and $\gamma_i \equiv \gamma_j\ (n)$ implies $\gamma_i > \gamma_j$. Then for $k \in \{0, 1, \ldots, n-1\}$ the subsequence of α consisting of the α_i such that $\gamma_i \equiv k - r$ is a partition $\mu^{(k)}$ (possibly empty), and one has [25]

$$(35) \qquad \langle p_n(s_\lambda)\ ;\ s_\mu \rangle = (-1)^{\ell(s)} \langle s_\lambda\ ;\ s_{\mu^{(0)}} \cdots s_{\mu^{(n-1)}} \rangle$$

where $\langle \cdot\,;\,\cdot \rangle$ denotes the standard scalar product of the algebra of symmetric functions for which the s_λ form an orthonormal basis. The n-tuple of partitions $(\mu^{(0)}, \ldots, \mu^{(n-1)})$ is called the n-quotient of μ and $(-1)^{\ell(s)}$ the n-sign of μ, denoted $\varepsilon_n(\mu)$. Conversely, provided that r is large enough, given an arbitrary n-tuple of partitions $(\mu^{(0)}, \ldots, \mu^{(n-1)})$ there exists a unique $\mu \in \mathbb{P}_r^+$ such that $\mu + \rho \in \widehat{\mathfrak{S}}_r\rho$ and μ has $(\mu^{(0)}, \ldots, \mu^{(n-1)})$ as n-quotient (see [33, 13]).

Example 3.2. Let $r = 8$, $n = 3$, and $\mu = (6, 6, 4, 4, 4, 3, 2, 1)$. Then

$$\mu + \rho = (13, 12, 9, 8, 7, 5, 3, 1) = (7, 6, 3, 5, 4, 2, 0, 1) + 3\,(2, 2, 2, 1, 1, 1, 1, 0).$$

Thus the 3-quotient of μ is

$$(\mu^{(0)}, \mu^{(1)}, \mu^{(2)}) = ((1,1), (2,2,1), (2,1)).$$

◇

Let us define the Littlewood-Richardson coefficient

$$\begin{aligned} c^{\lambda}_{\mu^{(0)},\dots,\mu^{(n-1)}} &:= \langle s_{\mu^{(0)}} \cdots s_{\mu^{(n-1)}} \; ; \; s_\lambda \rangle \\ &= [W(\mu^{(0)}) \otimes \cdots \otimes W(\mu^{(n-1)}) \; : \; W(\lambda)]\,. \end{aligned}$$

Combining (34) and (35), we have obtained

Theorem 3.3. *Let $\lambda, \mu^{(0)}, \dots, \mu^{(n-1)}$ be partitions and denote by μ the partition with n-quotient $(\mu^{(0)}, \dots, \mu^{(n-1)})$. Take $r \geq l(\mu)$, the number of parts of μ. Then,*

$$c^{\lambda}_{\mu^{(0)},\dots,\mu^{(n-1)}} = P^{-}_{\mu+\rho, n\lambda+\rho}(1)$$

where the right-hand side is a Kazhdan-Lusztig polynomial of parabolic type for $\widehat{\mathfrak{S}}_r$ with $m = -n$. In other words, setting

$$\nu = w(n\lambda + \rho, -n)^{-1}(n\lambda + \rho),$$

one has in terms of the (ordinary) Kazhdan-Lusztig polynomials for $\widehat{\mathfrak{S}}_r$

$$c^{\lambda}_{\mu^{(0)},\dots,\mu^{(n-1)}} = \sum_{z \in \mathfrak{S}_{\nu,-n}} (-1)^{\ell(z)} P_{w(\mu+\rho,-n)z\,,\, w(n\lambda+\rho,-n)}(1).$$

If $l(\lambda) > r$ the polynomial $P^{-}_{\mu+\rho,n\lambda+\rho}$ is not defined, but in this case $l(\lambda) > l(\mu)$ and it is easy to see that $c^{\lambda}_{\mu^{(0)},\dots,\mu^{(n-1)}} = 0$.

Note that if $w = \tau^k\sigma$, $w' = \tau^m\sigma'$ with $k, m \in \mathbf{Z}$, $\sigma, \sigma' \in \widetilde{\mathfrak{S}}_r$, then $P_{w,w'}$ is nonzero only if $k = m$ and then $P_{w,w'} = P_{\sigma,\sigma'}$. Thus the Kazhdan-Lusztig polynomials above are in fact polynomials for $\widetilde{\mathfrak{S}}_r$.

Example 3.3. Take $r = 3$ and $n = -m = 2$. The dominant weights occuring in the expansion of $C^{-}_{(6,3,0)}$ are

$$(6,3,0), \; (6,2,1), \; (5,4,0), \; (4,3,2),$$

with respective coefficients

$$1, \; -q^{-1}, \; -q^{-1}, \; q^{-2}.$$

This gives the following expressions for some Littlewood-Richardson coefficients (which are all equal to 1):

$$c^{(2,1)}_{(1),(2)} = P^-_{(6,3,0),(6,3,0)}(1), \qquad c^{(2,1)}_{\emptyset,(2,1)} = P^-_{(6,2,1),(6,3,0)}(1),$$

$$c^{(2,1)}_{(2),(1)} = P^-_{(5,4,0),(6,3,0)}(1), \qquad c^{(2,1)}_{(1),(1,1)} = P^-_{(4,3,2),(6,3,0)}(1).$$

In terms of ordinary Kazhdan-Lusztig polynomials for $\widetilde{\mathfrak{S}}_3$ we can write for example

$$c^{(2,1)}_{(1),(1,1)} = P_{s_2s_0s_2\,,\,s_2s_0s_2s_1s_2s_0s_2}(1) - P_{s_2s_0s_2s_1\,,\,s_2s_0s_2s_1s_2s_0s_2}(1) = 2-1.$$

$\diamond$

Example 3.4. Let us express the coefficient $c^{(3,2,1)}_{(2,1),(2,1)} = 2$ in terms of Kazhdan-Lusztig polynomials. We take $r=4$, $\lambda=(3,2,1)$ and $\mu=(4,4,2,2)$ so that μ has 2-quotient $((2,1);(2,1))$. It follows that

$$c^{(3,2,1)}_{(2,1),(2,1)} = P^-_{(7,6,3,2),(9,6,3,0)}(1).$$

This Kazhdan-Lusztig polynomial corresponds to the following elements of $\widetilde{\mathfrak{S}}_4$:

$$\begin{aligned} w((9,6,3,0),-2) &= s_1s_2s_1s_3s_2s_1s_0s_1s_3s_2s_1s_3s_0s_1s_3s_2s_0\tau^{-10}, \\ w((7,6,3,2),-2) &= s_1s_2s_1s_3s_2s_1s_0s_1s_3s_2s_0\tau^{-10}. \end{aligned}$$

$\diamond$

Observe that if $n \geq r$, then the n-quotient of the partition $n\mu = (n\mu_1,\ldots,n\mu_r)$ is just $((\mu_1),\ldots,(\mu_r),\emptyset,\ldots,\emptyset)$ up to reordering, and therefore Theorem 3.3 gives

$$P^-_{n\mu+\rho,n\lambda+\rho}(1) = c^\lambda_{(\mu_1),\ldots,(\mu_r)} = K_{\lambda,\mu},$$

the Kostka number. On the other hand, taking into account Lemma 2.3 and the fact that the weight $n\lambda+\rho$ is regular, one also has

$$P^-_{n\mu+\rho,n\lambda+\rho}(1) = P_{n_{\mu^*}\,\tau^{-r+1},n_{\lambda^*}\,\tau^{-r+1}}(1) = P_{n_{\mu^*},n_{\lambda^*}}(1).$$

Hence

$$P_{n_\mu,n_\lambda}(1) = K_{\lambda^*,\mu^*} = K_{\lambda,\mu}$$

since the weight multiplicities of the contragredient representation $W(\lambda^*)$ are equal to those of $W(\lambda)$, and we recover the expression of [28] for the weight multiplicities.

Thus we see that the modular Lusztig conjecture with its restriction $n \geq r$ is enough to express the weight multiplicities in terms of Kazhdan-Lusztig polynomials, but for what concerns the general tensor product multiplicities we need the case $n < r$ and the quantum Lusztig conjecture.

§4. Littlewood-Richardson coefficients and ribbon tableaux

4.1. Ribbon tableaux

Let us start from the well-known formula

$$h_\mu = \sum_\lambda |\mathrm{Tab}\,(\lambda, \mu)|\, s_\lambda, \tag{36}$$

where $h_\mu := h_{\mu_1} \dots h_{\mu_r}$ is a product of complete homogeneous symmetric functions and $\mathrm{Tab}\,(\lambda, \mu)$ denotes the set of semi-standard Young tableaux of shape λ and weight μ [33]. Let $n \in \mathbf{N}^*$. Semi-standard n-ribbon tableaux are combinatorial objects which replace ordinary Young tableaux when one substitutes the plethysm $p_n(h_\mu)$ in place of h_μ in (36). More precisely, denoting by $\mathrm{Tab}_n(\lambda, \mu)$ the set of n-ribbon tableaux of shape λ and weight μ (to be defined below), one has

$$p_n(h_\mu) = \sum_\lambda \varepsilon_n(\lambda)\, |\mathrm{Tab}_n(\lambda, \mu)|\, s_\lambda, \tag{37}$$

where $\varepsilon_n(\lambda)$ is the n-sign of λ.

A ribbon tableau of weight $\mu = (1, 1, \dots, 1)$ is called standard. Standard ribbon tableaux were introduced by Stanton and White [41] in relation with generalizations of the Robinson-Schensted correspondence for the complex reflection groups $G(n, 1, r) = (\mathbf{Z}/n\mathbf{Z}) \wr \mathfrak{S}_r$. In particular, the case $n = 2$ (domino tableaux) is related to Weyl groups of type B, C, D, and therefore to the geometry of flag manifolds for classical groups [23] and to the classification of the primitive ideals of classical enveloping algebras [1, 9]. Semi-standard domino tableaux were introduced in [2] for calculating the multiplicities of the symmetric and alternating square of an irreducible representation of $\mathfrak{gl}_r$ (see also [16, 24]). In an attempt to extend the results of [2] to higher degree plethysms, semi-standard n-ribbon tableaux were defined in [21] and several conjectures were formulated. We shall give a brief review of [21] refering to the paper for more detail.

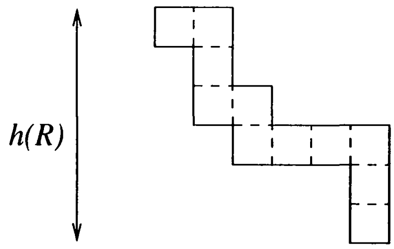

Fig. 2. An 11-ribbon of height $h(R) = 6$

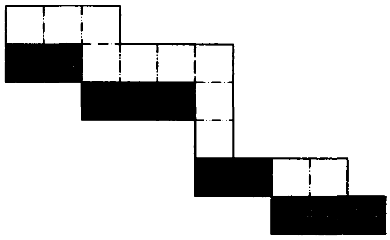

Fig. 3. A skew diagram θ with its subdiagram $\theta\!\downarrow$ shaded

A ribbon is a connected skew Young diagram of width 1, *i.e.* which does not contain any 2×2 square (see Figure 2). The rightmost and bottommost cell is called the origin of the ribbon. An n-ribbon is a ribbon made of n square cells. Let θ be a skew Young diagram, and let $\theta\!\downarrow$ be the horizontal strip made of the bottom cells of the columns of θ (see Figure 3). We say that θ is a horizontal n-ribbon strip of weight m if it can be tiled by m n-ribbons the origins of which lie in $\theta\!\downarrow$. One can check that if such a tiling exists, it is unique (see below Lemma 6.3 and Figure 7). Now, an n-ribbon tableau T of shape λ/ν and weight $\mu = (\mu_1, \dots, \mu_r)$ is defined as a chain of partitions

$$\nu = \alpha^0 \subset \alpha^1 \subset \cdots \subset \alpha^r = \lambda$$

such that α^i/α^{i-1} is a horizontal n-ribbon strip of weight μ_i. Graphically, T may be described by numbering each n-ribbon of α^i/α^{i-1} with the number i (see Figure 4). We denote by $\mathrm{Tab}_n(\lambda/\nu, \mu)$ the set of n-ribbon tableaux of shape λ/ν and weight μ. Define the spin of a ribbon R as $\mathrm{spin}(R) := h(R) - 1$ where $h(R)$ is the height of R, and the spin of a ribbon tableau T as the sum of the spins of its ribbons. Then the sign $(-1)^{\mathrm{spin}(T)}$ depends only on the shape λ/ν of T and is equal to the n-sign $\varepsilon_n(\lambda)$ when ν is empty. We denote it in general by $\varepsilon_n(\lambda/\nu)$.

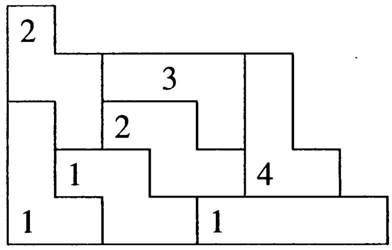

Fig. 4. A 4-ribbon tableau of shape $(8, 7, 6, 6, 1)$, weight $(3, 2, 1, 1)$ and spin 9

4.2. A q-analogue of the Littlewood-Richardson coefficients

Using a classical formula for multiplying a monomial symmetric function by a Schur function one can easily derive Eq. (37). Note that since $h_{s\mu} = h_\mu$ $(s \in \mathfrak{S}_r)$, (37) implies that

$$(38) \qquad |\mathrm{Tab}_n(\lambda, s\mu)| = |\mathrm{Tab}_n(\lambda, \mu)|, \qquad (s \in \mathfrak{S}_r).$$

Let φ_n denote the adjoint of the endomorphism $f \mapsto p_n(f)$ of the space of symmetric functions with respect to $\langle \cdot\,;\,\cdot \rangle$. Recall from Section 3.3 the definition of the n-quotient $(\lambda^{(0)}, \cdots, \lambda^{(n-1)})$ of a partition λ of length r such that $\lambda + \rho \in \widehat{\mathfrak{S}}_r \rho$ (for the action of $\widehat{\mathfrak{S}}_r$ on weights via π_n). Then (35) is equivalent to

$$(39) \qquad \varphi_n(s_\lambda) = \varepsilon_n(\lambda)\, s_{\lambda^{(0)}} \cdots s_{\lambda^{(n-1)}},$$

where we put $\varepsilon_n(\lambda) = 0$ if $\lambda + \rho \notin \widehat{\mathfrak{S}}_r \rho$. By (37) we have

$$|\mathrm{Tab}_n(\lambda, \mu)| = \varepsilon_n(\lambda)\, \langle p_n(h_\mu)\,;\, s_\lambda \rangle = \varepsilon_n(\lambda)\, \langle h_\mu\,;\, \varphi_n(s_\lambda) \rangle.$$

Recalling that the basis dual to $\{h_\mu\}$ is the basis $\{m_\mu\}$ of monomial symmetric functions, we thus have

$$(40) \qquad s_{\lambda^{(0)}} \cdots s_{\lambda^{(n-1)}} = \sum_{\mu \in P^+} |\mathrm{Tab}_n(\lambda, \mu)|\, m_\mu.$$

Hence, putting $x^T := x_1^{\alpha_1} \cdots x_r^{\alpha_r}$ for a ribbon tableau T of weight $\alpha = (\alpha_1, \dots, \alpha_r)$, we get using (38)

(41)

$$s_{\lambda^{(0)}} \cdots s_{\lambda^{(n-1)}} = \sum_{\mu \in P^+} \left(\sum_{\beta \in \mathfrak{S}_r \mu} \left(\sum_{T \in \mathrm{Tab}_n(\lambda, \beta)} x^T \right) \right) = \sum_{T \in \mathrm{Tab}_n(\lambda, \cdot)} x^T$$

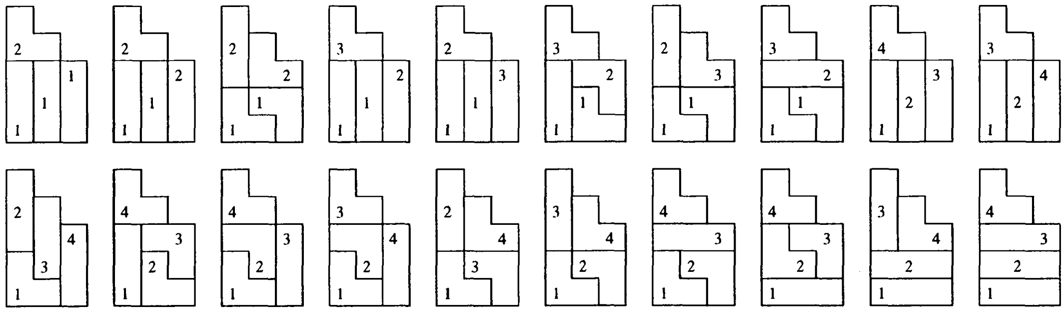

Fig. 5. The 3-ribbon tableaux of shape $(3,3,3,2,1)$ and dominant weight

where we denote by $\mathrm{Tab}_n(\lambda,\cdot)$ the set of n-ribbon tableaux of shape λ (and arbitrary weight).

Now we can introduce a q-analogue of (41) via the spin of ribbon tableaux and set

$$G(\lambda^{(0)},\dots,\lambda^{(n-1)};q,x) := \sum_{T\in \mathrm{Tab}_n(\lambda,\cdot)} q^{\mathrm{spin}(T)} x^T. \tag{42}$$

It was proved in [21] that this function is symmetric with respect to the variables x_i. (This is not clear a priori, and the proof will be recalled below (see Remark 6.5).) Thus, expanding on the basis of Schur functions we get

$$G(\lambda^{(0)},\dots,\lambda^{(n-1)};q,x) = \sum_{\nu} c^{\nu}_{\lambda^{(0)},\dots,\lambda^{(n-1)}}(q)\, s_\nu(x), \tag{43}$$

where the $c^{\nu}_{\lambda^{(0)},\dots,\lambda^{(n-1)}}(q) \in \mathbf{Z}[q]$ are some q-analogues of the Littlewood-Richardson coefficients. The symmetric function (43) is the function $\widetilde{G}^{(n)}_{\lambda}(x;q)$ of [21] up to the change of variable $q \mapsto q^{-2}$ and rescaling by an appropriate power of q.

Example 4.1. The partition having as 3-quotient $= ((1),(1,1),(1))$ is $\mu = (3,3,3,2,1)$. Thus the symmetric function $G((1),(1,1),(1);q)$ is calculated by enumerating the 3-ribbon tableaux of shape μ and dominant weight, and counting their spin (see Figure 5). One obtains

$$\begin{aligned} G((1),(1,1),(1);q) &= q^7 m_{(3,1)} + (q^7+q^5) m_{(2,2)} \\ &\quad + (2q^7+2q^5+q^3) m_{(2,1,1)} \\ &\quad + (3q^7+5q^5+3q^3+q) m_{(1,1,1,1)} \\ &= q^7 s_{(3,1)} + q^5 s_{(2,2)} + (q^5+q^3) s_{(2,1,1)} \\ &\quad + q s_{(1,1,1,1)}. \end{aligned}$$

◇

We can now state our main result, which is the q-analog of Theorem 3.3.

Theorem 4.1. *With the notation of Theorem* 3.3

$$\begin{aligned} c^{\lambda}_{\mu^{(0)},\dots,\mu^{(n-1)}}(q) &= P^{-}_{\mu+\rho,n\lambda+\rho}(q) \\ &= \sum_{z\in\mathfrak{S}_{\nu,-n}} (-q)^{\ell(z)} P_{w(\mu+\rho,-n)z\,,\,w(n\lambda+\rho,-n)}(q). \end{aligned}$$

The next two sections will be devoted to the proof of Theorem 4.1. This proof does not rely on the Lusztig conjecture and thus will give an independent proof of Theorem 3.3.

§5. Canonical bases and Kazhdan-Lusztig polynomials

5.1. Another basis of $\mathcal{P}$

The basis of $\mathcal{P}$ consisting of the weights λ is adapted to the Coxeter-type presentation of $\widehat{H}_r$ in terms of the generators $T_0,\dots,T_{r-1},\tau$. There is another natural basis adapted to the Bernstein presentation in terms of $T_1,\dots,T_{r-1},Y_1,\dots,Y_r$, which is defined as follows. Fix $m\in\mathbf{Z}^*$ and consider the action of $\widehat{H}_r$ via Π_m. Every $\lambda\in P$ has a unique expression as $\lambda = m\beta+\gamma$ $(\beta,\gamma\in P,\ \gamma\in\mathfrak{S}_r\mathcal{A}_{r,m})$. We define $V_\lambda := Y^\beta\gamma$. In other words, the basis $\{V_\lambda\}$ is characterized by

$$V_\gamma = \gamma \qquad (\gamma\in\mathfrak{S}_r\mathcal{A}_{r,m}), \tag{44}$$

$$Y^\beta V_\lambda = V_{\lambda+m\beta} \qquad (\lambda,\beta\in P). \tag{45}$$

Example 5.1. Take $r=2$ and $m=-2$. Then

$$\begin{aligned} V_{(-1,-2)} &= Y_2(-1,0) = T_1\tau(-1,0) = (-1,-2),\\ V_{(-2,-1)} &= Y_1(0,-1) = T_0^{-1}\tau(0,-1) = (-2,-1),\\ V_{(2,-1)} &= Y_1^{-1}(0,-1) = \tau^{-1}T_0(0,-1) = (2,-1),\\ V_{(-1,2)} &= Y_2^{-1}(-1,0) = \tau^{-1}T_1^{-1}(-1,0) = (-1,2)+(q-q^{-1})(0,1). \end{aligned}$$

Take $r=3$ and $m=-3$. Then

$$\begin{aligned} V_{(-2,-1,3)} &= Y_3^{-1}(-2,-1,0) = \tau^{-1}T_1^{-1}T_2^{-1}(-2,-1,0)\\ &= (-2,-1,3)+(q-q^{-1})(0,-1,1)+(q-q^{-1})(-2,0,2)\\ &\quad +(q-q^{-1})^2(-1,0,1). \end{aligned}$$

Remark 5.1. Let $n = |m|$. The basis $\{V_\lambda\}$ can be naturally identified with the basis of monomial tensors of a certain $U_q(\widehat{\mathfrak{sl}}_n)$-module (see Section 7.1). ◇

As illustrated by Example 5.1, in some cases the vectors V_λ and λ coincide. This is made more precise in the following

Proposition 5.2. *If $\lambda = m\beta + \gamma$ as above with $\beta \in P^-$ then $V_\lambda = \lambda$. In particular, if $m < 0$ and $\lambda \in P^+$, or $m > 0$ and $\lambda \in P^-$, then $V_\lambda = \lambda$.*

Proof — Put $s = w(\gamma, m)$ and $\nu = s^{-1}\gamma$. Then by (29) and Lemma 2.2

$$V_\lambda = Y^\beta \gamma = T_\beta T_s \nu = T_{y^\beta s}\, \nu.$$

For $\sigma \neq 1$ in $\mathfrak{S}_{\nu,m} \subset \mathfrak{S}_r$ one has $\ell(s\sigma) > \ell(s)$ (because s is minimal in its coset $s\mathfrak{S}_{\nu,m}$) and $s\sigma \in \mathfrak{S}_r$. Hence by Lemma 2.2

$$\ell(y^\beta s\sigma) = \ell(y^\beta) + \ell(s\sigma) > \ell(y^\beta) + \ell(s) = \ell(y^\beta s).$$

Therefore $y^\beta s$ is also minimal in its coset, that is $w(\lambda, m) = y^\beta s$, and

$$V_\lambda = T_{y^\beta s}\, \nu = T_{w(\lambda,m)}\, \nu = \lambda.$$

□

The next proposition gives a key relation between the bar involution and the basis V_λ. It will result from the following

Lemma 5.3. *Let $\beta \in P$ and $s \in \mathfrak{S}_r$. Then*

$$\overline{(Y^\beta T_s)} = T_{w_0}^{-1} Y^{w_0\beta} T_{w_0 s}.$$

Proof — Recall that $\ell(w_0 s) + \ell(s) = \ell(w_0)$, hence $T_{w_0 s} T_{s^{-1}} = T_{w_0}$ and $\overline{T_s} = T_{w_0}^{-1} T_{w_0 s}$. Write $\beta = \beta' - \beta''$ with β', $\beta'' \in P^+$. By (28) we have $\overline{Y^\beta} = T_{\beta'} T_{\beta''}^{-1}$. Hence, $\overline{(Y^\beta T_s)} = T_{\beta'} T_{\beta''}^{-1} T_{w_0}^{-1} T_{w_0 s}$. Now, using Lemma 2.2 we see that

$$T_{\beta'} T_{\beta''}^{-1} T_{w_0}^{-1} = T_{\beta'} T_{w_0}^{-1} T_{w_0\beta''}^{-1} = T_{w_0}^{-1} T_{w_0\beta'} T_{w_0\beta''}^{-1}$$

because β', $\beta'' \in P^+$. Now, $w_0\beta = (-w_0\beta'') - (-w_0\beta')$, with $-w_0\beta''$, $-w_0\beta' \in P^+$. Hence, using again (28), $\overline{(Y^\beta T_s)} = T_{w_0}^{-1} Y^{w_0\beta} T_{w_0 s}$. □

Proposition 5.4. *Let $\lambda \in P$ and let $\nu \in \mathcal{A}_{r,m}$ be the point congruent to λ. Then*

$$\overline{V_\lambda} = q^{-\ell(w_{0,\nu})} T_{w_0}^{-1} V_{w_0\lambda},$$

where $w_{0,\nu}$ is the longest element in the stabilizer $\mathfrak{S}_{\nu,m}$.

Proof — By Lemma 5.3, $\overline{V_\lambda} = \overline{(Y^\beta T_s)}\, \nu = T_{w_0}^{-1} Y^{w_0\beta} T_{w_0 s}\, \nu$. The minimal length of an element $\sigma \in \mathfrak{S}_r$ such that $\sigma\,\nu = (w_0 s)\nu$ is $\ell(w_0 s) - \ell(w_{0,\nu})$. Hence $T_{w_0 s}\, \nu = q^{-\ell(w_{0,\nu})}(w_0 s)\nu$, and this proves the proposition. □

Example 5.2. Take $m = -2$ and $\lambda = (2,0)$. Then,

$$\begin{aligned} \overline{V_{(2,0)}} &= \overline{Y_1^{-1}}(0,0) = \tau^{-1} T_0^{-1}(0,0) = (2,0) + (q - q^{-1})(0,2), \\ T_1^{-1} V_{(0,2)} &= T_1^{-1} Y_2^{-1}(0,0) = T_1^{-1}\tau^{-1} T_1^{-1}(0,0) = q(2,0) + (q^2 - 1)(0,2). \end{aligned}$$

◇

5.2. Action of $\widehat{H}_r$ on the basis V_λ

The next lemma allows to compute the action of $\widehat{H}_r$ on $\{V_\lambda\}$.

Lemma 5.5. *Let $i \in \{1, \dots, r-1\}$ and $k \in \mathbf{Z}$. There holds*

$$T_i Y_i^k = Y_{i+1}^k T_i + (q - q^{-1}) Y_{i+1} \frac{Y_i^k - Y_{i+1}^k}{Y_i - Y_{i+1}}.$$

In other words,

$$T_i Y_i^k = \begin{cases} Y_{i+1}^k T_i + (q - q^{-1}) \sum_{j=1}^{k} Y_i^{k-j} Y_{i+1}^j, & (k \geq 0), \\ Y_{i+1}^k T_i + (q^{-1} - q) \sum_{j=1}^{-k} Y_i^{-j} Y_{i+1}^{j+k}, & (k < 0). \end{cases}$$

Proof — It follows from (24) (20) by a straightforward computation. □

Let $\lambda \in P$ and $1 \leq i \leq r-1$. Write $\lambda = m\beta + \gamma$ with $\beta, \gamma \in P$ and $\gamma \in \mathfrak{S}_r \mathcal{A}_{r,m}$. Then $V_\lambda = (\prod_{j \neq i,i+1} Y_j^{\beta_j})(Y_i Y_{i+1})^{\beta_{i+1}} Y_i^{\beta_i - \beta_{i+1}} V_\gamma$. Since T_i commutes with Y_j ($j \neq i, i+1$) and $Y_i Y_{i+1}$, we have

$$T_i V_\lambda = \left(\prod_{j \neq i,i+1} Y_j^{\beta_j} \right) (Y_i Y_{i+1})^{\beta_{i+1}} T_i Y_i^{\beta_i - \beta_{i+1}} V_\gamma.$$

Thus to compute $T_i V_\lambda$ we can use the commutation relation of Lemma 5.5 with $k = \beta_i - \beta_{i+1}$ together with the fact that since $V_\gamma = \gamma$, we have

$$T_i V_\gamma = \begin{cases} V_{s_i\gamma} & \text{if } \operatorname{desc}(\gamma, i, m) = \operatorname{sgn}(m), \\ q^{-1} V_{s_i\gamma} & \text{if } \operatorname{desc}(\gamma, i, m) = 0, \\ V_{s_i\gamma} + (q^{-1} - q) V_\gamma & \text{if } \operatorname{desc}(\gamma, i, m) = -\operatorname{sgn}(m). \end{cases}$$

5.3. Projection on the positive Weyl chamber

From now on we fix $n \geq 2$ and we assume that $\widehat{H}_r$ acts on $\mathcal{P}_r$ via Π_{-n}. Introduce the $\mathbf{Z}[q, q^{-1}]$-submodule

$$\mathcal{J}_r := \sum_{i=1}^{r-1} \operatorname{im} C_i' \subset \mathcal{P}_r,$$

and define $\mathcal{F}_r := \mathcal{P}_r / \mathcal{J}_r$. The image of $\lambda \in P$ in $\mathcal{F}_r$ under the natural projection

$$\operatorname{pr} : \mathcal{P}_r \longrightarrow \mathcal{F}_r$$

will be denoted by $[\lambda] = [\lambda_1, \ldots, \lambda_r]$. For $v \in \mathcal{P}_r$ we have by definition

$$\operatorname{pr}(C_i' v) = 0 = \operatorname{pr}(T_i v) + q \operatorname{pr}(v).$$

Hence taking $v = \lambda \in P$, we obtain that if $\lambda_i < \lambda_{i+1}$ then $[\lambda] = -q^{-1}[s_i\lambda]$, and if $\lambda_i = \lambda_{i+1}$ then $[\lambda] = 0$. This implies that a spanning set of $\mathcal{F}_r$ is given by the $[\lambda]$ such that $\lambda_1 > \lambda_2 > \ldots > \lambda_r$. We put $P^{++} := \{\lambda \in P \mid \lambda_1 > \lambda_2 > \ldots > \lambda_r\}$.

Lemma 5.6. *$\{[\lambda] \mid \lambda \in P^{++}\}$ is a basis of $\mathcal{F}_r$.*

Proof — Suppose that $\sum_{\lambda \in P^{++}} a_\lambda [\lambda] = 0$. Then $\sum_{\lambda \in P^{++}} a_\lambda \lambda \in \mathcal{J}_r$. Recall that

$$C_{w_0} = \sum_{s \in \mathfrak{S}_r} (-q)^{\ell(s) - \ell(w_0)} T_s = \overline{C_{w_0}} = \sum_{s \in \mathfrak{S}_r} (-q)^{-\ell(s) + \ell(w_0)} T_s^{-1}$$

satisfies $C_{w_0} C_i' = 0$ $(1 \leq i \leq r-1)$. Hence $\mathcal{J}_r \subset \ker C_{w_0}$. Thus

$$C_{w_0}\Big(\sum_{\lambda \in P^{++}} a_\lambda \lambda\Big) = \sum_{\lambda \in P^{++},\ s \in \mathfrak{S}_r} a_\lambda (-q)^{-\ell(s) + \ell(w_0)} s\lambda = 0,$$

which implies that $a_\lambda = 0$ for all $\lambda \in P^{++}$. □

Note that for $v \in \mathcal{P}_r$, $\overline{C_i' v} = C_i' \overline{v}$. Hence $\overline{\mathcal{J}_r} \subset \mathcal{J}_r$ and one can define a semi-linear involution on $\mathcal{F}_r$ by

$$\overline{\operatorname{pr}(v)} := \operatorname{pr}(\overline{v}) \qquad (v \in \mathcal{P}_r). \tag{46}$$

Let us define

$$|\lambda\rangle := q^{-\ell(w_0)}\,\mathrm{pr}\,(V_\lambda). \tag{47}$$

Then, by Proposition 5.2, for $\lambda \in P^{++}$ we have $|\lambda\rangle = q^{-\ell(w_0)}\,[\lambda]$, so that $\{|\lambda\rangle \mid \lambda \in P^{++}\}$ is also a basis of $\mathcal{F}_r$. The next proposition shows that it is also useful to work with the vectors $|\lambda\rangle$ associated with arbitrary weights $\lambda \in P$, which can be thought of as some q-wedge products (see below Section 7.2).

Proposition 5.7. *For $\lambda \in P$, we have*

$$\overline{|\lambda\rangle} = (-1)^{\ell(w_0)} q^{\ell(w_0)-\ell(w_{0,\nu})}\,|w_0\lambda\rangle.$$

Proof — By Proposition 5.4 we have $\overline{V_\lambda} = q^{-\ell(w_{0,\nu})} T_{w_0}^{-1} V_{w_0\lambda}$. But for all $v \in \mathcal{P}_r$,

$$\mathrm{pr}\,(T_{w_0}^{-1} v) = (-q)^{-\ell(w_0)}\,\mathrm{pr}\,(v).$$

Thus,

$$\begin{aligned}\overline{|\lambda\rangle} = q^{\ell(w_0)}\,\mathrm{pr}\,(\overline{V_\lambda}) &= (-1)^{\ell(w_0)} q^{-\ell(w_{0,\nu})}\mathrm{pr}\,(V_{w_0\lambda}) \\ &= (-1)^{\ell(w_0)} q^{\ell(w_0)-\ell(w_{0,\nu})}\,|w_0\lambda\rangle.\end{aligned}$$

□

Remark 5.8. It is easy to check that the exponent $\ell(w_0)-\ell(w_{0,\nu})$ of q is equal to the number of pairs (i,j) with $1 \le i < j \le r$ such that $\lambda_i - \lambda_j$ is not divisible by n. ◇

The next proposition gives a set of straightening rules to express an element $|\mu\rangle$ with $\mu \notin P^{++}$ on the basis $\{|\lambda\rangle \mid \lambda \in P^{++}\}$.

Proposition 5.9. *Let $\mu \in P$ be such that $\mu_i < \mu_{i+1}$. Write $\mu_{i+1} = \mu_i + kn + j$ with $k \ge 0$ and $0 \le j < n$. Then*

$$|\mu\rangle = -|s_i\mu\rangle \quad \textit{if } j = 0, \tag{48}$$

$$|\mu\rangle = -q^{-1}|s_i\mu\rangle \quad \textit{if } k = 0, \tag{49}$$

$$|\mu\rangle = -q^{-1}|s_i\mu\rangle - |y_i^{-k}y_{i+1}^{k}\mu\rangle - q^{-1}|y_i^{k}y_{i+1}^{-k}s_i\mu\rangle \quad \textit{otherwise.} \tag{50}$$

Proof — To simplify the notation, let us write $l = \mu_i$ and $m = \mu_{i+1}$. Since the relations only involve components i and $i+1$ we shall also use the shorthand notations (k,l) and $|k,l\rangle$ in place of $V_{(\mu_1,\dots,\mu_{i-1},k,l,\mu_{i+2},\dots,\mu_r)} \in \mathcal{P}_r$ and $|(\mu_1,\dots,\mu_{i-1},k,l,\mu_{i+2},\dots,\mu_r)\rangle \in \mathcal{F}_r$.

Suppose $j = 0$. It follows from Section 5.2 that $T_i(l,l) = q^{-1}(l,l)$. Hence $(l,l) \in \operatorname{im} C_i'$. Since $(Y_i^{-k}+Y_{i+1}^{-k})C_i' = C_i'(Y_i^{-k}+Y_{i+1}^{-k})$ we also have $(Y_i^{-k} + Y_{i+1}^{-k})(l,l) = (m,l) + (l,m) \in \operatorname{im} C_i'$, and thus $|l,m\rangle + |m,l\rangle = 0$.

Suppose $k = 0$. Then $T_i(l,m) = (m,l)$ by Section 5.2, and $C_i'(l,m) = (m,l) + q(l,m) \in \operatorname{im} C_i'$, which gives $|l,m\rangle = -q^{-1}|m,l\rangle$.

Finally suppose that $j,k > 0$. By the previous case $(m, l+kn) + q(l+kn,m) \in \operatorname{im} C_i'$. Applying $Y_i^k + Y_{i+1}^k$ we get that $(m,l) + (m-kn, l+kn) + q(l,m) + q(l+kn, m-kn) \in \operatorname{im} C_i'$, which gives the third claim. □

Example 5.3. Take $r = 2$ and $n = 2$. Then

$$|1,4\rangle = -q^{-1}\,|4,1\rangle - |3,2\rangle - q^{-1}\,|2,3\rangle,$$

by Eq. (50), and $|2,3\rangle = -q^{-1}\,|3,2\rangle$ by Eq. (49). Thus

$$|1,4\rangle = -q^{-1}\,|4,1\rangle + (q^{-2}-1)\,|3,2\rangle.$$

Hence, by Proposition 5.7, $\overline{|4,1\rangle} = |4,1\rangle + (q-q^{-1})\,|3,2\rangle$. ◇

For $\mu \in P^{++}$ write $\overline{|\mu\rangle} = \sum_{\lambda\in P^{++}} a_{\lambda\mu}(q)\,|\lambda\rangle$. Using Proposition 5.7 and Proposition 5.9, we easily see that the coefficients $a_{\lambda\mu}(q)$ satisfy the following properties

Corollary 5.10. (i) *The coefficients $a_{\lambda\mu}(q)$ are invariant under translation of λ and μ by $\epsilon_1 + \cdots + \epsilon_r$. Hence it is enough to describe the $a_{\lambda\mu}(q)$ for which $\lambda - \rho$ and $\mu - \rho$ have non-negative components, i.e. $\lambda - \rho$ and $\mu - \rho$ are partitions.*

(ii) *If $a_{\lambda\mu}(q) \neq 0$ then $\lambda \in \widetilde{\mathfrak{S}}_r\mu$. In particular, if $\lambda - \rho$ and $\mu - \rho$ are partitions, they are partitions of the same integer k.*

(iii) *The matrix $\mathbf{A}_k$ with entries the $a_{\lambda\mu}(q)$ for which $\lambda - \rho$ and $\mu - \rho$ are partitions of k is lower unitriangular if the columns and rows are indexed in decreasing lexicographic order.*

Example 5.4. For $n = 2$ and $r = 3$, the matrices $\mathbf{A}_k$ for $k = 2,3,4$ are

$$\begin{array}{cc} (4,1,0) & (3,2,0) \\ 1 & 0 \\ q-q^{-1} & 1 \end{array} \qquad \begin{array}{ccc} (5,1,0) & (4,2,0) & (3,2,1) \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ q-q^{-1} & 0 & 1 \end{array}$$

$$\begin{array}{cccc}
(6,1,0) & (5,2,0) & (4,3,0) & (4,2,1) \\
1 & 0 & 0 & 0 \\
q-q^{-1} & 1 & 0 & 0 \\
q^{-2}-1 & q-q^{-1} & 1 & 0 \\
0 & q^2-1 & q-q^{-1} & 1
\end{array}$$

◇

5.4. Canonical bases of $\mathcal{F}_r$

Let $\mathcal{L}^+$ (resp. $\mathcal{L}^-$) be the $\mathbf{Z}[q]$ (*resp.* $\mathbf{Z}[q^{-1}]$)-lattice in $\mathcal{F}_r$ with basis $\{|\lambda\rangle \mid \lambda \in P^{++}\}$. The fact that the matrix of the bar involution is unitriangular on the basis $\{|\lambda\rangle \mid \lambda \in P^{++}\}$ implies by a classical argument (see [32], 7.10 and [6]) that

Theorem 5.11. *There exist bases $\{G_\lambda^+ \mid \lambda \in P^{++}\}$, $\{G_\lambda^- \mid \lambda \in P^{++}\}$ of $\mathcal{F}_r$ characterized by:*

(i) $\overline{G_\lambda^+} = G_\lambda^+$, $\overline{G_\lambda^-} = G_\lambda^-$,

(ii) $G_\lambda^+ \equiv |\lambda\rangle \bmod q\mathcal{L}^+$, $G_\lambda^- \equiv |\lambda\rangle \bmod q^{-1}\mathcal{L}^-$.

These bases were introduced in [22] (in the limit $r \to \infty$, *cf.* Section 7), using Proposition 5.7 as the definition of the bar involution on $\mathcal{F}$. Set

$$G_\mu^+ = \sum_\lambda c_{\lambda,\mu}(q)\,|\lambda\rangle\,, \qquad G_\lambda^- = \sum_\mu l_{\lambda,\mu}(-q^{-1})\,|\mu\rangle\,.$$

Let $\mathbf{C}_k$ and $\mathbf{L}_k$ denote respectively the matrices with entries the coefficients $c_{\lambda\mu}(q)$ and $l_{\lambda\mu}(q)$ for which $\lambda-\rho$ and $\mu-\rho$ are partitions of k.

Example 5.5. For $r=3$ and $n=2$ we have

$$\begin{array}{c|cccc}
 & (6,1,0) & (5,2,0) & (4,3,0) & (4,2,1) \\
\hline
 & 1 & 0 & 0 & 0 \\
\mathbf{C}_4 = & q & 1 & 0 & 0 \\
 & 0 & q & 1 & 0 \\
 & q & q^2 & q & 1
\end{array}$$

$$\begin{array}{c|cccc}
 & (6,1,0) & (5,2,0) & (4,3,0) & (4,2,1) \\
\hline
 & 1 & q & q^2 & 0 \\
\mathbf{L}_4 = & 0 & 1 & q & 0 \\
 & 0 & 0 & 1 & q \\
 & 0 & 0 & 0 & 1
\end{array}$$

◇

Clearly, if $c_{\lambda,\mu}$ or $l_{\lambda,\mu} \neq 0$, then λ and μ lie on the same orbit under $\widehat{\mathfrak{S}}_r$. Let ν be the point of $\mathcal{A}_{r,-n}$ on this orbit. Write $\widehat{w}_\lambda := w(w_0\lambda, -n)w_{0,\nu}$ and similarly $\widehat{w}_\mu := w(w_0\mu, -n)w_{0,\nu}$. The main result of this section is

Theorem 5.12 (Varagnolo, Vasserot [42]). *With the above notation, we have*

$$l_{\lambda,\mu} = P^-_{\mu,\lambda}, \tag{51}$$

a parabolic Kazhdan-Lusztig polynomial for the action of $\widehat{\mathfrak{S}}_r$ on P_r via π_{-n}, and

$$c_{\lambda,\mu} = \sum_{s\in\mathfrak{S}_r} (-q)^{\ell(s)} P_{s\widehat{w}_\lambda\,,\,\widehat{w}_\mu}. \tag{52}$$

Remark 5.13. (i) In view of Theorem 2.4, it follows from Eq. (52) that $c_{\lambda,\mu}$ is also a parabolic Kazhdan-Lusztig polynomial of negative type with respect to the parabolic subgroup $\mathfrak{S}_r$ of $\widehat{\mathfrak{S}}_r$, (but for the right $\widehat{H}_r$-module $\mathbf{1}_{q^{-1}} \otimes_{H_r} \widehat{H}_r$). This agrees with the expression obtained by Goodman and Wenzl when $\mu - \rho$ is a n-regular partition [10].
(ii) Let $\overline{\mathcal{F}}_r$ denote the specialization of $\mathcal{F}_r$ at $q = 1$. Define a $\mathbf{Z}$-linear map ι from the Grothendieck group of finite-dimensional representations of $U_\zeta(\mathfrak{gl}_r)$ to $\overline{\mathcal{F}}_r$ by

$$\iota[W(\lambda)] = |\lambda + \rho\rangle \qquad (\lambda \in P_r^+).$$

Then comparing Theorem 5.12 and the Lusztig conjecture (33) we see that $\iota[L(\lambda)] = G^-_{\lambda+\rho}$.

Proof — Consider the element $D_\lambda := \mathrm{pr}\,(C^-_\lambda) \in \mathcal{F}_r$. Then $\overline{D_\lambda} = D_\lambda$ by (46). Since $\lambda \in P^{++}$, $\mathrm{desc}(\lambda, i, -n) = 1$ for all $i = 1, \ldots, r-1$. Therefore using (31) we see that

$$D_\lambda = [r]! \sum_{\mu\in P^{++}} P^-_{\mu,\lambda}(-q^{-1})\,|\mu\rangle.$$

Hence $(1/[r]!)D_\lambda$ is bar invariant and congruent to $|\lambda\rangle$ modulo $q^{-1}\mathcal{L}^-$. Thus $D_\lambda = [r]!\,G^-_\lambda$ and (51) is proved.

Next put $E_\mu := \mathrm{pr}\,(C^+_{w_0\mu}) \in \mathcal{F}_r$. Then $\overline{E_\mu} = E_\mu$. We have

$$E_\mu = \mathrm{pr}\left(\sum_{\alpha\in\widehat{\mathfrak{S}}_r\nu} P^+_{\alpha,w_0\mu}\,\alpha\right) = \sum_{\lambda\in P^{++}} \left(\sum_{s\in\mathfrak{S}_r} (-q)^{-\ell(s)} P^+_{s\lambda,w_0\mu}\right) q^{\ell(w_0)}\,|\lambda\rangle.$$

This shows that $E_\mu \equiv (-1)^{\ell(w_0)}\,|\mu\rangle \bmod q\mathcal{L}^+$. Hence, $E_\mu = (-1)^{\ell(w_0)}\,G^+_\mu$. It follows that

$$\begin{aligned} c_{\lambda,\mu} &= \sum_{s\in\mathfrak{S}_r} (-q)^{\ell(w_0)-\ell(s)} P^+_{s\lambda,w_0\mu} \\ &= \sum_{\sigma\in\mathfrak{S}_r} (-q)^{\ell(\sigma)} P_{w(\sigma w_0\lambda,-n)w_{0,\nu}\,,\,w(w_0\mu,-n)w_{0,\nu}} \end{aligned}$$

by Theorem 2.4. Finally, since $w_0\lambda \in P^{--}$ we have $w(\sigma w_0\lambda,-n) = \sigma w(w_0\lambda,-n)$ for all $\sigma\in\mathfrak{S}_r$, and we get (52). $\square$

§6. A q-analogue of the tensor product theorem

6.1. Action of $Z(\widehat{H}_r)$ on $\mathcal{F}_r$

By a result of Bernstein (see [28], Th. 8.1), the center $Z(\widehat{H}_r)$ of $\widehat{H}_r$ is the algebra of symmetric Laurent polynomials in the elements Y_i. Clearly, $Z(\widehat{H}_r)$ leaves invariant the submodule $\mathcal{J}_r$. It follows that $Z(\widehat{H}_r)$ acts on $\mathcal{F}_r = \mathcal{P}_r/\mathcal{J}_r$. This action can be computed via (45) and (47). In particular $B_k = \sum_{i=1}^r Y_i^k$ acts by

$$B_k\,|\lambda\rangle = \sum_{j=1}^{r} |\lambda - nk\epsilon_j\rangle, \qquad (k\in\mathbf{Z}^*). \tag{53}$$

Note that the right-hand side of (53) may involve terms $|\mu\rangle$ with $\mu\notin P^+$ which have to be expressed on the basis $\{|\lambda\rangle \mid \lambda\in P^{++}\}$ by repeated applications of Proposition 5.9.

Example 6.1. Take $r=4$ and $n=2$. We have

$$B_{-2}\,|3,2,1,0\rangle = |7,2,1,0\rangle + |3,6,1,0\rangle + |3,2,5,0\rangle + |3,2,1,4\rangle.$$

By Proposition 5.9,

$$\begin{aligned} &|3,6,1,0\rangle = -q^{-1}\,|6,3,1,0\rangle + (q^{-2}-1)\,|5,4,1,0\rangle, \\ &|3,2,5,0\rangle = -q^{-1}\,|3,5,2,0\rangle + (q^{-2}-1)\,|3,4,3,0\rangle = q^{-1}\,|5,3,2,0\rangle, \\ &|3,2,1,4\rangle = -q^{-1}\,|3,2,4,1\rangle + (q^{-2}-1)\,|3,2,3,2\rangle = -q^{-2}\,|4,3,2,1\rangle, \end{aligned}$$

which yields

$$\begin{aligned} B_{-2}\,|3,2,1,0\rangle &= |7,2,1,0\rangle - q^{-1}|6,3,1,0\rangle \\ &\quad + (q^{-2}-1)|5,4,1,0\rangle + q^{-1}|5,3,2,0\rangle - q^{-2}|4,3,2,1\rangle. \end{aligned}$$

◇

The compatibility of the bar involution with this action is given by the next

Proposition 6.1. *For $u \in \mathcal{F}_r$ and $z \in Z(\widehat{H}_r)$ one has*

$$\overline{zu} = z\overline{u}.$$

Proof — Since z is a symmetric Laurent polynomial in the Y_i, we see using Lemma 5.3 that $\overline{z} = T_{w_0}^{-1} z T_{w_0} = z$. □

6.2. Action of $Z(\widehat{H}_r)$ and ribbon tableaux

We shall now show that the straightening relations can be avoided provided that one uses appropriate linear bases of $Z(\widehat{H}_r)$. For $d \in [1,r] := \{1,2,\ldots,r\}$ and $m \in \mathbf{N}^*$ define

$$\widetilde{\mathcal{U}}_d := \sum_{1\le i_1<i_2<\ldots<i_d\le r} Y_{i_1}Y_{i_2}\cdots Y_{i_d}, \tag{54}$$

$$\widetilde{\mathcal{V}}_d := \sum_{1\le i_1<i_2<\ldots<i_d\le r} Y_{i_1}^{-1}Y_{i_2}^{-1}\cdots Y_{i_d}^{-1}, \tag{55}$$

$$\mathcal{U}_m := \sum_{1\le i_1\le i_2\le\ldots\le i_m\le r} Y_{i_1}Y_{i_2}\cdots Y_{i_m}, \tag{56}$$

$$\mathcal{V}_m := \sum_{1\le i_1\le i_2\le\ldots\le i_m\le r} Y_{i_1}^{-1}Y_{i_2}^{-1}\cdots Y_{i_m}^{-1}. \tag{57}$$

For $\alpha \in [1,r]^s$ set $\widetilde{\mathcal{U}}_\alpha := \widetilde{\mathcal{U}}_{\alpha_1}\cdots\widetilde{\mathcal{U}}_{\alpha_s}$, $\widetilde{\mathcal{V}}_\alpha := \widetilde{\mathcal{V}}_{\alpha_1}\cdots\widetilde{\mathcal{V}}_{\alpha_s}$, and for $\beta \in \mathbf{N}^{*s}$ set $\mathcal{U}_\beta := \mathcal{U}_{\beta_1}\cdots\mathcal{U}_{\beta_s}$, $\mathcal{V}_\beta := \mathcal{V}_{\beta_1}\cdots\mathcal{V}_{\beta_s}$. In other words, using the notation of [33] for symmetric functions,

$$\widetilde{\mathcal{U}}_\alpha = e_\alpha(Y_1,\ldots,Y_r), \qquad \widetilde{\mathcal{V}}_\alpha = e_\alpha(Y_1^{-1},\ldots,Y_r^{-1}),$$
$$\mathcal{U}_\beta = h_\beta(Y_1,\ldots,Y_r), \qquad \mathcal{V}_\beta = h_\beta(Y_1^{-1},\ldots,Y_r^{-1}).$$

The following formulas were obtained in [21]. They will allow us to relate ribbon tableaux to Kazhdan-Lusztig polynomials. Put

$$L^{(n)}_{\lambda/\nu\,,\,\mu}(q) := \sum_{T\in \mathrm{Tab}_n(\lambda/\nu\,,\,\mu)} q^{\mathrm{spin}(T)}. \tag{58}$$

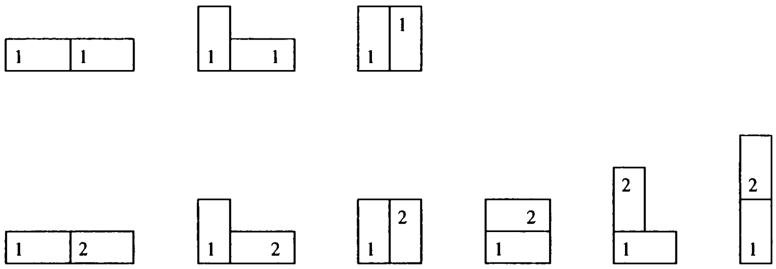

Fig. 6. The domino tableaux of weight (2) and (1, 1)

Theorem 6.2. *Let $\nu \in \mathbb{P}_r^+$ and $\alpha \in [1,r]^s$. Set $k = |\alpha| := \alpha_1 + \cdots + \alpha_s$. We have*

$$\widetilde{\mathcal{U}}_\alpha \, |\nu + \rho\rangle = (-q)^{-(n-1)k} \sum_{\mu \in \mathbb{P}_r^+} L^{(n)}_{\nu'/\mu', \alpha}(-q) \; |\mu + \rho\rangle, \tag{59}$$

$$\widetilde{\mathcal{V}}_\alpha \, |\nu + \rho\rangle = (-q)^{-(n-1)k} \sum_{\lambda \in \mathbb{P}_r^+} L^{(n)}_{\lambda'/\nu', \alpha}(-q) \; |\lambda + \rho\rangle, \tag{60}$$

where for $\lambda \in \mathbb{P}_r^+$, λ' denotes the conjugate partition.

Note that in (59) (60) λ', μ', ν' may be partitions of length $s > r$.

Example 6.2. Let us redo the calculation of Example 6.1 using domino tableaux. Clearly, $B_{-2} = \widetilde{\mathcal{V}}_{(1,1)} - 2\widetilde{\mathcal{V}}_{(2)}$. Now, applying Theorem 6.2 we have

$$\begin{aligned}
\widetilde{\mathcal{V}}_{(2)} \, |\rho\rangle &= q^{-2} \, |(1,1,1,1) + \rho\rangle - q^{-1} \, |(2,1,1) + \rho\rangle + |(2,2) + \rho\rangle, \\
\widetilde{\mathcal{V}}_{(1,1)} \, |\rho\rangle &= q^{-2} \, |(1,1,1,1) + \rho\rangle - q^{-1} \, |(2,1,1) + \rho\rangle \\
&\quad + (1 + q^{-2}) \, |(2,2) + \rho\rangle - q^{-1} |(3,1) + \rho\rangle + |(4) + \rho\rangle
\end{aligned}$$

as illustrated by Figure 6, and we recover the result of Example 6.1. $\diamond$

The proof of Theorem 6.2 is based on the following simple combinatorial lemma.

Lemma 6.3. *Let λ, $\nu \in \mathbb{P}_r^+$ and $k \in [1,r]$. Put $\beta = \epsilon_1 + \cdots + \epsilon_k$. The skew Young diagram λ'/ν' is a horizontal n-ribbon strip of weight k if and only if there exist $s, \sigma \in \mathfrak{S}_r$ such that $\nu + \rho + s(n\beta) = \sigma(\lambda + \rho)$. If this is the case,*

$$\ell(\sigma) = (n-1)k - \mathrm{spin}(\lambda'/\nu').$$

Proof — The proof is elementary and is left to the reader. $\square$

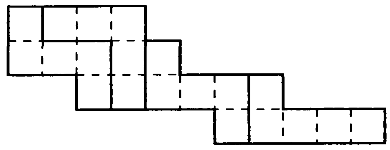

Fig. 7. A horizontal 5-ribbon strip of weight 4 and spin 7

Example 6.3. Take $r = 11$, $\lambda = (4,4,4,4,3,2,2,2,1,1,1)$ and $\nu = (2,2,1,1,1,1)$. Then $\lambda'/\nu' = (11,8,5,4)/(6,2)$ is a horizontal 5-ribbon strip of weight 4. Indeed

$$(12,11,14,13,7,6,9,3,2,1,5) = \nu + \rho + (0,0,5,5,0,0,5,0,0,0,5)$$

is a permutation of $\lambda + \rho$. This permutation has length 9, thus the spin of λ'/ν' is equal to $4.4 - 9 = 7$, as can be checked on Figure 7. $\diamond$

Proof of Theorem 6.2— Since $\widetilde{\mathcal{V}}_\alpha := \widetilde{\mathcal{V}}_{\alpha_1} \cdots \widetilde{\mathcal{V}}_{\alpha_s}$, it is enough to prove the theorem in the case $\alpha = (k)$. Let $\beta = \epsilon_1 + \cdots + \epsilon_k$. Observe that we can reformulate (55) as $\widetilde{\mathcal{V}}_k = \sum_{\zeta \in \mathfrak{S}_r \beta} Y^{-\zeta}$. Hence we have

$$\widetilde{\mathcal{V}}_k \, |\nu + \rho\rangle = \sum_{\gamma \in \mathfrak{S}_r n\beta} |\nu + \rho + \gamma\rangle .$$

If $\xi := \nu + \rho + \gamma \notin P^{++}$ we have to use the straightening relations of Proposition 5.9 to express $|\xi\rangle$ on the basis $\{|\lambda\rangle \mid \lambda \in P^{++}\}$. But if $\xi_i < \xi_{i+1}$ then clearly we must have $\xi_i < \xi_{i+1} < \xi_i + n$, and we need only the simple relation (49). It follows that $|\xi\rangle = (-q)^{-\ell(\sigma)} |\lambda + \rho\rangle$, where $\lambda + \rho$ is the decreasing reordering of ξ and σ is the permutation mapping ξ into $\lambda + \rho$. By Lemma 6.3, $\ell(\sigma) = (n-1)k - \mathrm{spin}(\lambda'/\nu')$ and we are done. The proof for $\widetilde{\mathcal{U}}_k$ is similar. $\square$

We now deduce from Theorem 6.2 similar formulas for the operators $\mathcal{U}_\beta$ and $\mathcal{V}_\beta$.

Theorem 6.4. *Let $\nu \in \mathbb{P}_r^+$ and $\beta \in \mathbf{N}^{*s}$. We have*

$$\mathcal{U}_\beta \, |\nu + \rho\rangle = \sum_{\mu \in \mathbb{P}_r^+} L^{(n)}_{\nu/\mu,\beta}(-q^{-1}) \; |\mu + \rho\rangle, \tag{61}$$

$$\mathcal{V}_\beta \, |\nu + \rho\rangle = \sum_{\lambda \in \mathbb{P}_r^+} L^{(n)}_{\lambda/\nu,\beta}(-q^{-1}) \; |\lambda + \rho\rangle. \tag{62}$$

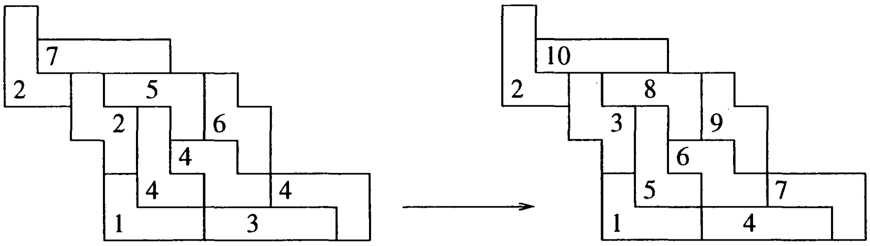

Fig. 8. Standardization $T \to \mathcal{T}$ of a ribbon tableau

Proof — Again, it is enough to prove this for $\beta = (k)$. Recall that a composition of $k \in \mathbf{N}$ is an ordered partition of k, that is, a sequence $\alpha = (\alpha_1, \dots, \alpha_s)$ of positive integers such that $\sum_i \alpha_i = k$. We denote this by $\alpha \models k$ and we call s the length $l(\alpha)$ of α. There is a classical formula for expressing the symmetric function h_k in terms of the e_α, namely

$$h_k = \sum_{\alpha \models k} (-1)^{k-l(\alpha)} e_\alpha .$$

Thus by Theorem 6.2, we have

$$\mathcal{V}_k \, |\nu + \rho\rangle = (-q)^{-(n-1)k} \sum_{\lambda} \left(\sum_{\alpha \models k} (-1)^{k-l(\alpha)} L^{(n)}_{\lambda'/\nu',\alpha}(-q) \right) |\lambda + \rho\rangle .$$

Recall that for a ribbon tableau T, $(-1)^{\mathrm{spin}(T)} = \varepsilon_n(\lambda/\nu)$ depends only on the shape λ/ν of T. It is clear that $\varepsilon_n(\lambda'/\nu') = (-1)^{(n-1)k}\varepsilon_n(\lambda/\nu)$, hence we are reduced to prove that

$$q^{-(n-1)k} \sum_{\alpha \models k} (-1)^{k-l(\alpha)} L^{(n)}_{\lambda'/\nu',\alpha}(q) = L^{(n)}_{\lambda/\nu,k}(q^{-1}).$$

To do this, we associate with each ribbon tableau T of weight α a standard ribbon tableau $\mathcal{T}$ of weight $(1, \dots, 1)$ as follows. Consider two ribbons R and R' of T, numbered i and i' respectively. We say that $R < R'$ if $i < i'$, or $i = i'$ and R is to the left of R'. Clearly this is a total order. Now $\mathcal{T}$ is the tableau with the same shape and inner ribbon structure as T, whose ribbons are numbered $1, 2, 3, \dots$ in the previous total order (see Figure 8).

Let us fix a skew shape λ'/ν' and consider the set $\mathcal{E}$ of all ribbon tableaux of this shape and of arbitrary weight $\alpha \models k$. For $T \in \mathcal{E}$ of weight α, write $\epsilon(T) := (-1)^{k-l(\alpha)}$. We want to prove that

(63)
$$\sum_{T\in\mathcal{E}} \epsilon(T)\, q^{\mathrm{spin}(T)} = \begin{cases} q^{(n-1)k}\, L^{(n)}_{\lambda/\nu\,,\,k}(q^{-1}) & \text{if } \lambda/\nu \text{ is a horizontal } n\text{-ribbon strip,} \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathcal{T} \in \mathcal{E}$ be a standard tableau, and let $\mathcal{E}_\mathcal{T} \subset \mathcal{E}$ denote the subset consisting of those tableaux T whose standardization gives $\mathcal{T}$. We say that $\mathcal{T}$ is a column if for all $i = 1, \dots, k-1$ the ribbon R_{i+1} numbered $i+1$ lies above the ribbon R_i numbered i, that is, if the origin of R_{i+1} lies in a row strictly above the origin of R_i. Eq. (63) will follow from the more precise statement

$$(64)\qquad \sum_{T\in\mathcal{E}_\mathcal{T}} \epsilon(T)\, q^{\mathrm{spin}(T)} = \begin{cases} q^{\mathrm{spin}(\mathcal{T})} & \text{if } \mathcal{T} \text{ is a column,} \\ 0 & \text{otherwise.} \end{cases}$$

Now this is very easy. First, by definition all $T \in \mathcal{E}_\mathcal{T}$ have the same inner ribbon structure, hence the same spin, and we can simplify the powers of q of both sides of (64). Then we only have to observe that a tableau $T \in \mathcal{E}_\mathcal{T}$ is specified by the numbering of its ribbons, *i.e.* by a map $f_T : \ [1,k] \longrightarrow [1,k]$ satisfying

(i) $f_T(i+1) = f_T(i)$ or $f_T(i+1) = f_T(i) + 1$,

(ii) if R_{i+1} lies above R_i in $\mathcal{T}$ then $f_T(i+1) = f_T(i)+1$.

Let $a(\mathcal{T})$ be the number of i's such that R_{i+1} is not above R_i. Then clearly $|\mathcal{E}_\mathcal{T}| = 2^{a(\mathcal{T})}$ and more precisely the number of $T \in \mathcal{E}_\mathcal{T}$ such that $f_T(k) = j$ (*i.e.* $\epsilon(T) = (-1)^{k-j}$) is equal to $\binom{a(\mathcal{T})}{j}$. Hence by the binomial theorem

$$\sum_{T\in\mathcal{E}_\mathcal{T}} \epsilon(T) = \begin{cases} 1 & \text{if } a(\mathcal{T}) = 0, \textit{ i.e. } \mathcal{T} \text{ is a column,} \\ 0 & \text{otherwise.} \end{cases}$$

To finish the proof we need only note that λ/ν is a horizontal n-ribbon strip if and only if there exists a (necessarily unique) column tableau $\mathcal{T}$ of shape λ'/ν', and in this case $\mathrm{spin}(\mathcal{T}) = (n-1)k - \mathrm{spin}(\lambda/\nu)$. □

Remark 6.5. Since the $\mathcal{V}_m$ commute, $\mathcal{V}_\beta$ is invariant under permutation of β. Hence Theorem 6.4 implies that $L^{(n)}_{\lambda/\nu\,,\,\beta}(q)$ is also invariant under permutation of β. This proves that the polynomial (42) is indeed symmetric. ◇

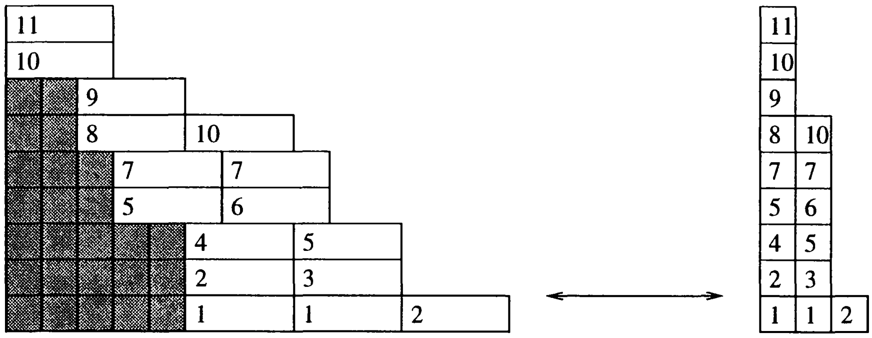

Fig. 9. Correspondence between n-ribbon tableaux of spin 0 and n-restricted inner shape, and ordinary tableaux

6.3. Action of $Z(\widehat{H}_r)$ on the canonical basis $\{G^-_\lambda\}$

For $\lambda \in \mathbb{P}^+_r$, define

$$S_\lambda := s_\lambda(Y_1^{-1}, \dots, Y_r^{-1}) \in Z(\widehat{H}_r),$$

where s_λ is the Schur function. The following theorem is a formal analogue of Theorem 3.2.

Theorem 6.6. *Let $\lambda \in P^+$. Write $\lambda = \lambda^{(0)} + n\lambda^{(1)}$, where $\lambda^{(0)}$ is n-restricted, that is,*

$$0 \leq \lambda_i^{(0)} - \lambda_{i+1}^{(0)} < n \qquad (1 \leq i \leq r-1),$$

and $\lambda_r^{(1)} \geq 0$. Then $G^-_{\lambda+\rho} = S_{\lambda^{(1)}}\, G^-_{\lambda^{(0)}+\rho}$.

Proof — By definition of the basis G^-, we have to prove that $F_\lambda := S_{\lambda^{(1)}}\, G^-_{\lambda^{(0)}+\rho}$ satisfies

$$\overline{F_\lambda} = F_\lambda \qquad \text{and} \qquad F_\lambda \equiv |\lambda + \rho\rangle \quad \bmod q^{-1}\mathcal{L}^-.$$

The first property is clear by Proposition 6.1. Indeed, S_λ is a $\mathbf{Q}$-linear combination of products of elements B_{-i}. To prove the second one we observe that by Theorem 6.4 for all $\nu \in \mathbb{P}^+_r$ and $\alpha \in \mathbf{N}^{*s}$, $\mathcal{V}_\alpha\,|\nu+\rho\rangle \in \mathcal{L}^-$. Since $G^-_{\lambda^{(0)}+\rho} \equiv |\lambda^{(0)} + \rho\rangle \bmod q^{-1}\mathcal{L}^-$, and $S_{\lambda^{(1)}}$ is a $\mathbf{Z}$-linear combination of operators $\mathcal{V}_\alpha$ we thus have

$$F_\lambda \equiv S_{\lambda^{(1)}}|\lambda^{(0)} + \rho\rangle \quad \bmod q^{-1}\mathcal{L}^-.$$

In fact Theorem 6.4 implies

$$\mathcal{V}_\alpha\,|\nu + \rho\rangle \equiv \sum_T |\mathrm{sh}(T) + \rho\rangle \quad \bmod q^{-1}\mathcal{L}^-,$$

where the sum is over the n-ribbon tableaux of weight α, spin 0 and inner shape ν, and $\mathrm{sh}(T)$ stands for the outer shape of T. Therefore for all α

$$\mathcal{V}_\alpha\,|\lambda^{(0)}+\rho\rangle \equiv \sum_{T'} |\mathrm{sh}(T')+\rho\rangle \mod q^{-1}\mathcal{L}^-,$$

where the sum is over the n-ribbon tableaux T' of weight α with inner shape $\lambda^{(0)}$ whose ribbons are all horizontal. Now $\lambda^{(0)}$ being n-restricted, there is an obvious bijection between the set of these tableaux T' and the set $\mathrm{Tab}\,(\cdot,\alpha)$ of ordinary Young tableaux of weight α (see Figure 9). Hence, for all α

$$\mathcal{V}_\alpha\,|\lambda^{(0)}+\rho\rangle \equiv \sum_{\beta} |\mathrm{Tab}\,(\beta,\alpha)|\,|\lambda^{(0)}+n\beta+\rho\rangle \mod q^{-1}\mathcal{L}^-.$$

Comparing with the well-known formula $h_\alpha = \sum_\beta |\mathrm{Tab}\,(\beta,\alpha)|\,s_\beta$ which yields

$$\mathcal{V}_\alpha = \sum_\beta |\mathrm{Tab}\,(\beta,\alpha)|\,S_\beta,$$

we deduce that for all β,

$$S_\beta\,|\lambda^{(0)}+\rho\rangle \equiv |\lambda^{(0)}+n\beta+\rho\rangle \mod q^{-1}\mathcal{L}^-,$$

and putting $\beta=\lambda^{(1)}$ we are done. □

6.4. Proof of Theorem 4.1

Let us write in the ring of symmetric functions $s_\lambda = \sum_\nu \kappa_{\lambda,\nu}\,h_\nu$. Then we also have $m_\nu = \sum_\lambda \kappa_{\lambda,\nu}\,s_\lambda$. Hence

$$G(\mu^{(0)},\dots,\mu^{(n-1)};q) := \sum_\nu L^{(n)}_{\mu,\nu}(q)\,m_\nu = \sum_\lambda \left(\sum_\nu \kappa_{\lambda,\nu}\,L^{(n)}_{\mu,\nu}(q)\right) s_\lambda,$$

which gives

$$c^\lambda_{\mu^{(0)},\dots,\mu^{(n-1)}}(q) = \sum_\nu \kappa_{\lambda,\nu}\,L^{(n)}_{\mu,\nu}(q).$$

On the other hand, by Theorem 6.6 and Theorem 5.12 we have

$$S_\lambda\,|\rho\rangle = G^-_{n\lambda+\rho} = \sum_\mu P^-_{\mu+\rho,n\lambda+\rho}(-q^{-1})\,|\mu+\rho\rangle.$$

Finally, using Theorem 6.4 we get

$$S_\lambda\,|\rho\rangle = \sum_\nu \kappa_{\lambda,\nu}\,\mathcal{V}_\nu\,|\rho\rangle = \sum_\mu \left(\sum_\nu \kappa_{\lambda,\nu}\,L^{(n)}_{\mu,\nu}(-q^{-1})\right)|\mu+\rho\rangle$$
$$= \sum_\mu c^\lambda_{\mu^{(0)},\dots,\mu^{(n-1)}}(-q^{-1})\,|\mu+\rho\rangle,$$

and by comparing the coefficients of $|\mu+\rho\rangle$ we have

$$c^\lambda_{\mu^{(0)},\dots,\mu^{(n-1)}}(q) = P^-_{\mu+\rho,n\lambda+\rho}(q).$$

□

§7. An inversion formula for Kazhdan-Lusztig polynomials

In this section we extend the coefficients to $\mathbf{Q}(q)$ and work with

$$\mathbf{P}_r := \mathbf{Q}(q)\otimes_{\mathbf{Z}[q,q^{-1}]}\mathcal{P}_r,\quad \mathbf{F}_r := \mathbf{Q}(q)\otimes_{\mathbf{Z}[q,q^{-1}]}\mathcal{F}_r,$$

$$\widehat{\mathbf{H}}_r := \mathbf{Q}(q)\otimes_{\mathbf{Z}[q,q^{-1}]}\widehat{H}_r.$$

7.1. Action of $U_q(\widehat{\mathfrak{sl}}_n)$ on the weight lattice of $\mathfrak{gl}_r$

Let $U_q(\widehat{\mathfrak{sl}}_n)$ be the quantum enveloping algebra of the affine Lie algebra $\widehat{\mathfrak{sl}}_n$. This is a $\mathbf{Q}(q)$-algebra with generators $e_i,\ f_i,\ q^{\pm h_i}$ $(0\le i\le n-1)$. The standard relations can be found for example in [20] and will be omitted. There is a canonical involution $x\mapsto\overline{x}$ of $U_q(\widehat{\mathfrak{sl}}_n)$ defined as the unique ring homomorphism such that $\overline{q}=q^{-1}$, $\overline{e_i}=e_i$, and $\overline{f_i}=f_i$.

Using the basis $\{V_\lambda\}$ for $m=-n$ one can define an action of $U_q(\widehat{\mathfrak{sl}}_n)$ on $\mathbf{P}_r$. First we start with the trivial case $r=1$, where $\mathbf{P}_r$ reduces to $\mathbf{P}_1=\bigoplus_{l\in\mathbf{Z}}\mathbf{Q}(q)V_l$. It is immediate to check that the formulas

$$f_iV_l := \delta_{l\equiv i}\,V_{l+1},\qquad e_iV_l := \delta_{l\equiv i+1}\,V_{l-1},\qquad q^{\pm h_i}V_l := q^{\pm(\delta_{l\equiv i}-\delta_{l\equiv i+1})}\,V_l,$$

extend to an action of $U_q(\widehat{\mathfrak{sl}}_n)$ on $\mathbf{P}_1$. Here, $a\equiv b$ means congruence modulo n and $\delta_{a\equiv b}$ is the Kronecker δ equal to 1 if $a\equiv b$ and to 0 otherwise. Then using the comultiplication

$$\begin{cases}\Delta f_i = f_i\otimes 1 + q^{h_i}\otimes f_i,\\ \Delta e_i = e_i\otimes q^{-h_i} + 1\otimes e_i,\\ \Delta q^{\pm h_i} = q^{\pm h_i}\otimes q^{\pm h_i},\end{cases}\tag{65}$$

and identifying $\mathbf{P}_r$ with $\mathbf{P}_1^{\otimes r}$ by $V_\lambda \mapsto V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_r}$, we obtain the following formulas

$$f_i V_\lambda := \sum_{\substack{j=1 \\ \lambda_j \equiv i}}^{r} q^{\sum_{k=1}^{j-1}(\delta_{\lambda_k \equiv i} - \delta_{\lambda_k \equiv i+1})} V_{\lambda + \epsilon_j}, \tag{66}$$

$$e_i V_\lambda := \sum_{\substack{j=1 \\ \lambda_j \equiv i+1}}^{r} q^{-\sum_{k=j+1}^{r}(\delta_{\lambda_k \equiv i} - \delta_{\lambda_k \equiv i+1})} V_{\lambda - \epsilon_j}. \tag{67}$$

Proposition 7.1. *This action of $U_q(\widehat{\mathfrak{sl}}_n)$ on $\mathbf{P}_r$ commutes with the action of $\widehat{\mathbf{H}}_r$ via Π_{-n}.*

Proof — It is clear from (66) (67) that

$$f_i Y^\mu V_\lambda = f_i V_{\lambda - n\mu} = Y_\mu f_i V_\lambda, \qquad e_i Y^\mu V_\lambda = e_i V_{\lambda - n\mu} = Y_\mu e_i V_\lambda,$$

that is, the action of $U_q(\widehat{\mathfrak{sl}}_n)$ commutes with the operators Y^μ. Hence, recalling the discussion of Section 5.2, we see that it is enough to prove that $f_i T_j V_\gamma = T_j f_i V_\gamma$ for $\gamma \in \mathfrak{S}_r \mathcal{A}_{r,-n}$ and $1 \leq j \leq r-1$. Moreover, since T_j only acts on components j and $j+1$ of γ, we can assume that $r = 2$. Then the claim is verified by a direct computation. For example on the one hand

$$f_0 T_1 V_{(1-n,0)} = f_0 V_{(0,1-n)} = V_{(1,1-n)},$$

and on the other hand

$$T_1 f_0 V_{(1-n,0)} = q^{-1} T_1 V_{(1-n,1)} = q^{-1} T_1 Y_2^{-1} V_{(1-n,1-n)}$$

$$= q^{-1}(Y_1^{-1} T_1 + (q - q^{-1}) Y_1^{-1}) V_{(1-n,1-n)} = V_{(1,1-n)}.$$

□

Remark 7.2. This action of $U_q(\widehat{\mathfrak{sl}}_n)$ does not commute with the positive level action Π_n of $\widehat{\mathbf{H}}_r$. For example if $r = 2$ and $n = 3$

$$f_2 \Pi_3(T_1)(V_{(2,0)}) = q^{-1} V_{(0,3)}, \qquad \Pi_3(T_1)(f_2 V_{(2,0)}) = q V_{(0,3)}.$$

However, one can easily obtain an action commuting with Π_n by simply replacing the comultiplication Δ of (65) by its opposite

$$\Delta^{\mathrm{op}} f_i = f_i \otimes q^{h_i} + 1 \otimes f_i, \quad \Delta^{\mathrm{op}} e_i = e_i \otimes 1 + q^{-h_i} \otimes e_i.$$

◇

The action of $U_q(\widehat{\mathfrak{sl}}_n)$ is compatible with the bar involution of $\mathbf{P}_r$ in the following sense.

Proposition 7.3. *For $x \in U_q(\widehat{\mathfrak{sl}}_n)$ and $v \in \mathbf{P}_r$ one has $\overline{(xv)} = \overline{x}\,\overline{v}$. In other words,*

$$\overline{f_i v} = f_i \overline{v}, \qquad \overline{e_i v} = e_i \overline{v} \qquad\qquad (0 \le i \le n-1).$$

Proof — We can assume that $v = V_\lambda$. Then by (70) and Proposition 5.7 we have

$$(68) \qquad \overline{f_i V_\lambda} := q^{-\ell(w_{0,\xi})}\, T_{w_0}^{-1} \left(\sum_{\substack{j=1 \\ \lambda_j \equiv i}}^{r} q^{\sum_{k=1}^{j-1} (\delta_{\lambda_k \equiv i} - \delta_{\lambda_k \equiv i+1})}\, V_{w_0(\lambda+\epsilon_j)} \right).$$

Here, $\xi \in \mathcal{A}_{r,-n}$ is the point congruent to $\lambda + \epsilon_j$, which does not depend on j because λ_j is required to be $\equiv i$. On the other hand, since f_i commutes with $T_{w_0}^{-1}$ by Proposition 7.1,

$$f_i \overline{V_\lambda} = q^{-\ell(w_{0,\nu})}\, T_{w_0}^{-1} \left(\sum_{\substack{j=1 \\ \lambda_{r+1-j} \equiv i}}^{r} q^{\sum_{k=1}^{j-1} (\delta_{\lambda_{r+1-k} \equiv i} - \delta_{\lambda_{r+1-k} \equiv i+1})}\, V_{w_0 \lambda + \epsilon_j} \right)$$

$$(69) \qquad = q^{-\ell(w_{0,\nu})}\, T_{w_0}^{-1} \left(\sum_{\substack{j=1 \\ \lambda_j \equiv i}}^{r} q^{\sum_{k=1}^{r-j} (\delta_{\lambda_{r+1-k} \equiv i} - \delta_{\lambda_{r+1-k} \equiv i+1})}\, V_{w_0(\lambda+\epsilon_j)} \right).$$

It remains to check that the coefficients of $T_{w_0}^{-1} V_{w_0(\lambda+\epsilon_j)}$ in (68) and (69) are equal, which is equivalent to

$$\sum_{k=1}^{r} (\delta_{\lambda_k \equiv i} - \delta_{\lambda_k \equiv i+1}) - 1 = \ell(w_{0,\nu}) - \ell(w_{0,\xi}).$$

This is elementary, using for instance Remark 5.8. The formula for e_i is similar and its proof is omitted. $\square$

7.2. Action of $U_q(\widehat{\mathfrak{sl}}_n)$ on $\mathbf{F}_r$

Since the action of $U_q(\widehat{\mathfrak{sl}}_n)$ on $\mathbf{P}_r$ commutes with the action of $\widehat{\mathbf{H}}_r$, the subspace $\mathbf{I}_r := \mathbf{Q}(q) \otimes_{\mathbf{Z}[q,q^{-1}]} \mathcal{J}_r$ is stable under $U_q(\widehat{\mathfrak{sl}}_n)$ and we obtain an induced action of $U_q(\widehat{\mathfrak{sl}}_n)$ on $\mathbf{F}_r$. As explained in Section 7.1, the vector V_λ should be regarded as a monomial tensor $V_\lambda \equiv V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_r}$. Hence its projection $|\lambda\rangle$ on $\mathbf{F}_r$ should be thought of as some q-wedge product $|\lambda\rangle \equiv V_{\lambda_1} \wedge_q \cdots \wedge_q V_{\lambda_r}$ with the anticommutation relations replaced by the straightening rules of Proposition 5.9. The action on $|\lambda\rangle$ of the generators of $U_q(\widehat{\mathfrak{sl}}_n)$ is obtained by projecting (66), (67):

$$f_i|\lambda\rangle := \sum_{\substack{j=1 \\ \lambda_j \equiv i}}^{r} q^{\sum_{k=1}^{j-1}(\delta_{\lambda_k \equiv i} - \delta_{\lambda_k \equiv i+1})} \, |\lambda + \epsilon_j\rangle, (0 \le i \le n-1), \tag{70}$$

$$e_i|\lambda\rangle := \sum_{\substack{j=1 \\ \lambda_j \equiv i+1}}^{r} q^{-\sum_{k=j+1}^{r}(\delta_{\lambda_k \equiv i} - \delta_{\lambda_k \equiv i+1})} \, |\lambda - \epsilon_j\rangle, (0 \le i \le n-1). \tag{71}$$

Note that if $\lambda \in P^{++}$ then $\lambda \pm \epsilon_j \in P^+$. It follows that either $|\lambda \pm \epsilon_j\rangle$ belongs to the basis $\{|\lambda\rangle \mid \lambda \in P^{++}\}$, or $|\lambda \pm \epsilon_j\rangle = 0$. Hence, Eq. (70) (71) require no straightening relation and are very simple to use in practice. The compatibility of the bar involution with this action is given by the next

Proposition 7.4. *For $u \in \mathbf{F}_r$ and $0 \le i \le n-1$ one has*

$$\overline{f_i u} = f_i \overline{u}, \qquad \overline{e_i u} = e_i \overline{u}.$$

Proof — This follows immediately from (46) and Proposition 7.3. □

7.3. The Fock space $\mathbf{F}_\infty$

For $s \ge r$ define a linear map $\varphi_{r,s} : \mathbf{F}_r \longrightarrow \mathbf{F}_s$ by

$$\varphi_{r,s}(|\lambda\rangle) := |\lambda_1, \dots, \lambda_r, -r, -r-1, \dots, -s+1\rangle \qquad (\lambda \in P_r^+).$$

Then clearly $\varphi_{s,t} \circ \varphi_{r,s} = \varphi_{r,t}$. Let $\mathbf{F}_\infty := \varinjlim \mathbf{F}_r$ be the direct limit of the vector spaces $\mathbf{F}_r$ with respect to the maps $\varphi_{r,s}$. Each $|\lambda\rangle$ in $\mathbf{F}_r$ has an image $\varphi_r(|\lambda\rangle) \in \mathbf{F}_\infty$, which should be thought of as some infinite q-wedge

$$\varphi_r(|\lambda\rangle) \equiv V_{\lambda_1} \wedge_q \cdots \wedge_q V_{\lambda_r} \wedge_q V_{-r} \wedge_q V_{-r-1} \wedge_q \cdots$$

Lemma 7.5. (i) *If* $\lambda_r \le -r$ *then* $\varphi_r(|\lambda\rangle) = 0$.
(ii) *If* $\lambda \in P_r^{++}$ *and* $\lambda_r > -r$ *then* $\varphi_r(|\lambda\rangle) \neq 0$.

Proof — (i) Write $\lambda_r = k \le -r$ and consider the element

$$\varphi_{r,-k+1}(|\lambda\rangle) = |\lambda_1, \dots, \lambda_r, -r, -r-1, \dots, k\rangle.$$

By applying Proposition 5.9 one checks easily that $|k, -r, -r-1, \dots, k\rangle$ straightens to 0 in F_{-k-r+2}. Therefore $\varphi_{r,-k+1}(|\lambda\rangle) = 0$, hence $\varphi_r(|\lambda\rangle) = 0$. (ii) By Lemma 5.6 if $\lambda \in P_r^{++}$ and $\lambda_r > -r$ then $\varphi_{r,s}(|\lambda\rangle) \neq 0$ for all $s > r$. Hence $\varphi_r(|\lambda\rangle) \neq 0$. □

Let $\mathbb{P}^+$ denote the set of all partitions, *i.e.* of all finite non-increasing sequences of positive integers. Put $\rho_r^* := (0, -1, \dots, -r+1)$, and for $\alpha = (\alpha_1, \dots, \alpha_s) \in \mathbb{P}^+$ define

$$|\alpha) := \varphi_s(|\alpha + \rho_s^*\rangle).$$

It readily follows from Lemma 5.6 and Lemma 7.5 that $\{|\alpha) \mid \alpha \in \mathbb{P}^+\}$ is a basis of $\mathbf{F}_\infty$. We define a grading on $\mathbf{F}_\infty$ by requiring that

$$\deg |\alpha) := \sum_{i=1}^{s} \alpha_i.$$

Then for all $\lambda \in P_r$, $\varphi_r(|\lambda\rangle)$ is homogeneous of degree

$$\deg \varphi_r(|\lambda\rangle) = \sum_{i=1}^{r} (\lambda_i + i - 1).$$

In particular, if $\sum_{i=1}^{r} (\lambda_i + i - 1) < 0$ then $\varphi_r(|\lambda\rangle) = 0$.

7.4. Action of $U_q(\widehat{\mathfrak{sl}}_n)$ on $\mathbf{F}_\infty$

Let $\lambda \in \mathcal{P}_r$. It follows easily from (70) that

$$f_i\, \varphi_{r,s}(|\lambda\rangle) = \varphi_{r+1,s}\, f_i\, \varphi_{r,r+1}\,(|\lambda\rangle) \tag{72}$$

for all $s > r$. Hence one can define an endomorphism f_i of $\mathbf{F}_\infty$ by

$$f_i\, \varphi_r(|\lambda\rangle) = \varphi_{r+1}\, f_i\, \varphi_{r,r+1}\,(|\lambda\rangle) \tag{73}$$

and thus get an action of $U_q^-(\widehat{\mathfrak{sl}}_n)$ on $\mathbf{F}_\infty$.

On the basis $\{|\alpha) \mid \alpha \in \mathbb{P}^+\}$ this action is expressed as follows. Let α and β be two Young diagrams such that β is obtained from α by adding a cell γ whose content is $\equiv i \bmod n$. Such a cell is called a removable

i-cell of β, or an indent i-cell of α. Let $I_i^r(\alpha,\beta)$ (*resp.* $R_i^r(\alpha,\beta)$) be the number of indent i-cells of α (*resp.* of removable i-cells of α) situated to the right of γ (γ not included). Set $N_i^r(\alpha,\beta) = I_i^r(\alpha,\beta) - R_i^r(\alpha,\beta)$. Then Eq. (70) gives

$$f_i|\alpha\rangle = \sum_{\beta} q^{N_i^r(\alpha,\beta)}|\beta\rangle, \tag{74}$$

where the sum is over all partitions β such that β/α is an i-cell.

Defining an action of $U_q^+(\widehat{\mathfrak{sl}}_n)$ is not as straightforward since there is no formula like (72) for e_i. For example if $n = 2$,

$$\begin{aligned}
&e_1\,|2\rangle = |1\rangle,\\
&e_1\,|2,-1\rangle = q^{-1}\,|1,-1\rangle,\\
&e_1\,|2,-1,-2\rangle = |1,-1,-2\rangle + |2,-1,-3\rangle,\\
&e_1\,|2,-1,-2,-3\rangle = q^{-1}\,|1,-1,-2,-3\rangle,
\end{aligned}$$

and in general

$$\begin{aligned}
&e_1\varphi_{1,2r}\,|2\rangle = q^{-1}\varphi_{1,2r}e_1|2\rangle,\\
&e_1\varphi_{1,2r+1}\,|2\rangle = \varphi_{1,2r+1}e_1|2\rangle + |2,-1,\dots,-2r+1,-2r-1\rangle.
\end{aligned}$$

However, one can check that putting $e_i\varphi_r(|\lambda\rangle) := q^{-\delta_{i\equiv r}}\,\varphi_r(e_i|\lambda\rangle)$ one gets a well-defined action of $U_q^+(\widehat{\mathfrak{sl}}_n)$ compatible with (73) (see [15]). Its combinatorial description is given by

$$e_i|\beta\rangle = \sum_{\alpha} q^{-N_i^l(\alpha,\beta)}|\alpha\rangle, \tag{75}$$

where the sum is over all partitions α such that β/α is an i-cell, and $N_i^l(\alpha,\beta)$ is defined as $N_i^r(\alpha,\beta)$ but replacing right by left.

In contrast to $\mathbf{F}_r$, the representation $\mathbf{F}_\infty$ has primitive vectors, *i.e.* vectors annihilated by all e_i. In particular the vector $|0\rangle$ labelled by the unique partition of 0 is primitive. In fact $\mathbf{F}_\infty$ is a level 1 highest weight integrable representation of $U_q(\widehat{\mathfrak{sl}}_n)$, while $\mathbf{F}_r$ is a level 0 representation (without highest weight). As shown by Kashiwara, Miwa and Stern [15], the decomposition of $\mathbf{F}_\infty$ into simple $U_q(\widehat{\mathfrak{sl}}_n)$-modules is obtained by considering the limit $r \to \infty$ of the action of $Z(\widehat{\mathbf{H}}_r)$ on $\mathbf{F}_r$.

7.5. Action of $\mathbf{H}_\infty$ on $\mathbf{F}_\infty$

Let $\lambda \in P_r$. It follows from the easily checked relations

$$\begin{cases} |-s,-r,-r-1,\dots,-s\rangle = 0,\\ |-r,-r-1,\dots,-s,-r\rangle = 0,\end{cases} \qquad (s \geq r \geq 0) \tag{76}$$

that the vector $\varphi_s\, B_k\, \varphi_{r,s}(|\lambda\rangle)$ is independent of s for $s > r$ large enough. Hence one can define endomorphisms B_k of $\mathbf{F}_\infty$ by

$$B_k\varphi_r(|\lambda\rangle) := \varphi_s\, B_k\, \varphi_{r,s}(|\lambda\rangle) \qquad (k \in \mathbf{Z}^*,\ s \gg 1). \tag{77}$$

By construction, these endomorphisms commute with the action of $U_q(\widehat{\mathfrak{sl}}_n)$ on $\mathbf{F}_\infty$. However they no longer generate a commutative algebra but a Heisenberg algebra. Indeed, it was proved by Kashiwara, Miwa and Stern [15] that

$$[B_k, B_l] = \begin{cases} k\,\dfrac{1-q^{-2nk}}{1-q^{-2k}} & \text{if } k = -l, \\ 0 & \text{otherwise.} \end{cases} \tag{78}$$

We shall denote this Heisenberg algebra by $\mathbf{H}_\infty$. The elements $\widetilde{\mathcal{U}}_\beta$, $\widetilde{\mathcal{V}}_\beta$, $\mathcal{U}_\beta$, $\mathcal{V}_\beta$ of $Z(\widehat{H}_r)$ also give rise to well-defined elements of $\mathbf{H}_\infty$ that we still denote by $\widetilde{\mathcal{U}}_\beta$, $\widetilde{\mathcal{V}}_\beta$, $\mathcal{U}_\beta$, $\mathcal{V}_\beta$. By Theorem 6.2 and Theorem 6.4, their action on the basis $\{|\nu\rangle,\ \nu \in \mathbb{P}^+\}$ of $\mathbf{F}_\infty$ is given by

$$\widetilde{\mathcal{U}}_\beta\,|\nu\rangle = q^{-(n-1)k} \sum_{\mu\in\mathbb{P}^+} L^{(n)}_{\nu'/\mu',\,\beta}(-q)\ |\mu\rangle, \tag{79}$$

$$\widetilde{\mathcal{V}}_\beta\,|\nu\rangle = q^{-(n-1)k} \sum_{\lambda\in\mathbb{P}^+} L^{(n)}_{\lambda'/\nu',\,\beta}(-q)\ |\lambda\rangle, \tag{80}$$

$$\mathcal{U}_\beta\,|\nu\rangle = \sum_{\mu\in\mathbb{P}^+} L^{(n)}_{\nu/\mu,\,\beta}(-q^{-1})\ |\mu\rangle, \tag{81}$$

$$\mathcal{V}_\beta\,|\nu\rangle = \sum_{\lambda\in\mathbb{P}^+} L^{(n)}_{\lambda/\nu,\,\beta}(-q^{-1})\ |\lambda\rangle, \tag{82}$$

where $k = |\beta|$. It was shown in [15] that $\mathbf{F}_\infty$ is irreducible under the commuting actions of $U_q(\widehat{\mathfrak{sl}}_n)$ and $\mathbf{H}_\infty$. It follows that $\{\mathcal{V}_\beta\,|0\rangle,\ \beta \in \mathbb{P}^+\}$ is a basis of the space of primitive vectors of $\mathbf{F}_\infty$ for $U_q(\widehat{\mathfrak{sl}}_n)$.

7.6. The bar involution of $\mathbf{F}_\infty$

Before introducing the involution we need the following lemmas.

Lemma 7.6. *Let $\mu \in P_{m+1}$ such that $\mu_i > -m$ $(i = 1,\dots,m+1)$ and $\sum_i(\mu_i + i - 1) \le m$. Then $|\mu\rangle = 0$.*

Proof — We have

$$|\mu\rangle = \sum_{\lambda\in P^{++}_{m+1}} x_\lambda\,|\lambda\rangle$$

for some coefficients x_λ. Because of the hypothesis $\mu_i > -m$ and of the form of the straightening relations, the components of the weights λ occuring in this sum must all be $> -m$. On the other hand, setting $\alpha_i = \lambda_i + i - 1$ we see that α is a partition with $|\alpha| \le m$, hence $l(\alpha) \le m$. Thus the last component of all the λ must be $= -m$, and the sum is empty. $\square$

Lemma 7.7. *Let $\lambda \in P_r$ and let $m \ge r$. Assume that $\lambda_i > -m$ $(i = 1, \ldots, r)$ and $\sum_i (\lambda_i + i - 1) \le m$. Then*

$$\begin{aligned} &| -m, \lambda_1, \ldots, \lambda_r, -r, \ldots, -m+1\rangle \\ &= (-1)^m q^{-a(\lambda)} \, |\lambda_1, \ldots, \lambda_r, -r, \ldots, -m+1, -m\rangle, \end{aligned}$$

where $a(\lambda) = \sharp\{j \le r \mid \lambda_j \not\equiv -m\} + \sharp\{-r \ge j \ge -m+1 \mid j \not\equiv -m\}$.

Proof — Consider the straightening of

$$\nu = | -m, \lambda_1, \ldots, \lambda_r, -r, \ldots, -m+1\rangle$$

computed by means of Proposition 5.9. At each step, if the third rule (50) has to be used, then only the first term of the right-hand side may be non-zero. Indeed the two other terms involve weights μ which satisfy the hypothesis of Lemma 7.6. Therefore the straightening of ν is simply obtained by reordering its components and multiplying by the appropriate sign and power of q. $\square$

If λ satisfies the hypothesis of Lemma 7.7, then repeated applications of this lemma show that for $p \ge m$,

$$\begin{aligned} &(-1)^{\binom{p}{2}} q^{b(\lambda,p)} \, | -p, \ldots, -r, \lambda_r, \ldots, \lambda_1\rangle \\ &= (-1)^{\binom{m}{2}} q^{b(\lambda,m)} \, | -m, \ldots, -r, \lambda_r, \ldots, \lambda_1, -m-1, \ldots, -p\rangle \end{aligned}$$

Here $b(\lambda, p)$ is the number of pairs (i, j) of components of the vector $(\lambda_1, \ldots, \lambda_r, -r, \ldots, -p)$ with $i \not\equiv j \mod n$. In other words, using Proposition 5.7 and Remark 5.8

$$\overline{\varphi_{r,p}(|\lambda\rangle)} = \varphi_{m,p}(\overline{\varphi_{r,m}(|\lambda\rangle)}).$$

Thus we can define a semi-linear involution on $\mathbf{F}_\infty$ by putting

$$(83) \quad \overline{\varphi_r(|\lambda\rangle)} := \varphi_m(\overline{\varphi_{r,m}(|\lambda\rangle)}) \quad (\lambda \in P_r, \ \deg \varphi_r(|\lambda\rangle) = m, \ \lambda_i > -m).$$

In particular, for $\alpha \in \mathbb{P}^+$ and $s \ge |\alpha|$, we have $\overline{|\alpha)} = \varphi_s(\overline{|\alpha + \rho_s^*\rangle})$.

Proposition 7.8. *For $\alpha \in \mathbb{P}^+$, $0 \le i \le n-1$ and $k \in \mathbf{N}^*$ we have*

$$\overline{f_i\,|\alpha)} = f_i\,\overline{|\alpha)}, \quad \overline{e_i\,|\alpha)} = e_i\,\overline{|\alpha)},$$

$$\overline{B_{-k}\,|\alpha)} = B_{-k}\,\overline{|\alpha)}, \quad \overline{B_k\,|\alpha)} = q^{2(n-1)k} B_k\,\overline{|\alpha)}.$$

Proof — For f_i and B_{-k}, the proof readily follows from Proposition 7.4, Proposition 6.1 and (73) (77) (83). (Note that the condition $\lambda_i > -m$ in (83) is preserved by the action of these lowering operators.) Let us prove the statement for B_k. We argue by induction on $\deg|\alpha)$. In degree 0, the unique basis vector is $|0)$ and we have $B_k|0) = B_k\overline{|0)} = 0$, so the claim is trivially verified. Let us assume that the result is proved for all $|\alpha)$ of degree $\le m$. Since the action of the operators B_{-l} and f_i on $|0)$ generates the whole Fock space, it is enough to prove that

$$\overline{B_k\,f_i v} = q^{2(n-1)k} B_k\,\overline{f_i v}, \qquad \overline{B_k\,B_{-l} v} = q^{2(n-1)k} B_k\,\overline{B_{-l} v}$$

for all v of degree $\le m$. Now B_k and f_i commute, so

$$\overline{B_k f_i v} = \overline{f_i B_k v} = f_i \overline{B_k v} = f_i(q^{2(n-1)k} B_k \overline{v}) = q^{2(n-1)k} B_k f_i \overline{v}$$

$$= q^{2(n-1)k} B_k \overline{f_i v}.$$

If $l \neq k$ we know that B_k and B_{-l} commute and we can argue similarly. Finally if $l = k$, by (78),

$$\begin{aligned}
\overline{B_k\,B_{-k} v} &= \overline{B_{-k}\,B_k v} + k\,\frac{1-q^{2(n-1)k}}{1-q^{2k}}\overline{v} \\
&= q^{2(n-1)k} B_{-k} B_k \overline{v} + k\,\frac{1-q^{2(n-1)k}}{1-q^{2k}}\overline{v} \\
&= q^{2(n-1)k}\left(B_{-k}B_k + k\,\frac{1-q^{-2(n-1)k}}{1-q^{-2k}}\right)\overline{v} = q^{2(n-1)k} B_k B_{-k}\overline{v} \\
&= q^{2(n-1)k} B_k \overline{B_{-k} v}.
\end{aligned}$$

The proof for e_i is similar, using the commutation relation

$$[e_i, f_j] = \delta_{ij}\frac{q^{h_i} - q^{-h_i}}{q - q^{-1}}.$$

□

Proposition 7.8 implies that for $|\beta| = k$,

$$\text{(84)} \qquad \overline{\widetilde{\mathcal{V}}_\beta\,|\alpha)} = \widetilde{\mathcal{V}}_\beta\,\overline{|\alpha)}, \qquad \overline{\mathcal{V}_\beta\,|\alpha)} = \mathcal{V}_\beta\,\overline{|\alpha)},$$

$$\text{(85)} \qquad \overline{\widetilde{\mathcal{U}}_\beta\,|\alpha)} = q^{2(n-1)k}\,\widetilde{\mathcal{U}}_\beta\,\overline{|\alpha)}, \qquad \overline{\mathcal{U}_\beta\,|\alpha)} = q^{2(n-1)k}\,\mathcal{U}_\beta\,\overline{|\alpha)}.$$

7.7. The scalar product of $\mathbf{F}_\infty$

Define a scalar product on $\mathbf{F}_\infty$ by $\langle |\alpha\rangle\,,\,|\beta\rangle\rangle = \delta_{\alpha\beta}$.

Proposition 7.9. *For $u, v \in \mathbf{F}_\infty$ one has*

$$\langle f_i u\,,\,v\rangle = \langle u\,,\,q^{h_i-1}e_i v\rangle, \quad \langle e_i u\,,\,v\rangle = \langle u\,,\,q^{-h_i-1}f_i v\rangle,$$
$$\langle \widetilde{\mathcal{V}}_\alpha u\,,\,v\rangle = \langle u\,,\,\widetilde{\mathcal{U}}_\alpha v\rangle, \qquad \langle \mathcal{V}_\alpha u\,,\,v\rangle = \langle u\,,\,\mathcal{U}_\alpha v\rangle.$$

Proof — This follows immediately from (74) (75) (79) (80) (81) (82).

7.8. Symmetry of the bar involution

Define a semi-linear involution $v \mapsto v'$ on $\mathbf{F}_\infty$ by $|\alpha\rangle' := |\alpha'\rangle$, where α' is the partition conjugate to $\alpha \in \mathbb{P}^+$.

Proposition 7.10. *For $u \in \mathbf{F}_\infty$ and $\beta \in \mathbb{P}^+$ with $|\beta| = k$, there holds*

$$(e_i u)' = q^{h_{-i}-1}e_{-i}u', \qquad (f_i u)' = q^{-h_{-i}-1}f_{-i}u',$$
$$(\mathcal{V}_\beta u)' = (-q)^{(n-1)k}\,\widetilde{\mathcal{V}}_\beta u', \qquad (\mathcal{U}_\beta u)' = (-q)^{(n-1)k}\,\widetilde{\mathcal{U}}_\beta u'.$$

Proof — This also follows from (74) (75) (79) (80) (81) (82). □

Let $S_\beta = \sum_\alpha \kappa_{\beta,\alpha}\mathcal{V}_\alpha$ be the element of $\mathbf{H}_\infty$ corresponding to the Schur function s_β. The third equation above implies that

$$(S_\beta u)' = (-q)^{(n-1)k}\,S_{\beta'}u'. \tag{86}$$

Theorem 7.11. *For $u, v \in \mathbf{F}_\infty$ we have*

$$\langle \overline{u}\,,\,v\rangle = \langle u'\,,\,\overline{v'}\rangle\,.$$

Proof — The proof is by induction on the degree d of u and v. If $d = 0$ this is clear. So let us assume that the theorem is proved in degree $d < m$. The operators e_i, f_i, $\widetilde{\mathcal{U}}_k$, $\widetilde{\mathcal{V}}_k$, $\mathcal{U}_k$, $\mathcal{V}_k$ are homogeneous of respective degree -1, $+1$, $-kn$, $+kn$, $-kn$, $+kn$. Since $\mathbf{F}_\infty$ is generated by the action of the operators f_i and $\mathcal{V}_k$ on the highest weight vector $|0\rangle$, it is enough to prove that

$$\langle \overline{(f_i u)}, v\rangle = \langle (f_i u)', \overline{v'}\rangle, \tag{87}$$
$$\langle \overline{(\mathcal{V}_k w)}, v\rangle = \langle (\mathcal{V}_k w)', \overline{v'}\rangle, \tag{88}$$

for all u, v, w with $\deg u = m-1$, $\deg v = m$, $\deg w = m - kn$.

Let us prove (87). We have

$$\langle\overline{(f_i u)}, v\rangle = \langle f_i\, \overline{u}, v\rangle = \langle \overline{u}, q^{h_i - 1} e_i v\rangle = \langle u', \overline{(q^{h_i-1} e_i v)'}\rangle.$$

The first equality comes from Proposition 7.8, the second from Proposition 7.9 and the third from the fact that $\deg u < m$. Now, by Proposition 7.8, 7.9 and 7.10

$$\begin{aligned}\langle u', \overline{(q^{h_i-1} e_i v)'}\rangle &= \langle u', \overline{e_{-i} v'}\rangle = \langle u', e_{-i} \overline{v'}\rangle \\ &= \langle q^{-h_{-i}-1} f_{-i} u', \overline{v'}\rangle = \langle (f_i u)', \overline{v'}\rangle,\end{aligned}$$

and (87) is proved.

The proof of (88) is similar. We have

$$\langle\overline{(\mathcal{V}_k w)}, v\rangle = \langle \mathcal{V}_k \overline{w}, v\rangle = \langle \overline{w}, \mathcal{U}_k v\rangle = \langle w', \overline{(\mathcal{U}_k v)'}\rangle.$$

The first equality comes from (84), the second from Proposition 7.9 and the third from the fact that $\deg w < m$. Then, using again Proposition 7.8, 7.9 and 7.10,

$$\begin{aligned}\langle w', \overline{(\mathcal{U}_k v)'}\rangle &= \langle w', \overline{(-q)^{(n-1)k} \widetilde{\mathcal{U}}_k(u')}\rangle = \langle w', (-q)^{(n-1)k} \widetilde{\mathcal{U}}_k(\overline{u'})\rangle \\ &= \langle (-q)^{(n-1)k} \widetilde{\mathcal{V}}_k w', \overline{v'}\rangle = \langle (\mathcal{V}_k v)', \overline{w'}\rangle,\end{aligned}$$

and (88) is proved. □

7.9. Canonical bases of $\mathbf{F}_\infty$

For $\beta \in \mathbb{P}^+$ write $\overline{|\beta)} = \sum_{\alpha\in\mathbb{P}^+} b_{\alpha,\beta}(q)\,|\alpha)$. Then, for $|\alpha| = |\beta| \le r$ it follows from (83) that we have

$$b_{\alpha,\beta}(q) = a_{\alpha+\rho_r^*,\,\beta+\rho_r^*}(q) = a_{\alpha+\rho_r,\,\beta+\rho_r}(q),$$

where the coefficients $a_{\lambda,\mu}(q)$ ($\lambda,\ \mu \in P_r^{++}$) have been defined in Section 5.3. Hence by Corollary 5.10 the matrix

$$\mathbf{B}_k := [b_{\alpha,\beta}(q)], \qquad (\alpha, \beta \vdash k)$$

is unitriangular, and one can define canonical bases $\{\mathcal{G}_\alpha^+ \mid \alpha \in \mathbb{P}^+\}$, $\{\mathcal{G}_\alpha^- \mid \alpha \in \mathbb{P}^+\}$ of $\mathbf{F}_\infty$ characterized by:

(i) $\overline{\mathcal{G}_\alpha^+} = \mathcal{G}_\alpha^+$, $\quad \overline{\mathcal{G}_\alpha^-} = \mathcal{G}_\alpha^-$,
(ii) $\mathcal{G}_\alpha^+ \equiv |\alpha) \bmod q\mathcal{L}_\infty^+$, $\quad \mathcal{G}_\alpha^- \equiv |\alpha) \bmod q^{-1}\mathcal{L}_\infty^-$,

where $\mathcal{L}_\infty^+$ (*resp.* $\mathcal{L}_\infty^-$) is the $\mathbf{Z}[q]$-submodule (*resp.* $\mathbf{Z}[q^{-1}]$-submodule) spanned by the vectors $|\alpha\rangle$. Set

$$\mathcal{G}_\beta^+ = \sum_\alpha d_{\alpha,\beta}(q)\,|\alpha\rangle\,, \qquad \mathcal{G}_\alpha^- = \sum_\beta e_{\alpha,\beta}(-q^{-1})\,|\beta\rangle\,,$$

and

$$\mathbf{D}_k := [d_{\alpha,\beta}(q)], \qquad \mathbf{E}_k := [e_{\alpha,\beta}(q)], \qquad (\alpha,\beta \vdash k).$$

Then, for $r \geq k$ we have

$$d_{\alpha,\beta}(q) = c_{\alpha+\rho_r,\,\beta+\rho_r}(q), \qquad e_{\alpha,\beta}(q) = l_{\alpha+\rho_r,\,\beta+\rho_r}(q).$$

Hence by Theorem 5.12 we get

$$e_{\alpha,\beta} = P^-_{\beta+\rho_r,\alpha+\rho_r}, \tag{89}$$

a parabolic Kazhdan-Lusztig polynomial for $\widehat{\mathfrak{S}}_r$ associated with the parabolic subgroup $\mathfrak{S}_{\nu,-n}$ which stabilizes the point $\nu \in \mathcal{A}_{r,-n}$ congruent to $\alpha+\rho_r$ and $\beta+\rho_r$. Also, putting $\widehat{u}_\alpha := w(w_0(\alpha+\rho_r),-n)\,w_{0,\nu}$ and $\widehat{u}_\beta := w(w_0(\beta+\rho_r),-n)\,w_{0,\nu}$, we have

$$d_{\alpha,\beta} = \sum_{s\in\mathfrak{S}_r} (-q)^{\ell(s)} P_{s\widehat{u}_\alpha\,,\,\widehat{u}_\beta}. \tag{90}$$

Note that by Theorem 2.4 this is also a parabolic Kazhdan-Lusztig polynomial of negative type associated with the subgroup $\mathfrak{S}_r \subset \widehat{\mathfrak{S}}_r$. It is interesting to give another expression of $d_{\alpha,\beta}$ in terms of the action π_n (instead of π_{-n}). Let $\underline{P}_r = P_r/\mathbf{Z}(1,\dots,1)$ and $\lambda \mapsto \underline{\lambda}$ be the natural projection $P_r \to \underline{P}_r$. The action π_n of $\widehat{\mathfrak{S}}_r$ on P_r induces an action $\underline{\pi}_n$ of $\widetilde{\mathfrak{S}}_r$ on $\underline{P}_r$ with fundamental alcove $\underline{\mathcal{A}}_{r,n} := \{\underline{\lambda} \in \underline{P}_r \mid \lambda_1 \geq \cdots \geq \lambda_r,\ \lambda_1 - \lambda_r \leq n\}$. Let ξ be the point of $\underline{\mathcal{A}}_{r,n}$ congruent to $\underline{\alpha+\rho_r}$ and $\underline{\beta+\rho_r}$ under $\underline{\pi}_n$, and let $w_{0,\xi}$ denote the longest element of its stabilizer. Consider the projection $\underline{\ \cdot\ } : \widehat{\mathfrak{S}}_r \to \widetilde{\mathfrak{S}}_r$ defined by $\underline{\sigma\tau^k} = \sigma$ ($k \in \mathbf{Z}$, $\sigma \in \widetilde{\mathfrak{S}}_r$), and the automorphism $\sharp$ of $\widehat{\mathfrak{S}}_r$ defined by $s_i^\sharp = s_{-i}$ ($i \in \mathbf{Z}/r\mathbf{Z}$). It is easy to check that, for $\lambda \in P_r^+$, $\underline{w(w_0\lambda,-n)} = \underline{(w(\lambda,n))^\sharp}$. It follows that

$$d_{\alpha,\beta} = \sum_{s\in\mathfrak{S}_r} (-q)^{\ell(s)} P_{s\widehat{v}_\alpha\,,\,\widehat{v}_\beta}, \tag{91}$$

where $\widehat{v}_\alpha$, $\widehat{v}_\beta$ are given by $\widehat{v}_\alpha = \underline{w(\alpha+\rho_r,n)}w_{0,\xi}$, $\widehat{v}_\beta = \underline{w(\beta+\rho_r,n)}w_{0,\xi}$.

Remark 7.12. Consider the $U_q(\widehat{\mathfrak{sl}}_n)$-submodule M of $\mathbf{F}_\infty$ generated by $|0)$. This is an irreducible integrable representation with highest weight Λ_0. By Proposition 7.8, the bar involution of $\mathbf{F}_\infty$ induces the Kashiwara involution of M, and it follows that the subset $\{\mathcal{G}_\alpha^+ \mid \alpha \text{ is } n\text{-regular } \}$ is the global lower crystal basis of M (see [20]). The expression (90) and (91) of the coefficients of this basis as Kazhdan-Lusztig polynomials have been obtained independently by Vasserot, Varagnolo [42] and by Goodman, Wenzl [10] respectively. ◇

It follows from Theorem 6.6 that the basis $\mathcal{G}_\alpha^-$ satisfies the following analogue of the Steinberg-Lusztig tensor product theorem. Let $\alpha \in \mathbb{P}^+$ of length r. Write $\alpha = \alpha^{(0)} + n\alpha^{(1)}$, where $\alpha^{(0)}$ is n-restricted, that is,

$$0 \le \alpha_i^{(0)} - \alpha_{i+1}^{(0)} < n \qquad (1 \le i \le r-1).$$

Then $\mathcal{G}_\alpha^- = S_{\alpha^{(1)}}\, \mathcal{G}_{\alpha^{(0)}}^-$. Taking $\alpha^{(0)} = (0)$ and writing $n\alpha$ in place of α we obtain that

$$\mathcal{G}_{n\alpha}^- = S_\alpha\, |0). \tag{92}$$

We can now prove the following symmetry of the basis $\{\mathcal{G}_\alpha^-\}$.

Theorem 7.13. *Let λ, $\mu^{(0)}, \ldots, \mu^{(n-1)}$ be partitions. Set $k = |\lambda|$. There holds*

(i) $\quad (\mathcal{G}_{n\lambda}^-)' = (-q)^{(n-1)k}\, \mathcal{G}_{n\lambda'}^-$,

(ii) $\quad c^\lambda_{\mu^{(0)},\ldots,\mu^{(n-1)}}(q^{-1}) = q^{-(n-1)k}\, c^{\lambda'}_{(\mu^{(n-1)})',\ldots,(\mu^{(0)})'}(q)$.

Proof — By (92) and (86) we have

$$(\mathcal{G}_{n\lambda}^-)' = (S_\lambda\,|0))' = (-q)^{(n-1)k}\, S_{\lambda'}\,|0) = (-q)^{(n-1)k}\, \mathcal{G}_{n\lambda'}^-.$$

The second equation follows now from the fact that if μ is the partition with n-quotient $(\mu^{(0)}, \ldots, \mu^{(n-1)})$ then the conjugate partition μ' has n-quotient $((\mu^{(n-1)})', \ldots, (\mu^{(0)})')$. □

Let $\{\mathcal{G}_\alpha^*\}$ denote the basis of $\mathbf{F}_\infty$ adjoint to $\{\mathcal{G}_\alpha^+\}$ for the above scalar product. In other words, $\langle \mathcal{G}_\alpha^*, \mathcal{G}_\beta^+ \rangle = \delta_{\alpha,\beta}$. Write

$$\mathcal{G}_\alpha^* = \sum_\beta g_{\alpha,\beta}(q)\,|\beta), \qquad \text{and} \qquad \mathbf{G}_k := [g_{\alpha,\beta}(q)], \qquad (\alpha, \beta \vdash k).$$

Since $\{|\alpha)\}$ is an orthonormal basis, we have $\mathbf{G}_k = \mathbf{D}_k^{-1}$.

Theorem 7.14. *For $\alpha \in \mathbb{P}^+$ one has $(\mathcal{G}_\alpha^*)' = \mathcal{G}_{\alpha'}^-$.*

Proof — We have to prove that $(\mathcal{G}^*_\alpha)'$ satisfies the two defining properties of $\mathcal{G}^-_{\alpha'}$, namely

$$(\mathcal{G}^*_\alpha)' \equiv \alpha' \bmod q^{-1}\mathcal{L}^-_\infty, \qquad \overline{(\mathcal{G}^*_\alpha)'} = (\mathcal{G}^*_\alpha)'.$$

The first property is obvious. Indeed by definition $\mathcal{G}^+_\alpha \equiv |\alpha) \bmod q\mathcal{L}^+_\infty$. Since $\mathbf{G}_k = \mathbf{D}_k^{-1}$, we deduce that $\mathcal{G}^*_\alpha \equiv |\alpha) \bmod q\mathcal{L}^+_\infty$, which implies that $(\mathcal{G}^*_\alpha)' \equiv \alpha' \bmod q^{-1}\mathcal{L}^-_\infty$. The second property is equivalent to

$$\langle \overline{(\mathcal{G}^*_\alpha)'}\,,\, (\mathcal{G}^+_\beta)' \rangle = \delta_{\alpha,\beta}, \qquad (\alpha,\ \beta \vdash k),$$

because $\{(\mathcal{G}^*_\alpha)'\}$ is the basis adjoint to $\{(\mathcal{G}^+_\beta)'\}$. Now, by Theorem 7.11,

$$\langle \overline{(\mathcal{G}^*_\alpha)'}\,,\, (\mathcal{G}^+_\beta)' \rangle = \langle \mathcal{G}^*_\alpha\,,\, \overline{\mathcal{G}^+_\beta} \rangle = \langle \mathcal{G}^*_\alpha\,,\, \mathcal{G}^+_\beta \rangle = \delta_{\alpha,\beta}.$$

□

Corollary 7.15. *Let* $\mathbf{J}_k = [j_{\alpha,\beta}(q)]_{\alpha,\beta\vdash k} := [e_{\alpha',\beta'}(-q)]^{-1}_{\alpha,\beta\vdash k}$. *Then* $\mathbf{J}_k = \mathbf{D}_k$. *In other words, we have*

$$\sum_{\gamma\vdash k} e_{\alpha',\gamma'}(-q)\, d_{\gamma,\beta}(q) = \delta_{\alpha,\beta},$$

where $e_{\alpha',\gamma'}$ *and* $d_{\gamma,\beta}$ *are the parabolic Kazhdan-Lusztig polynomials given by* (89) (90).

Remark 7.16. (i) Let α, β be two partitions of k and take $r \geq k$. By Lusztig's conjecture (33), it follows from Corollary 7.15 that

$$d_{\alpha,\beta}(1) = j_{\alpha,\beta}(1) = [W(\alpha') : L(\beta')],$$

the multiplicity of the simple $U_\zeta(\mathfrak{gl}_r)$-module $L(\beta')$ in the Weyl module $W(\alpha')$, as was conjectured in [22], Conjecture 5.2.
(ii) For $\lambda \in P^+_r$, let $T(\lambda)$ denote the indecomposable tilting $U_\zeta(\mathfrak{gl}_r)$-module with highest weight λ. By Proposition 8.2 of [7] which states that

$$[W(\alpha') : L(\beta')] = [T(\beta) : W(\alpha)],$$

we see that $[T(\beta) : W(\alpha)] = d_{\alpha,\beta}(1)$. Taking into account (91) we thus get another proof of the character formula of Soergel [40] in type A. Note that we do not need to deduce the formula for singular weights from that for regular weights (see [39], Remark 7.2). In particular, we see that the formula is also valid for $n < r$, when all integral weights are singular.

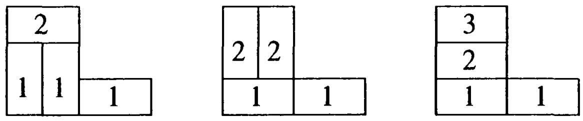

Fig. 10. The Yamanouchi domino tableaux of shape (42^2)

§8. Tables

We illustrate our results by giving some tables of q-Littlewood-Richardson coefficients and of polynomials $d_{\alpha,\beta}(q)$. These tables are q-analogues of those calculated by James in [12], which were the starting point of our investigation. They have been computed using the package `FOCK` written by the authors and available as a part of the environment `ACE` [43].

8.1. Canonical highest weight vectors of the Fock space representation of $U_q(\widehat{\mathfrak{sl}}_2)$

The following tables give the coefficients $e_{2\alpha,\beta}(-q^{-1})$ of the expansion of $\mathcal{G}^-_{2\alpha}$ on the standard basis $\{|\beta\rangle\}$ for $n = 2$ up to partitions of 10. They should be read by columns, *e.g.*

$$\mathcal{G}^-_{(4)} = |4\rangle - q^{-1}\,|3,1\rangle + q^{-2}\,|2,2\rangle.$$

These vectors form a basis of the subspace of primitive vectors of $\mathbf{F}_\infty$. Their coefficients are the q-analogues $c^{\lambda}_{\mu^{(0)},\mu^{(1)}}(-q^{-1})$ of the Littlewood-Richardson multiplicities for all partitions $\mu^{(0)}$, $\mu^{(1)}$ with $|\mu^{(0)}|+|\mu^{(1)}| \le 5$. They are easily computed using the combinatorial description of [2] in terms of Yamanouchi domino tableaux. For example the row labelled (42^2) is given by the tableaux of Figure 10.

	(2)
(2)	1
(1^2)	$-q^{-1}$

	(4)	(2^2)
(4)	1	0
(31)	$-q^{-1}$	0
(2^2)	q^{-2}	1
(21^2)	0	$-q^{-1}$
(1^4)	0	q^{-2}

	(6)	(42)	(2^3)
(6)	1	0	0
(51)	$-q^{-1}$	0	0
(42)	q^{-2}	1	0
(41^2)	0	$-q^{-1}$	0
(3^2)	$-q^{-3}$	$-q^{-1}$	0
(31^3)	0	q^{-2}	0
(2^3)	0	q^{-2}	1
(2^21^2)	0	$-q^{-3}$	$-q^{-1}$
(21^4)	0	0	q^{-2}
(1^6)	0	0	$-q^{-3}$

	(8)	(62)	(4^2)	(42^2)	(2^4)
(8)	1	0	0	0	0
(71)	$-q^{-1}$	0	0	0	0
(62)	q^{-2}	1	0	0	0
(61^2)	0	$-q^{-1}$	0	0	0
(53)	$-q^{-3}$	$-q^{-1}$	0	0	0
(51^3)	0	q^{-2}	0	0	0
(4^2)	q^{-4}	q^{-2}	1	0	0
(431)	0	0	$-q^{-1}$	0	0
(42^2)	0	q^{-2}	q^{-2}	1	0
(421^2)	0	$-q^{-3}$	0	$-q^{-1}$	0
(41^4)	0	0	0	q^{-2}	0
(3^22)	0	$-q^{-3}$	0	$-q^{-1}$	0
(3^21^2)	0	q^{-4}	q^{-2}	q^{-2}	0
(32^21)	0	0	$-q^{-3}$	0	0
(31^5)	0	0	0	$-q^{-3}$	0
(2^4)	0	0	q^{-4}	q^{-2}	1
(2^31^2)	0	0	0	$-q^{-3}$	$-q^{-1}$
(2^21^4)	0	0	0	q^{-4}	q^{-2}
(21^6)	0	0	0	0	$-q^{-3}$
(1^8)	0	0	0	0	q^{-4}

	(10)	(82)	(64)	(62^2)	(4^22)	(42^3)	(2^5)
(10)	1	0	0	0	0	0	0
(91)	$-q^{-1}$	0	0	0	0	0	0
(82)	q^{-2}	1	0	0	0	0	0
(81^2)	0	$-q^{-1}$	0	0	0	0	0
(73)	$-q^{-3}$	$-q^{-1}$	0	0	0	0	0
(71^3)	0	q^{-2}	0	0	0	0	0
(64)	q^{-4}	q^{-2}	1	0	0	0	0
(631)	0	0	$-q^{-1}$	0	0	0	0
(62^2)	0	q^{-2}	q^{-2}	1	0	0	0
(621^2)	0	$-q^{-3}$	0	$-q^{-1}$	0	0	0
(61^4)	0	0	0	q^{-2}	0	0	0
(5^2)	$-q^{-5}$	$-q^{-3}$	$-q^{-1}$	0	0	0	0
(532)	0	$-q^{-3}$	0	$-q^{-1}$	0	0	0
(531^2)	0	q^{-4}	q^{-2}	q^{-2}	0	0	0
(52^21)	0	0	$-q^{-3}$	0	0	0	0
(51^5)	0	0	0	$-q^{-3}$	0	0	0
(4^22)	0	q^{-4}	q^{-2}	q^{-2}	1	0	0
(4^21^2)	0	$-q^{-5}$	$-q^{-3}$	$-q^{-3}$	$-q^{-1}$	0	0
(43^2)	0	0	$-q^{-3}$	0	$-q^{-1}$	0	0
(431^3)	0	0	0	0	q^{-2}	0	0
(42^3)	0	0	q^{-4}	q^{-2}	q^{-2}	1	0
(42^21^2)	0	0	0	$-q^{-3}$	$-q^{-3}$	$-q^{-1}$	0
(421^4)	0	0	0	q^{-4}	0	q^{-2}	0
(41^6)	0	0	0	0	0	$-q^{-3}$	0
(3^31)	0	0	q^{-4}	0	q^{-2}	0	0
(3^22^2)	0	0	$-q^{-5}$	$-q^{-3}$	$-q^{-3}$	$-q^{-1}$	0
(3^221^2)	0	0	0	q^{-4}	0	q^{-2}	0
(3^21^4)	0	0	0	$-q^{-5}$	$-q^{-3}$	$-q^{-3}$	0
(32^21^3)	0	0	0	0	q^{-4}	0	0
(31^7)	0	0	0	0	0	q^{-4}	0
(2^5)	0	0	0	0	q^{-4}	q^{-2}	1
(2^41^2)	0	0	0	0	$-q^{-5}$	$-q^{-3}$	$-q^{-1}$
(2^31^4)	0	0	0	0	0	q^{-4}	q^{-2}
(2^21^6)	0	0	0	0	0	$-q^{-5}$	$-q^{-3}$
(21^8)	0	0	0	0	0	0	q^{-4}
(1^{10})	0	0	0	0	0	0	$-q^{-5}$

8.2. Basis $\{\mathcal{G}^+_\beta\}$ of the Fock space representation of $U_q(\widehat{\mathfrak{sl}}_2)$

The following tables give the coefficients $d_{\alpha,\beta}(q)$ of the expansion of $\mathcal{G}^+_\beta$ on the standard basis $\{|\alpha)\}$ for $n = 2$ up to partitions of 10. They should be read by columns, *e.g.*

$$\mathcal{G}^+_{(3,1)} = |3,1) + q\,|2,2) + q^2\,|2,1,1).$$

Each square matrix corresponds to a weight space of $\mathbf{F}_\infty$. (The weight space containing $|10)$ being too large, the corresponding matrix had to be displayed on two pages.) The 1-dimensional weight spaces corresponding to the partitions (1), (2,1), (3,2,1), (4,3,2,1) have been omitted.

(2)	1	0
(1^2)	q	1

(3)	1	0
(1^3)	q	1

(4)	1	0	0	0	0
(31)	q	1	0	0	0
(2^2)	0	q	1	0	0
(21^2)	q	q^2	q	1	0
(1^4)	q^2	0	0	q	1

(5)	1	0	0	0	0
(32)	0	1	0	0	0
(31^2)	q	q	1	0	0
(2^21)	0	q^2	q	1	0
(1^5)	q^2	0	q	0	1

(41)	1	0
(21^3)	q	1

(6)	1	0	0	0	0	0	0	0	0	0
(51)	q	1	0	0	0	0	0	0	0	0
(42)	0	q	1	0	0	0	0	0	0	0
(41^2)	q	q^2	q	1	0	0	0	0	0	0
(3^2)	0	0	q	0	1	0	0	0	0	0
(31^3)	q^2	q	q^2	q	q	1	0	0	0	0
(2^3)	0	0	q^2	q	q	0	1	0	0	0
(2^21^2)	0	q^2	q^3	q^2	q^2	q	q	1	0	0
(21^4)	q^2	q^3	0	q	0	q^2	0	q	1	0
(1^6)	q^3	0	0	q^2	0	0	0	0	q	1

(7)	1	0	0	0	0	0	0	0	0	0
(52)	0	1	0	0	0	0	0	0	0	0
(51^2)	q	q	1	0	0	0	0	0	0	0
(421)	0	q^2	q	1	0	0	0	0	0	0
(3^21)	0	0	0	q	1	0	0	0	0	0
(32^2)	0	0	q	q^2	q	1	0	0	0	0
(321^2)	0	q	q^2	q^3	q^2	q	1	0	0	0
(31^4)	q^2	q^2	q	0	0	0	q	1	0	0
(2^21^3)	0	q^3	q^2	0	0	q	q^2	q	1	0
(1^7)	q^3	0	q^2	0	0	0	0	q	0	1

(61)	1	0	0	0	0
(43)	0	1	0	0	0
(41^3)	q	q	1	0	0
(2^31)	0	q^2	q	1	0
(21^5)	q^2	0	q	0	1

(521)	1	0
(321^3)	q	1

(8)	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(71)	q	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(62)	0	q	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(61^2)	q	q^2	q	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(53)	0	0	q	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(51^3)	q^2	q	q^2	q	q	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(4^2)	0	0	0	0	q	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
(431)	0	0	q	0	q^2	0	q	1	0	0	0	0	0	0	0	0	0	0	0	0
(42^2)	0	0	q^2	q	0	0	0	q	1	0	0	0	0	0	0	0	0	0	0	0
(421^2)	0	q^2	$q+q^3$	q^2	q^2	q	q	q^2	q	1	0	0	0	0	0	0	0	0	0	0
(41^4)	q^2	q^3	q^2	q	q^3	q^2	q^2	0	0	q	1	0	0	0	0	0	0	0	0	0
$(3^2 2)$	0	0	0	0	0	0	q	q^2	q	0	0	1	0	0	0	0	0	0	0	0
$(3^2 1^2)$	0	0	q^2	0	0	0	q^2	q^3	q^2	q	0	q	1	0	0	0	0	0	0	0
$(32^2 1)$	0	0	q^3	q^2	q^2	q	$q+q^3$	q^4	$q+q^3$	q^2	0	q^2	q	1	0	0	0	0	0	0
(31^5)	q^3	q^2	q^3	q^2	0	q	0	0	0	q^2	q	0	q	0	1	0	0	0	0	0
(2^4)	0	0	0	0	q^3	q^2	q^2	0	0	0	0	0	0	q	0	1	0	0	0	0
$(2^3 1^2)$	0	0	q^3	q^2	q^4	q^3	q^3	0	q	q^2	q	0	q	q^2	0	q	1	0	0	0
$(2^2 1^4)$	0	q^3	q^4	q^3	0	q^2	0	0	q^2	q^3	q^2	0	q^2	0	q	0	q	1	0	0
(21^6)	q^3	q^4	0	q^2	0	q^3	0	0	0	0	q	0	0	0	q^2	0	0	q	1	0
(1^8)	q^4	0	0	q^3	0	0	0	0	0	0	q^2	0	0	0	0	0	0	0	q	1

(9)	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(72)	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(71^2)	q	q	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(621)	0	q^2	q	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(54)	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(531)	0	0	0	q	q	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(52^2)	0	0	q	q^2	0	q	1	0	0	0	0	0	0	0	0	0	0	0	0	0
(521^2)	0	q	q^2	q^3	q	q^2	q	1	0	0	0	0	0	0	0	0	0	0	0	0
(51^4)	q^2	q^2	q	0	q^2	0	0	q	1	0	0	0	0	0	0	0	0	0	0	0
(4^21)	0	0	0	0	q^2	q	0	0	0	1	0	0	0	0	0	0	0	0	0	0
(421^3)	0	q^3	q^2	q	q^3	q^2	q	q^2	q	q	1	0	0	0	0	0	0	0	0	0
(3^3)	0	0	0	0	0	q^2	q	0	0	q	0	1	0	0	0	0	0	0	0	0
(3^21^3)	0	0	0	q^2	0	q^3	q^2	0	0	q^2	q	q	1	0	0	0	0	0	0	0
(32^3)	0	0	0	0	q^2	q^3	q^2	q	0	q^2	0	q	0	1	0	0	0	0	0	0
(32^21^2)	0	0	q^2	q^3	q^3	q^4	$q+q^3$	q^2	q	q^3	q^2	q^2	q	q	1	0	0	0	0	0
(321^4)	0	q^2	q^3	q^4	0	0	q^2	q	q^2	0	q^3	0	q^2	0	q	1	0	0	0	0
(31^6)	q^3	q^3	q^2	0	0	0	0	q^2	q	0	0	0	0	0	0	q	1	0	0	0
(2^41)	0	0	0	0	q^4	0	0	q^3	q^2	0	0	0	0	q^2	q	0	0	1	0	0
(2^21^5)	0	q^4	q^3	0	0	0	q^2	q^3	q^2	0	0	0	0	0	q	q^2	q	0	1	0
(1^9)	q^4	0	q^3	0	0	0	0	0	q^2	0	0	0	0	0	0	0	q	0	0	1

(81)	1	0	0	0	0	0	0	0	0	0
(63)	0	1	0	0	0	0	0	0	0	0
(61^3)	q	q	1	0	0	0	0	0	0	0
(432)	0	0	0	1	0	0	0	0	0	0
(431^2)	0	q	0	q	1	0	0	0	0	0
(42^21)	0	q^2	q	q^2	q	1	0	0	0	0
(41^5)	q^2	q^2	q	0	q	0	1	0	0	0
(3^221)	0	0	0	q^3	q^2	q	0	1	0	0
(2^31^3)	0	q^3	q^2	0	q^2	q	q	0	1	0
(21^7)	q^3	0	q^2	0	0	0	q	0	0	1

(721)	1	0	0	0	0
(541)	0	1	0	0	0
(521^3)	q	q	1	0	0
(32^31)	0	q^2	q	1	0
(321^5)	q^2	0	q	0	1

(10)	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(91)	q	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(82)	0	q	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(81^2)	q	q^2	q	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(73)	0	0	q	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
(71^3)	q^2	q	q^2	q	q	1	0	0	0	0	0	0	0	0	0	0	0	0
(64)	0	0	0	0	q	0	1	0	0	0	0	0	0	0	0	0	0	0
(631)	0	0	q	0	q^2	0	q	1	0	0	0	0	0	0	0	0	0	0
(622)	0	0	q^2	q	0	0	0	q	1	0	0	0	0	0	0	0	0	0
(621^2)	0	q^2	$q+q^3$	q^2	q^2	q	q	q^2	q	1	0	0	0	0	0	0	0	0
(61^4)	q^2	q^3	q^2	q	q^3	q^2	q^2	0	0	q	1	0	0	0	0	0	0	0
(5^2)	0	0	0	0	0	0	q	0	0	0	0	1	0	0	0	0	0	0
(532)	0	0	0	0	0	0	q	q^2	q	0	0	0	1	0	0	0	0	0
(531^2)	0	0	q^2	0	q	0	$2q^2$	q^3	q^2	q	0	q	q	1	0	0	0	0
(52^21)	0	0	q^3	q^2	q^2	q	q^3	q^4	$q+q^3$	q^2	0	0	q^2	q	1	0	0	0
(51^5)	q^3	q^2	q^3	q^2	q^2	q	q^3	0	0	q^2	q	q^2	0	q	0	1	0	0
(4^22)	0	0	0	0	0	0	q^2	0	0	0	0	q	q	0	0	0	1	0
(4^21^2)	0	0	0	0	q^2	0	q^3	0	0	0	0	q^2	q^2	q	0	0	q	1
(43^2)	0	0	0	0	0	0	0	0	q	0	0	0	q^2	0	0	0	q	0
(431^3)	0	0	q^2	0	q^3	0	q^2	q	q^2	q	0	q	q^3	q^2	0	0	q^2	q
(42^3)	0	0	0	0	q^3	q^2	q^2	0	q^2	q	0	q	q^3	q^2	q	0	q^2	q
(42^21^2)	0	0	q^3	q^2	q^4	q^3	$2q^3$	q^2	$q+q^3$	$2q^2$	q	$2q^2$	q^4	q^3	q^2	0	q^3	q^2
(421^4)	0	q^3	q^4+q^2	q^3	q^3	q^2	q^4	q^3	q^2	$q+q^3$	q^2	q^3	0	q^2	0	q	0	q
(41^6)	q^3	q^4	q^3	q^2	q^4	q^3	0	0	0	q^2	q	0	0	q^3	0	q^2	0	q^2
(3^31)	0	0	0	0	0	0	0	0	q^2	0	0	q	q^3	q^2	q	0	q^2	q
(3^22^2)	0	0	0	0	0	0	q^3	0	q^3	q^2	0	$2q^2$	q^4	q^3	q^2	0	q^3	q^2
(3^221^2)	0	0	0	0	0	0	q^4	q^3	q^4+q^2	q^3	0	$2q^3$	q^5	q^4	q^3	0	q^4	$q+q^3$
(3^21^4)	0	0	q^3	0	0	0	0	q^4	q^3	q^2	0	0	0	0	0	0	0	q^2
(32^21^3)	0	0	q^4	q^3	q^3	q^2	q^4	q^5	q^4+q^2	$2q^3$	q^2	q^3	0	q^2	q	q	0	$q+q^3$
(31^7)	q^4	q^3	q^4	q^3	0	q^2	0	0	0	q^3	q^2	0	0	0	0	q	0	0
(2^5)	0	0	0	0	0	0	q^4	0	0	q^3	q^2	q^3	0	0	0	0	0	0
(2^41^2)	0	0	0	0	q^4	q^3	q^5	0	0	q^4	q^3	q^4	0	q^3	q^2	q^2	0	q^2
(2^31^4)	0	0	q^4	q^3	q^5	q^4	0	0	q^2	q^3	q^2	0	0	q^4	q^3	q^3	0	q^3
(2^21^6)	0	q^4	q^5	q^4	0	q^3	0	0	q^3	q^4	q^3	0	0	0	0	q^2	0	0
(21^8)	q^4	q^5	0	q^3	0	q^4	0	0	0	0	q^2	0	0	0	0	q^3	0	0
(1^{10})	q^5	0	0	q^4	0	0	0	0	0	0	q^3	0	0	0	0	0	0	0

(43^2)	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(431^2)	q	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(42^3)	q	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(42^21^2)	q^2	q	q	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
(421^4)	0	q^2	0	q	1	0	0	0	0	0	0	0	0	0	0	0	0	0
(41^6)	0	0	0	0	q	1	0	0	0	0	0	0	0	0	0	0	0	0
(3^31)	q	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0
(3^22^2)	q^2	0	q	0	0	0	q	1	0	0	0	0	0	0	0	0	0	0
(3^221^2)	$q+q^3$	q^2	q^2	q	0	0	q^2	q	1	0	0	0	0	0	0	0	0	0
(3^21^4)	q^2	q^3	0	q^2	q	0	0	0	q	1	0	0	0	0	0	0	0	0
(32^21^3)	q^3	q^4	q^2	$q+q^3$	q^2	0	0	q	q^2	q	1	0	0	0	0	0	0	0
(31^7)	0	0	0	0	q^2	q	0	0	0	q	0	1	0	0	0	0	0	0
(2^5)	0	0	q^2	q	0	0	0	q	0	0	0	0	1	0	0	0	0	0
(2^41^2)	0	0	q^3	q^2	0	0	0	q^2	0	0	q	0	q	1	0	0	0	0
(2^31^4)	0	0	0	q	q^2	q	0	0	0	q	q^2	0	0	q	1	0	0	0
(2^21^6)	0	0	0	q^2	q^3	q^2	0	0	0	q^2	0	q	0	0	q	1	0	0
(21^8)	0	0	0	0	0	q	0	0	0	0	0	q^2	0	0	0	q	1	0
(1^{10})	0	0	0	0	0	q^2	0	0	0	0	0	0	0	0	0	0	q	1

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A Weight Basis for Representations of Even Orthogonal Lie Algebras

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Abstract.

A weight basis for each finite-dimensional irreducible representation of the orthogonal Lie algebra $\mathfrak{o}(2n)$ is constructed. The basis vectors are parametrized by the D-type Gelfand–Tsetlin patterns. The basis is consistent with the chain of subalgebras $\mathfrak{g}_1 \subset \cdots \subset \mathfrak{g}_n$, where $\mathfrak{g}_k = \mathfrak{o}(2k)$. Explicit formulas for the matrix elements of generators of $\mathfrak{o}(2n)$ in this basis are given. The construction is based on the representation theory of the Yangians and extends our previous results for the symplectic Lie algebras.

§1. Introduction

In their pioneering works [3] and [4] Gelfand and Tsetlin proposed a combinatorial method to explicitly construct representations of the classical Lie algebras. For each finite-dimensional irreducible representation of the general linear Lie algebra $\mathfrak{gl}(N)$ and the orthogonal Lie algebra $\mathfrak{o}(N)$ they gave a parametrization of basis vectors and provided explicit formulas for the matrix elements of generators of the Lie algebras in the basis. Derivations of the matrix element formulas in the orthogonal case are given in [14, 17]; see also [5]. A number of different approaches to the problem of constructing representation bases for simple Lie algebras has been developed; see [9] for more references. Note also recent results by Donnelly [2] and Littelmann [6]. In [2] explicit combinatorial constructions of the fundamental representations of the B and C series Lie algebras and of their q-analogs are given; in [6] monomial bases parametrized by patterns of Gelfand–Tsetlin type are constructed for all simple complex Lie algebras. In [9] an analog of the Gelfand–Tsetlin basis for the symplectic Lie algebras $\mathfrak{sp}(2n)$ is constructed and explicit formulas for the matrix elements of generators of $\mathfrak{sp}(2n)$ in this basis are given. Bases for the finite-dimensional irreducible representations

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of the classical Lie algebras of B, C, and D series can be constructed in a uniform manner with the use of the representation theory of the Yangians, as in [9]. Here we extend the results of [9] to the case of the D series and hope to treat the remaining B series case in a forthcoming publication.

Our basis for $\mathfrak{o}(2n)$ is different from that of Gelfand and Tsetlin [4]. Their basis is consistent with the chain of subalgebras

$$\mathfrak{o}(2) \subset \mathfrak{o}(3) \subset \cdots \subset \mathfrak{o}(N).$$

The reductions $\mathfrak{o}(k) \downarrow \mathfrak{o}(k-1)$ are multiplicity-free which makes the basis orthogonal with respect to a natural contravariant bilinear form. However, the basis vectors are not weight vectors with respect to the Cartan subalgebra of $\mathfrak{o}(N)$. To get a weight (although non-orthogonal) basis we consider the following chain instead:

$$\mathfrak{o}(2) \subset \mathfrak{o}(4) \subset \cdots \subset \mathfrak{o}(2n)$$

so that all the subalgebras belong to the D series.

The reduction $\mathfrak{o}(2n) \downarrow \mathfrak{o}(2n-2)$ is not multiplicity free. This means that the subspace $V(\lambda)^+_\mu$ of $\mathfrak{o}(2n-2)$-highest vectors of a weight μ in an $\mathfrak{o}(2n)$-module $V(\lambda)$ is not necessarily one-dimensional. However, this space turns out to possess a natural structure of an irreducible representation of a large associative algebra $\mathrm{Y}^+(2)$ called the twisted Yangian (introduced by Olshanski in [13]) and can also be equipped with an action of the $\mathfrak{gl}(2)$-Yangian $\mathrm{Y}(2)$. This allows us to construct a Yangian Gelfand–Tsetlin basis in $V(\lambda)^+_\mu$ associated with an inclusion $\mathrm{Y}(1) \subset \mathrm{Y}(2)$; see [7, 11, 12, 15].

Our calculations are based on the relationship between the twisted Yangian $\mathrm{Y}^+(2)$ and the transvector algebra $\mathrm{Z}(\mathfrak{g}_n, \mathfrak{g}_{n-1})$, $\mathfrak{g}_n = \mathfrak{o}(2n)$. The transvector algebras (they are sometimes called the Mickelsson algebras or S-algebras) are studied in detail in [18, 19].

Although the constructions of the bases are very similar for the orthogonal and symplectic cases, there is a slight difference in the calculation of the matrix elements of generators of the Lie algebras in the basis. The Lie algebra $\mathfrak{sp}(2n)$ contains the "second diagonal" generators $F_{-k,k} = 2E_{-k,k}$ where the E_{ij} denote the standard generators of $\mathfrak{gl}(2n)$ (see Section 2 below). The action of these elements in the basis is rather simple and can be easily found. However, their counterparts do not exist in the orthogonal case. Instead, there are second degree elements $\Phi_{-k,k}$ of the universal enveloping algebra $\mathrm{U}(\mathfrak{g}_n)$ which belong to the centralizer of $\mathfrak{g}_{k-1}$ in $\mathrm{U}(\mathfrak{g}_k)$ and play the role similar to that of the elements $F_{-k,k}$ in the symplectic case.

§2. Notations and preliminary results

We shall enumerate the rows and columns of $2n \times 2n$-matrices over $\mathbb{C}$ by the indices $-n, \dots, -1, 1, \dots, n$. We let the E_{ij}, $i, j = -n, \dots, n$ denote the standard basis of the Lie algebra $\mathfrak{gl}(2n)$. We shall also assume throughout the paper that the index 0 is skipped in a sum or in a product. Introduce the elements

$$F_{ij} = E_{ij} - E_{-j,-i}. \tag{2.1}$$

We have $F_{-j,-i} = -F_{ij}$. In particular, $F_{-i,i} = 0$ for all i. The orthogonal Lie algebra $\mathfrak{g}_n := \mathfrak{o}(2n)$ can be identified with the subalgebra in $\mathfrak{gl}(2n)$ spanned by the elements F_{ij}, $i, j = -n, \dots, n$.

The subalgebra $\mathfrak{g}_{n-1}$ is spanned by the elements (2.1) with the indices i, j running over the set $\{-n+1, \dots, n-1\}$. Denote by $\mathfrak{h} = \mathfrak{h}_n$ the diagonal Cartan subalgebra in $\mathfrak{g}_n$. The elements $F_{11}, \dots, F_{nn}$ form a basis of $\mathfrak{h}$.

The finite-dimensional irreducible representations of $\mathfrak{g}_n$ are in a one-to-one correspondence with n-tuples $\lambda = (\lambda_1, \dots, \lambda_n)$ where all the entries λ_i are simultaneously integers or half-integers (elements of the set $\frac{1}{2} + \mathbb{Z}$) and the following inequalities hold:

$$-|\lambda_1| \geq \lambda_2 \geq \cdots \geq \lambda_n.$$

Such an n-tuple λ is called the highest weight of the corresponding representation which we shall denote by $V(\lambda)$. It contains a unique, up to a multiple, nonzero vector ξ (the highest vector) such that $F_{ii}\,\xi = \lambda_i\,\xi$ for $i = 1, \dots, n$ and $F_{ij}\,\xi = 0$ for $-n \leq i < j \leq n$.

Denote by $V(\lambda)^+$ the subspace of $\mathfrak{g}_{n-1}$-highest vectors in $V(\lambda)$:

$$V(\lambda)^+ = \{\eta \in V(\lambda) \mid F_{ij}\,\eta = 0, \qquad -n < i < j < n\}.$$

Given a $\mathfrak{g}_{n-1}$-highest weight $\mu = (\mu_1, \dots, \mu_{n-1})$ we denote by $V(\lambda)^+_\mu$ the corresponding weight subspace in $V(\lambda)^+$:

$$V(\lambda)^+_\mu = \{\eta \in V(\lambda)^+ \mid F_{ii}\,\eta = \mu_i\,\eta, \qquad i = 1, \dots, n-1\}.$$

Consider the extension of the universal enveloping algebra $\mathrm{U}(\mathfrak{g}_n)$

$$\mathrm{U}'(\mathfrak{g}_n) = \mathrm{U}(\mathfrak{g}_n) \otimes_{\mathrm{U}(\mathfrak{h})} \mathrm{R}(\mathfrak{h}),$$

where $\mathrm{R}(\mathfrak{h})$ is the field of fractions of the commutative algebra $\mathrm{U}(\mathfrak{h})$. Let J denote the left ideal in $\mathrm{U}'(\mathfrak{g}_n)$ generated by the elements F_{ij} with $-n < i < j < n$. Set

$$\mathrm{Z}(\mathfrak{g}_n, \mathfrak{g}_{n-1}) = \{x \in \mathrm{U}'(\mathfrak{g}_n)/\mathrm{J} \mid F_{ij}\,x \equiv 0, \quad -n < i < j < n\}.$$

Then $\mathrm{Z}(\mathfrak{g}_n, \mathfrak{g}_{n-1})$ is an algebra with the multiplication inherited from $\mathrm{U}'(\mathfrak{g}_n)$. We call it the *transvector algebra*; see [18, 19] for further details.

Set

$$f_i = F_{ii} - i + 1, \qquad f_{-i} = -f_i$$

for $i = 1, \dots, n$. Let p denote the *extremal projection* for the Lie algebra $\mathfrak{g}_{n-1}$; see [1, 19]. The projection p naturally acts in the space $\mathrm{U}'(\mathfrak{g}_n)/\mathrm{J}$ and its image coincides with $\mathrm{Z}(\mathfrak{g}_n, \mathfrak{g}_{n-1})$. The elements

$$pF_{ia}, \qquad a = -n, n, \qquad i = -n+1, \dots, n-1$$

are generators of $\mathrm{Z}(\mathfrak{g}_n, \mathfrak{g}_{n-1})$ [19]. They can be given by the following explicit formulas (modulo J):

$$pF_{ia} = \sum_{i > i_1 > \cdots > i_s > -n} F_{ii_1} F_{i_1 i_2} \cdots F_{i_{s-1} i_s} F_{i_s a} \frac{1}{(f_i - f_{i_1}) \cdots (f_i - f_{i_s})},$$

where $s = 0, 1, \dots$ (it is assumed that index 0 is excluded in the sum). We shall use the normalized generators of $\mathrm{Z}(\mathfrak{g}_n, \mathfrak{g}_{n-1})$ defined by

$$\begin{aligned} z_{ia} &= pF_{ia}(f_i - f_{i-1}) \cdots (\widehat{f_i - f_{-i}}) \cdots (f_i - f_{-n+1}), \\ z_{ai} &= pF_{ai}(f_i - f_{i+1}) \cdots (\widehat{f_i - f_{-i}}) \cdots (f_i - f_{n-1}), \end{aligned} \tag{2.2}$$

where the hats indicate the factors to be omitted if they occur. We obviously have $z_{ai} = (-1)^{n-i} z_{-i,-a}$. The following equivalent formula holds for z_{ai}:

$$\begin{aligned} z_{ai} = {} & (f_i - f_{i+1}) \cdots (\widehat{f_i - f_{-i}}) \cdots (f_i - f_{n-1}) \\ & \times \sum_{n > i_1 > \cdots > i_s > i} \frac{1}{(f_i - f_{i_1}) \cdots (f_i - f_{i_s})} F_{ai_1} F_{i_1 i_2} \cdots F_{i_{s-1} i_s} F_{i_s i}. \end{aligned} \tag{2.3}$$

The elements z_{ia} and z_{ai} naturally act in the space $V(\lambda)^+$ and are called the *raising* and *lowering operators*. One has for $i = 1, \dots, n-1$:

$$z_{ia} : V(\lambda)^+_\mu \to V(\lambda)^+_{\mu+\delta_i}, \qquad z_{ai} : V(\lambda)^+_\mu \to V(\lambda)^+_{\mu-\delta_i},$$

where $\mu \pm \delta_i$ is obtained from μ by replacing μ_i with $\mu_i \pm 1$.

Note the following relations between these operators; cf. [19]. For $a, b \in \{-n, n\}$ and $i + j \neq 0$ one has

$$z_{aj} z_{bi} (f_i - f_j + 1) = z_{bi} z_{aj} (f_i - f_j) + z_{ai} z_{bj}.$$

In particular, z_{ai} and z_{aj} commute for $i+j \neq 0$. One easily verifies that z_{ai} and z_{bi} also commute for all a, b. We shall use the following element which can be checked to belong to the algebra $\mathrm{Z}(\mathfrak{g}_n, \mathfrak{g}_{n-1})$

(2.4)

$$z_{n,-n} = \sum_{n>i_1>\cdots>i_s>-n} F_{ni_1} F_{i_1 i_2} \cdots F_{i_s,-n} \frac{(f_n - f_{j_1}) \cdots (f_n - f_{j_k})}{2f_n},$$

where $s = 1, 2, \ldots$ and $\{j_1, \ldots, j_k\}$ is the complement to the subset $\{i_1, \ldots, i_s\}$ in $\{-n+1, \ldots, n-1\}$. There is an equivalent formula for $z_{n,-n}$ which can either be proved directly (cf. [9, Section 2]), or can be deduced from (4.6) below (use the fact that $Z_{n,-n}(g_n) = Z_{n,-n}(-g_n)$):

$$z_{n,-n} = \sum_{n>i_1>\cdots>i_s>-n} F_{ni_1} F_{i_1 i_2} \cdots F_{i_s,-n} \frac{(f_{-n} - f_{j_1} - 1) \cdots (f_{-n} - f_{j_k} - 1)}{2f_{-n} - 2}.$$

This formula together with (2.3) is used in the derivation of the following relations from (2.2) (cf. [9, Proposition 2.1]): for $a = -n, n$

$$F_{n-1,a} = \sum_{i=-n+1}^{n-1} z_{n-1,i}\, z_{ia} \prod_{j=-n+1,\, j\neq \pm i}^{n-1} \frac{1}{f_i - f_j}, \tag{2.5}$$

where $z_{n-1,n-1} := 1$ and the equalities are considered in $\mathrm{U}'(\mathfrak{g}_n)$ modulo the ideal J.

Let us now introduce the $\mathfrak{gl}(2)$-*Yangian* $\mathrm{Y}(2)$ and the (*orthogonal*) *twisted Yangian* $\mathrm{Y}^+(2)$; see [10] for more details. The Yangian $\mathrm{Y}(2)$ is the complex associative algebra with the generators $t_{ab}^{(1)}, t_{ab}^{(2)}, \ldots$ where $a, b \in \{-n, n\}$, and the defining relations

$$[t_{ab}(u), t_{cd}(v)] = \frac{1}{u-v}\Big(t_{cb}(u)t_{ad}(v) - t_{cb}(v)t_{ad}(u)\Big), \tag{2.6}$$

where

$$t_{ab}(u) := \delta_{ab} + t_{ab}^{(1)} u^{-1} + t_{ab}^{(2)} u^{-2} + \cdots \in \mathrm{Y}(2)[[u^{-1}]].$$

Introduce the series $s_{ab}(u)$, $a, b \in \{-n, n\}$ by

$$s_{ab}(u) = t_{an}(u)t_{-b,-n}(-u) + t_{a,-n}(u)t_{-b,n}(-u).$$

Write $s_{ab}(u) = \delta_{ab} + s_{ab}^{(1)} u^{-1} + s_{ab}^{(2)} u^{-2} + \cdots$. The twisted Yangian $\mathrm{Y}^+(2)$ is defined as the subalgebra of $\mathrm{Y}(2)$ generated by the elements

$s_{ab}^{(1)}, s_{ab}^{(2)}, \dots$ where $a, b \in \{-n, n\}$. Note that $\mathrm{Y}^+(2)$ can be equivalently defined as an abstract algebra with these generators and certain linear and quadratic defining relations; see [10, Section 3].

The Yangian $\mathrm{Y}(2)$ is a Hopf algebra with the coproduct

$$\Delta(t_{ab}(u)) = t_{an}(u) \otimes t_{nb}(u) + t_{a,-n}(u) \otimes t_{-n,b}(u). \tag{2.7}$$

The twisted Yangian $\mathrm{Y}^+(2)$ is a left coideal in $\mathrm{Y}(2)$ with

$$\Delta(s_{ab}(u)) = \sum_{c,d \in \{-n,n\}} t_{ac}(u) t_{-b,-d}(-u) \otimes s_{cd}(u). \tag{2.8}$$

Given a pair of complex numbers (α, β) such that $\alpha - \beta \in \mathbb{Z}_+$ we denote by $L(\alpha, \beta)$ the irreducible representation of the Lie algebra $\mathfrak{gl}(2)$ with the highest weight (α, β) with respect to the upper triangular Borel subalgebra. We have $\dim L(\alpha, \beta) = \alpha - \beta + 1$. We may regard $L(\alpha, \beta)$ as a $\mathrm{Y}(2)$-module by using the algebra homomorphism $\mathrm{Y}(2) \to \mathrm{U}(\mathfrak{gl}(2))$ given by

$$t_{ab}(u) \mapsto \delta_{ab} + E_{ab} u^{-1}, \qquad a, b \in \{-n, n\}. \tag{2.9}$$

The coproduct (2.7) allows one to construct representations of $\mathrm{Y}(2)$ of the form

$$L = L(\alpha_1, \beta_1) \otimes \cdots \otimes L(\alpha_k, \beta_k).$$

For any $\gamma \in \mathbb{C}$ denote by $W(\gamma)$ the one-dimensional representation of $\mathrm{Y}^+(2)$ spanned by a vector w such that

$$s_{nn}(u)\, w = \frac{u + \gamma}{u + 1/2}\, w, \qquad s_{-n,-n}(u)\, w = \frac{u - \gamma + 1}{u + 1/2}\, w,$$

and $s_{a,-a}(u)\, w = 0$ for $a = -n, n$. By (2.8) we can regard the tensor product $L \otimes W(\gamma)$ as a representation of $\mathrm{Y}^+(2)$. Representations of this type essentially exhaust all finite-dimensional irreducible representations of $\mathrm{Y}^+(2)$ [8]. The vector space isomorphism

$$L \otimes W(\gamma) \to L, \qquad v \otimes w \mapsto v, \qquad v \in L \tag{2.10}$$

provides $L \otimes W(\gamma)$ with an action of $\mathrm{Y}(2)$.

§3. Construction of the basis

Introduce the following series with coefficients in the transvector algebra $\mathrm{Z}(\mathfrak{g}_n, \mathfrak{g}_{n-1})$: for $a, b \in \{-n, n\}$

$$(3.1)\quad Z_{ab}(u) = \Bigg(-\Big(\delta_{ab}(u - n + 3/2) + F_{ab}\Big) \prod_{i=-n+1}^{n-1} (u + g_i) + \sum_{i=-n+1}^{n-1} z_{ai} z_{ib} \, (u + g_{-i}) \prod_{j=-n+1,\, j \neq \pm i}^{n-1} \frac{u + g_j}{g_i - g_j} \Bigg) \frac{1}{2u+1},$$

where $g_i := f_i + 1/2$ for all i.

As we shall see below (Corollary 3.3) the space $V(\lambda)^+_\mu$ is nonzero only if there exist $\nu_1, \dots, \nu_{n-1}$ such that the inequalities (3.10) hold. We shall be assuming that this condition is satisfied.

Proposition 3.1. (i) *The mapping*

$$(3.2)\qquad s_{ab}(u) \mapsto -2u^{-2n+2} Z_{ab}(u), \qquad a, b \in \{-n, n\}$$

defines an algebra homomorphism $\mathrm{Y}^+(2) \to \mathrm{Z}(\mathfrak{g}_n, \mathfrak{g}_{n-1})$.

(ii) *The representation of* $\mathrm{Y}^+(2)$ *in the space* $V(\lambda)^+_\mu$ *defined via the homomorphism (3.2) is irreducible.*

Proof. We use the same arguments as for the proof of the corresponding statements in the symplectic case; see [9, Section 5]. So we shall only give a few key formulas; the details can be restored by using [9].

Introduce the $2n \times 2n$-matrix $F = (F_{ij})$ whose ijth entry is the element $F_{ij} \in \mathfrak{g}_n$ and set $F(u) = 1 + F(u + 1/2)^{-1}$. Denote by $\widehat{F}(u)$ the corresponding *Sklyanin comatrix*; see [8, Section 2]. (More precisely, one first considers the twisted Yangian $\mathrm{Y}^+(2n)$ for the Lie algebra $\mathfrak{g}_n = \mathfrak{o}(2n)$ and the corresponding *S-matrix* $S(u)$ [10, Section 3]. The Sklyanin comatrix $\widehat{S}(u)$ is defined by the relation $\operatorname{sdet} S(u) = \widehat{S}(u)\, S(u - 2n + 1)$, where $\operatorname{sdet} S(u)$ is the *Sklyanin determinant* of $S(u)$; see [10, Section 4] for its definition. Then $\widehat{F}(u)$ is defined as the image of $\widehat{S}(u)$ under the algebra homomorphism $\mathrm{Y}^+(2n) \to \mathrm{U}(\mathfrak{g}_n)$ such that $S(u) \mapsto F(u)$; see [8, Section 2] for more details.) The mapping

$$(3.3)\qquad s_{ab}(u) \mapsto c(u)\, \widehat{F}(-u + n - 1)_{ab}, \qquad a, b \in \{-n, n\},$$

where

$$c(u) = \prod_{k=1}^{n-1} (1 - (k - 1/2)^2 u^{-2}),$$

defines an algebra homomorphism from $\mathrm{Y}^+(2)$ to the centralizer C_n of $\mathfrak{g}_{n-1}$ in $\mathrm{U}(\mathfrak{g}_n)$ [8, Proposition 2.1]; cf. [13]. Further, a slight generalization of [9, Proposition 3.1] implies the following expression for the ab-entries of the matrix $\widehat{F}(u+1/2)$:

$$\widehat{F}(u+1/2)_{ab} = \left(\delta_{ab} - \sum_{k=1}^{\infty} F_{ab}^{(k)}\, u^{-k}\right) \cdot \operatorname{sdet} F^{(n-1)}(u-1/2).$$

Here $\operatorname{sdet} F^{(n-1)}(u)$ is the Sklyanin determinant of the matrix obtained from $F(u)$ by deleting the $(\pm n)$th rows and columns [10, Section 4], and

$$F_{ab}^{(k)} = \sum F_{ai_1} F_{i_1 i_2} \cdots F_{i_{k-1} b},$$

summed over the indices $i_m \in \{-n+1, \dots, n-1\}$. Finally, calculating the images of $F_{ab}^{(k)}$ and $\operatorname{sdet} F^{(n-1)}(u)$ with respect to the natural homomorphism $\pi : \mathrm{C}_n \to \mathrm{Z}(\mathfrak{g}_n, \mathfrak{g}_{n-1})$ (cf. [9, Section 5]), we find that the composition of π and (3.3) yields (3.2). Q.E.D.

The next theorem provides an identification of the $\mathrm{Y}^+(2)$-module $V(\lambda)^+_\mu$.

Theorem 3.2. *We have an isomorphism of* $\mathrm{Y}^+(2)$*-modules*

$$V(\lambda)^+_\mu \simeq L(\alpha_1, \beta_1) \otimes \cdots \otimes L(\alpha_{n-1}, \beta_{n-1}) \otimes W(-\alpha_0) \tag{3.4}$$

where $\alpha_1 = \min\{-|\lambda_1|, -|\mu_1|\} - 1/2, \quad \alpha_0 = \alpha_1 + |\lambda_1 + \mu_1|,$

$$\alpha_i = \min\{\lambda_i, \mu_i\} - i + 1/2, \qquad i = 2, \dots, n-1,$$
$$\beta_i = \max\{\lambda_{i+1}, \mu_{i+1}\} - i + 1/2, \qquad i = 1, \dots, n-1$$

with $\mu_n := -\infty$. *In particular,* $V(\lambda)^+_\mu$ *is equipped with an action of* $\mathrm{Y}(2)$ *defined by (2.10).*

Proof. Consider the following vector in $V(\lambda)^+_\mu$

$$\xi_\mu = \prod_{i=1}^{n-1} \left(z_{ni}^{\max\{\lambda_i,\mu_i\}-\mu_i}\, z_{i,-n}^{\max\{\lambda_i,\mu_i\}-\lambda_i} \right) \xi. \tag{3.5}$$

Repeating the arguments of the proof of Theorem 5.2 in [9] we show that ξ_μ is the highest vector of the $\mathrm{Y}^+(2)$-module $V(\lambda)^+_\mu$. That is, ξ_μ is annihilated by $s_{-n,n}(u)$, and ξ_μ is an eigenvector for $s_{nn}(u)$. Namely, $s_{nn}(u)\xi_\mu = \mu(u)\xi_\mu$, where the highest weight $\mu(u)$ is given by

$$\begin{aligned} \mu(u) = {} & (1-\alpha_0 u^{-1}) \cdots (1-\alpha_{n-1} u^{-1}) \\ & \times (1+\beta_1 u^{-1}) \cdots (1+\beta_{n-1} u^{-1})(1+\tfrac{1}{2}u^{-1})^{-1}. \end{aligned} \tag{3.6}$$

This is proved simultaneously with the following relations by induction on the degree of the monomial in (3.5): for $i = 1, \dots, n-1$

$$z_{in}\,\xi_\mu = -(m_i + \widetilde{\alpha}_1 + 1)\cdots(\widehat{m_i + \alpha_i} + 1)\cdots(m_i + \alpha_{n-1} + 1) \\ \times (m_i - \beta_0 + 1)\cdots(m_i - \beta_{n-1} + 1)\,\xi_{\mu+\delta_i},$$

and

$$z_{-ni}\,\xi_\mu = -(m_i - \widetilde{\alpha}_1)\cdots(m_i - \alpha_{n-1}) \\ \times (m_i + \beta_0)\cdots(\widehat{m_i + \beta_{i-1}})\cdots(m_i + \beta_{n-1})\,\xi_{\mu-\delta_i},$$

where we have used the notation

$$m_i = \mu_i - i + 1/2, \qquad i = 1, \dots, n-1, \\ \widetilde{\alpha}_1 = \min\{\lambda_1, \mu_1\} - 1/2, \qquad \beta_0 = \max\{\lambda_1, \mu_1\} + 1/2,$$

(note that $\{\widetilde{\alpha}_1, -\beta_0\} = \{\alpha_0, \alpha_1\}$). On the other hand, it follows from [8, Corollary 6.6] that the tensor product in (3.4) is an irreducible representation of $\mathrm{Y}^+(2)$. Its highest weight can be easily calculated and is given by the same formula (3.6). Q.E.D.

Set

$$T_{ab}(u) = u^{n-1}\,t_{ab}(u), \qquad a, b \in \{-n, n\}.$$

By (2.7) and (2.9), $T_{ab}(u)$, as an operator in $V(\lambda)^+_\mu$, is a polynomial in u:

$$T_{ab}(u) = \delta_{ab}u^{n-1} + t^{(1)}_{ab}\,u^{n-2} + \cdots + t^{(n-1)}_{ab}. \tag{3.7}$$

By (2.6), (2.8) and (3.2) we have an equality of operators in $V(\lambda)^+_\mu$:

(3.8)
$$Z_{n,-n}(u) = \frac{(u-\alpha_0)T_{n,-n}(-u)T_{nn}(u) + (u+\alpha_0)T_{n,-n}(u)T_{nn}(-u)}{(-1)^n\,2u}.$$

Therefore, $Z_{n,-n}(u)$ is a polynomial in u^2 of degree $n-2$. On the other hand, we find from (3.1) that $Z_{n,-n}(-g_i) = z_{ni}z_{i,-n}$. Thus, by the Lagrange interpolation formula, $Z_{n,-n}(u)$ can also be given by

$$Z_{n,-n}(u) = \sum_{i=1}^{n-1} z_{ni}z_{i,-n} \prod_{j=1,\, j\neq i}^{n-1} \frac{u^2 - g_j^2}{g_i^2 - g_j^2}. \tag{3.9}$$

Remark. To make the above evaluation $Z_{n,-n}(-g_i)$ well-defined we agree to consider the series $Z_{ab}(u)$ with $a, b \in \{-n, n\}$ as elements of the *right* module over the field of rational functions in $g_1, \dots, g_n, u$ generated by monomials in the z_{ia}. □

Theorem 3.2 implies that basis vectors of $V(\lambda)^+_\mu$ can be naturally parametrized by $(n-1)$-tuples $(\nu_1, \dots, \nu_{n-1})$, where all the entries are simultaneously integers or half-integers together with the λ_i and the μ_i, and the following inequalities hold:

$$\begin{aligned} &-|\lambda_1| \ge \nu_1 \ge \lambda_2 \ge \nu_2 \ge \lambda_3 \ge \cdots \ge \lambda_{n-1} \ge \nu_{n-1} \ge \lambda_n, \\ &-|\mu_1| \ge \nu_1 \ge \mu_2 \ge \nu_2 \ge \mu_3 \ge \cdots \ge \mu_{n-1} \ge \nu_{n-1}. \end{aligned} \tag{3.10}$$

For $i \ge 1$ set

$$\gamma_i = \nu_i - i + 1/2, \qquad l_i = \lambda_i - i + 1/2.$$

Introduce the vectors

$$\xi_{\nu\mu} = \prod_{i=1}^{n-1} Z_{n,-n}(\gamma_i - 1) \cdots Z_{n,-n}(\beta_i + 1) Z_{n,-n}(\beta_i)\, \xi_\mu.$$

Using (3.9) we can write an equivalent expression; cf. [9, Section 6]:

$$\xi_{\nu\mu} = \prod_{i=1}^{n-1} z_{ni}^{\nu_{i-1}-\mu_i} z_{i,-n}^{\nu_{i-1}-\lambda_i} \cdot \prod_{k=l_n+1}^{\gamma_{n-1}-1} Z_{n,-n}(k)\, \xi, \tag{3.11}$$

where $\nu_0 := \max\{\lambda_1, \mu_1\}$. The vectors $\xi_{\nu\mu}$ with ν satisfying (3.10) form a basis of the space $V(\lambda)^+_\mu$; see [9, Proposition 6.1]. We shall use the following normalized basis vectors

$$\zeta_{\nu\mu} = \prod_{1 \le i < j \le n-1} (-\gamma_i - \gamma_j)!\ \xi_{\nu\mu}.$$

The generators of the Yangian Y(2) act in the basis $\{\zeta_{\nu\mu}\}$ by the rule: for $i = 1, \dots, n-1$

$$\begin{aligned} T_{nn}(u)\, \zeta_{\nu\mu} &= (u + \gamma_1) \cdots (u + \gamma_{n-1})\, \zeta_{\nu\mu}, \\ T_{n,-n}(-\gamma_i)\, \zeta_{\nu\mu} &= \frac{1}{\gamma_i - \alpha_0}\, \zeta_{\nu+\delta_i,\mu}, \\ T_{-n,n}(-\gamma_i)\, \zeta_{\nu\mu} &= \prod_{k=0}^{n-1} (\alpha_k - \gamma_i + 1) \prod_{k=1}^{n-1} (\beta_k - \gamma_i)\, \zeta_{\nu-\delta_i,\mu}; \end{aligned} \tag{3.12}$$

cf. [9, Proposition 4.2]. The action of $T_{-n,-n}(u)$ can be found by using the *quantum determinant*

$$d(u) = T_{-n,-n}(u+1)T_{nn}(u) - T_{n,-n}(u+1)T_{-n,n}(u) \tag{3.13}$$
$$= T_{-n,-n}(u)T_{nn}(u+1) - T_{-n,n}(u)T_{n,-n}(u+1); \tag{3.14}$$

see, e.g. [10, Section 2]. The coefficients of the quantum determinant belong to the center of Y(2) and so, $d(u)$ acts in $V(\lambda)^+_\mu$ as a scalar which can be found by the application of (3.13) to the highest weight vector ξ_μ. So, we have

$$d(u)\,\zeta_{\nu\mu} = (u+\alpha_1+1)\cdots(u+\alpha_{n-1}+1) \\ \times (u+\beta_1)\cdots(u+\beta_{n-1})\,\zeta_{\nu\mu}.$$

Now, using (3.12) and (3.14) we obtain

$$T_{-n,-n}(u)\,\zeta_{\nu\mu} = \prod_{i=1}^{n-1}\frac{(u+\alpha_i+1)(u+\beta_i)}{u+\gamma_i+1}\,\zeta_{\nu\mu} \\ + \prod_{i=1}^{n-1}\frac{1}{u+\gamma_i+1}\,T_{-n,n}(u)T_{n,-n}(u+1)\,\zeta_{\nu\mu}. \tag{3.15}$$

The operators $T_{-n,n}(u)$ and $T_{n,-n}(u)$ are polynomials in u of degree $\leq n-2$; see (3.7). Therefore, their action can be found from (3.12) by using the Lagrange interpolation formula.

The following branching rule for the reduction $\mathfrak{g}_n \downarrow \mathfrak{g}_{n-1}$ is implied by Theorem 3.2; cf. [9, Corollary 5.3].

Corollary 3.3. *The restriction of $V(\lambda)$ to the subalgebra $\mathfrak{g}_{n-1}$ is isomorphic to the direct sum $\bigoplus c(\mu)V'(\mu)$ of finite-dimensional irreducible representations $V'(\mu)$ of $\mathfrak{g}_{n-1}$ where the multiplicity $c(\mu)$ equals the number of $(n-1)$-tuples ν satisfying the inequalities (3.10).*

Proof. We have $c(\mu) = \dim V(\lambda)^+_\mu$. By Theorem 3.2,

$$\dim V(\lambda)^+_\mu = \prod_{i=1}^{n-1}(\alpha_i - \beta_i + 1),$$

if there exists ν satisfying (3.10). Otherwise, the space $V(\lambda)^+_\mu$ is trivial. This is proved by comparison of the dimensions of $V(\lambda)$ and $\bigoplus c(\mu)V'(\mu)$ with the use of [16, Chapter VII, Section 9]. Q.E.D.

Applying the above construction of the vectors $\zeta_{\nu\mu}$ to the subalgebras of the chain

$$\mathfrak{g}_1 \subset \mathfrak{g}_2 \subset \cdots \subset \mathfrak{g}_n, \qquad \mathfrak{g}_k = \mathfrak{o}(2k)$$

we obtain a basis of $V(\lambda)$ parametrized by the *D-type Gelfand–Tsetlin patterns* (cf. [6]) which we denote by Λ:

$$\begin{array}{ccccccc}
\lambda_{n1} & & \lambda_{n2} & & \cdots & & \lambda_{nn} \\
& \lambda'_{n-1,1} & & \cdots & & \lambda'_{n-1,n-1} & \\
\lambda_{n-1,1} & & \cdots & & \lambda_{n-1,n-1} & & \\
& \cdots & & \cdots & & & \\
\lambda_{21} & & \lambda_{22} & & & & \\
& \lambda'_{11} & & & & & \\
\lambda_{11} & & & & & &
\end{array}$$

Here the upper row coincides with λ, all the entries are simultaneously integers or half-integers and the following inequalities hold

$$-|\lambda_{k1}| \geq \lambda'_{k-1,1} \geq \lambda_{k2} \geq \lambda'_{k-1,2} \geq \cdots \geq \lambda_{k,k-1} \geq \lambda'_{k-1,k-1} \geq \lambda_{kk},$$
$$-|\lambda_{k-1,1}| \geq \lambda'_{k-1,1} \geq \lambda_{k-1,2} \geq \lambda'_{k-1,2} \geq \cdots \geq \lambda_{k-1,k-1} \geq \lambda'_{k-1,k-1}$$

for $k = 2, \dots, n$. Set

$$(3.16) \qquad l_{ki} = \lambda_{ki} - i + 1, \qquad l'_{ki} = \lambda'_{ki} - i + 1, \qquad 1 \leq i \leq k \leq n$$

and introduce the vectors

$$\xi_\Lambda = \prod_{k=2,\dots,n}^{\rightarrow} \left(\prod_{i=1}^{k-1} z_{ki}^{\lambda'_{k-1,i-1} - \lambda_{k-1,i}} z_{i,-k}^{\lambda'_{k-1,i-1} - \lambda_{ki}} \prod_{q=l_{kk}+1}^{l'_{k-1,k-1}-1} Z_{k,-k}(q - \frac{1}{2}) \right) \xi$$

with $\lambda'_{k-1,0} := \max\{\lambda_{k1}, \lambda_{k-1,1}\}$. Finally, set

$$\zeta_\Lambda = N_\Lambda \, \xi_\Lambda, \qquad N_\Lambda = \prod_{k=2}^{n-1} \prod_{1 \leq i < j \leq k} (-l'_{ki} - l'_{kj} + 1)!$$

The following proposition is implied by Corollary 3.3.

Proposition 3.4. *The vectors ζ_Λ parametrized by the Gelfand–Tsetlin patterns Λ form a basis of the representation $V(\lambda)$.* $\square$

§4. Matrix element formulas

Introduce the following elements of $\mathrm{U}(\mathfrak{g}_n)$:

$$\Phi_{-k,k} = \sum_{i=1}^{k-1} F_{-k,i} F_{ik}, \qquad k = 2, \ldots, n.$$

We shall find the action of $\Phi_{-k,k}$ in the basis $\{\zeta_\Lambda\}$. This will be used later on. Since $\Phi_{-k,k}$ commutes with the subalgebra $\mathfrak{g}_{k-1}$ it suffices to consider the case $k = n$. The image of $\Phi_{-n,n}$ under the natural homomorphism $\pi : \mathrm{C}_n \to \mathrm{Z}(\mathfrak{g}_n, \mathfrak{g}_{n-1})$ coincides with the coefficient at u^{2n-4} of the polynomial $Z_{-n,n}(u)$; see the proof of Proposition 3.1. The following analog of (3.8) is obtained from (2.6), (2.8) and (3.2):

$$Z_{-n,n}(u) = \frac{(u-\alpha_0) T_{-n,-n}(-u) T_{-n,n}(u) + (u+\alpha_0) T_{-n,-n}(u) T_{-n,n}(-u)}{(-1)^n \, 2u}.$$

Therefore, we have an equality of operators in $V(\lambda)^+_\mu$:

$$\Phi_{-n,n} = -t^{(2)}_{-n,n} + t^{(1)}_{-n,n} t^{(1)}_{-n,-n} + (1+\alpha_0)\, t^{(1)}_{-n,n}. \tag{4.1}$$

The image of $s^{(1)}_{nn}$ under the homomorphism (3.3) is F_{nn}. On the other hand, by (2.8) we have

$$s^{(1)}_{nn} = t^{(1)}_{nn} - t^{(1)}_{-n,-n} - \alpha_0 - 1/2,$$

as operators in $V(\lambda)^+_\mu$. Therefore, (4.1) can be written as

$$\Phi_{-n,n} = -t^{(2)}_{-n,n} + t^{(1)}_{-n,n} t^{(1)}_{nn} - (F_{nn} + 3/2)\, t^{(1)}_{-n,n}.$$

Finally, relations (3.12) imply that

$$\Phi_{-n,n} \zeta_{\nu\mu} = \sum_{i=1}^{n-1} \theta_i \,(F_{nn} - \gamma_i + 3/2)\, \zeta_{\nu - \delta_i, \mu}, \tag{4.2}$$

where

$$\theta_i = -\prod_{k=0}^{n-1} (\alpha_k - \gamma_i + 1) \prod_{k=1}^{n-1} (\beta_k - \gamma_i) \prod_{j=1,\, j\neq i}^{n-1} (\gamma_j - \gamma_i)^{-1}.$$

Using (2.2) one easily computes the action of F_{nn} in $V(\lambda)^+_\mu$ so that

$$F_{nn}\,\zeta_{\nu\mu}=\left(2\sum_{i=0}^{n-1}\nu_i-\sum_{i=1}^{n}\lambda_i-\sum_{i=1}^{n-1}\mu_i\right)\zeta_{\nu\mu}.$$

Remark. One can introduce the elements $\Phi_{k,-k}$ by

$$\Phi_{k,-k}=\sum_{i=1}^{k-1}F_{ki}F_{i,-k}.$$

The action of $\Phi_{n,-n}$ on $V(\lambda)^+_\mu$ is found in the same way as that of $\Phi_{-n,n}$:

$$\Phi_{n,-n}\zeta_{\nu\mu}=\sum_{i=1}^{n-1}\prod_{j=1,\,j\neq i}^{n-1}\frac{1}{\gamma_j-\gamma_i}\,\zeta_{\nu+\delta_i,\mu},$$

although this will not be used. □

The operator $F_{n-1,-n}$ preserves the subspace of $\mathfrak{g}_{n-2}$-highest vectors in $V(\lambda)$. Therefore it suffices to calculate its action on the basis vectors of the form

$$\xi_{\nu\mu\nu'}=X_{\mu\nu'}\,\xi_{\nu\mu},\tag{4.3}$$

where $X_{\mu\nu'}$ denotes the operator

$$X_{\mu\nu'}=\prod_{i=1}^{n-2}z_{n-1,i}^{\nu'_{i-1}-\mu'_i}\,z_{i,-n+1}^{\nu'_{i-1}-\mu_i}\cdot\prod_{a=m_{n-1}+1}^{\gamma'_{n-1}-1}Z_{n-1,-n+1}(a),$$

ν' and μ' are $(n-2)$-tuples of integers or half-integers such that the inequalities (3.10) are satisfied with λ, ν, μ respectively replaced by μ, ν', μ'; we set $\gamma'_i=\nu'_i-i+1/2$ and $\nu'_0=\max\{\mu_1,\mu'_1\}$. The operator $F_{n-1,-n}$ is permutable with the elements $z_{n-1,i}$, $z_{i,-n+1}$ and $Z_{n-1,-n+1}(u)$ which follows from their explicit formulas. Hence, we can write

$$F_{n-1,-n}\,\xi_{\nu\mu\nu'}=X_{\mu\nu'}\,F_{n-1,-n}\,\xi_{\nu\mu}.\tag{4.4}$$

By (2.5) (with $a=-n$) we need to express

$$X_{\mu\nu'}\,z_{n-1,i}z_{i,-n}\,\xi_{\nu\mu},\qquad i=-n+1,\dots,n-1\tag{4.5}$$

as a linear combination of the vectors $\xi_{\nu\mu\nu'}$. If $i\neq\pm1$ then the calculation is exactly the same as in [9, Section 6] where one uses the relations

$$Z_{n,-n}(-g_n)=z_{n,-n},\qquad Z_{n,-n}(-g_i)=z_{ni}z_{i,-n},\tag{4.6}$$

which follow from (2.4) and (3.9). Now consider (4.5) with $i = -1$. We have

$$X_{\mu\nu'}\, z_{n-1,-1} z_{-1,-n}\, \xi_{\nu\mu} = -X_{\mu\nu'}\, z_{1,-n+1} z_{n1}\, \xi_{\nu\mu}.$$

If $\lambda_1 \geq \mu_1$ then $z_{n1}\, \xi_{\nu\mu} = \xi_{\nu,\mu-\delta_1}$ while for $\lambda_1 < \mu_1$ we derive from (4.6) that

$$z_{n1}\, \xi_{\nu\mu} = \sum_{i=1}^{n-1} \prod_{a=1,\, a\neq i}^{n-1} \frac{m_1^2 - \gamma_a^2}{\gamma_i^2 - \gamma_a^2}\, \xi_{\nu+\delta_i,\mu-\delta_1}.$$

Similarly, if $\mu_1' \geq \mu_1$ then $X_{\mu\nu'}\, z_{1,-n+1} = X_{\mu-\delta_1,\nu'}$ while for $\mu_1' < \mu_1$ one has

$$X_{\mu\nu'}\, z_{1,-n+1} = \sum_{r=1}^{n-2} \prod_{a=1,\, a\neq r}^{n-2} \frac{m_1^2 - \gamma_a'^2}{\gamma_r'^2 - \gamma_a'^2}\, X_{\mu-\delta_1,\nu'+\delta_r}.$$

Finally, take $i = 1$ in (4.5). If $\lambda_1 \leq \mu_1$ then $z_{1,-n}\, \xi_{\nu\mu} = \xi_{\nu,\mu+\delta_1}$, and if $\lambda_1 > \mu_1$ then

$$z_{1,-n}\, \xi_{\nu\mu} = \sum_{i=1}^{n-1} \prod_{a=1,\, a\neq i}^{n-1} \frac{(m_1+1)^2 - \gamma_a^2}{\gamma_i^2 - \gamma_a^2}\, \xi_{\nu+\delta_i,\mu+\delta_1}.$$

Similarly, if $\mu_1' \leq \mu_1$ then $X_{\mu\nu'}\, z_{n-1,1} = X_{\mu+\delta_1,\nu'}$ and if $\mu_1' > \mu_1$ then

$$X_{\mu\nu'}\, z_{n-1,1} = \sum_{r=1}^{n-2} \prod_{a=1,\, a\neq r}^{n-2} \frac{(m_1+1)^2 - \gamma_a'^2}{\gamma_r'^2 - \gamma_a'^2}\, X_{\mu+\delta_1,\nu'+\delta_r}.$$

The action of the elements $F_{n-1,n}$ on the vectors (4.3) can be expressed in two different ways. First we sketch a calculation similar to the one used above which leads to (rather complicated) explicit formulas for the matrix elements. Then we give slightly less explicit but more convenient formulas where $F_{n-1,n}$ is represented by a commutator-like expression of simpler operators.

We have the following analog of (4.4):

$$F_{n-1,n}\, \xi_{\nu\mu\nu'} = X_{\mu\nu'}\, F_{n-1,n}\, \xi_{\nu\mu}. \tag{4.7}$$

Now use (2.5) with $a = n$. Here we need to calculate $z_{in}\, \xi_{\nu\mu}$ instead of $z_{i,-n}\, \xi_{\nu\mu}$ in the previous case. Suppose that $i > 1$. We have

$$z_{in}\, \xi_{\nu\mu} = z_{in} z_{ni}\, \xi_{\nu,\mu+\delta_i} = z_{-n,-i} z_{-i,-n}\, \xi_{\nu,\mu+\delta_i} = Z_{-n,-n}(-g_{-i})\, \xi_{\nu,\mu+\delta_i};$$

see (3.1). However,

$$-g_{-i}\,\xi_{\nu,\mu+\delta_i} = (m_i+1)\,\xi_{\nu,\mu+\delta_i}.$$

To calculate $Z_{-n,-n}(m_i+1)\,\xi_{\nu,\mu+\delta_i}$ we use the following equality of operators in $V(\lambda)^+_\mu$:

$$Z_{-n,-n}(u) = \frac{(u-\alpha_0)T_{-n,n}(u)T_{n,-n}(-u)+(u+\alpha_0+1)T_{-n,-n}(u)T_{nn}(-u)}{(-1)^n\,(2u+1)},$$

see (2.6), (2.8) and (3.2); and then apply formulas (3.12) and (3.15).

To calculate $z_{-i,n}\,\xi_{\nu\mu}$ we first permute $z_{-i,n}$ with the generators z_{nj} and $z_{j,-n}$ with $j=1,\dots,i-1$ in (3.11). Further, we use the relation

$$z_{-i,n}z_{i,-n} = (-1)^{n-i}z_{-n,i}z_{i,-n} = (-1)^{n-i}Z_{-n,-n}(-g_i)$$

and complete the calculation in a similar manner. To find $z_{\pm 1,n}\,\xi_{\nu\mu}$ we need to consider a few different cases which depend on the relationship between the parameters λ_1, μ_1 and μ'_1 and then proceed exactly as above in the calculation of the action of $F_{n-1,-n}$.

We now give an alternative way of computing the action of $F_{n-1,n}$. The basic idea is to replace the operator z_{in} in the above calculation of $z_{in}\,\xi_{\nu\mu}$ by the following expression: for $i=-n+1,\dots,n-1$

$$z_{in} = [z_{i,-n},\Phi_{-n,n}]\,\frac{1}{f_i+F_{nn}} \tag{4.8}$$

and then use the formulas for the action of $z_{i,-n}$ and $\Phi_{-n,n}$; see (4.2). More precisely, we regard (4.8) as a relation in the transvector algebra $\mathrm{Z}(\mathfrak{g}_n,\mathfrak{g}_{n-1})$ which can be proved as follows. First, we calculate the commutator $[F_{i,-n},\Phi_{-n,n}]$ in $\mathrm{U}(\mathfrak{g}_n)$ then consider it modulo the ideal J and apply the extremal projection p (see Section 2).

We have $\Phi_{-n,n}F_{nn} = (F_{nn}+2)\,\Phi_{-n,n}$ and so, (2.5), (4.7) and (4.8) imply that

$$F_{n-1,n}\,\xi_{\nu\mu\nu'} = X_{\mu\nu'}\,\left(\Phi_{n-1,-n}(2)\,\Phi_{-n,n}-\Phi_{-n,n}\Phi_{n-1,-n}(0)\right)\xi_{\nu\mu}, \tag{4.9}$$

where

$$\Phi_{n-1,-n}(u) = \sum_{i=-n+1}^{n-1} z_{n-1,i}\,z_{i,-n}\prod_{a=-n+1,\,a\neq\pm i}^{n-1}\frac{1}{f_i-f_a}\cdot\frac{1}{u+f_i+F_{nn}}.$$

The action of $\Phi_{n-1,-n}(u)$ is found exactly as that of $F_{n-1,-n}$. Note that the operators $X_{\mu\nu'}$ and $\Phi_{-n,n}$ commute.
Remark. The operator $\Phi_{n-1,-n}(u)$ is a rational function in u which can have singularities at the values $u=0$ and $u=2$ in (4.9). However, the operator

$$\Phi_{n-1,-n}(u+2)\,\Phi_{-n,n}-\Phi_{-n,n}\Phi_{n-1,-n}(u)$$

is regular at $u=0$ and coincides with $F_{n-1,n}$. Note the similarity with the symplectic case [9], where the corresponding generator $F'_{n-1,n}$ is expressed as a commutator: $2F'_{n-1,n}=[F'_{n-1,-n},F'_{-n,n}]$. □

The elements $F_{k-1,-k}$, $F_{k-1,k}$ with $k=2,\dots,n$ and F_{21}, $F_{-2,1}$ generate $\mathfrak{g}_n$ as a Lie algebra. Summarizing the above calculations we obtain the following formulas for the matrix elements of the generators. Given a pattern Λ we use the notation (3.16) and set for $1\le i<k\le n$:

$$A_{ki}=\prod_{a=1,\,a\neq i}^{k-1}\frac{1}{l^2_{k-1,i}-l^2_{k-1,a}},$$

$$B_{ki}(x)=\prod_{a=1,\,a\neq i}^{k-1}\frac{(x+l'_{k-1,a})(x-l'_{k-1,a}+1)}{l'_{k-1,a}-l'_{k-1,i}},$$

and

$$C_{ki}=(\max\{\lambda_{k1},\lambda_{k-1,1}\}+l'_{k-1,i}-1)(\min\{\lambda_{k1},\lambda_{k-1,1}\}-l'_{k-1,i}+1)$$
$$\times\prod_{a=2}^{k}(l_{ka}-l'_{k-1,i}+1)\prod_{a=2}^{k-1}(l_{k-1,a}-l'_{k-1,i}+1)\prod_{a=1,\,a\neq i}^{k-1}\frac{1}{l'_{k-1,a}-l'_{k-1,i}}.$$

We denote by $\Lambda\pm\delta_{ki}$ and $\Lambda\pm\delta'_{ki}$ the arrays obtained from Λ by replacing λ_{ki} and λ'_{ki} by $\lambda_{ki}\pm1$ and $\lambda'_{ki}\pm1$ respectively. Consider the basis $\{\zeta_\Lambda\}$ of the representation $V(\lambda)$; see Proposition 3.4. We shall suppose that $\zeta_\Lambda=0$ if the array Λ is not a pattern.

Theorem 4.1. *The action of the generators of the Lie algebra $\mathfrak{o}(2n)$ in the basis $\{\zeta_\Lambda\}$ is given by the following formulas.*

$$F_{kk}\,\zeta_\Lambda=\left(2\sum_{i=1}^{k}\lambda'_{k-1,i-1}-\sum_{i=1}^{k}\lambda_{ki}-\sum_{i=1}^{k-1}\lambda_{k-1,i}\right)\zeta_\Lambda,$$

$$F_{k-1,-k}\,\zeta_\Lambda=\sum_{i=1}^{k-1}A_{ki}\left(\zeta^+_\Lambda(k,i)-\zeta^-_\Lambda(k,i)\right).$$

Here

$$\zeta_\Lambda^+(k,i) = \sum_{j=1}^{k-1}\sum_{m=1}^{k-2} B_{kj}(l_{k-1,i})B_{k-1,m}(l_{k-1,i})\,\zeta_{\Lambda+\delta'_{k-1,j}+\delta_{k-1,i}+\delta'_{k-2,m}}$$

for $i = 2,\dots,k-1$; *and for* $i = 1$ *if* $\lambda_{k-1,1} < \lambda_{k1}, \lambda_{k-2,1}$. *Otherwise,*

$$\zeta_\Lambda^+(k,1) = \zeta_{\Lambda+\delta_{k-1,1}} \qquad \text{if} \quad \lambda_{k-1,1} \geq \lambda_{k1}, \lambda_{k-2,1},$$

$$\zeta_\Lambda^+(k,1) = \sum_{j=1}^{k-1} B_{kj}(l_{k-1,1})\,\zeta_{\Lambda+\delta'_{k-1,j}+\delta_{k-1,1}} \qquad \text{if} \quad \lambda_{k-2,1} \leq \lambda_{k-1,1} < \lambda_{k1},$$

$$\zeta_\Lambda^+(k,1) = \sum_{m=1}^{k-2} B_{k-1,m}(l_{k-1,1})\,\zeta_{\Lambda+\delta_{k-1,1}+\delta'_{k-2,m}} \qquad \text{if} \quad \lambda_{k1} \leq \lambda_{k-1,1} < \lambda_{k-2,1}.$$

Furthermore,

$$\zeta_\Lambda^-(k,i) = \zeta_{\Lambda-\delta_{k-1,i}}$$

for $i = 2,\dots,k-1$; *and for* $i = 1$ *if* $\lambda_{k-1,1} \leq \lambda_{k1}, \lambda_{k-2,1}$. *Otherwise,*

$$\zeta_\Lambda^-(k,1) = \sum_{j=1}^{k-1} B_{kj}(l_{k-1,1}-1)\,\zeta_{\Lambda+\delta'_{k-1,j}-\delta_{k-1,1}} \qquad \text{if} \quad \lambda_{k1} < \lambda_{k-1,1} \leq \lambda_{k-2,1},$$

$$\zeta_\Lambda^-(k,1) = \sum_{m=1}^{k-2} B_{k-1,m}(l_{k-1,1}-1)\,\zeta_{\Lambda-\delta_{k-1,1}+\delta'_{k-2,m}} \qquad \text{if} \quad \lambda_{k-2,1} < \lambda_{k-1,1} \leq \lambda_{k1},$$

$$\zeta_\Lambda^-(k,1) = \sum_{j=1}^{k-1}\sum_{m=1}^{k-2} B_{kj}(l_{k-1,1}-1)B_{k-1,m}(l_{k-1,1}-1)\zeta_{\Lambda+\delta'_{k-1,j}-\delta_{k-1,1}+\delta'_{k-2,m}} \qquad \text{if} \quad \lambda_{k-1,1} > \lambda_{k1}, \lambda_{k-2,1}.$$

The action of $F_{k-1,k}$ *is found from the relation*

$$F_{k-1,k} = \Big[\Phi_{k-1,-k}(u+2)\,\Phi_{-k,k} - \Phi_{-k,k}\Phi_{k-1,-k}(u)\Big]_{u=0},$$

where

$$\Phi_{-k,k}\,\zeta_\Lambda = \sum_{i=1}^{k-1} C_{ki}\,(F_{kk} - l'_{k-1,i} + 2)\,\zeta_{\Lambda-\delta'_{k-1,i}}$$

and

$$\Phi_{k-1,-k}(u)\,\zeta_\Lambda =$$
$$\sum_{i=1}^{k-1} A_{ki}\left(\frac{1}{u + l_{k-1,i} + F_{kk} - 1}\,\zeta_\Lambda^+(k,i) - \frac{1}{u - l_{k-1,i} + F_{kk} - 1}\,\zeta_\Lambda^-(k,i)\right).$$

Example. Let $n = 2$. We have $z_{21} = F_{21}$, $z_{1,-2} = F_{1,-2}$ and $Z_{2,-2}(u) = F_{21}F_{1,-2}$. Therefore, the basis vectors are given by

$$\zeta_\Lambda = F_{21}^{\lambda'_{10}-\lambda_{11}}\,F_{1,-2}^{\lambda'_{10}-\lambda_{21}}\,(F_{21}F_{1,-2})^{\lambda'_{11}-\lambda_{22}}\,\xi,$$

where $\lambda'_{10} = \max\{\lambda_{21}, \lambda_{11}\}$. The Lie algebra $\mathfrak{o}(4)$ is isomorphic to the direct sum of two copies of $\mathfrak{sl}(2)$ and the action of their generators in the basis $\{\zeta_\Lambda\}$ is easily found. The resulting formulas also hold for the action of the elements of the subalgebra $\mathfrak{g}_2 \subset \mathfrak{g}_n$ in the basis $\{\zeta_\Lambda\}$ of the $\mathfrak{g}_n$-module $V(\lambda)$. $\square$

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Schur's Q-functions and Twisted Affine Lie Algebras

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To the memory of our friend and former colleague,
Nobuo Sasakura

Abstract.

Weight vectors of the basic representations of $A_{2\ell}^{(2)}$ and $D_{\ell+1}^{(2)}$ are studied. They are expressed in terms of Schur's Q-functions. The up and down motion along the string of the fundamental imaginary root is described as a combinatorial game.

§0. Introduction

The article deals with an explicit expression of the weight vectors of the twisted affine Lie algebras of type $A_{2\ell}^{(2)}$ and $D_{\ell+1}^{(2)}$.

In 1981 Date et al. introduced a KP like hierarchy of nonlinear differential equations, which has the infinite dimensional Lie algebra of type B_∞ as the infinitesimal transformation group of solutions, and named it the KP hierarchy of type B or the BKP hierarchy for short. In [DJKM] they investigated the reductions of the BKP hierarchy and related them with the basic representations of the twisted affine Lie algebras of type $A_{2\ell}^{(2)}$ and $D_{\ell+1}^{(2)}$. Using the principal realization of the basic representation on an algebra of polynomials with infinitely many variables, they expressed the polynomial solutions to the reduced BKP hierarchies by means of the Schur functions.

In this context the Schur function indexed by the partition λ of n is defined by

$$S_\lambda(x) = \sum_\rho \chi_\rho^\lambda \frac{x_1^{m_1} x_2^{m_2} \cdots}{m_1! m_2! \cdots}, \tag{0.1}$$

where the summation runs over the partitions $\rho = (1^{m_1} 2^{m_2} \cdots)$ of n and χ_ρ^λ denotes the irreducible character value of the symmetric group S_n. The original Schur function, as a symmetric function, is obtained

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from $S_\lambda(x)$ by putting $x_j = p_j/j$ ($j \geqslant 1$), where p_j is the j-th power sum symmetric function. A determinant expression is also known as the Jacobi-Trudi formula [Mac]. Utilizing this determinant formula, the solutions to the KP hierarchy or its reductions are beautifully expressed by means of the Schur functions.

However, theory of the BKP hierarchy is that of Pfaffians by nature and does not fit the Schur functions well. Instead one expects that the BKP hierarchy is related to Schur's Q-functions defined by, for a strict partition λ of n,

$$Q_\lambda(t) = \sum_\rho 2^{\frac{\ell(\lambda)-\ell(\rho)+\epsilon}{2}} \zeta_\rho^\lambda \frac{t_1^{m_1} t_3^{m_3} \cdots}{m_1! m_3! \cdots}, \tag{0.2}$$

where the summation runs over the partitions $\rho = (1^{m_1} 3^{m_3} \cdots)$ of n consisting of odd parts, ζ_ρ^λ denotes the irreducible spin character value of S_n, and $\epsilon = 0$ or 1 according to that $n - \ell(\lambda)$ is even or odd. The original Q-function, as a symmetric function, is recovered from $Q_\lambda(t)$ by putting $t_j = 2p_j/j$ ($j \geqslant 1$, odd).

You [Y] showed that $Q_\lambda(t)$ solves the BKP hierarchy for any strict partition λ, and later the present authors [NY1] investigated a relation between the reduced BKP hierarchies and the (reduced) Q-functions.

In the theory of the KP hierarchy, the r-reduction means the elimination of the time variables x_{rj} ($j \geqslant 1$). For example, the KdV hierarchy is the 2-reduction of the KP hierarchy and has the time variables x_j ($j \geqslant 1$, odd). As for the BKP hierarchy it has only the odd numbered time variables t_j ($j \geqslant 1$, odd) from the beginning. When the reduction number r is odd ($\geqslant 3$), the reduction procedure is achieved by eliminating the variables t_{rj} ($j \geqslant 1$, odd) in the BKP hierarchy. As a result the infinitesimal transformation group becomes $A_{2\ell}^{(2)}$ if $r = 2\ell + 1$. On the contrary, if $r = 2\ell + 2$, the representation space $V = \mathbb{C}[t_j; j \geqslant 1, \text{odd}]$ remains unchanged, but the infinitesimal transformation group reduced to $D_{\ell+1}^{(2)}$.

The weighted homogeneous polynomial solutions to the r-reduced BKP hierarchy appear as the maximal weight vectors of the basic representation of $A_{2\ell}^{(2)}$ ($r = 2\ell + 1$) or $D_{\ell+1}^{(2)}$ ($r = 2\ell + 2$) when one realizes the representation on V. They are expressed in terms of the (reduced) Q-functions [NY2].

In this article we will make a close investigation of the weight vectors of the basic representations of $A_{2\ell}^{(2)}$ and $D_{\ell+1}^{(2)}$. The up and down motion of the weights along the string of the fundamental imaginary root is described as a combinatorial game on an abacus representing the strict partitions.

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§1. Neutral free fermions and Schur's Q-functions

We first review some ingredients of operator formalism for Schur's Q-functions. Let $\mathbb{B}$ be the $\mathbb{C}$-algebra generated by ϕ_n $(n \in \mathbb{Z})$ with respect to the relations

$$\phi_n\phi_m + \phi_m\phi_n = (-1)^n\delta_{n+m,0} \quad (n, m \in \mathbb{Z}). \tag{1.1}$$

In the literature (*e.g.* [DJKM]) the generators ϕ_n are referred to as the neutral free fermions. Define the degree on $\mathbb{B}$ by $\deg \phi_n = 1$ $(n \in \mathbb{Z})$. If we let $\mathbb{B}_0$ (resp. $\mathbb{B}_1$) be the subspace consisting of the elements of even (resp. odd) degree, then $\mathbb{B} = \mathbb{B}_0 \oplus \mathbb{B}_1$ has a structure of a superalgebra. Let $\mathcal{F} = \mathcal{F}_0 \oplus \mathcal{F}_1 = \mathbb{B}_0|0\rangle \oplus \mathbb{B}_1|0\rangle$ (resp. $\mathcal{F}^* = \langle 0|\mathbb{B}_0 \oplus \langle 0|\mathbb{B}_1$) be the Fock space (resp. the dual Fock space), where the vacuum $|0\rangle$ (resp. $\langle 0|$) is defined by

$$\phi_n|0\rangle = 0 \quad (n < 0) \qquad (\text{resp. } \langle 0|\phi_n = 0 \quad (n > 0)). \tag{1.2}$$

The vacuum expectation value $\langle 0|a|0\rangle$ $(a \in \mathbb{B})$ is uniquely determined by setting $\langle 0|1|0\rangle = 1$, $\langle 0|\phi_0|0\rangle = 0$.

We construct a realization of $\mathcal{F}_0$. The normal ordering for the neutral free fermions is defined by

$$\overset{\circ}{\underset{\circ}{}}\phi_n\phi_m\overset{\circ}{\underset{\circ}{}} = \phi_n\phi_m - \langle 0|\phi_n\phi_m|0\rangle. \tag{1.3}$$

Define the Hamiltonian by

$$H(t) = \frac{1}{2}\sum_{\substack{j\geqslant 1\\ j:\text{odd}}}\sum_{n\in\mathbb{Z}}(-1)^{n+1}t_j\phi_n\phi_{-j-n}. \tag{1.4}$$

Let $V = \mathbb{C}[t_j;\ j \geqslant 1, \text{odd}]$ be the algebra of polynomials with infinitely many variables. There is an isomorphism from $\mathcal{F}_0$ to V defined by

$$a|0\rangle \longmapsto \langle 0|e^{H(t)}a|0\rangle \quad (a \in \mathbb{B}_0), \tag{1.5}$$

which is often called the boson-fermion correspondence. The subspace

$$B_\infty = \left\{ \sum_{n,m\in\mathbb{Z}} c_{nm}\overset{\circ}{\underset{\circ}{}}\phi_n\phi_m\overset{\circ}{\underset{\circ}{}};\, c_{nm} = 0 \text{ if } |n+m| \gg 0 \right\} \tag{1.6}$$

of a completion $\overline{\mathbb{B}}_0$ admits a structure of a Lie algebra isomorphic to the one dimensional central extension of $\mathfrak{o}(\infty)$. The Cartan subalgebra of B_∞ consists of the (infinite) linear combinations of ${}^\circ_\circ\phi_n\phi_{-n}{}^\circ_\circ$ ($n \in \mathbb{Z}$). Although certain infinite sums are allowed as elements of B_∞, the space V affords an action of B_∞. This Fock representation of B_∞ on V is best described by the vertex operator

$$Z_B(p,q) = \frac{p-q}{2(p+q)}\left(e^{\xi(t,p)+\xi(t,q)}e^{-2\xi(\tilde{\partial},p^{-1})+2\xi(\tilde{\partial},q^{-1})} - 1\right) \tag{1.7}$$

which corresponds to the action of ${}^\circ_\circ\phi(p)\phi(q){}^\circ_\circ$ on $\mathcal{F}_0$, where $\phi(p) = \sum_{n\in\mathbb{Z}}\phi_n p^n$. Here we have set $\xi(t,p) = \sum_{\substack{j\geqslant 1\\ j:\mathrm{odd}}} t_j p^j$ and $\xi(\tilde{\partial},p^{-1}) = \sum_{\substack{j\geqslant 1\\ j:\mathrm{odd}}} \frac{1}{j}\frac{\partial}{\partial t_j}p^{-j}$.

A strict partition is a sequence $\lambda = (\lambda_1, \ldots, \lambda_{2d})$ of non-negative integers with $\lambda_1 > \cdots > \lambda_{2d} \geqslant 0$. The number of positive parts is called the length of λ and denoted by $\ell(\lambda)$. When fermions are associated with the strict partition λ, we always append 0 in the tail of λ if $\ell(\lambda)$ is odd. For a strict partition $\lambda = (\lambda_1, \ldots, \lambda_{2d})$, put $v_\lambda = \phi_{\lambda_1}\cdots\phi_{\lambda_{2d}}|0\rangle \in \mathcal{F}_0$. It is shown in [DJKM] that $\{v_\lambda\ ;\ \lambda \text{ is a strict partition}\}$ forms a weight basis for the Fock representation of B_∞.

In order to express these weight vectors as elements in V via the boson-fermion correspondence, we recall Schur's Q-functions. Define polynomials $q_n(t) \in V$ by

$$e^{\xi(t,p)} = \sum_{n=0}^{\infty} q_n(t)p^n.$$

For positive integers m, n ($m > n \geqslant 0$) put

$$\begin{aligned} Q_{(m,n)}(t) &= q_m(t)q_n(t) + 2\sum_{k=1}^{n}(-1)^k q_{m+k}(t)q_{n-k}(t), \\ Q_{(n,m)}(t) &= -Q_{(m,n)}(t). \end{aligned}$$

And finally, for a strict partitions $\lambda = (\lambda_1, \ldots, \lambda_{2d})$, set

$$Q_\lambda(t) = \mathrm{Pf}(Q_{(\lambda_i,\lambda_j)}(t)),$$

where Pf stands for the Pfaffian of a skew-symmetric matrix. We refer to this polynomial $Q_\lambda(t)$ as the Q-function associated with the strict partition λ. The original Q-function [*e.g.* Mac], as a symmetric function, is obtained from $Q_\lambda(t)$ by putting $t_j = 2p_j/j$ ($j \geqslant 1$, odd), where p_j is the j-th power sum symmetric function.

Using anti-commutation relations of the neutral free fermions, it is easy to see that

$$[H(t),\phi(p)] = \xi(t,p)\phi(p),$$

from which one deduces

$$e^{H(t)}\phi(p)e^{-H(t)} = e^{\mathrm{ad}H(t)}\phi(p) = e^{\xi(t,p)}\phi(p).$$

Looking at the coefficient of p^n, one has

$$e^{H(t)}\phi_n e^{-H(t)} = \sum_{k=0}^{\infty} q_k(t)\phi_{n-k}$$

for any $n \in \mathbb{Z}$. Consequently we have

$$\begin{aligned}\langle 0|e^{H(t)}\phi_m\phi_n|0\rangle &= \frac{1}{2}q_m(t)q_n(t) + \sum_{k=1}^{n}(-1)^k q_{m+k}(t)q_{n-k}(t)\\ &= \frac{1}{2}Q_{(m,n)}(t)\end{aligned}$$

for $m > n \geqslant 0$. A standard fermion calculus shows that

$$\langle 0|e^{H(t)}\phi_{\lambda_1}\cdots\phi_{\lambda_{2d}}|0\rangle = \frac{1}{2^d}\mathrm{Pf}(Q_{(\lambda_i,\lambda_j)}(t)) = \frac{1}{2^d}Q_\lambda(t)$$

for a strict partition $\lambda = (\lambda_1, \dots, \lambda_{2d})$.

§2. Basic representation of $A_{2\ell}^{(2)}$

In this section we fix $\ell \geqslant 1$ and put $r = 2\ell + 1$. The Lie subalgebra $A_{2\ell}^{(2)}$ of B_∞ is realized by the following Chevalley generators:

(2.1a) $$e_0 = \sqrt{2}\sum_{k\in\mathbb{Z}}(-1)^{k+1} {}^{\circ}_{\circ}\phi_{kr}\phi_{-kr-1}{}^{\circ}_{\circ},$$

(2.1b) $$e_i = \sum_{k\in\mathbb{Z}}(-1)^{k+i} {}^{\circ}_{\circ}\phi_{kr-i-1}\phi_{-kr+i}{}^{\circ}_{\circ},$$

(2.1c) $$e_\ell = \sum_{k\in\mathbb{Z}}(-1)^{k+\ell+1} {}^{\circ}_{\circ}\phi_{kr+\ell}\phi_{-kr-\ell-1}{}^{\circ}_{\circ},$$

(2.1d) $$f_0 = \sqrt{2}\sum_{k\in\mathbb{Z}}(-1)^{k+1} {}^{\circ}_{\circ}\phi_{kr}\phi_{-kr+1}{}^{\circ}_{\circ},$$

(2.1e) $$f_i = \sum_{k\in\mathbb{Z}} (-1)^{k+i+1} {}^{\circ}_{\circ}\phi_{kr-i}\phi_{-kr+i+1}{}^{\circ}_{\circ},$$

(2.1f) $$f_\ell = \sum_{k\in\mathbb{Z}} (-1)^{k+\ell} {}^{\circ}_{\circ}\phi_{kr+\ell+1}\phi_{-kr-\ell}{}^{\circ}_{\circ},$$

(2.1g) $$\alpha_0^\vee = 2\sum_{k\in\mathbb{Z}} (-1)^{k+1} {}^{\circ}_{\circ}\phi_{kr-1}\phi_{-kr+1}{}^{\circ}_{\circ} + 1,$$

(2.1h) $$\alpha_i^\vee = \sum_{k\in\mathbb{Z}} (-1)^{k+i} \left({}^{\circ}_{\circ}\phi_{kr+i}\phi_{-kr+i}{}^{\circ}_{\circ} - {}^{\circ}_{\circ}\phi_{kr-i-1}\phi_{-kr+i+1}{}^{\circ}_{\circ}\right),$$

(2.1i) $$\alpha_\ell^\vee = \sum_{k\in\mathbb{Z}} (-1)^{k+\ell} {}^{\circ}_{\circ}\phi_{kr+\ell}\phi_{-kr-\ell}{}^{\circ}_{\circ},$$

for $1 \leqslant i \leqslant \ell - 1$. The fundamental imaginary root reads

$$\delta = 2\sum_{i=0}^{\ell-1} \alpha_i + \alpha_\ell.$$

As we have seen in the previous section, the polynomial ring $V = \mathbb{C}[\,t_j\ ;\ j \geqslant 1, \text{odd}]$ can be viewed as a B_∞-module via the boson-fermion correspondence. Moreover any Q-function $Q_\lambda(t)$ is a weight vector. The restriction to the Lie subalgebra $A_{2\ell}^{(2)}$ corresponds to the specialization $q^r = -p^r$ of the parameters in the vertex operator (1.7). The Lie algebra $A_{2\ell}^{(2)}$ stabilizes the ideal $I^{(r)}$ of V generated by t_{jr} ($j \geqslant 1$, odd), because $p^{jr} + q^{jr} = 0$ ($j \in \mathbb{Z}$). Hence $A_{2\ell}^{(2)}$ acts on the quotient algebra $V^{(r)} = V/I^{(r)}$, which will always be identified with the subalgebra $V = \mathbb{C}[\,t_j\ ;\ j \geqslant 1, \text{odd and } j \not\equiv 0 \pmod{r}]$. This action is shown to be irreducible and in fact isomorphic to the basic representation of $A_{2\ell}^{(2)}$ with the highest weight vector $1 \in V^{(r)}$.

For a polynomial $F(t) \in V$, denote the r-reduced polynomial of $F(t)$ by

$$F^{(r)}(t) = F(t)|_{t_r = t_{3r} = \cdots = 0} \in V^{(r)}$$

Since the Cartan subalgebra of $A_{2\ell}^{(2)}$ is contained in that of B_∞, all r-reduced Q-functions $Q_\lambda^{(r)}(t)$ are weight vectors under the action of the former.

To describe the weight of a given r-reduced Q-function, we need the following r-abacus:

$$\begin{array}{ccccccc}
 & 1 & 2 & 3 & \cdots & r-2 & r-1 \\
r & r+1 & r+2 & r+3 & \cdots & 2r-2 & 2r-1 \\
2r & 2r+1 & 2r+2 & 2r+3 & \cdots & 3r-2 & 3r-1 \\
3r & 3r+1 & 3r+2 & 3r+3 & \cdots & 4r-2 & 4r-1
\end{array}$$

$$\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot$$

For a strict partition we put a set of beads on the assigned positions. The above figure is the r-abacus representing the strict partition $(2r+1, 2r, r+2, r-1, 1)$.

Theorem 2.1. *Let $Q_\lambda^{(r)}(t)$ be an r-reduced Q-function of weight $wt(\lambda)$. Then a strict partition corresponding to weight $wt(\lambda)+\delta$ is obtained by one of the following:*

1. *Move a bead one position up along a runner.*
2. *Remove the bead at the position r.*
3. *Remove the two beads at positions s and $r-s$, simultaneously, for $1 \leqslant s \leqslant \ell$.*

Proof. Since each α_i $(0 \leqslant i \leqslant \ell)$ has degree 1 in this realization (the principal realization), $\delta = 2\sum_{i=0}^{\ell-1}\alpha_i + \alpha_\ell$ has degree r. Each move described above decreases the degree of the corresponding r-reduced Q-function by r. Accordingly it suffices to show that the fermion operators corresponding to the moves (1)–(3) commute with $\alpha_i^\vee$ $(0 \leqslant i \leqslant \ell)$. Let $v = \phi_{\lambda_1}\cdots\phi_{\lambda_{2d}}|0\rangle$ be the given weight vector in $\mathcal{F}_0$. Up to sign, the move (1) is achieved by multiplying v by ${}^{\circ}_{\circ}\phi_{k-r}\phi_{-k}{}^{\circ}_{\circ}$ $(k \geqslant r+1)$ from the left. The commutation relations

$$[\alpha_i^\vee, {}^{\circ}_{\circ}\phi_{k-r}\phi_{-k}{}^{\circ}_{\circ}] = 0$$

are verified by using the general formula:

(2.2)
$$\begin{aligned}
[{}^{\circ}_{\circ}\phi_a\phi_b{}^{\circ}_{\circ}, {}^{\circ}_{\circ}\phi_c\phi_d{}^{\circ}_{\circ}] &= (-1)^b\delta_{b+c,0}{}^{\circ}_{\circ}\phi_a\phi_d{}^{\circ}_{\circ} - (-1)^b\delta_{b+d,0}{}^{\circ}_{\circ}\phi_a\phi_c{}^{\circ}_{\circ} \\
&\quad - (-1)^a\delta_{a+c,0}{}^{\circ}_{\circ}\phi_b\phi_d{}^{\circ}_{\circ} + (-1)^a\delta_{a+d,0}{}^{\circ}_{\circ}\phi_b\phi_c{}^{\circ}_{\circ} \\
&\quad + (-1)^{a+b}(\delta_{a+c,0}\delta_{b+c,0} - \delta_{a+d,0}\delta_{b+d,0})\,(Y(-b) - Y(a))\,,
\end{aligned}$$

where

$$Y(a) = \begin{cases} 1 & a > 0 \\ \frac{1}{2} & a = 0 \\ 0 & a < 0. \end{cases}$$

Likewise the moves (2) and (3) are achieved by multiplying ${}^\circ_\circ\phi_{-r}\phi_0{}^\circ_\circ$ and ${}^\circ_\circ\phi_{-s}\phi_{s-r}{}^\circ_\circ$ $(1 \leqslant s \leqslant \ell)$, respectively, from the left. By a direct computation these operators are shown to commute with $\alpha_i^\vee$ $(0 \leqslant i \leqslant \ell)$. ∎

We should remark that each move in the above theorem corresponds to the removal of a r-bar from a (shifted) Young diagram introduced by Morris [Mo2].

A weight Λ is said to be maximal if $\Lambda + \delta$ is no longer a weight. It is known [K, Lemma 12.6] that Λ is a maximal weight if and only if Λ lies on the Weyl group orbit through the highest weight Λ_0. The maximal weight vectors are the r-reduced Q-functions $Q_\lambda^{(r)}(t)$ with λ obtained as the "stalemates" of the game described in Theorem 2.1. These r-reduced Q-functions $Q_\lambda^{(r)}(t)$ coincide with the (full) Q-functions $Q_\lambda(t)$ and solve the r-reduced BKP hierarchy [DJKM]. For the case $r = 3$, the Q-functions associated with the strict partitions

$$\{\emptyset, (3n+1, 3n-2, \ldots, 4, 1), (3n+2, 3n-1, \ldots, 5, 2) \quad (n \geqslant 0)\}$$

cover the maximal weight vectors for $A_2^{(2)}$.

The r-reduced Q-functions $\left\{Q_\lambda^{(r)}(t); \lambda \text{ is a strict partition}\right\}$ are linearly dependent. The linear relations satisfied by those functions are fully investigated in [NY2]. Here we only restate Proposition 2.1 in [NY2].

Proposition 2.2. *The set*

$$\left\{Q_\lambda^{(r)}(t); \lambda \textit{ is a strict partition with no part divisible by } r\right\}$$

forms a weight basis for $V^{(r)}$.

Another combinatorial way to compute the weight of a given strict partition λ is known as follows. Draw the Young diagram λ and fill each cell with a non-negative integer in such a way that, in each row the sequence $(0, 1, 2, \ldots, \ell-1, \ell, \ell-1, \ldots, 2, 1, 0)$ repeats from the left as long as possible. Let k_i $(0 \leqslant i \leqslant \ell)$ be the number of i's written in the

Young diagram. Then we have

$$\mathrm{wt}(\lambda) = \Lambda_0 - \sum_{i=0}^{\ell} k_i \alpha_i.$$

For example, let $\lambda = (9, 5, 4, 2, 1)$. Then the $A_2^{(2)}$-weight of $Q_\lambda(t)$ is

$$\mathrm{wt}(\lambda) = \Lambda_0 - 7\alpha_0 - 6\alpha_1 - 5\alpha_2 - 3\alpha_3$$

since we have the Young diagram

0	1	2	3	2	1	0	0	1
0	1	2	3	2				
0	1	2	3					
0	1							
0								

§3. Basic representation of $D_{\ell+1}^{(2)}$

In this section we fix $\ell \geqslant 2$ and put $r = 2\ell + 2$. We discuss the basic representation of $D_{\ell+1}^{(2)}$. For the most part the arguments are parallel to those in Section 2.

Let $\omega = \exp(2\pi i/r)$. Consider the reduced vertex operators

$$Z_B(p, -\omega^j p) = \sum_{i \in \mathbf{Z}} Z_i^{(j)} p^i \qquad (1 \leq j \leq \ell).$$

Then the homogeneous components $Z_i^{(j)}$, together with the Heisenberg Lie algebra, constitute a Lie algebra acting on $V = \mathbf{C}[t_j;\ j \geqslant 1, \text{odd}\ ]$, which is isomorphic to the basic representation of $D_{\ell+1}^{(2)}$.

The Chevalley generators and coroots of $D_{\ell+1}^{(2)}$ are described in terms of the fermion operators as follows:

$$e_0 = \sqrt{2} \sum_{k \in \mathbb{Z}} {}^{\circ}_{\circ} \phi_{kr-1} \phi_{-kr} {}^{\circ}_{\circ}, \tag{3.1a}$$

$$e_i = (-1)^i \sum_{k \in \mathbb{Z}} {}^{\circ}_{\circ} \phi_{kr+i} \phi_{-kr-i-1} {}^{\circ}_{\circ}, \tag{3.1b}$$

$$e_\ell = (-1)^{\ell+1}\sqrt{2}\sum_{k\in\mathbb{Z}} {}^{\circ}_{\circ}\phi_{kr+\ell}\phi_{-kr-\ell-1}{}^{\circ}_{\circ}, \tag{3.1c}$$

$$f_0 = -\sqrt{2}\sum_{k\in\mathbb{Z}} {}^{\circ}_{\circ}\phi_{kr}\phi_{-kr+1}{}^{\circ}_{\circ}, \tag{3.1d}$$

$$f_i = (-1)^i\sum_{k\in\mathbb{Z}} {}^{\circ}_{\circ}\phi_{kr+i+1}\phi_{-kr-i}{}^{\circ}_{\circ}, \tag{3.1e}$$

$$f_\ell = (-1)^{\ell}\sqrt{2}\sum_{k\in\mathbb{Z}} {}^{\circ}_{\circ}\phi_{kr+\ell+1}\phi_{-kr-\ell}{}^{\circ}_{\circ}, \tag{3.1f}$$

$$\alpha_0^\vee = -2\sum_{k\in\mathbb{Z}} {}^{\circ}_{\circ}\phi_{kr-1}\phi_{-kr+1}{}^{\circ}_{\circ} + 1, \tag{3.1g}$$

$$\alpha_i^\vee = (-1)^i\sum_{k\in\mathbb{Z}}\left({}^{\circ}_{\circ}\phi_{kr+i}\phi_{-kr-i}{}^{\circ}_{\circ} - {}^{\circ}_{\circ}\phi_{kr-i-1}\phi_{-kr+i+1}{}^{\circ}_{\circ}\right), \tag{3.1h}$$

$$\alpha_\ell^\vee = (-1)^{\ell}2\sum_{k\in\mathbb{Z}} {}^{\circ}_{\circ}\phi_{kr+\ell}\phi_{-kr-\ell}{}^{\circ}_{\circ}, \tag{3.1i}$$

for $1 \leqslant i \leqslant \ell - 1$. The fundamental imaginary root reads

$$\delta = \sum_{i=0}^{\ell}\alpha_i.$$

Note that the Cartan subalgebra of $D_{\ell+1}^{(2)}$ is the intersection of that of B_∞ and $D_{\ell+1}^{(2)}$. Hence any Q-function $Q_\lambda(t)$ for a strict partition is a $D_{\ell+1}^{(2)}$-weight vector. In contrast with the case of $A_{2\ell}^{(2)}$, we do not have to eliminate any variables of the Q-functions. Therefore these weight vectors are automatically linearly independent.

We will describe the weight of a given Q-function. To this end we need the following r-abacus:

	1	2	$\cdots$	ℓ	$\ell+2$	$\ell+3$	$\cdots$	$2\ell+1$
$\ell+1$								
$2\ell+2$	$2\ell+3$	$2\ell+4$	$\cdots$	$3\ell+2$	$3\ell+4$	$3\ell+5$	$\cdots$	$4\ell+3$
$3\ell+3$								
$4\ell+4$	$4\ell+5$	$4\ell+6$	$\cdots$	$5\ell+4$	$5\ell+6$	$5\ell+7$	$\cdots$	$6\ell+5$

For a strict partition we put a set of beads on the assigned positions. The above figure is the r-abacus representing the strict partition $(5\ell+7, 4\ell+3, 3\ell+5, 3\ell+3, 2\ell+3, \ell+1, 2)$.

Theorem 3.1. *Let $Q_\lambda(t)$ be a Q-function of weight $wt(\lambda)$. Then a strict partition corresponding to weight $wt(\lambda)+\delta$ is obtained by one of the following:*

(1) *Move a bead one position up along the leftmost runner.*
(2) *Remove the bead at the position $\ell+1$.*

A strict partition corresponding to weight $wt(\lambda)+2\delta$ is obtained by one of the following:

(3) *Iterate the procedures obtaining $wt(\lambda)+\delta$.*
(4) *Move a bead one position up along a runner except for the leftmost one.*
(5) *Remove the two beads at positions s and $r-s$, simultaneously, for $1 \leqslant s \leqslant \ell$.*

Proof. The idea is the same as in Theorem 2.1. In this case the degree of $\delta = \sum_{i=0}^{\ell} \alpha_i$ equals to $r/2 = \ell+1$. Since the moves (1) and (2) (*resp.* (3)–(5)) decreases the degree of corresponding Q-function by $r/2$ (*resp.* r), we only have to check that fermion operators corresponding to those moves commute with $\alpha_i^\vee$ $(0 \leqslant i \leqslant \ell)$. Let $v = \phi_{i_t} \cdots \phi_{i_1}|0\rangle$ be a given weight vector. Up to sign, the move (1) is achieved by multiplying ${}^\circ_\circ \phi_{k(\ell+1)}\phi_{-(k+1)(\ell+1)}{}^\circ_\circ$ from the left of v. Therefore it suffices to check that

$$[\alpha_i^\vee, {}^\circ_\circ \phi_{k(\ell+1)}\phi_{-(k+1)(\ell+1)}{}^\circ_\circ] = 0 \quad (0 \leqslant i \leqslant \ell),$$

which is easily verified by (2.2) and the the fermion expression (3.1) of $\alpha_i^\vee$. Likewise the moves (2), (4) and (5) are achieved by multiplying ${}^\circ_\circ \phi_{-(\ell+1)}\phi_0{}^\circ_\circ$, ${}^\circ_\circ \phi_{k(\ell+1)+s}\phi_{-(k+2)(\ell+1)-s}{}^\circ_\circ$ $(1 \leqslant s \leqslant \ell)$, and ${}^\circ_\circ \phi_s\phi_{r-s}{}^\circ_\circ$ $(1 \leqslant s \leqslant \ell)$, respectively, from the left. It is easily verified that these operators commute with $\alpha_i^\vee$ $(0 \leqslant i \leqslant \ell)$. ∎

The maximal weight vectors are the Q-functions $Q_\lambda(t)$ with λ corresponding to the "stalemates". These Q-functions solve the r-reduced BKP hierarchy [DJKM]. For example, the strict partitions whose Q-functions are maximal weight vectors for $D_4^{(2)}$ of degree up to 12 are

$$\{\emptyset, (1), (2), (3), (2,1), (3,1), (5), (3,2), (6), (5,1), (3,2,1), (7), (6,1), (5,2), (5,2,1), (7,2), (6,3), (9,1), (7,3), (6,3,1), (6,5), (10,2), (7,5), (9,2,1), (6,5,1), (7,3,2)\}$$

Another combinatorial way to compute the weight of a given strict partition λ is known as follows. Draw the Young diagram λ and fill each cell with a non-negative integer in such a way that, in each row the sequence $(0, 1, 2, \ldots, \ell-1, \ell, \ell, \ell-1, \ldots, 2, 1, 0)$ repeats from the left as long as possible. Let k_i $(0 \leqslant i \leqslant \ell)$ be the number of i's written in the Young diagram. In other words,

$$k_i = \sum_{j \geqslant 0} (\mu_{rj+i+1} + \mu_{rj+r-i}),$$

where $\lambda' = (\mu_1, \mu_2, \ldots)$ is the transpose of λ. Then we have

$$\mathrm{wt}(\lambda) = \Lambda_0 - \sum_{i=0}^{\ell} k_i \alpha_i.$$

For example, let $\lambda = (9, 5, 4, 2, 1)$. Then the $D_4^{(2)}$-weight of $Q_\lambda(t)$ is

$$\mathrm{wt}(\lambda) = \Lambda_0 - 7\alpha_0 - 5\alpha_1 - 4\alpha_2 - 5\alpha_3$$

since we have the Young diagram

$$\begin{array}{|c|c|c|c|c|c|c|c|c|}
\hline
0 & 1 & 2 & 3 & 3 & 2 & 1 & 0 & 0 \\
\hline
0 & 1 & 2 & 3 & 3 \\
\cline{1-5}
0 & 1 & 2 & 3 \\
\cline{1-4}
0 & 1 \\
\cline{1-2}
0 \\
\cline{1-1}
\end{array}$$

Finally we make a remark on the weight multiplicity for the basic representation of $D_{\ell+1}^{(2)}$. A converse move of (1), (2) (resp. (3), (4), (5)) makes a weight vector of weight $\mathrm{wt}(\lambda) - \delta$ (resp. $\mathrm{wt}(\lambda) - 2\delta$) from the given strict partition λ. Starting from the highest weight Λ_0, which corresponds to the empty partition $\emptyset$, the number of bead configurations for the strict partitions of $n(\ell+1)$ is equal to $b^{(\ell)}(n)$, defined by

$$\sum_{n=0}^{\infty} b^{(\ell)}(n) q^n = \frac{\phi(q^2)}{\phi(q)} \cdot \frac{1}{\phi(q^2)^\ell} = \frac{1}{\phi(q)\phi(q^2)^{\ell-1}}.$$

This enumeration is an easy exercise of combinatorics. Since the maximal weights are on the Weyl group orbit through Λ_0 and they are all of multiplicity one, we have

$$\mathrm{mult}(\Lambda - n\delta) = b^{(\ell)}(n)$$

for any maximal weight Λ [K].

§4. Application to the case of $A_1^{(1)}$

We now apply the results of the preceding section to the basic representation of $A_1^{(1)}$. To this end, we first recall the following theorem which is due to M. Wakimoto.

Theorem 4.1. *Let e_i and f_i $(i = 0, 1, 2, 3)$ be the Chevalley generators and $\alpha_i^\vee$ $(i = 0, 1, 2, 3)$ be the simple coroots of $D_4^{(2)}$. Put*

$$\begin{aligned} \tilde{e}_0 &= e_0 + e_3, & \tilde{e}_1 &= \sqrt{2}(e_1 + e_2), \\ \tilde{f}_0 &= f_0 + f_3, & \tilde{f}_1 &= \sqrt{2}(f_1 + f_2), \\ \tilde{\alpha}_0^\vee &= \alpha_0^\vee + \alpha_3^\vee, & \tilde{\alpha}_1^\vee &= 2(\alpha_1^\vee + \alpha_2^\vee). \end{aligned} \tag{4.1}$$

Then $\tilde{e}_i$ and $\tilde{f}_i$ $(i = 0, 1)$ generate a Lie subalgebra isomorphic to $A_1^{(1)}$. Moreover, the restriction of the basic representation of $D_4^{(2)}$ to this subalgebra remains irreducible and turns out to be the basic representation of $A_1^{(1)}$.

Proof. It is straightforward to check that $\tilde{e}_i$, $\tilde{f}_i$ and $\tilde{\alpha}_i^\vee$ $(i = 0, 1)$ satisfy the defining relations of $A_1^{(1)}$.

The latter can be proved by looking at the formal characters. The formal character of the basic representation of $D_4^{(2)}$ is given by the following infinite product:

$$\begin{aligned} \mathrm{ch}L(\Lambda_0; D_4^{(2)}) = e^{\Lambda_0} &\overset{(-)}{\prod_{j}}(1 + e^{j\delta}) \\ \times &\overset{(-)}{\prod_{j:\text{odd}}} (1 + e^{j\delta \pm \alpha_3})(1 + e^{j\delta \pm (\alpha_2 + \alpha_3)})(1 + e^{j\delta \pm (\alpha_1 + \alpha_2 + \alpha_3)}), \end{aligned}$$

where $\delta = \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3$. Here $\overset{(-)}{\prod_{j}}(1 + e^{j\delta + \alpha})$ denotes the product where j runs over the integers such that $j\delta + \alpha$ is a negative root. Under the restriction we have $\alpha_0 = \alpha_3$ and $\alpha_1 = \alpha_2$. Hence, putting $\delta = \alpha_0 + \alpha_1$, the "restricted character" reads $e^{\Lambda_0} \overset{(-)}{\prod_{j}}(1 + e^{j\delta})(1 + e^{2j\delta \pm \alpha_0})$, which is nothing but the formal character $\mathrm{ch}L(\Lambda_0; A_1^{(1)})$. ∎

In a similar fashion, one can embed $D_{2^n}^{(2)}$ into $D_{2^n+1}^{(2)}$ so that the basic representation is preserved under the restriction. Since the above theorem is thought of as the special case $n = 1$, we denote by $D_2^{(2)}$ the subalgebra isomorphic to $A_1^{(1)}$.

As we have seen in Section 3, the basic representation of $D_4^{(2)}$ is realized on $V = \mathbf{C}[t_j; j \geqslant 1, \text{odd}]$. Therefore V is viewed also as the space of the basic representation of $D_2^{(2)} \cong A_1^{(1)}$. The same argument as in Section 3 shows that the $D_2^{(2)}$-weight vectors are Schur's Q-functions for the strict partitions. In this section we need the 4-abacus in order to determine the weight of a given Q-function. Here we give the 4-abacus representing the strict partition $\lambda = (9, 5, 3, 2)$.

$$
\begin{array}{ccc}
 & 1 & 3 \\
2 & & \\
4 & 5 & 7 \\
6 & & \\
8 & 9 & 11 \\
\cdots & \cdots & \cdots
\end{array}
$$

Suppose we are given a Q-function $Q_\lambda(t)$ of weight $\text{wt}(\lambda)$. A strict partition corresponding to $\text{wt}(\lambda) + \delta$ is obtained by one of the following:

(1) Move a bead one position up along the leftmost runner.
(2) Remove the bead at the position 2.

A strict partition corresponding to $\text{wt}(\lambda) + 2\delta$ is obtained by one of the following:

(3) Iterate the procedure obtaining $\text{wt}(\lambda) + \delta$.
(4) Move a bead one position up along the first or third runner.
(5) Remove the two beads at the positions 1 and 3, simultaneously.

The strict partitions corresponding to the maximal weights are thus characterized by the "stalemates", which constitute the following set:

(4.2)
$$HC_4 := \{\emptyset, (4n+1, 4n-3, \ldots, 5, 1), (4n+3, 4n-1, \ldots, 7, 3) \quad (n \geqslant 0)\}.$$

Again as in Section 3, there is another way to compute the weight of a given Q-function. Let $\lambda' = (\mu_1, \mu_2, \ldots)$ be the transpose of the strict partition λ. Then the weight of $Q_\lambda(t)$ is equal to

$$\mathrm{wt}(\lambda) = \Lambda_0 - \sum_{j\geqslant 0}(\mu_{4j+1} + \mu_{4j+4})\alpha_0 - \sum_{j\geqslant 0}(\mu_{4j+2} + \mu_{4j+3})\alpha_1.$$

We remark here the equivalence of the 4-reduced BKP hierarchy and the 2-reduced KP hierarchy, i.e., the KdV hierarchy [DJKM]. The vertex operator of the former looks

$$Z_B(p, -ip) = \frac{1+i}{2(1-i)}\left\{\exp\Big(\sum_{\substack{j\geqslant 1\\ j:\mathrm{odd}}}(1-i^j)t_j p^j\Big)\exp\Big(-2\sum_{\substack{j\geqslant 1\\ j:\mathrm{odd}}}\frac{(1+i^j)}{j}\frac{\partial}{\partial t_j}p^{-j}\Big)-1\right\}.$$

Hence, by changing the variables

$$x_j = t_j \cos\left(\frac{j\pi}{4}\right) \qquad (j \geqslant 1, \mathrm{odd}) \tag{4.3}$$

and $k = (1-i)p/\sqrt{2}$, it reads

$$\frac{1+i}{2(1-i)}\left\{\exp\Big(2\sum_{\substack{j\geqslant 1\\ j:\mathrm{odd}}} x_j k^j\Big)\exp\Big(-2\sum_{\substack{j\geqslant 1\\ j:\mathrm{odd}}}\frac{1}{j}\frac{\partial}{\partial x_j}k^{-j}\Big) - 1\right\},$$

which equals, up to constant, the vertex operator for the KdV hierarchy.

The set of all the weighted homogeneous polynomial solutions coincides with the set of the maximal weight vectors. As for the 4-reduced BKP hierarchy this set consists of $Q_\lambda(t)$ with λ in HC_4.

Let DP_4 denote the set of partitions $\lambda = (1^{m_1}2^{m_2}\cdots)$ for which $m_i \leqslant 1$ when i is odd [LT]. Let DPR_4 be the subset of DP_4 consisting of partitions $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ such that $0 < \lambda_i - \lambda_{i+1} \leqslant 4$ for odd i and $0 \leqslant \lambda_i - \lambda_{i+1} < 4$ for even i, where $\lambda_{\ell+1} = 0$.

The "doubling" $d(\lambda)$ of $\lambda = (\lambda_1, \lambda_2, \ldots \lambda_\ell) \in DP_4$ is defined by

$$d(\lambda) = \left(\left[\frac{\lambda_1+1}{2}\right], \left[\frac{\lambda_1}{2}\right], \left[\frac{\lambda_2+1}{2}\right], \left[\frac{\lambda_2}{2}\right], \ldots \left[\frac{\lambda_\ell+1}{2}\right], \left[\frac{\lambda_\ell}{2}\right]\right)$$

(cf. [BO]). Let λ° denote the transpose of $d(\lambda)$ for $\lambda \in DP_4$. If $\lambda' = (\mu_1, \mu_2, \ldots)$ is the transpose of λ, then one sees that $\lambda^\circ = (\mu_1 + \mu_2, \mu_3 + \mu_4, \ldots)$. It can be verified that $\lambda^\circ \in DP_4$ and the operation $\circ$ is an involution in DP_4. Moreover the set of strict partitions is mapped onto DPR_4 by $\circ$. Put $\Delta_r = (r, r-1, \ldots, 2, 1)$ for $r \geqslant 1$ and $\Delta_0 = \emptyset$. These are called the staircase partitions. The involution $\circ$ gives a one-to-one correspondence between the set HC_4 and the set of the staircase partitions.

Let denote by ϖ the change of variables (4.3). More precisely, define an algebra isomorphism

$$\varpi : \mathbb{C}[t_j; j \geqslant 1, \mathrm{odd}] \longrightarrow \mathbb{C}[x_j; j \geqslant 1, \mathrm{odd}]$$

by $\varpi(t_j) = x_j / \cos(j\pi/4)$ for $j \geqslant 1$, odd.

Theorem 4.2. *For $\lambda \in HC_4$, we have*

$$\varpi(Q_\lambda) = 2^{\ell(\lambda)/2} S_{\lambda^\circ}.$$

In order to prove this theorem, we define the polynomial $B_\mu(x)$ for a strict partition μ by

$$B_\mu(x) = \sum_\rho b_\rho^\lambda \frac{x_1^{m_1} x_3^{m_3} \cdots}{m_1! m_3! \cdots},$$

where the summation runs over the partitions $\rho = (1^{m_1} 3^{m_3} \cdots)$ consisting of odd parts, and b_ρ^λ denotes the irreducible Brauer character value for the symmetric group S_n of characteristic 2 [JK]. One knows that $\{B_\mu(x); \mu \text{ is a strict partition}\}$ gives a weight basis for $\mathbb{C}[x_j; j \geqslant 1, \mathrm{odd}]$, the space of the basic representation of $A_1^{(1)}$ in the KdV picture. It is worth noting that

$$S_{\Delta_r}(x) = B_{\Delta_r}(x) \qquad (r \geqslant 0).$$

Let $\tilde{S}_n$ be the double cover of S_n, which is generated by $\tau_1, \ldots \tau_{n-1}$ and z with respect to the relations:

$$\begin{aligned} z^2 = 1, \quad z\tau_i = \tau_i z, \quad \tau_i^2 &= z \quad (1 \leqslant i \leqslant n-1), \\ (\tau_i \tau_{i+1})^3 &= z \quad (1 \leqslant i \leqslant n-2), \\ \tau_i \tau_j &= z \tau_j \tau_i \quad (|i-j| \geqslant 2). \end{aligned}$$

Let $\theta : \tilde{S}_n \longrightarrow S_n$ be the canonical epimorphism. A partition $\rho = (1^{m_1} 2^{m_2} \cdots)$ of n determines a conjugacy class C_ρ of S_n consisting of the elements of cycle type ρ. It is a classical result due to Schur that $\theta^{-1}(C_\rho)$ splits into two $\tilde{S}_n$-conjugacy classes $\tilde{C}_\rho^+$ and $z\tilde{C}_\rho^+$ if and only if either (1) the parts of ρ are all odd, or (2) ρ is a strict partition such that $n - \ell(\rho)$ is odd. An irreducible representation of $\tilde{S}_n$ is said to be "negative" if the central element z is mapped to -1. The character ζ of a negative representation satisfies $\zeta(z\tilde{C}_\rho^+) = -\zeta(\tilde{C}_\rho^+)$. For a partition $\rho = (\rho_1, \rho_2, \ldots, \rho_\ell)$ of n consisting of odd parts, define $\tau_\rho = \pi_1 \pi_2 \cdots \pi_\ell$, where $\pi_j = \tau_{a+1} \tau_{a+2} \cdots \tau_{a+\rho_j - 1}$ $(a = \rho_1 + \rho_2 + \cdots + \rho_{j-1})$

for $1 \leqslant j \leqslant \ell$. For example, if $\rho = (5,3,1)$, then $\tau_\rho = \tau_1\tau_2\tau_3\tau_4\tau_6\tau_7$. A direct computation shows that $\mathrm{ord}(\tau_\rho) \equiv f(\rho) + 1 \pmod 2$, where $f(\rho) = \sum_{j\equiv 3,5 \pmod 8} m_j$ for $\rho = (1^{m_1}3^{m_3}\cdots)$. The irreducible negative representations of $\tilde{S}_n$ are parametrized by the strict partitions of n. Let ζ_ρ^λ denote the character value of the irreducible negative representation $\langle\lambda\rangle$ corresponding to the strict partition λ, evaluated on the element τ_ρ. One finds the character tables (ζ_ρ^λ) for $n \leqslant 14$ in [HH]. The irreducible negative representation $\langle\lambda\rangle$ has a composition series under the reduction modulo 2. The irreducible 2-modular representations of $\tilde{S}_n$ are nothing but those of S_n, since the central element z is always mapped to 1 when the characteristic equals 2. Therefore they are also parametrized by the strict partitions of n. Denote by $\tilde{d}_{\lambda\mu}$ the number of occurrence of the irreducible 2-modular representation indexed by the strict partition μ in the composition series of $\langle\lambda\rangle$ ([B]).

Let $\tilde{Z}_n = \left((-1)^{f(\rho)}\zeta_\rho^\lambda\right)_{\lambda\rho}$, $\tilde{D}_n = (\tilde{d}_{\lambda\mu})_{\lambda\mu}$ and $B_n = (b_\rho^\mu)_{\mu\rho}$, where λ and μ are strict partitions of n and ρ is the partition of n consisting of odd parts. By a general theory of modular representations, one sees that

$$\tilde{Z}_n = \tilde{D}_n B_n.$$

The Brauer characters are defined on the elements of odd order. Hence b_ρ^λ is evaluated on $z\tau_\rho$ if τ_ρ is of even order. This is the reason why the signs in $\tilde{Z}_n$ appear.

Theorem 4.3. *For a strict partition λ of n, we have*

$$\varpi(Q_\lambda) = 2^{(\ell(\lambda)+\epsilon)/2} \sum_\mu \tilde{d}_{\lambda\mu} B_\mu,$$

where the summation runs over the strict partitions μ of n, and $\epsilon = 0$ or 1 according to that $n - \ell(\lambda)$ is even or odd.

Proof. Recall that the Q-functions are related with the power sum symmetric functions by

$$Q_\lambda(t) = \sum_\rho 2^{(\ell(\lambda)-\ell(\rho)+\epsilon)/2} \zeta_\rho^\lambda \frac{t_1^{m_1} t_3^{m_3}\cdots}{m_1! m_3! \cdots},$$

where the summation runs over $\rho = (1^{m_1}3^{m_3}\cdots)$. Here we have put $t_j = 2p_j/j$ as before. Therefore we have

$$\varpi(Q_\lambda) = 2^{(\ell(\lambda)+\epsilon)/2} \sum_\rho (-1)^{f(\rho)} \zeta_\rho^\lambda \frac{x_1^{m_1} x_3^{m_3}\cdots}{m_1! m_3! \cdots},$$

which implies the required formula. ∎

For $\lambda \in HC_4$, the decomposition number $\tilde{d}_{\lambda\mu}$ is equal to 1 if $\mu = \lambda^\circ$ and 0 otherwise [B]. This proves Theorem 4.2, since $n - \ell(\lambda)$ is even for $\lambda \in HC_4$.

Looking at the square matrices $\tilde{D}_n$, given in [B] for $n \leqslant 15$, one observes that $\det \tilde{D}_n$ is a power of 2. A proof of this fact will be given in a separate paper [TY].

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Capelli Elements in the Classical Universal Enveloping Algebras

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For any complex classical group $G = O_N, Sp_N$ consider the ring $\mathrm{Z}(\mathfrak{g})$ of G-invariants in the corresponding enveloping algebra $\mathrm{U}(\mathfrak{g})$. Let u be a complex parameter. For each $n = 0, 1, 2, \ldots$ and every partition ν of n into at most N parts we define a certain rational function $Z_\nu(u)$ which takes values in $\mathrm{Z}(\mathfrak{g})$. Our definition is motivated by the works of Cherednik and Sklyanin on the reflection equation, and also by the classical Capelli identity. The degrees in $\mathrm{U}(\mathfrak{g})$ of the values of $Z_\nu(u)$ do not exceed n. We describe the images of these values in the n-th symmetric power of $\mathfrak{g}$. Our description involves the plethysm coefficients as studied by Littlewood, see Theorem 3.4 and Corollary 3.6.

§1. Capelli elements in the algebra $\mathrm{U}(\mathfrak{gl}_N)$

We work with the general linear Lie algebra $\mathfrak{gl}_N$ over the complex field $\mathbb{C}$. In this section we recall the definition from [OO1, S] of the Capelli elements in the universal enveloping algebra $\mathrm{U}(\mathfrak{gl}_N)$. Here we also recall an explicit construction from [N2,O] of these elements.

Let the indices i, j run through the set $\{1, \ldots, N\}$. Let the vectors e_i form the standard basis in $\mathbb{C}^N$. We fix in the Lie algebra $\mathfrak{gl}_N$ the basis of the standard matrix units E_{ij}. We will also regard E_{ij} as generators of the universal enveloping algebra $\mathrm{U}(\mathfrak{gl}_N)$. Now choose the Borel subalgebra in $\mathfrak{gl}_N$ spanned by the elements E_{ij} with $i \leqslant j$. Then choose the basis $E_{11}, \ldots, E_{NN}$ in the corresponding Cartan subalgebra.

Let ν be any partition of n into at most N parts. We will write $\nu = (\nu_1, \ldots, \nu_N)$. Let U_ν be the irreducible $\mathfrak{gl}_N$-module of highest weight ν. The module U_ν appears in the decomposition of the n-th tensor power of the defining $\mathfrak{gl}_N$-module $\mathbb{C}^N$. It is called the *polynomial* $\mathfrak{gl}_N$-module corresponding to the partition ν.

There is a distinguished basis in the centre $\mathrm{Z}(\mathfrak{gl}_N)$ of the universal enveloping algebra $\mathrm{U}(\mathfrak{gl}_N)$, parametrized by the same partitions ν. The

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element C_ν of this basis is determined up to multipler from $\mathbb{C}$ by the following proposition. This proposition is due to Sahi [S, Theorem 1]. Consider the canonical ascending filtration on the algebra $\mathrm{U}(\mathfrak{gl}_N)$. With respect to this filtration the subspace $\mathfrak{gl}_N \subset \mathrm{U}(\mathfrak{gl}_N)$ has degree one.

Proposition 1.1. *There is an element C_ν in $\mathrm{Z}(\mathfrak{gl}_N)$ of degree at most n such that for any partition $\lambda = (\lambda_1, \ldots, \lambda_N)$ of not more than n we have $C_\nu \cdot U_\lambda \neq \{0\}$ if and only if $\lambda = \nu$.*

We will call $C_\nu \in \mathrm{Z}(\mathfrak{gl}_N)$ the *Capelli element* in the algebra $\mathrm{U}(\mathfrak{gl}_N)$ corresponding to the partition ν. The elements C_ν corresponding to the partitions $\nu = (1, \ldots, 1, 0, \ldots, 0)$ were studied by Capelli in [C]. In the case $\nu = (n, 0, \ldots, 0)$ they were studied in [N1]. An explicit formula for the eigenvalue of the central element C_ν in the $\mathfrak{gl}_N$-module U_λ for any λ and ν was given by Okounkov and Olshanski in [OO1]. Let us reproduce this formula, it will fix the multiplier from $\mathbb{C}$ up to which the element $C_\nu \in \mathrm{Z}(\mathfrak{gl}_N)$ has been determined so far.

Let $a = (a_1, a_2, \ldots)$ be an arbitrary sequence of complex numbers. For each $k = 0, 1, 2, \ldots$ introduce the k-th *generalized factorial power* $(u\,|\,a)^k = (u - a_1) \cdots (u - a_k)$ of the variable u. Consider the function in N independent variables $y_1, \ldots, y_N$

$$s_\nu(y_1, \ldots, y_N\,|\,a) = \frac{\det\big[\,(y_j|a)^{\nu_i+N-i}\,\big]}{\det\big[\,(y_j|a)^{N-i}\,\big]} \tag{1.1}$$

where the determinants are taken with respect to $i, j = 1, \ldots, N$. This function is a symmetric polynomial in $y_1, \ldots, y_N$ which is called the *generalized factorial Schur polynomial,* see [M, Example I.3.20]. Note that here the denominator

$$\det\big[\,(y_j|a)^{N-i}\,\big] \;=\; \prod_{i<j} (y_i - y_j)$$

is the Vandermonde determinant. Thus the denominator in (1.1) does not depend on the sequence a.

If $a = (0, 0, \ldots)$ the polynomial $s_\nu(y_1, \ldots, y_N\,|\,a)$ is the ordinary Schur polynomial $s_\nu(y_1, \ldots, y_N)$. For the general sequence a by (1.1)

$$s_\nu(y_1, \ldots, y_N\,|\,a) = s_\nu(y_1, \ldots, y_N) \;+\; \text{lower degree terms.}$$

Therefore all the polynomials $s_\nu(y_1, \ldots, y_N\,|\,a)$ where the partitions ν have not more than N parts, form a linear basis in the ring of symmetric polynomials in the variables $y_1, \ldots, y_N$ with complex coefficients.

Proposition 1.2. *The Capelli element $C_\nu \in \mathrm{Z}(\mathfrak{gl}_N)$ can be chosen so that its eigenvalues in the irreducible $\mathfrak{gl}_N$-modules U_λ are respectively*

$$s_\nu(\lambda_1+N-1,\lambda_2+N-2,\ldots,\lambda_N \mid 0,1,2,\ldots). \tag{1.2}$$

By the Harish-Chandra theorem [D, Theorem 7.4.5], the eigenvalue of any element from $\mathrm{Z}(\mathfrak{gl}_N)$ in the irreducible module U_λ is a symmetric polynomial in $\lambda_1+N-1,\lambda_2+N-2,\ldots,\lambda_N$ and all the symmetric polynomials arise in this way. The proof of Proposition 1.2 consists of a direct verification that when $\lambda_1+\ldots+\lambda_N \leqslant n$, the expression (1.2) vanishes unless $\lambda=\nu$. The details can be found in [OO1, Section 3].

An explicit formula for the element $C_\nu \in \mathrm{U}(\mathfrak{gl}_N)$ in terms of the generators E_{ij} was given by [N2, Theorem 5.3] and [O, Theorem 1.3]. It generalizes the formula from [C] for C_ν with $\nu=(1,\ldots,1,0,\ldots,0)$ and employs the classical results of Young [Y1,Y2] about the irreducible representations of the symmetric group S_n. Let us recall the relevant results from [Y1,Y2] here.

Let W_ν be the irreducible S_n-module corresponding to the partition ν. We identify the partition ν with its Young diagram. Fix the chain

$$S_1 \subset S_2 \subset \cdots \subset S_n \tag{1.3}$$

of subgroups with the standard embeddings. There is a decomposition of the space W_ν into the direct sum of one-dimensional subspaces associated with this chain. These subspaces are parametrized by the *standard tableaux* of shape ν. Each of these tableaux is a bijective filling of the boxes of the Young diagram ν with the numbers $1,\ldots,n$ such that in every row and column the numbers increase from left to right and from top to bottom respectively. Denote by $\mathcal{T}_\nu$ the set of these tableaux.

For every tableau $T \in \mathcal{T}_\nu$ define a one-dimensional subspace W_T in W_ν as follows. For any $p \in \{1,\ldots,n\}$ take the tableau obtained from T by removing each of the numbers $p+1,\ldots,n$. Let the Young diagram ω be its shape. The subspace W_T is contained in an irreducible S_p-submodule of W_ν corresponding to ω. Any basis of W_ν formed by vectors $w_T \in W_T$ is called a *Young basis*. Fix an S_n-invariant inner product $\langle\ ,\ \rangle_\nu$ in W_ν. The subspaces W_T are then pairwise orthogonal. We shall be assuming that $\langle w_T, w_T\rangle_\nu = 1$ for each tableau $T \in \mathcal{T}_\nu$.

For any tableau $T \in \mathcal{T}_\nu$ consider the normalized diagonal matrix element of the S_n-module W_ν corresponding to the vector w_T

$$\Phi_T = \frac{\dim W_\nu}{n!} \sum_{\sigma\in S_n} \langle w_T, \sigma\cdot w_T\rangle_\nu\, \sigma \in \mathbb{C}\cdot S_n. \tag{1.4}$$

There is an explicit formula for this element of the group ring $\mathbb{C}\cdot S_n$. This formula is the most simple when $T\in\mathcal{T}_\nu$ is the *column tableau.* This tableau is obtained by filling the boxes of the diagram ν with $1,\ldots,n$ by columns from left to right, downwards in each column. We shall denote this tableau by T_c. Let S_ν and S'_ν be the subgroups in S_n preserving the collections of numbers appearing respectively in every row and column of the tableau T_c. Take the elements of the group ring $\mathbb{C}\cdot S_n$

$$\Theta_\nu=\sum_{\sigma\in S_\nu}\sigma \quad\text{and}\quad \Theta'_\nu=\sum_{\sigma\in S'_\nu}\sigma\cdot\operatorname{sgn}\sigma\,.$$

As usual, we denote by $\nu'_1,\nu'_2,\,\ldots$ the column lenghts of the diagram ν. Then by [Y1]

$$\Phi_{T_c}=\frac{\dim W_\nu}{n!}\cdot\frac{\Theta'_\nu\Theta_\nu\Theta'_\nu}{\nu'_1!\,\nu'_2!\,\ldots}\;.$$

There is an alternative description of the one-dimensional subspace W_T in W_ν due to Jucys [J]. Consider the sum of transpositions

$$z_p=(1,p)+(2,p)+\ldots+(p-1,p)\,\in\,\mathbb{C}\cdot S_n\,.$$

The elements $z_1,\ldots,z_n\in\mathbb{C}\cdot S_n$ are called the *Jucys-Murphy elements* corresponding to the standard chain (1.3). They pairwise commute. Fix a tableau $T\in\mathcal{T}_\nu$. For every $r=1,\ldots,n$ put $c_p=k-l$ if the number p appears in the k-th column and l-th row of the tableau T. The number c_p is called the *content* of the box of the diagram ν occupied by p. Here on the left we show the column tableau of shape $\nu=(4,3,1)$:

1	4	6	8
2	5	7	
3			

0	1	2	3
-1	0	1	
-2			

On the right we have indicated the contents of the boxes of the Young diagram $\nu=(4,3,1)$. So here we get $(c_1,\ldots,c_8)=(0,\text{-}1,\text{-}2,1,0,2,1,3)$. Observe that the standard tableau $T\in\mathcal{T}_\nu$ can be always recovered from the sequence of contents $c_1,\ldots,c_n$. The next lemma is contained in [J].

Lemma 1.3. *We have $z_p\cdot w_T=c_p w_T$ in W_ν for any $p=1,\ldots,n$.*

Let us now reproduce the explicit formula from [N2,O] for the element $C_\nu\in\mathrm{U}(\mathfrak{gl}_N)$. Consider the permutational action of the symmetric group S_n in the tensor product $(\mathbb{C}^N)^{\otimes n}$. Denote by Y_T the linear operator in $(\mathbb{C}^N)^{\otimes n}$ corresponding of the element (1.4). The image of this operator is equivalent to U_ν as a $\mathfrak{gl}_N$-module, see [W, Section IV.4]. Moreover, by definition we have the equality $Y_T^2 = Y_T$.

Further, denote by ι_p the embedding of the algebra $\mathrm{End}(\mathbb{C}^N)$ into the tensor product $\mathrm{End}(\mathbb{C}^N)^{\otimes n}$ as the p-th tensor factor:

$$\iota_p(X) = 1^{\otimes (p-1)} \otimes X \otimes 1^{\otimes (n-p)}; \qquad p = 1, \ldots, n. \tag{1.5}$$

We will use this notation throughout the present article. Now put

$$E(u) = -u + \sum_{ij} E_{ij} \otimes E_{ji} \in \mathrm{End}(\mathbb{C}^N) \otimes \mathrm{U}(\mathfrak{gl}_N)[u], \tag{1.6}$$

$$E_p(u) = (\iota_p \otimes \mathrm{id})\big(E(u)\big) \in \mathrm{End}(\mathbb{C}^N)^{\otimes n} \otimes \mathrm{U}(\mathfrak{g})[u]. \tag{1.7}$$

Let $\mathrm{tr} : \mathrm{End}(\mathbb{C}^N) \to \mathbb{C}$ be the usual matrix trace, so that $\mathrm{tr}(E_{ij})$ equals the Kronecker delta δ_{ij}. Now consider the product

$$E_1(u_1) \cdots E_n(u_n) \in \mathrm{End}(\mathbb{C}^N)^{\otimes n} \otimes \mathrm{U}(\mathfrak{gl}_N)[u_1, \ldots, u_n].$$

Theorem 1.4. *For any standard tableau $T \in \mathcal{T}_\nu$ we have*

$$C_\nu = (\mathrm{tr}^{\otimes n} \otimes \mathrm{id})\big(Y_T \otimes 1 \cdot E_1(c_1) \cdots E_n(c_n)\big). \tag{1.8}$$

The proofs of this theorem given in [N2,O] were rather involved. A more elegant proof was later found by Molev [M2, Theorem 8.2]. All these results were based on the notion of a fusion procedure introduced by Cherednik in [C2]. We keep using this notion in the present article.

Consider again the group ring $\mathbb{C} \cdot S_n$. For every two distinct indices $p, q = 1, \ldots, n$ introduce the rational function of two complex variables u, v valued in $\mathbb{C} \cdot S_n$

$$\varphi_{pq}(u, v) = 1 - \frac{(p, q)}{u - v}.$$

As direct calculations show, these rational functions satisfy the equations

$$\varphi_{pq}(u,v)\, \varphi_{pr}(u,w)\, \varphi_{qr}(v,w) = \varphi_{qr}(v,w)\, \varphi_{pr}(u,w)\, \varphi_{pq}(u,v) \tag{1.9}$$

for all pairwise distinct indices p, q, r. Consider the rational function of u, v, w appearing at either side of (1.9). The factor $\varphi_{pr}(u,w)$ in (1.9) has a pole at $u = w$. However, we have the following lemma.

Lemma 1.5. *The restriction of* (1.9) *to the set of all* (u, v, w) *such that $v = w \pm 1$, is regular at $u = w$.*

Proof. Under the condition $v = w \pm 1$ the rational function (1.9) can be written as

$$\Big(1 - \frac{(p,q)+(p,r)}{u-w\mp 1}\Big)\cdot\big(1\mp(q,r)\big)$$

which is a rational function of u,w manifestly regular at $u=w$ $\square$

Using Lemma 1.5 one can prove the next proposition, for details see [N2, Proposition 2.12]. Let the superscript $^\vee$ denote the group embedding $S_n \to S_{n+1}$ determined by the assignment $(p,q)\mapsto(p+1,q+1)$.

Proposition 1.6. *We have the identity in the algebra* $\mathbb{C}\cdot S_{n+1}(u)$

$$\Big(1-\sum_{p=1}^{n}\frac{(1,p+1)}{u}\Big)\cdot\Phi_T^\vee = \varphi_{12}(u,c_1)\cdots\varphi_{1,n+1}(u,c_n)\cdot\Phi_T^\vee\,.$$

The proof of the next proposition is similar and will be also omitted.

Proposition 1.7. *We have the identity in the algebra* $\mathbb{C}\cdot S_{n+1}(u)$

$$\Big(1+\sum_{p=1}^{n}\frac{(p,n+1)}{u}\Big)\cdot\Phi_T = \varphi_{1,n+1}(-c_1,u)\cdots\varphi_{n,n+1}(-c_n,u)\cdot\Phi_T\,.$$

We will also use an alternative definition of the element $\Phi_T\in\mathbb{C}\cdot S_n$ due to Cherednik [C2]. Suppose the numbers $1,\dots,n$ appear respectively in the rows $l_1,\dots,l_n$ of the standard tableau T. Order the set of all pairs p,q with $1\leqslant p<q\leqslant n$ lexicographically.

Theorem 1.8. *The rational function of u defined as the ordered product in $\mathbb{C}\cdot S_n(u)$ of the elements $\varphi_{pq}(c_p+l_pu,c_q+l_qu)$ over the pairs p,q is regular at $u=0$, and takes at $u=0$ the value $\Phi_T\cdot n!/\dim W_\nu$.*

One can prove this theorem by again using Lemma 1.5. This proof is contained in [N2, Section 2]. Another proof can be found in [JKMO].

We will close this section with a generalization of Theorem 1.4. Let us consider for a standard tableau $T\in\mathcal{T}_\nu$ the element of $\mathrm{U}(\mathfrak{gl}_N)[u]$

$$(1.10)\qquad (\mathrm{tr}^{\otimes n}\otimes\mathrm{id})\big(Y_T\otimes 1\cdot E_1(u+c_1)\cdots E_n(u+c_n)\big)\,.$$

Corollary 1.9. *The element* (1.10) *belongs to* $\mathrm{Z}(\mathfrak{gl}_N)[u]$ *and does not depend on the choice of a tableau $T\in\mathcal{T}_\nu$. The eigenvalue of* (1.10) *in the irreducible $\mathfrak{gl}_N$-module U_λ is*

$$s_\nu(\lambda_1-u+N-1,\lambda_2-u+N-2,\dots,\lambda_N-u\mid 0,1,2,\dots)\,.$$

Proof. For any complex value of the parameter u consider the automorphism of the unital algebra $\mathrm{U}(\mathfrak{gl}_N)$ determined by the assignment $E_{ij} \mapsto E_{ij} - u \cdot \delta_{ij}$. The element (1.10) can be obtained by applying this automorphism to the left hand side of (1.8), see the definition (1.6). So the first statement of Corollary 1.9 follows from Theorem 1.4. By pulling back the $\mathfrak{gl}_N$-module U_λ through that automorphism we obtain the irreducible $\mathfrak{gl}_N$-module of highest weight $(\lambda_1 - u, \ldots, \lambda_N - u)$. The second statement of Corollary 1.9 now follows from Proposition 1.2 $\square$

The principal aim of this article is to introduce the analogues of the elements (1.10) for the remaining classical Lie algebras $\mathfrak{so}_N$ and $\mathfrak{sp}_N$.

§2. Traceless tensors in the space $(\mathbb{C}^N)^{\otimes m}$

We will regard the orthogonal and symplectic Lie algebras $\mathfrak{so}_N$ and $\mathfrak{sp}_N$ as subalgebras in $\mathfrak{gl}_N$. From now on we will let the indices i, j run through the set $\{-M, \ldots, -1, 1, \ldots, M\}$ if $N = 2M$ and through the set $\{-M, \ldots, -1, 0, 1, \ldots, M\}$ if $N = 2M + 1$. Let e_i be the elements of the standard basis in $\mathbb{C}^N$. We will realize the complex orthogonal group O_N as the subgroup in GL_N preserving the symmetric bilinear form $\langle e_i, e_j \rangle = \delta_{i,-j}$ on $\mathbb{C}^N$. The complex symplectic group Sp_N will be realized as the subgroup in GL_N preserving the alternating form $\langle e_i, e_j \rangle = \delta_{i,-j} \cdot \operatorname{sgn} i$.

Let G be any of the subgroups O_N, Sp_N in GL_N. Denote by $\mathfrak{g}$ the corresponding Lie subalgebra in $\mathfrak{gl}_N$. Put $\varepsilon_{ij} = \operatorname{sgn} i \cdot \operatorname{sgn} j$ if $G = Sp_N$ and $\varepsilon_{ij} = 1$ if $G = O_N$. The Lie subalgebra $\mathfrak{g} \subset \mathfrak{gl}_N$ is then spanned by the elements

$$F_{ij} = E_{ij} - \varepsilon_{ij} \cdot E_{-j,-i}\,.$$

We will also regard F_{ij} as generators of the universal enveloping algebra $\mathrm{U}(\mathfrak{g})$. We choose the Borel subalgebra in $\mathfrak{g}$ spanned by the elements F_{ij} with $i \leqslant j$. Let us fix the basis $F_{-M,-M}, \ldots, F_{-1,-1}$ in the corresponding Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$. Any weight $\mu = (\mu_1, \ldots, \mu_M)$ of $\mathfrak{h}$ will be taken with respect to this basis. The half-sum of the positive roots of $\mathfrak{h}$ is

$$\rho = (\varepsilon + M - 1, \varepsilon + M - 2, \ldots, \varepsilon)$$

where $\varepsilon = 0\,, \frac{1}{2}\,, 1$ for $\mathfrak{g} = \mathfrak{so}_{2M}\,, \mathfrak{so}_{2M+1}\,, \mathfrak{sp}_{2M}$ respectively.

Now we assume that μ is a partition of m with at most M parts. Then μ can be regarded as a dominant weight of $\mathfrak{h}$. Let V_μ be the irreducible $\mathfrak{g}$-module of the highest weight μ. Note that if $\mathfrak{g} = \mathfrak{so}_{2M}$ then $\mu^* = (\mu_1, \ldots, \mu_{M-1}, -\mu_M)$ is again a dominant weight of $\mathfrak{h}$. We will also consider the corresponding irreducible $\mathfrak{so}_{2M}$-module V_{μ^*}. It is obtained from the module V_μ via the conjugation in $\mathfrak{so}_{2M} \subset \mathrm{End}(\mathbb{C}^N)$ by

$$E_{1,-1} + E_{-1,1} + E_{22} + E_{-2,-2} + \ldots + E_{MM} + E_{-M,-M} \in O_N\,.$$

All these irreducible $\mathfrak{g}$-modules appear in the decomposition of the m-th tensor power of the identity $\mathfrak{g}$-module $\mathbb{C}^N$. Take any two distinct indices p and q from the set $\{1, \ldots, m\}$. By applying the G-invariant bilinear form $\langle\ ,\ \rangle$ on $\mathbb{C}^N$ to an element $t \in (\mathbb{C}^N)^{\otimes m}$ in the p-th and q-th tensor factors we obtain a certain element $t' \in (\mathbb{C}^N)^{\otimes (m-2)}$. Then the element t is called *traceless* if $t' = 0$ for all possible indices $p \neq q$.

Now fix any embedding of the irreducible $\mathfrak{gl}_N$-module U_μ to $(\mathbb{C}^N)^{\otimes m}$. The subspace $U_\mu \cap V$ in U_μ is preserved by the action of the subalgebra $\mathfrak{g} \subset \mathfrak{gl}_N$. For $\mathfrak{g} = \mathfrak{so}_{2M+1}, \mathfrak{sp}_{2M}$ this subspace is isomorphic to V_μ as $\mathfrak{g}$-module. For $\mathfrak{g} = \mathfrak{so}_{2M}$ it is isomorphic to V_μ only if $\mu_M = 0$. Otherwise $U_\mu \cap V$ splits into the direct sum of the $\mathfrak{so}_{2M}$-modules V_μ and V_{μ^*}. All these statements are contained in [W, Section V.9].

We denote by $\mathrm{Z}(\mathfrak{g})$ the ring of invariants in the universal enveloping algebra $\mathrm{U}(\mathfrak{g})$ with respect to the adjoint action of the group G. The ring $\mathrm{Z}(\mathfrak{g})$ coincides with the centre of $\mathrm{U}(\mathfrak{g})$ for $\mathfrak{g} = \mathfrak{so}_{2M+1}, \mathfrak{sp}_{2M}$ but is strictly contained in the centre of $\mathrm{U}(\mathfrak{g})$ when $\mathfrak{g} = \mathfrak{so}_{2M}$. Then any element of $\mathrm{Z}(\mathfrak{so}_{2M})$ acts in the irreducible $\mathfrak{so}_{2M}$-modules V_μ and V_{μ^*} by the same scalars.

There is a distinguished basis in the vector space $\mathrm{Z}(\mathfrak{g})$ analogous to the basis of the Capelli elements C_ν in $\mathrm{Z}(\mathfrak{gl}_N)$. This basis is labelled by the partitions μ and was introduced by Okounkov and Olshanski by generalizing Proposition 1.1. The element B_μ of this basis is determined up to multipler from $\mathbb{C}$ by the next proposition [OO2, Theorem 2.3]. Consider the canonical ascending filtration on the algebra $\mathrm{U}(\mathfrak{g})$. With respect to this filtration the subspace $\mathfrak{g} \subset \mathrm{U}(\mathfrak{g})$ has degree one.

Proposition 2.1. *There exists an element B_μ in $\mathrm{Z}(\mathfrak{g})$ of degree at most $2m$ such that for any partition $\lambda = (\lambda_1, \ldots, \lambda_M)$ of not more than m we have $B_\mu \cdot V_\lambda \neq \{0\}$ if and only if $\lambda = \mu$.*

Explicit formula for the eigenvalue of the element $B_\mu \in \mathrm{Z}(\mathfrak{g})$ in the irreducible $\mathfrak{g}$-module V_λ for any λ and μ has been also given in [OO2]. We will reproduce this formula, it fixes the multiplier from $\mathbb{C}$ up to which the element $B_\mu \in \mathrm{Z}(\mathfrak{g})$ is determined by Proposition 2.1. This formula again employs the definition (1.1).

Proposition 2.2. *The element $B_\mu \in \mathrm{Z}(\mathfrak{g})$ can be chosen so that* *its eigenvalues in the irreducible* $\mathfrak{g}$*-modules* V_λ *are respectively*

$$s_\mu\big((\lambda_1+\rho_1)^2,\ldots,(\lambda_M+\rho_M)^2\mid\varepsilon^2,(\varepsilon+1)^2,\ldots\big).$$

The proof of this proposition does not differ significantly from that of Proposition 1.2. For details see [OO2, Theorem 2.5]. A certain explicit expression for the element $B_\mu\in\mathrm{U}(\mathfrak{g})$ in terms of the generators F_{ij} has been recently given by Olshanski in [O2]. This is an analogue of the expression [OO1, Theorem 14.1] for the element $C_\nu\in\mathrm{U}(\mathfrak{gl}_N)$ which is more complicated than (1.8). An analogue of the formula (1.8) for B_μ with the general partition μ is unknown. For $\mu=(1,\ldots,1,0,\ldots,0)$ and $\mu=(m,0,\ldots,0)$ this analogue was given in [MN]. In the present article we will consider a natural generalization of the construction [MN]. But in general it yields elements of the ring $\mathrm{Z}(\mathfrak{g})$ different from B_μ.

Similarly to (1.5), for any element $X\in\mathrm{End}(\mathbb{C}^N)^{\otimes 2}$ and any two distinct indices $p,q\in\{1,\ldots,n\}$ with fixed n we will denote

$$X_{pq}=(\iota_p\otimes\iota_q)(X)\in\mathrm{End}(\mathbb{C}^N)^{\otimes n}.$$

Along with Lemma 1.5, we will use one more simple observation. Denote

$$F(u)=-u-\eta+\sum_{ij}E_{ij}\otimes F_{ji}\in\mathrm{End}(\mathbb{C}^N)\otimes\mathrm{U}(\mathfrak{gl}_N)[u] \tag{2.1}$$

where we set $\eta=\frac{1}{2},-\frac{1}{2}$ for $\mathfrak{g}=\mathfrak{so}_N,\mathfrak{sp}_N$ respectively. Let

$$\tilde{E}(u)=-u+\sum_{ij}\varepsilon_{ij}\cdot E_{ij}\otimes E_{-i,-j}\in\mathrm{End}(\mathbb{C}^N)\otimes\mathrm{U}(\mathfrak{gl}_N)[u]$$

be the element obtained from $E(u)$ by applying the transposition with respect to the bilinear $\langle\ ,\ \rangle$ in the tensor factor $\mathrm{End}(\mathbb{C}^N)$. Now consider

$$\frac{\tilde{E}(\eta-u)\,E(\eta+u)}{u-\eta}\in\mathrm{End}(\mathbb{C}^N)\otimes\mathrm{U}(\mathfrak{gl}_N)(u). \tag{2.2}$$

We have the standard representation $\mathrm{U}(\mathfrak{gl}_N)\to\mathrm{End}(\mathbb{C}^N)^{\otimes m}$ which makes the element (2.2) acting in the space $(\mathbb{C}^N)^{\otimes(m+1)}$. The element $F(u)\in\mathrm{End}(\mathbb{C}^N)\otimes\mathrm{U}(\mathfrak{g})[u]$ also acts in the space $(\mathbb{C}^N)^{\otimes(m+1)}$ and the latter action preserves the subspace $\mathbb{C}^N\otimes V$. Here is a simple lemma.

Lemma 2.3. *Action of the element* (2.2) *in the space* $(\mathbb{C}^N)^{\otimes(m+1)}$ *preserves the subspace* $\mathbb{C}^N\otimes V$. *The action of the element* $F(u)$ *in this subspace coincides with the action of* (2.2).

Proof. Consider the elements of the algebra $\mathrm{End}\,(\mathbb{C}^N)^{\otimes 2}$

$$P = \sum_{ij} E_{ij} \otimes E_{ji} \qquad \text{and} \qquad Q = \sum_{ij} \varepsilon_{ij} \cdot E_{ij} \otimes E_{-i,-j}\,.$$

The element P corresponds to the exchange operator $e_i \otimes e_j \mapsto e_j \otimes e_i$ in $(\mathbb{C}^N)^{\otimes 2}$. The element Q is obtained from P by applying to either tensor factor of $\mathrm{End}\,(\mathbb{C}^N)^{\otimes 2}$ transposition with respect to $\langle\ ,\ \rangle$. Observe that

$$PQ = QP = \begin{cases} \ \ Q & \text{if } \mathfrak{g} = \mathfrak{so}_N\,, \\ -Q & \text{if } \mathfrak{g} = \mathfrak{sp}_N\,. \end{cases} \tag{2.3}$$

Further, by the definition of a traceless tensor $t \in \mathrm{End}\,(\mathbb{C}^N)^{\otimes m}$ we have the equality $Q_{pq}\,t = 0$ for any two distinct indices $p,q \in \{1,\ldots,m\}$.

By definition the image of the element $F(u) \in \mathrm{End}\,(\mathbb{C}^N) \otimes \mathrm{U}(\mathfrak{g})\,[u]$ in $\mathrm{End}\,(\mathbb{C}^N)^{\otimes(m+1)}\,[u]$ under the representation $\mathrm{U}(\mathfrak{g}) \to \mathrm{End}\,(\mathbb{C}^N)^{\otimes m}$ is

$$P_{12} + \ldots + P_{1,m+1} - Q_{12} - \ldots - Q_{1,m+1} - u - \eta\,. \tag{2.4}$$

On the other hand, the image of the element (2.2) in $\mathrm{End}\,(\mathbb{C}^N)^{\otimes(m+1)}\,(u)$ under the representation $\mathrm{U}(\mathfrak{gl}_N) \to \mathrm{End}\,(\mathbb{C}^N)^{\otimes m}$ is the product

$$\left(1 + \frac{Q_{12} + \ldots + Q_{1,m+1}}{u - \eta}\right) \cdot \big(P_{12} + \ldots + P_{1,m+1} - u - \eta\big)\,.$$

By (2.3) and the definition of η this product equals (2.4) plus the sum

$$\sum_{p \neq q} \frac{Q_{1,q+1}\,P_{1,p+1}}{u - \eta} = \sum_{p \neq q} \frac{P_{1,p+1}\,Q_{p+1,q+1}}{u - \eta}\,.$$

But the action of the latter sum in $\mathbb{C}^N \otimes V$ is identically zero $\square$

Using this lemma we can easily prove the following proposition. It is a particular case of a more general result from [O1]. Denote

$$R(u,v) = 1 - \frac{P}{u - v} \qquad \text{and} \qquad \tilde{R}(u,v) = 1 + \frac{Q}{u + v}$$

in $\mathrm{End}\,(\mathbb{C}^N)^{\otimes 2}(u,v)$. The first of these two functions is the *rational Yang R-matrix*. For any two distinct indices $p,q \in \{1,\ldots,n\}$ the element $R_{pq}(u,v) \in \mathrm{End}\,(\mathbb{C}^N)^{\otimes n}(u,v)$ corresponds to $\varphi_{pq}(u,v) \in \mathbb{C} \cdot S_n(u,v)$ under the permutational action of the symmetric group S_n in $(\mathbb{C})^{\otimes n}$.

Similarly to (1.7), for any fixed n and every index $p = 1,\ldots,n$ let $F_p(u) \in \mathrm{End}\,(\mathbb{C}^N)^{\otimes n} \otimes \mathrm{U}(\mathfrak{g})[u]$ and $\tilde{E}_p(u) \in \mathrm{End}\,(\mathbb{C}^N)^{\otimes n} \otimes \mathrm{U}(\mathfrak{gl}_N)[u]$

be the images of $F(u)$ and $\tilde{E}(u)$ with respect to the embedding $\iota_p \otimes \mathrm{id}$. In the equations (2.5) to (2.9) below we will write $R(u,v)$ and $\tilde{R}(u,v)$ instead of $R(u,v)\otimes 1$ and $\tilde{R}(u,v)\otimes 1$ in $\mathrm{End}(\mathbb{C}^N)^{\otimes 2}\otimes \mathrm{U}(\mathfrak{g})$ for short.

Proposition 2.4. *We have the relation in* $\mathrm{End}(\mathbb{C}^N)^{\otimes 2}\otimes\mathrm{U}(\mathfrak{g})\,(u,v)$

$$R(u,v)\,F_1(u)\,\tilde{R}(u,v)\,F_2(v) = F_2(v)\,\tilde{R}(u,v)\,F_1(u)\,R(u,v)\,. \tag{2.5}$$

Proof. This proposition can be verified by direct calculation. Here we will give a conceptual proof which goes back to the origin [C2, S2] of the *reflection equation* (2.5). In the algebra $\mathrm{End}(\mathbb{C}^N)^{\otimes 2}\otimes \mathrm{U}(\mathfrak{gl}_N)\,(u,v)$

$$R(u,v)\,E_1(u)\,E_2(v) = E_2(v)\,E_1(u)\,R(u,v)\,, \tag{2.6}$$

$$R(u,v)\,\tilde{E}_1(-u)\,\tilde{E}_2(-v) = \tilde{E}_2(-v)\,\tilde{E}_1(-u)\,R(u,v)\,, \tag{2.7}$$

$$\tilde{E}_1(-u)\,\tilde{R}(u,v)\,E_2(v) = E_2(v)\,\tilde{R}(u,v)\,\tilde{E}_1(-u)\,, \tag{2.8}$$

$$E_1(u)\,\tilde{R}(u,v)\,\tilde{E}_2(-v) = \tilde{E}_2(-v)\,\tilde{R}(u,v)\,E_1(u)\,. \tag{2.9}$$

The relation (2.6) is well known and can be easily verified. The relation (2.7) is obtained from (2.6) by applying in the tensor factor $\mathrm{U}(\mathfrak{gl}_N)$ the automorphism $E_{ij}\mapsto -\varepsilon_{ij}\cdot E_{-j,-i}$. Applying to (2.6) transposition with respect to $\langle\ ,\ \rangle$ in the first tensor factor of $\mathrm{End}(\mathbb{C}^N)^{\otimes 2}$ we obtain (2.8). By applying to (2.7) transposition with respect to $\langle\ ,\ \rangle$ in the second tensor factor of $\mathrm{End}(\mathbb{C}^N)^{\otimes 2}$ we obtain (2.9).

Using (2.6) to (2.9) we get the equality in $\mathrm{End}(\mathbb{C}^N)^{\otimes 2}\otimes\mathrm{U}(\mathfrak{gl}_N)\,(u,v)$

$$R(u,v)\,\tilde{E}_1(\eta-u)(u)\,E_1(\eta+u)\,\tilde{R}(u,v)\,\tilde{E}_2(\eta-v)\,E_2(\eta+v) =$$
$$\tilde{E}_2(\eta-v)\,E_2(\eta+v)\,\tilde{R}(u,v)\,\tilde{E}_1(\eta-u)(u)\,E_1(\eta+u)\,R(u,v)\,.$$

The intersection of the kernels of all the representations $\mathrm{U}(\mathfrak{g})\mapsto \mathrm{End}\,V$ for $n=1,2,\ldots$ is zero [D, Theorem 2.5.7], therefore (2.5) follows from the above equality in $\mathrm{End}(\mathbb{C}^N)^{\otimes 2}\otimes \mathrm{U}(\mathfrak{gl}_N)\,(u,v)$ by Lemma 2.3 □

We will now introduce the main object of our study in this article. Let ν be any partition of n with at most N parts. Let T be any standard tableau of shape ν. It determines the sequence of contents $c_1,\ldots,c_n$. Consider the element of the algebra $\mathrm{End}(\mathbb{C}^N)^{\otimes n}\otimes \mathrm{U}(\mathfrak{g})(u)$

$$F_T(u) = (Y_T\otimes 1)\cdot \prod_{p=1}^{n}\Big(1+\frac{Q_{1p}\otimes 1+\ldots+Q_{p-1,p}\otimes 1}{2u+c_p}\Big)\,F_p(u+c_p)$$

where the (noncommuting) factors corresponding to $s=1,\ldots,n$ are arranged from the left to right. For example, for each of the partitions

$\nu = (2)$ and $\nu = (1,1)$ there is only one standard tableau of shape ν. For these partitions we get the elements of the algebra $\operatorname{End}(\mathbb{C}^N)^{\otimes 2} \otimes \mathrm{U}(\mathfrak{g})(u)$

$$F_T(u) = \bigl(1 \pm P \otimes 1\bigr) \cdot F_1(u) \cdot \Bigl(1 + \frac{Q \otimes 1}{2u \pm 1}\Bigr) \cdot F_2(u \pm 1)$$

respectively. Our main object of study is the rational function of u

$$Z_\nu(u) = (\operatorname{tr}^{\otimes n} \otimes \operatorname{id})\bigl(F_T(u)\bigr) \tag{2.10}$$

which by definition takes values in $\mathrm{U}(\mathfrak{g})$, cf. (1.10). As we will show later, this function does not depend on the choice of the tableau $T \in \mathcal{T}_\nu$.

Proposition 2.5. *The function $Z_\nu(u)$ takes values in the ring* $\mathrm{Z}(\mathfrak{g})$.

Proof. We regard the group G as a subgroup in $GL_N \subset \operatorname{End}(\mathbb{C}^N)$. Consider the adjoint action ad of the group G in the enveloping algebra $\mathrm{U}(\mathfrak{g})$. Observe that by the definition (2.1) for any element $g \in G$

$$(\operatorname{id} \otimes \operatorname{ad} g)\bigl(F(u)\bigr) = g \otimes 1 \cdot F(u) \cdot g^{-1} \otimes 1\,.$$

Elements $Y_T, Q_{1p}, \ldots, Q_{p-1,p} \in \operatorname{End}(\mathbb{C}^N)^{\otimes n}$ commute with $g^{\otimes n}$. So

$$(\operatorname{id} \otimes \operatorname{ad} g)\bigl(F_T(u)\bigr) = \bigl(g^{\otimes n} \otimes 1\bigr) \cdot F_T(u) \cdot \bigl((g^{-1})^{\otimes n} \otimes 1\bigr)\,.$$

Hence

$$(\operatorname{tr}^{\otimes n} \otimes \operatorname{ad} g)\bigl(F_T(u)\bigr) = (\operatorname{tr}^{\otimes n} \otimes \operatorname{id})\bigl(F_T(u)\bigr) \quad \square$$

We need one more formula for the element $F_T(u)$. It has motivated our definition of $Z_\nu(u)$. We will keep to the convention used in the definition of $F_T(u)$: in any product over a certain index the noncommuting factors are arranged from the left to the right, as this index increases.

Proposition 2.6. *Element $F_T(u) \in \operatorname{End}(\mathbb{C}^N)^{\otimes n} \otimes \mathrm{U}(\mathfrak{g})(u)$ equals*

$$(Y_T \otimes 1) \cdot \prod_{p=1}^{n} \Bigl(\prod_{q=1}^{p-1} \tilde{R}_{qp}(u + c_q, u + c_p) \otimes 1\Bigr) F_p(u + c_p)\,.$$

Proof. We use the induction on n. In the case $n = 1$ the required equality is tautological. Assume we have the required equality for some partition ν of $n \geqslant 1$. Take any standard tableau U with $n+1$ boxes and not more than N rows, such that by removing the box with number $n+1$ we get T. Let c be the content of the removed box. Consider the projector $Y_U \in \mathrm{End}(\mathbb{C}^N)^{\otimes(n+1)}$. It is divisible on the right by $Y_T \otimes \mathrm{id}$. So by definition the element $F_T(u) \in \mathrm{End}(\mathbb{C}^N)^{\otimes(n+1)} \otimes \mathrm{U}(\mathfrak{g})(u)$ equals

$$(Y_U \otimes 1) \cdot F_T(u) \cdot \Big(1 + \frac{Q_{1,n+1} \otimes 1 + \ldots + Q_{n,n+1} \otimes 1}{2u + c}\Big) \cdot F_{n+1}(u + c)$$

But the element $F_T(u) \in \mathrm{End}(\mathbb{C}^N)^{\otimes n} \otimes \mathrm{U}(\mathfrak{g})(u)$ is divisible on the right by $Y_T \otimes 1$. This follows from Theorem 1.8 and Proposition 2.4, see also [MNO, Section 4.2]. The alternative expression for $F_U(u)$ is now provided by inductive assumption and by the identity in $\mathrm{End}(\mathbb{C}^N)^{\otimes(n+1)}(v)$

$$\begin{aligned}(2.11)\qquad &(Y_T \otimes \mathrm{id}) \cdot \big(1 + (Q_{1,n+1} + \ldots + Q_{n,n+1})/v\big) = \\ &(Y_T \otimes \mathrm{id}) \cdot \tilde{R}_{1,n+1}(c_1, v) \cdots \tilde{R}_{n,n+1}(c_n, v)\end{aligned}$$

with $v = 2u + c$. Let us verify this identity. By applying to both sides of the equation (2.11) the transposition with respect to $\langle\ ,\ \rangle$ in each of the first n tensor factors in $\mathrm{End}(\mathbb{C}^N)^{\otimes(n+1)}$ we get

$$\begin{aligned}(2.12)\qquad &\big(1 + (P_{1,n+1} + \ldots + P_{n,n+1})/v\big) \cdot (Y_T \otimes \mathrm{id}) = \\ &R_{1,n+1}(-c_1, v) \cdots R_{n,n+1}(-c_n, v) \cdot (Y_T \otimes \mathrm{id})\,.\end{aligned}$$

We used the fact that the element $Y_T \otimes \mathrm{id} \in \mathrm{End}(\mathbb{C}^N)^{\otimes(n+1)}$ is invariant under this transposition. But (2.12) is provided by Proposition 1.7 □

Theorem 2.7. *$Z_\nu(u)$ does not depend on the choice of $T \in \mathcal{T}_\nu$.*

Proof. Any standard tableau of the shape ν can be obtained from the column tableau T_c by a chain of transformations $T \mapsto T'$ where the entries of the tableaux $T, T' \in \mathcal{T}_\nu$ differ by a single transposition $(r, r+1)$ such that $l_r > l_{r+1}$ for the tableau T. Let $c_1', \ldots, c_n'$ be the sequence of contents of the tableau T'. It is obtained from the sequence $c_1, \ldots, c_n$ by exchanging the terms c_r and c_{r+1}. Note that here we have $|c_r - c_{r+1}| > 1$, put $d = (c_r - c_{r+1})^{-1}$. Due to [Y2, Theorem IV] we have the relation

$$(2.13)\qquad \Phi_{T'} = \big((r, r+1) + d\big)\, \frac{\Phi_T}{1 - d^2}\, \big((r, r+1) + d\big)$$

in the group ring $\mathbb{C} \cdot S_n$, see the definition (1.4). Let X be the product $PR(c_{r+1}, c_r) = P + d$ in $\mathrm{End}(\mathbb{C}^N)^{\otimes 2}$, then the relation (2.13) implies

$$(2.14)\qquad Y_{T'} = X_{r,r+1}\, \frac{Y_T}{1 - d^2}\, X_{r,r+1}$$

in $\operatorname{End}(\mathbb{C}^N)^{\otimes n}$. On the other hand, by using Proposition 2.4 we obtain

$$X_{r,r+1}\cdot\prod_{p=1}^{n}\left(\prod_{q=1}^{p-1}\tilde{R}_{1p}(u+c_q',u+c_p')\otimes 1\right)F_p(u+c_p')=$$

$$\prod_{p=1}^{n}\left(\prod_{q=1}^{p-1}\tilde{R}_{1p}(u+c_q,u+c_p)\otimes 1\right)F_p(u+c_p)\cdot X_{r,r+1}\tag{2.15}$$

in $\operatorname{End}(\mathbb{C}^N)^{\otimes n}\otimes \mathrm{U}(\mathfrak{g})(u)$. Combining the relations (2.14),(2.15) we get

$$F_{T'}(u)=\bigl(X_{r,r+1}\otimes 1\bigr)\,\frac{F_T(u)}{1-d^2}\,\bigl(X_{r,r+1}\otimes 1\bigr)\,.$$

We have the relation $X^2=2dX+1-d^2$ in the algebra $\operatorname{End}(\mathbb{C}^N)^{\otimes 2}$. The vectors w_T and $w_{T'}$ of the Young basis in the S_n-module W_ν are orthogonal, so (2.14) implies the equality $Y_T\,X_{r,r+1}\,Y_T=0$. Therefore

$$Y_T\,X_{r,r+1}^2\,Y_T=(1-d^2)\cdot Y_T\,.$$

But the element $F_T(u)\in\operatorname{End}(\mathbb{C}^N)^{\otimes n}\otimes \mathrm{U}(\mathfrak{g})(u)$ is divisible by $Y_T\otimes 1$ on the right as well as on the left, see the proof of Proposition 2.6. Thus

$$(\mathrm{tr}^{\otimes n}\otimes\mathrm{id})\bigl(F_{T'}(u)\bigr)=(\mathrm{tr}^{\otimes n}\otimes\mathrm{id})\Bigl(\bigl(X_{r,r+1}\otimes 1\bigr)\,\frac{F_T(u)}{1-d^2}\,\bigl(X_{r,r+1}\otimes 1\bigr)\Bigr)$$
$$=(\mathrm{tr}^{\otimes n}\otimes\mathrm{id})\Bigl(\bigl(X_{r,r+1}^2\otimes 1\bigr)\,\frac{F_T(u)}{1-d^2}\Bigr)=(\mathrm{tr}^{\otimes n}\otimes\mathrm{id})\bigl(F_T(u)\bigr)\quad\square$$

§3. Leading terms of the element $Z_\nu(u)$

Throughout this section $\nu=(\nu_1,\ldots,\nu_N)$ will be any partition of n into not more than N parts. However, we will always have $n=2m$. We fixed the basis $F_{-M,-M},\ldots,F_{-1,-1}$ in the Cartan subalgebra $\mathfrak{h}\subset\mathfrak{g}$.

Consider again the standard ascending filtration of the algebra $\mathrm{U}(\mathfrak{g})$

$$\mathrm{U}_0(\mathfrak{g})\subset\mathrm{U}_1(\mathfrak{g})\subset\mathrm{U}_2(\mathfrak{g})\subset\ldots\subset\mathrm{U}(\mathfrak{g})\,.$$

Here $\mathrm{U}_0(\mathfrak{g})=\mathbb{C}\,,\mathrm{U}_1(\mathfrak{g})=\mathfrak{g}$. By definition the subspace $\mathrm{U}_n(\mathfrak{g})\subset\mathrm{U}(\mathfrak{g})$ consists of all the elements with degree not more than n. We will identify the quotient space $\mathrm{U}_n(\mathfrak{g})/\,\mathrm{U}_{n-1}(\mathfrak{g})$ with the subspace in the symmetric algebra $\mathrm{S}(\mathfrak{g})$ consisting of the homogeneous elements of degree n.

By (2.10) we get $Z_\nu(u)\in\mathrm{U}_n(\mathfrak{g})\otimes\mathbb{C}(u)$. The image of $Z_\nu(u)$ in

$$\bigl(\mathrm{U}_n(\mathfrak{g})/\,\mathrm{U}_{n-1}(\mathfrak{g})\bigr)\otimes\mathbb{C}(u)\;\subset\;\mathrm{S}(\mathfrak{g})\otimes\mathbb{C}(u)$$

is a homogeneous polynomial in $F_{ij} \in \mathfrak{g}$ of degree n with the coefficients from $\mathbb{C}(u)$. Due to Proposition 2.5 this polynomial is invariant under the adjoint action of the group G in $\mathrm{S}(\mathfrak{g})$. By the Chevalley theorem [D, Theorem 7.3.5] this polynomial is uniquely determined by its image

$$f_\nu(x_1, \ldots, x_M \,|\, u) \in \mathbb{C}[x_1, \ldots, x_M] \otimes \mathbb{C}(u) \tag{3.1}$$

with respect to the homomorphism $\eta : \mathrm{S}(\mathfrak{g}) \to \mathbb{C}[x_1, \ldots, x_M]$ defined by the assignment $F_{ij} \mapsto 0$ if $i \neq j$ or if $i = j = 0$, and by

$$F_{-M,-M} \mapsto x_1 \,, \ \ldots, \ F_{-1,-1} \mapsto x_M \,.$$

Moreover, the image (3.1) is a symmetric polynomial in $x_1^2, \ldots, x_M^2$. Our present aim is to determine the polynomial (3.1) for any partition ν of $n = 2m$. In particular, we will describe the partitions ν where the polynomial (3.1) is not identically zero.

Denote by Λ_M the ring of symmetric polynomials in $x_1, \ldots, x_M$ with complex coefficients. For any partition $\rho = (\rho_1, \rho_2, \ldots)$ put

$$p_\rho(x_1, \ldots, x_M) = \prod_{k=1}^{\ell(\rho)} \left(x_1^{\rho_k} + \ldots + x_M^{\rho_k}\right) \in \Lambda_M \,.$$

As usual, here $\ell(\rho)$ is the number of non-zero parts in the partition ρ.

We will use some elementary facts from the representation theory of the symmetric group S_{2n}. Consider the hyperoctahedral group H_n as the subgroup in S_{2n} that centralizes the product of transpositions $(1, n+1) \cdots (n, 2n) \in S_{2n}$. Thus $H_n = S_n \ltimes (\mathbb{Z}_2)^n$ where the subgroup $S_n \subset S_{2n}$ acts on $1, \ldots, 2n$ by simultaneous permutations of $1, \ldots, n$ and $n+1, \ldots, 2n$. Here the subgroup $(\mathbb{Z}_2)^n \subset S_{2n}$ is generated by the pairwise commuting transpositions $(1, n+1), \ldots, (n, 2n)$. Consider the one-dimensional representations χ_+ and χ_- of the group H_n which are trivial on its subgroup S_n while $\chi_\pm : (s, n+s) \mapsto \pm 1$ respectively. Take the corresponding minimal idempotents in the group ring $\mathbb{C} \cdot H_n$

$$h_\pm = \frac{1}{n!\, 2^n} \sum_{\sigma \in H_n} \chi_\pm(\sigma)\, \sigma \,.$$

Note that the intersection of $h_-(\mathbb{C} \cdot S_{2n}) h_+$ with $h_+(\mathbb{C} \cdot S_{2n}) h_-$ is zero.

Proposition 3.1. *We can uniquely determine two linear maps*

$$\mathrm{ch} \colon h_-(\mathbb{C} \cdot S_{2n}) h_+ \longrightarrow \Lambda_M \quad \text{and} \quad \mathrm{ch} \colon h_+(\mathbb{C} \cdot S_{2n}) h_- \longrightarrow \Lambda_M$$

by setting

$$\mathrm{ch}(h_- \sigma h_+) = \mathrm{ch}(h_+ \sigma h_-) = p_\rho(x_1^2, \ldots, x_M^2) \cdot 2^{\ell(\rho)}$$

for any permutation σ of $1,\ldots,n$ *with the cycle lengths* $2\rho_1, 2\rho_2, \ldots$.

Proof. Any double coset of the subgroup H_n in S_{2n} contains a permutation that acts on the numbers $n+1,\ldots,2n$ trivially. Moreover, all permutations σ of $1,\ldots,n$ with the same cycle lengths belong to the same double coset. If any of these lengths is odd then $h_-\sigma h_+ = h_+\sigma h_- = 0$. Now for each partition $\rho = (\rho_1, \rho_2, \ldots)$ of m choose a permutation σ with the cycle lengths $2\rho_1, 2\rho_2, \ldots$. All the corresponding elements $h_-\sigma h_+ \in \mathbb{C}\cdot S_{2n}$ are linearly independent. Therefore our definitions of two linear maps ch are self-consistent □

We will call the two linear maps in Proposition 3.1 the *characteristic maps,* see [M,Section VII.2]. Now fix any standard tableau $T \in \mathcal{T}_\nu$ and take the corresponding minimal idempotent $\Phi_T \in \mathbb{C}\cdot S_n$. Regard Φ_T as an element of the group ring $\mathbb{C}\cdot S_{2n}$ where the subgroup $S_n \subset S_{2n}$ acts on the numbers $n+1,\ldots,2n$ trivially. Consider the product

$$\Psi_T(u) = \prod_{p=1}^{n} \Big(1 + \frac{(1,n+p) + \ldots + (p-1,n+p)}{2u + c_p}\Big) \cdot \Phi_T \tag{3.2}$$

in $\mathbb{C}(u)\cdot S_{2n}$ where the (non-commuting) factors corresponding to the indices $p = 1,\ldots,n$ are as usual arranged from the left to the right. Computation of the homogeneous polynomial (3.1) in $x_1,\ldots,x_M$ hinges on the following observation.

Proposition 3.2. *For* $\mathfrak{g} = \mathfrak{so}_N$ *and* $\mathfrak{g} = \mathfrak{sp}_N$ *the polynomials* (3.1) *coincide with the images in* $\Lambda_M \otimes \mathbb{C}(u)$ *of* $h_-\Psi_T(u)\,h_+$ *and* $h_+\Psi_T(u)\,h_-$ *respectively under the characteristic maps.*

Proof. Take the permutational action of the group S_{2n} in the space $(\mathbb{C}^N)^{\otimes 2n}$. Then the image in $\mathrm{End}(\mathbb{C}^N)^{\otimes 2n}(u)$ of $\Psi_T(u)$ is the product

$$\prod_{p=1}^{n} \Big(1 + \frac{P_{1,n+p} + \ldots + P_{p-1,n+p}}{2u + c_p}\Big) \cdot \big(Y_T \otimes \mathrm{id}^{\otimes n}\big)\,.$$

Decompose this product with respect to the standard basis in the space $\mathrm{End}(\mathbb{C}^N)^{\otimes 2n}$. We get the sum

$$\sum_{i_1\ldots i_{2n}} \sum_{j_1\ldots j_{2n}} \psi^{i_1\ldots i_{2n}}_{j_1\ldots j_{2n}}(u) \cdot E_{i_1 j_1} \otimes \cdots \otimes E_{i_{2n} j_{2n}}$$

where the coefficients are certain rational functions of u valued in $\mathbb{C}$.

We will put $\varepsilon_i = \operatorname{sgn} i$ if $\mathfrak{g} = \mathfrak{sp}_N$ and set $\varepsilon_i = 1$ if $\mathfrak{g} = \mathfrak{so}_N$. Then $\varepsilon_{ij} = \varepsilon_i \varepsilon_j$ by definition. Denote

$$I_{ij}(u) = \varepsilon_i \cdot \bigl(F_{j,-i} - (u+\eta)\,\delta_{j,-i}\bigr) \in \mathrm{U}(\mathfrak{g})[u]\,, \qquad J_{ij} = \varepsilon_i \cdot \delta_{i,-j}\,.$$

By the definition of $F_T(u) \in \operatorname{End}(\mathbb{C}^N)^{\otimes n} \otimes \mathrm{U}(\mathfrak{g})(u)$ the element $Z_\nu(u)$ equals

$$\sum_{i_1\ldots i_{2n}} \sum_{j_1\ldots j_{2n}} \psi^{i_1\ldots i_{2n}}_{j_1\ldots j_{2n}}(u) \cdot I_{i_{n+1}i_1}(u+c_1)\, J_{j_{n+1}j_1} \cdots I_{i_{2n}i_n}(u+c_n)\, J_{j_{2n}j_n}$$

where we employed the definition (2.1) and the fact that the elements P, Q are obtained from each other by applying to the second tensor factor of $\operatorname{End}(\mathbb{C}^N)^{\otimes 2}$ the transposition with respect to $\langle\ ,\ \rangle$. This expression for $Z_\nu(u)$ shows that the polynomial $f_\nu(x_1, \ldots, x_M \,|\, u)$ equals the sum

$$\sum_{i_1\ldots i_n} \sum_{j_1\ldots j_n} \psi^{i_1\ldots i_n, -i_1\ldots -i_n}_{j_1\ldots j_n, -j_1\ldots -j_n}(u) \cdot \eta(F_{i_1i_1})\, \varepsilon_{i_1}\varepsilon_{j_1} \cdots \eta(F_{i_ni_n})\, \varepsilon_{i_n}\varepsilon_{j_n}\,.$$

The product $\eta(F_{i_1i_1})\, \varepsilon_{i_1}\varepsilon_{j_1} \cdots \eta(F_{i_ni_n})\, \varepsilon_{i_n}\varepsilon_{j_n}$ is invariant under the permutations of the indices $i_1, \ldots, i_n$ and of the indices $j_1, \ldots, j_n$. For $\mathfrak{g} = \mathfrak{so}_N$ it is also invariant under any substitution $j_p \mapsto -j_p$ with $p = 1, \ldots, n$ but changes the sign under the substitution $i_p \mapsto -i_p$. Inversely, for $\mathfrak{g} = \mathfrak{sp}_N$ this product is invariant under any substitution $i_p \mapsto -i_p$ but changes the sign under the substitution $j_p \mapsto -j_p$.

Now take any permutation of the first n tensor factors in $(\mathbb{C}^N)^{\otimes 2n}$ with the cycle lengths $2\rho_1, 2\rho_2, \ldots$ and decompose it in $\operatorname{End}(\mathbb{C}^N)^{\otimes 2n}$ as

$$\sum_{i_1\ldots i_{2n}} \sum_{j_1\ldots j_{2n}} \delta^{i_1\ldots i_{2n}}_{j_1\ldots j_{2n}} \cdot E_{i_1j_1} \otimes \cdots \otimes E_{i_{2n}j_{2n}}$$

where each of the coefficients equals 0 or 1. It remains to show that

$$\sum_{i_1\ldots i_n} \sum_{j_1\ldots j_n} \delta^{i_1\ldots i_n, -i_1\ldots -i_n}_{j_1\ldots j_n, -j_1\ldots -j_n} \cdot \eta(F_{i_1i_1})\, \varepsilon_{i_1j_1} \cdots \eta(F_{i_ni_n})\, \varepsilon_{i_nj_n} \tag{3.3}$$

then equals $p_\rho(x_1^2, \ldots, x_M^2) \cdot 2^{\ell(\rho)}$ which is evident. Indeed, the latter expression and the sum (3.3) both are multiplicative with respect to the decomposition of our permutation of the first n tensor factors in $(\mathbb{C}^N)^{\otimes 2n}$ into the product of cycles. So we can assume that there is one single cycle of length $n = 2m$. In this case

$$\delta^{i_1\ldots i_n, -i_1\ldots -i_n}_{j_1\ldots j_n, -j_1\ldots -j_n} = \begin{cases} 1 & \text{if } i_1 = j_1 = \ldots = i_n = j_n\,, \\ 0 & \text{otherwise} \end{cases}$$

and (3.3) equals

$$\sum_i \left(\eta(F_{ii})\right)^n = 2\left(x_1^n + \ldots + x_M^n\right) \quad \square$$

Consider the two elements $h_- \Psi_T(u) h_+$ and $h_+ \Psi_T(u) h_-$ of the ring $\mathbb{C}(u) \cdot S_{2n}$. According to Proposition 3.2, the first element corresponds to the case $\mathfrak{g} = \mathfrak{so}_N$ while the second corresponds to $\mathfrak{g} = \mathfrak{sp}_N$. We will evaluate the images of these two elements under the corresponding characteristic maps by studying their actions in irreducible S_{2n}-modules.

Let ω be any partition of $2n$. The irreducible S_{2n}-module W_ω contains a non-zero vector w_+ such that $\sigma \cdot w_+ = \chi_+(\sigma) w_+$ for any $\sigma \in H_n$, if and only if every row of the Young diagram of ω has even length. Then the vector $w_+ \in W_\nu$ is unique up to a scalar multiplier, and we will assume that $\langle w_+ , w_+ \rangle = 1$. The module W_ω contains a non-zero vector w_- with $\sigma \cdot w_- = \chi_-(\sigma) w_-$ for any $\sigma \in H_n$, if and only if every column of ω has even length. The vector $w_- \in W_\nu$ is then unique up to a scalar multiplier, and we will assume that $\langle w_- , w_- \rangle = 1$. All these facts are well known, see for instance [M, Section VII.2].

Now suppose that $\omega = (2\mu_1, 2\mu_1, 2\mu_2, 2\mu_2, \ldots)$ for a certain partition $\mu = (\mu_1, \mu_2, \ldots)$ of m. We do not impose any restriction on the number of parts in μ yet. The partition ω satisfies both conditions above, so we have non-zero vectors $w_+, w_- \in W_\omega$. Let $b_1, \ldots, b_m$ be the contents of the diagram μ ordered arbitrarily.

Let χ_ν be the character of the irreducible S_n-module W_ν, take the element

$$\mathrm{X}_\nu = \frac{1}{n!} \sum_{\sigma \in S_n} \chi_\nu(\sigma)\, \sigma \in \mathbb{C} \cdot S_n \,.$$

We will also regard X_ν as an element of the group ring $\mathbb{C} \cdot S_{2n}$ by using the standard embedding $S_n \to S_{2n}$. Where the double signs $\pm$ and $\mp$ appear in the next proposition, one should simultaneously take only the upper signs or only the lower signs. Recall that in this section $n = 2m$.

Proposition 3.3. *Action of the element* $h_\mp \Psi_T(u) h_\pm \in \mathbb{C}(u) \cdot S_{2n}$ *in the module* W_ω *coincides with the action of* $h_\mp \mathrm{X}_\nu h_\pm \in \mathbb{C} \cdot S_{2n}$ *times*

$$\frac{(u+b_1)(u+b_1 \pm 1/2) \cdots (u+b_m)(u+b_m \pm 1/2)}{(u+c_1/2)(u+c_2/2) \cdots (u+c_{n-1}/2)(u+c_n/2)} \in \mathbb{C}(u)\,. \tag{3.4}$$

Proof. For each $p = 1, \ldots, n$ consider the elements of the ring $\mathbb{C} \cdot S_{2n}$

$$z_p' = \sum_{q=1}^{p-1} (n+q, n+p) \quad \text{and} \quad z_p'' = \sum_{q=1}^{p-1} (n+q, n+p) + (q, n+p)\,.$$

The elements $z'_1, \ldots, z'_n$ are the images in $\mathbb{C} \cdot S_{2n}$ of the Jucys-Murphy elements $z_1, \ldots, z_n \in \mathbb{C} \cdot S_n$ under the embedding

$$S_n \to S_{2n} : (q,p) \mapsto (n+q, n+p)\,. \tag{3.5}$$

In particular, the elements $z'_1, \ldots, z'_n$ pairwise commute. Note that the elements $z''_1, \ldots, z''_n$ also pairwise commute. The definition (3.2) can be now rewritten as

$$\Psi_T(u) \;=\; \prod_{p=1}^{n} \left(1 + \frac{z''_p - z'_p}{2u + c_p}\right) \cdot \Phi_T\,. \tag{3.6}$$

But for any p the element z'_p commutes with each of $z''_{p+1}, \ldots, z''_n$. On the other hand, due to Lemma 1.3 we have the equalities in $\mathbb{C} \cdot S_{2n}$

$$z'_p \Phi_T\, h_\pm = \Phi_T\, z_p\, h_\pm = c_p \cdot \Phi_T\, h_\pm\,.$$

Therefore (3.6) implies the equality in the ring $\mathbb{C}(u) \cdot S_{2n}$

$$\Psi_T(u)\, h_\pm = \frac{(2u + z''_1) \cdots (2u + z''_n)}{(2u + c_1) \cdots (2u + c_n)}\, \Phi_T\, h_\pm\,. \tag{3.7}$$

The standard chain of subgroups (1.3) corresponds to the natural ordering of the numbers $1, \ldots, n$. Now consider the chain of subgroups

$$S_1 \subset S_2 \subset \ldots \subset S_{2n-1} \subset S_{2n}$$

corresponding the ordering $n+1, 1, n+2, 2, \ldots, 2n, n$. The elements $z''_1, \ldots, z''_n \in \mathbb{C} \cdot S_{2n}$ are the Jucys-Murphy elements corresponding to the latter chain with the indices $1, 3, \ldots, 2n-1$. Take the Young basis in W_ν corresponding to this chain of subgroups in S_{2n}. The vectors w_U of this basis are parametrized by standard tableaux U of shape ω with the entries $1, \ldots, 2n$. But by [BG, Theorem 3.4] the vector $w_- \in W_\omega$ is a linear combination of the vectors w_U where $1, 3, \ldots, 2n-1$ occupy the first, third, ... rows of the tableau U. The collection of contents of the boxes in these rows is $2b_1, 2b_1+1, \ldots, 2b_m, 2b_m+1$. By Lemma 1.3 action of $h_-\,(2u + z''_1) \cdots (2u + z''_n)$ in W_ω coincides with the action of

$$(2u + 2b_1)(2u + 2b_1 + 1) \ldots (2u + 2b_m)(2u + 2b_m + 1) \cdot h_-\,.$$

Similarly, the vector $w_+ \in W_\omega$ is a linear combination of the vectors w_U where $1, 3, \ldots, 2n-1$ occupy the first, third, ... columns of the tableau U. The collection of contents of the boxes in these columns is

$2b_1, 2b_1 - 1, \ldots, 2b_m, 2b_m - 1$. Again due to Lemma 1.3 the action of $h_+(2u + z_1'') \cdots (2u + z_n'')$ in W_ω coincides with the action of

$$(2u + 2b_1)(2u + 2b_1 - 1) \ldots (2u + 2b_m)(2u + 2b_m - 1) \cdot h_+ .$$

Thus by (3.7) the action of the element $h_\mp \Psi_T(u) h_\pm$ in the module W_ν coincides with the action of $h_\mp \Phi_T h_\pm$ multiplied by the product (3.4).

To complete the proof of Proposition 3.3 it remains to observe that for any $\sigma \in S_n \subset S_{2n}$ we have $h_\mp \Phi_T h_\pm = h_\mp \sigma \Phi_T \sigma^{-1} h_\pm$. Therefore

$$h_\mp \Phi_T h_\pm = \frac{1}{n!} \sum_{\sigma \in S_n} h_\mp \sigma \Phi_T \sigma^{-1} h_\pm = h_\mp \mathrm{X}_\nu h_\pm \quad \square$$

Let us now formulate the main result of this section. Consider again the ring Λ_N of symmetric polynomials in the variables $y_1, \ldots, y_N$. We assume that $x_1, \ldots, x_M$ are independent of those N variables. Equip the vector space Λ_N with the standard inner product, so that the Schur polynomials $s_\nu(y_1, \ldots, y_N)$ where ν runs through the set of partitions with not more than N parts, constitute an orthonormal basis in Λ_N.

Symmetric polynomial $s_\mu(y_1^2, \ldots, y_N^2)$ is the *plethysm* of the Schur polynomial $s_\mu(y_1, \ldots, y_N)$ with the power sum $y_1^2 + \ldots + y_N^2$. Expand

$$s_\mu(y_1^2, \ldots, y_N^2) = \sum_\nu L_{\mu\nu}\, s_\nu(y_1, \ldots, y_N) \tag{3.8}$$

in Λ_N with respect to the basis of Schur polynomials. The polynomials $p_\rho(y_1, \ldots, y_N)$ form an othogonal basis in Λ_N. If $\rho_1 + \rho_2 + \ldots = m$ and the number of permutations in S_m with the cycle lengths $\rho_1, \rho_2, \ldots$ is $m!/z_\rho$ then the squared norm of $p_\rho(y_1, \ldots, y_N)$ is z_ρ. Further, then

$$s_\mu(y_1, \ldots, y_N) = \sum_\rho \chi_\mu^\rho\, p_\rho(y_1, \ldots, y_N)/z_\rho \tag{3.9}$$

where χ_μ^ρ denotes the value of the irreducible character χ_μ of S_m on a permutation with the cycle lengths $\rho_1, \rho_2, \ldots$. Therefore we have

$$L_{\mu\nu} = \sum_\rho \chi_\mu^\rho \chi_\nu^{2\rho} / z_\rho . \tag{3.10}$$

As usual, we denote $2\rho = (2\rho_1, 2\rho_2, \ldots)$. Note that $z_{2\rho} = 2^{\ell(\rho)} z_\rho$ then.

Combinatorial description of the coefficients $L_{\mu\nu}$ in the expansion (3.8) has been provided in [L, Section 5]. Another description of these coefficients is given by [CL, Theorem 5.3]. In particular, if $L_{\mu\nu} \neq 0$ then the Young diagram of ν can be split into horizontal and vertical blocks of two boxes each. These blocks are called *dominoes*, see [BG].

We put $\eta = \frac{1}{2}$ if $\mathfrak{g} = \mathfrak{so}_N$ and put $\eta = -\frac{1}{2}$ if $\mathfrak{g} = \mathfrak{sp}_N$. Recall that $c_1, \ldots, c_n$ are the contents of a standard tableau T of shape ν. The contents $b_1, \ldots, b_m$ of the boxes of μ have been ordered arbitrarily.

Theorem 3.4. *The polynomial $f_\nu(x_1, \ldots, x_M \,|\, u)$ equals the sum over all partitions μ of m into not more than M parts, of the products*

$$\frac{(u+b_1)(u+b_1+\eta)\cdots(u+b_m)(u+b_m+\eta)}{(u+c_1/2)(u+c_2/2)\cdots(u+c_{n-1}/2)(u+c_n/2)}\, L_{\mu\nu}\, s_\mu(x_1^2, \ldots, x_M^2)\,.$$

Proof. In this proof the upper signs in $\pm$ and $\mp$ correspond to the case $\mathfrak{g} = \mathfrak{so}_N$ while the lower signs correspond to $\mathfrak{g} = \mathfrak{sp}_N$. Initially let μ run through the set of all partitions of m, without any restriction on the number of parts. The elements

$$\Gamma_\mu = \frac{\dim W_\omega}{(2n)!} \sum_{\tau \in S_{2n}} \langle\, w_\mp \,, \tau \cdot w_\pm \,\rangle_\omega \, h_\mp \tau h_\pm \;\in\; \mathbb{C} \cdot S_{2n} \tag{3.11}$$

form a basis in the vector space $h_\mp (\mathbb{C} \cdot S_{2n})\, h_\pm$. Let us expand

$$h_\mp \Psi_T(u)\, h_\pm = \sum_\mu f_{\mu\nu}(u)\, \Gamma_\mu \tag{3.12}$$

with respect to this basis and compute the coefficients $f_{\mu\nu}(u) \in \mathbb{C}(u)$. The element $h_\mp \tau h_\pm$ acts in the S_{2n}-module W_ω as the linear operator $\langle\, \tau \cdot w_\pm \,, w_\mp \,\rangle_\omega E$ where $E : w \mapsto \langle\, w \,, w_\pm \,\rangle_\omega \, w_\mp$ for any vector $w \in W_\omega$.

The element Γ_μ acts as the operator E in the module W_ω and vanishes in any other irreducible S_{2n}-module. Denote by $d_{\mu\nu}(u)$ the rational function (3.4). By Proposition 3.3 and by the definition of X_ν

$$f_{\mu\nu}(u) = \frac{d_{\mu\nu}(u)}{n!} \sum_{\sigma \in S_n} \chi_\nu(\sigma)\, \langle\, \sigma \cdot w_\pm \,, w_\mp \,\rangle_\omega \,. \tag{3.13}$$

Here the factor $\langle\, \sigma \cdot w_\pm \,, w_\mp \,\rangle_\omega$ may be non-zero only if the permutation σ has the cycle lengths $2\rho_1, 2\rho_2, \ldots$ for some partition ρ of m. Then

$$\langle\, \sigma \cdot w_\pm \,, w_\mp \,\rangle_\omega = I_\mu \cdot 2^{\ell(\rho)} \chi_\mu^\rho \tag{3.14}$$

where I_μ depends only on the choice of the vectors $w_+ \,, w_- \in W_\omega$ and

$$|\, I_\mu \,|^2 = \frac{(2n)!}{\dim W_\omega \cdot (2^n n!)^2}\,. \tag{3.15}$$

This result was independently obtained by Ivanov [I, Theorem 3.9] and Rains [R, Corollary 7.6]. Using (3.10) and (3.13) along with this result,

$$f_{\mu\nu}(u) = d_{\mu\nu}(u) I_\mu \cdot \sum_\rho \chi^\rho_\mu \chi^{2\rho}_\nu / z_\rho = d_{\mu\nu}(u) I_\mu L_{\mu\nu} . \tag{3.16}$$

To complete the proof of Theorem 3.4 it now remains to apply the characteristic map to each side of the equality (3.12). By Proposition 3.2 on the left-hand side we get the polynomial $f_\nu(x_1, \dots, x_M \,|\, u)$. There are exactly $(2^n n!)^2/(4^{\ell(\rho)} z_\rho)$ elements in the double coset of the subgroup H_n in S_{2n} containing the permutation of $1, \dots, n$ with the cycle lenghts $2\rho_1, 2\rho_2, \dots$. By the definition (3.11) and again by (3.14),(3.15)

$$\mathrm{ch}(\Gamma_\mu) = I_\mu^{-1} \cdot \sum_\rho \chi^\rho_\mu \, p_\rho(x_1^2, \dots, x_M^2)/z_\rho = I_\mu^{-1} \cdot s_\mu(x_1^2, \dots, x_M^2) ,$$

here we have also used Proposition 3.1 and the classical expansion (3.9). Thus the expression (3.16) for the coefficient in (3.12) shows that

$$f_\nu(x_1, \dots, x_M \,|\, u) = \sum_\mu d_{\mu\nu}(u) \, I_\mu \, L_{\mu\nu} \cdot \mathrm{ch}(\Gamma_\mu) =$$
$$\sum_\mu d_{\mu\nu}(u) \, L_{\mu\nu} \, s_\mu(x_1^2, \dots, x_M^2) .$$

The latter sum can be restricted to the partitions μ with not more than M parts, since for the other partitions we have $s_\mu(x_1^2, \dots, x_M^2) = 0$ □

Corollary 3.5. *If the polynomial* (3.1) *corresponding to* ν *is not identically zero, then the Young diagram of* ν *splits into dominoes.*

One can reformulate Theorem 3.4 as follows, cf. [OO2, Theorem 1.2]. Let us denote by $b_\mu(u)$ and $c_\nu(u)$ the numerator and the denominator of the fraction in (3.4). The upper signs in $b_\mu(u)$ correspond to $\mathfrak{g} = \mathfrak{so}_N$ while the lower signs correspond to $\mathfrak{g} = \mathfrak{sp}_N$.

Corollary 3.6. *For any fixed positive integers* m *and* N *we have*

$$\sum_\nu c_\nu(u) \, f_\nu(x_1, \dots, x_M \,|\, u) \, s_\nu(y_1, \dots, y_N) =$$
$$\sum_\mu b_\mu(u) \, s_\mu(x_1^2, \dots, x_M^2) \, s_\mu(y_1^2, \dots, y_N^2) . \tag{3.17}$$

where ν *and* μ *range respectively over all partitions of* $n = 2m$ *with at most* N *parts and all partitions of* m *with at most* $M = [N/2]$ *parts.*

Proof. By Theorem 3.4 for any partition ν of $n = 2m$ into not more than N parts the product $c_\nu(u)\, f_\nu(x_1, \ldots, x_M)\, s_\nu(y_1, \ldots, y_N)$ equals

$$\sum_\mu b_\mu(u)\, L_{\mu\nu}\, s_\mu(x_1^2, \ldots, x_M^2)\, s_\nu(y_1, \ldots, y_N)\,.$$

Taking here the sum over ν we obtain (3.17) by the definition (3.8) $\square$

We will complete this article with the following two examples. First, let us put $\mu = (m, 0, \ldots, 0)$ and $\nu = (2m, 0, \ldots, 0)$. Then the element $B_\mu \in \mathrm{Z}(\mathfrak{g})$ described in Section 2, coincides with the value of $Z_\nu(u)$ at $u = -m-\frac{1}{2}$ for $\mathfrak{g} = \mathfrak{so}_N$ and with the value of $(u+m-\frac{1}{2})/(u-\frac{1}{2}) \cdot Z_\nu(u)$ at the point $u = -m+\frac{1}{2}$ for $\mathfrak{g} = \mathfrak{sp}_N$; see [MN,Theorem 3.3].

Second, put $\mu = (1, \ldots, 1, 0, \ldots, 0)$ and $\nu = (1, \ldots, 1, 0, \ldots, 0)$ where the part 1 appears m and $2m$ times respectively. Then the element $(-1)^m B_\mu \in \mathrm{Z}(\mathfrak{g})$ coincides with the value of $Z_\nu(u)$ at $u = m + \frac{1}{2}$ for $\mathfrak{g} = \mathfrak{sp}_N$ and with the value of $(u - m + \frac{1}{2})/(u + \frac{1}{2}) \cdot Z_\nu(u)$ at the point $u = m - \frac{1}{2}$ for $\mathfrak{g} = \mathfrak{so}_N$; see [MN,Theorem 3.4].

It would be interesting to establish a link between our functions $Z_\nu(u)$ and the elements $B_\mu \in \mathrm{Z}(\mathfrak{g})$ with the general partitions μ.

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On Permutation Statistics and Hecke Algebra Characters

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Abstract.

Irreducible characters of Hecke algebras of type A may be represented as refined counts of simple statistics on suitable subsets of permutations. Such formulas have been generalized to characters of other Coxeter groups and their Hecke algebras and to coinvariant algebras. In this paper we present several formulas, applications to combinatorial identities, and related problems. New results are given with proofs.

§1. Introduction

Combinatorial properties of the Kazhdan-Lusztig basis and the study of coinvariant algebras have recently led to the discovery of a new family of combinatorial character formulas. Cf. [APR1, APR2, HLR, Ra2, Ste, Ro2, Ro5]. Here we survey some of these formulas with emphasis on permutation statistics. The goal of this paper is to present existing formulas and to study their role in deriving combinatorial identities.

A permutation $\pi \in S_n$ is called *unimodal* if there exists $1 \leq m \leq n$, such that

$$\pi(1) < \pi(2) < \cdots < \pi(m) > \pi(m+1) > \cdots > \pi(n).$$

Denote the set of all unimodal permutations in S_n by U_n.

For a permutation $\pi \in S_n$ define

$$\ell(\pi) := \#\{i < j | \pi(i) > \pi(j)\}.$$

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$$descent(\pi) := \#\{i | \pi(i) > \pi(i+1)\}.$$

$$major(\pi) := \sum_{\{i|\pi(i)>\pi(i+1)\}} i$$

By considering different representations of symmetric group characters we derive the following identity:

$$\sum_{\pi \in S_n} \omega^{major(\pi)} t^{\ell(\pi)} = \sum_{\pi \in U_n} (-1)^{descent(\pi)} t^{\ell(\pi)} \tag{1}$$

where ω is a root of unity of order n.

This identity is generalized using Hecke algebra characters:

$$\sum_{\pi \in S_n} \omega^{major(\pi^{-1})} q^{descent(\pi)} t^{major(\pi)} = \sum_{\pi \in U_n} (-q)^{descent(\pi)} t^{\ell(\pi)} \tag{2}$$

The rest of the paper is organized as follows. In the second section we give necessary background from representation theory of Hecke algebras, coinvariant algebras and permutation statistics. In the first part of Section 3 we present two different representations of the irreducible characters of the symmetric group as refined counts of descent number and major index of permutations. A new elementary proof is given. The second part of Section 3 contains formulas for characters of coinvariant algebras and applications. In Section 4 formulas for various characters of Hecke algebras are given with applications to permutation statistics identities. Section 5 concludes the paper with a brief sketch on other Coxeter groups and open problems.

§2. Background

2.1. Permutation Statistics

Let S_n be the symmetric group on n letters, and let $f_i : S_n \to Z_+$ $(1 \le i \le t)$ be (non-negative, integer valued) combinatorial parameters. Then one is interested in the *refined count* of permutations according to these parameters:

$$\sum_{\pi \in S_n} q_1^{f_1(\pi)} \cdots q_t^{f_t(\pi)}.$$

The study of permutation statistics started with Euler, who considered the number of descents. Netto, at the beginning of the century, considered the number of inversions, and MacMahon considered the major index. Multivariate refined counting was studied by [Ca, FS, GG] and many others.

In this section we define basic permutation statistics, which appear in the rest of the paper, and describe some of their properties. It should be noted that permutation statistics are connected to tableaux statistics via standard maps, such as Robinson-Schensted-Knuth correspondence and row (column) word tableaux (see below). For our purposes we prefer the permutation language.

The *length* of $\pi \in S_n$ is the number

$$\ell(\pi) := \#\{i < j | \pi(i) > \pi(j)\}.$$

This statistic is also called the *inversion number.* This number is well known to be equal to the minimal length of π as a product of simple reflections $s_i = (i, i+1)$. Using this approach the length is generalized naturally to arbitrary Coxeter groups.

The *descent number* of $\pi \in S_n$ is defined by

$$descent(\pi) := \#\{i | \pi(i) > \pi(i+1)\}.$$

This statistic has also a natural generalization to arbitrary Coxeter groups.

The *major index* of a permutation $\pi \in S_n$ is the sum (possibly zero)

$$major(\pi) := \sum\nolimits_{\{i | \pi(i) > \pi(i+1)\}} i$$

Generalization of this statistic to arbitrary Coxeter group is a challenging open problem. See Section 5.1.

A classical Theorem of MacMahon shows that the length function and the major index have the same generating function. The following well known identity refines this result [FS, Corollary 1].

Theorem 0.1

$$\sum\nolimits_{\pi \in S_n} q^{major(\pi)} t^{major(\pi^{-1})} = \sum\nolimits_{\pi \in S_n} q^{major(\pi)} t^{\ell(\pi^{-1})}.$$

The Robinson-Schensted correspondence is a bijection between permutations $\pi \in S_n$ and pairs of standard tableaux of same shape $(P(\pi), P(\pi^{-1}))$. See e.g. [Sa, Ch. 3.3]. A *Knuth class* in the symmetric group S_n is a set $\{\pi \in S_n | P(\pi) = Q\}$, where $P(\pi)$ is the left standard tableau corresponding to π under the Robinson-Schensted correspondence and Q is a fixed standard tableau. Define an *inverse Knuth class* as a set of the form $\mathcal{C}^{-1} = \{\pi^{-1} | \pi \in \mathcal{C}\}$, where $\mathcal{C}$ is a Knuth class in S_n.

The *descent set* of a permutation $\pi \in S_n$ is the set $\{i | \pi(i) > \pi(i+1)\}$.

Fact 0.2 *All permutations in an inverse Knuth class have a common descent set; So, have a common major index and a common descent number.*

It follows that the set of all permutations of a fixed descent number (major index) is a union of inverse Knuth classes. This fact is generalized to arbitrary Coxeter groups in the following way: The set of all elements of a fixed descent set is a union of Kazhdan-Lusztig left cells. See e.g. [Hu, Ch. 7.15].

Let $\mathcal{C}$ be a Knuth class in the symmetric group S_n. Denote by $descent(\mathcal{C}^{-1})$ the descent number of the permutations in $\mathcal{C}^{-1}$, and by $major(\mathcal{C}^{-1})$ the major index of the permutations in $\mathcal{C}^{-1}$.

The following hook formula of Stanley is very useful.

Theorem 0.3 *[ECII, Corollary 21.5]*
Let $\mathcal{C}$ be a Knuth class of shape λ. Then

$$\sum_{\pi \in \mathcal{C}} q^{major(\pi)} = q^{\sum_i \lambda_i (i-1)} \frac{\prod_{i=1}^{n} (q^i - 1)}{\prod_{(i,j)\in\lambda} (q^{h_{ij}} - 1)},$$

where $h_{i,j}$ are the hook lengths in the diagram of λ.

Any permutation may be considered as a sequence of positive integers. A less classical statistic on sequences (with repeats) is the *charge*. For definition see [Md, p. 242]. For permutation sequences the following folkloristic claim holds.

Claim 0.4 *For any permutation $\pi \in S_n$,*

$$charge(\pi) = major(w_0 \pi^{-1} w_0),$$

where $w_0 = n, n-1, n-2, \ldots, 1$ is the longest permutation in S_n.

To verify this claim recall that the charge of a permutation π is the sum of (weighted) lengths of increasing subsequences of consequent digits in π. The claim is based on the fact that the length of the i-th increasing subsequence of consequent digits in π equals to the difference between the i-th and the $i-1$-th descents of $w_0\pi^{-1}w_0$. Therefore, by an elementary observation the sum in the charge equals to the sum of the descents of $w_0\pi^{-1}w_0$. I.e, to the major index of $w_0\pi^{-1}w_0$.

The Kostka-Foulkes polynomials, denoted by $K_{\lambda,\mu}(q)$, are defined as the entries of the transition matrix between the Schur polynomial basis of the symmetric functions and the Hall-Littlewood q-polynomials. For more details see [Md, Ch. III, 6]. The Kostka-Foulkes polynomials may be represented as refined counts of charge of semi-standard tableaux [LS]. The following is a special case of Lascoux-Schützenberger Theorem [Md, Ch. III, (6.5)]. Let T be a standard tableau. Denote by $\pi(T)$ the permutation obtained by reading T from right to left in consecutive rows. $\pi(T)$ is called the *row-word of* T.

Theorem 0.5

$$K_{\lambda,1^n}(q) = \sum_T q^{charge(\pi(T))},$$

where the sum is taken over all standard tableaux of shape λ.

2.2. Representations

2.2.1. *Hecke Algebras and their Cellular Representations*

Hecke algebras. Let W be a Coxeter group, with a set of simple reflections S. The associated *Hecke algebra* $\mathcal{H}_W(q)$ is defined over the polynomial ring $Z[q]$ as follows. $\mathcal{H}_W(q)$ is spanned over the basis $\{T_w | w \in W\}$, where multiplication is defined by

$$T_w T_v = T_{wv}, \text{ if } \ell(wv) = \ell(w) + \ell(v)$$

$$(T_s - 1)(T_s + q) = 0, \ \forall\ s \in S$$

Here $\ell(w)$ is the length of w.

It should be noted that the last relation is slightly non-standard; this is done in order to get more elegant q-analogues. In order to shift to the standard version, one should replace each T_i by $-T_i$.

The elements $T_s, s \in S$ generate $\mathcal{H}_W(q)$.

Denote the Hecke algebra of the symmetric group S_n by $\mathcal{H}_n(q)$, and denote $T_{s_i} \in \mathcal{H}_n(q)$ by T_i. Then $\mathcal{H}_n(q)$ is generated by $T_1, \ldots, T_{n-1}$with respect to the following relations

$$T_i T_j = T_j T_i, \text{ if } |i-j| > 1$$

$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, \ \ 1 \le i < n-1$$

$$(T_i - 1)(T_i + q) = 0, \ \ \forall i$$

Let $\mu = (\mu_1, \ldots, \mu_t)$ be a partition of n. The element $T_\mu \in \mathcal{H}_n(q)$ is defined by

$$T_\mu := T_1 T_2 \cdots T_{\mu_1 - 1} T_{\mu_1 + 1} \cdots\cdots T_{\mu_1 + \ldots \mu_t - 1}$$

I.e. T_μ is the subproduct of $T_1 T_2 \cdots T_{\mu_1 + \cdots + \mu_t - 1}$ omitting $T_{\mu_1 + \cdots + \mu_i}$ for $1 \le i < t$.

In their fundamental paper [KL], Kazhdan and Lusztig present a distinguished basis for every Hecke algebra, and construct a rich family of representations of Coxeter groups and their Hecke algebras. In the case of the symmetric group, this construction gives a decomposition of the Hecke algebra into irreducible representations.

A cornerstone for this theory is the concept of cellular structure. Any Coxeter group can be partitioned into subsets called *Kazhdan-Lusztig cells.* The action of the group on each of these cells gives rise to a so-called *Kazhdan-Lusztig representation.* Denote the associated Kazhdan-Lusztig representation of a right (left) Kazhdan-Lusztig cell, $\mathcal{C}$, by $\rho^{\mathcal{C}}$, and the corresponding character by $\chi^{\mathcal{C}}$.

In the symmetric group case the Kazhdan-Lusztig right cells are the Knuth classes, while the Kazhdan-Lusztig left cells are the inverse Knuth classes [KL, §5, proof of Proposition 1.4]. The representation associated with a Kazhdan-Lusztig right (left) cell is the S_n-irreducible representation S^λ, where λ is the shape of Q. We say that λ is the *shape of the cell (Knuth class)* $\mathcal{C}$.

2.2.2. *The Coinvariant Algebra*

The symmetric group S_n acts on the polynomial ring $P_n = Q[x_1, \ldots, x_n]$ by permuting the variables. The *coinvariant algebra* is the quotient P_n/I_n, where I_n is the ideal generated by the symmetric (S_n-invariant) polynomials without a constant term. The coinvariant algebra of a finite Weyl group W is defined similarly. The group algebra of W and its coinvariant algebra are isomorphic as W-modules [Hu, Ch. 3.6]. Early work of Borel showed how to identify the coinvariant algebra with the cohomology ring of G/B, where G is a simple Lie group and B is a Borel subgroup. Schubert polynomials, constructed in the seminal papers [BGG] and [De], form a distinguished basis for the coinvariant algebra. These polynomials correspond to the Schubert cells in $H^*(G/B)$.

The coinvariant algebra has a natural grading to homogenous components, induced from the grading of the polynomial ring by total degree. Denote by R^k the k-th homogeneous component of the coinvariant algebra. and by χ^k its corresponding character as a W-module.

Decomposition into irreducibles. The decomposition of the coinvariant algebra into irreducible representations involves major indices [KW].

Theorem 0.6 *The multiplicity of the S_n-irreducible representation S^λ in the k-th homogeneous component of the coinvariant algebra, R^k, is equal to the number of Knuth classes of shape λ and $major(\mathcal{C}^{-1}) = k$.*

An elegant proof of this theorem, using the principal specialization of Schur functions, was given by Stanley and developed in [Ste, Ga, Reu, Ch. 8.3]. In [Ro5] we applied Foata and Schützenberger's work on the major index, together with properties of Kazhdan-Lusztig representations, to derive this result. An analogous rule for decomposing the coinvariant algebras of classical Weyl groups of type B and other wreath products was given in [Ste]. This analogue involves a new interpretation of the major index in terms of Coxeter elements [AR1]. Decomposition of other quotient rings is described by Kostka-Foulkes polynomials [GP].

Hecke algebra action. Explicit deformations of the symmetric group action on the coinvariant algebra is presented in [APR1]. Two different Hecke algebra actions on the polynomial ring P_n are defined.

For the first action, each generator T_i acts on P_n as the linear operator

$$R_i(x_i^\alpha x_{i+1}^\beta m) := \begin{cases} q x_i^\beta x_{i+1}^\alpha m, & \text{if } \alpha \geq \beta \\ (1-q)x_i^\alpha x_{i+1}^\beta m + x_i^\beta x_{i+1}^\alpha m, & \text{if } \alpha < \beta \end{cases}$$

Here m is a monomial involving neither x_i nor x_{i+1}. The action of R_i is extended to the full polynomial ring by linearity. For the second action each generator T_i acts on P_n as the q-commutator

$$A_i := \partial_i X_i - q X_i \partial_i$$

where ∂_i is the divided difference operator $\partial_i := (x_i - x_{i+1})^{-1}(1 - s_i)$, and X_i is multiplication by x_i.

The symmetric functions are invariant under these two actions. Therefore, the actions on the homogeneous components of the coinvariant algebra form Hecke algebra representations. It should be noted that the two actions form equivalent representations. The associated characters may represented as refined counts over subsets of permutations. Surprisingly, the Kazhdan-Lusztig characters of the Hecke algebra may be represented as refined counts of exactly the same statistic over different summation sets. See Theorems 8 and 9 below.

§3. Symmetric Group Characters

3.1. Irreducible Characters

The Murnaghan-Nakayama classical rule represents the symmetric group characters as a signed enumeration of the so called rim-hook tableaux. Cf. [Sa, Ch. 4.10]. For a representation theoretical interpretation of refined counts of rim-hook tableaux see [LLT]. In this section we represent symmetric group characters as refined counts of permutations. These representations are convenient for generalizations to Hecke algebras, coinvariant algebras, and other groups and algebras.

Definition. A sequence of positive integers $a = a_1, \dots, a_n$ is *unimodal* if there exists $1 \le m \le n$, such that

$$a_1 < a_2 < \cdots < a_m > a_{m+1} > \cdots > a_n.$$

Let $\mu = (\mu_1, \dots, \mu_t)$ be a partition of n. A sequence of n positive integers is *μ-unimodal* if the first μ_1 integers form a unimodal sequence, the next μ_2 integers form a unimodal sequence, and so on.

A permutation $\pi \in S_n$ is called a *μ-unimodal permutation* if the sequence $\pi(1), \dots, \pi(n)$ is μ-unimodal. For example, $\pi = 174239856$ is $(4,3,2)$-unimodal, but not $(5,4)$-unimodal.

Denote the set of all μ-unimodal permutations in S_n by U_μ. Let λ and μ be partitions of n, and let χ^λ_μ be the S_n-character value of the irreducible representation S^λ at a conjugacy class of type μ. The following Theorem is a special case of [Ro2, Theorem 4].

Theorem 1.

$$\chi^\lambda_\mu = \sum\nolimits_{\pi \in C \cap U_\mu} (-1)^{descent(\pi)},$$

where the sum runs over all μ-unimodal permutations in a Knuth class C of shape λ.

For an elementary combinatorial proof see [Ro2, Proof of Theorem 6].

The following lemma is an immediate consequence of Stanley's hook formula (Theorem 0.3).

Lemma 2. *Let C be a Knuth class of shape λ. Then*

$$\sum\nolimits_{\pi \in C} \omega^{major(\pi)} = \begin{cases} (-1)^k, & \text{if } \lambda = (n-k, 1^k) \\ 0, & \text{otherwise} \end{cases},$$

where ω is a root of unity of order n.

Proof. By Theorem 0.3 for any Knuth class of shape $(n-k, 1^k)$

$$\sum\nolimits_{\pi \in C} \omega^{major(\pi)} = \prod_{i=1}^{k} \omega^i \frac{\omega^{n-i} - 1}{\omega^i - 1} = (-1)^k.$$

If λ is not a hook then combining the fact that the refined count is a polynomial together with Theorem 0.3 implies that

$$\sum\nolimits_{\pi\in C} q^{major(\pi)} = \frac{q^n-1}{q^j-1} p(q),$$

for some $j < n$, where j is a divisor of n, and $p(q)$ is a polynomial in q. Hence,

$$\sum\nolimits_{\pi\in C} \omega^{major(\pi)} = (1+\omega^j+\omega^{2j}+\cdots+\omega^{n-j})\cdot p(\omega) = 0\cdot p(\omega) = 0.$$

□

Another combinatorial representation of irreducible characters follows from this lemma.

Theorem 3.

$$\chi^\lambda_{(n)} = \sum\nolimits_{\pi\in C} \omega^{major(\pi)},$$

where the sum is taken over a Knuth class C of shape λ, and ω is a root of unity of order n.

Proof. It follows from Theorem 1 that $\chi^\lambda_{(n)} = (-1)^k$ if $\lambda = (n-k, 1^k)$ and zero otherwise. Combining this fact with Lemma 2 gives the desired result.

□

Remark. Theorem 3 was first proved in a deep work of Stembridge. In this work it was shown that the summands in the right hand side are essentially the eigenvalues. See [Ste, Theorem 3.2].

The Kostka-Foulkes polynomials are defined as entries of transition matrices between bases of q-symmetric functions. See Section 2.2. Combining Theorem 3 with Theorem 0.5 we obtain

Corollary 4.

$$\chi^\lambda_{(n)} = K_{\lambda',1^n}(\omega),$$

where $K_{\lambda,\mu}(q)$ is the Kostka-Foulkes polynomial, ω is a root of unity of order n, and λ' is the conjugate partition of λ.

Proof. By Claim 0.4 and Theorem 0.5

$$K_{\lambda,1^n}(q) = \sum_T q^{charge(\pi(T))} = \sum_{\pi\in w_0 S_\lambda w_0} q^{major(\pi)}.$$

Here $S_\lambda := \{(\pi(T))^{-1} | T \text{ is a standard tableau of shape } \lambda\}$, where $\pi(T)$ is the row word of T.

But S_λ is a Knuth class of shape λ'. Moreover, for any Knuth class $\mathcal{C}$ of shape λ', $w_0\mathcal{C}$ and $\mathcal{C}w_0$ are Knuth classes of shape λ. Hence, $w_0 S_\lambda w_0$ is a Knuth class of shape λ'.

Theorem 3 completes the proof. □

3.2. Coinvariant Algebra Characters

Let χ^k_μ be the S_n-character value at a conjugacy class of type μ of the k-th homogeneous component of the coinvariant algebra of the symmetric group S_n. Denote the set $\{\pi \in S_n | \ell(\pi) = k\}$ by $L(k)$. An analogue of Theorem 1 is proved in [Ro5, Theorem 1].

Theorem 5.

$$\chi^k_\mu = \sum\nolimits_{\pi \in L(k) \cap U_\mu} (-1)^{descent(\pi)},$$

where the sum runs over all μ-unimodal permutations of length k.

In other words

Theorem 5'.

$$\sum\nolimits_k \chi^k_\mu t^k = \sum\nolimits_{\pi \in U_\mu} (-1)^{descent(\pi)} t^{\ell(\pi)}.$$

It follows from Theorem 3 that

Theorem 6.

$$\sum\nolimits_k \chi^k_{(n)} t^k = \sum\nolimits_{\pi \in S_n} \omega^{major(\pi)} t^{\ell(\pi)}.$$

Proof. Clearly,

$$\chi^k_{(n)} = \sum\nolimits_\lambda m_{\lambda,k} \chi^\lambda_{(n)},$$

where $m_{\lambda,k}$ is the multiplicity of the irreducible representation S^λ in the k-th homogeneous component of the coinvariant algebra.

Theorem 0.6 asserts that $m_{\lambda,k}$ equals to the number of Knuth classes $\mathcal{C}$ with $major(\mathcal{C}^{-1}) = k$. It follows that

$$\sum\nolimits_k \chi^k_{(n)} t^k = \sum\nolimits_{\mathcal{C}} \chi^{\mathcal{C}}_{(n)} t^{major(\mathcal{C}^{-1})}.$$

Here $\chi^{\mathcal{C}}$ is the irreducible character χ^λ, where λ is the shape of $\mathcal{C}$.

By Theorem 3 the right hand side equals to

$$\sum\nolimits_{\mathcal{C}} \sum\nolimits_{\pi \in \mathcal{C}} \omega^{major(\pi)} t^{major(\mathcal{C}^{-1})} = \sum\nolimits_{\pi \in S_n} \omega^{major(\pi)} t^{major(\pi^{-1})}.$$

Theorem 0.1 completes the proof.

□

Comparing Theorem 5' to Theorem 6 implies the following identity.

Corollary 7.

$$\sum\nolimits_{\pi \in S_n} \omega^{major(\pi)} t^{\ell(\pi)} = \sum\nolimits_{\pi \in U_n} (-1)^{descent(\pi)} t^{\ell(\pi)},$$

where U_n is the set of all unimodal permutations in S_n.

§4. Hecke Algebra Characters

Hecke algebra characters provide q-analogues of the above results. The following formula for the irreducible characters is proved in [Ro2, Theorem 4] and in [Ra2]. Recall the definition of $T_\mu \in \mathcal{H}_n(q)$ from Section 2.2.1, and let $\chi_q^\lambda(T_\mu)$ be the $\mathcal{H}_n(q)$-character value of the irreducible representation corresponding to λ at the element $T_\mu \in \mathcal{H}_n(q)$. Then

Theorem 8.
$$\chi_q^\lambda(T_\mu) = \sum\nolimits_{\pi \in C \cap U_\mu} (-q)^{descent(\pi)},$$

where the sum runs over all μ-unimodal permutations in a Knuth class C of shape λ.

Action of Hecke algebra of type A on coinvariant algebras is described in Section 2.1.2. Let χ_q^k be the character of the Hecke algebra $\mathcal{H}_n(q)$, defined by the action on the k-th homogeneous component of the coinvariant algebra. The following analogue of Theorem 5 is proved in [APR1, Theorems 5.1 and 6.6].

Theorem 9.

$$\chi_q^k(T_\mu) = \sum\nolimits_{\pi \in L(k) \cap U_\mu} (-q)^{descent(\pi)},$$

where $L(k)$ is the set of all permutations of length k in S_n.

Here is an alternative combinatorial description.

Theorem 10.

$$\chi_q^k(T_{(n)}) = \sum\nolimits_{\{\pi \in S_n \mid major(\pi) = k\}} \omega^{major(\pi^{-1})} q^{descent(\pi)}.$$

Proof. In order to prove this corollary we need the following lemma.

Lemma 11.

$$\chi_q^\lambda(T_{(n)}) = \sum_{\pi\in\mathcal{C}} \omega^{major(\pi)} q^{descent(\pi^{-1})},$$

where the sum is taken over a Knuth class $\mathcal{C}$ of shape λ.

Proof of Lemma 11. By Fact 0.2 all permutations in an inverse Knuth class $\mathcal{C}^{-1}$ have a common descent number, denoted by *descent*$(\mathcal{C}^{-1})$. Moreover, it is easy to verify that for any Knuth class $\mathcal{C}$ of hook shape $(n-k, 1^k)$, *descent*$(\mathcal{C}^{-1}) = k$. Combining these facts with Lemma 2 we obtain

$$\sum_{\pi\in\mathcal{C}} \omega^{major(\pi)} q^{descent(\pi^{-1})} = q^{descent(\mathcal{C}^{-1})} \sum_{\pi\in\mathcal{C}} \omega^{major(\pi)} =$$

$$= \begin{cases} (-q)^k, & \text{if } \lambda = (n-k, 1^k) \text{ for some } 0 \le k < n \\ 0, & \text{otherwise} \end{cases}.$$

On the other hand, by Theorem 8

$$\chi_q^\lambda(T_{(n)}) = \begin{cases} (-q)^k, & \text{if } \lambda = (n-k, 1^k) \text{ for some } 0 \le k < n \\ 0, & \text{otherwise} \end{cases}.$$

□

Using Theorem 0.6, as in the proof of Theorem 6, together with Lemma 11 yields

$$\sum_k \chi_q^k(T_{(n)}) t^k = \sum_{\mathcal{C}} \chi^{\mathcal{C}}(T_{(n)}) t^{major(\mathcal{C}^{-1})} =$$

$$= \sum_{\mathcal{C}} \sum_{\pi\in\mathcal{C}} \omega^{major(\pi)} q^{descent(\pi^{-1})} t^{major(\mathcal{C}^{-1})} =$$

$$= \sum_{\pi\in S_n} \omega^{major(\pi)} q^{descent(\pi^{-1})} t^{major(\pi^{-1})}.$$

We conclude that

$$\chi_q^k(T_{(n)}) = \sum_{\{\pi\in S_n \mid major(\pi^{-1})=k\}} \omega^{major(\pi)} q^{descent(\pi^{-1})}.$$

□

Comparing Theorem 9 with Theorem 10 we obtain

Corollary 12. *With the above notations*

$$\sum_{\pi\in S_n} \omega^{major(\pi^{-1})} q^{descent(\pi)} t^{major(\pi)} = \sum_{\pi\in U_n} (-q)^{descent(\pi)} t^{\ell(\pi)}.$$

§5. Final Remarks and Open Problems

5.1. Other Weyl Groups

Let H be a parabolic subgroup of an arbitrary Coxeter group W, which is isomorphic to a direct product of symmetric groups. In the following definition we refer to cycle type and $weight^q_\mu$ of elements in H under this isomorphism.

Definition. Let μ be a cycle type of an element in H. For any element $w = r \cdot \pi \in W$, where $\pi \in H$ and r is the representative of minimal length of the left coset of wH in W, define

$$weight^q_\mu(w) := \begin{cases} (-q)^{descent(\pi)}, & \text{if } \pi \text{ is } \mu\text{-unimodal} \\ 0, & \text{otherwise} \end{cases}.$$

Note that $weight^q_\mu$ is independent of the choice of H, provided that H is isomorphic to a direct product of symmetric groups and that μ is the cycle type of some element in H.

Let $v_\mu \in H$ have a cycle type μ, and let T_μ be the element in the Hecke algebra $\mathcal{H}_W(q)$ indexed by v_μ.

Theorem 13. *[Ro1, Corollary 3] Let $\mathcal{C}$ be a finite Kazhdan-Lusztig right cell in an arbitrary Coxeter group W, and let $\chi^{\mathcal{C}}$ be its associated Hecke algebra character. Then*

$$\chi^{\mathcal{C}}(T_\mu) = \sum\nolimits_{w \in \mathcal{C}} weight^q_\mu(w).$$

A formally similar result for coinvariant algebras is proved in [APR2]. Let R^k be the k-th homogeneous component of the coinvariant algebra of W. Denote by χ^k the W-character of R^k. Let $v_\mu \in H$ have cycle type μ. Then

Theorem 14. *[APR2, Theorem 4] Let W be an arbitrary finite Weyl group. With the above notations*

$$\chi^k(v_\mu) = \sum\nolimits_{\{w \in W : \ell(w) = k\}} weight^1_\mu(w).$$

So, coinvariant algebra characters of an arbitrary finite Weyl group W and Kazhdan-Lusztig characters of these groups may be represented as sums of exactly the same weights, but over different summation sets. This curious analogy seems to deserve further study.

Unfortunately, we do not know of an explicit Hecke algebra action on homogeneous components of coinvariant algebras of general type.

Problem 1. *Define an action of the Hecke algebra of an arbitrary finite Weyl group on its coinvariant algebra, which produces a natural q-analogue of Theorem 14.*

When it comes to Theorem 3, analogues are known for classical Weyl groups and wreath products [Ste].

Problem 2. *Give an analogue of Theorem 3 for an arbitrary finite Coxeter group.*

Stembridge proved that Theorem 3 describes the S_n-character $\chi^{\lambda}_{(n)}$ as a sum of the eigenvalues of the full cycle at the irreducible representation S^{λ}. Theorem 13 describes Kazhdan-Lusztig characters of a general Hecke algebra. Eigenvalues are described by Geck and Michel.

Theorem 15. *[GM, Proposition 1.3] Let W be a finite Coxeter group, and let ρ be an irreducible representation of its Hecke algebra $\mathcal{H}_W(q)$. Let $w \in W$ be an element of minimal length in some conjugacy class in W. Let ω be a root of unity of order d, where d is the order of w. Then there exist integers m_i and rational numbers r_i, such that the eigenvalues of $\rho(T_w), T_w \in \mathcal{H}_W(q)$, are*

$$\omega^{m_i} q^{r_i}, \qquad (1 \leq i \leq dim\rho).$$

Unfortunately, the problem of determining the integers m_i is not solved in general. In case of classical Weyl groups and related wreath products, these integers are determined by a generalized major index [Ste, Theorem 5.1]. In this perspective, the problem of determining the integers m_i (and so, solving Problem 2) is strongly related to the problem of defining major index on arbitrary Coxeter groups. Partial results appear in [Rei1-2, FC1-3, Stei ,Ste, AR1].

A closely related problem is the following: Recall that the sets of permutations of a fixed major index are unions of Kazhdan-Lusztig cells. This fact, together with Garsia-Gessel refined count of the major index of shuffles [GG], implies an extremely simple combinatorial rule for restricting the coinvariant algebra of type A to parabolic subgroups [Ro4]. This rule is an exact analogue of the Barbasch-Vogan rule for restricting Kazhdan-Lusztig representations of arbitrary Weyl groups. No such a rule is known for coinvariant algebras of other Weyl groups.

Geck and Michel give an algorithm for calculating the exponents r_i in Theorem 15 [GM, §4.3]. In light of Corollary 4, a combinatorial understanding of this algorithm may be helpful in the study of deformations of Kostka-Foulkes polynomials. See e.g. [GH].

5.2. Other Representation Theory Interpretations

Stanley's hook formula for refined counts of major index over standard tableaux of shape λ (Theorem 0.3) is identical with Olsson's hook formula, giving the dimensions of unipotent representations of $GL_n(q)$, the general linear group over a finite field [O]. Refined counts of charge on semi-standard tableaux of shape λ gives Kostka-Foulkes polynomials [LS]. These polynomials are equal to unipotent characters of $GL_n(q)$ [Lu3, Sh1, Ch. 2.7]. Therefore the charge essentially refines the major index, both as permutation statistics and in the representation theoretic interpretation. The charge gives also the eigenvalues of conjugacy classes of type $r^{n/r}$. Unfortunately, eigenvalues of conjugacy classes of general type are not given by charge. See [Ste].

Finally, it should be mentioned that Hecke algebra bitraces may be represented as refined counts of nonnegative integer matrices [HLR]. This result follows from Theorem 8. The problem of representing characters of Brauer and Birman-Wenzl algebras as refined counts is a promising open problem [Ra3].

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Bosonic Formula for Level-restricted Paths

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Abstract.

We prove a bosonic formula for the generating function of level-restricted paths for the nonexceptional affine Kac-Moody algebras. In affine type A this yields an expression for the level-restricted generalized Kostka polynomials.

§1. Introduction

Let $\mathfrak{g}$ be a nonexceptional affine Kac-Moody algebra, that is, one of type $A_n^{(1)}$ ($n \geq 1$), $B_n^{(1)}$ ($n \geq 3$), $C_n^{(1)}$ ($n \geq 2$), $D_n^{(1)}$ ($n \geq 4$), $A_{2n}^{(2)}$ ($n \geq 1$), $A_{2n-1}^{(2)}$ ($n \geq 3$) or $D_{n+1}^{(2)}$ ($n \geq 2$). Let $U_q(\mathfrak{g})$ be the quantized affine algebra and $U_q(\mathfrak{g})^+$ the "upper triangular" part of $U_q(\mathfrak{g})$. Let V be a $U_q(\mathfrak{g})^+$-submodule of a finite direct sum V' of irreducible integrable highest weight $U_q(\mathfrak{g})$-modules, and Π the limit of the Demazure operator for an element w of the Weyl group as $\ell(w) \to \infty$. The main theorem of this paper gives sufficient conditions on V so that the formula

$$\Pi \ \mathrm{ch}(V) = \mathrm{ch}(V') \tag{1}$$

holds, where $\mathrm{ch}(V)$ is the character of V. When V is the one-dimensional $U_q(\mathfrak{g})^+$-module generated by the dominant integral weight Λ then (1) is the Weyl-Kac character formula. The above result is well-known when V is a union of Demazure modules for any Kac-Moody algebra $\mathfrak{g}$.

Let $\mathfrak{g}'$ be the derived subalgebra of $\mathfrak{g}$. Consider the $U_q(\mathfrak{g}')$-module V given by a tensor product of finite-dimensional $U_q(\mathfrak{g}')$-modules that admit a crystal of level at most ℓ, with the one-dimensional subspace generated by a highest weight vector of an irreducible integrable highest weight $U_q(\mathfrak{g}')$-module of level ℓ. Such modules V can be given the structure of a $U_q(\mathfrak{g})^+$-module and as such, satisfy the above conditions.

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Then a special case of (1) is a bosonic formula for the q-enumeration of level-restricted inhomogeneous paths by the energy function. In type $A^{(1)}_{n-1}$ this formula was conjectured in [3], stated there as a q-analogue of the Goodman-Wenzl straightening algorithm for outer tensor products of irreducible modules over the type A Hecke algebra at a root of unity [4]. In the isotypic component of the vacuum, the bosonic formula coincides with half of the bose-fermi conjecture in [20, (9.2)].

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§2. Notation

Most of the following notation is taken from ref. [7]. Let X be a Dynkin diagram of affine type with vertices indexed by the set $I = \{0, 1, 2, \dots, n\}$ as in [7], Cartan matrix $A = (a_{ij})_{i,j\in I}$, $\mathfrak{g} = \mathfrak{g}(A)$ the affine Kac-Moody algebra, and $\mathfrak{h}$ the Cartan subalgebra. Let $\{\alpha_i^\vee : i \in I\} \subset \mathfrak{h}$ and $\{\alpha_j : j \in I\} \subset \mathfrak{h}^*$ be the simple coroots and roots, which are linearly independent subsets that satisfy $\langle \alpha_i^\vee, \alpha_j\rangle = a_{ij}$ for $i, j \in I$ where $\langle\cdot,\cdot\rangle : \mathfrak{h} \otimes \mathfrak{h}^* \to \mathbb{C}$ is the natural pairing. Let $Q = \bigoplus_{i\in I} \mathbb{Z}\alpha_i$ be the root lattice. Let the null root $\delta = \sum_{i\in I} a_i\alpha_i$ be the unique element of the positive cone $\bigoplus_{i\in I} \mathbb{Z}_{\geq 0}\, \alpha_i$ in Q, that generates the one-dimensional lattice $\{\beta \in Q | \langle \alpha_i^\vee, \beta\rangle = 0 \text{ for all } i \in I\ \}$. Let $K = \sum_{i\in I} a_i^\vee \alpha_i^\vee \in \mathfrak{h}$ be the canonical central element, where the integers $a_i^\vee$ are the analogues of the integers a_i for the dual algebra $\mathfrak{g}^\vee$ defined by the transpose ${}^tA$ of the Cartan matrix A. Let $d \in \mathfrak{h}$ (the degree derivation) be defined by the conditions $\langle d, \alpha_i\rangle = \delta_{i0}$ where δ_{ij} is the Kronecker delta; d is well-defined up to a summand proportional to K. Then $\{\alpha_0^\vee, \dots, \alpha_n^\vee, d\}$ is a basis of $\mathfrak{h}$. Let $\{\Lambda_0, \dots, \Lambda_n, \delta\}$ be the dual basis of $\mathfrak{h}^*$; the elements $\{\Lambda_0, \dots, \Lambda_n\}$ are called the fundamental weights. The weight lattice is defined by $P = \bigoplus_{i\in I} \mathbb{Z}\Lambda_i \bigoplus \mathbb{Z}a_0^{-1}\delta$; in the usual definition the scalar a_0^{-1} is absent. The weight lattice contains the root lattice since $\alpha_j = \sum_{i\in I} a_{ij}\Lambda_i$ for $j \in I$. Define $P^+ = \bigoplus_{i\in I} \mathbb{Z}_{\geq 0}\, \Lambda_i \bigoplus \mathbb{Z}a_0^{-1}\delta$. Say that a weight $\Lambda \in P^+$ has level ℓ if $\ell = \langle K, \Lambda\rangle$.

Let $(\cdot|\cdot)$ denote the standard symmetric bilinear form on $\mathfrak{h}^*$. Since $\{\alpha_0, \dots, \alpha_n, \Lambda_0\}$ is a basis of $\mathfrak{h}^*$, this form is uniquely defined by setting $(\alpha_i|\alpha_j) = a_i^\vee a_i^{-1} a_{ij}$ for $i, j \in I$, $(\alpha_i|\Lambda_0) = \delta_{i0}a_0^{-1}$ for $i \in I$ and $(\Lambda_0|\Lambda_0) = 0$. This form induces an isomorphism $\nu : \mathfrak{h} \to \mathfrak{h}^*$ defined by $a_i^\vee \nu(\alpha_i^\vee) = a_i\alpha_i$ for $i \in I$ and $\nu(d) = a_0\Lambda_0$. Also $\nu(K) = \delta$.

The Weyl group W is the subgroup of $GL(\mathfrak{h}^*)$ generated by the simple reflections r_i ($i \in I$) defined by $r_i(\beta) = \beta - \langle \alpha_i^\vee, \beta\rangle\alpha_i$. The form $(\cdot|\cdot)$ is W-invariant. Suppose $\alpha \in Q$ is a real root, that is, the

α-weight space of $\mathfrak{g}$ is nonzero and there is a simple root α_i and a Weyl group element $w \in W$ such that $\alpha = w(\alpha_i)$. Define $\alpha^\vee \in \mathfrak{h}$ by $w(\alpha_i^\vee)$. This is independent of the expression $\alpha = w(\alpha_i)$. Define $r_\alpha \in W$ by $r_\alpha(\beta) = \beta - \langle \alpha^\vee, \beta \rangle \alpha$ for $\beta \in \mathfrak{h}^*$.

Let $\mathfrak{g}'$ be the derived algebra of $\mathfrak{g}$, obtained by "omitting" the degree derivation d. Its weight lattice is $P_{cl} \cong P/\mathbb{Z}a_0^{-1}\delta$. Denote the canonical projection $P \to P_{cl}$ by cl. Write $\alpha_i^{cl} = \mathrm{cl}(\alpha_i)$ and $\Lambda_i^{cl} = \mathrm{cl}(\Lambda_i)$ for $i \in I$. The elements $\{\alpha_i^{cl} \mid i \in I\}$ are linearly dependent. Write af : $P_{cl} \to P$ for the section of cl given by $\mathrm{af}(\Lambda_i^{cl}) = \Lambda_i$ for all $i \in I$. Write $P_{cl}^+ = \bigoplus_{i\in I} \mathbb{Z}_{\geq 0} \Lambda_i^{cl}$. Define the level of $\mu \in P_{cl}^+$ to be $\langle K, \mathrm{af}(\mu) \rangle$.

Consider the Dynkin diagram $\overline{X}$ obtained by removing the vertex 0 from the diagram X, with corresponding Cartan matrix $\overline{A}$ indexed by the set $J = I - \{0\}$, and let $\overline{\mathfrak{g}} = \mathfrak{g}(\overline{A})$ be the simple Lie algebra. One has the inclusions $\overline{\mathfrak{g}} \subset \mathfrak{g}' \subset \mathfrak{g}$. Let $\{\overline{\alpha}_i : i \in J\}$ be the simple roots, $\{\overline{\Lambda}_i : i \in J\}$ the fundamental weights, and $\overline{Q} = \bigoplus_{i \in J} \mathbb{Z}\overline{\alpha}_i$ the root lattice for $\overline{\mathfrak{g}}$. The weight lattice of $\overline{\mathfrak{g}}$ is $\overline{P} = \bigoplus_{i\in J} \mathbb{Z}\overline{\Lambda}_i$ and $\overline{P} \cong P_{cl}/\mathbb{Z}\Lambda_0$. The image of $\Lambda \in P$ into $\overline{P}$ is denoted by $\overline{\Lambda}$. We shall use the section of the natural projection $P_{cl} \to \overline{P}$ given by the map $\overline{P} \to P_{cl}$ that sends $\overline{\Lambda}_i \mapsto \Lambda_i^{cl} - \Lambda_0^{cl}$ for $i \in J$. By abuse of notation, for $\Lambda \in P$, $\overline{\Lambda}$ shall also denote the image of the element $\overline{\Lambda}$ under the lifting map $\overline{P} \to P$ specified above.

Let $\overline{P}^+ = \bigoplus_{i\in J} \mathbb{Z}_{\geq 0} \overline{\Lambda}_i$. For $\lambda \in \overline{P}^+$, denote by $V(\lambda)$ the irreducible integrable highest weight $U_q(\overline{\mathfrak{g}})$-module of highest weight λ.

Let $\theta = \delta - a_0\alpha_0 = \sum_{i\in J} a_i\alpha_i \in \overline{Q}$. One has the formulas $(\theta|\theta) = 2a_0$, $\theta = a_0\nu(\theta^\vee)$, and $\alpha_0^\vee = K - a_0\theta^\vee$. Observe that

$$\mathrm{cl}(\alpha_0) = -a_0^{-1} \sum_{i\in J} a_i \alpha_i^{cl} = -\mathrm{cl}(\nu(\theta^\vee)).$$

For $\Lambda \in P^+$ let $\mathbb{V}(\Lambda)$ be the irreducible integral highest weight module of highest weight Λ over the quantized universal enveloping algebra $U_q(\mathfrak{g})$, $\mathbb{B}(\Lambda)$ the crystal base of $\mathbb{V}(\Lambda)$, and $u_\Lambda \in \mathbb{B}(\Lambda)$ the highest weight vector.

By restriction from $U_q(\mathfrak{g})$ to $U_q(\mathfrak{g}')$, the module $\mathbb{V}(\Lambda)$ is an irreducible integral highest weight module for $U_q(\mathfrak{g}')$ of highest weight $\mathrm{cl}(\Lambda)$, with crystal $\mathbb{B}(\Lambda)$ that is P_{cl}-weighted by composing the weight function $\mathbb{B}(\Lambda) \to P$ with the projection cl. Conversely, any integrable irreducible highest weight $U_q(\mathfrak{g}')$-module can be obtained this way.

§3. Short review of affine crystal theory

3.1. Crystals

A P-weighted I-crystal B is a colored graph with vertices indexed by $b \in B$, directed edges colored by $i \in I$, and a weight function wt : $B \to P$, satisfying the axioms below. First some notation is required. Denote an edge from b to b' colored i, by $b' = f_i(b)$ or equivalently $b = e_i(b')$. Write $\phi_i(b)$ (resp. $\epsilon_i(b)$) for the maximum index m such that $f_i^m(b)$ (resp. $e_i^m(b)$) is defined.

1. If $b' = f_i(b)$ then $\mathrm{wt}(b') = \mathrm{wt}(b) - \alpha_i$.
2. $\phi_i(b) - \epsilon_i(b) = \langle \alpha_i^\vee, \mathrm{wt}(b) \rangle$.

An element $u \in B$ is a highest weight vector if $e_i(u)$ is undefined for all $i \in I$. The i-string of $b \in B$ consists of all elements $e_i^m(b)$ $(0 \le m \le \epsilon_i(b))$ and $f_i^m(b)$ $(0 \le m \le \phi_i(b))$. The nondominant part of the i-string is comprised of all elements which admit e_i.

We also define the crystal reflection operator $s_i : B \to B$ by

$$s_i(b) = \begin{cases} f_i^{\phi_i(b)-\epsilon_i(b)}(b) & \text{if } \phi_i(b) > \epsilon_i(b) \\ b & \text{if } \phi_i(b) = \epsilon_i(b) \\ e_i^{\epsilon_i(b)-\phi_i(b)}(b) & \text{if } \phi_i(b) < \epsilon_i(b). \end{cases}$$

It is obvious that s_i is an involution. Observe that

$$\mathrm{wt}(s_i(b)) = r_i \mathrm{wt}(b) = \mathrm{wt}(b) - \langle \alpha_i^\vee, \mathrm{wt}(b) \rangle \alpha_i. \tag{2}$$

Define the notation $\phi(b) = \sum_{i \in I} \phi_i(b) \Lambda_i$ and $\epsilon(b) = \sum_{i \in I} \epsilon_i(b) \Lambda_i$.

If a $U_q(\mathfrak{g})$-module (resp. $U_q(\mathfrak{g}')$-module, resp. $U_q(\overline{\mathfrak{g}})$-module) has a crystal base then the latter is naturally a P-weighted (resp. P_{cl}-weighted, resp. $\overline{P}$-weighted) I-crystal (resp. I-crystal, resp. J-crystal).

3.2. Tensor products

Given crystals B_1 and B_2, contrary to the literature (but consistent with the Robinson-Schensted-Knuth correspondence in type A), define the following crystal structure on the tensor product $B_2 \otimes B_1$. The elements are denoted $b_2 \otimes b_1$ for $b_i \in B_i$ $(i \in \{1, 2\})$ and one defines

$$\begin{aligned} \phi_i(b_2 \otimes b_1) &= \phi_i(b_2) + \max(0, \phi_i(b_1) - \epsilon_i(b_2)) \\ \epsilon_i(b_2 \otimes b_1) &= \epsilon_i(b_1) + \max(0, -\phi_i(b_1) + \epsilon_i(b_2)). \end{aligned}$$

When $\phi_i(b_2 \otimes b_1) > 0$ (resp. $\epsilon_i(b_2 \otimes b_1) > 0$) one defines

$$f_i(b_2 \otimes b_1) = \begin{cases} b_2 \otimes f_i(b_1) & \text{if } \phi_i(b_1) > \epsilon_i(b_2) \\ f_i(b_2) \otimes b_1 & \text{if } \phi_i(b_1) \le \epsilon_i(b_2) \end{cases}$$

and respectively

$$e_i(b_2 \otimes b_1) = \begin{cases} b_2 \otimes e_i(b_1) & \text{if } \phi_i(b_1) \geq \epsilon_i(b_2) \\ e_i(b_2) \otimes b_1 & \text{if } \phi_i(b_1) < \epsilon_i(b_2). \end{cases}$$

An element of a tensor product of crystals is called a path.

3.3. Energy function

The definitions here follow [16]. Suppose that B_1 and B_2 are crystals of finite-dimensional $U_q(\mathfrak{g}')$-modules such that $B_2 \otimes B_1$ is connected. Then there is an isomorphism of P_{cl}-weighted I-crystals $B_2 \otimes B_1 \cong B_1 \otimes B_2$. This is called the local isomorphism. Let the image of $b_2 \otimes b_1 \in B_2 \otimes B_1$ under this isomorphism be denoted $b_1' \otimes b_2'$. Then there is a unique (up to a global additive constant) map $H : B_2 \otimes B_1 \to \mathbb{Z}$ such that

$$H(e_i(b_2 \otimes b_1)) = H(b_2 \otimes b_1) + \begin{cases} -1 & \text{if } i = 0,\ e_0(b_2 \otimes b_1) = e_0(b_2) \otimes b_1 \\ & \text{and } e_0(b_1' \otimes b_2') = e_0(b_1') \otimes b_2', \\ 1 & \text{if } i = 0,\ e_0(b_2 \otimes b_1) = b_2 \otimes e_0(b_1) \\ & \text{and } e_0(b_1' \otimes b_2') = b_1' \otimes e_0(b_2'), \\ 0 & \text{otherwise.} \end{cases}$$

This map is called the local energy function.

Let $B = B_L \otimes \cdots \otimes B_1$ with B_j the crystal of a finite-dimensional $U_q(\mathfrak{g}')$-module for $1 \leq j \leq L$. Assume that for all $1 \leq i < j \leq L$, $B_j \otimes B_i$ is a connected P_{cl}-weighted I-crystal. Given $b = b_L \otimes \cdots \otimes b_1 \in B$, denote by $b_j^{(i+1)}$ the $(i+1)$-th tensor factor in the image of b under the composition of local isomorphisms that switch B_j with B_k as k goes from $j-1$ down to $i+1$. Then define the energy function

$$E_B(b) = \sum_{1 \leq i < j \leq L} H_{j,i}(b_j^{(i+1)} \otimes b_i) \tag{3}$$

where $H_{j,i} : B_j \otimes B_i \to \mathbb{Z}$ is the local energy function. It satisfies the following property.

Lemma 1. [5, Prop. 1.1] *Suppose $i \in I$, $b \in B$ and $e_i(b)$ is defined. If $i \neq 0$ then $E_B(e_i(b)) = E_B(b)$. If $i = 0$ and b has the property that for any of its images $b' = b_L' \otimes \cdots \otimes b_1'$ under a composition of local isomorphisms, $e_0(b') = b_L' \otimes \cdots \otimes e_0(b_k') \otimes \cdots \otimes b_1'$ with $k \neq 1$, then $E_B(e_0(b)) = E_B(b) - 1$.*

3.4. Classically restricted paths

Say that $b \in B := B_L \otimes \cdots \otimes B_1$ is classically restricted if b is a $\overline{\mathfrak{g}}$-highest weight vector, that is, $e_i(b)$ is undefined for all $i \in J$. For $\lambda \in \overline{P}^+$ denote by $\mathcal{P}(B,\lambda)$ the set of classically restricted $b \in B$ of weight λ. Define the polynomial

$$K(B,\lambda)(q) = \sum_{b \in \mathcal{P}(B,\lambda)} q^{E_B(b)} \tag{4}$$

where E_B is the energy function on B. For $\mathfrak{g}$ of type $A_{n-1}^{(1)}$ $K(B,\lambda)(q)$ is the generalized Kostka polynomial [18, 19, 20].

3.5. Almost perfect crystals

Let B be the crystal of a finite-dimensional $U_q(\mathfrak{g}')$-module. Say that B is almost perfect of level ℓ [17] if it satisfies the following weakening of the definition of a perfect crystal [9, Def. 4.6.1]:

1. $B \otimes B$ is connected.
2. There is a $\Lambda' \in P_{cl}$ such that there is a unique $b' \in B$ such that $\mathrm{wt}(b') = \Lambda'$ and for every $b \in B$, $\mathrm{wt}(b) \in \Lambda' - \bigoplus_{i \in J} \mathbb{Z}_{\geq 0}\alpha_i$.
3. For every $b \in B$, $\langle K, \epsilon(b) \rangle \geq \ell$.
4. For every $\Lambda \in P_{cl}^+$ of level ℓ, there is a $b, b' \in B$ such that $\epsilon(b) = \phi(b') = \Lambda$.

B is said to be perfect if the elements b and b' in item 4 are unique.

3.6. Level restricted paths

From now on, fix a positive integer ℓ (the level).

For $1 \leq j \leq L$ let B_j be the crystal of a finite-dimensional $U_q(\mathfrak{g}')$-module, that is almost perfect of level at most ℓ.

Let $B = B_L \otimes \cdots \otimes B_1$, $\Lambda, \Lambda' \in P_{cl}^+$ weights of level ℓ, and $\mathcal{P}(B,\Lambda,\Lambda')$ the set of paths $b = b_L \otimes \cdots \otimes b_1 \in B$ such that $b \otimes u_\Lambda \in B \otimes \mathbb{B}(\Lambda)$ is a highest weight vector of weight Λ'.

In the special case that $\Lambda = \ell\Lambda_0$, the elements of $\mathcal{P}(B,\Lambda,\Lambda')$ are called the level-ℓ restricted paths of weight Λ'.

Theorem 2. [9] [13, Appendix A]. *Let $\mathfrak{g}$ be a nonexceptional affine Kac-Moody algebra, B the tensor product of crystals of finite-dimensional $U_q(\mathfrak{g}')$-modules that are almost perfect of level at most ℓ, and $\Lambda \in P_{cl}^+$ a weight of level ℓ. Then there is an isomorphism of P_{cl}-weighted I-crystals*

$$B \otimes \mathbb{B}(\Lambda) \cong \bigoplus_{\Lambda' \in P_{cl}^+} \bigoplus_{b \in \mathcal{P}(B,\Lambda,\Lambda')} \mathbb{B}(\Lambda') \tag{5}$$

where Λ' is of level ℓ.

This isomorphism of P_{cl}-weighted crystals can be lifted to one of P-weighted crystals by specifying an integer multiple of $a_0^{-1}\delta$ for each highest weight vector in $B\otimes\mathbb{B}(\Lambda)$. However for our purposes this should be done in a way that extends the definition of the energy function for B. To this end, choose a perfect crystal B_0 of level ℓ, and assume that for all $0\le i<j\le L$, $B_j\otimes B_i$ is connected. Let $b_0\in B_0$ be the unique element such that $\phi(b_0)=\Lambda$. Define the energy function $E:B\to\mathbb{Z}$ by $E(b)=E_{B,B_0}(b\otimes b_0)$ where $E_{B,B_0}:B\otimes B_0\to\mathbb{Z}$ is the energy function defined in (3). For $b\in\mathcal{P}(B,\Lambda,\Lambda')$, define an affine weight function $\mathrm{wt}(b\otimes u_\Lambda)=\mathrm{af}(\Lambda')-E(b)a_0^{-1}\delta$. This defines the P-weight of every highest weight vector in $B\otimes\mathbb{B}(\Lambda)$ and hence a P-weight function for all of $B\otimes\mathbb{B}(\Lambda)$.

Then one has the following P-weighted analogue of (5):

$$B\otimes\mathbb{B}(\mathrm{af}(\Lambda))\cong\bigoplus_{\Lambda'\in P_{cl}^+}\bigoplus_{b\in\mathcal{P}(B,\Lambda,\Lambda')}\mathbb{B}(\mathrm{wt}(b\otimes u_\Lambda)) \tag{6}$$

where Λ' is of level ℓ. This decomposition can be described by the polynomial

$$K(B,\Lambda,\Lambda',B_0)(q)=\sum_{b\in\mathcal{P}(B,\Lambda,\Lambda')}q^{E(b)}. \tag{7}$$

Our goal is to prove a formula for the polynomial $K(B,\Lambda,\Lambda',B_0)(q)$.

§4. General bosonic formula

Let J be the antisymmetrizer

$$J=\sum_{w\in W}\varepsilon(w)w.$$

Write

$$R=\prod_{\alpha\in\Delta_+}(1-\exp(-\alpha))^{\mathrm{mult}(\alpha)}$$

where Δ_+ is the set of positive roots of $\mathfrak{g}$ and $\mathrm{mult}(\alpha)$ is the dimension of the α-weight space in $\mathfrak{g}$.

Let $\rho\in P^+$ be the unique weight defined by $\langle\alpha_i^\vee,\rho\rangle=1$ for all $i\in I$ and $\langle d,\rho\rangle=0$. It satisfies $\langle\theta^\vee,\rho\rangle=a_0^{-1}\langle K-\alpha_0^\vee,\rho\rangle=a_0^{-1}(h^\vee-1)$ where $h^\vee=\sum_{i\in I}a_i^\vee$ is the dual Coxeter number. Define the operator

$$\Pi(p)=R^{-1}e^{-\rho}J(e^\rho p).$$

where R^{-1} makes sense by expanding the reciprocals of the factors of R in geometric series. The computation is defined in a suitable completion of $\mathbb{Z}[P]$. One has $\Pi\,(e^\Lambda) = \operatorname{ch}\mathbb{V}(\Lambda)$ for all $\Lambda \in P^+$, which is the Weyl-Kac character formula [7, Theorem 10.4].

Theorem 3. *Let $\mathfrak{g}$ be a nonexceptional affine Kac-Moody algebra, B' the crystal of a finite direct sum of irreducible integrable highest weight $U_q(\mathfrak{g})$-modules and $B \subset B'$ a subset such that:*

1. *B is closed under e_i for all $i \in I$.*
2. *B' is generated by B.*
3. *For all $b \in B$ and $i \in I$, if $\epsilon_i(b) > 0$ then the i-string of b in B' is contained in B.*

Then

$$\Pi\ \operatorname{ch}(B) = \operatorname{ch}(B'). \tag{8}$$

Proof. Without loss of generality it may be assumed that $B' = \mathbb{B}(\Lambda)$ for some $\Lambda \in P^+$. Multiplying both sides of (8) by $R\,e^\rho$, one obtains

$$\sum_{(w,b)\in W\times B} \varepsilon(w)w(e^{\mathrm{wt}(b)+\rho}) = \sum_{w\in W} \varepsilon(w)w(e^{\Lambda+\rho}).$$

Observe that both sides are W-alternating. The W-alternants have a basis given by $J(\Lambda+\rho)$ where $\Lambda \in P^+$. Taking the coefficient of $e^{\Lambda+\rho}$ on both sides,

$$\sum_{(w,b)\in\mathcal{S}} \varepsilon(w) = 1 \tag{9}$$

where $\mathcal{S}$ is the set of pairs $(w,b) \in W \times B$ such that

$$\mathrm{wt}(b) = w^{-1}(\Lambda+\rho) - \rho. \tag{10}$$

Observe that if $(w,b) \in \mathcal{S}$ is such that b is a highest weight vector, then $w = 1$ and $b = u_\Lambda$, for both of the regular dominant weights $\mathrm{wt}(b) + \rho$ and $\Lambda+\rho$ are in the same W-orbit and hence must be equal. Conditions 1 and 2 ensure that $u_\Lambda \in B$. Let $\mathcal{S}' = \mathcal{S} - \{(1,u_\Lambda)\}$. It is enough to show that there is an involution $\Phi : \mathcal{S}' \to \mathcal{S}'$ with no fixed points, such that if $\Phi(w,b) = (w',b')$ then w and w' have opposite sign. In this case Φ is said to be sign-reversing. Let $\mathcal{S}_i$ be the set of pairs $(w,b) \in \mathcal{S}'$ such that $\epsilon_i(b) > 0$. Define the map $\Phi_i : \mathcal{S}_i \to \mathcal{S}_i$ by $\Phi_i(w,b) = (wr_i, s_i e_i(b))$.

Note that $s_i e_i(b) \in B$ by condition 3. The condition (10) for $\Phi_i(w,b)$ is

$$\begin{aligned}(wr_i)^{-1}(\Lambda+\rho)-\rho &= r_i w^{-1}(\Lambda+\rho) - r_i\rho + r_i\rho - \rho \\ &= r_i(w^{-1}(\Lambda+\rho)-\rho) - \langle \alpha_i^\vee, \rho\rangle \alpha_i \\ &= r_i(\mathrm{wt}(b)) - \alpha_i = \mathrm{wt}(f_i s_i(b)) = \mathrm{wt}(s_i e_i(b)).\end{aligned}$$

Since $s_i e_i(b) = f_i s_i(b)$, $\epsilon_i(s_i e_i(b)) > 0$, so that $(wr_i, s_i e_i(b)) \in \mathcal{S}_i$. This shows that Φ_i is well-defined. It follows directly from the definitions that Φ_i is a sign-reversing involution.

Since $\mathcal{S}' = \bigcup_{i\in I} \mathcal{S}_i$ it suffices to define a global involutive choice of the canceling root direction for each pair $(w,b) \in \mathcal{S}'$, that is, a function $v : \mathcal{S}' \to I$ such that if $v(w,b) = i$ then

(V1) $(w,b) \in \mathcal{S}_i$.

(V2) $v(wr_i, s_i e_i(b)) = i$.

Let $\Lambda = \Lambda_{i_1} + \cdots + \Lambda_{i_\ell}$ be an expression of Λ as a sum of fundamental weights. By [6, Lemma 8.3.1], $\mathbb{B}(\Lambda)$ is isomorphic to the full subcrystal of $\mathbb{B}(\Lambda_{i_\ell}) \otimes \cdots \otimes \mathbb{B}(\Lambda_{i_1})$ generated by $u_{\Lambda_{i_\ell}} \otimes \cdots \otimes u_{\Lambda_{i_1}}$.

Given $(w,b) \in \mathcal{S}'$, let $b_\ell \otimes \cdots \otimes b_1$ be the image of b in the above tensor product of crystals of modules of fundamental highest weight. Let r be minimal such that $b_r \otimes b_{r-1} \otimes \cdots \otimes b_1$ is not a highest weight vector. Then $b_{r-1} \otimes \cdots \otimes b_1$ is a highest weight vector, say of weight Λ'.

Let $\mathcal{B}$ be a perfect crystal of the same level as Λ_{i_r}. Given any $L > 0$, the theory of perfect crystals [9, Section 4.5] gives an isomorphism of P-weighted crystals

$$\mathbb{B}(\Lambda_{i_r}) \cong \mathcal{B}^{\otimes L} \otimes \mathbb{B}(\Lambda_j)$$

where j is determined by i_r and L and $\mathcal{B}^{\otimes L}$ is P-weighted using the energy function.

Let $b_r \in \mathbb{B}(\Lambda_{i_r})$ have image $p_{-1} \otimes \cdots \otimes p_{-L} \otimes u'$ where $u' \in \mathbb{B}(\Lambda_j)$. Assume that L is large enough so that $u' = u_{\Lambda_j}$. If one takes the image of b_r in such a tensor product for $L' > L$ the tensor factors p_{-1} through p_{-L} do not change.

Let k be minimal such that $p_k \otimes \cdots \otimes p_{-L} \otimes u_{\Lambda_j} \otimes u_{\Lambda'}$ is not a highest weight vector. Observe that k is independent of L as long as L is big enough. Then $p_{k-1} \otimes \cdots \otimes p_{-L} \otimes u_{\Lambda_j} \otimes u_{\Lambda'}$ is a highest weight vector, say of weight Λ''.

So $p_k \in \mathcal{B}$ is such that $\epsilon_i(p_k) > \langle \alpha_i^\vee, \Lambda''\rangle$ for some $i \in I$; let I' be the set of such $i \in I$.

Fix an $i \in I'$. Consider the same constructions for $b' = s_i e_i(b)$. Let $b'_\ell \otimes \cdots \otimes b'_1$ be the image of b' in the above tensor product of irreducible crystals of fundamental highest weights. Then $b'_{r-1} \otimes \cdots \otimes$

$b'_1 = b_{r-1} \otimes \cdots \otimes b_1$ and $b'_r \otimes \cdots \otimes b'_1$ is not a highest weight vector; in particular it admits e_i. Take L large enough so that the image of b'_r in $\mathcal{B}^{\otimes L} \otimes \mathbb{B}(\Lambda_j)$ also has the form $p'_{-1} \otimes \cdots \otimes p'_{-L} \otimes u_{\Lambda_j}$. Then $p_{k-1} \otimes \cdots \otimes p_{-L} = p'_{k-1} \otimes \cdots \otimes p'_{-L}$ and $p'_k \otimes \cdots \otimes p'_{-L} \otimes u_{\Lambda_j} \otimes u_{\Lambda'}$ admits e_i.

The level of the fundamental weight Λ_i is $a_i^\vee$. For the affine algebras $A_n^{(1)}$ and $C_n^{(1)}$, $a_i^\vee = 1$ for all $i \in I$. For all others $1 \leq a_i^\vee \leq 2$. The theorem now follows from Lemma 4 below, applied with the affine highest weight vector $u_{\Lambda''}$, perfect crystal element $p_k \in \mathcal{B}$, and left tensor factor element $\cdots \otimes b_{r+2} \otimes b_{r+1} \otimes p_{-1} \otimes \cdots \otimes p_{k+1} \in \cdots \mathbb{B}(\Lambda_{i_{r+2}}) \otimes \mathbb{B}(\Lambda_{i_{r+1}}) \otimes \mathcal{B}^{\otimes 1-k}$. Q.E.D.

We remark that in Lemma 4, the function v constructed in the proof, is independent of Λ as well.

Lemma 4. *Let $\mathfrak{g}$ be a nonexceptional affine Kac-Moody algebra and ℓ' the level of some fundamental weight. Then there is a perfect crystal $\mathcal{B}$ of level ℓ' with the following properties.*

Let Λ be a dominant integral weight of level $\ell \geq \ell'$. Denote by S the set of elements $b_1 \in \mathcal{B}$ such that $b_1 \otimes u_\Lambda$ is not a highest weight vector in $\mathcal{B} \otimes \mathbb{B}(\Lambda)$.

Then there is a map $v : S \to I$ (depending only on Λ, $\mathcal{B}$, and $b_1 \in S \subset \mathcal{B}$) such that if $v(b_1) = i$ then

1. $\epsilon_i(b_1 \otimes u_\Lambda) > 0$.
2. *For any crystal B_2 and element $b_2 \in B_2$ such that the connected component of the element $b_2 \otimes b_1 \otimes u_\Lambda$ in $B_2 \otimes \mathcal{B} \otimes \mathbb{B}(\Lambda)$ is isomorphic to a crystal of the form $\mathbb{B}(\Lambda')$, and writing $b'_2 \otimes b'_1 \otimes u_\Lambda = s_i e_i(b_2 \otimes b_1 \otimes u_\Lambda)$, one has $b'_1 \in S$ and $v(b'_1) = i$.*

Proof. For the involutive property 2, it is sufficient that v is constant on the nondominant part of every string. Hence one only needs to consider

(11) elements b_1 that are on the nondominant part of at least two strings of length ≥ 2.

Perfect crystals of level one for $A_n^{(1)}$ $(n \geq 1)$, $B_n^{(1)}$ $(n \geq 3)$, $D_n^{(1)}$ $(n \geq 4)$, $A_{2n}^{(2)}$ $(n \geq 1)$, $A_{2n-1}^{(2)}$ $(n \geq 3)$ and $D_{n+1}^{(2)}$ $(n \geq 2)$ are listed in Table 1 (see [9, Section 6]). Note that there are no elements satisfying (11). This guarantees the existence of the map v with the desired properties.

The crystal $B(2\Lambda_1) \oplus B(0)$ is a level one perfect crystal for $C_n^{(1)}$ $(n \geq 2)$ [8]. The crystal graph corresponding to the integrable highest weight module $V(\Lambda_1)$ of $U_q(C_n)$ is given by [14, (4.2.4)]

$A_n^{(1)}$	$\boxed{1} \xrightarrow{1} \boxed{2} \xrightarrow{2} \boxed{3} \xrightarrow{3} \cdots \xrightarrow{n} \boxed{n+1}$; $\boxed{n+1} \xrightarrow{0} \boxed{1}$
$B_n^{(1)}$	$\boxed{1} \xrightarrow{1} \boxed{2} \xrightarrow{2} \cdots \xrightarrow{n-1} \boxed{n} \xrightarrow{n} \boxed{0} \xrightarrow{n} \boxed{\bar{n}} \xrightarrow{n-1} \cdots \xrightarrow{2} \boxed{\bar{2}} \xrightarrow{1} \boxed{\bar{1}}$; $\boxed{\bar{1}} \xrightarrow{0} \boxed{2}$; $\boxed{\bar{2}} \xrightarrow{0} \boxed{1}$
$D_n^{(1)}$	$\boxed{1} \xrightarrow{1} \boxed{2} \xrightarrow{2} \cdots \xrightarrow{n-2} \boxed{n-1}$; $\boxed{n-1} \xrightarrow{n-1} \boxed{n} \xrightarrow{n} \boxed{\overline{n-1}}$; $\boxed{n-1} \xrightarrow{n} \boxed{\bar{n}} \xrightarrow{n-1} \boxed{\overline{n-1}}$; $\boxed{\overline{n-1}} \xrightarrow{n-2} \cdots \xrightarrow{2} \boxed{\bar{2}} \xrightarrow{1} \boxed{\bar{1}}$; $\boxed{\bar{1}} \xrightarrow{0} \boxed{2}$; $\boxed{\bar{2}} \xrightarrow{0} \boxed{1}$
$A_{2n}^{(2)}$	$\boxed{1} \xrightarrow{1} \boxed{2} \xrightarrow{2} \cdots \xrightarrow{n-1} \boxed{n} \xrightarrow{n} \boxed{\bar{n}} \xrightarrow{n-1} \cdots \xrightarrow{2} \boxed{\bar{2}} \xrightarrow{1} \boxed{\bar{1}}$; $\boxed{\bar{1}} \xrightarrow{0} \emptyset \xrightarrow{0} \boxed{1}$
$A_{2n-1}^{(2)}$	$\boxed{1} \xrightarrow{1} \boxed{2} \xrightarrow{2} \cdots \xrightarrow{n-1} \boxed{n} \xrightarrow{n} \boxed{\bar{n}} \xrightarrow{n-1} \cdots \xrightarrow{2} \boxed{\bar{2}} \xrightarrow{1} \boxed{\bar{1}}$; $\boxed{\bar{1}} \xrightarrow{0} \boxed{2}$; $\boxed{\bar{2}} \xrightarrow{0} \boxed{1}$
$D_{n+1}^{(2)}$	$\boxed{1} \xrightarrow{1} \boxed{2} \xrightarrow{2} \cdots \xrightarrow{n-1} \boxed{n} \xrightarrow{n} \boxed{0} \xrightarrow{n} \boxed{\bar{n}} \xrightarrow{n-1} \cdots \xrightarrow{2} \boxed{\bar{2}} \xrightarrow{1} \boxed{\bar{1}}$; $\boxed{\bar{1}} \xrightarrow{0} \emptyset \xrightarrow{0} \boxed{1}$

TABLE 1. Level one perfect crystals

$$\boxed{1} \xrightarrow{1} \boxed{2} \xrightarrow{2} \cdots \xrightarrow{n-1} \boxed{n} \xrightarrow{n} \boxed{\bar{n}} \xrightarrow{n-1} \cdots \xrightarrow{2} \boxed{\bar{2}} \xrightarrow{1} \boxed{\bar{1}}.$$

The crystal $B(2\Lambda_1)$ is the connected component of $B(\Lambda_1) \otimes B(\Lambda_1)$ containing $u_{\Lambda_1} \otimes u_{\Lambda_1}$ (see [14, Section 4.4]) which fixes the action of e_i and f_i for $1 \leq i \leq n$. The edges in $B(2\Lambda_1) \oplus B(0)$ corresponding to f_0 are given by [8]

$$\boxed{i}\boxed{\bar{1}} \xrightarrow{0} \boxed{1}\boxed{i} \qquad \text{for } i \neq 1, \bar{1}$$
$$\boxed{\bar{1}}\boxed{\bar{1}} \xrightarrow{0} \emptyset$$
$$\emptyset \xrightarrow{0} \boxed{1}\boxed{1}.$$

There are the following strings of length greater than one

$$\begin{array}{lllll}
 & \boxed{k}\boxed{k} \xrightarrow{k} \boxed{k}\boxed{k+1} \xrightarrow{k} \boxed{k+1}\boxed{k+1} & \text{for } 1 \leq k < n \\
\text{(12a)} & \boxed{k}\boxed{\overline{k+1}} \xrightarrow{k} \boxed{k}\boxed{\bar{k}} \xrightarrow{k} \boxed{k+1}\boxed{\bar{k}} & \text{for } 1 \leq k < n \\
 & \boxed{\overline{k+1}}\boxed{\overline{k+1}} \xrightarrow{k} \boxed{\overline{k+1}}\boxed{\bar{k}} \xrightarrow{k} \boxed{\bar{k}}\boxed{\bar{k}} & \text{for } 1 \leq k < n \\
\text{(12b)} & \boxed{n}\boxed{n} \xrightarrow{n} \boxed{n}\boxed{\bar{n}} \xrightarrow{n} \boxed{\bar{n}}\boxed{\bar{n}} & \\
 & \boxed{\bar{1}}\boxed{\bar{1}} \xrightarrow{0} \emptyset \xrightarrow{0} \boxed{1}\boxed{1} &
\end{array}$$

Note that none of the elements satisfies (11).

For type $A^{(2)}_{2n-1}$ the crystal $B(2\Lambda_1)$ is perfect of level 2 [10, Sec. 1.6 and 6.7]. The elements are given by $\boxed{x}\boxed{y}$ with $x \leq y$ and $x, y \in \{1 < 2 < \cdots < n < \bar{n} < \cdots < \bar{2} < \bar{1}\}$. The action of f_i for $i = 1, 2, \ldots, n$ is the same as for the above $C^{(1)}_n$ crystal of level one, and $f_0 = \sigma \circ f_1 \circ \sigma$ where σ is the involution that exchanges 1 and $\bar{1}$ (with appropriate reorderings).

The strings of length greater than one are the same as in (12a) and (12b). In addition there are the following 0-strings of length 2

$$\begin{array}{ll}
 & \boxed{\bar{1}}\boxed{\bar{1}} \xrightarrow{0} \boxed{2}\boxed{\bar{1}} \xrightarrow{0} \boxed{2}\boxed{2} \\
\text{(13)} & \boxed{\bar{2}}\boxed{\bar{1}} \xrightarrow{0} \boxed{1}\boxed{\bar{1}} \xrightarrow{0} \boxed{1}\boxed{2} \\
 & \boxed{\bar{2}}\boxed{\bar{2}} \xrightarrow{0} \boxed{1}\boxed{\bar{2}} \xrightarrow{0} \boxed{1}\boxed{1} .
\end{array}$$

The only elements fulfilling (11) are $\boxed{1}\boxed{\bar{1}}$, $\boxed{1}\boxed{2}$, $\boxed{2}\boxed{\bar{1}}$, and $\boxed{2}\boxed{2}$ which belong to a 0-string and a 1-string of length two. It can be checked that setting $v(b) = 0$ for b one of these four elements guarantees the involutive condition of v.

For type $B^{(1)}_n$ the crystal $B(2\Lambda_1)$ is perfect of level 2 [10, Sec. 1.7 and 6.8]. It consists of the elements $\boxed{x}\boxed{y}$ with $x \leq y$ and $x, y \in \{1 < \cdots < n < 0 < \bar{n} < \cdots < \bar{1}\}$; $x = y = 0$ is excluded. The action of f_i for $i = 1, 2, \ldots, n$ is given by the tensor product rule using the action on the level 1 crystal of $B^{(1)}_n$ as given in Table 1, and $f_0 = \sigma \circ f_1 \circ \sigma$ where σ is the involution that exchanges 1 and $\bar{1}$ (with appropriate reorderings).

The strings of length greater than one are those of equations (12a) and (13) and in addition the following n-string of length four

$$\boxed{n}\boxed{n} \xrightarrow{n} \boxed{n}\boxed{0} \xrightarrow{n} \boxed{n}\boxed{\bar{n}} \xrightarrow{n} \boxed{0}\boxed{\bar{n}} \xrightarrow{n} \boxed{\bar{n}}\boxed{\bar{n}}\ . \tag{14}$$

The same four elements as for $A^{(2)}_{2n-1}$ satisfy (11) and again setting $v(b) = 0$ for these ensures the involutive property of v.

For type $D^{(1)}_n$ the crystal $B(2\Lambda_1)$ is perfect of level 2 [10, Sec. 1.8 and 6.9]. It consists of the elements $\boxed{x}\boxed{y}$ with $x \leq y$ and $x, y \in \{1 < 2 < \cdots < n, \bar{n} < \cdots < \bar{1}\}$, the cases $x = n, y = \bar{n}$ and $x = \bar{n}, y = n$ being excluded. The action of f_i for $i = 1, 2, \ldots, n$ is given by the tensor product rule using the action on the level 1 crystal of $D^{(1)}_n$ as given in Table 1, and $f_0 = \sigma \circ f_1 \circ \sigma$ where σ is the involution that exchanges 1 and $\bar{1}$ (with appropriate reorderings).

Again the strings of length greater than one are the same as in equations (12a) and (13) plus the following n-strings

$$\boxed{n}\boxed{n} \xrightarrow{n} \boxed{n}\boxed{\overline{n-1}} \xrightarrow{n} \boxed{\overline{n-1}}\boxed{\overline{n-1}}$$

$$\boxed{n-1}\boxed{n-1} \xrightarrow{n} \boxed{n-1}\boxed{\bar{n}} \xrightarrow{n} \boxed{\bar{n}}\boxed{\bar{n}}$$

$$\boxed{n-1}\boxed{n} \xrightarrow{n} \boxed{n-1}\boxed{\overline{n-1}} \xrightarrow{n} \boxed{\bar{n}}\boxed{\overline{n-1}}\ .$$

In addition to the four elements $\boxed{1}\boxed{\bar{1}}$, $\boxed{1}\boxed{2}$, $\boxed{2}\boxed{\bar{1}}$, and $\boxed{2}\boxed{2}$ also the elements $\boxed{\overline{n-1}}\boxed{\overline{n-1}}$, $\boxed{n-1}\boxed{\overline{n-1}}$, $\boxed{n}\boxed{\overline{n-1}}$, and $\boxed{\bar{n}}\boxed{\overline{n-1}}$ satisfy (11). The latter ones are contained in an $(n-1)$-string and an n-string. Setting $v(b) = 0$ for the first four elements and $v(b) = n$ for the last four elements ensures the involutive property of v.

The crystal $B(0) \oplus B(\Lambda_1) \oplus B(2\Lambda_1)$ is a level 2 perfect crystal for $D^{(2)}_{n+1}$ [10, Sections 1.9 and 6.10]. The elements of this crystal are $\emptyset$, $\boxed{x}$, and $\boxed{x}\boxed{y}$ with $x, y \in \{1 < 2 < \cdots < n < 0 < \bar{n} < \cdots < \bar{1}\}$ and $x \leq y$; $x = y = 0$ is excluded. The action of f_i for $i = 1, 2, \ldots, n$ is given by the tensor product rule using the action on the level 1 crystal of $D^{(2)}_{n+1}$

as given in Table 1, and the action of f_0 is given by

$$
\begin{array}{lll}
\emptyset \xrightarrow{0} \boxed{1} & & \\
\boxed{x} \xrightarrow{0} \boxed{1}\boxed{x} & & \text{for } x \neq \bar{1} \\
\boxed{\bar{1}} \xrightarrow{0} \emptyset & & \\
\boxed{x}\boxed{\bar{1}} \xrightarrow{0} \boxed{x} & & \text{for } x \neq 1
\end{array} \tag{15}
$$

and undefined otherwise.

The strings of length greater than one are given by (12a), (14) and

$$\boxed{\bar{1}}\boxed{\bar{1}} \xrightarrow{0} \boxed{\bar{1}} \xrightarrow{0} \emptyset \xrightarrow{0} \boxed{1} \xrightarrow{0} \boxed{1}\boxed{1} \tag{16}$$

$$\boxed{n} \xrightarrow{n} \boxed{0} \xrightarrow{n} \boxed{\bar{n}} \tag{17}$$

There are no elements with property (11).

The crystal $B(0) \oplus B(\Lambda_1) \oplus B(2\Lambda_1)$ is a level 2 perfect crystal for $A_{2n}^{(2)}$ [10, Sec. 1.10 and 6.11]. The elements of this crystal are $\emptyset$, $\boxed{x}$, and $\boxed{x}\boxed{y}$ with $x, y \in \{1 < 2 < \cdots < n < \bar{n} < \cdots < \bar{1}\}$ and $x \leq y$. The action of f_i for $i = 1, 2, \ldots, n$ is given by the tensor product rule using the action on the level 1 crystal of $A_{2n}^{(2)}$ as given in Table 1, and the action of f_0 is the same as in (15).

The strings of length greater than one are as in (12a) for $n \geq 2$, (12b) and (16). Again there are no elements with property (11). Q.E.D.

Remark 5. Suppose $\mathfrak{g}$ is of type $A_{n-1}^{(1)}$ in Lemma 4. The function v amounts to a canonical choice of a simple root i among those such that the given element admits e_i. Consider $b \in \mathbb{B}(\Lambda_r)$ such that $b \neq u_{\Lambda_r}$. In addition to the realization of the crystal $\mathbb{B}(\Lambda_r)$ by the space of homogeneous paths using the crystal given in the proof of Lemma 4, one may also consider the realization in [2] by n-regular partitions. Suppose λ is the partition corresponding to b. Then up to the Dynkin diagram automorphism that sends $r+i$ to $r-i$ modulo n, the choice of violation v corresponds to the corner cell of λ that is in the rightmost column of λ. This choice of corner cell is used in [15] to define the smallest Demazure crystal of $\mathbb{B}(\Lambda_r)$ containing b.

§5. Inhomogeneous paths

Theorem 6. *Let $\mathfrak{g}$ be as in Theorem 3, and B, Λ, and B_0 be as in* (6). *Suppose in addition that for all $1 \leq j \leq L$ and $b \in B_j$, if $b \otimes b_0 \mapsto b_0' \otimes b'$*

under the local isomorphism $B_j \otimes B_0 \to B_0 \otimes B_j$ and $e_0(b \otimes b_0) = e_0(b) \otimes b_0$ then $e_0(b'_0 \otimes b') = e_0(b'_0) \otimes b'$. Then

$$\Pi\,(\mathrm{ch}(B \otimes u_\Lambda)) = \mathrm{ch}(B \otimes \mathbb{B}(\Lambda)). \tag{18}$$

Proof. It is enough to verify the hypotheses of Theorem 3, applied to $B \otimes u_\Lambda \subset B \otimes \mathbb{B}(\Lambda)$. $B \otimes \mathbb{B}(\Lambda)$ is isomorphic to a direct sum of irreducible integrable highest weight modules by Theorem 2. $B \otimes u_\Lambda$ is obviously closed under the e_i. It follows from [11, Lemma 1] that $B \otimes u_\Lambda$ generates $B \otimes \mathbb{B}(\Lambda)$. To check the third condition of Theorem 3, let $b \in B$ and $i \in I$ be such that $\epsilon_i(b \otimes u_\Lambda) > 0$. Then $\epsilon_i(b) > \phi_i(u_\Lambda) = \langle \alpha_i^\vee, \Lambda \rangle$. This implies that the i-string of $b \otimes u_\Lambda$ inside $B \otimes \mathbb{B}(\Lambda)$, consists of vectors of the form $b' \otimes u_\Lambda$ where $b' \in B$.

Finally, Lemma 1 with B replaced by $B \otimes B_0$ guarantees that the affine weight function on $B \otimes \mathbb{B}(\Lambda)$ determined by its value on highest weight vectors, agrees on the subset $B \otimes u_\Lambda$ with the function $\mathrm{wt}(b) = \mathrm{af}(\mathrm{wt}'(b)) - E_{B,B_0}(b \otimes b_0) a_0^{-1} \delta$ where $\mathrm{wt}' : B \to P_{cl}$ is the original weight function. Q.E.D.

Remark 7. Observe that even without the extra hypothesis on the action of e_0 in Theorem 6, one obtains a bosonic formula. The extra condition is only needed to show that the energy function $b \mapsto E_{B,B_0}(b \otimes b_0)$ gives rise to the correct affine weight for all elements of the form $b \otimes u_\Lambda$ and not just on the highest weight vectors. Perhaps this extra condition is always a consequence of the other hypotheses.

Now the formula (18) is written more explicitly. Let $m \in \mathbb{Z}$ and $\Lambda, \Lambda' \in \mathrm{af}(P_{cl}^+)$ be of level ℓ. A formula equivalent to (18) is obtained by taking the coefficient of $\mathrm{ch}\mathbb{V}(\Lambda' - m a_0^{-1} \delta)$ on both sides:

$$[q^m] K(B, \Lambda, \Lambda', B_0)(q) = \sum_{(w,b) \in \mathcal{S}} \varepsilon(w)$$

where $\mathcal{S}$ is the set of pairs $(w, b) \in W \times B$ such that

$$w^{-1}(\Lambda' + \rho) - m a_0^{-1} \delta - \rho = \mathrm{wt}(b \otimes u_\Lambda). \tag{19}$$

Let M be the sublattice of $\overline{P}$ given by the image under ν of the $\mathbb{Z}$-span of the orbit $\overline{W}\theta^\vee$. Let $T \subset GL(\mathfrak{h}^*)$ be the group of translations by the elements of M, where $t_\alpha \in T$ is translation by $\alpha \in M$. Then $W \cong T \rtimes \overline{W}$ and $r_0 = t_{\nu(\theta^\vee)} r_\theta$. For $\alpha \in M$ and $\Lambda \in P$ of level ℓ, one has [7, (6.5.2)]

$$t_\alpha(\Lambda) = \Lambda + \ell\alpha - ((\Lambda|\alpha) + \tfrac{1}{2}\,|\alpha|^2 \ell)\delta. \tag{20}$$

The action of $\tau \in \overline{W}$ on the level ℓ weight Λ is given by

$$\tau(\Lambda) = \tau(\overline{\Lambda} + \ell\Lambda_0) = \tau(\overline{\Lambda}) + \ell\Lambda_0.$$

Now $\rho = h^\vee\Lambda_0 + \overline{\rho}$ where $h^\vee$ is the dual Coxeter number and $\overline{\rho}$ is the half-sum of the positive roots in $\overline{\mathfrak{g}}$.

Recall that $\overline{W}$ leaves δ invariant. In (19) write $w = t_\alpha\tau$ where $\tau \in \overline{W}$ and $\alpha \in M$, obtaining

$$\begin{aligned}
\mathrm{wt}(b \otimes u_\Lambda) =& \tau^{-1}t_{-\alpha}(\Lambda' + \rho) - ma_0^{-1}\delta - \rho \\
=& - ma_0^{-1}\delta - \rho + \tau^{-1}\{\Lambda' + \rho - (\ell + h^\vee)\alpha \\
& - \{(\Lambda' + \rho| - \alpha) + \tfrac{1}{2}|\alpha|^2(\ell + h^\vee)\}\delta\} \\
=& \ell\Lambda_0 - \overline{\rho} + \tau^{-1}(\overline{\Lambda'} + \overline{\rho} - (\ell + h^\vee)\alpha) \\
& + \{-ma_0^{-1} + (\overline{\Lambda'} + \overline{\rho}|\alpha) - \tfrac{1}{2}|\alpha|^2(\ell + h^\vee)\}\delta
\end{aligned}$$

Since both sides are weights of level ℓ, by equating coefficients of δ and projections into $\overline{P}$, one obtains the equivalent conditions

$$\overline{\mathrm{wt}(b)} = -\overline{\Lambda} - \overline{\rho} + \tau^{-1}(\overline{\Lambda'} - (\ell + h^\vee)\alpha + \overline{\rho}) \tag{21}$$

and

$$a_0^{-1}E(b) = a_0^{-1}m - (\overline{\Lambda'} + \overline{\rho}|\alpha) + \tfrac{1}{2}|\alpha|^2(\ell + h^\vee). \tag{22}$$

Therefore one has the equality

(23)
$$K(B, \Lambda, \Lambda', B_0)(q) = \sum_{\tau\in\overline{W}} \sum_{\alpha\in M} \sum_{b\in B} \varepsilon(\tau) q^{E(b) + a_0(\overline{\Lambda'}+\overline{\rho}|\alpha) - \frac{1}{2}a_0|\alpha|^2(\ell+h^\vee)}$$

where $b \in B$ satisfies

$$\mathrm{wt}(b) = -\overline{\Lambda} - \overline{\rho} + \tau^{-1}(\overline{\Lambda'} - (\ell + h^\vee)\alpha + \overline{\rho}).$$

§6. Type A

6.1. Conjecture of [3]

For simplicity let us assume that $\mathfrak{g}$ is of untwisted affine type, where $a_0 = 1$ and $(\overline{\rho}|\theta) = h^\vee - 1$ [7, Ex. 6.2].

Let $\Lambda \in P$ be a weight of level ℓ but not necessarily dominant. Consider the weight $\Lambda + \rho$. If it is regular (not fixed by any $w \in W$)

then there is a unique $w \in W$ such that $w(\Lambda+\rho) \in P^+$. It follows from the definition of Π that

$$(24) \quad \Pi\, e^{\Lambda} = \begin{cases} \varepsilon(w)\mathrm{ch}\mathbb{V}(w(\Lambda+\rho)-\rho) & \text{if } \Lambda+\rho \text{ is } W\text{-regular and} \\ & w(\Lambda+\rho) \in P^+ \\ 0 & \text{if } \Lambda+\rho \text{ is not } W\text{-regular.} \end{cases}$$

Then for all $i \in I$,

$$(25) \qquad -\Pi\, e^{\Lambda} = \Pi\, e^{r_i(\Lambda+\rho)-\rho}.$$

Suppose $i \neq 0$. Then

$$\begin{aligned} r_i(\Lambda+\rho)-\rho &= (\ell+h^\vee)\Lambda_0 + r_i(\overline{\Lambda}+\overline{\rho}) - (h^\vee\Lambda_0+\overline{\rho}) \\ &= \ell\Lambda_0 - \alpha_i + r_i(\overline{\Lambda}). \end{aligned}$$

For $i = 0$, recall that

$$r_0 = t_{\nu(\theta^\vee)} r_\theta = t_\theta r_\theta = r_\theta t_{-\theta}.$$

Then

$$\begin{aligned} t_{-\theta}(\Lambda+\rho) &= \Lambda+\rho-(\ell+h^\vee)\theta + \{(\Lambda+\rho|\theta) - \tfrac{1}{2}|\theta|^2(\ell+h^\vee)\}\delta \\ &= (\ell+h^\vee)\Lambda_0 + \overline{\rho} + \overline{\Lambda} - (\ell+h^\vee)\theta + \{(\overline{\Lambda}|\theta) - (1+\ell)\}\delta \end{aligned}$$

and

$$\begin{aligned} r_0(\Lambda+\rho)-\rho &= r_\theta\big\{(\ell+h^\vee)\Lambda_0 + \overline{\rho} + \overline{\Lambda} - (\ell+h^\vee)\theta \\ &\quad + \{(\overline{\Lambda}|\theta) - (1+\ell)\}\delta\big\} - \rho \\ &= (\ell+h^\vee)\Lambda_0 + \overline{\rho} - \langle\theta^\vee,\overline{\rho}\rangle\theta + r_\theta(\overline{\Lambda}) \\ &\quad + (\ell+h^\vee)\theta + \{(\overline{\Lambda}|\theta) - (1+\ell)\}\delta - (h^\vee\Lambda_0+\overline{\rho}) \\ &= \ell\Lambda_0 + r_\theta(\overline{\Lambda}) + (\ell+1)\theta + \{(\overline{\Lambda}|\theta) - (1+\ell)\}\delta. \end{aligned}$$

Now let $\mathfrak{g}$ be of type $A^{(1)}_{n-1}$. Let $\overline{P}$ be identified with the subspace of $\mathbb{Z}^n$ given by vectors with sum zero.

For $\alpha \in \overline{P}$ define the Demazure operator $\overline{\Pi}$ to be the linear operator on $\mathbb{Z}[\overline{P}]$ such that

$$s_\alpha := \overline{\Pi}(e^\alpha) = \overline{J}^{-1}(e^{\overline{\rho}})\overline{J}(e^{\overline{\rho}+\alpha})$$

where $\overline{J} = \sum_{\tau \in \overline{W}} \varepsilon(\tau)\tau$. Let $q = e^{-\delta}$. Then for $\alpha \in \overline{P}$,

$$(26) \qquad -\Pi\, e^{\ell\Lambda_0} e^{\alpha} = \begin{cases} \Pi\, e^{\ell\Lambda_0} e^{r_i(\alpha) - \alpha_i} & \text{for } i \neq 0 \\ \Pi\, e^{\ell\Lambda_0} e^{r_\theta(\alpha) + (\ell+1)\theta} q^{\ell+1-(\alpha|\theta)} & \text{for } i = 0. \end{cases}$$

These equations express the q-equivalence in [3]. Let $\mathbb{Z}^n$ have standard basis $\{\epsilon_i \mid 1 \leq i \leq n\}$ and $\overline{P}$ be the subspace of $\mathbb{Z}^n$ orthogonal to the vector $\sum_{i=1}^n \epsilon_i$. Then $\alpha_i = \epsilon_i - \epsilon_{i+1}$ for $1 \leq i \leq n-1$, $\theta = \epsilon_1 - \epsilon_n$, $(\cdot|\cdot)$ is the ordinary dot product in $\mathbb{Z}^n$, and $\overline{W}$ is the symmetric group on n letters acting on the coordinates of $\mathbb{Z}^n$. Since $\Pi \circ \overline{\Pi} = \Pi$ and $\overline{\Pi}$ is $\mathbb{Z}\Lambda_0$-linear, one may replace every term e^α by $s_\alpha := \overline{\Pi} e^\alpha$ in (26). Define the map $\mathbb{Z}[\overline{P}]^{\overline{W}}[q] \to \mathbb{Z}[\overline{P}]^{\overline{W}}[q]$ given by $s_\alpha \mapsto \Pi(e^{\ell\Lambda_0 + \alpha})e^{-\ell\Lambda_0}$. Define $f \equiv g$ in $\mathbb{Z}[\overline{P}]^{\overline{W}}[q]$ by the condition that the above linear map sends f and g to the same element. With this definition, we have

$$(27) \qquad -s_\alpha \equiv \begin{cases} s_{(\alpha_1,\ldots,\alpha_{i+1}-1,\alpha_i+1,\ldots,\alpha_n)} & \text{for } i \neq 0 \\ s_{(\ell+1+\alpha_n,\alpha_2,\ldots,\alpha_{n-1},-1-\ell+\alpha_1)} q^{\ell+1-\alpha_1+\alpha_n} & \text{for } i = 0. \end{cases}$$

It is not hard to see that this recovers the q-equivalence of Schur functions given in [3].

6.2. Bosonic conjecture of [20, (9.2)]

In this section it is assumed that $\mathfrak{g}$ is of type $A^{(1)}_{n-1}$, $\Lambda = \ell\Lambda_0$, and the tensor factors B_j are perfect crystals of the form B^{k_j,ℓ_j} in the notation of [10] with $\ell_j \leq \ell$ for all j. By restriction to $U_q(\overline{\mathfrak{g}})$, B_j is the crystal of the irreducible integrable $U_q(\overline{\mathfrak{g}})$-module of highest weight $\ell_j\overline{\Lambda}_{k_j}$. In this case B_0 is not needed. To see this, recall that B_j can be realized as the set of column-strict Young tableaux of the rectangular shape having k_j rows and ℓ_j columns with entries in the set $\{1,2,\ldots,n\}$. In [19] the P_{cl}-weighted I-crystal structure on the perfect crystals $B^{k,\ell}$ is computed explicitly. In particular, if $b \in B_j$ is a tableau then $\epsilon_0(b)$ is at most the number of ones in the tableau b, which is at most ℓ_j by column-strictness. Therefore $b \otimes u_{\ell\Lambda_0}$ never admits e_0. Thus the energy function E_B of (3) has the property that for any $b \in B = B_L \otimes \cdots \otimes B_1$ such that $e_0(b \otimes u_{\ell\Lambda_0}) = e_0(b) \otimes u_{\ell\Lambda_0}$, one has $E_B(e_0(b)) = E_B(b) - 1$. Thus one obtains the bosonic formula in this case.

Since $\mathfrak{g}$ is of type $A^{(1)}_{n-1}$, $a_0 = 1$ and $h^\vee = n$. Take $\Lambda = \Lambda' = \ell\Lambda_0$ in (23). The lattice M is given by the root lattice $\overline{Q}$ of $\overline{\mathfrak{g}}$, which may be realized by $\{\beta \in \mathbb{Z}^n \mid \sum_{i=1}^n \beta_i = 0\}$. Let $B_{\tau,\beta}$ be the set of paths $b \in B$

of weight $-\overline{\rho}+\tau^{-1}(-(\ell+n)\beta+\overline{\rho})$. Then

$$\begin{aligned}K(B,\ell\Lambda_0,\ell\Lambda_0)(q) &= \sum_{\tau\in\overline{W}}\sum_{\beta\in M}\sum_{b\in B_{\tau,\beta}} \varepsilon(\tau)q^{E_B(b)+(\overline{\rho}|\beta)-\frac{1}{2}|\beta|^2(\ell+n)} \\ &= \sum_{\tau\in\overline{W}}\sum_{\beta\in M}\sum_{b\in B_{\tau,\beta}} \varepsilon(\tau)q^{E_B(b)-\sum_{i=1}^n\{\frac{1}{2}(\ell+n)\beta_i^2+i\beta_i\}}.\end{aligned}$$

Notice that $\sum_{b\in B_{\tau,\beta}} q^{E_B(b)}$ is (up to an overall factor) the $q\to 1/q$ form of the supernomial S of ref. [20] so that $K(B,\ell\Lambda_0,\ell\Lambda_0)(q)$ equals the left-hand side of [20, (9.2)] up to an overall power of q. This shows that the left-hand side of [20, (9.2)] is indeed the generating function of level-ℓ restricted paths. To establish the equality [20, (9.2)] it remains to prove that also the right-hand side equals the generating function of level-restricted paths.

6.3. Identities for level one and level zero

As in the previous section let $\mathfrak{g}$ be of type $A_{n-1}^{(1)}$ and assume that $B=B^{k_L,1}\otimes\cdots\otimes B^{k_1,1}$. Fix $\ell=1$ and $\Lambda,\Lambda'\in P_{cl}^+$ weights of level 1. It is easy to verify that $\mathcal{P}(B,\Lambda,\Lambda')$ consists of at most one element p. Choose B,Λ,Λ' such that $p\in\mathcal{P}(B,\Lambda,\Lambda')$ exists. Then by (7) and (23) we find that

$$\sum_{\tau\in\overline{W}}\sum_{\beta\in M}\sum_{b\in B_{\tau,\beta,\Lambda,\Lambda'}} \varepsilon(\tau)q^{E(b)-\sum_{i=1}^n\{\frac{n+1}{2}\beta_i^2+i\beta_i\}} = q^{E(p)} \tag{28}$$

where $B_{\tau,\beta,\Lambda,\Lambda'}$ is the set of paths $b\in B$ of weight $-\overline{\Lambda}-\overline{\rho}+\tau^{-1}(\overline{\Lambda}'-(n+1)\beta+\overline{\rho})$.

A similar formula exists for $\ell=0$:

$$\sum_{\tau\in\overline{W}}\sum_{\beta\in M}\sum_{b\in B_{\tau,\beta}} \varepsilon(\tau)q^{E(b)-\sum_{i=1}^n\{\frac{n}{2}\beta_i^2+i\beta_i\}} = \delta_{B,\emptyset} \tag{29}$$

where $B_{\tau,\beta}$ is the set of paths $b\in B$ of weight $-\overline{\rho}+\tau^{-1}(-n\beta+\overline{\rho})$. The right-hand side is the generating function of paths in B of level zero since there are no level zero restricted paths unless B is empty. However, the arguments of Sections 4 and 5 do not imply that also the left-hand side is the generating function of level zero paths since it was assumed in the proof of Theorem 3 that the level of the crystals B_j does not exceed ℓ. We have assumed that $B_j=B^{k_j,1}$ which are crystals of level one. However, it is possible to define a sign-reversing involution directly on $B=B^{k_L,1}\otimes\cdots\otimes B^{k_1,1}$ without using the crystal isomorphisms that are used in the proof of Theorem 3. Let $b \in B$. There exists at least one $0 \leq i \leq n$ such that $e_i(b_1)$ is defined. Define $v(b) = \min\{i | e_i(b_1) \text{ is defined}\}$ which has the property that $v(b) = v(\Phi_i(b))$ where as before $\Phi_i = s_i e_i$. Hence define the involution $\Phi(b) = \Phi_{v(b)}(b)$. It is again sign-reversing and has no fixed points when $B \neq \emptyset$. This proves that the left-hand side of (29) is the generating function of level 0 restricted paths.

Equation (28) was conjectured in [20, 21]. For $n = 2$ identity (29) follows from the q-binomial theorem, for $n = 3$ it was proven in [1, Proposition 5.1] and for general n it was conjectured in [21].

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Length Functions for $G(r, p, n)$

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Abstract.

In this paper, we construct a length function $n(w)$ for the complex reflection group $W = G(r,p,n)$ by making use of certain partitions of the root system associated to $\widetilde{W} = G(r,1,n)$. We show that the function $n(w)$ yields the Poincaré polynomial $P_W(q)$. We give some characterization of this function in a way independent of the choice of the root system.

§1. Introduction

Let $\widetilde{W} = G(r,1,n)$ be an imprimitive complex reflection group. In [BM1], K. Bremke and G. Malle introduced a certain type of root system (and its partition into positive and negative roots) associated to $\widetilde{W}$, and defined a length function n_1 on $\widetilde{W}$ by making use of the root system. They showed that this function satisfies some good properties as a generalization of the length function of finite Coxeter groups. In particular, the polynomial $\sum_{w \in \widetilde{W}} q^{n_1(w)}$ coincides with the Poincaré polynomial $P_{\widetilde{W}}(q)$ of $\widetilde{W}$. In [RS], we studied further properties of n_1, and gave some characterization of it in a way independent of the choice of the root system, in connection with the usual length function defined by standard generators of $\widetilde{W}$.

In [BM2], a similar problem was studied for the reflection subgroup $G(r,r,n)$ of $\widetilde{W}$. They defined a length function $\tilde{n}_2$ on $\widetilde{W}$ by using a similar root system as above, but by using completely different partition into positive and negative roots. They defined a length function n_2 on $G(r,r,n)$ as the restriction of $\tilde{n}_2$, and showed that n_2 yields the Poincaré polynomial $P_{G(r,r,n)}(q)$.

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In this paper, we consider a more general group $W = G(r,p,n)$. The group W is a reflection subgroup of $\widetilde{W}$ containing $G(r,r,n)$. We construct some partitions of the root system, (in fact, we need two kinds of such partitions) and define a length function $\tilde{n}$ on $\widetilde{W}$ associated to the root system. We also define a function n on W as the restriction of $\tilde{n}$ on W. We then show that our length functions satisfy the property that

$$\frac{1}{p}\sum_{w\in\widetilde{W}} q^{\tilde{n}(w)} = \sum_{w\in W} q^{n(w)} = P_W(q),$$

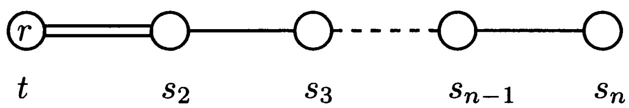
where $P_W(q)$ is the Poincaré polynomial associated to W. Our function $n(w)$ is much more complicated than the previous cases. But in some sense, it is the mixture of the functions n_1 and n_2. In fact, if $p = 1$, $n(w)$ coincides with $n_1(w)$, while if $p = r$, $n(w)$ coincides with $n_2(w)$.

We give a characterization of the function $\tilde{n}$ on $\widetilde{W}$ in a similar way as in [RS], in an independent way of the choice of the root system. This is done by making use of a certain length function on $\widetilde{W}$ defined without using the root data. However, in contrast to the case treated in [RS], it is not the function defined by generators of $\widetilde{W}$ or W.

§2. Length functions associated to a root system

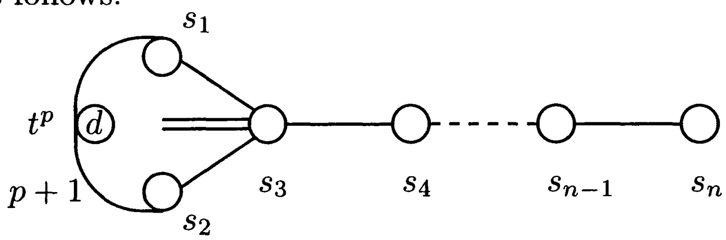
2.1 Let V be the unitary space $\mathbf{C}^n$ with the standard basis vectors $e_1, \dots, e_n$. We denote by $\widetilde{W} = G(r,1,n)$ the imprimitive complex reflection group generated by reflections $t, s_2, \dots, s_n$. Here s_i is the permutation of e_i and e_{i-1} for $i = 2, \dots, n$, and t is the complex reflection of order r defined by $te_1 = \zeta e_1$ and $te_i = e_i$ for $i \neq 1$, where ζ is a fixed primitive r-th root of unity. The group $\widetilde{W}$ has a Coxeter-like diagram with respect to the set $\widetilde{S} = \{t, s_2, \dots, s_n\}$ of generators as follows;

$B_n^{(r)}$:



For each factor p of r, we denote by $W = G(r,p,n)$ the reflection subgroup of $\widetilde{W}$ of index p generated by $S = \{t^p, s_1 = t^{-1}s_2t, s_2, \dots, s_n\}$. The special case where $p = r$, the group $W' = G(r,r,n)$ is generated by $S' = \{s_1, \dots, s_n\}$. We have $W' \subset W \subset \widetilde{W}$. We put $r = pd$. The presentation of the group W in terms of the set S is determined by

[BMR]. In particular, if $p \geq 3, d \neq 1$, the Coxeter-like diagram of W is given as follows.



2.2 Let Φ be a root system associated to $\widetilde{W}$ defined in [BM1]. Here we follow the description of Φ given in [RS]. Hence we consider a set $X = \{e_i^{(a)} \mid 1 \leq i \leq n, a \in \mathbf{Z}/r\mathbf{Z}\}$, and we express an element $(e_i^{(a)}, e_j^{(b)}) \in X \times X$ as $e_i^{(a)} - e_j^{(b)}$ whenever $i \neq j$. The root system Φ is defined by

$$\Phi = \Phi_l \coprod \Phi_s \quad \text{with}$$
$$\Phi_l = \{e_i^{(a)} - e_j^{(b)} \mid 1 \leq i, j \leq n, i \neq j, a, b \in \mathbf{Z}/r\mathbf{Z}\},$$
$$\Phi_s = X = \{e_i^{(a)} \mid 1 \leq i \leq n, a \in \mathbf{Z}/r\mathbf{Z}\}$$

An element in Φ_l (resp. in Φ_s) is called a long root (resp. a short root), respectively. The group $\widetilde{W}$ acts naturally on the set Φ in such a way that s_i permutes $e_i^{(a)}$ and $e_{i-1}^{(a)}$, and $te_1^{(a)} = e_1^{(a+1)}$, $te_j^{(a)} = e_j^{(a)}$ for $j \neq 1$.

For $\alpha = e_i^{(a)} - e_j^{(b)} \in \Phi_l$, we define $-\alpha \in \Phi_l$ by $-\alpha = e_j^{(b)} - e_i^{(a)}$. We shall define two types of partitions, $\Phi_l = \Phi_l^+ \cup \Phi_l^- = \Phi_l^{++} \cup \Phi_l^{--}$ such that $\Phi_l^- = -\Phi_l^+, \Phi_l^{--} = -\Phi_l^{++}$. In the following formulae, long roots $\alpha \in \Phi_l$ are always denoted as $\alpha = e_i^{(a)} - e_j^{(b)}$. Also for each $a \in \mathbf{Z}$, let $\bar{a}$ be the integer determined by the condition that $\bar{a} \equiv a \pmod{p}$ and that $-p/2 < \bar{a} \leq p/2$. The partition of the first type is given as follows.

(2.2.1)
$$\begin{aligned}
\Phi_l^+ &= \{\alpha \mid -p/2 < a \leq 0,\ i > j\} \\
&\cup \{\alpha \mid 0 < \bar{a} \leq p/2,\ p/2 < b \leq r - p/2,\ i > j\} \\
&\cup \{\alpha \mid -p/2 < \bar{b} \leq 0,\ 0 < b \leq r - p/2,\ i < j\} \\
&\cup \{\alpha \mid 0 < \bar{b} \leq p/2,\ -p/2 < a \leq p/2,\ i < j\}, \\
\Phi_l^- &= \{\alpha \mid -p/2 < b \leq 0,\ i < j\} \\
&\cup \{\alpha \mid 0 < \bar{b} \leq p/2,\ p/2 < a \leq r - p/2,\ i < j\} \\
&\cup \{\alpha \mid -p/2 \leq \bar{a} \leq 0,\ 0 < a \leq r - p/2,\ i > j\} \\
&\cup \{\alpha \mid 0 < \bar{a} \leq p/2,\ -p/2 < b \leq p/2,\ i > j\}.
\end{aligned}$$

The fact that $\Phi_l^- = -\Phi_l^+$, and that Φ_l is a disjoint union of Φ_l^+ and Φ_l^- is verified as follows. Set

$$\begin{aligned}
A &= \{\alpha \mid -p/2 < a \le 0,\ i > j\},\\
B &= \{\alpha \mid 0 < \bar{a} \le p/2,\ p/2 < b \le r - p/2,\ i > j\},\\
C &= \{\alpha \mid -p/2 < \bar{a} \le 0,\ 0 < a \le r - p/2,\ i > j\},\\
D &= \{\alpha \mid 0 < \bar{a} \le p/2,\ -p/2 < b \le p/2,\ i > j\}.
\end{aligned}$$

Then, it is easy to see that A, B, C and D are mutually disjoint, and $A \cup B \cup C \cup D$ coincides with the set $\{\alpha \in \Phi_l \mid i > j\}$. Moreover, we have

$$\Phi_l^+ = A \cup B \cup -C \cup -D, \qquad \Phi_l^- = -A \cup -B \cup C \cup D.$$

This shows the required property.

The partition of the second type is given as follows.

(2.2.2)
$$\begin{aligned}
\Phi_l^{++} &= \{\alpha \mid -p/2 < \bar{a} \le 0,\ i > j\} \cup \{\alpha \mid 0 < \bar{b} \le p/2,\ i < j\},\\
\Phi_l^{--} &= \{\alpha \mid 0 < \bar{a} \le p/2,\ i > j\} \cup \{\alpha \mid -p/2 < \bar{b} \le 0,\ i < j\}.
\end{aligned}$$

We also define a grading of Φ_s by modifying the grading of Φ_s given in [RS] as follows. Let $\Phi_s = \Phi_{s,0} \cup \Phi_{s,1} \cup \cdots \cup \Phi_{s,d-1}$, where

(2.2.3)
$$\Phi_{s,m} = \{e_i^{(a)} \mid mp - p/2 < a \le mp + p/2,\ 1 \le i \le n\} \quad (0 \le m < d).$$

Next we define a subset $\Omega = \Omega_l' \cup \Omega_l'' \cup \Omega_s$ of Φ as follows.

$$\begin{aligned}
\Omega_s &= \{e_i^{(0)} \mid 1 \le i \le n\},\\
\Omega_l' &= \{e_i^{(0)} - e_j^{(b)} \mid b \equiv 0 \pmod{p},\ i > j\},
\end{aligned}$$

and

$$\begin{aligned}
\Omega_l'' = \{e_i^{(a)} - e_j^{(mp-a)} &\mid -p/2 < a < 0,\ 0 \le m < d,\ i > j\}\\
\cup\{e_i^{(mp-b+\delta)} - e_j^{(b)} &\mid 0 < b \le p/2,\ 0 \le m < d,\ i < j\},
\end{aligned}$$

where

$$\delta = \begin{cases} 1 & \text{if } p \text{ is even},\\ 0 & \text{if } p \text{ is odd}.\end{cases}$$

We define functions $\tilde{n}'_l, \tilde{n}''_l, \tilde{n}_s : \widetilde{W} \to \mathbf{N}$ by

$$\tilde{n}'_l(w) = |w\Omega'_l \cap \Phi_l^-|, \qquad \tilde{n}''_l(w) = |w\Omega''_l \cap \Phi_l^{--}|,$$

and by

$$\tilde{n}_s(w) = \sum_{\alpha \in \Omega_s} \nu(w(\alpha)),$$

where for each $\alpha \in \Phi_s$, we put $\nu(\alpha) = k$ if $\alpha \in \Phi_{s,k}$. We define a length function $\tilde{n} : \widetilde{W} \to \mathbf{N}$ by $\tilde{n} = \tilde{n}'_l + \tilde{n}''_l + \tilde{n}_s$. We consider the restriction of these functions to W, and define n'_l, n''_l and n_s as the restriction of $\tilde{n}'_l, \tilde{n}''_l$, and $\tilde{n}_s$, respectively. Then we define a length function n of W by $n = n'_l + n''_l + n_s$.

Remark 2.3. In the case where $p = 1$, we have $\Omega''_l = \emptyset$. Moreover, $\Phi_l^+ = \{\alpha \mid a = 0\} \cup \{\alpha \mid b \neq 0\}$, and $\Phi_l^- = -\Phi_l^+$. This partition together with the set $\Omega'_l \cup \Omega_s$ coincide with the set $\Omega_l \cup \Omega_s$ of Φ_l given in [RS], and the grading of Φ_s also coincides with that of Φ_s given there. Hence the function n coincides with the length function of $G(r,1,n)$ defined in [BM1].

While in the case where $p = r$, we have $\Phi_s = \Phi_{s,0}$. Moreover $\Phi_l^+ = \Phi_l^{++}, \Phi_l^- = \Phi_l^{--}$, and this partition of Φ_l together with $\Omega'_l \cup \Omega''_l$ coincide essentially with those given in [BM2]. (Also note that Ω'_l coincides with the root system of the symmetric group S_n). Hence n agrees with the length function of $G(r,r,n)$ defined there.

2.4. Let W_I be the reflection subgroup of W generated by $I = \{t^p, s_1, s_2, \dots, s_m\}$ for some $m \leq n$. Then W_I is isomorphic to $G(r,p,m)$. It is clear from the definition that the restriction of n on W_I coincides with the function n_I defined similarly for $G(r,p,m)$. On the other hand, let $J = \{t^p, s_2, \dots, s_n\}$ be a subset of S, and W_J the subgroup of W generated by J. If $d > 1$, then W_J is isomorphic to $G(d,1,n)$, and J coincides with the standard set of generators of $G(d,1,n)$. While if $d = 1$, W_J is isomorphic to S_n. Let n_J be the length function of W_J as given in [RS]. In the case where $d > 1$, we denote by $n_{J,l}$ and $n_{J,s}$ the functions associated to long roots and short roots, respectively.

Lemma 2.5. *The restriction of n on W_J coincides with n_J.*

Proof. The case where $d = 1$ is easy. So, we assume that $d > 1$. Let $\Phi_{l,J}$ be the subset of Φ_l consisting of roots of the form $e_i^{(a)} - e_j^{(b)}$ with $p \mid a$, $p \mid b$. Then $\Phi_{l,J}$ is in a natural correspondence, via the map $e_i^{(a)} - e_j^{(b)} \mapsto e_i^{(a')} - e_j^{(b')}$ with $a' = a/p, b' = b/p$, with the set of long roots for $G(d,1,n)$, where $\Phi_{l,J} \cap \Phi_l^+$ (resp. $\Phi_{l,J} \cap \Phi_l^-$) corresponds to the set of positive (resp. negative) roots, respectively. Similarly, let $\Phi_{s,J}$ be the subset of Φ_s consisting of $e_i^{(a)}$ with $p \mid a$. Then $\Phi_{s,J}$ corresponds naturally to the set of short roots for $G(d,1,n)$, and the restriction of the grading of Φ_s to $\Phi_{s,J}$ coincides with the grading of the set of short roots. Note that the above correspondence is compatible with the actions of W_J. Under this correspondence, the sets Ω_l' and Ω_s are mapped to the sets Ω_l and Ω_s in the root system for $G(d,1,n)$. Since $w(\Omega_s) \subset \Phi_{s,J}$ (resp. $w(\Omega_l') \subset \Phi_{l,J}$) for each $w \in W_J$, we see that the restriction of n_s (resp. n_l') on W_J coincides with $n_{J,s}$ (resp. $n_{J,l}$), respectively. Hence in order to prove the lemma, it suffices to show that $n_l''(w) = 0$, i.e., $w(\Omega_l'') \subset \Phi_l^{++}$ for $w \in W_J$. Take an element $\alpha = e_i^{(a)} - e_j^{(b)} \in w(\Omega_l'')$. Then either $-p/2 < \bar{a} < 0$ and $\bar{b} = -\bar{a}$, or $0 < \bar{b} \le p/2$ and $\bar{a} = -\bar{b} + \delta$. This implies that $\alpha \in \Phi_l^{++}$ and the lemma follows. Q.E.D.

2.6. By applying Lemma 2.5, we can determine the values $n(s)$ for $s \in S$ as follows.

$$n(s) = \begin{cases} 1 & \text{if } s \in \{s_2, \dots, s_n\}, \\ 1 & \text{if } s = t^p \text{ with } d > 1, \\ d & \text{if } s = s_1 \text{ with } p \ge 3 \text{ or } d = 1, \\ 3d - 1 & \text{if } s = s_1 \text{ with } p = 2, d > 1. \end{cases} \tag{2.6.1}$$

In fact, the first two case follow from the lemma. We consider the remaining cases. We have $s_1(\Omega_l') \subset \Phi_l^+$ if $p \ge 3$ or $d = 1$. While if $p = 2$, and $d > 1$, then $s_1(e_2^{(0)} - e_1^{(b)}) < 0$ for $b \equiv 0 \ (p)$. On the other hand, $s_1(e_1^{(0)}) = e_2^{(1)}$ and $s_1(e_2^{(0)}) = e_1^{(-1)}$, and s_1 leaves other short roots fixed. Hence by (2.2.3), $s_1(\Omega_s) \subset \Phi_{s,0}$ if $p \ge 3$. While if $p = 2$, we have $s_1(e_2^{(0)}) \in \Phi_{l,d-1}$, and s_1 maps all other elements in Ω_s to $\Phi_{s,0}$. Moreover we have

$$\Omega_l'' \cap s_1(\Phi_l^{--}) = \begin{cases} \{e_1^{(mp)} - e_2^{(1)} \mid 0 \le m < d\} & \text{if } p \text{ is even}, \\ \{e_2^{(-f)} - e_1^{(mp+f)} \mid 0 \le m < d\} & \text{if } p \text{ is odd}, \end{cases}$$

where $p = 2f+1$. This implies that $n_l'(s_1) = 0$, $n_l''(s_1) = d$ and $n_s(s_1) = 0$ if $p \ge 3$ or $d = 1$, and $n_l'(s_1) = d, n_l''(s_1) = d$ and $n_s(s_1) = d - 1$ otherwise. So we have $n(s_1) = d$ or $3d - 1$ and (2.6.1) follows.

Let $\Phi_{l,J}$ be the subset of Φ_l defined in the beginning of the proof of Lemma 2.5. Set $\Phi_{l,J}^+ = \Phi_{l,J} \cap \Phi_l^+$. We define a subset $\widetilde{W}^J$ of $\widetilde{W}$ by

$$\widetilde{W}^J = \{w \in \widetilde{W} \mid w(\Phi_{l,J}^+) \subset \Phi_l^+, w(\Omega_s) \subset \Phi_{s,0}\}. \tag{2.6.2}$$

Then the following lemma holds.

Lemma 2.7. *Let $w \in \widetilde{W}^J, w' \in W_J$. Then we have*

$$\tilde{n}'_l(ww') = \tilde{n}'_l(w'), \tag{2.7.1}$$
$$\tilde{n}''_l(ww') = \tilde{n}''_l(w),$$
$$\tilde{n}_s(ww') = \tilde{n}_s(w').$$

In particular, $\tilde{n}(ww') = \tilde{n}(w) + \tilde{n}(w')$.

Proof. Since $\Omega'_l \subset \Phi^+_{l,J}$, it follows from (2.6.2) that $\tilde{n}'_l(w) = 0$. (2.6.2) implies also $\tilde{n}_s(w) = 0$. On the other hand, we know that $\tilde{n}''_l(w') = n''_l(w') = 0$ from the proof of Lemma 2.4. Hence the last formula follows from (2.7.1). We show (2.7.1). Since $w(\Phi^-_{l,J}) \subset \Phi^-_l$, $w'(\alpha)$ and $ww'(\alpha)$ have the same sign for $\alpha \in \Omega'_l$. This implies the first assertion of (2.7.1). Let

$$\widetilde{\Omega}''_l = \{e_i^{(a)} - e_j^{(b)} \mid -p/2 < \bar{a} < 0,\, \bar{a} + \bar{b} = 0,\, i > j\}$$
$$\cup \{e_i^{(a)} - e_j^{(b)} \mid 0 < \bar{b} \le p/2,\, \bar{a} + \bar{b} = \delta,\, i < j\}.$$

Since $w'(\Omega''_l) \subset \Phi^{++}_l$, we see that w' stabilizes $\widetilde{\Omega}''_l$. The second assertion follows from this if we notice that the definition of the sets Φ^{++}_l or Φ^{--}_l depends only on $\bar{a}$ and $\bar{b}$ for $\alpha = e_i^{(a)} - e_j^{(b)}$, and that $\widetilde{\Omega}''_l$ has the same pattern as Ω''_l for the action of w'. The last assertion is also immediate from (2.2.3). This proves the lemma. Q.E.D.

2.8. By modifying the definition in [BM2], we define an element $w(a, m) \in \widetilde{W}$ for $-p/2 < a \le p/2, 1 \le m \le n$ as follows.

$$w(a,m) = \begin{cases} s_m \cdots s_2 t^a & \text{if } 0 < a \le p/2, \\ s_m \cdots s_2 t^a s_2 \cdots s_m & \text{if } -p/2 < a \le 0. \end{cases} \tag{2.8.1}$$

Let us define a subset $\mathcal{N}$ of $\widetilde{W}$ by

$$\mathcal{N} = \{w(a_1, 1)w(a_2, 2) \cdots w(a_n, n) \mid -p/2 < a_i \le p/2\}.$$

We set $\mathcal{N}' = \mathcal{N} \cap W$. Then $\mathcal{N}'$ can be written as

(2.8.2)
$$\mathcal{N}' = \{w(a_1, 1)w(a_2, 2) \cdots w(a_n, n) \in \mathcal{N} \mid \textstyle\sum a_i \equiv 0 \pmod{p}\}.$$

Also we set $W^J = \widetilde{W}^J \cap W$. We have the following proposition.

Proposition 2.9. *The set $\mathcal{N}$ (resp. $\mathcal{N}'$) coincides with the set $\widetilde{W}^J$ (resp. W^J). Moreover, $\mathcal{N}$ (resp. $\mathcal{N}'$) gives rise to a system of complete representatives of left cosets $\widetilde{W}/W_J$ (resp. W/W_J), respectively.*

Proof. First we show that $\mathcal{N}$ is contained in $\widetilde{W}^J$. Take $\alpha = e_i^{(mp)} - e_j^{(m'p)} \in \Phi_{l,J}$. Then for $w \in \mathcal{N}$, $w(\alpha)$ is expressed as $w(\alpha) = e_k^{(mp+a_k)} - e_l^{(m'p+a_l)}$, where a_k and a_l satisfy the following condition;

$$\begin{array}{lll} -p/2 < a_k \le p/2, & 0 < a_l \le p/2 & \text{if } i > j, k < l, \\ -p/2 < a_k \le 0, & -p/2 < a_l \le p/2 & \text{if } i > j, k > l, \\ -p/2 < a_k \le p/2, & -p/2 < a_l \le 0 & \text{if } i < j, k < l, \\ 0 < a_k \le p/2, & -p/2 < a_l \le p/2 & \text{if } i < j, k > l. \end{array}$$

Then it is easy to see that $w(\alpha) \in \Phi_l^+$ exactly when $m = 0$ if $i > j$, and when $m' \ne 0$ if $i < j$. But this condition is equivalent to the condition that $\alpha \in \Phi_{l,J}^+$. It follows that $w(\Phi_{l,J}^+) \subset \Phi_l^+$. Next take $e_i^{(0)} \in \Omega_s$. Then we have $w(e_i^{(0)}) = e_j^{(a_j)}$ for some j with $-p/2 < a_j \le p/2$. This implies that $w(\Omega_s) \subset \Phi_{s,0}$. Hence we have $\mathcal{N} \subset W^J$.

Next we note that W^J is a subset of the set of left coset representatives of W by W_J. In fact assume that there exist $w_1, w_2 \in W^J$ such that $w_1 = w_2 x$ with $x \in W_J$. Then by (2.7.1) in the proof of Lemma 2.7, we have $n_l'(w_2 x) = n_l'(x)$ and $n_l'(w_1) = 0$. Hence $n_l'(x) = 0$. Since the restriction of n_l' on W_J is the length function on $W_J = G(d, 1, n)$, we have $x = 1$. So $w_1 = w_2$.

It follows from the above remark that $|\widetilde{W}^J| \le |\widetilde{W}/W_J| = p^n$. On the other hand, we have $|\mathcal{N}| = p^n$. (In fact, if $w = w(a_1, 1) \cdots w(a_n, n) \in \mathcal{N}$, then there exists $e_i^{(0)}$ such that $w(e_i^{(0)}) = e_n^{(a_n)}$. Hence the elements in $\mathcal{N}$ are parametrized by n-tuples $(a_1, \dots, a_n)$ with $-p/2 < a_i \le p/2$). This shows that $\mathcal{N} = \widetilde{W}^J$ gives a complete set of representatives for $\widetilde{W}/W_J$.

The statement for W follows from this by noticing that $|\mathcal{N}'| = |W/W_J| = p^{n-1}$. Q.E.D.

Remark 2.10. The above proposition shows that any element $w \in \widetilde{W}$ (resp. $w \in W$) can be expressed in a unique way as

$$w = w(a_1, 1) w(a_2, 2) \cdots w_n(a_n, n) w', \tag{2.10.1}$$

where $w' \in W_J$ (resp. and $\sum_i a_i \equiv 0 \pmod p$). The numbers $a_1, \dots, a_n$ occuring in the decomposition (2.10.1) can be interpreted directly as follows; since $\widetilde{W} \simeq S_n \ltimes (\mathbf{Z}/r\mathbf{Z})^n$, an element w in $\widetilde{W}$ can be written in a form $w = \sigma z$, with $\sigma \in S_n$ and $z \in (\mathbf{Z}/r\mathbf{Z})^n$. Note that z can be written uniquely as $z = (z_1, \dots, z_n)$ with $z_i \in \mathbf{Z}$ such that $-r/2 < z_i \le r/2$ for $i = 1, \dots, n$. Each z_i determines an integer $\bar{z}_i$ such that $-p/2 < \bar{z}_i \le p/2$, and that $\bar{z}_i \equiv z_i \pmod p$ as in 2.2. Under these notations, we have $a_i = \bar{z}_i$ for $i = 1, \dots, n$. See also 3.2 for more details.

We shall compute th values $\tilde{n}_l''(w)$ for $w \in \mathcal{N}$, and $n_l''(w)$ for $w \in \mathcal{N}'$.

Lemma 2.11. *The following formulae hold.*

$$\text{(i)} \qquad \tilde{n}_l''(w(a,m)) = \begin{cases} d(m-1)(2a-1) & \text{if } 0 < a \le p/2, \\ d(m-1)(-2a) & \text{if } -p/2 < a \le 0. \end{cases}$$

(ii) *For $w = w(a_1, 1)w(a_2, 2) \cdots w(a_n, n) \in \mathcal{N}$ we have,*

$$\tilde{n}_l''(w) = \sum_{i=1}^{n} \tilde{n}_l''(w(a_i, i)). \tag{2.11.1}$$

Moreover, the function $\tilde{n}_l''$ coincides with $\tilde{n}$ on $\mathcal{N}$. In particular, if $w \in \mathcal{N}'$, the value $n(w)$ is given by the right hand side of (2.11.1).

Proof. First we show (i). Let $w = w(a,m)$. Assume that $0 < a \le p/2$. Then $w = s_m s_{m-1} \cdots s_2 t^a$. Take $\alpha = e_i^{(b)} - e_j^{(kp-b)} \in \Omega_l''$, where $i > j$ and $-p/2 < b < 0$. Then $w(\alpha)$ becomes positive unless $j = 1, i \le m$. In that case we have $w(\alpha) = e_{i-1}^{(b)} - e_m^{(kp-b+a)}$, and $w(\alpha) < 0$ if and only if $-p/2 < \overline{-b+a} \le 0$. This condition is equivalent to $p/2 < a - b < p$, and we have

$$\begin{aligned} \sharp\{\alpha = e_i^{(b)} - e_j^{(kp-b)} &\in \Omega_l'' \mid w(\alpha) < 0\} \\ &= \sharp\{\alpha \mid p/2 < a-b < p,\ 0 \le k < d,\ 2 \le i \le m\} \\ &= \begin{cases} d(m-1)(a-1) & \text{if } p \text{ is even}, \\ d(m-1)a & \text{if } p \text{ is odd}. \end{cases} \end{aligned}$$

Next take $\alpha = e_i^{(mp-b+\delta)} - e_j^{(b)} \in \Omega_l''$, where $i < j$ and $0 < b \le p/2$. A similar consideration as above shows that $w(\alpha) < 0$ if and only if $i = 1$ and $0 < \overline{a - b + \delta} \le p/2$. Then we have

$$\begin{aligned} \sharp\{\alpha = e_i^{(mp-b+\delta)} - e_j^{(b)} &\in \Omega_l'' \mid w(\alpha) < 0\} \\ &= \sharp\{\alpha \mid 0 < a-b+\delta \le p/2,\ 0 \le k < d,\ 2 \le j \le m\}, \\ &= \begin{cases} d(m-1)a & \text{if } p \text{ is even}, \\ d(m-1)(a-1) & \text{if } p \text{ is odd}. \end{cases} \end{aligned}$$

It follows that $\tilde{n}_l''(w) = d(m-1)(2a-1)$.

Next assume that $-p/2 < a \le 0$. Then $w = s_m \cdots s_2 t^a s_2 \cdots s_m$. First take $\alpha = e_i^{(b)} - e_j^{(kp-b)}$, where $i > j$ and $-p/2 < b < 0$. Then $w(\alpha)$ is positive unless $i = m$. In that case, $w(\alpha) = e_m^{(a+b)} - e_j^{(kp-b)}$ and $w(\alpha) < 0$ if and only if $0 < \overline{a+b} \le p/2$. This implies that $-p < a+b \le -p/2$, and we have

$$\begin{aligned}\sharp\{\alpha = e_i^{(b)} - e_j^{(kp-a)} \in \Omega_l'' \mid w(\alpha) < 0\} \\ &= \sharp\{\alpha \mid -p < a+b \le -p/2, 0 \le k < d, 1 \le j < m\} \\ &= d(m-1)(-a).\end{aligned}$$

Next take $\alpha = e_i^{(kp-b+\delta)} - e_j^{(b)}$, where $i < j$ and $0 < b \le p/2$. Then $w(\alpha)$ is positive unless $j = m$. In that case $w(\alpha) = e_i^{(kp-b+\delta)} - e_m^{(a+b)}$, and $w(\alpha) < 0$ if and only if $-p/2 < \overline{a+b} \le 0$. Hence we have

$$\begin{aligned}\sharp\{\alpha = e_i^{(kp-b+\delta)} - e_j^{(b)} \in \Omega_l'' \mid w(\alpha) < 0\} \\ &= \{\alpha \mid -p/2 < a+b \le 0, 0 \le k < d, 1 \le i < m\} \\ &= d(m-1)(-a)\end{aligned}$$

It follows that $\tilde{n}_l''(w) = (m-1)d(-2a)$, and we get (i).

Next we show (ii). Take $\alpha = e_i^{(b)} - e_j^{(mp-b)} \in \Omega_l''$, with $i > j$, and assume that $w(\alpha) < 0$. Now $w(\alpha)$ can be written as $w(\alpha) = e_k^{(b+a_k)} - e_l^{(mp-b+a_l)}$ for some k, l. First consider the case where $k > l$. Let $w' = w(a_{k+1}, k+1) \cdots w(a_n, n)$. Then $w'(\alpha)$ can be written as $w'(\alpha) = e_k^{(b)} - e_{j'}^{(mp-b)}$ for some $j' < k$. It follows that $\beta = w'(\alpha) \in \Omega_l''$ and $w(a_k, k)\beta < 0$. If $k < l$, we consider $w'' = w(a_{l+1}, l+1) \cdots w(a_n, n)$ instead of w'. Then $w''(\alpha)$ can be written as $w''(\alpha) = e_{i'}^{(b)} - e_1^{(mp-b)}$ for some $i' > 1$. Hence $\beta = w''(\alpha) \in \Omega_l''$ and $w(a_l, l)\beta < 0$. Conversely, we take $\beta = e_{i'}^{(b)} - e_{j'}^{(mp-b)} \in \Omega_l''$ with $i' > j'$, and assume that $w(a_k, k)\beta < 0$. Then $i' = k$ or $j' = 1$. If we set $\alpha = w'^{-1}(\beta)$, then we see that $\alpha = e_i^{(b)} - e_j^{(mp-b)} \in \Omega_l''$ with $i > j$, and that $w(\alpha) < 0$.

A similar fact as above also holds for $\alpha = e_i^{(mp-b+\delta)} - e_j^{(b)} \in \Omega_l''$. (Here, $\beta = e_{i'}^{(mp-b+\delta)} - e_k^{(b)}$ with $i' < k$, or $\beta = e_1^{(mp-b+\delta)} - e_{j'}^{(b)}$ with $1 < j'$, and so $\beta \in \Omega_l''$). This proves (2.10.1).

Finally, assume that $w \in \mathcal{N}'$. Then $n(w) = \tilde{n}_l''(w)$ by (2.7.1). Hence (2.10.1) gives the value $n(w)$. Q.E.D.

Remark 2.12. If $p \ge 3$ or $d = 1$, then $s_1 = w(-1,1)w(1,2) \in \mathcal{N}'$. While if $p = 2, d \ne 1$, we have $s_1 = ww'$ with $w = w(1,1)w(1,2) \in \mathcal{N}'$

and $w' = s_2 t^{-2} s_2 \in W_J$. Here $n(w) = d$ and $n(w') = n_J(w') = 2d-1$. So, in this case we have $n(s_1) = 3d-1$ by Lemma 2.7. This justifies (2.6.1).

2.13. For a complex reflection group G, we denote by $P_G(q)$ the Poincaré polynomial associated to the coinvariant algebra of G. The Poincaré polynomial $P_W(q)$ for $W = G(r,p,n)$ is given as

$$P_W(q) = \prod_{i=1}^{n-1} \frac{q^{ri}-1}{q-1} \cdot \frac{q^{dn}-1}{q-1}. \tag{2.13.1}$$

Then the following proposition holds.

Proposition 2.14. *We have*

$$\frac{1}{p}\sum_{w\in\widetilde{W}} q^{\tilde{n}(w)} = \sum_{w\in W} q^{n(w)} = P_W(q).$$

Proof. We show the second equality. By Lemma 2.7 and Proposition 2.9, we have

$$\sum_{w\in W} q^{n(w)} = \sum_{w\in\mathcal{N}'} q^{(n(w)} \sum_{w\in W_J} q^{n(w)}. \tag{2.14.1}$$

Now W_J is isomorphic to $G(d,1,n)$ and the restriction of n on W_J coincides with n_J by Lemma 2.5. Hence by [BM1, Prop. 2.12] we have

$$\sum_{w\in W_J} q^{n(w)} = P_{G(d,1,n)}(q) = \prod_{i=1}^{n} \frac{q^{di}-1}{q-1}. \tag{2.14.2}$$

On the other hand, in the expression $w = \sum_i w(a_i, i) \in \mathcal{N}'$, we can choose $a_2, \dots, a_n$ freely, and a_1 is determined uniquely by $a_2, \dots, a_n$. Moreover, we have $\tilde{n}_l''(w(a,1)) = 0$ by Lemma 2.11. Hence again by using Lemma 2.10, we have

$$\sum_{w\in\mathcal{N}'} q^{n(w)} = \prod_{i=2}^{n}\sum_{k=0}^{p-1} q^{dk(i-1)} = \prod_{i=1}^{n-1} \frac{q^{ri}-1}{q^{di}-1}. \tag{2.14.3}$$

Substituting (2.14.2) and (2.14.3) into (2.14.1), we get the second equality. The formula $\frac{1}{p}\sum_{w\in\widetilde{W}} q^{\tilde{n}(w)} = P_W(q)$ can be proved in a similar way if one notices that

$$\sum_{w\in\mathcal{N}} q^{\tilde{n}(w)} = \prod_{i=1}^{n}\sum_{k=0}^{p-1} q^{dk(i-1)}.$$

Q.E.D.

§3. A characterization of the function $\tilde{n}$

3.1. In this section we shall characterize the length function $\tilde{n}$ in terms of a certain length function on $\widetilde{W}$, which is defined independent of the root system. We use the same notation as in Remark 2.10.

Let $\widetilde{W}_0 = G(2,1,n)$ be the Weyl group of type B_n. We define a map $\varphi : \widetilde{W} \to \widetilde{W}_0$ by $\varphi(w) = \sigma(\varepsilon_1, \dots, \varepsilon_n)$, where $w = \sigma(z_1, \dots, z_n)$ is as above, and ε_i is determined by

$$\varepsilon_i = \begin{cases} 1 & \text{if } \bar{z}_i > 0, \\ 0 & \text{if } \bar{z}_i \le 0. \end{cases}$$

(Here we use the same notation for $\widetilde{W}_0$ as the special case $r = 2$ for $G(r,1,n)$). Let us define a length function $\ell_1 : \widetilde{W} \to \mathbf{N}$ as follows. For $w = \sigma z$, we put $\ell_1(w) = \ell_0(\varphi(w))$, where ℓ_0 is the length function on $\widetilde{W}_0$ with respect to the long roots. (More precisely, using the basis $e_1, \dots, e_n$ of V, the set of long roots $\Phi_l \subset V$ associated to $\widetilde{W}_0$ is given as $\Phi_l = \{\pm e_i \pm e_j \mid 1 \le i,j \le n, i \ne j\}$, on which $\widetilde{W}_0$ acts naturally. Now the set Φ_l^+ of positive roots is given as $\Phi_l^+ = \{e_i \pm e_j \mid i > j\}$. For $w' \in \widetilde{W}_0$, we put $\ell_0(w') = |\Phi_l^+ \cap -w'(\Phi_l^+)|$). Next we define a function $\ell_2 : \widetilde{W} \to \mathbf{N}$ by $\ell_2(w) = \sum_{i=1}^n \hat{z}_i$, where

$$\hat{z}_i = \begin{cases} 2z_i - 1 & \text{if } z_i > 0, \\ -2z_i & \text{if } z_i \le 0. \end{cases}$$

Then we define a length function ℓ by $\ell = \ell_1 + \ell_2$. It is clear from the definition that if $r = 2$, ℓ_2 coincides with the length function of W_0 with respect to short roots, and so the function ℓ coincides with the usual length function of the Weyl group of type B_n.

3.2. Let $w = w(a_1, 1) \cdots w(a_n, n)$ be an element in $\mathcal{N}$. The expression $w = \sigma z$ of w as in 3.1 can be described as follows. Let $I = \{1 \le i \le n \mid a_i > 0\}$, and J the complement of I in $\{1, 2, \dots, n\}$. We write $I = \{i_1 > i_2 > \cdots > i_k\}$ with $k = |I|$, and $J = \{j_1 < j_2 < \cdots < j_l\}$ with $l = |J|$. Set

$$\sigma = \begin{pmatrix} 1 & 2 & \cdots & k & k+1 & \dots & n \\ i_1 & i_2 & \cdots & i_k & j_1 & \cdots & j_l \end{pmatrix}. \tag{3.2.1}$$

and

$$z = (a_{i_1}, \dots, a_{i_k}, a_{j_1}, \dots, a_{j_l}) \in \mathbf{Z}_{>0}^k \times \mathbf{Z}_{\le 0}^l. \tag{3.2.2}$$

Then we have $w = \sigma z$. Conversely, any element $w = \sigma z$ with σ, z defined as above in terms of I, J, together with the condition that $-p/2 < a_i \le p/2$, gives an element of $\mathcal{N}$. These facts can be checked by using the induction on n.

Now $\varphi(w) \in \widetilde{W}_0$ can be expressed as a signed permutation,

$$\varphi(w) = \begin{pmatrix} 1 & 2 & \cdots & k & k+1 & \dots & n \\ -i_1 & -i_2 & \cdots & -i_k & j_1 & \cdots & j_l \end{pmatrix}. \tag{3.2.3}$$

From this we see that the set $\{\varphi(w) \mid w \in \widetilde{W}\}$ coincides with the set of distinguished representatives for the set $\widetilde{W}_0/S_n$.

We have the following lemma.

Lemma 3.3. *Let $\mathcal{N}$ and W_J be as before. Then for each $w \in \mathcal{N}$, w is the unique minimal length element in the coset wW_J with respect to ℓ. In other words,*

$$\mathcal{N} = \{w \in \widetilde{W} \mid \ell(w) \le \ell(ww') \text{ for any } w' \in W_J\}.$$

Proof. Let $w = \sigma z \in \mathcal{N}$. To prove the lemma, it is enough to show $\ell(w) < \ell(ww')$ for any $w' \in W_J - \{1\}$. Since $w' \in W_J$, one can write $w' = \sigma' z'$ with $\sigma' \in S_n$ and $z' = (z'_1, \dots, z'_n)$ such that $z'_i \equiv 0 \pmod p$. Here $\sigma' \ne 1$ or $z' \ne 0$. Then $ww' = \sigma\sigma'{\sigma'}^{-1}(z)z'$, and ${\sigma'}^{-1}(z)_i = z_{\sigma'(i)}$. Since $z'_i \equiv 0 \pmod p$, we have $\overline{z_{\sigma'(i)} + z'_i} = \bar{z}_{\sigma'(i)}$. Hence $\varphi(ww') = \varphi(w)\sigma'$. But since $\varphi(w)$ is a distinguished representative for the cosets $\widetilde{W}_0/S_n$, we see that $\ell_1(w) < \ell_1(ww')$ if $\sigma' \ne 1$.

Next we show that $\ell_2(w) < \ell_2(ww')$ if $z' \ne 0$. We may assume that $r \ne p$. By our assumption, we have $-p/2 < z_{\sigma'(i)} \le p/2$, and $z'_i \equiv 0 \pmod p$. If $z_{\sigma'(i)}$ and z'_i have the same sign, clearly $|z_{\sigma'(i)} + z'_i| > |z_{\sigma'(i)}|$. (In this case, if $|z_{\sigma'(i)} + z'_i| > r/2$, one has to replace $z_{\sigma'(i)} + z'_i$ by $z_{\sigma'(i)} + z'_i \pm r$. But since $r \ne p$, still the inequality holds). Now assume that $z_{\sigma'(i)}$ and z'_i have the distinct sign. Then we have $|p - z_{\sigma'(i)}| \ge |z_{\sigma'(i)}|$, and the equality holds only when $z_{\sigma'(i)} = p/2$. So the only case we have to care about is the case that $z_{\sigma'(i)} = p/2$ and $z'_i = -p$. But in this case, $(z_{\sigma'(i)} + z'_i)^\wedge = p > \hat{z}_{\sigma'(i)} = p-1$. This shows that $\ell_2(w) < \ell_2(ww')$ if $z' \ne 0$. Hence we have $\ell(w) < \ell(ww')$ if $w' \ne 1$ as asserted. Q.E.D.

3.4. Let $I = \{t^p, s_1, s_2, \dots, s_{n-1}\}$ be a subset of S, and we consider the subgroup $\widetilde{W}_I$ of $\widetilde{W}$ generated by I. Hence $\widetilde{W}_I$ is isomorphic to $G(r,p,n-1)$. We set $\mathcal{D} = \{w(a,n) \mid -p/2 < a \le p/2\}$. Then we have the following lemma.

Lemma 3.5. (i) *The set $\mathcal{D}$ is a set of complete representatives of the double cosets $\widetilde{W}_I\backslash\widetilde{W}/W_J$.*

(ii) *For $w = w(a_1, 1)\cdots w(a_n, n) \in \mathcal{N}$, we have $\ell(w) = \sum_i \ell(w(a_i, i))$.*

(iii) *The set $\mathcal{D}$ is characterized as the set of elements $w \in \widetilde{W}$ such that w is the unique minimal length element in $\widetilde{W}_I w W_J$ with respect to ℓ.*

Proof. We know already by Remark 2.10 that $\widetilde{W} = \widetilde{W}_I \mathcal{D} W_J$. On the other hand, let $x = w(a, n) \in \mathcal{D}$. Then any element $y \in \widetilde{W}_I x W_J$ has the property that y maps some $e_i^{(0)}$ to $e_n^{(a')}$ with $a' \equiv a \pmod p$. This implies that the double cosets are disjoint for distinct elements in $\mathcal{D}$, and we get (i).

We show (ii). Let $w \in \mathcal{N}$. Then by using (3.2.3), one can check that $\varphi(w) = \varphi(w(a_1, 1))\cdots\varphi(w(a_n, n))$, and that $\varphi(w(a_n, n))$ is a distinguished representatives for the cosets $(\widetilde{W}_I)_0\backslash\widetilde{W}_0$. (Here $(\widetilde{W}_I)_0$ denotes the subgroup of $\widetilde{W}_0$ of type B_{n-1} obtained from $\widetilde{W}_I$). Hence the function ℓ_0 is additive with respect to the decomposition of $\varphi(w)$, and so we have $\ell_1(w) = \sum_i \ell(w(a_i, i))$. On the other hand, if we write $w = \sigma z$ as in 3.2, z is given as in (3.2.2). This implies that $\ell_2(w) = \sum_i \ell_2(w(a_i, i))$, and the assertion follows.

Finally we show (iii). Take $x = w(a, n) \in \mathcal{D}$. Then by Remark 2.10, any element $y \in \widetilde{W}_I x W_J$ can be written uniquely as $y = w_1 x w_2$ with $w_1 \in \mathcal{N}_I$ and $w_2 \in W_J$. (Here $\mathcal{N}_I = \mathcal{N} \cap \widetilde{W}_I$). Then by Lemma 3.3, $\ell(w_1 x) \le \ell(w_1 x w_2)$, where the equality holds only when $w_2 = 1$. On the other hand, by (ii), we have $\ell(w_1 x) = \ell(w_1) + \ell(x)$. Hence (iii) holds. Q.E.D.

Remark 3.6. The set $\mathcal{N}$ (resp. $\mathcal{D}$) is also characterized as the set of minimal length elements in each coset in $\widetilde{W}/W_J$ (resp. $\widetilde{W}_I\backslash\widetilde{W}/W_J$) by Proposition 2.9 and Lemma 2.11.

3.7. We now give a characterization of the function $\tilde{n}$ in terms of the function ℓ. In some sense this gives a characterization of the function n on W since $\tilde{n}|_W = n$. Note that by Lemma 3.3 and Lemma 3.5, the sets $\mathcal{N}$ and $\mathcal{D}$ are determined by the function ℓ independently of the choice of the root system.

Theorem 3.8. *Assume that $d \ne 1$. Then the function $\tilde{n} : \widetilde{W} \to \mathbf{N}$ is the unique function satisfying the following properties.*

(i) *The restriction of $\tilde{n}$ on W_J (resp. on $\widetilde{W}_I$) coincides with n_J (resp. $\tilde{n}_I$), where $\tilde{n}_I$ denotes the function on $\widetilde{W}_I = G(r, 1, n-1)$ defined in a similar way as $\tilde{n}$ on $\widetilde{W}$.*

(ii) *For $w \in \mathcal{N}$, $w' \in W_J$, we have $\tilde{n}(ww') = \tilde{n}(w) + \tilde{n}(w')$. For $w \in \mathcal{N}_I$, $w' \in \mathcal{D}$, we have $\tilde{n}(ww') = \tilde{n}(w) + \tilde{n}(w')$.*

(iii) *Let g be an element in $\widetilde{W}$ which is conjugate to t, with $g \neq t$. Set $\alpha = p/2$ if p is even, and $\alpha = -(p-1)/2$ if p is odd. Then we have*

$$0 < \tilde{n}(g) < \tilde{n}(g^{-1}) < \tilde{n}(g^2) < \tilde{n}(g^{-2}) < \cdots < \tilde{n}(g^{\alpha}),$$

(iv) $\displaystyle \frac{1}{p}\sum_{w \in \widetilde{W}} q^{\tilde{n}(w)} = P_W(q).$

Proof. We have already seen in section 2 that $\tilde{n}$ satisfies the condition (i), (ii) and (iv). We show that $\tilde{n}$ satisfies (iii). Take $g \in \widetilde{W}$ as in (iii). Then g can be written as $g = s_i s_{i-1} \cdots s_2 t s_2 \cdots s_{i-1} s_i$ for some $i \geq 2$. Hence we have

$$g^a = \begin{cases} w(a,i) s_i \cdots s_2 & \text{if } 0 < a \leq p/2, \\ w(a,i) & \text{if } -p/2 < a \leq 0. \end{cases} \tag{3.8.1}$$

Since $s_i \cdots s_2 \in W_J$, the length $\tilde{n}(g^a)$ can be computed by Lemma 2.7 and Lemma 2.11, as follows.

$$\tilde{n}(g^a) = \begin{cases} (i-1)\{d(2a-1)+1\} & \text{if } 0 < a \leq p/2, \\ (i-1)(-2ad) & \text{if } -p/2 < a \leq 0. \end{cases}$$

Since $d \neq 1$, the condition (iii) is verified by using the above formula.

Next we show the uniqueness of $\tilde{n}$. If $n = 1$, $\widetilde{W}$ is the cyclic group generated by t and W_J is the subgroup of $\widetilde{W}$ generated by t^p. Hence it is determined by the conditions (i) and (ii). So we assume that $n > 1$. By (i) and (ii), it is enough to see that $\tilde{n}(w)$ is determined uniquely for $w \in \mathcal{D}$. Let $w = w(a,n) \in \mathcal{D}$ and set $c(a) = \tilde{n}(w)/(n-1)$. Then by (iv), we have

$$\{c(a) \mid -p/2 < a \leq p/2\} = \{0, d, 2d, \ldots, (p-1)d\}. \tag{3.8.2}$$

Since $|\mathcal{D}| = p$, $c(a)$ are all distinct. On the other hand, let $g = s_n \cdots s_2 t s_2 \cdots s_n$. Then by (3.8.1) and (ii), we have

$$\tilde{n}(g^a) = \begin{cases} (n-1)(c(a)+1) & \text{if } 0 < a \leq p/2, \\ (n-1)c(a) & \text{if } -p/2 < a \leq 0. \end{cases}$$

Hence by using (iii), we have

$$c(i) + 1 < c(-i) < c(i+1) + 1$$

for $i = 1, 2, \ldots$. Since $c(a) \equiv 0 \pmod{d}$, and $d \neq 1$, we have $c(i) < c(-i) < c(i+1)$. It follows, by (3.8.2), that we have

$$c(a) = \begin{cases} (2a-1)d & \text{if } a > 0, \\ (-2a)d & \text{if } a \leq 0. \end{cases}$$

The function $\tilde{n}$ is now determined on $\mathcal{D}$, and so the theorem follows.

Q.E.D.

Remark 3.9. In the case where $d = 1$, the property (iii) in the theorem does not hold. Instead, we have the following relation.

(iii') $\quad 0 < \tilde{n}(g) = \tilde{n}(g^{-1}) < \tilde{n}(g^2) = \tilde{n}(g^{-2}) < \cdots \leq \tilde{n}(g^\alpha).$

Then the function $\tilde{n}$ is characterized by the properties (i) $\sim$ (iv), but replacing (iii) by (iii'). In fact, by a similar argument as above, we have

$$c(i) + 1 = c(-i) < c(i+1) + 1$$

for $i = 1, 2, \ldots$. Thus $c(i)$ is the smallest integer among all the $c(a)$ such that $|a| \geq i$. Since the set $\{c(a) \mid -p/2 < a \leq p/2\}$ coincides with the set $\{0, 1, \ldots, p-1\}$, this determines $c(i)$ and so $c(-i)$ successively for $i = 1, 2, \ldots$. Hence the function $\tilde{n}$ is determined uniquely.

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Representations of Degenerate Affine Hecke Algebra and $\mathfrak{gl}_n$

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Abstract.

We study the representation theory of the degenerate affine Hecke algebra H_ℓ of GL_ℓ using functors that connect the representation theory of H_ℓ and that of the Lie algebra $\mathfrak{gl}_n$. In particular, a new algebraic approach to the classification theorem of simple H_ℓ-modules is given.

Introduction

Let H_ℓ denote the degenerate (or graded) affine Hecke algebra of GL_ℓ introduced by Drinfeld [Dr] as a certain limit of the affine Hecke algebra. Lusztig [Lu1, Lu2] introduced the degenerate affine Hecke algebra associated to a general reductive group, and proved that the representation theory of the degenerate affine Hecke algebra and that of the corresponding affine Hecke algebra are very close, and one can be essentially recovered from the other.

The representation theory of the (degenerate) affine Hecke algebra has been developed by some methods. Zelevinsky [Ze1] classified simple admissible modules over $GL_\ell(F)$, where F is a p-adic field. This gives a classification of simple modules over the affine Hecke algebra of GL_ℓ through a theorem due to Bernstein, Borel and Matsumoto. In Zelevinsky's classification, the simple modules are constructed as unique simple quotient modules (resp. unique simple submodules) of certain induced modules called standard modules (resp. co-standard modules). In [Ze2, Ze3], Zelevinsky conjectured that the multiplicities of simple modules in the composition series of an induced module are described by Kazhdan-Lusztig polynomials of the symmetric group.

This conjecture was proved by Ginzburg [Gi1] (see also [CG]) through geometric methods. (In fact, Ginzburg gave the multiplicity formulas for general affine Hecke algebras in terms of intersection cohomologies. For

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degenerate affine Hecke algebras, the corresponding formulas were given by Lusztig [Lu3].)

As shown in [BB1, BK], the Kazhdan-Lusztig polynomials also occur in the multiplicity formulas for highest weight modules over semisimple Lie algebras. Consequently, the multiplicity formulas for H_ℓ-modules and those for $\mathfrak{gl}_\ell$-modules are both described by the Kazhdan-Lusztig polynomials of the symmetric group.

This observation led us to the study of a family of functors from the category $\mathcal{O}$ of $\mathfrak{gl}_n$-modules to the category of finite-dimensional H_ℓ-modules in [AS, Su]. It turned out that these functors, which arose from conformal field theory [AST], transform the composition series of a Verma module to the composition series of a standard module under certain conditions, and they connect multiplicity formulas in two categories directly. They give a new approach to the representation theory of H_ℓ. For example, some results for H_ℓ-modules can be deduced from the corresponding results for $\mathfrak{gl}_n$-modules through the functors.

The purpose of this paper is to survey the theory of the functors and to see how it is applied to the study of the representation theory of H_ℓ

After some preliminaries in §1 and §2, we define the functors in §3. It turns out that the functors map a Verma module over $\mathfrak{gl}_n$ to an induced module over H_ℓ, which we introduce in §4. One of the most important statement concerning induced modules is Theorem 5.3, which states that an induced module has a unique simple quotient under certain conditions. Using Theorem 5.3, we prove that a simple module over $\mathfrak{gl}_n$ is mapped to a simple module over H_ℓ (or zero) in §5. Theorem 5.3 also plays an essential role in §6, where we give a new proof for the classification of simple H_ℓ-modules. The functors reduce a part of the problem to the classification of simple modules in the category $\mathcal{O}$. In §7, we apply the functors to get some explicit consequences concerning a special class of simple modules parameterized by skew Young diagrams. §8 is on Kazhdan-Lusztig multiplicity formulas. We see that the multiplicity formulas for $\mathfrak{gl}_n$ (given in [BB1, BK]) imply those for H_ℓ (given in [Gi1, Lu3]) via the functors. We also obtain a refinement of the multiplicity formulas concerning the Jantzen filtration on the induced modules (Rogawski's conjecture).

We treat the degenerate affine Hecke algebra in this paper but it is not hard to extend the story to the non-degenerate case, where the degenerate affine Hecke algebra is replaced by the affine Hecke algebra, and $\mathfrak{gl}_n$ is replaced by its quantum enveloping algebra. In Appendix B, we give an action of the affine Hecke algebra on the tensor product of modules over the quantized enveloping algebra. A q-analogue of the functors is constructed from this action.

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§1. Root system and Lie algebra $\mathfrak{gl}_n$

Let $n \in \mathbb{Z}_{\geq 2}$. Let $\mathfrak{gl}_n$ denote the Lie algebra consisting of all $n \times n$ matrices with entries in $\mathbb{C}$. An inner product is defined on $\mathfrak{gl}_n$ by

$$(1.1) \qquad (x|y)_n = \mathrm{tr}(xy)$$

for $x, y \in \mathfrak{gl}_n$. Let $\mathfrak{t}_n$ be the Cartan subalgebra of $\mathfrak{gl}_n$ consisting of all diagonal matrices, and let $\mathfrak{t}_n^*$ be its dual space. The natural pairing is denoted by $\langle\ ,\ \rangle_n : \mathfrak{t}_n^* \times \mathfrak{t}_n \to \mathbb{C}$. Let $E_{i,j}$ $(1 \leq i, j \leq n)$ denote the matrix with only nonzero entries 1 at the (i,j)-th component. Define a basis $\{\epsilon_i\}_{i=1,\dots,n}$ of $\mathfrak{t}_n^*$ by $\epsilon_i(E_{j,j}) = \delta_{i,j}$, and define the roots by $\alpha_{ij} = \epsilon_i - \epsilon_j$ and the simple roots by $\alpha_i = \epsilon_i - \epsilon_{i+1}$.

Put

$$(1.2) \qquad R_n = \{\alpha_{ij} \mid 1 \leq i \neq j \leq n\},$$

$$(1.3) \qquad R_n^+ = \{\alpha_{ij} \mid 1 \leq i < j \leq n\}, \quad R_n^- = R_n \setminus R_n^+,$$

$$(1.4) \qquad \Pi_n = \{\alpha_i \mid i = 1, \dots, n-1\}.$$

Then $R_n \subseteq \mathfrak{t}_n^*$ is a root system of type A_{n-1}. Since the restriction of $(\ |\ )_n$ to $\mathfrak{t}_n$ is non-degenerate, we have an isomorphism $\mathfrak{t}_n^* \xrightarrow{\sim} \mathfrak{t}_n$, whose image of $\xi \in \mathfrak{t}_n^*$ is denoted by $\xi^\vee$. In particular we have $\epsilon_i^\vee = E_{i,i}$ and $\alpha_i^\vee = E_{i,i} - E_{i+1,i+1}$. We often identify $\mathfrak{t}_n^*$ with $\mathbb{C}^n$ by $\sum_{i=1}^n \lambda_i \epsilon_i \leftrightarrow (\lambda_1, \dots, \lambda_n)$.

Define

$$(1.5) \qquad Q_n = \bigoplus_{i=1}^{n-1} \mathbb{Z}\alpha_i,$$

$$(1.6) \qquad P_n = \bigoplus_{i=1}^{n} \mathbb{Z}\epsilon_i, \quad P_n^+ = \{\lambda \in P_n \mid \langle \lambda, \alpha^\vee \rangle_n \geq 0 \text{ for all } \alpha \in R_n^+\}.$$

An element of P_n (resp. P_n^+) is called a *integral* (resp. *dominant integral*) weight.

Putting $\mathfrak{n}_n^+ = \oplus_{i<j} \mathbb{C}E_{i,j}$, $\mathfrak{n}_n^- = \oplus_{i>j} \mathbb{C}E_{i,j}$, we have a triangular decomposition $\mathfrak{gl}_n = \mathfrak{n}_n^+ \oplus \mathfrak{t}_n \oplus \mathfrak{n}_n^-$. We put $\mathfrak{b}_n^\pm = \mathfrak{n}_n^\pm \oplus \mathfrak{t}_n$.

The Weyl group W_n associated to the root system (R_n, Π_n) is, by definition, a subgroup of $GL(\mathfrak{t}_n^*)$ generated by the reflections s_α $(\alpha \in R_n)$ defined by

$$s_\alpha(\lambda) = \lambda - \langle \lambda, \alpha^\vee \rangle_n \alpha \quad (\lambda \in \mathfrak{t}_n^*). \tag{1.7}$$

We often use another action of W_n on $\mathfrak{t}_n^*$, which is given by

$$w \circ \lambda = w(\lambda + \rho) - \rho \quad (w \in W_n,\ \lambda \in \mathfrak{t}_n^*), \tag{1.8}$$

where $\rho = (n-1, n-2 \ldots, 0) \in \mathfrak{t}_n^*$.

For a $\mathfrak{t}_n$-module X and $\lambda \in \mathfrak{t}_n^*$, put

$$X_\lambda = \{v \in X \mid hv = \langle \lambda, h \rangle_n v \text{ for all } h \in \mathfrak{t}_n\}, \tag{1.9}$$

$$X_\lambda^{\text{gen}} = \{v \in X \mid (h - \langle \lambda, h \rangle_n)^k v = 0 \text{ for all } h \in \mathfrak{t}_n, \text{ some } k \in \mathbb{Z}_{>0}\}, \tag{1.10}$$

$$P(X) = \{\lambda \in \mathfrak{t}_n^* \mid X_\lambda \neq 0\}. \tag{1.11}$$

The space X_λ (resp X_λ^{gen}) is called the *weight space* (resp. *generalized weight space*) of weight λ with respect to $\mathfrak{t}_n$, and an element of $P(X)$ is called a weight of X.

Let $U(\mathfrak{gl}_n)$ denote the universal enveloping algebra of $\mathfrak{gl}_n$. There is a unique anti-involution σ of $U(\mathfrak{gl}_n)$ such that $\sigma(E_{ij}) = E_{ji}$. For a $\mathfrak{gl}_n$-module X, a bilinear form $(\ |\ ) : X \times X \to \mathbb{C}$ is called a $\mathfrak{gl}_n$-contravariant form if $(u|xv) = (\sigma(x)u|v)$ for all $u, v \in X$ and $x \in \mathfrak{gl}_n$.

For $\lambda \in \mathfrak{t}_n^*$, let $M(\lambda) = U(\mathfrak{gl}_n) \otimes_{U(\mathfrak{b}_n^+)} \mathbb{C}v_\lambda$ denote the Verma module with highest weight λ, where v_λ denotes the highest weight vector. There is a unique $\mathfrak{gl}_n$-contravariant form on $M(\lambda)$ such that $(v_\lambda|v_\lambda) = 1$. It follows that the radical of $(\ |\ )$ is the unique maximal submodule of $M(\lambda)$. (See e.g. [Ja] for the proofs.) The unique simple quotient module of $M(\lambda)$ is denoted by $L(\lambda)$.

Let $\mathcal{O} = \mathcal{O}(\mathfrak{gl}_n)$ denote the category of $\mathfrak{gl}_n$-modules which are finitely generated over $U(\mathfrak{gl}_n)$, $\mathfrak{n}_n^+$-locally finite and $\mathfrak{t}_n$-semisimple (see [BGG]). The modules $M(\lambda)$ and $L(\lambda)$ are objects of $\mathcal{O}$. Let $\chi_\lambda : Z(U(\mathfrak{gl}_n)) \to \mathbb{C}$ denote the infinitesimal character of $M(\lambda)$ (i.e. $zv = \chi_\lambda(z)v$ for all $z \in Z(U(\mathfrak{gl}_n))$, $v \in M(\lambda)$). We introduce an equivalence relation in $\mathfrak{t}_n^*$ by

$$\lambda \sim \mu \Leftrightarrow \lambda = w \circ \mu \ \text{ for some } w \in \mathfrak{S}_n. \tag{1.12}$$

Then it follows that $\chi_\lambda = \chi_\mu$ if and only if $\lambda \sim \mu$. Define the full subcategory $\mathcal{O}^{\chi_\lambda}$ of $\mathcal{O}$ by

$$ob\, \mathcal{O}^{\chi_\lambda} = \{X \in ob\, \mathcal{O} \mid (\mathrm{Ker}\chi_\lambda)^k X = 0 \text{ for some } k\}. \tag{1.13}$$

Then any $X \in ob\,\mathcal{O}$ admits a decomposition

$$X = \bigoplus_{\lambda} X^{\chi_\lambda} \tag{1.14}$$

such that $X^{\chi_\lambda} \in ob\,\mathcal{O}^{\chi_\lambda}$, where λ runs over all representatives of $\mathfrak{t}_n^*/\sim$. The correspondence $X \mapsto X^{\chi_\lambda}$ gives an exact functor on $\mathcal{O}$.

For $X \in ob\,\mathcal{O}$, put

$$H^0(\mathfrak{n}_n^+, X) = \{v \in X \mid \mathfrak{n}_n^+ v = 0\}, \tag{1.15}$$

$$H_0(\mathfrak{n}_n^-, X) = X/\mathfrak{n}_n^- X. \tag{1.16}$$

Then these are finite-dimensional $\mathfrak{t}_n$-modules. By the universality of the Verma module and (1.14), we have $H^0(\mathfrak{n}_n^+, X)_\lambda \cong \mathrm{Hom}_{\mathfrak{gl}_n}(M(\lambda), X) = \mathrm{Hom}_{\mathfrak{gl}_n}(M(\lambda), X^{\chi_\lambda})) \cong H^0(\mathfrak{n}_n^+, X^{\chi_\lambda})_\lambda$. It also holds that $H_0(\mathfrak{n}_n^-, X)_\lambda \cong H_0(\mathfrak{n}_n^-, X^{\chi_\lambda})_\lambda$. Hence we have a natural injective (resp. surjective) map $H^0(\mathfrak{n}_n^+, X)_\lambda \to (X^{\chi_\lambda})_\lambda$, (resp. $(X^{\chi_\lambda})_\lambda \to H_0(\mathfrak{n}_n^-, X)_\lambda$). Set

$$D_n = \{\lambda \in \mathfrak{t}_n^* \mid \langle \lambda + \rho, \alpha^\vee \rangle_n \notin \mathbb{Z}_{<0} \text{ for all } \alpha \in R_n^+\}. \tag{1.17}$$

Lemma 1.1 ([AS]). *Let $\lambda \in D_n$. Then the maps defined above are both bijective: $H^0(\mathfrak{n}_n^+, X)_\lambda \cong (X^{\chi_\lambda})_\lambda \cong H_0(\mathfrak{n}_n^-, X)_\lambda$.*

§2. Symmetric group and degenerate affine Hecke algebra

Let $\ell \in \mathbb{Z}_{\geq 2}$. Let $\mathfrak{S}_\ell$ denote the symmetric group. Let s_i denote the simple reflection $(i, i+1)$. Then $\mathfrak{S}_\ell$ is generated by $s_1, \dots, s_{\ell-1}$, and the correspondence $s_i \mapsto s_{\alpha_i}$ gives an isomorphism from $\mathfrak{S}_\ell$ to the Weyl group W_ℓ of the root system (R_ℓ, Π_ℓ).

The length function $l : \mathfrak{S}_\ell \to \mathbb{Z}_{\geq 0}$ is defined by $l(w) = \sharp R_\ell(w)$ for $w \in \mathfrak{S}_\ell$, where

$$R_\ell(w) = R_\ell^+ \cap w^{-1}(R_\ell^-). \tag{2.1}$$

We write $w \to y$ if $y = s_\alpha w$ for some $\alpha \in R_\ell$ and $l(w) < l(y)$. Define $w < y$ if there is a sequence $w \to w_1 \to w_2 \to \cdots \to y$. The resulting relation $\leq$ in $\mathfrak{S}_\ell$ defines a partial order called the *Bruhat order*. Put

$$P_n(\ell) = \{\lambda \in P_n \mid \lambda_i \geq 0 \ (i = 1, \dots, n) \text{ and } \sum_{i=1}^n \lambda_i = \ell\},$$

$$P_n^+(\ell) = P_n(\ell) \cap P_n^+.$$

An element of $P_n(\ell)$ is called a partition of ℓ with n components. The set $P_n^+(\ell)$ is in one to one correspondence with the set of Young diagrams with ℓ boxes consisting of at most n rows.

Define a surjective map $P_n(\ell) \to P_n^+(\ell)$ by the correspondence $\lambda \mapsto \lambda^+$, where λ^+ denotes the unique element in $P_n^+(\ell) \cap \{w(\lambda) \mid w \in \mathfrak{S}_\ell\}$.

Let us recall that simple $\mathfrak{S}_\ell$-modules are parameterized by the set $P_\ell^+(\ell)$. Let S_λ denote the simple module corresponding to $\lambda \in P_\ell^+(\ell)$.

For $\lambda = (\lambda_1, \dots, \lambda_n) \in P_n(\ell)$, consider the parabolic subgroup $\mathfrak{S}_\lambda := \mathfrak{S}_{\lambda_1} \times \cdots \times \mathfrak{S}_{\lambda_n}$ of $\mathfrak{S}_\ell$ and set

$$\mathfrak{S}_\lambda^\perp = \{w \in \mathfrak{S}_\ell \mid l(ws) > l(w) \text{ for all } s \in \mathfrak{S}_\lambda \cap \{s_1, \dots, s_{\ell-1}\}\}. \tag{2.2}$$

Then an element w of $\mathfrak{S}_\lambda^\perp$ is the unique shortest element in the coset $w\mathfrak{S}_\lambda$.

The group $\mathfrak{S}_\ell$ acts on the set $\mathfrak{S}_\lambda^\perp \cong \mathfrak{S}_\ell/\mathfrak{S}_\lambda$ and thus the space $\mathbb{C}[\mathfrak{S}_\lambda^\perp]$ spanned by the elements in $\mathfrak{S}_\lambda^\perp$ is regarded as a $\mathbb{C}[\mathfrak{S}_\ell]$-module. The $\mathfrak{S}_\ell$-module structure of $\mathbb{C}[\mathfrak{S}_\lambda^\perp]$ depends only on the image $\lambda^+ \in P_n(\ell)$.

Let $\lambda \in P_n^+(\ell)$. It is known that the $\mathfrak{S}_\ell$-module $\mathbb{C}[\mathfrak{S}_\lambda^\perp]$ decomposes into

$$\mathbb{C}[\mathfrak{S}_\lambda^\perp] \cong S_\lambda \oplus \bigoplus_{\nu \in P_n^+(\ell),\ \nu \rhd \lambda} S_\nu^{\oplus K_{\nu,\lambda}}, \tag{2.3}$$

where $\rhd$ denotes the dominance order in the set of partitions, and $K_{\nu,\lambda}$ denotes some non-negative integer called Kostka number (see e.g. [Mac, Sa]).

Let $S(\mathfrak{t}_\ell)$ denote the symmetric algebra of $\mathfrak{t}_\ell$, which is isomorphic to the polynomial ring $\mathbb{C}[\epsilon_1^\vee, \dots, \epsilon_\ell^\vee]$.

Definition 2.1. The *degenerate* (*or graded*) *affine Hecke algebra* H_ℓ of GL_ℓ is the unital associative algebra over $\mathbb{C}$ defined by the following properties:

(i) As a vector space, $H_\ell \cong \mathbb{C}[\mathfrak{S}_\ell] \otimes S(\mathfrak{t}_\ell)$.

(ii) The subspaces $\mathbb{C}[\mathfrak{S}_\ell] \otimes \mathbb{C}$ and $\mathbb{C} \otimes S(\mathfrak{t}_\ell)$ are subalgebras of H_ℓ in a natural fashion (their images will be identified with $\mathbb{C}[\mathfrak{S}_\ell]$ and $S(\mathfrak{t}_\ell)$ respectively).

(iii) The following relations hold in H_ℓ:

$$s_i \cdot \xi - s_i(\xi) \cdot s_i = -\langle \alpha_i, \xi \rangle_\ell \quad (i = 1, \dots, \ell,\ \xi \in \mathfrak{t}_\ell). \tag{2.4}$$

Proposition 2.2. [Lu1] *The center of* H_ℓ *is*

$$S(\mathfrak{t}_\ell)^{\mathfrak{S}_\ell} := \{f \in S(\mathfrak{t}_\ell) \mid w(f) = f \quad \textit{for any } w \in \mathfrak{S}_\ell\}.$$

It is easy to verify that there exists a unique anti-involution ι on H_ℓ such that

$$\iota(s_i) = s_i \ (i = 1, \dots, \ell - 1), \quad \iota(\epsilon_i^\vee) = \epsilon_i^\vee \ (i = 1, \dots, \ell). \tag{2.5}$$

For an H_ℓ-module Y, a bilinear form $(\ |\ ) : Y \times Y \to \mathbb{C}$ is called an H_ℓ-contravariant form if $(u|xv) = (\iota(x)u|v)$ for all $u, v \in Y$ and all $x \in H_\ell$.

Let us introduce intertwining operators, which are useful tools for the investigation of representation theory of H_ℓ. In the rest of this section we refer to e.g. [Lu1, AST] for the proofs of statements.

For each $i \in \{1, \dots, \ell - 1\}$, we put

$$\phi_i = 1 + s_i \alpha_i^\vee \in H_\ell.$$

Then we have

$$\phi_i \cdot \xi = s_i(\xi) \cdot \phi_i \quad (\xi \in \mathfrak{t}_\ell).$$

Proposition 2.3. *The elements $\{\phi_i\}_i$ defined above satisfy the following relations*:

$$\phi_i \cdot \phi_{i+1} \cdot \phi_i = \phi_{i+1} \cdot \phi_i \cdot \phi_{i+1} \quad (i = 1, \dots, \ell - 2), \tag{2.6}$$

$$\phi_i \cdot \phi_j = \phi_j \cdot \phi_j \quad (|i - j| \neq 1), \tag{2.7}$$

$$\phi_i^2 = 1 - \alpha_i^{\vee 2} \quad (i = 1, \dots, \ell - 1)). \tag{2.8}$$

For $w \in \mathfrak{S}_\ell$, let $w = s_{j_1} \cdots s_{j_s} \in \mathfrak{S}_\ell$ be a reduced expression. Put

$$\phi_w = \phi_{j_1} \cdots \phi_{j_s} \in H_\ell.$$

Then the element ϕ_w does not depend on the choice of reduced expressions by Proposition 2.3, and it holds that

$$\phi_{wy} = \phi_w \cdot \phi_y \ \text{ if } l(wy) = l(w) + l(y). \tag{2.9}$$

By (2.6), we have

$$\phi_w \cdot \xi = w(\xi) \cdot \phi_w \quad (w \in \mathfrak{S}_\ell,\ \xi \in \mathfrak{t}_\ell)\,. \tag{2.10}$$

For an H_ℓ-module Y and $\zeta \in \mathfrak{t}_\ell^*$, we define Y_ζ, Y_ζ^{gen}, and $P(Y)$ by the same formulas as (1.9), (1.10), and (1.11) respectively.

Note that any finite-dimensional H_ℓ-module Y admits the decomposition $Y = \oplus_{\zeta \in \mathfrak{t}_\ell^*} Y_\zeta^{\mathrm{gen}}$.

Proposition 2.4. *Let Y be an H_ℓ-module. Let $\zeta \in \mathfrak{t}_\ell^*$ and $w \in \mathfrak{S}_\ell$. Then $\phi_w(Y_\zeta) \subseteq Y_{w(\zeta)}$ and $\phi_w(Y_\zeta^{\mathrm{gen}}) \subseteq Y_{w(\zeta)}^{\mathrm{gen}}$.*

The element ϕ_w is called the *intertwining operator* (of weight spaces).

Proposition 2.5. *Let $w \in \mathfrak{S}_\ell$. The following relations hold in H_ℓ :*
(i)

$$\phi_w = w \cdot \prod_{\alpha \in R_\ell(w)} \alpha^\vee + \sum_{y<w} y \cdot p_y,$$

for some $p_y \in S(\mathfrak{t}_\ell)$. Here $R_\ell(w) = R_\ell^+ \cap w^{-1}(R_\ell^-)$.
(ii)

$$\phi_{w^{-1}} \cdot \phi_w = \prod_{\alpha \in R_\ell(w)} (1 - \alpha^{\vee 2}).$$

§3. Functors F_λ

Let us recall the definition of the functor

$$F_\lambda : \mathcal{O}(\mathfrak{gl}_n) \to \mathcal{R}(H_\ell)$$

introduced in [AS]. Here $\mathcal{R}(H_\ell)$ denotes the category of finite-dimensional representations of H_ℓ. Let $V_n = \mathbb{C}^n$ denote the vector representation of $\mathfrak{gl}_n$.

Proposition 3.1 ([AS]). *For any $X \in \mathcal{O}(\mathfrak{gl}_n)$, there exists a unique homomorphism*

$$\theta : H_\ell \to \mathrm{End}_{U(\mathfrak{gl}_n)}(X \otimes V_n^{\otimes \ell}) \tag{3.1}$$

such that

$$\theta(s_i) = \Omega_{i\,i+1} \qquad (i = 1, \dots, \ell - 1), \tag{3.2}$$

$$\theta(\epsilon_i^\vee) = \sum_{0 \le j < i} \Omega_{ji} + n - 1 \quad (i = 1, \dots, \ell), \tag{3.3}$$

where Ω_{ji} denote the operator given by the element

$$\sum_{1 \le k,m \le n} 1^{\otimes j} \otimes E_{k,m} \otimes 1^{\otimes i-j-1} \otimes E_{m,k} \otimes 1^{\otimes \ell - i} \in \mathfrak{gl}^{\otimes \ell+1}. \tag{3.4}$$

Remark 3.2. The action of $\mathfrak{S}_\ell$ given by (3.2) is just the natural action of $\mathfrak{S}_\ell$ on $V_n^{\otimes \ell}$.

Let $\lambda \in D_n$ and $X \in ob\, \mathcal{O}(\mathfrak{gl}_n)$. We define

$$F_\lambda(X) = (X \otimes V_n^{\otimes \ell})_\lambda^{\chi_\lambda} \tag{3.5}$$

with an induced H_ℓ-module structure through the homomorphism θ. Obviously F_λ defines an exact functor from $\mathcal{O}(\mathfrak{gl}_n)$ to $\mathcal{R}(H_\ell)$.

Let $X, Y \in ob\,\mathcal{O}(\mathfrak{gl}_n)$ with $\mathfrak{gl}_n$-contravariant forms $(\ |\ )_X$, $(\ |\ )_Y$. Then the tensor product $X \otimes Y$ is equipped with a $\mathfrak{gl}_n$-contravariant bilinear form $(\ |\ )_X \times (\ |\ )_Y$.

The following Proposition immediately follows from the definition of the action θ.

Lemma 3.3 ([Su]). *Let X be a $\mathfrak{gl}_n$-module with a $\mathfrak{gl}_n$-contravariant form. The $\mathfrak{gl}_n$-contravariant form on $X \otimes V_n^{\otimes \ell}$ is also H_ℓ-contravariant, and it induces an H_ℓ-contravariant form on $(X \otimes V_n^{\otimes \ell})_\lambda^{\chi_\lambda} = F_\lambda(X)$.*

§4. Induced modules

Let $\lambda, \mu \in \mathfrak{t}_n^*$ be such that $\lambda - \mu \in P_n(\ell)$, and put

$$\ell_i = \lambda_i - \mu_i, \quad (i = 1, \dots, n). \tag{4.1}$$

Put $H_{\lambda-\mu} := H_{\ell_1} \otimes \cdots \otimes H_{\ell_n} = \mathbb{C}[\mathfrak{S}_{\lambda-\mu}] \otimes S(\mathfrak{t}_\ell)$ and regard it as a subalgebra of H_ℓ. There exists a one-dimensional representation $\mathbb{C}_{\lambda,\mu} = \mathbb{C}\mathbf{1}_{\lambda,\mu}$ of $H_{\lambda,\mu}$ such that

$$w\mathbf{1}_{\lambda,\mu} = \mathbf{1}_{\lambda,\mu} \quad (w \in \mathfrak{S}_{\lambda-\mu}), \tag{4.2}$$

$$\xi\mathbf{1}_{\lambda,\mu} = \langle \zeta_{\lambda,\mu}, \xi \rangle_\ell \mathbf{1}_{\lambda,\mu} \quad (\xi \in \mathfrak{t}_\ell), \tag{4.3}$$

where $\zeta_{\lambda,\mu} \in \mathfrak{t}_\ell^*$ is given by

(4.4)

$$\langle \zeta_{\lambda,\mu}, \epsilon_j^\vee \rangle_\ell = \mu_i + n - i + j - \sum_{k=1}^{i-1} \ell_k - 1 \quad \text{for} \quad \sum_{k=1}^{i-1} \ell_k < j \leq \sum_{k=1}^{i} \ell_k.$$

Note, in particular, that if we put $a_i = \sum_{k=1}^{i-1} \ell_k + 1$ and $b_i = \sum_{k=1}^{i} \ell_k$, then

$$\langle \zeta_{\lambda,\mu}, \epsilon_{a_i}^\vee \rangle_\ell = \langle \mu + \rho, \epsilon_i^\vee \rangle_n, \quad \langle \zeta_{\lambda,\mu}, \epsilon_{b_i}^\vee \rangle_\ell = \langle \lambda + \rho, \epsilon_i^\vee \rangle_n - 1, \tag{4.5}$$

$$\langle \zeta_{\lambda,\mu}, \alpha_i^\vee \rangle_\ell = -1 \text{ for } i \notin \{b_1, b_2, \dots, b_n\}. \tag{4.6}$$

Define an H_ℓ-module $\mathcal{M}(\lambda, \mu)$ by

$$\mathcal{M}(\lambda, \mu) = H_\ell \underset{H_{\lambda-\mu}}{\otimes} \mathbb{C}_{\lambda,\mu}. \tag{4.7}$$

It is obvious that $\mathcal{M}(\lambda, \mu) \downarrow_{\mathbb{C}[\mathfrak{S}_\ell]} \cong \mathbb{C}[\mathfrak{S}_{\lambda-\mu}^\perp]$ and thus its dimension is given by

$$\dim \mathcal{M}(\lambda, \mu) = \frac{\ell!}{\ell_1! \cdots \ell_n!}.$$

For $\zeta \in \mathfrak{t}_\ell^*$, let $\mathfrak{S}_\ell[\zeta]$ denote the stabilizer of ζ:

$$\mathfrak{S}_\ell[\zeta] = \{s \in \mathfrak{S}_\ell \mid w(\zeta) = \zeta\}. \tag{4.8}$$

Lemma 4.1. *For $\lambda, \mu \in \mathfrak{t}_n^*$ such that $\lambda - \mu \in P_n(\ell)$, we have*
(i) $P(\mathcal{M}(\lambda, \mu)) = \{w(\zeta_{\lambda,\mu}) \mid w \in \mathfrak{S}_{\lambda-\mu}^{\perp}\}$.
(ii) *For $\eta \in P(\mathcal{M}(\lambda, \mu))$, we have*

$$\dim \mathcal{M}(\lambda, \mu)_\eta^{\text{gen}} = \#\{w \in \mathfrak{S}_{\lambda-\mu}^{\perp} \mid w(\zeta_{\lambda,\mu}) = \eta\}.$$

In particular, $\dim \mathcal{M}(\lambda, \mu)_{\zeta_{\lambda,\mu}}^{\text{gen}} = \# \left(\mathfrak{S}_{\lambda-\mu}^{\perp} \cap \mathfrak{S}_\ell[\zeta_{\lambda,\mu}]\right)$.

Proof. First, note that $\{w\mathbf{1}_{\lambda,\mu} \mid w \in \mathfrak{S}_{\lambda-\mu}^{\perp}\}$ gives a basis of $\mathcal{M}(\lambda, \mu)$. For $\xi \in \mathfrak{t}_\ell$ and $w \in \mathfrak{S}_{\lambda-\mu}^{\perp}$, it follows from the relation (2.4) that

$$\begin{aligned} \xi \cdot w\mathbf{1}_{\lambda,\mu} &= w \cdot w^{-1}(\xi)\mathbf{1}_{\lambda,\mu} + \sum_{y<w} a_y y\mathbf{1}_{\lambda,\mu} \\ &= \langle w(\zeta_{\lambda,\mu}), \xi\rangle_\ell w\mathbf{1}_{\lambda,\mu} + \sum_{y<w} a_y y\mathbf{1}_{\lambda,\mu}, \end{aligned} \tag{4.9}$$

for some numbers a_y, where y runs over those elements of $\mathfrak{S}_{\lambda-\mu}^{\perp}$ such that $y < w$. Hence we have (i). Now (ii) is obvious. Q.E.D.

We extend the definition of $\mathcal{M}(\lambda, \mu)$ for any $\lambda, \mu \in \mathfrak{t}_n^*$ by

$$\mathcal{M}(\lambda, \mu) = 0 \text{ for } \lambda, \mu \in \mathfrak{t}_n^* \text{ such that } \lambda - \mu \notin P_n(\ell). \tag{4.10}$$

Theorem 4.2 ([AS]). *Let $\lambda \in D_n$ and $\mu \in \mathfrak{t}_n^*$. Then there is an isomorphism of H_ℓ-modules*

$$F_\lambda(M(\mu)) \cong \mathcal{M}(\lambda, \mu).$$

For $w \in \mathfrak{S}_n$, let w_μ^λ denote the unique longest element in the coset $\mathfrak{S}_n[\lambda + \rho]w\mathfrak{S}_n[\mu + \rho]$.

Lemma 4.3. *Let $\lambda, \mu \in D_n$ and $w \in \mathfrak{S}_n$ be such that $\lambda - w\circ\mu \in P_n(\ell)$. Then $\mathcal{M}(\lambda, w \circ \mu) \cong \mathcal{M}(\lambda, w_\mu^\lambda \circ \mu)$.*

Proof. We prove the statement using the fact known in the representation theory of $\mathfrak{gl}_n$; there exists an injective homomorphism $M(w_\mu^\lambda \circ \mu) \to M(w \circ \mu)$. By applying the exact functor F_λ, we have an injective homomorphism $\mathcal{M}(\lambda, w_\mu^\lambda \circ \mu) \to \mathcal{M}(\lambda, w \circ \mu)$. It is easy to see that $(\lambda - w \circ \mu)^+ = (\lambda - w_\mu^\lambda \circ \mu)^+$. This implies $\dim \mathcal{M}(\lambda, w_\mu^\lambda \circ \mu) = \dim \mathcal{M}(\lambda, w \circ \mu)$ and thus $\mathcal{M}(\lambda, w \circ \mu) \cong \mathcal{M}(\lambda, w_\mu^\lambda \circ \mu)$. Q.E.D.

§5. Simple quotient

We give a sufficient condition for an induced module to have a unique simple quotient (Theorem 5.3), which is an essential step to the classification of simple representations of H_ℓ. Theorem 5.3 has been obtained by Zelevinsky [Zel]. We give another proof and the key Lemma 5.2 seems to be new.

Lemma 5.1. *Let $\lambda, \mu \in \mathfrak{t}_n^*$ be such that $\lambda-\mu \in P_n(\ell)$, and suppose that $\langle \lambda+\rho, \alpha_i^\vee \rangle_n = 0$ or $\langle \lambda+\rho, \alpha_i^\vee \rangle_n \notin \mathbb{Z}$. Then $\mathcal{M}(\lambda, \mu) \cong \mathcal{M}(s_i \circ \lambda, s_i \circ \mu)$.*

Proof. If $\langle \lambda+\rho, \alpha_i^\vee \rangle_n = 0$, then the statement follows from Lemma 4.3.

Suppose $\langle \lambda+\rho, \alpha_i^\vee \rangle_n \notin \mathbb{Z}$. Put $\ell_j = \lambda_j - \mu_j$ $(j = 1, \ldots, n)$ and let w be the element of $\mathfrak{S}_{\ell_i+\ell_{i+1}}$ corresponding to the permutation $(1, 2, \ldots, \ell_i + \ell_{i+1}) \mapsto (\ell_i + 1, \ell_i + 2, \ldots, \ell_i + \ell_{i+1}, 1, 2, \ldots, \ell_i)$. Regard $\mathfrak{S}_{\ell_i+\ell_{i+1}}$ as a subgroup of $\mathfrak{S}_\ell$ via $\{1\} \times \mathfrak{S}_{\ell_i+\ell_{i+1}} \times \{1\} \subseteq \mathfrak{S}_{\ell_1+\cdots+\ell_{i-1}} \times \mathfrak{S}_{\ell_i+\ell_{i+1}} \times \mathfrak{S}_{\ell_{i+2}+\cdots+\ell_n} \subseteq \mathfrak{S}_\ell$. Then $\zeta_{s_i\circ\lambda, s_i\circ\mu} = w(\zeta_{\lambda,\mu})$ and there exists an H_ℓ-homomorphism $\mathcal{M}(s_i\circ\lambda, s_i\circ\mu) \to \mathcal{M}(\lambda, \mu)$ such that $\mathbf{1}_{s_i\circ\lambda, s_i\circ\mu} \mapsto \phi_w \mathbf{1}_{\lambda,\mu}$. It follows from Proposition 2.5-(ii) that $\phi_{w^{-1}}\phi_w \mathbf{1}_{\lambda,\mu}$ is nonzero and thus ϕ_w is invertible. Hence it gives an isomorphism. Q.E.D.

For $\eta \in \mathfrak{t}_n^*$, put $R_n[\eta] = \{\alpha \in R_n \mid \langle \eta, \alpha^\vee \rangle_n = 0\}$. It is not difficult to see that $R_n[\eta]$ is a root system and its Weyl group is the stabilizer $\mathfrak{S}_n[\eta]$ of η, i.e. $\mathfrak{S}_n[\eta] = \langle s_\alpha \mid \alpha \in R_n[\eta] \rangle$.

Put

$$(5.1) \qquad P_\eta^+ = \{\mu \in \mathfrak{t}_n^* \mid \langle \mu, \alpha^\vee \rangle_n \in \mathbb{Z}_{\geq 0} \quad \text{for any } \alpha \in R_n^+ \cap R_n[\eta]\},$$

$$(5.2) \qquad P_\eta^- = \{\mu \in \mathfrak{t}_n^* \mid \langle \mu, \alpha^\vee \rangle_n \in \mathbb{Z}_{\leq 0} \quad \text{for any } \alpha \in R_n^+ \cap R_n[\eta]\}.$$

The proof of the following important lemma is given in Appendix A.

Lemma 5.2. *Let $\lambda, \mu \in \mathfrak{t}_n^*$ be such that $\lambda - \mu \in P_n(\ell)$. Suppose the following conditions:*

(a) $\lambda \in D_n$. (b) $\mu + \rho \in P_{\lambda+\rho}^+$.

(c) *There exists numbers $1 = m_0 < m_1 < \cdots < m_k = \ell$ for which we have*

$$(5.3) \qquad \lambda_i - \lambda_j \in \mathbb{Z} \Leftrightarrow m_{r-1} < i, j \leq m_r \text{ for some } r \in \{1, \ldots, k\}.$$

Then, we have $\mathcal{M}(\lambda, \mu)_{\zeta_{\lambda,\mu}} = \mathbb{C}\mathbf{1}_{\lambda,\mu}$.

Theorem 5.3. *Let $\lambda, \mu \in \mathfrak{t}_n^*$ be such that $\lambda - \mu \in P_n(\ell)$. If $\lambda \in D_n$, then $\mathcal{M}(\lambda, \mu)$ has a unique simple quotient module, which is denoted by $\mathcal{L}(\lambda, \mu)$.*

Proof. By Lemma 5.1, it is enough to prove the statement assuming that λ satisfies the conditions in Lemma 5.2. Let N be a submodule of $\mathcal{M}(\lambda, \mu)$. If $N^{\text{gen}}_{\zeta_{\lambda,\mu}} \neq 0$, then $N_{\zeta_{\lambda,\mu}} \neq 0$. By Lemma 5.2, this implies $\mathbf{1}_{\lambda,\mu} \in N$ and thus $N = \mathcal{M}(\lambda, \mu)$. Hence a proper submodule N must satisfy $N \subseteq \oplus_{\eta \neq \zeta_{\lambda,\mu}} \mathcal{M}(\lambda, \mu)^{\text{gen}}_{\zeta_{\lambda,\mu}}$. The sum of all the proper submodules also satisfies this property and it is a unique maximal proper submodule. Q.E.D.

For $\lambda \in D_n$, we call $\mathcal{M}(\lambda, \mu)$ a *standard module.* The following lemma is also a consequence of Lemma 5.2.

Lemma 5.4. *Let $\lambda \in D_n$ and $\mu \in \lambda - P_n(\ell)$. Let $(\ |\ )$ be a non-zero H_ℓ-contravariant form on $\mathcal{M}(\lambda, \mu)$ and let N be a unique maximal submodule of $\mathcal{M}(\lambda, \mu)$. Then $N = \mathrm{rad}(\ |\ )$.*

Proof. It is obvious that $\mathrm{rad}(\ |\ ) \subseteq N$. To prove the opposite inclusion, first note that $\mathcal{M}(\lambda, \mu)^{\text{gen}}_{\eta} \perp \mathcal{M}(\lambda, \mu)^{\text{gen}}_{\zeta}$ with respect to $(\ |\ )$ unless $\eta = \zeta$. For any $u \in N$ and $x \in H_\ell$, we have $(u|x\mathbf{1}_{\lambda,\mu}) = (\iota(x)u|\mathbf{1}_{\lambda,\mu}) = 0$ because $\iota(x)u \in N \subseteq \oplus_{\eta \neq \zeta_{\lambda,\mu}} \mathcal{M}(\lambda, \mu)^{\text{gen}}_{\eta}$ and $\mathbf{1}_{\lambda,\mu} \in \mathcal{M}(\lambda, \mu)^{\text{gen}}_{\zeta_{\lambda,\mu}}$. This implies $N \subseteq \mathrm{rad}(\ |\ )$. Q.E.D.

By Lemma 3.3, the $\mathfrak{gl}_n$-contravariant form on $L(\mu)$ induces an H_ℓ-contravariant form on $\mathcal{L}(\lambda, \mu) = F_\lambda(L(\mu))$, and it turns out to be non-degenerate. Now, Lemma 5.4, implies that the H_ℓ-module $F_\lambda(L(\mu))$ is simple unless it is zero. More precisely, we have

Theorem 5.5 ([AS, Su]). *Let $\lambda \in D_n$ and $\mu \in \lambda - P_n(\ell)$.*

(i) *If $\mu + \rho \in P^-_{\lambda+\rho}$ then we have $F_\lambda(L(\mu)) \cong \mathcal{L}(\lambda, \mu)$.*

(ii) *If $\mu + \rho \notin P^-_{\lambda+\rho}$ then we have $F_\lambda(L(\mu)) = 0$.*

Remark 5.6. (i) One can express μ in Theorem 5.5 as $\mu = w \circ \tilde{\mu}$ with some $w \in \mathfrak{S}_n$ and $\tilde{\mu} \in D_n$. Then the condition $\mu + \rho \in P^-_{\lambda+\rho}$ is equivalent to

$$\mu = w^\lambda \circ \tilde{\mu} \quad \text{or equivalently} \quad \mu = w^\lambda_{\tilde{\mu}} \circ \tilde{\mu}.$$

Here w^λ (resp. $w^\lambda_{\tilde{\mu}}$) denotes the unique longest element in the coset $\mathfrak{S}_n[\lambda + \rho]w$ (resp. $\mathfrak{S}_n[\lambda + \rho]w\mathfrak{S}_n[\tilde{\mu} + \rho]$). (See [Su, Remark 3.2.3] for the proof.)

(ii) In [Su], we give a proof of Theorem 5.5 using the result by Zelevinsky [Zel, Theorem 6.1] that describes when two simple modules are isomorphic. In the following, we give a modified proof of Theorem 5.5 without referring to Zelevinsky's result. (See Theorem 6.5.)

Proof of Theorem 5.5. The statement (ii) follows from Lemma 4.3 easily (see [Su]).

Let us prove (i). It is enough to see that $F_\lambda(L(\mu))$ is nonzero under the condition $\mu+\rho \in P^-_{\lambda+\rho}$, by which we can write μ as $\mu = w^\lambda_{\tilde\mu} \circ \tilde\mu$, where $\tilde\mu \in D_n$ and $w^\lambda_{\tilde\mu}$ is the longest element in $\mathfrak{S}_n[\lambda+\rho]w\mathfrak{S}_n[\tilde\mu+\rho]$. In the Grothendieck group of $\mathcal{O}(\mathfrak{gl}_n)$, we write

$$M(w^\lambda_{\tilde\mu} \circ \tilde\mu) = L(w^\lambda_{\tilde\mu} \circ \tilde\mu) + \sum_{y_{\tilde\mu}} a_{y_{\tilde\mu}} L(y_{\tilde\mu} \circ \tilde\mu). \tag{5.4}$$

Here the sum runs over those elements $y_{\tilde\mu} \in W_n$ such that $y_{\tilde\mu}$ is longest in $y_{\tilde\mu}W_n[\tilde\mu+\rho]$ and $y_{\tilde\mu} > w^\lambda_{\tilde\mu}$. Note that this implies

$$y_{\tilde\mu} \notin \mathfrak{S}_n[\lambda+\rho]w^\lambda_{\tilde\mu}\mathfrak{S}_n[\tilde\mu+\rho]. \tag{5.5}$$

Applying F_λ to (5.4) we have

$$\mathcal{M}(\lambda, w^\lambda_{\tilde\mu} \circ \tilde\mu) = F_\lambda(L(w^\lambda_{\tilde\mu} \circ \tilde\mu)) + \sum_{y_{\tilde\mu}} a_{y_{\tilde\mu}} F_\lambda(L(y_{\tilde\mu} \circ \tilde\mu)) \tag{5.6}$$

in the Grothendieck group of $\mathcal{R}(H_\ell)$. Note that

$$F_\lambda(L(y_{\tilde\mu} \circ \tilde\mu))\downarrow_{\mathbb{C}[\mathfrak{S}_\ell]} \subseteq \mathcal{M}(\lambda, y_{\tilde\mu} \circ \tilde\mu)\downarrow_{\mathbb{C}[\mathfrak{S}_\ell]} = \oplus_{\nu \trianglerighteq (\lambda - y_{\tilde\mu}\circ\tilde\mu)^+} S_\nu^{\oplus a_\nu}$$

with some $a_\nu \in \mathbb{Z}_{\geq 0}$. By Lemma 5.7 below, it follows from (5.5) that $(\lambda - y_{\tilde\mu}\circ\tilde\mu)^+ \triangleright (\lambda - w^\lambda_{\tilde\mu}\circ\tilde\mu)$, and thus $F_\lambda(L(y_{\tilde\mu}\circ\tilde\mu))$ does not contain $S_{(\lambda - w^\lambda_{\tilde\mu}\circ\tilde\mu)^+}$, which must be contained in $\mathcal{M}(\lambda, w^\lambda_{\tilde\mu} \circ \tilde\mu)$. Therefore $F_\lambda(L(w^\lambda_{\tilde\mu} \circ \tilde\mu))$ cannot be zero. Q.E.D.

Lemma 5.7. *Let $\lambda, \mu \in D_n$ and $w, y \in \mathfrak{S}_n$ be such that $\lambda - w \circ \mu, \lambda - y \circ \mu \in P_n(\ell)$. If $y \notin \mathfrak{S}_n[\lambda+\rho]w\mathfrak{S}_n[\mu+\rho]$ and $y > w$, then $(\lambda - y\circ\mu)^+ \triangleright (\lambda - w\circ\mu)^+$*

Proof. First suppose that $y = s_\alpha w$ for $\alpha \in R^+_n$. Then $l(y) > l(w)$ implies $w^{-1}(\alpha) \in R^+_n$, and it follows that $\langle\lambda+\rho, \alpha^\vee\rangle_n \geq 0$ and $\langle w(\mu+\rho), \alpha^\vee\rangle_n = \langle\mu+\rho, w^{-1}(\alpha^\vee)\rangle \geq 0$. Hence we have

$$\begin{aligned} |\langle\lambda - y\circ\mu, \alpha^\vee\rangle_n| = &|\langle\lambda+\rho, \alpha^\vee\rangle_n + \langle w(\mu+\rho), \alpha^\vee\rangle_n| \geq \\ &|\langle\lambda+\rho, \alpha^\vee\rangle_n - \langle w(\mu+\rho), \alpha^\vee\rangle_n| = |\langle\lambda - w\circ\mu, \alpha^\vee\rangle_n|. \end{aligned}$$

This implies $(\lambda - y\circ\mu)^+ \trianglerighteq (\lambda - w\circ\mu)^+$. The equality holds only when $\langle\lambda+\rho, \alpha^\vee\rangle_n = 0$ or $\langle w(\mu+\rho), \alpha^\vee\rangle_n = 0$, that is, only when $y = s_\alpha w \in \mathfrak{S}_n[\lambda+\rho]w\mathfrak{S}_n[\mu+\rho]$.

Now let us consider the general case. Since $y > w$, there is a sequence $\alpha^{(1)}, \ldots, \alpha^{(m)}$ in R_n^+ such that $y = s_{\alpha^{(m)}} \cdots s_{\alpha^{(1)}} w$ and $l(w^{(k+1)}) > l(w^{(k)})$ $(k \geq 0)$, where $w^{(k)} = s_{\alpha^{(k)}} \cdots s_{\alpha^{(1)}} w$. Now the statement follows by the induction on m. Q.E.D.

In the proof of Theorem 5.5, we have also proved the following

Corollary 5.8. *Let $\lambda \in D_n$ and $\mu \in \lambda - P_n(\ell)$. Then*

$$\mathcal{L}(\lambda, \mu)\downarrow_{\mathbb{C}[\mathfrak{S}_\ell]} \cong S_{(\lambda-\mu)^+} \oplus \bigoplus_{\nu \rhd (\lambda-\mu)^+} S_\nu^{\oplus N_{\mu,\nu}^\lambda}$$

for some non-negative integers $N_{\mu,\nu}^\lambda$.

§6. Classification of simple modules

Let us consider the particular case where $\ell = n$. For $\zeta \in \mathfrak{t}_\ell^*$, we put

$$\mathcal{I}(\zeta) := \mathcal{M}(\zeta - \rho + \epsilon, \zeta - \rho), \quad \mathbf{1}_\zeta := \mathbf{1}_{\zeta-\rho+\epsilon, \zeta-\rho},$$

where $\epsilon = (1, \ldots, 1) \in \mathfrak{t}_\ell^*$. The H_ℓ-module $\mathcal{I}(\zeta)$ is called the principal series representation associated with ζ. As a $\mathbb{C}[\mathfrak{S}_\ell]$-module, $\mathcal{I}(\zeta)$ is isomorphic to the regular representation. Note also that $\xi \mathbf{1}_\zeta = \langle \zeta, \xi \rangle_\ell \mathbf{1}$ for $\xi \in \mathfrak{t}_\ell$, and that

$$\mathrm{Hom}_{H_\ell}(\mathcal{I}(\zeta), Y) = Y_\zeta$$

for any H_ℓ-module Y.

Lemma 6.1 ([Ro]). *Let $\zeta \in \mathfrak{t}_\ell^*$ and $w \in \mathfrak{S}_\ell$. Then $\mathcal{I}(\zeta)$ and $\mathcal{I}(w(\zeta))$ have the same composition factors.*

Proof. It is enough to prove the statement when w is the simple reflection, say s_i. The intertwining operator $\phi_i = 1 + s_i \alpha_i^\vee$ defines H_ℓ-homomorphisms $\Phi_i : \mathcal{I}(\zeta) \to \mathcal{I}(s_i(\zeta))$ and $\Phi^i : \mathcal{I}(s_i(\zeta)) \to \mathcal{I}(\zeta)$ such that $\mathbf{1}_\zeta \mapsto \phi_i \mathbf{1}_{s_i(\zeta)}$ and $\mathbf{1}_{s_i(\zeta)} \mapsto \phi_i \mathbf{1}_\zeta$ respectively. If $\langle \zeta, \alpha_i^\vee \rangle_\ell \neq \pm 1$, then by (2.8), Φ_i is an isomorphism. Now, it is enough to prove the statement in the case $\langle \zeta, \alpha_i^\vee \rangle_\ell = 1$. Through $\mathcal{I}(\zeta)\downarrow_{\mathbb{C}[\mathfrak{S}_\ell]} \cong \mathcal{I}(s_i(\zeta))\downarrow_{\mathbb{C}[\mathfrak{S}_\ell]} \cong \mathbb{C}[\mathfrak{S}_\ell]$, the Φ_i and Φ^i are regarded as the maps between $\mathbb{C}[\mathfrak{S}_\ell]$ given by $v \mapsto v(1 - s_i)$ and $v \mapsto v(1 + s_i)$ $(v \in \mathbb{C}[\mathfrak{S}_\ell])$ respectively. Therefore the sequence $\mathcal{I}(\zeta) \xrightarrow{\Phi_i} \mathcal{I}(s_i(\zeta)) \xrightarrow{\Phi^i} \mathcal{I}(\zeta)$ is exact. Hence, in the Grothendieck group of $\mathcal{R}(H_\ell)$, we have

$$\mathcal{I}(s_i(\zeta)) = (\mathcal{I}(\zeta)/\mathrm{Ker}(\Phi_i)) \oplus \mathrm{Im}(\Phi^i) = (\mathcal{I}(\zeta)/\mathrm{Ker}(\Phi_i)) \oplus \mathrm{Ker}(\Phi_i) = \mathcal{I}(\zeta),$$

as required. Q.E.D.

Lemma 6.1 implies the following

Proposition 6.2. *Any finite-dimensional irreducible H_ℓ-module is a composition factor of $\mathcal{I}(\zeta)$ for some $\zeta \in \rho + D_\ell$.*

Theorem 6.3. (*cf.* [Zel, Theorem 6.1] [Ch1])

Any finite-dimensional simple module over H_ℓ is isomorphic to $\mathcal{L}(\lambda, w \circ (\lambda - \epsilon))$ for some $\lambda \in D_\ell$ and $w \in \mathfrak{S}_\ell$ such that $\lambda - w \circ (\lambda - \epsilon) \in P_\ell(\ell)$.

Proof. Let L be a finite-dimensional simple H_ℓ-module. By Proposition 6.2, we can suppose that L is a composition factor of $\mathcal{I}(\zeta)$ for some $\zeta \in \rho + D_\ell$. Put $\lambda = \zeta - \rho + \epsilon \in D_\ell$. By Theorem 4.2 and Theorem 5.5, the functor F_λ transforms the composition series of $M(\lambda - \epsilon)$ to the composition series of $\mathcal{I}(\lambda + \rho - \epsilon) = \mathcal{I}(\zeta)$. Therefore L is of the form $\mathcal{L}(\lambda, w \circ (\lambda - \epsilon)) = F_\lambda(L(w^\lambda_{\lambda-\epsilon} \circ (\lambda - \epsilon))$ for some $w \in \mathfrak{S}_\ell$. Q.E.D.

We say that an H_ℓ-module Y is of *level* n if $Y\downarrow_{\mathbb{C}[\mathfrak{S}_\ell]} \cong \oplus_{\nu \in P_n^+(\ell)} S_\nu^{\oplus a_\nu}$ for some $a_\nu \in \mathbb{Z}_{\geq 0}$. The induced module $\mathcal{M}(\lambda, \mu)$ ($\lambda, \mu \in \mathfrak{t}_n^*$) is of level n. Any finite-dimensional H_ℓ-module is of level ℓ.

Corollary 6.4. *Any simple H_ℓ- module of level n is isomorphic to $\mathcal{L}(\lambda, \mu)$ for some $\lambda \in D_n$ and $\mu \in \lambda - P_n(\ell)$.*

Theorem 6.5. (*cf.* [Zel, Theorem 6.1]) *Suppose that $\lambda, \mu \in D_n$ and $w, y \in \mathfrak{S}_n$ satisfy $\lambda - w \circ \mu, \lambda - y \circ \mu \in P_n(\ell)$. Then the following are equivalent:*

(a) $y \in \mathfrak{S}_n[\lambda + \rho] w \mathfrak{S}_n[\mu + \rho]$,

(b) $\mathcal{M}(\lambda, w \circ \mu) \cong \mathcal{M}(\lambda, y \circ \mu)$,

(c) $\mathcal{L}(\lambda, w \circ \mu) \cong \mathcal{L}(\lambda, y \circ \mu)$.

Proof. (a)$\Rightarrow$(b) follows from Lemma 4.3. (b)$\Rightarrow$(c) is obvious.

(c)$\Rightarrow$(b): Suppose (c), then there is a weight vector $v \in \mathcal{M}(\lambda, y \circ \mu)$ whose weight is $\zeta_{\lambda, w\circ\mu}$. Let $s_i \in \mathfrak{S}_{\lambda - w \circ \mu}$. Then $\phi_i v$ is a weight vector of weight $s_i(\zeta_{\lambda, w\circ\mu})$. But $s_i(\zeta_{\lambda, w\circ\mu})$ does not belong to $P(\mathcal{L}(\lambda, y \circ \mu)) = P(\mathcal{L}(\lambda, w \circ \mu))$ because it does not belong to $P(\mathcal{M}(\lambda, w \circ \mu)) = \{x(\zeta_{\lambda, w\circ\mu}) \mid x \in \mathfrak{S}^\perp_{\lambda - w\circ\mu}\}$ (Lemma 4.1-(i)). Hence $\phi_i v = (1 - s_i)v = 0$ for any $s_i \in \mathfrak{S}_{\lambda - w \circ \mu}$. Therefore there exists an H_ℓ-homomorphism $f : \mathcal{M}(\lambda, w \circ \mu) \to \mathcal{M}(\lambda, y \circ \mu)$ such that $f(\mathbf{1}_{\lambda, w\circ\mu}) = v$. By Corollary 5.8, the image $f(\mathcal{M}(\lambda, w \circ \mu))$ contains $S_{(\lambda - w\circ\mu)^+} = S_{(\lambda - y\circ\mu)^+}$ as a $\mathbb{C}[\mathfrak{S}_\ell]$-submodule. Since $S_{(\lambda - y\circ\mu)^+}$ generates $\mathcal{M}(\lambda, y \circ \mu)$ over H_ℓ, the homomorphism f is surjective and thus bijective.

(b)$\Rightarrow$(a): We prove the statement only for the case $\lambda \in P_\ell$. The general case is reduced to this case. Suppose (b). It is enough to prove $w = y$ assuming that w (resp. y) is the shortest element in $\mathfrak{S}_n[\lambda + \rho] w \mathfrak{S}_n[\mu + \rho]$ (resp. $\mathfrak{S}_n[\lambda + \rho] y \mathfrak{S}_n[\mu + \rho]$). Note that this assumption implies $w \circ \mu + \rho \in P^+_{\lambda+\rho}$ and $y \circ \mu + \rho \in P^+_{\lambda+\rho}$ (see [Su, Remark 3.2.3] for the proof). First we prove $\zeta_{\lambda,w\circ\mu} = \zeta_{\lambda,y\circ\mu}$. For this purpose, we introduce a total order in P_ℓ by

$$\zeta > \eta \Leftrightarrow {}^\exists k \in \{1,\ldots,\ell\} \text{ such that } \zeta_k > \eta_k \text{ and } \zeta_i = \eta_i \text{ for } i \geq k+1.$$

Through some combinatorial argument, it follows from the assumption $w\circ\mu+\rho \in P^+_{\lambda+\rho}$ (resp. $y\circ\mu+\rho \in P^+_{\lambda+\rho}$) that $\zeta_{\lambda,w\circ\mu}$ (resp. $\zeta_{\lambda,y\circ\mu}$) is the minimal element in $P(\mathcal{M}(\lambda, w\circ\mu)) = \{x(\zeta_{\lambda,\mu}) \mid x \in \mathfrak{S}^\perp_{\lambda-w\circ\mu}\}$ (resp. in $P(\mathcal{M}(\lambda, y\circ\mu))$. Therefore (b) implies $\zeta_{\lambda,w\circ\mu} = \zeta_{\lambda,y\circ\mu}$.

Next, let us prove $\mathfrak{S}_{\lambda-w\circ\mu} = \mathfrak{S}_{\lambda-y\circ\mu}$. Let $s_i \in \mathfrak{S}_{\lambda-w\circ\mu}$. Then by the same argument we used in the proof of the implication (c)$\Rightarrow$(b), we have $s_i \mathbf{1}_{\lambda,y\circ\mu} = \mathbf{1}_{\lambda,y\circ\mu}$ for any $s_i \in \mathfrak{S}_{\lambda-w\circ\mu}$. This implies $\mathfrak{S}_{\lambda-w\circ\mu} \subseteq \mathfrak{S}_{\lambda-y\circ\mu}$. Similarly we have $\mathfrak{S}_{\lambda-y\circ\mu} \subseteq \mathfrak{S}_{\lambda-w\circ\mu}$, and thus $\mathfrak{S}_{\lambda-w\circ\mu} = \mathfrak{S}_{\lambda-y\circ\mu}$.

Finally, let us see $w\circ\mu = y\circ\mu$, that is equivalent to $w = y$. Put $p_i = \langle \lambda - w\circ\mu, \epsilon_i^\vee\rangle_n$ and $q_i = \langle \lambda - y\circ\mu, \epsilon_i^\vee\rangle_n$. Suppose $w\circ\mu \neq y\circ\mu$ and let $k \in \{1,\ldots,n\}$ be the largest number such that $p_k \neq q_k$. We may assume that $p_k \neq 0$. Then $\mathfrak{S}_{\lambda-w\circ\mu} = \mathfrak{S}_{\lambda-y\circ\mu}$ implies that there exists $j < k$ such that $q_i = 0$ for $i = j+1, j+2, \ldots, k$ and $p_k = q_j$. Put $m = \sum_{i=1}^k p_i = \sum_{i=1}^j q_i$. Now $\zeta_{\lambda,w\circ\mu} = \zeta_{\lambda,y\circ\mu}$ implies $\langle \lambda+\rho, \epsilon_k^\vee\rangle_n = \langle \zeta_{\lambda,w\circ\mu}, \epsilon_m^\vee\rangle_\ell + 1 = \langle \zeta_{\lambda,y\circ\mu}, \epsilon_m^\vee\rangle_\ell + 1 = \langle \lambda+\rho, \epsilon_j^\vee\rangle_n$, and thus $\alpha_{jk} \in R_n^+ \cap R_n[\lambda+\rho]$. But $\langle y\circ\mu, \alpha_{jk}^\vee\rangle_n = -q_j < 0$. This contradicts the assumption $y\circ\mu+\rho \in P^+_{\lambda+\rho}$. Hence $w\circ\mu = y\circ\mu$. Q.E.D.

§7. Skew shape representations

As remarked in [AS], the construction of the functors gives a generalization of the Frobenius-Schur-Weyl reciprocity. Let us recall the classical Frobenius-Schur-Weyl reciprocity between $\mathfrak{S}_\ell$ and $\mathfrak{gl}_n$. Let $\mathfrak{gl}_n$ and $\mathfrak{S}_\ell$ act on the space $V_n^{\otimes \ell}$ from the left naturally. Then each of the images of $U(\mathfrak{gl}_n)$ and $\mathbb{C}[\mathfrak{S}_\ell]$ in $\mathrm{End}_{\mathbb{C}}(V_n^{\otimes \ell})$ is the commutant of the other. This gives the following decomposition law:

$$V_n^{\otimes \ell} = \bigoplus_{\lambda \in P_n^+(\ell)} L(\lambda) \otimes S_\lambda, \tag{7.1}$$

as a $U(\mathfrak{gl}_n) \times \mathbb{C}[\mathfrak{S}_\ell]$-module.

Proposition 7.1. *Let $\mu \in P_n^+$.*

(i) *As a* $U(\mathfrak{gl}_n) \times H_\ell$*-module,*

$$L(\mu) \otimes V_n^{\otimes \ell} = \bigoplus_{\lambda \in P_n^+, \lambda - \mu \in P_n(\ell)} L(\lambda) \otimes \mathcal{L}(\lambda, \mu), \tag{7.2}$$

(ii) *Each of the images of* $U(\mathfrak{gl}_n)$ *and* H_ℓ *on* $\mathrm{End}_{\mathbb{C}}(L(\mu) \otimes V_n^{\otimes \ell})$ *is the commutant of the other.*

Proof. (i) Note that, for $\lambda \in P_n^+$ and a finite-dimensional $\mathfrak{gl}_n$-module X, we have

$$\mathrm{Hom}_{U(\mathfrak{gl}_n)}(L(\lambda), X) = \mathrm{Hom}_{U(\mathfrak{gl}_n)}(M(\lambda), X). \tag{7.3}$$

The right hand side is isomorphic to $H^0(\mathfrak{n}_n^+, X)_\lambda$. Hence, by Lemma 1.1, we have

$$\begin{aligned} L(\mu) \otimes V_n^{\otimes \ell} &= \bigoplus_{\lambda \in P_n^+} L(\lambda) \otimes \mathrm{Hom}_{U(\mathfrak{gl}_n)}(L(\lambda), L(\mu) \otimes V_n^{\otimes \ell}) \\ &= \bigoplus_{\lambda \in P_n^+} L(\lambda) \otimes F_\lambda(L(\mu)). \end{aligned}$$

Now, Theorem 5.5 implies the statement. (ii) follows from (i). Q.E.D.

Suppose $\lambda, \mu \in P_n^+$ and $\lambda - \mu \in P_n(\ell)$. Then λ/μ gives a skew Young diagram (skew shape) with ℓ boxes. The corresponding simple module $\mathcal{L}(\lambda, \mu)$ is called *a skew shape representation*, which has been studied e.g. in [Ch1, Ch2, Ch3, Ra]. We will recover some results on them as consequences of the applications of the functors.

Proposition 7.2 ([Ch3, Ra]). *Let* $\lambda, \mu \in P_n^+$ *such that* $\lambda - \mu \in P_n(\ell)$. *Then*

$$\mathcal{L}(\lambda, \mu)\downarrow_{\mathfrak{S}_\ell} \cong \bigoplus_{\nu \in P_n^+, \lambda - \nu \in P_n(\ell)} S_\nu^{\oplus c_{\mu\nu}^\lambda}, \tag{7.4}$$

where the coefficient is given by the Littlewood-Richardson number

$$c_{\mu\nu}^\lambda = \dim_{\mathbb{C}} \mathrm{Hom}_{U(\mathfrak{gl}_n)}(L(\lambda), L(\mu) \otimes L(\nu)).$$

Proof. Follows from $\mathcal{L}(\lambda, \mu) = \mathrm{Hom}_{U(\mathfrak{gl}_n)}(L(\lambda), L(\mu) \otimes V_n^{\otimes \ell})$ and (7.1). Q.E.D.

It is well-known that the *characteristic* (see [Mac]) of the $\mathbb{C}[\mathfrak{S}_\ell]$-module S_ν is given by the Schur function. Hence, Proposition 7.2 states that the characteristic of $\mathcal{L}(\lambda, \mu)$ (as a $\mathbb{C}[\mathfrak{S}_\ell]$-module) is given by the skew Schur function ([Mac]).

Proposition 7.3 ([Ch3]). *Let $\lambda, \mu \in P_n^+$ be such that $\lambda - \mu \in P_n(\ell)$. Then there exists an exact sequence*

$$0 \leftarrow \mathcal{L}(\lambda, \mu) \leftarrow \mathcal{C}_0 \leftarrow \mathcal{C}_1 \leftarrow \cdots \leftarrow \mathcal{C}_{n(n-1)/2} \leftarrow 0 \tag{7.5}$$

of H_ℓ-modules, where

$$\mathcal{C}_i = \bigoplus_{y \in \mathfrak{S}_n,\ l(y)=i} \mathcal{M}(\lambda, y \circ \mu).$$

Proof. Apply F_λ to the BGG resolution ([BGG]) for the finite-dimensional simple $\mathfrak{gl}_n$-module $L(\mu)$. Q.E.D.

Remark 7.4. By considering the characteristics as $\mathbb{C}[\mathfrak{S}_\ell]$-modules, one can see that the Jacobi-Trudi identity for a skew Schur function ([Mac]) follows from the sequence (7.5) (cf. [Ze4, Ak]).

§8. Multiplicity formulas

For a module M and a simple module L, let $[M : L]$ denote the multiplicity of L in the composition series of M.

Let $\mathfrak{S}_n^\mu$ denote the integral Weyl group of $\mu \in \mathfrak{t}_n^*$:

$$\mathfrak{S}_n^\mu = \{w \in \mathfrak{S}_n \mid \mu - w \circ \mu \in Q_n\}. \tag{8.1}$$

The following formula is a direct consequence of Theorem 4.2 and Theorem 5.5:

Theorem 8.1. *Let $\lambda, \mu \in D_n$ and let $w, y \in \mathfrak{S}_n^\mu$ such that $\lambda - w \circ \mu, \lambda - y \circ \mu \in P_n(\ell)$. Then we have*

$$[\mathcal{M}(\lambda, w \circ \mu) : \mathcal{L}(\lambda, y \circ \mu)] = [M(w \circ \mu) : L(y^\lambda \circ \mu)], \tag{8.2}$$

where y^λ denotes the longest element in $\mathfrak{S}_n[\lambda + \rho]y$.

Let $\lambda, \mu \in D_n$ and $w, y \in \mathfrak{S}_n^\mu$ be as in Theorem 8.1. The equality (8.2) has been known (at least in the case $\ell = n$) through the following two multiplicity formulas:

$$[M(w \circ \mu) : L(y \circ \mu)] \quad = P_{w, y_\mu}(1), \tag{8.3}$$

$$[\mathcal{M}(\lambda, w \circ \mu) : \mathcal{L}(\lambda, y \circ \mu)] = P_{w, y_\mu^\lambda}(1). \tag{8.4}$$

Here $P_{w,y}(q) \in \mathbb{Z}[q, q^{-1}]$ denotes the Kazhdan-Lusztig polynomial [KL] of the Hecke algebra associated to $\mathfrak{S}_n^\mu$ (we put $P_{w,y}(q) = 0$ for $w \not< y$ for convenience), and y_μ (resp. y_μ^λ) denotes the longest element in $y\mathfrak{S}_n[\mu+\rho]$ (resp. $\mathfrak{S}_n[\lambda + \rho]y\mathfrak{S}_n[\mu + \rho]$).

Remark 8.2. It follows from (8.3) and (8.4) that $P_{w,y_\mu}(1) = P_{w_\mu,y_\mu}(1)$ and $P_{w,y^\lambda_\mu}(1) = P_{w_\mu,y^\lambda_\mu}(1) = P_{w^\lambda_\mu,y^\lambda_\mu}(1)$. The last number is expressed in terms of the intersection cohomology concerning nilpotent orbits on the quiver variety [Ze3].

The formula (8.3) was conjectured by Kazhdan-Lusztig [KL] and proved by Beilinson-Bernstein [BB1] and Brylinski-Kashiwara [BK]. The formula (8.4) was conjectured by Zelevinsky [Ze2] (see also [Ze3]) and proved by Ginzburg [Gi1] (see also [CG]) and by Lusztig [Lu3]. The theory of perverse sheaves plays an essential role in these proofs.

Let us see that Theorem 8.1 (proved in a purely algebraic way) implies that the Kazhdan-Lusztig formula (8.3) is equivalent to its degenerate affine Hecke analogue (or its p-adic analogue) (8.4). The implication (8.3) $\Rightarrow$ (8.4) is obvious. The implication (8.4) $\Rightarrow$ (8.3) is proved as follows. Take any $\mu \in D_n$ and $w, y \in \mathfrak{S}^\mu_n$. Then we can find $\ell \in \mathbb{Z}_{\geq 2}$ and $\lambda \in D_n + \rho$ such that

$$\lambda - z \circ \mu \in P_n(\ell) \text{ for all } z \in \mathfrak{S}^\mu_n.$$

In this case $F_\lambda(L(z \circ \mu))$ never vanishes and thus it is isomorphic to $\mathcal{L}(\lambda, z \circ \mu)$. Now (8.4) implies (8.3).

Note that the formula (8.3) has an inverse formula, which expresses the character of $L(w \circ \mu)$ as a combination of the character of Verma modules. By applying the functor, we have the corresponding formula for H_ℓ-modules.

Corollary 8.3. *Let $\lambda, \mu \in D_n$ and let $y \in \mathfrak{S}^\mu_n$ such that $\lambda - y \circ \mu \in P_n(\ell)$. Then, in the Grothendieck group of $\mathcal{R}(H_\ell)$, we have*

$$\mathcal{L}(\lambda, y \circ \mu) = \mathcal{L}(\lambda, y^\lambda_\mu \circ \mu) =$$
$$\sum_{w^\lambda_\mu \in \mathfrak{S}^\mu_n} \left(\sum_{x \in \mathfrak{S}_n[\lambda+\rho] w^\lambda_\mu \mathfrak{S}_n[\mu+\rho]} (-1)^{l_\mu(x)+l_\mu(y^\lambda_\mu)} P_{x\pi, y^\lambda_\mu \pi}(1) \right) \mathcal{M}(\lambda, w^\lambda_\mu \circ \mu).$$

Here l_μ and π denote the length function and the longest element of $\mathfrak{S}^\mu_n$ respectively, and $\sum_{w^\lambda_\mu \in \mathfrak{S}^\mu_n}$ denotes the summation over those elements $w^\lambda_\mu \in \mathfrak{S}^\mu_n$ such that w^λ_μ is longest in $\mathfrak{S}_n[\lambda+\rho] w^\lambda_\mu \mathfrak{S}_n[\mu+\rho]$.

Next we will consider a refinement of the formula (8.4) concerning the Jantzen filtration. We fix a weight $\delta \in \mathfrak{t}^*_n$. Let $A = \mathbb{C}[[t]]$ denote the ring of formal power series in t. We use the notation: $\eta^t = \eta + \delta t \in \mathfrak{t}^*_n \otimes A$ for $\eta \in \mathfrak{t}^*_n$. For $\mu \in \mathfrak{t}^*_n$, let $M(\mu^t)$ be the Verma module of $\mathfrak{gl}_n \otimes A$ with highest weight μ^t:

$$M(\mu^t) = (U(\mathfrak{gl}_n) \otimes A) \underset{U(\mathfrak{b}_n^+) \otimes A}{\otimes} (Av_{\mu^t}).$$

The $\mathfrak{gl}_n$-contravariant bilinear form on $M(\mu)$ can be naturally extended to a $\mathfrak{gl}_n \otimes A$-contravariant form $(\ |\ )_{M(\mu^t)}$ on $M(\mu^t)$.

Define

$$M(\mu^t)_j = \{v \in M(\mu^t) \mid (v \mid u)_{M(\mu^t)} \in t^j A \text{ for all } u \in M(\mu^t)\}.$$

Putting $M(\mu)_j = M(\mu^t)_j \,/ (tM(\mu^t) \cap M(\mu^t)_j)$ we have a filtration

$$M(\mu) = M(\mu)_0 \supseteq M(\mu)_1 \supseteq M(\mu)_2 \supseteq \cdots$$

by $\mathfrak{gl}_n$-modules called the *Jantzen filtration* [Ja].

It is possible to define an analogous filtration (which we call the Jantzen filtration) on $\mathcal{M}(\lambda, \mu)$ associated to δ, although it is not straitforward (see [Ro, Su]). Let $\mathcal{M}(\lambda, \mu) = \mathcal{M}(\lambda, \mu)_0 \supseteq \mathcal{M}(\lambda, \mu)_1 \supseteq \cdots$ be the Jantzen filtration associated to δ. We refer [Su] for the proof of the following theorem.

Theorem 8.4 ([Su]). *Suppose that $\lambda \in D_n$ and $\mu \in \mathfrak{t}_n^*$ satisfy $\lambda - \mu \in P(V_n^{\otimes \ell})$ and $\mu + \rho \in P_{\lambda+\rho}^-$. Then $F_\lambda(M(\mu)_j) = \mathcal{M}(\lambda, \mu)_j$.*

A priori the Jantzen filtrations depend on the choice of the deformation direction $\delta \in \mathfrak{t}_n^*$. It has been known that the Jantzen filtration on $M(\mu)$ does not depend on the choice of δ for which $(\ |\ )_{M(\mu^t)}$ is non-degenerate [Ba]. Now Theorem 8.4 implies

Proposition 8.5. *Let λ and μ be as above. Then the Jantzen filtration on $\mathcal{M}(\lambda, \mu)$ does not depend on the choice of δ such that*

(8.5) $\quad \langle \delta, \alpha^\vee \rangle_n \neq 0$ *for any* $\alpha \in R_n^+$ *such that* $\langle \mu + \rho, \alpha^\vee \rangle_n \in \mathbb{Z}_{>0}$.

Let $\{M(\mu)_j\}_j$ and $\{\mathcal{M}(\lambda, \mu)_j\}_j$ be the Jantzen filtrations associated to same δ. As a direct consequence of Theorem 5.5 and Theorem 8.4, we have

Theorem 8.6. *Let $\lambda, \mu \in D_n$ and $w, y \in \mathfrak{S}_n^\mu$ be such that $\lambda - w \circ \mu, \lambda - y \circ \mu \in P_n(\ell)$. Then we have*

(8.6) $\quad [\mathcal{M}(\lambda, w \circ \mu)_j : \mathcal{L}(\lambda, y \circ \mu)] = [M(w^\lambda \circ \mu)_j : L(y^\lambda \circ \mu)],$

where w^λ and y^λ denote the longest element in $\mathfrak{S}_n[\lambda+\rho]w$ and $\mathfrak{S}_n[\lambda+\rho]y$ respectively.

Let $\lambda, \mu \in D_n$ and $w, y \in \mathfrak{S}_n^\mu$ be such that $\lambda - w \circ \mu$, $\lambda - y \circ \mu \in P_n(\ell)$. Suppose that w and y are the longest elements in $\mathfrak{S}_n[\lambda+\rho]w\mathfrak{S}_n[\mu+\rho]$ and $\mathfrak{S}_n[\lambda+\rho]y\mathfrak{S}_n[\mu+\rho]$, respectively. Let $\{M(w \circ \mu)_j\}_j$ and $\{\mathcal{M}(\lambda, w \circ \mu)_j\}_j$ be the Jantzen filtration associated to δ satisfying the condition (8.5).

The following formula was conjectured in [GJ2, GM], and proved in [BB2].

$$(8.7) \qquad \sum_{j \in \mathbb{Z}_{\geq 0}} [\mathrm{gr}_j M(w \circ \mu) : L(y \circ \mu)] q^{(l_\mu(y) - l_\mu(w) - j)/2} = P_{w,y}(q),$$

where $P_{w,y}(q)$ denotes the Kazhdan-Lusztig polynomial of $\mathfrak{S}_n^\mu$, and l_μ denotes the length function on $\mathfrak{S}_n^\mu$. Combining with Theorem 8.6, the improved Kazhdan-Lusztig formula (8.7) implies its degenerate affine Hecke analogue, which was conjectured in [Ro] and proved in [Gi2] (for the non-degenerate affine Hecke algebras).

Theorem 8.7. (cf. [Gi2, Theorem 2.6.1]) *We have*

$$(8.8) \quad \sum_{j \in \mathbb{Z}_{\geq 0}} [\mathrm{gr}_j \mathcal{M}(\lambda, w \circ \mu) : \mathcal{L}(\lambda, y \circ \mu)] q^{(l_\mu(y) - l_\mu(w) - j)/2} = P_{w,y}(q).$$

Remark 8.8. A similar result for affine Hecke algebras has been announced also by I. Grojnowski.

§A. Proof of Lemma 5.2

We proceed by two steps.

Step 1.

In the following we use notations $\mathcal{M}_\ell(\lambda, \mu)$ to denote H_ℓ-module $\mathcal{M}(\lambda, \mu)$, and $\rho^{(n)}$ (resp. $\epsilon^{(n)}$) to denote $\rho = (n-1, \ldots, 1, 0) \in \mathfrak{t}_n^*$ (resp. $\epsilon = (1, \ldots, 1) \in \mathfrak{t}_n^*$) when we want to clarify the rank. For positive integers ℓ and n such that n divides ℓ, we set

$$\begin{aligned} \mathcal{M}_{\ell,n} &= \mathcal{M}_\ell(-\rho^{(n)} + (\ell/n)\epsilon^{(n)}, -\rho^{(n)}), \\ \zeta_{\ell,n} &= \zeta_{-\rho^{(n)} + (\ell/n)\epsilon^{(n)}, -\rho^{(n)}} \in \mathfrak{t}_\ell^*, \quad \mathbf{1} = \mathbf{1}_{-\rho^{(n)} + (\ell/n)\epsilon^{(n)}, -\rho^{(n)}} \in \mathcal{M}_{\ell,n}. \end{aligned}$$

We will prove

Proposition A.1. *Under the notations given above, we have*

$$(\mathcal{M}_{\ell,n})_{\zeta_{\ell,n}} = \mathbb{C}\mathbf{1}.$$

In the case $\ell = n$, the module $\mathcal{M}_{n,n}$ is nothing but the principal series representation $\mathcal{I}(\mathbf{0}^{(n)})$ (see §6), where $\mathbf{0}^{(n)} = (0, \dots, 0) \in \mathfrak{t}_n^*$. In this case, Proposition A.1 has been proved by Rogawski (and also by Cherednik), and we will refer to this result later (in the proof of Lemma A.7):

Lemma A.2 ([Ro] [Ch4]). (i) $\dim (\mathcal{I}(\mathbf{0}^{(n)}))_{\mathbf{0}^{(n)}} = 1$.
(ii) $\mathcal{I}(\mathbf{0}^{(n)}) = (\mathcal{I}(\mathbf{0}^{(n)}))^{\mathrm{gen}}_{\mathbf{0}^{(n)}}$.
(iii) $\mathcal{I}(\mathbf{0}^{(n)})$ *is simple.*

Remark A.3. Similar statements hold for $\mathcal{I}(k\epsilon^{(n)})$ $(k \in \mathbb{C})$.

In order to prove Proposition A.1, we need some preparations. For $1 \leq r \leq \ell - 1$ and $1 \leq p \leq \ell - r$, let c_r^p denote the following cyclic permutation

$$c_r^p = s_{r+p-1} \cdots s_{r+1} s_r \in \mathfrak{S}_\ell.$$

Lemma A.4. *Let Y be an H_ℓ-module and suppose that $v \in Y$ is such that*

$$\alpha_k^\vee v = -v \quad (k = r+1, \dots, r+p-1), \tag{A.1}$$

$$\alpha_r^\vee v = pv, \quad s_{r+p} v = v. \tag{A.2}$$

Then $v \in \mathbb{C}[\mathfrak{S}_\ell] \phi_{c_r^p} v$.

Remark A.5. Since $\alpha_{r,r+p} \in R_n(c_r^p)$ and $\alpha_{r,r+p}^\vee v = v$, it follows from Proposition 2.5-(ii) that $\phi_{(c_r^p)^{-1}} \phi_{c_r^p} v = 0$.

Proof of Lemma A.4. We will construct an element $\psi \in H_\ell$ such that $\psi \phi_{c_r^p} v = v$ explicitly. Note that $\phi_{c_r^p} = \phi_{r+p-1} \phi_{c_r^{p-1}}$ by (2.9). Since $\alpha_{r+p-1}^\vee \phi_{c_r^{p-1}} v = \phi_{c_r^{p-1}} \alpha_{r,r+p}^\vee v = \phi_{c_r^{p-1}} v$, we have

$$\begin{aligned} \phi_{c_r^p} v = \phi_{r+p-1} \phi_{c_r^{p-1}} v &= (1 + s_{r+p-1} \alpha_{r+p-1}^\vee) \phi_{c_r^{p-1}} v \\ &= (1 + s_{r+p-1}) \phi_{c_r^{p-1}} v. \end{aligned} \tag{A.3}$$

It is clear that $s_{r+p} \phi_{c_r^{p-1}} v = \phi_{c_r^{p-1}} s_{r+p} v = \phi_{c_r^{p-1}} v$, from which we have

$$\begin{aligned} &\frac{1}{2}(1 + s_{r+p} - s_{r+p-1} s_{r+p}) \phi_{c_r^p} v \\ =&\frac{1}{2}(1 + s_{r+p} - s_{r+p-1} s_{r+p})(1 + s_{r+p-1}) \phi_{c_r^{p-1}} v = \phi_{c_r^{p-1}} v. \end{aligned}$$

On the other hand, since $R_\ell(c_r^{p-1}) = \{\alpha_{r,k} \mid r+1 \le k \le r+p-1\}$, we have

$$\phi_{(c_r^{p-1})^{-1}}\phi_{c_r^{p-1}}v = \prod_{k=2}^{p}(1-k^2)v \quad \text{(see Proposition 2.5-(ii)).}$$

Therefore we get

$$\text{(A.4)} \quad \frac{1}{2}\prod_{k=2}^{p}\frac{1}{(1-k^2)}\cdot\phi_{(c_r^{p-1})^{-1}}\cdot(1+s_{r+p}-s_{r+p-1}s_{r+p})\cdot\phi_{c_r^p}v = v$$

as required. Q.E.D.

Assume that n divides ℓ and put $m = \ell/n$. In the set $P(\mathcal{M}_{\ell,n})$ of weights of $\mathcal{M}_{\ell,n}$, there exists a unique anti-dominant element $\zeta_{\ell,n}^\circ$, that is given by

$$\text{(A.5)} \qquad \zeta_{\ell,n}^\circ = (\overbrace{0,\dots,0}^{n},\overbrace{1,\dots,1}^{n},\dots,\overbrace{m-1,\dots,m-1}^{n}).$$

Take an element $\tau \in (\mathfrak{S}_\ell)^{\perp}_{\lambda-\mu}$ such that $\tau(\zeta_{\ell,n}) = \zeta_{\ell,n}^\circ$, which is given by

$$\text{(A.6)} \qquad \tau = \omega^1\cdots\omega^{m-1} \in \mathfrak{S}_\ell.$$

Here

$$\text{(A.7)} \qquad \omega^p = \sigma_{n-1}^p\sigma_{n-2}^p\cdots\sigma_1^p,$$

with

$$\text{(A.8)} \qquad \sigma_k^p = c_{k(p+1)-(k-1)}^p\cdots c_{k(p+1)-1}^p c_{k(p+1)}^p.$$

Note that

$$l(\tau) = \sum_{p=1}^{m-1}\sum_{k=1}^{n-1}\sum_{j=0}^{k-1} l(c_{k(p+1)-j}^p),$$

and thus ϕ_τ is expressed as a product of $\phi_{c_r^p}$'s.

Iterated applications of Lemma A.4 imply the following

Lemma A.6. *The vector $\phi_\tau\mathbf{1}$ is a cyclic vector of $\mathcal{M}_{\ell,n}$:*

$$H_\ell\phi_\tau\mathbf{1} = \mathcal{M}_{\ell,n}.$$

Now we prove the following

Lemma A.7. $(\mathcal{M}_{\ell,n})_{\zeta_{\ell,n}^\circ} = \mathbb{C}\phi_\tau\mathbf{1}.$

Proof. For a subset I of $\{1,\ldots,\ell-1\}$, let $\mathfrak{S}_I$ denote the subgroup of $\mathfrak{S}_\ell$ generated by $\{s_i \mid i\in I\}$, and let $\mathfrak{t}_I$ denote the subspace of $\mathfrak{t}_\ell$ spanned by $\{\epsilon_i^\vee \mid i\in I \text{ or } i-1\in I\}$. Put $H_I=\mathbb{C}[\mathfrak{S}_I]\otimes S(\mathfrak{t}_I)$ and regard it as a subalgebra of H_ℓ.

Put

$$\begin{aligned} B_i &= \{(i-1)n+1,(i-1)n+2,\ldots,in-1\},\\ B &= B_1\sqcup\cdots\sqcup B_m. \end{aligned} \tag{A.9}$$

Consider subalgebras $H_{B_i}\cong H_n$ of H_ℓ corresponding to B_i $(i=1,\ldots,m)$, and their modules $K_i:=H_{B_i}\phi_\tau\mathbf{1}\subseteq\mathcal{M}_{\ell,n}$. By (A.5) and Lemma A.2, we have

$$K_i\cong\mathcal{I}((i-1)\epsilon^{(n)}). \tag{A.10}$$

The subspace

$$H_B\mathbf{1}=(H_{B_1}\otimes\cdots\otimes H_{B_m})\mathbf{1}=K_1\otimes\cdots\otimes K_m$$

of $\mathcal{M}_{\ell,n}$ is an $S(\mathfrak{t}_\ell)$-submodule. Lemma A.2 implies

$$(H_B\mathbf{1})^{\mathrm{gen}}_{\zeta^\circ_{\ell,n}}=(K_1)^{\mathrm{gen}}_{\mathbf{0}^{(n)}}\otimes(K_2)^{\mathrm{gen}}_{\epsilon^{(n)}}\otimes\cdots\otimes(K_m)^{\mathrm{gen}}_{(m-1)\epsilon^{(n)}},$$

and its dimension is $(n!)^m$. On the other hand, it follows from Lemma 4.1 that

$$\dim(\mathcal{M}_{\ell,n})^{\mathrm{gen}}_{\zeta^\circ_{\ell,n}}=(n!)^m,$$

and thus $(\mathcal{M}_{\ell,n})^{\mathrm{gen}}_{\zeta^\circ_{\ell,n}}=(H_B\mathbf{1})^{\mathrm{gen}}_{\zeta^\circ_{\ell,n}}$. Combining with Lemma A.2-(i), we have

$$(\mathcal{M}_{\ell,n})_{\zeta^\circ_{\ell,n}}=(H_B\mathbf{1})_{\zeta^\circ_{\ell,n}}=(K_1)_{\mathbf{0}^{(n)}}\otimes(K_2)_{\epsilon^{(n)}}\otimes\cdots\otimes(K_m)_{(m-1)\epsilon^{(n)}}=\mathbb{C}\phi_\tau\mathbf{1}.$$

Q.E.D.

Proof of Proposition A.1. Take any $v\in(\mathcal{M}_{\ell,n})_{\zeta_{\ell,n}}$. Lemma A.7 implies that $\phi_\tau v=c\cdot\phi_\tau\mathbf{1}$ for some $c\in\mathbb{C}$. Putting $v_0=v-c\mathbf{1}$, we have

$$\phi_\tau v_0=0. \tag{A.11}$$

Let $\alpha_i\in B$. Then we have $\langle\zeta_{\ell,n},\alpha_i^\vee\rangle_\ell=-1$, and thus $(1-s_i)v_0=\phi_i v_0\in(\mathcal{M}_{\ell,n})_{s_i(\zeta_{\ell,n})}$. Since the weight $s_i(\zeta_{\ell,n})$ does not belong to $P(\mathcal{M}_{\ell,n})$, it must be zero. Therefore $s_iv_0=v_0$.

Hence there exists an H_ℓ-homomorphism

$$f:\mathcal{M}_{\ell,n}\to\mathcal{M}_{\ell,n} \tag{A.12}$$

such that $f(\mathbf{1}) = v_0$. By (A.11), we have $\phi_\tau \mathbf{1} \in \mathrm{Ker} f$. This implies $\mathrm{Ker} f = \mathcal{M}_{\ell,n}$ by Lemma A.6. Therefore $v_0 = 0$ and thus $v \in \mathbb{C}\mathbf{1}$. Q.E.D.

Step 2.

We will reduce Lemma 5.2 to Proposition A.1. Fix $\lambda \in D_n$ and $\mu \in \lambda - P_n(\ell)$. Put $\ell_i = \lambda_i - \mu_i$ $(i = 1, \ldots, n)$ and $a_i = \sum_{k=1}^{i-1} \ell_k + 1$, $b_i = \sum_{k=1}^{i} \ell_k$ $(i = 1, \ldots, n)$. Recall that

$$\langle \zeta_{\lambda,\mu}, \alpha^\vee_{a_i,a_j} \rangle_\ell = \langle \mu + \rho, \alpha^\vee_{i,j} \rangle_n, \ \langle \zeta_{\lambda,\mu}, \alpha^\vee_{b_i,b_j} \rangle_\ell = \langle \lambda + \rho, \alpha^\vee_{i,j} \rangle_n.$$

The following lemma is easy to prove.

Lemma A.8. *Let $w \in \mathfrak{S}^\perp_{\lambda-\mu}$ and $k, k' \in \{1, \ldots, \ell - 1\}$. If $a_i \leq k < k' \leq b_i$ for some i, then $w(k) < w(k')$.*

By the conditions in Lemma 5.2, we can find integers

$$0 = n'_0 < n'_1 < n'_2 < \cdots < n'_r = n,$$

$$0 = n_0 < n_1 < n_2 < \cdots < n_s = n$$

such that

$$\{\alpha \in R_n \mid \langle \lambda + \rho, \alpha^\vee \rangle_n = 0\} = R_n \cap \sum_{i \neq n'_0, \ldots, n'_r} \mathbb{Z}\alpha_i,$$

$$\{\alpha \in R_n \mid \langle \lambda + \rho, \alpha^\vee \rangle_n = \langle \mu + \rho, \alpha^\vee \rangle_n = 0\} = R_n \cap \sum_{i \neq n_0, \ldots, n_s} \mathbb{Z}\alpha_i$$

respectively. Set

$$I'_p = \{a_{n'_{p-1}+1}, a_{n'_{p-1}+1} + 1, \ldots, b_{n'_p} - 1\} \ (p = 1, \ldots, r), \ I' = I'_1 \sqcup \cdots \sqcup I'_r,$$
$$I_p = \{a_{n_{p-1}+1}, a_{n_{p-1}+1} + 1, \ldots, b_{n_p} - 1\} \ (p = 1, \ldots, s), \ I = I_1 \sqcup \cdots \sqcup I_s.$$

Note that $\mathfrak{S}_{\lambda-\mu} \subseteq \mathfrak{S}_I \subseteq \mathfrak{S}_{I'}$ and

$$\mathfrak{S}_{I'}/\mathfrak{S}_{\lambda-\mu} \cong \mathfrak{S}^\perp_{\lambda-\mu} \cap \mathfrak{S}_{I'}, \quad \mathfrak{S}_I/\mathfrak{S}_{\lambda-\mu} \cong \mathfrak{S}^\perp_{\lambda-\mu} \cap \mathfrak{S}_I.$$

Lemma A.9. $\mathfrak{S}^\perp_{\lambda-\mu} \cap \mathfrak{S}_\ell[\zeta_{\lambda,\mu}] \subseteq \mathfrak{S}^\perp_{\lambda-\mu} \cap \mathfrak{S}_I$.

Proof. Let $w \in \mathfrak{S}^\perp_{\lambda-\mu} \cap \mathfrak{S}_\ell[\zeta_{\lambda,\mu}]$. First, we will prove $w \in \mathfrak{S}^\perp_{\lambda-\mu} \cap \mathfrak{S}_{I'}$. It is enough to prove that $w(\{1, 2, \ldots, b_{n'_k}\}) = \{1, 2, \ldots, b_{n'_k}\}$ for any $k = 1, 2, \ldots, r$. Suppose that $w(\{1, 2, \ldots, b_{n'_k}\}) \neq \{1, 2, \ldots, b_{n'_k}\}$ and let c be the largest number such that

(A.13) $\qquad c \notin \{1, 2, \ldots, b_{n'_k}\}$ and $w^{-1}(c) \in \{1, 2, \ldots, b_{n'_k}\}$.

Since $w \in \mathfrak{S}^{\perp}_{\lambda-\mu}$, it follows from Lemma A.8 that $w^{-1}(c) = b_i$ for some i. Let j be the number such that $a_j \leq c \leq b_j$. Note that $i \leq n'_k < j$ and thus

$$\langle \lambda + \rho, \alpha^{\vee}_{i,j} \rangle_n \neq 0. \tag{A.14}$$

Since $w \in \mathfrak{S}_\ell[\zeta_{\lambda,\mu}]$, we have

$$\langle \zeta_{\lambda,\mu}, w^{-1}(\epsilon^{\vee}_c) - \epsilon^{\vee}_c \rangle_\ell = \langle w(\zeta_{\lambda,\mu}) - \zeta_{\lambda,\mu}, \epsilon^{\vee}_c \rangle_\ell = 0.$$

On the other hand, we have

$$\begin{aligned} &\langle \zeta_{\lambda,\mu}, w^{-1}(\epsilon^{\vee}_c) - \epsilon^{\vee}_c \rangle_\ell = \langle \zeta_{\lambda,\mu}, \epsilon^{\vee}_{b_i} - \epsilon^{\vee}_c \rangle_\ell \\ =&\langle \zeta_{\lambda,\mu}, \epsilon^{\vee}_{b_i} - \epsilon^{\vee}_{b_j} \rangle_\ell + (b_j - c) = \langle \lambda + \rho, \alpha^{\vee}_{i,j} \rangle_n + (b_j - c). \end{aligned} \tag{A.15}$$

Hence we have $c = b_j$ and $\langle \lambda + \rho, \alpha^{\vee}_{i,j} \rangle_n = 0$, that contradicts (A.14). Therefore we proved $w \in \mathfrak{S}^{\perp}_{\lambda-\mu} \cap \mathfrak{S}_{I'}$.

Next, suppose that $w(\{1, 2, \ldots, b_{n_k}\}) \neq \{1, 2, \ldots, b_{n_k}\}$ for some k, and let c be the smallest number such that

$$w(c) \in \{1, 2, \ldots, b_{n_k}\} \text{ and } c \notin \{1, 2, \ldots, b_{n_k}\}. \tag{A.16}$$

Then Lemma A.8 implies $c = a_i$ for some i. Now, similar argument as above deduces a contradiction and thus shows $w \in \mathfrak{S}^{\perp}_{\lambda-\mu} \cap \mathfrak{S}_I$. Q.E.D.

Let $v \in \mathcal{M}(\lambda, \mu)_{\zeta_{\lambda,\mu}}$. For each $p \in \{1, \ldots, s\}$, we can write v as

$$v = \sum_j x^{(p)}_j \cdot z^{(p)}_j \mathbf{1}_{\lambda,\mu}, \tag{A.17}$$

where $\{x^{(p)}_j\}_j$ are linearly independent elements of $\mathbb{C}[\mathfrak{S}^{\perp}_{\lambda-\mu} \cap \mathfrak{S}_{I \setminus I_p}]$, and $z^{(p)}_j \in \mathbb{C}[\mathfrak{S}^{\perp}_{\lambda-\mu} \cap \mathfrak{S}_{I_p}]$.

Lemma A.10. $\xi z^{(p)}_k \mathbf{1}_{\lambda,\mu} = \langle \zeta_{\lambda,\mu}, \xi \rangle z^{(p)}_k \mathbf{1}_{\lambda,\mu}$ *for* $\xi \in \mathfrak{t}_{I_p}$.

Proof. We have

$$0 = (\xi - \langle \zeta_{\lambda,\mu}, \xi \rangle) v = \sum_j x^{(p)}_j \cdot (\xi - \langle \zeta_{\lambda,\mu}, \xi \rangle) \cdot z^{(p)}_j \mathbf{1}_{\lambda,\mu}. \tag{A.18}$$

Since $\mathfrak{S}_{I_p} \subseteq \mathfrak{S}_\ell$ is closed with respect to the Bruhat order, we have $\xi z^{(p)}_j \mathbf{1}_{\lambda,\mu} \in \mathbb{C}[\mathfrak{S}^{\perp}_{\lambda-\mu} \cap \mathfrak{S}_{I_p}] \mathbf{1}_{\lambda,\mu}$. Because $\{x^{(p)}_j\}_j$ are linearly independent, each $(\xi - \langle \zeta_{\lambda,\mu}, \xi \rangle) z^{(p)}_j \mathbf{1}_{\lambda,\mu}$ must be zero. Q.E.D.

Let $H_{I_p} = \mathbb{C}[\mathfrak{S}_{I_p}] \otimes S(\mathfrak{t}_{I_p}) \subseteq H_\ell$ be the subalgebra corresponding to $I_p \subseteq \Pi_\ell$. Obviously

$$H_{I_p} \cong H_d,$$

where $d = \#I_p$.

It is clear that H_{I_p}-module $H_{I_p}\mathbf{1}_{\lambda,\mu}$ is isomorphic to $\mathcal{M}_{d,n_p-n_{p-1}}$. Hence Proposition A.1 implies that $z_k^{(p)}\mathbf{1}_{\lambda,\mu} \in \mathbb{C}\mathbf{1}_{\lambda,\mu}$. Thus we have $v \in \mathbb{C}[\mathfrak{S}_{\lambda-\mu}^{\perp} \cap \mathfrak{S}_{I\setminus I_p}]$ for any p. This implies $v \in \mathbb{C}\mathbf{1}_{\lambda,\mu}$ and proves Lemma 5.2.

§B. q-analogue

Let $q \in \mathbb{C}^*$ and suppose that q is not a root of 1.

Definition B.1. The affine Hecke algebra $\mathcal{H}_\ell(q)$ of GL_ℓ is the associative algebra over $\mathbb{C}$ with generators

$$T_i^{\pm 1}\ (i = 1, \ldots, \ell-1), \quad Y_i^{\pm 1}\ (i = 1, \ldots, \ell),$$

and relations

$$\begin{gathered}
T_iT_i^{-1} = 1 = T_i^{-1}T_i, \quad (T_i + q)(T_i - q^{-1}) = 0, \\
T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}, \quad T_iT_j = T_jT_i \text{ (if } |i-j| > 1), \\
Y_iY_i^{-1} = 1 = Y_i^{-1}Y_i, \quad Y_iY_j = Y_jY_i, \\
T_iY_iT_i = Y_{i+1}, \quad T_iY_j = Y_jT_i \text{ (if } j \notin \{i, i+1\}).
\end{gathered}$$

The subalgebra $\bar{\mathcal{H}}_\ell(q) \subset \mathcal{H}_\ell(q)$ generated by $T_1, \ldots, T_{\ell-1}$ is called the Hecke algebra of GL_ℓ.

Let U_q denote the quantized enveloping algebra of $\mathfrak{gl}_n$ with a coproduct $\Delta : U_q \to U_q \otimes U_q$. (We refer to [Ji] for the definition.)

Let X and Y be objects of the BGG category $\mathcal{O}(U_q)$ (see e.g. [Jo]), and suppose that X or Y is finite-dimensional. Let $R_{XY} \in \operatorname{End}_{\mathbb{C}}(X \otimes Y)$ be the *R-matrix* on $X \otimes Y$ in the sense of [Ta]. (Actually, in [Ta], the R-matrix is considered only in the case where X and Y are both finite-dimensional. But it is easy to see that the same construction gives a well-defined operator on $X \otimes Y$ as long as X or Y is finite-dimensional. We also refer to [Ta] for the proof of the properties of the R-matrix below.) The operator R_{XY} is invertible and satisfies

$$\text{(B.1)} \qquad \Delta(u)\check{R}_{XY} = \check{R}_{XY}\Delta(u) \quad (u \in U_q),$$

where we set $\check{R}_{XY} = p \circ R_{XY}$ with p being the permutation $p(x \otimes y) = y \otimes x$. Let Z be another objects of $\mathcal{O}(U_q)$ such that at least two of $\{X, Y, Z\}$ are finite-dimensional. Then we have the *Yang-Baxter equation* on $X \otimes Y \otimes Z$:

$$\text{(B.2)} \qquad (\check{R}_{YZ} \otimes 1_X)(1_Y \otimes \check{R}_{XZ})(\check{R}_{XY} \otimes 1_Z) = (1_Z \otimes \check{R}_{XY})(\check{R}_{XZ} \otimes 1_Y)(1_X \otimes \check{R}_{YZ}).$$

Regard V_n as the vector representation of U_q. As proved by Jimbo [Ji], the correspondence

$$T_i \mapsto 1^{\otimes i-1} \otimes \check{R}_{V_n V_n} \otimes 1^{\otimes \ell - i - 1} \quad (i = 1, \ldots, \ell - 1)$$

gives an action of $\bar{\mathcal{H}}_\ell(q)$ on $V_n^{\otimes \ell}$. The following proposition is easy to prove using (B.1) and (B.2):

Proposition B.2. *There exists a unique homomorphism*

$$\mathcal{H}_\ell(q) \to \operatorname{End}_{U_q}(X \otimes V_n^{\otimes \ell})$$

such that

$$T_i \mapsto \check{R}_i \quad (i = 1, \ldots, \ell - 1),$$
$$Y_i \mapsto \check{R}_{i-1} \cdots \check{R}_1 \left((\check{R}_{V_n X} \check{R}_{X V_n}) \otimes 1^{\otimes \ell - 1} \right) \check{R}_1 \cdots \check{R}_{i-1} \quad (i = 1, \ldots, \ell),$$

where

$$\check{R}_i = 1^{\otimes i} \otimes \check{R}_{V_n V_n} \otimes 1^{\otimes \ell - i - 1} \quad (i = 1, \ldots, \ell - 1).$$

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A Recursion Formula of the Weighted Parabolic Kazhdan-Lusztig Polynomials

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Abstract.

In this article, we give a recursion formula of the weighted parabolic Kazhdan-Lusztig polynomials and describe a relationship between those polynomials and weighted Kazhdan-Lusztig polynomials introduced by G.Lusztig ([4]).

§1. Introduction

Our aim in this article is to give a recursion formula of the weighted parabolic Kazhdan-Lusztig polynomials introduced by H. Tagawa [5] as an extension of the parabolic Kazhdan-Lusztig polynomials and the weighted Kazhdan-Lusztig polynomials. Also, we describe a relationship between those polynomials and weighted Kazhdan-Lusztig polynomials, which is an extension of Deodhar's result on the parabolic Kazhdan-Lusztig polynomials and the Kazhdan-Lusztig polynomials (cf.[1]).

Let us give a brief review of known results. In 1982, G. Lusztig introduced the weighted Kazhdan-Lusztig polynomials, the special case of which has a representation theoretic interpretation (cf.[4]). Also, in 1987, V. Deodhar introduced two kinds of parabolic Kazhdan-Lusztig polynomials, one of which gives the dimensions of the intersection cohomology modules of Schubert varieties in G/P, where G is a Kac-Moody group and P is a "standard" parabolic subgroup of G (cf.[1]). Recently, H. Tagawa introduced the weighted parabolic Kazhdan-Lusztig polynomials and he obtained combinatorial formulas which were extensions of Deodhar's results on the parabolic Kazhdan-Lusztig polynomials (cf.[2]). But, unfortunately, the coefficients of the weighted parabolic Kazhdan-Lusztig polynomials are not always non-negative.

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This paper is organized as follows: In the next section, we recall the definition of the weighted parabolic R-polynomials and the weighted parabolic Kazhdan-Lusztig polynomials. Moreover, we show some interesting equalities used in the sequel. In Section 3, we give a recursion formula of the weighted parabolic Kazhdan-Lusztig polynomials which is an extension of Lusztig's result on the weighted Kazhdan-Lusztig polynomials (cf.[4]). In Section 4, we describe a relationship between weighted parabolic Kazhdan-Lusztig polynomials and weighted Kazhdan-Lusztig polynomials.

§2. Preliminaries and Notations

The purpose of this section is to define the weighted parabolic R-polynomials and the weighted parabolic Kazhdan-Lusztig polynomials. Throughout this article, (W, S) is an arbitrary Coxeter system, e is the unit element of W. Let $\mathbf{Z}$ be the set of integers, $\mathbf{N}$ the set of non-negative integers, and $\mathbf{P}$ the set of natural numbers.

First, we recall the definition of the Bruhat order.

Definition 2.1. We put $T := \{wsw^{-1}; s \in S,\ w \in W\}$. For $y, z \in W$, we denote $y <' z$ if and only if there exists an element t of T such that $\ell(tz) < \ell(z)$ and $y = tz$, where ℓ is the length function. Then the Bruhat order denoted by $\leq$ is defined as follows: For $x, w \in W$, $x \leq w$ if and only if there exists a sequence $x_0, x_1, \ldots, x_r$ in W such that $x = x_0 <' x_1 <' \cdots <' x_r = w$. We also use the notation $x \lessdot w$ if $x < w$ and $\ell(x) = \ell(w) - 1$.

The following is well known as the subword property. For $w \in W$, let $s_1 s_2 \cdots s_m$ be a reduced expression of w, i.e. $w = s_1 s_2 \cdots s_m$, $s_i \in S$ for all $i \in \{1, 2, \ldots, m\}$ and $\ell(w) = m$. For $x \in W$, $x \leq w$ if and only if there exists a sequence of natural numbers $i_1, i_2, \ldots, i_t$ such that $1 \leq i_1 < i_2 < \cdots < i_t \leq m$ and $x = s_{i_1} s_{i_2} \cdots s_{i_t}$. This expression of x is not reduced in general, i.e. it may happen that $\ell(x) < t$. However it is known that one can find a sequence of natural numbers $j_1, j_2, \ldots, j_k$ such that $1 \leq j_1 < j_2 < \cdots < j_k \leq m$, $x = s_{j_1} s_{j_2} \cdots s_{j_k}$ and $\ell(x) = k$.

From now on, the order on W is the Bruhat order. Next, we recall the definition of weights (cf.[4]).

Definition 2.2. Let Γ be an abelian group or a $\mathbf{Z}$-algebra of an abelian group with the unit element $\mathbf{e}$. φ is called a weight of W into Γ if and only if φ is a map of W into Γ satisfying the following conditions:

(i) $\varphi(e) = \mathbf{e}$,

(ii) $\varphi(s_1 s_2 \dots s_m) = \varphi(s_1)\varphi(s_2)\dots\varphi(s_m)$ for any reduced expression $s_1 s_2 \dots s_m$ in W.

(iii) $\varphi(s)$ is an invertible element in Γ for any $s \in S$.

In particular, any weight φ satisfies the following.

(ii)' For $s, t \in S$, if the order of st is odd, then $\varphi(s) = \varphi(t)$.

Conversely, a map $\widetilde{\varphi}$ of S into Γ satisfying (i), (ii)' and (iii) is uniquely extended to a weight of W into Γ.

From now on, Γ is an abelian group, $\mathbf{e}$ is the unit element of Γ, φ is a weight of W into Γ and we put $S = \{s_1, s_2, \dots, s_n\}$. For $w \in W$, we denote $\varphi(w)$ by $q_w^{\frac{1}{2}}$ and $(q_{s_1}^{\frac{1}{2}}, q_{s_2}^{\frac{1}{2}}, \dots, q_{s_n}^{\frac{1}{2}})$ by $\mathbf{q}$. Next, we recall the definition of the weighted Hecke algebras and the weighted R-polynomials (cf.[4]).

Definition 2.3. Let $\mathcal{H}_\varphi(W)$ be the free $\mathbf{Z}[\Gamma]$-module having the set $\{T'_w; w \in W\}$ as a basis and multiplication such that

$$T'_s T'_w = \begin{cases} T'_{sw} & \text{if } w < sw, \\ q_s T'_{sw} + (q_s - \mathbf{e})T'_w & \text{if } sw < w \end{cases}$$

for $w \in W$ and $s \in S$. We call $\mathcal{H}_\varphi(W)$ the weighted Hecke algebra (of W with respect to φ).

It is known that $\mathcal{H}_\varphi(W)$ is an associative algebra (see [3] Chapter 7 for more general theory). For $s \in S$, we can easily see that $(T'_s)^{-1} = (q_s^{-1} - \mathbf{e})T'_e + q_s^{-1}T'_s$.

Then, the weighted R-polynomial is defined as follows:

Definition 2.4. There exists a unique family of polynomials $\{R'_{x,w}(\mathbf{q}) \in \mathbf{Z}[\Gamma]; x, w \in W\}$ satisfying

$$\overline{T'_w} = q_w^{-1} \sum_{x \in W} (-1)^{\ell(x)+\ell(w)} R'_{x,w}(\mathbf{q}) T'_x \text{ for } w \in W,$$

where we put $\overline{T'_w} := T'^{-1}_{w^{-1}}$ for $w \in W$. We call these polynomials $R'_{x,w}(\mathbf{q})$ weighted R-polynomials of W.

Let J be a subset of S, W_J the subgroup of W generated by J and $W^J := \{y \in W; \ell(yz) = \ell(y) + \ell(z) \text{ for any } z \in W_J\}$. Then, it is well known that, for $w \in W$, there exist a unique element w^J in W^J and a unique element w_J in W_J such that $w = w^J w_J$ (cf.[3]).

Now, we can define weighted parabolic Hecke modules.

Definition 2.5. Let $A(\varphi)$ be the $\mathbf{Z}$-algebra of $\mathbf{Z}[\Gamma]$ generated by $\{q_s^{\frac{1}{2}}; s \in S\}$ and ψ a weight of W into $A(\varphi)$ with $\psi(s) = -\mathbf{e}$ or $\psi(s) = q_s$ for each $s \in S$. In the same way, for $w \in W$, we denote $\psi(w)$ by u_w. After this, for convenience, we denote $\mathbf{e}$ by 1. Also, for $s \in S$, we put $\widetilde{u}_s := q_s$ if $u_s = -1$ and $\widetilde{u}_s := -1$ if $u_s = q_s$. Note that the map $\widetilde{\psi}$ of W into $A(\varphi)$ defined as follows is also a weight.

$$\widetilde{\psi}(w) := \begin{cases} \mathbf{e} & \text{if } w{=}\mathrm{e}, \\ \widetilde{u}_{s_1}\widetilde{u}_{s_2}\cdots\widetilde{u}_{s_m} & \text{if } s_1 s_2 \ldots s_m \text{ is a reduced expression of } w. \end{cases}$$

Let $M^J_{\varphi,\psi}(W)$ be the free $\mathbf{Z}[\Gamma]$-module with basis $\{m'^J_w; w \in W^J\}$. For $s \in S$, we define $L'(s) \in \mathrm{Hom}_{\mathbf{Z}[\Gamma]}(M^J_{\varphi,\psi}(W))$ as follows:

$$L'(s)m'^J_w := \begin{cases} q_s m'^J_{sw} + (q_s - 1)m'^J_w & \text{if } sw < w, \\ m'^J_{sw} & \text{if } w < sw \in W^J, \\ u_s m'^J_w & \text{if } w < sw \notin W^J, \end{cases}$$

and linear extension.

Then, we call $M^J_{\varphi,\psi}(W)$ the weighted parabolic Hecke module (of W^J with respect to φ and ψ).

Let ρ'_J be a map from $\mathcal{H}_\varphi(W)$ to $M^J_{\varphi,\psi}(W)$ defined by

$$\rho'_J(\sum_{x\in W} a_x T'_x) := \sum_{x\in W} a_x u_{x_J} m'^J_{x^J},$$

where x^J and x_J are unique elements satisfying $x = x^J x_J$, $x^J \in W^J$ and $x_J \in W_J$. Then, the following is known (see [5]).

Lemma 2.6. ([5, Lemma 2.5])

(i) *ρ'_J is onto.*

(ii) *For $s \in S$ and $x \in W$, $L'(s)(\rho'_J(T'_x)) = \rho'_J(T'_s T'_x)$.*

(iii) *For $s \in S$, $L'(s)^2 = q_s L'(e) + (q_s - 1)L'(s)$, where $L'(e)$ is the identity map on $M^J_{\varphi,\psi}(W)$.*

(iv) *For $w \in W$ and $x \in W^J$, we can define*

$$T'_w \cdot m'^J_x := \begin{cases} m'^J_x \ \textit{if } w = e, \\ (L'(s_1)L'(s_2)\ldots L'(s_m))m'^J_x \\ \quad \textit{if } s_1 s_2 \ldots s_m \textit{ is a reduced expression of } w. \end{cases}$$

Namely, $M^J_{\varphi,\psi}(W)$ has an $\mathcal{H}_\varphi(W)$-module structure.

(v) *For* $w \in W$, $\rho'_J(T'_w) = T'_w \cdot m_e'^J$.

We define an operation $\bar{}$ on $M^J_{\varphi,\psi}(W)$ as follows:

$$\overline{\sum_{\gamma\in\Gamma} b_\gamma \gamma} := \sum_{\gamma\in\Gamma} b_\gamma \gamma^{-1} \text{ for } \sum_{\gamma\in\Gamma} b_\gamma\gamma \in \mathbf{Z}[\Gamma],$$

$$\overline{m_w'^J} := T'^{-1}_{w^{-1}} \cdot m_e'^J \text{ for } w \in W^J,$$

$$\overline{\sum_{w\in W^J} a_w m_w'^J} := \sum_{w\in W^J} \overline{a_w}\,\overline{m_w'^J} \text{ for } \sum_{w\in W^J} a_w m_w'^J \in M^J_{\varphi,\psi}(W).$$

We can see that the operation $\bar{}$ is an involution on $M^J_{\varphi,\psi}(W)$ by the following.

Lemma 2.7. ([5, Lemma 2.6]) *Let* $x \in W^J$ *and* $s \in S$. *Then, we have*

$$\overline{m_x'^J} = \rho'_J(\overline{T'_x}), \quad \overline{T'_s \cdot m_x'^J} = \overline{T'_s} \cdot \overline{m_x'^J}, \quad \overline{\overline{m_x'^J}} = m_x'^J.$$

Here, we describe the following interesting formula.

Proposition 2.8. *For* $w \in W$,

$$q_w^{-1} \sum_{x\in W} (-1)^{\ell(x)+\ell(w)} u_x R'_{x,w}(\mathbf{q}) = u_w^{-1}. \tag{1}$$

Proof. By the definition of the weighted R-polynomials, we can easily find a recursion formula of those polynomials. So, by direct calculation and the recursion formula, we can show this proposition by induction on $\ell(w)$. *q.e.d*

As a corollary of Proposition 2.8, we see the following.

Corollary 2.9. *For* $X \in \mathcal{H}_\varphi(W)$,

$$\overline{\rho'_J(X)} = \rho'_J(\overline{X}).$$

Proof. First, for $w \in W_J$, by Proposition 2.8, we have

$$\rho'_J(\overline{T'_w}) = q_w^{-1} \sum_{x\in W_J} (-1)^{\ell(x)+\ell(w)} R'_{x,w}(\mathbf{q}) u_x m_e'^J = u_w^{-1} m_e'^J.$$

Hence, for $w \in W$, by Lemma 2.6 and Lemma 2.7,

$$\overline{\rho'_J(T'_w)} = \overline{u_{w_J} m'^J_{w^J}} = u_{w_J}^{-1}(\overline{T'_{w^J}} \cdot m_e'^J) = \overline{T'_{w^J}} \cdot (\rho'_J(\overline{T'_{w_J}})) = \rho'_J(\overline{T'_w}),$$

where w^J and w_J are unique elements satisfying $w = w^J w_J$, $w^J \in W^J$ and $w_J \in W_J$. Hence, by definitions of the operation and ρ'_J, Corollary 2.9 holds. q.e.d

From now on, we denote $(u_{s_1}, u_{s_2}, \ldots, u_{s_n})$ by $\mathbf{u}$ and $(\widetilde{u}_{s_1}, \widetilde{u}_{s_2}, \ldots, \widetilde{u}_{s_n})$ by $\widetilde{\mathbf{u}}$. By using this operation, we can define the weighted parabolic R-polynomials as follows:

Definition 2.10. There exists a unique family of polynomials $\{R'^J_{x,w}(\mathbf{q})_{\mathbf{u}} \in \mathbf{Z}[\Gamma]; x, w \in W^J\}$ satisfying

$$\overline{m'^J_w} = q_w^{-1} \sum_{x \in W^J} (-1)^{\ell(x)+\ell(w)} R'^J_{x,w}(\mathbf{q})_{\mathbf{u}} m'^J_x \text{ for } w \in W^J.$$

We call these polynomials $R'^J_{x,w}(\mathbf{q})_{\mathbf{u}}$ weighted parabolic R-polynomials of W^J. For convenience, we put $R'^J_{x,w}(\mathbf{q})_{\mathbf{u}} := 0$ if $x \notin W^J$ or $w \notin W^J$.

For example, the following equalities are known.

Proposition 2.11. ([5, Lemma 3.4, Proposition 3.9])
Let $x, w \in W^J$.

(i) $(-1)^{\ell(x)+\ell(w)} q_w q_x^{-1} \overline{R'^J_{x,w}(\mathbf{q})_{\mathbf{u}}} = R'^J_{x,w}(\mathbf{q})_{\widetilde{\mathbf{u}}}$.

(ii) $\displaystyle\sum_{x \le y \le w} (-1)^{\ell(y)+\ell(w)} R'^J_{x,y}(\mathbf{q})_{\mathbf{u}} R'^J_{y,w}(\mathbf{q})_{\widetilde{\mathbf{u}}} = \delta_{x,w}$,
where $\delta_{x,w}$ is Kronecker delta.

(iii) *Let $s \in S$ with $sw < w$.*

$$R'^J_{x,w}(\mathbf{q})_{\mathbf{u}} = \begin{cases} R'^J_{sx,sw}(\mathbf{q})_{\mathbf{u}} & \text{if } sx < x, \\ q_s R'^J_{sx,sw}(\mathbf{q})_{\mathbf{u}} + (q_s - 1) R'^J_{x,sw}(\mathbf{q})_{\mathbf{u}} & \text{if } x < sx \in W^J, \\ \widetilde{u}_s R'^J_{x,sw}(\mathbf{q})_{\mathbf{u}} & \text{if } x < sx \notin W^J. \end{cases}$$

A relationship between the weighted parabolic R-polynomials and the weighted R-polynomials is the following.

Proposition 2.12. ([5, Proposition 3.11, Lemma 3.12])

(i) $R'^{\phi}_{x,w}(\mathbf{q})_{\mathbf{u}} = R'_{x,w}(\mathbf{q})$ *for $x, w \in W$.*

(ii) $\displaystyle R'^J_{x,w}(\mathbf{q})_{\mathbf{u}} = \sum_{y \in W_J} (-1)^{\ell(y)} u_y R'_{xy,w}(\mathbf{q})$ *for $x, w \in W^J$.*

We define some more notations.

Notation 2.13.

(i) Let r be the number of the different elements in $\{q_s; s \in S\}$, i.e. $r = \sharp\{q_s; s \in S\}$, and we put $\{q_{s_1}, q_{s_2}, \ldots, q_{s_r}\} = \{q_s; s \in S\}$, where $\sharp A$ is the cardinality of a set A. Put

$$\begin{aligned} \Gamma' &:= \{q_{s_1}^{\frac{n_1}{2}} q_{s_2}^{\frac{n_2}{2}} \cdots q_{s_r}^{\frac{n_r}{2}}; n_i \in \mathbf{Z} \text{ for } i \in [r]\}, \\ \Gamma'' &:= \Gamma'^2 (= \{\gamma^2; \gamma \in \Gamma'\}) \end{aligned}$$

where $[r] := \{1, 2, \ldots, r\}$.

(ii) For μ, $\gamma \in \Gamma''$, we denote $\mu \lhd \gamma$ if and only if there exist integers h_i and k_i with $h_i \leq k_i$, $\mu = q_{s_1}^{h_1} q_{s_2}^{h_2} \cdots q_{s_r}^{h_r}$ and $\gamma = q_{s_1}^{k_1} q_{s_2}^{k_2} \cdots q_{s_r}^{k_r}$.

In order to define the weighted parabolic Kazhdan-Lusztig polynomials, we define a total order on Γ' called a strong order.

Definition 2.14. We define a "strong order" on Γ' as a total order $<$ which satisfies the following conditions:

(i) For $\alpha, \beta, \gamma \in \Gamma'$, if $\alpha \leq \beta$, then $\alpha\gamma \leq \beta\gamma$.

(ii) For any $s \in S$, $\mathbf{e} < q_s^{\frac{1}{2}}$.

Example 2.15. If a weight φ of W into Γ satisfies that

$$q_{s_1}^{\frac{k_1}{2}} q_{s_2}^{\frac{k_2}{2}} \ldots q_{s_r}^{\frac{k_r}{2}} = \mathbf{e} \Leftrightarrow k_i = 0 \text{ for all } i \in [r].$$

Then, the lexicographic order with respect to $k_1, k_2, \ldots, k_r$ is a strong order on Γ'.

From now on, we assume that φ has a strong order on Γ' and we fix a strong order on Γ'. Put $\Gamma'_+ := \{\gamma \in \Gamma'; \mathbf{e} < \gamma\}$, $\Gamma'_- := \{\gamma \in \Gamma'; \gamma < \mathbf{e}\} (= (\Gamma'_+)^{-1})$ and $\Gamma''_+ := \{\gamma \in \Gamma''; \mathbf{e} \lhd \gamma\}$. Then, we can define weighted parabolic Kazhdan-Lusztig polynomials as follows:

Proposition 2.16. ([5, Proposition 4.4]) *There exists a unique family of polynomials $\{P'^J_{x,w}(\mathbf{q})_\mathbf{u} \in \mathbf{Z}[\Gamma''_+]; x, w \in W^J\}$ satisfying the following conditions:*

(i) $P'^J_{x,x}(\mathbf{q})_\mathbf{u} = 1$ *for all* $x \in W^J$.

(ii) $P'^J_{x,w}(\mathbf{q})_\mathbf{u} = 0$ *if* $x \not\leq w$.

(iii) $q_w^{-\frac{1}{2}} q_x^{\frac{1}{2}} P'^J_{x,w}(\mathbf{q})_\mathbf{u} \in \mathbf{Z}[\Gamma'_-]$ *if* $x < w$.

(iv)

$$q_w q_x^{-1} \overline{P'^J_{x,w}(\mathbf{q})_\mathbf{u}} = \sum_{x \leq y \leq w, y \in W^J} R'^J_{x,y}(\mathbf{q})_\mathbf{u} P'^J_{y,w}(\mathbf{q})_\mathbf{u}.$$

We define the "uniquely" determined polynomials from Proposition 2.16 as the weighted parabolic Kazhdan-Lusztig polynomials with respect to the strong order $<$. Note that we can easily see that $P'^{\phi}_{x,w}(\mathbf{q})_{\mathbf{u}} = P'_{x,w}(\mathbf{q})$ for $x, w \in W$, here $P'_{x,w}(\mathbf{q})$ is the weighted Kazhdan-Lusztig polynomials defined in Section 4. From now on, for convenience, we put $P'^{J}_{x,w}(\mathbf{q})_{\mathbf{u}} := 0$ if $x \notin W^J$ or $w \notin W^J$.

§3. A recursion formula

In this section, we define an extension of $\mu(x, w)$, which is the coefficient of $q^{\frac{\ell(w)-\ell(x)-1}{2}}$ in the Kazhdan-Lusztig polynomial $P_{x,w}(q)$, and get a recursion formula of the weighted parabolic Kazhdan-Lusztig polynomials.

Definition-Proposition 3.1. Let $s \in S$ and we put

$$c(s, \mathbf{u}) := \{x \in W^J; \begin{cases} sx < x \text{ or } sx \notin W^J & \text{if } u_s = q_s, \\ sx < x & \text{if } u_s = -1 \end{cases}\}.$$

Then, there exists a unique family of polynomials

$$\{M^{Js}_{x,w} \in \mathbf{Z}[\Gamma']; x, w \in W^J, x < w < sw, x \in c(s, \mathbf{u})\}$$

satisfying

$$\sum_{x \leq y < w, y \in c(s,\mathbf{u})} P^{*J}_{x,y}(\mathbf{q})_{\widetilde{\mathbf{u}}} M^{Js}_{y,w} - q_s^{\frac{1}{2}} P^{*J}_{x,w}(\mathbf{q})_{\widetilde{\mathbf{u}}} \in \mathbf{Z}[\Gamma'_{-}], \quad \overline{M^{Js}_{x,w}} = M^{Js}_{x,w},$$

where $P^{*J}_{x,w}(\mathbf{q})_{\mathbf{u}} := q_w^{-\frac{1}{2}} q_x^{\frac{1}{2}} P'^{J}_{x,w}(\mathbf{q})_{\mathbf{u}}$ for $x, w \in W^J$.

This is easily obtained by direct calculation and induction on $\ell(w) - \ell(x)$ and the proof is therefore omitted. Then, a recursion formula of the weighted parabolic Kazhdan-Lusztig polynomials is described as follows:

Theorem 3.2.

(i) *Let $x, w \in W^J$ and $s \in S$ with $sw < w$. Then, we have*

$$\begin{aligned} P'^{J}_{x,w}(\mathbf{q})_{\mathbf{u}} &= \begin{cases} q_s P'^{J}_{x,sw}(\mathbf{q})_{\mathbf{u}} + P'^{J}_{sx,sw}(\mathbf{q})_{\mathbf{u}} & \text{if } sx < x \\ P'^{J}_{x,sw}(\mathbf{q})_{\mathbf{u}} + q_s P'^{J}_{sx,sw}(\mathbf{q})_{\mathbf{u}} & \text{if } x < sx \in W^J \\ (\widetilde{u}_s + 1) P'^{J}_{x,sw}(\mathbf{q})_{\mathbf{u}} & \text{if } x < sx \notin W^J \end{cases} \\ &\quad - \sum_{x \leq y < sw, y \in c(s,\widetilde{\mathbf{u}})} q_y^{-\frac{1}{2}} q_w^{\frac{1}{2}} P'^{J}_{x,y}(\mathbf{q})_{\mathbf{u}} M^{Js}_{y,sw}. \end{aligned}$$

(ii) *Let $x, w \in W^J$. If there exists $s \in S$ such that $sw < w$ and $sx \in W^J$, then we have*

$$P'^J_{x,w}(\mathbf{q})_\mathbf{u} = P'^J_{sx,w}(\mathbf{q})_\mathbf{u}.$$

Note that if $sw < w$, then $x \le w \Leftrightarrow sx \le w$.

(iii) *Let $x, w \in W^J$. If there exists $s \in S$ such that $sw < w$, $x < sx \notin W^J$ and $u_s = q_s$, then we have*

$$P'^J_{x,w}(\mathbf{q})_\mathbf{u} = 0.$$

Before the proof of this theorem, we show some lemmas and propositions.

Lemma 3.3. *Let $x, w \in W^J$ and $s \in S$ with $w < sw \notin W^J$ and $sx \in W^J$. Then, we have*

$$R'^J_{x,w}(\mathbf{q})_\mathbf{u} = \begin{cases} \widetilde{u}_s^{-1} R'^J_{sx,w}(\mathbf{q})_\mathbf{u} & \text{if } sx < x, \\ \widetilde{u}_s R'^J_{sx,w}(\mathbf{q})_\mathbf{u} & \text{if } x < sx. \end{cases}$$

Proof. First, by Lemma 2.6 and Lemma 2.7, we can easily see that

$$\overline{q_s^{-\frac{1}{2}}(L'(s) + L'(e)) m'^J_w} = q_s^{-\frac{1}{2}}(L'(s) + L'(e)) \overline{m'^J_w}. \tag{2}$$

Hence, by (2) and our assumption that $w < sw \notin W^J$,

$$u_s^{-1}\overline{m'^J_w} + \overline{m'^J_w} = q_s^{-1} L'(s)\overline{m'^J_w} + q_s^{-1}\overline{m'^J_w}.$$

Hence, we have

$$L'(s)\overline{m'^J_w} = \sum_{x \in W^J} q_w^{-1}(-1)^{\ell(w)+\ell(x)} u_s R'^J_{x,w}(\mathbf{q})_\mathbf{u} m'^J_x.$$

On the other hand, by the definition of $R'^J_{x,w}(\mathbf{q})_\mathbf{u}$, we can see

$$\begin{aligned} & L'(s)\overline{m'^J_w} \\ = & \sum_{sy<y \in W^J} q_w^{-1}(-1)^{\ell(w)+\ell(y)}((q_s - 1)R'^J_{y,w}(\mathbf{q})_\mathbf{u} - R'^J_{sy,w}(\mathbf{q})_\mathbf{u}) m'^J_y \\ & - \sum_{y<sy \in W^J} q_w^{-1}(-1)^{\ell(w)+\ell(y)} q_s R'^J_{sy,w}(\mathbf{q})_\mathbf{u} m'^J_y \\ & + \sum_{y<sy \notin W^J} q_w^{-1}(-1)^{\ell(w)+\ell(y)} u_s R'^J_{y,w}(\mathbf{q})_\mathbf{u} m'^J_y. \end{aligned}$$

Thus, we have

$$u_s R'^J_{x,w}(\mathbf{q})_\mathbf{u} = \begin{cases} (q_s - 1)R'^J_{x,w}(\mathbf{q})_\mathbf{u} - R'^J_{sx,w}(\mathbf{q})_\mathbf{u} & \text{if } sx < x, \\ -q_s R'^J_{sx,w}(\mathbf{q})_\mathbf{u} & \text{if } x < sx \in W^J, \\ u_s R'^J_{x,w}(\mathbf{q})_\mathbf{u} & \text{if } x < sx \notin W^J. \end{cases}$$

By using this equality, we can obtain this lemma. q.e.d.

Lemma 3.4. *Let $x, y, w \in W^J$ and $s \in S$. If $sx < x \lessdot w < sw \notin W^J$, $sx < y < w$ and $x \neq y$, then $y \notin W^J$.*

We can easily obtain this lemma by the subword property and the proof is therefore omitted.

Then, we can show the following.

Proposition 3.5. *Let $x, w \in W^J$, $s \in S$, $w < sw \notin W^J$, $sx \in W^J$ and $u_s = -1$. Then, we have*

$$P'^J_{x,w}(\mathbf{q})_\mathbf{u} = P'^J_{sx,w}(\mathbf{q})_\mathbf{u}. \tag{3}$$

Note that the above equality does not always hold in case $u_s = q_s$.

Proof. We may assume that $sx < x$. Case 1. $x \not\leq w$. In this case, we can easily see that $sx \not\leq w$. So, both sides of (3) are equal to 0. Case 2. $x \leq w$. In this case, we show this theorem by induction on $\ell(w) - \ell(x)$. In case $\ell(w) - \ell(x) = 1$. Note that we may not consider the case that $\ell(w) - \ell(x) = 0$ by our assumption in this proposition. Let $q_w q_x^{-1} = q_t$ ($t \in S$) and $y \in W - \{x\}$ with $sx \lessdot y \lessdot w$. Then, by Lemma 3.4, $y \notin W^J$. So, by the fact that $R'^J_{x,w}(\mathbf{q})_\mathbf{u} = q_s - 1$ if $x \lessdot w$ and $q_w q_x^{-1} = q_s$, $P'^J_{x,w}(\mathbf{q})_\mathbf{u} = 1$ if $x \lessdot w$, we have

$$q_w q_{sx}^{-1} \overline{P'^J_{sx,w}(\mathbf{q})_\mathbf{u}} - P'^J_{sx,w}(\mathbf{q})_\mathbf{u} = q_s q_t - 1.$$

Hence,

$$\overline{P^{*J}_{sx,w}(\mathbf{q})_\mathbf{u}} - q_s^{\frac{1}{2}} q_t^{\frac{1}{2}} = P^{*J}_{sx,w}(\mathbf{q})_\mathbf{u} - q_s^{-\frac{1}{2}} q_t^{-\frac{1}{2}}. \tag{4}$$

Then, the left hand side of (4) is an element in $\mathbf{Z}[\Gamma'_+]$ and the right hand side of (4) is an element in $\mathbf{Z}[\Gamma'_-]$. So, by the fact that $\mathbf{Z}[\Gamma'_+] \cap \mathbf{Z}[\Gamma'_-] = \{0\}$, we have

$$P'^J_{sx,w}(\mathbf{q})_\mathbf{u} = 1.$$

On the other hand, since $\ell(w) - \ell(x) = 1$,

$$P'^J_{x,w}(\mathbf{q})_\mathbf{u} = 1.$$

We suppose that (3) holds when $\ell(w) - \ell(x) < k$ $(k \geq 2)$ and we will show this one in case $\ell(w) - \ell(x) = k$. For $y \in W^J$ with $sy < y$, by Proposition 2.11-(iii), we have

$$q_s R'^J_{x,y}(\mathbf{q})_{\mathbf{u}} - R'^J_{sx,y}(\mathbf{q})_{\mathbf{u}} = R'^J_{sx,sy}(\mathbf{q})_{\mathbf{u}} - q_s R'^J_{x,sy}(\mathbf{q})_{\mathbf{u}}.$$

Hence, by our inductive hypothesis, we have

$$\begin{aligned}&\sum_{sy<y\in W^J} (q_s R'^J_{x,y}(\mathbf{q})_{\mathbf{u}} - R'^J_{sx,y}(\mathbf{q})_{\mathbf{u}}) P'^J_{y,w}(\mathbf{q})_{\mathbf{u}} \\ &= P'^J_{x,w}(\mathbf{q})_{\mathbf{u}} - P'^J_{sx,w}(\mathbf{q})_{\mathbf{u}} \\ &- \sum_{z<sz\in W^J} (q_s R'^J_{x,z}(\mathbf{q})_{\mathbf{u}} - R'^J_{sx,z}(\mathbf{q})_{\mathbf{u}}) P'^J_{z,w}(\mathbf{q})_{\mathbf{u}}.\end{aligned}$$

So, we have

$$\begin{aligned}&\sum_{sy<y\in W^J \text{ or } y<sy\in W^J} (q_s R'^J_{x,y}(\mathbf{q})_{\mathbf{u}} - R'^J_{sx,y}(\mathbf{q})_{\mathbf{u}}) P'^J_{y,w}(\mathbf{q})_{\mathbf{u}} \\ &= P'^J_{x,w}(\mathbf{q})_{\mathbf{u}} - P'^J_{sx,w}(\mathbf{q})_{\mathbf{u}}.\end{aligned}$$

On the other hand, by Lemma 3.3,

$$\sum_{y\in W^J, y<sy\notin W^J} (q_s R'^J_{x,y}(\mathbf{q})_{\mathbf{u}} - R'^J_{sx,y}(\mathbf{q})_{\mathbf{u}}) P'^J_{y,w}(\mathbf{q})_{\mathbf{u}} = 0.$$

Thus, by the above equalities,

$$\sum_{y\in W^J} (q_s R'^J_{x,y}(\mathbf{q})_{\mathbf{u}} - R'^J_{sx,y}(\mathbf{q})_{\mathbf{u}}) P'^J_{y,w}(\mathbf{q})_{\mathbf{u}} = P'^J_{x,w}(\mathbf{q})_{\mathbf{u}} - P'^J_{sx,w}(\mathbf{q})_{\mathbf{u}}.$$

Hence, by Proposition 2.16-(iv), we have

$$q_s^{\frac{1}{2}} \overline{P^{*J}_{x,w}(\mathbf{q})_{\mathbf{u}}} - \overline{P^{*J}_{sx,w}(\mathbf{q})_{\mathbf{u}}} = q_s^{-\frac{1}{2}} P^{*J}_{x,w}(\mathbf{q})_{\mathbf{u}} - P^{*J}_{sx,w}(\mathbf{q})_{\mathbf{u}}.$$

So, we can see

$$q_s^{-\frac{1}{2}} P^{*J}_{x,w}(\mathbf{q})_{\mathbf{u}} - P^{*J}_{sx,w}(\mathbf{q})_{\mathbf{u}} = 0.$$

This completes the proof of Proposition 3.5. $\quad$ q.e.d

By Proposition 2.11, we can easily obtain the following.

Definition-Proposition 3.6. *For $w \in W^J$, we put*

$$\begin{aligned}C'^J_w &:= q_w^{-\frac{1}{2}} \sum_{x\leq w} P'^J_{x,w}(\mathbf{q})_{\bar{\mathbf{u}}} m'^J_x, \\ D'^J_w &:= \sum_{x\in W^J} (-1)^{\ell(x)+\ell(w)} q_w^{\frac{1}{2}} q_x^{-1} \overline{P'^J_{x,w}(\mathbf{q})_{\mathbf{u}}} m'^J_x.\end{aligned}$$

Then, we have

$$\overline{C'^J_w} = C'^J_w, \quad \overline{D'^J_w} = D'^J_w.$$

Then, as a corollary of Proposition 3.5, we can see the following.

Corollary 3.7. *Let $w \in W^J$, $s \in S$, $w < sw \notin W^J$ and $u_s = q_s$. Then, we have*

$$L'(s)C'^J_w = q_s C'^J_w.$$

The following lemma is easily obtained by direct calculation.

Lemma 3.8. *Let $w \in W^J$ and $s \in S$.*

(i) *If $w < sw$, we put*

$$q_s^{-\frac{1}{2}}(L'(s) + L'(e))C'^J_w - C'^J_{sw} - \sum_{y<w, y\in c(s,\mathbf{u})} M^{Js}_{y,w} C'^J_y = \sum_{x\in W^J} f_x \widetilde{m'}^J_x,$$

where $\widetilde{m'}^J_x := q_x^{-\frac{1}{2}} m'^J_x$ for $x \in W^J$. Then, we have

$$f_x = \begin{cases} q_s^{\frac{1}{2}} P^{*J}_{x,w}(\mathbf{q})_{\tilde{\mathbf{u}}} + P^{*J}_{sx,w}(\mathbf{q})_{\tilde{\mathbf{u}}} & \text{if } sx < x \\ q_s^{-\frac{1}{2}} P^{*J}_{x,w}(\mathbf{q})_{\tilde{\mathbf{u}}} + P^{*J}_{sx,w}(\mathbf{q})_{\tilde{\mathbf{u}}} & \text{if } x < sx \in W^J \\ q_s^{-\frac{1}{2}}(u_s + 1) P^{*J}_{x,w}(\mathbf{q})_{\tilde{\mathbf{u}}} & \text{if } x < sx \notin W^J \end{cases}$$
$$- P^{*J}_{x,sw}(\mathbf{q})_{\tilde{\mathbf{u}}} - \sum_{x\le y<w, y\in c(s,\mathbf{u})} P^{*J}_{x,y}(\mathbf{q})_{\tilde{\mathbf{u}}} M^{Js}_{y,w}.$$

(ii) *If $sw < w$, we put*

$$(q_s^{-\frac{1}{2}} L'(s) - q_s^{\frac{1}{2}} L'(e))C'^J_w = \sum_{x\in W^J} g_x \widetilde{m'}^J_x.$$

Then, we have

$$g_x = \begin{cases} P^{*J}_{sx,w}(\mathbf{q})_{\tilde{\mathbf{u}}} - q_s^{-\frac{1}{2}} P^{*J}_{x,w}(\mathbf{q})_{\tilde{\mathbf{u}}} & \text{if } sx < x, \\ P^{*J}_{sx,w}(\mathbf{q})_{\tilde{\mathbf{u}}} - q_s^{\frac{1}{2}} P^{*J}_{x,w}(\mathbf{q})_{\tilde{\mathbf{u}}} & \text{if } x < sx \in W^J, \\ q_s^{-\frac{1}{2}}(u_s - q_s) P^{*J}_{x,w}(\mathbf{q})_{\tilde{\mathbf{u}}} & \text{if } x < sx \notin W^J. \end{cases}$$

Then, we have the following.

Proposition 3.9. *For $w \in W^J$ and $s \in S$, we have*

$$q_s^{-\frac{1}{2}} L'(s) C'^J_w = \begin{cases} -q_s^{-\frac{1}{2}} C'^J_w + C'^J_{sw} + \sum_{y<w, y\in c(s,\mathbf{u})} M^{Js}_{y,w} C'^J_y & \text{if } w < sw \in W^J, \\ q_s^{\frac{1}{2}} C'^J_w & \text{if } sw < w. \end{cases}$$

Proof. We show this proposition by induction on $\ell(w)$. We can easily see that Proposition 3.9 holds in case $\ell(w) = 0$. So, we suppose that Proposition 3.9 holds when $\ell(w) < k$ $(k \geq 1)$ and we will show this one in case $\ell(w) = k$. Case 1. $w < sw \in W^J$. We put

$$q_s^{-\frac{1}{2}}(L'(s) + L'(e))C'^J_w - C'^J_{sw} - \sum_{y<w, y\in c(s,\mathbf{u})} M^{Js}_{y,w} C'^J_y = \sum_{x\in W^J} f_x \widetilde{m'}^J_x.$$

Note that $f_x = 0$ if $\ell(x) > \ell(sw)$. First, by Lemma 3.8, Definition-Proposition 3.1 and Corollary 3.7, we can see that $f_x \in \mathbf{Z}[\Gamma'_-]$. Next, we show that $f_x = 0$ for all $x \in W^J$. By Proposition 3.6 and the equality that $\overline{M^{Js}_{y,w}} = M^{Js}_{y,w}$, we can obtain

$$\begin{aligned}
&\overline{q_s^{-\frac{1}{2}} L'(s) C'^J_w + q_s^{-\frac{1}{2}} C'^J_w - C'^J_{sw} - \sum_{y<w, y\in c(s,\mathbf{u})} M^{Js}_{y,w} C'^J_y} \\
&= q_s^{-\frac{1}{2}}(L'(s) + L'(e))C'^J_w - C'^J_{sw} - \sum_{y<w, y\in c(s,\mathbf{u})} M^{Js}_{y,w} C'^J_y.
\end{aligned}$$

So, we have

$$\sum_{x\in W^J} f_x \widetilde{m'}^J_x = \sum_{x,y\in W^J, y\leq x} \overline{f_x} q_x^{-\frac{1}{2}} q_y^{\frac{1}{2}} (-1)^{\ell(x)+\ell(y)} R'^J_{y,x}(\mathbf{q})_{\mathbf{u}} \widetilde{m'}^J_y. \tag{5}$$

We suppose that there exists $x \in W^J$ satisfying $f_x \neq 0$. Let x_0 be an element in W^J such that $f_{x_0} \neq 0$ and $f_x = 0$ for any $x \in W^J$ with $\ell(x) > \ell(x_0)$. Then, we see that the coefficient of $\widetilde{m'}^J_{x_0}$ in the right hand side of (5) is $\overline{f_{x_0}}$. Hence, we have $f_{x_0} = \overline{f_{x_0}} \neq 0$. This contradicts that $f_{x_0} \in \mathbf{Z}[\Gamma_-]$. So, we have

$$f_x = 0 \text{ for } \forall x \in W^J$$

and we obtain

$$q_s^{-\frac{1}{2}} L'(s) C'^J_w = -q_s^{-\frac{1}{2}} C'^J_w + C'^J_{sw} + \sum_{y<w, y\in c(s,\mathbf{u})} M^{Js}_{y,w} C'^J_y.$$

Case 2. $sw < w$. By our inductive hypothesis, we can use

$$C'^J_w = q_s^{-\frac{1}{2}}(L'(s) + L'(e))C'^J_{sw} - \sum_{y<sw, y\in c(s,\mathbf{u})} M^{Js}_{y,sw} C'^J_y.$$

So, by Proposition 3.7, Lemma 2.6 and our inductive hypothesis, we can see that

$$q_s^{-\frac{1}{2}} L'(s) C'^J_w = q_s^{\frac{1}{2}} C'^J_w.$$

Therefore, this completes the proof of Proposition 3.9 q.e.d

At last, we can prove our main theorem.

Proof of Theorem 3.2. By Proposition 3.9 and Lemma 3.8-(i), we can easily see (i). Also, (ii) and (iii) are easily obtained by Proposition 3.9 and Lemma 3.8-(ii). q.e.d

§4. A relationship with weighted K-L polynomials

The purpose of this section is to show a relationship between weighted parabolic Kazhdan-Lusztig polynomials and weighted Kazhdan-Lusztig polynomials, which is an extension of Deodhar's result on a relationship between parabolic Kazhdan-Lusztig polynomials and Kazhdan-Lusztig polynomials ([1]). First, we recall the definition of the weighted Kazhdan-Lusztig polynomials.

Definition-Proposition 4.1. ([4]) There exists a unique family of polynomials $\{P'_{x,w}(\mathbf{q}) \in \mathbf{Z}[\Gamma''_+]; x, w \in W\}$ satisfying the following conditions:

(i) $P'_{x,x}(\mathbf{q}) = 1$ for all $x \in W$.

(ii) $P'_{x,w}(\mathbf{q}) = 0$ if $x \not\leq w$.

(iii) $q_w^{-\frac{1}{2}} q_x^{\frac{1}{2}} P'_{x,w}(\mathbf{q}) \in \mathbf{Z}[\Gamma'_-]$ if $x < w$.

(iv)
$$q_w q_x^{-1} \overline{P'_{x,w}(\mathbf{q})} = \sum_{x \leq y \leq w} R'_{x,y}(\mathbf{q}) P'_{y,w}(\mathbf{q}).$$

As the beginning of this section, we show the following.

Lemma 4.2. *Let $w \in W$. We put*

$$D'_w = \sum_{x \leq w} (-1)^{\ell(x)+\ell(w)} q_w^{\frac{1}{2}} q_x^{-1} \overline{P'_{x,w}(\mathbf{q})} T'_x.$$

(i) $\overline{D'_w} = D'_w$.

(ii) $\rho'_J(D'_w) = \sum_{x \in W^J} (-1)^{\ell(x)+\ell(w)} q_w^{\frac{1}{2}} q_x^{-1} \Big(\sum_{y \in W_J} \widetilde{u}_y^{-1} \overline{P'_{xy,w}(\mathbf{q})} \Big) m_x'^J.$

Proof. We can easily obtain this lemma by the direct calculation and the definition of the weighted Kazhdan-Lusztig polynomials. Note that $(-1)^{\ell(x)} q_x^{-1} u_y = \widetilde{u}_y^{-1}$. q.e.d

Then, we have the following.

Theorem 4.3. *Let $x, w \in W^J$.*

(i) *If $\widetilde{u}_y q_y^{-\frac{1}{2}} \in \mathbf{Z}[\Gamma'_-]$ for all $y \in W_J$ satisfying $xy \leq w$,*

$$P'^J_{x,w}(\mathbf{q})_\mathbf{u} = \sum_{y \in W_J} \widetilde{u}_y P'_{xy,w}(\mathbf{q}).$$

In particular, if $u_s = q_s$ for $\forall s \in S$,

$$P'^J_{x,w}(\mathbf{q})_\mathbf{u} = \sum_{y \in W_J} (-1)^{\ell(y)} P'_{xy,w}(\mathbf{q}).$$

(ii) *If $u_s = -1$ for all $s \in S$ and $\sharp W_J < +\infty$,*

$$P'^J_{x,w}(\mathbf{q})_\mathbf{u} = P'_{xz_0,wz_0}(\mathbf{q}),$$

where z_0 is the longest element in W_J.

Proof. (i) For $x, w \in W^J$, we put

$$G_{x,w} := \sum_{y \in W_J} \widetilde{u}_y P'_{xy,w}(\mathbf{q}) \in \mathbf{Z}[\Gamma''_+].$$

Then, we will show that a family of polynomials $\{G_{x,w}; x, w \in W^J\}$ satisfies conditions (i), (ii), (iii) and (iv) in Proposition 2.16. Let $x, w \in W^J$. By the fact that $\widetilde{u}_e^{-1} = 1$ and $P'_{x,x}(\mathbf{q}) = 1$, we have $G_{x,x} = 1$. So, (i) holds. If $x \not\leq w$, for $y \in W_J$, we can easily see that $xy \not\leq w$ by the subword property. Hence, (ii) holds. If $x < w$, by our assumption that $\widetilde{u}_y q_y^{-\frac{1}{2}} \in \mathbf{Z}[\Gamma'_-]$ for all $y \in W_J$ satisfying $xy \leq w$, we have

$$q_w^{-\frac{1}{2}} q_x^{\frac{1}{2}} G_{x,w} = \sum_{y \in W_J} \widetilde{u}_y q_y^{-\frac{1}{2}} q_w^{-\frac{1}{2}} q_{xy}^{\frac{1}{2}} P'_{xy,w}(\mathbf{q}) \in \mathbf{Z}[\Gamma'_-].$$

Hence, (iii) holds. By Lemma 4.2-(ii), we can see

$$\overline{\rho'_J(D'_w)} = \sum_{y \in W^J} (-1)^{\ell(y)+\ell(w)} q_w^{-\frac{1}{2}} \Big(\sum_{x \in W^J} R'^J_{y,x}(\mathbf{q})_\mathbf{u} G_{x,w} \Big) m'^J_y.$$

On the other hand, by Corollary 2.9 and Lemma 4.2-(i), we have

$$\overline{\rho'_J(D'_w)} = \rho'_J(D'_w).$$

Hence, we have

$$\begin{aligned} &\sum_{x \in W^J} (-1)^{\ell(x)+\ell(w)} q_w^{\frac{1}{2}} q_x^{-1} \overline{G_{x,w}} m'^J_x \\ &= \sum_{x \in W^J} (-1)^{\ell(x)+\ell(w)} q_w^{-\frac{1}{2}} \Big(\sum_{y \in W^J} R'^J_{x,y}(\mathbf{q})_\mathbf{u} G_{y,w} \Big) m'^J_x. \end{aligned}$$

Thus, we obtain

$$q_w q_x^{-1}\overline{G_{x,w}} = \sum_{y\in W^J} R'^J_{x,y}(\mathbf{q})_{\mathbf{u}} G_{y,w}$$

and (iv) holds. Therefore, by the uniqueness of the weighted parabolic Kazhdan-Lusztig polynomials, we have

$$P'^J_{x,w}(\mathbf{q})_{\mathbf{u}} = G_{x,w} = \sum_{y\in W_J} \widetilde{u}_y P'_{xy,w}(\mathbf{q}).$$

(ii) First, we can easily see that $P'_{x,w}(\mathbf{q}) = P'_{x^{-1},w^{-1}}(\mathbf{q})$ for $x, w \in W$. Moreover, it is shown by Lusztig [4] that $P'_{x,w}(\mathbf{q}) = P'_{sx,w}(\mathbf{q})$ for $x, w \in W$ and $s \in S$ satisfying $x \leq w$, $sx < x$, $sw < w$. So, we have

$$P'_{xy,wz_0}(\mathbf{q}) = P'_{xz_0,wz_0}(\mathbf{q}) \text{ for } \forall x, w \in W^J \text{ and } \forall y \in W_J.$$

Hence, by Lemma 4.2-(ii), we have

$$\begin{aligned}&\rho'_J(D'_{wz_0})\\ &= (-1)^{\ell(z_0)} q_{z_0}^{\frac{1}{2}} \sum_{y\in W_J} \widetilde{u}_y^{-1} \sum_{x\in W^J} (-1)^{\ell(x)+\ell(w)} q_w^{\frac{1}{2}} q_x^{-1} \overline{P'_{xz_0,wz_0}(\mathbf{q})} m'^J_x.\end{aligned}$$

Hence, by almost the same method to (i), we can obtain (ii). Note that $\overline{q_{z_0}^{\frac{1}{2}} \sum_{y\in W_J} q_y^{-1}} = q_{z_0}^{\frac{1}{2}} \sum_{y\in W_J} q_y^{-1}$. <u>*q.e.d.*</u>

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Special Polynomials and Generalized Painlevé Equations

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Abstract.

We will review recent developments on the special polynomials arising in Painlevé equations and their generalizations.

§1. Introduction

The following six equations are called Painlevé equations;

(P_I) $\quad y'' = 6y^2 + t,$

(P_{II})

$$y'' = 2y^3 + ty + \alpha,$$

(P_{III})

$$y'' = \frac{1}{y}y'^2 - \frac{1}{t}y' + \frac{1}{t}(\alpha y^2 + \beta) + \gamma y^3 + \frac{\delta}{y},$$

(P_{IV})

$$y'' = \frac{1}{2y}y'^2 + \frac{3}{2}y^3 + 4ty^2 + 2(t^2 - \alpha)y + \frac{\beta}{y},$$

(P_V)

$$\begin{aligned} y'' = &(\frac{1}{2y} + \frac{1}{y-1})y'^2 - \frac{y'}{t} \\ &+ \frac{(y-1)^2}{t^2}(\alpha y + \frac{\beta}{y}) + \frac{\gamma y}{t} + \frac{\delta y(y+1)}{y-1}, \end{aligned}$$

(P_{VI})

$$\begin{aligned} y'' =&\frac{1}{2}(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t})y'^2 - (\frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t})y' \\ &+ \frac{y(y-1)(y-t)}{t^2(t-1)^2}\left[\alpha + \beta\frac{t}{y^2} + \gamma\frac{t-1}{(y-1)^2} + \delta\frac{t(t-1)}{(y-t)^2}\right], \end{aligned}$$

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where $y = y(t)$ is the unknown function, $' = d/dt$ and $\alpha, \beta, \gamma, \delta$ are complex parameters. These equations have the Painlevé property i.e. any movable singularity (depending on initial data) is a pole. This property is known as a practical method to test the integrability of differential equations.

By the work of K. Okamoto, the following facts are known for the Painlevé equations P_J J =II,III,IV,V or VI (see [1,2] for example).

(1) P_J has affine Weyl group symmetry of type (II) $A_1^{(1)}$, (III) $C_2^{(1)}$, (IV) $A_2^{(1)}$, (V) $A_3^{(1)}$ or (VI) $D_4^{(1)}$.
(2) For special values of the parameters, P_J is solved by hypergeometric functions such as (II) Airy, (III) Bessel, (IV) Hermite, (V) Laguerre or (VI) Gauss.
(3) There are also other rational (or algebraic) solutions such as (II) Yablonskii-Vorob'ev, (IV) Okamoto or (III,V,VI) Umemura.

We will study the polynomials in (3) from the points of view of combinatorial structure and determinant formula.

These polynomials arise as the τ-functions for the special solutions of the Painlevé equations, and defined by some recurrence relations. The origin of such recurrence relations (Toda equations) is the Bäcklund symmetry of the Painlevé equations. We shall explain these in the simplest example of P_{II}.

The second Painlevé equation P_{II} is a hamiltonian system

$$q' = \frac{\partial H}{\partial p}, \quad p' = -\frac{\partial H}{\partial q}.$$

where

$$H = \frac{1}{2}p^2 - (q^2 + \frac{t}{2})p - bq, \quad (b = \alpha + \frac{1}{2}).$$

The τ-function is defined as

$$\tau = \exp(\int H dt), \quad H = (\log \tau)'.$$

The P_{II} equation has the symmetry given by the Bäcklund transformation such as

$$s_1(q) = q + \frac{b}{p}, \quad s_1(p) = p, \quad s_1(b) = -b.$$
$$r(q) = -q, \quad r(p) = -p + 2q^2 - t, \quad r(b) = 1 - b.$$

These transformations s_1, r and $s_0 = rs_1r$ generate the affine Weyl group $W = \langle s_1, s_0, r \mid s_1^2 = s_0^2 = r^2 = 1,\ s_0 r = r s_1 \rangle$ of type $A_1^{(1)}$. There is an translation $T = rs_1$ such as

$$T^{-1}(q) = -q - \frac{b}{p}, \quad T(p) = -p + 2q^2 + t, \quad T(b) = b - 1.$$

These transformations commute with the derivation. Given a solution (p, q, b) for P_{II}, one obtains a sequence of solutions

$$(p_n, q_n, b_n) = T^n(p, q, b), \quad n \in Z.$$

K. Okamoto proved that the corresponding τ-functions τ_n satisfy the Toda equation

$$(\log \tau_n)'' = c_n \frac{\tau_{n-1}\tau_{n+1}}{\tau_n^2},$$

where c_n is a nonzero constant.

§2. Special polynomials

2.1. Yablonskii-Vorob'ev polynomials for P_{II}

Recurrence relation : $(' = \frac{d}{dt})$

$$T_{m+1}T_{m-1} = tT_m^2 - 4(T''_m T_m - T'^2_m), \tag{2.1}$$

Initial condition : $T_0 = T_1 = 1$.

Examples.

$T_2 = t,$

$T_3 = 4 + t^3,$

$T_4 = -80 + 20t^3 + t^6,$

$T_5 = 11200t + 60t^4 + t^{10},$

$T_6 = -6272000 - 3136000t^3 + 78400t^6 + 2800t^9 + 140t^{12} + t^{15}.$

T_m is a monic polynomial of $\deg(T_m) = m(m-1)/2$. The rational function

$$y(t) = \frac{d}{dt}\left(\log \frac{T_{m+1}}{T_m}\right)$$

solves the second Painlevé equation P_{II} with parameter $\alpha = -m$.

We denote by $s_\lambda(t)$ the Schur function in terms of power sum variables $t_m = 1/m \sum x_i^m$,

$$s_\lambda(t) = \sum_{m_i \geq 0} \pi_\lambda(1^{m_1}2^{m_2}\cdots)\frac{t_1^{m_1}}{m_1!}\frac{t_2^{m_2}}{m_2!}\cdots.$$

It is known that the polynomials T_m are certain specialization of the Schur function [3],

Theorem 1. *We have*

$$T_n = c_n s_{(n-1,\dots,2,1)}(t_1, t_2, t_3, \dots),$$

where c_n is a constant and $t_1 = t, t_2 = 0, t_3 = -4/3, t_4 = \dots = 0$.

Example. $s_{(3,2,1)} = \frac{t_1^6}{45} - \frac{t_1^3 t_3}{3} + t_5 - t_3^2 \to \frac{1}{45}(t^6 + 20t^3 - 80)$.

T_m is also a similarity reduction of a special solution of the KdV equation $4u_t = u_{xxx} + 6uu_x$, $u = 2(\log\tau)_{xx}$ such that $\tau(zx, z^3 t) = z^{m(m-1)/2}\tau(x,t)$ and $T_m(x) = c\tau(x, t = -\frac{4}{3})$.

2.2. Okamoto polynomials for P_{IV}

Recurrence relation: $(' = \frac{d}{dx})$

$$Q_{n+1}Q_{n-1} = (x^2 + 2n - 1)Q_n^2 + Q_n'' Q_n - {Q_n'}^2. \tag{2.2}$$

Initial condition: $Q_0 = Q_1 = 1$.

Example.

$$Q_2 = x^2 + 1,$$
$$Q_3 = x^6 + 5x^4 + 5x^2 + 5,$$
$$Q_4 = x^{12} + 14x^{10} + 65x^8 + 140x^6 + 175x^4 + 350x^2 + 175.$$

These are also specialization of the Schur functions. [4,5]

Theorem 2. *We have*

$$Q_n(x) = c_n s_{(2n-2,\dots,6,4,2)}(t_1, t_2, \dots)$$

where c_n is a constant and $t_1 = x$, $t_2 = \frac{1}{2}$, $t_3 = t_4 = \dots = 0$.

Remark. The constant term of Q_m can be obtained by the formula

$$\pi_\lambda(r^k) = \pm\frac{r^k k!}{\prod_{h\equiv 0 (\mathrm{mod}\ r)} h} \quad \text{(or 0)}.$$

2.3. Umemura polynomials for P_V

Recurrence relation: $(' = \frac{d}{dt})$

$$T_{n+1}T_{n-1} = \left(\frac{t}{8} - v + \frac{3}{4}n\right) T_n^2 + T_n' T_n + t\left(T_n'' T_n - {T_n'}^2\right), \tag{2.3}$$

Initial conditions: $T_0 = T_1 = 1$.

The rational function

$$y(t) = -\frac{T_n(t, v + \frac{1}{2})T_{n+1}(t, v + \frac{1}{4})}{T_n(t, v)T_{n+1}(t, v + \frac{3}{4})}$$

solves the equation P_V with parameters $\alpha = 2v^2$, $\beta = -2(v - \frac{n}{2})^2$, $\gamma = n$ and $\delta = -\frac{1}{2}$. The Umemura polynomial T_m is also a specialization of the Schur function as follows, [6]

Theorem 3. *We have*

$$T_{n+1}(t, v) = c_n s_{(n,n-1,\dots,2,1)}(t_1, t_2, t_3, \dots)$$

where c_n is a constant and $t_m = t/2 + (-4v + n + 2)/m$.

Example.

$$\begin{aligned} 2^2 T_2 &= x + l_3, \\ 2^6 T_3 &= x^3 + 3l_4x^2 + 3l_3l_5x + l_3l_4l_5, \\ 2^{12} T_4 &= x^6 + 6l_5x^5 + 15l_4l_6x^4 + (10l_4l_5l_6 + 10l_3l_5l_7)\,x^3 + \\ &\quad 15l_3l_5^2l_7x^2 + 6l_3l_4l_5l_6l_7x + l_3l_4l_5^2l_6l_7, \end{aligned}$$

where $l_k = k - 4v$, $x = t/2$.

The polynomial T_m can be represented as a Wronskian determinant of Laguerre polynomials. It is interesting to note that the polynomial T_m is also a τ-function for discrete P_{II}, if the parameter v is regarded as discrete time.[7]

Theorem 4. *Put $T_n^m = T_{n+1}(t, v = (n + 1 - m)/4)$, then the rational function*

$$y_m = \frac{T_n^{m-1}T_{n+1}^{m-1}}{T_n^m T_{n+1}^m} - 1,$$

solves the second discrete Painlevé equation

$$(dP_{II}) \qquad y_{m+1} + y_{m-1} = \frac{4}{t}\,\frac{(m+1)y_m + (n+1)}{y_m^2 - 1}.$$

2.4. Umemura Polynomials for P_{VI}

Recurrence relation: $(' = \frac{d}{dv})$

(2.4)
$$U_{n-1}U_{n+1} = \{\frac{1}{4}(-2b_1^2 - 2b_2^2 + (b_1^2 - b_2^2)v) + (n - \frac{1}{2})\}U_n^2 + \frac{1}{4}(v^2-4)^2(U_n U_n'' - {U_n'}^2) + \frac{1}{4}(v^2-4)vU_nU_n'.$$

Initial condition: $U_0 = U_1 = 1$.

The following is conjectured in [2] and proved in [8].

Theorem 5.

$$2^{-n(n-1)}U_n = \sum_{I\subset[n-1]} \dim(V_{\lambda_I}^{GL_n}) \prod_{i\in I} c_i \prod_{j\in[n-1]-I} d_j.$$

where $[n-1] = \{1, 2, \ldots, n-1\}$,

$$c_i = \prod_{k=1}^{i}\left((- 4b_1^2 + (2k-1)^2)\frac{2-v}{4}\right),$$

$$d_i = \prod_{k=1}^{i}\left((- 4b_2^2 + (2k-1)^2)\frac{2+v}{4}\right),$$

and $\lambda_I = (I|I)$ *(Frobenius symbol).*

Example.

$$2^2T_2 = c_1 + d_1,$$
$$2^6T_3 = c_1c_2 + 3c_1d_2 + 3d_1c_2 + d_1d_2,$$
$$2^{12}T_4 = c_1c_2c_3 + 6d_1c_2c_3 + 15c_1d_2c_3 + (10d_1d_2c_3 + 10c_1c_2d_3) + 15d_1c_2d_3 + 6c_1d_2d_3 + d_1d_2d_3.$$

Remark. Note that the same coefficients appear in the Umemura polynomials U_m for P_{VI} and T_m for P_V. Such a relation was also observed for polynomials arising in the third Painlevé equation P_{III}.[2,14]

§3. Generalization for the root systems

We shall generalize the story above for other Painlevé type equations with affine Weyl group symmetry besides $A_1^{(1)}$, $A_2^{(1)}$, $A_3^{(1)}$, $C_2^{(1)}$ or $D_4^{(1)}$. To do this, first we generalize the representations of the affine Weyl groups in terms of root system data (Cartan matrix).

3.1. A representation of Weyl groups

Let $A = (a_{ij})_{i,j\in I}$ be a generalized Cartan Matrix such as

$$a_{ii} = 2, \quad a_{ij} \in Z_{\leq 0}, \quad a_{ij} = 0 \leftrightarrow a_{ji} = 0 \ (i \neq j).$$

The corresponding Weyl group $W = W(A)$ is defined as

$$W = \langle s_i(i \in I) \mid s_i^2 = 1, \ (s_i s_j)^{m_{ij}} = 1 \rangle,$$

where $m_{ij} = 2, 3, 4, 6$ or ∞ when $a_{ij}a_{ji} = 0, 1, 2, 3$ or ≥ 4. We introduce additional data $U = (u_{ij})_{i,j\in I}$ such that $u_{ii} = 0$, $u_{ij} = 0 \leftrightarrow a_{ij} = 0$ and $u_{ij}a_{ij} = -u_{ji}a_{ji}$, $(i \neq j)$. Let $R = C(\alpha_i \ ; f_i \ ; \ i \in I)$ be the field of rational functions. Then we have [9]

Theorem 6. *There is a representation of W on R such that*

$$s_i(\alpha_j) = \alpha_j - \alpha_i a_{ij}, \quad s_i(f_j) = f_j + \frac{\alpha_i}{f_i} u_{ij}.$$

This representation has the following applications

(1) Bäcklund transformations of known Painlevé equations.
(2) Discrete integrable dynamical systems
(3) Generalized Painlevé equations for root systems.

We shall explain these in the next subsections.

3.2. Symmetric Form

For the Painlevé equations P_{IV}, P_V and P_{VI}, the Bäcklund transformations take the universal form as above by suitable choice of dependent variables.

Example. Symmetric form of P_{IV} [4]

$$\begin{aligned} f_0' &= f_0(f_1 - f_2) + \alpha_0, \\ f_1' &= f_1(f_2 - f_0) + \alpha_1, \\ f_2' &= f_2(f_0 - f_1) + \alpha_2. \end{aligned} \tag{3.1}$$

3.3. Discrete integrable systems

(Extended) affine Weyl group is a semi-direct product of Lattice M and finite Weyl group W_0. Let $\{T_i\}$ be the generators of M. In the above representation, these are non trivially commuting bi-rational mapping.[9]

Example. $A = A_3^{(1)}$. Put $T_1 = \pi s_3 s_2 s_1$, then

$$\begin{aligned} T_1(f_2) &= \pi s_3 s_2 s_1(f_2) = \pi s_3 s_2(f_2 + \frac{\alpha_1}{f_1}) \\ &= \pi s_3(f_2 + \frac{\alpha_1 + \alpha_2}{f_1 - \frac{\alpha_2}{f_2}}) = \pi(f_2 - \frac{\alpha_3}{f_3} + \frac{\alpha_1 + \alpha_2 + \alpha_3}{f_1 - \frac{\alpha_2 + \alpha_3}{f_2 - \frac{\alpha_3}{f_3}}}) \\ &= f_3 - \frac{\alpha_0}{f_0} + \frac{\alpha_2 + \alpha_3 + \alpha_0}{f_2 - \frac{\alpha_3 + \alpha_0}{f_3 - \frac{\alpha_0}{f_0}}}. \end{aligned}$$

The commuting family of rational mappings are considered as discrete analogue of the Painlevé equations (see section 4).

3.4. Higher Painlevé equations

Next problem is to find differential equations for which the Weyl groups act as the Bäcklund transformation.

There exist a series of such equations for $A_l^{(1)}$. These series contain P_{IV} (for $l = 2$) and P_V (for $l = 3$) as simplest examples and expected to have the Painlevé property.

Case $l = 2n$:

$$\frac{df_j}{dt} = f_j \sum_{1 \le r \le n} (f_{j+2r-1} - f_{j+2r}) + \alpha_j. \tag{3.2}$$

Case $l = 2n + 1$:

$$\begin{aligned} \frac{df_j}{dt} = f_j &\sum_{1 \le r \le s \le n} (f_{j+2r-1} f_{j+2s} - f_{j+2r} f_{j+2s+1}) \\ &+ (\frac{k}{2} - \sum_{1 \le r \le n} \alpha_{j+2r}) f_j + \alpha_j (\sum_{1 \le r \le n} f_{j+2r}). \end{aligned} \tag{3.3}$$

For each case, $f_j = f_j(t)$ are the unknown functions and α_j are constants such as $\alpha_0 + \cdots + \alpha_l = k$ $(0 \le j \le l)$.

Remark.

(1) These equations can be obtained as a continuum limit of the discrete system previously discussed.
(2) These equations admit a hamiltonian formulation.[10]
(3) These systems also have a Lax formalism and can be considered as a similarity reduction of (modified) KP equations.

§4. Special polynomials arising from the representation

In $A_l^{(1)}$ case, for any $w \in W$, $w(f_i)$ is always factorized into four polynomials as

$$w(f_i) = \frac{PQ}{RS}.$$

Example. For $A_3^{(1)}$, we see

$$T_1(f_2) = \frac{(f_3f_0 - \alpha_0)(f_2f_3f_0 + \alpha_2f_0 - \alpha_0f_2)}{f_0(f_2f_3f_0 - \alpha_0f_2 - (\alpha_3 + \alpha_0)f_0)}.$$

These polynomials can be interpreted as the τ-function of our discrete system. Their polynomiality is closely related with the singularity confinement property, which is a discrete analog of the Painlevé property.

Remark. The following property of the difference system is called singularity confinement: "Any singularity depending on initial data will disappear after finite iteration of mapping and the initial data can be recovered after such iteration". [11].

Theorem 7. *The representation of Weyl groups can be extended to $C(\alpha_i\ ; f_i\ ; \tau_i\ ; i \in I)$ in such a way that*

$$s_i(\tau_i) = \frac{f_i}{\tau_i}\prod_{k \neq i}\tau_k^{|a_{ki}|}, \quad s_i(\tau_j) = \tau_j\ (i \neq j).$$

For any $w \in W$, $w(\tau_i)$ is factorized as $w(\tau_i) = \phi_{i,w}\prod_{j\in I}{\tau_j}^{m_j}$, where $m_j = \langle \alpha\check{}_j, w(\Lambda_i)\rangle \in Z$ and $\phi_{i,w} \in C(\alpha_i\ ; f_i\ ; i \in I)$. We observe that [9]

Conjecture. *$\phi_{i,w}$ is a polynomial.*

We have a proof of this conjecture in $A_l^{(1)}$ case, by using explicit determinant formulas.[12]

Theorem 8. *For any $w \in W$, the polynomial $\phi_{0,w}$ is given by the following determinant of Jacobi-Trudi type*

$$\phi_{0,w} = \frac{1}{N_w}\det\left[\pi^{1-j}(h_{\lambda_i - i + j})\right]_{1\leq i,j\leq l(\lambda)}.$$

Where $N_w \in Z[\alpha_i; i \in I]$, $h_j \in Z[\alpha_i, f_i; i \in I]$ and λ is a partition determined by $w \in W$.

Note that the determinant structure of polynomial ϕ is the same as the 9th variation of the Schur function.[13]

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A Duality of a Twisted Group Algebra of the Hyperoctahedral Group and the Queer Lie Superalgebra

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§1. Introduction

We establish a duality relation (Theorem 4.2) between one of the twisted group algebras of the hyperoctahedral group H_k (or the Weyl group of type B_k) and a Lie superalgebra $\mathfrak{q}(n_0) \oplus \mathfrak{q}(n_1)$ for any integers $k \geq 4$ and n_0, $n_1 \geq 1$. Here $\mathfrak{q}(n_0)$ and $\mathfrak{q}(n_1)$ denote the "queer" Lie superalgebras as called by some authors. The twisted group algebra $\mathcal{B}'_k$ in focus in this paper belongs to a different cocycle from the one $\mathcal{B}_k$ used by A. N. Sergeev in his work [8] on a duality with $\mathfrak{q}(n)$ and by the present author in a previous work [11]. This $\mathcal{B}'_k$ contains the twisted group algebra $\mathcal{A}_k$ of the symmetric group $\mathfrak{S}_k$ in a straightforward manner (cf. §1. 1. 1), and has a structure similar to the semidirect product of $\mathcal{A}_k$ and $\mathbb{C}[(\mathbb{Z}/2\mathbb{Z})^k]$. ($\mathcal{B}'_k$ and $\mathcal{B}_k$ were denoted by $\mathbb{C}^{[-1,+1,+1]}W_k$ and $\mathbb{C}^{[+1,+1,-1]}W_k$ respectively by J. R. Stembridge in [10].)

In §2, we construct the $\mathbb{Z}_2$-graded simple $\mathcal{B}'_k$-modules (where $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$) using an analogue of the little group method. These simple $\mathcal{B}'_k$-modules are slightly different from the non-graded simple $\mathcal{B}'_k$-modules constructed by Stembridge in [10] because of the difference between $\mathbb{Z}_2$-graded and non-graded theories, but they can easily be translated into each other. We will use the algebra $\mathcal{C}_k \dot{\otimes} \mathcal{B}'_k$, where $\mathcal{C}_k$ is the 2^k-dimensional Clifford algebra (cf. (3.2)) and $\dot{\otimes}$ denotes the $\mathbb{Z}_2$-graded tensor product (cf. [1], [2], [11, §1]), as an intermediary for establishing our duality, as we explain below. The construction of the simple $\mathcal{B}'_k$-modules leads to a construction of the simple $\mathcal{C}_k \dot{\otimes} \mathcal{B}'_k$-modules in §3.

In §4, we define a representation of $\mathcal{C}_k \dot{\otimes} \mathcal{B}'_k$ in the k-fold tensor product $W = V^{\otimes k}$ of $V = \mathbb{C}^{n_0+n_1} \oplus \mathbb{C}^{n_0+n_1}$, the space of the natural representation of the Lie superalgebra $\mathfrak{q}(n_0 + n_1)$. This representation

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of $\mathcal{C}_k \dot{\otimes} \mathcal{B}'_k$ depends on n_0 and n_1, not just $n_0 + n_1$. Note that $\mathcal{B}_k$ can be regarded as a subalgebra of $\mathcal{C}_k \dot{\otimes} \mathcal{B}'_k$, since $\mathcal{B}_k$ is isomorphic to $\mathcal{C}_k \dot{\otimes} \mathcal{A}_k$ by our previous result (cf. (3.3) of [11]). Under this embedding, our representation of $\mathcal{C}_k \dot{\otimes} \mathcal{B}'_k$ restricts to the representation of $\mathcal{B}_k$ in W defined by Sergeev (cf. Theorem A). We show that the centralizer of $\mathcal{C}_k \dot{\otimes} \mathcal{B}'_k$ in $\mathrm{End}(W)$ is generated by the action of the Lie superalgebra $\mathfrak{q}(n_0) \oplus \mathfrak{q}(n_1)$ (Theorem 4.1). Moreover we show that $\mathcal{B}'_k$ and $\mathfrak{q}(n_0) \oplus \mathfrak{q}(n_1)$ act on a subspace W^ε of W "as mutual centralizers of each other" (Theorem 4.2). Note that $\mathcal{A}_k$ and $\mathfrak{q}(n)$ act on the same space W^ε "as mutual centralizers of each other" (cf. Theorem B).

In Appendix, we include short explanations of some known results, which we use in the previous sections.

In this paper, all vector spaces, and associative algebras, and representations are assumed to be finite dimensional over $\mathbb{C}$ unless specified otherwise. The precise statements of the results sketched in the introduction use the formulation of $\mathbb{Z}_2$-graded representations of $\mathbb{Z}_2$-graded algebras (superalgebras) (cf. §1.1.3) as was used in [1] and [2].

1.1. Preliminaries

1.1.1. A twisted group algebra $\mathcal{B}'_k$. For any $k \geq 1$, let $\mathcal{B}'_k$ denote the associative algebra generated by τ' and the elements γ_i, $1 \leq i \leq k-1$, with relations

(1.1)
$$\begin{aligned}
&\tau'^2 = 1, \gamma_i^2 = -1 \quad (-1 \leq i \leq k-1), \quad (\gamma_i\gamma_{i+1})^3 = -1 \quad (1 \leq i \leq k-2),\\
&(\gamma_i\gamma_j)^2 = -1 \quad (|i-j| \geq 2), \quad (\tau'\gamma_i)^2 = 1 \quad (2 \leq i \leq k-1),\\
&(\tau'\gamma_1)^4 = 1.
\end{aligned}$$

If $k \geq 4$, then $\mathcal{B}'_k$ is isomorphic to a twisted group algebra of the hyperoctahedral group H_k with a non-trivial 2-cocycle (cf. [10, Prop. 1.1]). We regard $\mathcal{B}'_k$ as a $\mathbb{Z}_2$-graded algebra by giving the generator τ' (resp. the generator γ_i, $1 \leq i \leq k-1$) degree 0 (resp. degree 1). Note that this grading of $\mathcal{B}'_k$ is different from that of $\mathcal{B}_k$ in (3.1) or in [11].

Let $\mathcal{A}_k$ denote the $\mathbb{Z}_2$-graded subalgebra of $\mathcal{B}'_k$ generated by γ_i, $1 \leq i \leq k-1$. If $k \geq 4$, then $\mathcal{A}_k$ is isomorphic to a twisted group algebra of the symmetric group $\mathfrak{S}_k$ with a non-trivial 2-cocycle, with the $\mathbb{Z}_2$-grading as in [2] and [11].

1.1.2. Partitions and symmetric functions. Let P_k denote the set of all partitions of k, and put $P = \coprod_{k \geq 0} P_k$. For $\lambda \in P$, we write $l(\lambda)$ for the length of λ, namely the number of non-zero parts of λ.

Also we write $|\lambda| = k$ if $\lambda \in P_k$. Let DP_k and OP_k denote the distinct partitions (or strict partitions, namely partitions whose parts are distinct) and the odd partitions (namely partitions whose parts are all odd) of k respectively. Let DP_k^+ and DP_k^- be the sets of all $\lambda \in DP_k$ such that $(-1)^{k-l(\lambda)} = +1$ and -1 respectively. Note that $(-1)^{k-l(\lambda)}$ equals the signature of permutations with cycle type λ. We also put $DP = \coprod_{k\geq 0} DP_k$ and $OP = \coprod_{k\geq 0} OP_k$. Let $(DP^2)_k$ (resp. $(OP^2)_k$) denote the set of all $(\lambda, \mu) \in DP^2$ (resp. OP^2) such that $|\lambda| + |\mu| = k$. Let $(DP^2)_k^+$ and $(DP^2)_k^-$ be the sets of all $(\lambda, \mu) \in (DP^2)_k$ such that $(-1)^{k-l(\lambda)-l(\mu)} = +1$ and -1 respectively.

Let Λ_x denote the ring of the symmetric functions in the variables $x = \{x_1, x_2, \dots\}$ with coefficients in $\mathbb{C}$; namely our Λ_x is the scalar extension of the Λ_x in [6], which is $\mathbb{Z}$-algebra, to $\mathbb{C}$.

Let Ω_x denote the subring of Λ_x generated by the power sums of odd degrees, namely the $p_r(x)$, $r = 1, 3, 5, \dots$. Then $\{p_\mu(x) \mid \mu \in OP\}$ is a basis of Ω_x, where $p_\mu = \prod_{i\geq 1} p_{\mu_i}$. For $\lambda \in DP$, let $Q_\lambda(x) \in \Lambda_x$ denote Schur's Q-function indexed by λ (cf. [7], [9, §6]). Then $\{Q_\lambda(x) \mid \lambda \in DP\}$ is also a basis of Ω_x.

1.1.3. Semisimple superalgebras. This theory of semisimple superalgebras was developed by T. Józefiak in [1], which we mostly follow. A $\mathbb{Z}_2$-graded algebra A, which is called a **superalgebra** in this paper, is called **simple** if it does not have non-trivial $\mathbb{Z}_2$-graded two-sided ideals. If A is a simple superalgebra, then it is either isomorphic to $M(m, n)$ (denoted by $M(m|n)$ in [2]) for some m and n, or isomorphic to $Q(n)$ for some n (see [2], [11, §1] for the definitions of simple superalgebras $M(m, n)$, $Q(n)$).

Let V be an A-**module**, namely a $\mathbb{Z}_2$-graded vector space $V = V_0 \oplus V_1$ together with a representation ρ: $A \to \mathrm{End}(V)$ satisfying $\rho(A_\alpha)V_\beta \subset V_{\alpha+\beta}$ (α, $\beta \in \mathbb{Z}_2$). We simply write $\rho(a)v = av$ for $a \in A$ and $v \in V$. By an A-submodule of V we mean a $\mathbb{Z}_2$-graded $\rho(A)$-stable subspace of V. We say that V is **simple** if it does not have non-trivial A-submodules.

Let V and W be two A-modules. Let $\mathrm{Hom}_A^\alpha(V, W)$ ($\alpha \in \mathbb{Z}_2$) denote the subspace of $\mathrm{Hom}^\alpha(V, W) = \{f \in \mathrm{Hom}(V, W)\,;\, f(V_\beta) \subset W_{\alpha+\beta}\}$ consisting of all elements $f \in \mathrm{Hom}^\alpha(V, W)$ such that $f(av) = (-1)^{\alpha\cdot\beta} af(v)$ for $a \in A_\beta$ ($\beta \in \mathbb{Z}_2$), $v \in V$. Put $\mathrm{Hom}_A^{\cdot}(V, W) = \mathrm{Hom}_A^0(V, W) \oplus \mathrm{Hom}_A^1(V, W)$ and put $\mathrm{End}_A^{\cdot}(V) = \mathrm{Hom}_A^{\cdot}(V, V)$. We call $\mathrm{End}_A^{\cdot}(V)$ the **supercentralizer** of A in $\mathrm{End}(V)$. Two A-modules V and W are called **isomorphic** if there exists an invertible linear map $f \in \mathrm{Hom}_A^{\cdot}(V, W)$. If this is the case, we write $V \cong_A W$ (or simply write $V \cong W$). If V and W are simple A-modules, then $V \cong W$ if and only if there exists an invertible element in $\mathrm{Hom}_A^0(V,W)$ or $\mathrm{Hom}_A^1(V,W)$. Note that, in [11] we distinguished between V and the shift of V which is defined to be the same vector space as V with the switched grading. In this paper, however, we identify V and the shift of V.

If V is a simple A-module, then $\mathrm{End}_A^{\cdot}(V)$ is isomorphic to either $M(1,0) \cong \mathbb{C}$ or $Q(1) \cong \mathcal{C}_1$ (cf. [1, Prop. 2.17], [2, Prop. 2.5, Cor. 2.6]). In the former (resp. latter) case, we say that V is of **type** M (resp. of **type** Q). This gives the following theorem (see [1], [2], [11, §1] for the definition of the "supertensor product" of the superalgebras or modules).

Theorem 1.1. *Let $C = A \dot{\otimes} B$ be the supertensor product of superalgebras A and B and let $V = U \otimes W$ be the supertensor product of a simple A-module U and a simple B-module W.*

(a) *If U, W are of type M, then V is a simple C-module of type M.*
(b) *If one of U and W is of type M and the other is of type Q, then V is a simple C-module of type Q.*
(c) *If U and W are of type Q, then V is a sum of two copies of a simple C-module X of type M: $V = X \oplus X$.*

Moreover, the above construction gives all simple $A \dot{\otimes} B$-modules.

Using the above U, W, V and X, define an $A \dot{\otimes} B$-module $U \dot{\circ} W$ by

$$U \dot{\circ} W = \begin{cases} V & \text{if } U \text{ or } W \text{ is of type } M, \\ X & \text{if } U \text{ and } W \text{ are of type } Q. \end{cases} \tag{1.2}$$

Let $\mathrm{Irr}\, A$ denote the set of all isomorphism classes of simple A-modules for any superalgebra A.

Corollary 1.2. *We have a bijection*

$$\dot{\circ}\colon \mathrm{Irr}\, A \times \mathrm{Irr}\, B \ni (U,W) \overset{\sim}{\mapsto} U \dot{\circ} W \in \mathrm{Irr}\, A \dot{\otimes} B.$$

§2. Simple modules for $\mathcal{B}_k'$

The simple $\mathcal{A}_k$-modules are parametrized by DP_k (cf. [2], [7], [9]). For $\lambda \in DP_k$, let V_λ denote a simple $\mathcal{A}_k$-module indexed by λ. Then V_λ is of type M (resp. of type Q) if $\lambda \in DP_k^+$ (resp. $\lambda \in DP_k^-$). We construct a $\mathcal{B}_k'$-module $V_{\lambda,\mu}$ for $(\lambda,\mu) \in (DP^2)_k$ as follows. Define a surjective homomorphism of superalgebras $\pi_k\colon \mathcal{B}_k' \to \mathcal{A}_k$ (resp. $\pi_k'\colon \mathcal{B}_k' \to \mathcal{A}_k$) by $\pi_k(\tau') = 1$, $\pi_k|_{\mathcal{A}_k} = \mathrm{id}_{\mathcal{A}_k}$ (resp.$\pi_k'(\tau') = -1$, $\pi_k'|_{\mathcal{A}_k} = \mathrm{id}_{\mathcal{A}_k}$). The simple $\mathcal{A}_{k'}$ (resp. $\mathcal{A}_{k-k'}$)-module V_λ (resp. V_μ) can be lifted to a $\mathcal{B}_{k'}'$ (resp.

$\mathcal{B}'_{k-k'}$)-module via $\pi_{k'}$ (resp. $\pi'_{k-k'}$), where $k' = |\lambda|$. This (simple) $\mathcal{B}'_{k'}$ (resp. $\mathcal{B}'_{k-k'}$)-module is denoted by $V_{\lambda,\phi}$ (resp. $V_{\phi,\mu}$). Let $V_{\lambda,\mu}$ denote the $\mathcal{B}'_k$-module induced from the $\mathcal{B}'_{k'} \dot{\otimes} \mathcal{B}'_{k-k'}$-module $V_{\lambda,\phi} \dot{\circ} V_{\phi,\mu}$, namely

$$V_{\lambda,\mu} = \mathcal{B}'_k \otimes_{\mathcal{B}'_{k'} \dot{\otimes} \mathcal{B}'_{k-k'}} (V_{\lambda,\phi} \dot{\circ} V_{\phi,\mu})$$

(see the definition of $\dot{\circ}$ in (1.2)), where $\mathcal{B}'_{k'} \dot{\otimes} \mathcal{B}'_{k-k'}$ is embedded into $\mathcal{B}'_k$ via

$$\begin{aligned} &\tau' \dot{\otimes} 1 \mapsto \tau', \quad \gamma_i \dot{\otimes} 1 \mapsto \gamma_i \quad (1 \le i \le k'-1), \\ &1 \dot{\otimes} \tau' \mapsto \tau'_{k'+1}, \quad 1 \dot{\otimes} \gamma_j \mapsto \gamma_{k'+j} \quad (1 \le j \le k-k'-1) \end{aligned}$$

where $\tau'_i = \gamma_{i-1}\gamma_{i-2}\cdots\gamma_1\tau'\gamma_1\cdots\gamma_{i-2}\gamma_{i-1}$, $1 \le i \le k$.

Theorem 2.1. (cf. [10, Th. 7.1]) *$\{V_{\lambda,\mu} \mid (\lambda,\mu) \in (DP^2)_k\}$ is a complete set of the isomorphism classes of simple $\mathcal{B}'_k$-modules. $V_{\lambda,\mu}$ is of type M (resp. of type Q) if $(\lambda,\mu) \in (DP^2)^+_k$ (resp. $(\lambda,\mu) \in (DP^2)^-_k$)*

The proof is analogous to the little group method, and is omitted. It can also be shown that this parametrization coincides with that by Stembridge in [10, Th. 7.1] modulo the usual difference between $\mathbb{Z}_2$-graded and non-graded modules.

If $(\lambda,\mu) \in (DP^2)^-_k$, then fix a non-zero homogeneous element $x_{\lambda,\mu}$ of $\mathrm{End}_{\mathcal{B}'_k}(V_{\lambda,\mu}) \cong Q(1)$ of degree 1.

§3. The algebras $\mathcal{B}_k$ and $\mathcal{C}_k \dot{\otimes} \mathcal{B}'_k$

For any $k \ge 1$, let $\mathcal{B}_k$ denote the associative algebra generated by τ and the elements σ_i, $1 \le i \le k-1$, with relations

(3.1)
$$\begin{aligned} &\tau^2 = \sigma_i^2 = 1 \quad (1 \le i \le k-1), \quad (\sigma_i\sigma_{i+1})^3 = 1 \quad (1 \le i \le k-2), \\ &(\sigma_i\sigma_j)^2 = 1 \quad (|i-j| \ge 2), \quad (\tau\sigma_i)^2 = 1 \quad (2 \le i \le k-1), \\ &(\tau\sigma_1)^4 = -1. \end{aligned}$$

We regard $\mathcal{B}_k$ as a superalgebra by giving the generator τ' (resp. the generator σ_i, $1 \le i \le k-1$) degree 1 (resp. degree 0). The subgroup of $(\mathcal{B}_k)^\times$ generated by σ_i, $1 \le i \le k-1$, is isomorphic to the symmetric group of degree k and it is denoted by $\mathfrak{S}_k$.

Let $\mathcal{C}_k$ denote the 2^k-dimensional Clifford algebra, namely $\mathcal{C}_k$ is generated by $\xi_1, \ldots, \xi_k$ with relations

$$\xi_i^2 = 1, \quad \xi_i\xi_j = -\xi_j\xi_i \quad (i \ne j) \ . \tag{3.2}$$

We regard $\mathcal{C}_k$ as a superalgebra by giving the generator ξ_i, $1 \le i \le k$, degree 1. $\mathcal{C}_k$ is a simple superalgebra. Let X_k be a unique simple $\mathcal{C}_k$-module. If k is even (resp. odd), then X_k is of type M (resp. of type Q). If k is odd, then fix a non-zero element z_k of $\mathrm{End}^1_{\mathcal{C}_k}(X_k)$.

Define a linear map $\vartheta\colon \mathcal{B}_k \to \mathcal{C}_k \dot{\otimes} \mathcal{A}_k$ by

$$\begin{aligned} (3.3) \qquad & \vartheta(\tau_i) \mapsto \xi_i \otimes 1 \qquad (1 \le i \le k), \\ & \vartheta(\sigma_j) \mapsto \frac{1}{\sqrt{2}}(\xi_j - \xi_{j+1}) \otimes \gamma_j \ \ (1 \le j \le k-1) \end{aligned}$$

where $\tau_i = \sigma_{i-1} \cdots \sigma_1 \tau \sigma_1 \ldots \sigma_{i-1}$. Then ϑ is an isomorphism of algebras (cf. [11, Th. 3.2]). For $\lambda \in DP_k$, define a $\mathcal{B}_k$-module W_λ by $W_\lambda = X_k \circ V_\lambda$. By Corollary 1.2, $\{W_\lambda \mid \lambda \in DP_k\}$ is a complete set of isomorphism classes of simple $\mathcal{B}_k$-modules.

Let $\hat{\mathcal{B}}_k$ denote the supertensor product (cf. [1], [2], [11, §1]) of the algebras $\mathcal{C}_k$ and $\mathcal{B}'_k$, namely $\hat{\mathcal{B}}_k = \mathcal{C}_k \dot{\otimes} \mathcal{B}'_k$. Since $\mathcal{B}_k \cong \mathcal{C}_k \dot{\otimes} \mathcal{A}_k$, $\mathcal{B}_k$ can be regarded as a subalgebra of $\hat{\mathcal{B}}_k$. For $(\lambda, \mu) \in (DP^2)_k$, put $W_{\lambda,\mu} = X_k \circ V_{\lambda,\mu}$. By Theorem 1.1 and (1.2), $W_{\lambda,\mu}$ is of type M (resp. of type Q) if $l(\lambda) + l(\mu)$ is even (resp. odd). By Corollary 1.2, $\{W_{\lambda,\mu} \mid (\lambda,\mu) \in (DP^2)_k\}$ is a complete set of isomorphism classes of simple $\hat{\mathcal{B}}_k$-modules.

§4. A duality of $\mathcal{B}'_k$ and $\mathfrak{q}(n_0) \oplus \mathfrak{q}(n_1)$

Let $\mathfrak{q}(n)$ denote the Lie subsuperalgebra of $\mathfrak{gl}(n,n)$ (denoted by $l(n,n)$ in [5]) consisting of the matrices of the form $\begin{pmatrix} A & B \\ B & A \end{pmatrix}$. The Jacobi product $[\ ,\ ]\colon \mathfrak{q}(n) \times \mathfrak{q}(n) \to \mathfrak{q}(n)$ is defined by $[X,Y] = XY - (-1)^{\overline{X}\cdot\overline{Y}} YX$ for all homogeneous elements X, $Y \in \mathfrak{q}(n)$, where the symbol $^{-}$ expresses the degree of a homogeneous element. This Lie superalgebra is called the queer Lie superalgebra. Let $\mathcal{U}_n = \mathcal{U}(\mathfrak{q}(n))$ denote the universal enveloping algebra of $\mathfrak{q}(n)$, which can be regarded as a superalgebra. Let W denote the k-fold supertensor product of the $2n$-dimensional natural representation $V = V_0 \oplus V_1$, $\mathbf{dim}\, V = (n,n)$, namely $W = V^{\otimes k}$, where $\mathbf{dim}\, V$ denotes the pair $(\dim V_0, \dim V_1)$. We define a representation $\Theta\colon \mathcal{U}_n \to \mathrm{End}(W)$ by

$$\Theta(X)(v_1 \otimes \cdots \otimes v_k) = \sum_{j=1}^{k} (-1)^{\overline{X}\cdot(\overline{v_1}+\cdots+\overline{v_{j-1}})} v_1 \otimes \cdots \otimes \overset{j}{\overbrace{Xv_j}} \otimes \cdots \otimes v_k$$

for all homogeneous elements $X \in \mathfrak{q}(n)$ and $v_i \in V$ $(1 \le i \le k)$. Note that $\mathcal{U}_n$ is an infinite dimensional superalgebra. However, for a fixed number k, $\mathcal{U}_n$ acts on W through its finite dimensional image in $\mathrm{End}(W)$. Therefore we can use the results in §1.1.3 on finite dimensional superalgebras and their finite dimensional modules.

Let n_0 and n_1 be two positive integers such that $n_0 + n_1 = n$. The Lie superalgebra $\mathfrak{q}(n_0) \oplus \mathfrak{q}(n_1)$ can be embedded into $\mathfrak{q}(n)$ via

(4.1)

$$\mathfrak{q}(n_0) \oplus \mathfrak{q}(n_1) \ni \left(\begin{pmatrix} A & B \\ B & A \end{pmatrix}, \begin{pmatrix} C & D \\ D & C \end{pmatrix} \right) \mapsto \begin{pmatrix} A & 0 & B & 0 \\ 0 & C & 0 & D \\ B & 0 & A & 0 \\ 0 & D & 0 & C \end{pmatrix} \in \mathfrak{q}(n).$$

The universal enveloping algebra of $\mathfrak{q}(n_0) \oplus \mathfrak{q}(n_1)$ is isomorphic to $\mathcal{U}_{n_0} \dot{\otimes} \mathcal{U}_{n_1}$ which can be embedded into $\mathcal{U}_n$ as a subalgebra generated by the elements of $\mathfrak{q}(n_0) \oplus \mathfrak{q}(n_1)$.

Now we define a representation $\Psi\colon \hat{\mathcal{B}}_k \to \mathrm{End}(W)$, which depends on n_0 and n_1, by

(4.2)

$$\begin{aligned}
&\Psi(\xi_i \otimes 1)(v_1 \otimes \cdots \otimes v_k) = (-1)^{\overline{v_1} + \cdots + \overline{v_{i-1}}} v_1 \otimes \cdots \otimes P v_i \otimes \cdots \otimes v_k \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (1 \le i \le k), \\
&\Psi(1 \otimes \tau')(v_1 \otimes \cdots \otimes v_k) = (Q v_1) \otimes v_2 \otimes \cdots \otimes v_k, \\
&\Psi(1 \otimes \gamma_j)(v_1 \otimes \cdots \otimes v_k) \\
&= \frac{(-1)^{\overline{v_1} + \cdots + \overline{v_{j-1}}}}{\sqrt{2}} v_1 \otimes \cdots \otimes (P v_j) \otimes v_{j+1} \otimes \cdots \otimes v_k \\
&\qquad - \frac{(-1)^{\overline{v_1} + \cdots + \overline{v_{j-1}} + \overline{v_j}}}{\sqrt{2}} v_1 \otimes \cdots \otimes v_j \otimes (P v_{j+1}) \otimes \cdots \otimes v_k \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (1 \le j \le k-1)
\end{aligned}$$

for all homogeneous elements $v_j \in V$, $1 \le j \le k$, where

$$P = \begin{pmatrix} 0 & -\sqrt{-1} I_n \\ \sqrt{-1} I_n & 0 \end{pmatrix} \in M(n,n)_1,$$

$$Q = \begin{pmatrix} I_{n_0} & 0 & 0 & 0 \\ 0 & -I_{n_1} & 0 & 0 \\ 0 & 0 & I_{n_0} & 0 \\ 0 & 0 & 0 & -I_{n_1} \end{pmatrix} \in Q(n)_0.$$

Note that, by the isomorphism $\vartheta\colon \mathcal{B}_k \cong \mathcal{C}_k \dot{\otimes} \mathcal{A}_k \subset \hat{\mathcal{B}}_k$, W can be regarded as a $\mathcal{B}_k$-module and this $\mathcal{B}_k$-module was investigated by Sergeev in [8] (cf. Theorem A). Then, observing the actions of $\vartheta(\tau)$, $\vartheta(\sigma_i) \in$

$\mathcal{C}_k \dot{\otimes} \mathcal{A}_k$, $1 \leq i \leq k-1$, on W, we have

(4.3)
$$\Psi\left(\vartheta(\tau)\right)(v_1 \otimes \cdots \otimes v_k) = (Pv_1) \otimes \cdots \otimes v_k,$$
$$\Psi\left(\vartheta(\sigma_i)\right)(v_1 \otimes \cdots \otimes v_k) = (-1)^{\overline{v_i}\cdot\overline{v_{i+1}}} v_1 \otimes \cdots \otimes v_{i+1} \otimes v_i \otimes \cdots \otimes v_k$$

for all homogeneous elements $v_j \in V$, $1 \leq j \leq k-1$.

Let W' be a $\mathcal{U}_{n_0} \dot{\otimes} \mathcal{U}_{n_1}$-submodule of W. Since $(\mathfrak{q}(n_0) \oplus \mathfrak{q}(n_1))_0 \cong \mathfrak{gl}(n_0, \mathbb{C}) \oplus \mathfrak{gl}(n_1, \mathbb{C})$ as a Lie algebra, and V is a sum of two copies (V_0 and V_1) of the natural representation of $\mathfrak{gl}(n_0, \mathbb{C}) \oplus \mathfrak{gl}(n_1, \mathbb{C})$, $W'|_{(\mathfrak{q}(n_0)\oplus\mathfrak{q}(n_1))_0}$ is embedded into a sum of tensor powers of the natural representation, so that this representation of $\mathfrak{gl}(n_0, \mathbb{C}) \oplus \mathfrak{gl}(n_1, \mathbb{C})$ can be integrated to a polynomial representation $\theta_{W'}$ of $GL(n_0, \mathbb{C}) \times GL(n_1, \mathbb{C})$. Let $\mathrm{Ch}[W']$ denote the character of $\theta_{W'}$, namely

$$\begin{aligned}&\mathrm{Ch}[W'](x_1, x_2, \ldots, x_{n_0}, y_1, y_2, \cdots, y_{n_1})\\ &= \mathrm{tr}\, \theta_{W'}(\mathrm{diag}(x_1, x_2, \ldots, x_{n_0}), \mathrm{diag}(y_1, y_2, \ldots, y_{n_1})).\end{aligned}$$

The following theorem determines the supercentralizer of $\Psi(\hat{\mathcal{B}}_k)$ in $\mathrm{End}(W)$ and describes the characters of simple $\mathcal{U}_{n_0} \dot{\otimes} \mathcal{U}_{n_1}$-modules appearing in W.

Theorem 4.1. (1) *The two superalgebras $\Psi(\hat{\mathcal{B}}_k)$ and $\mathcal{U}_{n_0} \dot{\otimes} \mathcal{U}_{n_1}$ act on W as the mutual supercentralizers of each other*:

$$\text{(4.4)}\quad \mathrm{End}^{\cdot}_{\Theta(\mathcal{U}_{n_0}\dot{\otimes}\mathcal{U}_{n_1})}(W) = \Psi(\hat{\mathcal{B}}_k), \quad \mathrm{End}^{\cdot}_{\Psi(\hat{\mathcal{B}}_k)}(W) = \Theta(\mathcal{U}_{n_0} \dot{\otimes} \mathcal{U}_{n_1}).$$

(2) *The simple $\hat{\mathcal{B}}_k$-module $W_{\lambda,\mu}$ $((\lambda, \mu) \in (DP^2)_k)$ occurs in W if and only if $l(\lambda) \leq n_0$ and $l(\mu) \leq n_1$. Moreover we have*

$$\text{(4.5)}\qquad W \cong_{\hat{\mathcal{B}}_k \dot{\otimes} (\mathcal{U}_{n_0} \dot{\otimes} \mathcal{U}_{n_1})} \bigoplus_{\substack{(\lambda,\mu)\in(DP^2)_k \\ l(\lambda)\leq n_0, l(\mu)\leq n_1}} W_{\lambda,\mu} \dot{\circ} U_{\lambda,\mu}$$

where $U_{\lambda,\mu}$ denotes a simple $\mathcal{U}_{n_0} \dot{\otimes} \mathcal{U}_{n_1}$-module.

(3) *We have $U_{\lambda,\mu} \cong_{\mathcal{U}_{n_0}\dot{\otimes}\mathcal{U}_{n_1}} U_\lambda \dot{\circ} U_\mu$, where U_λ (resp. U_μ) denotes the simple $\mathcal{U}_{n_0}$ (resp. $\mathcal{U}_{n_1}$)-module corresponding to the simple $\mathcal{B}_{|\lambda|}$ (resp. $\mathcal{B}_{|\mu|}$)-module W_λ (resp. W_μ) in Sergeev's duality (cf. Theorem A).*

(4) *The character values of* $\mathrm{Ch}[U_{\lambda,\mu}]$ *are given as follows*:

(4.6)
$$\begin{aligned}&\mathrm{Ch}[U_{\lambda,\mu}](x_1, x_2, \ldots, x_{n_0}, y_1, y_2, \ldots, y_{n_1})\\ &\quad = (\sqrt{2})^{d(\lambda,\mu)-l(\lambda)-l(\mu)} Q_\lambda(x_1, x_2, \ldots, x_{n_0}) Q_\mu(y_1, y_2, \ldots, y_{n_1})\end{aligned}$$

where $d\colon (DP^2)_k \to \mathbb{Z}_2$ *denotes a map defined by* $d(\lambda,\mu) = 0$ *(resp.* $d(\lambda,\mu) = 1$*) if* $l(\lambda) + l(\mu)$ *is even (resp.* $l(\lambda) + l(\mu)$ *is odd).*

Proof. First we will show the second equality of (4.4). Then the first equality also follows from the double supercentralizer theorem (abbreviated as DSCT) for semisimple superalgebras (cf. [11, Th. 2.1]).

By Theorem A (1), we have $\mathrm{End}^{\cdot}_{\Psi(\vartheta(\mathcal{B}_k))}(W) \supset \Theta(\mathcal{U}_{n_0} \dot{\otimes} \mathcal{U}_{n_1})$, since $\Theta(\mathcal{U}_{n_0} \dot{\otimes} \mathcal{U}_{n_1})$ is a subsuperalgebra of $\Theta(\mathcal{U}_n)$. Hence $\Theta(X \otimes Y)$ commutes with $\Psi(\vartheta(\mathcal{B}_k))$ for any $X \in \mathfrak{q}(n_0)$, $Y \in \mathfrak{q}(n_1)$. By direct calculations, it can be shown that $\Theta(X \otimes Y)$ and $\Psi(1 \otimes \tau')$ also commute. Since $\hat{\mathcal{B}}_k$ is generated as an algebra by the elements $\vartheta(\tau_i)$, $1 \le i \le k$, the elements $\vartheta(\sigma_j)$, $1 \le j \le k-1$, and $1 \otimes \tau'$, we have $\mathrm{End}^{\cdot}_{\Psi(\hat{\mathcal{B}}_k)}(W) \supset \Theta(\mathcal{U}_{n_0} \dot{\otimes} \mathcal{U}_{n_1})$. We need only to show that

$$\mathrm{End}^{\cdot}_{\Psi(\hat{\mathcal{B}}_k)}(W) \subset \Theta(\mathcal{U}_{n_0} \dot{\otimes} \mathcal{U}_{n_1}). \tag{4.7}$$

We have $\mathrm{End}^{\cdot}_{\Psi(\hat{\mathcal{B}}_k)}(W) \subset \mathrm{End}^{\cdot}_{\Psi(\vartheta(\mathcal{B}_k))}(W) = \Theta(\mathcal{U}_n)$ by Theorem A (1). It can be easily checked that $\Theta(\mathcal{U}_n) \subset Q(n) \dot{\otimes} \cdots \dot{\otimes} Q(n)$, where $Q(n)$ denotes the underlying vector space of $\mathfrak{q}(n)$ (or the superalgebra it forms), so that we have $\mathrm{End}^{\cdot}_{\Psi(\hat{\mathcal{B}}_k)}(W) \subset Q(n) \dot{\otimes} \cdots \dot{\otimes} Q(n)$. We identify $\mathrm{End}(W)$ with $\overbrace{\mathrm{End}(V) \dot{\otimes} \cdots \dot{\otimes} \mathrm{End}(V)}^{k}$ by defining the action of $f_1 \otimes f_2 \otimes \cdots \otimes f_k \in \mathrm{End}(V)^{\dot{\otimes} k}$ on W by

$$\begin{aligned}&(f_1 \otimes f_2 \otimes \cdots \otimes f_k)(v_1 \otimes v_2 \otimes \cdots \otimes v_k)\\ &\quad = (-1)^{\overline{f_2}\cdot\overline{v_1} + \overline{f_3}\cdot(\overline{v_1}+\overline{v_2}) + \cdots + \overline{f_k}\cdot(\overline{v_1}+\cdots+\overline{v_{k-1}})} f_1 v_1 \otimes f_2 v_2 \otimes \cdots \otimes f_k v_k\end{aligned}$$

for all homogeneous elements $f_j \in \mathrm{End}(V)$ and $v_j \in V$, $1 \le j \le k$. Define a representation $\theta\colon \mathbb{C}[\mathfrak{S}_k] \to \mathrm{End}(\mathrm{End}(W))$ of $\mathbb{C}[\mathfrak{S}_k]$ by

$$\begin{aligned}&\theta(\sigma_i)(f_1 \otimes \cdots \otimes f_i \otimes f_{i+1} \otimes \cdots \otimes f_k)\\ &\quad = (-1)^{\overline{f_i}\cdot\overline{f_{i+1}}}(f_1 \otimes \cdots \otimes f_{i+1} \otimes f_i \otimes \cdots \otimes f_k)\end{aligned}$$

for all $1 \le i \le k-1$ and homogeneous elements f_j, $1 \le j \le k$, of $\mathrm{End}(V)$. Moreover, define elements T_i, $1 \le i \le k$, of $\mathrm{End}(\mathrm{End}(W))$ by

$$T_i(f_1 \otimes \cdots \otimes f_k) = f_1 \otimes \cdots \otimes Q f_i Q \otimes \cdots \otimes f_k$$

for all $f_j \in \mathrm{End}(V)$, $1 \le j \le k$. Furthermore put

$$S = \frac{1}{n!} \sum_{w \in \mathfrak{S}_k} \theta(w), \quad T = \prod_{i=1}^{k} \left(\frac{1}{2}(\mathrm{Id}_{\mathrm{End}(W)} + T_i) \right).$$

Note that, since $T_i \in \mathrm{End}^0(\mathrm{End}(W))$ for all i, the factors in the definition of T commute. If $f \in \mathrm{End}^{\cdot}_{\Psi(\hat{\mathcal{B}}_k)}(W)$, then it follows that $S(f) = f$ and $\frac{1}{2}(\mathrm{Id}_{\mathrm{End}(W)} + T_i)(f) = f$, $1 \leq i \leq k$, since $\theta(\sigma)(f) = \Psi(\vartheta(\sigma)) \circ f \circ \Psi(\vartheta(\sigma))^{-1}$ and $T_i(f) = \Psi(1 \otimes \tau_i') \circ f \circ \Psi(1 \otimes \tau_i')$. Therefore, any element f of $\mathrm{End}^{\cdot}_{\Psi(\hat{\mathcal{B}}_k)}(W)$ can be expressed as a linear combination of elements of the form

$$ST(f_1 \otimes \cdots \otimes f_k)$$

with $f_j \in Q(n)$, $1 \leq j \leq k$. Since

$$T(f_1 \otimes \cdots \otimes f_k) = \left(\frac{1}{2}\right)^k (f_1 + Qf_1Q) \otimes \cdots \otimes (f_k + Qf_kQ)$$

and $f + QfQ$ belongs to $Q(n_0) \oplus Q(n_1)$ for any $f \in Q(n)$, we have

$$T\left(Q(n) \dot{\otimes} \cdots \dot{\otimes} Q(n)\right) \subset (Q(n_0) \oplus Q(n_1)) \dot{\otimes} \cdots \dot{\otimes} (Q(n_0) \oplus Q(n_1)).$$

Hence it follows that

$$\mathrm{End}^{\cdot}_{\Psi(\hat{\mathcal{B}}_k)}(W) \subset S\big((Q(n_0) \oplus Q(n_1)) \dot{\otimes} \cdots \dot{\otimes} (Q(n_0) \oplus Q(n_1))\big).$$

By induction on k, it can be shown that

$$S\big(\overbrace{(Q(n_0) \oplus Q(n_1)) \dot{\otimes} \cdots \dot{\otimes} (Q(n_0) \oplus Q(n_1))}^{k}\big)$$

is generated as an algebra by elements of the form $S(X \otimes 1 \otimes \cdots \otimes 1) = \frac{1}{n}\Theta(X)$ with $X \in \mathfrak{q}(n_0) \oplus \mathfrak{q}(n_1)$. Therefore (4.7) follows.

Next we will show (2) and (3) simultaneously. Since V is a sum of the natural representations X and Y of $\mathfrak{q}(n_0)$ and $\mathfrak{q}(n_1)$ respectively: $V = X \oplus Y$, where $\mathbf{dim}\, X = (n_0, n_0)$, $\mathbf{dim}\, Y = (n_1, n_1)$, W can be decomposed into a sum of tensor powers of X and Y. Since a $\mathcal{U}_{n_0} \dot{\otimes} \mathcal{U}_{n_1}$-submodule of W of the form $\cdots \otimes X \otimes Y \otimes \cdots$ is isomorphic to that of the form $\cdots \otimes Y \otimes X \otimes \cdots$, we have

$$W \cong_{\mathcal{U}_{n_0} \dot{\otimes} \mathcal{U}_{n_1}} \bigoplus_{k'=0}^{k} \left(\overbrace{X \otimes \cdots \otimes X}^{k'} \otimes \overbrace{Y \otimes \cdots \otimes Y}^{k-k'} \right)^{\oplus \binom{k}{k'}}.$$

From Theorem A (2), we have

$$(4.8) \qquad X^{\otimes k'} \cong_{\mathcal{B}_{k'}\dot{\otimes}\mathcal{U}_{n_0}} \bigoplus_{\substack{\lambda\in DP_{k'} \\ l(\lambda)\le n_0}} W_\lambda \mathbin{\dot{\circ}} U_\lambda,$$

$$Y^{\otimes k-k'} \cong_{\mathcal{B}_{k-k'}\dot{\otimes}\mathcal{U}_{n_1}} \bigoplus_{\substack{\mu\in DP_{k-k'} \\ l(\mu)\le n_1}} W_\mu \mathbin{\dot{\circ}} U_\mu.$$

Therefore, it follows that simple $\mathcal{U}_{n_0} \dot{\otimes} \mathcal{U}_{n_1}$-modules which occur in W are of the form $U_\lambda \mathbin{\dot{\circ}} U_\mu$, $(\lambda, \mu) \in (DP^2)_k$, and that $U_\lambda \mathbin{\dot{\circ}} U_\mu$ occurs in W if and only if $l(\lambda) \le n_0$ and $l(\mu) \le n_1$. By (4.7) and DSCT, W can be decomposed into a sum of non-isomorphic simple $\hat{\mathcal{B}}_k \dot{\otimes} (\mathcal{U}_{n_0} \dot{\otimes} \mathcal{U}_{n_1})$-modules. In order to determine the simple $\hat{\mathcal{B}}_k$-module which is paired with the simple $\mathcal{U}_{n_0} \dot{\otimes} \mathcal{U}_{n_1}$-module $U_\lambda \mathbin{\dot{\circ}} U_\mu$, we consider the $\mathcal{B}_{k'} \dot{\otimes} \mathcal{B}_{k-k'}$-submodule $\overbrace{X\otimes\cdots\otimes X}^{k'} \otimes \overbrace{Y\otimes\cdots\otimes Y}^{k-k'}$ of W. Since $\tau_i' \in \hat{\mathcal{B}}_{k'}$, $1 \le i \le k'$ (resp. $\tau_j' \in \hat{\mathcal{B}}_{k-k'}$, $1 \le j \le k-k'$), acts on $X^{\otimes k'}$ (resp. $Y^{\otimes k-k'}$) as $\mathrm{Id}_{X^{\otimes k'}}$ (resp. $-\mathrm{Id}_{Y^{\otimes k-k'}}$), the $\mathcal{B}_{k'}$ (resp. $\mathcal{B}_{k-k'}$)-submodule W_λ (resp. W_μ) of $X^{\otimes k'}$ (resp. $Y^{\otimes k-k'}$) can be regarded as a $\hat{\mathcal{B}}_{k'}$ (resp. $\hat{\mathcal{B}}_{k-k'}$)-module and is isomorphic to $W_{\lambda,\phi}$ (resp. $W_{\phi,\mu}$). From (4.8), a simple $\hat{\mathcal{B}}_k$-submodule of W which corresponds to $U_\lambda \mathbin{\dot{\circ}} U_\mu$ contains $W_{\lambda,\phi} \otimes W_{\phi,\mu}$ as a $\hat{\mathcal{B}}_{k'} \dot{\otimes} \hat{\mathcal{B}}_{k-k'}$-submodule. This condition forces this simple $\hat{\mathcal{B}}_k$-module to be isomorphic to $W_{\lambda,\mu}$. Consequently, the result (2) and (3) follow.

The result (4) immediately follows from Theorem A (3) and the fact that

$$\mathrm{Ch}[U \mathbin{\dot{\circ}} U'](x_1,\ldots,x_{n_0},y_1,\ldots,y_{n_1}) = \begin{cases} \mathrm{Ch}[U](x_1,\ldots,x_{n_0})\,\mathrm{Ch}[U'](y_1,\ldots,y_{n_1}) & \text{if } U \text{ or } U' \text{ is of type } M, \\ \dfrac{1}{2}\,\mathrm{Ch}[U](x_1,\ldots,x_{n_0})\,\mathrm{Ch}[U'](y_1,\ldots,y_{n_1}) & \text{if } U,\ U' \text{ are of type } Q. \end{cases}$$

Q.E.D.

By Theorem 1.1, (1.2), Theorem 4.1 (3) and Theorem A, the simple $\mathcal{U}_{n_0} \dot{\otimes} \mathcal{U}_{n_1}$-module $U_{\lambda,\mu}$ is of type M (resp. of type Q) if $l(\lambda) + l(\mu)$ is even (resp. odd). If $l(\lambda) + l(\mu)$ is odd, then fix a non-zero element $u_{\lambda,\mu}$ of $\mathrm{End}^1_{\mathcal{U}_{n_0}\dot{\otimes}\mathcal{U}_{n_1}}(U_{\lambda,\mu})$.

We can rewrite (4.5) using the isomorphism $W_{\lambda,\mu} \cong X_k \mathbin{\dot{\circ}} V_{\lambda,\mu}$ as

$\hat{\mathcal{B}}_k$-modules. We have

$$W \cong \bigoplus_{(\lambda,\mu)\in(DP^2)_k} X_k \dot{\circ} V_{\lambda,\mu} \dot{\circ} U_{\lambda,\mu}.$$

Note that, if U, V and W are simple modules for superalgebras A, B and C respectively, then both $(U \dot{\circ} V) \dot{\circ} W$ and $U \dot{\circ} (V \dot{\circ} W)$ denote the unique (up to isomorphism) simple $(A \dot{\otimes} B \dot{\otimes} C)$-module occurring in $(U \otimes V) \otimes W \cong U \otimes (V \otimes W)$, so that, up to isomorphism, the operation $\dot{\circ}$ is associative. There are three cases where the $\mathcal{C}_k \dot{\otimes} \mathcal{B}'_k \dot{\otimes} (\mathcal{U}_{n_0} \dot{\otimes} \mathcal{U}_{n_1})$-module $X_k \dot{\circ} V_{\lambda,\mu} \dot{\circ} U_{\lambda,\mu}$ is different from the supertensor product $X_k \otimes V_{\lambda,\mu} \otimes U_{\lambda,\mu}$.

(1) If k is even and $(\lambda,\mu) \in (DP^2)_k^-$, then X_k, $V_{\lambda,\mu}$, $U_{\lambda,\mu}$ are of type M, Q, Q respectively. We have

$$X_k \dot{\circ} V_{\lambda,\mu} \dot{\circ} U_{\lambda,\mu} = X_k \otimes (V_{\lambda,\mu} \dot{\circ} U_{\lambda,\mu})$$

where $V_{\lambda,\mu} \dot{\circ} U_{\lambda,\mu}$ is one of the two eigenspaces of $x_{\lambda,\mu} \otimes u_{\lambda,\mu}$.

(2) If k is odd and $(\lambda,\mu) \in (DP^2)_k^+$, then X_k, $V_{\lambda,\mu}$, $U_{\lambda,\mu}$ are of type Q, M, Q respectively. We have

$$X_k \dot{\circ} V_{\lambda,\mu} \dot{\circ} U_{\lambda,\mu} = (X_k \otimes V_{\lambda,\mu}) \dot{\circ} U_{\lambda,\mu}$$

where $(X_k \otimes V_{\lambda,\mu}) \dot{\circ} U_{\lambda,\mu}$ is one of the two eigenspaces of $(z_k \otimes 1) \otimes u_{\lambda,\mu}$.

(3) If k is odd and $(\lambda,\mu) \in (DP^2)_k^-$, then X_k, $V_{\lambda,\mu}$, $U_{\lambda,\mu}$ are of type Q, Q, M respectively. We have

$$X_k \dot{\circ} V_{\lambda,\mu} \dot{\circ} U_{\lambda,\mu} = (X_k \dot{\circ} V_{\lambda,\mu}) \otimes U_{\lambda,\mu}$$

where $X_k \dot{\circ} V_{\lambda,\mu}$ is one of the two eigenspaces of $z_k \otimes x_{\lambda,\mu}$.

Put $r = \lfloor k/2 \rfloor$ and $\zeta_i = \sqrt{-1}\xi_{2i-1}\xi_{2i} \in \mathcal{C}_k$ for $1 \le i \le r$. Then the elements $\Psi(\zeta_i \otimes 1)$, $1 \le i \le r$, are commuting involutions of $\Psi((\mathcal{C}_k)_0 \dot{\otimes} 1) \subset \Psi((\hat{\mathcal{B}}_k)_0) = \mathrm{End}^0_{\Theta(\mathcal{U}_{n_0} \dot{\otimes} \mathcal{U}_{n_1})}(W)$. For each $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_r) \in \mathbb{Z}_2^r$, put $W^\varepsilon = \{w \in W \,|\, \Psi(\zeta_i \otimes 1)(w) = (-1)^{\varepsilon_i} w \quad (1 \le i \le r)\}$. Then we have $W = \bigoplus_{\varepsilon \in \mathbb{Z}_2^r} W^\varepsilon$. Since $\zeta_i \otimes 1$ commutes with $1 \dot{\otimes} \mathcal{B}'_k$ for each $1 \le i \le r$, W^ε is a $\mathcal{B}'_k \dot{\otimes} (\mathcal{U}_{n_0} \dot{\otimes} \mathcal{U}_{n_1})$-module.

Theorem 4.2. *For each $\varepsilon \in \mathbb{Z}_2^r$, the submodule W^ε is decomposed as a multiplicity-free sum of simple $\mathcal{B}'_k \dot{\otimes} (\mathcal{U}_{n_0} \dot{\otimes} \mathcal{U}_{n_1})$-modules as follows:*

$$W^\varepsilon \cong_{\mathcal{B}'_k \dot{\otimes} (\mathcal{U}_{n_0} \dot{\otimes} \mathcal{U}_{n_1})} \bigoplus_{(\lambda,\mu)\in(DP^2)_k} V_{\lambda,\mu} \dot{\circ} U_{\lambda,\mu}. \tag{4.9}$$

In the above decomposition, the simple $\mathcal{B}'_k$-modules are paired with the simple $\mathcal{U}_{n_0}\dot{\otimes}\mathcal{U}_{n_1}$-modules in a bijective manner. More precisely, we have the following results.

(1) *Assume that k is even. Then the simple $\mathcal{B}'_k\dot{\otimes}(\mathcal{U}_{n_0}\dot{\otimes}\mathcal{U}_{n_1})$-modules $V_{\lambda,\mu}\circ U_{\lambda,\mu}$ in W^ε are all of type M. Furthermore we have*

(4.10)
$$\mathrm{End}^{\cdot}_{\Theta(\mathcal{U}_{n_0}\dot{\otimes}\mathcal{U}_{n_1})}(W^\varepsilon)=\Psi(\mathcal{B}'_k),\quad \mathrm{End}^{\cdot}_{\Psi(\mathcal{B}'_k)}(W^\varepsilon)=\Theta(\mathcal{U}_{n_0}\dot{\otimes}\mathcal{U}_{n_1}).$$

(2) *Assume that k is odd. Then the simple $\mathcal{B}'_k\dot{\otimes}(\mathcal{U}_{n_0}\dot{\otimes}\mathcal{U}_{n_1})$-modules $V_{\lambda,\mu}\circ U_{\lambda,\mu}$ in W^ε are all of type Q. Furthermore we have*

(4.11)
$$\mathrm{End}^{\cdot}_{\Theta(\mathcal{U}_{n_0}\dot{\otimes}\mathcal{U}_{n_1})}(W^\varepsilon)\cong\mathcal{C}_1\dot{\otimes}\Psi(\mathcal{B}'_k),\quad \mathrm{End}^{\cdot}_{\Psi(\mathcal{B}'_k)}(W^\varepsilon)\cong\mathcal{C}_1\dot{\otimes}\Theta(\mathcal{U}_{n_0}\dot{\otimes}\mathcal{U}_{n_1}).$$

Proof. For each $\varepsilon=(\varepsilon_1,\dots,\varepsilon_r)\in\mathbb{Z}_2^r$, put $X_k^\varepsilon=\{\xi\in X_k\,|\,\zeta_i\xi=(-1)^{\varepsilon_i}\xi\quad(1\le i\le r)\}$. Then we have $X_k=\bigoplus_{\varepsilon\in\mathbb{Z}_2^r}X_k^\varepsilon$.

(1) Assume that k is even. Note that X_k^ε is one-dimensional. Let ξ^ε be a base of X_k^ε, namely $X_k^\varepsilon=\mathbb{C}\xi^\varepsilon$. Since the elements ζ_i are of degree 0, ξ^ε is a homogeneous element of X_k. Hence we have $X_k^\varepsilon\otimes(V_{\lambda,\mu}\circ U_{\lambda,\mu})\cong_{\mathcal{B}'_k\dot{\otimes}(\mathcal{U}_{n_0}\dot{\otimes}\mathcal{U}_{n_1})}V_{\lambda,\mu}\circ U_{\lambda,\mu}$.

If $(\lambda,\mu)\in(DP^2)_k^+$, then we have

$$\begin{aligned}X_k\circ V_{\lambda,\mu}\circ U_{\lambda,\mu}=X_k\otimes V_{\lambda,\mu}\otimes U_{\lambda,\mu}&=\bigoplus_{\varepsilon\in\mathbb{Z}_2^r}X_k^\varepsilon\otimes V_{\lambda,\mu}\otimes U_{\lambda,\mu}\\&=\bigoplus_{\varepsilon\in\mathbb{Z}_2^r}X_k^\varepsilon\otimes(V_{\lambda,\mu}\circ U_{\lambda,\mu}).\end{aligned}$$

If $(\lambda,\mu)\in(DP^2)_k^-$, then we have

$$X_k\circ V_{\lambda,\mu}\circ U_{\lambda,\mu}=\bigoplus_{\varepsilon\in\mathbb{Z}_2^r}X_k^\varepsilon\otimes(V_{\lambda,\mu}\circ U_{\lambda,\mu})$$

since the elements ζ_i, $1\le i\le r$, and $1\otimes x_{\lambda,\mu}\otimes u_{\lambda,\mu}$ commute. Consequently we have

$$\begin{aligned}W^\varepsilon&\cong_{\mathcal{B}'_k\dot{\otimes}(\mathcal{U}_{n_0}\dot{\otimes}\mathcal{U}_{n_1})}X_k^\varepsilon\otimes\left(\bigoplus_{(\lambda,\mu)\in(DP^2)_k}V_{\lambda,\mu}\circ U_{\lambda,\mu}\right)\\&\cong_{\mathcal{B}'_k\dot{\otimes}(\mathcal{U}_{n_0}\dot{\otimes}\mathcal{U}_{n_1})}\bigoplus_{(\lambda,\mu)\in(DP^2)_k}V_{\lambda,\mu}\circ U_{\lambda,\mu}\end{aligned}$$

Therefore (4.9) follows. By Theorem 1.1 and (1.2), the simple modules $V_{\lambda,\mu} \circ U_{\lambda,\mu}$ appearing in the above decomposition are of type M.

First we will show the second equality in (4.10). Then the first equality follows from DSCT. Since W^ε is a $\mathcal{B}'_k \otimes (\mathcal{U}_{n_0} \dot{\otimes} \mathcal{U}_{n_1})$-module, we have

$$\Theta(\mathcal{U}_{n_0} \dot{\otimes} \mathcal{U}_{n_1})|_{W^\varepsilon} \subset \dot{\mathrm{End}}_{\Psi(\mathcal{B}'_k)}(W^\varepsilon).$$

By DSCT, (4.5) and (4.9) (already proved for this case), we have

$$\dim \dot{\mathrm{End}}_{\Psi(\mathcal{B}'_k)}(W^\varepsilon) = \dim \dot{\mathrm{End}}_{\Psi(\hat{\mathcal{B}}_k)}(W)$$

since both equal $\sum_{(\lambda,\mu)\in(DP^2)_k^+} (\dim U_{\lambda,\mu})^2 + \sum_{(\lambda,\mu)\in(DP^2)_k^-} \frac{1}{2}(\dim U_{\lambda,\mu})^2$. By Theorem 4.1 (1), we have $\dim \dot{\mathrm{End}}_{\Psi(\hat{\mathcal{B}}_k)}(W) = \dim \Theta(\mathcal{U}_{n_0} \dot{\otimes} \mathcal{U}_{n_1})$. Define a linear map $\mathfrak{p}_\varepsilon \colon \Theta(\mathcal{U}_{n_0} \dot{\otimes} \mathcal{U}_{n_1}) \to \Theta(\mathcal{U}_{n_0} \dot{\otimes} \mathcal{U}_{n_1})|_{W^\varepsilon}$ by $\mathfrak{p}_\varepsilon(f) = f|_{W^\varepsilon}$ for $f \in \Theta(\mathcal{U}_{n_0} \dot{\otimes} \mathcal{U}_{n_1})$. It is clear that $\mathfrak{p}_\varepsilon$ is surjective. We claim that $\mathfrak{p}_\varepsilon$ is injective. Assume that $f \in \ker \mathfrak{p}_\varepsilon$, namely $f|_{W^\varepsilon} = 0 \in \mathrm{End}(W^\varepsilon)$. Since f and the elements ξ_{2j-1} commute, and a subgroup of $(\mathcal{C}_k)^\times$ generated by the elements ξ_{2j-1}, $1 \le j \le r$, transitively act on $\{W^{\varepsilon'}\,;\, \varepsilon' \in \mathbb{Z}_2^r\}$ as follows:

$$\xi_{2j-1} W^{(\varepsilon_1,\dots,\varepsilon_r)} = W^{(\varepsilon_1,\dots,\varepsilon_j+1,\dots,\varepsilon_r)} \quad (1 \le^\forall j \le r)$$

it follows that $f|_{W^{\varepsilon'}} = 0$ for all $\varepsilon' \in \mathbb{Z}_2^r$. Therefore $f = 0$ in $\mathrm{End}(W)$. Hence $\mathfrak{p}_\varepsilon$ is injective. Consequently we have $\dim \Theta(\mathcal{U}_{n_0} \dot{\otimes} \mathcal{U}_{n_1})|_{W^\varepsilon} = \dim \dot{\mathrm{End}}_{\Psi(\mathcal{B}'_k)}(W^\varepsilon)$. It follows that $\Theta(\mathcal{U}_{n_0} \dot{\otimes} \mathcal{U}_{n_1})|_{W^\varepsilon} = \dot{\mathrm{End}}_{\Psi(\mathcal{B}'_k)}(W^\varepsilon)$, as required.

(2) Assume that k is odd. Note that X_k^ε is 2-dimensional. Then $X_k^\varepsilon = \mathbb{C}\xi^\varepsilon \oplus \mathbb{C}z_k\xi^\varepsilon$.

If $(\lambda,\mu) \in (DP^2)_k^+$, then $V_{\lambda,\mu} \circ X_k = V_{\lambda,\mu} \otimes X_k$ and we regard the $\mathcal{B}'_k \dot{\otimes} \mathcal{C}_k$-module $V_{\lambda,\mu} \otimes X_k$ as a $\mathcal{C}_k \dot{\otimes} \mathcal{B}'_k$-module via $\omega_{\mathcal{C}_k,\mathcal{B}'_k}$, where $\omega_{\mathcal{C}_k,\mathcal{B}'_k} \colon \mathcal{C}_k \dot{\otimes} \mathcal{B}'_k \to \mathcal{B}'_k \dot{\otimes} \mathcal{C}_k$ denotes an isomorphism of superalgebras determined by $\omega_{\mathcal{C}_k,\mathcal{B}'_k}(a \otimes b) = (-1)^{\bar{a}\cdot\bar{b}} b \otimes a$ for all homogeneous elements $a \in \mathcal{C}_k$ and $b \in \mathcal{B}'_k$. An isomorphism $\theta \colon X_k \otimes V_{\lambda,\mu} \xrightarrow{\sim} V_{\lambda,\mu} \otimes X_k$ is defined by $\theta(\xi \otimes v) = (-1)^{\bar{\xi}\cdot\bar{v}} v \otimes \xi$ for all homogeneous elements $\xi \in X_k$ and $v \in V_{\lambda,\mu}$. Since $\theta \circ (z_k \otimes 1) = (1 \otimes z_k) \circ \theta$, we have

$$\begin{aligned} X_k \circ V_{\lambda,\mu} \circ U_{\lambda,\mu} &\cong_{\hat{\mathcal{B}}_k \dot{\otimes} \mathcal{U}_n} (V_{\lambda,\mu} \otimes X_k) \circ U_{\lambda,\mu} \\ &\cong_{\hat{\mathcal{B}}_k \dot{\otimes} \mathcal{U}_n} V_{\lambda,\mu} \otimes (X_k \circ U_{\lambda,\mu}) \end{aligned}$$

where $X_k \dot{\circ} U_{\lambda,\mu}$ denotes one of the two eigenspaces of $z_k \otimes u_{\lambda,\mu}$. Since the elements ζ_i and $z_k \otimes u_{\lambda,\mu}$ commute, we have

$$X_k \dot{\circ} U_{\lambda,\mu} = \bigoplus_{\varepsilon \in \mathbb{Z}_2^r} X_k^\varepsilon \dot{\circ} U_{\lambda,\mu}$$

where $X_k^\varepsilon \dot{\circ} U_{\lambda,\mu}$ denotes one of the two eigenspaces of $z_k|_{X_k^\varepsilon} \otimes u_{\lambda,\mu}$. Since $X_k^\varepsilon \dot{\circ} U_{\lambda,\mu}$ is a $\mathcal{U}_{n_0} \dot{\otimes} \mathcal{U}_{n_1}$-submodule of $X_k \dot{\circ} U_{\lambda,\mu} \cong_{\mathcal{U}_{n_0} \dot{\otimes} \mathcal{U}_{n_1}} U_{\lambda,\mu}^{\oplus 2^r}$ and $\dim(X_k^\varepsilon \dot{\circ} U_{\lambda,\mu}) = \dim U_{\lambda,\mu}$, it follows that $X_k^\varepsilon \dot{\circ} U_{\lambda,\mu} \cong_{\mathcal{U}_{n_0} \dot{\otimes} \mathcal{U}_{n_1}} U_{\lambda,\mu}$.

If $(\lambda, \mu) \in (DP^2)_k^-$, then we have

$$X_k \dot{\circ} V_{\lambda,\mu} \dot{\circ} U_{\lambda,\mu} = (X_k \dot{\circ} V_{\lambda,\mu}) \otimes U_{\lambda,\mu} = \bigoplus_{\varepsilon \in \mathbb{Z}_2^r} (X_k^\varepsilon \dot{\circ} V_{\lambda,\mu}) \otimes U_{\lambda,\mu}$$

since the elements ζ_i, $1 \leq i \leq r$, and $z_k \otimes x_{\lambda,\mu} \otimes 1$ commute, where $X_k^\varepsilon \dot{\circ} V_{\lambda,\mu}$ denotes one of the two eigenspaces of $z_k|_{X_k^\varepsilon} \otimes x_{\lambda,\mu}$. Since $X_k^\varepsilon \dot{\circ} V_{\lambda,\mu}$ is a $\mathcal{B}'_k$-submodule of $X_k \dot{\circ} V_{\lambda,\mu} \cong_{\mathcal{B}'_k} V_{\lambda,\mu}^{\oplus 2^r}$ and $\dim(X_k^\varepsilon \dot{\circ} V_{\lambda,\mu}) = \dim V_{\lambda,\mu}$, it follows that $X_k^\varepsilon \dot{\circ} V_{\lambda,\mu} \cong_{\mathcal{B}'_k} V_{\lambda,\mu}$.

Consequently we have

$$W^\varepsilon \cong \bigoplus_{(\lambda,\mu) \in (DP^2)_k} V_{\lambda,\mu} \otimes U_{\lambda,\mu}.$$

By Theorem 1.1 and (1.2), the simple modules $V_{\lambda,\mu} \otimes U_{\lambda,\mu}$ appearing in the above decomposition are of type Q and we have $V_{\lambda,\mu} \otimes U_{\lambda,\mu} = V_{\lambda,\mu} \dot{\circ} U_{\lambda,\mu}$. Therefore, (4.9) and the former statement of (2) follow.

The supercentralizer $\mathrm{End}^{\cdot}_{\Psi(\mathcal{B}'_k)}(W^\varepsilon)$ contains an invertible element $\Psi(\xi_k) \in \Psi(\mathcal{C}_k)$. The subsuperalgebra of $\mathrm{End}^{\cdot}_{\Psi(\mathcal{B}'_k)}(W^\varepsilon)$ generated by $\Psi(\xi_k)$ is isomorphic to $\mathcal{C}_1$. By the arguments similar to the proof of (4.10), the result (4.11) follows from DSCT (cf. [11, Cor. 2.2]). Q.E.D.

Let us mention a relation between the branching rule of the $\mathfrak{q}(n)$-modules to $\mathfrak{q}(n_0) \oplus \mathfrak{q}(n_1)$ and that of the $\hat{\mathcal{B}}_k$-modules to $\mathcal{B}_k$ (or that of the $\mathcal{B}'_k$-modules to $\mathcal{A}_k$).

Let A be a superalgebra and let B be a subsuperalgebra of A. If V is an A-module, then we can restrict it to a B-module, which we write as $V \downarrow_B^A$. Moreover, we write $[V : U]_A$ (or simply write $[V : U]$) for the multiplicity of a simple A-module U in an A-module V.

Corollary 4.3. *Put*

(4.12)
$$m^{\lambda}_{\mu,\nu} = [U_\lambda \downarrow^{\mathcal{U}_n}_{\mathcal{U}_{n_0} \dot{\otimes} \mathcal{U}_{n_1}} : U_{\mu,\nu}],$$

(4.13)
$$m'^{\lambda}_{\mu,\nu} = [W_{\mu,\nu} \downarrow^{\hat{\mathcal{B}}_k}_{\mathcal{B}_k} : W_\lambda] \quad \left(\text{resp.}[V_{\mu,\nu} \downarrow^{\mathcal{B}'_k}_{\mathcal{A}_k} : V_\lambda]\right).$$

Then we have

(4.14)
$$m'^{\lambda}_{\mu,\nu} = \begin{cases} \frac{1}{2} m^{\lambda}_{\mu,\nu} & \text{if both } U_{\mu,\nu} \text{ and } W_{\mu,\nu} \text{ (resp. } V_{\mu,\nu}\text{) are of type } M \\ & \quad \text{and both } U_\lambda \text{ and } W_\lambda \text{ (resp. } V_\lambda\text{) are of type } Q, \\ 2m^{\lambda}_{\mu,\nu} & \text{if both } U_{\mu,\nu} \text{ and } W_{\mu,\nu} \text{ (resp. } V_{\mu,\nu}\text{) are of type } Q \\ & \quad \text{and both } U_\lambda \text{ and } W_\lambda \text{ (resp. } V_\lambda\text{) are of type } M, \\ m^{\lambda}_{\mu,\nu} & \text{otherwise.} \end{cases}$$

Proof. Put

$$W' = W_\lambda \mathbin{\dot\circ} U_{\mu,\nu}, \quad W_1 = W \downarrow^{\mathcal{B}_k \dot{\otimes} \mathcal{U}_n}_{\mathcal{B}_k \dot{\otimes} (\mathcal{U}_{n_0} \dot{\otimes} \mathcal{U}_{n_1})}, \quad W_2 = W \downarrow^{\hat{\mathcal{B}}_k \dot{\otimes} (\mathcal{U}_{n_0} \dot{\otimes} \mathcal{U}_{n_1})}_{\mathcal{B}_k \dot{\otimes} (\mathcal{U}_{n_0} \dot{\otimes} \mathcal{U}_{n_1})}.$$

Since $W_1 \cong W_2$, we have $[W_1 : W'] = [W_2 : W']$. Moreover, put

$$W'_1 = (W_\lambda \mathbin{\dot\circ} U_\lambda) \downarrow^{\mathcal{B}_k \dot{\otimes} \mathcal{U}_n}_{\mathcal{B}_k \dot{\otimes} (\mathcal{U}_{n_0} \dot{\otimes} \mathcal{U}_{n_1})}, \; W'_2 = (W_{\mu,\nu} \mathbin{\dot\circ} U_{\mu,\nu}) \downarrow^{\hat{\mathcal{B}}_k \dot{\otimes} (\mathcal{U}_{n_0} \dot{\otimes} \mathcal{U}_{n_1})}_{\mathcal{B}_k \dot{\otimes} (\mathcal{U}_{n_0} \dot{\otimes} \mathcal{U}_{n_1})}.$$

From (4.5) and (A.2), we have $[W_1 : W'] = [W'_1 : W']$ and $[W_2 : W'] = [W'_2 : W']$. Using (4.12) and (4.13), we have

$$(W_\lambda \otimes U_\lambda) \downarrow^{\mathcal{B}_k \dot{\otimes} \mathcal{U}_n}_{\mathcal{B}_k \dot{\otimes} (\mathcal{U}_{n_0} \dot{\otimes} \mathcal{U}_{n_1})} \cong \bigoplus_{(\mu,\nu) \in (DP^2)_k} (W_\lambda \otimes U_{\mu,\nu})^{\oplus m^{\lambda}_{\mu,\nu}},$$

$$(W_{\mu,\nu} \otimes U_{\mu,\nu}) \downarrow^{\hat{\mathcal{B}}_k \dot{\otimes} (\mathcal{U}_{n_0} \dot{\otimes} \mathcal{U}_{n_1})}_{\mathcal{B}_k \dot{\otimes} (\mathcal{U}_{n_0} \dot{\otimes} \mathcal{U}_{n_1})} \cong \bigoplus_{\lambda \in DP_k} (W_\lambda \otimes U_{\mu,\nu})^{\oplus m'^{\lambda}_{\mu,\nu}}.$$

By Theorem 1.1 and (1.2), the above modules $(W_\lambda \otimes U_\lambda) \downarrow^{\mathcal{B}_k \dot{\otimes} \mathcal{U}_n}_{\mathcal{B}_k \dot{\otimes} (\mathcal{U}_{n_0} \dot{\otimes} \mathcal{U}_{n_1})}$, $(W_{\mu,\nu} \otimes U_{\mu,\nu}) \downarrow^{\hat{\mathcal{B}}_k \dot{\otimes} (\mathcal{U}_{n_0} \dot{\otimes} \mathcal{U}_{n_1})}_{\mathcal{B}_k \dot{\otimes} (\mathcal{U}_{n_0} \dot{\otimes} \mathcal{U}_{n_1})}$, $W_\lambda \otimes U_{\mu,\nu}$ are sums of two copies of W'_1, W'_2, W' if W_λ is of type Q, $W_{\mu,\nu}$ is of type Q, both W_λ and $U_{\mu,\nu}$ are of type Q, respectively. Note that U_λ (resp. $U_{\mu,\nu}$) is of the same type as W_λ (resp. $W_{\mu,\nu}$). Therefore we have

$$[W_1' : W'] = \begin{cases} \frac{1}{2} m^{\lambda}_{\mu,\nu} & \text{if } W_\lambda \text{ is of type } Q \text{ and } W_{\mu,\nu} \text{ is of type } M, \\ m^{\lambda}_{\mu,\nu} & \text{otherwise,} \end{cases}$$

$$[W_2' : W'] = \begin{cases} \frac{1}{2} m'^{\lambda}_{\mu,\nu} & \text{if } W_\lambda \text{ is of type } M \text{ and } W_{\mu,\nu} \text{ is of type } Q, \\ m'^{\lambda}_{\mu,\nu} & \text{otherwise.} \end{cases}$$

Comparing the above two equations, we obtain the result (4.14).

Next, using (4.9) and (B.1), we consider the multiplicities of the simple $\mathcal{A}_k \otimes (\mathcal{U}_{n_0} \otimes \mathcal{U}_{n_1})$-module $V_\lambda \circ U_{\lambda,\mu}$ in $W^\varepsilon \downarrow^{\mathcal{A}_k \otimes \mathcal{U}_n}_{\mathcal{A}_k \otimes (\mathcal{U}_{n_0} \dot{\otimes} \mathcal{U}_{n_1})}$ and $W^\varepsilon \downarrow^{\mathcal{B}_k' \dot{\otimes} (\mathcal{U}_{n_0} \dot{\otimes} \mathcal{U}_{n_1})}_{\mathcal{A}_k \dot{\otimes} (\mathcal{U}_{n_0} \dot{\otimes} \mathcal{U}_{n_1})}$ respectively. Then (4.14) similarly follows. Q.E.D.

Let H_k' be the subgroup of $(\mathcal{B}_k')^\times$ generated by -1, τ', $\gamma_1, \dots, \gamma_{k-1}$. Then H_k' is a double cover (a central extension with a $\mathbb{Z}_2$ kernel) of H_k. Let $w^{\kappa,\nu}$ denote the element of H_k' defined by

$$\begin{aligned} w^{\kappa,\nu} &= w_1 w_2 \cdots w_l w_1' w_2' \cdots w_{l'}' \quad (l = l(\kappa), l' = l(\nu)), \\ w_i &= \gamma_{a+1}\gamma_{a+2} \cdots \gamma_{a+\kappa_i - 1} \quad (a = \kappa_1 + \cdots + \kappa_{i-1}), \\ w_i' &= \gamma_{b+1}\gamma_{b+2} \cdots \gamma_{b+\nu_i - 1}\tau_{b+\nu_i}' \quad (b = |\kappa| + \nu_1 + \cdots + \nu_{i-1}). \end{aligned}$$

Note that the image of $w^{\kappa,\nu}$ in H_k is a representative of the conjugacy class of H_k indexed by (κ, ν).

Define a map ε: $(DP^2)_k \to \mathbb{Z}_2$ by $\varepsilon(\lambda,\mu) = 0$ (resp. $\varepsilon(\lambda,\mu) = 1$) if $(\lambda,\mu) \in (DP^2)_k^+$ (resp. $(\lambda,\mu) \in (DP^2)_k^-$).

We describe a formula for the character values of simple $\mathcal{B}_k'$-modules.

Corollary 4.4. *We have*

(4.15)

$$\begin{aligned} &2^{\frac{l(\kappa)+l(\nu)}{2}} p_\kappa(x,y) p_\nu(x,-y) \\ &\quad = \sum_{(\lambda,\mu) \in (DP^2)_k} \mathrm{Ch}[V_{\lambda,\mu}](w^{\kappa,\nu}) 2^{\frac{-l(\lambda)-l(\mu)-\varepsilon(\lambda,\mu)}{2}} Q_\lambda(x) Q_\mu(y) \end{aligned}$$

for all $(\kappa,\nu) \in (OP^2)_k$, where $p_\kappa(x,y) = p_\kappa(x_1,x_2,\dots,y_1,y_2,\dots)$ and $p_\nu(x,-y) = p_\nu(x_1,x_2,\dots,-y_1,-y_2,\dots)$ and $\mathrm{Ch}[V_{\lambda,\mu}]$ denotes the character of $V_{\lambda,\mu}$, namely $\mathrm{Ch}[V_{\lambda,\mu}](w) = \mathrm{tr}(w_{V_{\lambda,\mu}})$ for $w \in \mathcal{B}_k'$ where $w_{V_{\lambda,\mu}}$ denotes the action of $w \in \mathcal{B}_k'$ on $V_{\lambda,\mu}$.

Proof. As we have noted in the preceding paragraph to Theorem 4.1, any $\mathcal{B}'_k \dot{\otimes} (\mathcal{U}_{n_0} \dot{\otimes} \mathcal{U}_{n_1})$-submodule W' of W can be regarded as a $\mathcal{B}'_k$-module with a commuting polynomial representation $\theta_{W'}$ of $GL(n_0,\mathbb{C}) \times GL(n_1,\mathbb{C})$. Here we extend our notation in Theorem 4.1 to let $\mathrm{Ch}[W'](x \otimes g)$ denote the trace $\mathrm{tr}(x_{W'} \circ \theta_{W'}(g))$ for $x \in \mathcal{B}'_k$ and $g \in GL(n_0,\mathbb{C}) \times GL(n_1,\mathbb{C})$, where $x_{W'}$ denotes the action of $x \in \mathcal{B}'_k$ on W'.

For any ε, $\varepsilon' \in \mathbb{Z}_2^r$, we have $W^\varepsilon \cong_{\mathcal{B}'_k \dot{\otimes}(\mathcal{U}_{n_0} \dot{\otimes} \mathcal{U}_{n_1})} W^{\varepsilon'}$. Hence, for $(\kappa,\nu) \in (OP^2)_k$ and $E = \mathrm{diag}(x_1,\dots,x_{n_0},y_1,\dots,y_{n_1}) \in GL(n,\mathbb{C})$, we have

$$\mathrm{Ch}[W^\varepsilon]\,(w^{\kappa,\nu} \otimes E) = 2^{-r}\,\mathrm{Ch}[W]\,((1 \otimes w^{\kappa,\nu}) \otimes E) \tag{4.16}$$

where $1 \otimes w^{\kappa,\nu} \in \mathcal{C}_k \dot{\otimes} \mathcal{B}'_k = \hat{\mathcal{B}}_k$. We calculate the right hand side using the embedding $\vartheta\colon \mathcal{B}_k \hookrightarrow \hat{\mathcal{B}}_k$ (cf. (3.3)), namely $1 \otimes \gamma_j = \vartheta(\frac{1}{\sqrt{2}}(\tau_j - \tau_{j+1})\sigma_j)$. Put $k' = |\kappa|$ and $l = l(\kappa)$. Then $k - k' = |\nu|$. Moreover put $W' = V^{\otimes k'}$ and $W'' = V^{\otimes k-k'}$. We have $w^{\kappa,\nu} = w^{\kappa,\phi} w^{\phi,\nu}$, where $w^{\kappa,\phi} \in \mathcal{B}'_{k'}$, $w^{\phi,\nu} \in \mathcal{B}'_{k-k'}$. Define a representations of $\hat{\mathcal{B}}_{k'}$ on W' (resp. a representation of $\hat{\mathcal{B}}_{k-k'}$ on W'') by the same manner as the representation Ψ of $\hat{\mathcal{B}}_k$ in W. Then the action of $1 \otimes w^{\kappa,\phi}$ (resp. the action of $1 \otimes w^{\phi,\nu}$) on W can be expressed as (the action of $1 \otimes w^{\kappa,\phi}$ on W') $\otimes \overbrace{\mathrm{id} \otimes \cdots \otimes \mathrm{id}}^{k-k'}$ (resp. $\overbrace{\mathrm{id} \otimes \cdots \otimes \mathrm{id}}^{k'}$ (the action of $1 \otimes w^{\phi,\nu}$ on W'')). Hence we have

$$\begin{aligned} &\mathrm{Ch}[W]\,((1 \otimes w^{\kappa,\nu}) \otimes E) \\ &\quad = \mathrm{Ch}[W']\left((1 \otimes w^{\kappa,\phi}) \otimes E\right) \mathrm{Ch}[W'']\left((1 \otimes w^{\phi,\nu}) \otimes E\right). \end{aligned} \tag{4.17}$$

The element $1 \otimes w^{\kappa,\phi}$ of $\hat{\mathcal{B}}_{k'}$ is a product of $k' - l$ elements $1 \otimes \gamma_j = \vartheta(\frac{1}{\sqrt{2}}(\tau_j - \tau_{j+1})\sigma_j)$. This product can be rearranged into the following form:

$$\begin{aligned} &(\text{constant}) \times (\text{a product of the elements } \vartheta(\tau_p) - \vartheta(\tau_q)) \\ &\qquad\qquad \times (\text{a product of the elements } \vartheta(\sigma_j)). \end{aligned}$$

The product of the elements $\vartheta(\sigma_j)$ equals $\vartheta(\sigma^{\kappa,\phi})$. Expanding the product of $\vartheta(\tau_p) - \vartheta(\tau_q)$ into a sum of $2^{k'-l}$ elements, we have

$$1 \otimes w^{\kappa,\phi} = \left(\frac{1}{\sqrt{2}}\right)^{k'-l} \times \sum (\text{a product of the elements } \vartheta(\tau_p)) \times \vartheta(\sigma^{\kappa,\phi})$$

where $\sigma^{\kappa,\phi} = g_1 g_2 \cdots g_l$, $g_i = \sigma_{a+1}\sigma_{a+2}\cdots\sigma_{a+\nu_i-1}$ $(a = \sum_{j=1}^{i-1}\kappa_j)$. Then all terms in the summation are conjugate to $\vartheta(\sigma^{\kappa,\phi})$ in $\vartheta((\mathcal{B}_k)^\times)$. Therefore we have

$$\begin{aligned}\mathrm{Ch}[W']\left((1\otimes w^{\kappa,\phi})\otimes E\right) &= 2^{k'-l}(\sqrt{2})^{l-k'}\,\mathrm{Ch}[W']\left(\vartheta(\sigma^{\kappa,\phi})\otimes E\right)\\ &= (\sqrt{2})^{k'+l}p_\kappa(x_1,x_2,\dots,y_1,y_2,\dots).\end{aligned}$$

Put $l' = l(\nu)$. Similarly we have

$$\begin{aligned}\mathrm{Ch}[W'']\left((1\otimes w^{\phi,\nu})\otimes E\right) &= 2^{k-k'-l'}(\sqrt{2})^{l'-k+k'}\,\mathrm{Ch}[W''](\vartheta(\sigma'^{\phi,\nu})\otimes E)\\ &= (\sqrt{2})^{k-k'+l'}p_\nu(x_1,x_2,\dots,-y_1,-y_2,\dots)\end{aligned}$$

where $\sigma'^{\phi,\nu} = g'_1 g'_2\cdots g'_{l'}$, $g'_i = \sigma_{b+1}\sigma_{b+2}\dots\sigma_{b+\nu_i-1}\tau'_{b+\nu_i}$ $(b = \sum_{j=1}^{i-1}\nu_j)$. By (4.16) and (4.17), we have

$$\begin{aligned}&\mathrm{Ch}[W^\varepsilon]\left((1\otimes w^{\kappa,\nu})\otimes E\right)\\ &= \begin{cases}(\sqrt{2})^{l+l'}p_\kappa(x_1,x_2,\dots,y_1,y_2,\dots)p_\nu(x_1,x_2,\dots,-y_1,-y_2,\dots) & \text{if } k \text{ is even},\\ (\sqrt{2})^{l+l'+1}p_\kappa(x_1,x_2,\dots,y_1,y_2,\dots)p_\nu(x_1,x_2,\dots,-y_1,-y_2,\dots) & \text{if } k \text{ is odd}.\end{cases}\end{aligned}$$

On the other hand, by (4.6) and (4.9), if k is even, then we have

$$\begin{aligned}&\mathrm{Ch}[\bigoplus_{(\lambda,\mu)\in(DP^2)_k} V_{\lambda,\mu}\mathbin{\dot\circ} U_{\lambda,\mu}]\,(w^{\kappa,\nu}\otimes E)\\ &= \sum_{(\lambda,\mu)\in(DP^2)_k}\mathrm{Ch}[V_{\lambda,\mu}](w^{\kappa,\nu})\\ &\qquad\times(\sqrt{2})^{-\varepsilon(\lambda,\mu)-l(\lambda)-l(\mu)}Q_\lambda(x_1,\dots,x_{n_0})Q_\mu(y_1,\dots,y_{n_1}),\end{aligned}$$

and if k is odd, then we have

$$\begin{aligned}&\mathrm{Ch}[\bigoplus_{(\lambda,\mu)\in(DP^2)_k} V_{\lambda,\mu}\mathbin{\dot\circ} U_{\lambda,\mu}]\,(w^{\kappa,\nu}\otimes E)\\ &= \sqrt{2}\sum_{(\lambda,\mu)\in(DP^2)_k}\mathrm{Ch}[V_{\lambda,\mu}](w^{\kappa,\nu})\\ &\qquad\times(\sqrt{2})^{-\varepsilon(\lambda,\mu)-l(\lambda)-l(\mu)}Q_\lambda(x_1,\dots,x_{n_0})Q_\mu(y_1,\dots,y_{n_1}).\end{aligned}$$

Since these hold for all n_0 and n_1, the result follows. Q.E.D.

We review Stembridge's formula for the character values of simple $\mathcal{B}'_k$-modules, in a form adapted to the simple modules in the $\mathbb{Z}_2$-graded sense.

Theorem 4.5. (cf. [10, Lem. 7.5]) *We have*

$$2^{\frac{3(l(\kappa)+l(\nu))}{2}} p_\kappa(x) p_\nu(y) = \sum_{(\lambda,\mu)\in(DP^2)_k} \mathrm{Ch}[V_{\lambda,\mu}](w^{\kappa,\nu}) 2^{\frac{-l(\lambda)-l(\mu)-\varepsilon(\lambda,\mu)}{2}} Q_\lambda(x,y) Q_\mu(x,-y)$$

for all $(\kappa,\nu)\in(OP^2)_k$, where $Q_\lambda(x,y)=Q_\lambda(x_1,x_2,\dots,y_1,y_2,\dots)$ and $Q_\mu(x,-y)=Q_\mu(x_1,x_2,\dots,-y_1,-y_2,\dots)$.

The formula (4.15) is different from Stembridge's formula. Let us mention a relationship between the two formulas. Define an algebra endomorphism ι of $\Omega_x\otimes\Omega_y$ by $\iota(f\otimes 1)=f(x,y)=f(x_1,x_2,\dots,y_1,y_2,\dots)$ and $\iota(1\otimes g)=g(x,-y)=g(x_1,x_2,\dots,-y_1,-y_2,\dots)$ (since the y-part belongs to Ω_y, this "naïve notation" actually coincides with $g(x-y)$ in the Λ-ring notation). Note that $\{Q_\lambda(x,y)Q_\mu(x,-y) \mid (\lambda,\mu)\in(DP^2)\}$ is a basis of $\Omega_x\otimes\Omega_y$ (cf. [10, Th. 7.1, Lem. 7.5]). Since $\iota(Q_\lambda(x)Q_\mu(y)) = Q_\lambda(x,y)Q_\mu(x,-y)$, it follows that ι is an automorphism. Moreover, since $\iota(p_r(x,y))=2p_r(x)$ and $\iota(p_r(x,-y))=2p_r(y)$ for any odd r, it follows that the image of (4.15) under ι coincides with Stembridge's formula.

Appendix

A. Sergeev's duality. We review Sergeev's duality relation between $\mathcal{B}_k$ and $\mathcal{U}_n$ using DSCT. Define a map $d\colon DP_k\to\mathbb{Z}_2$ by $d(\lambda)=0$ (resp. $d(\lambda)=1$) if $l(\lambda)$ is even (resp. $l(\lambda)$ is odd).

Theorem A. [8] (1) *The two superalgebras $\Psi(\mathcal{B}_k)$ and $\mathcal{U}_n$ act on W as mutual supercentralizers of each other*:

$$\text{(A.1)} \qquad \mathrm{End}^{\cdot}_{\Theta(\mathcal{U}_n)}(W)=\Psi(\mathcal{B}_k), \quad \mathrm{End}^{\cdot}_{\Psi(\mathcal{B}_k)}(W)=\Theta(\mathcal{U}_n).$$

(2) *The simple $\mathcal{B}_k$-module W_λ ($\lambda\in DP_k$) occurs in W if and only if $l(\lambda))\le n$. Then we have*

$$\text{(A.2)} \qquad W \cong_{\mathcal{B}_k\dot{\otimes}\mathcal{U}_n} \bigoplus_{\substack{\lambda\in DP_k\\ l(\lambda)\le n}} W_\lambda \,\dot{\circ}\, U_\lambda$$

where U_λ denotes a simple $\mathcal{U}_n$-module corresponding to W_λ in W in the sense of DSCT.

(3) *The character values of* $\mathrm{Ch}[U_\lambda]$ *are given as follows*:

$$\mathrm{Ch}[U_\lambda](x_1, x_2, \ldots, x_n) = (\sqrt{2})^{d(\lambda)-l(\lambda)} Q_\lambda(x_1, x_2, \ldots, x_n). \tag{A.3}$$

B. A duality of $\mathcal{A}_k$ and $\mathfrak{q}(n)$. We established a duality relation between $\mathcal{A}_k$ and $\mathcal{U}_n$ on the same space W^ε as in Theorem 4.2.

Theorem B. [11, Th. 4.1] *The submodule* W^ε *is decomposed as a multiplicity-free sum of simple* $\mathcal{A}_k \dot{\otimes} \mathcal{U}_n$*-modules as follows*:

$$W^\varepsilon \cong_{\mathcal{A}_k \dot{\otimes} \mathcal{U}_n} \bigoplus_{\substack{\lambda \in DP_k \\ l(\lambda) \leq n}} V_\lambda \mathbin{\dot{\circ}} U_\lambda. \tag{B.1}$$

(1) *Assume that k is even. Then the simple* $\mathcal{A}_k \dot{\otimes} \mathcal{U}_n$*-modules* $V_\lambda \mathbin{\dot{\circ}} U_\lambda$ *in* W^ε *are of type* M. *Furthermore we have*

$$\mathrm{End}^{\cdot}_{\Theta(\mathcal{U}_n)}(W^\varepsilon) = \Psi(\mathcal{A}_k), \quad \mathrm{End}^{\cdot}_{\Psi(\mathcal{A}_k)}(W^\varepsilon) = \Theta(\mathcal{U}_n). \tag{B.2}$$

(2) *Assume that k is odd. Then the simple* $\mathcal{A}_k \dot{\otimes} \mathcal{U}_n$*-modules* $V_\lambda \mathbin{\dot{\circ}} U_\lambda$ *in* W^ε *are of type* Q. *Furthermore we have*

$$\mathrm{End}^{\cdot}_{\Theta(U_n)}(W^\varepsilon) \cong \mathcal{C}_1 \dot{\otimes} \Psi(\mathcal{A}_k), \quad \mathrm{End}^{\cdot}_{\Psi(\mathcal{A}_k)}(W^\varepsilon) \cong \mathcal{C}_1 \dot{\otimes} \Theta(\mathcal{U}_n). \tag{B.3}$$

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