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Weighted Chern-Mather classes and Milnor classes of hypersurfaces

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Abstract.

We introduce a class extending the notion of Chern-Mather class to possibly nonreduced schemes, and use it to express the difference between Schwartz-MacPherson's Chern class and the class of the virtual tangent bundle of a singular hypersurface of a nonsingular variety. Applications include constraints on the possible singularities of a hypersurface and on contacts of nonsingular hypersurfaces, and multiplicity computations.

§0. Introduction

The notion of Chern-Mather class was introduced by Robert MacPherson in [10], as one of the main ingredients in his definition of functorial Chern classes for possibly singular complex varieties. An equivalent notion had in fact already been given by Wentsün Wu; the two notions are compared in [15]. One way to think about Mather's class of Y as defined by MacPherson is the following: blow-up Y so that the pull-back of its sheaf of differentials is locally free modulo torsion; then mod out the torsion, dualize, and take Chern classes. The operation can in fact be performed for any sheaf; this is worked out in [9].

This definition ignores possible nilpotents on Y. We feel that it would be desirable to have a class in the spirit of Chern-Mather class, but in some way sensitive to possible nonreduced structures on Y: first, this is natural from the algebro-geometric standpoint; secondly, as we will see, a natural candidate carries useful information when applied to

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the *singularity subscheme* of a hypersurface (for which possibly non-reduced scheme structures play a fundamental rôle).

Our candidate is introduced in §1. Its definition is a suitable weighted sum of 'conventional' Chern-Mather classes of subvarieties of Y. The subvarieties are the supports of the components of the (intrinsic) normal cone of Y, and the weights are the lengths of the components of this cone. The class we obtain (trivially) agrees with Mather's if Y is a reduced local complete intersection.

If Y is the singularity subscheme of a hypersurface, we can relate the weighted Chern-Mather class with other natural classes defined in this case. For example, in [1] we have defined and studied a *μ-class* associated with the singularity subscheme of a hypersurface; in this paper, we answer a question which we could not address previously: how to give a reasonable definition for arbitrary schemes Y, from which the μ-class could be recovered if Y is the singularity subscheme of a hypersurface X. The weighted Chern-Mather class is precisely such a class (Corollary 1.4). We hope that this viewpoint will eventually give us the right hint on how to define a μ-class for the singularities of more general varieties X.

The main application of weighted Chern-Mather classes is to the computation of the difference between Schwartz-MacPherson's class of a hypersurface and the class of its virtual tangent bundle. A formula for the difference, in terms of the μ-class, is proved 'numerically' in [3], and at the level of Chow groups in [2] (Theorem I.5). Such differences have been named 'Milnor classes', as they generalize the fact that, for local complete intersections with isolated singularities, the Milnor *number* computes the difference between the (topological) Euler characteristic and the degree of the class of the virtual tangent bundle (see [13], [14], [12], [5], and references therein).

Weighted Chern-Mather classes allow us to recast the formula from [2]. We state this in §1 (Theorem 1.2), together with other facts about weighted Chern-Mather classes, such as their relation vis-a-vis a class appearing in [12] or their behavior under blow-ups. Proofs of these statements are sketched in §2, together with a few general considerations regarding Milnor classes. Theorem 1.2 is proved in full in §2.

The expression for the μ-class in terms of weighted Chern-Mather classes allows us in principle to compute the former for a wide class of examples. We give a couple of applications in this direction in §3, in the spirit of the examples worked out in [1], §4. For example, we prove that if two nonsingular hypersurfaces M_1, M_2 of degrees d_1, d_2 in projective space are tangent along a positive dimensional subvariety, then $d_1 = d_2$. This fact was proved in [2], but with a strong additional hypothesis on the contact locus of M_1 and M_2; the new formula for the μ-class shows that the extra hypothesis is unnecessary. We also collect in §3 a few explicit computations of weighted Chern-Mather classes.

The core of this paper is little more than a rewriting of a part of [12]. In that reference, Adam Parusiński and Piotr Pragacz give an alternative proof of the formula in [2] by a local computation of multiplicities, which relates it to a formula from [6] (over $\mathbb{C}$, and in homology) for the characteristic cycle of a hypersurface. For singularities of a hypersurface, a complex geometry analog of weighted Chern-Mather classes is introduced in [12]; the classes are compared here in Theorem 1.5. The proof of Theorem 1.2 given in §2 owes much to the approach of Parusiński and Pragacz: it is my attempt to produce a proof in the style of [12], but in a set-up closer to intersection theory in algebraic geometry (hence valid for rational equivalence; and potentially more amenable to algebraic generalizations, e.g., to positive characteristic). The reference to [6] is bypassed by an explicit computation of local Euler obstructions.

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§1. Weighted Chern-Mather classes.

All schemes in this note are of finite type over an algebraically closed field of characteristic 0, and (for simplicity) embeddable in an ambient nonsingular variety, which we will denote by M.

Assume that Y is reduced and irreducible, of dimension k. The *Chern-Mather* class of Y can be defined as follows. Let $G_k(TM)$ denote the Grassmann bundle whose fiber over $p \in M$ consists of the Grassmannian of k-planes in TM, and let Y° be the nonsingular locus in Y. Consider the map

$$Y^\circ \to G_k(TM)$$

defined by sending $p \in Y^\circ$ to $T_pY \subset T_pM$. The *Nash blow-up* of Y is the closure $\widetilde{Y}$ of the image of this map; it comes equipped with a proper map ν to Y, and with the restriction T of the tautological subbundle over $G_k(TM)$. This data is easily checked to be independent of the ambient variety M. The Chern-Mather class of Y is defined by

$$c_{\mathrm{Ma}}(Y) := \nu_* \left(c(T) \cap [\widetilde{Y}]\right)$$

in the Chow group A_*Y of Y. This class of course agrees with the total ('homology') class of the tangent bundle of Y if Y happens to be nonsingular to begin with.

Note that this definition assumes that Y is reduced, as it needs Y to be nonsingular at the general point, and ignores by construction the presence of nilpotents along subvarieties of Y. Our task is to modify this notion to take account of possible nilpotents on Y.

Let then $Y \subset M$ be arbitrary. We consider the normal cone $C_Y M$ of Y in M, and associate with Y the set $\{(Y_i, m_i)\}_i$, where the $Y_i \overset{j_i}{\hookrightarrow} Y$ are the supports of the irreducible components C_i of $C_Y M$, and m_i denotes the geometric multiplicity of C_i in $C_Y M$ (so $[C_i] = m_i[(C_i)_{\mathrm{red}}]$).

Lemma 1.1. *The data $\{(Y_i, m_i)\}$ is intrinsic of Y, i.e., independent of the ambient nonsingular variety.*

Proof. (Cf. [7], Example 4.2.6.) It is enough to compare embeddings $Y \hookrightarrow M$, $Y \hookrightarrow M'$, where both M, M' are nonsingular, and M is smooth over M'. In this case there is an exact sequence of cones

$$0 \to T_{M'|M} \to C_Y M \to C_Y M' \to 0$$

(where $T_{M'|M}$ is the relative tangent bundle) in the sense of [7], Example 4.1.6, and it follows that the supports of the irreducible components of the two cones coincide, as well as the geometric multiplicities of the components. Q.E.D.

By Lemma 1.1, the following definition is also intrinsic of Y:

Definition. The *weighted Chern-Mather class* of Y is

$$c_{\mathrm{wMa}}(Y) := \sum_i (-1)^{\dim Y - \dim Y_i} m_i j_{i*} c_{\mathrm{Ma}}(Y_i) \quad \text{in } A_*Y.$$

(Warning: we will henceforth neglect to indicate 'obvious' push-forwards such as j_{i*}, and pull-backs.)

Note that if Y is a reduced irreducible local complete intersection, then its normal cone is reduced and irreducible, so the class defined here agrees with the Chern-Mather class of Y. In particular, if Y is nonsingular then $c_{\mathrm{wMa}}(Y) = c(TY) \cap [Y]$ is the total homology class of the tangent bundle of Y.

A few examples of computations of weighted Chern-Mather classes can be found in §3. Our main motivation in introducing the class $c_{\mathrm{wMa}}(Y)$ is that we can prove it is particularly well-behaved if Y is the *singularity scheme of a hypersurface* X in a nonsingular variety M. By hypersurface here we mean the zero-scheme of a nonzero section of a line-bundle $\mathcal{L}$ on M; the singularity subscheme of X is the subscheme locally defined by the partial derivatives of an equation for X. (This scheme structure is independent of the ambient variety M.) In the rest of this section we survey a few facts about $c_{\mathrm{wMa}}(Y)$ under the hypothesis that Y is the singularity subscheme of a hypersurface. Proofs are given in §2.

Our motivation is to highlight apparently different contexts in which the class $c_{\mathrm{wMa}}(Y)$ manifests itself. Although these contexts will invoke other characters of the play, remember that $c_{\mathrm{wMa}}(Y)$ is a class *intrinsic of* Y, and which is defined regardless of whether Y is the singularity subscheme of a hypersurface. The challenge is to find extensions of these results which do not assume that Y is the singularity subscheme of a hypersurface.

For the first fact, let $c_{\mathrm{SM}}(X) \in A_*X$ denote Schwartz-MacPherson's Chern class of X, and let $c_{\mathrm{F}}(X) \in A_*X$ denote the class of its virtual tangent bundle; the subscript F is to remind us that this class agrees with the class introduced (for much more general schemes) by William Fulton, cf. Example 4.2.6 of [7].

Theorem 1.2. *Let $\mathcal{L} = \mathcal{O}(X)$, and let Y be the singularity subscheme of X. Then*

$$c_{wMa}(Y) = (-1)^{\dim X - \dim Y} c(\mathcal{L}) \cap (c_F(X) - c_{SM}(X)) \quad \text{in } A_*(X).$$

That is, $c_{\mathrm{wMa}}(Y)$ essentially measures the difference between the functorial homology Chern class $c_{\mathrm{SM}}(X)$ and the class of the virtual tangent bundle of X. The functoriality of the class $c_{\mathrm{SM}}(X)$ was proved by Robert MacPherson [10]; the class was later shown to agree with the class previously defined by Marie-Hélène Schwartz. For a treatment of Schwartz-MacPherson's classes over any algebraically closed field of characteristic 0, see [8]; this is the context we assume here. Also, we let $c_{\mathrm{SM}}(X) = c_{\mathrm{SM}}(X_{\mathrm{red}})$; with this proviso, Theorem 1.2 holds for nonreduced hypersurfaces X—remarkably, the drastic change in c_{F} when some component of X is replaced by a multiple is precisely compensated by the change in the weighted Mather class of the singularity subscheme.

For the next result, it is convenient to employ the following notations (a variation on the notations used in [2], [3]): for $a \in A_p$ and $\mathcal{L}$ a line bundle, set

$$a_\vee = (-1)^p a \quad , \quad a_{\mathcal{L}} = c(\mathcal{L})^p \cap a \quad .$$

(So $a_{\mathcal{L}} = c(\mathcal{L})^n \cap (a \otimes \mathcal{L})$, where the term in () uses the definition in [3], and n is the dimension of the ambient scheme). These notations behave well with respect to several natural operations, similarly to the notations introduced in [3]. For example, the formula on the right defines an action of Pic on the Chow group: that is, $a_{\mathcal{L}_1 \otimes \mathcal{L}_2} = (a_{\mathcal{L}_1})_{\mathcal{L}_2}$ for line bundles $\mathcal{L}_1$ and $\mathcal{L}_2$.

Proposition 1.3. *Let Y be the singularity subscheme of a section of a line bundle $\mathcal{L}$ on a nonsingular variety M. Then*

$$c_{wMa}(Y) = (-1)^{\dim Y} \left(c(T^*M \otimes \mathcal{L}) \cap s(Y, M)\right)_{\vee \mathcal{L}} \quad \textit{in } A_*Y.$$

Here $s(Y, M)$ denotes the *Segre class* of Y in M, in the sense of [7], Chapter 4. Note that this equality is completely false unless Y is a singularity subscheme of a hypersurface in M. However, if Y *is* a singularity subscheme of a hypersurface in M, then the right-hand-side must be independent of M: this was proved directly in [1], Corollary 1.7, and follows again as the left-hand-side is intrinsic of Y. Proposition 1.3 is significant in view of the consequence:

Corollary 1.4.

$$\mu_{\mathcal{L}}(Y) = (-1)^{\dim Y} c_{wMa}(Y)_{\vee \mathcal{L}} \quad .$$

The class $\mu_{\mathcal{L}}(Y)$ is the 'μ-class' defined and studied in [1]; it carries a notable amount of information about X, with applications to duality and to the study of contacts of hypersurfaces. Corollary 1.4 solves a puzzle left open in [1] (p. 326): to define a class for arbitrary schemes, specializing to $\mu_{\mathcal{L}}(Y)$ for singular schemes of hypersurfaces. It also clarifies the dependence of the μ-class on the line bundle $\mathcal{L}$: it follows from Corollary 1.4 that if $\mathcal{L}_1$, $\mathcal{L}_2$ are line bundles, then

$$\mu_{\mathcal{L}_2}(Y) = \mu_{\mathcal{L}_1}(Y)_{\mathcal{L}_1^\vee \otimes \mathcal{L}_2}$$

(this does *not* follow formally from the expression for the μ-class in terms of the Segre class of Y.) For applications of Proposition 1.3 and Corollary 1.4, see Examples 3.4, 3.5.

The next fact we list also requires some notations. We now assume that X is a reduced hypersurface, over $\mathbb{C}$. The question is whether, in this particularly 'geometric' case, $c_{\mathrm{wMa}}(Y)$ can be recovered from numerical invariants of X. The answer comes from [12]: define a function

$\mu : Y \to \mathbb{Z}$ by setting $\mu(y) = (-1)^{\dim X}(\chi(y) - 1)$, where $\chi(y)$ is the Euler characteristic of the Milnor fiber of X at y; μ is a constructible function on Y, so we can apply to it MacPherson's transformation c_{SM} (that is, write μ as a linear combination of characteristic functions 1_Z for subvarieties Z of Y, then replace each 1_Z in this combination by $c_{\mathrm{SM}}(Z)$).

Theorem 1.5.

$$c_{wMa}(Y) = (-1)^{\dim Y} c_{SM}(\mu) \quad \textit{in } A_*Y.$$

Equivalently, write μ as a linear combination of local Euler obstructions (also an ingredient in [10]): $\mu = \sum \ell_i \mathrm{Eu}_{Y_i}$; then the content of Theorem 1.5 is that in this situation the Y_i's are precisely the supports of the components of the normal cone of Y, and the numbers ℓ_i determined by μ agree (up to sign) with the multiplicities m_i used to define $c_{\mathrm{wMa}}(Y)$. Again, we would be very interested in extensions of this result to more general Y: what numerical invariants of a space X (not necessarily a hypersurface) determine the multiplicities of the components of the normal cone of its singularity subscheme? Can these multiplicities be computed for an arbitrary scheme Y, by a similar 'Milnor fiber' approach? Once more, note that the left-hand-side in Theorem 1.5 is defined for arbitrary Y; to what extent can the right-hand-side also be defined for arbitrary Y? We know of several problems in enumerative geometry for which finding these multiplicities is one of the main computational ingredients. For an explicit computation (not directly related to enumerative geometry) see Example 3.6.

Finally, it would be interesting to have results on the functoriality of the class $c_{\mathrm{wMa}}(Y)$; little is known about the functoriality of the ordinary Chern-Mather class. Again, something can be said if Y is the singularity subscheme of a hypersurface X (over an arbitrary algebraically closed field of characteristic zero, and possibly nonreduced). Let Z be a nonsingular subvariety of $Y \subset X \subset M$, and consider the blow-up $\widetilde{M}$ of M along Z:

$$\begin{tikzcd} & Y' \arrow[r] \arrow[d] & X' \arrow[r] \arrow[d, "\rho"] & \widetilde{M} \arrow[d, "\pi"] \\ Z \arrow[r] & Y \arrow[r] & X \arrow[r] & M \end{tikzcd} .$$

Here $X' = \pi^{-1}X$ is the scheme-theoretic inverse image of X, a hypersurface of $\widetilde{M}$, and Y' is the singularity subscheme of X'.

Proposition 1.6. *Assume Z has codimension d in M. Then*

$$\rho_* c_{wMa}(Y') = (-1)^{\dim X - \dim Y} c_{wMa}(Y) - (d-1) c_{wMa}(Z) \quad \textit{in } A_*X.$$

Here of course $c_{\mathrm{wMa}}(Z) = c(TZ) \cap [Z]$, as Z is nonsingular. Also note that by assumption X is singular along Z, hence Y' contains the exceptional divisor in $\widetilde{M}$.

Proofs of the statements made in this section are sketched in §2, with emphasis on Theorem 1.2, which relates the weighted Chern-Mather class of the singularity of a hypersurface with its *Milnor class.*

§2. The Milnor class of a hypersurface.

As is well known, for a compact complex hypersurface X with isolated singularities the sum of the Milnor numbers of the singularities measures the difference between the topological Euler characteristic of X and that of a nonsingular hypersurface linearly equivalent to X (if there is such a hypersurface). To my knowledge, the first who used this fact to define and study a generalization of the Milnor number to non-isolated hypersurface singularities is Adam Parusiński, [11].

Now, the functoriality of Schwartz-MacPherson's class implies that, for a hypersurface X as above, the Euler characteristic of X equals the degree of the (zero-dimensional component of the) class $c_{\mathrm{SM}}(X)$. On the other hand, the Euler characteristic of a nonsingular hypersurface linearly equivalent to X equals the degree of the class of the virtual tangent bundle of X (that is, of $c_{\mathrm{F}}(X)$ with notations as in §1). That is, Parusiński's Milnor number equals (up to a sign), the degree of the difference between the two classes:

$$\int (c_{\mathrm{F}}(X) - c_{\mathrm{SM}}(X)) \quad .$$

It is natural then to study the whole class $c_{\mathrm{F}}(X) - c_{\mathrm{SM}}(X)$; this (or slight variations of it) has been named the *Milnor class* of X by some authors (see [5], [12], [14]).

Note that nothing in the definition of the class $c_{\mathrm{F}}(X) - c_{\mathrm{SM}}(X)$ requires X to be a hypersurface: both Schwartz-MacPherson's and Fulton's classes can be defined for arbitrary varieties. For reduced compact complex local complete intersections, the Milnor class is computed in homology in [5] in terms of vector fields on X, an approach reminiscent of Schwartz's definition of $c_{\mathrm{SM}}(X)$.

In fact the class makes sense for arbitrary schemes X over any algebraically closed field of characteristic 0, and naturally lives in the Chow group A_*Y of the singular locus of X. We would like to pose the following question:

—*To what extent is the Milnor class of X determined by the singularity subscheme Y of X?* or, in more ambitious terms:

—Is there a natural definition of a class on an arbitrary scheme Y, from which the Milnor class of X can be computed if Y is the singularity subscheme of X?

In view of the results collected in §1, the situation is clear for hypersurfaces. The singular locus of a hypersurface has a natural scheme structure, given by the partial derivatives of local equations of X. Theorem 1.2 then asserts that (for arbitrary hypersurfaces X over an algebraically closed field of characteristic 0, and writing $\mathcal{L} = \mathcal{O}(X)_{|Y}$)

$$c_{\mathrm{F}}(X) - c_{\mathrm{SM}}(X) = (-1)^{\dim X - \dim Y} c(\mathcal{L})^{-1} \cap c_{\mathrm{wMa}}(Y) \quad \text{in } A_*X:$$

that is, *if two hypersurfaces have the same singularity subscheme Y and their line bundles restrict to the same bundle on Y, then they have the same Milnor class; and, further, this can be recovered from the class $c_{wMa}(Y)$, which can be defined for arbitrary schemes Y.*

Therefore, Theorem 1.2 answers the two questions posed above, for hypersurfaces. To our knowledge, the questions are completely open for more general schemes X. Milnor classes of local complete intersections (for which the singular locus also carries a natural scheme structure) have been studied in [5], but from a different viewpoint, which does not seem to address questions such as the ones posed above.

Theorem 1.2 could be deduced from results in the existing literature (particularly from [12] or [2]). However, while the main result in [2] is at the level of generality at which we are aiming, its proof is rather unenlightening. The approach in [12] is much more cogent, but it is stated in homology and relies on the complex geometry of the situation—for example, in [12] the hypersurface is assumed to be reduced and compact. The argument given below works for possibly nonreduced hypersurfaces, over arbitrary algebraically closed fields of characteristic 0, and gives the formula in rational equivalence; it only relies on the basic formalism of Schwartz-MacPherson's classes (as developed in [8]). We would like to stress that, anyway, at its core is a multiplicity computation we learned from [12].

Proof of Theorem 1.2. We consider the blow-up $\widetilde{M} \xrightarrow{\pi} M$ along Y, and let $\mathcal{X}$, $\mathcal{Y}$ be the pull-back of X and the exceptional divisor, respectively. Note that $\mathcal{Y} \subset \mathcal{X}$, so[1] there is an effective Cartier divisor in $\widetilde{M}$ whose cycle equals $\mathcal{X} - \mathcal{Y}$; we will denote this divisor by $\mathcal{X} - \mathcal{Y}$.

[1]A note of warning to non-algebraic geometers: here and in the following we are using common set-theoretic notations (such as $\subset$, $-$, etc.) in their scheme-theoretic sense. For example, $\mathcal{Y} \subset \mathcal{X}$ means that the ideal sheaf of $\mathcal{X}$ is contained in the ideal sheaf of $\mathcal{Y}$. Since both $\mathcal{X}$ and $\mathcal{Y}$ are Cartier divisors,

Now let p be a point of X. We have $\pi^{-1}(p) \subset \mathcal{X} - \mathcal{Y}$, so it makes sense to consider the Segre class of $\pi^{-1}(p)$ in $\mathcal{X} - \mathcal{Y}$.

Claim 2.1. *Denoting degree by* $\int$,

$$\int \frac{s(\pi^{-1}(p), \mathcal{X} - \mathcal{Y})}{1 + \mathcal{X} - \mathcal{Y}} = 1 \quad .$$

A preliminary result is in order before we prove this claim. We have

$$\pi^{-1}(p) \hookrightarrow (\mathcal{X} - \mathcal{Y}) \hookrightarrow \widetilde{M} \quad ,$$

where the second embedding is regular. We claim that

$$s(\pi^{-1}(p), \mathcal{X} - \mathcal{Y}) = c(N_{\mathcal{X}-\mathcal{Y}}\widetilde{M}) \cap s(\pi^{-1}(p), \widetilde{M}) \quad .$$

Note that this is *not* automatic in this situation, cf. Example 4.2.8 in [7]. In our case, it will follow from the following lemma:

Lemma 2.2. *Let D, E be hypersurfaces in a variety V. Assume that $D - E$ is positive and has no components in common with E. Then $s(E, D) = c(N_D V) \cap s(E, V)$.*

Proof of the lemma. By the hypothesis and Lemma 4.2 in [7],

$$\begin{aligned} s(E, D) &= s(E, E) + s(E \cap (D - E), D - E) = [E] + \frac{E \cdot (D - E)}{1 + E} \\ &= \frac{([E] + E \cdot E) + E \cdot (D - E)}{1 + E} = (1 + D) \cap \frac{[E]}{1 + E} \\ &= c(N_D V) \cap s(E, V) \quad . \end{aligned}$$

Q.E.D.

Proof of Claim 2.1. We apply Lemma 2.2 to the normalized blow-up V of $\widetilde{M}$ along $\pi^{-1}(p)$, with E =the exceptional divisor, and D =the inverse image of $\mathcal{X} - \mathcal{Y}$. To see that the hypotheses are satisfied, we have to show that every component of E appears with the same multiplicity in E and D.

For this[2], let $\gamma(t)$ be a germ of a nonsingular curve centered at the general point of a component of E, let $\tilde{\gamma}(t)$ be the composition to M, and let F be a local equation for X at p; also, choose local parameters $x_1, \ldots, x_n$ for M at p. The ideal of E is the pull-back of $(x_1, \ldots, x_n)$ to V, so the multiplicity m_E of the component in E equals the order of

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this just says that local equations for $\mathcal{X}$ are multiples of local equations for $\mathcal{Y}$. This is necessary for the statement that follows.

[2]This computation is essentially lifted from an analogous computation in the proof of Proposition 2.2 in [12].

vanishing of the pull-back $x_i(t) = \tilde{\gamma}^* x_i$ of a generic local parameter. The multiplicity m_D in D equals $m_{\mathcal{X}} - m_{\mathcal{Y}}$, where $m_{\mathcal{X}}$, $m_{\mathcal{Y}}$ are respectively the multiplicities in the pull-backs of $\mathcal{X}$, $\mathcal{Y}$.

Now $m_{\mathcal{X}}$ is the order of vanishing of

$$\tilde{\gamma}^* F = F(x_1(t), \ldots, x_n(t)) \quad ,$$

while $m_{\mathcal{Y}}$ is the order of vanishing of the pull-back of

$$\left(F, \frac{\partial F}{\partial x_1}, \ldots, \frac{\partial F}{\partial x_n}\right) \quad ,$$

that is, the order of vanishing of $\tilde{\gamma}^* \frac{\partial F}{\partial x_i}$ for a generic local parameter x_i. Now taking the derivative with respect to t gives (by the chain rule!)

$$m_{\mathcal{X}} - 1 = m_{\mathcal{Y}} + m_E - 1 \quad ,$$

from which the desired equality $m_E = m_D$ follows.

Applying Lemma 2.2, we get

$$s(E, D) = (1 + \mathcal{X} - \mathcal{Y}) \cap s(E, V) \quad ,$$

hence

$$s(\pi^{-1}(p), \mathcal{X} - \mathcal{Y}) = (1 + \mathcal{X} - \mathcal{Y}) \cap s(\pi^{-1}(p), \widetilde{M})$$

by the birational invariance of Segre classes ([7], Proposition 4.2). From this,

$$\pi_* \frac{s(\pi^{-1}, \mathcal{X} - \mathcal{Y})}{1 + \mathcal{X} - \mathcal{Y}} = s(p, M) = [p] \quad ,$$

again by the birational invariance of Segre classes, and the claim follows by taking degrees. Q.E.D.

We are finally ready to prove Theorem 1.2. Identify $\mathcal{Y}$ with the projective normal cone of Y in M, let $\mathcal{Y}_i$ be the reduced components of $\mathcal{Y}$, and let Y_i be their support in Y. Then

$$\begin{cases} \mathcal{X} & = \widetilde{X} + \sum n_i \mathcal{Y}_i \\ \mathcal{Y} & = \sum m_i \mathcal{Y}_i \end{cases}$$

for suitable m_i, n_i. By Claim 2.1,

$$\begin{aligned}
1 &= \int \frac{s(\pi^{-1}(p), \mathcal{X} - \mathcal{Y})}{1 + \mathcal{X} - \mathcal{Y}} \\
&= \int \frac{s(\pi^{-1}(p) \cap \widetilde{X}, \widetilde{X}) + \sum (n_i - m_i) s(\pi^{-1}(p) \cap \mathcal{Y}_i, \mathcal{Y}_i)}{1 + \mathcal{X} - \mathcal{Y}} \\
&\quad \text{by Lemma 4.2 in [7]} \\
&= \mathrm{Eu}_X(p) + \sum (n_i - m_i)(-1)^{\dim M + 1 - \dim Y_i} \mathrm{Eu}_{Y_i}(p)
\end{aligned}$$

using the formula for Euler obstructions due to Gonzalez-Sprinberg and Verdier, as computed in [8], Lemma 2 (as pointed out in [2], §1.3 and in [12], §3, the divisor $\mathcal{X} - \mathcal{Y}$ can be embedded in $\mathbb{P}(T^*M)$, and $1 + \mathcal{X} - \mathcal{Y}$ is then the restriction of the class of the tautological bundle in $\mathbb{P}(T^*M)$). Now, every relation between constructible functions yields a relation for characteristic classes. Here, this gives (using the formula for Mather's classes in [8], Lemma 1, going back to Claude Sabbah):

$$\begin{aligned}
c_{\mathrm{SM}}(X) &= c_{\mathrm{Ma}}(X) + \sum (n_i - m_i)(-1)^{\dim M + 1 - \dim Y_i} c_{\mathrm{Ma}}(Y_i) \\
&= c(TM) \cap \pi_* \left(\frac{[\widetilde{X}]}{1 + \mathcal{X} - \mathcal{Y}} + \sum (n_i - m_i) \frac{[\mathcal{Y}_i]}{1 + \mathcal{X} - \mathcal{Y}} \right) \\
&= c(TM) \cap \pi_* \left(\frac{[\mathcal{X}]}{1 + \mathcal{X}} - \frac{1}{1 + \mathcal{X}} \sum m_i \frac{[\mathcal{Y}_i]}{1 + \mathcal{X} - \mathcal{Y}} \right) \\
&= c_{\mathrm{F}}(X) + c(\mathcal{L})^{-1} \cap \sum m_i (-1)^{\dim M - \dim Y_i} c_{\mathrm{Ma}}(Y_i) \\
&= c_{\mathrm{F}}(X) + (-1)^{\dim M - \dim Y} c(\mathcal{L})^{-1} \cap c_{\mathrm{wMa}}(Y)
\end{aligned}$$

which is the desired formula. Q.E.D.

As observed in the proof, $\mathcal{X} - \mathcal{Y}$ can be naturally embedded in $\mathbb{P}(T^*M)$. The content of Claim 2.1 is that $\mathcal{X} - \mathcal{Y}$ gives then the *characteristic cycle* of X (corresponding to the characteristic function 1_X of X in M).

The other statements in §1 now follow easily, either by comparing the expression for c_{SM} with the expressions in [2] and [12], or by direct manipulations that can be extracted from those sources. The argument given here re-proves Theorem I.3 in [2]/Theorem 3.1 in [12]; and for example, §1 in [2] shows how to go directly from this form of the result to expressions in terms of Segre classes or μ-classes (thus proving Proposition 1.3, Corollary 1.4).

The details are left to the reader. Theorem 1.5 is our reading of Theorem 2.3 (iii) from [12]. The blow-up formula of Proposition 1.6 follows from Proposition IV.2 in [2].

§3. Examples and applications.

Normal cones behave well with respect to proper finite maps and with respect to flat maps, cf. Proposition 4.2 in [7]. For example, assume that Y is irreducible, and $\widetilde{M} \xrightarrow{\pi} M$ is a surjective birational map on the ambient space. Then there is an induced surjective map from the cone of $\pi^{-1}Y$ to the cone of Y. This can be used to obtain the data $\{(Y_i, m_i)\}$ of §1, for example by suitably blowing up an ambient space; this can lead to direct computations of weighted Chern-Mather classes.

Example 3.1. Suppose Y consists of a curve C, with an embedded multiple planar point at a point p. More precisely, assume C, Y have local ideals respectively $\mathcal{I}_C$, $\mathcal{I}_C \cdot (x,y)^m$, $m \geq 1$, near p in a nonsingular ambient surface S with local parameters x, y. Also, assume that C has multiplicity r at p. Then

$$c_{\mathrm{wMa}}(Y) = c_{\mathrm{Ma}}(C) - (m+r)[p] \quad .$$

Indeed, blow-up S at p; the total transform of Y consists of the proper transform of C, and of $(m+r)$ times the exceptional divisor. Therefore, the normal cone of Y contains a component with multiplicity $m+r$ over p. (But note there is no such component if $m = 0$.)

For example, take Y to be the union of two lines ℓ_1, ℓ_2 in $\mathbb{P}^2$, with an embedded planar point at the intersection $p = \ell_1 \cap \ell_2$; then $c_{\mathrm{wMa}}(Y) = [\ell_1] + [\ell_2] + [p]$. If the embedded point is on one of the lines, but not at p, then $c_{\mathrm{wMa}}(Y) = [\ell_1] + [\ell_2] + 2[p]$. If each line comes with multiplicity r, and the embedded point is at p, then the class is

$$r c_{\mathrm{Ma}}(\ell_1) + r c_{\mathrm{Ma}}(\ell_2) - (1+2r)[p] = r[\ell_1] + r[\ell_2] + (2r-1)[p] \quad .$$

Example 3.2. Example 3.1 can be easily generalized to the situation in which Y is a subscheme of a given ambient space M, and the residual to a Cartier divisor D in Y is a known scheme Y'. Then $c_{\mathrm{wMa}}(Y)$ can be written in terms of $c_{\mathrm{Ma}}(D)$, $c_{\mathrm{wMa}}(Y')$, and the multiplicity of D along the distinguished components of Y'; details are left to the reader. A very different expression can be obtained if Y' is the singularity subscheme of a hypersurface X in M, and D is the r-th multiple of X $(r \geq 0)$.

Claim 3.1. *Let $\mathcal{L} = \mathcal{O}(X)_{|Y'}$. Then*

$$c_{wMa}(Y) = r\, c_F(X) + (-1)^{\dim X - \dim Y'} \frac{c(\mathcal{L}^{\otimes(r+1)})}{c(\mathcal{L})} \cap c_{wMa}(Y') \quad .$$

The proof is an easy application of the results in §1, and is also left to the reader.

To contrast the two approaches, take again the example of the union of two lines ℓ_1, ℓ_2 in $\mathbb{P}^2$, each coming with multiplicity r, with an embedded planar point at the intersection. Since the planar point is the singularity subscheme of the union of two (simple) lines, Claim 3.1 computes the weighted Chern-Mather class of this scheme as

$$\begin{aligned} r\, c_{\mathrm{F}}(X) &+ (-1)^{\dim X - \dim Y'} \frac{c(\mathcal{L}^{\otimes(r+1)})}{c(\mathcal{L})} \cap c_{\mathrm{wMa}}(Y') \\ &= r \frac{c(T\mathbb{P}^2)}{c(\mathcal{O}_{\mathbb{P}^2}(2))} \cap ([\ell_1] + [\ell_2]) - [p] \\ &= r([\ell_1] + [p]) + r([\ell_2] + [p]) - [p] \end{aligned}$$

with the same result as before, but by a very different route.

It would be useful to have formulas such as Claim 3.1, but with less stringent hypotheses on X.

Example 3.3. If $X = X_1 \cup \cdots \cup X_r$ is a divisor with normal crossings, with all X_i supported on nonsingular hypersurfaces $(X_i)_{\mathrm{red}}$, and Y is its singularity subscheme, then

$$c_{\mathrm{wMa}}(Y) = \pm c(TM) \cap \left(1 - \frac{1 + [X]}{(1 + (X_1)_{\mathrm{red}}) \cdots (1 + (X_r)_{\mathrm{red}})}\right) \cap [M] \quad ,$$

taking the sign $+$, resp. $-$ according to whether X is reduced or not. The expression is interpreted by expanding it, which leaves a class naturally supported on Y; it follows from Proposition 1.3 and [2], §2.2 (Lemma II.2 in [2] computes the Segre class if X is reduced, and the computation in the proof of Lemma II.1 is used to cover the non-reduced case).

Example 3.4. What do we learn about hypersurfaces by studying their Milnor classes?

As shown in [1], the μ-class of a hypersurface X packs a good amount of information about X: for example, the multiplicity of X as a point of the discriminant of a linear system and the dimension of this discriminant can be recovered very easily from the μ-class (hence from the Milnor class). In the classical language, the μ-classes of hyperplane sections of an embedded nonsingular projective variety M give a localized analog of the *ranks* of M, and provide a natural tool to study projective duality.

In a different direction, the good behavior of the μ-class can be used to put restrictions on the possible singularities of a hypersurface in a given ambient space. Several examples of this phenomenon are illustrated in [1], §3, where the main tool was the observation that if the singularity subscheme Y of a hypersurface X is *nonsingular,* then

$$\mu_{\mathcal{L}}(Y) = c(T^*Y \otimes \mathcal{L}) \cap [Y] \quad .$$

Now, Corollary 1.4 from §1:

$$\mu_{\mathcal{L}}(Y) = (-1)^{\dim Y} c_{\mathrm{wMa}}(Y)_{\vee \mathcal{L}}$$

is a substantial upgrade of this formula, and this allows us to extend some of those results.

Claim 3.2. *If two smooth hypersurfaces of degree d_1, d_2 in projective space are tangent along a positive dimensional set, then $d_1 = d_2$.*

More generally, if two smooth hypersurfaces M_1, M_2 of a variety V are tangent along an irreducible (for simplicity) set Z, and $\dim Z > 0$, then we claim that

$$r\, M_1 \cdot [Z] = r\, M_2 \cdot [Z] \quad ,$$

where r is the order of tangency of M_1 and M_2 (for example, $r = 1$ if M_1, M_2 have simple contact). This is essentially Proposition IV.7 in [2], with all hypotheses on the contact locus (except the positive dimensionality) removed. The stronger statement given above follows from the results in §1. Indeed, in the situation of the statement, let $X = M_1 \cap M_2$; then X is a hypersurface in two distinct ways: with respect to $\mathcal{L}_2 = \mathcal{O}(M_2)_{|M_1}$ in M_1, and with respect to $\mathcal{L}_1 = \mathcal{O}(M_1)_{|M_2}$ in M_2. The contact locus is $Y = \operatorname{Sing} X$ (with the scheme structure specified in §1), and $[Y] = r[Z]$. By Theorem 1.2

$$c(\mathcal{L}_2)^{-1} \cap c_{\mathrm{wMa}}(Y) = c(\mathcal{L}_1)^{-1} \cap c_{\mathrm{wMa}}(Y) \quad ,$$

implying

$$c_1(\mathcal{L}_1) \cap [Y] = c_1(\mathcal{L}_2) \cap [Y] \quad ,$$

which is the statement.

Example 3.5. We say that a hypersurface X of a nonsingular variety M is (analytically) 'homogeneous at p' if the equation of X is homogeneous for some choice of system of parameters in the completion of the local ring for M at p. We are going to consider degree-d hypersurfaces X in $\mathbb{P}^n$, whose singular scheme Y has a connected component supported on a nonsingular curve C of genus g and degree r; we assume that Y has the reduced structure at all but finitely many points $q_1, \dots, q_s$, and that X is homogeneous at each of the q_i. In particular, X has multiplicity 2 at all other points of C; we let m_i be the multiplicity of X at q_i.

How constrained is this situation? Examples 3.4—3.6 in [1] deal with the case in which the singular scheme is reduced, that is, there are no points 'q_i' as above. This situation is then very rigid: for example, one sees that only quadrics can have singular scheme equal to a line, and no hypersurface in projective space can have singular scheme equal to a twisted cubic (cf. p. 347 in [1]).

The natural expectation would be that letting the singular scheme be nonreduced should allow many more examples. For instance, cones over nodal plane curves give examples of hypersurfaces in $\mathbb{P}^3$ of arbitrary degree ≥ 2 and singular scheme generically reduced, but with an embedded homogeneous point (at the vertex). However, the results in this paper show that the situation is still quite rigid:

Claim 3.3. *Under the hypotheses detailed above, $(n-1)$ must divide $4(g+r-1)$. In fact, necessarily*

$$(n-1)\Big((d-2)r - \sum(m_i-2)\Big) = 4(g+r-1) \quad .$$

For example, twisted cubics can support singularity subschemes as above only in dimensions $n = 3, 5, 9$, regardless of the number of embedded points allowed on them. (We do not know if such examples do exist.) The only situation in unconstrained dimension is for $g + r - 1 = 0$, that is, $g = 0$ and $r = 1$: lines are the only nonsingular curves in projective space which may support a generically reduced singularity subscheme in all dimensions (under the local homogeneity assumption). Further, if Y is supported on a line and only has *one* embedded homogeneous point, then the formula implies that the multiplicity of X at this point is d; therefore, X is necessarily a cone in this case.

For $\sum(m_i - 2) = 0$, the formula in Claim 3.3 recovers the formula at p. 347 of [1] (that is, the reduced case). For $n = 2$, the hypotheses imply that X is a plane curve consisting of a double component C and a residual curve of degree $(d - 2r)$; the formula then follows from the genus formula and Bézout's theorem. In higher dimensions, the following argument is the only proof we know.

Proof of the claim. We compute directly the weighted Chern-Mather class of Y and the Segre class $s(Y, \mathbb{P}^n)$. Proposition 1.3 gives a relation between these two classes, and the formula follows by taking degrees.

Explicitly, blow-up $\mathbb{P}^n$ at the 'special' points $q_1, \dots, q_s$, and then along the proper transform of the curve C. The homogeneity hypothesis implies that the (scheme-theoretic) inverse image of Y in the top blow-up is a Cartier divisor, with a component of multiplicity 1 dominating C, and s components with multiplicity $(m_1 - 1), \dots, (m_s - 1)$ dominating the q_i's. The Segre class of Y in $\mathbb{P}^n$ is then computed by using the birational invariance of Segre classes, and we get

$$\begin{aligned} &i_* s(Y, \mathbb{P}^n) \\ &\quad = r[\mathbb{P}^1] + \left(s(n-1) + 2 - 2g - r(n+1) + \sum_i ((m_i - 1)^n - n(m_i - 1)) \right) [\mathbb{P}^0] \end{aligned}$$

(where $i : Y \hookrightarrow \mathbb{P}^n$ is the inclusion).

On the other hand, the component dominating q_i maps to a corresponding component of the projective normal cone to Y in $\mathbb{P}^n$; computing differentials, we see that this map has degree $(m_i - 1)^{n-1} - 1$. This allows us to compute the weighted Chern-Mather class of Y:

$$c_{\mathrm{wMa}}(Y) = c_{\mathrm{Ma}}(C) - \sum_i \left((m_i - 1)^{n-1} - 1 \right) (m_i - 1) c_{\mathrm{Ma}}(q_i) \quad ,$$

from which

$$i_* c_{\mathrm{wMa}}(Y) = r[\mathbb{P}^1] + \left(2 - 2g - \sum_i ((m_i - 1)^n - (m_i - 1)) \right) [\mathbb{P}^0] \quad .$$

Now let h denote the hyperplane class in $\mathbb{P}^n$. The expression for the Segre class gives

$$\begin{aligned} i_* c(T^*\mathbb{P}^n \otimes \mathcal{O}(d)) \cap s(Y, \mathbb{P}^n) &= i_* \frac{(1 + (d-1)h)^{n+1}}{1 + dh} \cap s(Y, \mathbb{P}^n) = r[\mathbb{P}^1] \\ + &\left((s + rd - 2r)(n-1) + 2 - 2g - 4r + rd + \sum_i ((m_i - 1)^n - n(m_i - 1)) \right) [\mathbb{P}^0] \end{aligned}$$

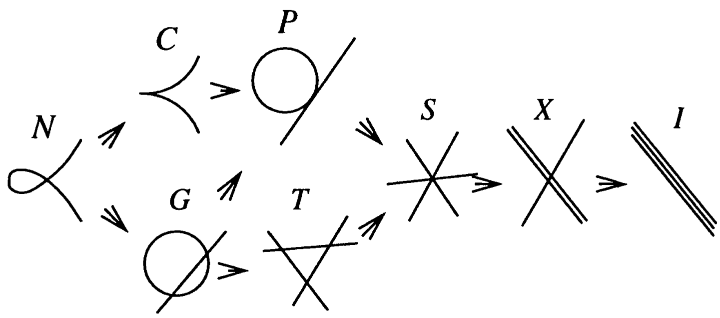
and therefore

$$\begin{aligned} &i_* (-1)^{\dim Y} \left(c(T^*M \otimes \mathcal{L}) \cap s(Y, M) \right)_{\vee \mathcal{L}} = r[\mathbb{P}^1] \\ &\quad + \left((2r - dr - s)(n-1) - 2 + 2g + 4r - \sum_i ((m_i - 1)^n - n(m_i - 1)) \right) [\mathbb{P}^0]. \end{aligned}$$

By Proposition 1.3, this class must equal $i_*c_{\mathrm{wMa}}(Y)$. Equating the two expressions gives the formula in the statement. Q.E.D.

Example 3.6. Finally, we give an example of the use of weighted Chern-Mather classes in the computation of the multiplicities of components of a normal cone. Such multiplicities are important for enumerative applications, and it would be very useful to develop tools to compute them. For singularity subschemes of hypersurfaces, the connection between weighted Chern-Mather classes and Milnor classes often lets us recover these multiplicities from computations of MacPherson's classes and local Euler obstructions. It would be interesting to extend such techniques to more general schemes.

Let D be the hypersurface of $\mathbb{P}^9$ parametrizing *singular* plane cubics, and let Y be its singularity subscheme. The following picture represents the natural stratification of D (with arrows denoting specialization):



The scheme Y is supported on the union of the closures $\overline{C}$, $\overline{G}$ of the loci parametrizing cuspidal cubics and binodal cubics. What are the multiplicities of the components of the normal cone of Y in $\mathbb{P}^9$? The point here is that we can compute $c_{\mathrm{wMa}}(Y)$ without knowing these multiplicities:

Claim 3.4. *Denote by i the inclusion of Y in $\mathbb{P}^9$. Then*

$$\begin{aligned} i_*c_{wMa}(Y) &= 69[\mathbb{P}^7] \\ &+ 120[\mathbb{P}^6] + 210[\mathbb{P}^5] + 252[\mathbb{P}^4] + 210[\mathbb{P}^3] + 120[\mathbb{P}^2] + 45[\mathbb{P}^1] + 10[\mathbb{P}^0]. \end{aligned}$$

Proof. This follows from Theorem 1.2 and the computations of characteristic classes for D in §4 of [4]. Q.E.D.

Now the task is to find the coefficients expressing the weighted Chern-Mather class of Y as a combination of the Chern-Mather classes of the loci C, G, etc. We first find the constructible function ν corresponding to $c_{\mathrm{wMa}}(Y)$ under MacPherson's transformation. For this, we use the result of the computation from [4] of Chern-Schwartz-MacPherson's classes of the strata of D. Writing $c_{\mathrm{wMa}}(Y) = c_{\mathrm{SM}}(\nu) = \nu(C) \cdot c_{\mathrm{SM}}(1_C) + \nu(G) \cdot$

$c_{\mathrm{SM}}(1_G) + \dots$ and solving the resulting system of linear equations, we find

$$\nu(C) = 2;\ \nu(G) = 1;\ \nu(P) = 0;\ \nu(T) = 1;$$
$$\nu(S) = 3;\ \nu(X) = 1;\ \nu(I) = 1.$$

(The paragraph preceding the statement of Theorem 1.5 gives a geometric interpretation of $\mu = -\nu$.) As pointed out in the discussion following Theorem 1.5, to find the multiplicities we now need to express this constructible function as a combination of local Euler obstructions of the strata. These are easy to compute in codimension one, and we proceed to the computation of the multiplicities for the components dominating the loci $\overline{C}$, $\overline{G}$, $\overline{P}$, $\overline{T}$. For these loci, we only need to observe that $\overline{C}$, $\overline{G}$ are nonsingular along P, and $\overline{G}$ has multiplicity 3 along T (these follows from easy local computations). As the local Euler obstruction agrees with the multiplicity in codimension one, this gives

$$\mathrm{Eu}_C = \begin{cases} & \dots \\ 0 & T \\ 1 & P \\ 0 & G \\ 1 & C \end{cases}, \quad \mathrm{Eu}_G = \begin{cases} & \dots \\ 3 & T \\ 1 & P \\ 1 & G \\ 0 & C \end{cases},$$

where we indicate the value of the function at the general point of the listed locus. Therefore

$$\nu = 2\mathrm{Eu}_C + \mathrm{Eu}_G - 3\mathrm{Eu}_P - 2\mathrm{Eu}_T + \dots$$

from which we read that the multiplicities of the components of the normal cone are: 2 over C, 1 over G, 3 over P, 2 over T.

Finding the multiplicities over the remaining three loci S, X, I requires computing the local Euler obstructions for all the strata of D. We leave this to the motivated reader.

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Cobordism of non-spherical knots

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From Chern classes to Milnor classes A history of characteristic classes for singular varieties

Jean-Paul Brasselet

Abstract.

In this paper, we give a survey and recent developments about the definitions of characteristic classes for possibly singular complex analytic (or algebraic) varieties. We recall the classical construction of characteristic classes in the case of manifolds, by obstruction theory and using Schubert cycles. Then, we present various generalizations of characteristic classes to singular varieties, due to M.H. Schwartz, W.T. Wu, J. Mather, R. MacPherson, W. Fulton and K. Johnson and we discuss relations among these definitions. More recent results concern the definition and properties of so-called Milnor classes, as developped by P. Aluffi, J.P. Brasselet-D. Lehmann-J. Seade-T. Suwa, A. Parusiński-P. Pragacz and S. Yokura.

§0. Introduction

The Euler-Poincaré characteristic was the first characteristic class (or number) to be introduced. For a triangulable (possibly singular) compact variety X without boundary, it can be defined, as

$$\chi(X) = \sum(-1)^i n_i\,,$$

where n_i is the number of i-dimensional simplices. It is also equal to $\sum(-1)^i b_i$ where $b_i = \mathrm{rk} H_i(X)$. The Poincaré-Hopf theorem says that, if X is a manifold and v a (continuous) vector field with a finite number of isolated singularities a_k of indices $I(v, a_k)$, then

$$\chi(X) = \sum I(v, a_k).$$

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This means that the Euler-Poincaré characteristic measures the obstruction to the existence of a non-zero vector field tangent to X.

On another hand, characteristic classes of projective varieties have been defined by Severi, Todd and others using polar varieties. Then Chern defined such characteristic clases for hermitian manifolds, in several ways, in particular as measuring the obstruction to the construction of complex r-frames tangent to the manifold, and using Schubert varieties (related to polar varieties). During some time, the attractiveness of the axiomatic properties of Chern classes caused the viewpoint of polar varieties to be somewhat forgotten.

For singular varieties, it appears that Wu and Mather classes can be defined in terms of polar varieties, with a formula similar to the non-singular case. On the other hand, the obstruction theory Chern's point of view has been generalized by M.H. Schwartz, and the axiomatic point of view by R. MacPherson. The Schwartz and MacPherson classes coincide, via Alexander duality.

The Fulton and Fulton-Johnson classes use Segre classes definition, without reference to the original definitions of Chern classes of varieties (for complete intersections they correspond to the Chern classes of the virtual bundle, generalization of the tangent bundle).

A natural question was to compare the Schwartz-MacPherson and the Fulton-Johnson classes. A result of Suwa shows that in the case of isolated singularities, the difference is given by the Milnor numbers in the singular points. It was natural to call Milnor classes the difference arising in the general case. This difference has been described by several authors by different means.

In this paper, cohomology classes will be constructed in the context of cell decompositions in order to keep things consistent with Poincaré duality. We will denote by M a complex manifold, by (K) a triangulation of M, (K') a barycentric subdivision of (K) and (D) the associated dual cell decomposition. The dual cell of a simplex $\sigma \in K$ will be denoted by $d(\sigma)$ or simply d if there is no possible confusion. The barycenter $\hat{\sigma}$ is the intersection point $\hat{\sigma} = \hat{d}(\sigma) = d(\sigma) \cap \sigma$. The (D)-cochain whose value is 1 at $d(\sigma)$ and 0 at other cells of (D) will be denoted by $\widetilde{d}(\sigma)$.

In the sequel, all homology and cohomology groups will be understood with integer coefficients. Recall that if M is a compact complex m-dimensional manifold, the Poincaré duality isomorphism

$$H^{2m-i}(M) \longrightarrow H_i(M),$$

the cap-product with the fundamental class $[M] \in H_{2m}(M)$, is represented at the chain level as the homomorphism $C^{2m-i}_{(D)}(M) \to C^{(K)}_i(M)$

sending the elementary (D)-cochain $\widetilde{d}(\sigma)$ to the elementary (K)-chain σ.

It was a great pleasure for me to participate in the Franco-Japanese congress on singularities in Sapporo. I want to thank all people who made remarks and comments about a preliminary version of this survey, especially P. Aluffi, G. Barthel, P. Pragacz, J. Seade, T. Suwa, B. Teissier and S. Yokura.

§1. Chern classes in the non-singular case.

In his original paper [Ch], Chern gave several constructions of characteristic classes for Hermitian manifolds: by obstruction theory, using the decomposition of the Grassmann manifold in Schubert cycles, using differential forms and by transgression cocycles. We will briefly recall the first two definitions, which extend to singular varieties. The paper [Ch] is highly recommended for the study of Chern classes.

1.1. Chern classes by obstruction theory.

Let us recall the idea of constructing Chern classes by obstruction theory (see [Ch]), following Steenrod [Ste], part III.

We denote by TM the complex tangent bundle to the complex m-dimensional manifold M and by T_rM the bundle of complex r-frames tangent to M. The fiber of T_rM over a point $x \in M$ is the Stiefel manifold $W_{r,m}$ of complex r-frames in $\mathbf{C}^m$. Let $d = d(\sigma)$ be a k-cell in a trivialization domain U of T_rM, i.e. $T_rM|_U \cong U \times W_{r,m}$. Let us suppose that we are given an r-frame $v^{(r)} = (v_1, \ldots, v_r)$ on the boundary ∂d of d. This defines a section of T_rM over ∂d and, by composition, a map

$$S^{k-1} \cong \partial d \xrightarrow{v^{(r)}} T_rM|_U \xrightarrow{\mathrm{pr}_2} W_{r,m}$$

where pr_2 denotes the projection to the second factor. We thus obtain an element

$$[v^{(r)}; \partial d] \in \pi_{k-1}(W_{r,m})$$

which vanishes if and only if the r-frame $v^{(r)}$ can be extended without singularity to all of d. We remark that if this element is non-zero, then we can extend the r-frame to the relative interior of d by homothety centered at the barycenter $\hat{d} = \hat{d}(\sigma)$, thus obtaining an isolated singularity of index $[v^{(r)}; \partial d]$.

Let us recall ([Ste], §25.7) that

$$\pi_i(W_{r,m}) = \begin{cases} 0 & \text{for } i < 2m-2r+1, \\ \mathbf{Z} & \text{for } i = 2m-2r+1. \end{cases}$$

This result implies that we can construct an r-frame $v^{(r)}$, i.e. a section of $T_r M$, by induction on the dimension of cells of the given cell decomposition of M without singularity up to the $(2m-2r+1)$-skeleton and with isolated singularities on the $2p = 2(m-r+1)$-skeleton. For each $2p$-cell $d(\sigma)$, the index of the complex r-frame $v^{(r)}$ at its only singular point $\hat{d} = d(\sigma) \cap \sigma$ in d is $I(v^{(r)}, \hat{d}) = [v^{(r)}; \partial d] \in \mathbf{Z}$. Associating to each p-cell $d(\sigma)$ the integer $I(v^{(r)}, \hat{d})$ defines a $2p$-cochain that actually is a cocycle, called the obstruction cocycle.

Definition ([Ch]). The p-th (cohomology) *Chern class* of M, $c^p(M) \in H^{2p}(M; \mathbf{Z})$ is the class of the obstruction cocycle.

By the Poincaré duality isomorphism, the image of $c^p(M)$ in $H_{2(r-1)}(M)$ is the $(r-1)$-st homology Chern class of M represented by the cycle

$$\sum_{\dim \sigma = 2(r-1)} I(v^{(r)}, \hat{d}(\sigma))\ \sigma. \tag{1}$$

In particular, the evaluation of $c^m(M)$ on the fundamental class $[M]$ of M yields the Euler-Poincaré characteristic.

1.2. Chern classes using Schubert cycles and polar varieties.

The construction of Chern classes using Schubert cycles was already present in Chern's original paper. This construction was emphasized by Gamkrelidze in [Ga1] and [Ga2]. A historical introduction and complete bibliography can be found in the Teissier's paper [T2].

The Schubert cell decomposition of the Grassmann manifold $\mathcal{G} = \mathcal{G}(n,m)$ of n-planes in $\mathbf{C}^m$ has been described by Ehresmann [Eh] and it was used by Chern to give an alternative definition of his characteristic classes. Let

$$\{0\} = D_m \subset D_{m-1} \subset \cdots \subset D_1 \subset D_0 = \mathbf{C}^m \tag{$\mathcal{D}$}$$

be a flag in $\mathbf{C}^m$, with $\mathrm{codim}_{\mathbf{C}} D_j = j$.

For each integer k, with $0 \leq k \leq n$, the k-th Schubert variety associated to $\mathcal{D}$, defined by

$$M_k(\mathcal{D}) = \{T \in \mathcal{G}(n,m) : \dim(T \cap D_{n-k+1}) \geq k\}$$

is an algebraic subvariety of $\mathcal{G}(n,m)$ of pure codimension k. The inequality condition is equivalent to saying that T and D_{n-k+1} do not span $\mathbf{C}^m$.

Let θ^n be the universal (sub)bundle over $\mathcal{G}(n,m)$. The cycle

$(-1)^k M_k(\mathcal{D})$ represents the image, under the Poincaré duality isomorphism, of the Chern class $c^k(\theta^n) \in H^{2k}(\mathcal{G}(n,m))$. If V is an n-dimensional complex analytic manifold and $f : V \to \mathcal{G}(n,m)$ is the classifying map for TV, i.e. such that $TV \cong f^*(\theta^n)$, then the cohomological Chern classes of V are $c^k(V) = c^k(TV) = f^*(c^k(\theta^n))$ (see [MS]).

Let us consider the projective situation. We denote by $G(n,m)$ the Grassmann manifold of n-dimensional linear subspaces in $\mathbf{P}^m$. We fix a flag of projective linear subspaces

$$(\mathcal{D}) \qquad L_m \subset L_{m-1} \subset \cdots \subset L_1 \subset L_0 = \mathbf{P}^m$$

where $\operatorname{codim}_{\mathbf{C}} L_j = j$. The k-th Schubert variety associated to $\mathcal{D}$ is defined by

$$M_k(\mathcal{D}) = \{\widetilde{T} \in G(n,m) : \dim(\widetilde{T} \cap L_{n-k+2}) \geq k-1\}$$

Let us remark that we always have $\dim(\widetilde{T} \cap L_{n-k+2}) \geq k-2$. The Schubert variety $M_k(\mathcal{D})$ has codimension k in $G(n,m)$.

Let us denote $N = nm = \dim_{\mathbf{C}} G(n,m)$ and fix $0 \leq s \leq m$. The Schubert variety

$$(2) \qquad \begin{aligned} M_k^{N-s} &= \{(x,T) : x \in L_{s-k},\ x \in T,\ \dim(T \cap L_{n-k+2}) \geq k-2\} \\ &= L_{s-k} \cap M_k(\mathcal{D}) \end{aligned}$$

is the intersection of $M_k(\mathcal{D})$ with a general $(s-k)$-codimensional plane and it has codimension s in $G(n,m)$. The (homological) Chern classes of $G(n,m)$ are

$$(3) \qquad c_{N-s}(G(n,m)) = \sum_{k=0}^{s} (-1)^k \binom{n-s+1}{n-k+1} M_k^{N-s}.$$

Let us now consider the case of an n-projective manifold $V \subset \mathbf{P}^m$. The k-th polar variety is defined by

$$P_k = \{x \in V : \dim(T_x(V) \cap L_{n-k+2}) \geq k-1\},$$

where $T_x(V)$ is the projective tangent space to V at x. For L_{n-k+2} sufficiently general, the codimension of P_k in V is equal to k. Also, the class $[P_k]$ of P_k modulo rational equivalence in the Chow group $A_{n-k}(V)$ does not depend on L_{n-k+2} for L_{n-k+2} sufficiently general. This class is called the k-th polar class of V.

Let $\gamma : V \to G(n,m)$ be the Gauss map, i.e. the map defined by

$$\gamma(x) = T_x(V) \subset \mathbf{P}^m.$$

Then

$$P_k = \gamma^{-1}(M^k(\mathcal{D})).$$

The relation between Chern classes and polar classes has been described by Gamkrelidze and Todd.

If $\mathcal{L} = \mathcal{O}_{\mathbf{P}^m}(1)|_V$, then we obtain the Todd formula (compare with (3)):

$$(4) \qquad c_{n-s}(V) = \sum_{k=0}^{s} (-1)^k \binom{n-s+1}{n-k+1} c^1(\mathcal{L})^{s-k} \cap [P_k]$$

where the cap-product with $c^1(\mathcal{L})^{s-k}$ is equivalent to the intersection with a general $(s-k)$-codimensional plane.

§2. Chern classes in the singular case.

In the singular case, there are different possible definitions of Chern classes, generalizing the ones in the non-singular case.

The Wu and Mather classes generalize the definitions by Schubert cycles and polar varieties. J. Zhou proved that Wu and Mather classes coincide.

The Schwartz classes use obstruction theory, and the MacPherson classes, defined in an algebraic geometry way, satisfy good functorial properties. J.P. Brasselet and M.H. Schwartz proved that Schwartz and MacPherson classes coincide, via Alexander duality.

The Fulton and Fulton-Johnson definitions of Chern classes use Segre classes and correspond to the class of the virtual tangent bundle in the case of local complete intersections (for example).

The relation between Wu-Mather classes and Schwartz-MacPherson classes appears in MacPherson's definition itself. The MacPherson construction uses Wu-Mather classes, taking into account the local complexity of the singular locus along Whitney strata. This is the role of the local Euler obstruction.

The difference between Schwartz-MacPherson and Fulton-Johnson classes is expressed, in the case of isolated singularities, in terms of the Milnor numbers (at the singularities) (Seade-Suwa). In the general case, this difference is called Milnor class and has been studied by several authors: P. Aluffi, J.P. Brasselet-D. Lehmann-J. Seade-T. Suwa, A. Parusiński-P. Pragacz and S. Yokura.

2.1. The Wu classes (1965).

In the singular case, Wu [Wu2] generalized Chern's and Gamkrelidze's constructions in the following way: Let $X^n \subset \mathbf{P}^m$ be a complex projective algebraic variety and let X' be a subvariety of X containing the singular part X_{sing}. Denoting by $A_*(X)$ the Chow group of classes of algebraic cycles of X and with $A_*(X, X')$ the subgroup of classes that have no component in X', there is a natural inclusion

$$J : A_*(X, X') \to A_*(X).$$

Wu defines a notion of transform of X, which coincides with the Nash transform (see [Z1]). We recall the definition of Nash transform, the original definition of Wu being slightly different.

Let us denote by $\nu : G \to \mathbf{P}^m$ the Grassmann bundle over $\mathbf{P}^m$ whose fibre over x is the Grassmann manifold $G(n, m)$ of n-linear subspaces in $T_x\mathbf{P}^m$. The Gauss map $\gamma : X_{\text{reg}} \to G$ is defined on the regular part $X_{\text{reg}} = X \setminus X_{\text{sing}}$ of X by

$$\gamma(x) = T_x(X_{\text{reg}}) \subset T_x\mathbf{P}^m$$

The *Wu* (or *Nash*) *transform* $\widetilde{X}$ is defined as the closure of the image of γ in G. In general $\widetilde{X}$ is singular; nevertheless, if X is an analytic variety, then $\widetilde{X}$ is also analytic, and the restriction $\nu : \widetilde{X} \to X$ of the projection $\nu : G \to \mathbf{P}^m$ is analytic. It induces a map

$$\nu_* : A_*(\widetilde{X}, \widetilde{X}') \to A_*(X, X')$$

where $\widetilde{X}' = \nu^{-1}(X')$.

The (transverse) intersection of cycles with $\widetilde{X}$ defines a map

$$A_{d-s}(G) \xrightarrow{I} A_{n-s}(\widetilde{X}, \widetilde{X}'),$$

with $\dim_{\mathbf{C}} G = d$.

Finally, let $D : A_s(G) \to A_{d-s}(G)$ be the duality map in G. The composition $W = J \circ \nu_* \circ I \circ D$ is a map

$$W_s : A_s(G) \to A_{n-s}(X)$$

In analogy to the formula (3), we have:

Definition ([Wu1]). The Wu classes are defined by

$$c^W_{n-s}(X) = \sum_{k=0}^{s} (-1)^k \binom{n-s+1}{n-k+1} W_s(M^s_k)$$

2.2. The Mather classes (1974).

R. MacPherson named Mather classes the classes that Mather described to him on a blackboard (see [M2]). Let us recall their definition.

Let X be an n-dimensional analytic complex subvariety X of an m-dimensional manifold M. We consider the Nash transform $\widetilde{X}$ and denote by E the tautological bundle over the Grassmann bundle G. The fiber of E over $P \in G$ is

$$E_P = \{\, v(x) \in T_x(M) : \ v(x) \in P, \ x = \nu(P)\,\}.$$

Let us denote by $\widetilde{E}$ the restriction of E to $\widetilde{X}$. We have a commutative diagram:

$$\begin{array}{ccc} \widetilde{E} & \hookrightarrow & E \\ \downarrow & & \downarrow \\ \widetilde{X} & \hookrightarrow & G \\ \downarrow & & \downarrow \\ X & \hookrightarrow & M \end{array}$$

Definition ([M2]). The Mather class of X is defined by

$$c^M(X) = \nu_*(c^*(\widetilde{E}) \cap [\widetilde{X}]),$$

where $c^*(\widetilde{E})$ denotes the usual (total) Chern class of the bundle $\widetilde{E}$ in $H^*(\widetilde{X})$ and the cap-product with $[\widetilde{X}]$ is the Poincaré duality homomorphism (in general not an isomorphism).

The Mather class can be defined by using polar varieties in the following way: First of all, let us consider the local situation. For a general flag $\mathcal{D}$ and an affine variety $X^n \subset \mathbf{C}^m$, we define

$$\begin{array}{ccccccc} & & \widetilde{X} & \hookrightarrow & \mathcal{G}(n,m) \times \mathbf{C}^m & \xrightarrow{\pi_1} & \mathcal{G}(n,m) \\ & \sigma \nearrow & \downarrow \nu & & \downarrow \pi_2 & & \\ X_{\text{reg}} & \hookrightarrow & X & \hookrightarrow & \mathbf{C}^m & & \end{array}$$

and we denote by $\widetilde{\gamma} = \pi_1|_{\widetilde{X}} : \widetilde{X} \to \mathcal{G}(n,m)$ the Gauss map.

Let us define the following analytic subspace of X [LT]:

$$N_k(\mathcal{D}) = \nu \circ \widetilde{\gamma}^{-1}(M_k(\mathcal{D})) = \overline{\nu(\widetilde{\gamma}^{-1}(M_k(\mathcal{D})) \cap \sigma(X_{\text{reg}}))}$$

If the flag $\mathcal{D}$ is good (sufficiently general), i.e. $\widetilde{\gamma}$ is transverse to the strata

$$M_{k,i}(\mathcal{D}) = \{W \in \mathcal{G}(n,m) : \operatorname{codim}(W + D_{n-k+i-1}) = k+1\}$$

of $M_k(\mathcal{D})$, then the cycle $N_k(\mathcal{D})$ is well defined and independent of the choice of the (good) flag. In that case, it is called the polar variety (Lê-Teissier).

If the flag $\mathcal{D}$ is good, and still in the local situation, let $\pi : X \to \mathbf{C}^{n-k+1}$ be the restriction to X of a linear projection with kernel D_{n-k+1}, then $N_k(\mathcal{D})$ is the closure (in X) of the critical locus of the restriction of π to X_{reg} [LT].

In the projective case, the polar variety is the closure of

$$\{x \in X_{\text{reg}} : \dim(T_x(X_{\text{reg}}) \cap L_{n-k+2}) \geq k-1\} \tag{5}$$

where $\operatorname{codim}_{\mathbf{C}^m} L_{n-k+2} = n-k+2$.

Now, if $X^n \subset \mathbf{P}^m$ is a projective variety, then (see (4) and [Pi2])

$$c^M_{n-s}(X) = \sum_{k=0}^{s} (-1)^k \binom{n-s+1}{n-k+1} c^1(\mathcal{L})^{s-k} \cap [N_k(\mathcal{D})]$$

where $\mathcal{L} = \mathcal{O}_{\mathbf{P}^m}(1)|_X$.

Theorem ([Z1]). *Let X be a projective variety. Then the Mather and Wu classes of X coincide.*

The Mather classes can be also expressed in terms of conormal space, notion which is strongly related to the one of polar variety (see [T1] and [S]). The conormal space is the subvariety of the cotangent bundle T^*M of M defined as the closure of

$$T^*_X M = \{(x,\xi) \in T^*M : x \in X_{\text{reg}}, \xi|_{T_x(X_{\text{reg}})} \equiv 0\}.$$

We denote by $C(X,M) \subset \mathbf{P}T^*M$ the projectivization of the conormal space and by τ the projection $\tau : C(X,M) \to X$, restriction of the projection $\mathbf{P}T^*M \to M$ to $C(X,M)$. By [S] (see also [PP4] and [Ke1]), we have

$$c^M_*(X) = (-1)^{m-n-1} c(TM|_X) \cap \tau_*\left(c(\mathcal{L})^{-1} \cap [C(X,M)]\right).$$

The Mather classes do not verify the Deligne-Grothendieck axioms that we recall below. That is the MacPherson's motivation for introducing the so-called Schwartz-MacPherson classes.

2.3. The Schwartz classes (1965).

The first definition of Chern class for singular varieties was given in 1965 by M.H. Schwartz in two "Notes aux CRAS" [Sc1]. We briefly recall her construction. Let $X \subset M$ be a singular n-dimensional complex variety embedded in a complex m-dimensional manifold. Let us consider a Whitney stratification $\{V_\alpha\}$ of M [Wh] such that X is a union of strata and denote by (K) a triangulation of M compatible with the stratification, i.e. each open simplex is contained in a stratum.

As before, we denote by (K') a barycentric subdivision of (K) and (D) the associated dual cell decomposition. Each cell of (D) is transverse to the strata. This implies that if d is a cell of dimension $2p = 2(m-r+1)$ and V_α is a stratum of dimension $2s$, then $d \cap V_\alpha$ is a cell such that

$$\dim(d \cap V_\alpha) = 2(s - r + 1)$$

This means that if d is a cell whose dimension is the dimension of obstruction to the construction of an r-frame tangent to M, then $d \cap V_\alpha$ is a cell whose dimension is exactly the dimension of obstruction to the construction of an r-frame tangent to the stratum V_α.

This fact leads M.H. Schwartz to the very nice construction of a stratified radial r-frame in the following way:

An r-frame $v^{(r)}$, defined on a part $A \subset M$, is called a *stratified r-frame* if at each point $x \in A$, $v^{(r)}(x)$ is tangent to the stratum V_α containing x. In the following we write $v^{(r)}$ as $(v^{(r-1)}, v_r)$, the last vector being individualized.

Proposition ([Sc1] [Sc2]). *One can construct, on the $2p$-skeleton $(D)^{2p}$, a stratified r-frame $v^{(r)}$, called radial frame, whose singularities satisfy the following properties:*

(i) *$v^{(r)}$ has only isolated singular points, which are zeroes of the last vector v_r. On $(D)^{2p-1}$, the r-frame $v^{(r)}$ has no singular point and on $(D)^{2p}$ the $(r-1)$-frame $v^{(r-1)}$ has no singular point.*

(ii) *Let $a \in V_\alpha \cap (D)^{2p}$ be a singular point of $v^{(r)}$ in the $2s$-dimensional stratum V_α. If $s > r-1$, the index of $v^{(r)}$ at a, denoted by $I(v^{(r)}, a)$, is the same as the index of the restriction of $v^{(r)}$ to $V_\alpha \cap (D)^{2p}$ considered as an r-frame tangent to V_α. If $s = r-1$, then $I(v^{(r)}, a) = +1$.*

(iii) *Inside a $2p$-cell d which meets several strata, the only singularities of $v^{(r)}$ are inside the lowest dimensional one (in fact located in the barycenter of d).*

(iv) *The r-frame $v^{(r)}$ is "pointing outward" a (particular) regular neighborhood U of X in M. It has no singularity on ∂U.*

The procedure of the construction of radial frames is made by induction on the dimension of the strata, using the properties of Whitney stratifications for proving the existence of frames "pointing outward" regular neighborhoods and satisfying property (ii). An r-frame already known on a neighborhood of the boundary of a stratum is extended with isolated singularities inside (a suitable skeleton) of the stratum and then extended with property (ii) to a regular neighborhood of this stratum.

Let us denote by $\mathcal{T}$ the tubular neighborhood of X in M consisting of the (D)-cells which meet X. Let us recall that $\widetilde{d}$ is the elementary (D)-cochain whose value is 1 at d and 0 at all other cells. We can define a $2p$-dimensional (D)-cochain in $C^{2p}(\mathcal{T}, \partial\mathcal{T})$ by:

$$\sum_{d \in \mathcal{T}} I(v^{(r)}, \hat{d})\, \widetilde{d}.$$

This cochain is a cocycle whose class lies in

$$H^{2p}(\mathcal{T}, \partial\mathcal{T}) \cong H^{2p}(\mathcal{T}, \mathcal{T} \setminus X) \cong H^{2p}(M, M \setminus X),$$

where the first isomorphism is given by retraction and the second by excision.

Definition ([Sc1] [Sc2]). The p-th Schwartz class $c^p(X)$ is the class obtained in $H^{2p}(M, M \setminus X)$.

2.4. The MacPherson classes (1974).

Let us recall firstly some basic definitions.

A *constructible set* in a variety X is a subset obtained by finitely many unions, intersections and complements of subvarieties. A *constructible function* $\alpha : X \to \mathbf{Z}$ is a function such that $\alpha^{-1}(n)$ is a constructible set for all n. The constructible functions on X form a group denoted by $\mathbf{F}(X)$. If $A \subset X$ is a subvariety, we denote by $\mathbf{1}_A$ the characteristic function whose value is 1 over A and 0 elsewhere.

If X is triangulable, α is a constructible function if and only if there is a triangulation (K) of X such that α is constant on the interior of each simplex of (K). Such a triangulation of X is called α-adapted.

The correspondence $\mathbf{F} : X \to \mathbf{F}(X)$ defines a contravariant functor when considering the usual pull-back $f^* : \mathbf{F}(Y) \to \mathbf{F}(X)$ for a morphism $f : X \to Y$. One interesting fact is that it can be made a covariant functor when considering the pushforward defined on characteristic functions by:

$$f_*(\mathbf{1}_A)(y) = \chi(f^{-1}(y) \cap A),$$

for all $y \in Y$, and linearly extended to elements of $\mathbf{F}(X)$.

The following result was conjectured by Deligne and Grothendieck in 1969 and proved by R. MacPherson [M2] in 1974.

Theorem ([M2]). *Let $\mathbf{F}$ be the covariant functor of constructible functions and let $H_*(\ ; \mathbf{Z})$ be the usual covariant $\mathbf{Z}$-homology functor.*

Then there exists a unique natural transformation

$$c_* : \mathbf{F} \to H_*(\ ; \mathbf{Z})$$

satisfying that $c_*(\mathbf{1}_X) = c^*(X) \cap [X]$ *if* X *is a manifold.*

The MacPherson's construction uses Mather classes and local Euler obstruction that we briefly recall.

The notion of *local Euler obstruction* was defined originally by R. MacPherson [M2] in 1974. It has been shown in [BDK] that the local invariant of singularities which appear in the Kashiwara formula for the index of holonomic modules [Ka] is equal to the local Euler obstruction. Definitions equivalent to MacPherson's have been given by several authors. We recall the one in [BS]: Let v be a radial vector field with an isolated singularity at $x \in V_\alpha$. Let B be a ball centered at x, small enough to be transversal to every stratum V_β with $V_\alpha \subset \overline{V_\beta}$, and such that x is the unique zero of v inside B. Using the Whitney conditions, it is possible to prove that there is a canonical lifting $\widetilde{v}$ of $v|_{\partial B \cap X}$ as a section of $\widetilde{E}|_{\nu^{-1}(\partial B \cap X)}$ (see [BS], Proposition 9.1). The obstruction to the extension of $\widetilde{v}$, on $\nu^{-1}(B \cap X)$, as a non-zero section of $\widetilde{E}$, evaluated on the corresponding fundamental class, is an integer denoted by $\mathrm{Eu}_x(X)$.

The local Euler obstruction is a constructible function Eu_X, constant on each stratum of the Whitney stratification. The relation between the local Euler obstruction and the polar varieties is given by Lê and Teissier [LT]:

Theorem ([LT]). *For a sufficiently general flag $\mathcal{D}$ in $\mathbf{C}^m$, the local Euler obstruction is expressed as*

$$\mathrm{Eu}_x(X) = \sum_{i=0}^{n-1} (-1)^{n-1-i} m_x(N_{n-1-i}(\mathcal{D}))$$

where $m_x(C)$ denotes the multiplicity of C at x.

For a Whitney stratification, we have the following lemma:

Lemma ([M1]). *There are integers n_α such that, for every point $x \in X$, we have:*

$$\sum_\alpha n_\alpha \mathrm{Eu}_x(\overline{V_\alpha}) = 1.$$

Definition ([M1]). The MacPherson class of X is defined by

$$c_*(X) = c_*(\mathbf{1}_X) = \sum_\alpha n_\alpha\ i_* c_M(\overline{V_\alpha})$$

where i denotes the inclusion $\overline{V_\alpha} \hookrightarrow X$.

Note that we have the following relation: $c^M(X) = c_*(\mathrm{Eu}_X)$.

In [BS] was proved the following result:

Theorem ([BS]). *The MacPherson class is the image of the Schwartz class by the Alexander duality isomorphism*

$$H^{2(m-r+1)}(M, M \setminus X) \xrightarrow{\cong} H_{2(r-1)}(X).$$

One of the consequences of this result is that the $(r-1)$-st MacPherson class $c_{r-1}(X)$ is represented by the cycle

$$\sum_{\sigma \in X} I(v^{(r)}, \hat{d}(\sigma))\ \sigma$$

where $\dim \sigma = 2(r-1)$ (see (1)).

The following theorem gives an expression of the MacPherson class in terms of Segre classes (see 2.5).

Theorem ([A3]). *If X is a hypersurface in a nonsingular variety M and Y is its singular scheme, then*

$$c_*(X) = c(TM) \cap s(X \setminus Y, M)$$

Following Sabbah [S] (see also [PP4]), we obtain a formula giving the Schwartz-MacPherson classes in terms of characteristic cycles. Denoting by $\mathbf{PCh}(\mathbf{1}_X) \subset T^*M$ the characteristic cycle associated to the constructible function $\mathbf{1}_X$ on M, we have (see the analogous formula for the Mather classes):

$$c_*(X) = (-1)^{n-1} c(TM|_X) \cap \tau_* \left(c(\mathcal{L})^{-1} \cap [\mathbf{PCh}(\mathbf{1}_X)] \right).$$

2.5. The Fulton classes (1984) ([Fu] exemple 4.2.6 (a)).

If X is a proper subvariety of a variety M, the Segre class $s(X, M)$ of X in M is the class in $A_*(X)$ defined as follows (see [F], §4): the normal cone to the closed subscheme X in the scheme M is defined as

$$C = C_X M = \mathrm{Spec}\left(\sum_{i=0}^{\infty} \mathcal{I}^i / \mathcal{I}^{i+1} \right)$$

where $\mathcal{I}$ is the ideal sheaf defining X in M. We denote by $P(C)$ the projectivized normal cone and p the projection $p : P(C) \to X$. Then

$$s(X, M) = \sum_{i \geq 0} p_*(c^1(\mathcal{O}(1))^i \cap [P(C)]).$$

When X is regularly imbedded in M, $C = N_X M$ is the normal vector bundle, and

$$s(X, M) = c(N_X M)^{-1} \cap [X].$$

The following Plücker formula, due to R. Piene, gives the relation between polar varieties (hence Mather classes) and Segre classes:

Theorem ([Pi1]). *Let X be an hypersurface of degree d in $\mathbf{P}^m$ and let $\mathcal{L} = \mathcal{O}_{\mathbf{P}^m}(1)|_X$. Then the polar variety N_k is given by*

$$[N_k] = (d-1)^k c^1(\mathcal{L})^k \cap [X] + \sum_{i=0}^{k-1} \binom{k}{i} (d-1)^i c^1(\mathcal{L})^i \cap s_{k-i}(X_{\text{sing}}, X).$$

The Fulton classes are defined by:

Definition ([Fu]). Let X be an algebraic scheme which can be imbedded as a closed subscheme of a non-singular variety M. We define the Fulton class of X in $A_*(X)$ by the formula

$$c^F(X) = c(TM|_X) \cap s(X, M),$$

where $c(TM|_X)$ is the total Chern class of the tangent bundle of M restricted to X and $s(X, M)$ is the Segre class of X in M.

This definition is independent of the choice of the embedding.

If X is a *local complete intersection*, then the normal bundle of X_{reg} in M extends canonically to X as a vector bundle $N_X M$ and

$$c^F(X) = c(TM|_X) c(N_X M)^{-1} \cap [X] = c(\tau_X) \cap [X]. \tag{6}$$

Here $\tau_X = TM|_X - N_X M$ denotes the virtual tangent bundle on X, defined in the Grothendieck group of vector bundles on X.

Let M be a non-singular compact complex analytic variety of pure dimension $n+1$ and let L be a holomorphic line bundle on M. Take $f \in H^0(M, L)$, a holomorphic section of L, such that the variety X of zeroes of f is a (nowhere dense) hypersurface in M. Then, the Fulton class of X is

$$c^F(X) = c(TM|_X - L|_X) \cap [X].$$

In [A1] P. Aluffi defines a notion of "thickening" of the scheme X along its singular subscheme Y: if $\mathcal{I}_Y$ denotes the ideal of Y and $\mathcal{I}$ the locally principal ideal of X, we denote by X^k the subscheme of M defined by the ideal $\mathcal{I}.\mathcal{I}_Y^k$. Then the Schwartz-MacPherson class and the Fulton class satisfy:

$$c_*(X) = c_F(X^{-1}) \qquad c^F(X) = c^F(X^0)$$

2.6. The Fulton-Johnson classes (1980) ([FJ], [Fu] exemple 4.2.6 (c)).

The definition (6) can also be generalized to arbitrary singular varieties in another way : for any coherent sheaf $\mathcal{F}$ on an algebraic scheme, one defines the Segre class $s(\mathcal{F})$ in the group $A_*(X)$ of cycles modulo rational equivalence as follows: Let $P(\mathcal{F}) = \mathrm{Proj}(\mathrm{Sym}(\mathcal{F}))$, with projection $p : P(\mathcal{F}) \to X$. Let us denote by $\mathcal{O}_{\mathcal{F}}(1)$ the canonical invertible sheaf which is the universal quotient of $p^*(\mathcal{F})$. If the support of $\mathcal{F}$ is X, define its Segre class $s(\mathcal{F})$ in $A_*(X)$ by the formula

$$\begin{aligned} s(\mathcal{F}) &= p_* \left(\sum_{i \geq 0} c^1(\mathcal{O}_{\mathcal{F}}(1))^i \cap [P(\mathcal{F})] \right) \\ &= p_* \left(c(\mathcal{O}_{\mathcal{F}}(-1))^{-1} \cap [P(\mathcal{F})] \right) \end{aligned}$$

For an arbitrary coherent sheaf $\mathcal{F}$ on X, define $s(\mathcal{F})$ to be $s(\mathcal{F} \oplus \mathcal{E}^1)$, where $\mathcal{E}^1$ is the trivial locally free sheaf of rank one on X.

Definition ([FJ]). If X is an algebraic scheme which may be imbedded in a non-singular scheme M, we define the Fulton-Johnson class of X in $A_*(X)$ by the formula

$$c^{FJ}(X) = c(TM|_X) \cap s(\mathcal{N}),$$

where $c(TM|_X)$ is the total Chern class of the tangent bundle of M restricted to X and $s(\mathcal{N})$ is the Segre class of the conormal sheaf of the embedding of X in M.

Remark. In the case of local complete intersection, the Fulton and Fulton-Johnson classes coincide and are equal to

$$c(TM|_X - N_X M) \cap [X].$$

§3. The Milnor classes.

The comparison between the Schwartz-MacPherson classes and the Fulton-Johnson classes can be viewed in two ways, which coincide in some classical situations. We observe that, in the case of isolated singularities, the difference is given by the Milnor numbers at the singular points. On the other hand, for a radial vector field tangent to the singular locus and with isolated singularity at a singular point, the difference between the "Schwartz" (classical) index and the "virtual" (GSV)-index is the Milnor number at this point. This observation motivates definition 2 below.

3.1. Definition and main properties of Milnor classes.

The following general definition is given by the corresponding authors in particular cases.

Definition 1 ([A3], [BLSS1], [PP4], [Y2]). The difference class

$$\mu_*(X) = (-1)^n(c^F(X) - c_*(X))$$

is called the Milnor class of X.

Let us consider the following situation ($\mathcal{H}$): X is an n-subvariety in the m-manifold M defined by a regular section, i.e. a holomorphic section generically transverse to the zero section, of a holomorphic vector bundle E (of rank $k = m - n$) over M [Su2]. We set $N = E|_X$. The virtual tangent bundle of X is denoted by

$$\tau_X = TM|_X \setminus N$$

Let us consider a compact connected subset $S \subset X$ (in particular a component of X_{sing}) and a neighborhood U of S in M such that $U \cap X - S \subset X_{\text{reg}}$. For each r-frame $v^{(r)}$ tangent to X_{reg} on $\partial U \cap X \cap D^{(2p)}$ with $2p = 2(m - r + 1)$ (see §2.3), we can define:

a) the localized Schwartz (usual) class $\text{Sch}(v^{(r)}, S) \in H_{2(r-1)}(S)$ which computes the obstruction to the extension of $v^{(r)}$ as a stratified r-frame inside $U \cap X \cap D^{(2p)}$. It is the contribution of S to $c_{r-1}(X) \in H_{2(r-1)}(X)$ ([BLSS1], Theorem 2.13),

b) the localized virtual class $\text{Vir}(v^{(r)}, S) \in H_{2(r-1)}(S)$ which computes the "obstruction to the extension of $v^{(r)}$ as linearly independent sections of τ_X", i.e. which is the contribution of S to $c_{r-1}(\tau_X) \in H_{2(r-1)}(X)$ ([BLSS1], Theorem 5.9).

Definition 2 ([BLSS1]). The $(r-1)$-st localized Milnor class of X at a compact component S of X_{sing} is defined by

$$\mu_{r-1}(X, S) = (-1)^{n-1}(\text{Sch}(v^{(r)}, S) - \text{Vir}(v^{(r)}, S)) \qquad \text{in } H_{2(r-1)}(S)$$

The total Milnor class is the sum over the components of X_{sing}:

$$\mu_{(r-1)}(X) = \sum_{S_\alpha \subset X_{\text{sing}}} (i_\alpha)_* \mu_{(r-1)}(X, S_\alpha) \in H_{2(r-1)}(X)$$

where i_α denotes the inclusion $S_\alpha \hookrightarrow X$.

The Milnor class $\mu_*(X)$ is supported on the singular locus of X. When $k = 1$ and $r = 1$, $\mu_0(X, S)$ is the Parusiński generalized Milnor number [Pa]. Also, if S is a point p and X a complete intersection near p, then $\mu_0(X, S)$ is the usual Milnor number.

The two definitions coincide in the case of local complete intersections, in particular in the case of hypersurfaces.

In the case $r = 1$, i.e. $v^{(1)} = v$, and for an isolated singularity p, the Schwartz index is the usual index and the virtual index coincides with the GSV-index (see [GSV], [LSS], [SS]). The difference of these indices is the Milnor number of X at p:

$$\mathrm{Sch}(v, p) - \mathrm{Vir}(v, p) = (-1)^{n+1}\mu(X, p).$$

Theorem ([SS]). *In the situation $\mathcal{H}$, suppose that X is compact and the singularities of X are isolated points $\{x_i\}$ where X is a local complete intersection. Then*

$$\mu_0(X) = (-1)^{n+1}\sum_{i=1}^{q}\mu(X, x_i)[x_i] \in H_0(X)$$

Theorem ([Su]). *In the previous situation, $\mu_i(X) = 0$ for $i > 0$.*

This result was also proved by [Pa] and [PP] for hypersurfaces with arbitrary singularities. It is generalized in the following way:

Theorem ([BLSS1] [BLSS2]). *Let X be a subvariety of a complex manifold in the situation $\mathcal{H}$, if X is compact, then we have, for each $r = 0, \dots, n-1$:*

$$c_r(X) = c_r(TM|_X - N) + (-1)^{n+1}\mu_r(X) \qquad \text{in } H_{2r}(X).$$

In other words, the difference between the total Schwartz-MacPherson class $c_*(X)$ of X and the total virtual class $c_*(TM|_X - N)$, regarded in homology, is the sum over the connected components of $\mathrm{Sing}(X)$ of the "total" localized Milnor classes $\mu_*(X, S) = \oplus_{i=0}^{n-1}\mu_i(X, S)$.

A similar formula for hypersurfaces is given by Aluffi (see [A1] for the notations):

Theorem ([A3]). *Let $X \subset M$ be a hypersurface with its singular subscheme Y and $\mathcal{L} = \mathcal{O}(X)$. Then we have*

$$c_*(X) = c^F(X) + c(\mathcal{L})^{\dim X} \cap (\mu_{\mathcal{L}}(Y)^\vee \otimes_M \mathcal{L}),$$

where $\mu_{\mathcal{L}}(Y) = c(T^*(M) \otimes \mathcal{L}) \cap s(Y, M)$.

We have the following Lefschetz-type formulae for the Milnor class:

Theorem ([BLSS1] [BLSS2]). *Let us denote by ℓ the complex dimension of S and let H be a complex $(m-\ell)$-dimensional plane transverse to S in M.*

a) *If X is a hypersurface in M, defined by a holomorphic section of a holomorphic line bundle E, and S a compact component of X_{sing}, then*

$$\mu_{r-1}(X,S) = (-1)^{\ell}\mu(X\cap H,p)\cdot[c(S)c(E)^{-1}]^{\ell-r+1}\cap[S]$$

b) *If $r=\ell+1$ and k is arbitrary, then*

$$\mu_{r-1}(X,S) = (-1)^{\ell}\mu(X\cap H,p)\cdot[S]$$

In the case where $\mu(X\cap H,p)=1$, the formula (a) is proved in [A3].

3.2. Description in terms of constructible functions [PP4].

Consider the function $\chi: X\to \mathbf{Z}$ defined by $\chi(x):=\chi(F_x)$, where F_x denotes the Milnor fibre at x and $\chi(F_x)$ its Euler characteristic. Define also the function $\mu: X\to\mathbf{Z}$ by $\mu=(-1)^{n-1}(\chi-\mathbf{1}_X)$.

Fix any stratification $\mathcal{S}$ of X such that μ is constant on the strata of $\mathcal{S}$, for instance any Whitney stratification of X. The topological type of the Milnor fibre is constant along the strata of any Whitney stratification of Z. Let us denote the value of μ on the stratum S by μ_S.

Let

$$\alpha(S)=\mu_S-\sum_{S'\neq S, S\subset\overline{S'}}\alpha(S')$$

be the numbers defined inductively on descending dimensions of S.

Theorem ([PP4]). *We have*

$$\mu_*(X)=\sum_{S\in\mathcal{S}}\alpha(S)c(L|_X)^{-1}\cap(i_{\overline{S},X})_*c_*(\overline{S})=c(L|_X)^{-1}\cap c_*(\mu),$$

where $i_{\overline{S},X}:\overline{S}\to X$ denotes the natural inclusion.

The formula was conjectured in [Y2] when X is projective. Under this last assumption, [PP2] proved earlier that

$$\int_X\mu_*(X)=\sum_{S\in\mathcal{S}}\alpha(S)\int_{\overline{S}}c(L|_{\overline{S}})^{-1}\cap c_*(\overline{S})$$

3.3. Description in terms of divisors [A3].

Let $B = Bl_Y M \to M$ be the blow-up of M along the singular subscheme Y of X. Let $\mathcal{X}$ and $\mathcal{Y}$ denote the total transform of X and the exceptional divisor in B, respectively.

Theorem ([A2]). *Let $\pi : \mathcal{X} \to X$ be the restriction of the blow-up to X. Then*

$$c_*(X) = c(TM|_X) \cap \pi_* \left(\frac{[\mathcal{X}] - [\mathcal{Y}]}{1 + \mathcal{X} - \mathcal{Y}} \right),$$

where, on the right hand side, $\mathcal{X}$ and $\mathcal{Y}$ mean the first Chern classes of the line bundles associated with $\mathcal{X}$ and $\mathcal{Y}$, i.e. those of $\pi^*(L|_X)$ and $\mathcal{O}_B(-1)$, the latter being the canonical line bundle on B.

Let us denote by $\mathcal{X}'$ the proper transform of X, the following formulae are also due to Aluffi [A3]

$$c^M(X) = c_*(\mathrm{Eu}_X) = c(TM|_X) \cap \pi_* \left(\frac{[\mathcal{X}']}{1 + \mathcal{X} - \mathcal{Y}} \right)$$

$$c^F(X) = c(TM|_X) \cap \pi_* \left(\frac{[\mathcal{X}]}{1 + \mathcal{X}} \right)$$

and we deduce [PP4]:

$$\mu_*(X) = (-1)^{n-1} c(TM|_X) \cap \pi_* \left(\frac{[\mathcal{Y}]}{(1 + \mathcal{X})(1 + \mathcal{X} - \mathcal{Y})} \right).$$

3.4. Specialization (the hypersurface case) [PP4].

Suppose that $X = f^{-1}(0)$ where f is a section of the line bundle L over M. Suppose that there exists a section $g \in H^0(M, L)$ such that $g^{-1}(0)$ is non-singular and transverse to the strata of a (fixed) Whitney stratification of X. For $t \in \mathbf{C}$ denote $f_t = f - tg$ and set $X_t = f_t^{-1}(0)$. We denote by $\mathbf{X}$ the following correspondence in $M \times \mathbf{C}$:

$$\mathbf{X} = \{(x, t) \in M \times \mathbf{C} | x \in X_t\}.$$

Denoting by $p : \mathbf{X} \to \mathbf{C}$ the restriction to $\mathbf{X}$ of the projection onto the second factor, then $X_t = p^{-1}(t)$ for $t \in \mathbf{C}$ and $X = X_0$. Denote by

$$\sigma_F : \mathbf{F}(\mathbf{X}) \to \mathbf{F}(X)$$

the specialization map on constructible functions and

$$\sigma_H : H_*(X_t) \to H_*(X)$$

the specialization map of homology classes (see [Ve]). For $\varphi \in \mathbf{F(X)}$ and t sufficiently small, one has $\sigma_H c_*(\varphi|_{X_t}) = c_*(\sigma_F \varphi)$.

The Fulton class $c^F(X)$ is given, in terms of MacPherson class as:

$$c^F(X) = c_*(\sigma_F(\mathbf{1_X}))$$

and the Milnor class as:

$$\mu_*(X) = c_*(\sigma_F(\mathbf{1_X}) - \mathbf{1}_X).$$

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Remarks on bivariant constructible functions

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Dedicated to the memory of Professor Nobuo Sasakura

§0. Introduction

The so-called *Chern-Schwartz-MacPherson class* (*or transformation*) is the unique natural transformation from the covariant functor of constructible functions to the integral homology covariant functor, satisfying a certain normalization condition (see [14], and also [3], [10]. [20].) The bivariant theory has been introduced by W.Fulton and R.MacPherson [9], and they conjectured (or posed as a question) the existence of a Grothendieck transformation from the bivariant theory of constructible functions to the bivariant homology theory in the category of complex algebraic varieties, which specializes to the original Chern-Schwartz-MacPherson transformation. The conjecture has been solved by Brasselet for a certain reasonable category [2] (see also [19] and [24]). In this paper we report some consequences of this Brasselet's theorem, concerning bivariant constructible functions (i.e., constructible functions satisfying the *local Euler condition*) and some related results and we also pose some problems.

§1. Constructible functions and Chern-Schwartz-MacPherson classes

A constructible set of an analytic variety X is obtained from analytic subvarieties of X by a finite number of unions, intersections and complements. A constructible function on a compact complex analytic

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variety X is an integer-valued function on X, $\alpha : X \to \mathbb{Z}$, such that for each integer n, $\alpha^{-1}(n)$ is a constructible set of X. We say that a cellular decomposition (K) of X is α-adapted if α is constant on the interior of each cell σ of (K), the value beeing denoted by $\alpha(\sigma)$.

Let $\mathcal{F}(X)$ denote the abelian group of constructible functions on X. Any constructible function can be expressed as a (finite) linear combination of the characteristic functions $1\!\!1_W$'s where W are reduced and irreducible subvarieties of X. Clearly the correspondence $\mathcal{F}$ assigning to each variety X the abelian group $\mathcal{F}(X)$ becomes a contravariant functor when we consider the usual (functional) pull-back $f^* : \mathcal{F}(Y) \to \mathcal{F}(X)$ for a morphism $f : X \to Y$; i.e., $f^*(\alpha)(x) := \alpha(f(x))$. An interesting feature of the correspondence $\mathcal{F}$ is that it can be made a covariant functor when we consider the following pushforward:

$$f_*(1\!\!1_W)(y) := \chi(f^{-1}(y) \cap W),$$

which is linearly extended with respect to the generators $1\!\!1_W$. Here $\chi(F)$ denotes the topological Euler-Poincaré characteristic of the space F. The proof of the covariant functoriality of $\mathcal{F}$ requires a stratification of the morphism f (see [14], [21]).

Deligne and Grothendieck (in 1969) conjectured the following in the algebraic category:

Let $\mathcal{F}$ be the above covariant functor of constructible functions and $H_*(\ :\mathbb{Z})$ be the usual $\mathbb{Z}$-homology covariant functor. Then there exists a unique natural transformation

$$C_* : \mathcal{F} \to H_*(\ :\mathbb{Z})$$

such that (normalization condition) if X is smooth, then

$$C_*(1_X) = c(T_X) \cap [X],$$

where $c(T_X)$ is the total Chern cohomology class of the tangent bundle T_X and $[X]$ is the fundamental homology class of X.

The conjecture was solved by MacPherson [14] (in 1974), using Chern-Mather classes, local Euler obstructions (which are constructible functions) and graph construction method. The folklore was that the above conjecture or theorem now was true in the analytic category also, and indeed in the analytic category MacPherson's proof works *mutatis mutandis*, except for the analyticity of the graph construction. However this analyticity was finally resolved affirmatively by M. Kwieciński in his thesis [13]. Thus the Chern-Schwartz-MacPherson transformation $C_* : \mathcal{F} \to H_*(\ :\mathbb{Z})$ can be considered in both the algebraic and analytic categories. The total homology class $C_*(1\!\!1_X)$ is called the *Chern-Schwartz-MacPherson class of X*. To avoid some possible confusion, we call the above transformation C_* the *Chern-Schwartz-MacPherson transformation*, emphasizing that it is a transformation. In fact, before the above conjecture was made M.-H.Schwartz [20] had already constructed characteristic cohomology classes of a (possibly singular) analytic variety embedded in a complex manifold, using the notion of radial vector field. For a given embedding X in a manifold M the Schwartz classes lie in $H^*_X(M) = H^*(M, M - X)$. It turned out that they are isomorphic to MacPherson's classes via Alexander duality isomorphism (see [3]).

§2. Bivariant theory of constructible functions

Let α be a constructible function on X. For $A \subset X$, we define

$$\chi(A; \alpha) = \sum_{n \in \mathbb{Z}} n\, \chi(A \cap \alpha^{-1}(n)),$$

which is the *Euler-Poincaré characteristic of A weighted by* α (*" pondérée par* α*"*) ([2], [14], [19]). With this notation, the pushforward $f_*\alpha$ of the constructible function α under a morphism $f : X \to Y$ is expressed as follows:

$$(f_*\alpha)(y) := \chi(f^{-1}(y); \alpha),$$

i.e., the Euler-Poincaré characteristic of the fiber $f^{-1}(y)$ weighted by α. Put it in another way, using the Chern-Schwartz-MacPherson transformation C_*, it can be rewritten as follows:

$$(f_*\alpha)(y) = \int_{f^{-1}(y)} C_*(\alpha|_{f^{-1}(y)}),$$

the degree of the 0-dimensional component of the total Chern-Schwartz-MacPherson class of the constructible function $\alpha|_{f^{-1}(y)}$ on the fiber $f^{-1}(y)$. This simple interpretation leads us to a naïve question of what one could say about these classes $C_*(\alpha|_{f^{-1}(y)})$ parameterized by the target variety Y. It turns out that for this we need the bivariant theory of constructible functions which has been introduced by Fulton and MacPherson [9].

For a technical reason, the category which we treat is the following one, denoted by $\mathcal{SC}$:
(i) The objects $\mathcal{O}bj(\mathcal{SC})$ consist of compact complex analytic varieties which are embeddable into smooth manifolds, and
(ii) The morphisms $\mathcal{H}om_{\mathcal{SC}}(X, Y)$ consist of analytic maps $f : X \to Y$ which are cellular, i.e., with (K) and (L) being cellular decompositions

of X and Y respectively, the image of each cell of (K) is a cell of (L) and the restriction of f to the interior of each cell is constant rank.
At the moment it is not known whether any analytic map is cellular. Conjecturally it would be so.

In the following "cell" will always mean a closed cell, the interior of σ will be denoted by σ^o and we define the star $St^o\sigma$ as the set of cells which meet the interior of σ.

In this category, the pushforward can be written as follows: Given cellular decompositions (K) and (L) of X and Y respectively, such that (K) is α-adapted and f cellular, then

$$(f_*\alpha)(y) = \sum_{\sigma \cap f^{-1}(y) \neq \phi} (-1)^{\dim_f \sigma} \alpha(\sigma) \tag{2.1.1}$$

where $\dim_f \sigma$ denotes the relative dimension of $\sigma \in (K)$. Here we note that the above formula (2.1.1) is due to the fact that the topological Euler-Poincaré characteristic of a CW-complex can be also defined to be the alternating sum of the number of cells of a (in fact, any) cellular decomposition of the CW-complex, and therefore that the Euler-Poincaré characteristic weighted by α is equal to the alternating sum of the number of cells multiplied by the weights "α".

Definition (2.1). Let α be a constructible function on X and let $f : X \to Y$ be an analytic map. We say that α satisfies the *local Euler condition with respect to* f if for any cellular decompositions (K) and (L) of X and Y respectively, such that (K) is α-adapted and f is cellular, and if for any $x \in X$, $x \in \sigma^o$, $\sigma \in (K)$, the following equality holds

$$\alpha(x) = \chi(St^o\sigma \cap f^{-1}(y); \alpha)$$

where $y \in St^o f(\sigma)$ is arbitrary.

Using the values of α on the cells of (K), the previous formula can be written

$$\alpha(x) = \sum_{\substack{\sigma' \subset St^o\sigma \\ \sigma' \cap f^{-1}(y) \neq \phi}} (-1)^{\dim_f \sigma'} \alpha(\sigma') \tag{2.1.2}$$

Remark 2.2. There is another definition of local Euler condition without refering to the cellular decomposition of a morphism (see [19]): $\alpha \in \mathcal{F}(X)$ satisfies the local Euler condition with respect to$f$ if for any point $x \in X$ and any local embedding $(X, x) \to (\mathbf{C}^N, 0)$ the following equality holds

$$\alpha(x) = \chi(B_\epsilon \cap f^{-1}(z); \alpha),$$

where B_ϵ is a sufficiently small open ball of the origin 0 with radius ϵ and z is any point close to $f(x)$.

Definition (2.3). The bivariant group of constructible functions is defined, for every morphism $f : X \to Y$, by:

$$\mathbb{F}(X \xrightarrow{f} Y) := \{\alpha \in \mathcal{F}(X) \mid \alpha \text{ satisfies the local Euler condition with respect to } f\}.$$

From this definition we see that

$$\mathbb{F}(X \xrightarrow{\mathrm{id}} X) = \{\alpha \in \mathcal{F}(X) | \alpha \text{ is locally constant}\}.$$

This fact will be used later.

For simplicity a constructible function satisfying the local Euler condition shall be called a *bivariant constructible function.* If $1\!\!1_X$ satisfies the local Euler condition with respect to the morphism $f : X \to Y$, i.e., $1\!\!1_X \in \mathbb{F}(X \xrightarrow{f} Y)$, then the morphism f is called an *Euler morphism.*

We can define the following three basic operations on $\mathbb{F}(X \to Y)$, which are called *bivariant operations.*

(BO-I) (Product operations): For morphisms $f : X \to Y$ and $g : Y \to Z$, the product operation

$$\odot : \mathbb{F}(X \xrightarrow{f} Y) \otimes \mathbb{F}(Y \xrightarrow{g} Z) \to \mathbb{F}(X \xrightarrow{gf} Z)$$

is defined, for $\alpha \in \mathbb{F}(X \xrightarrow{f} Y)$ and $\beta \in \mathbb{F}(Y \xrightarrow{g} Z)$, by:

$$(\alpha \odot \beta)(x) := \alpha(x) \cdot \beta(f(x)),$$

i.e., $\alpha \odot \beta := \alpha \cdot f^*\beta$. (To avoid some confusion, the symbol $\odot$ is used.)

(BO-II) (Pushforward operations): For morphisms $f : X \to Y$ (proper) and $g : Y \to Z$, the pushforward operation

$$f_* : \mathbb{F}(X \xrightarrow{gf} Z) \to \mathbb{F}(Y \xrightarrow{g} Z)$$

is defined, for $\alpha \in \mathbb{F}(X \xrightarrow{gf} Z)$, by:

$$(f_*\alpha)(y) := \chi(f^{-1}(y); \alpha),$$

which is the same as one described in §1.

(BO-III) (Pull-back operations): For a fiber square

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ f' \downarrow & & \downarrow f \\ Y' & \xrightarrow{g} & Y, \end{array}$$

where $X' = Y' \times_{g=f} X$ is the fiber product and $f' : X' \to Y'$ and $g' : X' \to X$ are the canonical projections, the pull-back operation

$$g^* : \mathbb{F}(X \xrightarrow{f} Y) \to \mathbb{F}(X' \xrightarrow{g} Y')$$

is defined, for $\alpha \in \mathbb{F}(X \xrightarrow{f} Y)$, by:

$$g^*\alpha := g'^*\alpha,$$

which is the usual (functional) pull-bak.

It is known that these three operations are well-defined, and we give a proof of this fact for the sake of completeness.

Proof of well-definedness.

Let (K), (L) and (M) be any cellular decompositions of X, Y and Z respectively adapted to the corresponding constructible functions and such that the corresponding morphisms are cellular.

(BO-I): Let x_o be a point of the interior of $\sigma_o \in (K)$. What we want to show is that

$$\alpha \odot \beta(x_o) = (\alpha \cdot f^*\beta)(x_o) = \alpha(x_o)\beta(f(x_o)) = \chi(St^o\sigma_o \cap (g \circ f)^{-1}(z); \alpha \odot \beta)$$

where $z \in St^o(g \circ f)\sigma_o$. We will denote by y a point in $St^o f(\sigma_o)$ and by $\tau_o = f(\sigma_o)$, so $z \in St^o g(\tau_o)$. We start with the last term:

$$\begin{aligned}
&\chi(St^o\sigma_o \cap (g \circ f)^{-1}(z); \alpha \odot \beta) \\
&= \sum_{\substack{\sigma \subset St^o\sigma_o \\ \sigma \cap (g \circ f)^{-1}(z) \neq \phi}} (-1)^{\dim_f \sigma} \alpha(\sigma) \cdot (-1)^{\dim_g f(\sigma)} \beta(f(\sigma)) \\
&= \sum_{\substack{\tau \subset St^o\tau_o \\ \tau \cap g^{-1}(z) \neq \phi}} (-1)^{\dim_g \tau} \beta(\tau) \cdot \sum_{\substack{\tau = f(\sigma) \\ \sigma \subset St^o\sigma_o \\ \sigma \cap f^{-1}(y) \neq \phi}} (-1)^{\dim_f \sigma} \alpha(\sigma) \\
&= \sum_{\substack{\tau \subset St^o\tau_o \\ \tau \cap g^{-1}(z) \neq \phi}} (-1)^{\dim_g \tau} \beta(\tau) \cdot \chi(St^o\sigma_o \cap f^{-1}(y); \alpha) \\
&= \beta(f(x_o)) \cdot \alpha(x_o) \\
&= \alpha \odot \beta(x_o).
\end{aligned}$$

Q.E.D

(BO-II): We must prove that, if $y \in \tau^o$ with $\tau \in (L)$, then

$$(f_*\alpha)(y) = \chi(St^o\tau \cap g^{-1}(z); f_*\alpha)$$

for any $z \in St^o g(\tau)$.

Denoting $h = g \circ f$, let us remark the following properties:

$$f^{-1}(g^{-1}(z) \cap St^o\tau) = \bigcup_{f(\sigma)=\tau} h^{-1}(z) \cap St^o\sigma$$

Let σ' be a cell of (K) such that $f(\sigma') \subset St^o\tau$, then

$$A(\sigma', \tau) = \{\sigma \in (K) \mid \sigma' \subset St^o\sigma, f(\sigma) = \tau\},$$

is the subset of the face $\sigma' \cap f^{-1}(\tau)$ of σ' consisting of cells whose image is τ. Its restriction to any fiber $f^{-1}(y)$, $y \in \tau^o$ is a cell whose Euler-Poincaré characteristic is

$$\chi(\sigma' \cap f^{-1}(y)) = \sum_{\sigma \in A(\sigma',\tau)} (-1)^{\dim_f \sigma} = 1. \tag{2.3.1}$$

This equality is a crucial observation which makes the proof of BO-II "the most fun" (cf [9, §6.1.2, the last two lines of p. 61]). We have:

$$\begin{aligned}
(f_*\alpha)(y) &= \chi(f^{-1}(y); \alpha) \\
&= \sum_{\sigma \cap f^{-1}(y) \neq \phi} (-1)^{\dim_f \sigma} \alpha(\sigma) && \text{(by 2.1.1)} \\
&= \sum_{\sigma \cap f^{-1}(y) \neq \phi} (-1)^{\dim_f \sigma} \sum_{\substack{\sigma' \subset St^o\sigma \\ \sigma' \cap h^{-1}(z) \neq \phi}} (-1)^{\dim_h \sigma'} \alpha(\sigma') && \text{(by 2.1.2)} \\
&= \sum_{\substack{\sigma' \subset St^o\sigma \\ \sigma' \cap h^{-1}(z) \neq \phi}} (-1)^{\dim_h \sigma'} \left(\sum_{\sigma \in A(\sigma',\tau)} (-1)^{\dim_f \sigma} \right) \alpha(\sigma') \\
&= \sum_{\substack{\sigma' \subset St^o\sigma \\ \sigma' \cap h^{-1}(z) \neq \phi}} (-1)^{\dim_h \sigma'} \alpha(\sigma') && \text{(by 2.3.1)} \\
&= \sum_{\substack{\sigma' \subset St^o\sigma \\ \sigma' \cap h^{-1}(z) \neq \phi}} (-1)^{\dim_g f(\sigma')} (-1)^{\dim_f \sigma'} \alpha(\sigma') \\
&= \sum_{\substack{\tau' \subset St^o\tau \\ \tau' \cap g^{-1}(z) \neq \phi}} (-1)^{\dim_g \tau'} \left(\sum_{f(\sigma')=\tau'} (-1)^{\dim_f \sigma'} \alpha(\sigma') \right) \\
&= \sum_{\substack{\tau' \subset St^o\tau \\ \tau' \cap g^{-1}(z) \neq \phi}} (-1)^{\dim_g \tau'} \chi(f^{-1}(\tau'); \alpha) \\
&= \sum_{\substack{\tau' \subset St^o\tau \\ \tau' \cap g^{-1}(z) \neq \phi}} (-1)^{\dim_g \tau'} (f_*\alpha)(\tau') \\
&= \chi(St^o\tau \cap g^{-1}(z); f_*\alpha). && \text{Q.E.D}
\end{aligned}$$

(BO-III): Let $x' \in \tau_o^o$ be a point in X'. Then for any $y' \in St^o f'(\tau_o)$ in Y', letting $\sigma_o = g'(\tau_o)$ and $y = g(y')$, we have

$$\begin{aligned}
\chi(St^o\tau_o \cap f'^{-1}(y'); g^*\alpha) &= \sum_{\substack{\tau \subset St^o\tau_o \\ \tau \cap f'^{-1}(y') \neq \phi}} (-1)^{\dim_{f'} \tau} g^*\alpha(\tau) \\
&= \sum_{\substack{\tau \subset St^o\tau_o \\ \tau \cap f'^{-1}(y') \neq \phi}} (-1)^{\dim_{f'} \tau} \alpha(g'(\tau)) \\
&= \sum_{\substack{\sigma \subset St^o\sigma_o \\ \sigma \cap f^{-1}(y) \neq \phi}} (-1)^{\dim_f \sigma} \alpha(\sigma) \\
&= \alpha(g'(x')) = (g^*\alpha)(x').
\end{aligned}$$

Q.E.D

It is easy to see that these bivariant operations enjoy the following seven properties.

(B-1) Product is associative : for a diagram $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$ and $\alpha \in \mathbb{F}(X \xrightarrow{f} Y)$, $\beta \in \mathbb{F}(Y \xrightarrow{g} Z)$ and $\gamma \in \mathbb{F}(Z \xrightarrow{h} W)$,

$$(\alpha \odot \beta) \odot \gamma = \alpha \odot (\beta \odot \gamma) \in \mathbb{F}(X \xrightarrow{hgf} W).$$

(B-2) Pushforward is functorial: for a diagram $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$ and $\alpha \in \mathbb{F}(X \xrightarrow{hgf} W)$,

$$(gf)_*(\alpha) = g_* f_*(\alpha) \in \mathbb{F}(Z \xrightarrow{h} W).$$

(B-3) Pullback is functorial: for a double fiber square

$$\begin{array}{ccccc}
X'' & \xrightarrow{h'} & X' & \xrightarrow{g'} & X \\
\downarrow f'' & & \downarrow f' & & \downarrow f \\
Y'' & \xrightarrow{h} & Y' & \xrightarrow{g} & Y
\end{array}$$

and $\alpha \in \mathbb{F}(X \xrightarrow{f} Y)$,

$$(gh)^*(\alpha) = h^* g^*(\alpha) \in \mathbb{F}(X'' \xrightarrow{f''} Y'').$$

(B-4) Product and pushforward commute: for a diagram $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$ and $\alpha \in \mathbb{F}(X \xrightarrow{gf} Z)$, $\beta \in \mathbb{F}(Z \xrightarrow{h} W)$,

$$f_*(\alpha \odot \beta) = f_*(\alpha) \odot \beta \in \mathbb{F}(Y \xrightarrow{hg} W).$$

(B-5) Product and pullback commute: for a double fiber square

$$\begin{array}{ccc} X' & \xrightarrow{h''} & X \\ \downarrow f' & & \downarrow f \\ Y' & \xrightarrow{h'} & Y \\ \downarrow g' & & \downarrow g \\ Z' & \xrightarrow{h} & Z \end{array}$$

and $\alpha \in \mathbb{F}(X \xrightarrow{f} Y)$, $\beta \in \mathbb{F}(Y \xrightarrow{g} Z)$,

$$h^*(\alpha \odot \beta) = h'^*(\alpha) \odot h^*(\beta) \in \mathbb{F}(X' \xrightarrow{g'f'} Z').$$

(B-6) Pushforward and pullback commute: for a double fiber square

$$\begin{array}{ccc} X' & \xrightarrow{h''} & X \\ \downarrow f' & & \downarrow f \\ Y' & \xrightarrow{h'} & Y \\ \downarrow g' & & \downarrow g \\ Z' & \xrightarrow{h} & Z \end{array}$$

and $\alpha \in \mathbb{F}(X \xrightarrow{gf} Z)$,

$$f'_*(h^*(\alpha)) = h^* f_*(\alpha) \in \mathbb{F}(Y' \xrightarrow{g'} Z').$$

(B-7) Projection formula: For a fiber square

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \downarrow f' & & \downarrow f \\ Y' & \xrightarrow{g} & Y, \end{array}$$

a morphism $Y \xrightarrow{h} Z$, $\alpha \in \mathbb{F}(X \xrightarrow{f} Y)$ and $\beta \in \mathbb{F}(Y' \xrightarrow{hg} Z)$,

$$g'_*((g^*\alpha) \odot \beta) = \alpha \odot g_*(\beta) \in \mathbb{F}(X \xrightarrow{hf} Z).$$

Before finishing this section we note that the well-definedness of the pushforward (BO-II) implies the following

Proposition (2.4). *Let $\alpha \in \mathbb{F}(X \xrightarrow{f} Y)$, then the pushforward $f_*\alpha$ is a locally constant function on Y.*

This can be seen as follows: Consider the pushforward on the following diagram:

$$X \xrightarrow{f} Y \xrightarrow{\mathrm{id}} Y.$$

Indeed, for $\alpha \in \mathbb{F}(X \xrightarrow{f} Y) = \mathbb{F}(X \xrightarrow{\mathrm{id}\cdot f} Y)$

$$f_*\alpha \in \mathbb{F}(Y \xrightarrow{\mathrm{id}} Y),$$

which implies that $f_*\alpha$ is locally constant since

$$\mathbb{F}(X \xrightarrow{\mathrm{id}} X) = \{\alpha \in \mathcal{F}(X) | \alpha \text{ is locally constant}\}.$$

In other words the local Euler condition posed on a constructible function may be a right local condition to guarantee such a strong requirement that the Euler-Poincaré characteristic of the fibers weighted by α are locally constant. This is certainly a strong requirement for a map.

§3. Bivariant Chern classes

In general, a bivariant theory B on a category $\mathcal{C}$ to abelian groups is an assignment to each morphism

$$X \xrightarrow{f} Y$$

in the category $\mathcal{C}$ an abelian group

$$B(X \xrightarrow{f} Y)$$

which is equipped with the three basic operations such as in (BO-I, BO-II, BO-III) above and satisfy the seven properties as in (B-1)-(B-7).

Let $\mathbb{H}(X \to Y)$ be the bivariant homology theory (see [2] and [9]). For a morphism $f : X \to Y$ and for any integer i, $H^i(X \to Y) := H^{i+2m}(Y \times M, Y \times M - \Phi(X))$, where $\phi : X \to M$ is an embedding into a smooth manifold of real dimension $2m$ and $\Phi := (f, \phi) : X \to Y \times M$ is an embedding. The definition is independent of the embedding $\phi : X \to M$. Then as in the case of the bivariant constructible function theory the three basic bivariant operations can be defined for the bivariant homology theory, namely we have the following (for details see Fulton-MacPherson's book [9]):

(BO-I:$\mathbb{H}$) (Product operations): For morphisms $f : X \to Y$ and $g : Y \to Z$, the product operation

$$\odot_{\mathbb{H}} : \mathbb{H}(X \xrightarrow{f} Y) \otimes \mathbb{H}(Y \xrightarrow{g} Z) \to \mathbb{H}(X \xrightarrow{gf} Z)$$

is defined.
(BO-II:$\mathbb{H}$) (Pushforward operations): For morphisms $f : X \to Y$ (proper) and $g : Y \to Z$, the pushforward operation

$$f_* : \mathbb{H}(X \xrightarrow{gf} Z) \to \mathbb{H}(Y \xrightarrow{g} Z)$$

is defined.
(BO-III:$\mathbb{H}$) (Pull-back operations): For a fiber square

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \downarrow f' & & \downarrow f \\ Y' & \xrightarrow{g} & Y \end{array}$$

the pull-back operation

$$g^* : \mathbb{H}(X \xrightarrow{f} Y) \to \mathbb{H}(X' \xrightarrow{g} Y')$$

is defined.

Fulton and MacPherson [9] conjectured (or posed as a question) the existence of a bivariant version of the Chern-Schwartz-MacPherson transformation $C_* : \mathcal{F} \to H_*(: \mathbb{Z})$, i.e., the existence of Grothendieck transformation (or "bivariant Chern class"), in the category of complex algebraic varieties. Brasselet [2] proved this conjecture in the category $\mathcal{SC}$ defined in §2. Also C. Sabbah [19] constructed a bivariant theory of cycles and J.Zhou [24] proved that Sabbah's bivariant Chern classes defined by bivariant cycles are the same as Brasselet's bivariant Chern classes.

Theorem (3.1). *(Brasselet [2, III, Théorème]) Let $\mathcal{SC}$ be the category to be considered. There exists a Grothendieck transformation*

$$\gamma : \mathbb{F} \to \mathbb{H}$$

such that if X is a smooth variety, then

$$\gamma(1\!\!1_\pi) = c(TX) \cap [X],$$

where $\pi : X \to pt$ is a map to a point pt and $1\!\!1_\pi := 1\!\!1_X \in \mathbb{F}(X \xrightarrow{\pi} pt)$. Namely, for each morphism $f : X \to Y$, γ gives rise to a homomorphism

$$\gamma : \mathbb{F}(X \xrightarrow{f} Y) \to \mathbb{H}(X \xrightarrow{f} Y)$$

such that γ *preserves the basic three operations, i.e.,* (i) $\gamma(\alpha \odot \beta) = \gamma(\alpha) \odot_{\mathbb{H}} \gamma(\beta)$, (ii) $\gamma(f_*\alpha) = f_*\gamma(\alpha)$ *and* (iii) $\gamma(f^*\alpha) = f^*\gamma(\alpha)$.

Remark 3.2 The uniqueness problem of γ is still open. We will discuss it a little later in the next section (Remark (4.10)).

Brasselet constructs the above transformation in such a way that the Chern-Schwartz-MacPherson classes $i_{y*}C_*(\alpha|_{f^{-1}(y)})$ of the fibers weighted by α are locally constant, where $i_y : f^{-1}(y) \to X$ is the inclusion map. Of course this is a much stronger requirement than the local constancy of the Euler-Poincaré characteristic of the fibers weighted by the constructible function α. In fact as a consequence of the above Brasselet's theorem we can say more and we see that this quite strong requirement is a necessity for a bivariant constructible function.

Theorem (3.3). (1) *Let* $\alpha \in \mathbb{F}(X \xrightarrow{f} Y)$ *and let* V_1, V_2 *be subvarieties of* Y *such that the Chern-Schwartz-MacPherson classes of* V_1 *and* V_2 *are homologous in* Y, *then the Chern-Schwartz-MacPherson classes* $C_*(\alpha|_{f^{-1}(V_1)})$ *and* $C_*(\alpha|_{f^{-1}(V_2)})$ *of the inverses* $f^{-1}(V_1), f^{-1}(V_2)$ *weighted with* α *are also homologous in* X. *Namely, if*

$$i_{1*}C_*(V_1) = i_{2*}C_*(V_2)$$

with $i_j : V_j \to Y$ *being the inclusion maps* $(j = 1, 2)$, *then*

$$e_{1*}C_*(\alpha|_{f^{-1}(V_1)}) = e_{2*}C_*(\alpha|_{f^{-1}(V_2)})$$

with $e_j : f^{-1}(V_j) \to X$ *being the inclusion maps* $(j = 1, 2)$.
(2) *In particular, if* $\alpha \in \mathbb{F}(X \xrightarrow{f} Y)$, *then the Chern-Schwartz-MacPherson classes* $i_{y*}C_*(\alpha|_{f^{-1}(y)})$ *of the fibers weighted by* α *are locally constant, where* $i_y : f^{-1}(y) \to X$ *is the inclusion map.*

Corollary (3.4). *Let* $f : X \to Y$ *be an Euler morphism. Then if* V_1, V_2 *be subvarieties of* Y *such that the Chern-Schwartz-MacPherson classes of* V_1 *and* V_2 *are homologous in* Y, *then the Chern-Schwartz-MacPherson classes* $C_*(f^{-1}(V_1))$ *and* $C_*(f^{-1}(V_2))$ *of the inverses* $f^{-1}(V_1), f^{-1}(V_2)$ *are also homologous in* X. *Namely, if*

$$i_{1*}C_*(V_1) = i_{2*}C_*(V_2)$$

with $i_j : V_j \to Y$ *being the inclusion maps* $(j = 1, 2)$, *then*

$$e_{1*}C_*(f^{-1}(V_1)) = e_{2*}C_*(f^{-1}(V_2))$$

with $e_j : f^{-1}(V_j) \to X$ being the inclusion maps $(j = 1, 2)$. In particular, the Chern-Schwartz-MacPherson classes ${i_y}_* C_*(f^{-1}(y))$ of the fibers are locally constant.

The proof of Theorem (3.3) goes as follows; for the sake of later use we give a detailed proof.

Proof of Theorem (3.3). The constructible function α induces the following homomorphism,

$$\alpha^{\mathbb{F}} : \mathcal{F}(Y) \to \mathcal{F}(X)$$

defined by

$$\alpha^{\mathbb{F}}(\beta) := \alpha \odot \beta = \alpha \cdot f^* \beta.$$

Then we can get the following commutative diagram:

(3.3.1)
$$\begin{array}{ccc} \mathcal{F}(Y) & \xrightarrow{\alpha^{\mathbb{F}}} & \mathcal{F}(X) \\ {\scriptstyle C_*} \downarrow & & \downarrow {\scriptstyle C_*} \\ H_*(Y;\mathbb{Z}) & \xrightarrow{\alpha^{\mathbb{H}}_{\gamma}} & H_*(X;\mathbb{Z}) \end{array}$$

Here $\alpha^{\mathbb{H}}_{\gamma} : H_*(Y;\mathbb{Z}) = \mathbb{H}(Y \to pt) \to H_*(X;\mathbb{Z}) = \mathbb{H}(X \to pt)$ is defined by

$$\alpha^{\mathbb{H}}_{\gamma}(a) := \gamma(\alpha) \odot_{\mathbb{H}} a,$$

where $\gamma : \mathbb{F} \to \mathbb{H}$ is a Grothendieck transformation and $\odot_{\mathbb{H}} : \mathbb{H}(X \xrightarrow{f} Y) \otimes \mathbb{H}(Y \to pt) \to \mathbb{H}(X \to pt)$ is the bivariant homology product operation. Here it should be noted that since the uniqueness of the Grothendieck transformation γ is not known yet the homomorphism $\alpha^{\mathbb{H}}_{\gamma}$ could depend on the transformation γ but that our statement is independent of the choice of γ. Of course the commutativity of the above diagram follows from the fact that the Grothendieck transformation preserves the three basic operations. First note that for a morphism $X \to pt$ the Grothendieck homomorphism $\gamma : \mathbb{F}(X \to pt) \to \mathbb{H}(X \to pt)$ is nothing but the Chern-Schwartz-MacPherson transformation $C_* : \mathcal{F}(X) \to H_*(X;\mathbb{Z})$. Then the commutativity can be seen as follows

$$\begin{aligned} C_*\alpha^{\mathbb{F}}(\beta) &= C_*(\alpha \odot \beta) \\ &= \gamma(\alpha \odot \beta) \\ &= \gamma(\alpha) \odot_{\mathbb{H}} \gamma(\beta) \\ &= \gamma(\alpha) \odot_{\mathbb{H}} C_*(\beta) \\ &= \alpha^{\mathbb{H}}_{\gamma} C_*(\beta). \end{aligned}$$

We call the commutative diagram (3.3.1) a *Verdier-type Riemann-Roch associated with the constructible function* α(cf. [22]). To finish the proof of (1), we just apply this Verdier-type Riemann-Roch to two constructible functions $1\!\!1_{V_1}, 1\!\!1_{V_2}$. First observe that for any subset $A \subset Y$ $\alpha^{\mathbb{F}}(1\!\!1_A) = \alpha \cdot f^* 1\!\!1_A = \alpha \cdot 1\!\!1_{f^{-1}(A)} = e_* \alpha|_{f^{-1}(A)}$, where $e : f^{-1}(A) \to X$ is the inclusion map. Now suppose thatV_1, V_2 are subvarieties of Y such that the Chern-Schwartz-MacPherson classes of V_1 and V_2 are homologous in X, i.e, $i_{1*}C_*(V_1) = i_{2*}C_*(V_2)$ with $i_j : V_j \to Y$ being the inclusion maps ($j = 1, 2$). Then we have

$$\begin{aligned}
e_{1*}C_*(\alpha|_{f^{-1}(V_1)}) &= C_*(e_{1*}\alpha|_{f^{-1}(V_1)}) \\
&= C_*\alpha^{\mathbb{F}}(1\!\!1_{V_1}) \\
&= \alpha^{\mathbb{H}}_{\gamma}(C_*(1\!\!1_{V_1})) \\
&= \alpha^{\mathbb{H}}_{\gamma}(i_{1*}C_*(V_1)) \\
&= \alpha^{\mathbb{H}}_{\gamma}(i_{2*}C_*(V_2)) \qquad (\text{since } i_{1*}C_*(V_1) = i_{2*}C_*(V_2)) \\
&= \alpha^{\mathbb{H}}_{\gamma}(C_*(1\!\!1_{V_2})) \\
&= C_*\alpha^{\mathbb{F}}(1\!\!1_{V_2}) \\
&= C_*(e_{2*}\alpha|_{f^{-1}(V_2)}) \\
&= e_{2*}C_*(\alpha|_{f^{-1}(V_2)}).
\end{aligned}$$

Thus (1) is proved and (2) is a special case of (1). Q.E.D

Remark (3.5). It follows from the definition of Eulerness that any local trivial fibration is always Euler. But Eulerness does not imply local triviality, as the following example (given by T. Ohmoto) shows. Let $X = \{(x, y, z) \in \mathbb{C}^3 | x^2 + y^2 + z = 0\} \cup \{\text{the } z\text{-axis}\}$ and let $f : X \to \mathbb{C}$ be the restriction to X of the projection $p : \mathbb{C}^3 \to \mathbb{C}$ to the third factor $\mathbb{C}$. The Milnor fiber at the origin is homotopic to the disjoint union of circle (i.e., the vanishing cycle) and one point, thus the Euler-Poincaré characteristic of a nearby fiber in a small neighborhood of the origin is equal to one. Hence at the origin it satisfies the local Euler condition, but it is not a local trivial fibration. At every point of X off the origin the map f is a local trivial fibration. Thus f is Euler but not a local trivial fibration. The map f is not proper, but the example can be modified into the following example of a map between compact varieties. Let us consider the following surface E in $\mathbb{P}^2 \times \mathbb{P}^1$:

$$\begin{aligned}
E : \;=\; &\{([x_0 : x_1 : x_2], [w_0 : w_1]) \in \mathbb{P}^2 \times \mathbb{P}^1 | \\
&\qquad w_0 x_0^2 + (w_0 + w_1) x_1^2 + w_1 x_2^2 = 0\}.
\end{aligned}$$

Let

$$X := E \cup ([1:0:0] \times \mathbb{P}^1) \cup ([0:1:0] \times \mathbb{P}^1) \cup ([0:0:1] \times \mathbb{P}^1).$$

and let $f : X \to \mathbb{P}^1$ be the restriction to the subvariety X of the projection $\mathbb{P}^2 \times \mathbb{P}^1 \to \mathbb{P}^1$ to the second factor. Then just like the above example, at the three distinguished points $([1:0:0],[0:1]), ([0:1:0],[1:-1]), ([0:0:1],[1:0])$ the Milnor fiber of f is homotopic to the union of the circle and one point and otherwise f is locally trivial off these three points. Hence $f : X \to \mathbb{P}^1$ is Euler but not a local trivial fibration.

In general, some other well-studied morphisms, such as flat, open, A_f, and triangulable morphisms, are not Euler. For example, consider a Kodaira's elliptic surface [12], i.e., a surjective holomorphic map

$$f : S \to C$$

of a smooth compact complex surface S onto a smooth compact complex curve C such that its generic fiber is a smooth elliptic curve and that it has only finitely many singular fibers. This Kodaira elliptic surface $f : S \to C$ is not Euler, because the topological Euler-Poincaré characteristics of the fibers are not constant; the topological Euler-Poincaré characteristic of the generic fiber is zero but that of the singular fibers are not zero. On the other hand, it follows from [11, §4] that the map $f : S \to C$ is flat since S and C are smooth and (locally) the fibers have the same dimension ($= 1$) and furthermore, since C is smooth, hence Cohen-Macauley, f is open due to the universal openness of the flat map. Since the target C of the map f is a smooth analytic curve, it follows from [11, Corollary 1, p.248] that f is A_f, i.e., there exists a Whitney stratification of f which satisfies Thom's A_f condition. It is not clear whether it is triangulable or not, which is left for the reader. However, as an example of a morhism which is triangulable but not Euler, we can consider the following simple situation:

$$X := (\mathbb{P}^1 \times [1:0]) \cup ([1:0] \times \mathbb{P}^1) \subset \mathbb{P}^1 \times \mathbb{P}^1.$$

Namely, if we let $[z_0 : z_1]$ and $[w_0 : w_1]$ be the homogeneous coordinates of the first and second factor $\mathbb{P}^1$, respectively, then X is defined by the equation $z_1 w_1 = 0$. Let $f : X \to \mathbb{P}^1$ be the restriction of the projection $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ (either to the first factor or to the second factor) to the subvariety X. Then f is obviously triangulable, but certainly not Euler because the topological Euler-Poincaré characteristics of the fibers are not constant; $\chi(f^{-1}([1:0])) = \chi(\mathbb{P}^1) = 2$ and $\chi(f^{-1}(x)) = 1$ if $x \neq [1:0]$.

At the moment a reasonable characterization of Eulerness is not known yet.

Before finishing this section we pose one problem. First, suggested by Proposition (2.4), one might be able to consider the following naïve group of constructible functions:

$$\mathbb{F}^{l.c.}(X \xrightarrow{f} Y) := \{\alpha \in \mathcal{F}(X) | f_*\alpha \text{ is a locally constant function on } Y\}$$

Then we can show the following

Proposition (3.6). *Let us consider only topologically connected compact complex analytic varieties. Then the above naïve group of constructible functions* $\mathbb{F}^{l.c.}(X \xrightarrow{f} Y)$ *becomes a bivariant theory with the same operations as ones for* $\mathbb{F}(X \xrightarrow{f} Y)$.

Proof. We have only to show that the three operations are well-defined. First we note that since our varieties are assumed to be topologically connected, that $f_*\alpha$ is locally constant on Y means that it is a constant function on Y. This constancy is needed only for the well-definedness of the product operation, as we see below.
(1) (BO-I) (Product operations) For morphisms $f : X \to Y$ and $g : Y \to Z$, let $\alpha \in \mathcal{F}(X)$ such that $f_*\alpha$ is a constant function on Y and $\beta \in \mathcal{F}(Y)$ such that $f_*\beta$ is a constant function on Z. Then we need to show that $(gf)_*(\alpha \odot \beta)$ is a constant function. First we note that$f_*(\alpha \odot \beta) = (f_*\alpha) \odot \beta$ by the commutativity of pushforward and product operation (B-4). Then since we can consider $f_*\alpha \in \mathbb{F}(Y \xrightarrow{\text{id}} Y)$, $(f_*\alpha) \odot \beta = (f_*\alpha) \cdot \beta = c \cdot \beta$, where $c = \chi(f^{-1}(y); \alpha)$ for any $y \in Y$ is a constant. Therefore $(gf)_*(\alpha \odot \beta) = g_*(f_*(\alpha \odot \beta)) = g_*(c \cdot \beta) = c \cdot g_*(\beta)$, which is a constant function because $g_*(\beta)$ is so. As we can see, the constancy of $f_*\alpha$ is crucial. (If it is not constant, we can easily get a counterexample.)
(2) (BO-II) (Pushforward operations) For morphisms $f : X \to Y$ and $g : Y \to Z$, and $\alpha \in \mathbb{F}^{l.c}(X \xrightarrow{gf} Z)$ we want to show that $f_*\alpha \in \mathbb{F}^{l.c}(Y \xrightarrow{g} Z)$. But this is obvious, because $(gf)_*\alpha = g_*(f_*\alpha)$.
(3) (BO-III) (Pull-back operations) For a fiber square

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ f' \downarrow & & \downarrow f \\ Y' & \xrightarrow{g} & Y, \end{array}$$

and $\alpha \in \mathcal{F}(X)$, we need to show that if $f_*\alpha$ is locally constant on Y, then $f'_*g'^*\alpha$ is locally constant. For this we can use the following lemma ([5, Proposition 3.5]):

Lemma (3.6.1). *The following diagram is commutative:*

$$\begin{array}{ccc}
\mathcal{F}(X) & \xrightarrow{g'} & \mathcal{F}(X') \\
f' \downarrow & & \downarrow f \\
\mathcal{F}(Y) & \xrightarrow{g} & \mathcal{F}(Y')
\end{array}$$

Since $f_*\alpha$ is locally constant, $g^*f_*\alpha$ is also locally constant. Then using the lemma, $g^*f_*\alpha = f'_*g'^*\alpha$ is locally constant, thus $g^*\alpha := g'^*\alpha \in \mathbb{F}^{l.c.}(X' \xrightarrow{f'} Y')$. Q.E.D

Note that

(1) $\mathbb{F}^{l.c.}(X \to pt) = \mathbb{F}(X \to pt) = \mathcal{F}(X)$,

(2) $\mathbb{F}^{l.c.}(X \xrightarrow{\text{id}} X) = \mathbb{F}(X \xrightarrow{\text{id}} X)$
$= \{\alpha \in \mathcal{F}(X) \mid \alpha \text{ is locally constant on} X\}$,

and
(3) in general, $\mathbb{F}(X \to Y) \subset \mathbb{F}^{l.c.}(X \to Y)$ and they are not necessarily equal as the following example shows: (Example) Let L_1 be the diagonal of the cartesian product $\mathbb{P}^1 \times \mathbb{P}^1$ of the 1-dimensional projective space $\mathbb{P}^1$. Choose a point z_0 in $\mathbb{P}^1$, and consider another line $L_2 : \{(z, z_0)|z \in P^1\} \subset \mathbb{P}^1 \times \mathbb{P}^1$. Set $L := L_1 \cup L_2 \subset \mathbb{P}^1 \times \mathbb{P}^1$. Let E be a smooth elliptic curve, so that its Euler characteristic $\chi(E) = 0$. Let $X := L \times E$. And let $f : X \to \mathbb{P}^1$ be the composite of the inclusion $X = L \times E \to \mathbb{P}^1 \times \mathbb{P}^1 \times E$, the projection to the first two factors $\mathbb{P}^1 \times \mathbb{P}^1 \times E \to \mathbb{P}^1 \times \mathbb{P}^1$ and the projection to the first factor $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$. Then the Euler-Poincaré characteristic of the fibers are clearly locally constant; in fact $\chi(f^{-1}(z)) = 0$ for any point z, which comes from the fact that $\chi(E) = 0$. Thus the pushforward $f_*\mathbb{1}_X$ is locally constant. However, it is easy to see that the map f is not Euler, i.e., $\mathbb{1}_X \notin \mathbb{F}(X \xrightarrow{f} Y)$. Because at every point of the fiber $f^{-1}(z_0) = \{(z_0, z_0)\} \times E$, $\mathbb{1}_X$ does not satisfy the local Euler condition with respect to f.

Let $\iota : \mathbb{F} \to \mathbb{F}^{l.c.}$ be the inclusion, i.e., $\iota(\alpha) = \alpha$.

Problem (3.7). *Can one construct a Grothendieck transformation* $\gamma^{l.c.} : \mathbb{F}^{l.c.} \to \mathbb{H}$ *such that* (1) $\gamma^{l.c.}(1_\pi) = c(T_X) \cap [X]$ *if* X *is smooth and* (2) $\gamma = \gamma^{l.c.} \circ \iota$ *?*

§4. Generalized Milnor numbers

Definition (4.1). (Parusiński's generalized Milnor number)([15], [16] and [17]) Let X be a local complete intersection variety of a smooth variety M. Let $n = \dim X$. Then the *Parusiński's generalized Milnor number* $\mu(X)$ is defined to be

$$\mu(X) := (-1)^{n+1}\left[\chi(X) - \int_X C^{FJ}(X)\right],$$

where $C^{FJ}(X)$ is *Fulton-Johnson's Chern class*, defined to be $c(TM|_X - N_XM) \cap [X]$ with N_XM being the normal bundle of X. Note that this class is independent of the embedding of X into a smooth variety M (see Fulton's book [7, Example 4.2.6]).

Since $\chi(X) = \int_X C_*(X)$, as a simple generalization of the generalized Milnor number we have the following

Definition (4.2). (see [1], [4], [18], [23]) Let the situation be as in Definition (4.1) . The Milnor class $\mathcal{M}(X)$ is defined by

$$\mathcal{M}(X) := (-1)^{n+1}\big[C_*(X) - C^{FJ}(X)\big].$$

With these definitions we can show the following theorem.

Theorem (4.3). *Let $f : X \xrightarrow{r} M \xrightarrow{p} Y$ be an Euler and local complete intersection morphism (i.e., $r : X \to M$ is a regular embedding and $p : M \to Y$ is a smooth morphism) such that over each point $y \in Y$, the restriction to the fibers $r_y : X_y \to M_y$ is also a regular embedding with $dimX_y = n$, i.e., the fiber $X_y := f^{-1}(y)$ is a local complete intersection variety of the smooth fiber $M_y := p^{-1}(y)$. Then the Milnor class $\mathcal{M}(X_y)$ of the fiber X_y (considered as classes of X) are locally constant. In particular, the generalized Milnor number of the fibers X_y are locally constant.*

Proof. Firstly we remark that the smoothness of the fiber M_y comes from the smoothness of the morphism p. Since $f : X \to Y$ is a local complete intersection morphism, we can define the following homomorphism

$$c(T_f) \cap f^* : H_*(Y;\mathbb{Z}) \to H_*(X;\mathbb{Z}),$$

where T_f is the virtual relative tangent bundle, defined to be

$$T_f := i^*T_p - N_XM,$$

and $f^* : H_*(Y;\mathbb{Z}) \to H_*(X;\mathbb{Z})$ is the Gysin homomorphism [7, Example 19.2.1]. Since we are in the homology theory, the homology classes

$c(T_f) \cap f^*([y])$ are certainly locally constant. Since $f : X \to Y$ is also Euler, it follows from Corollary (3.4) that the Chern-Schwartz-MacPherson classes ${i_y}_* C_*(X_y)$ of the fibers X_y are locally constant. So to prove the theorem we only need to prove the following equality

$$c(T_f) \cap f^*([y]) = {i_y}_* C^{FJ}(X_y)$$

for which we proceed as follows:

$$\begin{aligned} c(T_f) \cap f^*([y]) &= c(T_f) \cap i^*([M_y]) \\ &= c(T_f) \cap i^* {i_y}_*([M_y]) \\ &= c(T_f) \cap {i_y}_* i^*([M_y]) && \text{(by [5, Theorem (6.2)(a)])} \\ &= {i_y}_*\big(c({i_y}^* T_f) \cap i^*([M_y])\big) && \text{(by the projection formula)} \\ &= {i_y}_*\big(c(T_{f_y}) \cap [X_y]\big) && \text{(by [5, Example (6.2.1)])} \\ &= {i_y}_* C^{FJ}(X_y). \end{aligned}$$

Q.E.D

Motivated by this result, we can consider the following: Since we are mostly interested in homology classes determined by subvarieties of a variety, we consider the Chow group $A(X)$, i.e., the group of cycles modulo rational equivalence [7], and the following homology group, which shall be provisionally called the *"algebraic homology group"*, denoted by $AH_*(X;\mathbb{Z})$:

$$AH_*(X;\mathbb{Z}) := \operatorname{Image}(cl : A(X) \to H_*(X;\mathbb{Z})),$$

where $cl : A(X) \to H_*(X;\mathbb{Z})$ is the cycle map [6, §19.1].

Lemma (4.4). *For a variety X,*

$$AH_*(X;\mathbb{Z}) = \operatorname{Image}(C_* : \mathcal{F}(X) \to H_*(X;\mathbb{Z})).$$

Proof. First of all we note that MacPherson's proof [14] actually shows that $C_* : \mathcal{F}(X) \to H_*(X;\mathbb{Z})$ is the composite of the homomorphism $C_* : \mathcal{F}(X) \to A(X)$ into the Chow homology group and the cycle map $cl : A(X) \to H_*(X;\mathbb{Z})$. Here we use the same notation C_*, i.e.,

$$C_* = cl \circ C_*.$$

In fact it is easy to see by induction on dimension that the homomorphism $C_* : \mathcal{F}(X) \to A(X)$ is always surjective, because for any subvariety W $C_*(1\!\!1_W) = [W] +$ lower classes. Thus we get $AH_*(X;\mathbb{Z}) =$ Image $(C_* : \mathcal{F}(X) \to H_*(X;\mathbb{Z}))$. Q.E.D

Now consider a Verdier-type Riemann-Roch diagram associated with the bivariant constructible function $1\!\!1_X$:

$$\begin{array}{ccc}
\mathcal{F}(Y) & \xrightarrow{f^*=(\mathbf{1}_X)^{\mathbb{F}}} & \mathcal{F}(X) \\
{\scriptstyle C_*}\downarrow & & \downarrow{\scriptstyle C_*} \\
H_*(Y;\mathbb{Z}) & \xrightarrow{(\mathbf{1}_X)^{\mathbb{H}}_{\gamma}} & H_*(X;\mathbb{Z})
\end{array}$$

where $\gamma : \mathbb{F} \to \mathbb{H}$ is a Grothendieck transformation (cf. [22]). It follows from Lemma (4.4) that the restricted homomorphism $f^{Eu} := (1\!\!1_X)^{\mathbb{H}}_{\gamma}|_{AH_*(Y;\mathbb{Z})} : AH_*(Y;\mathbb{Z}) \to AH_*(X;\mathbb{Z})$ can be expressed as follows:

$$f^{Eu} := C_* f^* C_*^{-1} : AH_*(Y;\mathbb{Z}) \to AH_*(X;\mathbb{Z}),$$

which is well-defined because of the commutativity of the above Verdier-type Riemann-Roch diagram. And of course we have the following homomorphism

$$c(T_f) \cap f^* : AH_*(Y;\mathbb{Z}) \to AH_*(X;\mathbb{Z}).$$

These two homomorphisms coincide when $f : X \to Y$ is a smooth morphism, but in general they are not identical if f is not smooth ([22]). So it is quite natural to pose the problem of describing the difference between the two.

Problem (4.5). *Let $f : X \to Y$ be an Euler and local complete intersection morphism. Then give an explicit description of the following defect δ:*

$$f^{Eu} = c(T_f) \cap f^* + \delta.$$

Remark 4.6. For a hypersurface X Parusiński and Pragacz [17] give an interesting and promising formula for the generalized Milnor number $\mu(X)$, in terms of the Chern-Schwartz-MacPherson of the closure of the strata of a Whitney stratification of X. Suggested by their result and Theorem (4.3), we expect that the defect δ in the above looked-for formula will be possibly described using a Whitney stratification of a morphism f. After the preparation of the manuscript we learned that in the hypersurface case the Parusiński-Pragacz's formula for the Milnor number $\mu(X)$ has been generalized to a formula for the Milnor class $\mathcal{M}(X)$ in [18].

An interesting feature of this looked-for formula is that it implies some relationship among Fulton-Johnson's canonical class $\mathcal{C}(f) := c(T_f) \cap [X]$ of a local complete intersection morphism f [8] and the Chern-Schwartz-MacPherson class $C_*(X)$ of the source variety X and (hopefully) some kind of invariants of singularities of the morphism f. Here is a citation from [8, p.382]: "... It would be interesting to relate the canonical class of a mapping to its singularities."

Proposition (4.7). *Let $f : X \to Y$ be an Euler and local complete intersection morphism. Then we have the following formula:*

$$\mathcal{C}(f) = C_*(X) + \sum a_V C_*(f^{-1}(V)) - \delta([Y]),$$

where $[Y] = C_*(Y) + \sum_{\dim V < \dim Y} a_V C_*(V)$. *In particular, if f is smooth, then*

$$\mathcal{C}(f) = C_*(X) + \sum a_V C_*(f^{-1}(V)).$$

Proof. First we observe that since

$$[Y] = C_*(Y) + \sum_{\dim V < \dim Y} a_V C_*(V)$$

we can take

$$C_*^{-1}([Y]) = 1\!\!1_Y + \sum_{\dim V < \dim Y} a_V 1\!\!1_V.$$

Then

$$\begin{aligned}
\mathcal{C}(f) &= c(T_f) \cap [X] \\
&= c(T_f) \cap f^*([Y]) \\
&= f^{Eu}([Y]) - \delta([Y]) \\
&= C_* f^* C_*^{-1}([Y]) - \delta([Y]) \\
&= C_* f^* \left(1\!\!1_Y + \sum_{\dim V < \dim Y} a_V 1\!\!1_V \right) - \delta([Y]) \\
&= C_*(X) + \sum_{\dim V < \dim Y} a_V C_*(f^{-1}(V)) - \delta([Y]).
\end{aligned}$$

Q.E.D

Proposition (4.8). *Let $f : X \to Y$ be an Euler morphism and let Y be topologically connected. Then for any algebraic homology class*

$\alpha \in AH_*(Y;\mathbb{Z})$ we have

$$\int_X f^{Eu}(\alpha) = \chi_f \cdot \int_Y \alpha.$$

Here χ_f denotes the topological Euler-Poincaré characteristic of any fiber.

Proof. Since any homology class $\alpha \in AH_*(Y;\mathbb{Z})$ is generated by the Chern-Schwartz-MacPherson class $C_*(V) = C_*(1\!\!1_V)$ of subvarieties V, it suffices to show the formula for $\alpha = C_*(1\!\!1_V)$.

$$\begin{aligned}
\int_X f^{Eu}(C_*(1\!\!1_V)) &= \int_Y f_* f^{Eu}(C_*(1\!\!1_V)) \\
&= \int_Y f_* C_* f^* C_*^{-1}(C_*(1\!\!1_V)) \quad (\text{since } f^{Eu} = C_* f^* C_*^{-1}) \\
&= \int_Y C_* f_* f^*(1\!\!1_V) \quad (\text{since } f_* C_* = C_* f_*) \\
&= \int_Y C_*(\chi_f \cdot 1\!\!1_V) \quad (f_* f^*(1\!\!1_V) = \chi_f \cdot 1\!\!1_V) \\
&= \chi_f \cdot \int_Y C_*(1\!\!1_V).
\end{aligned}$$

Q.E.D

Problem (4.9).

1. *Let $f : X \to Y$ be a local complete intersection morphism and let Y be topologically connected. Let $\alpha \in AH_*(Y;\mathbb{Z})$. Describe the following number as in the above proposition*

$$\int_X c(T_f) \cap f^*(\alpha).$$

2. *Let $f : X \to Y$ be a local complete intersection morphism with Y being topologically connected such that $f : X \to M \to Y$ and that for each $y \in Y$ $r_y : X_y \to M_y$ is a regular embedding with $\dim X_y = n$, i.e., X_y is a local complete intersection of the smooth fiber M_y. Then is it true that*

$$\int_X c(T_f) \cap f^*(\alpha) = {\chi_f}^{FJ} \cdot \int_Y \alpha \quad ?$$

Here ${\chi_f}^{FJ} = \int_{X_y} C^{FJ}(X_y)$ is called the Fulton-Johnson's characteristic of the fiber.

If (2) of the above problem be true, then we will get the following formula:

$$\int_X \delta(\alpha) = (-1)^{n+1}\mu_f \cdot \int_Y \alpha,$$

where μ_f is the generalized Milnor number of the fiber.

Remark 4.10. Here we remark a little on the uniqueness problem of the bivariant Chern class. If we consider the algebraic homology group instead of the usual homology group, then to some extent we could see the "uniqueness" in the following sense. We want to see that if $\gamma, \gamma' : \mathbb{F} \to \mathbb{H}$ are two Grothendieck-Chern transformations, then for any bivariant constructible function α the bivariant homology $\gamma(\alpha) = \gamma'(\alpha)$. If we consider these two bivariant homology classes $\gamma(\alpha)$ and $\gamma'(\alpha)$ as *homological operators* $\alpha^{\mathbb{H}}_{\gamma}(a) = \gamma(\alpha) \odot_{\mathbb{H}} a$ and $\alpha^{\mathbb{H}}_{\gamma'}(a) = \gamma'(\alpha) \odot_{\mathbb{H}} a$, which both define the homomorphism

$$AH_*(Y;\mathbb{Z}) \to AH_*(X;\mathbb{Z}).$$

However, in the same argument as above we have the following equality:

$$\alpha^{\mathbb{H}}_{\gamma} = C_* \alpha^{\mathbb{F}} C_*^{-1} = \alpha^{\mathbb{H}}_{\gamma'}.$$

Thus all the Grothendieck transformations induce the same homological operators if they are restricted to the algebraic homological classes. In particular, in the case when the cycle map $cl : A_*(X) \to H_*(X)$ is an isomorphism, e.g., if X has a cellular decomposition (see [7, Example 1.9.1 and Example 19.1.11]), then the transformation $\gamma : \mathbb{F} \to \mathbb{H}$ is unique if it is considered as the homological operator $\alpha^{\mathbb{H}}_{\gamma}$. When the bivariant homology theory is replaced by the bivariant Chow homology theory ([6], [7]), see [6] for the uniqueness.

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Constructibilité de l'idéal de Bernstein

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Soit X une variété analytique, $f = (f_1, \dots, f_p)$ des fonctions analytiques sur X, $F = f_1 \dots f_p$ leur produit.
Soit M un $\mathcal{D}_X$-Module holonome régulier; C. Sabbah montre dans [Sab 1] [Sab 2] que toute section m de M satisfait localement des équations non triviales

$$(*) \qquad b(s_1, \dots, s_p) m f_1^{s_1} \dots f_p^{s_p} \in \mathcal{D}_X[s_1, \dots, s_p] m f_1^{s_1+1} \dots f_p^{s_p+1}$$

où $b(s_1, \dots, s_p)$ est un produit de formes affines. De plus, les coefficients de la partie linéaire de ces formes sont des entiers positifs ou nuls. En particulier, l'idéal $B(x, f_1, \dots, f_p, m)$ des polynômes $b(s_1, \dots, s_p)$ vérifiant au voisinage d'un point x une équation fonctionnelle $(*)$ est non réduit à zéro. Désignons par $\operatorname{car} M = \bigcup_{l \in L} T^*_{Y_l} X$ la variété caractéristique de M. Nous montrons que le germe de l'idéal $B(x, f_1, \dots, f_p, m)$ est constant le long des composantes d'une partition qui se détermine géométriquement à partir des restrictions de la seule fonction F aux Y_l. En particulier, nous en déduisons le résultat :

Théorème. *Soit M un $\mathcal{D}_X$-Module holonome régulier engendré par une section m et $\operatorname{car} M = \bigcup_{l \in L} T^*_{Y_l} X$ sa variété caractéristique. Soit $(V_\beta)_{\beta \in \Gamma}$ une stratification analytique de $\bigcup_{l \in L} Y_l$ compatible aux Y_l et à $F^{-1}(0)$, satisfaisant la condition de frontière et la condition a_F de Thom.*

Le germe de l'idéal de Bernstein de $f_1, \dots, f_p, m$ est constant le long des strates de la stratification $(V_\beta)_{\beta \in \Gamma}$.

Ce résultat généralise celui obtenu par J. Briançon et H. Maynadier ([B.M] théorème 3.3 page 14) dans le cas où $p = 1$ et m une fonction constante sur X.

Détaillons section par section les résultats que nous obtenons.

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Section 1 Soit Y un sous-espace analytique irréductible de X. On note T^*X le fibré cotangent à X, $T^*_Y X$ l'espace conormal à Y dans X. Soit A le sous-ensemble de $T^*X \times \mathbf{C}^p$ défini par :

$$A = \left\{ \left(x, \eta + s_1 \frac{df_1(x)}{f_1(x)} + \cdots + s_p \frac{df_p(x)}{f_p(x)}, s_1, \cdots, s_p \right) \right. \\ \left. ; F(x) \neq 0 \text{ et } (x, \eta) \in T^*_Y X \right\}$$

L'espace $W^\sharp_{f_1,\dots,f_p,Y}$, adhérence de A dans $T^*X \times \mathbf{C}^p$ a été introduit par T. Kawai et M. Kashiwara ([K.K]).

Nous donnons ici quelques propriétés de $W^\sharp_{f_1,\dots,f_p,Y}$. Désignons par π_2 la projection de $T^*X \times \mathbf{C}^p$ sur $\mathbf{C}^p$.

1. Les fibres réduites de la restriction de π_2 à $W^\sharp_{f_1,\dots,f_p,Y}$ sont des sous-espaces lagrangiens de T^*X. La fibre au-dessus de l'origine est en particulier un sous-espace lagrangien conique.
2. La projection par π_2 de la trace de $W^\sharp_{f_1,\dots,f_p,Y}$ sur l'hypersurface d'équation $F = 0$ est une réunion H d'hyperplans vectoriels de $\mathbf{C}^p$ dont les équations sont des formes linéaires à coefficients entiers positifs ou nuls.
3. La partie de $W^\sharp_{f_1,\dots,f_p,Y}$ au dessus de la droite vectorielle $s_1 = \ldots = s_p$ de $\mathbf{C}^p$ s'identifie à l'espace $W^\sharp_{F,Y}$.

Section 2 Nous étudions ici des propriétés générales des $\mathcal{D}_X[s_1,\dots,s_p]$-Modules cohérents. Nous dirons qu'un $\mathcal{D}_X[s_1,\dots,s_p]$-Module cohérent M est à fibre lagrangienne si $(\mathrm{car}_{\mathcal{D}_X[s_1,\dots,s_p]} M)(0)$, intersection de sa variété caractéristique et de $\pi_2^{-1}(0)$, est un sous-espace lagrangien de T^*X. Nous étudions plus particulièment les $\mathcal{D}_X[s_1,\dots,s_p]$-Modules cohérents à fibre lagrangienne. Ils se comportent bien par suite exacte. Nous montrons que l'idéal des polynômes de $\mathbf{C}[s_1,\dots,s_p]$ annulant le germe en un point d'un tel module est localement constant le long des strates d'une stratification associée à la variété lagrangienne $(\mathrm{car}_{\mathcal{D}_X[s_1,\dots,s_p]} M)$ (0).

Section 3 Soit m une section engendrant un $\mathcal{D}_X$-Module holonome régulier M. Désignons par $\mathrm{car}\, M = \bigcup_{l \in L} T^*_{Y_l} X$ sa variété caractéristique. Nous commençons par établir le résultat suivant :

Le $\mathcal{D}_X[s_1,\dots,s_p]$-Module $\mathcal{D}_X[s_1,\dots,s_p] m f_1^{s_1} \dots f_p^{s_p}$ est cohérent de variété caractéristique :

$$\mathrm{car}_{\mathcal{D}_X[s_1,\dots,s_p]}\, \mathcal{D}_X[s_1,\dots,s_p] m f_1^{s_1} \dots f_p^{s_p} = \bigcup_{F|_{Y_l} \neq 0} W^\sharp_{f_1,\dots,f_p,Y_l}$$

Ce résultat complète ceux de l'article [B.B.M.M] ; nous le montrons à l'aide du théorème de C. Sabbah sur les variétés caractéristiques de Modules relatifs [Sab 2].

On en déduit que $\mathcal{D}_X[s_1, \dots, s_p]mf_1^{s_1} \dots f_p^{s_p}$ est un $\mathcal{D}_X[s_1, \dots, s_p]$-Module cohérent à fibre lagrangienne. Donc, la variété caractéristique du $\mathcal{D}_X[s_1, \dots, s_p]$-Module à gauche cohérent :

$$N = \frac{\mathcal{D}_X[s_1, \dots, s_p]mf_1^{s_1} \dots f_p^{s_p}}{\mathcal{D}_X[s_1, \dots, s_p]mf_1^{s_1+1} \dots f_p^{s_p+1}}$$

est incluse dans $(\bigcup_{F|_{Y_l} \neq 0} W^\sharp_{f_1, \dots, f_p, Y_l}) \bigcap F^{-1}(0)$. Ce Module est encore à fibre lagrangienne; il en résulte que l'idéal $B(x, f_1, \dots, f_p, m)$ des polynômes $b(s_1, \dots, s_p)$ vérifiant au voisinage d'un point x de X une équation fonctionnelle

$$(*) \qquad b(s_1, \dots, s_p)mf_1^{s_1} \dots f_p^{s_p} \in \mathcal{D}_X[s_1, \dots, s_p]mf_1^{s_1+1} \dots f_p^{s_p+1}$$

est constant le long des strates d'une partition canoniquement associée à la variété lagrangienne

$$\Big(\bigcup_{F|_{Y_l} \neq 0} W^\sharp_{f_1, \dots, f_p, Y_l}\Big) \bigcap F^{-1}(0) \bigcap \pi_2^{-1}(0)$$

égale à la trace sur $F^{-1}(0)$ de l'espace conormal relatif à F sur $\bigcup_{F|_{Y_l} \neq 0} Y_l$.

TABLE DES MATIÈRES

§1. Famille de variétés lagrangiennes

Soit X une variété analytique complexe de dimension n. Soit $f = (f_1, \dots, f_p)$ des fonctions holomorphes sur X. Nous désignons par T^*X le fibré cotangent à X et par π_1 (resp. π_2) la projection de $T^*X \times \mathbf{C}^p$ sur

T^*X (resp. sur $\mathbf{C}^p$). On note F le produit $f_1 \dots f_p$. Soit $Y \subset X$ un sous espace analytique irréductible non contenu dans l'hypersurface $F^{-1}(0)$. On désigne par $T^*_Y X$ l'espace conormal à Y dans X, égal à l'adhérence dans T^*X du fibré conormal à la partie lisse de Y.

Notation 1. *Soit A le sous-ensemble de $T^*X \times \mathbf{C}^p$ défini par :*

$$A = \left\{ \left(x, \eta + s_1 \frac{df_1(x)}{f_1(x)} + \cdots + s_p \frac{df_p(x)}{f_p(x)}, s_1, \cdots, s_p \right) \right.$$
$$\left. ; \ F(x) \neq 0 \ \text{ et } \ (x, \eta) \in T^*_Y X \right\}$$

Nous notons $W^\sharp_{f_1,\cdots,f_p,Y}$ l'adhérence de A dans $T^*X \times \mathbf{C}^p$. L'ensemble $W^\sharp_{f_1,\cdots,f_p,Y}$ est un espace analytique complexe irréductible de dimension $n + p$. L'action de $\mathbf{C}^*$ sur $T^*X \times \mathbf{C}^p$ donnée par :

$$\lambda, (x, \xi, s) \longmapsto (x, \lambda\xi, \lambda s)$$

laisse stable A, donc $W^\sharp_{f_1,\cdots,f_p,Y}$. Le diviseur $F^{-1}(0)$ est également laissé stable par cette action.

Notation 2. *Pour tout $c \in \mathbf{C}^p$, nous noterons $W^\sharp_{f_1,\cdots,f_p,Y}(c)$ la fibre au-dessus de c de la restriction de π_2 à $W^\sharp_{f_1,\cdots,f_p,Y}$:*

$$W^\sharp_{f_1,\cdots,f_p,Y}(c) = W^\sharp_{f_1,\cdots,f_p,Y} \cap \pi_2^{-1}(c)$$

On identifie $W^\sharp_{f_1,\cdots,f_p,Y}(c)$ à un sous espace analytique de T^*X. Pour $c = 0$, c'est un sous-espace stable par l'action de $\mathbf{C}^*$ sur T^*X donnée par :

$$\lambda, (x, \xi) \longmapsto (x, \lambda\xi).$$

Proposition 1. *Pour tout $c \in \mathbf{C}^p$, l'espace $W^\sharp_{f_1,\cdots,f_p,Y}(c)$ est un sous-espace lagrangien de T^*X.*

Preuve. Soit $c = (c_1, \cdots, c_p) \in \mathbf{C}^p$. L'espace $W^\sharp_{f_1,\cdots,f_p,Y}$ étant un espace analytique irréductible de dimension $n + p$, les composantes irréductibles de l'espace analytique réduit sous jacent à $W^\sharp_{f_1,\cdots,f_p,Y}(c)$ sont de dimension supérieure ou égale à n. Pour établir la proposition, il suffira donc de montrer que $W^\sharp_{f_1,\cdots,f_p,Y}(c)$ est isotrope. Désignons par α la 1-forme canonique sur T^*X. Nous avons donc à montrer que la restriction de $d\alpha$ à la partie lisse de toute composante de $W^\sharp_{f_1,\cdots,f_p,Y}(c)$ est nulle. Pour cela, considérons l'éclatement normalisé $E : \widetilde{W^\sharp_{f_1,\cdots,f_p,Y}} \to$

$W^\sharp_{f_1,\dots,f_p,Y}$ de l'idéal engendré par $(s_1 - c_1, \dots, s_p - c_p)$ dans l'anneau structural de $W^\sharp_{f_1,\dots,f_p,Y}$. On a le diagramme commutatif :

$$\begin{array}{lcll} \widetilde{C} = E^{-1}(C) & \hookrightarrow & \widetilde{W}^\sharp_{f_1,\dots,f_p} & \\ \downarrow E & & \downarrow E & \\ C = W^\sharp_{f_1,\dots,f_p,Y}(c) & \hookrightarrow & W^\sharp_{f_1,\dots,f_p,Y} & \xrightarrow{\pi_1'} T^*X \end{array}$$

dans lequel $\widetilde{C}$ est le diviseur exceptionnel et π_1' la restriction de π_1 à $W^\sharp_{f_1,\dots,f_p,Y}$.

On a $d\alpha|_C = ((\pi_1')^* d\alpha)|_C$. Il suffira donc de montrer que la restriction de $E^*((\pi_1')^* d\alpha)$ à la partie lisse de toute composante de $\widetilde{C}$ est nulle. Plaçons nous au voisinage d'un point générique e d'une composante de $\widetilde{C}$. Le point e est un point lisse de $\widetilde{C}$ et de $\widetilde{W}^\sharp_{f_1,\dots,f_p,Y}$ (car $\widetilde{C}$ est un diviseur d'un espace normal) ; ainsi, il existe une fonction holomorphe ψ telle que $\widetilde{C}$ soit défini au voisinage de e par l'équation $\psi = 0$. Pour tout $j \in \{1, \dots, p\}$, il existe donc un entier naturel m_j strictement positif et une unité u_j tels que :

$$s_j - c_j = u_j \psi^{m_j}$$

et il existe un entier n_j et une unité v_j tels que :

$$f_j = v_j \psi^{n_j}$$

Notons Ω l'ouvert dense de Y des points lisses où F n'est pas nulle. Notons $U = {\pi_1'}^{-1}(\pi^{-1}(\Omega))$, si $\pi : T^*X \to X$ désigne la projection canonique. C'est un ouvert lisse dense de $W^\sharp_{f_1,\dots,f_p,Y}$. Par définition de A (sachant que la restriction de α à la variété lagrangienne $T^*_Y X$ est nulle), nous avons :

$$((\pi_1')^*\alpha)|_U = \left(\sum_{j=1}^{p} s_j \frac{df_j}{f_j}\right)\Bigg|_U$$

$$((\pi_1')^*d\alpha)|_U = \left(\sum_{j=1}^{p} ds_j \wedge \frac{df_j}{f_j}\right)\Bigg|_U$$

Ainsi, au voisinage de e :

$$(E^*(\pi_1')^* d\alpha)|_{E^{-1}(U)}$$

$$= \left(\sum_{j=1}^{p} \psi^{m_j} du_j \wedge \frac{dv_j}{v_j} + \sum_{j=1}^{p} \psi^{m_j - 1}\left(n_j du_j - m_j u_j \frac{dv_j}{v_j}\right) \wedge d\psi\right)\Bigg|_{E^{-1}(U)}$$

La restriction de cette forme à $\psi = 0$ est nulle, d'où le résultat.

Proposition 2. *Soit* $W^\sharp_{f_1,\dots,f_p,Y} \cap F^{-1}(0)$ *la trace de l'hypersurface* $F^{-1}(0)$ *sur* $W^\sharp_{f_1,\dots,f_p,Y}$. *L'espace :*

$$\pi_2(W^\sharp_{f_1,\dots,f_p,Y} \cap F^{-1}(0))$$

est une réunion d'hyperplans de $\mathbf{C}^p$ *définis par des équations à coefficients entiers positifs. La famille de ces hyperplans est localement finie sur* $F^{-1}(0)$.

Preuve. Considérons la normalisation $G : \overline{W} \to W^\sharp_{f_1,\dots,f_p,Y}$ de $W^\sharp_{f_1,\dots,f_p,Y}$. On a le diagramme commutatif :

$$\begin{array}{lll} \bar{Z} = G^{-1}(Z) & \hookrightarrow & \overline{W} \\ \downarrow G & & \downarrow G \\ Z = W^\sharp_{f_1,\dots,f_p,Y} \cap F^{-1}(0) & \hookrightarrow & W^\sharp_{f_1,\dots,f_p,Y} \stackrel{\pi_1'}{\to} T^*X \end{array}$$

Soit T une composante de Z. Soit e un point générique d'une composante de $\bar{Z}$ se projetant surjectivement sur T. Le point e est un point lisse de l'hypersurface $\bar{Z}$ et de $\overline{W}$. Soit $\psi = 0$ une équation réduite de $\bar{Z}$ au voisinage de e. Pour tout $j \in \{1,\dots,p\}$, il existe un entier positif n_j (stictement positif pour au moins un indice) et v_j une unité tels qu'au voisinage de e on ait :

$$f_j = v_j \psi^{n_j}$$

On désigne par U le même ouvert que dans la preuve de la proposition précédente. On a au voisinage de e :

$$\begin{aligned} (G^*(\pi_1')^*\alpha)|_{G^{-1}(U)} &= \left(G^*\left(\sum_{j=1}^p s_j df_j/f_j\right)\right)\Big|_{G^{-1}(U)} \\ &= \sum_{j=1}^p s_j n_j d\psi/\psi + \sum_{j=1}^p s_j dv_j/v_j \end{aligned}$$

Cette forme est la restriction de la forme holomorphe $G^*(\pi_1')^*\alpha$. Il faut donc que $\sum_{j=1}^p s_j n_j$ soit un multiple de ψ. Ainsi, $\pi_2(T)$ est contenu dans l'hyperplan H_T d'équation :

$$\sum_{j=1}^p s_j n_j = 0$$

Considérons la restriction $\pi_2|_T : T \to H_T$. Pour tout $c \in \mathbf{C}^p$ les fibres :

$$(\pi_2|_T)^{-1}(c)$$

sont incluses dans $W^\sharp_{f_1,\dots,f_p,Y}(c)$ et sont donc d'après la proposition 1 de dimension inférieure ou égale à n. La dimension de T est $n+p-1$ et la dimension de H_T est $p-1$, les fibres de $\pi_2|_T$ sont donc équidimensionnelles de dimension n.

T étant stable sous l'action de $\mathbf{C}^*$ sur $T^*X \times \mathbf{C}^p$ et fermée dans ce dernier espace, elle contient donc des points de la forme $(x, 0, 0)$. La fibre de $\pi_2|_T$ au dessus de 0 est donc non vide et de plus isotrope et conique. Comme H_T est lisse, le morphisme $\pi_2|_T$ est ouvert en un tel point $(x, 0, 0) \in T$ et son image contient donc un voisinage de l'origine de H_T. Comme cette image est conique, $\pi_2(T) = H_T$. On en déduit la proposition 2 et la remarque suivante :

Remarque 1. Soit $c \in \mathbf{C}^p$. Si $W^\sharp_{f_1,\dots,f_p,Y}(c) \cap F^{-1}(0)$ n'est pas vide, c'est une réunion de composantes irréductibles de $W^\sharp_{f_1,\dots,f_p,Y}(c)$, donc un espace lagrangien.

Corollaire 1. *L'espace* $W^\sharp_{f_1,\dots,f_p,Y} \cap \{s_1 = \dots = s_p\}$ *s'identifie (par le plongement diagonal de* $\mathbf{C}$ *dans* $\mathbf{C}^p$*) au sous-espace* $W^\sharp_{F,Y}$ *de* $T^*X \times \mathbf{C}$.

Preuve. $W^\sharp_{F,Y}$ est l'adhérence de

$$\{x, \eta + t(dF(x)/F(x)), t)\,;\ (x, \eta) \in T^*_Y X\,;\ F(x) \neq 0\}$$

On a clairement l'inclusion :

$$W^\sharp_{F,Y} \subset W^\sharp_{f_1,\dots,f_p,Y} \cap (s_1 = \dots = s_p)$$

En dehors de l'hypersurface $F^{-1}(0)$, cette inclusion est une égalité. Il résulte de la proposition 1 que $W^\sharp_{f_1,\dots,f_p,Y} \cap (s_1 = \dots = s_p)$ est équidimensionnelle de dimension $n+1$, donc de même dimension que $W^\sharp_{F,Y}$. Pour montrer le corollaire, il suffit donc de montrer qu'aucune composante irréductible de $W^\sharp_{f_1,\dots,f_p,Y} \cap (s_1 = \dots = s_p)$ n'est contenue dans $F^{-1}(0)$. Supposons le contraire : soit Z une composante contenue dans $F^{-1}(0)$; $\pi_2(Z)$ est contenue d'après la proposition 2 dans un hyperplan à coefficients entiers positifs. Z est donc contenue dans $W^\sharp_{f_1,\dots,f_p,Y}(0)$ qui est, d'après la proposition 1, de dimension n. C'est impossible, puisque Z est de dimension $n+1$.

§2. $\mathcal{D}[s_1, \dots, s_p]$-Module cohérent à fibre lagrangienne

2.1. Définitions

On reprend les notations du début de la section 1.

Désignons par $\mathcal{D}_X$, le faisceau des opérateurs différentiels sur la variété X. Pour $k \in \mathbf{N}$, notons $\mathcal{D}_X(k)$ le k^{ième} terme de la filtration de $\mathcal{D}_X$: si $(x_1, \ldots, x_n)$ désigne un système de coordonnées locales de X et $\beta = (\beta_1, \ldots, \beta_n) \in \mathbf{N}^n$, notons :

$$\partial^\beta = \left(\frac{\partial}{\partial x_1}\right)^{\beta_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\beta_n} \quad \text{et} \quad |\beta| = \beta_1 + \ldots + \beta_n$$

Un opérateur P de $\mathcal{D}_X(k)$, défini localement, s'écrit :

$$P = \sum_{|\beta| \leq k} c_\beta(x) \partial^\beta$$

On appelle degré de P et on note $\deg P$ l'entier $\sup \{|\beta| ; c_\beta \neq 0\}$. Le symbole principal d'ordre k de P est l'élément de $\mathcal{O}_{\mathbf{C}^n}[\xi_1, \ldots, \xi_n]$:

$$\sigma_k(P) = \sum_{|\beta| = k} c_\beta(x) \xi^\beta$$

et se recolle en une fonction $\sigma_k(P)$ sur le fibré cotangent T^*X. Considérons $\mathcal{D}_X[s_1, \ldots, s_p] = \mathbf{C}[s_1, \ldots, s_p] \otimes_{\mathbf{C}} \mathcal{D}_X$. Pour $j \in \mathbf{N}$, notons par $\mathbf{C}[s_1, \ldots, s_p](j)$ le sous-espace vectoriel de $\mathbf{C}[s_1, \ldots, s_p]$ constitué des polynômes de degré inférieur ou égal à j. L'anneau $\mathcal{D}_X[s_1, \ldots, s_p]$ est alors naturellement filtré : pour $l \in \mathbf{N}$, le terme d'ordre l de cette filtration est

$$\mathcal{D}_X[s_1, \ldots, s_p](l) = \sum_{j+k=l} \mathbf{C}[s_1, \ldots, s_p](j) \otimes_{\mathbf{C}} \mathcal{D}_X(k)$$

C'est une filtration croissante. Pour tout $l \in \mathbf{N}$, $\mathcal{D}_X[s_1, \ldots, s_p](l)$ est un $\mathcal{O}_X$-Module localement libre de type fini.

Pour $\alpha = (\alpha_1, \ldots, \alpha_p) \in \mathbf{N}^p$, notons $s^\alpha = s_1^{\alpha_1} \ldots s_p^{\alpha_p}$. Dans un système de coordonnées locales $(x_1, \ldots, x_n)$ de X, un opérateur P de $\mathcal{D}_X[s_1, \ldots, s_p](l)$ s'ecrit localement :

$$P = \sum_{|\alpha| + \deg P_\alpha \leq l} s^\alpha P_\alpha$$

avec P_α dans $\mathcal{D}_X$. Le symbole principal d'ordre l de P est l'élément de $\mathcal{O}_{\mathbf{C}^n}[\xi_1, \ldots, \xi_n, s_1, \ldots, s_p]$:

$$\sigma_l(P) = \sum_{|\alpha| + \deg P_\alpha = l} s^\alpha \sigma_{\deg P_\alpha}(P_\alpha)$$

et se recolle en une fonction sur $T^*X \times \mathbf{C}^p$, encore notée $\sigma_l(P)$, homogène sur les fibres de la projection $\pi \circ \pi_1$ sur X. On appelle degré de P et on note $\deg P$ l'entier $\sup\{|\alpha| + \deg P_\alpha \ ; \ P_\alpha \neq 0\}$. On vérifie que si P (resp. Q)est une section de $\mathcal{D}_X[s_1, \dots, s_p](l)$ (resp. $\mathcal{D}_X[s_1, \dots, s_p](m)$) l'opérateur :

$$PQ - QP \in \mathcal{D}_X[s_1, \dots, s_p](l + m - 1)$$

De plus, le symbole d'ordre $l + m - 1$ de $PQ - QP$ est, dans un système de coordonnées locales :

$$\{\sigma_l(P), \sigma_m(Q)\} = \sum_{i=1}^{n} \frac{\partial\sigma_m(Q)}{\partial x_i}\frac{\partial\sigma_l(P)}{\partial\xi_i} - \frac{\partial\sigma_m(Q)}{\partial\xi_i}\frac{\partial\sigma_l(P)}{\partial x_i}$$

Cette formule est une extension du crochet de Poisson associé à deux symboles d'opérateurs différentiels de $\mathcal{D}_X$.

D'après ce qui précède, le gradué $\mathrm{gr}\mathcal{D}_X[s_1, \dots, s_p]$ de $\mathcal{D}_X[s_1, \dots, s_p]$ est un anneau commutatif. Il s'identifie au sous-faisceau de $(\pi \circ \pi_1)_*(\mathcal{O}_{T^*X \times \mathbf{C}^p})$ des fonctions homogènes relativement aux variables (ξ, s). Les faisceaux d'anneaux $\mathrm{gr}\mathcal{D}_X[s_1, \dots, s_p]$ et $\mathcal{D}_X[s_1, \dots, s_p]$ sont cohérents.

Donnons maintenant quelques propriétés de la catégorie des $\mathcal{D}_X[s_1, \dots, s_p]$-Modules cohérents à gauche, qui généralisent les propriétés des $\mathcal{D}_X$-Modules cohérents (leurs démonstrations sont les mêmes, voir par exemple [G.M]). Soit M un $\mathcal{D}_X[s_1, \dots, s_p]$-Module cohérent à gauche; localement M admet une bonne filtration $(M_k)_{k \in \mathbf{N}}$. Le faisceau $\sqrt{\mathrm{ann}_{\mathrm{gr}\mathcal{D}_X[s_1, \dots, s_p]} \mathrm{gr} M}$ définit un idéal $\mathcal{J}(M)$ de $\mathrm{gr}\mathcal{D}_X[s_1, \dots, s_p]$ indépendant des bonnes filtrations locales. Il en est de même de la multiplicité de $\mathrm{gr}M$ en un point générique d'une composante irréductible de son support.

La variété des zéros de l'idéal $\mathcal{J}(M)$ est $\mathrm{car}_{\mathcal{D}_X[s_1, \dots, s_p]} M$, un sous-ensemble analytique de $T^*X \times \mathbf{C}^p$ appelé variété caractéristique de M. On appelle *cycle caractéristique* de M le cycle associé au module $\mathrm{gr}M$. On le note $\mathrm{Car}_{\mathcal{D}_X[s_1, \dots, s_p]} M$.

Le théorème de Gabber s'énonce dans notre situation ([G]) :

Soit M un $\mathcal{D}_X[s_1, \dots, s_p]$-Module cohérent à gauche. Si σ et τ sont deux sections de $\mathcal{J}(M)$, leur crochet $\{\sigma, \tau\}$ est une section de $\mathcal{J}(M)$.

Notation 3. *Soit* $\pi_2 : T^*X \times \mathbf{C}^p \to \mathbf{C}^p$ *la projection sur* $\mathbf{C}^p$. *Soit* M *un* $\mathcal{D}_X[s_1, \dots, s_p]$*-Module cohérent à gauche. On notera* $(\mathrm{car}_{\mathcal{D}_X[s_1, \dots, s_p]} M)(c)$ *la fibre du point* $c = (c_1, \dots, c_p)$ *de la restriction de* π_2 *à* $\mathrm{car}_{\mathcal{D}_X[s_1, \dots, s_p]} M$. *En particulier*

$$(\mathrm{car}_{\mathcal{D}_X[s_1, \dots, s_p]} M)(0) = (\mathrm{car}_{\mathcal{D}_X[s_1, \dots, s_p]} M) \bigcap \pi_2^{-1}(0)$$

Tout point de $(\mathrm{car}_{\mathcal{D}_X[s_1,\dots,s_p]}M)(c)$ est limite de points lisses de la variété caractéristique de M en lesquels la restriction de π_2 est de rang localement constant. Les fibres de π_2 en ces points sont lisses réduites. Il résulte alors du théorème de Gabber qu'au voisinage d'un de ses points génériques, cette fibre, identifiée à un sous espace de T^*X, est involutive au sens de la 2-forme canonique sur l'espace cotangent à X. Elle est donc de dimension supérieure ou égale à la dimension de X. La semi-continuité de la dimension des fibres implique alors :

Proposition 3. *Soit M un $\mathcal{D}_X[s_1,\dots,s_p]$-Module cohérent à gauche. Les fibres $(\mathrm{car}_{\mathcal{D}_X[s_1,\dots,s_p]}M)(c)$ non vides ont leurs composantes irréductibles de dimensions supérieures ou égales à la dimension de X.*

Compte-tenu du caractère conique des variétés caractéristiques, on a en identifiant X à la section nulle de $T^*X \times \mathbf{C}^p$:

$$\begin{aligned}\mathrm{Supp}(M) &= \mathrm{car}_{\mathcal{D}_X[s_1,\dots,s_p]}M \bigcap \{s=\xi=0\}\\ &= (\mathrm{car}_{\mathcal{D}_X[s_1,\dots,s_p]}M)(0)\bigcap\{\xi=0\}\end{aligned}$$

D'où :

Remarque 2. Soit M un $\mathcal{D}_X[s_1,\dots,s_p]$-Module cohérent à gauche : $M=0$ si et seulement si $(\mathrm{car}_{\mathcal{D}_X[s_1,\dots,s_p]}M)(0)=\emptyset$

Définition 1. Soit M un $\mathcal{D}_X[s_1,\dots,s_p]$-Module cohérent à gauche. On dira que M est à fibre lagrangienne si $(\mathrm{car}_{\mathcal{D}_X[s_1,\dots,s_p]}M)(0)$, est une sous-variété lagrangienne de T^*X.

Soit $0\to M'\to M\to M''\to 0$ une suite exacte de $\mathcal{D}_X[s_1,\dots,s_p]$-Modules cohérents à gauche. Comme dans le cadre des $\mathcal{D}_X$-Modules, on a :

$$\mathrm{car}_{\mathcal{D}_X[s_1,\dots,s_p]}M = \mathrm{car}_{\mathcal{D}_X[s_1,\dots,s_p]}M' \bigcup \mathrm{car}_{\mathcal{D}_X[s_1,\dots,s_p]}M''$$

On en déduit en particulier la proposition suivante :

Proposition 4. *Soit $0\to M'\to M\to M''\to 0$ une suite exacte de la catégorie des $\mathcal{D}_X[s_1,\dots,s_p]$-Modules cohérents à gauche. Le Module M est à fibre lagrangienne si et seulement si M' et M'' le sont.*

2.2. Constructibilité de l'idéal associé

Dans ce paragraphe, M désignera un $\mathcal{D}_X[s_1,\dots,s_p]$-Module à gauche cohérent à fibre lagrangienne. Soit $(X_\alpha)_{\alpha\in A}$ les projections des composantes irréductibles de la variété lagrangienne conique $(\mathrm{car}_{\mathcal{D}_X[s_1,\dots,s_p]}M)(0)$:

$$(\mathrm{car}_{\mathcal{D}_X[s_1,\dots,s_p]}M)(0) = \bigcup_{\alpha\in A} T^*_{X_\alpha}X$$

L'espace $\bigcup_{\alpha\in A} X_\alpha$ est le support de M et la famille $(X_\alpha)_{\alpha\in A}$ de sous-ensembles irréductibles est localement finie. Pour $J = (\alpha_1, \dots, \alpha_l)$ un l-uplet d'éléments de A, on note :

$$U_J = X_{\alpha_1} \bigcap \cdots \bigcap X_{\alpha_l} - \bigcup_{\beta\notin J} X_\beta \bigcap X_{\alpha_1} \bigcap \cdots \bigcap X_{\alpha_l}$$

Soit $\mathcal{A} = \mathcal{P}^f(A)$ l'ensemble des parties finies J de A pour lesquelles $U_J \neq \emptyset$. Alors, $\{U_J\}_{J\in\mathcal{A}}$ est une partition de $\bigcup_{\alpha\in A} X_\alpha$. En effet pour $J \in \mathcal{A}$: $x \in U_J$ si et seulement si $J = \{\alpha \,;\, x \in X_\alpha\}$

Définition 2. La partition $\{U_J\}_{J\in\mathcal{A}}$ est appelée la partition associée à la variété lagrangienne $\bigcup_{\alpha\in A} T^*_{X_\alpha} X$.

Notons pour $\alpha \in A$:

$$U'_\alpha = X_\alpha - \bigcup_{X_\alpha \not\subset X_\beta} X_\beta \cap X_\alpha$$

Si $J(\alpha) = \{\gamma \in A \,;\, X_\alpha \subset X_\gamma\}$, $J(\alpha) \in \mathcal{A}$ et $U'_\alpha = U_{J(\alpha)}$; c'est un ouvert connexe dense de X_α.

Notation 4. *Nous notons $B(x,M)$ l'idéal de $\mathbf{C}[s_1, \dots, s_p]$ des polynômes annulant le germe de M en x.*

Proposition 5. *Soit M un $\mathcal{D}_X[s_1, \dots, s_p]$-Module à gauche cohérent à fibre lagrangienne. Pour tout $J \in \mathcal{A}$, l'idéal $B(x,M)$ est constant pour $x \in U_J$, et est noté $B_J(M)$. En particulier, $B_\alpha(M) = B_{J(\alpha)}(M)$ est constant sur U'_α et on a :*

$$B_J(M) = \bigcap_{\alpha\in J} B_\alpha(M)$$

Preuve. Soit $x \in U_J$. Considérons le sous $\mathcal{D}_X[s_1, \dots, s_p]$-Module de M :

$$L = B(x,M)M$$

La variété caractéristique $\mathrm{car}_{\mathcal{D}_X[s_1,\dots,s_p]} L$ de L est contenue dans la variété caractéristique de M. Comme cette dernière est supposée à fibre lagrangienne, il résulte de la proposition 3 que l'ensemble des composantes irréductibles de $(\mathrm{car}_{\mathcal{D}_X[s_1,\dots,s_p]} L)(0)$ est contenu dans l'ensemble des composantes irréductibles de $(\mathrm{car}_{\mathcal{D}_X[s_1,\dots,s_p]} M)(0)$: $\{T^*_{X_\alpha} X\}_{\alpha\in A}$. Par définition de l'idéal $B(x,M)$, L est nul au voisinage de x, donc $(\mathrm{car}_{\mathcal{D}_X[s_1,\dots,s_p]} L)(0)$ est vide au voisinage de x. Ainsi, pour $\gamma \in J$, la variété $T^*_{X_\gamma} X$ n'est pas une composante irréductible de $(\mathrm{car}_{\mathcal{D}_X[s_1,\dots,s_p]} L)$ (0). Si $y \in U_J$, comme $J = \{\gamma \,;\, x \in X_\gamma\}$, on obtient que $(\mathrm{car}_{\mathcal{D}_X[s_1,\dots,s_p]} L)$

(0) est vide au voisinage de y. Ainsi, L est nul en restriction à U_J. L'idéal $B(x, M)$ est donc égal à l'idéal de $\mathbf{C}[s_1, \ldots, s_p]$ annulant la restriction de M à U_J, $B_J(M)$.
D'autre part, pour $x \in U_J$, le module L est nul au voisinage de x. Donc, pour $\alpha \in J$, le Module L est nul en un point de U'_α. On a donc :

$$B(x, M) \subset \bigcap_{\alpha \in J} B_\alpha(M)$$

Inversement pour $\gamma \in J$, $T^*_{X_\gamma} X$ n'est pas une composante irréductible de la variété caractéristique de $(\bigcap_{\alpha \in J} B_\alpha(M)).M$. Et donc $(\bigcap_{\alpha \in J} B_\alpha(M)).M$ est nul au voisinage de $x \in U_J$. L'inclusion précédente est donc une égalité.

§3. $\mathcal{D}[s_1, \ldots, s_p]$-Modules et équations fonctionnelles associés à p fonctions holomorphes

Soit $f = (f_1, \ldots, f_p)$ des fonctions holomorphes sur X. On désigne par F le produit de ces p fonctions. Soit M un $\mathcal{D}_X$-Module holonome. La variété caractéristique de M s'écrit $\operatorname{car} M = \bigcup_{l \in L} T^*_{Y_l} X$ où $Y_l \subset X$ est un sous-espace analytique irréductible de X. Considérons :

$$\mathcal{O}_X[s_1, \ldots, s_p, 1/F] f_1^{s_1} \ldots f_p^{s_p}$$

$\mathcal{O}_X[s_1, \ldots, s_p, 1/F]$-Module libre de rang 1 engendré par $f_1^{s_1} \ldots f_p^{s_p}$. Le produit tensoriel $M \otimes_{\mathcal{O}_X} \mathcal{O}_X[s_1, \ldots, s_p, 1/F] f_1^{s_1} \ldots f_p^{s_p}$ est muni de la structure de $\mathcal{D}_X$-Module obtenue en posant :

$$\frac{\partial}{\partial x_i}(m \otimes a f_1^{s_1} \ldots f_p^{s_p})$$
$$= \frac{\partial}{\partial x_i} m \otimes a f_1^{s_1} \ldots f_p^{s_p} + m \otimes \frac{\partial a}{\partial x_i} f_1^{s_1} \ldots f_p^{s_p} + \sum_{j=1}^{p} s_j m \otimes \frac{\frac{\partial f_j}{\partial x_i} a}{f_j} f_1^{s_1} \ldots f_p^{s_p}$$

pour tout $i \in \{1, \ldots, n\}$ et pour toute section locale m (resp. a) de M (resp. de $\mathcal{O}_X[s_1, \ldots, s_p, 1/F]$). Si m est une section de M, on notera $m f_1^{s_1} \ldots f_p^{s_p} = m \otimes f_1^{s_1} \ldots f_p^{s_p}$.

3.1. Rappels et compléments

Soit m une section de M. A l'aide du critère usuel sur les bonnes filtrations (voir [G.M]), on montre que $\mathcal{D}_X[s_1, \ldots, s_p] m f_1^{s_1} \ldots f_p^{s_p}$ est $\mathcal{D}_X[s_1, \ldots, s_p]$-cohérent.

Théorème 1. *Soit m une section engendrant un $\mathcal{D}_X$-Module holonome régulier M de variété caractéristique $\bigcup_{l\in L} T^*_{Y_l}X$. Le module $\mathcal{D}_X[s_1,\dots,s_p]mf_1^{s_1}\dots f_p^{s_p}$ est un $\mathcal{D}_X[s_1,\dots,s_p]$-Module cohérent de variété caractéristique :*

$$\mathrm{car}_{\mathcal{D}_X[s_1,\dots,s_p]}\, \mathcal{D}_X[s_1,\dots,s_p]mf_1^{s_1}\dots f_p^{s_p} = \bigcup_{F|_{Y_l}\neq 0} W^\sharp_{f_1,\dots,f_p,Y_l}$$

Preuve. Ce théorème a été demontré dans le cas $m=1$ et $M=\mathcal{O}_X$ à l'aide d'un théorème de C. Sabbah ([Sab 2] théorème 3.2., page 228)(voir aussi [B.B.M.M]). Commençons par établir un corollaire direct du théorème de C. Sabbah.

Soit $\phi : \mathcal{X} \to S$ une submersion entre deux espaces analytiques lisses. Désignons par $\mathcal{D}_{\mathcal{X}/S}$ l'anneau des opérateurs relatifs au morphisme ϕ. Soit $T^*\mathcal{X}/S$ le fibré cotangent relatif. A tout $\mathcal{D}_{\mathcal{X}/S}$-Module cohérent $\mathcal{N}$, on associe sa variété caractéristique $\mathrm{car}_{\mathcal{D}_{\mathcal{X}/S}}\mathcal{N} \subset T^*\mathcal{X}/S$. Si $\mathcal{Y}\subset\mathcal{X}$ est un sous-espace analytique, on désigne par $T^*_{\phi|_\mathcal{Y}}(\mathcal{X}/S)$ l'espace conormal relatif à la restriction de ϕ à $\mathcal{Y}$. Il s'agit de l'adhérence dans $T^*\mathcal{X}/S$ de l'ensemble des vecteurs conormaux nuls sur les espaces tangents aux fibres de la restriction de ϕ à $\mathcal{Y}$. Nous dirons que ϕ est non caractéristique pour $\mathcal{Y}$, si l'intersection de l'image du morphisme naturel $T^*S\times_\mathcal{X}\mathcal{X}\to T^*_\mathcal{Y}\mathcal{X}$ est contenue dans la section nulle.

Lemme 1. *Soit $\mathcal{M}$ un $\mathcal{D}_\mathcal{X}$-Module holonome régulier de variété caractéristique* $\mathrm{car}\,\mathcal{M} = \bigcup_{l\in L} T^*_{\mathcal{Y}_l}\mathcal{X}$*, et $F:\mathcal{X}\longrightarrow \mathbf{C}$ une fonction non triviale qui s'annule identiquement sur tout $\mathcal{Y}_l$ dont l'image par ϕ ne contient pas un ouvert non vide de S. Soit $\mathcal{N}$ un $\mathcal{D}_{\mathcal{X}/S}$-Module cohérent qui engendre $\mathcal{M}$. Supposons de plus que $\mathcal{M}$ soit sans F-torsion. La variété caractéristique de $\mathcal{N}$ est alors donnée par*

$$\mathrm{car}_{\mathcal{D}_{\mathcal{X}/S}}\mathcal{N} = \bigcup_{F|_{\mathcal{Y}_l}\neq 0} T^*_{\phi|_{\mathcal{Y}_l}}(\mathcal{X}/S)$$

Preuve du lemme. Le théorème de C. Sabbah dit exactement que si Σ est une composante irréductible de la variété caractéristique de $\mathcal{N}$, il existe $l\in L$ tel que $\Sigma = T^*_{\phi|_{\mathcal{Y}_l}}(\mathcal{X}/S)$.

Supposons que F ne s'annule pas identiquement sur $\mathcal{Y}_l$. En un point générique de $\mathcal{Y}_l$, le morphisme ϕ, transverse aux $\mathcal{Y}_j$ passant par ce point, n'est donc pas caractéristique pour $\mathcal{M}$. Au voisinage de ce point, $\mathcal{M}$ est alors cohérent comme $\mathcal{D}_{\mathcal{X}/S}$-Module. Par des arguments simples, on peut déterminer la variété caractéristique de $\mathcal{M}$ comme $\mathcal{D}_{\mathcal{X}/S}$-Module. Cette variété est la même que celle de $\mathcal{N}$ ([Sch] lemme 1.3.3, page 125). Cela prouve que $T^*_{\phi|_{\mathcal{Y}_l}}(\mathcal{X}/S)$ est contenu dans la variété caractéristique de $\mathcal{N}$.

Si F s'annule identiquement sur $\mathcal{Y}_l$, alors $T^*_{\phi|\mathcal{Y}_l}(\mathcal{X}/S)$ est de dimension strictement inférieure à $\dim \mathcal{X}$. Supposons que la variété caractéristique de $\mathcal{N}$ ne soit pas de dimension pure $\dim\mathcal{X}$. Soit $\mathcal{F}(\mathcal{N})$ le plus grand sous-module cohérent de $\mathcal{N}$ de dimension strictement inférieure à $\dim\mathcal{X}$. D'après [Bj] (Chap. 2.7 et Chap. 5.6), $\mathcal{F}(\mathcal{N})$ est alors non nul. Il engendre sur $\mathcal{D}_{\mathcal{X}}$ un sous-Module de $\mathcal{M}$. D'après nos hypothèses et le théorème de C. Sabbah, ce Module serait annulé par une puissance de F, d'où la contradiction.

Terminons la preuve du théorème. Considérons l'application :

$$X \times \mathbf{C}^p \xrightarrow{i} X \times \mathbf{C}^p \times \mathbf{C}^p = \mathcal{X}$$

$$(x, y) \mapsto (x, y, t_1 = e^{y_1} f_1(x), \ldots, t_p = e^{y_p} f_p(x))$$

Soit $p : X \times \mathbf{C}^p \to X$ la projection sur X. Notons $M' = \mathcal{O}_{X\times\mathbf{C}^p} \otimes_{p^{-1}\mathcal{O}_X} p^{-1}M$ l'image inverse par p de M. L'image directe $\mathcal{M} = i^+(M')$ est un $\mathcal{D}_{\mathcal{X}}$-Module régulier de variété caractéristique (voir par exemple [G.M] page 130.)

$$\bigcup_{l\in L} T^*_{i(Y_l\times\mathbf{C}^p)}\mathcal{X}$$

Considérons $\phi : \mathcal{X} \to S = \mathbf{C}^p$, $(x, y, t) \mapsto t$. Pour démontrer le théorème, quitte à remplacer M par $M[1/F]$, on peut supposer que M est sans F-torsion. Comme ϕ est submersif en restriction à $i(Y_l \times \mathbf{C}^p)$ si et seulement si F est non nulle sur Y_l , le lemme permet alors de calculer la variété caractéristique de $\mathcal{D}_{\mathcal{X}/S}(1 \otimes m)(e^{y_1} f_1)^{s_1} \ldots (e^{y_p} f_p)^{s_p}$. Le théorème s'en déduit par le même principe que dans le cas $m = 1$ et $M = \mathcal{O}_X$ (voir [B.B.M.M] page 126).

Il résulte de la proposition 1 que $\mathcal{D}_X[s_1, \ldots, s_p]mf_1^{s_1} \ldots f_p^{s_p}$ est à fibre lagrangienne. Comme conséquence directe du théorème 1, nous obtenons :

Corollaire 2. *Sous les hypothèses du théorème 1, la variété caractéristique du $\mathcal{D}_X[s_1, \ldots, s_p]$-Module à gauche cohérent*

$$N = \frac{\mathcal{D}_X[s_1, \ldots, s_p]mf_1^{s_1} \ldots f_p^{s_p}}{\mathcal{D}_X[s_1, \ldots, s_p]mf_1^{s_1+1} \ldots f_p^{s_p+1}}$$

est contenue dans $(\bigcup_{F|_{Y_l}\neq 0} W^\sharp_{f_1,\ldots,f_p,Y_l}) \bigcap F^{-1}(0)$.

Remarque 3. On peut montrer (ce sera fait dans un prochain travail) que cette inclusion est en fait une égalité. Nous n'utiliserons pas ce fait ici.

3.2. Idéal de Bernstein

Dans ce paragraphe M désigne un $\mathcal{D}_X$-Module holonome régulier engendré par une section m. Soit $\operatorname{car} M = \bigcup_{l\in L} T^*_{Y_l}X$ sa variété caractéristique. Pour tout $x \in X$, l'idéal $B(x, f_1, \ldots, f_p, m)$ des polynômes de $\mathbf{C}[s_1, \ldots, s_p]$ annulant la fibre en x du $\mathcal{D}_X[s_1, \ldots, s_p]$-Module à gauche :

$$N = \frac{\mathcal{D}_X[s_1, \ldots, s_p] m f_1^{s_1} \ldots f_p^{s_p}}{\mathcal{D}_X[s_1, \ldots, s_p] m f_1^{s_1+1} \ldots f_p^{s_p+1}}$$

est appelé idéal de Bernstein de $f_1, \ldots, f_p, m$ en x. Il est montré dans [Sab 1] [Sab 2] que cet idéal contient un polynôme non nul qui s'écrit comme produit de formes linéaires affines à coefficients rationnels positifs. Le module N, quotient d'un $\mathcal{D}_X[s_1, \ldots, s_p]$-Module à fibre lagrangienne, est donc à fibre lagrangienne. Et on a l'inclusion (corollaire 2) :

$$\operatorname{car}_{\mathcal{D}_X[s_1,\ldots,s_p]} N \subset \Big(\bigcup_{F|_{Y_l}\neq 0} W^\sharp_{f_1,\ldots,f_p,Y_l} \Big) \bigcap F^{-1}(0)$$

D'où l'inclusion :

$$(\operatorname{car}_{\mathcal{D}_X[s_1,\ldots,s_p]} N)(0) \subset \bigcup_{F|_{Y_l}\neq 0} (W^\sharp_{f_1,\ldots,f_p,Y_l})(0) \bigcap F^{-1}(0)$$

Il résulte de la remarque 1 que $(W^\sharp_{f_1,\ldots,f_p,Y_l})(0) \bigcap F^{-1}(0)$ est une variété lagrangienne contenue dans $(W^\sharp_{f_1,\ldots,f_p,Y_l})(0)$. De plus, d'après le corollaire 1, $(W^\sharp_{f_1,\ldots,f_p,Y_l})(0) = (W^\sharp_{F,Y_l})(0)$. Traduisons la proposition 5 :

Théorème 2. *Soit M un $\mathcal{D}_X$-Module holonome régulier engendré par une section m et $\operatorname{car} M = \bigcup_{l\in L} T^*_{Y_l}X$ sa variété caractéristique. L'idéal de Bernstein de $f_1, \ldots, f_p, m$ en x est constant le long des strates de la partition associée à la variété lagrangienne :*

$$\Big(\bigcup_{F|_{Y_l}\neq 0} W^\sharp_{F,Y_l} \Big)(0) \bigcap F^{-1}(0)$$

Soit l tel que $F|_{Y_l} \neq 0$. L'espace $(W^\sharp_{F,Y_l})(0)$ est la réunion de $T^*_{Y_l}X$ et de $(W_{F,Y_l})(0)$, trace de $F^{-1}(0)$ sur l'espace conormal relatif de la restriction de F à Y_l. Soit $(V_\beta)_{\beta\in\Gamma_l}$ une stratification analytique de Y_l (partition localement finie par des strates lisses connexes) compatible à $F^{-1}(0)$ et satisfaisant la condition de frontière et la condition $a_{F|_{Y_l}}$ de Thom. Cette condition entraine l'inclusion :

$$W^\sharp_{F,Y_l}(0) \bigcap F^{-1}(0) \subset \bigcup_{\beta\in\Gamma_l} T^*_{V_\beta}X$$

Donc, si $T^*_{X_\alpha}X$ est une composante irréductible de $(W^\sharp_{F,Y_l})(0) \bigcap F^{-1}(0)$, il existe $\beta \in \Gamma_l$ tel que X_α soit l'adhérence de V_β. Donc, grâce à la condition de frontière, X_α est réunion de strates de la stratification $(V_\beta)_{\beta \in \Gamma_l}$. On obtient ainsi le corollaire :

Corollaire 3. *Soit M un $\mathcal{D}_X$-Module holonome régulier engendré par une section m et* car $M = \bigcup_{l \in L} T^*_{Y_l}X$ *sa variété caractéristique. Soit $(V_\beta)_{\beta \in \Gamma}$ une stratification analytique de $\bigcup_{l \in L} Y_l$ compatible aux Y_l et à $F^{-1}(0)$ satisfaisant la condition de frontière et la condition a_F de Thom. Alors l'idéal de Bernstein de $f_1, \ldots, f_p, m$ en x est constant le long des strates de la stratification $(V_\beta)_{\beta \in \Gamma}$.*

Remarque 4 (après [B.M.M] (théorème 4.2.1 page 541))**.** Si la stratification $(V_\beta)_{\beta \in \Gamma}$ est de Whitney, nous savons qu'elle satisfait alors la condition a_F de Thom et la conclusion reste valable.

Dans le cas $p = 1$, N est un $\mathcal{D}_X[s_1]$-Module qui est holonome en tant que $\mathcal{D}_X$-Module. Dans ce cas, la preuve de la proposition 5 est plus simple : on peut montrer directement par la même méthode que le polynôme minimal de la fibre en un point d'un endomorphisme du $\mathcal{D}_X$-Module holonome M est constant le long des strates de la partition associée à sa variété caractéristique. On obtient ainsi une autre preuve de la proposition de [B.M]. Grâce à la correspondance de Riemann-Hilbert [M] [K], cette preuve se transcrit dans la catégorie des faisceaux pervers, où l'on dispose également de la notion de variété caractéristique, de la manière suivante :

Remarque 5. Soit $f : X \to \mathbf{C}$ une application analytique. Soit $\mathcal{F}$ un faisceau pervers sur X de variété caractéristique $\bigcup_{l \in L} T^*_{Y_l}X$ et $\psi_f \mathcal{F}$ le faisceau des cycles proches muni de son automorphisme de monodromie (voir [D.K]).

On obtient ainsi, par exemple, que le polynome minimal de la monodromie du germe du faisceau pervers $\psi_f \mathcal{F}$ est constant le long des strates d'une stratification compatible aux Y_l et à $f^{-1}(0)$, satisfaisant de plus la condition de frontière et la condition a_f de Thom.

On pourra se reporter à [B.M.M] [Sab 3] [Gn] pour le calcul de la variété caractéristique de $\psi_f \mathcal{F}$.

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Assigned base conditions and geometry of foliations on the projective plane

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Generalized Enriques diagrams and characteristic cones

Gerard Gonzalez-Sprinberg

Abstract.

Generalized Enriques diagrams are combinatorial data associated with constellations of infinitely near points and proximity relations. Classically they were introduced to deal with linear systems of curves with base conditions. We present a survey on some aspects and new results on this diagrams, examples and applications to relative characteristic cones and Zariski's complete ideal theory.

§1. Introduction

In [6] (Libro Quarto: "Le singolarità delle curve algebriche", I. 12 et II. 17), Enriques and Chisini consider systems of plane curves passing, with assigned multiplicities, through an assigned set of points or infinitely near points to a point of the plane. They found that there exist curves with such prescribed multiplicities (with no conditions on the degree of the curves) if and only if some inequalities, on these virtual multiplicities, hold for the given points, the so-called *proximity* relations. Enriques associates a graph ("schema grafico") to the constellation of infinitely near points appearing in the desingularisation of a plane curve and equiped this graph with the data of the proximity relations which keep track of the incidence between points and the exceptional divisors obtained by blowing-up precedent points. Du Val also considers these proximities relations (see [5]) and defines the proximity matrix.

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Some years later Zariski introduces the notion of complete ideals to give a new algebraic setup of the previous geometric theory ([14], [15]), where complete (i.e. integrally closed) ideals are the local avatars of complete linear systems . One of the main results is that any complete ideal $\mathcal{I}$ in a *two* dimensional local ring has a unique factorization into simple ideals, which corresponds to the factorization of the general element of $\mathcal{I}$ into analytically irreducible factors. This fact may be reformulated as the *regularity* of the relative characteristic cone associated with the minimal blowing-up of (infinitely near) points for which $\mathcal{I}$ becomes locally principal. The condition of regularity of a cone is considered in the sense of rational cones of toric varieties, i.e. with primitive integral extremal points forming a subset of a basis of the lattice. The characteristic and the quasi-ample cone of a proper morphism have been considered by Hironaka, Mumford and Kleiman [9] in the 60's.

In higher dimension than two, as Zariski had noted, the theory is much more involved and the main results do not extend, or not in the same terms. First, one has to restrict to *finitely supported* complete ideals in order to deal with constellations of closed points. But even with this restriction, the characteristic cone is not regular, or polyhedral, in the general case (see [4] or [1] for some examples). Lipman extended this theory for two dimensional local rings with rational singularities [10], and for higher dimensional regular local rings he obtained a unique factorization result by allowing negative exponents [11].

The preceding lines do not pretend to be an exhaustive account on the history of the subject, but only a sketch to situate it; many other important contributions have been made to this "theory in search of theorems", as Lipman says.

In this work we summarize *two* generalizations of Enriques diagrams in higher dimensions; there are two natural generalizations because the *dimension* one and *codimension* one conditions that coincide in the two-dimensional classic case are not equivalent when the ambient dimension is at least three.

First we characterize the so called proximity P-Enriques diagrams and determine, in terms of numerical invariants of such a diagram, the minimal dimension of a constellation of infinitely near points which induces a given diagram.

Then, we consider a case where the characteristic and the quasi-ample cones are equal, namely the toric constellations, and characterize the P-Enriques diagrams associated with them.

Finally we consider the so called linear proximity LP-Enriques diagrams, which determine the characteristic cone in the toric constellation case, and show a converse Zariski theorem: the characteristic cone is regular if and only if the LP-Enriques diagram comes from a *two-dimensional* constellation.

"Qui oserait dire que ce que nous avons détruit
valait cent fois mieux que ce que nous avions rêvé
et transfiguré sans relâche en murmurant aux ruines ?"
René Char

§2. Constellations of infinitely near points and P-Enriques diagrams

2.1. Let X be a regular variety over an algebraically closed field $\mathbb{K}$, of dimension $d \geq 2$. Consider varieties obtained from X by a finite sequence of closed points blowing-ups. Any point in such a variety is called an infinitely near point of X.
A point P is infinitely near $Q \in X$ if Q is the image of P under the composition of the blowing-ups; $P \geq Q$ in symbol.

A *constellation of infinitely near points* (in short, a constellation, if there is no confusion with other astronomical objects) is a set $\mathcal{C} = \{Q_0, \dots, Q_n\}$, with $Q_i \geq Q_0 \in X_0 = X$, such that $Q_i \in Bl_{Q_{i-1}} X_{i-1} =: X_i \xrightarrow{\sigma_{i-1}} X_{i-1}$, for $1 \leq i \leq n$; where $Bl_{Q_{i-1}} X_{i-1}$ denotes the blowing-up of X_{i-1} with center Q_{i-1} .

The point Q_0 is called the *origin* of the constellation $\mathcal{C}$. We call also the dimension of X the *dimension* of $\mathcal{C}$.

Let $\sigma_{\mathcal{C}} = \sigma_0 \circ \cdots \circ \sigma_n : X_{\mathcal{C}} \to X_0$ denote the composition of the blowing-ups of all the points of $\mathcal{C}$, where $X_{\mathcal{C}} = X_{n+1}$. Two constellations $\mathcal{C}$ and $\mathcal{C}'$ over X are identified if there is an automorphism π of X and an isomorphism $\pi' : X_{\mathcal{C}} \to X_{\mathcal{C}'}$ such that $\sigma_{\mathcal{C}'} \circ \pi' = \pi \circ \sigma_{\mathcal{C}}$.

The relation $Q_j \geq Q_i$, meaning that a composition of blowing-ups sends Q_j to Q_i in X_i, is a partial ordering on the points of $\mathcal{C}$. If this ordering is total, i.e. $Q_n \geq \cdots \geq Q_0$, we say that $\mathcal{C}$ is a *chain* constellation.

For example, for any constellation $\mathcal{C}$ and any $Q \in \mathcal{C}$, the set $\mathcal{C}^Q := \{P \in \mathcal{C} \mid Q \geq P\}$ of preceding points is a chain constellation. The number of points in $\mathcal{C}^Q$, different from Q, is called the *level* of Q.

For each point $Q \in \mathcal{C}$ let Q^+ be the set of points of $\mathcal{C}$ *consecutive* to Q, i.e. the points following Q such that there is no strict intermediate point; write $|Q^+|$ for the cardinal of this set. If Q^+ has only one point, it denotes this point, by a slight abuse of notation.

For each point $Q = Q_i$, call B_Q (or B_i) the *exceptional* divisor $\sigma_i^{-1}(Q)$ on X_{i+1}, and E_Q (or E_i) its *strict* (or *proper*) successive transforms on any X_j (which will be specified if necessary) with $Q_j \geq Q$, in particular on $X_{\mathcal{C}}$. The *total* transforms are denoted E_Q^* or E_i^*.

The sets of divisors, $\{E_Q \mid Q \in \mathcal{C}\}$ and $\{E_Q^* \mid Q \in \mathcal{C}\}$, considered in $X_{\mathcal{C}}$, are two basis of the lattice $N^1 = \bigoplus_{Q\in\mathcal{C}} \mathbb{Z}E_Q \cong \mathbb{Z}^{n+1}$ of divisorial cycles with *exceptional* support in $X_{\mathcal{C}}$.

Definition 2.2. A point $Q_j \geq Q_i$ is *proximate* to Q_i if $Q_j \in E_i$ in X_j; notation : $Q_j \to Q_i$ (or $j \to i$).

The *proximity index* of a point Q_j is defined as the number $\mathrm{ind}(Q_j)$ of points in $\mathcal{C}$ approximated by Q_j, i.e. $\mathrm{ind}(Q_j) := \#\{Q_i \in \mathcal{C} \mid Q_j \to Q_i\}$.

If $R \in Q^+$ then $R \to Q$, these are the so called trivial proximities ; if R belongs to the intersection of several exceptional divisors produced by blowing-up precedent points then R is proximate to all these points. In fact, if the dimension of C is *at least three*, then $R \to Q$ if and only $R \geq Q$ and $E_R \cap E_Q \neq \emptyset$ in $X_{\mathcal{C}}$.

If $R \to Q$ then $R \geq Q$; the converse does not hold in general. The proximity relation $(\to)$ is a binary relation on the set of points of a constellation, but not an ordering one.

Remark 2.3. For each point Q_i, the only exceptional divisors, besides E_i, appearing in the total transform E_i^*, in $X_{\mathcal{C}}$, are exactly those produced by blowing-up the points proximate to Q_i. Therefore $E_i = E_i^* - \sum_{j\to i} E_j^*$. The so called *proximity matrix* $((p_{ji}))$, with $p_{ii} = 1$, $p_{ji} = -1$ if $j \to i$ and 0 otherwise, is the basis change matrix from the E_i's to the E_j^*'s

Definition 2.4. The (proximity) *P-Enriques diagram* of a constellation $\mathcal{C}$ is the rooted tree $\Gamma_{\mathcal{C}}$ equiped with the binary relation $(\rightsquigarrow)$, whose vertices are in one to one correspondence with the points of $\mathcal{C}$, the edges with the couples of points (R, Q) such that $R \in Q^+$, the root with the origin of $\mathcal{C}$, and the relation $(\rightsquigarrow)$ with the proximity relation $(\to)$.

Any (finite) rooted tree may be obtained in this way, but not with the data of a binary relation . Next we characterize the P-Enriques diagrams, i.e. the rooted trees, equiped with a binary relation on the set of vertices, which are induced by some constellation.

Given a rooted tree Γ, denote by $(\succeq)$ the natural partial ordering on the set $\mathcal{V}(\Gamma)$ of its vertices : $p \succeq q$ if q belongs to the chain from p to the root; similarly, if $(\rightsquigarrow)$ is a binary relation on $\mathcal{V}(\Gamma)$, let $\operatorname{ind}(q) = \#\{p \in \mathcal{V}(\Gamma) \mid q \rightsquigarrow p\}$.

For each vertex q, let q^+ be the set of consecutive vertices to q with respect to the ordering $(\succeq)$.

Theorem 2.5. *Let Γ be a finite rooted tree equiped with a binary relation $(\rightsquigarrow)$ on the set of its vertices. Then Γ is the graph associated with a constellation of infinitely near points $\mathcal{C}$ and $(\rightsquigarrow)$ is induced by the the proximity relation on $\mathcal{C}$ if and only if , for any vertices p, q, r of Γ, the following conditions are satisfied:*

(a) $q \rightsquigarrow p \quad \Longrightarrow \quad q \succeq p\ ,\ q \neq p$

(b) $q \in p^+ \quad \Longrightarrow \quad q \rightsquigarrow p$

(c) $r \succeq p \succeq q$ *and* $r \rightsquigarrow q \quad \Longrightarrow \quad p \rightsquigarrow q$

If these conditions hold, then the minimum dimension $d_{\mathcal{P}}$ of a constellation whose P-Enriques diagram is the given one is at most $\max(2, \max_{q \in \mathcal{V}(\Gamma)}(\operatorname{ind}(q)) + 1)$.

Proof. The necessity of the conditions follows easily. For the sufficiency, proceed by induction on the number $|\mathcal{V}(\Gamma)|$ of vertices.

If $|\mathcal{V}(\Gamma)| > 1$, let r be a maximal vertex of Γ, and assume that a constellation $\mathcal{C}'$ of dimension d works for $\Gamma' = \Gamma \setminus \{r\}$. Let $r \in q^+$, and Q be the point of $\mathcal{C}'$ corresponding to q. The set $Y := \{P \in \mathcal{C}' \mid r \rightsquigarrow p\}$ is contained in $\mathcal{C}'^Q$ by (a) and $Q \in Y$ by (b).

By (c) one has $Q \to P$ for each $P \in Y \setminus \{Q\}$, so that $Q \in F := \bigcap_{P \in Y, P \neq Q} E_P$. It follows that $F \neq \emptyset$ and $\dim(F) = d + 1 - |Y|$, by the normal crossing of the divisors E_P, and on the other hand $\operatorname{ind}(Q) \geq |Y \setminus \{Q\}| = |Y| - 1$.

Now, we need a point R (in $X_{\mathcal{C}'}$) corresponding to r, having the corresponding proximities, i.e. a point $R \in B_Q \bigcap F$ but not in $(Q^+ \bigcup_{P \in \mathcal{C}^Q \setminus Y} E_P)$. Such a point exists if $d \geq \max_{p \in \mathcal{V}(\Gamma)} \operatorname{ind}(p) + 1$ (and at least 2), which is not less than $\max_{p \in \mathcal{V}(\Gamma')} \operatorname{ind}(p) + 1$ so the inductive hypothesis applies. This number is attained. Q.E.D.

Remark 2.6. The minimum dimension $d_{\mathcal{P}}$ of constellations inducing a given P-Enriques diagram may be one less than in the general case if there are no two maximal vertices r, with maximum indices, say r_1 and r_2, both in q^+, such that $\operatorname{ind}(r_i) = \operatorname{ind}(q) + 1$. Precisely, the minimum dimension is

$$d_{\mathcal{P}} = \max(2, \max_{q \in \mathcal{V}(\Gamma)}(\operatorname{ind}(q) + t(q)),$$

where $t(q){=}s(q)$ (resp. $t(q){=}2$) if $s(q){:=}\#\{r \in q^+ | \operatorname{ind}(r){>} \operatorname{ind}(q)\} \leq 1$ (resp. if $s(q) \geq 2$).

§3. Toric constellations and proximity

We begin by recalling some definitions and fixing notations for toric varieties (for a detailed treatement see some of the basic references on this subject, chapter 1 of [13] or [8]).

3.1. Let $N \cong \mathbb{Z}^d$ be a lattice of dimension $d \geq 2$ and Σ a *fan* in $N_\mathbb{R} = N \otimes_\mathbb{Z} \mathbb{R}$, i.e. a finite set of strongly convex polyhedral cones such that every face of a cone of Σ belongs to Σ and the intersection of two cones of Σ is a face of both. Denote by X_Σ the toric variety over a field $\mathbb{K}$ associated with Σ, equiped with the action of an algebraic torus $T \cong (\mathbb{K}^*)^d$. There is a one to one canonical correspondence between the T-orbits in X_Σ and the cones of Σ. Two basic facts of this correspondence are that the dimension of a T-orbit is equal to the codimension of the corresponding cone, and that a T-orbit is contained in the *closure* of another T-orbit if and only if the cone associated with the first one contains the cone associated with the second one.

The *morphisms* of toric varieties are the equivariant maps induced by the maps of fans $\varphi : (N', \Sigma') \to (N, \Sigma)$ such that $\varphi : N' \to N$ is a $\mathbb{Z}$-linear homomorphism whose scalar extension $\varphi : N'_\mathbb{R} \to N_\mathbb{R}$ has the property that for each $\sigma' \in \Sigma'$ there exists $\sigma \in \Sigma$ such that $\varphi(\sigma') \subset \sigma$; (see [13], 1.5).

Let $X_0 := X_{\Sigma_0} \cong \mathbb{K}^d$ be the d-dimensional affine toric variety associated with the fan Σ_0 formed by all the faces of a regular d-dimensional rational cone Δ in $N_\mathbb{R}$. Recall that a rational cone is called *regular* (or nonsingular) if the primitive integral extremal points form a subset of a basis of the lattice.

A *toric constellation* of infinitely near points is a constellation $\mathcal{C} = \{Q_0, \dots, Q_n\}$ such that each Q_j is a fixed point for the action of the torus in the toric variety X_j obtained by blowing-up X_{j-1} with center Q_{j-1}, $1 \leq j \leq n$. If a toric constellation is a chain, it is called a *toric chain*. The identification of constellations defined in 2.1 is the same in the toric case, but considering equivariant isomorphisms.

3.2. By choosing a fixed *ordered basis* $\mathcal{B} = \{v_1, ..., v_d\}$ of the lattice N we obtain a *codification* of the toric constellations, as well as criteria for proximity and (as shown in the following section) linear proximity.

Let $\Delta = \langle \mathcal{B} \rangle$ be the (regular) cone generated by the basis $\mathcal{B}$. The blowing-up $\sigma_i : X_i \to X_{i-1}$ of the closed orbit Q_{i-1}, is described as an elementary subdivision of a fan, as follows.

The variety X_1 is the toric variety associated with the fan Σ_1, obtained as the minimal subdivision of Σ_0 which contains the ray through

$u = \sum_{1 \le j \le d} v_j$.
For each integer i, $1 \le i \le d$, let $\mathcal{B}_i$ be the ordered basis of N obtained by replacing v_i by u in the basis $\mathcal{B}$; and let $\Delta_i := \langle \mathcal{B}_i \rangle$. The exceptional divisor B_0 is the closure in X_1 of the T-orbit defined by the ray through u, and each T-fixed point in X_1 corresponds to a maximal cone Δ_i of the fan Σ_1, $1 \le i \le d$.

The choice of the point $Q_1 \ge Q_0$ is thus equivalent to the choice of an integer a_1, $1 \le a_1 \le d$, which determines a cone Δ_{a_1} of the fan Σ_1.

The subdivision Σ_2 of Σ_1 corresponding to the blowing-up of Q_1 is obtained by replacing Δ_{a_1} (and its faces) in Σ_1 by the cones $\Delta_{a_1 i} := \langle \mathcal{B}_{a_1 i} \rangle$ (and their faces), where $\mathcal{B}_{a_1 i}$ is the ordered basis of N obtained from $\mathcal{B}_{a_1}$ by the substitution of its i-th vector by $\sum_{v \in \mathcal{B}_{a_1}} v$.

The choice of $Q_2 \in B_1$ is equivalent to the choice of an integer a_2 ,$1 \le a_2 \le d$, which determines a (regular) cone $\Delta_{a_1 a_2}$.

Proceeding by induction on n we obtain a *codification* of toric chains and also constellations, since for each $Q \in \mathcal{C}$, the constellation $\mathcal{C}^Q$ is a chain.

The codification is given by trees with weighted edges, where the weights are integers a, $1 \le a \le d$, which give the *direction* in which the following blowing-up is done. The precise description follows.

Definition 3.3. Let Γ be a tree, $\mathcal{E}(\Gamma)$ the set of edges of Γ, d an integer, d ≥ 2.
A *d-weihgting* of Γ is a map $\alpha : \mathcal{E}(\Gamma) \to \{1, \dots ,\mathrm{d}\}$ which associates to each edge of Γ a positive integer not greater than d, such that two edges with a common origin have different weights. A couple (Γ, α) is called a *d-weighted tree.*

Proposition 3.4. *Let $\mathcal{B}$ be an ordered basis of the lattice N and n a positive integer.*

(a) *The map which associates to each sequence of integers $\{a_1, \dots, a_n\}$ such that $1 \le a_i \le d$, $1 \le i \le n$, the toric chain $\{Q_0, \dots ,Q_n\}$ where Q_0 is the T-orbit corresponding to the cone $\Delta = \langle \mathcal{B} \rangle$, and where Q_i, $1 \le i \le n$, is the T-orbit in X_i corresponding to the cone $\Delta_{a_1 \dots a_i}$ of the fan Σ_i, is a bijection between the set of such sequences and the set of d-dimensional toric* chains *with $n+1$ points.*
(b) *A natural bijection between the set of d-dimensional toric constellations and the set of d-weighted trees is induced by the correspondence* (a).

Remark 3.5. Note that in a d-weighted tree each vertex is the origin of at most d edges. A d-weighting of a tree Γ induces a partition of the set $\mathcal{E}(\Gamma)$ of edges, where two edges are in the same class if they have the same weight. To each class of isomorphism of d-dimensional toric constellations is associated a unique class of isomorphism of trees equiped with a partition of the set of edges, partition with at most d classes of edges [7].

3.6. Given a toric constellation by a d-weighted graph, a vertex following q through a chain with edges weighted by a sequence $(a_1, \dots, a_k)$ is denoted by $q(a_1, \dots, a_k)$; if Q is the point corresponding to q, then the point corresponding to $q(a_1, \dots, a_k)$ is written in a similar way $Q(a_1, \dots, a_k)$.

Proposition 3.7 (*Criterion* for proximity in terms of a codification). *$Q(a_1, \dots, a_k) \to Q$ if and only if $a_1 \neq a_j$ for $2 \leq j \leq k$.*

Proof. The criterion follows from the fact that this is the condition to obtain, by elementary subdivisions of a regular fan, an adjacent maximal cone $\Delta_{a_1 \dots a_k}$ (corresponding to a 0-dimensional orbit) to the central ray of Δ_{a_1} (corresponding to the exceptional divisor) of the first subdivision of the cone Δ corresponding to Q. This is equivalent to saying that $Q(a_1, \dots, a_k) \in E_Q$, i.e. $Q(a_1, \dots, a_k) \to Q$. Q.E.D.

Theorem 3.8. *A P-Enriques diagram $(\Gamma, (\rightsquigarrow))$ is toric, i.e. may be induced by a toric constellation, if and only if:*

(a) *The proximity index is non-decreasing, i.e.* $\mathrm{ind}(r) \geq \mathrm{ind}(q)$ *if* $r \succeq q$.

(b) *If r is proximate to q, then there is at most one vertex s consecutive to r and not proximate to q, i.e. if $r \rightsquigarrow q$ then $\#\{s \in r^+ \mid s \not\rightsquigarrow q\} \leq 1$.*

If these conditions hold, then the minimum dimension $dt_{\mathcal{P}}(\Gamma, (\rightsquigarrow))$ of a toric constellation inducing the given P-Enriques diagram $(\Gamma, (\rightsquigarrow))$ is

$\max(2, \max_{q \in \Gamma}(\mathrm{ind}(q) + s(q)))$, *where* $s(q) := \#\{r \in q^+ \mid \mathrm{ind}(r) > \mathrm{ind}(q)\}$ *is the number of consecutive points to q whose proximity index is greater than the proximity index of q.*

Proof. In fact, if $R \in Q^+$, then $\mathrm{ind}(R) \leq \mathrm{ind}(Q) + 1$ for any constellation, since, by (c) of Theorem 2.5, $R \to P$ implies $Q \to P$; in the toric case one also has $\mathrm{ind}(Q) \leq \mathrm{ind}(R)$ because R may loose at most one proximity to a point approximated by Q, but on the other hand $R \to Q$. Indeed, restricting to the chain from the origin to R and by using the codification and the criterion of proximity 3.7, let $R = Q(a)$ and assume $Q \to P$, $Q \to P'$, $R \not\to P$; then $P^+ = P(a)$. Let $P'^+ = P'(a')$. If $P \geq P'$ (resp. if $P' \geq P$), then $a \neq a'$, because $Q \to P'$ (resp. $Q \to P$); then $R \to P'$. This shows the necessity of (a).

To prove (b), recall that two edges with origin R have necessarily different weights, so there is at most one whose weight is equal to the weight of the edge, with origin Q, in the chain from Q to R.

To prove the sufficiency, proceed by induction on the number of vertices and use the proximity criterion. Assume $|\mathcal{V}(\Gamma)| > 1$, let r be a maximal vertex of Γ, say $r \in q^+$. By the inductive hypothesis, the full subgraph $\Gamma' = \Gamma \setminus q^+$ equiped with the binary relation restricted to $\mathcal{V}(\Gamma')$, is also a P-Enriques diagram satisfying the conditions (a) and (b), so may be induced by a toric constellation $\mathcal{C}'$, codified by a d-weighted tree with d $\leq dt_{\mathcal{P}}(\Gamma', (\rightsquigarrow))$. Let $q^+ = \{r_1, \dots, r_s, \dots, r_t\}$; by (i) one has $\mathrm{ind}(r_j) \geq \mathrm{ind}(q)$, for $1 \leq j \leq t$. Let $r_1, \dots, r_s$ be vertices, in q^+, whose index is greater than $\mathrm{ind}(q)$; one needs $s = s(q)$ new weights to codify these vertices; but for each r_j with $s{+}1 \leq j \leq t$, the weight is determined by the lost proximity among the vertices approximated by q, and these weights are all different by the condition (ii). This shows the inductive step and the existence of a toric constellation associated with the given $(\Gamma, (\rightsquigarrow))$ with the dimension $dt_{\mathcal{P}}$. This is the minimum dimension since for any such constellation of dimension d, one has $d \geq \mathrm{ind}(q) + s(q)$, for each $q \in \mathcal{V}(\Gamma)$. Q.E.D.

Remark 3.9. The minimum dimension $dt_{\mathcal{P}}$ may be greater than $d_{\mathcal{P}}$, the dimension in the not necessarily toric case (Theorem 2.5), because there are less points available, so one needs to add $s(q)$ to the proximity index, not just 1 as in the general case.

Corollary 3.10. *A P-Enriques diagram* $(\Gamma, (\rightsquigarrow))$ *whose graph* Γ *is a* chain, *is toric if and only if the proximity index is not decreasing. In this case, the minimum dimension of an associated constellation is the index of the terminal point* (*and at least* 2).

Proof. In the toric chain case the condition (b) of the theorem is automatically satisfied and $\max_{q\in\Gamma}(\mathrm{ind}(q) + s(q))) = \max_{q\in\Gamma}(\mathrm{ind}(q))$ holds. Q.E.D.

Examples 3.11. (1) The simplest example of a non-toric P-Enriques diagram is a chain with four vertices, say q_0, q_1, q_2, q_3 such that, besides the trivial proximities of consecutive vertices, the only other proximity is $q_2 \rightarrow q_0$. In this example one has $\mathrm{ind}(q_2) = 2$ and $\mathrm{ind}(q_3) = 1$; condition (a) fails.

(2) Another example of a non-toric case is a graph of type $\mathbb{D}_4$, with a non-central vertex as the root, and with only the proximities of consecutive vertices. In this case condition (b) fails.

Remark that both cases may be induced by two dimensional constellations.

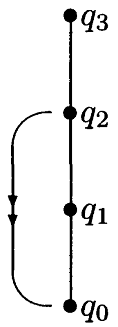

Figure (1).

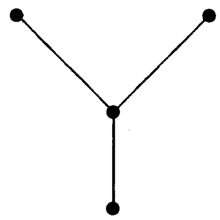

Figure (2).

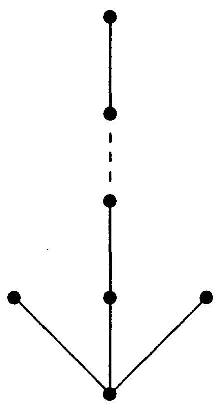

Figure (3).

(3) If the central vertex is the root in a graph of type $\mathbb{D}_n$, with $n \geq 4$, and if the only proximities are those of consecutive vertices, then conditions (a) and (b) hold; the minimal dimension of a constellation inducing this P-Enriques diagram is three for toric constellations and two for non-toric ones. If q_0 is the root, then $\operatorname{ind}(q_0) = 0$, $s(q_0) = 3$, $t(q_0) = 2$, and $\operatorname{ind}(q) = 1$, $s(q) = 0$ for each $q \neq q_0$.

(See Figures (1), (2) and (3)).

§4. Linear proximity and characteristic cones

4.1. In dimension two, the exceptional divisors appearing in the definition of the proximity relations are (rational) curves; in higher dimension we introduce, in the toric case, a condition involving curves, which will be finer, in general, than the proximity.

Definition 4.2. Let $\mathcal{C} = \{Q_0, \dots, Q_n\}$ be a toric constellation. A point Q_j is *linear proximate* to a point Q_i with respect to a one dimensional T-orbit $\ell \subset B_i$ if Q_j belongs to the strict transform in X_j of the closure of ℓ.

This relation is denoted by $Q_j \twoheadrightarrow Q_i$, or $Q_j \overset{\ell}{\twoheadrightarrow} Q_i$ if we need to specify the line ℓ involved.

If $R \twoheadrightarrow Q$ then $R \rightarrow Q$, but the converse does not hold in general.

Proposition 4.3 (*Criterion* for the linear proximity in terms of a codification). *Let Q be a point in a toric constellation of dimension* *d. Each 1-dimensional orbit ℓ in the exceptional divisor B_Q contains in its closure only two fixed points, say $Q(a)$ and $Q(b)$, which determine uniquely ℓ.*

$R \overset{\ell}{\twoheadrightarrow} Q$ if and only if there are integers a, b and m such that $a \neq b$, $1 \leq a \leq d$, $1 \leq b \leq d$, $0 \leq m$ and $R = Q(a, b^{[m]})$ or $R = Q(b, a^{[m]})$, where $x^{[m]}$ means x repeated m times.

Proof. The wall running between the cones corresponding to $Q(a)$ and $Q(b)$ is the cone corresponding to the line defined by this two points in B_Q. The only maximal cones, obtained by elementary subdivisions, having this wall as a face are those corresponding to the points $Q(a, b^{[m]})$ or $Q(b, a^{[m]})$ for some $m \geq 0$. Q.E.D.

In dimension two, proximity and linear proximity are equivalent. One inclusion may be generalized for toric *chains* in any dimension.

Proposition 4.4. *If $\mathcal{C}$ is toric chain (in any dimension), the proximity relation determines the linear proximity relation.*

Proof. If $R \twoheadrightarrow Q$, then $P \rightarrow Q$ for any P such that $R \geq P \geq Q$, $P \neq Q$, and these are the only proximities, for the intermediate points in the chain from Q to R, besides the proximities of consecutive points. Conversely, assuming this property, then $R \overset{\ell}{\twoheadrightarrow} Q$ for the line ℓ determined by the point Q^+ and the direction Q^{++} in the projective space B_Q, if Q^{++} is defined and precedes R, or any line through Q otherwise. Indeed, this assumption forces the code of R to be $Q(a, b^{[m]})$ for some weights a and b, $m \geq 0$. Q.E.D.

On the other hand, in general the linear proximity does not determine the proximity, even for chains.

4.5. We introduce now some definitions leading to the notion of the so called (linear proximity) LP-Enriques diagrams.

Given a rooted tree Γ, a sub graph formed by two chains with a common root and no common edge is called a *bi-chain*.

If Γ is the rooted tree associated with a toric constellation $\mathcal{C}$, q the vertex corresponding to $Q \in \mathcal{C}$ and ℓ is a 1-dimensional orbit in B_Q, then $\Gamma_q(\ell)$ denotes the full subgraph of Γ with vertices corresponding to Q and to the points $R \in \mathcal{C}$ such that $R \overset{\ell}{\twoheadrightarrow} Q$. Let $\Gamma(q)$ be the family of the maximal $\Gamma_q(\ell)$ when ℓ describes the set of one dimensional orbits in B_Q.

A vertex $q \in \Gamma$ is called *simple* (resp. *ramified*) if $|q^+| = 1$ (resp. if $|q^+| > 1$).

The following properties are easily checked with the linear proximity criterion (Proposition 4.3).

Proposition 4.6. *Let C be a toric constellation, Γ the associated tree.*

1. (a) *For each $q \in \Gamma$, the family $\Gamma(q)$ is non-empty and the elements of $\Gamma(q)$ are chains or bi-chains with root q.*
   (b) *If γ, $\gamma' \in \Gamma(q)$ and $\gamma \subset \gamma'$, then $\gamma = \gamma'$.*
2. (a) *Two distinct elements of $\bigcup_q \Gamma(q)$ have at most one common edge.*
   (b) *Two edges with common ramification root vertex q (resp. the edge with the simple root vertex q) belong (resp. belongs) to one and only one element of $\Gamma(q)$.*
3. (a) *For each $q \in \Gamma$ and $r \in q^+$ there is at most one vertex $s \in r^+$ such that the chain (q, r, s) is not contained in any element of $\Gamma(q)$.*
   (b) *If $(p, \dots, q, r)$ is a chain contained in a $\gamma \in \Gamma(p)$ and $s \in r^+$ satisfies* 3. (a), *then the chain $(p, \dots, q, r, s)$ is contained in γ.*

Definition 4.7. The *LP-Enriques diagram* of a toric constellation C is the associated graph Γ_C equiped with the linear proximity structure formed by the family of full subgraphs $\{\Gamma_C(q) \mid q \in \Gamma_C\}$.

Theorem 4.8. *The couple $(\Gamma, \{\Gamma(q) \mid q \in \Gamma\})$, given by a tree Γ and a family of full subgraphs $\Gamma(q)$, is the LP-Enriques diagram of a toric constellation C if and only if the properties* 1, 2 *and* 3 *hold.*

The minimum dimension of the constellations with given PL-Enriques diagram is $d_{\mathcal{PL}} = \max(2, \max_{q \in \Gamma}(|q^+| + n_q))$, where

$$n_q = \max_{r \in q^+} \#\{\gamma \in \Gamma(q) \mid r \in \gamma \text{ and } \gamma \text{ is a chain of length } > 1\}$$

Proof. The proof uses the codification of toric constellations, the linear proximity criterion 4.3 and proceeds by induction on the level of the vertices. Let's check that the dimension given in the statement is the minimum possible dimension; this will also show the essential part of the inductive step. Recall that the length of a chain graph is the number of its edges.

Now assume that the LP-Enriques diagram of a d-dimensional toric constellation C is the given one, and consider a d-weighting of Γ defining C. For each $q \in \Gamma$ one needs d distinct weights for the $|q^+|$ edges with root q, hence d$\geq |q^+|$. Furthermore, for each $r \in |q^+|$ one needs another weight for the second edge of each chain γ of length at least 2 such that

$r \in \gamma \in \Gamma(q)$, and this weight must be different from the $|q^+|$ weights of the edges with root q in order not to get a bi-chain and a contradiction with the conditions 2(a) and 2(b), and also different from the weights of the second edge of the other chains of the same type containing r. This shows that d$\geq (|q^+| + n_q)$.

On the other hand, the bi-chains with root q are automatically weighted once the first edges are weighted, and the second edges of chains with second vertex in q^+ different from r may have the same weights as those of the second vertices of chains through r. The maximality of the elements of $\Gamma(q)$ for each $q \in \Gamma$ and the conditions 3(a) and 3(b) insure that the codification in the inductive step is coherent with the preceding weights, and that the minimum dimension is attained. Q.E.D.

Remark 4.9. A PL-Enriques diagram may be induced by two non-isomorphic constellations. In some cases, for instance if for each vertex q the family $\Gamma(q)$ has only bi-chains or is reduced to the vertex, then the constellation inducing the given PL-Enriques diagram is unique (up to isomorphism of constellations), and its dimension is $|q_0^+|$ if q_0 denotes the root.

The maximum posible linear proximity dimension d_{LP} of a fixed tree, by changing its LP structure, is the number of edges. In this case all the chains (resp. bi-chains) have only one edge (resp. two edges) or are reduced to a vertex, for the maximal ones.

4.10. We recall now some definitions and resume relevant facts on *complete ideal theory* and *characteristic cones* (see [15], [10], [11], [9]), then we give some examples and applications of the results on Enriques diagrams.

Let $\mathcal{I}$ be an ideal in any commutative ring $\mathcal{R}$; an element $x \in \mathcal{R}$ is *integral over* $\mathcal{I}$ if x satisfies a condition of the form $x^n + r_1x^{n-1} + r_2x^{n-2} + \cdots + r_n = 0$ for some $n > 0$ and some $r_j \in \mathcal{I}^j$, $1 \leq j \leq n$. The set of all such x, denoted $\overline{\mathcal{I}}$, is called the *integral closure* or *completion* of $\mathcal{I}$; it is itself an ideal and we have $\mathcal{I} \subset \overline{\mathcal{I}} = \overline{\overline{\mathcal{I}}}$.

The ideal $\mathcal{I}$ is *integrally closed* or *complete* if $\mathcal{I} = \overline{\mathcal{I}}$.

For any two ideals $\mathcal{I}$, $\mathcal{J}$ in $\mathcal{R}$, define the $*$-product $\mathcal{I} * \mathcal{J} := \overline{\mathcal{I}\mathcal{J}}$ (the completion of the product $\mathcal{I}\mathcal{J}$). Then we have $\mathcal{I} * \mathcal{J} = \overline{\mathcal{I}} * \overline{\mathcal{J}}$.

The set of non-zero complete ideals in $\mathcal{R}$ with the $*$-product form a commutative monoid with cancellation (i.e. $\mathcal{I}_1 * \mathcal{J} = \mathcal{I}_2 * \mathcal{J} \Rightarrow \mathcal{I}_1 = \mathcal{I}_2$).

In the following let $\mathcal{R}$ be the local ring $\mathcal{O}_{X,Q_0}$ of the regular variety X at the origin $Q_0 \in X$ of the constellations of infinitely points to consider. In this context, the ideals in $\mathcal{R}$ that we consider are the $\mathcal{M}$-primary ideals , where $\mathcal{M}$ denotes the maximal ideal of R.

An ideal $\mathcal{I}$ is *finitely supported* if $\mathcal{I} \neq (0)$ and if there exists a constellation $\mathcal{C}$ such that $\mathcal{I}\mathcal{O}_{X_\mathcal{C}}$ is a locally principal ideal. The points Q_i, $0 \leq i \leq n$, of the *minimal* constellation $\mathcal{C}_\mathcal{I}$ with this property are called the *base points* of the ideal $\mathcal{I}$.

If $\mathcal{I}$ is a non-zero ideal in $\mathcal{O}_{X_i,Q_i}$, let $\mathrm{ord}_{Q_i}\mathcal{I} = \mathrm{ord}_{Q_i} f$ for a general $f \in \mathcal{I}$.

Define recursively the *weak transform* $\mathcal{I}_i = \mathcal{I}_{Q_i}$ and *the strict multiplicity* (called *point base* in [11]) $m_i = m_{Q_i}$ of $\mathcal{I}$ at a base point Q_i:

$\mathcal{I}_0 = \mathcal{I}$, $m_0 = \mathrm{ord}_{Q_0}\mathcal{I}$ and for $R \in Q^+$ let $\mathcal{I}_R = (x)^{-m_Q}\mathcal{I}_Q\mathcal{O}_{X_R,R}$ and $m_R = \mathrm{ord}_R\mathcal{I}_R$, where $x = 0$ is a local equation of B_Q at R.

The divisor $D_\mathcal{I} = \sum_i m_i E_i^*$ is the divisor defined by $\mathcal{I}$ in $X_\mathcal{C}$, i.e.the divisor $D_\mathcal{I}$ such that $\mathcal{I}\mathcal{O}_{X_\mathcal{C}} = \mathcal{O}_{X_\mathcal{C}}(-D_\mathcal{I})$.

If $\sigma = \sigma_\mathcal{C} : X_\mathcal{C} \to X$ is the composition of the blowing-ups, then the completion $\overline{\mathcal{I}}$ of $\mathcal{I}$ is nothing but the stalk at Q_0 of $\sigma_*(\mathcal{O}_{X_\mathcal{C}}(-D_\mathcal{I}))$. Therefore, if we consider the set of finitely supported *complete* ideals $\mathcal{I}$ with base points contained in $\mathcal{C}$, the map $\mathcal{I} \overset{\alpha}{\longmapsto} D_\mathcal{I}$ from this set into the set of exceptional divisors of $X_\mathcal{C}$, is injective.

On the other hand, the image of the map α is the set of effective exceptional divisors D in $X_\mathcal{C}$, such that $\mathcal{O}_{X_\mathcal{C}}(-D)$ is generated by its global sections in a neighborhood of the support of D (such divisors are called σ-generated), i.e. such that the natural morphism $\sigma^*\sigma_*\mathcal{O}_{X_\mathcal{C}}(-D) \to \mathcal{O}_{X_\mathcal{C}}(-D)$ is surjective.

The map α is actually a monoid isomorphism onto its image, with respect to the $*$-product and the addition of divisors, i.e. $\alpha(\mathcal{I}_1 * \mathcal{I}_2) = \alpha(\mathcal{I}_1) + \alpha(\mathcal{I}_2)$.

Remark 4.11. This map is analyzed in [2] in terms of *clusters*, i.e. weighted constellations, where the weights are the strict multiplicities $\underline{m} = (m_i \mid Q_i \in \mathcal{C})$ defined for a finitely supported (complete) ideal. The clusters corresponding to images of α are called idealistic clusters.

4.12. Let $N_1 = N_1(X_\mathcal{C}/X)$ (resp.$N^1 = N^1(X_\mathcal{C}/X)$) be the abelian group of exceptional – i.e. whose support contracts to Q_0 – one dimensional cycles on $X_\mathcal{C}$ (resp. (Cartier) divisors on $X_\mathcal{C}$) modulo numerical equivalence. A one dimensional cycle C (resp. a divisor D) is numerically equivalent to 0 if the intersection number $(C \cdot D) = 0$, for all divisors D (resp. all exceptional complete curves) on $X_\mathcal{C}$.

Set $A_1 = A_1(X_\mathcal{C}/X) = N_1 \otimes_\mathbb{Z} \mathbb{R}$, $A^1 = A^1(X_\mathcal{C}/X) = N^1 \otimes_\mathbb{Z} \mathbb{R}$. The exceptional fiber $\sigma^{-1}(Q_0)$ of σ is a projective scheme over $\mathbb{K}$ and the vector space A^1 maps injectively into $A^1(\sigma^{-1}(Q_0))$, so the dimension of A^1 is finite and the intersection pairing makes A_1 and A^1 dual vector spaces.

Now let $NE(X_{\mathcal{C}}/X)$ be the convex cone generated by the effective exceptional curves in A_1. Consider, in the dual space A^1, two cones:

Definition 4.13. Let $P(X_{\mathcal{C}}/X)$ be the dual cone of $-NE(X_{\mathcal{C}}/X)$, i.e. the cone formed by the classes d such that $(c \cdot d) \leq 0$ for every class c of exceptional effective curve in $X_{\mathcal{C}}$. In other words, $P(X_{\mathcal{C}}/X)$ is minus the *semiample relative cone* for the morphism σ at Q_0 (see [9]).

Let $\tilde{P}(X_{\mathcal{C}}/X)$ be the convex cone generated by the classes of divisors D such that $\mathcal{O}_{X_{\mathcal{C}}}(-D)$ is generated by their global sections; this cone is called the *characteristic cone* for σ at Q_0; this terminology has been introduced by H. Hironaka. We'll say that this cone is associated with $\mathcal{C}$.

Remark 4.14. The set of lattice points of the characteristic cone $\tilde{P}(X_{\mathcal{C}}/X)$ (i.e. its intersection with the lattice N^1) is the monoid of the classes of σ-generated divisors on $X_{\mathcal{C}}$. This monoid generates the characteristic cone and is canonically isomorphic to the monoid of finitely supported complete ideals in $\mathcal{O}_{X,Q_0}$ with base points contained in the constellation $\mathcal{C}$, via the map α introduced above. (In [2] this monoid is called the *galaxy* of the constellation).

Definition 4.15. Following Lipman [11], define a $*$-simple ideal as an ideal $\mathcal{P}$ in $\mathcal{R}$ if $\mathcal{P} \neq \mathcal{R}$ and if $\mathcal{P}$ does not have a non-trivial $*$-factorization, i.e. if $\mathcal{P} = \mathcal{I}_1 * \mathcal{I}_2$ then either $\mathcal{I}_1 = \mathcal{R}$ or $\mathcal{I}_2 = \mathcal{R}$.

4.16. The central result on unique factorization in [11] may be formulated as follows.

For each point $Q \in \mathcal{C}$ there is a (unique complete) $*$-simple ideal $\mathcal{P}_Q$ whose base points are the points of the chain from Q to the origin Q_O, such that the set of strict multiplicities is minimal for the inverse lexicographic order with respect to the natural ordering ($\geq$) in $\mathcal{C}$, and with $m_Q = 1$. Let's call $\mathcal{P}_Q$ the *special $*$-simple ideal* associated with Q

It follows from this fact, that every finitely supported complete $\mathcal{I}$ with base points contained in $\mathcal{C}$ has a *unique factorization* as a $*$-product of the special $*$-simple ideals $\mathcal{P}_Q$, allowing negative exponents. This means that there is a $*$-product of $\mathcal{I}$ by special $*$-simple ideals, equal to a product of special $*$-simple ideals, with different factors appearing in each side of the equality; i.e. there are unique integers r_Q such that $\mathcal{I} = \overset{*}{\prod}_{Q \in \mathcal{C}} \mathcal{P}_Q^{r_Q}$.

Indeed, it is straightforward to see, from the properties of the strict multiplicities, of the $\mathcal{P}_Q$, that the set $\alpha(\mathcal{P}_Q)$, $Q \in \mathcal{C}$, is a basis of the lattice N^1.

4.17. The above factorisation for any finitely generated complete ideal with base points in $\mathcal{C}$ has non negative exponents if and only if the $\alpha(\mathcal{P}_Q)$, with $Q \in \mathcal{C}$ generate the characteristic cone of $\sigma_{\mathcal{C}}$, and this is equivalent to saying that this cone is *regular*, in the sense of toric varieties, i.e. rational and with primitive integral extremal points forming (a subset of) a basis of the lattice.

The determination of the constellations with *regular* characteristic cone is an interesting open question.

The result of Zariski (see [14], [15] and [12] for a recent presentation) is formulated in this language by saying that in dimension two the characteristic cone is always regular. Furthermore, the $*$-product is just the product of ideals, and the $*$-simple ideals are the simple ideals.

Examples 4.18. There are chain constellations whose characteristic cone is polyhedral but not simplicial, or even non polyhedral, i.e. the simplicial cone is not closed (see for instance [4] example 3, [1] examples 4.1, 4.2, 4.3).

Even for chain constellations in dimension three the characteristic cone may be non regular. One could hope that for this "simple" case life would be easy, since for the toric chains it is always regular (see 4.25), as it follows from the linear proximities.

For example, a chain constellation consisting of six points with the second point Q_1 in a non degenerate conic in the plane B_0 and the following points in the strict transform of the conic, has a characteristic cone (in dimension six) with seven maximal faces and nine edges generated by the six integral points associated to the special $*$-simple ideals and three others.

Another example is given by a regular non inflexion point on a rational plane cubic curve in B_0 and eight following points on the strict transform of the cubic; in this case the monoid of integral points of the characteristic cone in dimension ten is not finitely generated.

4.19. For toric constellations the characteristic cone may be explicitely obtained (see [2], theorem 2.10). Note that in this case the characteristic cone coincides with the semiample cone (see [8], page 47).

The natural ideals to consider are the invariant ideals for the toric action, so that the constellations of base points are toric.

The conditions that such an ideal $\mathcal{I}$ is finitely generated and complete are formulated in terms of the Newton polyhedron $\mathcal{N}$ of $\mathcal{I}$ relative to the local system of parameters of the local ring, induced by a basis of the lattice where the fan lives.

The first condition is that the fan associated to the Newton polyhedron (which gives the normalized blowing-up of center $\mathcal{I}$) admits a regular subdivision obtained by elementary subdivisions of the regular cone Δ corresponding to Q_0; and the second one is that every monomial corresponding to an integral point of $\mathcal{N} + \Delta^\vee$ is in $\mathcal{I}$, where $\Delta^\vee$ denotes the dual cone of Δ.

The following result generalizes, for toric constellations in any dimension, the two dimensional proximity inequalities found by Enriques. Recall Proposition 4.3.

Theorem 4.20. *Let C be a toric constellation of dimension d. The characteristic cone associated with C is the cone generated by the classes of the divisors $D_{\underline{m}} = \sum_{Q\in C} m_Q E_Q^*$ such that $\underline{m}$ verifies the linear proximity inequalities $m_Q \geq \sum_{P \overset{\ell}{\twoheadrightarrow} Q} m_P$ for each $Q \in C$ and each $\ell = \ell(Q(a), Q(b))$, $a \neq b$ $1 \leq a \leq d$, $1 \leq b \leq d$.*

Proof. The linear proximity inequalities are necessary, since they are equivalent to $(D_{\underline{m}} \cdot \bar{\ell}) \leq 0$ for a semiample divisor $-D_{\underline{m}}$ and the closure $\bar{\ell}$ of each one dimensional orbit $\ell(Q(a), Q(b))$. Conversely, if these inequalities hold, then $-D_{\underline{m}}$ is semiample since the classes of the closures of the one dimensional orbits generate the cone of the numerically effective curves NE, and then the divisor is σ-generated because σ is a toric morphism. Q.E.D.

Remark 4.21. A constructive proof giving the Newton polyhedron of the unique complete ideal associated to such a divisor $D_{\underline{m}}$ (or the corresponding *idealistic cluster*) is presented in [2] theorem 2.10 (ii).

Corollary 4.22. *We keep the notations of the theorem. Let $C = \{Q_0, \dots, Q_n\}$ be a toric* chain.

(*a*) *The characteristic cone asociated with C is given by*

$$m_i \geq \sum_{j \twoheadrightarrow i} m_j \ , \ 0 \leq i \leq n.$$

(*b*) *The divisor $D_n = \sum_{0 \leq i \leq n} m_{i,n} E_i^*$ associated to the special $*$-simple ideal $\mathcal{P}_{Q_n}$ is given by $m_{n,n} = 1$, $m_{i,n} = \sum_{j \twoheadrightarrow i} m_{j,n}$, for $0 \leq i \leq n$.*

Proof. (a) follows from the Theorem and the fact that for each point there is only one relevant inequality, since C is a chain.

(b) follows from (a) since the minimality property of $\underline{m}$ is obtained if $m_{n,n} = 1$ and if every inequality involving an index $i \neq n$ becomes an equality. Q.E.D.

The special $*$-simple ideals, and the exponents of the factorizations are determined by the linear proximities:

Theorem 4.23. *Let $\mathcal{C}$ be a toric constellation.*

(a) Let $(D_Q)_{Q\in\mathcal{C}}$, be the basis of N_1 corresponding to the special $*$-simple ideals with base points in $\mathcal{C}$. Then $D_Q = \sum_{P\in\mathcal{C}} m_{PQ}E_P^*$, where $m_{PQ} = 0$ if $P \not\leq Q$, $m_{QQ} = 1$ and $m_{PQ} = \sum_{R\in\mathcal{C}\ \mid\ Q\geq R\twoheadrightarrow P} m_{RQ}$ if $P \leq Q$.

(b) Let $\mathbb{P}_L = ((l_{PQ}))$ be the linear proximity matrix *defined by $l_{PP} = 1$, $l_{PQ} = -1$ if $P \twoheadrightarrow Q$ and 0 otherwise.*

Then ${}^t\mathbb{P}_L$ is the basis change matrix from (E_Q^*) to (D_Q).

(c) Let $\mathcal{I}$ be a toric finitely generated ideal with base points in $\mathcal{C}$.

Then the exponents of its factorisation in terms of special $*$-simple ideals are:

$r_Q = m_Q - \sum_{P\twoheadrightarrow Q} m_P$.

Proof. (a) follows from 4.22, (b).

(b) and (c) follow from (a) and linear algebra. Q.E.D.

Recall the definition of the LP structure of the tree Γ associated with $\mathcal{C}$ (Proposition 4.6).

Corollary 4.24. *Let $P_{\mathcal{C}} = P(X_{\mathcal{C}}/X)$ be the characteristic cone associated with $\mathcal{C}$. The following conditions are equivalent:*

(a) The cone $P_{\mathcal{C}}$ is regular.

(b) $(D_Q)_{Q\in\mathcal{C}}$ is a basis of the semigroup $P_{\mathcal{C}} \bigcap N^1$.

(c) The cone $P_{\mathcal{C}}$ is simplicial.

(d) The special $*$-simple factorizations have only non negative exponents.

(e) For each $Q \in \mathcal{C}$ there is only one (maximal) chain or bichain in $\Gamma(q)$.

Proof. The conditions (a), (b), (c) and (d) are equivalent since the divisors D_Q form a basis of N^1. The equivalence between (e) and (c) follows from the preceding theorem, and the fact that the supporting hyperplanes of the maximal faces of the cone $P_{\mathcal{C}}$ are those associated with the maximal elements of Γ_Q for each $Q \in \mathcal{C}$. Q.E.D.

Remark 4.25. In particular, every toric chain constellation in any dimension has a regular characteristic cone. There are also non-chain constellations with this property.

We conclude with an application of the LP-Enriques diagrams for a converse Zariski theorem for toric constellations. Recall the definition of the minimal LP-dimension, $d_{\mathcal{LP}}$ of a LP-Enriques diagram (Theorem 4.8).

Theorem 4.26. *The characteristic cone of a toric constellation is* regular *if and only if its LP-Enriques diagram is induced by a two dimensional constellation.*

Proof. The characteristic cone of any two dimensional constellation is regular, by Zariski. Conversely, assume that the characteristic cone is regular. Then $\Gamma(q)$ has only one element for each $q \in \Gamma$, by the last Corollary. It follows necessarily that $0 \leq |q^+| \leq 2$. Now, $0 \leq |q^+| \leq 1$ implies that $0 \leq n_q \leq 1$ and $|q^+| = 2$ implies that $n_q = 0$. It follows that the minimal dimension $d_{\mathcal{LP}}$ of a constellation inducing the given LP-Enriques diagram is two. Q.E.D.

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The quotients of log-canonical singularities by finite groups

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Abstract.

In this paper we study the quotient of an isolated strictly log-canonical singularity by a finite group. As a result, we obtain the boundedness of indices of these singularities of dimension 3 and determine all possible indices. We also determine the ramification indices of the quotient map of a 2-dimensional strictly log-canonical singularities by a finite group.

§1. Introduction

A log-canonical, non-log-terminal singularity is called strictly log-canonical. Let (X, x) be an isolated strictly log-canonical singularity over $\mathbb{C}$. If its dimension is 2, then the index is 1, 2, 3, 4 or 6. This is observed by checking the list of the weighted dual graphs of all strictly log-canonical singularities. This is also proved by Shokurov [21] by means of complements and by Okuma [18] by means of plurigenera. In the 3-dimensional case, the author heard that the boundedness of indices of such singularities is proved by Shokurov in [22]. In this paper, we study the quotient of isolated strictly log-canonical singularities by finite group actions. First, in case the group acts freely in codimension 1, we obtain a formula for the indices of the quotient singularity (Lemma 3.3). By this formula, we obtain a different proof of the above fact on indices for dimension 2. We then prove that the index of 3-dimensional strictly log-canonical singularity is less than or equal to 66. More precisely, a positive integer r can be the index of such a singularity if and only if $\varphi(r) \leq 20$ and $r \neq 60$, where φ is the Euler function. This is related to the finite automorphisms on $K3$-surfaces, Abelian surfaces and elliptic

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curves. Next we study finite groups which act non-freely in codimension 1. For the 2-dimensional case, we determine the quotients by these groups with the branch divisors. Thus it follows that the ramification index of each ramification divisor is 2, 3, 4 or 6.

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§2. Isolated strictly log-canonical singularities.

2.1. Isolated strictly log-canonical singularities are studied in [6]. In this section we summarize those results and add some basic facts on these singularities.

Definition 2.2. Let (X, x) be a germ of normal singularity. If there is an integer r such that $\omega_X^{[r]}$ is invertible, the singularity is called a $\mathbb{Q}$-*Gorenstein singularity*. We call the minimum positive such number r the *index* of (X, x) and denote by $\mathrm{Ind}(X, x)$.

Definition 2.3. A $\mathbb{Q}$-Gorenstein singularity (X, x) is called a *log-canonical singularity* (resp. *log-terminal singularity*) if for a good resolution $f : Y \to X$ the canonical divisor on Y has an expression in $\mathrm{Div}(Y) \otimes \mathbb{Q}$:

$$K_Y = f^*K_X + \sum_i m_i E_i$$

with $m_i \geq -1$ (resp. $m_i > -1$) for every irreducible exceptional divisor E_i with $x \in f(E_i)$. Here a good resolution means a resolution whose exceptional set is a normally crossing divisor with the non-singular irreducible components. We call m_i the *discrepancy* over X at E_i or the *discrepancy* for f at E_i for each irreducible component E_i.

2.4. In the case of index 1, a strictly log-canonical singularity is equivalent to a purely elliptic singularity ([6]). In this case we define the essential divisor in the exceptional divisor of a good resolution. It actually plays an essential role in the exceptional divisor (cf. Lemma 3.7 [6]).

Definition 2.5. Let (X, x) be an isolated strictly log-canonical singularity of index 1 and $f : Y \to X$ a good resolution. Then one has a representation

$$K_Y = f^*K_X + \sum_{i\in I} m_i E_i - \sum_{j\in J} E_j,$$

with $m_i \geq 0$, $I \cap J = \emptyset$ and $J \neq \emptyset$. The divisor $E_J := \sum_{j\in J} E_j$ is called the *essential divisor* for a good resolution f.

2.6. Let (X, x) be an n-dimensional isolated strictly log-canonical singularity of index 1 and $f : Y \to X$ a good resolution with the essential divisor E_J. Since E_J is a complete variety with normal crossings,

$$H^{n-1}(E_J, \mathcal{O}_{E_J}) \simeq Gr_F^0 H^{n-1}(E_J, \mathbb{C}) = \bigoplus_{i=0}^{n-1} H_{n-1}^{0,i}(E_J),$$

where F is the Hodge filtration and $H_m^{i,j}(*)$ is the (i,j)-Hodge-component of $H^m(*, \mathbb{C})$. As the left hand side is a 1-dimensional $\mathbb{C}$-vector space (Lemma 3.7 [6]), it must coincide with one of $H_{n-1}^{0,i}(E_J)$ $(i = 0, 1, 2, \ldots, n-1)$.

Definition 2.7. An n-dimensional isolated strictly log-canonical singularity (X, x) of index 1 is said to be of type $(0, i)$, if $H^{n-1}(E_J, \mathcal{O}_{E_J}) = H_{n-1}^{0,i}(E_J)$.

2.8. The type is independent of the choice of a good resolution (Proposition 4.2 in [6]).

Example 2.9. A 2-dimensional srictly log-canonical singularity (X, x) of index 1 is of type $(0, 1)$ if and only if (X, x) is a simple elliptic singularity and of type $(0, 0)$ if and only if it is a cusp singularity.

Proposition 2.10. *Let (X, x) be a 3-dimensional isolated strictly log-canonical singularity of index 1 and of type $(0, 2)$ and $f : Y \to X$ the canonical model, i.e. Y has at worst canonical singularities and K_Y is f-ample. Let D be the exceptional divisor of f with the reduced structure. Then Y has at worst terminal singularities and D is isomorphic to either a normal $K3$-surface or an Abelian surface. Here a normal $K3$-surface is a normal surface whose minimal resolution is a $K3$-surface.*

Proof. First note that E_J is irreducible by Lemma 6, [8]. Since the discrepancy for f at each exceptional component is negative (the proof of Lemma 3.7 [8]), D is irreducible. Let $g : Y' \to Y$ be a proper birational morphism whose composite $f \circ g : Y' \to X$ is a good resolution. One sees that Y has at worst terminal singularities. Indeed, if not, there exists an exceptional divisor E_0 which is crepant for g. Then the discrepancy at E_0 for $f \circ g$ is less than 0, so E_0 becomes another component of the essential divisor, which is a contradiction. Now one can prove that Y is non-singular away from finite points. If D has 1-dimensional singular locus, then by the blowing-up at a 1-dimensional irreducible component of the singular locus one obtains a component E_1 whose discrepancy for $f \circ g$ is $-m+1<0$, where m is the multiplicity of D at a general point on the curve. It implies that E_1 is another component of the essential divisor, which is a contradiction. Therefore D is non-singular away from finite points. On the other hand, since $\omega_Y \simeq \mathcal{O}_Y(-D)$ is Cohen-Macaulay, so is D. Hence by Serre's criterion D is normal. The condition $\omega_Y \simeq \mathcal{O}_Y(-D)$ yields that $\omega_D \simeq \mathcal{O}_D$. A normal surface with this condition and $H^2(E_J, \mathcal{O}_{E_J}) = \mathbb{C}$, where E_J is a resolution of D, is either a normal $K3$-surface or an Abelian surface ([23]). Q.E.D.

§3. Finite groups which act freely in codimension 1.

Definition 3.1. Let G be a group and (X,x) a germ of a singularity. We say that G acts on (X,x) if G acts on a neighbourhood of x and fixes the point x. We say that G acts on (X,x) freely in codimension 1, if there exists a closed subset S of codimension greater than or equal to 2 on a neighbourhood X such that G acts freely on $X \setminus S$.

3.2. We denote the set of non-singular points of X by X_{reg}. Let (X,x) be a $\mathbb{Q}$-Gorenstein singularity of index m and a group G act on (X,x). We denote the germ $(X/G, x')$ by $(X,x)/G$, where $x' \in X/G$ is the image of x. Denote the maximal ideal of x by $\mathfrak{m}_x$. Then it induces a canonical representation

$$\rho : G \to GL(\omega_X^{[m]}/\mathfrak{m}_x\omega_X^{[m]}) \simeq \mathbb{C}^*.$$

because G fixes the point x.

Lemma 3.3. *Let (X,x) be a $\mathbb{Q}$-Gorenstein normal singularity of index m. Let G be a finite group which acts on (X,x) freely in codimension 1 and $\rho : G \to GL(\omega_X^{[m]}/\mathfrak{m}_x\omega_X^{[m]}) \simeq \mathbb{C}^*$ the canonical representation. Then*

$$\mathrm{Ind}((X,x)/G) = m|\operatorname{Im}\rho|.$$

In particular,

$$\mathrm{Ind}((X,x)/G) \le m|G|.$$

Proof. Denote the order of G by d, $|\operatorname{Im}\rho|$ by r and $\operatorname{Ind}((X,x)/G)$ by I. Let g be a generator of $\operatorname{Im}\rho$ and ϵ the primitive r-th root of 1 which corresponds to g. Let ω be a generator of $\omega_X^{[m]}$.

By the pull-back of a generator of $\omega_{X/G}^{[I]}$, one has a G-invariant I-ple n-form θ which is holomorphic and does not vanish on X_{reg}. Therefore $I = mm'$ for some $m' \in \mathbb{N}$ and $\theta = h\omega^{\otimes m'}$, where h is a nowhere vanishing holomorphic function on X. Since $\theta^g = \theta$ as an element of $\omega_X^{[I]}/\mathfrak{m}_x\omega_X^{[I]}$, one obtains that $\epsilon^{m'}h(x)\omega^{\otimes m'} = h(x)\omega^{\otimes m'}$. Hence $\epsilon^{m'} = 1$. This shows $I \geq mr$. Next, to prove $I \leq mr$, we construct a G-invariant mr-ple n-form which is holomorphic and does not vanish on X_{reg}. Denote an element of G which corresponds to $g \in \operatorname{Im}\rho$ by the same symbol g. Let θ be an mr-ple n-form $\omega \otimes \omega^g \ldots \otimes \omega^{g^{r-1}}$ and $\tilde{\theta}$ be $(1/d)\sum_{\sigma\in G}\theta^\sigma$. Then $\tilde{\theta}$ is an invariant mr-ple n-form. Let $\rho(\sigma) = g^i$ for $\sigma \in G$. Then in $\omega_X^{[mr]}/\mathfrak{m}_x\omega_X^{[mr]}$, $\theta^\sigma = \epsilon^{ri+(1+2+\ldots+r-1)}\omega^{\otimes r}$ which is $\omega^{\otimes r}$ if r is odd and $-\omega^{\otimes r}$ if r is even. Therefore $\tilde{\theta} = \pm\omega^{\otimes r} + \lambda$, where $\lambda \in \mathfrak{m}_x\omega_X^{[mr]}$. Since $\tilde{\theta} \notin \mathfrak{m}_x\omega_X^{[mr]}$, $\tilde{\theta}$ does not vanish on X_{reg}, which shows that $\tilde{\theta}$ is a required form. Q.E.D.

Corollary 3.4. *Let (X,x) be an isolated strictly log-canonical singularity of index 1 on which a finite group G acts. Let $f : \tilde{X} \to X$ be a G-equivariant resolution of the singularities and $\rho : G \to GL(\omega_X/f_*\omega_X) \simeq \mathbb{C}$ the induced representation. Then* $\operatorname{Ind}((X,x)/G) = |\operatorname{Im}\rho|$.

Proof. For an isolated strictly log-canonical singularity of index 1, it follows that $\mathfrak{m}_x\omega_X = f_*\omega_{\tilde{X}}$. Q.E.D.

Corollary 3.5. *Let (X,x) be an n-dimensional isolated strictly log-canonical singularity of index 1 on which a finite group G acts. Assume that there exists the canonical model $\varphi : X' \to X$ and let E be the reduced exceptional divisor. Then the action induces a representation $\rho : G \to GL(H^{n-1}(E,\mathcal{O}_E))$ and* $\operatorname{Ind}(X,x)/G = |\operatorname{Im}\rho|$.

Proof. Take a G-equivariant resolution $f : \tilde{X} \to X$. Then $\bigoplus_{m\geq 0} f_*\omega_{\tilde{X}}^{\otimes m}$ admits the action of G. So the canonical model admits the equivariant action of G, therefore the exceptional divisor E also does. Since $\omega_{X'} \simeq \mathcal{O}_{X'}(-E)$ (proof of Lemma 7 of [8]) and X' is Gorenstein in codimension 2, E is Cohen-Macaulay and $\omega_E \simeq \mathcal{O}_E$. These yield that $H^{n-1}(E,\mathcal{O}_E) = \mathbb{C}$. As $R^{n-1}\varphi_*\mathcal{O}_{X'} \simeq R^{n-1}f_*\mathcal{O}_{\tilde{X}} \simeq \mathbb{C}$, the surjection $R^{n-1}\varphi_*\mathcal{O}_{X'} \to H^{n-1}(E,\mathcal{O}_E)$ is an isomorphism. On the other hand $R^{n-1}f_*\mathcal{O}_{\tilde{X}}$ is dual to $\omega_X/f_*\omega_{\tilde{X}}$, on which one can apply Corollary 3.4. Q.E.D.

Corollary 3.6. *Let (X,x) be an n-dimensional isolated strictly log-canonical singularity of index 1 on which a finite group G acts. Let $f : Y \to X$ be a G-equivariant good resolution and E_J the essential divisor. Then the action induces a representation $\rho : G \to GL(H^{n-1}(E_J, \mathcal{O}_{E_J}))$ and* $\mathrm{Ind}(X,x)/G = |\,\mathrm{Im}\,\rho|$.

Proof. It is clear that G acts on E_J. Since E_J is the essential divisor, $R^{n-1}f_*\mathcal{O}_{X'} \simeq H^{n-1}(E_J, \mathcal{O}_{E_J})$ by Lemma 3.7 [6]. On the other hand $R^{n-1}f_*\mathcal{O}_{\tilde{X}}$ is dual to $\omega_X/f_*\omega_{\tilde{X}}$, on which one can apply Corollary 3.4. Q.E.D.

§4. Index of isolated strictly log-canonical singularities

4.1. In this section, one proves that the indices of isolated strictly log-canonical singularities of dimension 2 and 3 are determined. Here one should note that the boundedness of indices does not hold for log-terminal singularities and non-log-canonical singularities even for 2-dimensional case.

Example 4.2. (1) Let (Z_m, z_m) be the cyclic quotient singularity $\mathbb{C}^2/G$, where G is generated by

$$\begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon \end{pmatrix}.$$

Here ϵ is a primitive m-th root of unity. Then the exceptional curve on the minimal resolution is $\mathbb{P}^1$ and its self-intersection number is $-m$. Therefore the index of (Z_m, z_m) is m if m is odd and $m/2$ if m is even. This shows that the indices of log-terminal singularities are not bounded.

(2) Let $(X,x) \subset (\mathbb{C}^3, 0)$ be a hypersurface singularity defined by $x^4+y^4+z^4 = 0$ and (Z_m, z_m) is its quotient by the cyclic group generated by

$$\begin{pmatrix} \epsilon & 0 & 0 \\ 0 & \epsilon & 0 \\ 0 & 0 & \epsilon \end{pmatrix},$$

where ϵ is a primitive m-th root of unity. Then the index of (Z_m, z_m) is m. This shows that the indices of non-log-canonical singularities are not bounded.

4.3. Let $\pi : (X,x) \to (Z,z)$ be a finite morphism étale in codimension 1. Then (X,x) is strictly log-canonical if and only if (Z,z) is (see for example Proposition 1.7, [7]). Hence by the canonical cover, an arbitrary strictly log-canonical singularity is regarded as the quotient of such a singularity of index 1 by a finite group which acts on the singularity freely in codimension 1.

Definition 4.4. An isolated strictly log-canonical singularity is called of type $(0, i)$, if its canonical cover is of type $(0, i)$.

Theorem 4.5. *An arbitrary dimensional isolated strictly log-canonical singularity of type* $(0,0)$ *has index either* 1 *or* 2.

Proof. This is proved in Theorem 3.10, [7]. One can also prove it by using 3.6. Let $\pi : (X, x) \to (Z, z)$ be the canonical cover of an n-dimensional isolated strictly log-canonical singularity (Z, z) and $G = \langle g \rangle$ the associated cyclic group. Let $f : \tilde{X} \to X$ be a G-equivariant good resolution of (X, x) such that $\pi \circ f$ factors through a good resolution $g : \tilde{Z} \to Z$ of (Z, z). Denote the essential divisor for f by E_J and its dual complex by Γ. Then g induces an automorphism g^* on $H^{n-1}(\Gamma, \mathbb{Z})$. Since (X, x) is of type $(0,0)$, $\mathbb{C} \simeq H^{0,0}_{n-1}(E_J)$ and this is isomorphic to $H^{n-1}(\Gamma, \mathbb{C})$ by 2.5, [12]. Therefore $H^{n-1}(\Gamma, \mathbb{Z})$ is of rank 1. Let λ be a free generator of $H^{n-1}(\Gamma, \mathbb{Z})$ Then $g^*(\lambda) = \pm\lambda + (torsion)$ in $H^{n-1}(\Gamma, \mathbb{Z})$. Therefore $g^*(\lambda) = \pm\lambda$ in $H^{n-1}(\Gamma, \mathbb{C})$. Hence the order of the action of G on $H^{n-1}(E_J, \mathcal{O}_{E_J})$ is 1 or 2. Now apply 3.6. Q.E.D.

4.6. A non-singular projective variety X is called a Calabi-Yau variety, if it satisfies that $\omega_X \simeq \mathcal{O}_X$. It is well known that a 1-dimensional Calabi-Yau variety is an elliptic curve and 2-dimensional one is either a $K3$-surface or an Abelian surface. An automorphism g on X induces a linear automorphism g^* on $\Gamma(X, \omega_X) = \mathbb{C}$ which is dual to $H^n(X, \mathcal{O}_X)$, where $n = \dim X$. Now let us introduce a conjecture on finite automorphisms on Calabi-Yau varieties, which is essential to our problem.

Conjecture 4.7. For $n \in \mathbb{N}$, there is a number B_n such that n-dimensional Calabi-Yau variety X and a finite automorphism g on X, the order of the induced automorphism g^* on $H^n(X, \mathcal{O}_X) = \mathbb{C}$ is bounded by B_n.

For $n = 1, 2$, the conjecture holds true.

Proposition 4.8. *For an arbitrary elliptic curve* X*, denote the order* $|\operatorname{Im}\rho|$ *by* r*, where* $\rho : \operatorname{Aut}(X) \to GL(H^1(X, \mathcal{O}_X)) = \mathbb{C}^*$ *is the induced representation. Then* $\varphi(r) \leq 2$*, which means* $r =$ 1, 2, 3, 4 *or* 6.

Proof. This is a classical result and proved in various ways. For example, note that an automorphism of X is the composite of a group homomorphism and a translation. Since the translation has no effect on $H^1(X, \mathcal{O}_X) = \mathbb{C}$, $\operatorname{Im}\rho$ is $\rho(\operatorname{Aut}(X, 0))$, where $\operatorname{Aut}(X, 0)$ is the group of automorphisms. Since $\mathrm{Aut}(X, 0)$ fixes the zero element of the group, it is a finite group of order 1, 2, 4 or 6 (see, for example, IV, 4.7, [5]). Q.E.D.

Proposition 4.9. (i) (10.1.2, [16]) *For an arbitrary $K3$-surface X, denote the order $|\mathrm{Im}\,\rho|$ by r, where $\rho : \mathrm{Aut}(X) \to GL(H^2(X, \mathcal{O}_X)) = \mathbb{C}^*$ is the induced representation. Then $\varphi(r) \leq 20$, in particular $r \leq 66$. Here φ is the Euler function.*

(ii) (3.2, [4]) *For an arbitrary Abelian surface X, the order r of a finite automorphism on X satisfies $\varphi(r) \leq 4$, which means that $r = 1$, 2, 3, 4, 5, 6, 8, 10, 12.*

Now one obtains a new proof of the following result.

Theorem 4.10. *A 2-dimensional strictly log-canonical singularity has index $1, 2, 3, 4$ or 6.*

Proof. Let $\pi : (X, x) \to (Z, z)$ be the canonical cover of the strictly log-canonical singularity (Z, z) and G be the associated cyclic group. By 4.5, it is sufficient to prove the case that (X, x) is of type $(0, 1)$. Let $f : Y \to X$ be the minimal resolution and E the exceptional curve. Then f is a G-equivariant good resolution with the essential divisor E which is an elliptic curve. By 4.8, $|\mathrm{Im}\,\rho| = 1$, 2, 3, 4 or 6, where $\rho : G \to GL(H^1(E, \mathcal{O}_E)) = \mathbb{C}^*$ is the induced representation. Now apply 3.6. Q.E.D.

Theorem 4.11. *An isolated 3-dimensional strictly log-canonical singularity of type $(0, 2)$ has index r, where $\varphi(r) \leq 20$.*

Proof. Let $\pi : (X, x) \to (Z, z)$ be the canonical cover of a 3-dimensional strictly log-canonical singularity (Z, z) and G the associated cyclic group. Let E be the exceptional divisor on the canonical model of X. Then by 2.10 E is either a normal $K3$-surface or an Abelian surface. Note that the action of G on E is lifted onto the minimal resolution $\tilde{E}$ of E. Since the singularities on E are at worst rational double, one obtains that $\Gamma(E, \omega_E) = \Gamma(\tilde{E}, \omega_{\tilde{E}})$. By the Serre duality, the action of G on $H^2(E, \mathcal{O}_E)$ is the same as the one on $H^2(\tilde{E}, \mathcal{O}_{\tilde{E}})$. Therefore by 3.5 and 4.9 $r = \mathrm{Ind}(Z, z)$ satisfies $\varphi(r) \leq 20$. Q.E.D.

Theorem 4.12. *An isolated 3-dimensional strictly log-canonical singularity of type $(0, 1)$ has index $1, 2, 3, 4$ or 6.*

4.13. For the proof of Theorem 4.12 one needs the discussion on the following divisor: Let E_J be a simple normal crossing divisor on a non-singular 3-fold. Assume $E_J = E_1 + E_2 + \ldots + E_s$ is a cycle of elliptic ruled surfaces E_i and every intersection curve is a section on the ruled surfaces. Decompose E_J into two connected chains $E^{(i)}$ $(i = 1, 2)$

with no common components. Let C_1 and C_2 be the irreducible curves of $E^{(1)} \cap E^{(2)}$. Let $p : E^{(1)} \to C$ and $q : E^{(2)} \to C$ be the rulings and $p_i : C_i \to C$ be the restriction of p on C_i. Then one obtains the Mayer-Vietoris exact sequence:

$$\begin{aligned} H^1(E^{(1)}, \mathbb{C}) \oplus H^1(E^{(2)}, \mathbb{C}) \to H^1(C_1, \mathbb{C}) &\oplus H^1(C_2, \mathbb{C}) \\ &\to H^2(E_J, \mathbb{C}) \to 0, \end{aligned}$$

which is an exact sequence of mixed Hodge structure. By taking Gr_F^0, where F is the Hodge filtration, one obtains the following:

$$\begin{aligned} H^1(E^{(1)}, \mathcal{O}) \oplus H^1(E^{(2)}, \mathcal{O}) \xrightarrow{\Phi} H^1(C_1, \mathcal{O}) &\oplus H^1(C_2, \mathcal{O}) \\ &\xrightarrow{\Psi} H^2(E_J, \mathcal{O}) \to 0. \end{aligned}$$

Lemma 4.14. *Assume that $H^2(E_J, \mathcal{O}) = \mathbb{C}$. Let $\Phi|_{H^1(E^{(i)}, \mathcal{O})} = \varphi_i$ and $\Psi|_{H^1(C_i, \mathcal{O})} = \psi_i$. Then the following hold:*

(i) $\operatorname{Im} \varphi_1 = \operatorname{Im} \varphi_2 = \operatorname{Im} \Phi$;

(ii) *ψ_i is an isomorphism for $i = 1, 2$ and* $\operatorname{Ker} \Psi \circ (p_1^* \oplus p_2^*) = \Delta$, *where Δ is the diagonal subspace of $H^1(C, \mathcal{O}) \oplus H^1(C, \mathcal{O})$;*

(iii) *fix C_1, then the isomorphism ψ_1 is independent of the choice of the decomposition of E_J as in* 4.13.

Proof. If (i) does not hold, then $\operatorname{Im} \Phi \neq \operatorname{Im} \varphi_1$, where $\operatorname{Im} \varphi_1$ is of dimension 1, because φ_1 is a non-zero map from 1-dimensional vector space. Therefore Φ becomes surjective, a contradiction to $H^2(E_J, \mathcal{O}_{E_J}) \neq 0$. For (ii), consider the composite:

$$\begin{aligned} H^1(E^{(i)}, \mathcal{O}_{E^{(i)}}) \xrightarrow{\varphi_i} H^1(C_1, \mathcal{O}_{C_1}) &\oplus H^1(C_2, \mathcal{O}_{C_2}) \\ \xrightarrow{p_1^{*-1} \oplus p_2^{*-1}} \; &H^1(C, \mathcal{O}_C) \oplus H^1(C, \mathcal{O}_C). \end{aligned}$$

One obtains that $\operatorname{Im}((p_1^{*-1} \oplus p_2^{*-1}) \circ \varphi_i) = \Delta$. Therefore ψ_i is not a zero map. For (iii), take another C_2' and $E^{(i)'}$ ($i = 1, 2$) such that $E^{(1)'} \cap E^{(2)'} = C_1 \amalg C_2'$. One may assume that $C_2' \subset E^{(1)}$ and $E^{(1)'} \subset E^{(1)}$ and $E^{(2)} \subset E^{(2)'}$. Let $E^{(3)}$ be a subchain of E_J such that $E^{(1)} \cap E^{(2)'} = C_1 \amalg E^{(3)}$. Then C_2, $C_2' \subset E^{(3)}$. By these inclusions, we obtain the commutative diagram:

$$\begin{array}{ccccccc} H^1(E^{(1)}) \oplus H^1(E^{(2)}) & \to & H^1(C_1) \oplus H^1(C_2) & \xrightarrow{\Psi} & H^2(E_J) & \to 0 \\ \| \qquad \uparrow \wr & & \| \qquad \uparrow \wr & & \| & \\ H^1(E^{(1)}) \oplus H^1(E^{(2)'}) & \to & H^1(C_1) \oplus H^1(E^{(3)}) & \to & H^2(E_J) & \to 0 \\ \downarrow \wr \qquad \| & & \| \qquad \downarrow \wr & & \| & \\ H^1(E^{(1)'}) \oplus H^1(E^{(2)'}) & \to & H^1(C_1) \oplus H^1(C_2') & \xrightarrow{\Psi'} & H^2(E_J) & \to 0. \end{array}$$

So the restrictions of Ψ and Ψ' on $H^1(C_1, \mathcal{O})$ are the same. Q.E.D.

Proof of Theorem 4.12. Let (Z, z) be an isolated strictly log-canonical singularity of type (0, 1), $\pi : (X, x) \to (Z, z)$ the canonical cover and G the associated cyclic group. Let $f : Y \to X$ be a G-equivariant good resolution and E_J the essential divisor. Then E_J is either as in (i) or (ii) of Theorem 6.8 in Appendix.

Case 1. The case that E_J is as in (ii) of Theorem 6.8.

Let $E_J = E^{(-)} + E^{(0)} + E^{(+)}$ be the decomposition as in (ii). Then there is a ruling $p : E^{(0)} \to C$ over an elliptic curve C. Since each fiber of p is mapped to a fiber of p by the action of G, C admits the action of G and p becomes a G-equivariant morphism. Now by the Mayer-Vietoris exact sequence:

$$H^1(E^{(-)} + E^{(0)}, \mathcal{O}) \oplus H^1(E^{(0)} + E^{(+)}, \mathcal{O}) \to H^1(E^{(0)}, \mathcal{O})$$
$$\to H^2(E_J, \mathcal{O}) \to H^2(E^{(-)} + E^{(0)}, \mathcal{O}) \oplus H^2(E^{(0)} + E^{(+)}, \mathcal{O}) = 0,$$

one obtains a G-equivariant isomorphism $H^1(E^{(0)}, \mathcal{O}) \simeq H^2(E_J, \mathcal{O})$. On the other hand there is a G-equivariant isomorphism $p^* : H^1(C, \mathcal{O}) \to H^1(E^{(0)}, \mathcal{O})$. Since the action of G on $H^1(C, \mathcal{O})$ is induced from that on C, the order of the action on G on $H^1(C, \mathcal{O})$ is 1, 2, 3, 4, 6 by Proposition 4.8.

Case 2. The case that E_J is as in (i) of Theorem 6.8.

If the intersection curves are all fixed under the action of G, the generater g of G induces an automorphism of each intersection curve. Take C_i and $E^{(i)}$ $(i = 1, 2)$ as in 4.13. Then one obtains the commutative diagram of isomorphisms:

$$\begin{array}{ccc} H^1(C_1) & \xrightarrow{\psi_1} & H^2(E_J) \\ g|^*_{C_1} \downarrow & & \downarrow g^* \\ H^1(C_1) & \xrightarrow{\psi_1} & H^2(E_J). \end{array}$$

Since $g|^*_{C_1}$ is of order 1, 2, 3, 4, 6 by Proposition 4.8, so is g^*.

If $g(C_1) = C_2$ for $C_1 \neq C_2$, then under the notation in 4.13 let $h : C \to C$ be an automorphism $p_2 \circ g|_{C_1} \circ p_1^{-1}$. By the definition of h, we obtain the commutative diagram of isomorphisms:

$$\begin{array}{ccccc} H^1(C) & \xrightarrow{p_2^*} & H^1(C_2) & \xrightarrow{\psi_2'} & H^2(E_J) \\ \downarrow h^* & & g|^*_{C_1} \downarrow & & \downarrow g^* \\ H^1(C) & \xrightarrow{p_1^*} & H^1(C_1) & \xrightarrow{\psi_1} & H^2(E_J), \end{array}$$

where ψ_2' is induced from ψ_1 through g. Here, note that $H^2(E_J, \mathcal{O}) = \mathbb{C}$ by the assumption of the singularity. So one can apply Lemma 4.14, (iii), obtaining that $\psi_2' = \psi_2$. On the other hand, as $\operatorname{Ker} \Psi \circ (p_1^* \oplus p_2^*) = \Delta$ by Lemma 4.14, (ii), it follows that $\psi_1 \circ p_1^* = -\psi_2 \circ p_2^*$. Hence, by the diagram above, the order of g^* is 1, 2, 3, 4, 6 since that of h^* is 1, 2, 3, 4, 6 by 4.8. Q.E.D.

Theorem 4.15. *For a positive integer r the following are equivalent:*

(i) *r is the index of a 3-dimensional strictly log-canonical singularity;*
(ii) *$\varphi(r) \leq 20$ and $r \neq 60$, where φ is the Euler function.*

Proof. First assume (i), then by theorems 4.5, 4.11 and 4.12, it follows that $\varphi(r) \leq 20$. If there exists a 3-dimensional strictly log-canonical singularity (Z, z) of index 60, then by 4.5 and 4.12, (Z, z) must be of type $(0, 2)$. Let E be the exceptional divisor on the canonical model of the canonical cover (X, x), then E is a normal $K3$-surface. Let G be the corresponding group of the canonical cover, then G acts on E whose induced action on $H^2(E, \mathcal{O}_E)$ is of order 60. Since this action is lifted to the minimal resolution $\tilde{E}$ of E, one obtains a $K3$-surface $\tilde{E}$ which admits an automorphism whose action on $H^2(\tilde{E}, \mathcal{O}_{\tilde{E}})$ is of order 60. However, it is proved by Machida-Oguiso [13] that there is no $K3$-surface with such an automorphism.

Next assume (ii), then by [11] and [17], there is a $K3$-surface E with an automorphism $g : E \to E$ whose order and the order of induced automorphism on $H^2(E, \mathcal{O}_E)$ are both r. Let $G = \langle g \rangle$, $\pi : E \to E' = E/G$ the quotient map and $\mathcal{L}$ an ample invertible sheaf on E'. Let Y' and Y be the line bundles $\operatorname{Spec} \bigoplus_{m \geq 0} \mathcal{L}^{\otimes m}$ and $\operatorname{Spec} \bigoplus_{m \geq 0} \pi^* \mathcal{L}^{\otimes m}$ on E' and on E, respectively. Then $Y \to E$ has the zero section E_0 whose normal bundle is $\pi^* \mathcal{L}^{-1}$, so there is a contraction $f : (Y, E_0) \to (X, x)$ of E_0. Since the exceptional divisor E_0 is a $K3$-surface, the singularity (X, x) is strictly log-canonical of index 1 and of type $(0, 2)$ by [8]. One defines an action of G on (X, x) in the following way: Let σ be the action of G on E. On the other hand there is also an action τ of G on Y' which is trivial on E', because Y' admits a canonical action of $\mathbb{C}^*$ and G is considered as a subgroup of $\mathbb{C}^*$. Since Y is the fiber product $E \times_{E'} Y'$, one obtains the action of G on Y which is compatible with σ and τ. It is clear that this action is free on $Y \setminus E_0$ and E_0 is G-invariant. Therefore one can introduce the action of G on (X, x). The quotient $(Z, z) = (X, x)/G$ is strictly log-canonical of index r by Corollary 3.6. Q.E.D.

4.16. The boundedness of indices of higher dimensional strictly log-canonical singularities is also expected to follow from Conjecture 4.7. On the contrary, if indices of n-dimensional strictly log-canonical singularities are bounded, then Conjecture 4.7 holds for $(n-1)$-dimensional

Calabi-Yau varieties. Indeed, as in the proof of Theorem 4.15, for every Calabi-Yau $(n-1)$-fold E and a finite order automorphism g, one can construct a strictly log-canonical singularity of index r, where r is the order of the induced automorphism g^* on $H^{n-1}(E, \mathcal{O}_E)$. Hence the boundedness of indices implies Conjecture 4.7.

§5. Finite groups which act non-freely in codimension one.

5.1. Terminologies in [10] are used in this section. Here one considers a finite group action on a 2-dimensional strictly log-canonical singularity. If the action is not free in codimension 1, the index of the quotient is not bounded.

Example 5.2. Let $\pi : C \to \mathbb{P}^1$ be a double covering from an elliptic curve C. Then π is the quotient map by a group $G = \mathbb{Z}/(2)$. Let $\tilde{Z}_m$ and $\tilde{X}_m$ be $\operatorname{Spec} \bigoplus_{i \geq 0} \mathcal{O}_{\mathbb{P}^1}(mi)$ and $\operatorname{Spec} \bigoplus_{i \geq 0} \pi^* \mathcal{O}_{\mathbb{P}^1}(mi)$ respectively, then $\tilde{X}_m$ admits the canonical action of G and the induced morphism $\tilde{\pi} : \tilde{X}_m \to \tilde{Z}_m$ is the quotient map. Since the zero sections of $\tilde{X}_m$ and $\tilde{Z}_m$ are G-invariant, one obtains the quotient map $\pi' : X_m \to Z_m$, where X_m and Z_m are the contracted space of zero sections in $\tilde{X}_m$ and $\tilde{Z}_m$, respectively. Here the singularity of X_m is strictly log-canonical of index 1 and the singularity of Z_m has the index m if m is odd and $m/2$ if m is even as one sees in Example 4.2, which shows that the indices of the quotients $\{Z_m\}_{m \in N}$ are not bounded.

5.3. Let (X, x) be an n-dimensional normal singularity and G a finite group which acts on (X, x) non-freely in codimension 1. Let $\pi : (X, x) \to (Z, z) = (X, x)/G$ be the quotient map, then π ramifies at divisors on X. Let B_i $(i = 1, \dots, s)$ be the branch divisors of π and R_{ij} $(j = 1, \dots n_i)$ the ramification divisors over B_i. Then the ramification index of R_{ij} depends only on i, denote it by e_i, because the generic points of R_{ij}'s $(j = 1, \dots, n_i)$ are mapped to each other transitively by the action of G. As for a Weil divisor D on Z the pull-back $\pi^*(D)$ by finite morphism π is defined (see for example 1.8 in [2]), one obtains the formula of $\mathbb{Q}$-divisors:

$$K_X = \pi^* \left(K_Z + \sum_{i=1}^{s} \frac{e_i - 1}{e_i} D_i \right).$$

Lemma 5.4. *Under the notation of* 5.3, (X, x) *is strictly log-canonical, if and only if the pair* $(Z, \sum_{i=1}^{s}(1 - 1/e_i)D_i)$ *is log-canonical, non-klt around* z.

Proof. By 3.16 of [10] $(X, \emptyset)$ is log-canonical, non-klt around x, if and only if $(Z, \sum_{i=1}^{s}(1 - 1/e_i)D_i)$ is log-canonical, non-klt around z. Here note that $(X, \emptyset)$ is klt around x, if and only if (X, x) is log-terminal. Q.E.D.

Lemma 5.5. *Let Z be a normal surface and D an effective $\mathbb{Q}$-divisor on Z such that* $\mathrm{Supp}(D)$ *contains a point $z \in Z$. If (Z, D) is log-canonical, then (Z, z) is a quotient singularity.*

Proof. Let $f : \tilde{Z} \to Z$ be a resolution of singularities on Z. First one will prove that $\omega_Z = f_*\omega_{\tilde{Z}}$ around z. Take a positive integer m such that mD is an integral divisor and $\omega_Z^{[m]}(mD)$ is trivial around z. Represent $mD = \sum_{i=1}^{u} r_i D_i$, where D_i's are the irreducible components. Let ω be a generator of $\omega_Z^{[m]}(mD)$, then $\nu_{D_i}(\omega) = -r_i < 0$ for every i. Since (Z, D) is log-canonical, one obtains

$$K_{\tilde{Z}} = f^*(K_Z + D) + \sum_{j=1}^{v} m_j E_j - D',$$

with $m_j \geq -1$ for every j, where D' is the proper transform of D and E_j's are the irreducible exceptional curves. Therefore

$$\omega_{\tilde{Z}}^{m}\left(-\sum m m_j E_j + mD'\right) = f^*(\omega_Z^{[m]}(mD)).$$

Hence $\nu_{E_j}(\omega) = m m_j \geq -m$ for every j. If an element $\theta \in \omega_Z$ satisfies $\nu_{E_j}(\theta) < 0$ for some E_j with $f(E_j) = \{z\}$, then $\nu_{E_j}(\theta^m) \leq -m$. Since $\theta^m \in \omega_Z^{[m]} \subset \omega_Z^{[m]}(mD)$, it follows that $\theta^m = h\omega$ with $h \in \mathcal{O}_Z$. Then $-m \leq \nu_{E_j}(\omega) \leq \nu_{E_j}(\theta^m) \leq -m$, and therefore $\nu_{E_j}(h) = 0$. Hence h does not vanish at z, from which one may assume that h does not vanish on Z by deleting Z sufficiently. But this yields a contradiction $\nu_{D_i}(\theta^m) = \nu_{D_i}(\omega) = -r_i < 0$. Now one obtains that $\omega_Z = f_*\omega_{\tilde{Z}}$ around z. Since Z is a normal surface, this equality implies that (Z, z) is a rational singularity, hence a $\mathbb{Q}$-Gorenstein singularity. So one can represent

$$K_{\tilde{Z}} = f^*K_Z + \sum n_j E_j,$$

with $n_j = m_j + m'_j$, where $f^*D = D' + \sum m'_j E_j$. By $z \in \mathrm{Supp}(D)$, it follows that $m'_j > 0$ for every E_j with $f(E_j) = \{z\}$, which yields that $n_j > -1$ for these j. A 2-dimensional log-terminal singularity is a quotient singularity. Q.E.D.

Theorem 5.6. *Let (X,x) be a 2-dimensional strictly log-canonical singularity and a finite group G act on (X,x) non-freely in codimension 1. Then the number of the branch divisors is at most 4 and the combination of the ramification indices of the quotient map $\pi : (X,x) \to (X,x)/G$ are* (6), (4,4), (3,3), (3,3,3), (2,2), (2,2,2), (2,2,2,2), (6,2), (4,2), (3,2), (6,3,2), (4,4,2), (4,2,2), (3,3,2), (3,2,2).

Proof. Use the notation of 5.3. By Lemma 5.4 $(Z, \sum(1-1/e_i)D_i)$ is log-canonical, not klt and by Lemma 5.5 (Z,z) is a quotient singularity. Let $\rho : \mathbb{C}^2 \to Z$ be the quotient map. Since ρ is étale in codimension 1, $K_{\mathbb{C}^2} = \rho^* K_Z$. Then by Lemma 5.4 $(\mathbb{C}^2, \sum(1-1/e_i)\rho^* D_i)$ is log-canonical, non-klt. In the following classification theorem of such pairs, one can see that the number of the branch divisors is at most 4 and the combination of the values of e_i's are (6), (4, 4), (3, 3), (3, 3, 3), (2, 2), (2, 2, 2), (2, 2, 2, 2), (6, 2), (4, 2), (3, 2), (6, 3, 2), (4, 4, 2), (4, 2, 2), (3, 3, 2), (3, 2, 2). Q.E.D.

Theorem 5.7. *The pair $(\mathbb{C}^2, \sum(1-1/e_i)D_i)$ is log-canonical, non-klt around 0 if and only if (e_i) and (D_i) are as follows up to analytic isomorphism around 0:*

(1.1) $e_1 = 6$, $D_1 = (x^2 + g = 0)$, *where* $g = \sum_{a+b\geq 3} \alpha_{ab} x^a y^b$ $(\alpha_{03} \neq 0)$;

(1.2) $(e_1, e_2) = (4,4)$, $D_1 = (x = 0)$, $D_2 = (x + y^2 + g = 0)$, *where* $g = \sum_{2a+b\geq 3} \alpha_{ab} x^a y^b$;

(1.3) $(e_1, e_2) = (3,3)$, $D_1 = (x = 0)$, $D_2 = (x + y^3 + g = 0)$, *where* $g = \sum_{3a+b\geq 4} \alpha_{ab} x^a y^b$;

(1.4) $(e_1, e_2, e_3) = (3,3,3)$, $D_1 = (x = 0)$, $D_2 = (y = 0)$, $D_3 = (x + y = 0)$;

(1.5) $(e_1, e_2) = (2,2)$, $D_1 = (x^2 + g = 0)$, $D_2 = (y^2 + h = 0)$, *where* $g = \sum_{na+b\geq 2n+1} \alpha_{ab} x^a y^b$ $(\alpha_{02n+1} \neq 0,\ n \geq 1)$, $h = \sum_{a+mb\geq 2m+1} \beta_{ab} x^a y^b$ $(\beta_{2m+1,0} \neq 0,\ m \geq 1)$;

(1.6) $(e_1, e_2, e_3) = (2,2,2)$, $D_1 = (x = 0)$, $D_2 = (x + y^2 + g = 0)$, $D_3 = (x + \beta y^n + h = 0)$, *where* $g = \sum_{2a+b\geq 3} \alpha_{ab} x^a y^b$,
$h = \sum_{na+b\geq n+1} \beta_{ab} x^a y^b$ $(n \geq 2)$, $\beta \neq 0$ *and if* $n = 2$, $\beta \neq 1$;

(1.7) $(e_1, e_2, e_3) = (2,2,2)$, $D_1 = (x = 0)$, $D_2 = (x + y^n + g = 0)$, $D_3 = (y^2 + h = 0)$, *where* $g = \sum_{na+b\geq n+1} \alpha_{ab} x^a y^b$ $(n \geq 1)$, $h = \sum_{a+mb\geq 2m+1} \beta_{ab} x^a y^b$ $(\beta_{2m+1,0} \neq 0,\ m \geq 1)$;

(1.8) $(e_1, e_2, e_3, e_4) = (2,2,2,2)$, $D_i = (x + \alpha_i y + h_i = 0)$ *for* $i = 1, \dots, 4$, *where* $\deg h_i \geq 2$ *and* $\alpha_i \neq \alpha_j$ $(i \neq j)$;

(1.9) $(e_1, e_2, e_3, e_4) = (2,2,2,2)$, $D_1 = (x = 0)$, $D_2 = (y = 0)$, $D_3 = (x+y = 0)$, $D_4 = (x+\alpha y^n + g = 0)$, *where* $g = \sum_{na+b \geq n+1} \alpha_{ab} x^a y^b$ $(n \geq 2)$ *and* $\alpha \neq 0$;

(1.10) $(e_1, e_2, e_3, e_4) = (2,2,2,2)$, $D_1 = (x = 0)$, $D_2 = (x + y^n + g = 0)$, $D_3 = (y = 0)$, $D_4 = (y + x^m + h = 0)$, *where* $g = \sum_{na+b \geq n+1} \alpha_{ab} x^a y^b$ $(n \geq 2)$, $h = \sum_{a+mb \geq m+1} \beta_{ab} x^a y^b$ $(m \geq 2)$;

(2.1) $(e_1, e_2) = (6,2)$, $D_1 = (x = 0)$, $D_2 = (x + y^3 + g = 0)$, *where* $g = \sum_{3a+b \geq 4} \alpha_{ab} x^a y^b$;

(2.2) $(e_1, e_2) = (4,2)$, $D_1 = (x = 0)$, $D_2 = (x + y^4 + g = 0)$, *where* $g = \sum_{4a+b \geq 5} \alpha_{ab} x^a y^b$;

(2.3) $(e_1, e_2) = (3,2)$, $D_1 = (x = 0)$, $D_2 = (x + y^6 + g = 0)$, *where* $g = \sum_{6a+b \geq 7} \alpha_{ab} x^a y^b$;

(2.4) $(e_1, e_2) = (3,2)$, $D_1 = (x = 0)$, $D_2 = (x^2 + g = 0)$, *where* $g = \sum_{a+b \geq 3} \alpha_{ab} x^a y^b$ $(\alpha_{03} \neq 0)$;

(2.5) $(e_1, e_2) = (2,3)$, $D_1 = (x = 0)$, $D_2 = (y^2 + g = 0)$, *where* $g = \sum_{a+b \geq 3} \alpha_{ab} x^a y^b$ $(\alpha_{30} \neq 0)$;

(2.6) $(e_1, e_2, e_3) = (6,3,2)$, $D_1 = (x = 0)$, $D_2 = (y = 0)$, $D_3 = (x + y = 0)$;

(2.7) $(e_1, e_2, e_3) = (4,4,2)$, $D_1 = (x = 0)$, $D_2 = (y = 0)$, $D_3 = (x + y = 0)$;

(2.8) $(e_1, e_2, e_3) = (4,2,2)$, $D_1 = (x = 0)$, $D_2 = (y = 0)$, $D_3 = (x + y^2 + g = 0)$, *where* $g = \sum_{2a+b \geq 3} \alpha_{ab} x^a y^b$;

(2.9) $(e_1, e_2, e_3) = (3,3,2)$, $D_1 = (x = 0)$, $D_2 = (y = 0)$, $D_3 = (x + y^2 + g = 0)$, *where* $g = \sum_{2a+b \geq 3} \alpha_{ab} x^a y^b$;

(2.10) $(e_1, e_2, e_3) = (3,2,2)$, $D_1 = (x = 0)$, $D_2 = (x + y^3 + g)$, $D_3 = (y = 0)$, *where* $g = \sum_{3a+b \geq 4} \alpha_{ab} x^a y^b$.

Proof. Denote $\sum D_i$ by D. Since $(1 - 1/e_i) \geq 1/2$, $(\mathbb{C}^2, 1/2D)$ is log-canonical around 0. Therefore $1/2 \leq \mathrm{lcth}(\mathbb{C}^2, D, 0)$, where $\mathrm{lcth}(\mathbb{C}^2, D, 0)$ is the log-canonical threshold of $(\mathbb{C}^2, D)$ around 0. On the other hand $\mathrm{lcth}(\mathbb{C}^2, D, 0) \leq 2/\operatorname{mult}_0 D$ by 8.10 of [10]. Hence $\operatorname{mult}_0 D \leq 4$.

Case 1. $\#\{e_i\} = 1$.

In this case $e = e_i \leq 6$, because $(e-1)/e = \mathrm{lcth}(\mathbb{C}^2, D, 0)$ and the right hand side is shown to be $\leq 5/6$ by 8.16 of [10].

Subcase 1.1. $\mathrm{mult}_0 D = 2$.

First consider the case that D is analytically irreducible. Let n be the number of successive blowing-ups of $\mathbb{C}^2$ at the singular point of the proper transforms of D to get the resolution of D. Then by two more blowing-ups at the suitable centers, one obtains a log-resolution of $(\mathbb{C}^2, D)$. Let E_i $(i = 1, \ldots, n+2)$ be the exceptional curve of the i-th blowing-up and m_i the log-discrepancy of $(\mathbb{C}^2, (1-1/e)D)$ at E_i, which means:

$$K_{\tilde{\mathbb{C}^2}} + \frac{e-1}{e}\tilde{D} = f^*\left(K_{\mathbb{C}^2} + \frac{e-1}{e}D\right) + \sum_{i=1}^{n+2} m_i E_i,$$

where $f : \tilde{\mathbb{C}^2} \to \mathbb{C}^2$ is the log-resolution and $\tilde{D}$ is the proper transform of D. It follows that $m_i = i - (1-1/e)2i$ for $i = 1, \ldots, n$, $m_{n+1} = n+1-(1-1/e)(2n+1)$ and $m_{n+2} = 2n+2-(1-1/e)(4n+2)$. Therefore if $e = 2$, $(\mathbb{C}^2, (1-1/e)D)$ is klt for every n. If $e = 3$, it is klt for $n = 1, 2$ and non-log-canoninal for $n \geq 3$. If $e = 4$ and $e = 5$, it is klt for $n = 1$ and non-log-canonical for $n \geq 2$. If $e = 6$, it is non-log-canonical for $n \geq 2$ and log-canonical, non-klt for $n = 1$. Now one obtains that $(\mathbb{C}^2, (1-1/e)D)$ is log-canonical, non-klt, if and only if $e = 6$ and D has a double cusp at 0 which can be resolved by the blowing-up at 0. By Lemma 5.8 below one obtains the defining equation of D and this case turns out to be (1.1).

Lemma 5.8. *Let $(D, 0) \subset (\mathbb{C}^2, 0)$ be a double cusp defined by an equation $f = 0$. Let n be the number of successive blowing-ups of $\mathbb{C}^2$ at the singular point of the proper transforms of D to get the resolution of D. Then $f = x^2 + g$, where $g = \sum_{na+b \geq 2n+1} \alpha_{ab} x^a y^b$, $\alpha_{0 2n+1} \neq 0$ by a suitable coordinate transformation.*

Next consider the remaining case that D is the union of two non-singular curves. Let n be as above, then the successive n-blowing-ups give a log-resolution. Define E_i and m_i $(i = 1, \ldots, n)$ in the same way as above. Then $m_i = i - (1-1/e)2i$ for $i = 1, \ldots, n$. Therefore $(\mathbb{C}^2, (1-1/e)D)$ is log-canonical, non-klt, if and only if $e = 4$ and $n = 2$ or $e = 3$ and $n = 3$. By Lemma 5.9 below, the former is (1.2) and the latter is (1.3).

Lemma 5.9. *Let $D \subset \mathbb{C}^2$ be the union of two non-singular curves D_1 and D_2 defined by equations $f_1 = 0$ and $f_2 = 0$. Let n be the number of successive blowing-ups of $\mathbb{C}^2$ at the singular point of the proper* *transforms of D to get the resolution of D. Then $f_1 = x$ and $f_2 = x + y^n + g$, where $g = \sum_{na+b\geq n+1} \alpha_{ab}x^a y^b$ by a suitable coordinate transformation.*

Subcase 1.2. $\operatorname{mult}_0 D = 3$.

In this case, $(\mathbb{C}^2, (1-1/e)D)$ is log-canonical, non-klt, if and only if (1.4) or (1.6) holds. It is proved in the same way as in Subcase 1.1, and the proof is omitted.

Subcase 1.3. $\operatorname{mult}_0 D = 4$.

In this case, $(\mathbb{C}^2, (1-1/e)D)$ is log-canonical, non-klt, if and only if (1.5), (1.7), (1.8), (1.9) or (1.10) holds. The proof is omitted.

Case 2. $\#\{e_i\} > 0$.

In this case $\operatorname{mult}_0 D \leq 3$. Indeed, if $\operatorname{mult}_0 D = 4$, then $\operatorname{lcth}(\mathbb{C}^2, D, 0) = 1/2$ by the inequalities in the beginning of the proof of the theorem. This is a contradiction to the fact that $(\mathbb{C}^2, \sum(1-1/e_i)D_i)$ is log-canonical around 0 with $\sum(1-1/e_i)D_i > 1/2D$.

Subcase 2.1. $\operatorname{mult}_0 D = 2$.

Since D is reducible, D is the union of two non-singular curves. Let n, E_i and m_i be as in Subcase 1.1. Then $m_i = i\{1-(e_1-1)/e_1 - (e_2-1)/e_2\}$. Therefore $(\mathbb{C}^2, \sum(1-1/e_i)D)$ is log-canonical, non-klt, if and only if $(n, e_1, e_2) = (3, 6, 2)$, $(4, 4, 2)$ or $(6, 3, 2)$. These are the cases (2.1), (2.2) and (2.3), by Lemma 5.8 and Lemma 5.9.

Subcase 2.2. $\operatorname{mult}_0 D = 3$.

One can devide into two cases:

(1) $\operatorname{mult}_0 D_1 = 1$ and $\operatorname{mult}_0 D_2 = 2$ and

(2) $\operatorname{mult}_0 D_i = 1$ for $i = 1, 2, 3$.

Under the first case, $(\mathbb{C}^2, \sum(1-1/e_i)D)$ is log-canonical, non-klt, if and only if (2.4) or (2.5) holds, and under the second case, if and only if (2.6), (2.7), (2.8), (2.9) or (2.10) holds. The proof is in the same way as in Subcase 2.1. Q.E.D.

5.10. More generally, 2-dimensional log-canonical pairs are classified in [15] by the terminology of dual graphs of minimal good resolutions.

§6. Appendix : The essential divisors of type (0, 1)

In this section one studies the configurations of the essential divisors of strictly log-canonical singularities of index 1 and of type (0, 1). The configurations of such divisors were studied in [7]. But the proof skipped some steps and in ℓ.10, p.186, [7] it used a contraction criterion stated in p.61, §4, [20] which has a counter example (Proposition 3, Example, [3]). So in this appendix, we give a new proof including complete steps for the structure of the essential divisors. As a consequence we obtain a weaker result than stated in [7], but it is sufficient for our discussion in the preceding sections of this paper.

Definition 6.1. Let (X, x) be a normal singularity which admits an action of a group G. A birational proper morphism $g : Y \to X$ is called a G-equivariant $G\mathbb{Q}$-factorial terminal model of (X, x), if

(1) G acts on Y and g is G-equivariant,
(2) Y has at worst terminal singularities,
(3) every G-invariant divisor on Y is a $\mathbb{Q}$-Cartier divisor and
(4) K_Y is nef.

If (X, x) is of dimension 3, there exists a G-equivariant $G\mathbb{Q}$-factorial terminal model (relative version of 7.6 [1]).

Some parts of the following lemmas are proved in [7], but for the reader's convenience we give here the proofs.

Lemma 6.2. *Let (X, x) be a 3-dimensional isolated strictly log-canonical singularity of index 1 and of type $(0,1)$, $f : \tilde{X} \to X$ a good resolution and E_J the essential divisor on $\tilde{X}$. Then*

(i) *E_J is not irreducible,*
(ii) *every intersection curve of E_J has positive genus and*
(iii) *there is no triple point on E_J.*

Proof. If E_J is irreducible, then $H^2(E_J, \mathcal{O}_{E_J}) = \mathbb{C}$ consists of $(0,2)$-component, which is a contradiction. Take an irreducible component E_j of E_J and put $E_j^\vee = E_J - E_j$. Consider the exact sequence:

$$H^1(E_j, \mathcal{O}_{E_j}) \oplus H^1(E_j^\vee, \mathcal{O}_{E_j^\vee}) \to H^1(E_j \cap E_j^\vee, \mathcal{O}) \to H^2(E_J, \mathcal{O}_{E_J}) \to 0,$$

induced from the Mayer-Vietoris exact sequence and Proposition 3.8 of [6]. Since $H^2(E_J, \mathcal{O}_{E_J})$ consists of the $(0,1)$-component, there is $(0,1)$-component in $H^1(E_j \cap E_j^\vee, \mathcal{O})$. Therefore $E_j \cap E_j^\vee$ contains at least one curve of positive genus. Note that this holds for an arbitrary good resolution. Here, if ℓ is a rational intersection curve of E_J, take the blowing-up $\sigma : \tilde{X}' \to \tilde{X}$ with the center ℓ. Then the divisor $E_0 = \sigma^{-1}(\ell)$ is an essential component on $\tilde{X}'$ and the intersection curves of E'_J on E_0 are all rational, where E'_J is the essential divisor on $\tilde{X}'$. This is a contradiction to the fact proved above. If there is a triple point p on E_J, take the blowing-up at p. Then one also has an essential component with only rational double curves on it. Q.E.D.

Lemma 6.3. *Let $g : Y \to X$ be a G-equivariant $G\mathbb{Q}$-factorial terminal model of a 3-dimensional isolated strictly log-canonical singularity (X, x) of index 1 and D the reduced inverse image $g^{-1}(x)_{red}$. Then*

(i) *$K_Y = -D$,*

(ii) *the singularities of D are normal crossings except for finite points and*

(iii) *D is Cohen-Macaulay, therefore isolated singularities on D are normal.*

Proof. By the proof of Lemma 7 of [8], $K_Y = g^*K_X - \sum a_i D_i$ with $a_i > 0$ for all irreducible component D_i of D. Here by the assumption on the singularity, the negative discrepancy is -1, which yields (i). Let C be an irreducible component of 1-dimensional singular locus of D and m the multiplicity of D at a general point of C. Take the blowing-up $\sigma : Y' \to Y$ at the center C and denote the exceptional divisor over C by D_0. Then the discrepancy for $g \circ \sigma$ at D_0 is $1 - m$, because Y and C are both non-singular at a general point of C. Then by the assumption on the singularity (X, x), m must be 2. If the singularity of D is not ordinary at a general point of C, then, by successive blowing-ups of Y with suitable curves as centers, one obtains a partial resolution $g'' : Y'' \to X$ factored through g with $K_{Y''} = -D'_1 - D'_2 - D'_3 -$(other terms), where D'_1, D'_2 and D'_3 are components of ${g''}^{-1}(x)_{red}$ and intersect at a curve C'. By passing through the blowing-up of Y'' with center C', one obtains a good resolution $f : \tilde{X} \to X$, which has a discrepancy -2 at one component, a contradiction. (iii) follows from the fact that D is $\mathbb{Q}$-Cartier and the discussion as in 0.5 of [9]. Then by Serre's criterion, isolated singularities of D are normal. Q.E.D.

Definition 6.4. An irreducible component of 1-dimensional singular locus of D is called a double curve of D. If a double curve is the intersection of two irreducible components, it is called an intersection curve.

Proposition 6.5. *Let (X, x) be a 3-dimensional isolated strictly log-canonical singularity of index 1 and of type $(0, 1)$ and G a finite group acting on (X, x). Let $g : Y \to X$ be a G-equivariant $G\mathbb{Q}$-factorial terminal model of (X, x) and D the reduced inverse image $g^{-1}(x)_{red}$. Let $\sigma : D' \to D$ be the normalization. Then the structure of D is as follows:*

(i) *the case D is irreducible then D is one of the following:*

(i-1) *a normal elliptic ruled surface with two simple elliptic singularities or*

(i-2) *a normal rational surface with a simple elliptic singularity or*

(i-3) *a rational surface with a double curve C such that $\sigma^{-1}(C)$ is an elliptic curve or* (i-4) *an elliptic ruled surface with a simple elliptic singularity and a double curve C such that $\sigma^{-1}(C)$ is an elliptic curve or*

(i-5) *an elliptic ruled surface with two double curves C_1 and C_2 such that $\sigma^{-1}(C_1)$, $\sigma^{-1}(C_2)$ are disjoint elliptic curves or*

(i-6) *an elliptic ruled surface with a double curve C such that $\sigma^{-1}(C)$ consists of two disjoint elliptic curves;*

(ii) *the case D is not irreduible then D is one of the following:*

(ii-1) *a cycle of elliptic ruled surfaces with sections as double curves or*

(ii-2) *a chain of surfaces $D = D_1 + \ldots + D_s (s \geq 2)$ with elliptic intersection curves, where $D_2, \ldots, D_{s-1}$ are elliptic ruled surfaces and each of D_1 and D_s is as follows; rational surface or elliptic ruled surface with a simple elliptic singularity or elliptic ruled surface with a double curve C such that $\sigma^{-1}(C)$ is an elliptic curve.*

(iii) *the singularities of D' are at worst rational double points except for simple elliptic singularities appeared in* (i-1), (i-2), (i-4) *and* (ii-2). *Moreover, D' is non-singular along $\sigma^{-1}(C)$, where C is a double curve.*

Proof. First of all, note that the singularities on Y are isolated, because Y has at worst terminal singularities. By (i) of 6.3, the equality $\omega_D = \mathcal{O}_D$ holds away from finite points. Since D is Cohen-Macaulay by 6.3, this equality holds whole on D. Therefore

$$K_{D'} = -\sigma^{-1}(\text{double curves of } D).$$

Let $\varphi : \tilde{D} \to D'$ be the minimal resolution, then one obtains

$$K_{\tilde{D}} = \varphi^* K_{D'} - \Delta$$

with $\Delta \geq 0$, where $\varphi^* K_{D'}$ is the numerical pull-back defined in [19]. Now it follows that $-K_{\tilde{D}}$ is an effective divisor on each component of $\tilde{D}$. Denote an irreducible component of D by D_i and the corresponding component of D' and $\tilde{D}$ by D'_i and $\tilde{D}_i$, respectively. Then by [23], a pair $(\tilde{D}_i, \Gamma)$ $\Gamma \in |-K_{\tilde{D}_i}|$ is one of the following:

(1) $\tilde{D}_i$ is a rational surface and Γ is an elliptic curve;

(2) $\tilde{D}_i$ is a rational surface and Γ is a cycle of rational curves;

(3) $\tilde{D}_i$ is an elliptic ruled surface and Γ is two disjoint sections;

(4) $\tilde{D}_i$ is a ruled surface of genus ≥ 2 and $\Gamma = 2C_0 +$rational curves, where C_0 is a section.

But in our situation, (2) and (4) do not occur. Indeed, assume D_i is a component such that $\tilde{D}_i$ and Γ are as in (4). Take a good resolution $f : \tilde{X} \to Y$ isomorphic on points which are non-singular on

D and on Y. Let E_k be the proper transform of D_k on $\tilde{X}$. Represent $K_{\tilde{X}} = -\sum_k E_k + \sum_{F_j:f-\text{exceptional}} m_j F_j$. Then

$$K_{E_i} = -\sum_{k \neq i} E_k|_{E_i} + \sum_{F_j:f-\text{exceptional}} m_j F_j|_{E_i}. \tag{6.5.1}$$

Here non-empty $F_j|_{E_i}$ is either corresponding to a double curve of D or a point on D, while $E_k|_{E_i}$ corresponds to a double curve of D. Note that $f|_{E_i}$ factors through $\tilde{D}_i$ and an irreducible component of Γ is either corresponding to a double curve or a point on D. Therefore

$$K_{E_i} = -2C_0' + \sum n_j e_j, \tag{6.5.2}$$

where C_0' is the proper transform of C_0 and e_j is either corresponding to a double curve or a point on D. By the uniqueness of the representation, (6.5.1) and (6.5.2) coincide, which shows that there is a component F_j with $m_j = -2$, a contradiction to the condition on the singularity (X, x). Next if D_i is a component such that $\tilde{D}_i$ and Γ are as in (2). Then in the same way as above one can prove that there exists an essential component F_j which intersects E_i at a rational curve, which is a contradiction to Lemma 6.2.

Now one has only to consider the case (1) or (3). Note that each component of Γ corresponds to either a double curve or a point on D.

First assume that D is irreducible. Consider the case that $\tilde{D}$ and Γ are as in (1). If Γ corresponds to a double curve, then one obtains (i-3). If Γ corresponds to a point, then one obtains (i-2). Next consider the case that $\tilde{D}$ and Γ are as in (3). If both components of Γ correspond to points, then one obtains (i-1). If both components of Γ correspond to double curves, then one obtains (i-5) and (i-6). If one component of Γ corresponds to a double curve and the other to a point, then one obtains (i-4).

Next assume that D is reducible. Then at least one component of Γ of $\tilde{D}_i$ corresponds to a double curve of D. Hence the structure of D is either (ii-1) or (ii-2).

For the statement (iii), take any point $p \in D'$ which is not the simple elliptic singularity stated in (i-1), (i-2), (i-4) and (ii-2). If p is not in the curve corresponding to a double curve of D, then p is rational double, because $K_{\tilde{D}} = \varphi^* K_{D'}$ around p. Assume p is on the curve $C' \subset D_i'$ corresponding to a double curve of D and φ is not isomorphic at p. As $K_{D_i'} = -C'$ around p, it follows that $K_{\tilde{D}_i} = -\tilde{C} - \Delta$, where $\tilde{C}$ is the proper transform of C' on $\tilde{D}_i$, $\Delta > 0$ and $\Delta \cap \tilde{C} \neq \emptyset$, which is a contradiction to the configuration of Γ. Therefore this point p is non-singular. Q.E.D.

In order to prove the structure theorem of the essential part of 3-dimensional isolated strictly log-canonical singularities of index 1 and of type 1), one need the following lemmas.

Lemma 6.6. *Let X_i $(i = 1,2)$ be non-singular 3-folds, E an irreducible non-singular divisor with $K_{X_1} = -E$, C a non-singular curve on E. and $f : X_2 \to X_1$ a proper birational morphism isomorphic away from C. Denote the proper transform of E by E' and represent*

$$K_{X_2} = -E' + \sum_{F_j : f-\text{exceptional}} m_j F_j.$$

Then $m_j \geq 0$ for an irreducible component F_j with $f(F_j) = C$, and $m_j = 0$ for such F_j with moreover $f(F_j \cap E') = C$.

Proof. By replacing X_1 by a small analytic neighbourhood of a point on C, one obtains a smooth morphism $\pi : X_1 \to \Delta \subset \mathbb{C}$ such that $H_t \cap C$ is one point $\{p_t\}$, where $H_t = \pi^{-1}(t)$ for $t \in \Delta$. Denote $f^{-1}(H_t)$ by $\tilde{H}_t$, $\tilde{H}_t \cap E'$ by $\tilde{e}$ and $H_t \cap E$ by e. Then for a general $t \in \Delta$, $\tilde{H}_t$ is irreducible, non-singular and the intersection $\tilde{H}_t \cap F_j = e_j$ is a reduced curve for F_j with $f(F_j) = C$. Therefore by $K_{X_1}|_{H_t} = K_{H_t}$ and $K_{X_2}|_{\tilde{H}_t} = K_{\tilde{H}_t}$, it follows that

$$K_{H_t} = -e,$$

$$K_{\tilde{H}_t} = -\tilde{e} + \sum_{f(F_j)=C} m_j e_j.$$

Here $f|_{\tilde{H}_t} : \tilde{H}_t \to H_t$ is a proper birational morphism between non-singular surfaces, therefore the composite of blowing-ups at points. Hence $m_j \geq 0$ for all e_j and $m_j = 0$ for e_j with $e_j \cap \tilde{e} \neq \emptyset$. Q.E.D.

Lemma 6.7. *Let X_i $(i = 1,2)$ be non-singular 3-folds, E_1 and E_2 irreducible non-singular divisors which cross normally at a curve C and $K_{X_1} = -E_1 - E_2$. Let $f : X_2 \to X_1$ be a proper birational morphism such that $E'_1 \cap E'_2 = \emptyset$ and $E'_1 + E'_2 + \sum F_j$ is of normal crossings, where E'_i's are the proper transforms of E_i's and F_j's are exceptional divisors. Represent*

$$K_{X_2} = -E'_1 - E'_2 + \sum m_j F_j.$$

Then there exist ruled surfaces $F_1, \ldots, F_r$ over C such that $E'_1 + F_1 + \ldots + F_r + E'_2$ is a chain whose intersection curves are all sections of F_j's and $m_j = -1$ for $j = 1, \ldots, r$, $m_j \geq 0$ for $j \neq 1, \ldots, r$ and $f(F_j) = C$.

Proof. Take the same π as in the previous lemma and use the same notation H_t, $\tilde{H}_t$, p_t, e_j. Denote $H_t \cap E_i$ by e_i and $\tilde{H}_t \cap E'_i$ by e'_i. Then for general $t \in \Delta$,

$$K_{H_t} = -e_1 - e_2,$$

$$K_{\tilde{H}_t} = -e'_1 - e'_2 + \sum_{f(F_j)=C} m_j e_j.$$

Since $f|_{\tilde{H}_t}$ is a composite of blowing-ups at points, there exist $e_1, \ldots, e_r$ such that $\{e'_1, e_1, \ldots, e_r, e'_2\}$ forms a chain of rational curves in some order and $m_j = -1$ for $j = 1, \ldots, r$ and $m_j \geq 0$ for $j \neq 1, \ldots, r$ such that $f(F_j) = C$. For the assertion on F_j's ($j = 1, \ldots, r$), note first that the general fiber of $F_j \to C$ is a disjoint union of non-singular rational curves, therefore F_j is a ruled surface. Next take F_1 such that $F_1 \cap E'_1 \neq \emptyset$. Then $f|_{F_1} : F_1 \to C$ is the projection of ruled surface, because F_1 intersects E'_1 at a curve isomorphic to C. Therefore e_1 is irreducible. Then take F_2 such that $F_1 \cap F_2 \neq \emptyset$. If $F_1 \cap F_2$ is not a section of $f|_{F_1}$, $e_1 \cap e_2$ consists of more than one point, which contradicts to that $\{e'_1, e_1, e_2, \ldots\}$ forms a chain. So $f|_{F_2} : F_2 \to C$ has a section $F_1 \cap F_2$, which shows that it is a projection of a ruled surface over C and e_2 is irreducible. Inductively one obtains the assertion on F_j's for $j = 1, \ldots, r$. Q.E.D.

Theorem 6.8. *Let (X, x) be a 3-dimensional isolated strictly log-canonical singularity of index 1 and of type 1) and a finite group G act on (X, x). Then either:*

(i) *there is a G-equivariant good resolution $f : \tilde{X} \to X$ such that the essential divisor E_J is a cycle $E_1 + E_2 + \ldots + E_s$, $(s \geq 2)$ of elliptic ruled surfaces, where E_i and E_{i+1} intersect at a section on each component for $i = 1, \ldots, s$ $(E_{s+1} = E_1)$ or*

(ii) *there is a G-equivariant good resolution $f : \tilde{X} \to X$ such that the essential divisor E_J contains a G-invariant chain $E^{(0)} = E_1 + \ldots + E_s$ $(s \geq 1)$ of elliptic ruled surfaces, where E_i and E_{i+1} intersect at a section on each component for $i = 1, \ldots, s-1$. There are mutually disjoint subdivisors $E^{(-)}$ and $E^{(+)}$ of E_J such that $E_J = E^{(-)} + E^{(0)} + E^{(+)}$, where $E^{(-)} \cap E^{(0)}$ is a section of E_1 and $E^{(+)} \cap E^{(0)}$ is a section of E_s.*

Proof. Let $g : Y \to X$ be a G-equivariant $G\mathbb{Q}$-factorial terminal model of (X, x) and D the reduced inverse image $g^{-1}(x)_{red}$.

Assume that D is as in (i-1) of Proposition 6.5. Then there are two simple elliptic singularities p_1, p_2 on D. Take a G-equivariant good resolution $f : \tilde{X} \to Y$ and denote the proper transform of D by E and f-exceptional divisors by F_j's. Represent $K_{\tilde{X}} = -E + \sum m_j F_j$. Since

$m_j = 0$ for non-$f|_E$-exceptional curve $F_j|_E$ by Lemma 6.6, it follows that

$$(6.8.1) \qquad K_E = \sum_{F_j : f|_E-\text{exceptional}} m_j F_j|_E.$$

On the other hand, recall that $K_{\tilde{D}} = -C_1 - C_2$, where C_i's are the fibers of the simple elliptic singularities and disjoint sections of the elliptic ruled surface. Hence denoting the proper transform of C_i by $\tilde{C}_i$ and the canonical morphism $E \to \tilde{D}$ by ψ, one obtains:

$$(6.8.2) \qquad K_E = -\tilde{C}_1 - \tilde{C}_2 + \sum_{e_j : \psi-\text{exceptional}} n_j e_j,$$

where $n_j \geq 0$ because $\psi : E \to \tilde{D}$ is a composite of blowing-ups at points.

Noting that an $f|_E$-exceptional divisor is either $\tilde{C}_i$ or ψ-exceptional, compare (6.8.1) and (6.8.2). Then one obtains that there are components F_1 and F_2 such that $F_i|_E = \tilde{C}_i$ with $m_1 = m_2 = -1$ and $m_j \geq 0$ for every F_j $(j \neq 1, 2)$. Let $E^{(-)}$ be the sum of the essential components in $f^{-1}(p_1)$, $E^{(+)}$ that in $f^{-1}(p_2)$. If one puts $E^{(0)} = E$, then these satisfy the condition in (ii) of the theorem.

Assume that D is as in (i-2) of Proposition 6.5. In the same way as above, one obtains that there exists only one essential component F which intersects E. Since the intersection curve $F \cap E$ is G-invariant elliptic curve, one obtains another G-equivariant good resolution with the properties in (ii) of the theorem by compositing the blowing-up at $F \cap E$. In this case, the exceptional divisor of the blowing-up becomes $E^{(0)}$.

Assume D is as in (i-3) of Proposition 6.5. Let $f : \tilde{X} \to Y$ be a G-equivariant good resolution passing through the blowing-up at the double curve C which is G-invariant. Denote the proper transform of D on $\tilde{X}$ by E and the elliptic curve on E corresponding to C by $\tilde{C}$. Then there exists an f-exceptional curve F_1 such that $F_1|_E = \tilde{C}$. Represent $K_{\tilde{X}} = -E + \sum m_j F_j$. Then by Lemma6.6, it follows that

$$(6.8.3) \qquad K_E = m_1 F_1|_E + \sum_{F_j : f|_E-\text{exceptional}} m_j F_j|_E.$$

Since $K_{\tilde{D}} = -C'$, where C' is an elliptic curve corresponding to the double curve C, it follows that

$$(6.8.4) \qquad K_E = -\tilde{C} + \sum_{e_j : \psi-\text{exceptional}} n_j e_j.$$

Here one obtains $n_j \geq 0$, because $\psi : E \to \tilde{D}$ is a composite of blowing-ups at points. Noting that an $f|_E$ -exceptional curve is ψ-exceptional, compare (6.8.3) and (6.8.4). Then it follows that $m_1 = -1$ and $m_j \geq 0$ for $j \neq 1$ such that $F_j|_E \neq \emptyset$. Therefore there exists only one essential component F_1 which intersects E and the intersection $F_1 \cap E$ is G-equivariant elliptic curve. By taking the blowing-up at $F_1 \cap E$, one obtains $E^{(0)}$ which satisfies the conditions in (ii) of the theorem

Assume that D is as in (i-4) or (i-5) of Proposition 6.5. In the same way as in (i-1), one obtains that the conditions in (ii) of the theorem hold by denoting the proper transform of D by $E^{(0)}$.

Asssume that D is as in (i-6) or (ii-1) of Proposition 6.5. Take a G-equivariant good resolution $f : \tilde{X} \to Y$, decompose D into the irreducible components $D_1 + \ldots + D_s$ ($s \geq 1$) and denote the proper transform of D_i on $\tilde{X}$ by E_i. By Lemma 6.7, the essential divisor E_J on $\tilde{X}$ contains a subdivisor E'_J with the property in (i) of the theorem. Represent $E'_J = \sum_{i=1}^{s} E_i + \sum_{j=1}^{t} F_j$. Let F be an f-exceptional divisor not contained in E'_J. Suppose first $F|_{F_j} \neq \emptyset$ for some $j = 1, \ldots, t$. If $f(F|_{F_j})$ is a point, then it is contained in a fiber of the ruling of F_j, therefore it is rational. Then by (ii) of Lemma 6.2, F is not an essential component. If $f(F|_{F_j})$ is a curve, then by Lemma 6.7, F is not essential. Next suppose that $F|_{E_i} \neq \emptyset$ for some $i = 1, \ldots, s$. If $F|_{E_i}$ is $f|_{E_i}$-exceptional, then it is rational, because the singularities on the normalization D'_i of D_i are all rational by (iii) of Proposition 6.5. Therefore by (ii) of Lemma 6.2, F is not essential. If $F|_{E_i}$ is not $f|_{E_i}$-exceptional, then by Lemma 6.6, F is not essential. Now it follows that $E_J = E'_J$ by connectedness of the essential divisor.

Assume that D is as in (ii-2) of Proposition 6.5. Decompose D into irreducible components $D_1 + \ldots + D_s$. For the case $s = 2$, by taking the blowing-up at $D_1 \cap D_2$ one can reduce into the case $s = 3$. So one may assume that $s \geq 3$. Let $f : \tilde{X} \to Y$ be a G-equivariant good resolution and E_i the proper transform of D_i. Then by Lemma 6.7 in the essential divisor E_J on $\tilde{X}$ there exists a chain of elliptic ruled surfaces starting with E_2, including E_i ($2 < i < s-1$) and finishing with E_{s-1} such that the intersection curves are all sections on ruled surfaces. Note that this chain is G-invariant, because $D_2 + \ldots + D_{s-1}$ is G-invariant. In the same way as in the case (ii-1), one obtains that there are only two essential components which intersect this chain, and the intersection is sections of E_2 and of E_{s-1}. Denote this chain by $E^{(0)}$ and the sum of the essential components in $f^{-1}(D_1)$ by $E^{(-)}$ and that in $f^{-1}(D_s)$ by $E^{(+)}$. Then these satisfy the conditions in (ii) of the theorem. Q.E.D.

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Geometry of complex surface singularities

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§ Introduction.

In the local study of complex analytic spaces, it is natural to investigate the behaviour of the tangent spaces near a singular point. In the general case of equidimensional singularities, after choosing a local embedding of the singular space into a complex affine space, B. Teissier and the author have given the structure of the limit of tangent hyperplanes, i.e. hyperplanes containing a tangent space at a non-singular point, in terms of a family of cones contained in the tangent cone of the singularity and called the Auréole of the singularity (see [LT2]).

In the case of surface singularities, the Auréole is given by the tangent cone and a finite number of generatrices of the tangent cone called the exceptional tangents. Recent works of J. Snoussi showed that these exceptional tangents coincide with the special generatrices of Gonzalez and Lejeune ([GL]). His result is based on the fact that, after choosing a local embedding of the surface into a complex affine space, a hyperplane is not a limit of tangent hyperplanes if and only if its intersection with the normal surface singularity is a curve with a Milnor number (in the sense of Buchweitz and Greuel [BG]) which is minimum. This work enhances the interest in the local geometry of complex surface singularities that we began in [L3] and [LT1].

This paper is essentially a survey of results about the limits of tangent hyperplanes of a normal surface singularity. It gives a geometrical approach in the study of a normal surface singularity and suggests new research interests in effective resolutions of normal surface singularities. In particular, it should lead to effective bounds for the number of normalized blowing-up needed to solve the singularity.

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§1. An example.

1.1. Let $f\colon U \to \mathbf{C}$ be a complex analytic function defined on an open neighbourhood U of 0 in $\mathbf{C}^3$. We assume that $f(0) = 0$ and the function f has an isolated critical point at 0. The function f defines a complex analytic surface X closed in U. The analytic local ring $\mathcal{O}_{X,0}$ of X at 0 is

$$\mathcal{O}_{X,0} = \mathbf{C}\{X, Y, Z\}/(f)$$

quotient of the local ring of convergent series $\mathbf{C}\{X, Y, Z\}$ at 0 by the principal ideal generated by f in $\mathbf{C}\{X, Y, Z\}$.

Since we have assumed that 0 is an isolated critical point of f, the element f is irreducible in $\mathbf{C}\{X, Y, Z\}$, i.e. the principal ideal (f) generated by f in the ring $\mathbf{C}\{X, Y, Z\}$ is prime. Therefore, the ring $\mathcal{O}_{X,0}$ is an integral domain, i.e. it has no zero divisors. Furthermore, since it is the local ring of a hypersurface whose singularities are in codimension 2, a criterion of J.P. Serre (see [S] (IV D) §4) implies that the ring $\mathcal{O}_{X,0}$ is normal, i.e. it is integrally closed in its field of fractions.

1.2. As B. Teissier did in [T] (Chap. 1), we can associate to the germ $(X, 0)$ of the surface X at 0 the following invariants.

First, to any hypersurface V with an isolated singularity at the point 0, J. Milnor has associated an integer ([M] §7) called the Milnor number $\mu(V, 0)$ of V at 0. In our case, the Milnor number $\mu(X, 0)$ of the surface X at 0 is given by the complex dimension of the vector space

$$\mathcal{M}_{X,0} := \mathbf{C}\{X, Y, Z\}/(\partial f/\partial X, \partial f/\partial Y, \partial f/\partial Z)$$

quotient of $\mathbf{C}\{X, Y, Z\}$ by the ideal generated by the partial derivatives of f. Hilbert-Rückert Nullstellensatz (see [N] Chap. III §2 Theorem 2) implies that the $\mathbf{C}$-vector space $\mathcal{M}_{X,0}$ is finite dimensional over the field $\mathbf{C}$ if and only if f has an isolated critical point at 0. We have:

Lemma 1.2.1. *The Milnor number is a topological invariant of the hypersurface X at 0, in the sense that, for any hypersurface Y of $\mathbf{C}^3$ which has a singularity at the point y and for which there is a germ of homeomorphism of $(\mathbf{C}^3, 0)$ onto $(\mathbf{C}^3, y)$ which sends $(X, 0)$ onto (Y, y), we have that Y has an isolated singularity at 0 and $\mu(X, 0) = \mu(Y, y)$.*

Proof. Actually, the result is true in any dimension, but we give a proof for hypersurfaces in $\mathbf{C}^3$.

We need a topological interpretation of the Milnor number. In the case of isolated singularities, following J. Milnor [M] (Corollary 2.9), one can prove that there is $\epsilon_0 > 0$, such that, for any ϵ, $\epsilon_0 > \epsilon > 0$, the real sphere $S_\epsilon(0)$ (boundary of the open ball $B_\epsilon(0)$) of $\mathbf{C}^3$ centered at 0 with radius ϵ is transverse to the hypersurface $X := \{f = 0\}$. Let us fix ϵ, $\epsilon_0 > \epsilon > 0$. By the openness of the transversality, there is $\eta(\epsilon) > 0$, such that for any $t \in \mathbf{C}, 0 < |t| < \eta(\epsilon)$, the hypersurface $\{f = t\}$ intersects $S_\epsilon(0)$ transversally. So, the space $\{f = t\} \cap B_{\epsilon_0}(0)$ is a smooth manifold of real dimension 4. In [M] (Theorem 5.11 and Theorem 6.5), it is proven that the homotopy type of $\{f = t\} \cap B_{\epsilon_0}(0)$ is the one of a bouquet of $\mu(X, 0)$ 2-spheres, i.e. a space union of $\mu(X, 0)$ 2-spheres with one point in common. The space $\{f = t\} \cap B_{\epsilon_0}(0)$ is called a **Milnor fiber** of X at 0. On the other hand, Ehresmann Lemma (see e.g. [D] §20.8 Problème 4) implies that, for any η, $0 < \eta < \eta(\epsilon)$, the function f induces a locally trivial smooth fibration of $f^{-1}(\delta D_\eta) \cap B_\epsilon(0)$ onto δD_η, where δD_η is the circle of $\mathbf{C}$ centered at 0 with radius η. In [L2], we show that the homotopy class of this fibration is a topological invariant of the hypersurface X at 0. We call this fibration the **Milnor fibration** of X at 0.

Now let Y be a complex analytic surface closed in an open neighborhood V of $y \in Y$ in $\mathbf{C}^3$. Assume that we have a homeomorphism φ of a neighborhood U_1 of 0 in U onto V_1 of y in V, such that $\varphi(X \cap U_1) = Y \cap V_1$ and $\varphi(0) = y$. First, we prove that Y has an isolated singularity at y. Let x be a non-singular point of $X \cap U_1$. The homeomorphism φ induces a germ of homeomorphism of the germ (X, x) onto $(Y, \varphi(x))$. To prove that Y has an isolated singularity at y, it is enough to show that the point $\varphi(x)$ is non-singular on Y. This fact is a consequence of a Theorem of A'Campo ([AC] Théorème 3) which states that the Lefschetz number of the monodromy of a Milnor fibration is not zero if and only if the hypersurface is non-singular. In fact, it is easy to see that the Milnor fibration of X at a non-singular point x is trivial and its fiber is contractible. Therefore the Milnor fibration of Y at $\varphi(x)$, which is homotopically isomorphic to the Milnor fibration of X at x, has contractible fibers and is trivial ([M] Lemma 2.13). By A'Campo's theorem this implies that Y is not singular at $\varphi(x)$. On the other hand, since the Milnor fibers of X at 0 and Y at y have the same homotopy type, their Milnor numbers are equal.

We put

$$\mu^3(X, 0) := \mu(X, 0).$$

Secondly, one can prove that there is an open Zariski dense subset Ω_2 of the space $\check{\mathbf{P}}^2$ of complex hyperplanes through 0 in $\mathbf{C}^3$ such that, for any $H \in \Omega_2$, the Milnor number $\mu(H \cap X, 0)$ does not depend on H. Then, for $H \in \Omega_2$,

$$\mu^{(2)}(X, 0) := \mu(X \cap H, 0).$$

Third, we consider the multiplicity $m(X, 0)$ of X at 0:

$$\mu^{(1)}(X, 0) := m(X \cap H, 0).$$

As it is easily seen for hypersurfaces, there is an open dense Zariski subset Ω_1 of the space $\mathbf{P}^2$ of complex lines through 0 in $\mathbf{C}^3$, such that, for any $\ell \in \Omega_1$, the Milnor number $\mu(X \cap \ell, 0)$ is finite and does not depend on ℓ. But in this case, $X \cap \ell$ is a zero dimensional hypersurface in ℓ, so that $\mu(X \cap \ell, 0) + 1$ is nothing but the multiplicity of X at 0.

Let

$$f = f_m + f_{m+1} + \dots$$

be the Taylor expansion of f at 0, where f_k is a homogeneous polynomial of degree k and m is the multiplicity of f at 0. It is known that, for a hypersurface X, the multiplicity of the function f defining X equals the multiplicity at 0 of X, i.e. the multiplicity of the local ring $\mathcal{O}_{X,0}$ (see [S] V A) §2).

Then, it is easy to show that one can choose

$$\Omega_1 = \mathbf{P}^2 - Proj|C_{X,0}|$$

where $Proj|C_{X,0}|$ is the projective curve associated to the reduced tangent cone of X at 0.

B. Teissier showed in [T] (Chap. 1 §2), that the 3-uple

$$\mu^*(X, 0) = (\mu^{(3)}(X, 0), \mu^{(2)}(X, 0), \mu^{(1)}(X, 0))$$

is an analytic invariant of the germ of hypersurface $(X, 0)$, i.e. if the local rings $\mathcal{O}_{X,0}$ and $\mathcal{O}_{X',0}$ of two 2-dimensional hypersurfaces X and X' at 0 are isomorphic, then, we have

$$\mu^*(X, 0) = \mu^*(X', 0).$$

Notice that, since $\mu^{(3)}(X, 0)$ is a topological invariant of $(X, 0)$, it is obviously an analytic invariant of the germ of hypersurface $(X, 0)$.

Recall that a hyperplane H is a limit of tangent hyperplanes of the hypersurface X at 0, if there is a sequence x_n of non singular points of X which converges to 0 such that the sequence of tangent hyperplanes T_{X,x_n} converges to H. Now, in [T] (Consequence of Proposition 2.9 of Chap. 1, see also [HL] Théorème 2.2), B. Teissier proves:

Theorem 1.2.2. *Let $X, 0$ be a germ of complex hypersurface in $\mathbf{C}^{n+1}, 0$ with an isolated singularity at 0. Let H be a complex hyperplane through 0. Then, the hyperplane is not a limit of tangent hyperplanes to X at 0 if and only if $X \cap H$ has an isolated singularity and $\mu(X \cap H, 0)$ is minimal.*

The preceding theorem allows us to define the open Zariski set Ω_2 considered above as the complement in $\check{\mathbf{P}}^2$ of the set of limits of tangent hyperplanes to X at 0. In the case of complex surfaces X in $\mathbf{C}^3$ having an isolated singularity at 0, this shows that a complex plane H of $\mathbf{C}^3$ through 0 is not a limit of tangent hyperplanes to X at 0 if and only if $X \cap H$ has an isolated singularity at 0 and $\mu(X \cap H, 0) = \mu^{(2)}(X, 0)$.

Proposition 1.2.3. *Let H be a complex plane of $\mathbf{C}^3$ through 0 so that $\mu(X \cap H, 0) = \mu^{(2)}(X, 0)$. The multiplicity of $X \cap H$ at 0 equals the multiplicity $\mu^{(1)}(X, 0)$ of X at 0.*

Proof. To prove this fact, it is enough to apply a result of the author in [L1] (see also [LR] §3), showing that, in an analytic family of plane curves having an isolated singularity at 0 with their Milnor numbers at 0 constant, the topology of these plane curves at 0 and, hence, their multiplicity at 0, do not vary. We obtain the assertion of our proposition by considering the analytic family of plane sections parametrized by the set Ω_2 of general planes of $\mathbf{C}^3$ through 0. Q.E.D.

Remark 1.2.4. In fact, a remarkable result of B. Teissier shows that, for any germ of complex hypersurface $X, 0$ in $\mathbf{C}^{n+1}$ with an isolated singularity, if the Milnor number $\mu(X \cap H, 0)$ is minimal, for any general flag

$$\{0\} \subset H_1 \subset \ldots \subset H_n \subset \mathbf{C}^{n+1} = H_{n+1}$$

in which the Milnor number $\mu(X \cap H_i)$ is minimal among i-dimensional sections of X at 0, we have

$$\mu^*(X \cap H, 0) = (\mu(X \cap H_n, 0), \ldots, \mu(X \cap H_1, 0)).$$

1.3. In the case of complex analytic surfaces, we can summarize the results of the preceding section by:

Proposition 1.3.1. *Let $X, 0$ be a germ of complex analytic surface in $\mathbf{C}^3$ with an isolated singularity at 0. A plane H of $\mathbf{C}^3$ through 0 is not a limit of tangent planes to X at 0 if and only if we have*

$$\mu(X \cap H, 0) = \mu^{(2)}(X, 0),$$

in which case, the plane H is not contain in the tangent cone $C_{X,0}$ of X at 0.

Using this result, J.P.G. Henry and the author prove in [HL] (Théorème 3.8):

Theorem 1.3.2. *Let $X, 0$ be a germ of complex analytic surface in $\mathbf{C}^3$. There are a finite number of complex generatrices of the tangent cone $C_{X,0}$ of X at 0, such that the set of limits of tangent planes $\mathcal{T}_{X,0}$ to X at 0 is the union of the set of limits of tangent planes to $C_{X,0}$ at 0 and the pencils of planes $\mathcal{L}_i$ $(1 \le i \le k)$ through these generatrices:*

$$\mathcal{T}_{X,0} = Proj|C_{X,0}|\check{}\cup \mathcal{L}_1 \cup \ldots \cup \mathcal{L}_k.$$

We call these generatrices the **exceptional tangents** of X at 0.

There are several ways to find the exceptional tangents of a complex analytic surface X at 0. One of the most useful ways is:

Proposition 1.3.3. *Let Ω be the set of finite projections of $X, 0$ into $\mathbf{C}^2, 0$ induced by linear projections of $\mathbf{C}^3$ onto $\mathbf{C}^2$ with a local degree equal to the multiplicity $m(X, 0)$ at 0. Let $p \in \Omega$ and $\Gamma(p)$ be the critical curve of p. The set of exceptional tangents of X at 0 is the set of tangents of $\Gamma(p)$ which do not depend on $p \in \Omega$.*

We shall give below generalizations of these Propositions in the case of normal surfaces.

§2. Tangents on Normal surfaces.

In all this paragraph, we shall consider a normal surface singularity (X, x) (this means that the local ring $\mathcal{O}_{X,x}$ of the germ is an integral domain and is integrally closed in its field of fractions). The criterion of Serre already used above shows that a surface singularity is normal if and only if it is isolated and its local ring is Cohen-Macaulay. We choose a representative X of of the germ (X, x) such that $X - \{x\}$ is non-singular and X is closed in an open neighbourhood of x in $\mathbf{C}^N$.

2.1. In [GL] (Définition 2.1) G. Gonzalez and M. Lejeune-Jalabert gave a definition of a **general hyperplane section** of X at x.

Definition 2.1.1. Let $\sigma\colon \overline{X}_1 \to X$ be the normalized blowing-up of the maximal ideal M which defines x on X. Let H be a hyperplane of $\mathbf{C}^N$ through x. We say that H is a general hyperplane for X at x (or $X \cap H$ is a general hyperplane section), if the strict transform of $X \cap H$ by σ does not go through the singular points of $\overline{X}_1$, intersects transversally in $\overline{X}_1$ the reduced exceptional divisor $\overline{E}$ of $\overline{X}_1$ and does not contain any non-singular point of $\overline{E}$ where the restriction of the normalisation of the blowing-up of X at x is critical.

Remark 2.1.2. Let $e\colon X_1 \to X$ be the blowing-up of X at x, call E the exceptional divisor of X_1. Call n the normalisation

$$n\colon \overline{X}_1 \to X_1$$

of X_1. Then, $X \cap H$ is a general hyperplane section if and only if strict transform of $X \cap H$ by e does not contain the images by n of the singular points of $\overline{X}_1$ and the ramification points of the map from $\overline{E}$ to E induced by n, and the hyperplane $Proj(H)$ intersects $Proj|C_{X,x}|$ at non-singular points transversally in $Proj(\mathbf{C}^N) = \mathbf{P}^{N-1}$.

In [GL] (§2) G. Gonzalez and M. Lejeune-Jalabert called **special generatrices** the generatrices of the tangent cone $C_{X,x}$ which correspond to the images by n of the singular points of $\overline{X}_1$ and the ramification points of the map from $\overline{E}$ to E induced by n.

Now, recall that a hyperplane H is a tangent hyperplane at a non-singular point y of X, if it contains the tangent plane $T_{X,y}$. Then the hyperplane H is a limit of tangent hyperplanes of the surface X at x, if there is a sequence x_n of non singular points of X which converges to x and a sequence of complex hyperplanes H_n tangent to X at x_n such that H_n converges to H.

Of course, the set of limits of tangent hyperplanes of X at x is algebraic. In fact, one considers the closure $C(X)$ in $X \times \check{\mathbf{P}}^{N-1}$ of the set of points (y, H), where y is a non-singular point of X and H is a hyperplane tangent to X at y. Using a classical result of Remmert (see [RS] Satz 13), one can prove that $C(X)$ is a complex analytic space. The projection onto X induces a morphism

$$\kappa \colon C(X) \to X$$

which is analytic and proper. A result of Chow ([C], see [GR] Chapter 9 §5) implies that the fiber of κ over x which is analytic and closed in $\check{\mathbf{P}}^{N-1}$ is actually algebraic. The space $C(X)$ is called the conormal space of X in $\mathbf{C}^N$.

In his thesis, Jawad Snoussi proved:

Theorem 2.1.3. *A hyperplane H of $\mathbf{C}^N$ is general for X at x if and only if it is not a limit of tangent hyperplanes.*

In view of Teissier's theorem 1.2.2, J. Snoussi proves:

Theorem 2.1.4. *A hyperplane H of $\mathbf{C}^N$ is general for X at x if and only if the number of points in $H \cap Proj|C_{X,x}|$ equals the degree of $Proj|C_{X,x}|$ and the generalized Milnor number of Buchweitz and Greuel of the curve $H \cap X$ at x is minimal.*

In [BG] R. Buchweitz and G.-M. Greuel have defined a generalized Milnor number for any reduced curve. Namely let C be a reduced curve and O be a point of C. Denote the local ring of C at O by $\mathcal{O}_{C,O}$ and let

$\overline{\mathcal{O}}_{C,O}$ be its normalisation. Define $\delta(C,O)$ to be the dimension of the complex vector space $\overline{\mathcal{O}}_{C,O}/\mathcal{O}_{C,O}$

$$\delta(C,O) := \dim_{\mathbf{C}} \overline{\mathcal{O}}_{C,O}/\mathcal{O}_{C,O}.$$

Then, the generalized Milnor number of C at O is

$$\mu(C,O) := 2\delta(C,O) - r(C,O) + 1$$

where $r(C,O)$ is the number of analytic branches of C at O.

In [M] (Theorem 10.5), J. Milnor proved this relation between the Milnor number and $\delta(C,O)$, when C,O is the germ of a reduced plane curve.

A topological interpretation of the Milnor number for a curve (see [BG]) on a normal surface singularity defined by one equation is the following. We may assume that the singularity is locally embedded in some non-singular space $\mathbf{C}^N$ and that the curve is given by $\varphi = 0$, where φ is a holomorphic function defined in a neighbourhood of the singularity in $\mathbf{C}^N$. Then, there is $\epsilon_0 > 0$, such that, for any ϵ, $\epsilon_0 > \epsilon > 0$, there is $\eta_\epsilon > 0$, such that, for any complex number t, $\eta_\epsilon > |t| > 0$, the Milnor number of the curve singularity is equal to the first Betti number of the Riemann surface $B_\epsilon(0) \cap \{\varphi = t\}$, where $B_\epsilon(0)$ is the open ball of $\mathbf{C}^N$ centered at 0 with radius ϵ.
The key points to prove 2.1.3 and 2.1.4 are results of R. Buchweitz and G.-M. Greuel who show the semi-continuity of their generalized Milnor number in analytic families of curves and the equiresolution of analytic families with generalized Milnor number constant when these families are non-singular outside a section (see [BG]).
2.2. An important consequence of Snoussi's result (compare with Theorem 1.3.2) is:

Theorem 2.2.1. *Let (X,x) be a germ of normal complex analytic surface in $\mathbf{C}^N$. The set of limits of tangent hyperplanes $\mathcal{H}_{X,x}$ to X at x is the union of the set of limits of tangent hyperplanes to the tangent cone $C_{X,x}$ of X at x and the linear systems $\mathcal{L}_i$ $(1 \le i \le k)$ of hyperplanes through the special generatrices of $C_{X,x}$:*

$$\mathcal{H}_{X,x} = Proj|C_{X,x}|\check{} \cup \mathcal{L}_1 \cup \ldots \cup \mathcal{L}_k.$$

As a consequence, following the terminology already used in [LT1] (1.3.2), it will be more convenient to call the special generatrices of $C_{X,x}$ **the exceptional tangents** of (X,x).
In fact, J. Snoussi also obtains the description of the set of limits of tangent spaces to X at x.

Theorem 2.2.2. *Let (X,x) be a germ of normal complex analytic surface in $\mathbf{C}^N$. The set of limits of tangent spaces to $\mathcal{T}_{X,x}$ to X at x is the union of the set of limits of tangent spaces to the tangent cone $C_{X,x}$ at its vertex and of 1-dimensional subspaces $\mathcal{G}_i$ $(1 \leq i \leq k)$ of the grassmanian space $G(2,N)$ of 2-planes in $\mathbf{C}^N$ through x which contain the exceptional tangents ℓ_i $(1 \leq i \leq k)$.*

Note that the set of limits of tangent spaces to X at x is algebraic. This was predictable since it is the fiber of the Nash modification

$$\nu\colon \tilde{X} \to X$$

where $\tilde{X}$ is the closure in $X \times G(2,N)$ of the set of points $(y, T_{X,y})$, where y is a non-singular point of X and $T_{X,y}$ is the tangent space to X at y, and ν is induced by the projection onto X. A theorem of Remmert (see [RS] Satz 13) implies that $\tilde{X}$ is an analytic space.

2.3. From the results of [LT2], we can generalize the result of 1.3.3. Namely, we have:

Theorem 2.3.1. *Let Ω be the set of finite projections of $(X,0)$ into $(\mathbf{C}^2, O)$ induced by linear projections of $\mathbf{C}^N$ onto $\mathbf{C}^2$ and which have a local degree equal to $m(X,x)$ at x. Let $p \in \Omega$ and $\Gamma(p)$ be the critical curve of p. The set of exceptional tangents of X at x is the set of tangent lines of $\Gamma(p)$ which do not depend on $p \in \Omega$.*

Proof. We shall adapt the proof of [LT2] (Théorème 2.1.1) to the case of dimension 2.

Consider the blowing-up

$$e'\colon EC(X) \to C(X)$$

of the analytic subspace $\kappa^{-1}(x)$ in the conormal space $C(X)$ of X. Of course, it factors through the blowing-up

$$e\colon X_1 \to X$$

of the point x in X. We have the following commutative diagram

$$\begin{array}{ccccccc}
X \times \check{\mathbf{P}}^{N-1} \times \mathbf{P}^{N-1} & \supset & EC(X) & \xrightarrow{e'} & C(X) & \subset & X \times \check{\mathbf{P}}^{N-1} \\
 & & \downarrow \kappa' & & \downarrow \kappa & & \\
X \times \mathbf{P}^{N-1} & \supset & X_1 & \xrightarrow{e} & X & &
\end{array}$$

We may consider that $E_1 := (e' \circ \kappa)^{-1}(x) = (\kappa' \circ e)^{-1}(x)$ is embedded in $\check{\mathbf{P}}^{N-1} \times \mathbf{P}^{N-1}$. The spaces $D := e^{-1}(x)$ and $D_1 := \kappa^{-1}(x)$ are contained in $\check{\mathbf{P}}^{N-1}$ and $\mathbf{P}^{N-1}$ respectively. Let $E_1(\alpha)$, $\alpha \in A$, be the irreducible components of E_1. Let $D_1(\alpha)$ and $D(\alpha)$ be the images of $E_1(\alpha)$ by κ' and e'. We have (see [LT2] Théorème 2.1.1)

Lemma 2.3.2. *For each $\alpha \in A$, the variety $D(\alpha)$ is the dual of $D_1(\alpha)$ and the correspondence is given by $E_1(\alpha)$.*

Proof. In fact, this is a consequence of a lemma of Whitney (see [L3]) which states that, for any $(H, \ell) \in E_1$, we have

$$\ell \subset H.$$

The dimension of the components of E_1 is $N-2$. If $D_1(\alpha)$ has dimension 0, $E_1(\alpha)$ is isomorphic to $D(\alpha)$ and consists of all the hyperplanes which contain the point $\{D_1(\alpha)\}$. If $D_1(\alpha)$ has dimension 1, it is a component of the Projective set associated to the tangent cone and at a general point l_1 of $D_1(\alpha)$, the points (H, l_1) in $\kappa^{-1}(l_1)$ consists of hyperplanes containing the tangent plane to the tangent cone $|C_{X,x}|$ along the line l_1 (see [L3] (Théorème 1.2.1)). Then, the image of $E_1(\alpha)$ by e' is the closure of the set of hyperplanes which contain the tangent planes at non-singular points of the component $D_1(\alpha)$ of $Proj(|C_{X,x}|)$ which contains l_1. By definition $E_1(\alpha)$ is the correspondence variety of $D_1(\alpha)$ and its dual variety $e'(E_1(\alpha)$.

Now, we can prove Theorem 2.3.1. Let $\mathcal{D}(A)$ be the projective subvariety of $\check{\mathbf{P}}^{N-1}$ which consists of the hyperplanes through x which contain a codimension 2 space A through x. Then, $\mathcal{D}(A)$ is isomorphic to a projective space of dimension 1. Let p be the projection on $\mathbf{C}^2$ induced by the linear projection p_A of $\mathbf{C}^N$ onto $\mathbf{C}^2$ with Kernel A. When A is sufficiently general, say if A belongs to an open Zariski subset Ω' of the Grassmannian manifold of codimension 2 projective subspaces in $\mathbf{P}^{N-1}$, the projection p_A restricted to (X, x) has local degree equal to the multiplicity $m(X, x)$, so $p_A \in \Omega$. With p_2 being the map from $EC(X)$ into $\check{\mathbf{P}}^{N-1}$, we observe that the subspace $p_2^{-1}(\mathcal{D}(A))$ of $EC(X)$ is non-empty, if $m(X, x) \geq 2$, so it is a curve. It is easy to show that the curve $\Gamma(p)$ is

$$\Gamma(p) = e \circ \kappa'(p_2^{-1}(\mathcal{D}(A)).$$

Therefore the curve $\kappa'(p_2^{-1}(\mathcal{D}(A))$ is the strict transform of $\Gamma(p)$ by e and it intersects the exceptional divisor at points which correspond to the tangents of $\Gamma(p)$ at x. Since $\mathcal{D}(A)$ has dimension 1 it meets the set of hyperplanes $E(\alpha) := e'(E_1(\alpha))$ which contain $D_1(\alpha)$ when $D_1(\alpha)$ has dimension 0. Therefore $\kappa'(p_2^{-1}(\mathcal{D}(A))$ contain the sets $D_1(\alpha)$ of dimension 0. The other points of $\kappa'(p_2^{-1}(\mathcal{D}(A))$ contained in the exceptional divisor of e are in the dual of the intersections of $\mathcal{D}(A)$ and the sets $D(\alpha)$ dual to the components of $Proj(|C_{X,x}|)$. Since $|C_{X,x}|$ is a cone, these points are the generatrices of $|C_{X,x}|$ which are the closure of the components of the critical locus of the restriction to the non-singular part of $|C_{X,x}|$ of the linear projection p_A. These latter generatrices depend on the projection p_A.

Hence, this shows that the tangent lines in the tangent cone of $\Gamma(p)$ consist of lines of $|C_{X,x}|$ which depend on $p \in \Omega$ and of the exceptional tangents which do not depend on $p \in \Omega$.

§3. Resolutions of Normal surfaces.

3.1. Let (X, x) be a normal complex surface singularity. We choose a representative X of (X, x) such that $X - \{x\}$ is non-singular.

Definition 3.1.1. We say that a complex analytic map $\pi\colon Z \to X$ is a resolution of singularity of (X, x), if

i) the space Z is non-singular;
ii) the map π is proper;
iii) the map π induces an isomorphism of $Z - \pi^{-1}(x)$ onto $X - \{x\}$ and $Z - \pi^{-1}(x)$ is dense in Z.

An important result by R. Walker and O. Zariski ([W] and [Z] VI §21) is:

Theorem 3.1.2. *Any normal surface singularity has a resolution obtained by composing a finite number of compositions of a point blowing-up and a normalisation.*

In [BPV] (III §6), one can find a more geometrical construction of a resolution of a normal complex surface singularity (due to Jung [J], see [Hi]) by using the embedded resolution of the discriminant of a finite projection of the singularity onto a 2-dimensional complex plane.

Remark 3.1.3. Notice that there are many resolutions of the singularity (X, x). For instance, the identity is a resolution of the non-singular germ $(\mathbf{C}^2, 0)$ and the blowing-up of the point 0 in $(\mathbf{C}^2, 0)$ is also a resolution.

We shall see below that all resolutions are obtained from one of them.

3.2. Given a resolution π of the complex analytic normal surface singularity (X, x), most of the topological information of (X, x) is obtained from the geometry of the space $\pi^{-1}(x)$. In fact, we have first:

Theorem 3.2.1 (Main theorem of Zariski). *Let π be a resolution of a complex analytic normal surface singularity. The space $\pi^{-1}(x)$ is connected.*

One may find proof of this result in [H] (Chap. III, Corollary 11.4). The main argument comes from the fact that, since (X, x) is a normal singularity, there are regular neighbourhoods U of x in X whose boundary ∂U (called the **local link** of X at x) is a connected 3-manifold.

Theorem 3.2.1 shows that, if π is a resolution of a complex analytic normal surface singularity the fiber is either a point or a connected curve. It is a point only if the surface is non-singular and π is the identity, as a consequence of the following theorem of D. Mumford (see [Mu])

Theorem 3.2.2. *The local link of a normal surface singularity (X, x) is simply connected if and only if X is non-singular at the point x.*

Therefore, if the normal surface singularity (X, x) is really singular, for any resolution π of (X, x), the fiber $\pi^{-1}(x)$ is a connected curve.
There is another important theorem of D. Mumford ([Mu], see §1) which characterizes the fiber $\pi^{-1}(x)$ (the **exceptional fiber** of π) of a resolution $\pi\colon Z \to X$ of (X, x), when it is a curve. Let $E_1, \ldots, E_k$ be the irreducible components of $\pi^{-1}(x)$.

Theorem 3.2.3. *The intersection matrix $(E_i.E_j)_{1\leq i,j\leq k}$ is definite negative.*

This fact allows us to associate some important combinatorial invariants to a resolution π of (X, x). For instance, there is a theorem of Zariski (see [A] Proposition 2) which states

Theorem 3.2.4. *Let I be negative definite bilinear form on a free abelian group G generated by $e_1, \ldots, e_k$, there are elements $z \neq 0$*

$$z = \sum_1^k m_i e_i$$

of this group such that

$$I(z, e_i) \leq 0$$

for any i, $1 \leq i \leq k$. Furthermore, these elements make a semi-group $E^+(I)$ which has a smallest element $z_0 = \sum_1^k a_i e_i$, such that $a_i \geq 1$, for any i, $1 \leq i \leq k$. We shall call z_0 the fundamental element of I.

Definition 3.2.5. The fundamental element of the intersection form I on the free abelian group generated by the components of a resolution π of a complex normal surface is called the **fundamental cycle** of this resolution. The semi-group $E^+(I)$ is called the **Lipman semi-group** of the resolution π and is also denoted by $E^+(\pi)$.

An important remark is that, given a resolution $\pi\colon Z \to X$ of (X, x), any function $\varphi \in \mathcal{O}_{X,x}$ defines a divisor $(\varphi \circ \pi)$ on Z and the compact part of this divisor is an element of $E^+(\pi)$. For example, in a resolution π for which the inverse image $\pi^* M_{X,x}$ of the maximal ideal $M_{X,x}$ of $\mathcal{O}_{X,x}$ is invertible, the maximal cycle of the resolution is given by the compact part of the divisor given by a general element of $M_{X,x}$ (see [Y]).

In [Li] (§18), J. Lipman proved that the semi-group $E^+(\pi)$ of a resolution π of a rational singularity is given by the general elements of ideals I of $\mathcal{O}_{X,x}$ whose inverse images $\pi^* I$ are invertible.

3.3. It is useful to recall the notion of minimal resolution.

Definition 3.3.1. A resolution of a surface singularity (X, x) is called **minimal** if its exceptional divisor does not contained a non-singular rational curve of self-intersection -1. Such a curve is called an **exceptional curve of the first kind**.

The basic theorem about surface resolutions is (see [La] Chapter 5):

Theorem 3.3.2. *Minimal resolutions of a surface singularity (X, x) are isomorphic, i.e. if X is a representative of (X, x) such that $X - \{x\}$ is non-singular, $\pi_1\colon Z_1 \to X$ and $\pi_2\colon Z_2 \to X$ are two minimal resolutions of (X, x), there is an isomorphism $\varphi\colon Z_1 \to Z_2$, such that $\pi_2 \circ \varphi = \pi_1$.*

A consequence is the factorization theorem:

Corollary 3.3.3. *Let $\pi\colon Z \to X$ be a resolution of the surface singularity (X, x) and $\pi_0\colon Z_0 \to X$ be a minimal resolution of the surface singularity (X, x). There is a unique holomorphic map $\psi\colon Z \to Z_0$, such that $\pi = \pi_0 \circ \psi$, and ψ is the composition of a finite sequence of point blowing-ups.*

§4. General sections and Tjurina-Spivakovsky components.

4.1. Given a resolution $\pi\colon Z \to X$ of a normal surface singularity (X, x), let $E_1, \ldots, E_k$ be the components of the exceptional divisor $\pi^{-1}(x)$ of π. We consider a cycle a in the Lipman semi-group E^+ of π, so, for $1 \le i \le k$, we have $a.E_i \le 0$. The **Tjurina-Spivakovsky components** of a (compare with [Sp] Chap. III, Definition 3.1) are the maximal connected curves contained in the exceptional divisor $\pi^{-1}(x)$ whose components are components E_i such that $a.E_i = 0$. Therefore, the components of $\pi^{-1}(x)$ which are not contained in a Tjurina-Spivakovsky component of a are the components E_j of $\pi^{-1}(x)$ such that $a.E_j < 0$.

Consider an ideal I of the local ring $\mathcal{O}_{X,x}$ such that $I\mathcal{O}_Z$ is locally invertible. Following Lipman (see [Li] §18), the ideal $I\mathcal{O}_Z$ defines an element a_I in the semi-group E^+. One can check that a_I is the compact part of the divisor on Z defined by a general element of the ideal I. The following lemma shows the interest of Tjurina-Spivakovsky components:

Lemma 4.1.1. *Let $\pi\colon Z \to X$ be a resolution of a normal surface singularity (X, x) such that the maximal ideal M of $\mathcal{O}_{X,x}$ defines a locally invertible ideal $M\mathcal{O}_Z$. Let $q\colon Z \to \overline{X_1}$ be the factorisation of π through the normalized blowing-up $\overline{e}\colon \overline{X_1} \to X$ of the point $\{x\}$ in X. The connected curves of the exceptional divisor of π which are mapped by q to the singular points of $\overline{X}_1$ are Tjurina-Spivakovsky components of the cycle defined by $M\mathcal{O}_Z$.*

Proof. Since $M\mathcal{O}_Z$ is locally invertible, the resolution π factorizes through the blowing-up of M, i.e. the blowing-up of the point $\{x\}$ in X. Since Z is non-singular, it is also normal, so this factorisation lifts to the normalized blowing-up of the point $\{x\}$ in X.

$$\begin{tikzcd} Z \arrow[d, "q"] \arrow[dr, "\pi"] & \\ \overline{X_1} \arrow[r, "\overline{e}"] & X \end{tikzcd}$$

Let l be a general element of the maximal ideal. As we have noticed above, the cycle defined by M on Z coincide with the compact part of the divisor defined by l on Z. One can see that the components E_i of the exceptional divisor of π which are not in a Tjurina-Spivakovsky component, i.e. such that $a_M.E_i < 0$, are the components of the exceptional divisor which are intersected by the strict transform of the general element l. These components are in fact the strict transforms by π of the components of the projective set associated to the tangent cone of X at x. So, the images by the map q of the components contained in a Tjurina-Spivakovsky component of a_M must be points. Therefore q is obtained by contracting the Tjurina-Spivakovsky components of a_M. Since q is a resolution of the singularities of $\overline{X_1}$, the images of the Tjurina-Spivakovsky components of a_M contain the singular points of $\overline{X_1}$. If Z is an arbitrary resolution of (X, x), it is possible that, by contracting a Tjurina-Spivakovsky component of a_M, one obtains a non-singular point. However, if π is the minimal resolution of (X, x) in which π^*M is invertible, the Tjurina-Spivakovsky components of a_M all contract in a singular point of X_1. This fact is consequence of the observation that in such resolution none of the components in a Tjurina-Spivakovsky component of a_M is a curve of the first kind (see 3.3.3).

Remark 4.1.2. In [Tj], G. Tjurina found that, in the case of a rational singularity (X, x), the Tjurina-Spivakovsky components of the fundamental cycle of the minimal resolution of (X, x) contract into the singular points of the blowing-up of X at $\{x\}$. The reason is that, for any resolution π of a rational singularity, the inverse image $\pi^* M$ of the maximal ideal of x in X is invertible and that the blowing-up X_1 of the point x in X is already normal.

M. Spivakovsky extended the notion of Tjurina-Spivakovsky components to any cycle in the Lipman semi-group of a resolution of a rational singularity (see [Sp] Chap. III, Definition 3.1).

We have naturally generalized the definition of Spivakovsky to resolutions of general normal surface singularities, but unlike the case of rational singularities, the Lipman semi-group of a resolution π might be different from the semi-group of ideals I of the local ring $\mathcal{O}_{X,x}$ whose lifting $\pi^* I$ is invertible on Z.

Recall that J. Snoussi proved that the exceptional tangents of a normal surface singularity are the special generatrices of Gonzalez and Lejeune (see above in 2.1.2). So the result of Lemma 4.1.1 says that the images of the singular points of X_1 under normalisation are exceptional tangents. On the other hand, images of the singular points of the exceptional set of the normalized blowing-up under normalisation give also special generatrices, these images are also exceptional tangents. In particular, singular generatices of tangent cones of rational singularities are exceptional tangents, so that Snoussi gives a positive answer to a question of M. Spivakovsky in [Sp] (Chap. III, Remark 3.12).

An interesting corollary is the following:

Proposition 4.1.3. *If a normal surface singularity has no exceptional tangent, the normalized blowing-up of its singular point is non-singular.*

In [LT2], we proved that an isolated singularity of surface which has a reduced tangent cone and which has no exceptional tangent is equisingular to its tangent cone. In the case of germs of hypersurface in $\mathbf{C}^3$, we proved that if there are no exceptional tangents, the tangent cone is reduced. Of course, in these two cases, the normalized blowing-up of the singular point is non-singular, since the blowing-up of the singular point is already non-singular.

4.2. Examples. In general the inverse image of the maximal ideal by a resolution of a normal surface singularity might not be invertible (see [Y]). For example, consider the minimal resolution of the hypersurface $x^2 + y^3 + z^6 = 0$ (see [GL]). One can show that the exceptional divisor of the minimal resolution is a non-singular elliptic curve of self intersection -1. In order to obtain a resolution in which the inverse image of the maximal ideal is invertible, we need to blow-up a point in this elliptic curve. In this new resolution where the inverse of the maximal ideal is invertible the Tjurina-Spivakovsky component of the maximal cycle is the elliptic curve. This elliptic curve contracts to a singularity whose minimal resolution has an exceptional divisor whose unique component is this elliptic curve with self-intersection -2. It is easy to show that there is only one exceptional tangent in this example.

Another interesting example is the hypersurface $x^2 + y^4 + z^4 = 0$. For this case, in the minimal resolution the inverse image of the maximal ideal is an elliptic curve with self-intersection -2. In fact, the normalized blowing-up of the singular point is non-singular. However, we have 4 exceptional tangents which correspond to the ramification points of the projection of the elliptic curve on the non-singular rational curve which is the projective curve associated to the tangent cone of the singularity.

The two preceding examples are simple elliptic singularities in the sense of K. Saito (see [Sa]). The singularity of $x^2+y^3+z^6=0$ is simple elliptic of type $\tilde{E}_8$ and the singularity of $x^2+y^4+z^4=0$ is of type $\tilde{E}_7$. It is interesting to see that the blowing-up of $x^2+y^3+z^6=0$ is normal and contains one singularity which is equisingular to $x^2+y^4+z^4=0$, but not analytically isomorphic to it. In fact, using the deformation of $\tilde{E}_8$ type singularities given in Satz 1.9 of [Sa], one can find a deformation of $x^2+y^3+z^6=0$ which gives $x^2+y^4+z^4=0$ in its blowing-up.

4.3. A natural question is to decide if, in terms of limits of tangent hyperplanes or in terms of limits of tangent spaces, the singularities of the normalized blowing-up are simpler than the given one.

Another interesting problem related to the preceding question is to find an effective bound on the number of normalized blowing-up necessary to solve a normal surface singularity. The description given above is a first step in an attempt to understand the geometry of the normalized blowing-up.

It is interesting to notice that the geometry involved in a normalized blowing-up is similar to the one used by M. Spivakovsky in [Sp] to resolve a normal surface singularity by a finite composition of normalized Nash modifications. In some sense, these two processes of resolutions are dual. The problem of giving an effective bound on the number of normalized Nash modifications needed to solve a normal surface singularity is also not solved. In [Sp], M. Spivakovsky gives a detailed study of the complexity of the resolution graph of the minimal resolution of a minimal rational singularity and the behaviour of this complexity after a normalized Nash modification. This is the key step to obtain the resolution of normal surface singularities by composition of a finite number of normalized Nash modifications.

Since rational surface singularities are absolutely isolated (see [Tj]), the complexity of the resolution graph of the minimal resolution decreases strictly after each blowing-up. However, it is not trivial to get a bound of the number of point blowing-ups needed in order to reach a resolution from the local ring of the singularity without having to calculate the resolution graph of the minimal resolution.

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A Chern-Weil theory for Milnor classes

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Abstract.

Dans un travail antérieur ([BLSS]), en collaboration avec Brasselet, Suwa et Seade, nous avons presenté une théorie des classes de Milnor pour les ensembles analytiques complexes V qui sont localement des intersections complètes dans une variété holomorphe ambiante M sans singularité. Le principe consistait à comparer, dans l'homologie $H_{2*}(V)$, deux théories des classes de Chern de V, les classes de Schwartz-MacPherson $c_*^{\mathrm{SMP}}(V)$ et les classes virtuelles $c_*^{\mathrm{vir}}(V)$ (encore appelées de Fulton-Johnson): ces deux théories sont égales lorsque V est lisse, et coïncident alors avec l'image des classes de Chern usuelles par la dualité de Poincaré. Dans le cas général, leur différence se "localise" près de la partie singulière S de V: il existe un élément $\mu_*(V,S) \in H_{2*}(S)$, défini naturellement, dont l'image dans $H_{2*}(V)$ est égale à $(-1)^n\left[c_*^{\mathrm{Vir}}(V) - c_*^{\mathrm{SMP}}(V)\right]$. En outre, si $(S_\alpha)_\alpha$ désigne la famille des composantes connexes de S, la composante $\mu_0(V,S_\alpha)$ de $\mu_0(V,S)$ sur $H_0(S_\alpha)$ est égale au nombre de Milnor de S_α dans tous les cas où celui-ci a déjà été défini.

Dans [BLSS], nous utilisions à la fois des méthodes de Topologie et de Géométrie différentielle. Nous proposons ici une version de pure Géométrie différentielle.
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§1. Introduction

In a joint work with J.P. Brasselet, J. Seade and T. Suwa, we presented in [BLSS] a theory of Milnor classes for singular compact subvarieties V which are locally complete intersections in an analytic complex manifold M. The principle was to compare, in the homology $H_{2*}(V)$ of V, two different theories for Chern classes of V, namely the Schwartz-MacPherson classes $c_*^{\mathrm{SMP}}(V)$ and the virtual classes $c_*^{\mathrm{Vir}}(V)$, both of

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them coinciding with the Poincaré dual of usual Chern classes $c^{n-*}(V)$ in cohomology, when V is non-singular of complex dimension n. The difference $c_*^{\mathrm{Vir}}(V)-c_*^{\mathrm{SMP}}(V)$ of these two Chern classes is in fact localized near the singular part S of V, i.e. there exists a well defined element $\mu_*(V,S)$ in $H_{2*}(S)$, whose image in $H_{2*}(V)$ is equal to $(-1)^n\left[c_*^{\mathrm{Vir}}(V)-c_*^{\mathrm{SMP}}(V)\right]$. Furthermore, denoting by $(S_\alpha)_\alpha$ the family of connected components of S, the component $\mu_0(V,S_\alpha)$ of $\mu_0(V,S)$ on $H_0(S_\alpha)$ is equal to the Milnor number of S_α any time this one has already been defined (i.e. for S_α being an isolated point by Milnor ([Mi]) in case of hypersurfaces and Hamm ([H]) in any codimension, and for V being a hypersurface with general compact S_α by Parusiński ([P])). Notice also that such a theory for Milnor classes has been suggested by Yokura ([Y]), and given for complex hypersurfaces by Aluffi ([A2]) and Parusiński-Pragacz ([PP3]).

Both methods of topology and differential geometry were mixed in [BLSS]. In this paper, we wish to present the theory from a unified point of view, only in differential geometry. [This implies in particular that we use real coefficients in cohomology and homology, in fact as in [BLSS] while the theory with integral coefficients could have been defined there].

Most of the ideas in this paper are already in [BLSS], to which we refer also for examples. The main novelty is the explicit and systematical use of the Čech-de Rham complex with three kinds of open sets: the "ambiant" open set $\tilde{U}_A = M - V$, a tubular neighborhood $\tilde{U}_0$ of the regular part V_0 of V, and regular neighborhoods $\tilde{U}_\alpha$ of the S_α's. In fact, because it may happen that the differential forms that we are going to consider have the required properties only near some skeleton of a convenient cellular structure of M, we preferably use the image by integration of this Čech-de Rham complex into the cellular cochains (see [Le]). Furthermore, at least in a first step, instead of comparing the two theories of SMP and virtual classes in the homology $H_{2*}(V)$, it seems to us more natural to work in $H^{2(m-*)}(M, M-V)$ where $2m = dim_{\mathbb{R}} M$ (as originally in fact for SMP classes in [MHS]), Alexander duality $A : H^{2(m-*)}(M, M-V) \to H_{2*}(V)$ being an isomorphism when V is compact. There are two reasons for this: first it makes sense even if V or S is not compact, and secondly we do it implicitely in any case, because of the factorization $P_V = A \circ \tau$ at the chain and cochain level for V and S compact, where $P_V : H^{2n-*}(V) \to H_*(V)$ denotes the Poincaré homomorphism ($2n = dim_{\mathbb{R}} V$), and $\tau : H^{2n-*}(V) \to H^{2m-*}(M, M-V)$ the Thom-Gysin homomorphism (see [Br]). It turns out that this Thom-Gysin homomorphism is very easy to write down and compute in our framework, V being not necessarily compact.

Therefore, the organisation of the paper is the following: we describe in section 2 the geometrical situation that we are going to study. Main tools, such as the integration over suitable subcomplexes of the Čech-de Rham complex, or the Chern-Weil theory, are recalled in section 3. Section 4 is devoted to the computation of the Thom-Gysin homomorphism using differential geometry, section 5 to that of virtual classes, and section 6 to that of SMP classes. The Milnor classes are defined in section 7, using only radial frame fields such as in [Scl] for the original definition of SMP classes. In section 8, finally, we sketch a transcription of the point of view adopted in [BLSS], using more general frame fields.

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§2. Locally complete intersections.

Let $E \to M$ be a holomorphic vector bundle of rank k on acomplex manifold M of complex dimension $m = n + k$. Let s be a holomorphic section of E, and V be the zero set of s. If we assume furthermore s to be generically transverse to the zero section, the section s is then automatically regular, and the components of s with respect to a local trivialization generate the ideal of (local) holomorphic functions vanishing on V (after [T]). Thus, V is a **locally complete intersection in** M. The restriction of E to the regular part V_0 of V may be canonically identified with the normal bundle of V_0 in M. Thus $E|_V$ is an extension to all of V of this normal bundle . We still call it **normal bundle** to V as in the non singular case. The bundle $E|_V$ depends only on V and not on (E, s).

The natural projection $\pi_0 : TM|_{V_0} \to E|_{V_0}$ may be extended as a (smooth) projection $\pi : TM|_{\tilde{U}_0} \to E|_{\tilde{U}_0}$ (no more unique but it does not matter) on any tubular neighborhood $\tilde{U}_0$ of V_0, the kernel H of π being a smooth bundle on $\tilde{U}_0$ extending TV_0.

Let Σ be an analytic subset of V containing the singular part of V. After Lojasiewicz, there exists a smooth triangulation (K) of M adapted to V and Σ, (i.e. having V and Σ as subcomplexes). Denote respectively by (K') and $(K")$ the first and the second barycentric subdivision of K, and by (D) a smooth cellular structure dual to $(K")$.

Denoting by $(S_\alpha)_\alpha$ the set of connected components of Σ, we shall make the following assumption: each S_α is either included in the regular part V_0 of V or is a connected component of the singular part $Sing(V)$, but none of them intersects simultaneously V_0 and $Sing(V)$. In fact, once fixed the homological dimension $*$ in which we wish to compute $\mu_*(V,S)$ it is sufficient to assume that the intersection $S_\alpha \cap (D)^{2(m-*)}$ of S_α with the $2(m-*)$ skeleton of (D) does not intersect simultaneously V_0 and $Sing(V)$.

Let $\tilde{U}_A = M - V$ (the index "A" meaning "ambiant"). Let $\tilde{U}_\alpha$ be the interior of the link of S_α for (K'), and $\tilde{U}_1 = \bigcup_\alpha \tilde{U}_\alpha$. From now on, choose for tubular neighborhood $\tilde{U}_0$ of V_0 the interior of the link of V_0 for (K'). Then, $\tilde{\mathcal{U}} = (\tilde{U}_A, \tilde{U}_0, \tilde{U}_1)$ is a covering of M by open sets, such that $\tilde{U}_\alpha$ is a regular neighborhood of S_α, $\tilde{U}_0$ is a tubular neighborhood of V_0, and $U(V) = \tilde{U}_0 \cup \tilde{U}_1$ is a regular neighborhood of V, which is covered by $\mathcal{U} = (\tilde{U}_0, \tilde{U}_1)$. Furthermore, $\tilde{U}_\alpha \cap \tilde{U}_\beta = \emptyset$ for $\alpha \neq \beta$.

We have $V_0 = \tilde{U}_0 \cap V$. Let $U_\alpha = \tilde{U}_\alpha \cap V$ and $U_1 = \tilde{U}_1 \cap V$.

We define now a honeycomb system of cells $(\tilde{R}_A, \tilde{R}_0, \tilde{R}_1 = \bigcup_\alpha \tilde{R}_\alpha)$ (see the definition in [Le]) adapted to the open covering $\tilde{\mathcal{U}}$ of M, in the following way:
Let $\tilde{R}_A$ be the union of the $(K")$-simplices which do not intersect V.
Let $\tilde{R}_0$ be the union of the $(K")$-simplices which intersect V_0 but not Σ.
Let $\tilde{R}_\alpha$ be the union of the $(K")$-simplices which intersect S_α.

As usually, we denote by $\tilde{R}_{A0}$, $\tilde{R}_{A1} = \bigcup_\alpha \tilde{R}_{A\alpha}$, $\tilde{R}_{01} = \bigcup_\alpha \tilde{R}_{0\alpha}$, and $\tilde{R}_{A01} = \bigcup_\alpha \tilde{R}_{A0\alpha}$ the intersections of the above honeycombs, with suitable orientations. In fact, we shall often omit the *tilde* any time that the given set does not intersect V (i.e. when A does not occur in the indices). If it does, the omission of the *tilde* means that we take the intersection with V: for instance, $R_A = \tilde{R}_A$, $R_{A0} = \tilde{R}_{A0}$ and $R_{A\alpha} = \tilde{R}_{A\alpha}$, while $R_0 = \tilde{R}_0 \cap V$, $R_\alpha = \tilde{R}_\alpha \cap V$ and $R_{0\alpha} = \tilde{R}_{0\alpha} \cap V$...

We also write $\tilde{R} = \tilde{R}_0 \cup \tilde{R}_1$, with $\partial \tilde{R} = \tilde{R}_{A0} \cup \tilde{R}_{A1}$. Let $b\tilde{R} = \partial \tilde{R} \cup \tilde{R}_{01} = \tilde{R}_{A0} \cup \tilde{R}_{A1} \cup \tilde{R}_{01}$.

For any $(K")$-subcomplex X of M, we denote by $\mathcal{T}_D(X)$ the union of the (D) cells intersecting X. If Y is a subcomplex of X, $\mathcal{T}_D(X - Y)$ denotes the union of the (D) cells intersecting X but not Y.

For instance:
$\mathcal{T}_D(V)$ has V for deformation retract,
$\mathcal{T}_D(\Sigma)$ (resp. $\mathcal{T}_D(S_\alpha)$) has Σ (resp. S_α) for deformation retract,
$\mathcal{T}_D(M - V)$ is a deformation retract of $M - V$,
$\mathcal{T}_D(M - \Sigma)$ is a deformation retract of $M - \Sigma$,
$\mathcal{T}_D(V - \Sigma)$ has the homotopy type of $V - \Sigma$,
and $\mathcal{T}_D(b\tilde{R})$ has $b\tilde{R}$ for deformation retract.

We shall respectively denote by $C^*_D(M), C^*_D(V), C^*_D(\Sigma), C^*_D(M-V)$, $C^*_D(M-\Sigma)$ and $C^*_D(V-\Sigma)$ the complexes of cellular cochains with coefficients in $\mathbb{C}$ for (D)-cells respectively in M, $\mathcal{T}_D(V)$, $\mathcal{T}_D(\Sigma)$, $\mathcal{T}_D(M-V)$, $\mathcal{T}_D(M-\Sigma)$, and $\mathcal{T}_D(V-\Sigma)$. The corresponding cohomology algebras are respectively canonically isomorphic to $H^*(M), H^*(V), H^*(\Sigma), H^*(M-V)$, $H^*(M-\Sigma)$ and $H^*(V-\Sigma)$.

Denote also respectively by $C^*_D(M, M-V)$, $C^*_D(M, M-\Sigma)$ and $C^*_D(V, V-\Sigma)$ the kernels of the surjections $C^*_D(M) \to C^*_D(M-V)$, $C^*_D(M) \to C^*_D(M-\Sigma)$ and
$C^*_D(V) \to C^*_D(V-\Sigma)$. Their cohomology are respectively canonically isomorphic to $H^*(M, M-V)$, $H^*(M, M-\Sigma)$ and $H^*(V, V-\Sigma)$.

Notice that V, (resp. ∂R_A, $\partial\tilde{R}_0$ and $\partial\tilde{R}_\alpha$) is a subcomplex of $(K")$. Thus, (D)-cells of dimension j are transversal to them, and intersect them therefore in dimension $j-2k$ (resp. $2m-1$).

Frame fields and radial frame fields

Let r be an integer $(1 \leq r \leq n)$. We set $p = n - r + 1$, and $q = p + k = m - r + 1$. We shall denote by $\tilde{F}^{(r)} = (\tilde{F}^{(r-1)}, \tilde{v}_r)$ a field of smooth non singular r frames tangent to M near $\mathcal{T}_D(b\tilde{R}) \cap (D)^{2q}$, ($\tilde{F}^{(r-1)}$ denoting the $r-1$ frame generated by the $r-1$ first vectors, and $\tilde{v}_r$ denoting the last vector field of the frame), and having the following properties:
(i) Its restriction $F^{(r)} = (F^{(r-1)}, v_r)$ to V_0 is tangent to V_0. More generally $\tilde{F}^{(r)}$ remains in H over $\mathcal{T}_D(\partial\tilde{R}_0) \cap (D)^{2q}$.
(ii) A smooth non singular extension of $\tilde{F}^{(r-1)}$ is given in $\mathcal{T}_D(\tilde{R}_0) \cap (D)^{2q}$, still in H.

After usual obstruction theory, there always exists such frame fields: in fact, $b\tilde{R}$ is a deformation retract of $\mathcal{T}_D(b\tilde{R})$, and $b\tilde{R} \cap (D)^{2q}$ is $2q-1$ dimensional.

Among all frame fields having the above properties, there are in particular after [MHS] **radial** frame fields, denoted by $\tilde{F}^{(r)}_0$ in the sequel. (For a precise definition of a radial frame field, see [MHS] or [BS]). Notice that the properties (i) (ii) are far from characterizing the frame fields which are radial. For instance, in the case $r = 1$, if $\tilde{v}_1$ is radial, it is possible to choose the honeycombs such that $\tilde{v}_1$ be transversal to bR. After [MHS] all radial frame fields are homotopic.

Particular connections:

We shall call <u>*s connection*</u> every connection $\nabla^{s,E}$ on E over M, which is s trivial ($\nabla^{s,E} s = 0$) off $\mathcal{T}_D(V)$ and in particular over ∂R_A.

For any frame field $\tilde{F}^{(r)}$ satisfying the properties (i) and (ii) above, we shall call <u>$\tilde{F}^{(r)}$ *connection*</u> every connection ∇^M_F on TM over M, preserving the subbundle H of TM over $\mathcal{T}_D(\tilde{R}_0)$, which is $\tilde{F}^{(r)}$ trivial over $\mathcal{T}_D(b\tilde{R}) \cap (D)^{2q}$, the induced connection ∇^H over H being $\tilde{F}^{(r-1)}$ trivial over $\mathcal{T}_D(\tilde{R}_0) \cap (D)^{2q}$. Notice that the connection ∇_F^M, while having particular properties only over some subspace of M, has been extended over all of M.

Lemma 1. *There always exists a pair of connections $(\nabla_F^M, \nabla^{s,E})$, compatible with the projection $\pi : TM \to E$ over $\mathcal{T}_D(\tilde{R}_0)$, where ∇_F^M is an $\tilde{F}^{(r)}$ connection, and $\nabla^{s,E}$ an s connection.*

Such a pair will be called a **compatible** $(\tilde{F}^{(r)}, s)$ **pair**.

Proof. Obvious, using partition of unity. Q.E.D.

§3. Backgrounds and notations

A) Recall that the Čech de Rham complex $CDR^*(\tilde{\mathcal{U}})$ is the differential graded algebra of elements

$$\omega = \begin{pmatrix} \omega_A & \omega_0 & \omega_1 = (\omega_\alpha) \\ \omega_{A0} & \omega_{A1} = (\omega_{A\alpha}) & \omega_{01} = (\omega_{0\alpha}) \\ & \omega_{A01} = (\omega_{A0\alpha}) & \end{pmatrix},$$

(where ω_A, ω_0, ω_α, ω_{A0}, $\omega_{A\alpha}$, $\omega_{0\alpha}$, $\omega_{A0\alpha}$ denote respectively de Rham forms on the open sets $\tilde{U}_A$, $\tilde{U}_0$, $\tilde{U}_\alpha$, $\tilde{U}_{A0} = \tilde{U}_A \cap \tilde{U}_0$, $U_{A\alpha} = \tilde{U}_A \cap \tilde{U}_\alpha$, $\tilde{U}_{0\alpha} = \tilde{U}_0 \cap \tilde{U}_\alpha$, $\tilde{U}_{A0\alpha} = \tilde{U}_A \cap \tilde{U}_0 \cap \tilde{U}_\alpha$, and the parenthesis denote families of forms indexed by α),
with the differential

$$D\omega = \begin{pmatrix} d\omega_A & d\omega_0 & (d\omega_\alpha) \\ -d\omega_{A0} + \omega_0 - \omega_A & (-d\omega_{A\alpha} + \omega_\alpha - \omega_A) & (-d\omega_{0\alpha} + \omega_\alpha - \omega_0) \\ & (d\omega_{A0\alpha} + \omega_{0\alpha} - \omega_{A\alpha} + \omega_{A0}) & \end{pmatrix}.$$

This differential is a derivation

$$D(\omega \smile \eta) = D\omega \smile \eta + (-1)^{\dim \omega} \omega \smile D\eta$$

for the following product (which is not graded commutative):

$$\begin{pmatrix} \omega_A & \omega_0 & (\omega_\alpha) \\ \omega_{A0} & (\omega_{A\alpha}) & (\omega_{0\alpha}) \\ & (\omega_{A0\alpha}) & \end{pmatrix} \smile \begin{pmatrix} \eta_A & \eta_0 & (\eta_\alpha) \\ \eta_{A0} & (\eta_{A\alpha}) & (\eta_{0\alpha}) \\ & (\eta_{A0\alpha}) & \end{pmatrix} =$$

$$\begin{pmatrix} \omega_A \wedge \eta_A & \omega_0 \wedge \eta_0 & (\omega_\alpha \wedge \eta_\alpha) \\ (-1)^p \omega_A \wedge \eta_{A0} + \omega_{A0} \wedge \eta_0 & ((-1)^p \omega_A \wedge \eta_{A\alpha} + \omega_{A\alpha} \wedge \eta_\alpha) & ((-1)^p \omega_0 \wedge \eta_{0\alpha} + \omega_{0\alpha} \wedge \eta_\alpha) \\ (\omega_A \wedge \eta_{A0\alpha} & + \quad (-1)^{p-1} \omega_{A0} \wedge \eta_{0\alpha} & + \quad \omega_{A0\alpha} \wedge \eta_\alpha) \end{pmatrix}:$$

The cohomology algebra of $CDR^*(\tilde{\mathcal{U}})$ is naturally isomorphic to the de Rham cohomology of M (with complex coefficients), while the differential subalgebras $CDR^*(\tilde{\mathcal{U}}, M - V)$ (resp. $CDR^*(\tilde{\mathcal{U}}, M - \Sigma)$, resp. $CDR^*(\tilde{\mathcal{U}}, \tilde{U}_A \cup (\cup_\alpha \tilde{U}_\alpha))$ of elements ω such that $\omega_A = 0$ (resp. $\omega_A = 0$, $\omega_0 = 0$, and $\omega_{A0} = 0$), (resp. $\omega_A = 0$, $\omega_0 = 0$, and $\omega_{A0} = 0$) provide respectively the relative cohomology $H^*(M, M - V)$, $H^*(M, M - \Sigma)$ and $H^*(M, \tilde{U}_A \cup (\cup_\alpha \tilde{U}_\alpha))$ with complex coefficients. We shall write [0] instead of 0 when we wish to insist that some ω is taken in the subalgebra $CDR^*(\tilde{\mathcal{U}}, M - V)$, $CDR^*(\tilde{\mathcal{U}}, M - \Sigma)$ or $CDR^*(\tilde{\mathcal{U}}, \tilde{U}_A \cup \tilde{U}_1)$, and not in $CDR^*(\tilde{\mathcal{U}})$ itself, writing respectively such elements

$$\begin{pmatrix} [0] & \omega_0 & (\omega_\alpha) \\ \omega_{A0} & (\omega_{A\alpha}) & (\omega_{0\alpha}) \\ & (\omega_{A0\alpha}) & \end{pmatrix}, \quad \begin{pmatrix} [0] & [0] & (\omega_\alpha) \\ [0] & (\omega_{A\alpha}) & (\omega_{0\alpha}) \\ & (\omega_{A0\alpha}) & \end{pmatrix}$$

or

$$\begin{pmatrix} [0] & \omega_0 & [0] \\ \omega_{A0} & [0] & (\omega_{0\alpha}) \\ & (\omega_{A0\alpha}) & \end{pmatrix}.$$

Since the honeycombs R_A, $\tilde{R}_0$, and $\tilde{R}_\alpha$ are subcomplexes of $(K")$, the cells of (D) are transversal to these honeycombs, so that we may integrate elements $\omega \in CDR^j(\tilde{\mathcal{U}})$ along j cells γ of (D) (cf. [Le]): recall that $\int_\gamma \omega$ is equal to

$$\int_{\gamma \cap R_A} \omega_A + \int_{\gamma \cap \tilde{R}_0} \omega_0 + \int_{\gamma \cap\ R_{A0}} \omega_{A0}$$
$$+ \sum_\alpha \left[\int_{\gamma \cap \tilde{R}_\alpha} \omega_\alpha + \int_{\gamma \cap R_{A\alpha}} \omega_{A\alpha} + \int_{\gamma \cap \tilde{R}_{0\alpha}} \omega_{0\alpha} + \int_{\gamma \cap R_{A0\alpha}} \omega_{A0\alpha} \right]$$

with suitable orientations of the domains $R_A \cdots R_{A0} \cdots R_{A0\alpha}$.

The integration defines therefore a morphism from $CDR^*(\tilde{\mathcal{U}})$ into the cellular cochains $C^*_{(D)}(M)$, which commutes with the differentials and induces an algebra isomorphism in cohomology (see [Le]). We shall denote by

$$\left(\!\!\begin{pmatrix} \omega_A & \omega_0 & (\omega_\alpha) \\ \omega_{A0} & (\omega_{A\alpha}) & (\omega_{0\alpha}) \\ & (\omega_{A0\alpha}) & \end{pmatrix}\!\!\right)$$

the image of

$$\begin{pmatrix} \omega_A & \omega_0 & (\omega_\alpha) \\ \omega_{A0} & (\omega_{A\alpha}) & (\omega_{0\alpha}) \\ & (\omega_{A0\alpha}) & \end{pmatrix} \text{ in } C^*_{(D)}(M).$$

Similarly

$$\left(\!\!\begin{pmatrix} [0] & \omega_0 & (\omega_\alpha) \\ \omega_{A0} & (\omega_{A\alpha}) & (\omega_{0\alpha}) \\ & (\omega_{A0\alpha}) & \end{pmatrix}\!\!\right), \left(\!\!\begin{pmatrix} [0] & [0] & (\omega_\alpha) \\ [0] & (\omega_{A\alpha}) & (\omega_{0\alpha}) \\ & (\omega_{A0\alpha}) & \end{pmatrix}\!\!\right)$$

or

$$\left(\!\!\begin{pmatrix} [0] & \omega_0 & [0] \\ \omega_{A0} & [0] & (\omega_{0\alpha}) \\ & (\omega_{A0\alpha}) & \end{pmatrix}\!\!\right)$$

will denote elements in $C^j_{(D)}(M, M - V)$, in $C^j_{(D)}(M, M - \Sigma)$ or in $C^j_{(D)}(M, \mathcal{T}_D(M - V) \cup \mathcal{T}_D(\Sigma))$.

The notation

$$\left(\!\!\begin{pmatrix} [0] & [0] & (\omega_\alpha) \\ [0] & (\omega_{A\alpha}) & (\omega_{0\alpha}) \\ & (\omega_{A0\alpha}) & \end{pmatrix}\!\!\right) + \left(\!\!\begin{pmatrix} [0] & \omega'_0 & [0] \\ \omega'_{A0} & [0] & (\omega'_{0\alpha}) \\ & (\omega'_{A0\alpha}) & \end{pmatrix}\!\!\right)$$

will denote in fact the sum

$$\left(\!\!\begin{pmatrix} [0] & 0 & (\omega_\alpha) \\ 0 & (\omega_{A\alpha}) & (\omega_{0\alpha}) \\ & (\omega_{A0\alpha}) & \end{pmatrix}\!\!\right) + \left(\!\!\begin{pmatrix} [0] & \omega'_0 & 0 \\ \omega'_{A0} & 0 & (\omega'_{0\alpha}) \\ & (\omega'_{A0\alpha}) & \end{pmatrix}\!\!\right)$$

of the images in $C^*_{(D)}(M, M - V)$.

The Čech-de Rham complex $CDR^*(\mathcal{U})$ is the differential graded algebra of elements

$$\omega = \big(\omega_0, \omega_1 = (\omega_\alpha), \omega_{01} = (\omega_{0\alpha})\big)$$

(where ω_0, ω_α, $\omega_{0\alpha}$ denote respectively de Rham forms on the open sets $\tilde{U}_0$, $\tilde{U}_\alpha$, $\tilde{U}_{0\alpha}$, and the parenthesis denote families of forms indexed by α), with the differential

$$D\omega = \big(d\omega_0\ ,\ (d\omega_\alpha)\ , (-d\omega_{0\alpha} + \omega_\alpha - \omega_0)\big),$$

which is a derivation with respect to the (non graded commutative) product

$$\begin{aligned}(\omega_0\ ,\ (\omega_\alpha)\ ,\ (\omega_{0\alpha})) &\smile (\eta_0\ ,\ (\eta_\alpha)\ ,\ (\eta_{0\alpha}))\\ &= \big(\omega_0\wedge\eta_0\ ,\ (\omega_\alpha\wedge\eta_\alpha)\ ,\ ((-1)^p\omega_0\wedge\eta_{0\alpha}+\omega_{0\alpha}\wedge\eta_\alpha)\big).\end{aligned}$$

The cohomology algebra of $CDR^*(\mathcal{U})$ is naturally isomorphic to the de Rham cohomology of V (with complex coefficients), while the differential subalgebra $CDR^*(\mathcal{U}, V-\Sigma)$ of elements ω such that $\omega_0 = 0$ provide the relative cohomology $H^*(V, V-\Sigma)$. We shall write $\big([0], (\omega_\alpha), (\omega_{0\alpha})\big)$ the elements of $CDR^*(\mathcal{U}, V-\Sigma)$, and $\big(\omega_A, [0], (\omega_{O\alpha})\big)$ those of $CDR^*(\mathcal{U}, \Sigma)$.

We may integrate elements $\omega\in CDR^j(\mathcal{U})$ along j cells γ of $\mathcal{T}_D(V)$, and define $\int_\gamma \omega$ as being equal to

$$\int_{\gamma\cap R_0}\omega_0+\sum_\alpha\left[\int_{\gamma\cap R_\alpha}\omega_\alpha+\int_{\gamma\cap R_{0\alpha}}\omega_{0\alpha}\right].$$

The integration defines therefore a morphism from $CDR^*(\mathcal{U})$ into the cellular cochains $C^*_{(D)}(V)$ on V with complex coefficients, which commutes with the differentials and induces an algebra isomorphism in cohomology. We shall denote by $\langle\!\langle\omega_0, (\omega_\alpha), (\omega_{0\alpha})\rangle\!\rangle$ the image of $(\omega_0, (\omega_\alpha), (\omega_{0\alpha}))$ in $C^*_{(D)}(V)$.

Similarly $\langle\!\langle[0], (\omega_\alpha), (\omega_{0\alpha})\rangle\!\rangle$ will denote elements in $C^j_{(D)}(V, V-\Sigma)$, and $\langle\!\langle\omega_0, [0], (\omega_{0\alpha})\rangle\!\rangle$ elements in $C^j_{(D)}(V, \Sigma)$.

The notation $\langle\!\langle[0], (\omega_\alpha), (\omega_{0\alpha})\rangle\!\rangle + \langle\!\langle\omega_0', [0], (\omega_{0\alpha}')\rangle\!\rangle$ will denote in fact the sum $\langle\!\langle 0, (\omega_\alpha), (\omega_{0\alpha})\rangle\!\rangle + \langle\!\langle\omega_0', 0, (\omega_{0\alpha}')\rangle\!\rangle$ in $C^j_{(D)}(V)$.

Remark. When ω and γ are j dimensional, $\int_\gamma \omega$ depends only on the behaviour of ω near the j skeleton $(D)^j$ of (D), (i.e the behaviour of ω_A near $R_A\cap(D)^j$, etc....). Thus, it is sufficient that

$$\omega=\begin{pmatrix}\omega_A & \omega_0 & (\omega_\alpha)\\ \omega_{A0} & (\omega_{A\alpha}) & (\omega_{0\alpha})\\ & (\omega_{A0\alpha}) & \end{pmatrix}$$

be defined near $(D)^j$, for

$$\langle\!\langle\omega\rangle\!\rangle=\left(\!\!\begin{pmatrix}\omega_A & \omega_0 & (\omega_\alpha)\\ \omega_{A0} & (\omega_{A\alpha}) & (\omega_{0\alpha})\\ & (\omega_{A0\alpha}) & \end{pmatrix}\!\!\right)$$

to make sense. However, be careful to the fact that, in this case, the Stokes formula $d\langle\!\langle\omega\rangle\!\rangle = \langle\!\langle D\omega\rangle\!\rangle$ does not hold any more necessarily.

A similar remark holds for $\langle\!\langle \omega_0, (\omega_\alpha), (\omega_{0\alpha}) \rangle\!\rangle$.

B) In general, for a Chern polynomial φ (i.e., a polynomial of the Chern classes), and a connection ∇ on a complex C^∞ vector bundle, $C \to X$, we denote by $\varphi(\nabla)$ the cocycle on the base which is the image of φ by the Chern-Weil homomorphism asoociated to ∇. Thus it is a closed form whose cohomology class in the de Rham cohomology is the (real) characteristic class $\varphi(C)$ of the bundle associated to φ. In particular, the class of $c^i(\nabla)$ is the real i^{th} Chern class of A. If $(\nabla_0, \nabla_1, \ldots, \nabla_r)$ is a family of $r+1$ connections on a same vector bundle C, $\varphi(\nabla_0, \nabla_1, \ldots, \nabla_r)$ will denote more generally the Bott difference operator ([B]), so that

$$d\varphi(\nabla_0, \nabla_1, \ldots, \nabla_r) = \sum_{i=0}^{r} (-1)^i \varphi(\nabla_0, \nabla_1, \ldots, \widehat{\nabla}_i, \cdots, \nabla_r).$$

In particular, for $r = 1$, $d\varphi(\nabla_0, \nabla_1) = \varphi(\nabla_1) - \varphi(\nabla_0)$.

Denoting by c^i and by c'^j the Chern classes of some smooth complex bundles C and C', of ranks $n+q$ and q respectively, over a same manifold X, recall that the h-th Chern class $c''^h = c^h([C - C'])$ of the virtual bundle $[C - C'] \in KU(X)$ is a polynomial with respect to the c^i's and the c'^j's, defined as the coefficient of t^h in the expansion of the expression $(1 + \sum_i t^i c^i) \cdot (1 + \sum_j t^j c'^j)^{-1}$. This polynomial may be written as a finite sum

$$c''^h = \sum_\ell \varphi_\ell(c^1, \ldots, c^{n+q}) \cdot \psi_\ell(c'^1, \ldots, c'^q),$$

for some polynomials φ_ℓ and ψ_ℓ.

Let ∇ and ∇' be connections on C and C' respectively. Denoting by $\nabla^\bullet$ the pair (∇, ∇'), we set

$$c^h(\nabla^\bullet) = \sum_\ell \varphi_\ell(\nabla) \wedge \psi_\ell(\nabla') \ .$$

Then $c^h(\nabla^\bullet)$ is a closed $2h$-form on X which defines the class $c^h([C - C'])$.

If $\nabla_1^\bullet = (\nabla_1, \nabla_1')$ and $\nabla_2^\bullet = (\nabla_2, \nabla_2')$ are two such pairs, we set

$$c^h(\nabla_1^\bullet, \nabla_2^\bullet) = \sum_\ell \left(\psi_\ell(\nabla_1') \cdot \varphi_\ell(\nabla_1, \nabla_2) + \psi_\ell(\nabla_1', \nabla_2') \cdot \varphi_\ell(\nabla_2)\right).$$

Then we have:

$$d\,c^h(\nabla_1^\bullet, \nabla_2^\bullet) = c^h(\nabla_2^\bullet) - c^h(\nabla_1^\bullet).$$

If $\nabla_1^\bullet = (\nabla_1, \nabla_1')$, $\nabla_2^\bullet = (\nabla_2, \nabla_2')$ and $\nabla_3^\bullet = (\nabla_3, \nabla_3')$ are three such pairs, we denote by $c^h(\nabla_1^\bullet, \nabla_2^\bullet, \nabla_3^\bullet)$ the form

$$\sum_\ell \big(\psi_\ell(\nabla_1') \cdot \varphi_\ell(\nabla_1, \nabla_2, \nabla_3) \\ + \psi_\ell(\nabla_1', \nabla_2') \cdot \varphi_\ell(\nabla_2, \nabla_3) + \psi_\ell(\nabla_1', \nabla_2', \nabla_3') \cdot \varphi_\ell(\nabla_3)\big).$$

Then we have

$$d\,c^h(\nabla_1^\bullet, \nabla_2^\bullet, \nabla_3^\bullet) = c^h(\nabla_2^\bullet, \nabla_3^\bullet) - c^h(\nabla_1^\bullet, \nabla_3^\bullet) + c^h(\nabla_1^\bullet, \nabla_2^\bullet).$$

§4. Thom-Gysin homomorphism

The complex $CDR^*(\mathcal{U})$ is a quotient of $CDR^*(\tilde{\mathcal{U}})$, and we already observed in [Le] that the cup product

$$CDR^*(\tilde{\mathcal{U}}, M - V) \smile \big[\ker : CDR^*(\tilde{\mathcal{U}}) \to CDR^*(\mathcal{U})\big]$$

is identically zero, defining therefore a multiplication

$$CDR^*(\tilde{\mathcal{U}}, M - V) \times CDR^*(\mathcal{U}) \xrightarrow{\smile} CDR^*(\tilde{\mathcal{U}}, M - V),$$

which induces the product $H^*(M, M-V) \times H^*(V) \to H^*(M, M-V)$. Similarly, we get multiplications

$$CDR^*(\tilde{\mathcal{U}}, M - V) \times CDR^*(\mathcal{U}, V - \Sigma) \xrightarrow{\smile} CDR^*(\tilde{\mathcal{U}}, M - \Sigma),$$

and $CDR^*(\tilde{\mathcal{U}}, M - V) \times CDR^*(\mathcal{U}, \bigcup_\alpha U_\alpha) \xrightarrow{\smile} CDR^*(\tilde{\mathcal{U}}, \tilde{U}_A \cup (\bigcup_\alpha \tilde{U}_\alpha))$,

inducing respectively the products $H^*(M, M-V) \times H^*(V, V-\Sigma) \to H^*(M, M-\Sigma)$, and $H^*(M, M-V) \times H^*(V, \Sigma) \to H^*(M, M - \tilde{R}_0)$.

For $V = s^{-1}(0)$ as in section 2, the data of the section s, non vanishing on $M - V$, defines a natural lift $c^k(E, s)$ of the Chern class $c^k(E)$ by the morphism $H^{2k}(M, M-V) \to H^{2k}(M)$. It is proved in [Su2] that $c^k(E,s)$ corresponds to the fundamental class $[V]$ by the Alexander duality. Therefore, the cup product by the so-called "Thom class" $c^k(E,s)$ induces in cohomology the Thom-Gysin morphism τ such that $A \circ \tau = P_V$.

Let ∇^E be any C^∞ connection on E, and ∇'^E any s trivial connection on $E|_{M-V}$ (s trivial means : $\nabla'^E s = 0$). Then the Thom class $c^k(E,s)$ of E is represented by the cocycle

$$\begin{pmatrix} [0] & c^k(\nabla_E) & (c^k(\nabla_E)) \\ c^k(\nabla'^E, \nabla_E) & (c^k(\nabla'^E, \nabla_E)) & 0 \\ & 0 & \end{pmatrix} \in CDR^*(\tilde{\mathcal{U}}, M - V),$$

and the Thom Gysin morphism is induced by the map
$\tau : CDR^*(\mathcal{U}) \longrightarrow CDR^*(\tilde{\mathcal{U}}, M - V)$ such that

$$\tau\big(\eta_0\ ,\ (\eta_\alpha)\ ,\ (\eta_{0\alpha})\big) = \begin{pmatrix} [0] & c^k(\nabla^E)\wedge\eta_0 & \big(c^k(\nabla^E)\wedge\eta_\alpha\big) \\ c^k(\nabla'^E,\nabla^E)\wedge\eta_0 & \big(c^k(\nabla'^E,\nabla^E)\wedge\eta_\alpha\big) & \big(c^k(\nabla^E)\wedge\eta_{0\alpha}\big) \\ & \big(-c^k(\nabla'^E,\nabla^E)\wedge\eta_{0\alpha}\big) & \end{pmatrix}.$$

This Thom-Gysin morphism may then be refined by the maps

$$\tau_\Sigma : CDR^*(\mathcal{U}, V-\Sigma) \longrightarrow CDR^*(\tilde{\mathcal{U}}, M-\Sigma)$$
$$\text{and}\quad \tau_0 : CDR^*(\mathcal{U}, (\bigcup_\alpha U_\alpha)) \longrightarrow CDR^*(\tilde{\mathcal{U}}, U_A \cup (\bigcup_\alpha \tilde{U}_\alpha)),$$

respectively defined by the formulas

$$\tau_\Sigma\big([0]\ ,\ (\eta_\alpha)\ ,\ (\eta_{0\alpha})\big) = \begin{pmatrix} [0] & [0] & \big(c^k(\nabla^E)\wedge\eta_\alpha\big) \\ [0] & \big(c^k(\nabla'^E,\nabla^E)\wedge\eta_\alpha\big) & \big(c^k(\nabla^E)\wedge\eta_{0\alpha}\big) \\ & \big(-c^k(\nabla'^E,\nabla^E)\wedge\eta_{0\alpha}\big) & \end{pmatrix},$$

and

$$\tau_0\big(\eta_0\ , [0] ,\ (\eta_{0\alpha})\big) = \begin{pmatrix} [0] & c^k(\nabla^E)\wedge\eta_0 & [0] \\ c^k(\nabla'^E,\nabla^E)\wedge\eta_0 & [0] & \big(c^k(\nabla^E)\wedge\eta_{0\alpha}\big) \\ & \big(-c^k(\nabla'^E,\nabla^E)\wedge\eta_{0\alpha}\big) & \end{pmatrix}.$$

These maps do not depend in cohomology on the choices of ∇^E and ∇'^E. In fact, if ∇^E_1 and ∇^E_2 denote two connections on E, then

$$\begin{pmatrix} [0] & c^k(\nabla^E_2) & \big(c^k(\nabla^E_2)\big) \\ c^k(\nabla'^E,\nabla^E_2) & \big(c^k(\nabla'^E,\nabla^E_2)\big) & 0 \\ & 0 & \end{pmatrix} - \begin{pmatrix} [0] & c^k(\nabla^E_1) & \big(c^k(\nabla^E_1)\big) \\ c^k(\nabla'^E,\nabla^E_1) & \big(c^k(\nabla'^E,\nabla^E_1)\big) & 0 \\ & 0 & \end{pmatrix}$$
$$= D\begin{pmatrix} [0] & c^k(\nabla^E_1,\nabla^E_2) & \big(c^k(\nabla^E_1,\nabla^E_2)\big) \\ c^k(\nabla'^E,\nabla^E_1,\nabla^E_2) & \big(c^k(\nabla'^E,\nabla^E_1,\nabla^E_2)\big) & 0 \\ & 0 & \end{pmatrix},$$

i.e. is a coboundary in $CDR^*(\tilde{\mathcal{U}}, M-V)$.

Similarly, if ${\nabla'}^E_1$ and ${\nabla'}^E_2$ denote two connections on $E|_{M-V}$, both preserving s, then

$$\begin{pmatrix} [0] & c^k(\nabla^E) & (c^k(\nabla^E)) \\ c^k({\nabla'}^E_2, \nabla^E) & (c^k({\nabla'}^E_2, \nabla^E)) & 0 \\ & 0 & \end{pmatrix}$$

$$- \begin{pmatrix} [0] & c^k(\nabla^E) & (c^k(\nabla^E)) \\ c^k({\nabla'}^E_1, \nabla^E) & (c^k({\nabla'}^E_1, \nabla^E)) & 0 \\ & 0 & \end{pmatrix}$$

$$= D \begin{pmatrix} [0] & 0 & 0 \\ c^k({\nabla'}^E_1, {\nabla'}^E_2, \nabla^E) & (c^k({\nabla'}^E_1, {\nabla'}^E_2, \nabla^E))) & 0 \\ & 0 & \end{pmatrix},$$

(because $c^k({\nabla'}^E_1, {\nabla'}^E_2) = 0$, since ${\nabla'}^E_1$ and ${\nabla'}^E_2$ are both preserving the same s): we still get a coboundary in $CDR^*(\tilde{\mathcal{U}}, M - V)$.

Remark. If we take for ∇^E a s connection, then $c^k({\nabla'}^E, \nabla^E) = 0$ off $\mathcal{T}_D(V)$, and in particular over R_{A0} and $R_{A\alpha}$.

§5. Virtual classes

They are characteristic classes of the virtual tangent bundle $TV = [TM - E]|_V$ in $KU(V)$. Let $\nabla^\bullet = (\nabla^M, \nabla^E)$ be a pair of connections ∇^M on TM and ∇^E on E. Then the p^{th} Chern class $c^p_{\mathrm{vir}}(V)$ of the above virtual TV may be represented, in the Chern-Weil theory by the de Rham form $c^p(\nabla^\bullet) = \left[c(\nabla^M)/c(\nabla^E)\right]_p$ on $U(V) = \tilde{U}_0 \cup \tilde{U}_1$, (where $[...]_p$ denotes the homogeneous component of dimension $2p$), or equivalently by the element $\left(c^p(\nabla^\bullet), (c^p(\nabla^\bullet)), 0\right)$ in $CDR^{2p}(\mathcal{U})$. It does not depend on $\nabla^\bullet$ since, for two choices $\nabla^\bullet$ and $\bar{\nabla}^\bullet$ of the pair of connections, we have: $c^p(\bar{\nabla}^\bullet) - c^p(\nabla^\bullet) = dc^p(\nabla^\bullet, \bar{\nabla}^\bullet)$.

Let $\tilde{F}_0^{(r)}$ be a **radial** frame field. Let $\nabla^M_{F_o}$ be any $\tilde{F}_0^{(r)}$ connection on TM, and denote by ∇^H the induced connection on H over $\tilde{R}_0$. Set $\nabla^\bullet_{F_o} = (\nabla^M_{F_o}, \nabla^E)$, and define

$$\mathrm{Vir}^p_0 = \left(\!\left(c^p(\nabla^\bullet_{F_o}), [0], 0\right)\!\right) \text{ and } \mathrm{Vir}^p_\Sigma = \left(\!\left([0], c^p(\nabla^\bullet_{F_o}), 0\right)\!\right).$$

Proposition 1. (i) Vir^p_0 *and* Vir^p_Σ *are relative cocycles modulo* $\mathcal{T}_D(\Sigma)$ *and* $\mathcal{T}_D(R_0)$ *respectively.*

(ii) *Their cohomology class* $c^p_{0,\mathrm{vir}}(V, \tilde{F}_o^{(r)}) \in H^{2p}(\mathcal{T}_D(V), \mathcal{T}_D(\Sigma)) \cong H^{2p}(V, \Sigma)$*, and* $c^p_{\Sigma,\mathrm{vir}}(V, \tilde{F}_o^{(r)}) \in H^{2p}(\mathcal{T}_D(V), \mathcal{T}_D(R_0)) \cong H^{2p}(V, V - \Sigma)$.

[Notice that $\left(c^p(\nabla^\bullet_{F_o}), [0], 0\right)$ and $\left([0], c^p(\nabla^\bullet_{F_o}), 0\right)$ might not be cocycles!]

For any $2p$ dimensional (D)-cell σ in $\mathcal{T}_D(V)$, $\langle \mathrm{Vir}^p_\Sigma, \sigma\rangle$ is equal to $\int_{\tilde{R}_1\cap\sigma} c^p(\nabla^\bullet_{F_o})$.
If σ is in $\mathcal{T}_D(R_0)$, then $\tilde{R}_1\cap\sigma$ is empty or is in $\mathcal{T}_D(\tilde{R}_{01})\cap(D)^{2p}$ where $c^p(\nabla^\bullet_{F_o}) = c^p(\nabla^H) = 0$. Thus, $\langle \mathrm{Vir}^p_\Sigma, \sigma\rangle = 0$, which proves that Vir^p_Σ vanishes on $\mathcal{T}_D(R_0)$. Similarly, Vir^p_0 vanishes on $\mathcal{T}_D(\Sigma)$.

On the other hand, for any $2p+1$ dimensional (D)-cell τ, $\langle d\,\mathrm{Vir}^p_\Sigma, \tau\rangle = \langle \mathrm{Vir}^p_\Sigma, \partial\tau\rangle$ is equal to $\int_{\tilde{R}_1\cap\partial\tau} c^p(\nabla^\bullet_{F_0})$, that is $\int_{\tilde{R}_{01}\cap\tau} c^p(\nabla^\bullet_{F_0})$ after Stokes formula. If $\tilde{R}_{01}\cap\tau$ is not empty, it is included in $\mathcal{T}_D(\tilde{R}_{01})\cap(D)^{2p}$, where $c^p(\nabla^\bullet_{F_o}) = 0$: thus Vir^p_Σ is a cocycle. A similar proof works for Vir^p_0.

For two different $\tilde{F}^{(r)}_0$ connections $\nabla^\bullet_{1,F_o}$ and $\nabla^\bullet_{2,F_o}$, we have:

$$\big(\!\big([0], c^p(\nabla^\bullet_{2,F_o}), 0\big)\!\big) - \big(\!\big([0], c^p(\nabla^\bullet_{1,F_o}), 0\big)\!\big) = d\big(\!\big([0], 0, c^p(\nabla_{1,F_o}, \nabla_{2,F_o}\big)\!\big),$$

since $c^p(\nabla_{1,F_o}, \nabla_{2,F_o}) = 0$ near $\mathcal{T}_D R_{01}\cap(D)^{2q}$, both connections ∇_{1,F_o} and ∇_{2,F_o} preserving there a same $\tilde{F}^{(r)}_0$. Since two radial frame fields are always homotopic, these classes do not depend neither of the choice of the radial frame field, as far as it is radial.

After section 4, if we assume furthermore that $\nabla^{s,E}$ is a s connection, this decomposition has for image by the Thom-Gysin homomorphism

$$\left(\!\!\left(\begin{matrix}[0] & c^k(\nabla^{s,E})\wedge c^p(\nabla^\bullet_{F_o}) & [0]\\ 0 & [0] & 0\\ & 0 & \end{matrix}\right)\!\!\right) + \left(\!\!\left(\begin{matrix}[0] & [0] & \big(c^k(\nabla^{s,E})\wedge c^p(\nabla^\bullet_{F_o})\big)\\ [0] & 0 & 0\\ & 0 & \end{matrix}\right)\!\!\right).$$

§6. SMP classes

Let r, p and q be as above.

Proposition 2. *Let $\nabla^M_{F_o}$ denote some $\tilde{F}^{(r)}_0$ connection on TM, for a* **radial** *frame field $\tilde{F}^{(r)}_0$.*
(i) *Then*

$$\mathrm{SMP}^{2q} = \left(\!\!\left(\begin{matrix}[0] & c^q(\nabla^M_{F_o}) & \big(c^q(\nabla^M_{F_o})\big)\\ 0 & 0 & 0\\ & 0 & \end{matrix}\right)\!\!\right),$$

is a cocycle in $C^{2q}\big(M, \mathcal{T}_D(R_A)\big)$.
(ii) *Its cohomology class is well defined in*

$$H^{2q}\big(M, \mathcal{T}_D(R_A)\big) \cong H^{2q}(M, M-V).$$

(iii) *This cohomology class $c^{2q}_{\mathrm{SMP}}(V)$ is equal to the image in the cohomology with real coefficients of the* SMP *class defined in* [MHS] *and* [BS] *with integral coefficients.*

In fact, for any $2q$ (D)-cell σ, we have:
$\langle \mathrm{SMP}^{2q}, \sigma\rangle = \int_{\tilde{R}\cap\sigma} c^q(\nabla^M_{F_o})$.

For σ in $\mathcal{T}_D(R_A)$, we get $\langle \mathrm{SMP}^{2q}, \sigma\rangle = 0$ because $c^q(\nabla^M_{F_o}) = 0$ if σ intersects ∂R_A, and $\tilde{R}\cap\sigma = \emptyset$ if it doesn't. Thus SMP^{2q} is a relative cochain modulo $\mathcal{T}_D(R_A)$. On the other hand, for any $2q+1$ dimensional (D)-cell τ, $\langle d\,\mathrm{SMP}^{2q}, \tau\rangle$ is equal to $\int_{\tilde{R}\cap\partial\tau} c^q(\nabla^M_{F_o})$. If τ intersects $\partial\tilde{R}$, then $c^q(\nabla^M_{F_o}) = 0$. If not, then $\tilde{R}\cap\partial\tau = \partial\tau$ and $\int_{\tilde{R}\cap\partial\tau} c^q(\nabla^M_{F_o}) = 0$ after Stokes formula. Thus $d\,\mathrm{SMP}^{2q} = 0$, and we get part (i) of the proposition.

For two different $\tilde{F}_0^{(r)}$ connections ∇^M_{1,F_o} and ∇^M_{2,F_o}, we have:

$$\left(\!\!\left(\begin{matrix}[0] & c^q(\nabla^M_{2,F_o}) & (c^q(\nabla^M_{2,F_o}))\\ 0 & 0 & 0\\ & 0 & \end{matrix}\right)\!\!\right) - \left(\!\!\left(\begin{matrix}[0] & c^q(\nabla^M_{1,F_o}) & (c^q(\nabla^M_{1,F_o}))\\ 0 & 0 & 0\\ & 0 & \end{matrix}\right)\!\!\right)$$
$$= d\left(\!\!\left(\begin{matrix}[0] & c^q(\nabla^M_{1,F_o},\nabla^M_{2,F_o}) & (c^q(\nabla^M_{1,F_o},\nabla^M_{2,F_o}))\\ 0 & 0 & 0\\ & 0 & \end{matrix}\right)\!\!\right),$$

since $c^p(\nabla^M_{1,F_o},\nabla^M_{2,F_o}) = 0$ near $\mathcal{T}_D(\partial\tilde{R})\cap(D)^{2q}$, both connections ∇^M_{1,F_o} and ∇^M_{2,F_o} preserving there a same $\tilde{F}_0^{(r)}$. Two radial frame fields being always homotopic, these classes do not depend neither on the choice of the frame field, as far as it is radial, hence part (ii) of the proposition.

Part (iii) results that the above definition is just a differential geometric transcription of the definition given in [MHS] and [BS].

Remarks. (i) For the moment, as far that we wish only define $c^{2q}_{\mathrm{SMP}}(V)$, we do not need the covering $\tilde{\mathcal{U}}$ with 3 open sets $M-V, \tilde{U}_0$ and $\tilde{U}_1$: we could as well work with the 2 open sets $M-V$ and $\tilde{U}_0\cup\tilde{U}_1$. But we shall need it soon, when decomposing $c^{2q}_{\mathrm{SMP}}(V)$ into the contributions $c^{2q}_{\Sigma,\mathrm{SMP}}(V)$ and $c^{2q}_{0,\mathrm{SMP}}(V)$ of the regular and the singular part of V.

(ii) Notice that

$$\begin{pmatrix}[0] & c^q(\nabla^M) & (c^q(\nabla^M))\\ c^q(\nabla^M_{F_o},\nabla^M) & (c^q(\nabla^M_{F_o},\nabla^M)) & 0\\ & 0 & \end{pmatrix}$$

might not be a cocycle, because $c^q(\nabla^M_{F_o})$ vanishes only over $\mathcal{T}_D(\partial R_A)\cap(D)^{2q}$, and may be not on all of U_{A0} and $U_{A\alpha}$.

(iii) The SMP class is an obstruction for the radial frame field $\tilde{F}_0^{(r)}$ to be extended to all of $U(V)\cap(D)^{2q}$. In fact, if such an extension exists, then $c^q(\nabla^M_{F_o}) = 0$ on all of the above domain, so that the previous cocycle is equal to

$$d\left(\!\!\left(\begin{matrix}[0] & c^q(\nabla^M_{F_o},\nabla^M) & (c^q(\nabla^M_{F_o},\nabla^M))\\ 0 & 0 & 0\\ & 0 & \end{matrix}\right)\!\!\right).$$

In the definition above of the SMP class, we used only that $\nabla^M_{F_o}$ preserves $\tilde{F}_0^{(r)}$ over $\mathcal{T}_D(\partial R_A)\cap(D)^{2q}$. If we remember that it is still true over $\mathcal{T}_D(\tilde{R}_{01}\cap(D)^{2q}$, the above cocycle providing $c^q_{\mathrm{SMP}}(V)$ may be decomposed into

$$\left(\!\!\left(\begin{matrix}[0] & c^q(\nabla^M_{F_o}) & [0]\\ 0 & [0] & 0\\ & 0 & \end{matrix}\right)\!\!\right)+\left(\!\!\left(\begin{matrix}[0] & [0] & (c^q(\nabla^M_{F_o}))\\ [0] & 0 & 0\\ & 0 & \end{matrix}\right)\!\!\right),$$

which are still relative cocycles respectively in $C^{2q}(M,\mathcal{T}_D(\tilde{R}_A\cup\tilde{R}_1)$ and $C^{2q}(M,\mathcal{T}_D(\tilde{R}_A\cup\tilde{R}_0)$, whose relative cohomology classes, respectively denoted by $c^q_{0,\mathrm{SMP}}(V,\tilde{F}_o^{(r)})$ and $c^q_{\Sigma,\mathrm{SMP}}(V,\tilde{F}_o^{(r)})$ in $H^{2q}(M,\mathcal{T}_D(\tilde{R}_A\cup\tilde{R}_1)$ and $H^{2q}(M,\mathcal{T}_D(\tilde{R}_A\cup\tilde{R}_0)\cong H^{2q}(M,M-\Sigma)$ still do not depend on the choices of the various connections (similar proof).

§7. Milnor classes

Lemma 2. *The relative cohomology classes* $\tau_0(c^p_{0,\mathrm{vir}}(V,\tilde{F}_o^{(r)}))$ *and* $c^q_{0,\mathrm{SMP}}(V,\tilde{F}_o^{(r)})$ *are equal in* $H^{2q}(M\ ,\ \mathcal{T}_D(R_A\cup\bigcup_\alpha\tilde{R}_\alpha)$.

Proof. Choose a compatible $(\tilde{F}_0^{(r)},s)$ pair $(\nabla^M_{F_o},\nabla^{s,E})$ of connections, and let ∇^H be the connection induced by $\nabla^M_{F_o}$ on H over $\tilde{R}_0$.

$$\left(\!\!\left(\begin{matrix}[0] & c^k(\nabla^{s,E})\wedge c^p(\nabla^\bullet_{F_o}) & [0]\\ 0 & [0] & 0\\ & 0 & \end{matrix}\right)\!\!\right)=\left(\!\!\left(\begin{matrix}[0] & c^q(\nabla^M_{F_o}) & [0]\\ 0 & [0] & 0\\ & 0 & \end{matrix}\right)\!\!\right).$$

In fact, $c^q(\nabla^M_{F_o})=c^k(\nabla^{s,E})\wedge c^p(\nabla^\bullet_{F_o})+\sum_{j>0}[c^{k-j}(\nabla^{s,E})\wedge c^{p+j}(\nabla^\bullet_{F_o})]$. But $c^{p+j}(\nabla^\bullet_{F_o})=0$ over $\mathcal{T}_D(\tilde{R}_0)\cap(D)^{2q}$, since it is equal to $c^{p+j}(\nabla^H)$ (because of the compatibility of $(\nabla^H,\nabla^M_{F_o},\nabla^{s,E})$ with the exact sequence), and since $c^{p+j}(\nabla^H)=0$ over $\mathcal{T}_D(\tilde{R}_0)\cap(D)^{2q}$ for $j>0$ (because ∇^H preserves there the $r-1$ frame $\tilde{F}^{(r-1)}$). Q.E.D.

We deduce the

Theorem. (i) *The cohomology class* $\tau_\Sigma(c^p_{\Sigma,\mathrm{vir}}(V,\tilde F^{(r)}_o)) - c^q_{\Sigma,\mathrm{SMP}}(V,\tilde F^{(r)}_o)$ *of the cocycle*

$$\left(\!\!\left(\begin{matrix} [0] & [0] & \left(c^k(\nabla^{s,E}) \wedge c^p(\nabla^\bullet_{F_o}) - c^q(\nabla^M_{F_o})\right) \\ [0] & 0 & 0 \\ & 0 & \end{matrix}\right)\!\!\right)$$

is well defined in $H^{2q}(M, M-\Sigma)$, *i.e. does not depend on the choice of the* $(\tilde F^{(r)}_0, s)$ *pair* $(\nabla^M_{F_o}, \nabla^{s,E})$ *of connections.*
(ii) *It is a "localization" of* $\tau(c^p_{\mathrm{vir}}(V,\tilde F^{(r)}_o)) - c^q_{\mathrm{SMP}}(V,\tilde F^{(r)}_o)$, *which means:*

$$\tau(c^p_{\mathrm{vir}}(V,\tilde F^{(r)}_o)) - c^q_{\mathrm{SMP}}(V,\tilde F^{(r)}_o) = \beta\left[\tau_\Sigma(c^p_{\Sigma,\mathrm{vir}}(V,\tilde F^{(r)}_o)) - c^q_{\Sigma,\mathrm{SMP}}(V,\tilde F^{(r)}_o)\right],$$

where $\beta : H^{2q}(M, M-\Sigma) \to H^{2q}(M, M-V)$ *denotes the natural map.*
(iii) *The* α *component* $\mu^q(V, S_\alpha)$ *of* $(-1)^n\left[\tau_\Sigma(c^p_{\Sigma,\mathrm{vir}}(V,\tilde F^{(r)}_o)) - c^q_{\Sigma,\mathrm{SMP}}(V,\tilde F^{(r)}_o)\right]$ *in* $H^{2q}(M, M-S_\alpha)$, *defined by the cocycle*

$$(-1)^n \left(\!\!\left(\begin{matrix} [0] & [0] & \left(c^k(\nabla^{s,E}) \wedge c^p(\nabla^\bullet_{F_o}) - c^q(\nabla^M_{F_o})\right)_\alpha \\ [0] & 0 & 0 \\ & 0 & \end{matrix}\right)\!\!\right),$$

corresponds by Alexander duality to the homological Milnor class $\mu_{m-q}(V, S_\alpha) \in H_{2(m-q)}(S_\alpha)$ *defined in* [BLSS].

Proof. The parts (i) and (ii) have already been proved, part (ii) resulting from Lemma 2. On the other hand, the image of

$$\left(\!\!\left(\begin{matrix} [0] & [0] & \left(c^k(\nabla^{s,E}) \wedge c^p(\nabla^\bullet_{F_o})\right)_\alpha \\ [0] & 0 & 0 \\ & 0 & \end{matrix}\right)\!\!\right)$$

by Alexander duality $A : H^{2q}(M, M-S_\alpha) \to H_{2(m-q}(S_\alpha)$ is still equal to the image of $\left(\!\left(c^p(\nabla^\bullet_{F_o}), [0], 0\right)\!\right)$ by the Poincaré morphism $P_V : H^{2p}(V, V-S_\alpha) \to H_{2(m-q}(S_\alpha)$: this is exactly the definition given in [BLSS] for the virtual index $\mathrm{Vir}(\tilde F^{(r)}_0, S_\alpha)$ of $\tilde F^{(r)}_0$ at S_α. Similarly,

$$(-1)^n \left(\!\!\left(\begin{matrix} [0] & [0] & \left(c^q(\nabla^M_{F_o})\right)_\alpha \\ [0] & 0 & 0 \\ & 0 & \end{matrix}\right)\!\!\right)$$

has for image by A the Schwartz index $\mathrm{Sch}(\tilde F^{(r)}_0, S_\alpha)$ of $\tilde F^{(r)}_0$ at S_α. Thus, $A(\mu^q(V, S_\alpha) = (-1)^n(\mathrm{Vir}(\tilde F^{(r)}_0, S_\alpha) - \mathrm{Sch}(\tilde F^{(r)}_0, S_\alpha)$: this corresponds to the definition of the homological Milnor class $\mu_{2(m-q)}(V, S_\alpha)$ given in [BLSS]. Q.E.D.

Remarks. 1) The Milnor class $\mu^q(V, S_\alpha)$ vanishes for any α such that $S_\alpha \cap (D)^{2q}$ is included in V_0: in fact, for such α's, the definition of H over $\tilde{R}_0$ may be extended to $\tilde{R}_\alpha$, so that we can add $\tilde{R}_\alpha$ to $\tilde{R}_0$ in Lemma 2.

Therefore, $\mu^q(V)$ arises in fact from a well defined element of $H^{2q}(M, M - Sing(V))$.

2) For $r = 1$, it results from the theorem that the Milnor number of V at S_α, such as defined in [BLSS] (the usual Milnor number if S_α is an isolated point ([M], [H]), or such as defined in [P] when V is an hypersurface in M), is given by:

$$\mu_0(V, S_\alpha) = \int_{\tilde{R}_\alpha} \left[c^k(\nabla^{s,E}) \wedge c^n(\nabla^\bullet_{F_o}) - c^m(\nabla^M_{F_o})\right].$$

§8. Virtual and Schwartz indices

Let more generally $\tilde{F}^{(r)}$ be a frame field satisfying properties (i) and (ii) of the end of section 2, but not necessarily radial.

Replacing the $(\tilde{F}_0^{(r)}, s)$ pair of connections $(\nabla^M_{F_o}, \nabla^{s,E})$ by a $(\tilde{F}^{(r)}, s)$ pair $(\nabla^M_F, \nabla^{s,E})$, we can even take the same $\nabla^{s,E}$ in both pairs. Then everything works in the same way as in section 5, for the definitions of $c^p_{0,\mathrm{vir}}(V, \tilde{F}^{(r)}))$ and $c^p_{\Sigma,\mathrm{vir}}(V, \tilde{F}^{(r)}))$. We get the following decomposition of $\tau c^p_{\mathrm{vir}}(V)$:

$$\tau_0(c^p_{0,\mathrm{vir}}(V, \tilde{F}^{(r)})) = \left(\!\!\left(\begin{matrix} [0] & c^k(\nabla^{s,E}) \wedge c^p(\nabla^\bullet_F) & [0] \\ 0 & [0] & 0 \\ & 0 & \end{matrix}\right)\!\!\right)$$

and $$\tau_\Sigma(c^p_{\Sigma,\mathrm{vir}}(V, \tilde{F}^{(r)})) = \left(\!\!\left(\begin{matrix} [0] & [0] & (c^k(\nabla^{s,E}) \wedge c^p(\nabla^\bullet_F)) \\ [0] & 0 & 0 \\ & 0 & \end{matrix}\right)\!\!\right).$$

However, as we already mentionned, we would not get the SMP classes if we just replace $\tilde{F}_0^{(r)}$ by $\tilde{F}^{(r)}$ in sections 6 and 7. Thus, we still define $c^p_{0,\mathrm{SMP}}(V, \tilde{F}^{(r)}))$ by the similar procedure:

$$c^p_{0,\mathrm{SMP}}(V, \tilde{F}^{(r)})) = \left(\!\!\left(\begin{matrix} [0] & c^q(\nabla^M_F) & [0] \\ 0 & [0] & 0 \\ & 0 & \end{matrix}\right)\!\!\right).$$

Therefore, we still have, as in Lemma 2 (similar proof):

$$c^p_{0,\mathrm{vir}}(V, \tilde{F}^{(r)})) = c^p_{0,\mathrm{SMP}}(V, \tilde{F}^{(r)})).$$

But, now, as a transcription of what we did in [BLSS], we define

$$c^p_{\Sigma,\mathrm{SMP}}(V,\tilde F^{(r)})) = c^p_{\Sigma,\mathrm{SMP}}(V,\tilde F_0^{(r)}) + \tau_\Sigma\big[(c^p_{\Sigma,\mathrm{vir}}(V,\tilde F^{(r)}) - c^p_{\Sigma,\mathrm{vir}}(V,\tilde F_0^{(r)})\big].$$

More precisely, we define the "difference" of the two frames, as the cohomology class $\delta^p(\tilde F_0^{(r)},\tilde F^{(r)})$ of $\big(\!\big(0,0,\big(c^p(\nabla^\bullet_{F_o},\nabla^\bullet_F)\big)\big)\!\big) \in H^{2p}(V)$.
Since $\big(\!\big(c^p(\nabla^\bullet_F) - c^p(\nabla^\bullet_{F_o}),[0],-c^p(\nabla^\bullet_{F_0},\nabla^\bullet_F)\big)\!\big) = D\big(\!\big(c^p(\nabla^\bullet_{F_0},\nabla^\bullet_F),[0],0\big)\!\big)$, then $\big[(c^p_{0,\mathrm{vir}}(V,\tilde F^{(r)}) - c^p_{0,\mathrm{vir}}(V,\tilde F_0^{(r)})\big]$ and $\big(\!\big(0,[0],\big(c^p(\nabla^\bullet_{F_o},\nabla^\bullet_F)\big)\big)\!\big)$ are equal in $H^{2p}(V,\Sigma)$.
Similarly $\big[(c^p_{\Sigma,\mathrm{vir}}(V,\tilde F^{(r)}) - c^p_{\Sigma,\mathrm{vir}}(V,\tilde F_0^{(r)})\big]$ and $\big(\!\big([0],0,\big(-c^p(\nabla^\bullet_{F_o},\nabla^\bullet_F)\big)\big)\!\big)$ are equal in $H^{2p}(V,V-\Sigma)$.

By the Thom-Gysin homomorphism, we get:

$$\tau\delta^p(\tilde F_0^{(r)},\tilde F^{(r)}) = \left(\!\!\left(\begin{matrix} [0] & 0 & 0 \\ 0 & 0 & c^k(\nabla^{\,s,E})\wedge c^p(\nabla^\bullet_{F_o},\nabla^\bullet_F) \\ & 0 & \end{matrix}\right)\!\!\right),$$

whose cohomology class is defined as well in $H^{2q}\big(M\ ,\ \mathcal{T}_D(R_A\cup\bigcup_\alpha \tilde R_\alpha\big)$ as in $H^{2q}(M,M-\Sigma)$. Thus, we get:

$$c^q_{\Sigma,\mathrm{SMP}}(V,\tilde F^{(r)}) = \left(\!\!\left(\begin{matrix} [0] & 0 & c^q(\nabla^M_{F_o}) \\ 0 & 0 & -c^k(\nabla^{\,s,E})\wedge c^p(\nabla^\bullet_{F_o},\nabla^\bullet_F) \\ & 0 & \end{matrix}\right)\!\!\right).$$

Of course, we have done what we needed for still guetting

$$\mu^q(V) = (-1)^n\big[c^q_{\Sigma,\mathrm{vir}}(V,\tilde F^{(r)}) - c^q_{\Sigma,\mathrm{SMP}}(V,\tilde F^{(r)})\big],$$

which does not depend on the frame field $\tilde F^{(r)}$. In particular, we have:

$$\mu^q(V,S_\alpha) = (-1)^n \left(\!\!\left(\begin{matrix} [0] & [0] & \big(c^k(\nabla^{\,s,E})\wedge c^p(\nabla^\bullet_F) - c^q(\nabla^M_{F_o})\big)_\alpha \\ [0] & 0 & \big(c^k(\nabla^{\,s,E})\wedge c^p(\nabla^\bullet_{F_o},\nabla^\bullet_F)\big)_\alpha \\ & 0 & \end{matrix}\right)\!\!\right).$$

On the other hand, as for $\tilde F_0^{(r)}$, $c^q_{\mathrm{SMP}}(V)$ is still equal to the sum of the images of $c^q_{0,\mathrm{SMP}}(V,\tilde F^{(r)})$ and $c^q_{\Sigma,\mathrm{SMP}}(V,\tilde F^{(r)})$ in $H^{2q}(M,M-V)$. This is an obvious corollary of the generalization of Lemma 2 to $\tilde F^{(r)}$.

The virtual index (resp. the Schwartz index) of $\tilde F^{(r)}$ at S_α such as defined in [BLSS] is nothing else but the image by the Alexander duality of the α component $c^q_{S_\alpha,\mathrm{vir}}(V,\tilde F^{(r)})$ (resp. $c^q_{S_\alpha,\mathrm{SMP}}(V,\tilde F^{(r)})$) of $c^q_{\Sigma,\mathrm{vir}}(V,\tilde F^{(r)})$ (resp. $c^q_{\Sigma,\mathrm{SMP}}(V,\tilde F^{(r)})$).

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The Milnor fiber as a virtual motive

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In this text, which correponds to our talk at the Conference "Singularities in Geometry and Topology" held in Sapporo in July 1998, we present our results, obtained in collaboration with Jan Denef, on the virtual motive associated to the Milnor fiber.

§1. Introduction

1.1. Let X be a smooth and connected complex algebraic variety and consider $f : X \to \mathbf{C}$ a non constant morphism. For any singular point x of $f^{-1}(0)$, the Milnor fiber at x is defined as

$$F_x := B(x, \varepsilon) \cap f^{-1}(t),$$

for $0 < |t| \ll \varepsilon \ll 1$, with $B(x, \varepsilon)$ the open ball of radius ε centered at x. There is some abuse of notation here, since, strictly speaking, F_x depends on the choice of ε and t, but all the invariants we shall consider will not.

Maybe the most natural invariants of the Milnor fiber to look at first are the Betti numbers

$$b_i(F_x) := \mathrm{rk} H^i(F_x, \mathbf{C}).$$

In fact, these numbers are in general very difficult to compute as soon as the singularity of $f = 0$ at x is not isolated. Much more easy to determine is the Euler characteristic

$$\chi(F_x) := \sum_i (-1)^i b_i(F_x).$$

When X is of dimension n and the singularity of $f = 0$ at x is isolated, $\chi(F_x) = 1 + (-1)^{n-1} b_{n-1}(F_x)$, and $b_{n-1}(F_x)$ is nothing else but the Milnor number.

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1.2. Of course, the information given by this Euler characteristic is quite weak. An already better invariant may be obtained by taking in account the natural monodromy action on the cohomology of F_x. The action of the monodromy operator M gives a canonical decomposition

$$H^i(F_x, \mathbf{C}) = \bigoplus_{\lambda \in \mathbf{C}^\times} H^i(F_x, \mathbf{C})_\lambda, \tag{1.2.1}$$

with $H^i(F_x, \mathbf{C})_\lambda$ the part where the eigenvalues of M are equal to λ. Hence one can refine the invariant $\chi(F_x)$ by defining

$$\chi(F_x)_\lambda := \sum_i (-1)^i \mathrm{rk} H^i(F_x, \mathbf{C})_\lambda.$$

By A'Campo's formula [1] (in fact a direct consequence of the commutation of the nearby cycle functor with the direct image with proper support functor [22]), the following simple formula for $\chi(F_x)_\lambda$ in terms of a resolution of $f = 0$ holds:

$$\chi(F_x)_\lambda = \sum_{\lambda^m = 1} \chi(S_m \cap \pi^{-1}(x)).$$

Here the notation is the following: we are given a resolution $\pi : \tilde{X} \to X$of $f = 0$, $\tilde{X}$ is smooth, π is proper and birational, the preimage E of the singular locus of $f^{-1}(0)$ is a divisor with (strict) normal crossings, and π is an isomorphism onto its image outside E, and S_m denotes the open subvariety of E where $\pi^{-1}(f^{-1}(0))$ is locally given by $z^m = 0$, z being a local coordinate.

Since the cohomology groups $H^i(F_x, \mathbf{C})$ carry a natural mixed Hodge structure [19] [21] [12] [13] [14], one can consider generalized Hodge numbers

$$e^{p,q} := \sum (-1)^i h^{p,q} H^i(F_x, \mathbf{C})$$

and

$$e_\lambda^{p,q} := \sum (-1)^i h^{p,q} H^i(F_x, \mathbf{C})_\lambda.$$

In fact the data of the $e_\lambda^{p,q}$'s is equivalent to that of the Hodge spectrum defined in [19] [21] [20] [14]. (For an analogue of A'Campo's formula for the Hodge spectrum see Remark 4.2.2.)

The commun feature for all these invariants is that they all may be defined as some kind of Euler characteristics. The main object of this paper is to provide, in some sense, universal invariants of Euler characteristic type for the Milnor fiber.

§2. Universal Euler characteristics and motives

2.1. Universal Euler characteristics

Invariants of Euler characteristic type take usually their values in a ring and satisfy relations of the type $F(A \cup B) = F(A) + F(B) - F(A \cap B)$ and $F(A \times B) = F(A)F(B)$. Consider now Sch, the category of reduced and separated schemes of finite type over $\mathbf{C}$ (i.e. varieties) and define the abelian group $K_0(\mathrm{Sch})$ as the quotient of the free abelian group generated by symbols $[S]$, S in Sch, by the relations

$$[S] = [S'],$$

for S' isomorphic to S and

$$[S] = [S'] + [S \setminus S'],$$

for S' closed in S. There is a natural product on $K_0(\mathrm{Sch})$ such that

$$[S][S'] = [S \times S'],$$

which provides $K_0(\mathrm{Sch})$ with a ring structure. To any constructible subset W of a variety S one can naturally associate an element $[W]$ in $K_0(\mathrm{Sch})$ such that

$$[W \cup W'] = [W] + [W'] - [W \cap W']$$

(just write W as the disjoint union of a finite family of varieties S_i and set $[W] = \sum [S_i]$; this is independent of the choice of the S_i's). Clearly $S \mapsto [S]$ is the "universal Euler characteristic" of algebraic varieties.

2.2. Motives

In our situation we are interested in keeping track of the monodromy action, in particular we want to have some analogue of the eigenvalue decomposition (1.2.1). This is in fact one of the reasons why motives enter in the picture: if a finite group G acts on a smooth projective variety X, there is a direct sum decomposition $h(X) = \bigoplus h(X)_\alpha$ of the motive $h(X)$ associated to X, with α running over the set of irreducible characters of G. The notion of motives being maybe not so familiar to people in singularity theory (though they are in fact easy to define, natural, and, we hope to convince the reader, useful), we shall give now some basic definitions (a good recent reference is [18]). Let $\mathcal{V}$ denote the category of smooth and projective $\mathbf{C}$-schemes. For an object X in $\mathcal{V}$ and an integer d, $\mathcal{Z}^d(X)$ denotes the free abelian group generated by irreducible subvarieties of X of codimension d. We define the rational Chow group $A^d(X)$

as the quotient of $\mathcal{Z}^d(X) \otimes \mathbf{Q}$ modulo rational equivalence. For X and Y in $\mathcal{V}$, we denote by $\operatorname{Corr}^r(X,Y)$ the group of correspondences of degree r from X to Y. If X is purely d-dimensional, $\operatorname{Corr}^r(X,Y) = A^{d+r}(X \times Y)$, and if $X = \coprod X_i$, $\operatorname{Corr}^r(X,Y) = \oplus \operatorname{Corr}^r(X_i, Y)$. The category Mot of $\mathbf{C}$-motives may be defined as follows (cf. [18]). Objects of Mot are triples (X,p,n) where X is in $\mathcal{V}$, p is an idempotent (i.e. $p^2 = p$) in $\operatorname{Corr}^0(X,X)$, and n is an integer in $\mathbf{Z}$. If (X,p,n) and (Y,q,m) are motives, then

$$\operatorname{Hom}_{\mathrm{Mot}}((X,p,n),(Y,q,m)) = q \operatorname{Corr}^{m-n}(X,Y)\, p.$$

Composition of morphisms is given by composition of correspondences. The category Mot is additive, $\mathbf{Q}$-linear, and pseudo-abelian. There is a natural tensor product on Mot, defined on objects by

$$(X,p,n) \otimes (Y,q,m) = (X \times Y, p \otimes q, n+m).$$

We denote by h the functor $h : \mathcal{V}^{\circ} \to \mathrm{Mot}$ which sends an object X to $h(X) = (X, \mathrm{id}, 0)$ and a morphism $f : Y \to X$ to its graph in $\operatorname{Corr}^0(X,Y)$. This functor is compatible with the tensor product and the unit motive $1 = h(\operatorname{Spec}\mathbf{C})$ is the identity for the product. We denote by $\mathbf{L}$ the Lefschetz motive $\mathbf{L} = (\operatorname{Spec}\mathbf{C}, \mathrm{id}, -1)$. One can prove there is a canonical isomorphism

$$h(\mathbf{P}^1) \simeq 1 \oplus \mathbf{L},$$

so, in some sense, $\mathbf{L}$ corresponds to $H^2(\mathbf{P}^1)$. We denote by $^\vee$ the involution $^\vee : \mathrm{Mot}^{\circ} \to \mathrm{Mot}$, defined on objects by $(X,p,n)^\vee = (X, {}^tp, d-n)$ if X is purely d-dimensional, and as the transpose of correspondences on morphisms. For X in $\mathcal{V}$ purely of dimension d, $h(X)^\vee = h(X) \otimes \mathbf{L}^{-d}$ (Poincaré duality). For any field E containing $\mathbf{Q}$ one defines similarly the category $\mathrm{Mot} \otimes E$ of motives with coefficients in E, by replacing the Chow groups $A^{\cdot}$ by $A^{\cdot} \otimes_{\mathbf{Q}} E$.

Since algebraic correspondences naturally act on cohomology, any cohomology theory on the category $\mathcal{V}$ factors through Mot and $\mathrm{Mot} \otimes E$, for E an extension of $\mathbf{Q}$, hence motives have canonical Betti and Hodge realizations.

Consider $K_0(\mathrm{Mot})$, the Grothendieck group of the pseudo-abelian category Mot. It is the abelian group associated to the monoid of isomorphism classes of motives with respect to $\oplus$. The tensor product on Mot induces a natural ring structure on $K_0(\mathrm{Mot})$. One defines similarly the ring $K_0(\mathrm{Mot} \otimes E)$ for E an extension of $\mathbf{Q}$. Of particular interest to us will be the case when E is the extension $\mathbf{Q}(\mu_\infty)$ of $\mathbf{Q}$ generated by all roots of unity in $\mathbf{C}$. To simplify notation we set $A := K_0(\mathrm{Mot} \otimes \mathbf{Q}(\mu_\infty))$.

Realization functors on Mot induce realization morphisms on the level on Grothendieck groups. In particular, we shall consider the Hodge realization morphism

$$H : A \longrightarrow K_0(\mathrm{MHS}_{\mathbf{C}}),$$

with $K_0(\mathrm{MHS}_{\mathbf{C}})$ the Grothendieck group of the abelian category of complex mixed Hodge structures.

By the following result of Gillet and Soulé [9] and Guillén and Navarro Aznar [10] one can assign to any algebraic variety a natural Euler characteristic (with proper suppports) with value into the ring $K_0(\mathrm{Mot})$ of virtual motives.

Theorem 2.2.1. *There exists a unique morphism of rings*

$$\chi_c : K_0(\mathrm{Sch}) \longrightarrow K_0(\mathrm{Mot})$$

such that $\chi_c([X]) = [h(X)]$ for X projective and smooth.

Remark that $\chi_c([\mathbf{A}^1]) = \mathbf{L}$. From now on we shall also denote by $\mathbf{L}$ the element $[\mathbf{A}^1]$ in $K_0(\mathrm{Sch})$.

Let G be a abelian finite group (in fact the assumption that G is abelian is irrelevant). Let X be an algebraic variety over $\mathbf{C}$ endowed with a G-action. We say X is a G-variety if the G-orbit of any closed point in X is contained in an affine open scheme (this condition is always satisfied when X is quasi-projective). One defines in the usual way isomorphisms and closed immersions of G-varieties and so one may define a ring $K_0(\mathrm{Sch}, G)$, the Grothendieck ring of G-varieties over k, similarly as we defined $K_0(\mathrm{Sch})$.

For any character α of G, let us denote by p_α the corresponding idempotent in $\mathbf{Q}(\mu_\infty)[G]$. Let X be a smooth projective variety on which G acts. There is a natural ring morphism μ from $\mathbf{Q}(\mu_\infty)[G]$ to the ring of correspondences on X with coefficients in $\mathbf{Q}(\mu_\infty)$ sending a group element g onto the graph of multiplication by g. Let us denote by $h(X, \alpha)$ the motive $(X, \mu(p_\alpha), 0)$ in $\mathrm{Mot} \otimes \mathbf{Q}(\mu_\infty)$.

The following equivariant analogue of Theorem 2.2.1 is proved in [6].

Theorem 2.2.2. *For any character α of G, there exists a unique morphism of rings*

$$\chi_c(\_, \alpha) : K_0(\mathrm{Sch}, G) \longrightarrow A$$

such that $\chi_c([X], \alpha) = [h(X, \alpha)]$ for X projective and smooth with G-action.

2.3. An example: Fermat Hypersurfaces and Jacobi motives

An important and classical example of varieties with group action giving rise to interesting motives are Fermat hypersurfaces. These motives will also occur naturally in our motivic analogue of the Thom-Sebastiani formula (Theorem 5.4.2). For $n \geq 1$, we consider the affine Fermat variety F_d^n defined by the equation $x_1^d + \cdots + x_n^d = 1$ in $\mathbf{A}^n$. The action of μ_d, the group of d-th roots of unity, on each coordinate induces a natural action of the group μ_d^n on F_d^n. Hence, for $\alpha_1, \ldots, \alpha_n$ characters of μ_d, one defines the Jacobi motive $J(\alpha_1, \ldots, \alpha_n)$ as the element

$$J(\alpha_1, \ldots, \alpha_n) := \chi_c(F_d^n, (\alpha_1, \ldots, \alpha_n))$$

in A. It is clear that $J(\alpha_1, \ldots, \alpha_n)$ is symmetric in the α_i's. In fact, as is quite clasical, one can recover from $J(\alpha_1, \ldots, \alpha_n)$ the usual Jacobi sums (via étale realization using the Galois action) and the Beta function (via the period pairing for the Hodge realization) (cf., *e.g.*, [2]).

The following identities which are analogues of classical identities for Jacobi sums and Beta functions are proved in [8].

Proposition 2.3.1. (1) *We have* $J(1,1) = \mathbf{L}$.

(2) *We have* $J(1,\alpha) = 0$ *if* $\alpha \neq 1$.

(3) *If* $\alpha \neq 1$, $J(\alpha, \alpha^{-1}) = -1$.

(4) *We have*

$$J(\alpha_1, \alpha_2)[J(\alpha_1\alpha_2, \alpha_3) - \varepsilon] = J(\alpha_1, \alpha_2, \alpha_3) - \delta,$$

with $\varepsilon = \delta = 0$ *if* $\alpha_1\alpha_2 \neq 1$, $\varepsilon = 1$, $\delta = (\mathbf{L} - 1)$, *if* $\alpha_1\alpha_2 = 1$ *and* $\alpha_1 \neq 1$, *and* $\varepsilon = 1$, $\delta = \mathbf{L}$, *if* $\alpha_1 = \alpha_2 = 1$.

§3. An interlude: Motivic Igusa Zeta functions

3.1. Let p be a prime number and let K be a finite extension of $\mathbf{Q}_p$. Let R be the valuation ring of K, P the maximal ideal of R, and $\bar{K} = R/P$ the residue field of K. Let q denote the cardinality of $\bar{K}$, so $\bar{K} \simeq \mathbf{F}_q$. For z in K, let $\operatorname{ord} z$ denote the valuation of z, and set $|z| = q^{-\operatorname{ord} z}$. Let f be a non constant element of $K[x_1, \ldots, x_m]$. The p-adic Igusa local zeta function $Z(s)$ associated to f (relative to the trivial multiplicative character) is defined as the p-adic integral

$$Z(s) = \int_{R^m} |f(x)|^s |dx|, \tag{3.1.1}$$

for $s \in \mathbf{C}$, $\mathrm{Re}(s) > 0$, where $|dx|$ denotes the Haar measure on K^m normalized in such of way that R^m is of volume 1. For n in $\mathbf{N}$, set $Z_n = \{x \in R^m \mid \operatorname{ord} f(x) = n\}$. We may express $Z(s)$ as a series

$$Z(s) = \sum_{n \geq 0} \operatorname{vol}(Z_n)\, q^{-ns}. \tag{3.1.2}$$

Now, if we denote by X_n the image of Z_n in $(R/P^{n+1})^m$, we may rewrite the series as

$$Z(s) = \sum_{n \geq 0} \operatorname{card}(X_n)\, q^{-ns-(n+1)m} \tag{3.1.3}$$

since $\operatorname{vol}(Z_n) = \operatorname{card}(X_n)\, q^{-(n+1)m}$.

3.2. Now let X be a smooth and connected complex algebraic variety and consider $f : X \to \mathbf{C}$ a non constant morphism. We denote by $\mathcal{L}(X)$ the space of formal arcs on X: there is a natural bijection between the space of $\mathbf{C}$-points of $\mathcal{L}(X)$, $\mathcal{L}(X)(\mathbf{C})$, and $X(\mathbf{C}[[t]])$. There is a natural structure of $\mathbf{C}$-scheme on $\mathcal{L}(X)$, but we shall always consider $\mathcal{L}(X)$ with its reduced structure. Similarly, for $n \geq 0$, we can consider the space $\mathcal{L}_n(X)$ of arcs modulo t^{n+1}: a $\mathbf{C}$-point of $\mathcal{L}_n(X)$ corresponds to a $\mathbf{C}[t]/t^{n+1}\mathbf{C}[t]$-point on X. The space $\mathcal{L}_n(X)$ may be endowed with a natural structure of $\mathbf{C}$-scheme of finite type, and there is a natural morphism

$$\pi_n : \mathcal{L}(X) \longrightarrow \mathcal{L}_n(X)$$

given by truncation. In this setting $\mathcal{L}(\mathbf{A}^m)$ and $\mathcal{L}_n(\mathbf{A}^m)$ may be considered as analogues of R^m and $(R/P^{n+1})^m$. Pursuing this analogy further, one considers the reduced subscheme Z_n of $\mathcal{L}(\mathbf{A}^m_k)$ whose points are the series φ such that $\operatorname{ord}_t f(\varphi) = n$ and the image X_n of Z_n in $\mathcal{L}_n(\mathbf{A}^m_k)$, which has a natural structure of variety over $\mathbf{C}$. More generally, for W closed in X, we shall denote by $Z_{W,n}$ the closed subscheme of Z_n whose points are arcs φ with $\varphi(0)$ in W and by $X_{W,n}$ its image in $\mathcal{L}_n(\mathbf{A}^m_k)$.

A natural analogue of the right-hand side of (3.1.3), which is a series in $\mathbf{Z}[p^{-1}][[p^{-s}]]$, is the following series in $K_0(\mathrm{Sch}_k)[\mathbf{L}^{-1}][[\mathbf{L}^{-s}]]$

$$Z_{\mathrm{geom}}(s) = \sum_{n \geq 0} [X_n]\, \mathbf{L}^{-ns-(n+1)m}. \tag{3.2.1}$$

Here $\mathbf{L}^{-s}$ is just the name for a formal variable which could as well be written $T = \mathbf{L}^{-s}$.

3.3. More generally, p-adic Igusa local zeta functions involve multiplicative characters. Let π be a fixed uniformizing parameter of R and set $\mathrm{ac}(z) = z\pi^{-\operatorname{ord} z}$ for z in K. For any character $\alpha : R^\times \to \mathbf{C}^\times$ (i.e.

a group morphism with finite image), one defines the p-adic Igusa local zeta function $Z(s,\alpha)$ as the integral

$$Z(s,\alpha) = \int_{R^m} \alpha(\mathrm{ac}(f(x)))|f(x)|^s|dx|, \tag{3.3.1}$$

for $s \in \mathbf{C}$, $\mathrm{Re}(s) > 0$ (see [11], [4]). To extend definition (3.2.1) to the more general situation involving characters, we shall use motives in the following way.

We fix an integer $d \geq 1$. Let $g : W \to \mathbf{C}^\times$ be a morphism of $\mathbf{C}$-varieties. For any character α of μ_d, one may define an element $[W]_{g,\alpha}$ of $K_0(\mathrm{Mot}_k \otimes \mathbf{Q})$ as follows.

The morphism $[d] : \mathbf{C}^\times \to \mathbf{C}^\times$ given by $x \mapsto x^d$ is a Galois covering with Galois group μ_d. We consider the fiber product

$$\begin{array}{ccc} \widetilde{W}_{g,d} & \longrightarrow & W \\ \big\downarrow & & \big\downarrow{\scriptstyle g} \\ \mathbf{C}^\times & \xrightarrow{[d]} & \mathbf{C}^\times. \end{array}$$

The scheme $\widetilde{W}_{g,d}$ is endowed with an action of μ_d, so we can define

$$[W]_{g,\alpha} := \chi_c(\widetilde{W}_{g,d}, \alpha).$$

In our setting we can consider the morphism $f_n : X_n \to \mathbf{C}^\times$ whose value at a series φ is the coefficient of order n of $f(\varphi)$. When d divides d' we have a canonical surjective morphism of groups $\mu_{d'} \to \mu_d$ given by $x \mapsto x^{d'/d}$ which dualizes to a injective morphism of character groups $\widehat{\mu}_d \to \widehat{\mu}_{d'}$. We set $\widehat{\mu} := \varinjlim \widehat{\mu}_d$. We shall identify $\widehat{\mu}_d$ with the subgroup of elements of order dividing d in $\widehat{\mu}$.

Now let α be in $\widehat{\mu}$ an element of order d. Viewing α as a character of μ_d, we may now define the series

$$Z_{\mathrm{mot}}(s,\alpha) = \sum_{n \geq 0} [X_n]_{f_n,\alpha} \mathbf{L}^{-ns-(n+1)m} \tag{3.3.2}$$

in $K_0(\mathrm{Mot}_k \otimes \mathbf{Q})[[\mathbf{L}^{-s}]]$. More generally, for W a closed subvariety of X, one defines similarly a series $Z_{\mathrm{mot},W}(s,\alpha)$ by replacing in the previous definition X_n by the variety $X_{W,n}$.

3.4. Rationality and formula on a resolution

Let D be the divisor defined by $f = 0$ in X. Let (Y, h) be a resolution of f. By this, we mean that Y is a smooth and connected k-scheme of finite type, $h : Y \to X$ is proper, that the restriction

$h : Y \setminus h^{-1}(D) \to X \setminus D$ is an isomorphism, and that $(h^{-1}(D))_{\mathrm{red}}$ has only normal crossings as a subscheme of Y. Let E_i, $i \in J$, be the irreducible (smooth) components of $(h^{-1}(D))_{\mathrm{red}}$. For each $i \in J$, denote by N_i the multiplicity of E_i in the divisor of $f \circ h$ on Y, and by $\nu_i - 1$ the multiplicity of E_i in the divisor of h^*dx, where dx is a local non vanishing volume form, i.e. a local generator of the sheaf of differential forms of maximal degree. For $i \in J$ and $I \subset J$, we consider the schemes $E_i^\circ := E_i \setminus \cup_{j \neq i} E_j$, $E_I := \cap_{i \in I} E_i$, and $E_I^\circ := E_I \setminus \cup_{j \in J \setminus I} E_j$. When $I = \emptyset$, we have $E_\emptyset = Y$.

Now denote by J_d the set of $I \subset J$ such that $d \,|\, N_i$ for all i in I and by U_d the union of the E_I°'s, with I in J_d. Let Z be locally closed in U_d. For any character α of $\mu_d(k)$ of order d, we will construct an element $[Z]_{f,\alpha}$ in $K_0(\mathrm{Mot}_k \otimes \mathbf{Q})$ as follows. If on Z we may write $f \circ h = uv^d$ with u non vanishing on Z, we set $[Z_{f,\alpha}] = [Z]_{u,\alpha}$. In general, one covers Z by a finite set of Z_r's for which the previous condition holds, and we set

$$[Z_{f,\alpha}] = \sum_r [(Z_r)_{f,\alpha}] - \sum_{r_1 \neq r_2} [(Z_{r_1} \cap Z_{r_2})_{f,\alpha}] + \cdots .$$

One can check this definition does not depend of any choice.

We can now state the following result which is proved in [6]:

Theorem 3.4.1. *For any element α of $\hat{\mu}$ of order d,*

$$Z_{\mathrm{mot},W}(s,\alpha) = \mathbf{L}^{-m} \sum_{I \in J_d} [(E_I^\circ \cap h^{-1}(W))_{f,\alpha}] \cdot \prod_{i \in I} \frac{(\mathbf{L}-1)\,\mathbf{L}^{-N_i s - \nu_i}}{1 - \mathbf{L}^{-N_i s - \nu_i}} \tag{3.4.1}$$

in $A\,[[\mathbf{L}^{-s}]]$.

In particular it follows that $Z_{\mathrm{mot},W}(s,\alpha)$ is a rational series in $\mathbf{L}^{-s}$. It also follows that if the order of the character α does not divide any of the N_i's, then $Z_{\mathrm{mot},W}(s,\alpha)$ is identically zero (hence only of finite number of the functions $Z_{\mathrm{mot},W}(s,\alpha)$ are not identically zero).

The proof of Theorem 3.4.1 is based on the following geometric lemma which is a special case of Lemma 3.4 in [7].

Let X, Y and F be algebraic varieties over $\mathbf{C}$, and let A, *resp.* B, be a constructible subset of X, *resp.* Y. We say that a map $\pi : A \to B$ is *piecewise trivial fibration with fiber F*, if there exists a finite partition of B in subsets S which are locally closed in Y such that $\pi^{-1}(S)$ is locally closed in X and isomorphic to $S \times F$, with π corresponding under the isomorphism to the projection $S \times F \to S$. We say that the map π is a *piecewise trivial fibration over* some constructible subset C of B, if the restriction of π to $\pi^{-1}(C)$ is a piecewise trivial fibration.

Lemma 3.4.2. *Let X and Y be connected smooth schemes over $\mathbf{C}$ and let $h : Y \to X$ be a birational morphism. For e in $\mathbf{N}$, let Δ_e be the reduced subscheme of $\mathcal{L}(Y)$ defined by*

$$\Delta_e := \{\varphi \in Y(\mathbf{C}[[t]]) \mid \mathrm{ord}_t \det \mathcal{J}_\varphi = e\},$$

where $\mathcal{J}_\varphi$ is the jacobian of h at φ. For n in $\mathbf{N}$, let $h_{n*} : \mathcal{L}_n(Y) \to \mathcal{L}_n(X)$ be the morphism induced by h, and let $\Delta_{e,n}$ be the image of Δ_e in $\mathcal{L}_n(Y)$. If $n \geq 2e$, the following holds.

a) *The set $\Delta_{e,n}$ is a union of fibers of h_{n*}.*
b) *The restriction of h_{n*} to $\Delta_{e,n}$ is a piecewise trivial fibration with fiber $\mathbf{A}^e$ onto its image.*

Remark 3.4.3. These motivic Igusa functions specialize, by considering the trace of the Frobenius on their étale realization, in the p-adic case with good reduction, to the usual p-adic Igusa local zeta functions. They also specialize, by considering Euler characteristic of their Betti realization, to the topological zeta functions $Z_{\mathrm{top}}(s)$ introduced in [5], which were, heuristically, obtained as a limit as q goes to 1 of p-adic Igusa local zeta functions. We refer to [6] for details.

§4. The virtual motive attached to the Milnor fiber

4.1. Since $Z_{\mathrm{mot},W}(s,\alpha)$ is an A-linear combination of rational series of the form $\mathbf{L}^{-Ns-n}/(1-\mathbf{L}^{-Ns-n})$, with N and n in $\mathbf{N} \setminus \{0\}$, one can consider its limit as $s \to -\infty$, by defining

$$\lim_{s \to -\infty} \frac{\mathbf{L}^{-Ns-n}}{1-\mathbf{L}^{-Ns-n}} = -1.$$

One easily checks that one obtains in this way a well defined element

$$\lim_{s \to -\infty} Z_{\mathrm{mot},W}(s,\alpha)$$

in A. It follows from Theorem 3.4.1 that we have the following expression for $\lim_{s \to -\infty} Z_{\mathrm{mot},W}(s,\alpha)$ in terms of a resolution of $f = 0$:

$$\lim_{s \to -\infty} Z_{\mathrm{mot},W}(s,\alpha) = \mathbf{L}^{-m} \sum_{I \in J_d} [(E_I^\circ \cap h^{-1}(W))_{f,\alpha}](1-\mathbf{L})^{|I|}. \tag{4.1.1}$$

Note that it is not a priori clear that the right hand side of (4.1.1) is independent of the resolution, but it follows from the fact that the left hand side is canonical.

4.2. We assume from now on that W is contained in $f^{-1}(0)$. In that case, it follows from (4.1.1) that $\lim_{s \to -\infty} Z_{\mathrm{mot},W}(s,\alpha)$ is divisible by $1 - \mathbf{L}$, so we may define

$$S^{\psi}_{\alpha,W,f} := \frac{\mathbf{L}^m}{1 - \mathbf{L}} \lim_{s \to -\infty} Z_{\mathrm{mot},W}(s,\alpha).$$

Strictly speaking $S^{\psi}_{\alpha,W,f}$ is only defined up to $(\mathbf{L}-1)$-torsion in A, but this is not a serious problem, since $(\mathbf{L}-1)$-torsion is killed by realization functors. (In fact we do not know whether there exists or not any non trivial $(\mathbf{L}-1)$-torsion element in A.)

By the following result, proved in [6], the Hodge realization of $S^{\psi}_{\alpha,\{x\},f}$ is equal to the virtual Hodge structure defined by the Milnor fiber at x for the eigenvalue $\alpha(e^{2\pi i/d})$. Hence it is very natural to consider $S^{\psi}_{\alpha,\{x\},f}$ as the virtual motive associated to the Milnor fiber at x for the eigenvalue $\alpha(e^{2\pi i/d})$.

Theorem 4.2.1. *Let x be a point of $f^{-1}(0)$. Denote by $[H^i(F_x,\mathbf{C})_{\alpha(e^{2\pi i/d})}]$ the class in $K_0(\mathrm{MHS}_{\mathbf{C}})$ of $H^i(F_x,\mathbf{C})_{\alpha(e^{2\pi i/d})}$ with its canonical Hodge structure. The equality*

$$H(S^{\psi}_{\alpha,\{x\},f}) = \sum_i (-1)^i [H^i(F_x,\mathbf{C})_{\alpha(e^{2\pi i/d})}]$$

holds in $K_0(\mathrm{MHS}_{\mathbf{C}})$.

Remark 4.2.2. As a consequence of (4.1.1) and Theorem 4.2.1, one deduces an analogue of A'Campo's formula for the Hodge spectrum.

§5. Exponential integrals and a motivic Thom-Sebastiani Theorem

5.1. We begin by reviewing exponential integrals in the p-adic case, so we use again the notations of 3.1.

Let $f \in R[x_1, \ldots, x_m]$ be a non constant polynomial. Let $\Phi : R^m \to \mathbf{C}$ be a locally constant function with compact support. Let α be a character of $R^{\times}$. For i in $\mathbf{N}$, we set

$$Z_{\Phi,f,i}(\alpha) := \int_{\{x \in R^m \mid \mathrm{ord} f(x) = i\}} \Phi(x)\alpha(\mathrm{ac} f(x))|dx|.$$

We denote by Ψ the standard additive character on K, defined by

$$z \longmapsto \Psi(z) = \exp(2i\pi \mathrm{Tr}_{K/\mathbf{Q}_p} z).$$

For i in $\mathbf{N}$, we consider the exponential integral

$$E_{\Phi,f,i} := \int_{R^m} \Phi(x)\Psi(\pi^{-(i+1)} f(x))|dx|. \tag{5.1.1}$$

For α a character of $R^\times$, the conductor of α, $c(\alpha)$, is defined as the smallest $c \geq 1$ such that α is trivial on $1 + P^c$, and one associates to α the Gauss sum

$$g(\alpha) = q^{1-c(\alpha)} \sum_{v \in (R/P^{c(\alpha)})^\times} \alpha(v)\Psi(v/\pi^{c(\alpha)}).$$

The following result is a consequence of § 1 of [4].

Proposition 5.1.1. *For any i in $\mathbf{N}$,*

$$\begin{aligned} E_{\Phi,f,i} &= \int_{\{x \in R^m \mid \operatorname{ord} f(x) > i\}} \Phi(x)|dx| \\ &\quad + (q-1)^{-1} \sum_{\alpha} g(\alpha^{-1}) Z_{\Phi,f,i-c(\alpha)+1}(\alpha). \end{aligned} \tag{5.1.2}$$

Here $i - c(\alpha) + 1 \geq 0$. If moreover the critical locus of f in $\operatorname{Supp}\Phi$ *is contained in $f^{-1}(0)$, then, for all except a finite number of characters α, the integrals $Z_{\Phi,f,j}(\alpha)$ are zero for all j.*

Using Theorem 3.3 of [4], one deduces from Proposition 5.1.1 that, assuming that Φ is residual, i.e. that $\operatorname{Supp}\Phi$ is contained in R^m and that $\Phi(x)$ depends only on x modulo P, that the critical locus of f in $\operatorname{Supp}\Phi$ is contained in $f^{-1}(0)$ and that the divisor $f = 0$ has good reduction (in the sense that the conditions in Theorem 3.3 of [4] are satisfied), then

$$\begin{aligned} E_{\Phi,f,i} &= \int_{\{x \in R^m \mid \operatorname{ord} f(x) > i\}} \Phi(x)|dx| \\ &\quad + (q-1)^{-1} \sum_{\substack{\alpha \\ c(\alpha)=1}} g(\alpha^{-1}) Z_{\Phi,f,i}(\alpha). \end{aligned} \tag{5.1.3}$$

5.2. Exponential integrals

Let X be a smooth connected variety over $\mathbf{C}$ of dimension m and let $f : X \to \mathbf{C}$ be a morphism. If one is looking for a motivic analogue of p-adic exponential integrals, a hint is given by formula (5.1.3) which expresses p-adic exponential integrals as linear combinations of p-adic integrals involving multiplicative characters with Gauss sums as coefficients. Though Gauss sums are not motivic themeselves, they are related to Jacobi sums by the familiar relation

$$g(\alpha)g(\beta) = g(\alpha\beta)j(\alpha,\beta), \tag{5.2.1}$$

when α, β and $\alpha\beta$ are not equal to 1 and have conductor 1, with

$$j(\alpha,\beta)=\sum_{x\in\bar{K}\setminus\{0,1\}}\alpha(x)\beta(1-x).$$

But the Jacobi sums $j(\alpha,\beta)$ are motivic, being equal to the trace of the Frobenius on the étale realization of a Jacobi motive, hence we may follow the idea, introduced by Greg Anderson in [2], of enlarging the world of motives by adding Gauss sums motives related to Jacobi motives by a relation similar to 5.2.1. More precisely, one considers the free A-module U with basis G_α, α in $\widehat{\mu}(k)$. We define an A-algebra structure on U by putting the following relations:

$$G_1=-1 \tag{5.2.2}$$

$$G_\alpha G_{\alpha^{-1}}=\mathbf{L} \qquad \text{for} \quad \alpha\neq 1 \tag{5.2.3}$$

$$G_{\alpha_1}G_{\alpha_2}=J(\alpha_1,\alpha_2)\,G_{\alpha_1\alpha_2} \qquad \text{for} \quad \alpha_1,\alpha_2,\alpha_1\alpha_2\neq 1. \tag{5.2.4}$$

It follows from Proposition 2.3.1 that U is a commutative and associative algebra.

For m in $\mathbf{Z}$, let F^mA denote the subgroup of A generated by $h(S,f,i)$, with $i-\dim S\geq m$. This gives a filtration on the ring A; we denote by $\widehat{A}$ the completion of A with respect to this filtration and we set $\widehat{U}:=U\otimes_A\widehat{A}$. We shall also consider the subring A_{loc} of $\widehat{A}$ generated by the image of A in $\widehat{A}$ and the series $(1-\mathbf{L}^{-n})^{-1}$, $n\in\mathbf{N}\setminus\{0\}$. We denote by U_{loc} the tensor product $U\otimes_A A_{\mathrm{loc}}$, which is naturally a subring of $\widehat{U}$.

Let W be a closed subvariety of $f^{-1}(0)$. We define, for $i\geq 0$, the motivic analogue $E_{i,W,f,\mathrm{mot}}$ of $E_{\Phi,f,i}$ as the series

$$E_{i,W,f,\mathrm{mot}}:=\sum_{n>i}\frac{\chi_c([X_{W,n}])}{\mathbf{L}^{(n+1)m}}+\sum_{\alpha\in\widehat{\mu}(k)}\frac{1}{\mathbf{L}-1}G_{\alpha^{-1}}\frac{[X_{W,i}]_{f_i,\alpha}}{\mathbf{L}^{(i+1)m}} \tag{5.2.5}$$

in $\widehat{U}$. Remark that, since only a finite number of the functions $Z_{\mathrm{mot},W}(s,\alpha)$ are non zero, the second sum in (5.2.5) is finite. Furthermore, one can deduce from Theorem 3.4.1 that $E_{i,W,f,\mathrm{mot}}$ belongs in fact to the ring U_{loc}. In [8] the definition of $E_{i,W,f,\mathrm{mot}}$ is extended to the case where X is no longer smooth.

Now the standard multiplicativity property of exponential integrals is no longer trivial. In fact the following result is one of the main results in [8]:

Theorem 5.2.1. *Let X and X' be irreducible complex algebraic varieties over $\mathbf{C}$, let $f:X\to\mathbf{C}$ and $f':X'\to\mathbf{C}$ be morphisms of $\mathbf{C}$-varieties. Denote by $f\oplus f':X\times X'\to\mathbf{C}$ the morphism $(x,x')\mapsto$*

$f(x)+f'(x')$. Let W (resp. W') be a reduced subvariety of $f^{-1}(0)$ (resp. $f'^{-1}(0)$). For every $i \geq 0$,

$$E_{i,f\oplus f',W\times W',\mathrm{mot}} = E_{i,f,W,\mathrm{mot}} \cdot E_{i,f',W',\mathrm{mot}}. \tag{5.2.6}$$

5.3. An algebraic lemma on power expansions of rational functions

Denote by B the ring $\mathcal{U}_{\mathrm{loc}}$. We consider the ring of Laurent polynomials $B[T,T^{-1}]$ and its localisation $B[T,T^{-1}]_{\mathrm{rat}}$ obtained by inverting the multiplicative family generated by the polynomials $1-\mathbf{L}^aT^b$, a, b in $\mathbf{Z}$, $b\neq 0$. Remark that, in this definition, we could restrict to $b>0$ or to $b<0$. Hence, by expanding denominators into formal series, there are canonical embeddings of rings

$$\exp_T : B[T,T^{-1}]_{\mathrm{rat}} \hookrightarrow B[T^{-1},T]]$$

and

$$\exp_{T^{-1}} : B[T,T^{-1}]_{\mathrm{rat}} \hookrightarrow B[[T^{-1},T].$$

Here $B[T^{-1},T]]$ (resp. $B[[T^{-1},T]$) denotes the ring of series $\sum_{i\in\mathbf{Z}} a_iT^i$ with $a_i=0$ for $i\ll 0$ (resp. $i\gg 0$). By taking the difference $\exp_T - \exp_{T^{-1}}$ of the two expansions one obtains an embedding

$$\tau : B[T,T^{-1}]_{\mathrm{rat}}/B[T,T^{-1}] \hookrightarrow B[[T^{-1},T]],$$

where $B[[T^{-1},T]]$ is the group of formal Laurent series with coefficients in B.

Let $\varphi=\sum_{i\in\mathbf{Z}} a_iT^i$ and $\psi=\sum_{i\in\mathbf{Z}} b_iT^i$ be series in $B[[T^{-1},T]]$. We define their Hadamard product as the series

$$\varphi * \psi := \sum_{i\in\mathbf{Z}} a_ib_i\,T^i.$$

A basic elementary result (see Proposition 5.1.1 of [8] for a proof) states that if two series φ and ψ in $B[[T^{-1},T]]$ belong to the image of τ, then their Hadamard product $\varphi*\psi$ is also in the image of τ. It follows in particular that the intersection of $B[[T]]$ with the image of $\exp_T$, which we shall denote by $B[[T]]_{\mathrm{rat}}$, is stable under Hadamard product.

Let $\varphi=\exp_T(P)$ be in $B[[T]]_{\mathrm{rat}}$. We denote by $\lambda(\varphi)$ the constant term in the expansion of $\exp_{T^{-1}}(P)$.

We shall need the following lemma, whose proof is completely elementary (see Proposition 5.1.2 of [8] for the proof).

Funny Lemma 5.3.1. *Let φ and ψ be series in $TB[[T]]_{\mathrm{rat}}$. Then*

$$\lambda(\varphi * \psi) = -\,\lambda(\varphi)\cdot\lambda(\psi).$$

5.4. A motivic stationary phase formula

Now we consider the Poincaré series

$$E_{W,f}(T) := \sum_{i>0} E_{i,W,f,\mathrm{mot}} T^i.$$

Note that $E_{W,f}(T)$ has no constant term. One may deduce from Theorem 3.4.1 that the series $E_{W,f}(T)$ belongs in fact to $U_{\mathrm{loc}}[[T]]_{\mathrm{rat}}$.

We shall now consider $\mathcal{S}^{\psi}_{\alpha,W,f}$ as an element of A_{loc} (note there is no $(\mathbf{L}-1)$-torsion in A_{loc}), and we define $\mathcal{S}^{\phi}_{\alpha,W,f} = \mathcal{S}^{\psi}_{\alpha,W,f}$ for $\alpha \neq 1$, and $\mathcal{S}^{\phi}_{\alpha,W,f} = \mathcal{S}^{\psi}_{\alpha,W,f} - \chi_c([W])$, for $\alpha = 1$, in A_{loc}. Remark that, since $\mathcal{S}^{\psi}_{\alpha,W,f}$ corresponds to motivic Euler characteristic of nearby cycles, $\mathcal{S}^{\phi}_{\alpha,W,f}$ corresponds to motivic Euler characteristic of vanishing cycles.

One easily gets the following formula, which may be viewed as a motivic analogue of the stationary phase formula:

Motivic stationary phase formula 5.4.1. *The following relation holds in A_{loc}:*

$$\lambda(E_{W,f}(T)) = -\mathbf{L}^{-m} \sum_{\alpha \in \widehat{\mu}(k)} G_{\alpha^{-1}} \mathcal{S}^{\phi}_{\alpha,W,f}.$$

The following Motivic Thom-Sebastiani Theorem follows directly from the motivic analogue stationary phase formula and the Funny Lemma 5.3.1.

Theorem 5.4.2. *Let X and X' be smooth and connected algebraic varieties over $\mathbf{C}$ of pure dimension m and m'. Let $f : X \to \mathbf{A}^1_k$ and $f' : X' \to \mathbf{A}^1_k$ be morphisms of k-varieties. Let W (resp. W') be a reduced subscheme of $f^{-1}(0)$ (resp. $f'^{-1}(0)$). Then*

$$\sum_{\alpha} G_{\alpha^{-1}} \mathcal{S}^{\phi}_{\alpha,W\times W',f\oplus f'} = \Big(\sum_{\alpha} G_{\alpha^{-1}} \mathcal{S}^{\phi}_{\alpha,W,f}\Big) \cdot \Big(\sum_{\alpha} G_{\alpha^{-1}} \mathcal{S}^{\phi}_{\alpha,W',f'}\Big). \tag{5.4.1}$$

One can observe that the appearance in the Thom-Sebastiani formula of vanishing cycles instead of nearby cycles is explained here by the Funny Lemma 5.3.1 which is only valid for series without constant terms!

5.5. We now explain how one can deduce from Theorem 5.4.2 a Thom-Sebastiani Theorem for the Hodge spectrum.

Since a $\mathbf{C}$-Hodge structure of weight n is just a finite dimensional bigraded vector space $V = \bigoplus_{p+q=n} V^{p,q}$, or, equivalently, a finite dimensional vector space V with decreasing filtrations $F^{\cdot}$ and $\overline{F}^{\cdot}$ such that $V = F^p \oplus \overline{F}^q$ when $p+q = n+1$, one can define similarly a *rational* $\mathbf{C}$-Hodge structure of weight n, by allowing p and q to belong to $\mathbf{Q}$ but still requiring $p+q \in \mathbf{Z}$.

We denote by $K_0(\mathrm{RMHS}_{\mathbf{C}})$ the Grothendieck group of the abelian category of rational $\mathbf{C}$-Hodge structures. For $d \geq 1$, there is an embedding of $\widehat{\mu}_d(\mathbf{C})$ in $\mathbf{Q}/\mathbf{Z}$ given by $\alpha \mapsto a$ with $\alpha(e^{2\pi i/d}) = e^{2\pi i a}$. This gives an isomorphism $\widehat{\mu}(\mathbf{C}) \simeq \mathbf{Q}/\mathbf{Z}$. We denote by γ the section $\mathbf{Q}/\mathbf{Z} \to [0,1)$.

The morphism $H : A \to K_0(\mathrm{MHS}_{\mathbf{C}})$ may be extended to a morphism $H : U \to K_0(\mathrm{RMHS}_{\mathbf{C}})$ as follows. For p and q in $\mathbf{Q}$ with $p+q$ in $\mathbf{Z}$, we denote by $H^{p,q}$ the class of the rank 1 vector space with bigrading (p,q). We set $H(G_1) = -1$ and $H(G_\alpha) = -H^{1-\gamma(\alpha),\gamma(\alpha)}$ for $\alpha \neq 1$. This is compatible with the relations 5.2.2–5.2.4 since, by a standard calculation,

$$H(J_{\alpha_1,\alpha_2}) = -H^{1-(\gamma(\alpha_1)+\gamma(\alpha_2)-\gamma(\alpha_1+\alpha_2)),\gamma(\alpha_1)+\gamma(\alpha_2)-\gamma(\alpha_1+\alpha_2)},$$

when $\alpha_1 \neq 1$, $\alpha_2 \neq 1$ and $\alpha_1\alpha_2 \neq 1$.

By using a weight argument one can prove that $H : A \to K_0(\mathrm{MHS}_{\mathbf{C}})$ is zero on the kernel of the morphism $A \to \widehat{A}$. Hence H vanishes also on the kernel of the morphism $U \to \widehat{U}$, and we can extend it to the image of this morphism.

Assume now X is smooth and let x be a closed point of $f^{-1}(0)$. We shall denote by $\mathrm{Sp}(f,x)$ the Hodge spectrum as defined in [19] and [14] (which differs from that of [20] by multiplication by t).

By applying H to both sides of (5.4.1), when X and X' are smooth and W and W' are points one obtains the following Thom-Sebastiani Theorem for the Hodge spectrum, which was first proved by A. Varchenko in [21] when f and f' have isolated singularities (see also [17]), the general case being due to M. Saito [20], [15], [16]).

Theorem 5.5.1. *Let X and X' be smooth and connected complex algebraic varieties. Let $f : X \to \mathbf{A}^1_{\mathbf{C}}$ and $f' : X' \to \mathbf{A}^1_{\mathbf{C}}$ be morphisms of algebraic varieties. Let x and x' be closed points in $f^{-1}(0)$ and $f'^{-1}(0)$. Then*

$$\mathrm{Sp}(f \oplus f', (x,x')) = \mathrm{Sp}(f,x) \cdot \mathrm{Sp}(f',x').$$

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Elementary topology of stratified mappings

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Dedicated to Professor Takuo Fukuda on his sixtieth birthday

Let T be a triangulation of a manifold N and T^1 the first barycentric subdivision of T. Stiefel [29] conjectured that the union of i-simplices of T^1 is a (possibly infinite) $\mathbb{Z}_2$-cycle, which is the Poincaré dual of the $(n-i)$-th Stiefel-Whitney class of N ($n = \dim N$). This was proved by Whitney (see [36]), but his proof was not published. Later a proof was sketched by Cheeger [4] and a complete proof was given by Halperin and Toledo [11]. Sullivan showed (Corollary 2 in [30]) that this can be generalized to define a Stiefel-Whitney homology class of singular spaces with the *local mod 2 Euler characteristic condition* (Eu) (for the definition, see §2) such as real algebraic varieties. Sullivan defined that a mapping $f : N \to P$ is *semi-triangulable* if the extended mapping cylinder

$$\text{(M)} \qquad M_f = N \times [0,1] \underset{(x,1)\sim(f(x),1)}{\cup} P \times [1,2]$$

is triangulable, and he proved the formula

$$f_* W_*(N) = W_*(P)$$

for such f on the condition that all fibers of f have odd Euler characteristics.

This was immediately generalized by Grothendieck and Deligne for real algebraic mappings and semialgebraic constructible functions (semialgebraic stratifications with weight in $\mathbb{Z}_2$). Namely, given a semialgebraic constructible function h on a real algebraic variety X (with the local mod 2 Euler characteristic condition (Eu) defined as in §2), a total Stiefel-Whitney homology class $W_*(h) \in H_*(X;\mathbb{Z}_2)$ was defined and the following properties were proved (see cf. [19]):

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(1) $f_*W_*(h) = W_*(f_*h)$ for triangulated mappings $f : X \to Y$, where f_*h is the direct image of h (for the definition, see §2),
(2) $W_*(h+k) = W_*(h) + W_*(k)$,
(3) $W_*(1_X) = W_0(1_X) + W_1(1_X) + \cdots$ is the Poincaré dual of the total Stiefel-Whitney class $W^*(X)$ if X is a manifold.

The local mod 2 Euler characteristic condition was not referred in the paper [19]. The condition was stated by Fulton and MacPherson for triangulated constructible functions in the paper [6], where they generalized the idea for families of fibers of triangulated mappings in their bivariant theory. Recently Fu and McCrory [5] proved the above properties for proper real analytic mappings and subanalytic constructible functions without using the mapping cylinder method. The homology class $W_*(h)$ with these properties was shown to be unique (cf. [6]) by using Thom's representation theorem of $\mathbb{Z}_2$-cycles by images of triangulated maps of manifolds in [31].

Deligne and Grothendieck conjectured the existence of an integral homology class in the complex algebraic category with similar properties. It was proved by MacPherson [20]. Namely he defined the so-called Chern-Schwartz-MacPherson class $C_*(h) \in H_*(X; Z)$ with the following properties for a complex algebraic constructible function h on X.

(1′) $f_*C_*(h) = C_*(f_*h)$ for a proper complex algebraic map f,
(2′) $C_*(h+k) = C_*(h) + C_*(k)$,
(3′) $C_*(1_X) = C_0(1_X) + C_1(1_X) + \cdots$ is the Poincaré dual of the total Chern class $C^*(X)$ if X is smooth.

In proving these properties, MacPherson showed the direct image f_*1_X of the characteristic function of X by an $f : X \to Y$ decomposes naturally into a sum

$$f_*1_X = \Sigma a_i \ \mathrm{Eu}(V_i)$$

with reduced subvarieties V_i of P, where $\mathrm{Eu}(V_i)$ denotes the local Euler obstruction of V_i (see [20] for the definition.) And he proved the image $f_*C_*(X)$ is equal to the weighted sum of the Chern-Mather classes of those V_i (the Poincaré duals of Chern classes of the tautological bundles of Nash blow ups of V_i projected to Y.) These subvarieties V_i are nothing but the projections to Y of the irreducible components of the Nash blow up of the complex mappig cylinder M_f of $f : X \to Y$ at $Y \times 1$. (The coefficient a_i is determined by the multiplicity of the irreducible component and the Euler characteristic of the fiber of the projection.) Recently Kwiecinski [17] generalized the theory for the complex analytic category. It is remarkable that in a different vein this had been studied by Schwartz [2, 27].

In these generalizations, either triangulability or analyticity of mappings was assumed. Now we recall some results on the stratification of mappings. By Thom-Mather theory [21], generic proper C^∞-smooth mappings can be canonically stratified, and the canonical stratifications are A_f-regular. By the theory of subanalytic sets, all proper subanalytic mappings (the graphs are subanalytic) can be also stratified [13]. Before the notion of subanalytic sets was introduced by Hironaka, Sullivan [30] had already suggested that all proper real analytic mappings are semi-triangulable and all proper stratified mappings are also. This seems to be true, but the author does not know any satisfactory reference. The various regularity conditions of the mapping cylinder of stratified mappings were discussed by many authors (c.f. [3], [14]). Cappell and Shaneson suggested, in [3], that the natural stratification of the mapping cylinder of a smooth stratified mapping is not necessarily topologically tame.

In this paper we introduce a new regularity condition, *B_f-regularity condition* of stratified mappings in §1. We prove that Whitney-regular and B_f-regular proper stratified mappings are semi-triangulable (Proposition 2.2). Shiota [28] proved that proper A_f-regular stratified mappings are triangulable and hence semi-triangulable. We apply to those A_f-regular or B_f-regular mappings a weighted version of the mapping cylinder method due to Sullivan in §3 (Theorems 2.3, 2.4). Secondly we apply Theorem 2.4 to some special mappings (Morin mapping) in §5.

The author proved in the paper [23] that for a (locally) subanalytic constructible function h on a C^∞-manifold N, a generic proper smooth mapping $f : N \to P$ admits a canonical A_f-regular, Whitney-regular and B_f-regular stratification compatible with h. This result tells that the direct image of a subanalytic constructible function by a generic proper smooth mapping is constructible. If h satisfies the local mod 2 Euler characteristic condition (Eu), the direct image satisfies also the condition and the above formula (1) holds (Theorem 2.4).

§1. Extended Mapping cylinder and the B_f-regularity.

A *stratification* Σ of a subset K of a manifold N is a set of mutually disjoint locally compact and locally finite submanifolds of N such that $\bigcup_{X\in\Sigma} X = K$. The stratification Σ is *Whitney-regular* if the following condition holds: By using local coordinates assume $N = \mathbb{R}^n$, or N is embedded into an $\mathbb{R}^r$. Let $X, Y \in \Sigma$, and let $x_i \in X, y_i \in Y$ be sequences convergent to a $y \in Y$. Assume that the line $\overline{x_i y_i}$ and the tangent space $T_{x_i}X$ are convergent to a line ℓ and a subspace T respectively in the Grassmann manifolds of lines and dim X-planes. Then $\ell \subset T$ holds.

This condition is independent of the choice of the local coordinates or the embedding, hence well defined.

A *stratification of a mapping* $f : N \to P$ is a pair of stratifications (Σ_N, Σ_P) of the source and target such that f restricts on each $X \in \Sigma_N$ to a submersion to a $Y \in \Sigma_P$.

The stratification Σ_N (or f) is *A_f-regular at a $y \in N$* if the following condition holds: Assume a sequence $x_i \in X, X \in \Sigma_N$, is convergent to $y \in Y, Y \in \Sigma_N$ and $\ker d(f \mid X)_{x_i}$ is convergent to a subspace $K \subset T_y N$. Then $\ker d(f \mid Y)_y \subset K$ holds. We say f is *A_f-regular at* a subset $A \subset N$ if f is A_f-regular at every $y \in A$, and we say simply f is *A_f-regular* if $A = N$. Roughly stating, f is A_f-regular if and only if the fibers are almost parallel to each other (see cf. [21]). If (Σ_N, Σ_P) is Whitney-regular and the restrictions of f to the closures of the strata of Σ_N are proper, the fibers $f^{-1}(y)$ are locally topologically trivial over each stratum of Σ_P by Thom's isotopy theorem [21, 33], and furthermore if Σ_N is A_f-regular, f is locally topologically trivial as a mapping along the strata of Σ_P.

Here we introduce a new regularity condition of stratified mappings.

Definition. A stratum $X \in \Sigma_N$ is *B_f-regular over a stratum $Y \in \Sigma_P$* at a $y \in Y$ if the following condition holds: Assume $P = \mathbb{R}^p$ by using local coordinates. Let $x_i \in X$ be a sequence convergent to an $x \in N$ such that $f(x) = y$, and let $y_i \in Y$ be a sequence convergent to $y \in Y$. Assume that the line $\overline{f(x_i)y_i}$ is convergent to a line ℓ. Then there exists a sequence $v_i \in T_{x_i}X$ such that $\|v_i\| \to 0$ and

$$\frac{df(v_i) - (y_i - f(x_i))}{\min\{\|df(v_i)\|,\ \|y_i - f(x_i)\|\}} \to 0$$

as $i \to \infty$. We say the stratification (Σ_N, Σ_P) (or the mapping f) is *B_f-regular over Y* if all strata of Σ_N are B_f-regular over Y at every point of Y, the stratification is *B_f-regular at* a union K of the strata of Σ_P if it is B_f-regular over all strata in K, and *B_f-regular* if $K = P$.

It is easily seen that the B_f-regularity of f implies the Whitney-regularity of the restriction of Σ_P to the image of f.

Remark. Let (X, Σ) be a smooth complex analytic Whitney-regular stratified space, and $\pi : (\tilde{X}, \tilde{\pi}) \to (X, \pi)$ a strict transformation with a nonsingular closed center $Y \in \Sigma$. In general π does not admit an A_π-regular stratification as it is not flat. (It is well known that non flat mappings of complex analytic varieties are not triangulable.) On the other hald the pair of Σ and its strict transform $\tilde{\Sigma}$ admits a refinement, which is B_π-regular at Y by Theorem 1.3.

The *extended mapping cylinder* of $f : N \to P$ is the topological space defined by (M) in the beginning of the paper. To define a differentiable structure on this space, assume N and P are embedded in $\mathbb{R}^r$ and $\mathbb{R}^s$. Then the extended mapping cylinder is homeomorphic to the subset of $\mathbb{R}^r \times \mathbb{R}^s \times \mathbb{R}$ consisting of

$$\begin{array}{ll} ((1-t)x, tf(x), t), & x \in N,\ t \in [0,1], \\ (0, y, t), & y \in P,\ t \in [1,2]. \end{array}$$

We denote this subset also by M_f. When f admits a stratification (Σ_N, Σ_P), the extended mapping cylinder M_f decomposes naturally into the union of the strata of $\Sigma_N \times 0 \times 0$, $0 \times \Sigma_P \times 1$, $0 \times \Sigma_P \times (1,2)$, $\Sigma_P \times 2$, and

$$\{((1-t)x, tf(x), t) \mid x \in X, t \in (0,1)\}, \quad X \in \Sigma_N.$$

Denote this decomposition by ΣM_f. For a union $K \subset N$ of some strata of Σ_N, denote by $M_{f|K}$ the extended mapping cylinder of the restriction $f : K \to P$, and by $\Sigma M_{f|K}$ its natural decomposition.

The next lemma is easily seen.

Lemma 1.1. *Assume the restriction of f to K is proper. Then the extended mapping cylinder $M_{f|K}$ is locally compact and the decomposition $\Sigma M_{f|K}$ is a locally finite (but not necessarily Whitney-regular) stratification. And if the embeddings of N, P into $\mathbb{R}^r, \mathbb{R}^s$ are proper, the extended mapping cylinder $M_{f|K}$ is closed in $\mathbb{R}^r \times \mathbb{R}^s \times \mathbb{R}$.*

Proposition 1.2. *Assume the restriction of f to K is proper. Then the stratification $\Sigma M_{f|K}$ is Whitney-regular over a stratum $0 \times Y \times 1$ at a $y_0 = (0, y, 1)$ $(y \in Y, Y \in \Sigma_P)$ if and only if the restriction $f : K \to P$ is B_f-regular over Y at y.*

Proof. First we prove the "if " part. Assume the B_f-regularity. Let

$$F(x,t) = ((1-t)x, tf(x), t) : N \times [0,1] \to \mathbb{R}^r \times \mathbb{R}^s \times \mathbb{R}.$$

Let $X \in \Sigma_N$ be a stratum in K. It suffices to prove the Whitney-regularity of the stratum $F(X \times (0,1))$ over $0 \times Y \times 1$ at y_0. Let $x_i \in X$, $y_i \in Y$ and $\bar{x}_i = F(x_i, t_i)$, $\bar{y}_i = (0, y_i, 1) \to y_0$. Assume the tangent space $T_i = T_{\bar{x}_i} F(X \times (0,1))$ and the line ℓ_i passing through $\bar{x}_i$, $\bar{y}_i$ are convergent respectively to a subspace T of dimension $\dim X + 1$ and a line ℓ. We show that $\ell \subset T$. Clearly T_i is spanned by the vector

$$\text{(i)} \qquad dF_{(x_i,t_i)}\left(\frac{\partial}{\partial t}\right) = (-x_i, f(x_i), 1)$$

and the subspace

(ii) $$dF_{(x_i,t_i)}(T_{x_i}X \times 0) = \{((1-t_i)v, t_i\, df_{x_i}(v), 0) \mid v \in T_x X\}$$

and ℓ_i is spanned by the vector

$$\left(-x_i, f(x_i) + \frac{y_i - f(x_i)}{1-t_i}, 1\right) = (-x_i, f(x_i), 1) + \left(0, \frac{y_i - f(x_i)}{1-t_i}, 0\right).$$

Since $f \mid K$ is proper, passing to a subsequence, we may assume x_i is convergent to an $x \in \bar{X} \cap f^{-1}(y)$. Clearly $(-x_i, f(x_i), 1) \to (-x, y, 1)$. Since $\ell_i \to \ell$, $w_i = (y_i - f(x_i))/(1-t_i)$ is convergent to a vector $w \in \mathbb{R}^s$ or it is divergent but its linear span is convergent to the line ℓ in $0 \times \mathbb{R}^s \times 0$. If w_i is convergent to 0, then $\ell \subset T$ holds. So assume $w \neq 0$ or w_i is divergent. By the B_f-regularity of the restriction $f \mid K$ over Y at y, there exists a sequence $t_i(1-t_i)v_i \in T_{x_i}X$ such that $t_i(1-t_i)\|v_i\| \to 0$ and

$$\frac{df_{x_i}(t_i(1-t_i)v_i) - (y_i - f(x_i))}{\min\{\|df_{x_i}(t_i(1-t_i)v_i)\|,\ \|y_i - f(x_i)\|\}} \to 0,$$

from which it follows

$$\frac{t_i\, df_{x_i}(v_i) - w_i}{\min\{\|t_i\, df_{x_i}(v_i)\|,\ \|w_i\|\}} \to 0.$$

Since $t_i \to 1$ and $t_i(1-t_i)\|v_i\| \to 0$, it follows $(1-t_i)\|v_i\| \to 0$ and by the above convergence the line generated by the vector

$$((1-t_i)v_i, t_i\, df_{x_i}(v_i), 0) \in dF_{(x_i,t_i)}(T_{x_i}X \times 0)$$

is convergent to that of $(0, w, 0)$ or the line $\ell \subset 0 \times \mathbb{R}^s \times 0$. Therefore $\ell \subset T$ mod $(-x, y, 1)$. Since $(-x, y, 1)$ is contained in the limit T, it follows $\ell \subset T$.

Next consider the "only if " part. So conversely assume $F(X \times (0,1))$ is Whitney-regular over $0 \times Y \times 1$ at y_0. Let $x_i \in X \to x$, $y_i \in Y \to y = f(x)$ and assume the line $\overline{f(x_i)y_i}$ is convergent a line $L \subset \mathbb{R}^s$. Let $1 - t_i = \|y_i - f(x_i)\|$. Then $w_i = (y_i - f(x_i))/(1-t_i)$ is convergent to a unit vector $w \in \mathbb{R}^s$ generating L. Recall that the distance of a line and a linear subspace is defined by the norm of the difference of a unit vector in the line and its orthogonal projection of to the subspace. By the Whitney-regularity, the distance $d(T_i, \ell_i)$ of the tangent space T_i at $\bar{x}_i = F(x_i, t_i)$ and the line ℓ_i passing through $\bar{x}_i$ and $\bar{y}_i = (0, y_i, 0)$ tends to 0 as $i \to \infty$, (since if the distance is not convergent, then there is a subsequence T_{a_j} convergent to a subspace

T such that $d(T_{a_j}, \ell_{a_j}) \to \epsilon \neq 0$ by the compactness of Grassmann manifolds and the boundedness of the distance of subspaces). Since $(0, w, 0)$, $(-x, y, 1)$ are linearly independent and $(-x_i, f(x_i), 1) \in T_i$, the distance of the subspace in (ii) and the line generated by w_i tends to 0 as $i \to \infty$. Define a tangent vector v_i of X at x_i so that

$$((1 - t_i)v_i, t_i\, df_{x_i}(v_i), 0)$$

is the orthogonal projection of $(0, w_i, 0)$ to the subspace in (ii). Then $(1 - t_i)\ \|v_i\| \to 0$ and

$$\|t_i\, df_{x_i}(v_i) - w_i\| \to 0.$$

Since $t_i \to 1$,

$$\frac{df_{x_i}((1 - t_i)v_i) - (y_i - f(x_i))}{\min\{\|df_{x_i}((1 - t_i)v_i)\|,\ \|y_i - f(x_i)\|\}} \to 0.$$

Therefore X is B_f-regular over Y at y. This completes the proof of Proposition 1.2. Q.E.D.

Theorem 1.3. *All complex analytic or subanalytic proper mappings admit Whitney-regular and B_f-regular stratifications.*

Proof. A Whitney-regular stratification of an analytic or subanalytic mapping is constructed by the induction on the codimension of the strata in the target space. Namely assuming that there exist a closed (complex analytic or subanalytic) subset $K \subset P$ of dimension i and a Whitney-regular and B_f-regular stratification (Σ_N, Σ_P) of the restriction $f_K : N - f^{-1}(K) \to P - K$, we construct a (possibly disconnected) stratum Y of dimension i in K and a stratification of $f^{-1}(Y)$ such that $K - Y$ is of dimension $\leq i - 1$, Σ_N, Σ_P are Whitney-regular over the strata of $f^{-1}(Y)$ and Y respectively and the restriction $f : f^{-1}(Y) \to Y$ is a stratified submersion. To construct a B_f-regular stratification, delete from the stratum Y the closure of the set of those $y \in Y$ such that the extension ΣM_{f_K} is not Whitney-regular over $0 \times Y \times 1$ at $(0, y, 1)$. Then go to the next step to define a stratum of dimension $\leq i - 1$ in the complement $K - Y$. By the construction, the resulting stratification is B_f-regular. Q.E.D.

Remark. By Thom-Mather theory [21], a generic C^∞-smooth mapping f admits a canonical A_f-regular and Whitney-regular stratification. All map germs f_x of such an f at x admit versal unfoldings F_x, which are smoothly conjugate with polynomial map germs. Versal unfoldings admit good representatives such that their restrictions to the singular point sets are proper and finite-to-one. The mapping cylinder of such good representatives admit canonical B_{F_x}-regular stratifications refining the Thom-Mather canonical stratifications of F_x by Proposition 1.2 and the theory of semialgebraic sets. It is seen that the germs at $f(x)$ of such B_{F_x}-regular stratifications of the target are determined by the germs f_x choosing sufficiently small domains of definition for good representatives of F_x. Imposing a generic condition on f, we may assume the natural inclusions of the germs f_x into their versal unfoldings F_x are transverse to the canonical B_{F_x}-regular stratifications. By the transverse pull back define the stratifications of the germs f_x. By genericity we may assume also the stratifications of the targets of those germs are in general position. By their transverse refinement, define a germ of stratification at each point in the target space. Those germs are characterized by the restriction of f to the singular point set $\Sigma(f)$, which is proper and finite-to-one by genericity. Now by an argument similar to the construction of the canonical stratification of a generic smooth mapping in [21], it is seen that those germs glue together to form a stratification Σ_P of P. Similarly define the stratification Σ_N of the source N by gluing the germs of stratifications defined by the intersection refinements of the B_{f_x}-regular stratifications of the germs f_x induced from that of F_x and the pull back of Σ_P by f_x. Then the resulting stratification (Σ_N, Σ_P) is B_f-regular and a refinement of the canonical stratification due to Mather.

§2. A generalization of Sullivan's result

From now on in this section manifolds and mappings are C^2-smooth unless otherwise stated. Most statements dealing with constructible functions in this section remain valid if $\mathbb{Z}_2$ is changed to $\mathbb{Z}$.

A $\mathbb{Z}_2$-valued function h on N is a *constructible function* if there exists a Whitney-regular stratification Σ of N such that the level sets of h are unions of some strata of Σ. Then we say Σ is *compatible with h.* Write h in two ways with $\mathbb{Z}_2$-coefficients n_X, m_X as follows:

$$h = \sum_{X\in\Sigma} m_X 1_X = \sum_{X\in\Sigma} n_X 1_{\bar{X}},$$

where 1_X, $1_{\bar{X}}$ denote the characteristic functions of the stratum X of Σ and its closure, respectively. The *integration of h over N* is defined by

$$\chi(h) = \int_N h = \sum_{X\in\Sigma} m_X \chi(\bar{X}, \partial X) = \sum_{X\in\Sigma} n_X \chi(\bar{X}) \in \mathbb{Z}_2.$$

Here χ stands for the mod 2 Euler characteristic define by the infinite homology, and $\chi(\bar{X}, \partial X) = \chi(\bar{X}) - \chi(\partial X)$ is equal to the Euler characteristic $\chi(X)$ of X.

It is not difficult to see that the integration is independent of the stratification Σ compatible with h. Assume a mapping $f : N \to P$ admits a Whitney-regular stratification (Σ_N, Σ_P) such that Σ_N is compatible with h. The *direct image* f_*h of h is then defined by

$$f_*h(y) = \int_N h \cdot 1_{f^{-1}(y)},$$

where the integrand is constructible as f restricts to a submersion on each stratum in N to some stratum in P. By Thom's isotopy theorem [21, 33], f is locally topologically trivial over the strata of Σ_P and the direct image f_*h is a constructible function constant on the strata of Σ_P. In general the direct image is not necessarily constructible even for C^∞-smooth mappings f.

In the real analytic case, if the restriction of f to the support of h is proper, the direct image f_*h is constructible by the stratification theory of subanalytic sets. Let $g : P \to Q$ be a real analytic mapping and assume the restriction of g to the support of f_*h is proper. Then the direct image g_*f_*h is also constructible and we obtain the following functoriality

$$(g \circ f)_*h = g_*\, f_*\, h.$$

In the smooth case this holds when the composite (f, g) admits a triple of Whitney-regular stratifications $(\Sigma_N, \Sigma_P, \Sigma_Q)$ such that f, g are stratified mappings and Σ_N is compatible with h. This is the case when g is generic with respect to the stratification of f compatible with h and f_*h [23].

Now assume N and P are compact, f_*h is constructible and let g be a mapping of P to a point. The composite (f, g) is naturally stratified by the stratification of f compatible with the h, f_*h and the trivial stratification of the target of g, and then the direct image $(g \circ f)_*h$ is nothing but the constant function which asigns the integration of h over N to the point. In particular we obtain the following well known result (c.f. [15, 34]).

Corollary 2.1. *Let N, P be compact and let $f : (N, \Sigma_N) \to (P, \Sigma_P)$ be a Whitney-regular stratified mapping. Let $h : N \to \mathbb{Z}_2$ be a constructible function constant on each stratum of Σ_N. Then*

$$\int_N h = \int_P f_*h.$$

Remark. The statement holds also for non compact N, P if the restriction of f to the support of h is proper and the integration is finite.

Let Σ be a Whitney-regular stratification of N compatible with h. By the Whitney-regularity, a sphere $S_p \subset N$ of codimension 1 centered at a $p \in X$, $X \in \Sigma$ with a sufficiently small radius (in a Riemannian metric) is transverse to Σ. By Thom's isotopy theorem the germ of Σ at p is homeomorphic to the cone of $\Sigma \cap S_p$.

The following condition was first stated by Fulton and MacPherson [6] for triangulated constructible functions.

Definition. The *local mod 2 Euler characteristic condition* of h at a stratum X of Σ is

$$(\mathrm{Eu}_X) \qquad \int_{S_p} h \equiv 0 \mod 2 \qquad (p \in X).$$

This condition is independent of the choice of $p \in X$, and equivalent to the condition

$$\int_L h \equiv 0 \mod 2$$

for the link L of $X \in \Sigma$. We say that *Condition* (Eu) holds if Condition (Eu_X) holds for all strata $X \in \Sigma$.

It is known that Whitney-regular stratified sets are triangulable [8, 16]. So we may present h as a sum of characteristic functions of (closed) simplices of a triangulation T compatible with h,

$$h = \sum_{X \in T} n_X 1_X.$$

Denote the union of all j-simplices of the first barycentric subdivision of a simplex $X \in T$ by $W_j(X)$. The following definition is seen in the paper by Fulton and MacPherson [6].

Definition. Let

$$W_j(h) := \sum_{X \in T} n_X W_j(X).$$

By Lemma 3.2, if h satisfies Condition (Eu), then this is an infinite $\mathbb{Z}_2$-cycle of dimension j. The $\mathbb{Z}_2$-homology class defined by this cycle is called *the Stiefel-Whitney homology class* of h and denoted also by $W_j(h) \in H_j(N; \mathbb{Z}_2)$. The *total Stiefel-Whitney homology class of* h is

$$W_*(h) = W_0(h) + W_1(h) + \cdots.$$

Let k be another constructible function on N and Σ' a Whitney-regular stratification compatible with k. If Σ and Σ' are transverse, the intersection refinement $\Sigma \cap \Sigma'$ is Whitney-regular, compatible with $h+k$, and $h+k$ satisfies Condition (Eu). In this situation

$$W_*(h+k) = W_*(h) + W_*(k)$$

holds. However in general the theory of Stiefel-Whitney homology class for smooth constructible functions does not fit the categorical frameworks such as in [6].

Definition. For a function h on N, the *natural extension* h_f on M_f is defined by: $h_f(x,t) = h(x)$ for $(x,t) \in N \times [0,1)$ and $h_f(y,t) = f_*h(y)$ for $(y,t) \in P \times [1,2]$.

Proposition 2.2. *Assume the restriction of a stratified mapping f to the support of h is proper and B_f-regular at the support of f_*h. Then the extended mapping cylinder $M_{f|\mathrm{supp}\, h}$ admits a triangulation compatible with h_f.*

Proof. It is known that a Whitney-regular stratified closed subset of a smooth manifold admits a triangulation such that all strata are unions of simplices [8, 16]. Assume N, P are properly embedded respectively in some Euclidean spaces $\mathbb{R}^r$, $\mathbb{R}^s$. By Lemma 1.1 and Proposition 1.2, the extended mapping cylinder $M_{f|\mathrm{supp}\, h}$ is a closed Whitney-regular stratified subset of $\mathbb{R}^r \times \mathbb{R}^s \times \mathbb{R}$. Thus it admits a triangulation. Q.E.D.

Theorem 2.3. *Let $f : N \to P$ be a continuous mapping and h a $\mathbb{Z}_2$-valued function on N. Assume the restriction of f to the support of h is proper and the extended mapping cylinder $M_{f|\mathrm{supp}\, h}$ of the restriction admits a triangulation compatible with h_f, and h_f satisfies Condition* (Eu) *in the interior $M_f - (N \times 0 \cup P \times 2)$. Then h and f_*h on P satisfy Condition* (Eu)*, and*

$$f_*W_*(h) = W_*(f_*h). \tag{$*$}$$

Let $f : (N, \Sigma_N) \to (P, \Sigma_P)$ be a proper stratified mapping and h a constructible function constant on each strata of Σ_N. It is not difficult to see that if h satisfies Condition (Eu), then the direct image f_*h also satisfies Condition (Eu). (This was proved in [5, 6] for triangulated mappings, and in [34] for some more general mappings.)

The above theorem is proved in the next section. The next theorem is a generalization of the formula due to Sullivan, Fulton, MacPherson and McCrory [5, 6, 30].

Theorem 2.4. *Let $f : (N, \Sigma_N) \to (P, \Sigma_P)$ be a Whitney-regular stratified mapping, and let h be a $\mathbb{Z}_2$-valued constructible function on N such that Σ_N is compatible with h. Assume h satisfies Condition* (Eu), *the restriction of f to the support of h is proper, and Σ_N is A_f-regular at the support of h or Σ_P is B_f-regular at the support of f_*h, then the above formula $(*)$ holds.*

Proof. First assume $\Sigma_N \mid \operatorname{supp} h$ is A_f-regular. Then by the result of Shiota [28], the restriction of f to $\operatorname{supp} h$ admits a triangulation (S, T) refining (Σ_N, Σ_P). It follows that the extended mapping cylinder of the restriction admits a triangulation compatible with h_f. Secondly assume Σ_P is B_f-regular at the support of f_*h. Then the extended mapping cylinder of the restriction of f to the support of h admits a triangulation compatible with h_f by Proposition 2.2. In both cases, the formula $(*)$ holds by Theorem 2.3. Q.E.D.

§3. Mapping cylinder method and the proof of Theorem 2.3.

Proposition 3.1. *Assume the restriction of f to the support of h is proper, h satisfies Condition* (Eu) *and f is B_f-regular at the support of f_*h. Then h_f satisfies Condition* (Eu) *at the complement of the boundaries $N \times 0$, $P \times 2 \subset M_f$, and h_f satisfies Condition* (Eu) *at a stratum X on the boundaries if and only if h_f is zero on X.*

Proof. Clearly Condition (Eu) holds off the boundaries and $0 \times P \times 1$. First we prove the condition at a $(0, y, 1) \in 0 \times P \times 1$. Assume N, P are properly embedded into $\mathbb{R}^r$, $\mathbb{R}^s$ respectively. By Lemma 1.1 the mapping cylinder $M_{f|\operatorname{supp} h}$ is closed, and by Proposition 1.2 the natural stratification $\Sigma M_{f|\operatorname{supp} h}$ is Whitney-regular. A transverse intersection of a sphere of codimension 1 in $\mathbb{R}^r \times \mathbb{R}^s \times \mathbb{R}$ centered at $(0, y, 1)$ and the stratification $\Sigma M_{f|\operatorname{supp} h}$ is (transversely) isotopic to

$$\begin{aligned} S = f^{-1}(B_y) \times 0 \times 0 \cup (f^{-1}(S_y) \times 0 \times (0,1]) \\ \cup_f (0 \times S_y \times [1,2)) \cup 0 \times B_y \times 2, \end{aligned}$$

where B_y is a closed ball in P centered at y, S_y is the boundary of the ball transverse to Σ_P, and $\cup_f$ denotes the identifying space by the restriction $f : f^{-1}(S_y) \times 0 \times 1 \to 0 \times S_y \times 1$. By Thom's isotopy theorem, the integration of h_f over the intersection is equal to

$$\begin{aligned} \int_S h_f &= \int_{f^{-1}(B_y)} h + \int_{f^{-1}(S_y) \times 0 \times (0,1)} h \\ &\quad + \int_{S_y} f_*h + \int_{0 \times S_y \times (1,2)} f_*h + \int_{B_y} f_*h. \end{aligned}$$

Since $\int_{f^{-1}(S_y)} h \equiv 0$ and $\int_{S_y} f_* h \equiv 0 \mod 2$ as the direct image satisfies Condition (Eu) ([6]),

$$\equiv \int_{f^{-1}(B_y)} h + \int_{B_y} f_* h$$

and by Corollary 2.1

$$\equiv \; 0 \mod 2.$$

Next we consider Condition ($\mathrm{Eu}_{X\times 0\times 0}$) for an $X \in \Sigma_N$. Let B_x be a small ball in N centered at an $x \in X$ such that the boundary S_x is transverse to Σ_N. An intersection of the stratification $\Sigma M_{f|\mathrm{supp}\, h}$ with a small transverse sphere of codimension 1 in $\mathbb{R}^r \times \mathbb{R}^s \times \mathbb{R}$ centered at $(x, 0, 0)$ is homeomorphic to the stratified set

$$S = S_x \times 0 \cup S_x \times (0,1) \cup B_x \times 1 \subset N \times [0,1].$$

By Thom's isotopy theorem, the integration of h_f over the transverse sphere is equal to

$$\int_S h_f = \int_{S_x} h + \int_{S_x \times (0,1)} h + \int_{B_x} h \equiv h(x) \mod 2.$$

This tells that Condition ($\mathrm{Eu}_{X\times 0\times 0}$) holds if and only if $h = 0$ on X. A similar argument holds for the strata in $0 \times P \times 2$. Q.E.D.

Lemma 3.2. *Let T be a triangulation of the extended mapping cylinder M_f compatible with h_f. The coefficient in $\partial W_j(h)$ of a $(j-1)$-simplex $\Sigma = (\tau_0 \subset \tau_1 \subset \cdots \subset X)$ of the first barycentric subdivision of a simplex $X \in T$ (with the barycenter of X as a vertex of Σ) is 0 if and only if Condition* (Eu_X) *of h holds.*

Proof. Write $h_f = \sum_{X\in T} n_X 1_{\bar{X}}$. Let Y be a simplex of T of dimension $\geq j$. Then

$$\partial W_j(Y) = W_{j-1}(\partial Y) = W_{j-1}(1_{\partial Y}) = W_{j-1}\left(\sum_{Z\in T,\, Z\subset \partial Y} 1_{\bar{Z}}\right),$$

which is the $(j-1)$-th skeleton of the first barycentric subdivision of the boundary ∂Y. This formula tells that the coefficient of Σ in $\partial W_j(Y)$ is 1 (mod 2) if and only if $X \subset \partial Y$. Therefore the coefficient of Σ in $\partial W_j(h_f)$ is the number (with weight n_Y) of those closed simplices $Y \in T$ containing X in their boundaries. This number is nothing but the integration of h_f over the link of the stratum X. Q.E.D.

Now we prove Theorem 2.3. It is easily seen that Condition (Eu) of h_f on the interior of $M_{f|\mathrm{supp}\,h}$ implies the condition of h, f_*h. By Proposition 2.2, the extended mapping cylinder $M_{f|\mathrm{supp}\,h}$ admits a triangulation T compatible with h_f, and all simplices of the first barycentric subdivision of T are of the form Σ in Lemma 3.2. By Proposition 3.1 and Lemma 3.2, we obtain the boundary formula

$$\partial W_j(h_f) = W_{j-1}(h) + W_{j-1}(f_*h),$$

from which follows the formula $(*)$ of Theorem 2.3.

§4. Canonical decomposition of the direct image.

It would be worth to apply the theory to the following elementary case. Let $f : (N^n, \Sigma_N) \to (P^p, \Sigma_P)$, $n = p$, be a proper and finite-to-one stratified mapping of manifolds. The local mod 2 mapping degree D_{loc} is locally constant on the strata of Σ_N by Thom's isotopy theorem, hence it is a constructible function on N. By definition

$$(**) \qquad f_* D_{\mathrm{loc}} = D\,1_P,$$

D being the mod 2 mapping degree of f.

Proposition 4.1. *Let f be a proper and finite-to-one stratified mapping of manifolds. The local mapping degree D_{loc} satisfies Condition (Eu).*

Proof. Let $f(X) \subset Y$, $X \in \Sigma_N$, $Y \in \Sigma_P$. Let K denote the link of Y. Then $f^{-1}(K)$ is the link of X. By Corollary 2.1,

$$\int_{f^{-1}(K)} D_{\mathrm{loc}} = \int_K f_* D_{\mathrm{loc}} = \int_K D \cdot 1_P = 0.$$

Q.E.D.

Since f is finite-to-one, f is A_f-regular, and by the proposition and Theorem 2.4, we obtain

Proposition 4.2. *Let $f : (N^n, \Sigma_N) \to (P^n, \Sigma_P)$ be a proper and finite-to-one Whitney-regular stratified mapping. Then*

$$f_* W_*(D_{\mathrm{loc}}) = D\, W_*(P),$$

where D is the mod 2 mapping degree of f.

We generalize this in Theorem 4.7 for smooth Thom-Mather stratified mappings $f : N^n \to P^p$ with $n \geq p$.

Now let $f : N^n \to P^p$, $n \geq p$, be a complex analytic mapping with isolated singularities. The local Milnor fiber $f^{-1}(y')$ of a germ of a complex analytic mapping $f : N \to P$ at a singular point $x \in N$ is homotopic to the bouquet of μ spheres S^{n-p}, all of which vanish as y' tends to $y = f(x)$ [12, 22]. In other words, Euler characteristic of the fiber decreases $(-1)^{n-p}\mu$ from that of a generic local fiber when $y' = y$. It is known that the Milnor number μ of the germ f_x is algebraically determined by the finite jet of f at x. So $\mu = \mu(x)$ is a complex analytic constructible function on N. The difference of Euler characteristic of a (global) singular fiber $f^{-1}(y)$ and that of a non singular fiber is the sum of the Milnor number $(-1)^{n-p}\mu(x)$ at all singular points x of the singular fiber. Therefore we obtain the following generalization of the formula by Yomdin [38].

Proposition 4.3. *Assume $f : N \to P$ is a proper complex analytic mapping with isolated singularities. Then*

$$f_* \left(1_N + (-1)^{n-p}\mu\right) = D\, 1_P,$$

where D is the Euler characteristic of a generic fiber of f and μ is Milnor number.

Remark. $D_{\mathrm{loc}} = \mu + 1$ in the case when $n = p$ and f is complex analytic. The mod 2 Milnor number μ is defined for generic C^∞-smooth map germs later in this section, and it is seen that D_{loc} has the same parity as $\mu + 1$.

Mather [21] proved a generic proper C^∞-smooth mapping admits a canonical Whitney-regular and A_f-regular stratification. (It is known the complement of the set of those generic mappings has positive codimension in the proper mapping space with Whitney topology.) For those generic mappings f, germs at all $x \in N$ are conjugate with polynomial map germs by local continuous coordinate change of the source and target, and the singularities of the fibers are isolated. The mod 2 Milnor number $\mu(x)$ of the germ of f at x is well defined: $\mu(x) = \chi(f^{-1}f(x') \cap B_\epsilon) - 1$ with a small ball B_ϵ centered at x such that the boundary is transverse to the fiber passing through x and an x' sufficiently close to x such that $f^{-1}f(x')$ is also transverse to the boundary. We call those generic mappings *smooth Thom-Mather stratified mappings.*

Remark. In the following we present the various results on smooth Thom-Mather stratified mappings. The condition of being a smooth

Thom-Mather stratified mapping seems to be too strong for those statements. However we postpone to state those results in optimal manner, since their proofs are more technical.

Proposition 4.4. *Let f be a proper smooth Thom-Mather stratified mapping. Then*

$$(***) \qquad f_*(1_N + \mu) \equiv D\,1_P \mod 2,$$

where D is the mod 2 Euler characteristic of a generic fiber of f.

Proof. By Corollary 2.1

$$\begin{aligned} f_*(1_N+\mu)(y) &= \int_{f^{-1}(y)} (1_N+\mu) \\ &= \int_{f^{-1}(y)} 1_N + \int_{f^{-1}(y)} \mu \\ &= \chi(f^{-1}(y)) + \sum_{x \in \operatorname{Sing} f^{-1}(y)} \mu(x) \end{aligned}$$

since $f^{-1}(y)$ is locally contractible at all singular points x

$$\equiv \quad D \mod 2.$$

Q.E.D.

Proposition 4.5. *Let f be a proper smooth Thom-Mather stratified mapping. Then $1+\mu$ satisfies Condition* (Eu).

Proof. This follows from Proposition 4.6. Here we give a direct proof by integrating $1+\mu$ on a small traisverse sphere of codimension 1. Let D° be a small open ball centered at $f(x)$ such that the boundary $S' = \partial D^\circ$ is transverse to Σ_P. Then f is transverse to S' hence $f^{-1}(S')$ is smooth and transverse to Σ_N. Let D be a small closed ball centered at x. Since f has isolated singularities, we may assume the boundary $S = \partial D$ is transverse to the fibers over D'. The integration of $1+\mu$ on a transverse sphere of codimension 1 centered at x is equal to

$$\int_{f^{-1}(\bar{D}^\circ)\cap S} (1+\mu) + \int_{f^{-1}(S')\cap D} (1+\mu).$$

Since $f^{-1}(\bar{D}^\circ)\cap S$ is a fiber bundle over the link $K = f^{-1}f(x)\cap S$ with the fiber $\bar{D}^\circ$ and $\mu = 0$ on the fiber bundle, we see the first term is

$$\int_K \left(\int_{\bar{D}^\circ} 1 \right) = \int_K 1.$$

Since the germ of f at x is topologically conjugate with a polynomial map germ, the link K has even Euler characteristic by a result of Sullivan [30]. Therefore the first term is 0 mod 2. By Corollary 2.1 and Proposition 4.4, the second term is

$$\int_{S'} d' \equiv 0 \mod 2,$$

where d' is the mod 2 Euler characteristic of a generic fiber of the restriction $f : f^{-1}(S') \cap D \to S'$. This completes the proof. Q.E.D.

Let $M_1 \subset N$ be the support of μ, and define $M_i \subset M_{i-1}$, $i = 2, 3, \ldots$ inductively by the support of the restriction of $\mu - i + 1$ to M_{i-1}. By definition, M_i is closed and

$$\mu = \sum 1_{M_i}$$

in the complex analytic case, and

$$(* * **) \qquad \mu \equiv \sum 1_{M_i} \mod 2$$

in the smooth case. For polynomial map germs f, all M_i are algebraic [38], hence their characteristic functions satisfy Condition (Eu). By a result of Wall [35], the mod 2 Milnor number is invariant under topological conjugacy of map germs. Therefore all germs of smooth Thom-Mather stratified mappings possess the same properties. We proved

Proposition 4.6. *Condition* (Eu) *of* 1_{M_i} *holds for proper smooth Thom-Mather stratified mappings.*

By Proposition 4.4 and Proposition 4.6, we obtain

Proposition 4.7. *For a proper smooth Thom-Mather stratified mapping* $f : N \to P$*,*

$$f_* W_*(N) + f_* W_*(\mu) = f_* W_*(N) + \sum_{1 \leq i} f_* W_*(M_i) = DW_*(P).$$

Thom [32] proved that for a generic and proper $f : N \to P$, the critical point set $\Sigma(f)$ carries a $\mathbb{Z}_2$-fundamental class, and its Poincaré dual can be written in a polynomial of the Stiefel-Whitney (cohomology) classes of the difference bundle $TN - f^*TP$: the virtual tangent bundle of the fibers of f. The polynomial was explicitly calculated by Porteous (see c.f. [7]). Passing to the target space via Gysin homomorphism $f_!$, we obtain the following relation

$$f_! W^{n-p+1}(N) + f_!(\mathrm{Dual}[\Sigma(f)]) = DW^1(P),$$

where W^* stands for the Stiefel-Whitney cohomology class (for the proof, see [24]). This formula is the dimension $(p-1)$ part of the formula in Theorem 4.7, as $M_1 = \Sigma(f)$ for a generic f. Thom showed also, for a singularity type I (of the contact equivalence relation in [21]), the set of the closure of the set $\Sigma^I(f)$ of those $x \in N$ where the germ of f is contact equivalent to I carries a $\mathbb{Z}_2$-fundamental class. And its Poincaré dual is written in a polynomial of Stiefel-Whitney classes of the difference bundle. The polynomial is called *Thom polynomial*, but it is explicitly calculated only for very special singularity types.

§5. The canonical stratification of Morin mappings.

Let $p \leq n$. We say a C^∞-smooth mapping $f : N^n \to P^p$ is a *Morin mapping* if the following two conditions hold:
(1) f is locally equivalent to the following normal form A_k (or Thom-Boardman $\Sigma^{n-p+1,\overbrace{1,\dots,1}^{k-1}}$ singularity [1, 21, 32]) for some $k = 1, \dots, p$:

$$(f_u(x,y),u) = \left(x^{k+1} + Q(y) + \sum_{i=1}^{k-1} u_i x^i, u_1, \dots, u_{p-1}\right) : \mathbb{R}^n \to \mathbb{R}^p,$$

$x \in \mathbb{R}$, $y \in \mathbb{R}^{n-p}$, $Q(y)$ being a non degenerate quadratic form of y. Here two map germs $f_i : (N_i, x_i) \to (P_i, y_i)$, $i = 1, 2$ are *equivalent* if there exist germs of C^∞-diffeomorphisms $\phi : (N_1, x_1) \to (N_2, x_2)$, $\psi : (P_1, y_1) \to (P_2, y_2)$ such that $f_2 \circ \phi = \psi \circ f_1$ holds.
(2) Let $A_k(f)$ denote the set of those $x \in N$ where the germ of f is equivalent to the above normal form. (By the above normal form, $A_k(f)$ is smooth of dimension $p-k$.) The restrictions of f to $A_k(f), k = 1, \dots, p$, are multi transverse: the germs of $f \mid A_k(f)$, $k = 1, \dots, p$ at all finite point sets of N are in general position in P.

Denote by Σ_f the stratification of N consisting of $A_1(f), A_2(f), \dots, A_p(f)$ and their complement $N - \Sigma(f)$.

In the complex analytic case we define Morin mappings in a similar manner. (See [1, 37] for alternative definitions.)

The *canonical stratification* $f_*\Sigma_f$ (denoted Σ_P) of the discriminant of a Morin mapping f is defined as follows. A stratum of $f_*\Sigma_f$ consists of those y such that the fiber $f^{-1}(y)$ has a singularity type

$$A_{k_1} + A_{k_2} + \cdots + A_{k_\ell},$$

which is smooth of codimension $k_1 + \cdots + k_\ell \leq p$ in P. The *canonical stratification* $\Sigma_f \cap f^{-1} f_* \Sigma_f$ $(= \Sigma_N)$ of the source space N is defined by the intersection refinement of Σ_f and $f^{-1} f_* \Sigma_f$. By the normal form of

Morin singularity it is seen that if f is multi transverse with respect to Σ_f, then

$$f : (N, \Sigma_f \cap f^{-1} f_* \Sigma_f) \to (P, f_* \Sigma_f)$$

is an A_f-regular and Whitney-regular stratified mapping. The closure of the critical set $A_k(f)$ is also a smooth submanifold of N of dimension $p-k$. Let $1_{\bar{A}_k(f)}$ denote the characteristic function of the closure.

The following proposition is a spacial case of $(* * **)$ and can be verified by using the local normal form of Morin singularities.

Proposition 5.1. *Let $f : N^n \to P^p$, $p \leq n$, be a proper Morin mapping. Then*

$$\mu = \sum_{k=1}^{p} 1_{\bar{A}_k(f)}$$

in the complex analytic case, and

$$\mu \equiv \sum_{k=1}^{p} 1_{\bar{A}_k(f)} \mod 2$$

in the smooth case.

§6. Application to Morin mappings.

For complex analytic mappings, we obtain the following generalization of a result of Levine [18] by the properties of the Chern-Schwartz-MacPherson class and Proposition 5.1.

Theorem 6.1. *Let $f : N^n \to P^p$, $p \leq n$, be a proper complex analytic Morin mapping. Then*

$$f_* C_*(N) + (-1)^{n-p}\{f_* C_*(\bar{A}_1(f)) + \cdots + f_* C_*(\bar{A}_p(f))\} = D\,C_*(P).$$

In the paper [24] the author proved

Proposition 6.2. *Let f be as above. Then*

$$\mathrm{Eu}(D(f)) = f_* \mu,$$

where $D(f)$ denotes the discriminant (critical value set) of f.

From this formula and the definition of the total Chern-Schwartz-MacPherson class of $f_*\mu$ as in the introduction, we obtain the following interpretation of the formula in Proposition 4.4.

Theorem 6.3. *Let $f : N^n \to P^p$, $p \leq n$, be a proper complex analytic Morin mapping. Then*

$$f_* C_*(N) + (-1)^{n-p} C_M(D(f)) = D C_*(P),$$

where C_M stands for the Chern-Mather class.

From Theorem 6.1 and Theorem 6.3 it follows

Theorem 6.4. *Let f be as above. Then*

$$\frac{\mathrm{Dual}(C_M(D(f)))}{C^*(P)} = \frac{f_! C^*(\bar{A}_1(f))}{C^*(P)} + \cdots + \frac{f_! C^*(\bar{A}_p(f))}{C^*(P)},$$

where C^* stands for the total Chern class.

The left hand side of the above equality is nothing but the total Chern class of the "normal bundle" of the Nash blow up of the discriminant set $D(f)$. Since the discriminant is of codimension one, the "normal bundle" is a rank one vector bundle over the Nash blow up. Therefore we obtain

Proposition 6.5. *Let f be as above. then*

$$\frac{\mathrm{Dual}(C_M(D(f)))}{C^*(P)} = \mathrm{Dual}(D(f)) + C^1(N(f)),$$

where $C^1(N(f))$ denotes the first Chern class of the "normal bundle" of the Nash blow up of the discriminant.

This tells certain formulas of the discriminant and the image of the cusp point set $f(\bar{A}_1(f))$ as in the end of §4, and also

Theorem 6.6. *For a proper complex analytic Morin mapping f the cohomology class*

$$\frac{f_! C^*(\bar{A}_1(f))}{C^*(P)} + \cdots + \frac{f_! C^*(\bar{A}_p(f))}{C^*(P)}$$

vanishes in dimension ≥ 3.

This can be generalized to complex analytic mappings with arbitrary generic singularities as follows. If a proper mapping $f : N^n \to P^p$ has only generic singularities of corank $\leq c$, then the cohomology class

$$\sum_{1 \leq i} \frac{\mathrm{Dual}(f_* C_*(M_i(f)))}{C^*(P)}$$

vanishes in certain dimensions depending on c. Full account of the general theory will appear elsewhere.

In the real smooth case we obtain from Theorem 2.4 and Proposition 5.1, the following theorem.

Theorem 6.7. *Let $f : N^n \to P^p$, $p \leq n$, be a proper smooth Morin mapping. Then*

$$f_*W_*(N) + \{f_*W_*(\bar{A}_1(f)) + \cdots + f_*W_*(\bar{A}_p(f))\} = D\,W_*(P),$$

where W_* denotes the total Stiefel-Whitney homology class: Poincaré dual of the Stiefel-Whitney cohomology class.

This theorem was proved by the author in [24] with purely geometric argument based on the definition of Thom-Boardman singularities.

By the dimension $(p - i)$ part of the formula in the theorem, we obtain

Corollary 6.8. *Let $f : N^n \to P^p$, $p \leq n$, be a proper smooth Morin mapping. Then*

$$\begin{aligned} f_*W_{p-i}(N) + f_*W_{p-i}(\bar{A}_1(f)) + \cdots + f_*W_{p-i}(\bar{A}_{i-1}(f)) + f_*[\bar{A}_i(f)] \\ = D\,W_{p-i}(P). \end{aligned}$$

In the real case the mod 2 reductions of Theorem 6.4, Proposition 6.5 and Theorem 6.6 seem to remain valid, while the notion of Nash blow up is not defined yet for constructible functions as well as triangulated subsets.

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Geometry of cuspidal sextics and their dual curves

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§1. Introduction

Let C be a given irreducible plane curve of degree n defined by $f(x,y) = 0$ where $f(x,y)$ is an irreducible polynomial. C is called a *torus curve of type* (p,q) if $p,q|n$ and $f(x,y)$ is written as $f(x,y) = f_{n/p}(x,y)^p + f_{n/q}(x,y)^q$ for some polynomials $f_{n/p}, f_{n/q}$ of degree n/p and n/q respectively. This terminology is due to Kulikov, [K2]. Torus curves have been studied by many authors ([Z], [O1],[K2], [D],[T]).

In the process of studying Zariski pairs in the moduli of plane curves of degree 6 with 3 cusps of type $y^4 - x^3 = 0$, we have observed that *there exist two irreducible components* $\mathcal{N}_{3,1}/PSL(3,\mathbf{C})$ *and* $\mathcal{N}_{3,2}/PSL(3,\mathbf{C})$ *which corresponds to torus curves and non-torus curves respectively* (Lemma 25). *Their dual curves are sextics with six cusps and three nodes.* Starting from this observation, we study the moduli space of sextic with 6 cusps and 3 nodes which we denote by $\mathcal{M}$ and the moduli of their dual curves. It turns out that $\mathcal{M}$ has a beautiful symmetry. The "regular part" (=Plücker curves) of $\mathcal{M}$ is stable by the dual curve operation and the moduli of 3 (3,4)-cuspidal sextics $\mathcal{N}_3$ is on the "boundary" of $\mathcal{M}$ in a nice way (Theorem 18). By the dual operation, this moduli is isomorphic to a "singular" stratum $\mathcal{M}_3$ of $\mathcal{M}$, which consists of 6 cuspidal 3 nodal sextics with 3 flexes of order 2. The moduli space $\mathcal{M}$ is a disjoint union of torus curves and non-torus curves. The generic Alexander polynomial $\Delta_C(t)$ of $\mathbf{P}^2 - C$ is determined by the type of C. Namely if C is a torus curve, $\Delta_C(t) = t^2 - t + 1$ and $\pi_1(\mathbf{P}^2 - C) = \mathbf{Z}_2 * \mathbf{Z}_3$, while for non-torus curve C, $\Delta_C(t) = 1$. Moreover we show that *the dual curve C^* is a torus curve if and only if C is a torus curve.* This is striking, as it implies also that the topology of the complement is preserved by the dual operation for a torus curve in $\mathcal{M}$.

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This paper is composed as follows. In section 2, we study dual curves and their singularities. We show a simple lemma which enable us to compute the defining polynomials of the dual curves explicitly (Lemma 4) and then we introduce a stratification, which is called *flex stratification*, in the local moduli space of a germ of singularity. This stratification enjoys the following property. The defect of a singularity to the number of flexes is constant on each stratum and the image of a stratum by the dual map is again a stratum. Thus the topological structure of the dual singularity is also constant along a stratum (Theorem 14).

In section 3, we study the moduli space $\mathcal{M}$ and other moduli spaces which appear on the canonical stratification of the "closure" $\widehat{\mathcal{M}}$ of $\mathcal{M}$ (Theorem 18). In section 4, we compute the moduli space of sextics with 3 (3,4)-cusps. In sections 5 and 6, we compute the fundamental groups of the complements of 3 (3,4)-cuspidal sextics of torus type and non-torus type. In section 7, we give a new Zariski triple of plane curves of degree 12 with 12 (3,4)-cusps, as an application of Theorem 18.

§2. Dual curves

In this section, we first recall some basic properties of the dual curves. For general references, refer to [W1], [B-K] and [N]. We will also present several new results on dual curves which will be used in later sections (Lemma 4, Theorem 14).

Let C be an irreducible plane curve in $\mathbf{P}^2$ and let $F(X,Y,Z)=0$ be an irreducible polynomial which defines C and let $f(x,y)=F(x,y,1)$. Here X,Y,Z are homogeneous coordinates and (x,y) are affine coordinates given by $x=X/Z, y=Y/Z$. At a simple point $P=(\alpha,\beta)\in C\cap \mathbf{C}^2$, the tangent line T_PC is given by $\frac{\partial f}{\partial x}(\alpha,\beta)(x-\alpha)+\frac{\partial f}{\partial y}(\alpha,\beta)(y-\alpha)=0$.

Let $\mathbf{P}^{*2}$ be the dual projective space with homogeneous coordinates U,V,W. The dual curve C^* of C is the closure of the image of the mapping $P\mapsto T_PC$ for the regular points $P\in C$. More explicitly it is given by $(X,Y,Z)\mapsto (U,V,W)$ where $U=F_X$, $V=F_Y$ and $W=F_Z$. Thus a defining homogeneous polynomial of C^*, denoted by $G(U,V,W)$, can be obtained by eliminating X,Y,Z from the above equalities and $F(X,Y,Z)=0$.

Let $\phi:\tilde{C}\to C$ be a normalization of C and let t be the (local) coordinate of $\tilde{C}$. Let $(x(t),y(t))$ be the affine parameterization of C. Then the tangent line is given by $y-y(t)=\frac{y'(t)}{x'(t)}(x-x(t))$. Thus the dual curve is parameterized in homogeneous coordinates as follows:

$$(1)\qquad U(t)=y'(t),\quad V(t)=-x'(t),\quad W(t)=x'(t)y(t)-x(t)y'(t)$$

Applying (1) to C^* again, we see easily that the dual curve operation enjoys the reciprocity law $C^{**} = C$ and thus C and C^* are birationally equivalent.

2.1. Action of the automorphism

The group G:= PSL(3, **C**) acts on $\mathbf{P}^2$ from the right side as: $\mathbf{P}^2 \times G \to \mathbf{P}^2$, $((X,Y,Z),A) \mapsto (X,Y,Z)A$. Let $A \in G$ and we denote by φ_A the automorphism induced by the right multiplication. Then the image $\varphi_A(C)$ of the curve is defined by the polynomial $\varphi_{A^{-1}}^* F(X,Y,Z) = F((X,Y,Z)A^{-1})$. Put $C^A := \varphi_A(C)$. The following is easy to be proved.

Lemma 2. *We have $(C^A)^* = (C^*)^{{}^tA^{-1}}$. Thus if C^* is defined by $G(U,V,W) = 0$, $(C^A)^*$ is defined by $\varphi_{{}^tA}^* G(U,V,W) = G((U,V,W)^tA)$. In particular, if C^* is a torus curve, $(C^A)^*$ is a torus curve for any $A \in PSL(3,\mathbf{C})$.*

2.2. Class formula

Assume that C is an irreducible curve of degree n with k singularities P_i for $i = 1,\dots,k$. Let m_i be the multiplicity, let μ_i be the Milnor number and let r_i be the number of irreducible components of C at P_i respectively and let g be the genus of the normalization $\tilde{C}$. The degree n^* of the dual curve is called *the class number* of C and n^* is given by the formula:

$$(3) \quad n^* = 2(g-1+n) - \sum_{i=1}^{k}(m_i - r_i) = n(n-1) - \sum_{i=1}^{k}(\mu_i + m_i - 1)$$

The second equality follows from the (modified) Plücker formula:
$2 - 2g = 3n - n^2 + \sum_{i=1}^{k}(\mu_i + r_i - 1)$.

2.3. Defining polynomial of the dual curve.

Let $F(X,Y,Z)$, $f(x,y)$ and C be as before. Let $G(U,V,W)$ be the defining homogeneous polynomial of C^* and let $g(u,v)$ be the affine equation, given by $g(u,v) = G(u,v,1)$. G is given by eliminating X,Y,Z from $F(X,Y,Z)$ and $F_X - U$, $F_Y - V$, $F_Z - W$. However this elimination involves a tremendous computation. We prefer the following simple formula.

Lemma 4. *Assume that the line $Z = 0$ cuts C transversely. Let $P_i = (\alpha_i, \beta_i)$, $i = 1,\dots,k$ be the singular points of C and let μ_i be the Milnor number and let m_i be the multiplicity of C at P_i. Put $f_1(x_1,p,y_1) := f(x_1 - py_1, y_1)$ and let $h(x_1,p) := \Delta_{y_1}(f_1)$ be the discriminant polynomial of f_1 with respect to y_1. Then $h(x_1,p)$ is a polynomial of degree $n(n-1)$. Put $\tilde{g}(u,v) = h(-1/u, v/u)u^{n(n-1)}$. Then*

$\tilde{g}(u,v)$ can be written as $\tilde{g}(u,v) = g(u,v)L(u,v)$ where L is given by $L(u,v) = \prod_{i=1}^{k}(\alpha_i u + \beta_i v + 1)^{\mu_i+m_i-1}$ and the polynomial $g(u,v)$ is a defining polynomial of the dual curve in the affine coordinates $u = U/W, v = V/W$.

Remark 1. *This formula also holds without the genericity assumption of the line at infinity with a slight modification $\tilde{g}(u,v) = h(-1/u, v/u)u^{\deg(h)}$. The defining polynomial $g(u,v)$ is obtained by throwing away all the multiple factors from $\tilde{g}(u,v)$. Therefore for the determination of g, we only need an elimination of one variable. Thus the computation is very easy.*

Proof. Let $f^*(u,v)$ be the defining polynomial of the dual curve. Consider p as a fixed constant. (We consider p as a variable later.) First observe that $h(a,p) = 0$ with $a \neq -(\alpha_i + p\beta_i)$, $i = 1,\dots,k$, if and only if $x + py - a = 0$ is tangent to C. Thus $(-1/a, -p/a) \in C^*$ when $h(a,p) = 0$. Thus $g(u,v) = 0$ defines C^* as a set. By a standard argument of discriminant, $\deg_{x_1} h(x_1,p) = n(n-1)$ and the solutions of $h(x_1,p) = 0$ in x_1 are all simple except $x_1 = \alpha_i + \beta_i p$, while the contribution from the singular point P_i is given by $(x_1 - (\alpha_i + \beta_i p))^{\nu_i}$ where ν_i is the intersection multiplicity of C and $\frac{\partial f_1}{\partial y_1} = 0$ at P_i, considering p as a constant (see for example, [O5]). Furthermore we have the equality: $\nu_i = \mu_i + m_i - 1$ by [Le] for a generic p. We need to show $\deg\ h(x_1,p) = n(n-1)$ as a polynomial of two variables x_1, p.

Step 1. Assume that C is a smooth curve. Then it is well-known that $f^*(u,v)$ is an irreducible polynomial of degree $n(n-1)$. Let $h^*(x_1,p) := f^*(-1/x_1, -p/x_1)x_1^{n(n-1)}$. Then h^* is also an irreducible polynomial of degree $n(n-1)$ and by the above consideration, $h^*(x_1,p)$ divides $h(x_1,p)$. So we conclude that $h(x_1,p) = h^*(x_1,p)$ up to a multiplication of a constant.

Step 2. Our case. Let $f_t(x,y) = f(x,y) - t$. Then for $t \neq 0$, sufficiently small, $C_t := \{f_t(x,y) = 0\}$ is a smooth curve of degree n. Let $h_t(x_1,p)$ be the discriminant polynomial of $f_t(x_1 - py, y)$ in y. Then $h_t(x_1,p)$ has degree $n(n-1)$ as a polynomial of x_1, p. Thus as $h_0 = h$, $\deg\ h(x_1,p) \leq n(n-1)$. As we have already seen that $\deg_{x_1} h(x_1,p) = n(n-1)$, we conclude that $\deg\ h(x_1,p) = n(n-1)$. Q.E.D.

2.4. Flex points

Let C be an irreducible plane curve of degree n defined by a homogeneous polynomial $F(X,Y,Z) = 0$ and put $f(x,y) = F(x,y,1)$ as before. A regular point $P \in C$ is called a *flex of order r* if the intersection multiplicity $I(C, T_PC; P)$ of C and the tangent line T_PC at P is $r + 2$. We simply say a flex in the sense of a flex of order 1. It is well-known that flex points are defined by $H(X,Y,Z)=0$ on C where $H(X,Y,Z)$ is the Hessian of F which is a homogeneous polynomial of degree $3(n-2)$. Using the Euler equality $nF = XF_X + YF_Y + ZF_Z$, we can easily obtain $Z^2H \equiv -(n-1)^2(F_{X,X}F_Y^2 - 2F_{X,Y}F_YF_X + F_{Y,Y}F_X^2)$ modulo (F). We consider the polynomial $\mathcal{F}(f) := f_{x,x}f_y^2 - 2f_{x,y}f_yf_x + f_{y,y}f_x^2$ (see also §5, [O5]) and let J be the plane curve defined by $\mathcal{F}(f)(x,y)=0$. Note that $\deg \mathcal{F}(f) = 3n-4$. We define *the flex defect at P of C* by the intersection multiplicity $I(C,H;P)$ of C and H at P and we denote this integer by $\delta(P;f)$ or $\delta(P;C)$. By the above equality, the flex defect $\delta(P;f)$ is equal to the intersection number $I(C,J;P)$ for $P \in C \cap \mathbf{C}^2$. Let $P_1,\dots,P_k$ be the singular points of C. Thus we have

Proposition 5. *The number of flexes $i(C)$, counted with the multiplicity, is given by $3(n-2)n - \sum_{i=1}^k \delta(P_i;C)$.*

Remark 2. *The multiplicity of a flex point is counted by the flex defect, which turns out to be equal with the order by Corollary 9.*

2.5. Flex defect formula and flex stratification

Let σ be an equivalence class of an isolated plane curve singularity germ. Here two germs (C,O) and (C',O) at the origin are equivalent if they are joined by an equisingular family (i.e., μ-constant family). We define *the generic flex defect of σ* by $\min\{\delta(f;O); f \in \sigma\}$ and we denote the generic flex defect by $\bar\delta(\sigma)$. Let $f(x,y)$ be a polynomial and let $C(f)$ be the plane curve $\{f(x,y)=0\}$. We say that $f(x,y)$ or $C(f)$ is *generic* (at O) in σ if $(C(f),O) \in \sigma$ and $\delta(O;f) = \bar\delta(\sigma)$.

Let $\mathcal{P} = \{(m_1,n_1),\dots,(m_\ell,n_\ell)\}$ be a given set of Puiseux pairs and let $\sigma(\mathcal{P})$ be the equivalent class of the irreducible curve singularity having $\mathcal{P}$ as Puiseux pairs. Assume that $(C,O) \in \sigma(\mathcal{P})$ is defined by $f(x,y)=0$ and $y=0$ is the tangential direction. Then y can be expanded in a Puiseux series as $y = \varphi(x^{1/N})$,

$$(6)\qquad \varphi(x^{1/N}) = \sum_{i=s}^{k_0} a_i x^i + h_1(x^{1/N_1}) + \cdots + h_\ell(x^{1/N_\ell}), \\ N_j := n_1\cdots n_j,\ N = N_\ell$$

where $h_j(x^{1/N_j}) = c_j x^{m_j/N_j} + \sum_{m_j<k<m_{j+1}/n_{j+1}} c_{j,k}x^{k/N_j}$ and c_1, c_2, $\dots$, $c_\ell \neq 0$, $k_0 := [m_1/n_1]$, $\gcd(n_i,m_i)=1$ and $m_i > m_{i-1}n_i$. Note that $a_1 = 0$ by the assumption on the tangent direction. Let $\mathcal{S} = \{j; a_i \neq 0, j \geq 2\}$. We call the order s of $\varphi(x)$ in x *the Puiseux order* of f and we denote it by Puiseux ord(f). Note that $sN = I(C, y=0; O)$ where $y=0$ is the tangential direction. Thus the Puiseux order does not depend on the choice of linear coordinates.

Let s = Puiseux order(f). By the definition, $s = \min\{j \in \mathcal{S}\}$ if $\mathcal{S} \neq \emptyset$ and $s = m_1/n_1$ if $\mathcal{S} = \emptyset$. As a function of x, φ is well-defined in the region, say $-\pi \le \arg(x) < \pi$, when a branch $x^{1/N}$ is fixed. We fix a branch of $x^{1/N}$ hereafter. We consider the canonical stratification of $\sigma(\mathcal{P})$ given by $\{\sigma(\mathcal{P};2)\dots,\sigma(\mathcal{P};[m_1/n_1]),\sigma(\mathcal{P};m_1/n_1)\}$ where

$$\sigma(\mathcal{P};s) = \{(C(f),O) \in \sigma(\mathcal{P}); \text{Puiseux order}(f) = s\}.$$

We call this stratification *the flex stratification* of $\sigma(\mathcal{P})$.

Theorem 7. *Assume that $f(x,y) \in \sigma(\mathcal{P};s)$. Then we have*

$$(8) \qquad \delta(O;f) = (s-2)n_1\cdots n_\ell + \textstyle\sum_{j=1}^{\ell} 3(n_j-1)m_j(n_{j+1}\cdots n_\ell)^2$$

and f is generic if and only if $s \le 2$, namely if either $s = 2$ or $m_1/n_1 \le 2$ and $s = m_1/n_1$.

The formula (8) seems to be equivalent to Satz 2, p. 780, [B-K].

Proof. Put $\omega = \exp(2\pi\sqrt{-1}/N)$ and consider functions of $x^{1/N}$ defined by $\varphi_j(x^{1/N}) := \varphi(x^{1/N}\omega^j)$ for $j = 0,\dots,N-1$. Note $\varphi_0 = \varphi$. The defining function $f(x,y)$ is given by the product $f(x,y) = Ug$ where U is a unit and $g(x,y) = (y-\varphi_0(x^{1/N}))\cdots(y-\varphi_{N-1}(x^{1/N}))$. The intersection number $I(C,J;O)$ is given as $\mathrm{val}_t\mathcal{F}(f)(x(t),y(t))$, using the parameterization $x^{1/N} = t$ (so $x(t) = t^N$) and $y(t) = \varphi_0(t)$. First it is easy to show:

Assertion 1. *$\mathcal{F}(f)(x(t),y(t)) = U^3\mathcal{F}(g)(x(t),y(t))$ and $val_t(\mathcal{F}(f)$ $(x(t),y(t)))$ is equal to $val_t(\mathcal{F}(g)(x(t),y(t)))$.*

Composing the parameterization mapping $t \mapsto \psi(t) := (x(t),y(t))$, we have:

$$\begin{aligned}
g_{x,x}g_y^2(\psi(t)) &= \Big(2\textstyle\sum_{j=1}^{N-1}\frac{\partial\varphi_0}{\partial x}\frac{\partial\varphi_j}{\partial x}\prod_{k\neq 0,j}(\varphi_0-\varphi_k)\\
&\quad -\frac{\partial^2\varphi_0}{\partial x^2}\prod_{j=1}^{N-1}(\varphi_0-\varphi_j)\Big) \times \Big(\prod_{j=1}^{N-1}(\varphi_0-\varphi_j)\Big)^2(\psi(t))\\
-2g_{x,y}g_xg_y(\psi(t)) &= 2\Big(\textstyle\sum_{j=1}^{N-1}\frac{\partial\varphi_0}{\partial x}\prod_{k\neq 0,j}(\varphi_0-\varphi_k)\\
&\quad +\textstyle\sum_{j=1}^{N-1}\frac{\partial\varphi_j}{\partial x}\prod_{k\neq 0,j}(\varphi_0-\varphi_k)\Big)\times\left(-\frac{\partial\varphi_0}{\partial x}\right)\Big(\prod_{j=1}^{N-1}(\varphi_0-\varphi_j)\Big)^2(\psi(t))\\
g_{y,y}g_x^2 &= \Big(2\textstyle\sum_{j=1}^{N-1}\prod_{k\neq 0,j}(\varphi_0-\varphi_k)\Big)\left(\frac{\partial\varphi_0}{\partial x}\right)^2\Big(\prod_{j=1}^{N-1}(\varphi_0-\varphi_j)\Big)^2(\psi(t))
\end{aligned}$$

Thus by an easy computation we get $\mathcal{F}(f)(x(t),y(t)) = -U^3(\psi(t))\frac{\partial^2\varphi_0}{\partial x^2}$ $(x(t))\prod_{i=1}^{N-1}(\varphi_0(x(t))-\varphi_i(x(t))^3$. As the number of $\{0 < k < N; k \equiv 0(N_{j-1})\}$ is $n_j\cdots n_\ell - 1$, the assertion follows from the equalities $\mathrm{val}_t\frac{\partial^2\varphi_0}{\partial x^2}$

$(x(t)) = (s-2)N$ and $\mathrm{val}_t(\varphi_0 - \varphi_k)(x(t)) = m_j N/N_j$, if $k \equiv 0\ (N_{j-1})$ and $k \not\equiv 0\ (N_j)$. Q.E.D.

Corollary 9. *For flex point P of order k, we have $\ell = 0$ and $s = k+2$. Thus $\delta(P; f) = k$.*

We can generalize Theorem 7 to reducible singularities. To avoid the complexity of notations, we do this only for the class of singularity which is equivalent to the Brieskorn singularity $B_{p,q}: y^p - x^q = 0$ with $2 \le p \le q$ at the origin. We denote this equivalence class by $\beta_{p,q}$. Put $r = \gcd(p,q)$ and write $p = rn_1$ and $q = rm_1$. Then each irreducible component has the unique Puiseux pair $\{(m_1, n_1)\}$. Take a function germ $f(x,y)$ which defines such a singularity at the origin. As the resolution complexity of a Brieskorn singularity is one ([L-O]), after a linear change of coordinates, we may assume that $f(x,y) = f_1(x,y) \cdots f_r(x,y)$ where $f_j(x,y) = (y - \sum_{2 \le i < [q/p]} a_i x^i)^{n_1} + c_{j,m_1} x^{m_1} +$ (higher terms) where a_i, $2 \le i < [q/p]$, are independent of j and $c_{1,m_1}, \ldots, c_{r,m_1}$ are mutually distinct non-zero complex numbers. In particular, the Puiseux orders of $f_j, j = 1, \ldots, r$, are the same. Let $\sigma(\beta_{p,q}; s)$ be the set of $f \in \beta_{p,q}$ whose irreducible components have the Puiseux order s.

Theorem 10. *Assume that $p < q$ and $f \in \sigma(\beta_{p,q}; s)$. Then the flex defect and the generic flex defect are given as follows.*

$$\delta(O; f) = 3pq - 3q + (s-2)p, \quad \bar{\delta}(\beta_{p,q}) = \begin{cases} 3pq - 3q, & q > 2p \\ 3pq - 2(p+q), & q \le 2p \end{cases}$$

Proof. Let $y = \varphi_j(x^{1/n_1})$ be the Puiseux expansion of y in x for $f_j(x,y) = 0$. By the assumption, it is written as $\varphi_j(x^{1/n_1}) = \sum_{s \le i < [q/p]} a_i x^i + \sum_{k=m_1}^{\infty} c_{j,k} x^{k/n_1}$ where $(c_{1,m_1})^{n_1}, \ldots, (c_{r,m_1})^{n_1}$ are mutually distinct complex numbers. Let us consider $\varphi_{j,k}(x^{1/n_1}) = \varphi_j(x^{1/n_1}\omega^k)$ with $\omega = \exp(2\pi\sqrt{-1}/n_1)$. Then $f_j(x,y)$ is given by the product $(y - \varphi_{j,0}) \cdots (y - \varphi_{j,n_1-1})$. Denote the i-th branch $f_i(x,y) = 0$ by C_i. To compute the intersection number $I(C_1, J; O)$, we consider the parameterization $x(t) = t^{n_1}$ and $y(t) = \varphi_1(t)$. Then by the same computation as in the proof of Theorem 7, we obtain $\mathcal{F}(f)(x(t), y(t)) = -\frac{\partial^2 \varphi_{1,0}}{\partial x^2}(x(t), y(t)) \prod_{(i,k) \ne (1,0)} (\varphi_{1,0} - \varphi_{i,k})(x(t))^3$. Therefore we obtain the formula

$$(11) \quad I(C_1, J; O) = \mathrm{val}_t(\mathcal{F}(f)(x(t), y(t))) = 3rn_1m_1 - 3m_1 + (s-2)n_1$$

The other intersection numbers $I(C_j, J; O), j = 2, \dots, r$, are the same. Thus as $\delta(O; C)$ is the sum $I(C_1, J; O) + \cdots + I(C_r, J; O)$, the assertion follows immediately. Q.E.D.

Now we consider the case $p = q$. Then $r = p$ and we may assume that $f_j(x, y) = y - \sum_{k=1}^{\infty} c_{j,k} x^k$ where $\{c_{1,1}, \dots, c_{p,1}\}$ are mutually distinct complex numbers. Put $\mathcal{S}_i = \{j; j \geq 2, c_{i,j} \neq 0\}$. We assume that $\mathcal{S}_i \neq \emptyset$ for each $i = 1, \dots, p$. (Otherwise, C contains a line and it is contained in J.) Put s_i be the minimum of $\mathcal{S}_i$. Unlike the previous cases, $\delta(O; f)$ is not bounded.

Corollary 12. *Assume that $(C, O) \in \beta_{p,p+1}$, i.e., a cusp singularity of type $(p, p+1)$ at the origin. Then $\delta(O; C) = \bar{\delta}(\beta_{p,p+1}) = 3p^2 - p - 2$. For $A_{2p-1} = \beta_{2,2p}$, we have $\bar{\delta}(A_{2p-1}) = 6p$ for $p = 1, 2$ and $8p - 4$ for $p \geq 3$.*

By a similar computation, we have

Theorem 13. *The flex defect of the singularity $(C(f), O) \in \beta_{p,p}$ is given by $\delta(O; f) = 3p^2 - 3p + \sum_{i=1}^{p} (s_i - 2)$ and $\bar{\delta}(\beta_{p,p}) = 3p^2 - 3p$.*

Let σ_i, $i = 1, \dots, k$, be equivalence classes of plane curve singularity and let $\Sigma = \{\sigma_1, \dots, \sigma_k\}$. Consider the set of plane curves $\mathcal{M}(n; \Sigma)$ of degree n with k singularities which are equivalent to σ_i, $i = 1, \dots, k$. Take a curve $C \in \mathcal{M}(n; \Sigma)$ and let $P_1, \dots, P_k$ be the singular points of C. C is called *generic* in $\mathcal{M}(n; \Sigma)$ if the following conditions (1),(2),(3) are satisfied.
(1) (C, P_i) is a generic in σ_i and the tangent lines at P_i intersect C transversely except at P_i. (2) The flexes are of order one. (3) The multi-tangent lines are ordinary bi-tangent lines.

A Plücker curve is a generic curve in the case that Σ contains only nodes or cusps. The set of generic curves is an open subset of $\mathcal{M}(n; \Sigma)$ but it might be empty. See Example 17.

2.6. Dual singularity

Let $P \in C$ and P^* be the corresponding point of C^*. As is well-known, P is a $(k-1, k)$-cusp if and only if P^* is a flex of order $k - 2$. If P is a generic node, P^* consists of two tangent points with a bi-tangent line. We study the correspondence for other singularities. Take a point $O \in C$ and let $C_1, \dots, C_k$ be the local irreducible components at O and let $\ell_1, \dots, \ell_r$ be the corresponding tangent line at O. Then the dual image C_i^* of C_i passes through $\ell_i \in \mathbf{P}^{*2}$ for $i = 1, \dots, k$. In the case of C being irreducible at O, we simply denote ℓ_1 by O^*. We call the germ (C_i^*, ℓ_i) *the dual singularity* of the germ of (C_i, O).
(1) **Irreducible case.** Let $\mathcal{P} = \{(m_1, n_1), \dots, (m_\ell, n_\ell)\}$ and let $N_j = n_1 \cdots n_j$ $(N = N_\ell)$ and assume that $(C, O) \in \sigma(\mathcal{P}; s)$ is an irreducible

germ at O defined by $f(x,y) = 0$ whose Puiseux series is given by $\varphi(x^{1/N}) = \sum_{i\ge 2}^{k_0} c_{0,i}x^i + h_1(x^{1/N_1}) + \cdots + h_\ell(x^{1/N_\ell})$ where $k_0 < m_1/n_1$ and $h_j(x^{1/N_j}) = \sum_{m_j \le k < m_{j+1}/n_{j+1}} c_{j,k}x^{k/N_j}$, $c_{1,m_1}, c_{2,m_2}, \dots, c_{\ell,m_\ell} \neq 0$. Let s be the Puiseux order. Let $\mathcal{S} = \{j; c_{i,0} \neq 0, j \ge 2\}$. The dual singularity is described by the following. The case $\ell = 0$ with $s \ge 3$ (a flex of order $s-2$) is also contained in the argument.

Theorem 14. **(Local Duality)** *Let* $\sigma(\mathcal{P}; s)^* := \{(C^*, O^*); (C, O) \in \sigma(\mathcal{P}; s)\}$. *Then the dual operation gives a well-defined mapping on the set of the strata of the flex stratification. More precisely,*

(1) *Assume that* $\mathcal{S} \neq \emptyset$. *Then* $\sigma(\mathcal{P}; 2)^* = \sigma(\mathcal{P}, 2)$ *and* $\sigma(\mathcal{P}; s)^* = \sigma(\mathcal{P}^+; \frac{s}{s-1})$ *if* $s > 2$ *where* $\mathcal{P}^+ := \{(s, s-1), (m_1, n_1), \dots, (m_\ell, n_\ell)\}$. *The first equality says that the dual map* $*$ *gives an involution on* $\sigma(\mathcal{P}; 2)$.

(2) *Assume that* $\mathcal{S} = \emptyset$. *Then* $s = m_1/n_1$ *and* $\sigma(\mathcal{P}; \frac{m_1}{n_1})^* = \sigma(\mathcal{P}^*; \frac{m_1}{m_1 - n_1})$, *if* $m_1 - n_1 > 1$ *and* $\sigma(\mathcal{P}; \frac{m_1}{n_1})^* = \sigma(\mathcal{P}^-; m_1)$, *if* $m_1 = n_1 + 1$, *where* $\mathcal{P}^* := \{(m_1, m_1 - n_1), (m_2, n_2), \dots, (m_\ell, n_\ell)\}$ *and* $\mathcal{P}^- := \{(m_2, n_2), \dots, (m_\ell, n_\ell)\}$.

There is a related result by Wall [W2]. The cases $\ell = 0$, $s \ge 3$ or $\ell = 1$ and $m_1 = n_1 + 1$ are special cases of (1) and (2) respectively. It follows from (2) that a cusp of type $(k, k+1)$ and a flex of order $k-1$ corresponds each other by the dual operation.

Proof. Put $N_j = n_1 \cdots n_j$, $N^{(j)} = n_j \cdots n_\ell$ and $N = N_\ell$. Putting $x^{1/N} = t$, we can parameterize C as $x(t) = t^N$ and $y(t) = \varphi(t) = \sum_j b_j t^j$ where the coefficients are given by $b_k = c_{j,k/N^{(j+1)}}$, if $m_j \le k/N^{(j+1)} < m_{j+1}/n_{j+1}$ and $k/N^{(j+1)} \in \mathbf{Z}$. Otherwise $b_k = 0$. By (1), we can parameterize C^* as $u(t) = -\sum_j \frac{jb_j}{N} t^{j-N}$, $\quad w(t) = \sum_j (\frac{j}{N} - 1) b_j t^j$ where (u, w) is the affine coordinates defined by $u = U/V$, $w = W/V$. Note that $\mathrm{val}_t u(t) = (s-1)N$. We take a change of parameter τ so that $u(\tau) = \tau^{(s-1)N}$. Write $t = \tau \sum_{k=0}^\infty \lambda(k)\tau^k$. The coefficients $\lambda(0), \lambda(1), \lambda(2), \dots$ are inductively determined from the equality $u(t(\tau)) = \tau^{(s-1)N}$ after fixing $\lambda(0)$ which satisfies $\lambda(0)^{(s-1)N} = -1/sb_{sN}$.

Assertion 2. *For* $p < m_k N^{(k+1)} - sN$, $\lambda(p) = 0$ *if* $p \not\equiv 0$ *modulo* $N^{(k)}$. *The first non-trivial coefficient* $\lambda(p)$ *with* $p \not\equiv 0$ $(N^{(k)})$ *is* $\lambda(m_k N^{(k+1)} - sN)$ *and it is given by*

$$(15)\quad \lambda(m_k N^{(k+1)} - sN) = -\frac{m_k N^{(k+1)}}{s(s-1)N^2} \frac{b_{m_k N^{(k+1)}}}{b_{sN}} \lambda(0)^{m_k N^{(k+1)} - sN + 1}$$

Proof. Assume that we have shown $\lambda(p) = 0$ for $p \not\equiv 0$ modulo $N^{(k)}$ and $p < p'$ for some $p' \le m_k N^{(k+1)} - sN$. Consider the equality:

$(PC):\ \tau^{(s-1)N} = -\sum_{j\geq sN} \frac{jb_j}{N}\tau^{j-N}(\sum_q \lambda(q)\tau^q)^{j-N}$. We compare the coefficients of $\tau^{p'+sN-N}$. Assume first that $p' \not\equiv 0\ (N^{(k)})$ and $p' < m_kN^{(k+1)} - sN$. Then the term $\tau^{p'+sN-N}$ in the right side comes only from the first term ($j = sN$) of the summation and the coefficient is $-s(s-1)Nb_{sN}\lambda(0)^{sN-N-1}\lambda(p')$. Thus $\lambda(p') = 0$. By an induction, we get $\lambda(p) = 0$ for $p < m_kN^{(k+1)} - sN$ with $p \not\equiv 0$ modulo $N^{(k)}$.

Now we consider the case $p'=m_kN^{(k+1)}-sN$. The term $\tau^{m_kN^{(k+1)}-N}$ in the right side summation comes from $j = sN$ and $j = m_kN^{(k+1)}$. Comparing the coefficient of $\tau^{m_kN^{(k+1)}-N}$ in (PC), we have

$$-s(s-1)Nb_{sN}\lambda(0)^{sN-N-1}\lambda(m_kN^{(k+1)} - sN) \\ -\frac{m_kN^{(k+1)}}{N}b_{m_kN^{(k+1)}}\lambda(0)^{m_kN^{(k+1)}-N} = 0$$

and the assertion follows from this equality. Q.E.D.

The other coefficients $\lambda(j)$'s are complicated but they are not important. To determine the Puiseux pairs of the dual curve, we write $w(\tau) = \sum_j d(j)\tau^j$. Then by a similar argument,

Assertion 3. *(1) The coefficient $d(j)$ vanishes for any $j < sN$ and $d(sN) = (s-1)b_{sN}\lambda(0)^{sN}$.*
(2) The coefficient $d(j)$ for $j \not\equiv 0\ (N^{(k)})$ vanishes for $j < m_kN^{(k+1)}$ and the first non-vanishing coefficient $d(j)$ with $j \not\equiv 0\ (N^{(k)})$ is $d(m_kN^{(k+1)})$, which is given by $d(m_kN^{(k+1)}) = -b_{m_kN^{(k+1)}}\lambda(0)^{m_kN^{(k+1)}}$.

Proof. As $w(t) = \sum_j(\frac{j}{N} - 1)b_jt^j$, the first assertion of (2) follows immediately from Assertion 2. The second equality follows from

$$\begin{aligned} d(m_jN^{(j+1)}) &= (\frac{sN}{N} - 1)b_{sN}\lambda(0)^{sN-1}sN\lambda(m_jN^{(j+1)} - sN) \\ &\quad +(\frac{m_jN^{(j+1)}}{N} - 1)b_{m_jN^{(j+1)}}\lambda(0)^{m_jN^{(j+1)}} \\ &= -b_{m_jN^{(j+1)}}\lambda(0)^{m_jN^{(j+1)}} \quad \text{Q.E.D.} \end{aligned}$$

Assume that $\mathcal{S} \neq \emptyset$. Assume first that $s = 2$. Then $u = \tau^N$ and $(C^*, O^*) \in \sigma(\mathcal{P}; 2)$. If $s > 2$, $u(\tau) = \tau^{(s-1)N}$ and the assertion follows from $\frac{m_jN^{(j+1)}}{(s-1)N} = \frac{m_j}{(s-1)n_1\cdots n_j}$. Assume that $\mathcal{S} = \emptyset$ and $s = m_1/n_1$. Then $u(\tau) = \tau^{(m_1-n_1)N^{(2)}}$ and $\gcd(m_1 - n_1, n_1) = 1$. Thus the assertion follows. This completes the proof.

(2) **Reducible case**. A similar assertion can be proved for reducible curve germs. We do this for Brieskorn singularities. Let us consider a germ of a Brieskorn singularity $(C, O) \in \beta_{p,q}$ defined by a polynomial

$f(x,y)$ with the tangential direction $y=0$. Let $r=\gcd(p,q)$ and write $p=rn_1$ and $q=rm_1$. Let $f=f_1\cdots f_r$ be the factorization and let C_j be the irreducible component of C defined by $f_j(x,y)=0$. Recall that the Puiseux expansions of $f_j(x,y)$ in x for $i=1,\dots,r$ are the same up to the term x^{m_1/n_1}.

Theorem 16. **(Local Duality-bis)** *Assume that $p<q$ and $(C,O)\in\sigma(\beta_{p,q};s)$. Then $s=q/p$ and $(C^*,O^*)\in\sigma(\beta_{q-p,q};\frac{q}{q-p})$ if $q\le 2p$. If $2p<q$ and $s=2$, then $(C^*,O^*)\in\sigma(\beta_{p,q};2)$. If $2p<q$ and $s>2$, $(C^*,O^*)=\cup_{i=1}^r(C_i^*,O^*)$ and $(C_i^*,O^*)\in\sigma(\mathcal{P}^+;\frac{s}{s-1})$ with $\mathcal{P}^+=\{(s,s-1),(m_1,n_1)\}$. The Puiseux expansions of C_i^* in $u^{1/(s-1)n_1}, i=1,\dots,r$ coincide up to the term $u^{m_1/(s-1)n_1}$.*

Proof. Assume first that $m_1>2n_1$. Then C_j is defined by a polynomial $f_j(x,y)$ of the form $f_j(x,y)=(y-\sum_{i=s}^{k_0}a_ix^i)^{n_1}-c_j^{n_1}x^{m_1}+$ (higher terms) where $k_0=[m_1/n_1]$, $a_s\ne 0$ and $s\ge 2$. Here $a_s,\dots,a_{k_0}$ are independent of j. In the proof of Theorem 14, we have shown that $(C_j^*,O^*)\in\sigma(\mathcal{P}^+;\frac{s}{s-1})$ with $\mathcal{P}^+=\{(s,s-1),(m_1,n_1)\}$ and C_j^* is parameterized as $u(t)=\tau^{(s-1)n_1}$ and $w(t)=\sum_{s\le i<m_1/n_1}d(i)\tau^{in_1}+\sum_{i=m_1}^{\infty}d(j,i)\tau^i$. Thus the assertion follows from the observation $d(s)\ne 0$ and $d(s),\dots,d(k_0)$ are independent of j and $d(j,m_1)=c_j\times\lambda(0)^{m_1}$. In particular, this implies that if $s=2$, $(C_j^*,O^*)\in\sigma(\beta_{m_1,n_1};2)$ and C_j^* is defined by a polynomial of the type $g_j(u,w)=(w-\sum_{2\le i<m_1/n_1}d(i)u^i)^{n_1}$ $-d(j,m_1)^{n_1}\omega^Au^{m_1}+$ (higher terms) where $\omega=\exp(2\pi\sqrt{-1}/n_1)$ and $A=n_1(n_1-1)m_1/2$. Thus the assertion follows immediately. The case $m_1\le 2n_1$ can be treated similarly. Q.E.D.

Example 17. *Let us consider a rational curve $C=\{f(x,y)=0\}$ of degree 6 where $f(x,y)=(x^2+y^3)^2-4y^3x^3$. In the affine coordinate $(u,v)=(Z/X,Y/X)$, C is defined by $(u+v^3)^2-4v^3=0$. Thus C is a Jung transform of the rational curve $u^2-4v^3=0$ (See Example (6.6), [O4]). C has two singularities. One (2,3) cusp at $P:=(1,0,0)$ and one irreducible singularity of Puiseux pairs $\{(3,2),(9,2)\}$ at $Q:=(0,0,1)$. By Theorem 7, the flex defect at Q is 61 and the Milnor number is 18. Thus the dual curve should have three cusps and 3 nodes if C is generic in the moduli. The dual curve is given by $C^*=\{g(x,y)=0\}$ where $g(x,y)=16y^6+27y^3+540y^3x-216y^3x^2+729x+2187x^2+2187x^3+729x^4$. Thus C^* is a rational curve of degree 6 and it has three cusps and one singularity at $Q^*:=(1,0,0)$ of Milnor number 8 which is in the moduli $\sigma(\{(9,2)\};3)$ by Theorem 14. The discriminant of g in y is given by $cx^2(x+1)^6(8x-1)^9$, $c\ne 0$ and C^* has a $\beta_{3,3}$ singularity at $(-1,0,1)$. C is not generic as C^* does not have three nodes but a $\beta_{3,3}$. The reason* *is, C has a tri-tangent line $x = -1$. In fact, by computing the moduli space explicitly, we can show that there does not exist any generic curve in the moduli of C but every member has a tri-tangent line.*

§3. Moduli of certain sextics and their dual

In this section, we consider various moduli spaces of sextics. Unless otherwise stated, n, n^*, g are the degree, the class number and the genus of the curve in discussion respectively.

3.1. Moduli space $\mathcal{M} := \mathcal{M}(6;\Sigma)$.

Let $\Sigma = \{3\beta_{2,2}, 6\beta_{2,3}\}$ and consider the moduli space $\mathcal{M} := \mathcal{M}(6;\Sigma)$ of sextics with 6 cusps and 3 nodes. Let us denote the subset of $\mathcal{M}$ whose curves are generic (i.e., Plücker) by $\mathcal{M}'$. It is easy to see that $g(C) = 1$ for any $C \in \mathcal{M}$ by the modified Plücker formula. By the class formula (3), the dual curves C^* has degree 6. By Theorem 10 and Proposition 5, the dual curve C^* has also 6 cusps for $C \in \mathcal{M}'$. As $g(C^*) = 1$, they have 3 nodes. Thus we have the self-duality: $\mathcal{M}'^* = \mathcal{M}'$. However $\mathcal{M}^* \neq \mathcal{M}$. The reason is that there exists an interesting degeneration in this moduli as we will see below. First, the number of flexes on $C \in \mathcal{M}$ is 6 counting the multiplicity by Proposition 5. Thus the possible types of flexes are (0) 6 flexes of order 1, (i) 4 flexes of order 1 and one flex of order 2, (ii) 2 flexes of order 1 and 2 flexes of order 2, (iii) 3 flexes of order 2 and (iv) 3 flexes of order 1 and one flex of order 3. There do not exist other types as the dual curve has genus 1 and the sum of Milnor numbers of the singular points of C^* is less than or equal to 18 by the modified Plücker formula. The moduli space with these flex types are difficult to study directly. So we consider their dual moduli spaces.

(1) Let $\Sigma_1 = \{2\beta_{2,2}, 4\beta_{2,3}, \beta_{3,4}\}$ and let $\mathcal{N}_1 := \mathcal{M}(6;\Sigma_1)$. The genus of a curve in $\mathcal{N}_1$ is 1 and the class number is 6. Thus we have the inclusion: $\mathcal{N}_1'^* \subset \mathcal{M}$. Here we denote by $\mathcal{N}_1'$ the submoduli of $\mathcal{N}_1$ which consists of the generic curves. A curve $C \in \mathcal{N}_1'^*$ is not a Plücker curve but it has 4 flexes of order 1 and a flex of order 2. We put $\mathcal{M}_1 := \mathcal{N}_1'^*$. By reciprocity law, $C \in \mathcal{M}$ is in $\mathcal{M}_1$ if and only if C has 4 flexes of order 1, one flex of order 2 and two bi-tangents.

(2) Let $\Sigma_2 = \{\beta_{2,2}, 2\beta_{2,3}, 2\beta_{3,4}\}$ and $\mathcal{N}_2 := \mathcal{M}(6;\Sigma_2)$. For $C \in \mathcal{N}_2$, the genus $g(C) = 1$ and $n^* = 6$. The generic dual $\mathcal{N}_2'^*$ consists of curves C with 6 cusps and 3 nodes and 2 flexes of order 1 and 2 flexes of order 2. We denote this dual image $\mathcal{N}_2'^*$ by $\mathcal{M}_2$.

(3) Let $\Sigma_3 = \{3\beta_{3,4}\}$ and let $\mathcal{N}_3 := \mathcal{M}(6;\Sigma_3)$. We see that $g(C) = 1$ for any $C \in \mathcal{N}_3$ and the generic dual $\mathcal{N}_3'^*$ is again 6 cuspidal 3 nodal sextics with 3 flexes of order 2. The moduli of such curves is denoted by $\mathcal{M}_3$.

(4) Finally let $\Sigma_4 = \{\beta_{4,5}, 3\beta_{2,3}\}$ and let $\mathcal{N}_4 = \mathcal{M}(6;\Sigma_4)$. We see that $g = 1, n^* = 6$ and the generic dual $\mathcal{N}_3'^*$ is again 6 cuspidal 3 nodal sextics with 3 ordinary flexes and one flex of order 3. Put $\mathcal{M}_4 := \mathcal{N}_4'^*$.

Let $\mathcal{T}$ be the moduli space of (2,3)-torus curves of degree 6 and of type (2,3). The respective submoduli of torus type $\mathcal{M}\cap\mathcal{T}$, $\mathcal{M}_i\cap\mathcal{T}$ and $\mathcal{N}_i\cap\mathcal{T}$ are denoted simply by $\mathcal{M}_{torus}, \mathcal{M}_{i,torus}, \mathcal{N}_{i,torus}$ respectively. Non-torus moduli are denoted as $\mathcal{M}_{gen}, \mathcal{M}_{i,gen}, \mathcal{N}_{i,gen}$ respectively. The main result about the structure of the moduli spaces $\mathcal{M}$ is:

Theorem 18. *1. The union $\widehat{\mathcal{M}} := \mathcal{M}' \cup_{i=1}^4 \mathcal{M}_i \cup_{i=1}^4 \mathcal{N}_i'$ is invariant by the dual operation. Namely the dual operation $C \mapsto C^*$ gives an involution on $\widehat{\mathcal{M}}$. Furthermore the dual operation preserves curves of torus type and non-torus type. Namely ${\mathcal{M}'_\alpha}^* = \mathcal{M}'_\alpha$, ${\mathcal{N}'_{i,\alpha}}^* = \mathcal{M}_{i,\alpha}$ and ${\mathcal{M}_{i,\alpha}}^* = \mathcal{N}'_{i,\alpha}$ for $i = 1,\dots,4$ and α=torus or gen.*
2. (Stratification)$\mathcal{M}_{torus}=\mathcal{M}'_{torus} \cup_{i=1}^3 \mathcal{M}_{i,torus}$ *and* $\mathcal{M}_{gen}=\mathcal{M}'_{gen} \cup_{i=1}^4 \mathcal{M}_{i,gen}$. *Thus $\mathcal{M}_4 = \mathcal{M}_{4,gen}$ and $\mathcal{N}_4 = \mathcal{N}_{4,gen}$. The moduli spaces $\mathcal{M}'_{torus}, \mathcal{M}_{i,torus}, \mathcal{N}_{i,torus}$, $i = 1,2,3$ and $\mathcal{N}_{3,gen}$ are irreducible. For the moduli of the curves of torus type, we have the adherence relation:*

$$\overline{\mathcal{M}'_{torus}} \supset \overline{\mathcal{M}_{1,torus}} \supset \overline{\mathcal{M}_{2,torus}} \supset \mathcal{M}_{3,torus},$$
$$\overline{\mathcal{M}'_{torus}} \supset \overline{\mathcal{N}'_{1,torus}} \supset \overline{\mathcal{N}'_{2,torus}} \supset \mathcal{N}'_{3,torus}$$

3. (Alexander polynomial) *For $C \in \widehat{\mathcal{M}}_{torus}$, the Alexander polynomial $\Delta_C(t)$ is given by $t^2 - t + 1$* ([Li1],[D]). *For non-torus curve $C \in \widehat{\mathcal{M}}_{gen}$, it is given by 1.*
4. (Fundamental groups) *$\pi_1(\mathbf{P}^2 - C) \cong \mathbf{Z}_2 * \mathbf{Z}_3$ or $\pi_1(\mathbf{P}^2 - C) \cong \mathbf{Z}_6$ according to $C \in \widehat{\mathcal{M}}_{torus}$ or $C \in \mathcal{M}_{3,gen}$ respectively.*

Remark 3. *We do not know if the other moduli spaces of non-torus type are irreducible. If this is the case, the adherence relations and the commutativity of the fundamental group holds for other non-torus type sextics $\mathcal{M}_{i,gen}, \mathcal{N}_{i,gen}$, $i = 1,2,3$. The moduli space $\mathcal{N}_4$ seems to be irreducible.*

3.2. Alexander polynomial

Let C be an irreducible plane curve of degree n and L_∞ be the line at infinity. We assume that L_∞ intersects C transversely. We consider the Alexander polynomial $\Delta_C(t)$ with respect to L_∞ and we call it *the generic Alexander polynomial.* It has integral coefficients. For the definition of the Alexander polynomial, we refer to [Li2]. We recall several basic properties of $\Delta_C(t)$.
(1) $\Delta_C(t)$ divides the Alexander polynomial at infinity $(t^n - 1)^{n-2}(t-1)$

and also the product of the local Alexander polynomials at singular points of C ([Li2] and [Li1]).

Let $p : Y \to \mathbf{P}^2$ be the embedded resolution of the singularity of $C \cup L_\infty$. Let $q_m : X_m \to \mathbf{P}^2$ be the m-cyclic covering branched along $C \cup L_\infty$ and let $p_m : Z_m \to Y$ be the desingularization of the pull-back of q_m by p. Let $\Lambda := \mathbf{Q}[t, t^{-1}]$. Then $H_1(X_\infty; \mathbf{Q})$ is a Λ-module where t acts as the Deck transformation. Thus there are polynomials $\lambda_1(t), \dots, \lambda_k(t)$ with $\lambda_i | \lambda_{i+1}$, $i = 1, \dots, k-1$, such that $H_1(X_\infty; \mathbf{Q})$ is isomorphic to the direct sum $\sum_{i=1}^{k} \Lambda/(\lambda_i)$ and $\Delta_C(t) = \lambda_1(t) \cdots \lambda_k(t)$.
(2) The first Betti number $b_1(Z_m)$ of Z_m is equal to the sum $\sum_{i=1}^{k} \alpha_i$ where α_i is the the number of different m-th roots of unity in the roots of $\lambda_i(t) = 0$ ([Li2]).
(3) $\Delta_C(t)$ is a cyclotomic polynomial and $\Delta_C(1) = \pm 1$ (see for example, [R]).

Consider the case $m = n$ and we write $Z := Z_n$ for simplicity. Combining these properties, the determination of the Alexander polynomial is reduced to the calculation of the first Betti number of Z, or equivalently to the calculation of the irregularity of Z.

For the calculation of the irregularity $q(Z)$, the method by Esnault ([E]) and Artal ([A1]) is convenient. Let us recall it. Let $P_1, \dots, P_\nu$ be the singular points of C. Let $L^{(k)}$ be the divisor on Y introduced in [E]. Then $b_1(Z) = 2q(Z) = 2\sum_{k=0}^{n-1} \dim H^1(Y; \mathcal{O}(L^{(k)}))$ by [E] and $H^1(Y; \mathcal{O}(L^{(k)}))$ can be identified by the cokernel of the natural homomorphism $\sigma_{k-3,k} : H^0(\mathbf{P}^2; \mathcal{O}(k-3)) \to \sum_{P_i} \mathcal{O}_{\mathbf{P}^2, P_i}/\mathcal{I}_{P_i,k,n}$ where $\mathcal{I}_{P_i,k,n}$ is an ideal described as follows ([A1]). Let $E_{i,1}, \dots, E_{i,\ell_i}$ be the exceptional divisors over P_i and let $m_{i,j}$ be the multiplicity of $p^* f$ along $E_{i,j}$. Let $K = -3L + \sum_{i,j} k_{i,j} E_{i,j}$ be a canonical divisor, where L is a generic line, not passing through any of $P_1, \dots, P_\nu$. Then the ideal $\mathcal{I}_{P_i,k,n}$ is generated by the function germs g such that the pull-back $p^* g$ vanishes along $E_{i,j}$ at least with the multiplicity $-k_{i,j} + [k m_{i,j}/n]$.

Now we are ready to compute the Betti number of Z_6 for the sectics in $\widehat{\mathcal{M}}$. For the computation, we use canonical toric modifications at singular points ([O6]). Assume that the singularity P_i is non-degenerate and the restriction of $p : Y \to \mathbf{P}^2$ to a neighbourhood of P_i is a toric modification. Let Σ_i^* be a regualr fan subdividing the dual Newton diagram $\Gamma^*(f; P_i)$ at P_i which is used to construct the toric modification and let $P_{i,j} = {}^t(a_{i,j}, b_{i,j})$, $j = 1, \dots, \ell_i$ be the primitive covectors which generate 1-dimensional cones and let $\widehat{E}(P_{i,j})$ be the corresponding exceptional divisor. Then using the equality $\frac{dx}{x} \wedge \frac{dy}{y} = \frac{dx_\sigma}{x_\sigma} \wedge \frac{dy_\sigma}{y_\sigma}$, we have a simple formula: $K = -3L + \sum_{i,j}(a_{i,j} + b_{i,j} - 1)\widehat{E}(P_{i,j})$. Here (x_σ, y_σ)

are the toric coordinates of the coordinate chart $\mathbf{C}^2_\sigma$ and $\widehat{E}(P_{i,j})$ is the exceptional divisor corresponding to $P_{i,j}$. Refer to Chapter III, in [O6] for detail.

(a) For a cusp, $y^2 - x^3 +$ (higher terms) $= 0$, the exceptional divisors correspond to covectors $Q_1 = {}^t(1,1), Q_2 = {}^t(2,3)$, $Q_3 = {}^t(1,2)$. We have $K = -3L + \widehat{E}(Q_1) + 4\widehat{E}(Q_2) + 2\widehat{E}(Q_3)$ and $(p^*f) = C' + 2\widehat{E}(Q_1) + 6\widehat{E}(Q_2) + 3\widehat{E}(Q_3)$ (locally at each P_i). Here C' is the strict transform of C. Recall the equivalence: *a curve $C \in \mathcal{M}'$ is of torus type if and only if six cusps are on a conic* (see [D]). Let $C \in \mathcal{M}$ and let $P_1, \dots, P_6$ be the cusps. The nodes have nothing to do with the Alexander polynomial. The non-trivial case is $H^1(Y; \mathcal{O}(L^{(5)}))$. The kernel of $\sigma_{2,5} : H^0(\mathbf{P}^2; \mathcal{O}(2)) \to \sum_{i=1}^6 \mathcal{O}_{\mathbf{P}^2, P_i}/\mathcal{I}_{P_i,5,6}$ consists of conics passing through $P_1, \dots, P_6$. Thus $\dim \operatorname{Ker}(\sigma_{2,5}) = 1$ or 0 and therefore $b_1(Z_6) = 2$ or 0 depending on whether C is of torus type or not. By (1), this also implies $\Delta_C(t) = (t^2 - t + 1)^\alpha$, $\alpha \geq 1$, or 1 respectively.

(b) Now we consider (3,4)-cusp, $y^3 - x^4 +$ (higher terms) $= 0$. We have four exceptional divisors, corresponding to $Q_1 = {}^t(1,1), Q_2 = {}^t(3,4)$, $Q_3 = {}^t(2,3)$, $Q_4 = {}^t(1,2)$. $K = -3L + \widehat{E}(Q_1) + 6\widehat{E}(Q_2) + 4\widehat{E}(Q_3) + 2\widehat{E}(Q_4)$ and $(p^*f) = C' + 3\widehat{E}(Q_1) + 12\widehat{E}(Q_2) + 8\widehat{E}(Q_3) + 4\widehat{E}(Q_4)$.

Let $C \in \mathcal{N}_3$ be a sextic with 3 (3,4)-cusps. The non-trivial case is again $\sigma_{2,5} : H^0(\mathbf{P}^2; \mathcal{O}(2)) \to \sum_{i=1}^3 \mathcal{O}_{\mathbf{P}^2, P_i}/\mathcal{I}_{P_i,5,6}$. Locally $\mathcal{I}_{P_i,5,6}$ is generated by function germs $g(x,y)$ such that either it has no linear term in a coordinate centered at P_i or the conic $g = 0$ is tangent to the tangent cone of C at P_i. Thus $\dim \mathcal{O}_{\mathbf{P}^2, P_i}/\mathcal{I}_{P_i,5,6} = 2$. q is in the kernel of $\sigma_{2,5}$ if and only if the conic $q = 0$ passes through P_1, P_2, P_3 and is tangent to (the tangent cones of) C at $P_i, i = 1,2,3$. Thus $b_1(Z_6) = 2$ ($\Delta_C(t) = (t^2 - t + 1)^\beta$, $\beta \geq 1$) if and only if C is of torus type (cf. Corollary 24). Otherwise $b_1(Z_6) = 0$. To show $\alpha = \beta = 1$, we need a little more discussion but in our case, this follows immediately from the assertion on the fundamental group (see §5) and the Fox calculus. The computation of $b_1(Z_6)$ for curves in $\mathcal{N}_1, \mathcal{N}_2$ are similar.

(c) We consider a (4,5)-cusp, $y^4 - x^5 +$ (higher terms) $= 0$. We need 5 exceptional divisors, corresponding to the covectors $Q_1 = {}^t(1,1), Q_2 = {}^t(4,5)$, $Q_3 = {}^t(3,4)$, $Q_4 = {}^t(2,3)$ and $Q_5 = {}^t(1,2)$. The canonical divisor is locally given by $K = \widehat{E}(Q_1) + 8\widehat{E}(Q_2) + 6\widehat{E}(Q_3) + 4\widehat{E}(Q_4) + 2\widehat{E}(Q_5)$ and $(p^*f) = C' + 4\widehat{E}(Q_1) + 20\widehat{E}(Q_2) + 15\widehat{E}(Q_3) + 10\widehat{E}(Q_4) + 5\widehat{E}(Q_5)$.

Now we compute the Alexander polynomial of $C \in \mathcal{N}_4$. Thus C has a (4,5)-cusp singularity at P_1 and 3 (2,3)-cusps at P_2, P_3, P_4. Observe first that any two of $P_i, i = 2,3,4$ can not be colinear with P_1 by the

Bezout theorem. Again we only need to compute $\mathrm{Ker}(\sigma_{2,5})$. We can see easily that $\mathcal{I}_{P_1,5,6}$ is generated by the functions without any linear term at P_1 and $\mathcal{I}_{P_i,5,6}$ is generated by functions vanishing at P_i for $i = 2, 3, 4$. Thus the dimension of the target is also 6. A conic $q = 0$ is in the kernel of $\sigma_{2,5}$ if $q = 0$ has multiplicity 2 at P_1 and passes through P_2, P_3, P_4. This is impossible. Thus $\Delta_C(t)$ is trivial. See also Proposition 27. We thank to Anatoly Libgober for communicating us that the computation can be also made using quasiadjunction formula as in [Li1].

3.3. Moduli space $\mathcal{M}$

We first compute the moduli space $\mathcal{M}_{torus} = \mathcal{M} \cap \mathcal{T}$ where $\mathcal{M} = \mathcal{M}(6; 6\beta_{2,3} + 3\beta_{2,2})$. We start from the expression $f(x,y) = f_2(x,y)^3 + f_3(x,y)^2$ where

$$f_2(x,y) = y^2 + y(a_{1,0} + a_{1,1}x) + a_{0,0} + a_{0,1}x + a_{0,2}x^2 \text{ and}$$
$$f_3(x,y) = b_{3,0}y^3 + y^2(b_{2,0} + b_{2,1}x) + y(b_{1,0} + b_{1,1}x + b_{1,2}x^2) + b_{0,0} + b_{0,1}x + b_{0,2}x^2 + b_{0,3}x^3$$

First we may assume that the nodes are at $O = (0,0), A = (1,1), B = (1,-1)$ by the action of $\mathrm{PSL}(3,\mathbf{C})$. The submoduli of $\mathcal{M}_{torus}$ consisting of curves with three nodes at O, A, B is denoted by $\mathcal{M}_{torus}^{\#}$. As $\mathrm{PSL}(3,\mathbf{C})$ orbit of $\mathcal{M}_{torus}^{\#}$ is $\mathcal{M}$, it is enough to see the irreducibility of $\mathcal{M}_{torus}^{\#}$. Introducing the variables t_0, t_1, t_2 such that $f_2(O) = -t_0^2$, $f_2(A) = -t_1^2$ and $f_2(B) = -t_2^2$, we can explicitly solve the equations $f(Q) = \frac{\partial f}{\partial x}(Q) = \frac{\partial f}{\partial y}(Q) = 0$, $Q = O, A, B$ as they are linear conditions. We can solve these equations, one by one so that the moduli has 6 free parameters $a_{1,0}, a_{0,2}, b_{2,1}, t_0, t_1, t_2$ and the other coefficients are uniquely determined as follows.

$$\begin{aligned}
a_{0,0} &= -t_0^2, \\
a_{0,1} &= -1 - \tfrac{1}{2}t_1^2 - \tfrac{1}{2}t_2^2 + t_0^2 - a_{0,2}, \\
a_{1,1} &= -a_{1,0} - \tfrac{1}{2}t_1^2 + \tfrac{1}{2}t_2^2, \\
b_{0,0} &= t_0^3 \\
b_{0,1} &= -\tfrac{3}{2}t_0(-1 - \tfrac{1}{2}t_1 - \tfrac{1}{2}t_2 + t_0 - a_{0,2}), \\
b_{1,0} &= -\tfrac{3}{2}t_0 a_{1,0}, \\
b_{0,2} &= b_{2,1} + \tfrac{3}{2}t_2 - 3t_0 + \tfrac{3}{2}t_1 - \tfrac{3}{2}t_0 a_{1,0} + \tfrac{15}{16}t_1^3 - 3t_0 a_{0,2} \\
&\quad -\tfrac{9}{4}t_0 t_1^2 + \tfrac{3}{4}t_1 t_0^2 + \tfrac{3}{4}t_1 a_{0,2} + \tfrac{3}{4}t_1 a_{1,0} + \tfrac{3}{2}t_1(-a_{1,0} - \tfrac{1}{2}t_1^2 + \tfrac{1}{2}t_2^2) \\
&\quad -\tfrac{3}{2}t_0(-a_{1,0} - \tfrac{1}{2}t_1^2 + \tfrac{1}{2}t_2^2) + \tfrac{3}{16}t_2 - \tfrac{3}{16}t_1 t_2 \\
&\quad -\tfrac{3}{4}t_0 t_2^2 + \tfrac{3}{4}t_2 t_0^2 + \tfrac{3}{4}t_2 a_{0,2} + \tfrac{3}{4}t_2 a_{1,0} + \tfrac{9}{16}t_2 t_1^2,
\end{aligned}$$

$$\begin{aligned}
b_{0,3} &= \tfrac{1}{8}t_2^3 - \tfrac{3}{8}t_1t_2^2 + \tfrac{3}{4}t_0t_2^2 - \tfrac{3}{4}t_2t_0^2 - \tfrac{3}{4}t_2a_{0,2} - \tfrac{3}{4}t_2a_{1,0} \\
&\quad -\tfrac{3}{4}t_2 - \tfrac{3}{4}t_1t_0^2 - \tfrac{3}{4}t_1a_{0,2} + \tfrac{3}{4}t_1a_{1,0} + \tfrac{3}{2}t_0a_{0,2} \\
&\quad +\tfrac{1}{2}t_0^3 - \tfrac{3}{8}t_2t_1^2 + \tfrac{1}{8}t_1^3 - \tfrac{3}{4}t_1 - b_{2,1} + \tfrac{3}{4}t_0t_1^2 + \tfrac{3}{2}t_0, \\
b_{1,1} &= -\tfrac{3}{4}t_2 + \tfrac{3}{4}t_1 + \tfrac{3}{2}t_0a_{1,0} + \tfrac{3}{8}t_1^3 - \tfrac{3}{4}t_0t_1^2 + \tfrac{3}{4}t_1t_0^2 \\
&\quad +\tfrac{3}{4}t_1a_{0,2} - \tfrac{3}{4}t_1a_{1,0} - \tfrac{3}{2}t_0(-a_{1,0} - \tfrac{1}{2}t_1^2 + \tfrac{1}{2}t_2^2) - \tfrac{3}{8}t_2^3 + \tfrac{3}{8}t_1t_2^2 \\
&\quad +\tfrac{3}{4}t_0t_2^2 - \tfrac{3}{4}t_2t_0^2 - \tfrac{3}{4}t_2a_{0,2} - \tfrac{3}{4}t_2a_{1,0} - \tfrac{3}{8}t_2t_1^2, \\
b_{1,2} &= -\tfrac{9}{16}t_1^3 + \tfrac{3}{4}t_0t_1^2 - \tfrac{3}{4}t_1t_0^2 - \tfrac{3}{4}t_1a_{0,2} - \tfrac{3}{4}t_1a_{1,0} \\
&\quad -\tfrac{3}{2}t_1(-a_{1,0} - \tfrac{1}{2}t_1^2 + \tfrac{1}{2}t_2^2) + \tfrac{3}{2}t_0(-a_{1,0} - \tfrac{1}{2}t_1^2 + \tfrac{1}{2}t_2^2) - \tfrac{3}{16}t_2^3 \\
&\quad +\tfrac{9}{16}t_1t_2^2 - \tfrac{3}{4}t_0t_2^2 + \tfrac{3}{4}t_2t_0^2 + \tfrac{3}{4}t_2a_{0,2} + \tfrac{3}{4}t_2a_{1,0} + \tfrac{3}{16}t_2t_1^2, \\
b_{2,0} &= \tfrac{3}{16}t_2^3 - \tfrac{3}{16}t_1t_2^2 - \tfrac{3}{16}t_2t_1^2 - \tfrac{3}{4}t_2 + \tfrac{3}{16}t_1^3 - \tfrac{3}{4}t_1 - b_{2,1}, \\
b_{3,0} &= \tfrac{1}{16}t_2^3 - \tfrac{3}{16}t_1t_2^2 + \tfrac{3}{16}t_2t_1^2 + \tfrac{3}{4}t_2 - \tfrac{1}{16}t_1^3 - \tfrac{3}{4}t_1
\end{aligned}$$

Thus the moduli space $\mathcal{M}_{torus}^{\#}$ is a Zariski-open subset of $\mathbf{C}^6$ and this proves the irreducibility of the moduli $\mathcal{M}_{torus}^{\#}$ and $\mathcal{M}_{torus}$.

Remark 4. *Let $\mathcal{M}_{torus,col}$ be the submoduli space of $\mathcal{M}_{torus}$ for which three nodes are colinear. $\mathcal{M}_{torus,col}$ is a* codimention one sub-variety *of $\mathcal{M}_{torus}$ and $\mathcal{M}_{torus} - \mathcal{M}_{torus,col}$ is* Zariski dense *in $\mathcal{M}$. To see this, first we consider the submoduli $\mathcal{M}_{torus,col}^{\#}$ whose curves have theree nodes on O and $D := (1,0)$ and $E = (0,1)$. They are defined by $h(x,y) = 0$ where $h(x,y) = h_2(x,y)^3 - h_3(x,y)^2$ and $h_2(x,y) := y^2 + (A_{10} + A_{11}x)y + T_0^2 + 3T_0^2x^2$ and $h_3(x,y) := B_{30}y^3 + y^2(B_{20} + B_{21}x) + y(\frac{3}{2}T_0A_{10} - 3xT_0A_{11} - \frac{9}{2}T_0A_{10}x^2) + T_0^3 - 9T_0^3x^2$.*

For a given generic curve $C_0 \in \mathcal{M}_{torus,col}^{\#}$, we can explicitly find a family of curves $C_s := \{f(x,y,s) = 0\}$ in $\mathcal{M}$ such that three nodes of C_s are at D, E and $O_s := (0,s)$. We omit the explicit polynomial equation as it is long and the computation is boring. Instead we give an example. $f := f_2^3 - f_3^2$ where $f_2 := 1 + y^2 - s^2 + 3x^2 + \frac{2}{9}syx + s^2x^2$ and $f_3 = 2y^3 + 6y^2 + 3s^2 - 2yx^2s^2 + 1 - 9x^2 + y^2x - \frac{2}{3}syx - 3s^2x^2 + 9yx^2s - 9ys - 4y^2s + 2ys^2$.

3.4. Moduli spaces $\mathcal{N}_{i,torus}$ and the degeneration

We consider the moduli spaces $\mathcal{N}_{1,torus}$, $\mathcal{N}_{2,torus}$ and $\mathcal{N}_{3,torus}$. Let $O = (0,0), A = (1,1), B = (1,-1)$ be as above. We compute the sub-moduli spaces $\mathcal{N}_{1,torus}^{\#}, \mathcal{N}_{2,torus}^{\#}, \mathcal{N}_{3,torus}^{\#}$.

(1) Moduli space $\mathcal{N}_{1,torus}$. Consider first $\mathcal{N}_{1,torus}^{\#}$, the moduli of torus sextics $f_2(x,y)^3 + f_3(x,y)^2 = 0$ with a (3,4)-cusp singularity at O and 2 nodes at A, B and four ordinary cusps. As the sum of the intersection multiplicity of $f_2 = f_3 = 0$ is 6, it is necessary that $f_2(O) =$

0 and four other cusps are also on the conic $f_2(x,y)=0$. The condition for O to be a (3,4)-cusp is given by the following four linear equations: $f_2(O)=f_3(O)=\frac{\partial f_3}{\partial x}(O)=\frac{\partial f_3}{\partial y}(O)=0$.

Proposition 19. *The above (3,4)-cusp condition is the same as the limit of the node condition at O for $t_0\to 0$: $f(O)=\frac{\partial f}{\partial x}(O)=\frac{\partial f}{\partial y}(O)=0$.*

Proof. In fact, using $f_2(O)=-t_0^2$ and $f_3(O)=t_0^3$, we have $\frac{\partial f}{\partial x}(O)=t_0^3(3t_0\frac{\partial f_2}{\partial x}(O)+2\frac{\partial f_3}{\partial x}(O))$. Thus the limit for $t_0\to 0$ gives $\frac{\partial f_3}{\partial x}(O)=0$. The same argument applies for $\frac{\partial f_3}{\partial y}(O)$. Q.E.D.

Therefore the moduli is given by substituting $t_0=0$ in $\mathcal{M}$ and it has 5 free parameters $a_{1,0},a_{0,2},b_{2,1},t_1,t_2$ where $f_2(A)=-t_1^2$ and $f_2(B)=-t_2^2$. We see that $\mathcal{N}_{1,torus}^{\#}$ and (thus $\mathcal{N}_{1,torus}$ also) is irreducible. Geometrically this implies the following. Let $f_t(x,y)$ be the family given by fixing generic $a_{1,0},a_{0,2},b_{2,1},t_1,t_2$ and $t_0=t$ in the moduli space $\mathcal{M}_{torus}$. Then the conic $f_{2,t}(x,y)=0$ approches to the node at O when $t\to 0$. Actually one can see by a direct computation that there are two cusps among six cusps on a conic which approach to O so that they produce a (3,4)-cusps on $C_0=\{f_0=0\}$.

(2) Moduli space $\mathcal{N}_{2,torus}$. Now we consider the moduli space $\mathcal{N}_{2,torus}^{\#}$. The curves in this moduli have 2 (3,4)-cusps at A and B (and 2 other cusps) on the conic $f_2(x,y)=0$ and a node at O. By Proposition 19, the conditions at A,B are replaced by $t_1=t_2=0$ in $\mathcal{M}$. Thus it has 4 free parameters $a_{1,0},a_{2,0},b_{2,1},t_0$ where $f_2(O)=-t_0^2$. and the moduli space coincides again to the one which is obtained by substituting $t_1=t_2=0$ in the moduli space $\mathcal{M}_{torus}^{\#}$. Thus we see that $\mathcal{N}_{2,torus}^{\#}$ and $\mathcal{N}_{2,torus}$ are irreducible.

(3) Moduli space $\mathcal{N}_{3,torus}$. Finally the moduli space $\mathcal{N}_{3,torus}^{\#}$ with three (3,4)-cusps are given by $\mathcal{M}\cap\{t_0=t_1=t_2=0\}$. The corresponding polynomials are given by $f=f_2^3+f_3^2$ where $f_2=y^2+y(a_{1,0}-a_{1,0}x)+(-1-a_{0,2})x+a_{0,2}x^2$ and $f_3=b_{2,1}(y^2-x^2)(x-1)$. This is equal to the subspace of $\mathcal{M}_{torus}^{\#}$ given by $\mathcal{M}_{torus}^{\#}\cap\{t_0=t_1=t_2=0\}$.

We have shown in the above argument that $\mathcal{N}_{i,torus}$ is on the boundary of $\mathcal{M}_{torus}$. By the same argument, we can see that $\overline{\mathcal{N}_{i,torus}}\supset\mathcal{N}_{i+1,torus}$ for $i=1,2$. This proves the stratification assertion in Theorem 18. The fact $\mathcal{N}_{4,torus}=\emptyset$ will be proved in 4.2.

3.5. Proof of $(\widehat{\mathcal{M}_{torus}})^*=\widehat{\mathcal{M}_{torus}}$.

A polynomial $f(x,y)$ is called *even* in y if $f(x,y)=f(x,-y)$ for any (x,y). To prove the assertion, it is enough to show that there is a $C_0 \in \mathcal{M}'_{torus}$ such that $C_0^* \in \mathcal{M}'_{torus}$. In fact, assuming this for a moment and taking $C \in \widehat{\mathcal{M}}_{torus}$, we can connect C and C_0 by a piecewise analytic path $C_{\tau(t)}$, $0 \le t \le 1$ such that $C_{\tau(0)} = C_0$, $C_{\tau(1)} = C$ and $C_{\tau(t)} \in \mathcal{M}'_{torus}$ for any $t < 1$. For $0 \le t < 1$, the topology of the complements $\mathbf{C}^2 - C_{\tau(t)}, t < 1$ and $\mathbf{C}^2 - C^*_{\tau(t)}$ is independent of t as they are locally μ-constant family at every singular point. Thus they have the same topology and therefore they have the same Alexander polynomial. In particular, they are torus curves. By Lemma 4, the polynomial $g_t(u,v)$ which defines the dual curves $C^*_{\tau(t)}$ can be assumed to be analytic in t at $t = 1$. Thus this implies that $g_1(u,v)$ is also a torus curve. By the reciprocity law, this implies that the dual of a non-torus sextic in $\widehat{\mathcal{M}}$ is again a non-torus curve. Now we prove the existence of C_0. In fact, we can take any torus curve C defined by an even polynomial $f(x,y) \in \mathcal{M}'_{torus}$. Even curves are given by putting $a_{1,0} = 0$ and $t_2 = t_1$ in the moduli parameters. It is easy to see that the dual curve C^* is also even. Thus it has six cusps which are symmetric with respect to the y-axis and generically these 6 cusps are not on the x-axis. Thus there exists a conic which passes through these 6 points. Now by [D], C^* is a torus curve. Or more directly, we can give C_0 as the following curve. Q.E.D.

Example 20. *For example, we take an even polynomial $f = f_2^3 + f_3^2$ where $f_2(x,y) = y^2 - 1 - 2x + x^2$ and $f_3(x,y) = 1 + y^2(-\frac{5}{2} + x) + 3x - \frac{1}{2}x^2 - x^3$. The dual curve is defined by Lemma 4 by the polynomial $g(x,y) = 484x^6 + 720y^2x^4 + 357y^4x^2 + 59y^6 + 2068x^5 + 962y^2x^3 - 24y^4x - 761x^4 + 11516y^2x^2 - 1486y^4 - 14078x^3 + 14620y^2x - 24661x^2 + 12699y^2 - 21924x - 6728$. Now the torus decomposition is obtained as follows: $g(x,y) = 59g_2(x,y)^3 - \frac{1}{3481}g_3(x,y)^2$ where $g_2(x,y) = y^2 + \frac{241}{59} + \frac{86}{59}x + \frac{122}{59}x^2$ an $g_3(x,y) = -6117 - 7463x - 4639x^2 + 362x^3 + 2773y^2 + 177y^2x$.*

§4. Moduli space of three cuspidal sextics of type (3,4)

In this section , we study the moduli space $\mathcal{N}_3$ of plane curves of degree 6 with 3 $(3,4)$ cusps which are not necessarily of torus type. To study the moduli of sextics with 3 (3,4)-cusps, we may assume hereafter that the cusps are on $O = (0,0)$, $A = (1,1)$ and $B = (1,-1)$.

Lemma 21. *Let $\mathcal{Q}$ be the set of smooth conics which pass through O, A, B and let $\pi : \mathcal{Q} \to \mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$ be the mapping defined by $\pi(Q) = (T_OQ, T_AQ, T_BQ)$, $Q \in \mathcal{Q}$. Here T_PQ is the tangent line of Q at P. Then π is an embedding and the image $\pi(\mathcal{Q})$ is characterized as follows. Let $\alpha, \beta, \gamma \in \mathbf{P}^1$ be the respective tangent directions of Q at O, A*

and B. Then we can write $\beta = (b, 1)$, $\gamma = (c, 1)$ and $\alpha = (a_1, a_2)$ and they satisfy the equality: $(b+c)a - (2-b+c) = 0$ (respectively $b+c = 0$) if $a_2 \neq 0$ with $a := a_1/a_2$ (resp. if $a_2 = 0$). The corresponding conic is defined by $q(x, y) = y^2 + y(c+b)(1-x) + (-2-c+b)x + (1+c-b)x^2$.

Lemma 22. *Assume that $C = \{(x, y) \in \mathbf{C}^2; r(x, y) = 0\}$ be a reduced plane curve of degree 3 which has singularities at O, A, B. Then C is the union of 3 lines $(x-1)(y^2 - x^2) = 0$.*

The proofs of Lemma 21 and Lemma 22 are elementary and omitted.

Lemma 23. *Assume that $C_1 = \{(x, y) \in \mathbf{C}^2; f(x, y) = 0\}$ a germ of a smooth curve at the origin. Let $C_2 = \{(x, y) \in \mathbf{C}; g(x, y) = 0\}$ be another germ of a curve at the origin. Let d be the multiplicity of g at the origin and let $g_d(x, y)$ be the homogeneous part of g of degree d, which defines the tangent cone of C_2. Let p, q be positive integers such that $p < dq$. Consider the germ of a plane curve $C = \{(x, y) \in \mathbf{C}^2; f(x, y)^p - g(x, y)^q = 0\}$. Assume that each irreducible component of $g_d(x, y) = 0$ intersects C_1 transversely at the origin. Then $(C, O) \in \beta_{p,dq}$ and the tangential direction at the origin coincides with that of $f = 0$.*

Proof. Changing local coordinates if necessary, we may assume that $f(x, y) = y$ and $g_d(x, y) = \sum_{i=0}^{d} a_i y^i x^{d-i}$. The assumption implies that $a_0 \neq 0$. Thus $f^p(x, y) = y^p$ and $g^q(x, y) = g_d(x, y)^q + R$ where order$R \geq dq+1$. Thus we can write $f(x, y)^p - g(x, y)^q = y^p - a_0^q x^{dq} + R'(x, y)$ where the order of $R'(x, y)$ with respect to the weight $\mathrm{wt}(x) = p$ and $\mathrm{wt}(y) = dq$ is strictly larger than pqd. Thus the assertion follows. Q.E.D.

Corollary 24. *Let $C = \{(x, y) \in \mathbf{C}^2; f(x, y) = 0\}$ be a reduced sextic with 3 (3,4)-cusps at O, A, B. The following conditions are equivalent.*
(1) $f(x, y)$ is written as $c_1 x^2 (y^2 - x^2)^2 + c_2 q(x, y)^3$ for non-zero $c_1, c_2 \in \mathbf{C}^*$ and the conic $q(x, y) = 0$ is smooth and passes through O, A, B.
(2) There exists a conic $q(x, y) = 0$ which passes throuh O, A, B such that the respective tangent line of the conic is equal to that of C at O, A, B.
(3) C is a torus curve of type (2,3).

Proof. The implication (2) $\Longrightarrow$ (3) follows from Degtyarev [D] or Tokunaga [T]. Q.E.D.

4.1. Moduli space $\mathcal{N}_3$.

Now we compute the moduli space $\mathcal{N}_3^{\#}$ of sextics with 3 (3,4)-cusps at O, A, B. Assume that $C \in \mathcal{N}_3^{\#}$. By Bezout theorem, the tangent cone at O is not $y \pm x = 0$. The stabilizer $H^{\#}$ of $\mathcal{N}_3^{\#}$ in $\mathrm{PSL}(3, \mathbf{C})$ has dimensioin two. Thus under the action of $H^{\#}$, we may assume also that *the tangent cone of* C *at* O *is given by* $x = 0$. So we compute the submoduli $\mathcal{N}_3^{\#\#}$ of $\mathcal{N}_3^{\#}$ whose tangent cone at O is $x = 0$. Let $H^{\#\#}$ be the stabilizer of $\mathcal{N}_3^{\#\#}$. It has dimension one. We start from the expression $f(x,y) = \sum_{i+j\le 6} a_{i,j}y^i x^j$. We can normalize the coefficient $a_{6,0} = 1$ and we have 27 coefficients. The multiplicities of f at O, A, B are 3 by the assumption. Thus at each of these three points, the partial derivatives of order ≤ 2 must vanish. This gives $3 \times 6 = 18$ linear relations and we can eliminate 18 coefficients and we have still 9 coefficients left. For the other computation, we consider the projection $\pi : \mathcal{M} \to \mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$ which is defined by the tangent cone directions at O, A, B. We fix $(\alpha, \beta, \gamma) \in \mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$ and we study the fiber $\pi^{-1}(\alpha, \beta, \gamma)$. First we observe that $\beta, \gamma \ne (1,0)$, i.e., β and γ are transverse to the vertical line $x = 1$ by Bezout theorem. Thus we can put $\beta = (b,1)$, $\gamma = (c,1)$. By the assumption, $\alpha = (1,0)$. Let $h_3(f)(Q)(u,v)$ be the following homogeneous polynomial of degree 3: $\frac{1}{6}\frac{\partial^3 f}{\partial x^3}(Q)u^3 + \frac{1}{2}\frac{\partial^3 f}{\partial x^2 \partial y}(Q)u^2 v + \frac{1}{2}\frac{\partial^3 f}{\partial x \partial y^2}(Q)uv^2 + \frac{1}{6}\frac{\partial^3 f}{\partial y^3}(Q)v^3$.

The condition for O, A, B to be (3,4)-cusps with the above tangent cones is $h_3(f)(A) = c_A(v-bu)^3$, $h_3(f)(B) = c_B(v-cu)^3$ and $h_3(f)(O) = c_O u^3$ for some non-zero constants $c_A, c_B, c_O \in \mathbf{C}^*$. By an easy computation, we have $c_A = 8$, $c_B = -8$. Solving $h_3(f)(A) = 8(v - bx)^3$, $h_3(f)(B) = -8(v - cu)^3$ and $h_3(f)(O) = c_O u^3$, we can eliminate the remaining coefficients so that the moduli space $\mathcal{N}_3^{\#\#}$ is given by

$$\begin{aligned}\mathcal{N}_3^{\#\#} := \pi^{-1}(\{((1,0),(b,1),(c,1)) \in \mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1;\\ (b+c)(b^2 - 3b - bc + 3 + 3c + c^2) = 0\})\end{aligned}$$

The other coefficients are given by

$$\begin{aligned}
a_{5,0} &= 3(b+c),\ a_{5,1} = -3(b+c),\\
a_{4,0} &= -1 + a_{0,6} - 6(b^2+c^2) - 4(b^3-c^3) + 3(b-c),\\
a_{4,1} &= -4 - 2a_{0,6} + 12(b^2+c^2) + 3(c-b) - 8(b^3-c^3),\\
a_{4,2} &= 2 + a_{0,6} - 6(b^2+c^2), +4(b^3-c^3),\\
a_{3,1} &= -12(b+c) + 6(b^2-c^2),\ a_{3,2} = 18(b+c) - 12(b^2-c^2),\\
a_{3,3} &= -6(b+c) + 6(b^2-c^2),\\
a_{2,2} &= 14 - 18(b-c) + 18(b^2+c^2) - 8(b^3-c^3) - 2a_{0,6},\\
a_{2,3} &= -16 + 4a_{0,6} - 36(b^2+c^2) + 30(b-c) + 16(b^3-c^3),\\
a_{2,4} &= 5 + 8(c^3-b^3) + 18(b^2+c^2) + 12(c-b) - 2a_{0,6},\\
a_{1,2} &= 12(b+c) + 12(c^2-b^2) + 4(b^3+c^3),\\
a_{1,3} &= -24(b+c) + 30(c^2-b^2) - 12(b^3+c^3),
\end{aligned}$$

$$\begin{aligned}
a_{1,4} &= 15(b+c) - 24(b^2 - c^2) + 12(b^3 + c^3),\\
a_{1,5} &= -3(b+c) + 6(b^2 - 6c^2) - 4(b^3 + c^3),\\
a_{0,3} &= -8 - 4(c^3 - b^3) - 12(b^2 + c^2) + 12(b - c),\\
a_{0,4} &= 11 + a_{0,6} + 24(b^2 + c^2) + 21(c - b) + 8(c^3 - b^3),\\
a_{0,5} &= -4 - 2a_{0,6} - 12(b^2 + c^2) - 9(c - b) - 4(c^3 - b^3)
\end{aligned}$$

where $a_{0,6}$ is a free parameter. The quotient of the moduli $\mathcal{N}_3^{\#\#}/H^{\#\#}$ has two irreducible components, given by the respective quotients of $\mathcal{N}_{3,1}^{\#\#} := \pi^{-1}(\{b+c=0\})$ and $\mathcal{N}_{3,2}^{\#\#} := \pi^{-1}(\{b^2 - 3b - bc + 3 + 3c + c^2 = 0\})$. Therefore the quotient of moduli space $\mathcal{N}_3/\mathrm{PSL}(3;\mathbf{C})$ has also 2 irreducible components $\mathcal{N}_{3,1}/\mathrm{PSL}(3;\mathbf{C})$ and $\mathcal{N}_{3,2}/\mathrm{PSL}(3;\mathbf{C})$.

Remark 5. *The moduli space $\mathcal{N}_{3,2}^{\#\#}$ consists of two irreducible components $L_\pm := \pi^{-1}(\{(a_{0,6}, b, c); c - (b-3)/2 \pm (b-1)\sqrt{3}I/2\})$. However taking a $\psi \in H^{\#\#}$ such that $\psi(O) = A, \psi(A) = O$ and $\psi(B) = B$, we can easily see that $\psi(L_+) = L_-$, $\psi(L_-) = L_+$ and thus $\mathcal{N}_{3,2}^{\#\#}/H^{\#\#}$ is irreducible.*

Lemma 25. *The component $\mathcal{N}_{3,1}^{\#}$ coincides with the submoduli of sextics of torus type $C \in \mathcal{N}_{3,torus}$ which has 3 (3,4)-cusps at O, A, B. $\mathcal{N}_{3,2}^{\#}$ coincides with $\mathcal{N}_{3,gen}^{\#}$ defined in the section 3.*

Proof. The assertion follows from Lemma 21 and Corollary 24. In fact, for f corresponding to the above parameters and $c = -b$, the torus decomposition is given by $f(x,y) = f_2(x,y)^3 + kf_3(x,y)^2$ where $f_2(x,y) = y^2 + (2b-2)x + (1-2b)x^2$, $f_3(x,y) = (y^2 - x^2)(x-1)$ and $k = 6b - 1 + 8b^3 - 12b^2 + a_{0,6}$.

4.2. Moduli space $\mathcal{N}_4$.

We consider the moduli space of sextics with one (4,5)-cusp at the origin and 3 (2,3)-cusps. First we will show that $\mathcal{N}_{4,torus} = \emptyset$. In fact, assume that there exists a sextic $f(x,y) = f_2(x,y)^3 + f_3(x,y)^2 = 0$ in $\mathcal{N}_4$. It can be easily observed that O must be on the conic $f_2(x,y) = 0$. As the multiplicity of f at O is 4, f_3 has multiplicity at least 2 at the origin and thus f_2 also has multiplicity 2 at O. Thus $f_2(x,y)^3$ has multiplicity 6 at O and O can not be a (4,5)-cusp.

By Bezout theorem, any two of 3 cusps and the origin can not be colinear. Therefore by the action of $\mathrm{PSL}(3,\mathbf{C})$, we can assume that the locus of 3 cusps are either $A = (1,1), B = (1,-1)$ and $C = (1,0)$ if they are colinear or $A = (1,1), C = (1,0)$ and $C' = (0,1)$. The moduli space $\mathcal{N}_4$ seems to be irreducible but we only give examples in this paper.

Example 26. *1. Let* $C_0 = \{f(x,y) = 0\}$ *where*
$f(x,y) := y^6 - 6y^5 + 6y^5x + 16y^4 - 22y^4x + 4y^4x^2 - 32y^3x + 68y^3x^2 - 36y^3x^3 + 24y^2x^2 - 58y^2x^3 + 35y^2x^4 - 8yx^3 + 18yx^4 - 10yx^5 + x^4 - 2x^5 + x^6$.
C_0 *has a (4,5)-cusp singularity at the origin and 3 (2,3)-cusps at* $A = (1,1), B = (1,-1)$ *and* $C = (1,0)$.
2. Let $C_1 \in \mathcal{N}_4$ *be defined by* $f(x,y) = 0$ *where*
$f(x,y) = y^6 + y^4 - 2y^5 - 2x^5 + 6y^5x - 10y^4x - 5y^4x^2 + 4y^3x - 4y^3x^2 + 12y^3x^3 + 6y^2x^2 - 4y^2x^3 - 5y^2x^4 + 4yx^3 - 10yx^4 + 6yx^5 + 4Iy^5x - 4Iy^4x - 8Iy^4x^2 + 12Iy^3x^2 - 12Iy^2x^3 + 8Iy^2x^4 + 4Iyx^4 - 4Iyx^5 + x^6 + x^4$ *where* $I = \sqrt{-1}$. *Then* C_1 *has three cusps at* A, C, C'.

We can check that the dual curve has 6 cusps and 3 nodes in both examples. We assert that

Proposition 27. *For any* C *in the irreducible component of* $\mathcal{N}_4$ *containing* C_1, $\pi_1(\mathbf{P}^2 - C) \cong \mathbf{Z}_6$.

Proof. We show that $\pi_1(\mathbf{P}^2 - C_1 \cup \{x = 0\}) \cong \mathbf{Z}$, using a pencil lines through O where C_1 is in 2 of Example 26. Identifying $\mathbf{P}^2 - \{x = 0\}$ with $\mathbf{C}^2$, the generic pencil line intersect the affine curve $C_1 \cap \mathbf{C}$ 2 at two points and therefore $\pi_1(\mathbf{P}^2 - C_1 \cup \{x = 0\})$ is generated by two generators. Thus it is enought to show the existence of a pencil line which is tangent to C. This can be done by taking $y = 2/7x$ or $y = (-3 + 4i)/5\ x$ respectively. Now the surjectivity $\pi_1(\mathbf{P}^2 - C_1 \cup \{x = 0\} \to \pi_1(\mathbf{P}^2 - C_1)$ proves the commutativity of $\pi_1(\mathbf{P}^2 - C_1)$. Q.E.D.

We thank to Artal Bartolo for the suggestion of this choice of the pencil.

§5. Fundamental group of torus curves

In this section, we prove that

Theorem 28. $\pi_1(\mathbf{P}^2 - C) \cong \mathbf{Z}_2 * \mathbf{Z}_3$ *and* $\pi_1(\mathbf{C}^2 - C) \cong B_3$ *for a generic* $C \in \mathcal{N}_{3,1}$.

Here B_3 is the braid group of three strings. This theorem implies the next stronger assertion.

Theorem 29. $\pi_1(\mathbf{P}^2 - C) \cong \mathbf{Z}_2 * \mathbf{Z}_3$ *and* $\pi_1(\mathbf{C}^2 - C) \cong B_3$ *for any* $C \in \mathcal{M}'_{torus}, \mathcal{N}'_{i,torus}$ *and* $\mathcal{M}_{i,torus}$ *for* $i = 1,2,3$.

Proof. This can be proved by a direct computation. Here is another proof. Take $C \in \mathcal{M}'_{torus}$ for example. Then we can take a family C_t so that $C_0 = C$ and C_t is a 6 cuspidal sextic (without nodes) for $t \neq 0$. We can also find another family D_t such that $D_1 = C$ and

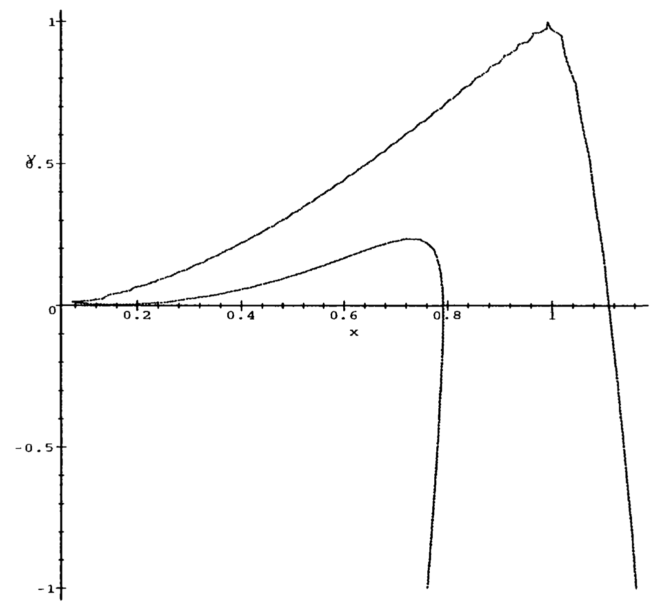

Fig. 1. Graph of $g = 0$

$D_0 \in \mathcal{N}_{3,1}$ and $D_t \in \mathcal{M}'_{torus}$ for $t \neq 0$. By a standard argument, we have surjective homomorphisms $\psi_1 : \pi_1(\mathbf{P}^2 - D_0) \to \pi_1(\mathbf{P}^2 - C)$ and $\psi_2 : \pi_1(\mathbf{P}^2 - C) \to \pi_1(\mathbf{P}^2 - C_1)$ which are isomorphisms on the first homology groups. Thus they induce surjections on the commutator groups. On the other hand, we know that $\pi_1(\mathbf{P}^2 - C_1) \cong \mathbf{Z}_2 * \mathbf{Z}_3$ and the commutator group $D(\mathbf{Z}_2 * \mathbf{Z}_3)$ is a free group $F(2)$ of rank two ([Z],[O1]). Thus we obtain a surjective homomorphism $\psi_2 \circ \psi_1 : F(2) = D(\pi_1(\mathbf{P}^2 - D_0)) \to F(2) = D(\pi_1(\mathbf{P}^2 - C_1))$. This implies that the kernel of $\psi_2 \circ \psi_1$ is trivial by Theorem 2.13, [M-K-S]. Thus ψ_1, ψ_2 are isomorphisms. Q.E.D.

Proof of Theorem 28. For the proof, we take the following sextic curve $C_1 := \{(x,y) \in \mathbf{C}^2; f(x.y) = 0\} \in \mathcal{N}_{3,1}$ where $f(x,y) = f_2(x,y)^3 + \frac{103}{2} f_3(x,y)^2$ and $f_2(x,y) = y^2 + x^2 - 2x$ and $f_3(x,y) = (x-1)(x^2 - y^2)$. Our curve C_1 is even in y. Let us consider a polynomial $g(x,y)$ defined by $g(x,y) := f(x, \sqrt{y})$. Then C_1 is obtained by the double covering branched along $y = 0$ of the curve $g(x,y) = 0$ and the singular fiber for $g(x,y) = 0$ with respect to the pencil $\{x = \eta; \eta \in \mathbf{C}\}$ is defined by the roots of $\Delta_y(g) = -42436x^3(130x - 103)(x-1)^8 = 0$. The graph of the real curve $C(g) := \{g(x,y) = 0\}$ is given in Figure 1. It has two compact components in its real graph. By Lemma 2.2 of [O5] and by the equality $g(x,0) = 1/2 \cdot x^3(-16 + 127x - 218x^2 + 105x^3)$, we get $\Delta_y(f)(x) = cx^9(-16 + 127x - 218x^2 + 105x^3)(130x - 103)^2(x-1)^{16}$, with some constant $c \in \mathbf{C}^*$. Let $p : \mathbf{C}^2 \to \mathbf{C}$ be the first projection and

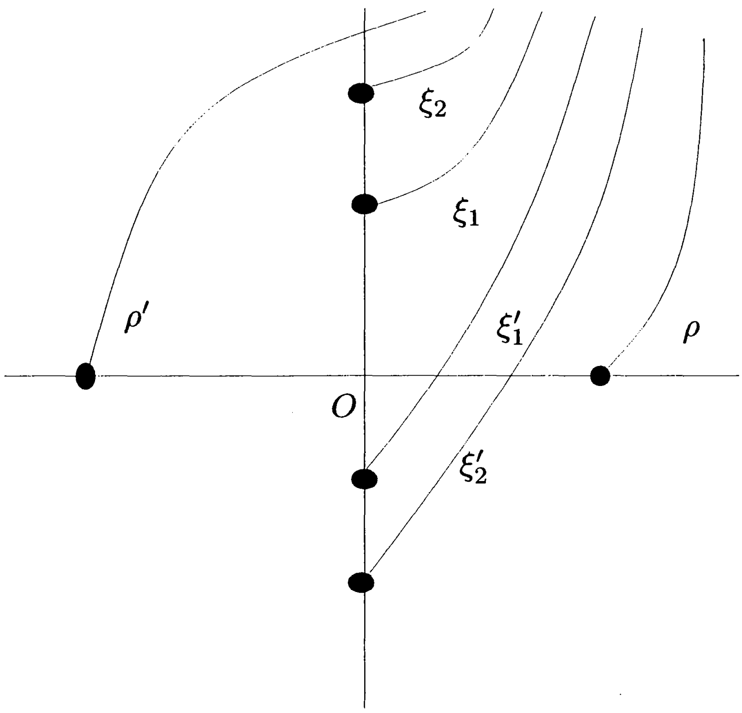

Fig. 2. Generators ($x = \beta_3 - \varepsilon$)

we consider the pencil given by $L_\eta = p^{-1}(\eta)$ as usual. We have chosen f so that the singular pencil lines are all real and given by $\beta_0 < \cdots < \beta_5$ where $\beta_0 = 0$, $\beta_1 = 0.173\cdots$, $\beta_2 = 0.792\cdots$, $\beta_3 = 103/130$, $\beta_4 = 1$, $\beta_5 = 1.110\cdots$. Here $\beta_i, i = 1, 2, 5$ are non-zero roots of $g(x, 0) = 0$ and the corresponding line $x = \beta_i$ is simply tangent to C at $(\beta_i, 0)$ for $i = 1, 2, 5$. Hereafter ε is assumed to be a sufficiently small positive number. We use the notation $\{\sigma, \tau\} := \sigma\tau\sigma\tau^{-1}\sigma^{-1}\tau^{-1}$. Thus $\{\sigma, \tau\} = e$ is equivalent to $\sigma\tau\sigma = \tau\sigma\tau$ where e is the unit. We often use the equivalence: $\{\sigma, \tau\} = e \iff \{\sigma, \sigma\tau\sigma^{-1}\} = e \iff \{\sigma, \sigma^{-1}\tau\sigma\} = e$. We compute the fundamental group $\pi_1(\mathbf{C}^2 - C_1)$ by Zariski's pencil method. We first take generators $\rho, \xi_1, \xi_2, \rho', \xi_1', \xi_2'$ of $\pi_1(L_{\beta_3-\varepsilon} - L_{\beta_3-\varepsilon} \cap C_1)$ as in Figure 2. In the following figures, for simplicity of drawing pictures, we denote a small lasso oriented counterclockwise by a path ending by a bullet ——• as in [O5]. As the monodromy relation at $x = \beta_3$, we get tangent relations $\xi_1 = \xi_2$, $\xi_1' = \xi_2'$. At $x = \beta_2$, we also get a tangent relation $\xi_1 = \xi_1'$. Thus we can put $\xi := \xi_1 = \xi_2 = \xi_1' = \xi_2'$. The generators are reduced to ξ, ρ, ρ'. For further computation, we freely use the relations which have been obtained. Figure 3 shows the situation of our generators at $x = \beta_4 - \varepsilon$. We get the monodromy relations at $x = \beta_4$: $\xi = (\xi^2\rho)(\xi\rho\xi^{-1})(\xi^2\rho)^{-1}$ and $\xi\rho\xi^{-1} = (\xi^2\rho)\xi(\xi^2\rho)^{-1}$ at $(1, 1)$ and $\xi = (\rho'\xi^2)(\xi^{-1}\rho'\xi)(\rho'\xi^2)^{-1}$ and $\xi^{-1}\rho'\xi = (\rho'\xi^2)\xi(\rho'\xi^2)^{-1}$ at $(1, -1)$

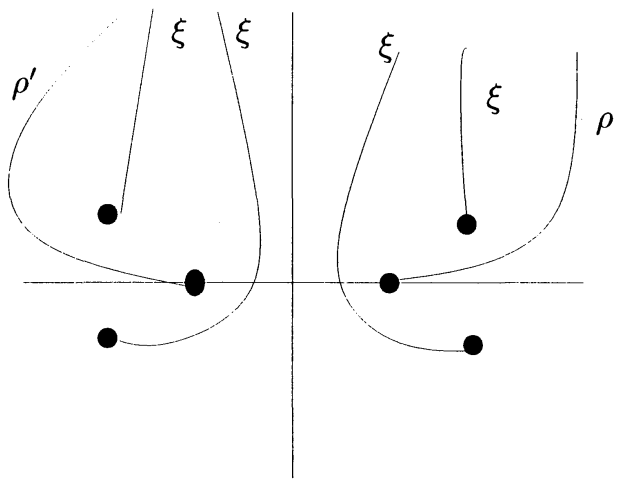

Fig. 3. $x = \beta_4 - \varepsilon$

which reduce to:

$$(30) \qquad \{\xi, \rho\} = e, \qquad \{\xi, \rho'\} = e$$

At $x = \beta_5$ we get a tangent relation: $(\xi^2\rho)\xi(\xi^2\rho)^{-1} = (\rho'\xi^2)^{-1}\xi(\rho'\xi^2)$ which reduces to

$$(31) \qquad \xi\rho\xi^{-1} = \xi^{-1}\rho'\xi$$

Put $\hat{\rho} = \xi\rho\xi^{-1}$. Then we can take $\xi, \hat{\rho}$ as new generators. The relation (30) gives the relation $\{\xi, \hat{\rho}\} = e$. We can see that the monodromy relation at $x = \beta_0$ is derived from the above relations. Thus we have shown that

$$(32) \qquad \pi_1(\mathbf{C}^2 - C_1) = \langle \xi, \hat{\rho}; \xi\hat{\rho}\xi = \hat{\rho}\xi\hat{\rho} \rangle \cong B_3$$

The fundamental group $\pi_1(\mathbf{P}^2 - C_1)$ is obtained by adding the relation $\rho'\xi^4\rho = e$ which is equivalent to $(\xi\hat{\rho}\xi)^2 = e$. Thus this group is isomorphic to $\mathbf{Z}_2 * \mathbf{Z}_3$. See [O3] for the proof.

§6. Non-torus sextic with three (3,4)-cusps

In this section, we will show that the fundamental groups $\pi_1(\mathbf{C}^2 - C)$ and $\pi_1(\mathbf{P}^2 - C)$ are isomorphic to cyclic groups $\mathbf{Z}$ and $\mathbf{Z}_6$ respectively for a generic member C of $\mathcal{N}_{3,2}$. The main difficulty is that, it seems, there does not exist a generic curve in $\mathcal{N}_{3,2}$ which is defined over real numbers for which the singular points are real and the singular fibers are all real. Thus we have to admit some singular points which are not

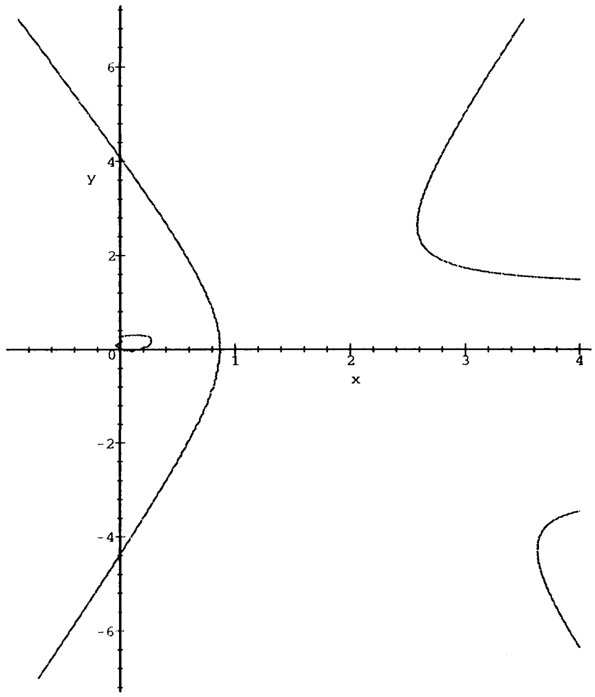

Fig. 4. Graph of C_2

real points or some non-real singular fibers. We take the following curve C_2 defined by

$$\begin{aligned} f(x,y) &= y^6 + y^4(18 - 30x + 9x^2) \\ &+y^3(3\sqrt{3}I - 9\sqrt{3}Ix + 9\sqrt{3}Ix^2 - 3\sqrt{3}Ix^3) \\ &+y^2(9x - 51x^2 + 63x^3 - 18x^4) \\ &+y(-3\sqrt{3}Ix^2 + 9\sqrt{3}Ix^3 - 9\sqrt{3}Ix^4 + 3\sqrt{3}Ix^5) - x^3 + 9x^4 - 9x^5 \end{aligned}$$

where $I = \sqrt{-1}$. We can easily see that $C_2 \in \mathcal{N}_{3,2}$. By the construction, C_2 has three (3,4)-cusps at O, A, B. Now we change the affine coordinates by $(x, y) \mapsto (x, yI)$, to make the defining polynomial to have real coefficients. Thus in the new coordinates, C_2 has three cusps at O, A', B' where $A' = (1, I), B' = (1, -I)$ and the defining polynomial $F(x, y)$ is a real polynomial given by $F(x, y) = f(x, yI)$. The discriminant of $F(x, y)$ in y, $\Delta_y(F)(x)$, which describes the singular fibers is given by $cx^8(9463x^6 + 135838x^5 - 1346423x^4 + 3270132x^3 - 2370951x^2 + 364014x + 22599)(x-1)^{16}$ with some $c \neq 0$. The singular pencil lines are on the real line and correspond to $x = \eta_i$, $i = 1, \dots, 8$, where $\eta_1 < \eta_2 < \eta_3 < \eta_4 < \eta_5 < \eta_6 < \eta_7 < \eta_8$ and $\eta_1 = -21.678\cdots$, $\eta_2 = -0.468\cdots$, $\eta_3 = 0$, $\eta_4 = 0.287\cdots$ $\eta_5 = 0.872\cdots$, $\eta_6 = 1$, $\eta_7 = 2.580\cdots$ and $\eta_8 = 3.629\cdots$. The real graph is given as in Figure 4.

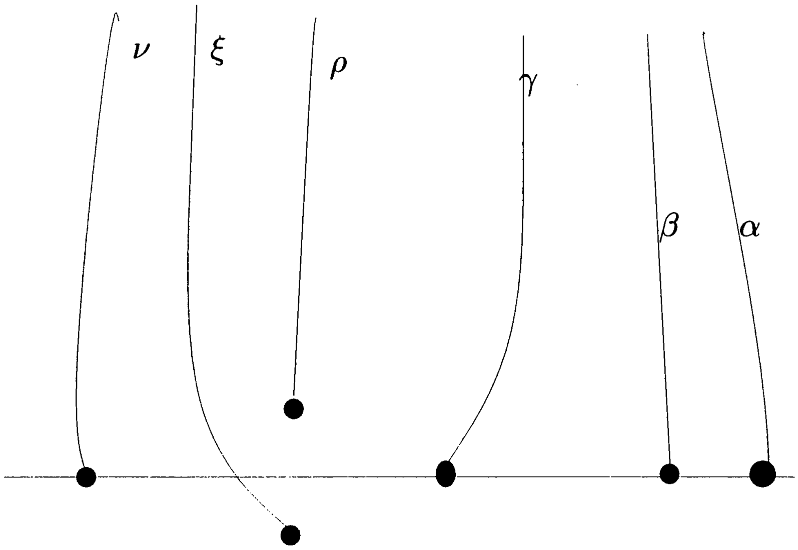

Fig. 5. Generators of $\pi_1(\mathbf{C}^2 - C_2)$

We observe that in the real graph of F, there is a small oval passing through the origin and 4 non-compact components. (One branch is far left outside of the figure.) The singular fibers $x = \eta_1, \eta_2, \eta_4, \eta_5, \eta_7, \eta_8$ are tangent to C_2 in the real graph. The lines $x = \eta_2, \eta_4$ are tangent to the oval. The singular fiber $x = \eta_3$ passes through a cusps at the origin and $x = \eta_6$ passes through two cusps at A', B'. By an easy computation, the principal part of the defining polynomial at three cusps O, A', B' (with respect to the coordinates centered at the singular points) are given by $(\sqrt{3}y - x)^3 + 16x^4 = 0$ at O, $-8(2x + yI)^3 + (54 - 6\sqrt{3}I)x^4 = 0$ at A' and $8(2x - yI)^3 + (54 + 6\sqrt{3}I)x^4 = 0$ at B'. First we take generators $\alpha, \beta, \gamma, \rho, \xi, \nu$ in the fiber $x = \eta_3 + \varepsilon = \varepsilon$ as in Figure 5.

The monodromy relations at $x = \eta_2, \eta_4$ are tangential relations and they are given by
$(R1)$: $\beta = \xi$, $\beta = \gamma$. Eliminating the generators γ, ξ using (R1), the monodromy relation at $x = \eta_3$ is given by $\beta(\beta\rho\beta) = (\beta\rho\beta)\rho$, $\rho(\beta\rho\beta) = (\beta\rho\beta)\xi$ which reduces to the cusp relation: $(R2)$: $\beta\rho\beta = \rho\beta\rho$. To read the monodromy relations at $x = \eta_5$ and η_6, we need to know how the six roots $y_1(x), \cdots, y_6(x)$ of $F(x, y) = 0$ in y move when x moves on the real axis from $x = \eta_4 + \varepsilon \to \eta_5 - \varepsilon$ and then on the circle $|x - \eta_5| = \varepsilon$ clockwise to $x = \eta_5 + \varepsilon$ and then on the real line from $x = \eta_5 + \varepsilon$ to $x = \eta_6 - \varepsilon$. Here we have chosen $y_i(x)$ to be continuous on x so that

1. the imaginary parts $\Im(y_1(x)), \Im(y_2(x))$ are positive and $y_3(x) = \overline{y_1}(x)$ and $y_4(x) = \overline{y_2}(x)$ on $\eta_4 + \varepsilon \le x \le \eta_5 - \varepsilon$ and $\eta_5 + \varepsilon \le x \le \eta_6 - \varepsilon$. We assume that $\Im(y_1(\eta_4 + \varepsilon)) < \Im(y_2(\eta_4 + \varepsilon))$.

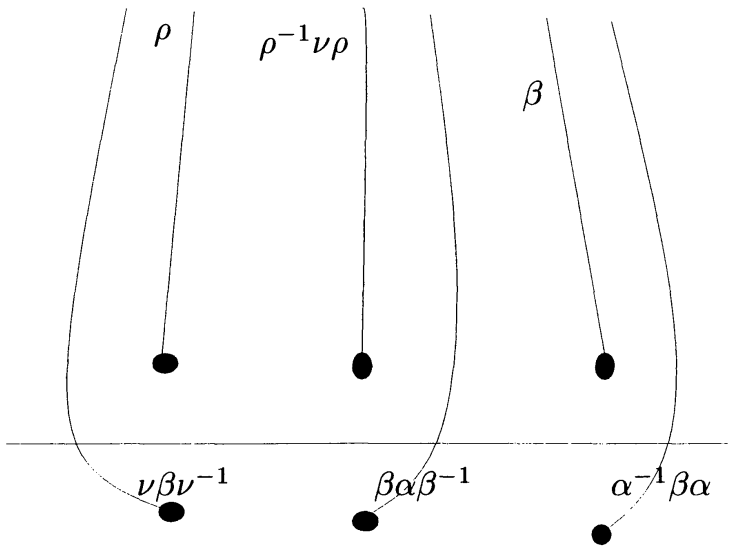

Fig. 6. Generators in $x = \eta_5 + \varepsilon$

2. $y_5(x)$ and $y_6(x)$ are real and $y_5(x) < y_6(x)$ for $\eta_4 + \varepsilon \le x \le \eta_5 - \varepsilon$ and
3. $\Im(y_5(x)) > 0$ and $y_6(x) = \overline{y_5}(x)$ for $\eta_5 + \varepsilon \le x \le \eta_6 - \varepsilon$.

The most delicate part of the argument is the determination of the braid of these six roots $y_j(x), j = 1, \dots, 6$ over $\eta_4 + \varepsilon \le x \le \eta_5 - \varepsilon$ and over $\eta_5 + \varepsilon \le x \le \eta_6 - \varepsilon$. We claim that

Assertion 4. *The ordering by the real part on non-real solutions is preserved on* $\eta_4 + \varepsilon \le x \le \eta_5 - \varepsilon$ *and* $\eta_5 + \varepsilon \le x \le \eta_6 - \varepsilon$. *Namely we have*

$$\Re(y_1(x)) < \Re(y_2(x)), \quad \eta_4 + \varepsilon \le x \le \eta_5 - \varepsilon \tag{33}$$

$$\Re(y_1(x)) < \Re(y_5(x)) < \Re(y_2(x)), \quad \eta_5 + \varepsilon \le x \le \eta_6 - \varepsilon \tag{34}$$

We assume this for a while. Then braids over the intervals $(\eta_4 + \varepsilon, \eta_5 - \varepsilon)$ and $(\eta_5 + \varepsilon, \eta_6 - \varepsilon)$ are uniquely determined. Thus in the fiber of $x = \eta_5 + \varepsilon$, the generators are deformed as in Figure 6. Then the monodromy relation at $x = \eta_5$ is given by
$(R3) :\ \rho^{-1}\nu\rho = \beta\alpha\beta^{-1}$. Now we have to read the monodromy relations at $x = \eta_6 (= 1)$. Thus we start from the fiber $x = \eta_5 + \varepsilon$ as in Figure 6. The local equation of our curve at A', B' are given by the equations $-8(2x + yI)^3 + (54 - 6\sqrt{3}I)x^4$ and $-8(2x - yI)^3 + (54 + 6\sqrt{3}I)x^4$. Thus the topological behaviors of three roots y_1, y_2, y_5 or y_3, y_4, y_6 over the circle $|x - \eta_6| = \varepsilon$ look like satellites going arround the earth ($= \pm 2xI$). The generators are deformed as in Figure 7 on the fiber $x = \eta_6 - \varepsilon$ and the monodromy relations are given by $\theta(\rho\beta\theta) = (\rho\beta\theta)\beta, \quad \beta(\rho\beta\theta) =$

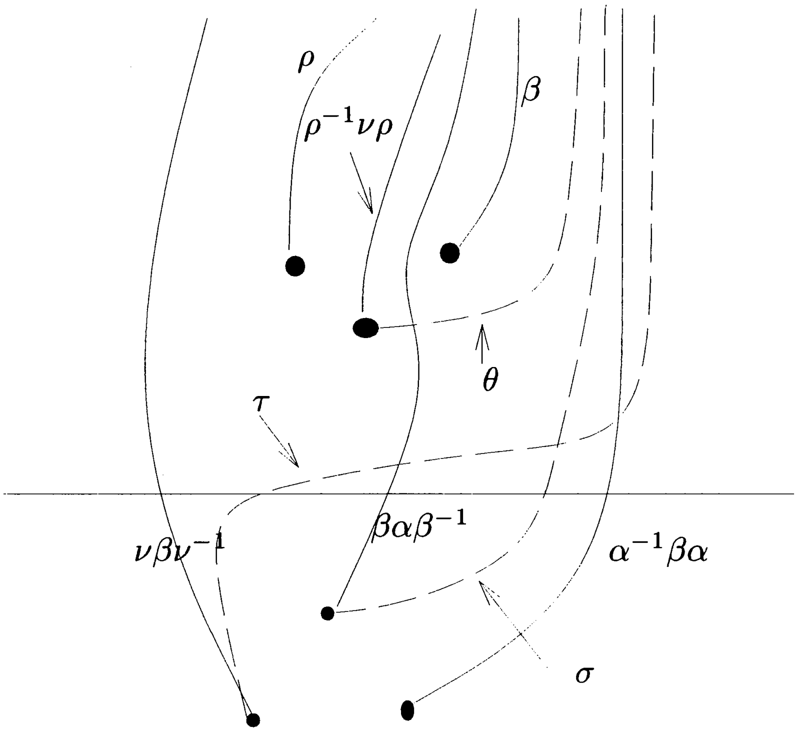

Fig. 7. Generators at $x = \eta_6 - \varepsilon$

$(\rho\beta\theta)\rho$ at A' and $(\alpha^{-1}\beta\alpha)(\tau\sigma\alpha^{-1}\beta\alpha) = (\tau\sigma\alpha^{-1}\beta\alpha)\sigma$, $\sigma(\tau\sigma\alpha^{-1}\beta\alpha) = (\tau\sigma\alpha^{-1}\beta\alpha)\tau$, at B'. As $\theta = \beta^{-1}\rho^{-1}\nu\rho\beta = \alpha$ by (R3), $\sigma = \alpha$ and $\tau = (\nu\rho\beta)^{-1}\nu\beta\nu^{-1}(\nu\rho\beta) = \beta^{-1}\rho^{-1}\beta\rho\beta = \rho$ by (R2), the above relations reduces to:

$$\alpha(\rho\beta\alpha) = (\rho\beta\alpha)\beta, \quad \beta(\rho\beta\alpha) = (\rho\beta\alpha)\rho \tag{35}$$

$$(\alpha^{-1}\beta\alpha)(\rho\beta\alpha) = (\rho\beta\alpha)\alpha, \quad \alpha(\rho\beta\alpha) = (\rho\beta\alpha)\rho \tag{36}$$

The second relation of (35) reduces to $\rho\alpha = \alpha\rho$ by (R2). By the last relation, the first relation of (35) reduces to the braid type relation: $\alpha\beta\alpha = \beta\alpha\beta$. As $\alpha(\rho\beta\alpha) = \rho\alpha\beta\alpha = \rho\beta\alpha\beta$, we get from (36) that $\beta = \rho$. Thus $\beta\alpha = \alpha\beta$ by (35). Combining the last braid relation, we get $\alpha = \beta$. By (R3), we obtain the relation $\nu = \alpha$. Therefore $\pi_1(\mathbf{C}^2 - C)$ is generated by a single generator α and thus $\pi_1(\mathbf{C}^2 - C) \cong \mathbf{Z}$ and therefore $\pi_1(\mathbf{P}^2 - C) \cong \mathbf{Z}_6$. Q.E.D.

Appendix. Outline of the proof of Assertion 4. The following proof is essentially due to Maple. We consider the polynomial $h(x,u,v) := F(x, u+vI)$ for x,u,v real and let $F_e(x,u,v)$ and $F_o(x,u,v)$ be the real and the imaginary part of $h(x,u,v)$ respectively. They are given by

$$\begin{aligned} F_e(x,u,v) &:= v^6 + b_4v^4 + b_2v^2 + b_0, \\ F_o(x,u,v) &:= d_5v^5 + d_3v^3 + d_1v \end{aligned} \tag{37}$$

where the coefficients are polynomials of x, u. We omit their explicit forms.

Suppose that there exists an $x_0 \in (\eta_4+\varepsilon, \eta_5-\varepsilon) \cup (\eta_5+\varepsilon, \eta_6-\varepsilon)$ so that either $\Re(y_1(x_0)) = \Re(y_2(x_0))$ or $\Re(y_2(x_0)) = \Re(y_3(x_0))$. We may assume $\Re(y_1(x_0)) = \Re(y_2(x_0))$ for example and put $u_0 = \Re(y_1(x_0)) \in \mathbf{R}$. This implies that the equation $h(x_0, u_0, v)$ for v has four real solutions $\pm\Im(y_1(x_0))$, $\pm\Im(y_2(x_0))$. Therefore the equation $F_e(x_0, u_0, v) = F_o(x_0, u_0, v) = 0$ has four real solutions. As $\Delta_y(F)(x) = 0$ has no solutions on the intervals $(\eta_4+\varepsilon, \eta_5-\varepsilon)\cup(\eta_5+\varepsilon, \eta_6-\varepsilon)$, v can not be 0. Thus putting $F_o'(x,u,v) = F_o(x,u,v)/v$, $F_e(x_0,u_0,v) = F_o'(x_0,u_0,v) = 0$ has four real solutions $\pm\Im(y_1(x_0)), \pm\Im(y_2(x_0))$. As $F_o'(x_0,u_0,v)$ has degree 4 in v, this implies that $F_o'(x_0,u_0,v)$ divides $F_e(x_0,u_0,v)$. Thus the remainder $R(x,u,v)$ of F_e by F_o' as a polynomial of v must be identically zero for $x = x_0, u = u_0$. Put $R = c_2v^2 + c_0$. c_2 and c_0 are polynomials of x, u. Thus (x_0, u_0) is a common real solution of $c_2 = c_0 = 0$. Let $S(x)$ be the resultant of c_2, c_0 as polynomials of u. We do not give the explicit forms of $c_0(x,u), c_2(x,u), S(x)$ here but $S(x)$ is a polynomial of degree 48 and $(x-1)$ has the multiplicity 27. Note that $S(x_0) = 0$ is a necessary condition to have a real partner u_0 so that $c_2(x_0,u_0) = c_0(x_0,u_0) = 0$ but it is not a sufficient condition as the possible partner u_0 might be not real. Similarly even if we have a real solution $(x_0,u_0) \in \mathbf{R}^2$ of $c_2 = c_0 = 0$, the four roots of $F_o'(x_0,u_0,v) = 0$ might not be real numbers. Anyway Maple gives the unique real solution on the interval $(0,1)$: $x_0 = .29572934753\cdots$. We check the solutions of $F(x_0,y) = 0$. We see that this does satisfy our requirement. Q.E.D.

§7. Application

In our previuos paper, we have constructed a Zariski's triple for plane curves of degree 12 with 27 cusps. In this section, we construct a new example of Zariski's triple $\{F_1, F_2, F_3\}$. They have degree 12 and 12 (3,4)-cusps.

(1) Let F_1 be a torus curve of type (3,4) defined by $f_3(x,y)^4 + f_4(x,y)^3 = 0$ where f_3 and f_4 are generic polynomials of degree 3 and 4 respectively. The Alexander polynomial $\Delta_{F_1}(t)$ is given by $(t^2-t+1)(t^4-t^2+1)$. The fundamental groups are given by

$$\pi_1(\mathbf{C}^2 - F_1) \cong \langle \rho_1, \rho_2, \rho_3; \rho_1(\rho_3\rho_2\rho_1) = (\rho_3\rho_2\rho_1)\rho_2, \\ \rho_2(\rho_3\rho_2\rho_1) = (\rho_3\rho_2\rho_1)\rho_3\rangle$$

and $\pi_1(\mathbf{P}^2 - F_1) \cong \mathbf{Z}_3 * \mathbf{Z}_4$ by [O1].

(2) Let F_2 be a generic cyclic (2,2)-covering $\mathcal{C}_{2,2}(C_1)$ where C_1 is a torus sextic of type (2,3) with three (3,4)-cusps which is, for example, defined by $f(x,y)$ used in the proof of Theorem 28. Then F_2 is defined by $f((x-a)^2+a,(y-b)^2+b)$ for generic a,b. The Alexander polynomial $\Delta_{F_2}(t)$ is given by t^2-t+1 by Theorem 3.4 of [O4]. The fundamental group $\pi_1(\mathbf{C}^2-F_2)$ is isomorphic to the braid group B_3 and $\pi_1(\mathbf{P}^2-F_2)$ is a central extention of $\mathbf{Z}_2 * \mathbf{Z}_3$ by $\mathbf{Z}_2$ (Theorem 3.4 , [O4]).

(3) Let F_3 be a generic cyclic (2,2)-covering of non-torus three (3,4)-cuspidal sextic C_2, constructed in Section 4. The fundamental groups $\pi_1(\mathbf{C}^2-F_3)$ and $\pi_1(\mathbf{P}^2-F_3)$ are isomorphic to cyclic groups $\mathbf{Z}$, $\mathbf{Z}_{12}$ respectively.

Thus there are at least three connected components in the moduli of 12 (3,4)-cuspidal plane curves of degree 12.

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Characteristic classes of coherent sheaves on singular varieties

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Dedicated to Professor Takuo Fukuda on his sixtieth birthday

For a compact singular variety V, there are several definitions of Chern classes, the Mather class, the Schwartz-MacPherson class, the Fulton-Johnson class and so forth ([BrSc], [F], [FJ], [M], [Sc1], see also [Al], [BLSS], [PP] and [Y] for recent developements). They are in the homology of V and, if V is non-singular, they all reduce to the Poincaré dual of the Chern class $c^*(TV)$ of the tangent bundle TV of V. On the other hand, for a coherent sheaf $\mathcal{F}$ on V, the (cohomology) Chern character $\mathrm{ch}^*(\mathcal{F})$ or the Chern class $c^*(\mathcal{F})$ makes sense if either V is non-singular or $\mathcal{F}$ is locally free. In this article, we propose a definition of the homology Chern character $\mathrm{ch}_*(\mathcal{F})$ or the Chern class $c_*(\mathcal{F})$ for a coherent sheaf $\mathcal{F}$ on a possibly singular variety V. In this direction, the homology Chern character or the Chern class is defined in [Sc2] (see also [K]) using the Nash type modification of V relative to the linear space associated to the coherent sheaf $\mathcal{F}$. Also, the homology Todd class $\tau(\mathcal{F})$ is introduced in [BFM] to describe their Riemann-Roch theorem. Our class is closely related to the latter.

The variety V we consider in this article is a local complete intersection defined by a section of a holomorphic vector bundle over the ambient complex manifold M. If $\mathcal{F}$ is a locally free sheaf on V, then the class $\mathrm{ch}_*(\mathcal{F})$ coincides with the image of $\mathrm{ch}^*(\mathcal{F})$ by the Poincaré homomorphism $H_*(V) \to H^*(V)$. This fact follows from the Riemann-Roch theorem for the embedding of V into M, which we prove at the level of Čech-de Rham cocycles. We also compute the Chern character and the Chern class of the tangent sheaf of V when V has only isolated singularities.

In section 1, we discuss characteristic cocycles in the Čech-de Rham complex and define local Chern classes and characters in the Čech-de

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Rham cohomology. We prove a lemma which gives an explicit relation between the cocycle for the product of two symmetric series and the product of cocycles for these series (Lemma 1.5, also Proposition 1.6), which is fundamental in the proof of the Riemann-Roch theorem at the cocycle level. In section 2, we describe the Thom class of the variety V in M as above and, in section 3, we prove the Riemann-Roch theorem mentioned above (Theorem 3.1, Corollaries 3.4 and 3.5). In section 4, we introduce the homology Chern character for a coherent sheaf on V (Definition 4.1). For this definition, we only need that V be a local complete intersection. Finally in section 5, we compute the Chern character and the Chern class of the tangent sheaf of V (Theorem 5.1).

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§1. Local Chern classes and characters in the Čech-de Rham cohomology

As to the theory of characteristic classes, we use the Chern-Weil theory modified to fit in the framework of Čech-de Rham cohomology. For the Chern-Weil theory of characteristic classes of vector bundles, we refer to [BB], [Bo] and [MS]. For the background on the Čech-de Rham cohomology, we refer to [BT]. The integration and characteristic classes in this cohomology theory are first studied in [Le1-4]. See also [Su2] for these material. They are also briefly summarized in the section 1 of [Su3] and we freely use the notation and facts there, except we indicate cohomology Chern classes by superscripts in this article.

(A) Characteristic forms

Let M be a C^∞ manifold of dimension m and let $(T^\vee_{\mathbb{R}}M)^c$ be the complexified cotangent bundle of M. For a C^∞ complex vector bundle E over M, we denote by $A^p(E)$ the vector space of sections of the bundle $\Lambda^p(T^\vee_{\mathbb{R}}M)^c \otimes E$ on M. Recall that a connection ∇ for E is a linear map $A^0(E) \to A^1(E)$ satisfying the Leibniz rule. Let K be the curvature of ∇, which is an element in $A^2(\mathrm{End}(E))$. We set $A = (\sqrt{-1}/2\pi)K$ and define

$$\begin{aligned} c^*(\nabla) &= \det(I + A), \\ \mathrm{ch}^*(\nabla) &= \mathrm{tr}(e^A), \\ \mathrm{td}(\nabla) &= \det\left(\frac{A}{I - e^{-A}}\right). \end{aligned} \tag{1.1}$$

Note that $I - e^{-A}$ is divisible by A and the result is invertible so that

$$\mathrm{td}^{-1}(\nabla) = \det\left(\frac{I - e^{-A}}{A}\right)$$

also makes sense. If we denote by $c^i(\nabla)$ the homogeneous piece in $c^*(\nabla)$ of degree i in the entries of A, it is a closed $2i$-form on M and its class $[c^i(\nabla)]$ in the de Rham cohomology $H^{2i}(M;\mathbb{C})$ is the i-th Chern class $c^i(E)$ of E. The class of $c^*(\nabla)$ in $H^*(M;\mathbb{C})$ is the total (cohomology) Chern class $c^*(E)$ of E. If we set $s^i(\nabla) = \operatorname{tr}(A^i)$, then it is a closed $2i$-form on M. Denoting by r the rank of E, we have

$$c^*(\nabla) = 1 + \sum_{i=1}^{r} c^i(\nabla) \qquad \text{and} \qquad \operatorname{ch}^*(\nabla) = r + \sum_{i\geq 1} \frac{s^i(\nabla)}{i!}.$$

The forms $c^i = c^i(\nabla)$ and $s^i = s^i(\nabla)$ are related by Newton's formula :

$$(1.2) \quad s^i - c^1 s^{i-1} + c^2 s^{i-2} - \cdots + (-1)^i i\, c^i = 0, \qquad i \geq 1.$$

The class of $\operatorname{ch}^*(\nabla)$ in $H^*(M;\mathbb{C})$ is the (cohomology) Chern character $\operatorname{ch}^*(E)$ of E. Each homogeneous piece of $\operatorname{td}(\nabla)$ is also closed and the class of $\operatorname{td}(\nabla)$ in $H^*(M;\mathbb{C})$ is the Todd class $\operatorname{td}(E)$ of E. Note that the constant term in $\operatorname{td}(\nabla)$ is 1 and that $\operatorname{td}(\nabla)$ can be expressed as a series (in fact a polynomial) in $c^i(\nabla)$. We have the following fundamental formula [HL, III, Corollary 5.4] :

$$(1.3) \qquad \sum_{i=0}^{r} (-1)^i \operatorname{ch}^*(\Lambda^i \nabla^\vee) = \operatorname{td}^{-1}(\nabla) \cdot c^r(\nabla),$$

where $\nabla^\vee$ denotes the connection for $E^\vee$ dual to ∇ and $\Lambda^i\nabla^\vee$ the connection for $\Lambda^i E^\vee$ induced by $\nabla^\vee$. Here we set $\Lambda^0 E^\vee = M \times \mathbb{C}$ (the trivial line bundle) and $\Lambda^0 \nabla^\vee = d$. See, e.g., [H, Theorem 10.1.1] for the above formula in cohomology.

Let $\xi = \sum_{i=0}^{q} (-1)^i E_i$ be a virtual bundle and $\nabla^\bullet = (\nabla^{(q)}, \dots, \nabla^{(0)})$ a family of connections, each $\nabla^{(i)}$ being a connection for E_i. We set

$$c^*(\nabla^\bullet) = \prod_{i=0}^{q} c^*(\nabla^{(i)})^{\epsilon(i)} \qquad \text{and} \qquad \operatorname{ch}^*(\nabla^\bullet) = \sum_{i=0}^{q} (-1)^i \operatorname{ch}^*(\nabla^{(i)}),$$

where $\epsilon(i) = (-1)^i$. If we denote by $c^i = c^i(\nabla^\bullet)$ and $s^i/i! = s^i(\nabla^\bullet)/i!$ the homogeneous pieces of degree $2i$ in $c^*(\nabla^\bullet)$ and $\operatorname{ch}^*(\nabla^\bullet)$, respectively, they are again related by (1.2). More generally, if $\varphi = \varphi(c^1, c^2, \dots)$ is a series in c^i (we call such a series a symmetric series), we set $\varphi(\nabla^\bullet) = \varphi(c^1(\nabla^\bullet), c^2(\nabla^\bullet), \dots)$. Then it is a closed form and its class $\varphi(\xi)$ in the cohomology ring $H^*(M;\mathbb{C})$ is the characteristic class of ξ with respect to φ. Suppose further that we have two families of connections

$\nabla^\bullet_\nu = (\nabla^{(q)}_\nu, \dots, \nabla^{(0)}_\nu)$, $\nu = 0, 1$, for ξ. Then, we have a form $\varphi(\nabla^\bullet_0, \nabla^\bullet_1)$ alternating in $(\nabla^\bullet_0, \nabla^\bullet_1)$ such that

$$(1.4) \qquad d\varphi(\nabla^\bullet_0, \nabla^\bullet_1) = \varphi(\nabla^\bullet_1) - \varphi(\nabla^\bullet_0),$$

which shows that the class $\varphi(\xi)$ does not depend on the choice of family of connections. We recall the construction of $\varphi(\nabla^\bullet_0, \nabla^\bullet_1)$ for later use ([Bo, p.65], [Su2, Ch.II, (8.2)]). Thus, for each $i = 0, \dots, q$, we consider the vector bundle $E_i \times \mathbb{R} \to M \times \mathbb{R}$ and let $\tilde\nabla^{(i)}$ be the connection for it given by $\tilde\nabla^{(i)} = (1-t)\nabla^{(i)}_0 + t\nabla^{(i)}_1$. We set $\tilde\nabla^\bullet = (\tilde\nabla^{(q)}, \dots, \tilde\nabla^{(0)})$. Denoting by π_* the integration along the fibers of the projection $\pi : M \times [0,1] \to M$, we define $\varphi(\nabla^\bullet_0, \nabla^\bullet_1) = \pi_*\varphi(\tilde\nabla^\bullet)$. Note that the "higher difference forms" for more than two families of connections are constructed similarly.

Now we prove a lemma which will be used in the next paragraph to describe explicitly the difference between the cocycle for the product of two symmetric series and the product of cocycles for these series. Note that $\varphi\psi(\nabla^\bullet) = \varphi(\nabla^\bullet)\cdot\psi(\nabla^\bullet)$, for symmetric series φ and ψ and a family of connections $\nabla^\bullet$.

Lemma 1.5. *In the above situation, for two symmetric series φ and ψ, we have*

$$\varphi\psi(\nabla^\bullet_0, \nabla^\bullet_1) = \varphi(\nabla^\bullet_0)\cdot\psi(\nabla^\bullet_0, \nabla^\bullet_1) + \varphi(\nabla^\bullet_0, \nabla^\bullet_1)\cdot\psi(\nabla^\bullet_1) - d\tau_{01},$$

where

$$\tau_{01} = \pi_*(\varphi(\pi^*\nabla^\bullet_0, \tilde\nabla^\bullet)\cdot d\psi(\pi^*\nabla^\bullet_1, \tilde\nabla^\bullet)).$$

Proof. By definition, the left hand side is equal to $\pi_*(\varphi(\tilde\nabla^\bullet)\cdot\psi(\tilde\nabla^\bullet))$ and the sum of the first two terms in the right hand side is equal to

$$\pi_*(\varphi(\pi^*\nabla^\bullet_0)\cdot\psi(\tilde\nabla^\bullet) + \varphi(\tilde\nabla^\bullet)\cdot\psi(\pi^*\nabla^\bullet_1)).$$

We have

$$\begin{aligned}\varphi(\tilde\nabla^\bullet)\cdot\psi(\tilde\nabla^\bullet) &- (\varphi(\pi^*\nabla^\bullet_0)\cdot\psi(\tilde\nabla^\bullet) + \varphi(\tilde\nabla^\bullet)\cdot\psi(\pi^*\nabla^\bullet_1)) \\ &= d\varphi(\pi^*\nabla^\bullet_0, \tilde\nabla^\bullet)\cdot d\psi(\pi^*\nabla^\bullet_1, \tilde\nabla^\bullet) - \pi^*(\varphi(\nabla^\bullet_0)\cdot\psi(\nabla^\bullet_1)).\end{aligned}$$

If we denote by i the embedding of the boundary $\{0,1\}$ of $[0,1]$ into $[0,1]$ and by $\partial\pi$ the restriction of π to $\{0,1\}$, the lemma follows from the identities $\pi_* \circ \pi^* = 0$,

$$\pi_* \circ d + d \circ \pi_* = (\partial\pi)_* \circ i^*$$

[Bo, (3.10) Theorem] and

$$(\partial\pi)_* \circ i^*(\varphi(\pi^*\nabla_0^\bullet, \tilde{\nabla}^\bullet) \cdot d\psi(\pi^*\nabla_1^\bullet, \tilde{\nabla}^\bullet))$$
$$= \varphi(\nabla_0^\bullet, \nabla_1^\bullet) \cdot d\psi(\nabla_1^\bullet, \nabla_1^\bullet) - \varphi(\nabla_0^\bullet, \nabla_0^\bullet) \cdot d\psi(\nabla_1^\bullet, \nabla_0^\bullet) = 0.$$

Q.E.D.

(B) Characteristic cocycles in the Čech-de Rham complex

Let M be as above. For an open covering $\mathcal{U}$ of M, we denote by $(A^*(\mathcal{U}), D)$ the Čech-de Rham complex associated to $\mathcal{U}$ [Su2, Ch.II,3]. The complex defines the Čech-de Rham cohomology $H^*(A^*(\mathcal{U}))$, which is canonically isomorphic with the de Rham cohomology $H^*(M;\mathbb{C})$. We recall this cohomology when $\mathcal{U}$ consists of two open sets U_0 and U_1 (the "Mayer-Vietoris situation"). In this case, a cochain σ in $A^p(\mathcal{U})$ is written as

$$\sigma = (\sigma_0,\, \sigma_1,\, \sigma_{01}),$$

where σ_0 and σ_1 are r-forms on U_0 and U_1, respectively, and σ_{01} is an $(r-1)$-form on $U_{01} = U_0 \cap U_1$, and the differential $D : A^p(\mathcal{U}) \to A^{p+1}(\mathcal{U})$ is given by

$$D\sigma = (d\sigma_0,\, d\sigma_1,\, \sigma_1 - \sigma_0 - d\sigma_{01}).$$

The Čech-de Rham cohomology is also equipped with the cup product, which is defined on the cochain level by assigning to σ in $A^p(\mathcal{U})$ and τ in $A^q(\mathcal{U})$ the cochain $\sigma \smile \tau$ in $A^{p+q}(\mathcal{U})$ given by

$$\sigma \smile \tau = (\sigma_0 \cdot \tau_0,\, \sigma_1 \cdot \tau_1,\, (-1)^p \sigma_0 \cdot \tau_{01} + \sigma_{01} \cdot \tau_1),$$

where the product is the exterior product. The cup product is compatible with the usual one in $H^*(M;\mathbb{C})$.

If $\xi = \sum_{i=0}^{q}(-1)^i E_i$ is a virtual bundle, we take a family of connections $\nabla_\nu^\bullet = (\nabla_\nu^{(q)}, \dots, \nabla_\nu^{(0)})$ for ξ on each U_ν, $\nu = 0, 1$, and for the collection $\nabla_\star^\bullet = (\nabla_0^\bullet, \nabla_1^\bullet)$ and a symmetric series φ, we define the cochain $\varphi(\nabla_\star^\bullet)$ in $A^*(\mathcal{U})$ by

$$\varphi(\nabla_\star^\bullet) = (\varphi(\nabla_0^\bullet),\, \varphi(\nabla_1^\bullet),\, \varphi(\nabla_0^\bullet, \nabla_1^\bullet)).$$

Then by (1.4), $\varphi(\nabla_\star^\bullet)$ is a cocycle and defines a class $[\varphi(\nabla_\star^\bullet)]$ in $H^*(A^*(\mathcal{U}))$. It does not depend on the choice of the collection of families of connections $\nabla_\star^\bullet$ and corresponds to the class $\varphi(\xi)$ under the isomorphism $H^*(A^*(\mathcal{U})) \simeq H^*(M;\mathbb{C})$.

From Lemma 1.5, we have the following :

Proposition 1.6. *For two symmetric series φ and ψ, we have, in $A^*(\mathcal{U})$,*

$$\varphi\psi(\nabla_\star^\bullet) = \varphi(\nabla_\star^\bullet) \smile \psi(\nabla_\star^\bullet) + D\tau,$$

where $\tau = (0, 0, \tau_{01})$ *with* τ_{01} *a form on* U_{01} *as given in* Lemma 1.5.

In the sequel, we use the above formula only for a collection $\nabla_\star = (\nabla_0, \nabla_1)$ of connections for a single vector bundle.

(C) Localization

In this paper, we consider the following two types of localizations :
(I) localization of the top Chern class of a vector bundle by a non-vanishing section,
and
(II) localization of the Chern classes of a virtual bundle by exactness.

To describe these, let M be as above and let V be a closed set in M. Letting $U_0 = M \setminus V$ and U_1 a neighborhood of V in M, we consider the covering $\mathcal{U} = \{U_0, U_1\}$ of M. We set

$$A^p(\mathcal{U}, U_0) = \{\, \sigma \in A^p(\mathcal{U}) \mid \sigma_0 = 0 \,\}.$$

Then $A^*(\mathcal{U}, U_0)$ is a subcomplex of $A^*(\mathcal{U})$ and the cohomology it defines is canonically isomorphic with the relative cohomology $H^*(M, M \setminus V; \mathbb{C})$. Note that the cup product of a cochain in $A^*(\mathcal{U})$ and a cochain in $A^*(\mathcal{U}, U_0)$ is in $A^*(\mathcal{U}, U_0)$ and this induces a natural $H^*(M;\mathbb{C})$-module structure on $H^*(M, M \setminus V; \mathbb{C})$.

Remark 1.7. In the situation of Proposition 1.6, if $\psi(\nabla_\star^\bullet)$ is in $A^*(\mathcal{U}, U_0)$, i.e., if $\psi(\nabla_0^\bullet) = 0$, then so is $\varphi\psi(\nabla_\star^\bullet)$, since $\varphi\psi(\nabla_0^\bullet) = \varphi(\nabla_0^\bullet) \cdot \psi(\nabla_0^\bullet)$. The proposition shows that the class $\varphi\psi(\xi)$ coincides with $\varphi(\xi) \smile \psi(\xi)$ in $H^*(M, M \setminus V; \mathbb{C})$, since τ is also in $A^*(\mathcal{U}, U_0)$.

We start with the type (I). Thus let E be a vector bundle of rank r over M and s a non-vanishing section of E on U_0. We say that a connection ∇ for E is s-trivial if $\nabla s = 0$. Recall that, for an s-trivial connection ∇, we have $c^r(\nabla) = 0$ [Su2, Ch.II, Proposition 9.1]. Let ∇_0 be an s-trivial connection for E on U_0 and ∇_1 an arbitrary connection for E on U_1. The top Chern class $c^r(E)$ of E is represented by the cocycle

$$c^r(\nabla_\star) = (c^r(\nabla_0),\, c^r(\nabla_1),\, c^r(\nabla_0, \nabla_1))$$

in $A^{2r}(\mathcal{U})$. Since ∇_0 is s-trivial, we have $c^r(\nabla_0) = 0$ and $c^r(\nabla_\star)$ is in fact in $A^{2r}(\mathcal{U}, U_0)$. Thus it defines a class in $H^{2r}(M, M \setminus V; \mathbb{C})$, which we denote by $c^r(E, s)$. It is sent to the class $c^r(E)$ by the canonical homomorphism

$$j^* : H^{2r}(M, M \setminus V; \mathbb{C}) \to H^{2r}(M; \mathbb{C}).$$

It does not depend on the choice of the s-trivial connection ∇_0 or on the choice of the connection ∇_1. We call $c^r(E, s)$ the localization of $c^r(E)$ with respect to the section s.

For the type (II), let

$$0 \longrightarrow E_q \xrightarrow{h_q} \cdots \xrightarrow{h_1} E_0 \longrightarrow 0 \tag{1.8}$$

be a complex of C^∞ complex vector bundles over M which is exact on U_0. Then we will see below that, for each $i > 0$, there is a canonical localization $c_V^i(\xi)$ in $H^{2i}(M, M \setminus V; \mathbb{C})$ of the Chern class $c^i(\xi)$ in $H^{2i}(M; \mathbb{C})$ of the virtual bundle $\xi = \sum_{i=0}^q (-1)^i E_i$.

Following [BB], we say that a family of connections $\nabla^\bullet = (\nabla^{(q)}, \ldots, \nabla^{(0)})$ for ξ is compatible with the sequence (1.8) if, for each $i = 1, \ldots, q$, the following diagram is commutative :

$$\begin{array}{ccc} A^0(E_i) & \xrightarrow{\nabla^{(i)}} & A^1(E_i) \\ {\scriptstyle h_i}\big\downarrow & & \big\downarrow{\scriptstyle 1\otimes h_i} \\ A^0(E_{i-1}) & \xrightarrow{\nabla^{(i-1)}} & A^1(E_{i-1}). \end{array}$$

Note that for a given exact sequence, there is always a family $\nabla^\bullet$ of connections compatible with the sequence. We have the following "vanishing theorem" [BB, Lemma (4.22)] :

Lemma 1.9. *If $\nabla_0^\bullet$ is a family of connections on U_0 compatible with* (1.8), *then, for each $i > 0$,*

$$c^i(\nabla_0^\bullet) = 0.$$

In fact, the above holds for a finite number of families of connections $\nabla_{0,0}^\bullet, \ldots, \nabla_{p,0}^\bullet$ on U_0 compatible with (1.8), i.e., $c^i(\nabla_{0,0}^\bullet, \ldots, \nabla_{p,0}^\bullet) = 0$. Thus, for a symmetric series φ without constant term, we also have $\varphi(\nabla_{0,0}^\bullet, \ldots, \nabla_{p,0}^\bullet) = 0$.

Let $\nabla_0^\bullet$ be a family of connections compatible with (1.8) on U_0 and $\nabla_1^\bullet$ an arbitrary family of connections for $\xi = \sum_{i=0}^q (-1)^i E_i$ on U_1. Then the class $c^i(\xi)$ is represented by the cocycle

$$c^i(\nabla_\star^\bullet) = (c^i(\nabla_0^\bullet), c^i(\nabla_1^\bullet), c^i(\nabla_0^\bullet, \nabla_1^\bullet))$$

in $A^{2i}(\mathcal{U})$. By Lemma 1.9, we have $c^i(\nabla_0^\bullet) = 0$ and thus the cocycle is in $A^{2i}(\mathcal{U}, U_0)$ and it defines a class $c_V^i(\xi)$ in $H^{2i}(M, M \setminus V; \mathbb{C})$. It is sent to $c^i(\xi)$ by the canonical homomorphism j^*. It is not difficult to see that the class $c_V^i(\xi)$ does not depend on the choice of the family of connections $\nabla_0^\bullet$ compatible with (1.8) or on the choice of the family of connections $\nabla_1^\bullet$.

If φ is a symmetric series without constant term, we may also define the localized class $\varphi_V(\xi)$ of $\varphi(\xi)$. In particular, noting that the alternating sum of the ranks of E_i is zero, if $M \setminus V \neq \emptyset$, we have the localized Chern character $\mathrm{ch}_V^*(\xi)$ in the relative cohomology $H^*(M, M \setminus V; \mathbb{C})$, which is sent to $\mathrm{ch}^*(\xi)$ by the homomorphism j^*. It is the class of the cocycle

$$\mathrm{ch}^*(\nabla_\star^\bullet) = (0,\, \mathrm{ch}^*(\nabla_1^\bullet),\, \mathrm{ch}^*(\nabla_0^\bullet, \nabla_1^\bullet))$$

in $A^*(\mathcal{U}, U_0)$.

Let E be another vector bundle over M and ∇ a connection for E on M. Then its Chern character $\mathrm{ch}^*(E)$ is the class of the cocycle

$$\mathrm{ch}^*(\nabla) = (\mathrm{ch}^*(\nabla),\, \mathrm{ch}^*(\nabla),\, 0)$$

in $A^*(\mathcal{U})$. The complex

$$0 \longrightarrow E \otimes E_q \longrightarrow \cdots \longrightarrow E \otimes E_0 \longrightarrow 0$$

is exact on U_0 and the family $\nabla \otimes \nabla_0^\bullet = (\nabla \otimes \nabla_0^{(q)}, \dots, \nabla \otimes \nabla_0^{(0)})$ of connections is compatible with the above sequence on U_0. We set $E \otimes \xi = \sum_{i=0}^{q} (-1)^i E \otimes E_i$ and let $\nabla \otimes \nabla_1^\bullet$ denote the family $(\nabla \otimes \nabla_1^{(q)}, \dots, \nabla \otimes \nabla_1^{(0)})$. Then $\mathrm{ch}^*(E \otimes \xi)$ is the class of the cocycle

$$\mathrm{ch}^*(\nabla \otimes \nabla_\star^\bullet) = (0,\, \mathrm{ch}^*(\nabla \otimes \nabla_1^\bullet),\, \mathrm{ch}^*(\nabla \otimes \nabla_0^\bullet, \nabla \otimes \nabla_1^\bullet)).$$

We have

$$\begin{aligned}
\mathrm{ch}^*(\nabla \otimes \nabla_1^\bullet) &= \mathrm{ch}^*(\nabla) \cdot \mathrm{ch}^*(\nabla_1^\bullet), \\
\mathrm{ch}^*(\nabla \otimes \nabla_0^\bullet, \nabla \otimes \nabla_1^\bullet) &= \mathrm{ch}^*(\nabla) \cdot \mathrm{ch}^*(\nabla_0^\bullet, \nabla_1^\bullet).
\end{aligned}$$

Hence, recalling the definition of the cup product, we have

$$\mathrm{ch}^*(\nabla \otimes \nabla_\star^\bullet) = \mathrm{ch}^*(\nabla) \smile \mathrm{ch}^*(\nabla_\star^\bullet). \tag{1.10}$$

in $A^*(\mathcal{U}, U_0)$. In particular, we have

$$\mathrm{ch}_V^*(E \otimes \xi) = \mathrm{ch}^*(E) \smile \mathrm{ch}_V^*(\xi).$$

Remark 1.11. The local Chern characters defined as above have all the necessary properties and should coincide with the ones in [I]. Hence they are in the cohomology $H^*(M, M \setminus V; \mathbb{Q})$ with $\mathbb{Q}$ coefficients. Also, the local Chern classes above are in the image of $H^*(M, M \setminus V; \mathbb{Z}) \to H^*(M, M \setminus V; \mathbb{C})$. See also [BFM] for local Chern characters.

Now let M be a complex manifold and denote by $\mathcal{O}_M$ and $\mathcal{A}_M$, respectively, the sheaves of germs of holomorphic functions and of real analytic functions on M. If U is a relatively compact open set in M and if $\mathcal{S}$ is a coherent $\mathcal{O}_U$-module, there is a complex of real analytic vector bundles on U as (1.8) such that at the sheaf level

$$0 \longrightarrow \mathcal{A}_U(E_q) \longrightarrow \cdots \longrightarrow \mathcal{A}_U(E_0) \longrightarrow \mathcal{A}_U \otimes_{\mathcal{O}_U} \mathcal{S} \longrightarrow 0 \tag{1.12}$$

is exact [AH1]. We call such a sequence a resolution of $\mathcal{S}$ by vector bundles. We define the Chern character $\mathrm{ch}^*(\mathcal{S})$ of $\mathcal{S}$ by $\mathrm{ch}^*(\mathcal{S}) = \mathrm{ch}^*(\xi)$, $\xi = \sum_{i=0}^{q}(-1)^i E_i$. Then it does not depend on the choice of the resolution. If we denote by V the support of $\mathcal{S}$, then it is an analytic set in U and on $U \setminus V$, the sequence (1.8) is exact. Thus we have the localized Chern character $\mathrm{ch}_V^*(\mathcal{S})$ in $H^*(U, U \setminus V; \mathbb{C})$. If E is a vector bundle over U, the characteristic classes of $E \otimes \mathcal{S}$ are those of $E \otimes \xi$. Hence, from (1.10), we have

$$\mathrm{ch}_V^*(E \otimes \mathcal{S}) = \mathrm{ch}^*(E) \smile \mathrm{ch}_V^*(\mathcal{S}). \tag{1.13}$$

Note that the above equality also holds if we replace E by a virtual bundle over U.

§2. Thom class

Let M be a complex manifold of dimension $n + k$ and V a compact analytic subvariety (reduced analytic subspace) of pure dimension n in M. We denote by i the embedding $V \hookrightarrow M$. If $V = \bigcup_{\alpha=1}^{\ell} V_\alpha$ is the irreducible decomposition of V, we set $[V] = \sum_{\alpha=1}^{\ell}[V_\alpha]$ in $H_n(V; \mathbb{C})$.

We define the *Thom homomorphism* $T : H^p(V; \mathbb{C}) \to H^{p+2k}(M, M \setminus V; \mathbb{C})$ by $T = A^{-1} \circ P$ so that we have the commutative diagram

$$\begin{array}{ccc}
H^p(V;\mathbb{C}) & \xrightarrow{\ T\ } & H^{p+2k}(M, M\setminus V;\mathbb{C}) \\
\downarrow{\scriptstyle P} & & \wr\downarrow{\scriptstyle A} \\
H_{2n-p}(V;\mathbb{C}) & \xrightarrow{\ =\ } & H_{2n-p}(V;\mathbb{C}),
\end{array}$$

where A and P denote, respectively, the Alexander isomorphism and the Pioncaré homomorphism [Su2, Ch.VI, 4]. Recall that P is given by the cap product with the class $[V]$. For the class [1] in $H^0(V; \mathbb{C})$, we denote $T([1])$ in $H^{2k}(M, M \setminus V; \mathbb{C})$ by Ψ_V, and call it the *Thom class* of V in M.

Remark 2.1. In [Br], these homomorphisms are defined in cohomology with $\mathbb{Z}$ coefficients by a combinatorial method. See [Ab] for a related work.

Let U be a regular neighborhood of V in M with continuous retraction $\rho : U \to V$. We have, by excision, $H^*(M, M \setminus V; \mathbb{C}) \simeq H^*(U, U \setminus V; \mathbb{C})$. Note that for σ in $H^*(U; \mathbb{C})$ and τ in $H^*(U, U \setminus V; \mathbb{C})$, we have

$$A(\sigma \smile \tau) = i^*\sigma \frown A(\tau).$$

Hence the Thom homomorphism T is given, for a class α in $H^p(V; \mathbb{C})$, by

$$T(\alpha) = \rho^*(\alpha) \smile \Psi_V. \tag{2.2}$$

We define the Gysin homomorphism $i_* : H^p(V; \mathbb{C}) \to H^{p+2k}(M; \mathbb{C})$ by $i_* = j^* \circ T$. Note that, if M is compact, we have the commutative diagram

$$\begin{array}{ccccc}
H^p(V;\mathbb{C}) & \xrightarrow{T} & H^{p+2k}(M, M \setminus V; \mathbb{C}) & \xrightarrow{j^*} & H^{p+2k}(M;\mathbb{C}) \\
\downarrow P_V & & \wr\downarrow A & & \wr\downarrow P_M \\
H_{2n-p}(V;\mathbb{C}) & \xrightarrow{=} & H_{2n-p}(V;\mathbb{C}) & \xrightarrow{i_*} & H_{2n-p}(M;\mathbb{C}).
\end{array}$$

In this and the subsequent sections, we consider the following two cases :

(i) V is non-singular,

(ii) V is a local complete intersection defined by a section (see Definition 2.3 below).

First, suppose V is non-singular and let $p : N_V \to V$ be the normal bundle of V in M. In this case, P and T are isomorphisms. We may take as U above a tubular neighborhood so that ρ is C^∞. Then $\rho : U \to V$ is isomorphic with $p : W \to V$ for a neighborhood W of the zero section in N_V, which we identify with V. The bundle $\rho^* N_V$ is also isomorphic with $p^* N_V$. Thus we have an isomorphism

$$H^*(M, M \setminus V; \mathbb{C}) \simeq H^*(N_V, N_V \setminus V; \mathbb{C}).$$

The Thom class Ψ_V of V corresponds to the Thom class Ψ_{N_V} of the bundle N_V under this isomorphism and the Thom homomorphism corresponds to the Thom isomorphism

$$T_{N_V} : H^p(V; \mathbb{C}) \xrightarrow{\sim} H^{p+2k}(N_V, N_V \setminus V; \mathbb{C}).$$

Note that, if we denote by s_Δ the diagonal section of the bundle $p^* N_V$ over N_V, its zero set is V and we have [Su2, Ch.III, Theorem 4.4]

$$\Psi_{N_V} = c^k(p^* N_V, s_\Delta).$$

Second, recall that a subvariety V of codimension k in M is a local complete intersection (abbreviated as LCI) in M if the ideal sheaf $\mathcal{I}_V$ in $\mathcal{O}_M$ of functions vanishing on V is locally generated by k functions. In this case, the normal sheaf $\mathcal{N}_V = \mathcal{H}om_{\mathcal{O}_V}(\mathcal{I}_V/\mathcal{I}_V^2, \mathcal{O}_V)$ is a locally free $\mathcal{O}_V$-module, $\mathcal{O}_V = \mathcal{O}_M/\mathcal{I}_V$. We denote by N_V the associated vector bundle.

Definition 2.3. We say that a subvariety V of codimension k in M is an *LCI defined by a section* if there exist a holomorphic vector bundle N of rank k over M and a holomorphic section s of N such that the local components of s generate $\mathcal{I}_V$.

Thus a subvariety V in M is an LCI defined by a section if and only if there exist a holomorphic vector bundle N over M and a holomorphic section s of N such that (*) s is regular [F, B.3] and the analytic subspace defined by s is reduced and is equal to V. Furthermore, the condition (*) is equivalent to saying that s is generically transverse to the zero section and V is the zero set of s ([T], [Lo, VI.1.6], see also [Su3, Remark 4.10.3]). In this case, we have $N_V = N|_V$. Note that an LCI defined by a section is a "strong" local complete intersection in the sense of [LS]. Note also that for any hypersurface ($k = 1$) V in M, there is a natural line bundle N such that V is an LCI defined by a section of N.

We recall the following theorem, which is proved in [Su2]. See [F, §14.1] for the algebraic case.

Theorem 2.4. *Let V be a compact LCI defined by a section s of a bundle N over M. Then the localization $c^k(N, s)$ in $H^{2k}(M, M \setminus V; \mathbb{C})$ of $c^k(N)$ with respect to s corresponds to $[V]$ under the Alexander duality $H^{2k}(M, M \setminus V; \mathbb{C}) \xrightarrow{\sim} H_{2n}(V; \mathbb{C})$.*

Thus, if V is an LCI defined by a section, Theorem 2.4 shows that

$$\Psi_V = c^k(N, s). \tag{2.5}$$

§3. Riemann-Roch theorem for embeddings

Let V be a compact subvariety in a complex manifold M, which is either of type (i) or (ii) in the previous section. Let U be a regular neighborhood of V in M with a continuous retraction $\rho : U \to V$. In the case (ii), suppose V is defined by a section s of a vector bundle N over M. In the case (i), (M, V) is C^∞ diffeomorphic with (N_V, V) and, in the latter, V is defined by the diagonal section s_Δ of the bundle p^*N_V over N_V. In what follows we write N_V by M anew and set $N = p^*N_V$ and $s = s_\Delta$. Thus in either case we may express the Thom class Ψ_V as

(2.5). In the case (i), we may take as U a tubular neighborhood and we may assume that ρ is the restriction of p to U.

Let $U_0 = M \setminus V$ and U_1 a neighborhood of V as before. Also, let ∇_0 be an s-trivial connection for N on U_0 and ∇_1 an arbitrary connection for N on U_1. We consider the vector bundle $N \times \mathbb{R}$ over $U_{01} \times \mathbb{R}$ and let $\tilde{\nabla}$ be the connection for it given by $\tilde{\nabla} = (1-t)\nabla_0 + t\nabla_1$. Let $\Lambda^\bullet \nabla_\nu^\vee$ denote the family of connections $(\Lambda^k \nabla_\nu^\vee, \dots, \Lambda^0 \nabla_\nu^\vee)$ on U_ν, for $\nu = 0, 1$. Also denote by $\Lambda^\bullet \tilde{\nabla}^\vee$ the family $(\Lambda^k \tilde{\nabla}^\vee, \dots, \Lambda^0 \tilde{\nabla}^\vee)$. Let $\pi : U_{01} \times [0,1] \to U_{01}$ be the projection. Recall that, in $A^*(\mathcal{U})$,

$$\mathrm{ch}^*(\Lambda^\bullet \nabla_\star^\vee) = (\mathrm{ch}^*(\Lambda^\bullet \nabla_0^\vee),\ \mathrm{ch}^*(\Lambda^\bullet \nabla_1^\vee),\ \mathrm{ch}^*(\Lambda^\bullet \nabla_0^\vee, \Lambda^\bullet \nabla_1^\vee))$$

whose class in $H^*(M; \mathbb{C})$ is $\mathrm{ch}^*(\lambda_{N^\vee})$, $\lambda_{N^\vee} = \sum_{i=0}^k (-1)^i \Lambda^i N^\vee$.

Theorem 3.1. *The cocycle* $\mathrm{ch}^*(\Lambda^\bullet \nabla_\star^\vee)$ *is in* $A^*(\mathcal{U}, U_0)$ *and is given by*

$$\mathrm{ch}^*(\Lambda^\bullet \nabla_\star^\vee) = \mathrm{td}^{-1}(\nabla_\star) \smile c^k(\nabla_\star) + D\tau,$$

where $\tau = (0, 0, \tau_{01})$, $\tau_{01} = \pi_*(\mathrm{td}^{-1}(\pi^* \nabla_0, \tilde{\nabla}) \cdot d\, c^k(\pi^* \nabla_1, \tilde{\nabla}))$.

Proof. By (1.3), we have

$$\begin{aligned}
\mathrm{ch}^*(\Lambda^\bullet \nabla_0^\vee) &= \mathrm{td}^{-1}(\nabla_0) \cdot c^k(\nabla_0) = 0, \\
\mathrm{ch}^*(\Lambda^\bullet \nabla_1^\vee) &= \mathrm{td}^{-1}(\nabla_1) \cdot c^k(\nabla_1), \\
\mathrm{ch}^*(\Lambda^\bullet \nabla_0^\vee, \Lambda^\bullet \nabla_1^\vee) &= \pi_* \mathrm{ch}^*(\Lambda^\bullet \tilde{\nabla}^\vee) \\
&= \pi_*(\mathrm{td}^{-1}(\tilde{\nabla}) \cdot c^k(\tilde{\nabla})) = (\mathrm{td}^{-1} \cdot c^k)(\nabla_0, \nabla_1).
\end{aligned}$$

Hence we see that

$$\mathrm{ch}^*(\Lambda^\bullet \nabla_\star^\vee) = (\mathrm{td}^{-1} \cdot c^k)(\nabla_\star)$$

and the theorem follows from Proposition 1.6 (see also Remark 1.7). Q.E.D.

Note that $\tau = 0$ when $k = 1$.

Remark 3.2. Consider the Koszul complex associated to s [F, B.3]:

$$(3.3) \qquad 0 \longrightarrow \Lambda^k N^\vee \longrightarrow \cdots \longrightarrow \Lambda^1 N^\vee \longrightarrow \Lambda^0 N^\vee \longrightarrow 0,$$

which is exact on $U_0 = M \setminus V$. It is not difficult to see that the family $\Lambda^\bullet \nabla_0^\vee$ is compatible with the sequence (3.3) on U_0. The fact that $\mathrm{ch}^*(\Lambda^\bullet \nabla_0^\vee) = 0$ also follows from this (cf. Lemma 1.9).

Let $\mathcal{F}$ be a coherent $\mathcal{O}_V$-module. The direct image $i_!\mathcal{F}$ is a coherent $\mathcal{O}_M$-module, which is simply $\mathcal{F}$ extended by zero on $M \setminus V$, and thus we have the localized Chern character $\mathrm{ch}_V^*(i_!\mathcal{F})$ in $H^*(M, M \setminus V; \mathbb{C})$.

In the case (i), we take a resolution of $\mathcal{F}$ of the form (1.12) on V. Then we have $\mathrm{ch}^*(\mathcal{F}) = \mathrm{ch}^*(\xi)$, $\xi = \sum_{i=0}^q (-1)^i E_i$. Let $\nabla^{(i)}$ be a connection for E_i, $i = 0, \ldots, q$, and denote by $\nabla^{\mathcal{F}}$ the family of connections $(\rho^*\nabla^{(0)}, \ldots, \rho^*\nabla^{(q)})$, for the virtual bundle $\rho^*\xi$ over U.

In the case (ii), we assume that $\mathcal{F}$ is locally free and thus $\mathcal{F} = \mathcal{O}_V(F)$ for some vector bundle F over V. Since the classification of continuous vector bundles and that of C^∞ vector bundles coincide over paracompact manifolds, we may assume that ρ^*F is a C^∞ vector bundle and let $\nabla^{\mathcal{F}}$ be a connection for ρ^*F on U.

In either case, let $\mathrm{ch}^*(\nabla_\star^{\mathcal{F}})$ denote the cocycle

$$\mathrm{ch}^*(\nabla_\star^{\mathcal{F}}) = (\mathrm{ch}^*(\nabla^{\mathcal{F}}), \mathrm{ch}^*(\nabla^{\mathcal{F}}), 0)$$

in $A^*(\mathcal{U})|_U$, whose class in $H^*(U; \mathbb{C})$ is $\rho^* \mathrm{ch}^*(\mathcal{F})$.

Corollary 3.4. *In the above situation, we have*

$$\mathrm{ch}^*(\nabla_\star^{\mathcal{F}}) \smile \mathrm{ch}^*(\Lambda^\bullet \nabla_\star^\vee) = \mathrm{ch}^*(\nabla_\star^{\mathcal{F}}) \smile \mathrm{td}^{-1}(\nabla_\star) \smile c^k(\nabla_\star) + D(\mathrm{ch}^*(\nabla_\star^{\mathcal{F}}) \smile \tau)$$

in $A^*(\mathcal{U}, U_0)|_U$.

Corollary 3.5. *Let V be a compact subvariety in M and $\mathcal{F}$ a coherent $\mathcal{O}_V$-module. We have the following formulas in either one of the cases:*
(i) *V is non-singular,*
(ii) *V is an LCI defined by a section and $\mathcal{F}$ is locally free.*

$$\begin{aligned} \mathrm{ch}_V^*(i_!\mathcal{F}) &= T(\mathrm{ch}^*(\mathcal{F}) \smile \mathrm{td}^{-1}(N_V)) \quad && \text{in } H^*(M, M \setminus V; \mathbb{C}), \\ \mathrm{ch}^*(i_!\mathcal{F}) &= i_*(\mathrm{ch}^*(\mathcal{F}) \smile \mathrm{td}^{-1}(N_V)) \quad && \text{in } H^*(M; \mathbb{C}). \end{aligned}$$

Proof. The Koszul complex (3.3) gives a locally free resolution of $i_!\mathcal{O}_V$:

$$0 \longrightarrow \mathcal{O}_M(\Lambda^k N^\vee) \longrightarrow \cdots \longrightarrow \mathcal{O}_M(\Lambda^0 N^\vee) \longrightarrow i_!\mathcal{O}_V \longrightarrow 0.$$

If we compute the local class $\mathrm{ch}_V^*(i_!\mathcal{O}_V)$ using this resolution, we see that it is represented by $\mathrm{ch}^*(\Lambda^\bullet \nabla_\star^\vee)$. We have, by (1.13),

$$\mathrm{ch}_V^*(i_!\mathcal{F}) = \begin{cases} \mathrm{ch}^*(\rho^*\xi \otimes i_!\mathcal{O}_V) = \mathrm{ch}^*(\rho^*\xi) \smile \mathrm{ch}_V^*(i_!\mathcal{O}_V), & \text{in the case (i)} \\ \mathrm{ch}^*(\rho^*F \otimes i_!\mathcal{O}_V) = \mathrm{ch}^*(\rho^*F) \smile \mathrm{ch}_V^*(i_!\mathcal{O}_V), & \text{in the case (ii).} \end{cases}$$

Recall that either $\mathrm{ch}^*(\rho^*\xi)$ or $\mathrm{ch}^*(\rho^*F)$ is represented by $\mathrm{ch}^*(\nabla_\star^{\mathcal{F}})$. Recalling also that $N|_U \simeq \rho^* N_V$ and $c^k(N,s) = \Psi_V$ (the Thom class), by Corollary 3.4, we get

$$\mathrm{ch}_V^*(i_!\mathcal{F}) = \rho^*(\mathrm{ch}^*(\mathcal{F}) \smile \mathrm{td}^{-1}(N_V)) \smile \Psi_V.$$

By (2.2), we get the first formula. The second follows from the first. Q.E.D.

Remarks 3.6. 1. The equalities in Corollary 3.5 hold in cohomology with $\mathbb{Q}$ coefficients (cf. Remarks 1.11 and 2.1).
2. In the case V is non-singular, the formulas are proved in [AH2]. If, furthermore, V is algebraic, the second formula in Corollary 3.5 is a special case of the Grothendieck-Riemann-Roch theorem [BoSe].
3. In [I], a similar formula is proved for the Thom class of a vector bundle. Namely, let $p : E \to X$ be a complex vector bundle of rank r over a topological space X. Then, in our natation,

$$\mathrm{ch}_X^*(\lambda_{E^\vee}) = p^*\,\mathrm{td}^{-1}(E) \smile \Psi_E,$$

where $\lambda_{E^\vee} = \sum_{i=0}^r (-1)^i \Lambda^i p^* E^\vee$ and Ψ_E denotes the Thom class of E. When X is a C^∞ manifold, this formula can be proved at the level of Čech-de Rham cocycles as above; in the situation of Theorem 3.1, simply let $M = E$, $V = X$ (identified with the zero section of E), $N = p^*E$ and $s = s_\Delta$ and note that $\Psi_E = c^r(p^*E, s_\Delta)$.
4. In the algebraic category, the formulas are proved for a locally free $\mathcal{O}_V$-module on an LCI by analyzing the graph construction in [BFM, 3. Proposition]. Note that their general Riemann-Roch theorem does not directly imply the formulas.
5. These formulas are also proved at the level of differential forms and currents in [HL]. See also [Bi].

§4. Homology Chern characters and classes

Let V be a subvariety of pure codimension k in a complex manifold M. Suppose that V is an LCI. Thus the ideal sheaf $\mathcal{I}_V$ of functions vanishing on V is locally generated by k functions and the normal sheaf $\mathcal{N}_V = \mathcal{H}om_{\mathcal{O}_V}(\mathcal{I}_V/\mathcal{I}_V^2, \mathcal{O}_V)$ is locally free. We denote by N_V the associated vector bundle and let $\tau_V = TM|_V - N_V$ be the virtual tangent bundle of V. Note that it does not depend on the embedding $i : V \hookrightarrow M$.

Definition 4.1. For a coherent $\mathcal{O}_V$-module $\mathcal{F}$, we define the homology Chern character $\mathrm{ch}_*(\mathcal{F})$ by

$$\mathrm{ch}_*(\mathcal{F}) = \mathrm{td}\, N_V \frown A(\mathrm{ch}_V^*(i_!\mathcal{F})).$$

Remarks 4.2. 1. If V is an LCI defined by a section of a vector bundle N over M, we may write

$$\mathrm{ch}_*(\mathcal{F}) = A(\mathrm{td}\, N \smile \mathrm{ch}_V^*(i_!\mathcal{F})).$$

2. The above definition is related to the (homology) Todd class $\tau(\mathcal{F})$ of $\mathcal{F}$ in [BFM] by

$$\mathrm{ch}_*(\mathcal{F}) = (\mathrm{td}^{-1}\, \tau_V) \frown \tau(\mathcal{F}).$$

In [BFM], $\tau(\mathcal{F})$ is defined using an embedding of V, but it is shown that $\tau(\mathcal{F})$ is independent of the embedding for a projective variety V. Thus $\mathrm{ch}_*(\mathcal{F})$ is also independent of the embedding in this case.

The following directly follows from the definition.

Proposition 4.3. (1) *For an exact sequence of coherent $\mathcal{O}_V$-modules*

$$0 \longrightarrow \mathcal{F}_q \longrightarrow \cdots \longrightarrow \mathcal{F}_0 \longrightarrow 0,$$

we have

$$\sum_{i=0}^{q}(-1)^i \,\mathrm{ch}_*(\mathcal{F}_i) = 0.$$

(2) *For a vector bundle E over V and a coherent $\mathcal{O}_V$-module $\mathcal{F}$,*

$$\mathrm{ch}_*(E \otimes \mathcal{F}) = \mathrm{ch}^*(E) \frown \mathrm{ch}_*(\mathcal{F}).$$

The following is a direct consequence of Corollary 3.5.

Proposition 4.4. *Suppose either V is non-singular or V is defined by a section and $\mathcal{F}$ is locally free. Then we have*

$$\mathrm{ch}_*(\mathcal{F}) = \mathrm{ch}^*(\mathcal{F}) \frown [V].$$

In particular, for the structure sheaf $\mathcal{O}_V$,

$$\mathrm{ch}_*(\mathcal{O}_V) = [V].$$

If $\mathrm{ch}_*(\mathcal{F})$ is in the image of the Poincaré homomorphism $H^*(V) \to H_*(V)$, we may define the homology Chern class $c_*(\mathcal{F})$ via Newton's formula. Namely, suppose

$$\mathrm{ch}_*(\mathcal{F}) = \sigma^* \frown [V],$$

for some σ^* in $H^*(V)$ and write $\sigma^* = \sum_{i\geq 0} \sigma^i/i!$ with σ^i in $H^{2i}(V)$. Then we define $\gamma^* = 1 + \sum_{i\geq 1} \gamma^i$ with γ^i in $H^{2i}(V)$ by

$$\sigma^i - \gamma^1\sigma^{i-1} + \gamma^2\sigma^{i-2} - \cdots + (-1)^i i\gamma^i = 0, \qquad i \geq 1.$$

If we define the homology Chern class $c_*(\mathcal{F})$ of $\mathcal{F}$ by

$$c_*(\mathcal{F}) = \gamma^* \frown [V],$$

then it is not difficult to check that the definition does not depend on the choice of σ^*.

Example 4.5. Suppose either V is non-singular or V is defined by a section and $\mathcal{F}$ is locally free. Then, from Proposition 4.4,

$$c_*(\mathcal{F}) = c^*(\mathcal{F}) \frown [V].$$

In particular,

$$c_*(\mathcal{O}_V) = [V].$$

§5. Characteristic classes of the tangent sheaf

Let V be an LCI defined by a section of a vector bundle N over a complex manifold M. Denoting by Ω_M and Ω_V the sheaves of holomorphic 1-forms on M and V, respectively, we have the exact sequence

$$0 \longrightarrow \mathcal{I}_V/\mathcal{I}_V^2 \longrightarrow \Omega_M \otimes_{\mathcal{O}_M} \mathcal{O}_V \longrightarrow \Omega_V \longrightarrow 0.$$

Let $\Theta_M = \mathcal{O}_M(TM)$ be the tangent sheaf of M. We define the tangent sheaf Θ_V of V by $\Theta_V = \mathcal{H}om_{\mathcal{O}_V}(\Omega_V, \mathcal{O}_V)$, which is independent of the embedding $V \hookrightarrow M$. From the above sequence, we have the exact sequence

$$0 \longrightarrow \Theta_V \longrightarrow \Theta_M \otimes_{\mathcal{O}_M} \mathcal{O}_V \longrightarrow \mathcal{N}_V \longrightarrow \mathcal{E}xt^1_{\mathcal{O}_V}(\Omega_V, \mathcal{O}_V) \longrightarrow 0.$$

Setting $\mathcal{E} = \mathcal{E}xt^1_{\mathcal{O}_V}(\Omega_V, \mathcal{O}_V)$, we get, from Propositions 4.3 and 4.4,

$$\mathrm{ch}_*(\Theta_V) = \mathrm{ch}^*(\tau_V) \frown [V] + \mathrm{ch}_*(\mathcal{E}).$$

If p is an isolated singular point of V, by the Riemann-Roch theorem for the embedding $p \hookrightarrow M$, we have $\mathrm{ch}_*(\mathcal{E}) = \tau(V,p)[p]$, where $\tau(V,p) = \dim \mathcal{E}xt^1_{\mathcal{O}_V}(\Omega_V, \mathcal{O}_V)_p$ is the Tjurina number of V at p. Thus we have the following :

Theorem 5.1. *Let V be an LCI of dimension n (≥ 1) defined by a section with isolated singularities $p_1, \dots, p_s$. For the tangent sheaf Θ_V of V, we have*

$$\mathrm{ch}_*(\Theta_V) = \mathrm{ch}^*(\tau_V) \frown [V] + \sum_{i=1}^{s} \tau(V, p_i)\, [p_i],$$

$$c_*(\Theta_V) = c^*(\tau_V) \frown [V] + (-1)^{n+1}(n-1)! \sum_{i=1}^{s} \tau(V, p_i)\, [p_i].$$

Note that the class $c^*(\tau_V) \frown [V]$ coincides with the canonical class of [F, Example 4.2.6], [FJ] in this case.

Let (V, p) be an isolated complete intersection singularity. If it admits a good $\mathbb{C}^*$-action in the sense of [Loo, 9.B], $\tau(V, p) = \mu(V, p)$, the Milnor number of V at p ([G, 3. Satz], [Loo, (9.10) Proposition]). On the other hand, for a variety as in Theorem 5.1, the Schwartz-MacPherson class $c_*(V)$ of V is given by [Su1]

$$c_*(V) = c^*(\tau_V) \frown [V] + (-1)^{n+1} \sum_{i=1}^{s} \mu(V, p_i)\,[p_i].$$

Hence we have

Corollary 5.2. *Let V be as in* Theorem 5.1 *with $n = 1$ or 2. If V admits a good $\mathbb{C}^*$-action near each singular point p_i, then*

$$c_*(\Theta_V) = c_*(V).$$

Remark 5.3. It would be an interesting problem to compare the class $\mathrm{ch}_*(\mathcal{F})$ with the homology Chern character of $\mathcal{F}$ as defined in [Sc2].

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Local types of singularities of plane curves and the topology of their complements

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§ Introduction

Let B be a reduced plane curve in $\mathbf{P}^2 = \mathbf{P}^2_{\mathbf{C}}$. After Zariski's famous article [37], there have been many results on the topology of $\mathbf{P}^2 \setminus B$(see References of [8], for example). The main purpose of this article is to survey some of recent progress on the topology of $\mathbf{P}^2 \setminus B$ with a special emphasis on the case of $\deg B = 6$, including a new example of a Zariski pair. Throughout this article, our fundamental question is the following:

Problem 0.1. What one can say about the topology of $\mathbf{P}^2 \setminus B$ just from the data of local types of singularities of B?

Hereafter we simply say *the configuration of singularities* in the place of *the data of local topological types of singularities.*

As Problem 0.1 seems to be rather vague, we consider more specific problem:

Problem 0.2. Under what condition on the configuration of singularities of B, can one determine the (non-) commutativity of $\pi_1(\mathbf{P}^2 \setminus B)$?

Even Problem 0.2 is still by no means easy. To know how subtle this problem is, let us recall Zariski's famous example:

Example 0.3 (Zariski [37], [38]). Let (B_1, B_2) be a pair of sextic curves with 6 cusps such that

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(i) there exists a conic, C, passing through the 6 cusps for B_1, while
(ii) there exists no such conic as in (i) for B_2.
For these sextic curves, $\pi_1(\mathbf{P}^2 \setminus B_1) \ncong \pi_1(\mathbf{P}^2 \setminus B_2)$.

Remark 0.4. More precisely, $\pi_1(\mathbf{P}^2 \setminus B_1) \cong \mathbf{Z}/2\mathbf{Z} * \mathbf{Z}/3\mathbf{Z}$. For B_2, Oka found an explicit example such that $\pi_1(\mathbf{P}^2 \setminus B_2) \cong \mathbf{Z}/6\mathbf{Z}$ in [22]. It is, however, still unknown whether $\pi_1(\mathbf{P}^2 \setminus B_2) \cong \mathbf{Z}/6\mathbf{Z}$ always holds for any sextic curve of second type.

As Zariski's example shows, in general, just the configuration of singularities is not enough to determine whether $\pi_1(\mathbf{P}^2 \setminus B)$ is abelian or non-abelian. Nevertheless, under some particular conditions, we are able to determine it. Let us begin with the cases when $\pi_1(\mathbf{P}^2 \setminus B)$ is abelian. The first statement is

Theorem 0.5 (Deligne-Fulton [7], [12]**).** *If B has only nodes, then $\pi_1(\mathbf{P}^2 \setminus B)$ is abelian.*

After this statement, Nori generalized it for irreducible plane curves having only nodes and cusps.

Theorem 0.6 (Nori, [15]**).** *Suppose that B is an irreducible curve of degree d and has only nodes and cusps. Let a and b be the numbers of nodes and cusps, respectively. If $2a+6b < d^2$, then $\pi_1(\mathbf{P}^2 \setminus B)$ is abelian.*

Note that Example 0.3 shows that the inequality in Theorem 0.6 is sharp. Shimada recently gave another kind of statement as follows:

Theorem 0.7 (Shimada [28]**).** *Under the same notations and assumption as in Theorem 0.6, if $2a \geq d^2 - 5d + 8$, then $\pi_1(\mathbf{P}^2 \setminus B)$ is abelian.*

All of these statements assure that $\pi_1(\mathbf{P}^2 \setminus B)$ is abelian. Although there are many results on reduced plane curves whose complements have non-abelian fundamental groups (see References of [8], for example), most of them are given by explicit equations; and the defining equations give much more information on curves than just the configuration of singularities does. Our main concern in this article is:

(i) To find some condition on the configuration of singularities which assures that $\pi_1(\mathbf{P}^2 \setminus B)$ is *non-abelian.*

(ii) To look into how good the given condition in (i) is.

To state our result, let us introduce some notation as follows:

(a) For $x \in \mathrm{Sing}(B)$, we denote its Milnor number by μ_x. We define the total Milnor number of B by

$$\mu_B = \sum_{x \in \mathrm{Sing}(B)} \mu_x.$$

(b) Let p be an odd prime. For B, we define a non-negative integer l_p as follows:

If $p = 3$, $l_3 =$ the number of singularities of type A_{3k-1} $(k \geq 1)$ and E_6.

If $p \geq 5$, $l_p =$ the number of singularities of type A_{pk-1} $(k \geq 1)$.

Now we are in position to state our result.

Theorem 0.8. *Let B be a reduced plane curve of even degree with at most simple singularities. Suppose that there exists an odd prime p such that*

$$l_p + \mu_B > d^2 - 3d + 3.$$

Then $\pi_1(\mathbf{P}^2 \setminus B)$ is non-abelian.

A straightforward, but interesting corollary to Theorem 0.8 is:

Corollary 0.9. *Let B be a plane curve of even degree with only nodes and cusps. Let a and b be the number of nodes and cusps, respectively. If $a + 3b > d^2 - 3d + 3$, then $\pi_1(\mathbf{P}^2 \setminus B)$ is non-abelian.*

Note that Corollary 0.9 gives a nice contrast to Theorem 0.6. In fact, the inequality Corollary 0.9 is equivalent to $2a + 6b > 2d^2 - 6d + 6$; and the left hand side is the same as that of the inequality in Theorem 0.6. We give examples of plane curves satisfying the conditions in Theorem 0.8 in §3.

Now our next question is:

Question 0.10. *Is the inequality in Theorem* 0.8 *best possible?*

As we see in §2, our proof for Theorem 0.8 is based on the existence of non-abelian Galois covering branched along B. Hence the inequality does not seem to be sharp. Nevertheless, it is best possible when $d = 6$. In fact, Oka proved the following result in [23].

Theorem 0.11 (Oka [23]). *There exists a pair of irreducible sextic curves (B_1, B_2) satisfying the following conditions*:

(i) *The configuration of singularities of B_1 and B_2 are the same; and they are either $3E_6$ or $3A_1 + 6A_2$.*

(ii) $\pi_1(\mathbf{P}^2 \setminus B_1) \cong \mathbf{Z}/2\mathbf{Z} * \mathbf{Z}/3\mathbf{Z}$, *while* $\pi_1(\mathbf{P}^2 \setminus B_2) \cong \mathbf{Z}/6\mathbf{Z}$.

A pair of plane curves as in Example 0.3 and Theorem 0.9 is called a Zariski pair, precise definition of which is as follows:

Definition 0.12 (cf. [1]**).** A pair of irreducible plane curves of the same degree, (B_1, B_2), is called a Zariski pair if (i) the configuration of singularities of B_1 and B_2 are the same, and (ii) $\mathbf{P}^2 \setminus B_1$ is not homeomorphic to $\mathbf{P}^2 \setminus B_2$.

As we see in §4, there are several examples for Zariski pairs of sextic curves satisfying the equality $l_3 + \mu_B = d^2 - 3d + 3$ (Theorem 4.1). All of these are possible candidates showing that the inequality in Theorem 0.8 is sharp. It might be interesting to determine the fundamental groups of the complements of such curves.

This article consists of five sections. In §1, we give a summary on Galois coverings. In §2, we explain how we prove Theorem 0.8. §3 and §4 are devoted to examples. In §5, we give a method to obtain sextic curves with the desired properties.

Notations and conventions

Throughout this article, the ground field is always the complex number field $\mathbf{C}$. We always understand (unless otherwise explicitly stated) by *variety* (*resp. surface*) a smooth projective variety (resp. surface) defined over $\mathbf{C}$. We denote the rational function field of X by $\mathbf{C}(X)$.

Let X be a normal variety, and let Y be a variety. Let $\pi : X \to Y$ be a finite morphism from X to Y. We define the branch locus of f, which we denote by $\Delta(X/Y)$ or $\Delta(f)$, as follows:

$$\Delta(X/Y) = \{y \in Y | \sharp(\pi^{-1}(y)) < \deg \pi\}.$$

For a divisor D on Y, $\pi^{-1}(D)$ denotes the set-theoretic inverse image of D, while $\pi^*(D)$ denotes the ordinary pullback. Also, $\operatorname{Supp} D$ means the supporting set of D.

Let $\pi : X \to Y$ be a $\mathcal{D}_{2p}$ covering of Y. Morphisms, β_1 and β_2, and the variety $D(X/Y)$ always mean those defined in §1.

Let W be a finite double covering of a surface Σ. The "*canonical resolution*" of W always means the resolution given by Horikawa in [13].

Let S be an elliptic surface over B. We call S *minimal* if the fibration is relatively minimal. In this paper, we always assume that an elliptic surface is minimal. For singular fibers of an elliptic surface, we use the notation of Kodaira [14], and for its configuration, we use the notation as in [25].

Let D_1, D_2 be divisors.

$D_1 \sim D_2$: linear equivalence of divisors.

$D_1 \approx D_2$: algebraic equivalence of divisors.

$D_1 \approx_{\mathbf{Q}} D_2$: $\mathbf{Q}$-algebraic equivalence of divisors.

For simple singularities of a plane curve, we use the same notation as that in [2].

§1. Preliminaries

1. Galois coverings of algebraic varieties

Let Y be a normal projective variety, and let X be a normal variety with a finite morphism $\pi : X \to Y$. Then $\mathbf{C}(X)$ is a finite extension of $\mathbf{C}(Y)$.

Definition 1.1. We call $\pi : X \to Y$ a Galois covering if $\mathbf{C}(X)$ is a Galois extension of $\mathbf{C}(Y)$.

Remark 1.2. Let X' be the $\mathbf{C}(X)$-normalization of Y. Then $X \cong X'$ over Y by the uniqueness for the $\mathbf{C}(X)$-normalization of Y.

The following proposition is fundamental in connecting branched coverings with $\pi_1(Y \setminus B)$. For its proof, see [30].

Proposition 1.3. *Let Y be a variety, X be a normal variety with a finite morphism $\pi : X \to Y$, and let B be the branch locus of π. If $\mathbf{C}(X)$ is a Galois extension of $\mathbf{C}(Y)$ with the Galois group, G, then there exists a surjective homomorphism $\pi_1(Y \setminus B) \to G$.*

Corollary 1.4. *Let Y be a variety, and let B be a reduced divisor on Y. If there exists a Galois covering $\pi : X \to Y$ branched along B with non-abelian Galois group, then $\pi_1(Y \setminus B)$ is non-abelian.*

2. $\mathcal{D}_{2p}$ coverings

Let p be an odd prime. Let $\pi : X \to Y$ be a Galois covering. We call X a $\mathcal{D}_{2p}$ covering if $\mathrm{Gal}(\mathbf{C}(X)/\mathbf{C}(Y))$ is a dihedral group of order $2p$. In this subsection, we give a summary on $\mathcal{D}_{2p}$ coverings. For details, see [29] and [33].

Let $\pi : X \to Y$ be a $\mathcal{D}_{2p}$ covering of a variety Y. Put $\mathcal{D}_{2p} = \langle \sigma, \tau | \sigma^2 = \tau^p = (\sigma\tau)^2 = 1 \rangle$. The invariant subfield, $\mathbf{C}(X)^\tau$, of $\mathbf{C}(X)$ is a quadratic extension of $\mathbf{C}(Y)$. Let $D(X/Y)$ be the $\mathbf{C}(X)^\tau$-normalization of Y. Then $D(X/Y)$ is a double covering of Y satisfying the following commutative diagram:

$$\begin{array}{ccc} X & & \\ & \searrow \beta_2 & \\ \downarrow \pi & & D(X/Y) \\ & \swarrow \beta_1 & \\ Y & & \end{array}$$

where $\beta_1 : D(X/Y) \to Y$ is a double covering of Y and $\beta_2 : X \to D(X/Y)$ is a p-fold cyclic covering of $D(X/Y)$.

With these notation, we have the following result in constructing a $\mathcal{D}_{2p}$ covering of Y.

Proposition 1.5. *Let $f : Z \to Y$ be a smooth finite double covering of a smooth projective variety Y. Let σ be the involution determined by the covering transformation of f. Suppose that there exist three effective divisors D_1, D_2, and D_3 on Z satisfying the following conditions:*

(i) *D_1 is positive. D_1 and $\sigma^* D_1$ have no common component.*

(ii) *If $D_1 = \sum_i a_i D_i^{(1)}$ denotes the decomposition into irreducible components, then $0 < a_i \leq (p-1)/2$ for every i.*

(iii) *$D_1 + pD_2 \sim \sigma^* D_1 + pD_3$.*

Then there exists a $\mathcal{D}_{2p}$ covering, X, of Y such that (i) $Z = D(X/Y)$ *and* (ii) $\Delta(X/Y) = \Delta(Z/Y) \cup f(\mathrm{Supp}(D_1))$.

We modify Proposition 1.5 slightly so that it is rather convenient for our purpose. Let B be as in Theorem 0.8. Let $f' : Z' \to \mathbf{P}^2$ be a double covering with $\Delta(f') = B$. Since B has at most simple singularities, Z' has at most rational double points. Let $\mu : Z \to Z'$ be the canonical resolution of Z' (see [2] III, §7 or [13] §2 for its definition). By the definition, we have the following diagram:

$$\begin{array}{ccc} Z' & \stackrel{\mu}{\leftarrow} & Z \\ f' \downarrow & & \downarrow f \\ \mathbf{P}^2 & \stackrel{q}{\leftarrow} & \Sigma \end{array}$$

where q is a sequence of blowing-ups and f is a double covering branched along the proper transform of B and (possibly empty) some irreducible component of the exceptional divisor of q. We put $\tilde{f} = q \circ f$. Then:

Proposition 1.6. *Let $f : Z \to \Sigma$ be as above, and let σ be the covering transformation. Suppose that there exists a pair of a positive divisor D and a line bundle $\mathcal{L}$ satisfying the condition as follows:*

(i) *If we let $D = \sum_i a_i D_i$ be the irreducible decomposition, then* $\gcd(\{a_i\}, p) = 1$; *and D and $\sigma^* D$ have no common component.*

(ii) *$D - \sigma^* D \sim p\mathcal{L}$.*

Then there exists a p-cyclic covering $g : S \to Z$ such that

(i) $\Delta(g) \subset \mathrm{Supp}(D + \sigma^* D)$ *and*

(ii) *the composition $f \circ g$ gives rise to a $\mathcal{D}_{2p}$ covering of Σ.*

For a proof, see [33] Proposition 1.1.

Corollary 1.7. *With the same notation as in Proposition* 1.6, *if* $\mathrm{Supp}(D + \sigma^* D)$ *is contained in the supporting set of the exceptional* *divisor of μ, then there exists a $\mathcal{D}_{2p}$ covering, S', of $\mathbf{P}^2$ branched along B.*

Proof. Let S' be the Stein factorization of $q \circ f$; and we denote the induced morphism by $\pi : S' \to \mathbf{P}^2$. Since $\mathbf{C}(S') \cong \mathbf{C}(S)$ and $\mathbf{C}(\mathbf{P}^2) \cong \mathbf{C}(\Sigma)$, π is a $\mathcal{D}_{2p}$ covering of $\mathbf{P}^2$. Hence it is enough to show $\Delta(\pi) = B$. By the assumption in the construction of S, the branch locus of $f \circ g$ is contained in the supporting set of the proper transform, $\bar{B}$, of B and the exceptional divisor of μ. As the the image of the exceptional set of q is a subset of $\mathrm{Sing}(B)$, we have our statement. Q.E.D.

§2. A sketch of a proof of Theorem 0.8

We keep the same notation as those in §1. The goal of this section is to show the following theorem.

Theorem 2.1. *Let B be as Theorem 0.8. Suppose that there exists an odd prime p such that*

$$l_p + \mu_B > d^2 - 3d + 3.$$

Then there exists a $\mathcal{D}_{2p}$ covering branched along B.

Note that Theorem 0.8 easily follows from Theorem 2.1 and Corollary 1.4. To prove Theorem 2.1, it is enough to show that the inequality assures the existence of a pair of a divisor and a line bundle, $(D, \mathcal{L})$, on Z satisfying the conditions in Proposition 1.6 and Corollary 1.7. The rest of this section is devoted to it.

Let $\mathrm{NS}(Z)$ be the Néron-Severi group of Z. As $\pi_1(\mathbf{P}^2) = \{1\}$ and B has at most simple singularities, by [3], [4] and [6], $\pi_1(Z) = \{1\}$. Hence $H^2(Z, \mathbf{Z})$ is a unimodular lattice with respect to the intersection pairing. In particular, $\mathrm{NS}(Z) = \mathrm{Pic}(Z)$ and it is a sublattice of $H^2(Z, \mathbf{Z})$. Let T be the subgroup of $\mathrm{NS}(Z)$ generated by the pull-back of a line of $\mathbf{P}^2$ and irreducible components of the exceptional divisor of μ. As we can easily see, T has a direct decomposition

$$T = \mathbf{Z}L \oplus \bigoplus_{x \in \mathrm{Sing}(B)} T_x,$$

where L is the pull-back of a line of $\mathbf{P}^2$ and T_x is the subgroup of $\mathrm{NS}(Z)$ generated by irreducible components of the exceptional divisor arising from the singularity $f'^{-1}(x)$. Note that the direct decomposition as above is orthogonal with respect to the intersection pairing.

Suppose that the effective divisor D as in Corollary 1.7 exists. Then this implies that $\mathrm{NS}(Z)/T$ has a p-torsion. In constructing $\mathcal{D}_{2p}$ coverings, what is important is that the converse of this holds.

Theorem 2.2. *If* $\mathrm{NS}(Z)/T$ *has a p-torsion, then there exists an effective divisor D and a line bundle on* $\mathcal{L}$ *satisfying the conditions in Proposition* 1.6 *and Corollary* 1.7.

We give here a rough explanation. For details, see [33].

Let $T^\sharp = \{D \in \mathrm{NS}(Z) | nD \in T \text{ for some } n \in \mathbf{N}\}$ and let $T^\vee = \mathrm{Hom}_{\mathbf{Z}}(T, \mathbf{Z})$. Then:

(i) $T^{\perp\perp} = T^\sharp$ and $T^\sharp/T \cong (\mathrm{NS}(Z)/T)_{\mathrm{tor}}$. Here for a subgroup, M, of $H^2(Z, \mathbf{Z})$, we denote its orthogonal complement with respect to the intersection pairing by $M^\perp$.

(ii) By using intersection pairing, one can identify $T^\sharp$ with a subgroup of $T^\vee$. Hence $T^\sharp/T \subset T^\vee/T \cong \mathbf{Z}/2\mathbf{Z} \oplus \oplus_{x \in \mathrm{Sing}(B)} T_x^\vee/T_x$. Also, as $T^\vee \otimes \mathbf{Q} = T \otimes \mathbf{Q}$, we can use a $\mathbf{Q}$-divisor in $T_x \otimes \mathbf{Q}$ as a representative for an element in $T^\vee/T$. For example, if the singularity x is of A_n type, then $T^\vee/T \cong \mathbf{Z}/n\mathbf{Z}$ and we can choose a representative of a generater of $T^\vee/T$ as follows:

$$\frac{1}{n+1} D_x,$$

where if n is even,

$$D_x = n(\Theta_1 - \Theta_n) + (n-1)(\Theta_2 - \Theta_{n-1}) + \cdots + \frac{n}{2}(\Theta_{n/2} - \Theta_{n/2+1}),$$

and if n is odd,

$$\begin{aligned} D_x &= n(\Theta_1 - \Theta_n) + (n-1)(\Theta_2 - \Theta_{n-1}) + \cdots \\ &+ \frac{n-1}{2}(\Theta_{(n-1)/2} - \Theta_{(n+3)/2}) + \frac{n+1}{2}\Theta_{(n+1)/2}, \end{aligned}$$

where Θ_i's are irreducible components of the exceptional divisor labeled in such way that $\Theta_i \Theta_{i+1} = 1$ $(1 \le i \le n-1)$. Note that $\sigma^*\Theta_i = \Theta_{n+1-i}$ with respect to the covering transformation of f.

Let $\mathcal{L}'$ be any element of $\mathrm{NS}(Z)$ that gives rise to a p-torsion element, α, in $\mathrm{NS}(Z)/T$. Then we may assume that $\mathcal{L}' \in T^\sharp$; and we have

$$\alpha = \left(\alpha_L, (\alpha_x)_{x \in \mathrm{Sing}(B)}\right), \quad \alpha_L \in \mathbf{Z}/2\mathbf{Z},\ \alpha_x \in T_x^\vee/T_x.$$

Since $p | \sharp(T_x^\vee/T_x)$ if and only if either x is of type A_{pk-1} or x is of type E_6 and $p = 3$, we may assume that $\alpha_x = 0$ for other type of singularities. For x with type A_{pk-1}, by (ii) as above, we may assume that $\alpha_x = i/pD_x \bmod T$ for some $0 < i < p$. By the above explicit formula, we can show $\alpha_x = 1/p(D' - \sigma^* D') \bmod T$, where D' is an effective divisor satisfying the condition (i) in Proposition 2.1. For x with type E_6, the situation is similar (see [30] or [33]). Thus, by replacing

$\mathcal{L}'$ if necessary, we can see there exists an effective divisor D and a line bundle $\mathcal{L}$ on Z satisfying the conditions in Proposition 1.6 and Corollary 1.7.

We now go on to show that the inequality in Theorem 0.8 implies the existence of p-torsion.

Lemma 2.3. *Let $b_i(Z)$ be the i-th Betti number of Z. Then we have*

$$b_2(Z) = d^2 - 3d + 4.$$

Proof. The statement easily follows from Lemma 6, [13] and the Noether formula.

In the following, we make use of some Nikulin theory ([21]). This argument is a modification of Miranda-Persson's in §4, [18]. A similar argument is also found in [35].

Suppose that there exists no p-torsion in $T^{\perp\perp}/T$. Then

$$S_p(T^\vee/T) \cong S_p\left((T^{\perp\perp})^\vee/T^{\perp\perp}\right),$$

where $S_p(G)$ denote the p-Sylow group of G. On the other hand, by Proposition 1.2 in [11], we have

$$(T^\perp)^\vee/T^\perp \cong (T^{\perp\perp})^\vee/T^{\perp\perp}.$$

Hence the number of generators, l_1, of $S_p(G_{T^{\perp\perp}}) \leq \operatorname{rank} T^\perp = b_2(Z) - \operatorname{rank} T = d^2-3d+4-(\mu_B+1)$. On the other hand, by the assumption we have $l_1 \geq l_p > d^2-3d+3-\mu_B$. This leads us to a contradiction. Q.E.D.

§3. Examples

In this section, we give some examples of plane curves satisfying the inequality in Theorem 0.8. Since $\pi_1(\mathbf{P}^2\setminus B)$ is always abelian for conics, we start with the case of $\deg B = 4$.

Example 3.1. $\deg B = 4$. In this case, $d^2 - 3d + 3 = 7$.

(i) Let B be a quartic curve with $3A_2$ singularities. Then $\mu_B = 6$, $l_3 = 3$. Hence the inequality in Theorem 0.8 is satisfied. This implies that $\pi_1(\mathbf{P}^2\setminus B)$ is non-abelian.

As it is well-known, $\pi_1(\mathbf{P}^2\setminus B)$ is a finite non-abelian group of order 12. In fact, $\pi_1(\mathbf{P}^2\setminus B) \cong B_3(\mathbf{P}^1)$ (see [37] or [8])

(ii) Let B be a quartic curve having two irreducible components; one is a cuspidal cubic, C, and the other is a tangent line, l, at an inflection point of C. In this case, the singularities of B are of type A_5 and A_2. Hence $\mu_B = 7$, $l_3 = 2$. Hence the inequality in Theorem 0.8

is satisfied; and $\pi_1(\mathbf{P}^2 \setminus B)$ is non-abelian. In [19], one can find more detailed description.

We now go on to the case of $\deg B = 6$.

Example 3.2. $\deg B = 6$. In this case, $d^2 - 3d + 3 = 21$. There exists a sextic curve B for every case in the following table. In each case, the inequality in Theorem 0.8 is satisfied. Hence $\pi_1(\mathbf{P}^2 \setminus B)$ is non-abelian.

	types of singularities of B	μ_B	l_3
1	$6A_2 + 4A_1$	16	6
2	$3E_6 + A_1$	19	3
3	$2E_6 + 2A_2 + 2A_1$	18	4
4	$E_6 + 4A_2 + 3A_1$	17	5
5	$E_6 + A_5 + 4A_2$	19	6
6	$E_6 + A_{11} + A_2$	19	3
7	$E_6 + A_8 + A_3 + A_2$	19	3
8	$E_6 + A_8 + 2A_2 + A_1$	19	4
9	$E_6 + A_5 + A_4 + 2A_2$	19	3
10	$D_5 + A_8 + 3A_2$	19	4
11	$E_6 + A_5 + A_3 + 2A_2 + A_1$	19	4
12	$E_6 + 2A_5 + A_3$	19	3
13	$D_5 + 2A_5 + 2A_2$	19	4
14	$D_4 + 3A_5$	19	3
15	$D_4 + A_{11} + 2A_2$	19	3
16	$3A_5 + 4A_1$	19	3

What is problem here is the existence of curves as above. We here explain it for No. 1, 2 and 15. For the others, we give a sketch how we show it in §5. Also, for those with $\mu_B = 19$, one can check it in [36]

No. 1: $6A_2 + 4A_1$. One obtains such a sextic curve as a generic plane section of the discriminant variety, $\mathrm{Disc}\big(H^0(\mathbf{P}^1, \mathcal{O}(4)), \mathcal{O}(4)\big)$ (see [10] for details).

No. 2, 16: $3E_6 + A_1$ and $3A_5 + 4A_1$. These two cases are closely related to each other. Let C be a nodal cubic curve and let l_1, l_2, and l_3 be three tangent lines at three inflection points of C (C has exactly three inflection points). A sextic curve for No. 16 is given by $C + l_1 + l_2 + l_3$. Next, consider a Cremona transformation given by these three tangent lines. Then the image of C gives a sextic curve for No. 2.

Remark 3.3. By Proposition 5.6 in [34] and [23], one can see that sextic curves for No. 1 - 10 are irreducible torus curves of type $(2, 3)$ (see [23] for torus curves). It might be interesting to study them systematically as in [23].

Remark 3.4. The author does not know any single example of B with $\deg B \geq 8$ satisfying the inequality in Theorem 0.8. The condition may be too strong for curves of higher degree. In fact, in [26], Sakai proved:

Theorem 3.5 (Sakai). *Let b be the number of cusps. Then*

$$b \leq \frac{5}{16}d^2 - \frac{3}{8}d.$$

Suppose that B has only cusps. Then Sakai's inequality implies that there is no B with $3b > d^2 - 3d + 3$ if $d \geq 29$. Hence our inequality is too strong for curves of higher degree. This is something one can expect, since Theorem 0.8 comes from Theorem 2.1, which gives very rough information on $\pi_1(\mathbf{P}^2 \setminus B)$. Nevertheless, as we see in next section, the inequality in Theorem 0.8 is very nice estimate for sextic curves when $p = 3$.

§4. 4 Some sextic curves with $l_3 + \mu_B = 21$

We look into what happens for sextic curves when the equality $l_3 + \mu_B = 21$ holds. For such cases, as we have already seen Theorem 0.9, we are not able to determine whether $\pi_1(\mathbf{P}^2 \setminus B)$ is abelian or not. In this section, we give other examples of Zariski pairs with equality $l_3 + \mu_B = 21$. More precisely, we give two kinds of sextic curves, B_1 and B_2, such that (i) both of them have the same configuration of singularities, (ii) B_1 is the branch locus for some $\mathcal{D}_6$ covering, while B_2 can never be. This means that $\mathcal{D}_6$ is a homomorphic image of $\pi_1(\mathbf{P}^2 \setminus B_1)$, while there is no homomorphism from $\pi_1(\mathbf{P}^2 \setminus B_2)$ to $\mathcal{D}_6$. Now we give a list for the configurations of singularities.

Theorem 4.1. *For each case in the following table, there exists a pair of irreducible sextic curves (B_1, B_2) with the properties* (i) *and* (ii) *as above.*

	Configuration of singularities of B
1	$E_6 + A_8 + A_2 + 2A_1$
2	$E_6 + A_5 + 2A_2 + 2A_1$
3	$E_6 + 4A_2 + 2A_1$
4	$2E_6 + A_5 + A_1$
5	$2E_6 + 2A_2 + A_1$

Remark 4.2. (i) Note that No 2 is not contained in the examples in [31] and [32]. We show that the example does exist in §5.

(ii) For all cases, one of geometric differences between B_1 and B_2 is the existence of a conic, C, as in Example 0.3. Namely, for B_1 there exists a conic, C, with properties (i) $C \cap B_1 \subset \mathrm{Sing}(B_1)$; and the type of singularities in $C \cap B_1$ are either A_{3k-1} or E_6, and (ii) the intersection multiplicity at A_{3k-1} (resp. E_6) is $2k$ (resp. 4), while there exists no such conic for B_2. In [9], Degtyarev conjectured that there exist exact one rigid isotopy class for a sextic curve having the configuration of singularities No 1, 2 and 4 in Theorem 4.1. Our examples show that his conjecture is false for these cases.

§5. Existence of sextic curves

The main purpose of this section is to explain how one gets sextic curves with the prescribed properties as in §3 and §4. The method that we explain here is the one in [31] and [32].

Let $\varphi : \mathcal{E} \to \mathbf{P}^1$ be an elliptic $K3$ surface with a section s_0, i.e., a Jacobian elliptic $K3$ surface. It is well-known that such surfaces are always obtained in the following way(cf. [17]):

Let $\mathbf{F}_4$ be the Hirzebruch surface of degree 4, i.e., $\mathbf{F}_4 = \mathbf{P}(\mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}(4))$. Let Δ_0 and Δ_∞ be the negative and positive section, respectively. Let T be a reduced divisor on $\mathbf{F}_4$ such that (i) $T \sim 3\Delta_\infty$ and (ii) T has at most simple singularities. As $\Delta_0 + T \sim 3\Delta_\infty \sim 4\Delta_0 + 12f$, where f denotes the class of a fiber $\mathbf{F}_4 \to \mathbf{P}^1$, there exists a double covering, $\mathcal{E}'$, of $\mathbf{F}_4$ branched along $\Delta_0 + T$. Let $\mu : \mathcal{E} \to \mathcal{E}'$ be the canonical resolution, which satisfies the following diagram:

$$\begin{array}{ccc} \mathcal{E}' & \stackrel{\mu}{\leftarrow} & \mathcal{E} \\ \downarrow & & \downarrow \\ \mathbf{F}_4 & \leftarrow & \Sigma, \end{array}$$

where $\Sigma \to \mathbf{F}_4$ is a composition of blowing-ups so that the branch locus of $\mathcal{E} \to \Sigma$ is smooth. Then $\mathcal{E}$ is a $K3$ surface with a Jacobian elliptic fibration induced by the ruling $\mathbf{F}_4 \to \mathbf{P}^1$; and its section s_0 comes from Δ_0.

We can also explain the above construction in another way as follows:

Since $\varphi : \mathcal{E} \to \mathbf{P}^1$ is a Jacobian elliptic fibration, the generic fiber of φ is an elliptic curve, $\mathcal{E}_{\mathbf{C}(\mathbf{P}^1)}$, over $\mathbf{C}(\mathbf{P}^1)$ (s_0 gives a reference point). Considering s_0 as the zero, we can equip $\mathcal{E}_{\mathbf{C}(\mathbf{P}^1)}$ with additive group structure. Let σ denote the inverse morphism with respect to the group law on $\mathcal{E}_{\mathbf{C}(\mathbf{P}^1)}$. It induces a fiber preserving involution on $\mathcal{E}$, which we also denote by σ. Consider the quotient surface $\mathcal{E}/\langle\sigma\rangle$. $\mathcal{E}/\langle\sigma\rangle$ is nothing but Σ in the above diagram, and it is not minimal in general. Blowing down (-1) curves contained in fibers not meeting Δ_0 in an appropriate order, we have $\mathbf{F}_4$. Let $MW(\mathcal{E})$ be the Mordell-Weil group of $\mathcal{E}$, i.e., the group of sections of φ. Now we can easily see:

Lemma 5.1. (i) *$\Delta_0 + T$ is the image of the locus of 2-torsions, $T_2(\mathcal{E})$, with respect to the group law.*

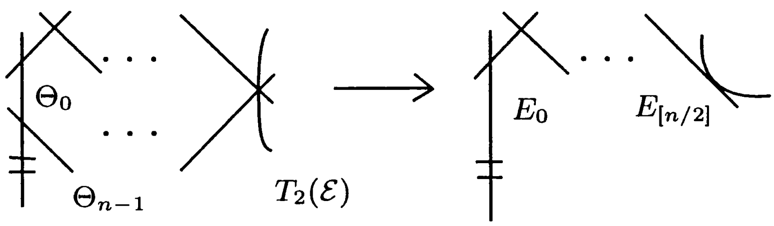
(ii) *T is irreducible if and only if the Mordell-Weil group, $MW(\mathcal{E})$, has no 2-torsion point.*

We now consider when one can blow down $\Sigma(= \mathcal{E}/\langle\sigma\rangle)$ to $\mathbf{P}^2$, not to $\mathbf{F}_4$, in such a way that the image of $T_2(\mathcal{E})$ is a sextic curve. There are several ways to do it ([24]), and we here explain one of them.

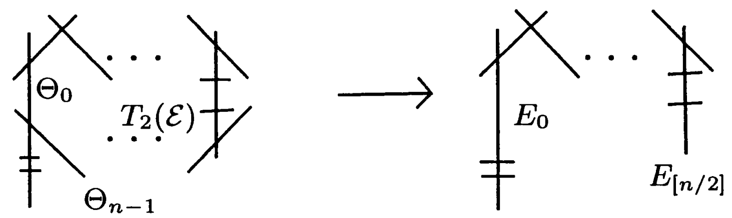
Lemma 5.2. *If $\varphi : \mathcal{E} \to \mathbf{P}^1$ has a singular fiber of type I_n $(n \geq 6)$, then one can blow down Σ to $\mathbf{P}^2$; and the image of $T_2(\mathcal{E})$ is a sextic curve with an E_6 singularity.*

Proof. The action of σ on an I_n fiber is as follows (cf. [5], [20]): Label irreducible components of an I_n fiber in such a way that

$$\Theta_0\Theta_1 = \cdots = \Theta_{n-1}\Theta_0 = 1, \quad \Theta_0 s_0 = 1.$$



n: odd



n: even

(Figure 1)

Then $\sigma^*\Theta_i = \Theta_{n-i}$ and $\sigma^*\Theta_0 = \Theta_0$. Hence the image of an I_n fiber in Σ is a tree of $([n/2] + 1)$ $\mathbf{P}^1$'s, E_i $(i = 0, ..., [n/2])$, such that

$$E_iE_{i+1} = 1, \quad \left(0 \leq i \leq \left[\frac{n}{2}\right] - 1\right), \quad E_0\bar{s}_0 = 1,$$

where $\bar{s}_0$ is the image of s_0, and

$$E_0^2 = E_{[n/2]}^2 = -1, \quad E_i^2 = -2, \quad 1 \leq i \leq \left[\frac{n}{2}\right] - 1.$$

In blowing down Σ to $\mathbf{F}_4$, we first blow down $E_{[n/2]}$, then $E_{[n/2]-1}$, $E_{[n/2]-1}$ and so on. In order to blow down Σ to $\mathbf{P}^2$, we do it in a different way. Namely, we first blow down E_0, then E_1 and E_2 in this order. Then $\bar{s}_0$ becomes a (-1) curve; and one can blow down it to a point, x. Then we blow down $E_{[n/2]}$, $E_{[n/2]-1}, \ldots$, E_4 in this order. Blowing down (-1) curves in the other fibers in the same way as we do in blowing down Σ to $\mathbf{F}_4$, we have $\mathbf{P}^2$. Since (i) the image of $T_2(\mathcal{E})$ has an E_6 singularity at x, (ii) the image of a general fiber for elliptic fibration is a line through x, we infer that the image of $T_2(\mathcal{E})$ is a sextic curve, $B_{\mathcal{E}}$, with an E_6 singularity. Q.E.D.

Remark 5.3. In a similar manner, one can also blow down Σ to $\mathbf{P}^2$ if φ has $3I_2$ (resp. I_4 and I_2) singular fibers. In this case, the corresponding triple point is D_4 (resp. D_5).

Corollary 5.4. *$B_{\mathcal{E}}$ is irreducible if and only if $MW(\mathcal{E})$ has no 2-torsion point.*

Definition 5.5. We call the singular fibers as in Lemma 5.2 and Corollary 5.3 the *preferred fibers.*

As one can easily see from its construction $B_{\mathcal{E}}$, the type of a singularity of $B_{\mathcal{E}}$ other than E_6, D_4 and D_5 as in Lemma 5.2 and Remark 5.3 has something to do with that of the corresponding singular fiber of φ. We give a table for its correspondence (cf. [17]):

Lemma 5.6. *The relation between the type of a non-preferred singular fiber of φ and that of the corresponding singularity of $B_{\mathcal{E}}$ is as follows;*

Type of a singular fiber	I_n $(n \geq 2)$	I_1	I_n^*
Type of a singular point	A_{n-1}	*a smooth point*	D_{n+4}

II	II^*	III	III^*	IV	IV^*
a smooth point	E_8	A_1	E_7	A_2	E_6

With the argument so far, the existence of the sextic curves as in Example 3-15 is reduce to that of Jacobian elliptic $K3$ surfaces with the prescribed configuration of singular fibers. Here we give a table for that.

Lemma 5.7. *A sextic curve with singularities as in one of the left column exists if a Jacobian elliptic $K3$ surface with the configuration of singular fibers in the same row of the right column exists.*

3	$2E_6 + 2A_2 + 2A_1$	I_6, IV^*, $2I_3$, $2I_2$
4	$E_6 + 4A_2 + 3A_1$	I_6, $4I_3$, $3I_2$
5	$E_6 + A_5 + 4A_2$	$2I_6$, $4I_3$
6	$E_6 + A_{11} + A_2$	I_6, I_{12}, I_3, $3I_1$
7	$E_6 + A_8 + A_3 + A_2$	I_6, I_9, I_4, I_3, $2I_1$
8	$E_6 + A_8 + 2A_2 + A_1$	I_6, I_9, $2I_3$, I_2, I_1
9	$E_6 + A_5 + A_4 + 2A_2$	$2I_6$, I_5, $2I_3$, I_1
10	$D_5 + A_8 + 3A_2$	I_4, I_2, I_9, $3I_3$
11	$E_6 + A_5 + A_3 + 2A_2 + A_1$	$2I_6$, I_4, $2I_3$, I_2
12	$E_6 + 2A_5 + A_3$	$3I_6$, I_4, $2I_1$
13	$D_5 + 2A_5 + 2A_2$	I_2, I_4, $2I_6$, $2I_3$
14	$D_4 + 3A_5$	$3I_2$, $3I_6$
15	$D_4 + A_{11} + 2A_2$	$3I_2$, I_{12}, $2I_3$

For No. 4 - 15, such elliptic K3 surfaces exist by [18]. For No. 3, one obtains it in the same way as in Lemma 4.2, [32]. Hence, by Lemma 5.7, there exist sextic curves for No. 3 - 15 in Example 3.2.

Now we go on to Theorem 4.1. An easy but key lemma to obtain a pair of sextic curves having the same configuration of singularities is as follows:

Lemma 5.8. *Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be Jacobian elliptic $K3$ surfaces such that*

(i) *the configurations of non semi-stable singular fibers of $\mathcal{E}_1$ and $\mathcal{E}_2$ are the same, and*

(ii) *the configurations of semi-stable singular fibers of $\mathcal{E}_1$ is I_6, $I_{n_1},\dots,$ I_{n_s} $(n_i \geq 2)$, $r\,I_1$, while that of $\mathcal{E}_2$ is I_7, $I_{n_1},\dots,I_{n_s}$ $(n_i \geq 2)$, $(r-1)\,I_1$.*

Then the configuration of singularities of $B_{\mathcal{E}_1}$ is the same as that of $B_{\mathcal{E}_2}$.

Proof. From Lemmas 5.2 and 5.6, the statement follows. Q.E.D.

Corollary 5.9. *Let $\varphi_1 : \mathcal{E}_1 \to \mathbf{P}^1$ and $\varphi_2 : \mathcal{E}_2 \to \mathbf{P}^1$ be elliptic $K3$ surfaces having the configurations of singular fibers as in the table below. Then the configurations of singularities of $B_{\mathcal{E}_1}$ and $B_{\mathcal{E}_2}$ are the right column in the table.*

	Singular fibers of $\mathcal{E}_1$	*Singular fibers of $\mathcal{E}_2$*	*Singularities of $B_{\mathcal{E}_i}$ $(i = 1, 2)$*
1	I_6, I_9, I_3, $2I_2$, $2I_1$	I_7, I_9, I_3, $2I_2$, I_1	$E_6 + A_8 + A_2 + 2A_1$
2	$2I_6$, $2I_3$, $2I_2$, $2I_1$	I_7, I_6, $2I_3$, $2I_2$, I_1	$E_6 + A_5 + 2A_2 + 2A_1$
3	I_6, $4I_3$, $2I_2$, $2I_1$	I_7, $4I_3$, $2I_2$, I_1	$E_6 + 4A_2 + 2A_1$
4	$2I_6$, IV^*, I_2, $2I_1$	I_7, IV^*, I_2, I_1	$2E_6 + A_5 + A_1$
5	I_6, IV^*, $2I_3$, I_2, $2I_1$	I_7, IV^*, $2I_3$, I_2, I_1	$2E_6 + 2A_2 + A_1$

In order to prove Theorem 4.1, the following is crucial.

Proposition 5.10. *Let $B_{\mathcal{E}_1}$ and $B_{\mathcal{E}_2}$ as in Corollary* 5.9. *There exists a $\mathcal{D}_6$ covering branched along $B_{\mathcal{E}_i}$ if and only if $MW(\mathcal{E}_i)$ has a* 3*-torsion.*

We give here an idea for our proof. Let T_{φ_i} be the subgroup of $\mathrm{NS}(\mathcal{E}_i)$ generated by the zero section, a general fiber and irreducible components of singular fibers not meeting the zero section. Then $MW(\mathcal{E}_i) \cong \mathrm{NS}(\mathcal{E}_i)/T_{\varphi}$ by [27]. Hence this indicates our proof is done in a similar way to that of Theorem 2.1. For details, see [31], [32].

Now Theorem 4.1 easily follows from the below:

Proposition 5.11. *For each case in Corollary* 5.9, *there exist elliptic surfaces $\mathcal{E}_1$ and $\mathcal{E}_2$ satisfying the following properties:*

(i) *$MW(\mathcal{E}_i)$ has no* 2*-torsion. In particular, $B_{\mathcal{E}_i}$ is irreducible.*

(ii) *$MW(\mathcal{E}_2)$ has no* 3*-torsion.*

(iii) *$MW(\mathcal{E}_1)$ has a* 3*-torsion.*

Proof. For all the statements except those for No. 2, one can find their proof in [31] and [32]. Hence we give a proof for No. 2 only. By [18], there exists $\mathcal{E}_2$ and it satisfies (i) and (ii) by Lemma 1.7 in [32]. For $\mathcal{E}_1$, we construct it in the following way: Let $\psi : Y \to \mathbf{P}^1$ be a rational elliptic surface with singular fibers $3I_3$, I_2, I_1. Let v_1 and v_2 be points of $\mathbf{P}^1$ such that $\psi^{-1}(v_i)$ $(i = 1, 2)$ are I_3 fibers. Let g be a degree 2 map from $\mathbf{P}^1$ to $\mathbf{P}^1$ branched at v_1 and v_2. Consider an elliptic $K3$ surface, $\mathcal{E}_1$, obtained as the pull-back surface of Y by g, i.e., the relatively minimal smooth model of the fiber product $Y \times_g \mathbf{P}^1$. Then:

Claim. *$MW(\mathcal{E}_1)$ has* (i) *a* 3*-torsion, and* (ii) *no* 2*-torsion.*

Proof of Claim. Since $MW(Y)$ has a 3-torsion (see [25], for example), so does $MW(\mathcal{E}_1)$. Since the covering transformation of g commutes with the inverse morphism, σ, $T_2(\mathcal{E}_1)$ is a double covering of $T_2(Y)$, and it is branched at two points on $T_2(Y)$. Hence $T_2(\mathcal{E}_1)$ is irreducible. In particular, $MW(\mathcal{E}_1)$ has no 2-torsion by Corollary 5.4. Q.E.D.

Thus we have $\mathcal{E}_1$ with the desired properties.

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