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Polyhedral Algebras, Arrangements of Toric Varieties, and their Groups

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§1. Introduction

In our previous paper [BG] we generalized standard properties of the group $\mathrm{GL}_n(k)$ of graded automorphisms of the polynomial ring $k[x_1, \dots, x_n]$ over a field k to the group $\operatorname{gr.aut}(k[S_P])$ of graded automorphisms of a polytopal k-algebra $k[S_P]$ associated with a lattice polytope P. The generators of the k-algebra $k[S_P]$ correspond bijectively to the lattice points in P, and their relations are the binomials representing the affine dependencies of the lattice points. (See Bruns, Gubeladze, and Trung [BGT] for polytopal algebras.) Thus $k[x_1, \dots, x_n]$ can be viewed as the polytopal algebra $k[S_{\Delta_{n-1}}]$ for the unit $(n-1)$-simplex Δ_{n-1}, and the fact that every invertible matrix can be reduced to a diagonal one by elementary row transformations is then a special case of our theorem [BG, Theorem 3.2] that every element of $\operatorname{gr.aut}(k[S_P])$ is a composition of elementary automorphisms, toric automorphisms, and affine symmetries of the polytope. (The symmetries are only needed if $\operatorname{gr.aut}(k[S_P])$ is not connected.) Polytopal algebras and their normalizations are special instances of affine semigroup algebras; more generally, we have described the group of graded automorphisms of an arbitrary normal affine semigroup algebra [BG, Remark 3.3(c)].

In [BG] an application to toric geometry is a description of the automorphism group of a projective toric variety over an algebraically closed field of arbitrary characteristic. Our approach avoids the theory of linear algebraic groups, and for projective toric varieties we have strengthened the classical theorem of Demazure [De] and its recent generalizations by Cox [Co] and Bühler [Bu].

The main issue of this paper is a generalization from the case of a single polytope to algebras $k[\Pi]$ corresponding to *lattice polyhedral*

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complexes Π of type as general as possible; these algebras will be called *polyhedral algebras.* Thus we are concerned with the graded automorphisms of fiber products of polytopal algebras, labeled naturally by lattice polyhedral complexes. In plain terms, the set of monomials of $k[\Pi]$ is the union of the set of monomials of the algebras $k[S_P]$ where P runs through the facets of Π, and the product of two monomials is their product in $k[S_P]$ if there exists P with both monomials belonging to $k[S_P]$, and zero otherwise. The simplest representatives of such algebras are Stanley-Reisner rings of simplicial complexes, whose graded automorphisms have recently been considered by Müller [Mu]. Combinatorial aspects of algebras defined by polyhedral complexes have been discussed by Stanley [Sta].

There is a natural hierarchy of lattice polyhedral complexes

$$\begin{aligned}
&\{\textit{abstract simplicial complexes}\} \subset \\
&\quad \{\textit{boundary lattice polyhedral complexes}\} \subset \\
&\qquad \{\textit{Euclidean lattice polyhedral complexes}\} \subset \\
&\qquad\quad \{\textit{quasi-Euclidean lattice polyhedral complexes}\} \subset \\
&\qquad\qquad \{\textit{oriented lattice polyhedral complexes}\} \subset \\
&\qquad\qquad\quad \{\textit{general lattice polyhedral complexes}\},
\end{aligned}$$

which appears in the subsequent sections; each of these classes constitutes just a small subclass in the next class, as illustrated by examples.

Boundary lattice complexes are obtained as subcomplexes of the set of faces of a single lattice polytope, whereas Euclidean complexes are formed by a collection of lattice polytopes in a Euclidean space $\mathbb{R}^n$ whose lattice structures are induced from the lattice $\mathbb{Z}^n$. For a quasi-Euclidean complex Π we relax the last requirement: the lattice providing the semigroup associated with each face of Π may vary among the facets of the complex. The definition of an oriented lattice polyhedral complex is more technical; roughly speaking, it permits us to define *elementary automorphisms* in terms of so-called *column structures.*

The group $\operatorname{gr.aut}(k[\Pi])$ is a linear algebraic group in a natural way. We will show that the elementary automorphisms together with the toric automorphisms generate its unity component if Π is oriented; if Π is even quasi-Euclidean, then the whole group is generated by elementary automorphisms, diagonal automorphisms and symmetries of the underlying complex. Here an automorphism α is called *diagonal* if each monomial is an eigenvector for α, and the *toric* automorphisms are the members of the unity component of the group of diagonal automorphisms (in the case of a single polytope this group is always connected). Moreover, under a certain combinatorial condition on the complex, one can provide a normal form for the representation of a general automorphism. This is the first main result of the paper (Theorem 5.2).

The combinatorial treatment that was successful in the case of a single polytope [BG] becomes exceedingly complicated for polyhedral complexes. Instead we will invoke Borel's theorem on maximal algebraic tori and other algebro-geometric arguments.

Polytopal algebras are related to graded normal affine semigroup algebras in the same way as polyhedral algebras are related to graded algebras defined by *rational polyhedral complexes* (Section 2). Analogously to the situation of a single polytope, our arguments apply to this class of algebras as well, yielding a description of their graded automorphism groups. An even more general class is constituted by the algebras described in terms of *weak fans* (Section 2). They are analogues of general, non-graded normal affine semigroup rings and are useful in the description of affine charts for *arrangements* of projective toric varieties; see Section 6. The analogy is limited, though: neither is the normalization of a polyhedral algebra combinatorially well-behaved in general, nor do all algebras given by weak fans come from rational polyhedral complexes (i.e. carry a graded structure such that monomials are homogeneous and of positive degree).

The second main result (Theorem 9.1) concerns the automorphism group of an arrangement of projective toric varieties, i.e. the Proj of a polyhedral algebra. Here the situation is more complicated than it was for projective toric varieties themselves: no longer can one give a natural one-to-one 'polyhedral interpretation' of very ample line bundles, which exists for single polytopes (Teissier [Te]). However, using once again Borel's theorem on maximal tori, we show that there are still reasonable polyhedral 'images' of the spaces of global sections for certain very ample line bundles. This suffices for the computation of the unity component of the automorphism group of an arrangement defined by a quasi-Euclidean complex and of the whole group for an arrangement defined by a *projectively quasi-Euclidean* complex Π; such a complex is distinguished by the fact that every complex projectively equivalent to Π is also quasi-Euclidean. Not all quasi-Euclidean complexes are projectively quasi-Euclidean, but in Section 8 we describe two natural big classes of such complexes; one of them includes the simplicial complexes.

In conjunction with [BG] this paper establishes a polyhedral generalization of classical K-theoretical objects – the general linear group $\mathrm{GL}_n(k)$ and its elementary subgroup $E_n(k)$. Naturally there arises a question: is there a further analogy with K-theory that might lead to a *polyhedral* K-theory? Already for low dimensional K-groups this question suggests challenging open problems.

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§2. Polyhedral complexes and polyhedral algebras

A polytope in a real vector space $\mathbb{R}^n$ is the convex hull of finitely many points. The vertices of a lattice polytope belong to the integral lattice $\mathbb{Z}^n \subset \mathbb{R}^n$.

We recall that for a lattice polytope $P \subset \mathbb{R}^n$ the sub-semigroup $S_P \subset \mathbb{Z}^{n+1}$ is by definition generated by $\{(x,1) |\ x \in \mathbb{Z}^n \cap P\} \subset \mathbb{Z}^{n+1}$. For a field k the semigroup algebra $k[S_P]$ is called the polytopal algebra of P over k ([BGT], [BG]).

Definition 2.1. A *lattice polyhedral complex* Π consists of

(a) an *abstract (finite) polyhedral complex* Π_X, that is a finite set X of *vertices* and a system Π_X of subsets of X such that $P \cap Q \in \Pi_X$ whenever $P, Q \in \Pi_X$,
(b) an embedding $P \to \mathbb{R}^{n_P}$ for each $P \in \Pi_X$ such that the image of P constitutes the vertex set of an n_P-dimensional lattice polytope $P^* \subset \mathbb{R}^{n_P}$,
(c) an embedding $\iota_{PQ} : P^* \to Q^*$ for each inclusion $P \subset Q$, $P, Q \in \Pi_X$ such that ι_{PQ} is an isomorphism of P^* with a face of Q^* as lattice polytopes.

Furthermore we require the following compatibility conditions:

(i) $\iota_{QR} \circ \iota_{PQ} = \iota_{PR}$ for $P, Q, R \in \Pi_X$, $P \subset Q \subset R$,
(ii) for every element $Q \in \Pi_X$ and each face F of the polytope Q^* there is an element $P \in \Pi_X$ such that $P \subset Q$ and $\iota_{PQ}(P^*) = F$.

(The condition $\dim(P^*) = n_P$ is useful for convenience of notation when we define projectively equivalent polyhedral complexes in Section 7.)

Let Π be a lattice polyhedral complex. For $P \in \Pi_X$ the set of lattice points of P^* will be denoted by $\mathrm{L}(P^*)$. We want to identify lattice points $x \in \mathrm{L}(P^*)$ and $y \in \mathrm{L}(Q^*)$ if $\iota_{PQ}(x) = y$. More precisely, we introduce the equivalence relation $\sim$ on the disjoint union of the sets of lattice points $\mathrm{L}(P^*)$, $P \in \Pi_X$, that is spanned by the relations $x \sim y$ for all x, y such that there exist $P, Q \in \Pi_X$ with $x \in \mathrm{L}(P^*)$, $y \in \mathrm{L}(Q^*)$, $P \subset Q$, and $\iota_{PQ}(x) = y$. The set of equivalence classes with respect to $\sim$ is denoted by $\mathrm{L}(\Pi)$ and they are called *lattice points of* Π. For simplicity of notation we will identify $\mathrm{L}(P^*)$ with its image in $\mathrm{L}(\Pi)$.

Let k be a field and Π a lattice polyhedral complex. Then it is easy to show that there exists a unique k-algebra $k[\Pi]$, satisfying the following conditions:

(1) $k[\Pi]$ is generated by $\mathrm{L}(\Pi)$;
(2) for any element $P \in \Pi_X$ the subalgebra of $k[\Pi]$ generated by $\mathrm{L}(P^*)$ is naturally isomorphic to the polytopal algebra $k[S_{P^*}]$;
(3) if there exists no $Q \in \Pi_X$ such that $x_1, \dots, x_s \in \mathrm{L}(\Pi)$ all belong to $\mathrm{L}(Q^*)$, then $x_1 \cdots x_s = 0$.

The algebra $k[\Pi]$ will be called the *polyhedral algebra of* Π. Condition (2) just means that for each $P \in \Pi_X$ the elements of $\mathrm{L}(P) \subset \mathrm{L}(\Pi)$ satisfy the binomial relations corresponding to their affine relations as lattice points in P^*. Furthermore these binomial relations together with the monomial relations in (3) define $K[\Pi]$.

Convention: The polytopes P^* will simply be denoted by P and they will be called *faces* of Π. We will write $P \prec \Pi$. Moreover, for $P, Q \in \Pi$, $P \subset Q$, we indicate by $P \prec Q$ that P is considered as a face of Q via ι_{PQ}.

The elements of the semigroups S_P, $P \prec \Pi$, will be called *monomials*; elements of the form αx, $\alpha \in k^*$, $x \in S_P$ are called *terms*.

Let Π^{face} be the poset (with respect to $\prec$) of the faces of Π, and Π^{facet} the subset consisting of all faces that can be written as an intersection of *facets*, i.e. maximal faces, of Π. Our conditions imply that we have a contravariant functor to (commutative) k-algebras:

$$\mathrm{alg}_k^{\mathrm{face}} : \Pi^{\mathrm{face}} \to k\text{-alg}$$

for which

$$\mathrm{alg}_k^{\mathrm{face}}(P) = k[S_P]$$

and $$\mathrm{alg}_k^{\mathrm{face}}(P \prec Q) = (\text{the `face projection' } k[Q] \to k[P]).$$

('Face projection' here means the unique k-algebra homomorphism under which $\mathrm{L}(Q) \setminus \mathrm{L}(P)$ is mapped to $0 \in k$ and each $x \in \mathrm{L}(P)$ to itself.) The restriction of $\mathrm{alg}_k^{\mathrm{face}}$ to Π^{facet} will be denote by $\mathrm{alg}_k^{\mathrm{facet}}$.

The following is the universal characterization of $k[\Pi]$:

$$k[\Pi] = \varprojlim \mathrm{alg}_k^{\mathrm{face}} = \varprojlim \mathrm{alg}_k^{\mathrm{facet}}.$$

Definition 2.2. (a) A polyhedral subcomplex of the complex of all faces of some lattice polytope is called a *boundary polyhedral lattice complex*.

(b) A lattice polyhedral complex that can be realized as a polyhedral complex of lattice polytopes (with respect to $\mathbb{Z}^n$) in some real vector space $\mathbb{R}^n$ is called *Euclidean.*

(c) A lattice polyhedral complex Π, realizable as a polyhedral complex of rational polytopes in some real vector space, is called *quasi-Euclidean.*

Proposition 2.3. (a) *If Π is the lattice polyhedral complex of all faces of some lattice polytope P(including P itself), then* $k[\Pi] = k[S_P]$.

(b) *If Π is a lattice simplicial complex consisting of unit lattice simplices, then $k[\Pi]$ is exactly the Stanley-Reisner ring of Π_X. Any Stanley-Reisner ring can be realized in this way.*

(c) *The inclusions {abstract simplicia complexes} $\subset$ {boundary lattice complexes} $\subset$ {Euclidean complexes} $\subset$ {quasi-Euclidean complexes} are strict.*

(See Bruns and Herzog [BH, Ch.5] for Stanley-Reisner rings.)

Proof. The claims (a) and (b) are obvious, as is the first inclusion in (c). It is evidently strict. So we only need to construct a Euclidean, but not boundary, lattice polyhedral complex and a quasi-Euclidean, but not Euclidean, one.

Consider the polyhedral complexes in Figure 1 where Π_1 consists

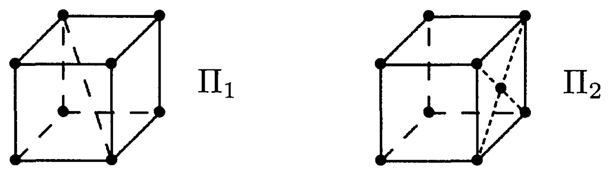


FIGURE 1.

of the 6 two-dimensional facets forming the surface of the lattice unit cube and one more facet given by a space diagonal, and Π_2 has 6 two-dimensional facets of which 5 are lattice unit squares and the 6th has an additional lattice point in its barycenter.

We claim that Π_1 is not a boundary complex. In fact, assume to the contrary that there exists a lattice polytope P in $\mathbb{R}^m$ whose boundary complex contains Π_1. Then there is a linear mapping from $\mathbb{R}^m$ to $\mathbb{R}$ that is positive on Π_1 outside the 1-dimensional facet and is 0 on it. Now observe that the affine hull of Π_1 in $\mathbb{R}^m$ is 3-dimensional and that the 2-dimensional facets of Π_1 must form the boundary of a 3-dimensional parallelepiped Ξ in $\mathbb{R}^m$ – this is an obvious rigidity property of the boundary complex of the unit 3-cube. The space diagonal, except its end-points, consists of interior points of Ξ. Hence any linear form positive on the boundary of Ξ (except the endpoints of the space diagonal) must also be positive in the interior of this diagonal – a contradiction.

It is easy to check that, like in the previous case, a Euclidean realization of Π_2 must form the boundary of some lattice parallelepiped. In particular, the lattice structures of each of the opposite pairs of facets must be naturally isomorphic. But this is not the case for Π_2 and, hence, there is no Euclidean realization of Π_2. (That Π_2 is quasi-Euclidean is obvious.) Q.E.D.

To a lattice polyhedral complex Π one can also associate a semigroup (commutative, with unity) S_Π, which is generated by $\mathrm{L}(\Pi)$ and one extra element ∞ in such a way that

(1) S_P is a sub-semigroup of S_Π for every face $P \in \Pi$,
(2) $x \cdot \infty = \infty \cdot \infty = \infty$ and $x_1 \cdots x_s = \infty$ whenever $x_1 \cdots x_s = 0$ in $k[\Pi]$.

Of course, this definition is independent of the field k. The kernel of the natural surjection $k[S_\Pi] \to k[\Pi]$ is the ideal (∞) $(\dim_k(\infty) = 1)$. Moreover, S_Π is mapped isomorphically to the multiplicative sub-semigroup of $k[\Pi]$ generated by $\mathrm{L}(\Pi)$ and 0.

Observe that $k[\Pi]$ is equipped with a natural grading:

$$k[\Pi] = k \oplus A_1 \oplus A_2 \oplus \cdots, \qquad A_1 = k\mathrm{L}(\Pi).$$

The group of graded k-automorphisms of $k[\Pi]$, denoted by $\Gamma_k(\Pi)$ later on, is called the *polyhedral linear group associated with* Π. Clearly, If Π is a lattice polyhedral complex determined by a lattice polytope P, then $\Gamma_k(\Pi)$ is the polytopal linear group $\Gamma_k(P)$ of [BG]. As for polytopal groups, one observes easily that polyhedral linear groups are affine k-groups: $\Gamma_k(\Pi)$ is a closed subgroup of $\mathrm{GL}_N(k)$, $N = \#\mathrm{L}(\Pi)$, whose defining equations are derived from the relations between the degree 1 monomials of $k[\Pi]$ by use of an obvious, simple algorithm.

The group of semigroup automorphisms of S_Π will be denoted by $\Sigma(\Pi)$. It is a finite group embedded into $\Gamma_k(\Pi)$ in a natural way, and we will identify $\Sigma(\Pi)$ with its image.

Next we introduce the notion of a *rational polyhedral complex*. The corresponding graded algebras are related to polyhedral algebras in the same way as graded normal affine semigroup rings are related to polytopal algebras.

Definition 2.4. A *rational polyhedral complex* Π_{rat} consists of the following data:

(a) an abstract polyhedral complex Π_X,
(b) an embedding $P \to \mathbb{R}^{n_P}$ for each $P \in \Pi_X$ such that the image is the vertex set of a *rational* polytope P^* (with respect to $\mathbb{Q}^{n_P} \subset \mathbb{R}^{n_P}$) whose faces correspond to the sets $\{R \in \Pi_X |\ R \subset P\}$ so that if $P \subset Q$ are two elements of Π_X, then the polytope P^* and the face P' of Q^* corresponding to P are naturally isomorphic as rational polytopes.

Furthermore we require that the isomorphism of P^* and P' induces a bijection between the sets of lattice points of $cP^* \cap \mathbb{Z}^{n_P}$ and $cP' \cap \mathbb{Z}^{n_Q}$ for each $c \in \mathbb{N}$. (Here cP^* and cP' denote the c-th homothetic images.)

It may happen that the faces of a finite rational polyhedral complex Π_{rat} are actually lattice polytopes, but the subscript $-_{\text{rat}}$ emphasizes that we are considering the rational structure.

For a face $P \in \Pi_{\text{rat}}$ we let $C(P)$ denote the finite rational convex cones in $\mathbb{R}^{n_P+1}$ with apex 0 that is spanned by $\{(x,1) |\ x \in P\}$; moreover, we let $\hat{S}_P$ denote the sub-semigroup $\mathbb{Z}^{n_P+1} \cap C(P) \subset \mathbb{Z}^{n_P+1}$. The algebra $k[\Pi_{\text{rat}}]$ is defined as the unique algebra satisfying the following conditions:

(1) $k[\hat{S}_P]$ is a subalgebra of $k[\Pi_{\text{rat}}]$ for every face $P \in \Pi_{\text{rat}}$ and if $P \prec Q$, then $k[\hat{S}_P] \subset k[\hat{S}_Q]$ in a natural way,
(2) $x_1 \cdots x_s = 0$ whenever $x_i \in \hat{S}_{P_i}$ for some faces $P_i \in \Pi_{\text{rat}}$, $i \in [1,s]$, and there is no face $R \in \Pi_{\text{rat}}$ such that $x_1, \dots, x_s \in \hat{S}_R$,
(3) $k[\Pi_{\text{rat}}] = \sum_P k[\hat{S}_P]$ as k-spaces, where P runs through the faces of Π_{rat}.

Here we adopt a convention on terminology and notation similar to that we have introduced for lattice polyhedral complexes. In particular, we can speak of a monomial in $k[\Pi_{\text{rat}}]$.

Proposition 2.5. (a) *The class of affine normal semigroup k-algebras coincides with the class of algebras of type $k[\Pi_{\text{rat}}]$ where Π_{rat} is the rational complex (of all faces) of a rational polytope.*
(b) *For a rational polyhedral complex Π_{rat} and a field k the algebra $k[\Pi_{\text{rat}}]$ carries a graded structure where all monomials are homogeneous of positive degree given by the last component of (the exponent vector of) x in $\mathbb{Z}^{n_P+1}$ for $x \in \hat{S}_P$. The group $\Gamma_k(\Pi_{\text{rat}})$ of graded automorphisms of $k[\Pi_{\text{rat}}]$ is an affine k-group.*
(c) *In general a lattice polyhedral complex Π does not define a rational polyhedral complex in a natural way, i.e. if we pass to the normalizations $\{y \in \operatorname{gp}(S_P) \mid y^m \in S_P \text{ for some } m \in \mathbb{N}\}$ of the S_P where P runs through the faces of Π, then the new system of*

semigroups may not satisfy the compatibility condition required in 2.4.

Proof. (a) just says that all affine normal semigroup k-algebras can be equipped with a graded structure such that monomials become homogeneous elements of positive degree (see [BH, Ch. 6]). (b) is an obvious analogue of the corresponding observations for lattice polyhedral complexes.

For (c) consider a 4-dimensional lattice polytope $P \subset \mathbb{R}^4$ such that

(1) its lattice points span $\mathbb{Z}^4$ (as an additive group),
(2) one of its facets is a 3-simplex $\delta \subset P$ whose vertices are the only lattice points in δ, but do not span the whole 3-dimensional affine sublattice $\operatorname{Aff}(\delta) \cap \mathbb{Z}^4 \subset \mathbb{Z}^4$.

($\operatorname{Aff}(\delta)$ is the affine hull of δ in $\mathbb{R}^4$.) The existence of such P is clear: just take a non-unimodular lattice 3-simplex δ in $\mathbb{R}^3$ whose vertices are the only lattice points in δ, and then complete it to a sufficiently big 4-polytope in the upper halfspace (with respect to an embedding $\mathbb{R}^3 \to \mathbb{R}^4$ as a coordinate hyperplane).

Now consider the lattice polyhedral complex having just two facets: P and a unit 4-simplex Δ (in its own ambient Euclidean space) which meet along δ. Then the normalizations of S_P and that of S_Δ do not agree along the cone spanned by δ. (The complex just considered is quasi-Euclidean, but not Euclidean. In fact, a Euclidean complex defines a rational polyhedral complex, since the semigroups of its faces are derived from the same lattice; the corresponding algebras have been considered by Stanley [Sta].) Q.E.D.

In our general setting the rôle of all normal affine semigroup rings is played by the algebras determined by *weak fans*. These algebras are useful in the description of the affine chart of $\operatorname{Proj}(k[\Pi])$ (see Section 6).

Definition 2.6. A *weak fan* $\mathcal{WF}$ consists of the following data:

(a) an abstract polyhedral complex Π_X,
(b) for each $P \in \Pi_X$ a rational strictly convex polyhedral cone $C_P \subset \mathbb{R}^{n_P}$ whose extremal rays are labeled by the elements of P in such a way that the faces of C_P correspond to the faces of P.

Furthermore we require that this correspondence induces an isomorphism of the lattice structures of C_P (with respect to $\mathbb{Z}^{n_P}$) and that of the corresponding face of C_Q (with respect to $\mathbb{Z}^{n_Q}$) if $P, Q \in \Pi_X$, $P \subset Q$.

Observe that a rational polyhedral complex Π_{rat} defines in a natural way a weak fan $\mathcal{WF}(\Pi_{\mathrm{rat}})$: one just considers the cones $C_P \subset \mathbb{R}^{n_P+1}$,

$P \in \Pi_{\text{rat}}$. Likewise, any (finite) fan Φ in the sense of toric geometry (for example, see Fulton [Fu]) gives rise to a weak fan $\mathcal{WF}(\Phi)$.

To a field k and a weak fan $\mathcal{WF}$ one associates a k-algebra $k[\mathcal{WF}]$ by patching the semigroup algebras $k[\mathbb{Z}^{n_P} \cap C_P]$ along the facets of the cones C_P. Again, one has the equalities (in self-explanatory notation):

$$k[\mathcal{WF}] = \varprojlim \operatorname{alg}_k^{\text{face}} = \varprojlim \operatorname{alg}_k^{\text{facet}}.$$

As mentioned already, any normal affine semigroup ring (without non-trivial units) can be equipped with a graded structure so that its monomials become homogeneous and of positive degree. However, not all algebras of type $k[\mathcal{WF}]$ carry a graded structure.

Example 2.7. There exists a complete fan Φ in $\mathbb{R}^3$ such that $\mathcal{WF}(\Phi)$ is not of type $\mathcal{WF}(\Pi_{\text{rat}})$ for some rational polyhedral complex Π_{rat}, i.e. there is no graded structure on $k[\mathcal{WF}(\Phi)]$ (k a field) such that its monomials are homogeneous of positive degree.

First observe that a weak fan $\mathcal{WF}$ is of type $\mathcal{WF}(\Pi_{\text{rat}})$ if and only if $k[\mathcal{WF}]$ carries a graded structure $k[\mathcal{WF}] = k \oplus A_1 \oplus A_2 \oplus \cdots$ such that all monomials are homogeneous of positive degree.

Choose 6 rational non-coplanar points in $\mathbb{R}^3$ as shown in Figure 2

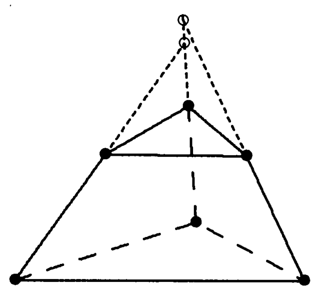


FIGURE 2.

where the top and bottom triangles are in parallel planes and the 'hidden' quadrangles are flat polygons whereas the frontal quadrangle is not a flat figure. Suppose that $0 \in \mathbb{R}^3$ lies in the interior of the convex hull of these 6 points; then the cones with common apex 0 that are spanned by the 2 triangles and the 3 quadrangles form a complete fan Φ of rational cones in $\mathbb{R}^3$. We claim that $\mathcal{WF}(\Phi)$ is not of type $\mathcal{WF}(\Pi_{\text{rat}})$.

In fact, this could only be the case if all 3 quadrangles were flat. We leave the proof of this general statement to the reader and content ourselves with a concrete example. Choose the 6 points as follows:

$$\begin{aligned} u &= (1,1,1), \quad & w &= (-1,1,1), \quad & y &= (0,-1,1), \\ v &= (1,1,0), \quad & x &= (-1,1,0), \quad & z &= (1,-3,0). \end{aligned}$$

Then we have the binomial relations

$$ux = vw, \qquad u^4 z = v^5 y^4, \qquad w^2 z = xy^2$$

in $k[\mathcal{WF}(\Phi)]$. It is easy to show by hand that one cannot assign positive degrees to the 6 elements such that these relations become homogeneous.

§3. Diagonal and toric automorphisms

Let Π be a lattice polyhedral complex. The rôle of the embedded torus of an affine toric variety is played by the subgroup of $\Gamma_k(\Pi)$ whose elements multiply the monomials $x \in \mathrm{L}(\Pi)$ by scalars from k^*. This subgroup is denoted by $\mathbb{D}_k(\Pi)$ and its elements are called *diagonal automorphisms.* It becomes a diagonal subgroup of $\mathrm{GL}_N(k)$, $N = \#\mathrm{L}(\Pi)$, in the natural realization of $\Gamma_k(\Pi)$ as an affine subgroup of $\mathrm{GL}_N(k)$.

One can give a more explicit description of $\mathbb{D}_k(\Pi)$. Consider the finitely generated Abelian group

$$A(\Pi) = \mathbb{Z}^{\mathrm{L}(\Pi)}/U(\Pi)$$

where $\mathbb{Z}^{\mathrm{L}(\Pi)}$ is the free Abelian group generated by the lattice points in Π and $U(\Pi)$ represents the affine relations between the elements of $\mathrm{L}(\Pi)$, i.e. $U(\Pi)$ is generated by all linear combinations

$$\sum_{x \in \mathrm{L}(P)} a_x e_x, \qquad a_x \in \mathbb{Z}, \ \sum_{x \in \mathrm{L}(P)} a_x x = 0,$$

where P runs through the facets of Π and e_x represents the base element corresponding to x. (Here $x \in \mathrm{L}(P)$ is to be considered as an element of $\mathbb{R}^{n_P+1}$ with last coordinate 1 so that $\sum_{x \in \mathrm{L}(P)} a_x x = 0$ implies $\sum_{x \in \mathrm{L}(P)} a_x = 0$.)

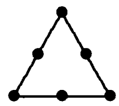
Let $\gamma \in \mathbb{D}_k(\Pi)$ and set $\lambda_x = \gamma(x)/x$ for all $x \in \mathrm{L}(\Pi)$. Then it is clear that $\sum_{x \in \mathrm{L}(P)} a_x x = 0$ implies $\prod_{x \in \mathrm{L}(P)} \lambda_x^{a_x} = 1$, and, conversely, every choice of $\lambda_x \in k^*$, $x \in \mathrm{L}(\Pi)$, satisfying these relations induces a diagonal automorphism of $k[\Pi]$. Therefore one has

Lemma 3.1. *For every lattice polyhedral complex Π and any field k*

$$\mathbb{D}_k(\Pi) = \mathrm{Hom}_{\mathbb{Z}}(A(\Pi), k^*).$$

Clearly, $\mathbb{D}_k(\Pi)$ contains a distinguished copy of k^* – the automorphisms which multiply the elements of $\mathrm{L}(\Pi)$ by a fixed scalar. When k^* is considered as a subgroup of $\Gamma_k(\Pi)$, we always mean the subgroup just specified.

In general $A(\Pi)$ is not torsionfree, not even if Π is Euclidean. Consider for example the complex Π below. It is easy to see that $A(\Pi) \approx \mathbb{Z}^3 \oplus (\mathbb{Z}/(2))$. Therefore $\mathbb{D}_k(\Pi)$ is not connected (if $\operatorname{char} k \neq 2$).



It is well known that the connected component of the diagonalizable group $\mathbb{D}_k(\Pi)$ is a torus. We denote it by $\mathbb{T}_k(\Pi)$ and call its elements *toric automorphisms*. Moreover we set $\Lambda(\Pi) = A(\Pi)/(\text{torsion})$. Then it is easy to show

Lemma 3.2. $\mathbb{T}_k(\Pi) \approx \operatorname{Hom}_{\mathbb{Z}}(\Lambda(\Pi), k^*) \approx \operatorname{Hom}_{\mathbb{Z}}(A(\Pi), \mathbb{Z}) \otimes k^*$.

Remark 3.3. (a) Let Π be a quasi-Euclidean complex, $\Pi \subset \mathbb{R}^n$, and $\mathrm{L}(\Pi) \subset \mathbb{Z}^n$. Then the elements $x \in \mathrm{L}(\Pi) \subset \mathbb{Z}^n$ satisfy all the affine relations that define $A(\Pi)$. Consequently one has an induced $\mathbb{Z}$-linear map $\Lambda(\Pi) \to \mathbb{Z}^n$. It is clear that the residue classes $\bar{e}_x$, $x \in \mathrm{L}(\Pi)$, span a quasi-Euclidean complex isomorphic to Π (in the vector space $\Lambda(\Pi) \otimes \mathbb{R}$). This realization is the *maximal embedding* of Π; every other embedding into a vector space factors through it.

(b) While $A(\Pi)$ may have torsion if Π is quasi-Euclidean, the subgroup generated by the elements $\bar{e}_x$, $x \in \mathrm{L}(P)$, is torsionfree for every face $P \prec \Pi$. In fact, the map described in part (a) sends this subgroup isomorphically onto the group $A(P)$.

For an arbitrary complex this does not necessarily hold; for example, it fails for the 'Möbius strip' Π_7 below (see Example 4.1).

The next lemma describes the subgroup of those elements of $\Gamma_k(\Pi)$ that map monomials to terms.

Lemma 3.4. (a) *If $\gamma \in \Gamma_k(\Pi)$ maps monomials to terms, then $\gamma = \delta \circ \sigma$ for some $\delta \in \mathbb{D}_k(\Pi)$ and $\sigma \in \Sigma(\Pi)$.*

(b) *For $\delta \in \mathbb{D}_k(\Pi)$ and $\sigma \in \Sigma(\Pi)$ one has $\sigma^{-1} \circ \delta \circ \sigma \in \mathbb{D}_k(\Pi)$; moreover, the subgroup of $\Gamma_k(\Pi)$ generated by $\mathbb{D}_k(\Pi)$ and $\Sigma(\Pi)$ is their semi-direct product.*

Proof. (a) is checked as easily as in the case of a single polytope treated in [BG, Section 4], and (b) is obvious. Q.E.D.

The next lemma provides a crucial argument.

Lemma 3.5. *Suppose Π is quasi-Euclidean and k is an infinite field.*

(a) *For any pair of different monomials* $x_1, x_2 \in k[\Pi]$ *there exists* $\tau \in \mathbb{T}_k(\Pi)$ *such that* $\tau(x_1) = a_1x_1$ *and* $\tau(x_2) = a_2x_2$ *for some distinct elements* $a_1, a_2 \in k^*$.

(b) $\mathbb{T}_k(\Pi)$ *is a maximal torus of* $\Gamma_k(\Pi)$.

Proof. We may assume that Π consists of rational polytopes in $\mathbb{R}^n$. By homothetic blowing up we can further assume that all the lattice points of Π have integral coordinates in $\mathbb{R}^n$. Clearly, we have a natural action of the torus $\mathbb{Z}^n \otimes k^* = (k^*)^n$ on $k[\Pi]$ – the restriction of the action on $k[\mathbb{Z}^n]$ to the monomials of $k[\Pi]$. This gives rise to an algebraic homomorphism $\phi : (k^*)^n \to \mathbb{D}_k(\Pi)$. By reasons of connectivity, $\phi((k^*)^n)$ is contained in $\mathbb{T}_k(\Pi)$. Now (a) becomes obvious.

Assume there is a torus $T \subset \Gamma_k(\Pi)$ that contains $\mathbb{T}_k(\Pi)$. Then $\alpha^{-1} \circ \beta \circ \alpha(x) = \beta(x)$ for all $\alpha \in \mathbb{T}_k(\Pi)$, $\beta \in T$ and $x \in k[\Pi]$. By running α through $\mathbb{T}_k(\Pi)$ and x through the monomials of $k[\Pi]$, and using (a), we conclude that β must map monomials to terms, i.e. $\beta \in \mathbb{D}_k(\Pi) \rtimes \Sigma(\Pi)$ by 3.4. But, since k is infinite, there is no torus in $\mathbb{D}_k(\Pi) \rtimes \Sigma(\Pi)$ strictly containing the unity component $\mathbb{T}_k(\Pi)$. Hence $T = \mathbb{T}_k(\Pi)$. Q.E.D.

§4. Column structures and elementary automorphisms

We recall from [BG] that a non-zero element $v \in \mathbb{Z}^n$ is called a *column vector* for a lattice polytope $P \subset \mathbb{R}^n$ if there exists a facet $F \prec P$ such that $x + v \in P$ for every lattice point $x \in P \setminus F$ [BG]. The pair (P, v) is a *column structure* and the facet F its *base facet.* We use the notation P_v for F. Figure 3 illustrates this notion.

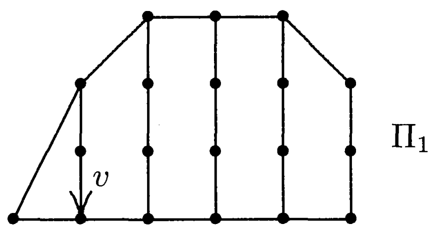


FIGURE 3.

Let (P, v) be a column structure. Then for any $x \in S_P$ there is a uniquely determined non-negative integer $\mathrm{ht}_v(x)$ such that $x + \mathrm{ht}_v(x)v \in S_P$ and $x + (\mathrm{ht}_v(x) + 1)v \notin S_P$ [BG, Lemma 2.2]. Clearly, if $C(P)$ denotes the cone in $\mathbb{R}^{n+1}$ spanned by S_P and $C(P_v)$ is its facet corresponding to the facet $P_v \prec P$ then $x + \mathrm{ht}_v(x)v \in C(P_v)$.

Let k be a field. The element $v \in \mathbb{Z}^n$ can be thought of as an element of the quotient field $\mathrm{Q.F.}(k[S_P])$ after the identification of $\mathbb{Z}^n$ with $\mathbb{Z}^n \oplus 0$ $(\subset \mathbb{Z}^{n+1})$. Choose $\lambda \in k$. Then the semigroup homomorphism

$$S_P \to \mathrm{Q.F.}(k[S_P]), \qquad x \mapsto (1+\lambda v)^{\mathrm{ht}_v(x)} x,$$

gives rise to a k-algebra homomorphism $k[S_P] \to \mathrm{Q.F.}(k[S_P])$. This homomorphism is actually a graded automorphism of $k[S_P]$ [BG, Section 3]. We denote it by e_v^λ and call it an *elementary automorphism* of $k[S_P]$. If P is a unimodular lattice n-simplex, then $\Gamma_k(P)$ is just $\mathrm{GL}_{n+1}(k)$ and the e_v^λ are exactly the standard elementary matrices [BG, Section 3]; this explains our terminology.

Now we extend these notions to lattice polyhedral complexes Π. For $x \in \mathrm{L}(\Pi)$ we let $\mathrm{Supp}(x)$ denote the set off all *facets* of Π that contain x – the set of supporting facets.

Consider the set of all column structures (P, v), $P \prec \Pi$, satisfying the condition

$$(\#_1) \qquad \mathrm{Supp}(x+v) \subset \mathrm{Supp}(x)$$

for every lattice point $x \in P \setminus P_v$. Here the sum $x + v$ is understood 'locally', i.e. with respect to the column structure (P, v).

We have the following relation on this set of column structures: $(P, v) \sim (Q, w)$ if $Q \prec P$ and $w = v$ on Q. Consider the equivalence relation spanned by $\sim$. Among the corresponding equivalence classes $[P, v]$ there are distinguished ones, namely those satisfying the condition:

($\#_2$) If $(Q, w) \in [P, v]$ and $R \prec \Pi$ is a face such that (Q, w) restricts to a column structure on $Q \cap R$, then there is a column structure (R, u) satisfying ($\#_1$) and restricting to the same column structure on $Q \cap R$.

Observe that ($\#_2$) is equivalent to the condition:

($\#_2'$) $[P, v]$ induces (i.e. contains) a column structure on at least one facet and if $(Q, w) \in [P, v]$ is a column structure for some *facet* $Q \prec \Pi$ and $R \prec \Pi$ is another facet such that (Q, w) restricts to a column structure on $Q \cap R$ then there is a column structure (R, u) satisfying ($\#_1$) and restricting to the same column structure on $Q \cap R$.

A *column vector for* Π is defined as such a distinguished equivalence class. For a column vector V the pair (Π, V) will be called a *column structure* (on Π).

We let $\mathrm{Col}(\Pi)$ denote the set of column structures on Π.

Example 4.1. The Figures 4 and 5 show several polyhedral complexes and their column structures.

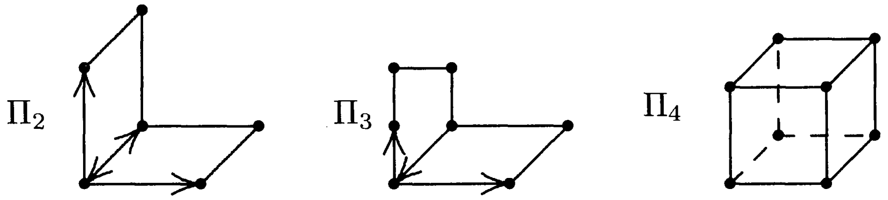


FIGURE 4.

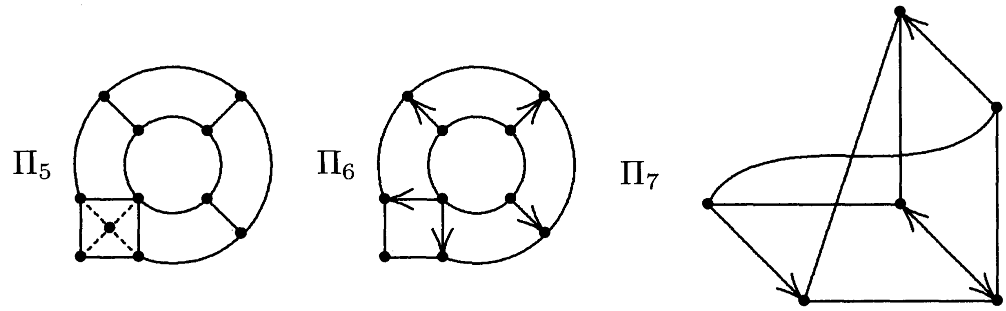


FIGURE 5.

- Π_2 and Π_3 have two 2-dimensional facets, $\#\mathrm{Col}(\Pi_2) = 4$ and $\#\mathrm{Col}(\Pi_3) = 3$,
- Π_4 is the boundary of the unit lattice cube, $\#\mathrm{Col}(\Pi_4) = 0$,
- Π_5 has five 2-dimensional facets four of which are unit squares and the fifth is a lattice square with a lattice point in its barycenter, $\#\mathrm{Col}(\Pi_5) = 0$,
- Π_6 has five unit squares as facets, as shown in the picture, $\#\mathrm{Col}(\Pi_6) = 1$,
- Π_7 is a Möbius strip consisting of 3 unit squares, $\#\mathrm{Col}(\Pi_7) = 1$, and the only column structure on Π_7 includes 2 'opposite' column structures on 3 edges.

Next we introduce the notion of an *oriented polyhedral complex.* This includes the class of quasi-Euclidean polyhedral complexes.

Definition 4.2. A lattice polyhedral complex Π is called oriented if $(P, v) \in V$ and $(P, w) \in V$ imply $v = w$ for any column structure (Π, V).

Lemma 4.3. *Every quasi-Euclidean lattice polyhedral complex is oriented, but not conversely.*

Proof. Assume Π is a quasi-Euclidean complex realized by a polyhedral complex of rational polytopes in $\mathbb{R}^n$, $n \in \mathbb{N}$. Let (Π, V) be a column structure and $(P,v),(P,w) \in V$. Then there is a finite sequence of column structures

$$(P,v) = (P_0,v_0),(P_1,v_1),\dots,(P_s,v_s) = (P,w),$$

where the P_i are faces of Π such that for each $i \in [1, s-1]$ either P_i is a face of P_{i+1} and $v_i = v_{i+1}$ on P_i or P_{i+1} is a face of P_i and $v_i = v_{i+1}$ on P_{i+1}. In particular, the v_i define the same vector in $\mathbb{R}^n$. Hence all quasi-Euclidean lattice polyhedral complexes are oriented.

An example of an oriented, but not quasi-Euclidean lattice polyhedral complex is provided by Π_5 above. In fact, easy geometric arguments show that if it were quasi-Euclidean, then the two adjacent edges of the square with barycenter would have to coincide. (One just uses that any affine realization of a unit lattice square must be a parallelogram.) Q.E.D.

Let k be a field, Π an oriented lattice polyhedral complex and V its column vector. For any element $\lambda \in k$ we define the map

$$e_V^\lambda : \mathrm{L}(\Pi) \to k[\Pi]$$

as follows. For $x \in \mathrm{L}(\Pi)$ there are two possibilities: either there is a column structure (P,v) such that $x \in \mathrm{L}(P)$, $P \prec \Pi$ and $V = [P,v]$, or such a column structure does not exist. In the first case we put

$$e_V^\lambda(x) = e_v^\lambda(x),$$

where e_v^λ is the corresponding elementary automorphism of $k[S_P]$ ($\subset k[\Pi]$), and in the second case x is mapped to itself. It follows from the definitions of a column vector and an oriented complex that this map is well defined. We claim that it gives rise to a (uniquely determined) graded k-algebra homomorphism of $k[\Pi]$. One only needs to check the following implication

$$\forall x_1,\dots,x_s \in \mathrm{L}(\Pi) \quad x_1 \cdots x_s = 0 \Rightarrow e_V^\lambda(x_1)\cdots e_V^\lambda(x_s) = 0;$$

in fact, $e_V(\lambda)$ respects the binomial relations since it restricts to an automorphism on $k[S_P]$ for each $P \prec \Pi$. Straightforward arguments show that condition $(\#_1)$ together with

$$\bigcap_1^s \mathrm{Supp}(x_i) = \emptyset$$

implies that none of the monomials in the canonical k-linear expansion of $e_V^\lambda(x_i)$ shares a supporting facet with those in the k-linear expansion of $e_V^\lambda(x_j)$ for $i \neq j$. This means that $e_V^\lambda(x_1)\cdots e_V^\lambda(x_s) = 0$, as claimed.

Next we define a pairing

$$\mathbb{D}_k(\Pi) \times \mathrm{Col}(\Pi) \to k^*, \quad (\delta, V) \mapsto \delta(V),$$

for an oriented polyhedral complex Π. Choose $\delta \in \mathbb{D}_k(\Pi)$, $V = [P, v] \in \mathrm{Col}(\Pi)$, and a face $P \prec \Pi$. Then δ restricts to a toric automorphism of $k[S_P]$. The latter extends to a toric automorphism of $k[\mathrm{gp}(S_P)]$. In particular, the image of $v \in \mathrm{gp}(S_P)$ under this automorphism equals $a_v v$ for some $a_v \in k^*$. We set $\delta(V) = a_v$. It is easily checked that this is a well defined mapping. Moreover, we have the equality $(a\cdot\delta)(v) = a(\delta(v))$ for δ and v as above and $a \in k^*$ $(\subset \mathbb{D}_k(\Pi))$.

Lemma 4.4. *Let (Π, V) be a column structure, where Π is an oriented lattice polyhedral complex. Then*

(a) *$e_V^\lambda \in \Gamma_k(\Pi)$, and the assignment $\lambda \mapsto e_V^\lambda$ defines an embedding of algebraic groups $\mathbb{A}_k^1 \to \Gamma_k(\Pi)$;*

(b) *the equation*

$$\delta \circ e_V^\lambda \circ \delta^{-1} = e_V^{\delta(V)\lambda}$$

holds for all $\delta \in \mathbb{D}_k(\Pi)$ and all elementary automorphism e_V^λ of $k[\Pi]$.

Proof. (a) follows from the analogous fact for a single polytope [BG, Lemma 3.1] and (b) is immediate from direct calculation. Q.E.D.

Let $\mathbb{E}_k(\Pi)$ denote the subgroup of $\Gamma_k(\Pi)$, generated by the elementary automorphisms. By Lemma 4.4(a) $\mathbb{E}_k(\Pi)$ is a connected subgroup of $\Gamma_k(\Pi)$ (see Borel [Bo, Proposition 2.2]). Therefore, we arrive at the following

Lemma 4.5. *$\mathbb{E}_k(\Pi)$ is a connected affine k-subgroup of the connected component of unity $\Gamma_k(\Pi)^0 \subset \Gamma_k(\Pi)$.*

Remark 4.6. One can define the notion of a column structure for a rational polyhedral complex Π_{rat} and the appropriate notion of an elementary automorphism for algebras of type $k[\Pi_{\mathrm{rat}}]$ in a natural way (along the lines of the definition for a single polytope [BG, Remark 3.3(c)]). One just has to work with monomials of arbitrary degrees. Then all the facts we have observed for lattice polyhedral complexes remain true in this situation as well. The details are left to the reader.

Remark 4.7. One could introduce the notion of *commutative* lattice polyhedral complexes which are more general than the oriented ones and for which one can still define the notion of an elementary automorphism so that the exact analogue of Lemma 4.4(a) is valid. (But we are not able to prove the analogue of Theorem 5.2 below for them.) Namely, a lattice polyhedral complex Π is called k-commutative (k a field) if for every column structure (Π, V) and every face $P \prec \Pi$ the following implication holds:

$$\forall \lambda, \mu \in k \quad ((P, v), (P, w) \in V) \Rightarrow (e_v^\lambda \text{ and } e_w^\mu \text{ commute}).$$

If Π is k-commutative for all fields, then it is called commutative.

We do not know whether k-commutative complexes are always commutative.

Observe that the complex Π_6 of Example 4.1 is commutative (easy) and its only column structure includes 2 column structures on one of the facets. In particular, Π_6 is not oriented. On the other hand the complex Π_7, Example 4.1, is apparently not a k-commutative lattice polyhedral complex for any field k: looking at the edges on which the only global column structure on Π_7 induces two 'opposite' column structures we get non-commutativity since $e_{12}^1 e_{21}^1 \neq e_{21}^1 e_{12}^1$, where e_{12}^1 and e_{21}^1 are the standard elementary matrices in $\mathrm{GL}_2(k)$.

§5. The main result: affine case

Before we state the first main result let us single out the following class of polytopes.

Definition 5.1. A polytope P is *facet-separated* if for every facet $F \prec P$ there is a facet $G \prec P$ such that $F \cap G = \emptyset$.

Typical representatives of non-facet-separated polytopes are *pyramids* – the polytopes whose vertices all but one live in some affine proper subspace of the ambient Euclidean space. However, starting from dimension 4, facet-separated polytopes and pyramids do not exhaust the class of all polytopes.

Theorem 5.2. *Let k be a field and Π be a lattice polyhedral complex.*

(a) *If Π is oriented and $\mathrm{char}(k) = 0$, then the unity component $\Gamma_k(\Pi)^0 \subset \Gamma_k(\Pi)$ consists precisely of those elements $\gamma \in \Gamma_k(\Pi)$ which admit a representation of type $\gamma = \varepsilon \circ \tau$ for some $\varepsilon \in \mathbb{E}_k(\Pi)$ and $\tau \in \mathbb{T}_k(\Pi)$; we have $\dim \Gamma_k(\Pi) = \#\mathrm{Col}(\Pi) + \mathrm{rank}(\Lambda(\Pi))$.*

(b) *If Π is quasi-Euclidean and $\mathrm{char}(k) = 0$, then every element $\gamma \in \Gamma_k(\Pi)$ admits a representation of type $\gamma = \varepsilon \circ \delta \circ \sigma$ for some $\varepsilon \in \mathbb{E}(\Pi)$, $\delta \in \mathbb{D}_k(\Pi)$ and $\sigma \in \Sigma(\Pi)$; furthermore $\mathbb{T}_k(\Pi)$ is a maximal torus of $\Gamma_k(\Pi)$.*

(c) *If all facets of* Π *are facet-separated polytopes then the exact analogues of* (a) *and* (b) *hold for any infinite field* k; *moreover, for any enumeration* $\operatorname{Col}(\Pi) = \{V_1, \ldots, V_s\}$ *and every element* $\gamma \in \Gamma_k(\Pi)^0$ (*under the hypothesis of* (a)) *and* $\gamma \in \Gamma_k(\Pi)$ (*under the hypothesis of* (b)) *there is a representation*

$$\gamma = e_{V_1}^{\lambda_1} \circ \cdots \circ e_{V_s}^{\lambda_s} \circ \delta \circ \sigma,$$

where $\lambda_1, \ldots, \lambda_s \in k$, $\delta \in \mathbb{D}_k(\Pi)$ *and* $\sigma \in \Sigma(\Pi)$.

Remark 5.3. The proof we present below yields the same result for algebras of type $k[\Pi_{\mathrm{rat}}]$ where Π_{rat} is a rational polyhedral complex of the appropriate type (see Remark 4.6).

We need some preparation. Throughout this section k is a field and Π is an oriented lattice polyhedral complex.

A convention: for an element $z \in k[\Pi]$ let $\operatorname{Supp}(z)$ denote the set of the facets $P \prec \Pi$ such that S_P contains a monomial appearing in the canonical k-linear expansion of z. (This notation is compatible with the previous one for lattice points).

For any face $P \in \Pi$ the canonical split epimorphism

$$k[\Pi] \to k[S_P], \quad x \mapsto 0 \text{ for } x \in \mathrm{L}(\Pi) \setminus \mathrm{L}(P),$$

will be denoted by π_P. Thus π_P is split by the inclusion $\iota_P : k[S_P] \to k[\Pi]$. Note that

$$(\dagger) \qquad \operatorname{Supp}(z) = \{P \mid \pi_P(z) \neq 0\}.$$

Lemma 5.4. $\{\operatorname{Ker}(\pi_P) \mid P \prec \Pi$ *a facet*$\}$ *is the set of minimal prime ideals of* $k[\Pi]$.

The proof is straightforward.

Lemma 5.5. *Let* $\gamma \in \Gamma_k(\Pi)$. *Then there is a permutation of the set of facets* $P \in \Pi$, *say* ρ_γ, *such that* $\gamma(\operatorname{Ker}(\pi_P)) = \operatorname{Ker}(\pi_{\rho_\gamma(P)})$ *for all facets* P *of* Π. *The assignment* $\gamma \mapsto \rho_\gamma$ *defines a group homomorphism from* $\Gamma_k(\Pi)$ *to the permutation group of the set of facets of* Π. *Its kernel is a closed subgroup of* $\Gamma_k(\Pi)$ *containing* $\Gamma_k(\Pi)^0$.

Proof. The first and second assertion follow immediately from Lemma 5.4, and that the kernel of the assignment $\gamma \mapsto \rho_\gamma$ is a closed subgroup eventually boils down to the statement that the stabilizer of a vector subspace is a closed subgroup of a linear algebraic group acting algebraically on a finite-dimensional vector space. Q.E.D.

Since γ maps $\operatorname{Ker}(\pi_P)$ onto $\operatorname{Ker}(\pi_{\gamma_\rho(P)})$, it induces a (unique) isomorphism γ_P fitting into the commutative diagram

$$\begin{tikzcd} k[\Pi] \arrow[r, "\gamma"] \arrow[d, "\pi_P"'] & k[\Pi] \arrow[d, "\pi_{\rho_\gamma(P)}"] \\ k[S_P] \arrow[r, "\gamma_P"'] & k[S_{\rho_\gamma(P)}]. \end{tikzcd}$$

One obviously has $\gamma_P = \pi_{\rho_\gamma(P)} \circ \gamma \circ \iota_P$.

More generally, let $P_1, \dots, P_s \prec \Pi$ be facets and $Q = P_1 \cap \dots \cap P_s$. We set

$$\rho_\gamma(Q) = \rho_\gamma(P_1) \cap \dots \cap \rho_\gamma(P_s).$$

As above, γ induces an isomorphism γ_Q fitting into the same commutative diagram as above where we only replace P by Q; furthermore $\gamma_Q = \pi_Q \circ \gamma \circ \iota_Q$.

Lemma 5.6. *Suppose γ_P maps monomials to terms for every facet $P \prec \Pi$. Then γ does so as well.*

Proof. Let z be a monomial, $\operatorname{Supp}(z) = \{P_1, \dots, P_s\}$, and $Q = P_1 \cap \dots \cap P_s$. Then $z \in k[S_Q]$, and therefore $\pi_Q(z) \neq 0$. It follows that $\pi_{\rho_\gamma(Q)}(\gamma(z)) \neq 0$ as well. Therefore the canonical k-linear expansion of $\gamma(z)$ must contain a monomial x with $\operatorname{Supp}(x) = \{\rho_\gamma(P_1), \dots, \rho_\gamma(P_s)\}$.

By equation (†) above we likewise have $\operatorname{Supp}(\gamma(z)) = \{\rho_\gamma(P_1), \dots, \rho_\gamma(P_s)\}$. Now the hypothesis implies that x is the only monomial appearing in the k-linear expansion of $\gamma(z)$. Q.E.D.

We also need several facts from [BG, Lemma 4.1, 4.2, 4.3 and Theorem 3.2(b)]. For the reader's convenience we collect them in the following proposition.

Let P be a lattice polytope. As usual, $\bar{S}_P$ stands for the normalization of the semigroup S_P, i.e. $\bar{S}_P = \{x \in \operatorname{gp}(S_P) \mid cx \in S_P \text{ for some } c \in \mathbb{N}\}$. Then $k[\bar{S}_P]$ is a Noetherian normal domain. For any facet $F \prec P$ one has the monomial height 1 prime ideal

$$\operatorname{Div}(F) \subset k[\bar{S}_P]$$

generated by the monomials of $k[\bar{S}_P]$ that do not belong to the facet of the cone $C(\bar{S}_P)$ corresponding to F.

One more observation: since any graded automorphism of $k[S_P]$ extends to a unique graded automorphism and $k[S_P]$ and $k[\bar{S}_P]$ coincide in degree 1, the two rings $k[S_P]$ and $k[\bar{S}_P]$ have the same group $\Gamma_k(P)$ of graded automorphisms.

Proposition 5.7. (a) *An automorphism $\gamma \in \Gamma_k(P)$ inducing a permutation of the set $\{\mathrm{Div}(F) \mid F \prec P$ a facet$\}$ maps monomials to terms.*

(b) *Let $v_1, \dots, v_s$ be column vectors of P with the common base facet $F = P_{v_i}$, $\lambda_1, \dots, \lambda_s \in k$, and $G \neq F$ another facet. Then*

$$e_{v_1}^{\lambda_1} \circ \cdots \circ e_{v_s}^{\lambda_s}(\mathrm{Div}(F)) = (1 + \lambda_1 v_1 + \cdots + \lambda_s v_s)\mathrm{Div}(F),$$
$$e_{v_1}^{\lambda_1} \circ \cdots \circ e_{v_s}^{\lambda_s}(\mathrm{Div}(G)) = \mathrm{Div}(G).$$

(c) *Let $F \prec P$ be a facet, $\lambda_1, \dots, \lambda_s \in k \setminus \{0\}$ and $v_1, \dots, v_s \in \mathrm{gp}(S_P)$ $(\subset \mathrm{Q.F.}(k[\bar{S}_P]))$ be pairwise different nonconstant Laurent monomials of degree 0. Suppose $(\lambda_1 v_1 + \cdots + \lambda_s v_s)\mathrm{Div}(F) \subset k[\bar{S}_P]$. Then $v_1, \dots, v_s$ are column vectors for P with the common base facet F.*

(d) *The connected component of unity $\Gamma_k(P)^0 \subset \Gamma_k(P)$ consists of those graded automorphisms of $k[S_P]$ which induce (by extension to $k[\bar{S}_P]$) the identity map on the divisor class group $\mathrm{Cl}(k[\bar{S}_P])$.*

We will also need the following facts.

Lemma 5.8. *Let $M \subset \mathbb{Z}^m$ be a finite system of Laurent monomials of $k[\mathbb{Z}^m]$ (k is a field and $m \in \mathbb{N}$) and $f, g \in k[\mathbb{Z}^m]$. Assume the k-subspaces of $k[\mathbb{Z}^m]$ generated by $\{xf \mid x \in M\}$ and $\{xg \mid x \in M\}$ coincide. Then $f = ag$ for some $a \in k^*$.*

Proof. The case $\#(M) = 1$ is trivial, and for the general case we use induction as follows. There is a $\mathbb{Z}$-linear form ϕ such that ϕ attains its maximal value on each of the following polytopes in a single point: the Newton polytopes $\mathrm{N}(f)$, $\mathrm{N}(g)$ and the convex hull $P(M)$ of M; in $P(M)$ let $\phi(z)$ be the maximum. Then z is a vertex of $P(M)$, hence $z \in M$, and we can pass to $M \setminus \{z\}$. Q.E.D.

Lemma 5.9. *Let G be an algebraic $\mathbb{C}$-group and $X \subset G$ be a Zariski closed subset with $\dim X < \dim G$. Then there is an element $g \in G$ such that none of the powers of g is in X.*

Proof. Passing to G^0 we may assume that G is connected and therefore irreducible. For any natural number c the algebraic mapping $\mathrm{pow}_c : G \to G$, $g \mapsto g^c$, is not globally degenerate since it is not degenerate in a small neighborhood of $1 \in G$ (the differential at 1 is the multiplication by c on the tangent space). In particular, $\mathrm{pow}_c^{-1}(X) \subset G$ is a Zariski closed subset of dimension strictly less then $\dim G$. (Otherwise we would have $\mathrm{pow}_c^{-1}(X) = G$, and pow_c would be degenerate everywhere.) Therefore, $\bigcup_1^\infty \mathrm{pow}_c^{-1}(X) \subset G$ is a proper subset. Q.E.D.

Proof of Theorem 5.2(a). Choose $\gamma \in \Gamma_k(\Pi)^0$. By Lemma 5.5 $\rho_\gamma = 1_{\text{facet}}$. As seen above, γ induces a graded k-automorphisms $\gamma_P : k[S_P] \to k[S_P]$ for each facet $P \prec \Pi$. More generally, for a finite system of facets $P_1, \dots, P_s \prec \Pi$ there is a graded automorphism $\gamma_Q : k[S_Q] \to k[S_Q]$, $Q = P_1 \cap \dots \cap P_s$ induced by γ. We let $\bar\gamma_Q$ denote the unique automorphic extension of γ_Q to $k[\bar S_Q]$. Clearly, the assignment $\gamma \mapsto \gamma_Q$ defines an algebraic group homomorphism $\Gamma_k(\Pi) \to \Gamma_k(Q)$. In particular, if $\gamma \in \Gamma_k(\Pi)^0$ then, by Proposition 5.7(d), the automorphism $\bar\gamma_Q$ induces the identity map on $\mathrm{Cl}(k[\bar S_Q])$.

For a pair of faces $P_1 \prec P_2 \prec \Pi$

$$\pi_{P_2P_1} : k[S_{P_2}] \to k[S_{P_1}], \quad \mathrm{L}(P_2) \setminus \mathrm{L}(P_1) \to 0,\ x \mapsto x \text{ for } x \in \mathrm{L}(P_1)$$

will denote the 'face' projection. Further, we let $\bar S_{P_2P_1}$ denote the subsemigroup of $\bar S_{P_2}$ that corresponds to the face $P_1 \prec P_2$ and let $\bar\pi_{P_2P_1}$ denote the corresponding face projection from $k[\bar S_{P_2}]$ to $k[\bar S_{P_2P_1}]$. In particular, $\pi_{P_2P_1}$ and $\bar\pi_{P_2P_1}$ coincide on $\mathrm{L}(P_2)$. By Proposition 2.5(c) the inclusion $\bar S_{P_1} \subset \bar S_{P_2P_1}$ may be strict.

Step 1. Let P be a face of Π, F facet of P and $\gamma \in \Gamma_k(\Pi)^0$. By Lemma 5.7(d) $\bar\gamma_P$ leaves the class of $\mathrm{Div}(F) \subset k[\bar S_P]$ invariant, i.e.

$$\bar\gamma_P(\mathrm{Div}(F)) = d\,\mathrm{Div}(F) \tag{1}$$

for some $d \in \mathrm{Q.F.}(k[\bar S_P])$. Since $\bar\gamma_P$ is a graded automorphism, d must be a homogeneous element of degree 0. Moreover, since $\mathrm{Div}(F)$ is a monomial ideal, d is a sum of degree 0 Laurent terms of $\mathrm{Q.F.}(k[\bar S_P])$. Say $d = a_1\mu_1 + \dots + a_s\mu_s$, where $a_1, \dots, a_s \in k^*$ and $\mu_1, \dots, \mu_s$ are pairwise different degree 0 Laurent monomials of $\mathrm{gp}(S_P)$. Assume that $\mu_i \neq 1$. Then by Proposition 5.7(c) (P, μ_i) is a column structure.

We claim that (P, μ_i) gives rise to a column vector for Π.

First we must show that if μ_i is a column vector for some face $Q \prec P$ and there is a face $R \prec \Pi$ containing Q, then there is a column structure on R restricting to the same column structure on Q. By enlarging Q to the intersection $P \cap R$ we may assume without loss of generality that $Q = P \cap R$.

We have a column structure (Q, μ_i) with the base facet Q_{μ_i} (of Q). There clearly exists a facet $G \prec R$ such that $G \cap Q = Q_{\mu_i}$. Fix any such a facet G (below it will become clear that G is unique) and consider the height 1 prime ideal $\mathrm{Div}(G) \subset k[\bar S_R]$. (Figure 6 illustrates the relation between P, Q, Q_{μ_i}, R, F, and G.) By the same reasons as for P one has

$$\bar\gamma_R(\mathrm{Div}(G)) = (b_1\nu_1 + \dots + b_t\nu_t)\mathrm{Div}(G) \tag{2}$$

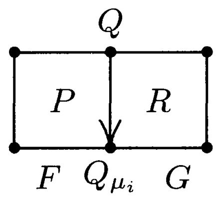


FIGURE 6.

for *uniquely* determined pairwise different degree 0 Laurent monomials $\nu_1, \dots, \nu_t \in \operatorname{gp}(S_Q)$ and $b_1, \dots, b_t \in k^*$. We have the following set-theoretical inclusions:

$$\operatorname{Div}(Q_{\mu_i}) \subset \bar{\pi}_{PQ}(\operatorname{Div}(F)) \quad \text{and} \quad \operatorname{Div}(Q_{\mu_i}) \subset \bar{\pi}_{RQ}(\operatorname{Div}(G)),$$

where $\operatorname{Div}(Q_{\mu_i})$ is the corresponding height 1 prime ideal of $k[\bar{S}_Q]$.

We also know that there is a representation

$$\bar{\gamma}_Q(\operatorname{Div}(Q_{\mu_i})) = (c_1\kappa_1 + \cdots + c_r\kappa_r)\operatorname{Div}(Q_{\mu_i}), \tag{3}$$

where $\kappa_1, \dots, \kappa_r$ are pairwise different degree 0 Laurent monomials from $\operatorname{gp}(S_Q)$ and $c_1, \dots, c_r \in k^*$.

By the construction of γ_Q and γ_P we have

$$\pi_{PQ} \circ \gamma_P = \gamma_Q \quad \text{and} \quad \pi_{RQ} \circ \gamma_R = \gamma_R.$$

It is clear that S_P coincides with $\bar{S}_P$ in degree 1, and similarly this holds for S_R and S_Q. Hence the equalities (1), (2) and (3) imply

$$\begin{aligned}
\bar{\pi}_{PQ} \circ \bar{\gamma}_P(\operatorname{Div}(F)_1) &= (\textstyle\sum_j a_{i_j}\mu_{i_j})\pi_{PQ}(\operatorname{Div}(F)_1)\\
&= (c_1\kappa_1 + \cdots + c_r\kappa_r)\operatorname{Div}(Q_{\mu_i})_1\\
\bar{\pi}_{RQ} \circ \bar{\gamma}_R(\operatorname{Div}(G)_1) &= (\textstyle\sum_l b_{k_l}\nu_{k_l})\pi_{RQ}(\operatorname{Div}(G)_1)\\
&= (c_1\kappa_1 + \cdots + c_r\kappa_r)\operatorname{Div}(Q_{\mu_i})_1,
\end{aligned}$$

where $\operatorname{Div}(-)_1$ refers to the corresponding degree 1 homogeneous component, and the summations are considered for

$$\mu_{i_j} \in \operatorname{gp}(S_Q) \cap \{\mu_1, \dots, \mu_s\} \quad \text{and} \quad \nu_{k_l} \in \operatorname{gp}(S_Q) \cap \{\nu_1, \dots, \nu_t\};$$

of course, the first intersection is taken in $\operatorname{gp}(S_P)$ ($\supset \operatorname{gp}(S_Q)$) and the second one in $\operatorname{gp}(S_R)$ ($\supset \operatorname{gp}(S_Q)$). By Lemma 5.8 we see that in the representation $\bar{\gamma}_Q(\mathrm{Div}(G)) = (b_1\nu_1 + \cdots + b_t\nu_t)\mathrm{Div}(G)$ one of the ν_k is μ_i.

Next we show that each μ_i satisfies the condition

$$\mathrm{Supp}(x\mu_i) \subset \mathrm{Supp}(x)$$

for every $x \in \mathrm{L}(P) \setminus \mathrm{L}(F)$. Assume to the contrary that there are a point $x \in \mathrm{L}(P) \setminus \mathrm{L}(F)$ and a facet $T \prec \Pi$ such that $\mu_i x \in T$ and $x \notin T$. We have

$$z = x(a_1\mu_1 + \cdots + a_s\mu_s) \in \bar{\gamma}_P(\mathrm{Div}(F)).$$

Since $k[\bar{S}_P]$ and $k[S_P]$ have the same degree 1 components, $z \in \mathrm{Div}(F) \cap k[S_P]$; by assumption $T \in \mathrm{Supp}(\gamma(z))$. Let I be the annihilator of $\mathrm{Ker}(\pi_T)$; then I is spanned by the monomials $\mu \in S_T$ that do not belong to S_R for any other facet $R \prec \Pi$. We have $\gamma(z) \cdot I \neq 0$. On the other hand, $z \cdot I = 0$. Thus we get the desired contradiction, because $\gamma(I) = I$, as follows from $\gamma(\mathrm{Ker}(\pi_T)) = \mathrm{Ker}(\pi_T)$.

Finally, assume (P, μ_i) restricts to a column structure on $P \cap R$ for some $R \prec \Pi$. Then we know already that there is a column structure (R, ν) restricting to the same column structure on $P \cap R$. But what we have shown is more. Namely,

(4) the column vector ν for the face $R \prec \Pi$ is derived from γ exactly in the same way as μ_i for P.

Thus the above arguments apply to the column structure (R, ν) as well, yielding condition $(\#_1)$ for it.

Step 2. We fix an enumeration of the facets of Π, say $P_1, P_2, \ldots$. For each facet $P_p \prec \Pi$ we also fix an enumeration of the facets of P_p, say $F_{p1}, F_{p2}, \ldots$. Consider a total ordering of the pairs (p, q). Then for each (p, q) the subgroup

$$\begin{aligned}\Gamma_{pq} = \{\gamma \in \Gamma_k(\Pi) \mid \bar{\gamma}_{P_r}(\mathrm{Div}(F_{rs})) = \mathrm{Div}(F_{rs}) \text{ for all } (r,s) \le (p,q)\} \\ \subset \Gamma_k(\Pi)\end{aligned}$$

is (Zariski) closed. In fact, it is the intersection of the stabilizers of finitely many vector subspaces. By Lemma 5.6 and Proposition 5.7(a) we have the equality

$$\mathbb{D}_k(\Pi) = \Gamma_{pq_{\max}}$$

where $(p,q)_{\max}$ is the maximal pair. Now we enlarge the set of pairs (p,q) by one element $(0,0)$, declare it as the smallest element of the new system and set $\Gamma_{00} = \Gamma_k(\Pi)$. We then have the sequence of affine groups

$$(*) \qquad \mathbb{D}_k(\Pi) = \Gamma_{pq_{\max}} \subset \ldots \subset \Gamma_{pq} \subset \ldots \subset \Gamma_{rs} \subset \ldots \subset \Gamma_{00} = \Gamma_k(\Pi)$$

for $(p,q) > (r,s)$.

Claim. *If $(p,q) > (r,s)$ are consecutive pairs and $\gamma \in \Gamma_{rs}$, then $E \circ \gamma^c \in \Gamma_{pq}$ for some natural number c and some element $E \in \mathbb{E}_k(\Pi)$.*

Without loss of generality we can assume (by passing to some power) that $\gamma \in \Gamma_k(\Pi)^0$.

Recall that we have identified $\Gamma_k(P)$ with the corresponding closed subgroup of $\mathrm{GL}_N(k)$, $N = \#\mathrm{L}(P) - 1$ (see Section 1). There is a finitely generated *subring* $\Lambda \subset k$ such that $\gamma, \gamma^{-1} \in \mathrm{GL}_N(\Lambda)$. Let k_0 denote any residue field of Λ. Thus k_0 is a finite field. Let γ_0 denote the reduction of γ in $\mathrm{GL}_N(k_0)$, a finite group. So there exists a natural number c such that $(\gamma_0)^c$ is the identity map of $k_0[\Pi]$, i.e. the identity matrix of $\mathrm{GL}_N(k_0)$. We will show that c is the desired number.

First we want to show that if $P \prec \Pi$ is any facet and $F \prec P$ is a facet of P, then

$$(\bar{\gamma}_P)^c(\mathrm{Div}(F)) = (1 + a_1 m_1 + \cdots + a_n m_n)\mathrm{Div}(F)$$

for some degree zero non-constant monomials $m_1, \dots, m_n \in \mathrm{gp}(S_P)$ and $a_1, \dots, a_n \in k^*$. (We do not exclude the case $n = 0$.)

All we need for this assertion is that $((\gamma_0)_P)^c$ is the identity map of $k_0[P]$ and

$$(\bar{\gamma}_P)^c(\mathrm{Div}(F)) = (\mu_1 + \cdots + \mu_n)\mathrm{Div}(F)$$

for some degree zero Laurent terms $\mu_1, \dots, \mu_n \in \mathrm{Q.F.}(k[S_P])$ (see the previous step). Now assume to the contrary that none of the μ_i is an element of k^*. Looking at the homogeneous degree 1 component (as we did in Step 1) we get

$$(\gamma_P)^c(\mathrm{Div}(F)_1) = (\mu_1 + \cdots + \mu_n)\mathrm{Div}(F)_1.$$

By Proposition 5.7(c) each of the $\mu_i \neq 1$ is a column vector for P with base facet F. The corresponding semigroup homomorphisms $\mathrm{ht}_{\mu_i} : S_P \to \mathbb{Z}_+$ are all the same. Let $x \in \mathrm{L}(P)$ be any point with the maximal possible value of $\mathrm{ht}_{\mu_i}(x)$. Clearly, $x \in \mathrm{Div}(F)$. By our assumption none of the elements of $(\mu_1 + \cdots + \mu_n)\mathrm{Div}(F)_1$ may involve the monomial x in its canonical k-linear expansion. But this contradicts the condition that $\mathrm{Div}(F)_1$ and $(\mu_1 + \cdots + \mu_n)\mathrm{Div}(F)_1$ have the same images in $k_0[S_P]$.

In particular we have

$$(\bar{\gamma}_{P_p})^c(\mathrm{Div}(F_{pq})) = (1 + a_1 m_1 + \cdots + a_n m_n)\mathrm{Div}(F_{pq})$$

for $a_1, \dots, a_n$ and $m_1, \dots, m_n$ as above. By Step 1 each of the monomials m_i defines a column structure on Π. Consider the automorphism

$$E = e_{V_1}^{a_1} \circ \cdots \circ e_{V_n}^{a_n} \in \mathbb{E}_k(\Pi),$$

where $V_1 = [P, m_1], \dots, V_n = [P, m_n]$. By Proposition 5.7(b) we get

$$(\overline{E^{-1} \circ \gamma^c})_{P_p}(\mathrm{Div}(F_{pq})) = \mathrm{Div}(F_{pq}).$$

Clearly, $(\overline{\gamma^c})_{P_t}(\mathrm{Div}(F_{tu})) = \mathrm{Div}(F_{tu})$ for $(t, u) < (p, q)$. Therefore, by (4) in Step 1 and 5.7(b) $(\overline{E^{-1} \circ \gamma^c})_{P_r}$ also leaves $\mathrm{Div}(F_{rs})$ untouched for any pair $(r, s) < (p, q)$. The claim has been proved.

We record a property of the column structures V_i that will be important below:

(5) (p, q) is the smallest (with respect to $<$) among all pairs (t, u) such that V_i contains (P_t, v) with base facet F_{tu}.

This follows from the construction of V_i in Step 1: if $\gamma(\mathrm{Div}(F_{tu})) = \mathrm{Div}(F_{tu})$, then V_i cannot contain a column vector with base facet F_{tu}.

Step 3. For two subsets $A, B \subset G$ of a group G we let $A \cdot B$ denote the subset $\{ab \mid a \in A, b \in B\} \subset G$ and $\big(AB\big)$ the subgroup of G generated by A and B.

Consider the special case $k = \mathbb{C}$. Assume $(p, q) > (r, s)$ are consecutive pairs. We will show the equality of the two connected groups

$$(**) \qquad \big(\mathbb{E}_k(\Pi)\Gamma^0_{rs}\big) = \big(\mathbb{E}_k(\Pi)\Gamma^0_{pq}\big),$$

where $-^0$ refers to the corresponding unity component. That these groups are in fact connected follows from Lemma 4.5 and [Bo, Proposition 2.2].

Consider the partition into right cosets

$$\Gamma_{pq} = \Gamma^0_{pq} g_1 \cup \dots \cup \Gamma^0_{pq} g_t \qquad g_1, \dots, g_t \in \Gamma_{pq}.$$

We have

$$\mathbb{E}_k(\Pi) \cdot \Gamma_{pq} \subset Y g_1 \cup \dots \cup Y g_t,$$

where $Y = \big(\mathbb{E}_k(\Pi)\Gamma^0_{pq}\big)$. By Step 2 there is a natural number c for any $\gamma \in \Gamma^0_{rs}$ such that

$$\gamma^c \in (\Gamma^0_{rs} \cap Y g_1) \cup \dots \cup (\Gamma^0_{rs} \cap Y g_t).$$

Omitting some g_i if necessary we get a disjoint union $Y g_1 \cup \dots \cup Y g_t = Y \cup Y g_{l_2} \cup Y g_{l_3} \cup \cdots$ of right cosets of Y. By Lemma 5.11

$$\dim \Gamma^0_{rs} = \dim\, (\Gamma^0_{rs} \cap Y) \cup (\Gamma^0_{rs} \cap Y g_{l_2}) \cup (\Gamma^0_{rs} \cap Y g_{l_3}) \cdots .$$

Hence, by the irreducibility of Γ^0_{rs} and the fact that $\Gamma^0_{rs} \cap Y \neq \emptyset$, we arrive at the inclusion $\Gamma^0_{rs} \subset Y$. Therefore, $\big(\mathbb{E}_k(\Pi)\Gamma^0_{rs}\big) \subset Y$. The opposite inclusion is obvious, hence the equality $(**)$.

The equality (**) and the sequence (*) in Step 2 imply

$$\Gamma_k(\Pi)^0 = \big(\mathbb{E}_k(\Pi)\mathbb{T}_k(\Pi)\big).$$

Now the same equality holds for an arbitrary subfield $k \subset \mathbb{C}$ because of the following general observations. Since $\Gamma_{\mathbb{C}}(\Pi)$ is defined over k, so is its unity component $\Gamma_{\mathbb{C}}(\Pi)^0$ [Bo, Proposition 1.2]. By the Lemmas 3.2 and 4.4(b) the connected subgroup $\big((\mathbb{E}_{\mathbb{C}}(\Pi)\mathbb{T}_{\mathbb{C}}(\Pi)\big) \subset \Gamma_{\mathbb{C}}(\Pi)$ is likewise defined over k. If the two irreducible affine k-varieties were different, then they would remain so after the scalar extension $k \to \mathbb{C}$, which is not the case.

Consider the case of an arbitrary field k of characteristic 0. If $\gamma \in \Gamma_k(\Pi)$, then γ is defined over a finitely generated subfield $k_0 \subset k$. Choosing any embedding $k_0 \to \mathbb{C}$ we fall in the previous case.

Finally, by Lemma 4.4(b) we have the equality $\big(\mathbb{E}_k(\Pi)\mathbb{T}_k(\Pi)\big) = \mathbb{E}_k(\Pi) \cdot \mathbb{T}_k(\Pi)$.

Step 4. We have to compute the dimension. As in Step 3 we may assume $k = \mathbb{C}$. For each facet $P \prec \Pi$ fix an interior monomial $x \in \operatorname{int}(S_P)$, i.e. a monomial corresponding to an interior point of the cone C_P. Let (P, v) be a column structure defining a column vector V_1 for Π. Assume $\operatorname{Col}(\Pi) = \{V_1, V_2, \ldots, V_s\}$. Then for arbitrary elements $\lambda_1, \lambda_2, \ldots, \lambda_s \in k^*$ the set of monomials appearing in the canonical k-linear expansion of $e_{V_1}^{\lambda_1}(x)$ is not covered by those appearing in the k-linear expansions of $e_{V_2}^{\lambda_2}(x)$, $e_{V_3}^{\lambda_3}(x)$ and so on (just look at the projection of x through v into the base facet P_v). This shows that we have $\#\operatorname{Col}(\Pi)$ linearly independent tangent vectors at $1 \in \Gamma_k(\Pi)$. Since the tangent vectors corresponding to the elements of $\mathbb{T}_k(\Pi)$ clearly belong to a complementary dimension and $\Gamma_k(\Pi)^0$ is a smooth variety, by Lemma 3.2 we conclude

$$\dim \Gamma_k(\Pi)^0 \geq \#\operatorname{Col}(\Pi) + \operatorname{rank}(\Lambda(\Pi)).$$

The opposite inequality is derived as follows. For any pair (r, s) we let $\mathbb{E}_{rs}$ denote the subgroup of $\mathbb{E}_k(\Pi)$ generated by elementary automorphisms of type e_V^λ where $V = [P_r, v]$ and such that (r, s) is the smallest (with respect to $<$) pair for which F_{rs} appears as a base facet of V (in particular $(P_r)_v = F_{rs}$). Let $\{V_1, \ldots, V_m\}$ denote the set of column vectors for Π that contribute to $\mathbb{E}_{rs}$. Essentially the same arguments as in the proof of Lemma 3.1 in [BG] show that the assignment

$$(\lambda_1, \ldots, \lambda_m) \mapsto e_{v_1}^{\lambda_1} \circ \cdots \circ e_{v_m}^{\lambda_m}$$

establishes the isomorphism of the abelian affine groups $\mathbb{A}_k^m$ and $\mathbb{E}_{rs}$.

We claim that for all consecutive pairs $(r,s) < (p,q)$ there is an element $g_{pq} \in \Gamma_k(\Pi)$ such that the subset

$$(\mathbb{E}_{pq} \cdot \Gamma^0_{pq})g_{pq} \subset \Gamma_k(\Pi)$$

contains a Zariski open subset of Γ^0_{rs}. In fact, by the property (5) of the automorphism E in Step 2, Lemma 5.9 and the irreducibility of Γ^0_{pq} we have

$$\Gamma^0_{rs} \subset \overline{\mathbb{E}_{pq} \cdot \Gamma_{pq}},$$

where the bar on the right hand side means the Zariski closure (in $\Gamma_k(\Pi)$). Now the claim follows from the facts that Γ_{pq} decomposes into finite number of right cosets of Γ^0_{pq} and that for each of these cosets, say $\Gamma^0_{pq}g$, the subset $(\mathbb{E}_{pq} \cdot \Gamma^0_{pq})g \subset \Gamma_k(\Pi)$ is constructible (and, hence, contains a Zariski open set of its closure). In particular we have the equality $\dim(\mathbb{E}_{pq} \cdot \Gamma^0_{pq}) \geq \dim \Gamma_{rs}$. In view of the sequence $(*)$ of the groups Γ_{tu} we get

$$\dim \Gamma_k(\Pi) \leq \dim \mathbb{T}_k(\Pi) + \Sigma_{(p,q)\neq(0,0)} \dim \mathbb{E}_{pq} = \operatorname{rank}(\Lambda(\Pi)) + \#\mathrm{Col}(\Pi).$$

Q.E.D.

Proof of Theorem 5.2(b). Suppose Π is a quasi-Euclidean lattice polyhedral complex, $\operatorname{char}(k) = 0$ and $\gamma \in \Gamma_k(\Pi)$. By Lemma 3.5(b) $\mathbb{T}_k(\Pi)$ is a maximal torus of $\Gamma_k(\Pi)$. By Theorem 5.2(a) and Lemma 3.4(a) it suffices to show that there is an element $\alpha \in \Gamma_k(\Pi)^0$ such that $\alpha \circ \gamma$ maps monomials to terms.

Consider the closed subgroup

$$\mathbb{D} = \gamma \mathbb{D}_k(\Pi)\gamma^{-1} \subset \Gamma_k(\Pi).$$

Its unity component is $\mathbb{D}^0 = \gamma \mathbb{T}_k(\Pi)\gamma^{-1}$. In particular, $\mathbb{D}^0$ is a maximal torus of $\Gamma_k(\Pi)$. By [Bo, Corollary 11.3(1)] there is an element $\alpha \in \Gamma_k(\Pi)^0$ such that $\alpha^{-1}\mathbb{T}_k(\Pi)\alpha = \mathbb{D}^0$. We get

$$(\alpha \circ \gamma)^{-1}\mathbb{T}_k(\Pi)(\alpha \circ \gamma) = \mathbb{T}_k(\Pi). \tag{1}$$

We claim that α is the desired element.

Assume to the contrary that there is a monomial $x \in k[\Pi]$ such that in the canonical k-linear expansion of $\alpha \circ \gamma(x)$ there occur two distinct monomials $y_1, y_2 \in k[\Pi]$.

By Lemma 3.5(a) there exist $\tau \in \mathbb{T}_k(\Pi)$ and two distinct elements $a_1, a_2 \in k^*$ such that $\tau(y_1) = a_1y_1$ and $\tau(y_2) = a_2y_2$. Therefore, there does not exist $a \in k^*$ for which $\tau \circ \alpha \circ \gamma(x) = a \cdot (\alpha \circ \gamma(x))$, or equivalently, there does not exist $a \in k^*$ such that $(\alpha \circ \gamma)^{-1} \circ \tau \circ (\alpha \circ \gamma)(x) = ax$ and this contradicts (1). Q.E.D.

Proof of Theorem 5.2(c). Let P be a facet-separated lattice polytope and $F \prec P$ be one of its facets. Suppose $m_1, \dots, m_s \in \mathrm{gp}(S_P)$ are pair-wise different Laurent monomials of degree 0 and $a_1, \dots, a_s \in k^*$ such that $(a_1m_1 + \cdots + a_sm_s)\mathrm{Div}(F)$ is a height 1 prime ideal of $k[\bar{S}_P]$ (notation as in Step 1); we claim that either one of the m_i is $1 \in k$ or

$$(a_1m_1 + \cdots + a_sm_s)\mathrm{Div}(F) = \mathrm{Div}(G)$$

for some facet $G \prec P$.

Indeed, if none of the m_i is $1 \in k$ then $(a_1m_1 + \cdots + a_sm_s)\mathrm{Div}(F) \subset \mathrm{Div}(G)$ for any facet $G \prec P$ such that $F \cap G = \emptyset$, which exists by assumption. We get an inclusion of two height 1 prime ideals. Hence the desired equality.

Note that the lower dimensional faces of a facet-separated polytopes need not be facet-separated again. Therefore one has to modify Step 1 slightly by working only with the facets of Π and condition $(\#_2')$. The fact just proved implies that in Step 2 above one can then take $c = 1$. Thus the restriction to fields of characteristic 0, which entered the proof only via Lemma 5.9, becomes superfluous, and all the arguments go through for an arbitrary field. We only need k to be infinite in order to be able to apply Lemma 3.5.

The existence of the normal forms, claimed in Theorem 5.2(c), follows immediately since we deal just once with each column vector during the whole process and since the process can be carried out in an arbitrary order of the column vectors. Q.E.D.

§6. Arrangements of projective toric varieties

In this section we develop some notions similar to those in [BG, Section 5], generalized from single polytopes to the new situation of polyhedral complexes. (For standard facts on toric varieties we refer to Danilov [Da], Fulton [Fu], Oda [Oda].)

Throughout this section k denotes an algebraically closed field.

A lattice polytope $P \subset \mathbb{R}^n$ is called *very ample* if for every vertex $v \in P$ the affine semigroup in $\mathbb{Z}^n$, defined by the $\dim(P)$-dimensional cone spanned by P at its corner v and then shifted by $-v$, is generated by the set

$$\{x - v \mid x \in \mathbb{Z}^n \cap P\}.$$

All normal lattice polytopes (i.e. those for which $k[S_P]$ is normal) are very ample, but not conversely [BG, Example 5.5].

A lattice polyhedral complex is called *very ample* if all its faces are very ample. Observe that it would suffice to require very ampleness only for the facets: the property is inherited by the lower-dimensional faces.

Suppose Π is a very ample lattice polyhedral complex and $z \in \Pi$ is a vertex. Then we define the *associated weak fan* $\Pi(z)$ as follows. Consider the faces $P \prec \Pi$ containing z. For any such face we have the rational polyhedral cone $C(P, z) \subset \mathbb{R}^{n_P}$ spanned by P at its vertex z and shifted (in $\mathbb{R}^{n_P}$) by $-z$. Due to the very ampleness of Π the system of these cones forms a weak fan in a natural way which we denote by $\Pi(z)$. Therefore, the cones of $\Pi(z)$ are naturally labeled by the faces $P \prec \Pi$ such that $z \in P$. We will denote them by $\mathcal{W}_P^{(z)}$ correspondingly.

Now assume we are given a finite system of vertices $z_1, \dots, z_k \prec \Pi$. For each face $P \prec \Pi$ we define the convex (but not necessarily strictly convex) rational cone $C(P, z_1, \dots, z_s) \subset \mathbb{R}^{n_P}$ as follows. If $\{z_1, \dots, z_s\}$ is not a subset of P, then we put $C(P, z_1, \dots, z_s) = \{0\} \subset \mathbb{R}^{n_P}$. If $\{z_1, \dots, z_s\} \subset P$ then there are two possibilities – either there is a supporting halfspace for P (in $\mathbb{R}^{n_P}$) that contains $\{z_1, \dots, z_s\}$ in its boundary, or such does not exist. In the first case we let $C(P, z_1, \dots, z_s)$ be the intersection (in $\mathbb{R}^{n_P}$) of all these supporting halfspaces for P, shifted after that by one of the $-z_i$ (all these parallel translates coincide). In the second case we put $C(P, z_1, \dots, z_n) = \mathbb{R}^{n_P}$.

Observe that if $P \prec Q$, then $C(P, z_1, \dots, z_s)$ is a face (in the obvious sense) of $C(Q, z_1, \dots, z_s)$. In particular, we can patch the semigroup algebras

$$k[C(P, z_1, \dots, z_s) \cap \mathbb{Z}^{n_P}], \quad P \prec \Pi,$$

using these 'face identifications' for all pairs $P \prec Q$ as we did for weak fans in Section 2. The resulting k-algebra will be denoted by $k[\Pi(z_1, \dots, z_n)]$. It is a common localization of the $k[\Pi(z_i)]$.

In the following we will use the notations $\mathcal{Z}_\Pi = \operatorname{Proj}(k[\Pi])$ and $\mathcal{Z}_P = \operatorname{Proj}(k[S_P])$ for $P \prec \Pi$. Thus $\mathcal{Z}_P$ is a normal projective toric variety (the normality follows from the very ampleness of P), and all normal projective toric varieties arise in this way. For each face $P \prec \Pi$ we have fixed an embedded torus of $\mathcal{Z}_P$ – namely, the one that respects the monomial structure of S_P. Let $\mathbb{T}(\mathcal{Z}_P)$ denote this torus. Thus $\mathbb{T}(\mathcal{Z}_P) = \mathbb{T}_k(P)/k^*$. If $P \prec Q \prec \Pi$ then we have the closed embeddings $\mathcal{Z}_P \subset \mathcal{Z}_Q \subset \mathcal{Z}_\Pi$ given by the 'face' projections of the corresponding homogeneous rings. We get a diagram of toric varieties and a corresponding diagram of their embedded tori,

$$\mathcal{D}_\Pi = \{\mathcal{Z}_P \subset \mathcal{Z}_Q \mid P \prec Q \prec \Pi\}$$

and $$\mathcal{D}_T = \{\mathbb{T}(\mathcal{Z}_Q) \xrightarrow{\text{rest}} \mathbb{T}(\mathcal{Z}_P) \mid P \prec Q \prec \Pi\}$$

where 'rest' denotes the restriction map. We set

$$\mathbb{D}(\mathcal{Z}_\Pi) = \varprojlim \mathcal{D}_T \quad \text{and} \quad \mathbb{T}(\mathcal{Z}_P) = \mathbb{D}(\mathcal{Z}_\Pi)^0.$$

In general, $\mathbb{D}(\mathcal{Z}_\Pi) \neq \mathbb{T}(\mathcal{Z}_\Pi)$, as can be seen as follows: if Π' is the cone over Π (adding exactly one more lattice point corresponding to a new variable), then $\mathbb{D}(\mathcal{Z}_{\Pi'}) = \mathbb{D}_k(\Pi)$, and in Section 3 we have given an complex Π with $\mathbb{D}_k(\Pi) \neq \mathbb{T}_k(\Pi)$.

One has the following easily verified description of $\mathcal{Z}_\Pi$ (see also [BG, Section 5]).

Proposition 6.1. *Let Π be a very ample lattice polyhedral complex.*

(a) *The projective variety $\mathcal{Z}_\Pi \subset \mathbb{P}^N$, $N = \#\mathrm{L}(\Pi) - 1$, is obtained by patching the affine schemes* $\mathrm{Spec}(k[\Pi(z)])$ *along their open subschemes*

$$\mathrm{Spec}(k[\Pi(z, z_1, \dots, z_s)]) \subset \mathrm{Spec}(k[\Pi(z)]),$$

where $z, z_1, \dots, z_s$ are vertices of Π.

(b) *The irreducible components of $\mathcal{Z}_\Pi$ are precisely the normal projective toric varieties $\mathcal{Z}_P = \mathrm{Proj}(k[\bar{S}_P]) \subset \mathbb{P}^{N_P}$, $N_P = \#\mathrm{L}(P) - 1$, where P runs through the facets of Π. Moreover,*

$$\mathcal{Z}_\Pi = \varinjlim \mathcal{D}_\Pi.$$

(c) *$\mathbb{D}(\mathcal{Z}_\Pi)$ is a diagonalizable group and, hence, $\mathbb{T}(\mathcal{Z}_\Pi)$ is a torus; they act algebraically on $\mathcal{Z}_\Pi$ so that for each face $P \prec \Pi$ the action restricts to the original one of $\mathbb{T}(\mathcal{Z}_P)$ on $\mathcal{Z}_P$.*

$\mathbb{D}(\mathcal{Z}_\Pi)$ is diagonalizable since it is a subgroup of the product of the $\mathbb{T}(\mathcal{Z}_P)$, $P \prec \Pi$.

Projective varieties of type $\mathcal{Z}_\Pi$ with Π very ample are called *arrangements of projective toric varieties* and the affine charts, described in Proposition 6.1(a), will be called Π-*affine charts.*

One easily observes the exact sequence of algebraic groups

$$0 \to (k^*)^{\pi_0(\Pi)} \to \Gamma_k(\Pi) \xrightarrow{\mathrm{pr}_\Pi} \mathrm{Aut}_k(\mathcal{Z}_\Pi),$$

where pr_Π is the canonical anti-homomorphism and $\pi_0(\Pi)$ refers to the (number of) connected components of Π (viewed as a CW-complex in a natural way).

It is clear that $\mathbb{D}_k(\Pi)$ is mapped to $\mathbb{D}(\mathcal{Z}_\Pi)$ by pr_Π. However, $\mathrm{pr}_\Pi(\mathbb{D}_k(\Pi))$ is in general smaller than $\mathbb{D}(\mathcal{Z}_\Pi)$; likewise $\mathrm{pr}_\Pi(\Gamma_k(\Pi))$ need not exhaust $\mathrm{Aut}_k(\mathcal{Z}_\Pi)$.

Example 6.2. Let Π be the complex of three unit segments forming the boundary of a triangle. Then $\mathcal{Z}_\Pi$ is an arrangement of three copies of the projective line $\mathbb{P}^1_k$ meeting each other pairwise in three different points. It follows from Theorem 5.2 and easy observations that $\Gamma_k(\Pi) = \Gamma_k(\Pi)^0 = \mathbb{T}_k(\Pi)$. But one has $\mathbb{T}_k(\Pi)/k^* = (k^*)^2$ and $\mathbb{D}(\mathcal{Z}_\Pi) = (k^*)^3$.

Next we introduce the notion of *projectively equivalent* lattice polyhedral complexes. Recall that the *normal fan* $\mathcal{N}(P)$ of a polytope $P \subset \mathbb{R}^n$ is defined as the complete fan in the dual space $(\mathbb{R}^n)^* = \operatorname{Hom}(\mathbb{R}^n, \mathbb{R})$ given by the system of cones

$$(\{\phi \in (\mathbb{R}^n)^* \mid \operatorname{Max}_P(\phi) = F\},\ F \text{ a face of } P).$$

Two polytopes $P, Q \subset \mathbb{R}^n$ are called *projectively equivalent* if $\mathcal{N}(P) = \mathcal{N}(Q)$. In other words, P and Q are projectively equivalent if and only if they have the same dimension, the same combinatorial type, and the faces of P are parallel translates of the corresponding ones of Q.

Lattice polyhedral complexes Π and Π' are called *projectively equivalent* if the following conditions are satisfied:

(a) there is an isomorphism between the underlying abstract polyhedral complexes $\psi : \Pi_X \to \Pi_{X'}$, i.e. there is a bijection ψ between the vertex sets X and X' inducing a bijection of the polyhedral complexes,
(b) $n_P = n_{\psi(P)}$,
(c) the lattice polytopes P^* and $(\psi(P))^* \subset \mathbb{R}^{n_P}$, are projectively equivalent for all $P \subset \Pi_X$ so that if $F \subset P^*$ and $G \subset (\psi(P))^*$ correspond each other under this projective equivalence, then $F = Q^*$ and $G = (\psi(Q))^*$ for some $Q \subset \Pi_X$.

(Here we use the same notation as in the definition of a lattice polyhedral complex in Section 2.) The isomorphism $\psi : \Pi_X \to \Pi_{X'}$ is called a *projective equivalence.*

The next lemma explains the name 'projectively equivalent'.

Lemma 6.3. *Let Π and Π' be projectively equivalent very ample lattice polyhedral complexes. Then there is a natural isomorphism $\mathcal{Z}_\Pi \approx \mathcal{Z}_{\Pi'}$ transforming the Π-affine chart into the Π'-affine chart; furthermore the sets* $\operatorname{Col}(\Pi)$ *and* $\operatorname{Col}(\Pi')$ *of column vectors are in natural one-to-one correspondence.*

Proof. The isomorphism $\mathcal{Z}_\Pi \approx \mathcal{Z}_{\Pi'}$ exists due to the tautological identification of the two affine charts. The claim on column vectors follows easily from the analogous fact for single polytopes [BG, Section 2]. Q.E.D.

We will need the following standard

Lemma 6.4. *Let V be a k-variety and G a connected k-group acting algebraically on V. Then G leaves the irreducible components of V invariant.*

Let Π be an oriented lattice polyhedral complex. An automorphism of $\mathcal{Z}_\Pi$ is called *elementary* if it is of type $\mathrm{pr}_\Pi(e_V^\lambda)$ for some elementary automorphism $e_V^\lambda \in \Gamma_k(\Pi)$ ($\lambda \in k$). For a column vector $V \in \mathrm{Col}(\Pi)$ the assignment

$$e_V^\lambda \mapsto \mathrm{pr}_\Pi(e_V^\lambda), \ \ \lambda \in k,$$

defines an algebraic homomorphism $\mathbb{A}_k^1 \to \mathrm{Aut}(\mathcal{Z}_\Pi)$. It follows from the exact sequence above that this is an injective mapping. The subgroup of $\mathrm{Aut}_k(\mathcal{Z}_\Pi)$ generated by the elementary automorphisms will be denoted by $\mathbb{E}(\mathcal{Z}_\Pi)$. Thus $\mathbb{E}(\mathcal{Z}_\Pi) = \mathrm{pr}_\Pi(\mathbb{E}_k(\Pi))$ is a connected group, spanned by one-parameter unipotent subgroups forming affine lines in $\mathbb{E}(\mathcal{Z}_\Pi)$.

Lemma 6.5. *Let Π and Π' be two projectively equivalent, oriented, and very ample polyhedral complexes. Then*

(a) $\mathbb{E}(\mathcal{Z}_\Pi) = \mathbb{E}(\mathcal{Z}_{\Pi'})$;

(b) $\delta \circ \varepsilon \circ \delta^{-1}$ *is an elementary automorphism for any* $\delta \in \mathbb{D}(\mathcal{Z}_\Pi)$ *and any elementary automorphism* ε *of* $\mathcal{Z}_\Pi$.

(Here $\mathbb{E}(\mathcal{Z}_\Pi)$ and $\mathbb{E}(\mathcal{Z}_{\Pi'})$ are regarded as subgroups of the same group $\mathrm{Aut}_k(\mathcal{Z}_\Pi)$ by virtue of Lemma 6.3.)

Proof. (a) It is enough to show that if $V \in \mathrm{Col}(\Pi)$ and $V' \in \mathrm{Col}(\Pi')$ are corresponding column vectors (in the sense of Lemma 6.3) and $\lambda \in k$ then $e_V^\lambda \in \Gamma_k(\Pi)$ and $e_{V'}^\lambda \in \Gamma_k(\Pi')$ define the same elements in $\mathrm{Aut}_k(\mathcal{Z}_\Pi)$. In fact, we get two elements from the unity component $\mathrm{Aut}_k(\mathcal{Z}_\Pi)^0$ and, hence, by Lemma 6.4 they both leave the irreducible components of $\mathcal{Z}_\Pi$ invariant. Therefore, by Proposition 6.1(b) the problem reduces to the special case of single polytopes and here [BG, Lemma 5.1] applies.

(b) follows from the case of a single polytope, which is covered by 4.4(b), and patching arguments. Q.E.D.

Next we define the finite subgroup $\Sigma(\Pi)_{\mathrm{Proj}} \subset \mathrm{Aut}_k(\mathcal{Z}_\Pi)$ for a very ample lattice polyhedral complex, which generalizes the symmetry group of the normal fan $\mathcal{N}(P)$ of a polytope P [BG, Section 5].

For each vertex $z \prec \Pi$ and each face $P \prec \Pi$ we have introduced the corresponding weak fan $\Pi(z)$ and normal fan $\mathcal{N}(P)$ (the latter defined in the dual space $(\mathbb{R}^{n_P})^*$). The cone of $\mathcal{N}(P)$, corresponding to a face $F \prec P$, will be denoted by $\mathcal{N}_F^{(P)}$.

For a face $P \prec \Pi$ and a vertex $z \prec P$ we have

$$\mathcal{N}_z^{(P)} = (\mathcal{W}_P^{(z)})^*,$$

where the star on the right hand side denotes the dual cone in $(\mathbb{R}^{n_P})^*$.

It follows that if $\mathcal{W}_P^{(z)} \in \Pi(z)$, $\mathcal{W}_Q^{(y)} \in \Pi(y)$ ($y, z \prec \Pi$ vertices, $z \prec P \prec \Pi$, $y \prec Q \prec \Pi$) and

$$\alpha : \mathbb{Z}^{n_P} \cap \mathcal{W}_P^{(z)} \to \mathbb{Z}^{n_Q} \cap \mathcal{W}_Q^{(y)}$$

is a semigroup homomorphism, then one has the corresponding naturally defined semigroup homomorphism

$$\alpha^* : (\mathbb{Z}^{n_Q})^* \cap \mathcal{N}_y^{(Q)} \to (\mathbb{Z}^{n_P})^* \cap \mathcal{N}_z^{(P)},$$

and vice versa. Moreover, $\alpha^{**} = \alpha$ and $(\alpha \circ \beta)^* = \beta^* \circ \alpha^*$. In particular, isomorphisms are mapped to isomorphisms.

We recall that an isomorphism of complete fans in Euclidean spaces means a integral linear isomorphism of the ambient spaces transforming one fan into the other.

We shall say that two weak fans are *isomorphic* if their underlying abstract polyhedral complexes are isomorphic and the corresponding affine semigroups are isomorphic in such a way that the involved isomorphisms agree on common 'face' sub-semigroups.

Now an element of $\Sigma(\Pi)_{\mathrm{Proj}}$ by definition is a triple $(\rho, \mathcal{A}, \mathcal{B})$, where

(1) ρ is an automorphism of the abstract polyhedral complex X_Π,

(2) $\mathcal{A}$ is a set of isomorphisms $\alpha_P^{(z)} : \mathbb{Z}^{n_P} \cap \mathcal{W}_P^{(z)} \to \mathbb{Z}^{n_{\rho(P)}} \cap \mathcal{W}_{\rho(P)}^{(\rho(z))}$, where z runs through the vertices and P through the faces of Π with $z \prec P$,

(3) $\mathcal{B}$ is a set of isomorphisms $\beta_z^{(P)} : (\mathbb{Z}^{n_{\rho(P)}})^* \cap \mathcal{N}_{\rho(z)}^{(\rho(P))} \to (\mathbb{Z}^{n_P})^* \cap \mathcal{N}_z^{(P)}$, z and P as above,

so that the following conditions are satisfied:

(4) for each vertex $z \prec \Pi$ the subset $\{\alpha_P^{(z)} \mid z \prec P \prec \Pi\} \subset \mathcal{A}$ establishes an isomorphism between the weak fans $\Pi(z)$ and $\Pi(\rho(z))$,

(5) for each face $P \prec \Pi$ the subset $\{\beta_z^{(P)} \mid z \prec P \text{ a vertex}\} \subset \mathcal{B}$ establishes an isomorphism between the normal fans $\mathcal{N}(\rho(P))$ and $\mathcal{N}(P)$,

(6) $\mathcal{B} = \{\alpha^* \mid \alpha \in \mathcal{A}\}$.

The group structure of $\Sigma(\Pi)_{\mathrm{Proj}}$ is defined by taking the appropriate compositions. It follows readily from Proposition 6.1(a) that we can consider the finite group $\Sigma(\Pi)_{\mathrm{Proj}}$ as a subgroup of $\mathrm{Aut}_k(\mathcal{Z}_\Pi)$ in a natural way, provided Π is very ample – the elements of $\Sigma(\Pi)_{\mathrm{Proj}}$ can naturally be thought of as automorphisms of the Π-affine chart on $\mathcal{Z}_\Pi$. Now a straightforward verification shows the following

Lemma 6.6. *Let Π be a very ample lattice polyhedral complex.*

(a) *If Π' is a very ample lattice polyhedral complex, projectively equivalent to Π, then $\Sigma(\Pi)_{\mathrm{Proj}}$ and $\Sigma(\Pi')_{\mathrm{Proj}}$ coincide (in the sense of Lemma 6.3);*

(b) *Let $\delta \in \mathbb{D}(\mathcal{Z}_\Pi)$, $\sigma \in \Sigma(\Pi)_{\mathrm{Proj}}$, and $\varepsilon \in \mathbb{E}(\mathcal{Z}_\Pi)$; then $\sigma^{-1} \circ \delta \circ \sigma \in \mathbb{D}(\mathcal{Z}_\Pi)$ and $\sigma^{-1} \circ \varepsilon \circ \sigma \in \mathbb{E}(\mathcal{Z}_\Pi)$;*

(c) *pr_Π embeds $\Sigma(\Pi)$ into $\Sigma(\Pi)_{\mathrm{Proj}}$.*

§7. Very ample line bundles on arrangements

In this section we first give an overview of known results ([Oda, Ch. 2], Teissier [Te]) on very ample line bundles on projective toric varieties. The generalization to arrangements of toric varieties, discussed later on, will be needed in the proof of Theorem 9.1.

Let n be a natural number and $P \subset \mathbb{R}^n$ be a very ample lattice n-polytope. We let $\mathcal{P}$ denote the set of lattice polytopes $Q \subset \mathbb{R}^n$, which are very ample and projectively equivalent to P. Then $\mathcal{P}$ carries the following semigroup structure (without unity):

$$Q + R = \{q + r \mid q \in Q, r \in R\}, \quad Q, R \in \mathcal{P}.$$

Thus $Q + R$ is the Minkowski sum of Q and R (very ampleness is preserved by Minkowski sums). Any element $Q \in \mathcal{P}$ defines a normal projective toric variety $\mathcal{Z}_Q = \mathrm{Proj}(k[\bar{S}_Q])$ (Proposition 6.1(b)) and the very ample line bundle $\mathcal{L}_Q$, the preimage of the structural line bundle $\mathcal{O}(1)$ under the natural closed embedding

$$\mathcal{Z}_Q \to \mathbb{P}^N_k, \ N = \#\mathrm{L}(Q) - 1.$$

We shall identify all the $\mathcal{Z}_Q$ for $Q \in \mathcal{P}$ via the natural isomorphism mentioned in Lemma 6.3.

The torus $(k^*)^n = \mathrm{Hom}(\mathbb{Z}^n, k^*)$ operates on all the algebras $K[\bar{S}_Q]$; furthermore it can be identified with the embedded torus $\mathbb{T}(\mathcal{Z}_P)$ of $\mathcal{Z}_P$. Thus the line bundle $\mathcal{L}_Q$ carries an $\mathbb{T}(\mathcal{Z}_P)$-equivariant structure, i.e. an action of the embedded torus $\mathbb{T}(\mathcal{Z}_P)$ which is compatible with the structural projection $\mathcal{L} \to \mathcal{Z}_P$ and is fiber-wise linear. (This action is obtained as the restriction of the action of $\mathbb{T}(\mathcal{Z}_P) \subset (k^*)^N$ on $\mathcal{O}(1)$.) Of course, any such action can be modified by a character χ of $\mathbb{T}(\mathcal{Z}_P)$, i.e. one replaces the linear map $\tau_x : \mathcal{L}_x \to \mathcal{L}_{\tau(x)}$ by $\chi(\tau)\tau_x$.

Furthermore, any equivariant $\mathbb{T}(\mathcal{Z}_P)$-structure on $\mathcal{L}_Q$ induces a corresponding action of $\mathbb{T}(\mathcal{Z}_P)$ on the canonical algebra

$$\mathcal{O}(\mathcal{L}_Q) = \bigoplus_{i \geq 0} H^0(\mathcal{Z}_P, \mathcal{L}_Q^{\otimes i}),$$

given by $\tau(f) = \tau^{-1} \circ f \circ \tau$ for $\tau \in \mathbb{T}(\mathcal{Z}_P)$ and a global section $f : \mathcal{Z}_P \to L_Q^{\otimes i}$ ($i \in \mathbb{N}$). With respect to this action $H^0(\mathcal{Z}_P, \mathcal{L}_Q^{\otimes i})$ decomposes into a direct sum of one-dimensional representations of $\mathbb{T}(\mathcal{Z}_P)$.

Lemma 7.1. (a) *With respect to the equivariant structure induced by the action of $(k^*)^n$ on $k[\bar{S}_Q]$, the characters of $(k^*)^n = \mathbb{T}(\mathcal{Z}_P)$ corresponding to the one-dimensional representations of $\mathbb{T}(\mathcal{Z}_P)$ in $H^0(\mathcal{Z}_P, \mathcal{L}_Q^{\otimes i})$ are pairwise different; under the identification* $\operatorname{Hom}((k^*)^n, k) = \mathbb{Z}^n$ *they are the lattice points of the i-th homothetic blow up iQ of Q.*

(b) *Any two equivariant structures on $\mathcal{L}_Q$ differ by a character of $\mathbb{T}(\mathcal{Z}_P)$. Thus the decomposition of $H^0(\mathcal{Z}_P, \mathcal{L}_Q^{\otimes i})$ is independent of the equivariant structure, and if one multiplies the equivariant structure in* (a) *by χ, than Q has to be replaced by $Q - \chi$.*

By letting k^* act trivially on $\mathcal{Z}_P$ we can extend the action of $\mathbb{T}(\mathcal{Z}_P)$ to an action of $k^{n+1} = \mathbb{T}_k(P) = \mathbb{T}(\mathcal{Z}_P) \times k^*$ on $\mathcal{Z}_P$. Moreover, any $\mathbb{T}(\mathcal{Z}_P)$-equivariant structure on $\mathcal{L}_Q$ can be extended to an action of $\mathbb{T}_k(P)$ if we let k^* act on $\mathcal{L}_Q$ by fiber-wise multiplication. This gives rise to an action of $\mathbb{T}_k(P)$ on the canonical algebra of $\mathcal{L}_Q$; of course, $\mathbb{T}_k(P)$ also acts naturally on $k[\bar{S}_Q]$.

Lemma 7.2. *For the equivariant structure on $\mathcal{L}_Q$ induced by the action of $(k^*)^n$ on $k[\bar{S}_Q]$ we have a graded k-algebra isomorphism $\mathcal{O}(\mathcal{L}_Q) \approx k[\bar{S}_Q]$ that respects the two $\mathbb{T}_k(P)$-actions.*

The assignment $Q \mapsto \mathcal{L}_Q$ induces a mapping $\mathcal{P} \to \operatorname{Pic}(\mathcal{Z}_P)$ which obviously factors through the quotient $\mathcal{P}/\!\sim$ where $Q \sim R$ if and only if R is a parallel translate of Q. This equivalence relation defines a congruence on the semigroup $\mathcal{P}$.

Lemma 7.3. *Let* $\operatorname{VALB}(\mathcal{Z}_P)$ ($\operatorname{EVALB}(\mathcal{Z}_P)$) *denote the sets of isomorphism classes of* (*equivariant*) *very ample line bundles on $\mathcal{Z}_P$. One has a commutative diagram*

$$\begin{array}{ccc} \mathcal{P} & \longrightarrow & \operatorname{EVALB}(\mathcal{Z}_P) \\ \big\downarrow & & \big\downarrow \\ \mathcal{P}/\!\sim & \longrightarrow & \operatorname{VALB}(\mathcal{Z}_P) \end{array}$$

where the horizontal maps are semigroup isomorphisms and the right vertical map 'forgets' the equivariant structure.

We can summarize this discussion as follows. For an ample line bundle $\mathcal{L}$ and a $\mathbb{T}(\mathcal{Z}_P)$-equivariant structure on $\mathcal{L}$, the decomposition of the canonical algebra of $\mathcal{L}$ into one-dimensional representations depends only on $\mathcal{L}$. Furthermore the representations appearing in $H^0(\mathcal{Z}_P, \mathcal{L})$ can naturally be labeled by the lattice points of Q where $Q \in \mathcal{P}$ is chosen such that $\mathcal{L} \approx \mathcal{L}_Q$. We denote them by $V_{\mathcal{L},x}$, $x \in \mathrm{L}(Q)$. These observations will be used in the next definition.

In what follows, the subalgebra of the canonical algebra of a line bundle $\mathcal{L}$ generated by its global sections will be called the *subcanonical algebra* of $\mathcal{L}$ and it will be denoted by $\mathrm{Alg}(\mathcal{L})$.

Definition 7.4. Let $\mathcal{L}$ be a very ample line bundle on $\mathcal{Z}_P$, $\mathcal{L} \approx \mathcal{L}_Q$. A system of global sections $(f_x)_{x\in\mathrm{L}(Q)} \subset H^0(\mathcal{Z}_P, \mathcal{L})$ is called *polytopal* if $f_x \in V_{\mathcal{L},x}$ for all x and there is a k-algebra isomorphism between $\mathrm{Alg}(\mathcal{L})$ and $k[S_Q]$, mapping f_x, $x \in \mathrm{L}(Q)$, to $x \in k[S_Q]$.

Roughly speaking, the next lemma says that two polytopal systems of sections in a line bundle only differ by a toric automorphism of $\mathcal{Z}_P$.

Lemma 7.5. *Let $\mathcal{L}$ and $\mathcal{L}'$ be very ample line bundles on $\mathcal{Z}_P$, $\mathcal{L} \approx \mathcal{L}' \approx \mathcal{L}_Q$ for some $Q \in \mathcal{P}$. Suppose $(f_x)_{x\in\mathrm{L}(Q)}$ and $(f'_x)_{x\in\mathrm{L}(Q)}$ are polytopal systems of global sections. Then there is a unique commutative diagram with vertical structural projections*

$$\begin{array}{ccc} \mathcal{L}' & \xrightarrow{T} & \mathcal{L} \\ \downarrow & & \downarrow \\ \mathcal{Z}_P & \xrightarrow[\tau]{} & \mathcal{Z}_P \end{array}$$

such that T is an algebraic fiber-wise linear map, $\tau \in \mathbb{T}(\mathcal{Z}_P)$ and $T^{-1} \circ f_x \circ \tau = f'_x$ for all $x \in \mathrm{L}(Q)$.

Proof. Fix $\mathbb{T}(\mathcal{Z}_P)$-equivariant structures on $\mathcal{L}$ and $\mathcal{L}'$. Observe that for any commutative diagram of the type considered the mapping T is automatically equivariant. It follows that T is $\mathbb{T}_k(P)$-equivariant for the induced $\mathbb{T}_k(P)$-structures on $\mathcal{L}$ and $\mathcal{L}'$ as well. It is also clear, that such a mapping T induces a graded k-algebra isomorphism between the canonical algebras $\mathcal{O}(\mathcal{L})$ and $\mathcal{O}(\mathcal{L}')$ which respects the $\mathbb{T}_k(P)$ actions – the diagram above is a pull-back diagram with equivariant horizontal isomorphisms.

Conversely, if we are given a graded $\mathbb{T}_k(P)$-equivariant isomorphism between the two canonical algebras, then this isomorphism gives rise (by projectivization) to a commutative diagram of the type considered.

Therefore, in view of Lemma 7.2, Lemma 7.5 is equivalent to the following obvious claim, which finishes the proof: for a lattice polytope Q and two systems of degree 1 terms $\alpha_x x$ and $\beta_x x$, $\alpha_x, \beta_x \in k^*$, $x \in \mathrm{L}(Q)$, satisfying the same relations in $k[\bar{S}_Q]$ as the $x \in \mathrm{L}(Q)$, there is a unique toric automorphism $\tau \in \mathbb{T}_k(Q)$ transforming one system into the other. Q.E.D.

For a very ample polyhedral complex Π we let $[\Pi]$ denote the class of such lattice polyhedral complexes Π' that there exists a projective equivalence $\psi : \Pi_X \to \Pi_{X'}$ (see above) and a system of semigroup isomorphisms $\phi_P : S_P \to S_{\psi(P)}$, $P \prec \Pi$, compatible on 'face' sub-semigroups.

Let Π be a very ample polyhedral complex. The set

$$\bigl\{[\Pi'] \mid \Pi' \text{ a very ample lattice polyhedral complex, projectively equivalent to } \Pi\bigr\}$$

carries a natural semigroup structure (without unity). Assume $[\Pi_1]$ and $[\Pi_2]$ belong to it. Then a face of Π_1 and the corresponding face of Π_2 can be realized as projectively equivalent very ample lattice polytopes in the same Euclidean space. The pairwise Minkowski sums naturally form a very ample lattice polyhedral complex Π_3 which is projectively equivalent to Π (one uses fixed projective equivalences $\Pi_X \xrightarrow{\approx} \Pi_{X_1}$, $\Pi_X \xrightarrow{\approx} \Pi_{X_2}$ and systems of the corresponding semigroup isomorphisms). It is clear that the class $[\Pi_3]$ is well defined. We put $[\Pi_1] + [\Pi_2] = [\Pi_3]$.

Assume $\mathcal{L}$ is a very ample line bundle on $\mathcal{Z}_\Pi$. For each face $P \prec \Pi$ the restricted line bundle $\mathcal{L}|_{\mathcal{Z}_P}$ is very ample on $\mathcal{Z}_P$, and we pick a lattice polytope $Q(\mathcal{L}, P)$ such that $\mathcal{L}|_{\mathcal{Z}_P} \approx \mathcal{L}_{Q(\mathcal{L},P)}$. It is clear from the discussion preceding Lemma 7.5 and the obvious isomorphisms

$$\mathcal{L}|_{\mathcal{Z}_P} \approx (\mathcal{L}|_{\mathcal{Z}_{P'}})|_{\mathcal{Z}_P}$$

for any faces $P \prec P' \prec \Pi$, that the polytopes $Q(\mathcal{L}, P)$ naturally form a very ample lattice polyhedral complex, which is projectively equivalent to Π. The class of this complex will be denoted by $\Pi(\mathcal{L})$. (Different choices of the polytopes $Q(\mathcal{L}, P)$ give rise to the same class.) If $\mathcal{L}'$ is another very ample line bundle on $\mathcal{Z}_\Pi$, then $\Pi(\mathcal{L}) + \Pi(\mathcal{L}') = \Pi(\mathcal{L} \otimes \mathcal{L}')$ (this reduces to the case of single polytopes; see 7.3).

Observe that for *any* very ample lattice polyhedral complex Π', which is projectively equivalent to Π, there is a very ample line bundle $\mathcal{L}$ on $\mathcal{Z}_\Pi$ such that $[\Pi'] = \Pi(\mathcal{L})$. In fact, the desired line bundle is provided by the restriction of $\mathcal{O}(1)$ under the canonical closed embedding $\mathcal{Z}_\Pi = \mathcal{Z}_{\Pi'} \to \mathbb{P}_k^N$, $N = \#\mathrm{L}(\Pi') - 1$.

Definition 7.6. Let Π and $\mathcal{L}$ be as above. A system $\bar{f} = \{f_1, \dots, f_s\}$ of global sections of $\mathcal{L}$ is called *polyhedral* if it satisfies the following conditions:

(i) $f_i \neq 0$ for all i, and for each face $P \prec \Pi$ the set of restrictions

$$\{f_i|_{\mathcal{Z}_P} \mid f_i \in \bar{f} \text{ and } f_i|_{\mathcal{Z}_P} \neq 0\}$$

is a polytopal system of global sections of the line bundle $\mathcal{L}|_{\mathcal{Z}_P}$ on $\mathcal{Z}_P$;

(ii) if $f_i|_{\mathcal{Z}_P} \neq 0$ and $f_i|_{\mathcal{Z}_Q} \neq 0$ for faces P and Q of Π, then $f_i|_{\mathcal{Z}_P \cap \mathcal{Z}_Q} \neq 0$ (in particular $\mathcal{Z}_P \cap \mathcal{Z}_Q \neq \emptyset$).

Caution. In general a very ample line bundle on $\mathcal{Z}_\Pi$ does not have a polyhedral system of global sections; see Example 9.3(a) below.

Lemma 7.7. *Let Π be a very ample complex and $\mathcal{L}$ be a very ample line bundle on $\mathcal{Z}_\Pi$ possessing a polyhedral system of global sections $\bar{f}$.*

(a) *$\bar{f}$ is a basis of the k-vector space $H^0(\mathcal{Z}_\Pi, \mathcal{L})$.*

(b) *Let $\Pi' \in \Pi(\mathcal{L})$. Then there is a k-algebra isomorphism $\Theta : \mathrm{Alg}(\mathcal{L}) \to k[\Pi']$ mapping the elements of $\bar{f}$ to elements of $\mathrm{L}(\Pi')$. Moreover, for every face $Q \prec \Pi'$ there is a commutative diagram*

$$\begin{tikzcd} \mathrm{Alg}(\mathcal{L}) \arrow[r, "\Theta"] \arrow[d, "\mathrm{rest}_Q"'] & k[\Pi'] \arrow[d, "\pi_Q"] \\ \mathrm{Alg}(\mathcal{L}|_{\mathcal{Z}_Q}) \arrow[r] & k[S_Q], \end{tikzcd}$$

where $\mathcal{Z}_Q$ denotes the projective toric subvariety of $\mathcal{Z}_\Pi = \mathcal{Z}_{\Pi'}$ naturally associated to Q, rest_Q is the restriction map and, as usual, π_Q is the corresponding face-projection.

(c) *If another very ample line bundle $\mathcal{L}'$ on $\mathcal{Z}_\Pi$ also has a polyhedral system of global sections, then so does $\mathcal{L} \otimes \mathcal{L}'$.*

Proof. We may assume $\Pi = \Pi'$. The essential point is that the elements of $\bar{f}$ correspond uniquely to the lattice points $x \in \mathrm{L}(\Pi)$: if $f_i|_{\mathcal{Z}_P}$ corresponds to $x \in \mathrm{L}(P)$, P a face of Π, then $f_i|_{\mathcal{Z}_Q}$ corresponds to the same lattice point x for all faces Q with $x \in \mathrm{L}(Q)$, as follows from condition (i) in the definition above. Condition (ii) implies that $f_i|_{\mathcal{Z}_Q} = 0$ if $x \notin \mathrm{L}(Q)$.

Now (a) is easily verified, and (b) and (c) follow from the analogous observations for single polytopes. It is important for (c) that in the case of a single polytope P the family $(f_i \otimes f'_j)$ formed from polytopal systems of global sections (f_i) and (f'_j) for $\mathcal{L}$ and $\mathcal{L}'$ has a unique extension to a polytopal system of global sections for $\mathcal{L} \otimes \mathcal{L}'$. Therefore a patching argument yields (c) for polyhedral complexes as well. Q.E.D.

The next lemma extends 7.5 to polyhedral complexes.

Lemma 7.8. *Let $\mathcal{L}$ and $\mathcal{L}'$ be very ample line bundles on $\mathcal{Z}_\Pi$, where Π is a very ample lattice polyhedral complex. Assume that $\Pi(\mathcal{L}) = \Pi(\mathcal{L}')$ and that $\mathcal{L}$ and $\mathcal{L}'$ both have polyhedral systems of global sections. Then there is an element $\delta \in \mathbb{D}(\mathcal{Z}_\Pi)$ such that $\mathcal{L}' = \delta^*(\mathcal{L})$.*

Remark 7.9. It is in general not true that $\mathcal{L} \approx \mathcal{L}'$ under the assumptions of Lemma 7.8. Moreover, the failure of the analogue of Lemma 7.3 for line bundles with polyhedral systems of global sections is measured precisely by the difference between $\mathbb{D}(\mathcal{Z}_\Pi)$ and the image of $\mathbb{D}_k(\Pi)$ in it. Consider, for instance, the lattice polyhedral complex Π of Example 6.2. Let $\mathcal{L}$ be the very ample line bundle on $\mathcal{Z}_\Pi$ obtained by the restriction of $\mathcal{O}(1)$ under the standard embedding $\mathcal{Z}_\Pi \to \mathbb{P}^3_k$. Now choose some $\delta \in \mathbb{D}(\mathcal{Z}_\Pi)$ and set $\mathcal{L}' = \delta^*(\mathcal{L})$. It is clear that $\Pi(\mathcal{L}) = \Pi(\mathcal{L}') = \Pi$. Moreover, $\mathcal{L}$ has a polyhedral system of global sections and, hence, so does $\mathcal{L}'$. But $\mathcal{L}$ and $\mathcal{L}'$ cannot be isomorphic line bundles for any δ, for otherwise any element of $\mathbb{D}(\mathcal{Z}_\Pi)$ would be liftable (via pr_Π) to an element of $\Gamma_k(\Pi)$, which is not the case according to Example 6.2. Indeed, for a k-variety $\mathcal{Z}$ and a very ample line bundle $\mathcal{L}$ on it the group of automorphisms $\alpha \in \mathrm{Aut}_k(\mathcal{Z})$, that are liftable to $\mathrm{gr.aut}_k(\mathrm{Alg}(\mathcal{L}))$, coincides with the group of automorphisms $\beta \in \mathrm{Aut}_k(\mathcal{Z})$ preserving $\mathcal{L}$ [Ha, II.6].

Proof of Lemma 7.8. Let $\bar{f} = \{f_1, f_2, \dots\}$ and $\bar{g} = \{g_1, g_2, \dots\}$ be polyhedral systems of global sections of $\mathcal{L}$ and $\mathcal{L}'$. Then for any face $P \prec \Pi$ the restrictions of the f_i and g_j form polytopal systems of global sections of $\mathcal{L}|_{\mathcal{Z}_P}$ and $\mathcal{L}'|_{\mathcal{Z}_P}$. By Lemma 7.3 we have $\mathcal{L}|_{\mathcal{Z}_P} \approx \mathcal{L}'|_{\mathcal{Z}_P}$. Therefore, by Lemma 7.5 for each $P \prec \Pi$ there is a unique commutative diagram

$$\begin{array}{ccc}
\mathcal{L}|_{\mathcal{Z}_P} & \xrightarrow{T_P} & \mathcal{L}'|_{\mathcal{Z}_P} \\
\downarrow & & \downarrow \\
\mathcal{Z}_P & \xrightarrow[\tau_P]{} & \mathcal{Z}_P,
\end{array}$$

where $\tau_P \in \mathbb{T}(\mathcal{Z}_P)$ and T_P is an algebraic fiber-wise linear map. The uniqueness of these squares guarantees that we can patch them to a commutative square

$$\begin{tikzcd} \mathcal{L} \arrow[r, "D"] \arrow[d] & \mathcal{L}' \arrow[d] \\ \mathcal{Z}_\Pi \arrow[r, "\delta"'] & \mathcal{Z}_\Pi, \end{tikzcd}$$

where $\delta \in \mathbb{D}(\mathcal{Z}_\Pi)$ and D is an algebraic fiber-wise linear map. Hence the claim. Q.E.D.

§8. Projectively quasi-Euclidean complexes

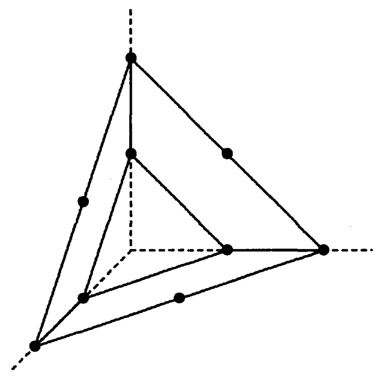
The following class of lattice polyhedral complexes is relevant in the description of $\mathrm{Aut}_k(\mathcal{Z}_\Pi)$.

Definition 8.1. A lattice polyhedral complex Π is *projectively quasi-Euclidean* if it is quasi-Euclidean and every lattice polyhedral complex projectively equivalent to Π is quasi-Euclidean as well.

Below we describe two big classes of projectively quasi-Euclidean complexes. However the following example shows that not all quasi-Euclidean complexes are projectively quasi-Euclidean, not even boundary ones.

Example 8.2. Consider the boundary lattice polyhedral complex in $\mathbb{R}^3$ as shown in the figure. It has three trapezoid facets with vertex sets

$$\{(1,0,0),(2,0,0),(0,1,0),(0,2,0)\},$$
$$\{(0,1,0),(0,2,0),(0,0,1),(0,0,2)\},$$
$$\{(0,0,1),(0,0,2),(1,0,0),(2,0,0)\}.$$



Now we change the last facet with the trapezoid spanned by $\{$(0,0,2),(0,0,3),(2,0,0),(3,0,0)$\}$ and leave the first two trapezoids untouched. It is clear that the new system of trapezoids again defines a lattice polyhedral complex, which is projectively equivalent to the original one. But the latter complex is not quasi-Euclidean. In fact, if it were so then any (rational) Euclidean realization would fit into $\mathbb{R}^3$ and the two triangles, spanned respectively by the short and long edges of the trapezoids, would be homothetic. This would imply that the ratios of lengths of the two parallel edges in our trapezoids are all the same. This, of course, is not the case: we have the ratios $\frac{1}{2}$, $\frac{1}{2}$, $\frac{2}{3}$.

Let $P \subset \mathbb{R}^n$ ($n \in \mathbb{N}$) be a (not necessarily lattice) polytope. We shall say that P is *affine-normal* if for any polytope $Q \subset \mathbb{R}^n$ that is projectively equivalent to P there exists an affine automorphism $\psi : \mathbb{R}^n \to \mathbb{R}^n$ such that ψ transforms P into Q and respects faces corresponding to each other under normal equivalence. Clearly, such an affine automorphism is *uniquely* determined if $\dim(P) = n$, and a face of an affine-normal polytope is affine-normal as well.

Recall that a polytope $P \subset \mathbb{R}^{n_P}$ is called a *join* of two polytopes $Q \subset \mathbb{R}^{n_Q}$ and $R \subset \mathbb{R}^{n_R}$ if there are affine embeddings $\phi_Q : \mathbb{R}^{n_Q} \to \mathbb{R}^{n_P}$ and $\phi_R : \mathbb{R}^{n_R} \to \mathbb{R}^{n_P}$ such that:

(1) $\mathrm{Im}(\phi_Q) \cap \mathrm{Im}(\phi_R) = \emptyset$,
(2) the affine hull of $\mathrm{Im}(\phi_Q) \cup \mathrm{Im}(\phi_R)$ is an (n_Q+n_R+1)-dimensional affine subspace of $\mathbb{R}^{n_P}$,
(3) P is the convex hull of $\phi_Q(Q) \cup \phi_R(R)$.

In particular, a join of a polytope P and a point is a pyramid of dimension $\dim(P)+1$ with base P. A join of Q and R is denoted by $\mathrm{J}(Q,R)$. It is easily observed that a join is unique up to a non-degenerate affine transformation.

Lemma 8.3. *If $P \subset \mathbb{R}^{n_P}$ and $Q \subset \mathbb{R}^{n_Q}$ are affine-normal polytopes, then so are their product $P \times Q \subset \mathbb{R}^{n_P+n_Q}$ and any join $\mathrm{J}(P,Q)$.*

Proof. Suppose $R \subset \mathbb{R}^{n_P+n_Q}$ is projectively equivalent to $P \times Q$. Without loss of generality we can assume that R is obtained from $P \times Q$ by a parallel translatation of one of the facets – the general case is obtained by induction over the set of facets. The facets of $P \times Q$ are of type either $F \times Q$ or $P \times G$ for some facets $F \prec P$ and $G \prec Q$. Consider the case $F \times Q$. Then R must have the type $P' \times Q$ for some P' projectively equivalent to P. Let $\psi : \mathbb{R}^{n_P} \to \mathbb{R}^{n_P}$ be the affine automorphism transforming P into P'. Then $\psi \times 1 : \mathbb{R}^{n_P+n_Q} \to \mathbb{R}^{n_P+n_Q}$ is the desired affine automorphism.

As for joins, it just suffices to observe that if R is projectively equivalent to $\mathrm{J}(P,Q)$ then $R = \mathrm{J}(P',Q')$ for some $P' \subset \mathbb{R}^{n_P}$ and $Q' \subset \mathbb{R}^{n_Q}$, projectively equivalent to P and Q respectively. Q.E.D.

In particular we see that all simplices, cubes, their joins, or more generally, joins of products of simplices etc., are affine-normal polytopes.

The *incidence graph* of a lattice polyhedral complex Π is defined as the graph, whose vertices are labeled by facets of Π and in which two vertices are connected by an edge if and only if the corresponding facets share a face.

Proposition 8.4. *A quasi-Euclidean lattice polyhedral complex is projectively quasi-Euclidean in either of the cases:*

(a) *the facets of* Π *are affine-normal polytopes,*

(b) *the incidence graph of* Π *is a tree.*

Proof. (a) Let Π be realized as a complex of rational polytopes in $\mathbb{R}^n$ for some $n \in \mathbb{N}$. Assume Π' is a lattice polyhedral complex projectively equivalent Π. Then each face $P \prec \Pi$ is projectively equivalent to the corresponding face $P' \prec \Pi'$ (they both are polytopes in $\mathbb{R}^{n_P}$). Let

$$\{\psi_P : \mathbb{R}^{n_P} \to \mathbb{R}^{n_P} \mid P \prec \Pi \text{ a face}\}$$

be the corresponding system of affine automorphisms transforming faces of Π' into those of Π. Since these maps are unique (as pointed out above), they are compatible on common faces. Therefore we can patch them to get a global bijective transformation

$$\Psi : \Pi \to \Pi'$$

as CW-complexes, which is face-wise affine. Observe that we are done once we know that $\Psi^{-1}(\mathrm{L}(\Pi'))$ consists of rational points of $\mathbb{R}^n$. But this follows readily from the facts that the vertices of Π are rational and that the ψ_P preserve barycentric coordinates.

For (b) one has an even stronger result – all lattice polyhedral complexes whose incidence graphs are trees are Euclidean. In fact, we can construct the Euclidean realization adding facets step by step and, at each step, sufficiently many new dimensions. Q.E.D.

In particular all simplicial complexes are projectively quasi-Euclidean.

§9. The main result: projective case

Theorem 9.1. *Let* k *be an algebraically closed field and* Π *be a very ample lattice polyhedral complex.*

(a) *If* $\mathrm{char}(k) = 0$ *and* Π *is quasi-Euclidean, then the unity component* $\mathrm{Aut}_k(\mathcal{Z}_\Pi)^0 \subset \mathrm{Aut}_k(\mathcal{Z}_\Pi)$ *consists precisely of those elements* $\alpha \in \mathrm{Aut}_k(\mathcal{Z}_\Pi)^0$ *that admit a representation* $\alpha = \varepsilon \circ \tau$ *for some* $\varepsilon \in \mathbb{E}(\mathcal{Z}_\Pi)$ *and* $\tau \in \mathbb{T}(\mathcal{Z}_\Pi)$.

(b) *If* $\mathrm{char}(k) = 0$ *and* Π *is projectively quasi-Euclidean, then every element* $\alpha \in \mathrm{Aut}_k(\mathcal{Z}_\Pi)$ *admits a representation* $\varepsilon \circ \delta \circ \sigma$ *for some* $\varepsilon \in \mathbb{E}(\mathcal{Z}_\Pi)$, $\delta \in \mathbb{D}(\mathcal{Z}_\Pi)$ *and* $\sigma \in \Sigma(\Pi)_{\mathrm{Proj}}$.

(c) *If all facets of* Π *are facet-separated polytopes, then the exact analogues of* (a) *and* (b) *hold in arbitrary characteristic; moreover, the elements of* $\mathrm{Aut}_k(\mathcal{Z}_\Pi)$ *have a normal form analogous to that in Theorem* 5.2(c).

Before setting out for the proof we formulate one more auxiliary result.

Lemma 9.2. *Let* $P \subset \mathbb{R}^n$ *be a very ample lattice polytope and* $\mathcal{L}$ *be a very ample line bundle on* $\mathcal{Z}_P$*. Then* $\alpha^*(\mathcal{L}) \approx \mathcal{L}$ *for every element* $\alpha \in \mathrm{Aut}_k(\mathcal{Z}_P)^0$.

Proof. It follows from Lemma 5.2 and Theorem 5.3 of [BG] that the natural antihomomorphism $\mathrm{gr.\,aut}_k(\mathrm{Alg}(\mathcal{L}))^0 \to \mathrm{Aut}_k(\mathcal{Z}_P)^0$ is surjective. But an automorphism liftable to $\mathrm{gr.\,aut}_k(\mathrm{Alg}(\mathcal{L}))$ preserves $\mathcal{L}$ as an element of $\mathrm{Pic}(\mathcal{Z}_P)$. Q.E.D.

Proof of Theorem 9.1. (a) Let $\mathcal{L}$ be the very ample line bundle on $\mathcal{Z}_\Pi$ obtained by the restriction of $\mathcal{O}(1)$ under the canonical embedding

$$\mathcal{Z}_\Pi \to \mathbb{P}_k^N, \ N = \#\mathrm{L}(\Pi) - 1.$$

Let $\alpha \in \mathrm{Aut}_k(\mathcal{Z}_\Pi)^0$. By Proposition 6.1(b) and Lemma 6.4 α restricts to an element of $\mathrm{Aut}_k(\mathcal{Z}_P)$ for each facet $P \prec \Pi$. It is clear that $\alpha|_{\mathcal{Z}_P} \in \mathrm{Aut}_k(\mathcal{Z}_P)^0$. By Lemma 9.2 we have an isomorphism

$$\mathcal{L}|_{\mathcal{Z}_P} \approx \alpha^*(\mathcal{L})|_{\mathcal{Z}_P} \tag{1}$$

of very ample line bundles on $\mathcal{Z}_P$. By Lemma 7.3, $\Pi(\mathcal{L}) = \Pi(\alpha^*(\mathcal{L}))$. Assume we have shown that $\alpha^*(\mathcal{L})$ has a polyhedral system of global sections. Then Lemma 7.8 yields $\delta \in \mathbb{D}(\mathcal{Z}_\Pi)$ with $\alpha^*(\mathcal{L}) = \delta^*(\mathcal{L})$. In particular, the element $\alpha \circ \delta^{-1} \in \mathrm{Aut}_k(\mathcal{Z}_\Pi)$ leaves the line bundle $\mathcal{L}$ invariant. But then (as mentioned in Remark 7.9) the image of the canonical anti-homomorphism

$$\Gamma_k(\Pi) = \mathrm{gr.\,aut}(\mathrm{Alg}(\mathcal{L})) \xrightarrow{\mathrm{pr}_\Pi} \mathrm{Aut}_k(\mathcal{Z}_\Pi)$$

contains $\alpha \circ \delta^{-1}$; in fact $\mathrm{Alg}(\mathcal{L}) = k[\Pi]$ by definition. So Theorem 5.2(b) applies:

$$\alpha = \sigma \circ \delta' \circ \varepsilon \circ \delta$$

for some $\delta' \in \mathbb{D}(\mathcal{Z}_\Pi)$, $\varepsilon \in \mathbb{E}(\mathcal{Z}_\Pi)$ and $\sigma \in \Sigma(\Pi)_{\mathrm{Proj}}$ (of course, we use that $\mathbb{D}_k(\Pi)$ maps to $\mathbb{D}(\mathcal{Z}_\Pi)$ and $\Sigma(\Pi)$ to $\Sigma(\Pi)_{\mathrm{Proj}}$). Next, Lemma 6.5(b) implies that

$$\mathbb{E}(\mathcal{Z}_\Pi) \cdot \mathbb{D}(\mathcal{Z}_\Pi) = \big(\mathbb{E}(\mathcal{Z}_\Pi)\mathbb{D}(\mathcal{Z}_\Pi)\big)$$

and

$$\mathbb{E}(\mathcal{Z}_\Pi) \cdot \mathbb{T}(\mathcal{Z}_\Pi) = \big(\mathbb{E}(\mathcal{Z}_\Pi)\mathbb{T}(\mathcal{Z}_\Pi)\big).$$

(See Step 3 in the proof of Theorem 5.2 for this notation.) Using that $\mathrm{Aut}_k(\mathcal{Z}_\Pi)^0$ has finite index in $\mathrm{Aut}_k(\mathcal{Z}_\Pi)$ and $\big(\mathbb{E}(\mathcal{Z}_\Pi)\mathbb{T}(\mathcal{Z}_\Pi)\big)$ has finite index in $\big(\mathbb{E}(\mathcal{Z}_\Pi)\mathbb{D}(\mathcal{Z}_\Pi)\big)$, we easily conclude that the connected subgroup $\big(\mathbb{E}(\mathcal{Z}_\Pi)\mathbb{T}(\mathcal{Z}_\Pi)\big) \subset \mathrm{Aut}_k(\mathcal{Z}_\Pi)^0$ has finite index and, hence, coincides with $\mathrm{Aut}_k(\mathcal{Z}_\Pi)^0$. Using the equality $\mathbb{E}(\mathcal{Z}_\Pi) \cdot \mathbb{T}(\mathcal{Z}_\Pi) = \big(\mathbb{E}(\mathcal{Z}_\Pi)\mathbb{T}(\mathcal{Z}_\Pi)\big)$ once more, we are done.

So everything amounts to showing that $\alpha^*(\mathcal{L})$ has a polyhedral system of global sections. (Observe that so far we did not use the quasi-Euclideaness of Π.) We solve this problem by first fixing polytopal global sections of $\alpha^*(\mathcal{L})|_{\mathcal{Z}_P}$ for each facet P of Π, and then correcting these systems so that they agree on the intersections $\mathcal{Z}_P \cap \mathcal{Z}_Q$. It is here where we need the quasi-Euclideaness of Π – we will make use of Borel's theorem on maximal tori in the very same way as in the proof of Theorem 5.2(b).

For the facets $P \prec \Pi$ we let $\bar{f}_P$ be arbitrary polytopal systems of global sections of $\alpha^*(\mathcal{L})|_{\mathcal{Z}_P}$. Fix a disjoint system of lattice polytopes $\hat{P}$ isomorphic to the P. By (1) we can think of the elements of $\bar{f}_P$ as lattice points of the polytopes $\hat{P}$. Let $P, Q \prec \Pi$ be facets and $\{x_1, \dots, x_s\} = \mathrm{L}(P) \cap \mathrm{L}(Q)$. Suppose $\{x_{P1}, \dots, x_{Ps}\} \subset \bar{f}_P$ and $\{x_{Q1}, \dots, x_{Qs}\} \subset \bar{f}_Q$ are the corresponding elements. By Lemma 7.7(b) (applied to the special case of a single polytope) these two systems restrict to polytopal systems of the same line bundle $\alpha^*(\mathcal{L})|_{\mathcal{Z}_{P\cap Q}}$ on $\mathcal{Z}_{P\cap Q}$. We will denote them by $\{x_{PQ1}, \dots, x_{PQs}\}$ and $\{x_{QP1}, \dots, x_{QPs}\}$. In particular, there is a unique toric automorphism $\tau_{PQ} \in \mathbb{T}_k(P\cap Q)$ transforming $\{x_{PQ1}, \dots, x_{PQs}\}$ into $\{x_{QP1}, \dots, x_{QPs}\}$ (by the natural action).

By the same token we get a system of elements $\tau_{PQ} \in \mathbb{T}_k(P \cap Q)$ for all facets $P, Q \prec \Pi$, satisfying the conditions

$$\tau_{PQ} = \tau_{QP}^{-1} \quad \text{and} \quad \tau_{PQ} = \tau_{PR} \circ \tau_{RQ}$$

on $P \cap Q$ and $P \cap Q \cap R$ respectively. The system τ_{PQ} will be called the *twisted structure* corresponding to the family $\bar{f}_P$, and it will be denoted by $\mathcal{T}$.

For any system of toric automorphisms $\{\tau_P \in \mathbb{T}_k(P) \mid P \prec \Pi$ a facet$\}$ we get a new family of polytopal global sections of the $\alpha^*(\mathcal{L})|_{\mathcal{Z}_P}$ – we just apply these toric automorphisms to the $\bar{f}_P$ correspondingly. Therefore, we obtain a new twisted structure. Our goal is to show that there is such a family $\{\tau_P \in \mathbb{T}_k(P) \mid P \prec \Pi$ a facet$\}$ that the resulting twisted structure totally consists of the identity automorphisms. To this end we consider the subcanonical algebra $\mathrm{Alg}(\alpha^*(\mathcal{L}))$. First of all we have the k-algebra isomorphism

$$k[\Pi] = \operatorname{Alg}(\mathcal{L}) \xrightarrow{\alpha^*} \operatorname{Alg}(\alpha^*(\mathcal{L})). \tag{2}$$

Consider a point $x \in \mathrm{L}(\Pi)$ and the corresponding set of lattice points

$$\{x_P \in \mathrm{L}(\hat{P}) \mid P \in \operatorname{Supp}(x)\}.$$

('Supp' has the same meaning as in Section 4.) By Lemma 7.7(b) (applied to the special situation of a single polytope) each of the x_P restricts to the zero global section of $\mathcal{L}|_{\mathcal{Z}_{P\cap R}}$ whenever $R \notin \operatorname{Supp}(x)$. On the other hand it follows from the observations above that for $P, Q \in \operatorname{Supp}(x)$ there is a uniquely determined element $c^{(x)}_{QP} \in k^*$ such that $c^{(x)}_{QP} x_{QP} = x_{PQ}$, where x_{PQ} denotes the restriction of x_P to $\mathcal{Z}_{P\cap Q}$ and similarly for x_{QP}. Therefore, we can patch the sections $c^{(x)}_{QP} x_Q$, $P, Q \in \operatorname{Supp}(x)$, which are defined on $\mathcal{Z}_Q$, and extend them by 0 on $\mathcal{Z}_R$ for $R \notin \operatorname{Supp}(x)$ to obtain a global section of $\alpha^*(\mathcal{L})$.

It follows that the quotient algebra

$$k[\Pi, \mathcal{T}] = k[S_{\hat{P}_1}] \times_k \cdots \times_k k[S_{\hat{P}_r}]/(\{c^{(x)}_{QP} x_Q - x_P \mid P, Q \in \operatorname{Supp}(x)\})$$

maps naturally to $\operatorname{Alg}(\alpha^*(\mathcal{L}))$ as a graded algebra, where $\{P_1, \dots, P_r\}$ is the set off all facets of Π (and the x_P are identified with the corresponding elements in the fiber product over k). The isomorphism (2) and Hilbert function arguments show that this mapping is actually a graded k-algebra isomorphism. The twisted structure $\{\tau_{PQ} \mid P, Q \prec \Pi \text{ facets}, P \cap Q \neq \emptyset\}$ is, of course, encoded in the scalars $c^{(x)}_{QP}$ – one has $\tau_{PQ} = 1$ if and only if $c^{(x)}_{QP} = 1$ for all $x \in \mathrm{L}(P) \cap \mathrm{L}(Q)$. We will in the following identify $\operatorname{Alg}(\alpha^*(\mathcal{L}))$ with $k[\Pi, \mathcal{T}]$.

The algebra $k[\Pi, \mathcal{T}]$ can be thought of as a 'twisted' polyhedral algebra built up of the same polytopal 'facet' algebras as $k[\Pi]$, but the identifications along common faces are carried out according to the twisted structure $\mathcal{T}$. The residue class in $k[\Pi, \mathcal{T}]$ of a term of $k[S_{\hat{P}_i}]$, $i \in [1, r]$, will be called a *twisted term.* (There is no appropriate notion of a twisted monomial.)

Let $\mathcal{S}_{\mathcal{T}}$ denote the multiplicative semigroup consisting of the twisted terms and 0, and $\mathcal{S}$ the corresponding one formed by ordinary terms and 0 (the latter live in $k[\Pi]$). One observes easily that there is a natural isomorphism

$$\Psi : \mathcal{S}_{\mathcal{T}}/k^* \xrightarrow{\approx} \mathcal{S}/k^* \approx S_\Pi.$$

Now we define the action of $\mathbb{T}_k(\Pi)$ on $k[\Pi, \mathcal{T}]$ by first setting

$$\tau(z) = \frac{\tau(z')}{z'} z, \quad z \in \mathcal{S}_{\mathcal{T}},\ z' \in \mathcal{S},\ \Psi([z]) = [z'],$$

and then extending it to the whole algebra $k[\Pi, \mathcal{T}]$ by k-linearity. The crucial point is that this action is well-defined. Since Π is quasi-Euclidean, by Lemma 3.5(a) we get an (evidently algebraic) embedding of affine groups

$$\mathbb{T}_k(\Pi) \to \operatorname{gr.aut}_k(k[\Pi, \mathcal{T}]).$$

Let $\mathbb{T}_1$ denote the image of $\mathbb{T}_k(\Pi)$. But we have yet another embedding of the same torus into $\operatorname{gr.aut}_k(k[\Pi, \mathcal{T}])$, namely

$$\mathbb{T}_k(\Pi) \to \operatorname{gr.aut}_k(k[\Pi, \mathcal{T}]), \quad \tau \mapsto \alpha^* \circ \tau \circ (\alpha^*)^{-1}.$$

Let $\mathbb{T}_2$ be the image of the second embedding. By Lemma 3.5(b) we know that $\mathbb{T}_2$ is a maximal torus. Hence $\mathbb{T}_1$ is maximal as well, and the two tori are conjugate in $\operatorname{gr.aut}_k(k[\Pi, \mathcal{T}])$ ([Bo, Corollary 11.3(1)]). Suppose that $\beta^{-1} \circ \mathbb{T}_1 \circ \beta = \mathbb{T}_2$ for some $\beta \in \operatorname{gr.aut}_k(k[\Pi, \mathcal{T}])$. Then

$$(\beta \circ \alpha^*)^{-1} \mathbb{T}_1 (\beta \circ \alpha^*) = \mathbb{T}_k(\Pi).$$

Using Lemma 3.5(a) and the very same arguments as in the proof of Theorem 5.2(b), one concludes that the isomorphism

$$\beta \circ \alpha^* : k[\Pi] \xrightarrow{\approx} k[\Pi, \mathcal{T}]$$

maps terms to terms. In particular, for each facet $P \prec \Pi$ we get two polytopal systems of global sections of the line bundle $\alpha^*(\mathcal{L})|_{\mathcal{Z}_P}$, namely $\beta \circ \alpha^*(\mathrm{L}(P))$ and $\bar{f}_P$. Then there must exist an element $\tau_P \in \mathbb{T}_k(P)$ transforming $\bar{f}_P$ into $\beta \circ \alpha^*(\mathrm{L}(P))$. A straightforward verification shows that this is the desired family of toric automorphisms.

(b) Let Π and k be as in the theorem. We again start with the very ample line bundle $\mathcal{L}$ on $\mathcal{Z}_\Pi$ obtained by restriction of $\mathcal{O}(1)$ under the closed embedding $\mathcal{Z}_\Pi \to \mathbb{P}_k^N$, $N = \#\mathrm{L}(\Pi) - 1$. Choose $\alpha \in \operatorname{Aut}_k(\mathcal{Z}_\Pi)$. Since Π is a projectively quasi-Euclidean complex and any representative of $\Pi(\alpha^*(\mathcal{L}))$ is projectively equivalent to Π we see that $\Pi(\alpha^*(\mathcal{L}))$ consists of quasi-Euclidean lattice polyhedral complexes. Now the same arguments as in the proof of (a) show that the very ample line bundle $\alpha^*(\mathcal{L})$ has a polyhedral system of global sections (though we may have $\Pi(\alpha^*(\mathcal{L})) \neq \Pi(\mathcal{L})$ now). Consider $\Pi_1 \in \Pi(\alpha^*(\mathcal{L}))$. Then Lemma 7.8 shows that there is an element $\delta_1 \in \mathbb{D}(\mathcal{Z}_\Pi)$ with

$$\delta_1^*(\mathcal{L}_1) = \alpha^*(\mathcal{L})$$

for the line bundle $\mathcal{L}_1$ on $\mathcal{Z}_\Pi$ obtained by the restriction of $\mathcal{O}(1)$ under the closed embedding

$$\mathcal{Z}_\Pi = \mathcal{Z}_{\Pi_1} \to \mathbb{P}_k^{N_1}, \ N_1 = \#\mathrm{L}(\Pi_1) - 1.$$

(We identify $\mathcal{Z}_\Pi$ and $\mathcal{Z}_{\Pi_1}$ via Lemma 6.3.) By Lemma 7.3 we have $\Pi(\mathcal{L}_1) = \Pi(\alpha^*(\mathcal{L}))$. Now we carry out the same process for Π_1 and $\mathcal{L}_1$ as we did for Π and $\mathcal{L}$, and so on. We will get a sequence of projectively equivalent very ample lattice polyhedral complexes $\Pi_0 = \Pi, \Pi_1, \Pi_2, \dots$ such that the very ample line bundles $\mathcal{L}_i$ on $\mathcal{Z}_\Pi$, obtained by restrictions of the $\mathcal{O}(1)$ under the closed embeddings $\mathcal{Z}_\Pi = \mathcal{Z}_{\Pi_i} \to \mathbb{P}_k^{N_i}$, $N_i = \#\mathrm{L}(\Pi_i) - 1$, satisfy the following conditions for $i \geq 0$:

(1) $\alpha^*(\mathcal{L}_i)$ has a polyhedral system of global sections,
(2) $\Pi_i \in \Pi(\mathcal{L}_i)$,
(3) $\Pi(\mathcal{L}_{i+1}) = \Pi(\alpha^*(\mathcal{L}_i))$,
(4) $\#\mathrm{L}(\Pi_i) = \#\mathrm{L}(\Pi)$, $i \geq 0$,

Equation (4) holds because

$$\begin{aligned}\#\mathrm{L}(\Pi_{i+1}) &= \dim_k H^0(\mathcal{Z}_\Pi, \mathcal{L}_{i+1})\\ &= \dim_k H^0(\mathcal{Z}_\Pi, \alpha^*(\mathcal{L}_i)) = \dim_k H^0(\mathcal{Z}_\Pi, \mathcal{L}_i) = \#\mathrm{L}(\Pi_i).\end{aligned}$$

Easy inductive arguments guarantee that the number of the classes $[\Pi']$ such that Π' is projectively equivalent to Π and $\#\mathrm{L}(\Pi') = \#\mathrm{L}(\Pi)$ is *finite*. Therefore, by the conditions (2), (3), and (4) there exist natural numbers p and q such that

$$\Pi(\mathcal{L}_{i+qj}) = \Pi(\mathcal{L}_i)$$

for all $i \geq p$ and all $j \geq 0$. Consider the very ample line bundle

$$\hat{\mathcal{L}} = \mathcal{L}_p \otimes \cdots \otimes \mathcal{L}_{p+q-1},$$

which has a polyhedral system of global sections by Lemma 7.7(c). We fix a complex $\hat{\Pi} \in \Pi(\hat{\mathcal{L}})$.

By Lemma 7.7(c) and condition (1) above the line bundle

$$\alpha^*(\hat{\mathcal{L}}) = \alpha^*(\mathcal{L}_p) \otimes \cdots \otimes \alpha^*(\mathcal{L}_{p+q-1})$$

has a polyhedral system of global sections as well, and we have the equalities

$$\Pi(\alpha^*(\hat{\mathcal{L}})) = \Pi(\mathcal{L}_p) + \cdots + \Pi(\mathcal{L}_{p+q-1}) = \Pi(\hat{\mathcal{L}}).$$

So by Lemma 7.8 there exists $\delta \in \mathbb{D}(\mathcal{Z}_{\hat{\Pi}})$ such that $(\alpha \circ \delta)^*(\hat{\mathcal{L}}) = \hat{\mathcal{L}}$. In this situation the automorphism $\alpha \circ \delta \in \mathrm{Aut}_k(\mathcal{Z}_{\hat{\Pi}})$ is in the image of the canonical anti-homomorphism

$$\Gamma_k(\hat{\Pi}) = \mathrm{gr.\,aut}(\mathrm{Alg}(\hat{\mathcal{L}})) \xrightarrow{\mathrm{pr}} \mathrm{Aut}_k(\mathcal{Z}_{\hat{\Pi}}).$$

Since Π is a projectively quasi-Euclidean complex and $\hat{\Pi}$ is projectively equivalent to Π, we can apply Theorem 5.2(b) to obtain the equality

$$\alpha \circ \delta = \sigma \circ \delta' \circ \varepsilon$$

for some $\sigma \in \operatorname{pr}(\Sigma(\hat{\Pi})) \subset \Sigma(\hat{\Pi})_{\mathrm{Proj}} = \Sigma(\Pi)_{\mathrm{Proj}}$ (Lemma 6.6(a),(c)), $\delta' \in \operatorname{pr}(\mathbb{D}_k(\hat{\Pi})) \subset \mathbb{D}(\mathcal{Z}_\Pi)$, $\varepsilon \in \mathbb{E}(\mathcal{Z}_{\hat{\Pi}}) = \mathbb{E}(\mathcal{Z}_\Pi)$ (Lemma 6.5(a)). Now the Lemmas 6.5(b) and 6.6(b) complete the proof.

(c) It is clear that the arguments presented above apply to part (c) as well, once one has observed that a polytope projectively equivalent to a facet-separated polytope is itself facet separated. Q.E.D.

Example 9.3. In the previous proof a 'twisted' structure was derived from $\alpha^*(\mathcal{L})$, and then 'untwisted' to a polyhedral system of global sections of $\alpha^*(\mathcal{L})$. In general a twisted structure cannot be untwisted: (a) it may happen that a line bundle $\mathcal{L}$ on $\mathcal{Z}_\Pi$ does not have a polyhedral system of global sections; at least in the quasi-Euclidean case $\operatorname{Alg}(\mathcal{L})$ and $k[\Pi]$ are then not isomorphic as graded algebras, as can be shown by arguments similar to those in the proof of 5.2(b); (b) if we define a 'twisted polyhedral algebra' abstractly for a given polyhedral complex Π, it may even happen that $\operatorname{Proj}(k[\Pi])$ is not isomorphic to $\operatorname{Proj}(k[\Pi, \mathcal{T}])$:

(a) The polyhedral complex Π_1 in Figure 7 consists of 4 polygons: a 2×1 rectangle as the bottom and three trapezoids. (The two triangles are open.) Let $\mathcal{T}$ denote the 'abstract' twisted structure indicated in

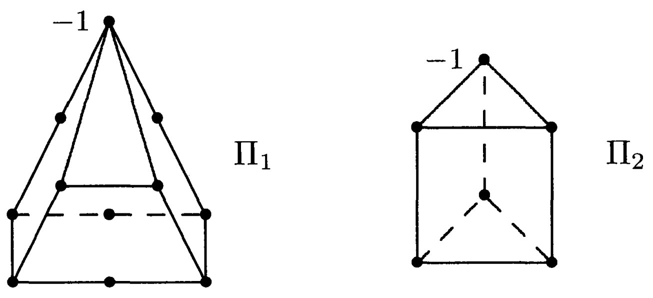


FIGURE 7.

the figure. Then it is easy to see that the second Veronese algebras of $k[\Pi]$ and $k[\Pi, \mathcal{T}]$ coincide. Therefore they define the same projective varieties, namely $\mathcal{Z}_\Pi$. However, the very ample line bundle coming from $k[\Pi, \mathcal{T}]$ does not have a polyhedral system of sections.

(b) The polyhedral complex Π_2 consists of 3 unit squares forming a triangular box without bottom and lid. In this case $k[\Pi]$ and $k[\Pi, \mathcal{T}]$ even define different projective varieties.

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Algebraic Shifting and Spectral Sequences

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Abstract.

There is a canonical spectral sequence associated to any filtration of simplicial complexes. Algebraically shifting a finite filtration of simplicial complexes produces a new filtration of shifted complexes.

We prove that certain sums of the dimensions of the limit terms of the spectral sequence of a filtration weakly decrease by algebraically shifting the filtration. A key step is the combinatorial interpretation of the dimensions of the limit terms of the spectral sequence of a filtration consisting of near-cones.

§1. Introduction

The key step of Björner and Kalai's characterization [BK] of f-vectors and Betti numbers of simplicial complexes was that algebraically shifting a simplicial complex K produces a new complex $\Delta(K)$ whose homology Betti numbers are the same as those of K, *i.e.*,

$$\beta^q(K) = \beta^q(\Delta(K)). \tag{1}$$

But the Betti numbers of $\Delta(K)$ are much easier to compute, because $\Delta(K)$ is shifted and hence a near-cone.

Relative homology is a little less straightforward. First note that if $L \subseteq K$ are a pair of simplicial complexes, then $\Delta(L) \subseteq \Delta(K)$ [Ka2, Theorem 2.2]. The equality (1) above becomes merely an inequality for relative homology,

$$\beta^q(K, L) \leq \beta^q(\Delta(K), \Delta(L));$$

in other words, relative Betti numbers (weakly) increase in each dimension [Du2] (see also [Rö], where a more general result, on generic initial

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ideals and Gröbner bases, was subsequently proved). As with the Betti numbers of a single near-cone, the relative Betti numbers of a pair of near-cones are easy to compute.

We now examine what happens when a finite filtration

$$\mathcal{K}\colon \emptyset = K_0 \subseteq K_1 \subseteq \cdots \subseteq K_m = K \tag{2}$$

of a simplicial complex K is algebraically shifted, *i.e.*, when each subcomplex in the filtration is algebraically shifted, giving a new filtration

$$\Delta(\mathcal{K})\colon \emptyset = \Delta(K_0) \subseteq \Delta(K_1) \subseteq \cdots \subseteq \Delta(K_m) = \Delta(K).$$

In particular, we will be concerned with a cohomology spectral sequence of this filtration whose limit terms $E^1_\infty, \ldots, E^m_\infty$ filter the cohomology $\widetilde{H}^*(K, \boldsymbol{k})$ of K over a field $\boldsymbol{k}$. That is, $\dim E^1_\infty + \cdots + \dim E^m_\infty = \beta^*(K) = \widetilde{H}^*(K, \boldsymbol{k})$; we can think of E^s_∞ as providing the contribution of $K_s \backslash K_{s-1}$ to the cohomology of K. Our main result (Theorem 6.1) is that the quantity $\dim E^1_\infty + \cdots + \dim E^p_\infty$ (weakly) decreases, and hence $\dim E^{p+1}_\infty + \cdots + \dim E^m_\infty$ (weakly) increases, by applying algebraic shifting. In some sense then, algebraic shifting moves more of the cohomology to later in the filtration of K. Relative homology is just the $n = 2, p = 1$ case, as $E^2_\infty = \widetilde{H}^*(K_2, K_1)$ for the filtration $\emptyset \subseteq K_1 \subseteq K_2$.

As with Betti numbers and relative Betti numbers, the quantity $\dim E^1_\infty + \cdots + \dim E^p_\infty$ is easy to compute for near-cones, and this is an important step of the proof.

Section 2 reviews the necessary background for simplicial complexes, including the exterior face ring, in which all our subsequent calculations take place. In Section 3, we first construct the spectral sequence corresponding to $\mathcal{K}$, and then use elementary manipulations to replace $\dim E^1_\infty + \cdots + \dim E^p_\infty$ by an expression not using spectral sequences. Then in Section 4 we interpret this expression combinatorially for near-cones; this combinatorial interpretation resembles and complements the combinatorial interpretations of the Betti numbers of a near-cone and the relative Betti numbers of a pair of near-cones. In Section 5, we briefly review algebraic shifting, and then modify arguments from [Du2] to prove the key inequality. Section 6 proves Theorem 6.1, which merely consists of tying together the results of the previous three sections.

§2. Simplicial complexes

For any subset S of a simplicial complex K, let S_q denote the set of q-dimensional faces of S. In particular, K_q is the set of q-dimensional faces of K itself; context should distinguish between K_q, for the q-dimensional faces of K, and K_s, for a member of the filtration (2).

Let $\boldsymbol{k}$ be a field, fixed throughout the paper. The qth *Betti number* of a simplicial complex K is $\beta^q = \beta^q(K) = \dim_{\boldsymbol{k}} \widetilde{H}^q(K)$, where $\widetilde{H}^q(K)$ is the qth reduced cohomology group of K (with respect to $\boldsymbol{k}$). Recall that over a field $\boldsymbol{k}$, $\dim_{\boldsymbol{k}} \widetilde{H}^q(K;\boldsymbol{k}) = \dim_{\boldsymbol{k}} \widetilde{H}_q(K;\boldsymbol{k})$, so that Betti numbers measure reduced homology as well as reduced cohomology.

Definition. Let K be a $(d-1)$-dimensional simplicial complex on vertex set $[n] := \{1, \ldots, n\}$. Let $V = \{e_1, \ldots, e_n\}$, and let $\Lambda(\boldsymbol{k}V)$ denote the exterior algebra of the vector space $\boldsymbol{k}V$; it has a $\boldsymbol{k}$-vector space basis consisting of all the monomials $e_S := e_{i_0} \wedge \cdots \wedge e_{i_q}$, where $S = \{i_0 < \cdots < i_q\} \subseteq [n]$ (and $e_\emptyset = 1$). Note that $\Lambda(\boldsymbol{k}V) = \oplus_{q=-1}^{n-1} \Lambda^{q+1}(\boldsymbol{k}V)$ is a graded $\boldsymbol{k}$-algebra, and that $\Lambda^{q+1}(\boldsymbol{k}V)$ has basis $\{e_S\colon |S| = q+1\}$. Let $(I_K)_q$ be the subspace of $\Lambda^{q+1}(\boldsymbol{k}V)$ generated by the basis $\{e_S\colon |S| = q+1,\ S \notin K\}$. Then $I_K := \oplus_{q=-1}^{d-1} (I_K)_q$ is the homogeneous graded ideal of $\Lambda(\boldsymbol{k}V)$ generated by $\{e_S\colon S \notin K\}$. Let $\Lambda^q[K] := \Lambda^{q+1}(\boldsymbol{k}V)/(I_K)_q$. Then the graded quotient algebra $\Lambda[K] := \oplus_{q=-1}^{d-1} \Lambda^q[K] = \Lambda(\boldsymbol{k}V)/I_K$ is called the *exterior face ring* of K (over $\boldsymbol{k}$).

The exterior face ring is the exterior algebra analogue to the Stanley-Reisner face ring of a simplicial complex [St]. For $x \in \boldsymbol{k}V$, let $\widetilde{x}$ denote the image of x in $\Lambda[K]$. For $S \subseteq K$, let

$$\widetilde{S} = \operatorname{span}\{\widetilde{e}_F\colon F \in S\}.$$

As with $I = I_K$ above, I_q will denote the q-dimensional part of *any* homogeneous graded subspace I contained in $\Lambda[K]$.

It is not hard to verify (or see equation (3) below) that the usual coboundary operator $\delta\colon \Lambda^q[K] \to \Lambda^{q+1}[K]$ used to compute cohomology may be given by $\delta\colon \widetilde{x} \mapsto \widetilde{f} \wedge \widetilde{x}$, where $f = e_1 + \cdots + e_n$. However, it will be necessary (see Section 5) to use a more "generic" coboundary operator, which will not change cohomology. Let $\hat{\boldsymbol{k}} = \boldsymbol{k}(\alpha_{11}, \alpha_{12}, \ldots, \alpha_{nn})$ be the field extension over $\boldsymbol{k}$ by n^2 transcendentals, $\{\alpha_{ij}\}_{1 \le i,j \le n}$, algebraically independent over $\boldsymbol{k}$. We will consider $\Lambda[K]$ as being over $\hat{\boldsymbol{k}}$ instead of $\boldsymbol{k}$ from now on. We are, in effect, simply adjoining these α_{ij}'s to our field of coefficients.

For now, we will only need the first n transcendentals, $\alpha_{11}, \ldots, \alpha_{1n}$. Let $f_1 = \alpha_{11}e_1 + \cdots + \alpha_{1n}e_n$. Then define the *weighted coboundary* *operator* δ: $\Lambda[K] \to \Lambda[K]$ by δ: $\widetilde{x} \mapsto \widetilde{f}_1 \wedge \widetilde{x}$, so

$$\delta(\widetilde{e}_S) = \widetilde{f}_1 \wedge \widetilde{e}_S = \sum_{j=1}^{n} \alpha_{1j} \widetilde{e}_j \wedge \widetilde{e}_S = \sum_{\substack{j \notin S \\ S \cup \{j\} \in K}} \pm \alpha_{1j} \widetilde{e}_{S \cup \{j\}} \tag{3}$$

(hence the name weighted coboundary operator). Betti numbers may be computed using this δ, *i.e.*, $\beta^q(K) = \dim_{\boldsymbol{k}}(\ker \delta)_q / (\operatorname{im} \delta)_q$ [BK, pp. 289–290].

§3. Spectral sequences

The filtration (2) in Section 1 naturally gives rise to a filtration of ideals in $\Lambda[K]$, as follows. For $0 \leq s \leq m$, define

$$Q^s = K \backslash K_s$$

so the ideals $\widetilde{Q}^s$ form a filtration

$$\Lambda[K] = \widetilde{Q}^0 \supseteq \cdots \supseteq \widetilde{Q}^m = \widetilde{0} = I_K. \tag{4}$$

By *e.g.* [Sp, p. 493], there is a convergent spectral sequence E_r corresponding to this filtration. By this we mean that there is a sequence of pairs $\{(E_r, d_r)\}_{r \geq 1}$, where: E_r is a bigraded vector space over a field $\boldsymbol{k}$; d_r is a differential on E_r of bidegree $(r, 1-r)$ (so d_r: $E_r^{s,t} \to E_r^{s+r,t-r+1}$); $H(E_r) := (\ker d_r / \operatorname{im} d_r) \cong E_{r+1}$; $E_1^{s,t} \cong \widetilde{H}^{s+t}(K_s \backslash K_{s-1})$; and E_∞ is associated to a filtration on $\widetilde{H}^*(K)$, in that $E_\infty^{s,t} \cong \ker(H^{s+t}(K) \to H^{s+t}(K_{s+1})) / \ker(H^{s+t}(K) \to H^{s+t}(K_s))$. For every $E_r^{s,t}$ expression in this section, the "total degree" $s+t$ is fixed, at say q, so we will suppress the "complementary degree" t, and write E_r^s to mean $E_r^{s,q-s}$ (s is called the "filtered degree"). Similarly, every subspace of $\Lambda[K]$ is understood to be just the q-dimensional component, and so we will write I to mean I_q. For further details on spectral sequences of filtrations, see, *e.g.*, [Sp, Section 9.1].

It is straightforward to verify that

$$E_r^s = \frac{Z_r^{s-1}}{Z_{r-1}^s + \delta Z_{r-1}^{s-r}} = \frac{Z_r^{s-1} + \widetilde{Q}^s}{\delta Z_{r-1}^{s-r} + \widetilde{Q}^s}$$

and $d_r = \delta$ form a spectral sequence corresponding to the filtration (4) as described above, where

$$Z_r^s = \{c \in \widetilde{Q}^s \colon \delta c \in \widetilde{Q}^{s+r}\}.$$

(The verification is analagous to that for the homology spectral sequence of a filtration [Sp, pp. 469–470].) Then, letting $r \to \infty$,

$$
\begin{aligned}
E_\infty^s &= \frac{Z_\infty^{s-1}}{Z_\infty^s + (\operatorname{im}\delta \cap \widetilde{Q}^{s-1})} \\
&= \frac{Z_\infty^{s-1} + \widetilde{Q}^s}{(\operatorname{im}\delta \cap \widetilde{Q}^{s-1}) + \widetilde{Q}^s},
\end{aligned} \tag{5}
$$

where

$$Z_\infty^s = \{c \in \widetilde{Q}^s \colon \delta c = \widetilde{0}\} = \widetilde{Q}^s \cap \ker\delta.$$

Lemma 3.1. *For the spectral sequence defined above,*

$$\dim E_\infty^{1,q-1} + \cdots + \dim E_\infty^{p,q-p} = \dim \frac{(\ker\delta + \widetilde{Q}^p)_q}{(\operatorname{im}\delta + \widetilde{Q}^p)_q}.$$

Proof. Recall that the total degree $s+t$ of every $E_r^{s,t}$ is fixed at q, as is the dimension of every subspace of $\Lambda[K]$, and so we suppress the q's in the proof.

By equation (5),

$$
\begin{aligned}
E_\infty^s &= \frac{(\ker\delta \cap \widetilde{Q}^{s-1}) + \widetilde{Q}^s}{(\operatorname{im}\delta \cap \widetilde{Q}^{s-1}) + \widetilde{Q}^s} \\
&= \frac{(\ker\delta + \widetilde{Q}^s) \cap \widetilde{Q}^{s-1}}{(\operatorname{im}\delta + \widetilde{Q}^s) \cap \widetilde{Q}^{s-1}}.
\end{aligned} \tag{6}
$$

The result now follows by an easy induction on p. For $p = 1$, by equation (6),

$$E_\infty^1 = \frac{(\ker\delta + \widetilde{Q}^1) \cap \widetilde{Q}^0}{(\operatorname{im}\delta + \widetilde{Q}^1) \cap \widetilde{Q}^0} = \frac{\ker\delta + \widetilde{Q}^1}{\operatorname{im}\delta + \widetilde{Q}^1}.$$

If $p > 1$, then

$$\begin{aligned}
\dim E_\infty^1 + \cdots + \dim E_\infty^p &=^{(1)} (\dim E_\infty^1 + \cdots + \dim E_\infty^{p-1}) + \dim E_\infty^p \\
&=^{(2)} \dim \frac{\ker\delta + \widetilde{Q}^{p-1}}{\operatorname{im}\delta + \widetilde{Q}^{p-1}} \\
&\qquad + \dim \frac{(\ker\delta + \widetilde{Q}^{p}) \cap \widetilde{Q}^{p-1}}{(\operatorname{im}\delta + \widetilde{Q}^{p}) \cap \widetilde{Q}^{p-1}} \\
&=^{(3)} \dim \frac{(\ker\delta + \widetilde{Q}^{p}) + \widetilde{Q}^{p-1}}{(\operatorname{im}\delta + \widetilde{Q}^{p}) + \widetilde{Q}^{p-1}} \\
&\qquad + \dim \frac{(\ker\delta + \widetilde{Q}^{p}) \cap \widetilde{Q}^{p-1}}{(\operatorname{im}\delta + \widetilde{Q}^{p}) \cap \widetilde{Q}^{p-1}} \\
&=^{(4)} \dim \frac{\ker\delta + \widetilde{Q}^{p}}{\operatorname{im}\delta + \widetilde{Q}^{p}}.
\end{aligned}$$

Equality $=^{(2)}$ above is by induction and equation (6), equality $=^{(3)}$ follows from $\widetilde{Q}^p \subseteq \widetilde{Q}^{p-1}$, and equality $=^{(4)}$ is a routine exercise in linear algebra (or see [Du2, Lemma 5.1]). Q.E.D.

§4. Near-cones

Let v be a vertex of a simplicial complex K. Let

$$\operatorname{del}_K v = \operatorname{del} v := \{F \in K \colon v \cup F \notin K\}$$

be the *deletion* of v (in K), let

$$\operatorname{lk}_K v = \operatorname{lk} v := \{F \in K \colon v \notin F,\ v \,\dot{\cup}\, F \in K\}$$

be the *link* of v (in K), and let the *star* of v (in K) be

$$v * \operatorname{lk} v = \{F \in K \colon v \cup F \in K\}$$

the cone over $\operatorname{lk} v$. Then K may be partitioned

$$K = (v * \operatorname{lk} v) \,\dot{\cup}\, \operatorname{del} v.$$

The link and star of v are subcomplexes of K.

We will say K is a *near-cone* with *apex* v if every face F in $\operatorname{del}_K v$ has its entire boundary $\{F \backslash w \colon w \in F\}$ contained in $v * \operatorname{lk}_K v$. In this case, every face of $\operatorname{del}_K v$ is a facet (*i.e.*, is maximal in K), since $v * \operatorname{lk} v$ is a subcomplex. If we contract the subcomplex $v * \operatorname{lk} v$ to v, what remains is a sphere for every face in $\operatorname{del} v$; therefore

$$\beta^q(K) = \#\{F \in \operatorname{del}_K v \colon \dim F = q\} \tag{7}$$

when K is a near-cone with apex v [BK, Theorem 4.3].

Lemma 4.1. *If K is a near-cone with apex v, then $\delta(\operatorname{lk} v) = \{\delta\widetilde{e}_F \colon F \in \operatorname{lk} v\}$ is a basis for* $\operatorname{im}\delta$.

Proof. The members of $\delta(\operatorname{lk} v)$ are linearly independent because if $F \in \operatorname{lk} v$ then $\delta\widetilde{e}_F$ has nontrivial support on $\widetilde{e}_{v\dot{\cup}F}$, but if $G \in \operatorname{lk} v$ and $G \neq F$, then $\delta\widetilde{e}_G$ has no support on $\widetilde{e}_{v\dot{\cup}F}$. Thus for each member of $\delta(\operatorname{lk} v)$ there is a face on which it alone has nontrivial support; linear independence follows immediately.

On the other hand, we will show that if $G \notin \operatorname{lk} v$, then $\delta\widetilde{e}_G$ is in the span of $\delta(\operatorname{lk} v)$. If $G \in \operatorname{del} v$, then $\delta\widetilde{e}_G = 0$, since G is a facet. The only possibility remaining is that $G = v \dot{\cup} F$ for $F \in \operatorname{lk} v$. In that case

$$\delta\widetilde{e}_F = \pm\alpha_{1v}\widetilde{e}_G + \sum_{\substack{w \neq v \\ F\dot{\cup}w \in K}} \pm\alpha_{1w}\widetilde{e}_{F\dot{\cup}w}$$

so

$$0 = \delta^2\widetilde{e}_F = \pm\alpha_{1v}\delta\widetilde{e}_G + \sum_{\substack{w \neq v \\ F\dot{\cup}w \in K}} \pm\alpha_{1w}\delta\widetilde{e}_{F\dot{\cup}w},$$

and so

$$\delta\widetilde{e}_G = \sum_{\substack{w \neq v \\ F\dot{\cup}w \in K}} \pm\left(\frac{\alpha_{1w}}{\alpha_{1v}}\right)\delta(F \dot{\cup} w).$$

Now, if $v \dot{\cup} (F \dot{\cup} w) \notin K$, then $F \dot{\cup} w \in \operatorname{del} v$, so $F \dot{\cup} w$ is a facet and so $\delta\widetilde{e}_{F\dot{\cup}w} = 0$. But if $v\dot{\cup}(F\dot{\cup}w) \in K$, then $F\dot{\cup}w \in \operatorname{lk} v$, so $\delta\widetilde{e}_{F\dot{\cup}w} \in \delta(\operatorname{lk} v)$. Thus $\delta\widetilde{e}_G$ is in the span of $\delta(\operatorname{lk} v)$. Q.E.D.

Lemma 4.2. *If K is a near-cone with apex v, then*

$$\ker\delta = \widetilde{D} + \operatorname{im}\delta,$$

where $D = \operatorname{del}_K v$.

Proof. Since every face in $D = \operatorname{del}_K v$ is a facet, $\widetilde{D} \subseteq \ker\delta$, so $\widetilde{D} + \operatorname{im}\delta \subseteq \ker\delta$.

By equation (7),

$$\dim(\ker\delta) - \dim(\operatorname{im}\delta) = \beta^*(K) = |\operatorname{del} v| = \dim\widetilde{D}.$$

Thus

$$\begin{aligned}\dim(\widetilde{D} + \operatorname{im}\delta) &= \dim \widetilde{D} + \dim(\operatorname{im}\delta) - \dim(\widetilde{D} \cap \operatorname{im}\delta)\\ &= \dim(\ker\delta) - \dim(\widetilde{D} \cap \operatorname{im}\delta).\end{aligned}$$

So now it only remains to show that

$$\widetilde{D} \cap \operatorname{im}\delta = 0. \tag{8}$$

To this end, recall from the proof of Lemma 4.1 that each $\delta\widetilde{e}_F$ in $\delta(\operatorname{lk} v)$ is the unique element of $\delta(\operatorname{lk} v)$ with nonzero support on $\widetilde{e}_{v\dot{\cup}F}$, but now note further that $v \dot{\cup} F \notin \operatorname{del} v$. Thus any nonzero element of $\operatorname{im}\delta = \delta(\widetilde{\operatorname{lk} v})$ has nontrivial support outside $\operatorname{del} v$, which establishes equation (8), and hence the lemma. Q.E.D.

Lemma 4.3. *If $K = L \dot{\cup} Q$ is a partition of the faces of a near-cone K into two disjoint subsets, then*

$$\dim \frac{(\ker\delta + \widetilde{Q})_q}{(\operatorname{im}\delta + \widetilde{Q})_q} = \#\{F \in L_q\colon v \notin F,\ v \dot{\cup} F \notin K\}.$$

Proof. Again let $D = \operatorname{del}_K v$. Then

$$\begin{aligned}\frac{\ker\delta + \widetilde{Q}}{\operatorname{im}\delta + \widetilde{Q}} &= \frac{\widetilde{D} + \operatorname{im}\delta + \widetilde{Q}}{\operatorname{im}\delta + \widetilde{Q}} && \text{by Lemma 4.2}\\ &\cong \frac{\widetilde{D}}{\widetilde{D} \cap (\operatorname{im}\delta + \widetilde{Q})}\\ &= \frac{\widetilde{D}}{\widetilde{D} \cap \widetilde{Q}} && \text{by equation (8)}\\ &= \frac{\widetilde{D}}{\widetilde{D \cap Q}}.\end{aligned}$$

Thus,

$$\begin{aligned}\dim \frac{(\ker\delta + \widetilde{Q})_q}{(\operatorname{im}\delta + \widetilde{Q})_q} &= |D_q| - |(D \cap Q)_q|\\ &= |\operatorname{del}_K v \cap L_q|\\ &= \#\{F \in L_q\colon v \notin F,\ v \dot{\cup} F \notin K\}.\end{aligned}$$

Q.E.D.

§5. Algebraic shifting

Algebraic shifting transforms a simplicial complex into a shifted simplicial complex with many of the same algebraic properties of the original complex. Algebraic shifting was introduced by Kalai in [Ka1]; our exposition is summarized from [BK] and included for completeness.

Definition. If $R = \{r_0 < \cdots < r_q\}$ and $S = \{s_0 < \cdots < s_q\}$ are $(q+1)$-subsets of $[n] = \{1, \ldots, n\}$, then:

- $R \leq_P S$ under the standard *partial order* if $r_i \leq s_i$ for all i; and
- $R <_L S$ under the *lexicographic order* if there is a j such that $r_j < s_j$ and $r_i = s_i$ for $i < j$.

Lexicographic order is a total order which refines the partial order.

Definition. A collection $\mathcal{C}$ of $(q+1)$-subsets of $[n]$ is *shifted* if $R \leq_P S$ and $S \in \mathcal{C}$ together imply that $R \in \mathcal{C}$. A simplicial complex Δ is *shifted* if the set of q-dimensional faces of Δ is shifted for every q.

It is not hard to see that shifted simplicial complexes are near-cones with apex 1.

Recall (see Section 2) that $\{\alpha_{ij}\}_{1 \leq i,j \leq n}$ are n^2 transcendentals adjoined to our field of coefficients.

Definition (Kalai). For $1 \leq i \leq n$, let

$$f_i = \sum_{j=1}^{n} \alpha_{ij} e_j,$$

so $\{f_1, \ldots, f_n\}$ forms a "generic" basis of $\hat{\boldsymbol{k}}V$. (Note this is consistent with our definition of f_1 in Section 2.) Define $f_S := f_{i_0} \wedge \cdots \wedge f_{i_q}$ for $S = \{i_0 < \cdots < i_q\}$ (and set $f_\emptyset = 1$). Let

$$\Delta(K, \boldsymbol{k}) := \{S \subseteq [n] \colon \widetilde{f_S} \notin \operatorname{span}\{\widetilde{f_R} \colon R <_L S\}\}$$

be the *algebraically shifted complex* obtained from K; we will write $\Delta(K)$ instead of $\Delta(K, \boldsymbol{k})$ when the field is understood to be $\boldsymbol{k}$. In other words, the $(q+1)$-subsets of $\Delta(K)$ can be chosen by listing all the $(q+1)$-subsets of $[n]$ in lexicographic order and omitting those that are in the span of earlier subsets on the list, modulo I_K and with respect to the f-basis.

The algebraically shifted complex $\Delta(K)$ is (as its name suggests) shifted, and is independent of the numbering of the vertices of K [BK, Theorem 3.1].

Recall from Section 1 that if $L \subseteq K$ is a pair of simplicial complexes, then $\Delta(L) \subseteq \Delta(K)$. Thus for $Q = K \backslash L$, we may define $\Delta(Q) = \Delta(K) \backslash \Delta(L)$.

Lemma 5.1. *Let $L \subseteq K$ be a pair of simplicial complexes and $Q = K \backslash L$. Then*

$$\dim \frac{(\ker \delta + \widetilde{Q})_q}{(\operatorname{im} \delta + \widetilde{Q})_q} \geq \#\{F \in \Delta(L)_q \colon 1 \notin F,\ 1 \dot{\cup} F \notin \Delta(K)\}.$$

Proof. This is implicit in the proof of [Du2, Theorem 5.2]. As it is not stated there explicitly, we reproduce here some of the details. From [Du2, Lemma 4.4]

$$\dim(\operatorname{im} \delta \cap \widetilde{Q})_{q+1} \leq \#\{F \in \Delta(K)_q \colon 1 \notin F,\ 1 \dot{\cup} F \in \Delta(Q)\},$$

and from [Du2, Lemma 4.5]

$$\dim(\delta \widetilde{Q})_{q+1} \geq \#\{F \in \Delta(Q)_q \colon 1 \notin F,\ 1 \dot{\cup} F \in \Delta(Q)\}.$$

Then, since $L = K \backslash Q$,

$$\dim \frac{(\operatorname{im} \delta \cap \widetilde{Q})_{q+1}}{(\delta \widetilde{Q})_{q+1}} \leq \#\{F \in \Delta(L)_q \colon 1 \notin F,\ 1 \dot{\cup} F \in \Delta(Q)\}.$$

By equations (1) and (7), respectively,

$$\beta^q(L) = \beta^q(\Delta(L)) = \#\{F \in \Delta(L)_q \colon 1 \notin F,\ 1 \dot{\cup} F \notin \Delta(L)\}.$$

But, with the notation $\delta^{-1}\widetilde{Q} := \{\widetilde{x} \in \Lambda[K] \colon \delta\widetilde{x} \in \widetilde{Q}\}$, we also have

$$\begin{aligned}
\beta^q(L) &= \dim \frac{(\delta^{-1}\widetilde{Q})_q}{(\operatorname{im} \delta + \widetilde{Q})_q} && \text{by [Du1, Lemma 3.3]} \\
&= \dim \frac{(\delta^{-1}\widetilde{Q})_q}{(\ker \delta + \widetilde{Q})_q} + \dim \frac{(\ker \delta + \widetilde{Q})_q}{(\operatorname{im} \delta + \widetilde{Q})_q} \\
&= \dim \frac{(\operatorname{im} \delta \cap \widetilde{Q})_{q+1}}{(\delta \widetilde{Q})_{q+1}} + \dim \frac{(\ker \delta + \widetilde{Q})_q}{(\operatorname{im} \delta + \widetilde{Q})_q} && \text{by [Du1, Lemma 3.6],}
\end{aligned}$$

and so

$$\begin{aligned}
\dim \frac{(\ker \delta + \widetilde{Q})_q}{(\operatorname{im} \delta + \widetilde{Q})_q} &= \beta^q(L) - \dim \frac{(\operatorname{im} \delta \cap \widetilde{Q})_{q+1}}{(\delta \widetilde{Q})_{q+1}} \\
&\geq \#\{F \in \Delta(L)_q \colon 1 \notin F,\ 1 \dot{\cup} F \notin \Delta(L)\} \\
&\quad - \#\{F \in \Delta(L)_q \colon 1 \notin F,\ 1 \dot{\cup} F \in \Delta(Q)\} \\
&= \#\{F \in \Delta(L)_q \colon 1 \notin F,\ 1 \dot{\cup} F \notin \Delta(K)\}.
\end{aligned}$$

Q.E.D.

§6. Proof of Main Theorem

Given a filtration $\mathcal{K}$ of a simplicial complex K, let $E_r^{s,t}$ refer to the terms of the corresponding spectral sequence given in Section 3, and let

$$e^{s,t}(\mathcal{K}) = \dim E_\infty^{s,t}(\mathcal{K}).$$

Theorem 6.1. *For all* p, q,

$$e^{1,q-1}(\mathcal{K}) + \cdots + e^{p,q-p}(\mathcal{K}) \geq e^{1,q-1}(\Delta(\mathcal{K})) + \cdots + e^{p,q-p}(\Delta(\mathcal{K})).$$

Proof. For $0 \leq s \leq m$, let $\Sigma^s = \Delta(K) \backslash \Delta(K_s)$, so

$$\Lambda[\Delta(K)] = \widetilde{\Sigma}^0 \supseteq \widetilde{\Sigma}^1 \supseteq \cdots \supseteq \widetilde{\Sigma}^m = I_{\Delta(K)}$$

is the filtration of ideals of $\Lambda[\Delta(K)]$ corresponding to the filtration $\Delta(\mathcal{K})$. By Lemmas 3.1 and 5.1,

$$\begin{aligned} e^{1,q-1}(\mathcal{K}) + \cdots + e^{p,q-p}(\mathcal{K}) &= \dim \frac{(\ker \delta_K + \widetilde{Q}^p)_q}{(\operatorname{im} \delta_K + \widetilde{Q}^p)_q} \\ &\geq \#\{F \in \Delta(K_p)_q \colon 1 \notin F,\ 1 \dot\cup F \notin \Delta(K)\}. \end{aligned}$$

On the other hand, because $\Delta(K)$ is shifted and hence a near-cone, Lemmas 3.1 and 4.3 give

$$\begin{aligned} e^{1,q-1}(\Delta(\mathcal{K})) + \cdots + e^{p,q-p}(\Delta(\mathcal{K})) &= \dim \frac{(\ker \delta_{\Delta(K)} + \widetilde{\Sigma}^p)_q}{(\operatorname{im} \delta_{\Delta(K)} + \widetilde{\Sigma}^p)_q} \\ &= \#\{F \in \Delta(K_p)_q \colon 1 \notin F,\ 1 \dot\cup F \notin \Delta(K)\}. \end{aligned}$$

Q.E.D.

Note that $e^{1,q-1}(\mathcal{K}) + \cdots + e^{m,q-m}(\mathcal{K}) = \beta^q(K)$, which, by equation (1) is unchanged under algebraic shifting. Thus, Theorem 6.1 says that algebraic shifting puts less of the fixed sum of the $e^{s,q-s}$'s into the earlier part of the filtration, and hence puts more into the later part. In particular,

$$\begin{aligned} &e^{p+1,q-p-1}(\mathcal{K}) + \cdots + e^{m,q-m}(\mathcal{K}) \\ &\qquad \leq e^{p+1,q-p-1}(\Delta(\mathcal{K})) + \cdots + e^{m,q-m}(\Delta(\mathcal{K})). \end{aligned}$$

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Coordinate Subspace Arrangements and Monomial Ideals

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Abstract.

We relate the (co)homological properties of real coordinate subspace arrangements and of monomial ideals.

§1. Introduction

In [PRW] we describe the cohomological properties of a real diagonal subspace arrangement via a minimal free resolution over a certain quotient of a polynomial ring by a monomial ideal. Here we relate the (co)homological properties of two objects: square-free monomial ideals and real coordinate subspace arrangements. The interest in studying such arrangements comes from the facts that they provide examples of arbitrary torsion in the cohomology of the complement of the arrangement [Bj] and the complements provide examples of manifolds with properties similar to toric varieties [DJ], and toric varieties as quotients (see for example [BCo]). A comparison of our formula [GPW, Theorem 2.1] for monomial ideals with the Goresky-MacPherson Formula [GM, III.1.5. Theorem A] for the cohomology of the complement of a subspace arrangement leads to Theorem 3.1. This result states that the i-dimensional cohomology of the complement of a real coordinate subspace arrangement is computed by the Betti numbers in the i-strand in the minimal free resolution of a certain square-free monomial ideal. In Corollaries 3.3 and 3.4 we show how this reveals an equivalence of results, which on the one hand were proved for subspace arrangements by Björner [Bj] and on the other hand were recently proved for monomial ideals by Eagon-Reiner and Terai [ER, Te]: Very recently Terai obtained

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a formula which expresses the regularity of a square-free monomial ideal in terms of the projective dimension of another monomial ideal and which immediately implies that the regularity of a monomial ideal is bounded by its arithmetic degree; Corollary 3.3 shows that Terai's formula is equivalent to Björner's result [Bj, Theorem 11.2.1(ii)].

Motivated by our work Babson-Chan [BCh] proved that the cohomology algebra of the complement of the complexification of a coordinate subspace arrangement is isomorphic to the Tor-algebra of a monomial ideal, see Theorem 3.6.

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§2. Multigraded Betti numbers

In this section we recall how to obtain the multigraded Betti numbers of a monomial ideal using the lcm-lattice. Consider the polynomial ring $S = k[x_1, \dots, x_n]$ over a field k as $\mathbf{N}^n$-graded by letting $\deg(x_i)$ be the i^{th} standard basis vector in $\mathbf{R}^n$. Let I be a monomial ideal minimally generated by monomials $m_1, \dots, m_d$. The ideal I and the minimal free resolution of S/I over S are $\mathbf{N}^n$-graded. Therefore we have $\mathbf{N}^n$-graded Betti numbers

$$b_{i,\mathbf{x}^\alpha}(S/I) = \dim_k \operatorname{Tor}^S_{i,\alpha}(S/I, k)$$

for $i \geq 0$, $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbf{N}^n$ and $\mathbf{x}^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. The *lcm-lattice* L_I of I is the partially ordered set on the set of least common multiples $\operatorname{lcm}(B)$ of all subsets $B \subseteq \{m_1, \dots, m_r\}$ ordered by divisibility. Clearly, L_I is a lattice (i.e., infima and suprema exist) with $1 = \operatorname{lcm}(\emptyset)$ as its minimal element and $\operatorname{lcm}(m_1, \dots, m_r)$ as its maximal element. Taylor's resolution (cf. [Ei, p. 439]) shows that $b_{i,m}(S/I) = 0$ if $m \notin L_I$.

Let L be a lattice with minimal element $\hat{0}$ and $p \in L$. We write $(\hat{0}, p)_L$ for the *open interval* $\{q \in L \mid \hat{0} < q < p\}$ in L. In particular, for $m \in L_I$ we denote by $(\hat{0}, m)_{L_I}$ the open lower interval in L_I below m.

Theorem 2.1([GPW, Theorem 2.1]). *For $i \geq 1$ and $m \in L_I$ we have*

$$b_{i,m}(S/I) = \dim \widetilde{\mathrm{H}}_{i-2}((\hat{0}, m)_{L_I}; k).$$

We present a short application of the above result. We apply Theorem 2.1 to a class of specific monomial ideals similar to those studied in [BH, Section 6]. If I is square-free, then L_I can be identified with a join-sublattice of the Boolean lattice on an n-element set generated by the supports of the monomials generating I. Let $M_{n,\ell}$ be the monomial ideal generated by the monomials with support $[i, i+\ell-1]$ for $i = 1, \ldots, n-\ell+1$. Then $L_{M_{n,\ell}}$ is isomorphic to the interval generated sublattice of the Boolean lattice generated by $[i, i+\ell-1]$ for $i = 1, \ldots, n-\ell+1$. By Björner-Wachs [BW, Corollary 8.4] we get

$$\Delta(L^\circ_{M_{n,\ell}}) \simeq \begin{cases} S^{2n/(\ell+1)-2}, & \text{if } n \equiv 0 \pmod{\ell+1} \\ S^{2(n+1)/(\ell+1)-3}, & \text{if } n \equiv -1 \pmod{\ell+1} \\ \mathbf{pt.}, & \text{otherwise.} \end{cases}$$

If $m \in L_{M_{n,\ell}}$, then the support set of m is the disjoint union of intervals $A_i = [j_i, l_i]$, $i = 1, \ldots, r$ with $l_i + 2 \leq j_{i+1}$. For $n_i = l_i - j_i + 1$ we then have $(\hat{0}, m)_{L_{M_{n,\ell}}} \cong \left(L_{n_1,\ell} \times \cdots \times L_{n_q,\ell}\right)^\circ$. Set

$$\begin{aligned} p = 2 \, + & \sum_{n_i \equiv 0 \bmod \ell+1} (2n/(\ell+1) - 2) \\ + & \sum_{n_i \equiv -1 \bmod \ell+1} (2(n+1)/(\ell+1) - 2) + 2(\ell-1). \end{aligned}$$

We conclude that $b_{i,m}(S/M_{n,\ell}) = 0$ if and only if there exists a j such that $n_j \not\equiv 0, -1 \pmod{\ell+1}$ or $i \neq p$; otherwise $b_{i,m} = 1$.

Proposition 2.2. *Let $m = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ be a monomial in L_I. For $m' = x_1^{\beta_1} \cdots x_n^{\beta_n}$ strictly dividing m we define $s(m') = \{i \mid \alpha_i = \beta_i\} \subseteq [n]$. Let $T(m)$ be the poset of subsets of $[n]$ obtained from $(\hat{0}, m)_{L_I}$ by applying the map s. Then*
(a) *$(\hat{0}, m)_{L_I}$ and $T(m)$ are homotopy equivalent.*
(b) *$\widetilde{H}_i((\hat{0}, m)_{L_I}; k) = 0$ for $i > n-2$.*
(c) *Suppose that there exists a minimal monomial generator g of I such that for each $1 \leq i \leq n$ we have that x_i^p divides g implies that x_i^{p+1} divides m. Then $\widetilde{H}_i((\hat{0}, m)_{L_I}; k) = 0$ for $i \geq 0$.*

In view of Theorem 2.1, we see that Proposition 2.2(b) gives a combinatorial proof of Hilbert's Syzygy Theorem for monomial ideals (cf.

[Ei, Corollary 19.7]), and Proposition 2.2(c) is an analogue to [BPS, Theorem 3.2].

Proof. Let $A \in T(m)$ be a set. Then the lower fiber $s^{-1}(\{B \in T(m) \mid B \subseteq A\})$ has the lcm of all minimal generators m' of I with $s(m') \subseteq A$ as its maximal element. In particular, the lower fiber is contractible. Applying Quillen's Fiber Lemma [Bj, Theorem 10.5] we conclude that $(\hat{0}, m)_{L_I}$ and $T(m)$ are homotopy equivalent. This proves (a).

The claim (b) holds since the order complex $\Delta(T(m))$ has dimension $\leq n-2$. Finally, note that under the assumption of (c), $T(m)$ has the empty set $\emptyset = s(g)$ as its least element and therefore is contractible. $\square$

§3. (Co)homology of real coordinate subspace arrangements and square-free monomial ideals

In this section we relate the (co)homological properties of real coordinate subspace arrangements and of square-free monomial ideals.

Let Δ be a simplicial complex on the vertex set $[n]$ and $F(\Delta)$ the set of facets (i.e., maximal faces) of Δ. Fix an orthonormal basis $e_1, \dots, e_n$ of $\mathbf{R}^n$. The *real coordinate subspace arrangement* defined by Δ is

$$\mathcal{K}_\Delta = \{\mathrm{span}(e_j \mid j \in \sigma) \mid \sigma \in F(\Delta)\}.$$

The *union* $\mathcal{V}_\Delta = \bigcup_{\sigma \in F(\Delta)} \mathrm{span}(e_j \mid j \in \sigma)$ is a real algebraic variety. We denote by $\widehat{\mathcal{V}}_\Delta$ the one-point compactification of $\mathcal{V}_\Delta$ inside the unit n-sphere (which is the one-point compactification of $\mathbf{R}^n$) and by $\mathcal{M}_\Delta = \mathbf{R}^n \setminus \mathcal{K}_\Delta$ the set-theoretic complement of the arrangement in $\mathbf{R}^n$. Furthermore, we denote by $\mathcal{L}_\Delta$ the intersection lattice of the arrangement $\mathcal{K}_\Delta$; it consists of all intersections $\{\bigcap_{V \in \mathcal{B}} V \mid \mathcal{B} \subseteq \mathcal{K}_\Delta\}$ ordered by reversed inclusion. In particular, the intersection corresponding to $\mathcal{B} = \mathcal{K}_\Delta$ serves as the maximal element and the intersection corresponding to $\mathcal{B} = \emptyset$ is regarded as $\mathbf{R}^n$ and serves as the minimal element in $\mathcal{L}_\Delta$.

On the other hand consider the polynomial ring $S = k[x_1, \dots, x_n]$ over a field k. Let I be a monomial ideal minimally generated by monomials $m_1, \dots, m_d$. The role of the intersection lattice is played by the lcm-lattice L_I with elements the least common multiples of $m_1, \dots, m_d$ ordered by divisibility. We define the total degree of x_i by $\mathrm{tdeg}(x_i) = 1$ for $1 \leq i \leq n$. The $\mathbf{N}$-graded Betti numbers lead to *bigraded Betti numbers* $b_{i,j}(S/I) = \dim_k \mathrm{Tor}_{i,j}^S(S/I, k) = \sum_{\substack{\mathrm{tdeg}(m)=j \\ m \in L_I}} b_{i,m}(S/I)$. The *total Betti numbers* are defined by $b_i(S/I) = \sum_{j\geq 0} b_{i,j}(S/I)$ for $i \geq 0$. The complexity of the resolution is measured by its length $\mathrm{pd}(S/I) = \max\{\, i \mid b_i(S/I) \neq 0\,\}$ and also by the invariant $\mathrm{reg}(S/I) = \max\,\{\, j - i \mid b_{i,j}(S/I) \neq 0\,\}$, called the *regularity*.

Let $\Delta^\vee = \{A \mid [n] \setminus A \notin \Delta\}$ be the Alexander dual complex of Δ. The Stanley-Reisner ideal of $\Delta^\vee$ is

$$\begin{aligned} I_{\Delta^\vee} &= \Big(\Big\{\, \frac{x_1 \cdots x_n}{x_{j_1} \cdots x_{j_p}} \;\Big|\; \{j_1, \dots, j_p\} \in \Delta\Big\}\Big) \\ &= \Big(\Big\{\, \text{monomial } t \;\Big|\; \gcd(t,t') \neq 1 \text{ for each monomial } t' \in I_\Delta \,\Big\}\Big). \end{aligned}$$

Note that we have

$$I_{\Delta^\vee} = \Big(\Big\{\, x_{j_1} \cdots x_{j_p} \;\Big|\; x_{j_1} = \cdots = x_{j_p} = 0 \text{ defines a subspace in } \mathcal{K}_\Delta\Big\}\Big).$$

Theorem 3.1. *Let Δ be a simplicial complex on the vertex set $[n]$, $\mathcal{K}_\Delta$ the real coordinate subspace arrangement defined by Δ, and $I_{\Delta^\vee}$ the Stanley-Reisner monomial ideal associated to $\Delta^\vee$. We have*

$$\dim \widetilde{\mathrm{H}}_{n-1-i}(\widehat{\mathcal{V}}_\Delta; k) = \dim \widetilde{\mathrm{H}}^i(\mathcal{M}_\Delta; k) = \sum_{j\geq 0} b_{j,i+j}(S/I_{\Delta^\vee}) \quad \textit{for } i \geq 0,$$

$$\max\{\, j \mid \widetilde{\mathrm{H}}^j(\mathcal{M}_\Delta; k) \neq 0\,\} = \mathrm{reg}(S/I_{\Delta^\vee}).$$

Thus $\dim \widetilde{\mathrm{H}}^i(\mathcal{M}_\Delta; k)$ picks up the cohomology of the i-strand in the minimal free resolution of $S/I_{\Delta^\vee}$. Therefore, the dimensions of the cohomology groups can be computed in concrete examples by the computer algebra system Macaulay 2 [GS] by computing the bigraded Betti numbers of $S/I_{\Delta^\vee}$ and then applying Theorem 3.1. Also, Theorem 3.1 yields $\dim \widetilde{\mathrm{H}}^*(\mathcal{M}_\Delta; k)$ for some special types of arrangements when explicit formulas for the Betti numbers of the corresponding monomial ideals are known; for example, a result of Bayer-Peeva-Sturmfels gives the Betti numbers for polarizations of generic monomial ideals and a result of Aramova-Herzog-Hibi provides the Betti numbers for weakly stable arrangements.

Proof of Theorem 3.1. For a lattice L with minimal element $\hat{0}$ we write $L_{>\hat{0}}$ for the poset obtained from L by removing the minimal element $\hat{0}$.

The equalities $\dim \widetilde{\mathrm{H}}_{n-1-i}(\widehat{\mathcal{V}}_\Delta; k) = \dim \widetilde{\mathrm{H}}^i(\mathcal{M}_\Delta; k)$ are well known and are proved by Alexander duality. Applying a formula of Goresky-MacPherson [GM, III.1.5. Theorem A] to the intersection lattice $\mathcal{L}_\Delta$ of

$\mathcal{K}_\Delta$ we get

$$\dim \widetilde{\mathrm{H}}^i(\mathcal{M}_\Delta; k) = \sum_{m \in (\mathcal{L}_\Delta)_{>\hat{0}}} \dim \widetilde{\mathrm{H}}_{\mathrm{codim}(m)-2-i}((\hat{0}, m)_{\mathcal{L}_\Delta}; k).$$

Note that $\mathcal{L}_\Delta$ is the lattice of all non-empty intersections of facets of Δ ordered by reversed inclusion and enlarged by an additional minimal element $\hat{0}$ and maximal element $\hat{1}$. By Proposition 2.3, the facets of Δ correspond bijectively to the minimal monomial generators of $I_{\Delta^\vee}$. Furthermore, if $\sigma_1, \dots, \sigma_r$ are facets of Δ, then we identify

$$\text{(3.2)} \qquad \bigcap_{1 \le i \le r} \sigma_i \in \mathcal{L}_\Delta \quad \longleftrightarrow \quad \mathrm{lcm}\Big(\frac{x_1 \cdots x_n}{\mathbf{x}_{\sigma_i}} \,|\, 1 \le i \le r\Big) \in L_{I_{\Delta^\vee}}.$$

Thus $\mathcal{L}_\Delta$ coincides with the lcm-lattice $L_{I_{\Delta^\vee}}$ of the monomial ideal $I_{\Delta^\vee}$. Also, (3.2) yields that

$$\begin{aligned}\mathrm{codim}\,\Big(\bigcap_{1 \le i \le r} \mathcal{K}_{\sigma_i} \Big) &= n - \Big| \bigcap_{1 \le i \le r} \sigma_i \Big| \\ &= \Big| \mathrm{supp}\Big(\mathrm{lcm}\Big(\frac{x_1 \cdots x_n}{\mathbf{x}_{\sigma_i}} \,\Big|\, 1 \le i \le r \Big) \Big) \Big|.\end{aligned}$$

Therefore, $\dim \widetilde{\mathrm{H}}_{\mathrm{codim}(m)-2-i}\,((\hat{0}, m)_{\mathcal{L}_\Delta}; k) = \widetilde{\mathrm{H}}_{\mathrm{tdeg}(m)-2-i}\,((\hat{0}, m)_{L_{I_{\Delta^\vee}}}; k)$, where m is considered as an element in $(\mathcal{L}_\Delta)_{>\hat{0}}$ on the left-hand side of the formula and m is considered as an element in $(L_{I_{\Delta^\vee}})_{>\hat{0}}$ on the right-hand side of the formula. By Theorem 2.1 there are equalities $\dim \widetilde{\mathrm{H}}_{j-2}((\hat{0}, m)_{L_{I_{\Delta^\vee}}}; k) = b_{j,m}(S/I_{\Delta^\vee})$ for $j \ge 1$; also note that $b_{0,m} = 0$ for $m \in (L_{I_{\Delta^\vee}})_{>\hat{0}}$. Combining this with the Goresky-MacPherson formula above we obtain the equalities

$$\dim \widetilde{\mathrm{H}}^i(\mathcal{M}_\Delta; k) = \sum_{m \in (L_{I_{\Delta^\vee}})_{>\hat{0}}} b_{\mathrm{tdeg}(m)-i,m}(S/I_{\Delta^\vee})$$

for $i \ge 0$. Taylor's (possibly non-minimal) resolution of $S/I_{\Delta^\vee}$ (cf. [Ei, p. 439]) implies that $b_{i,m}(S/I_{\Delta^\vee}) = 0$ if $m \notin L_{I_{\Delta^\vee}}$. Therefore,

$$\sum_{m \in (L_{I_{\Delta^\vee}})_{>\hat{0}}} b_{\mathrm{tdeg}(m)-i,m}(S/I_{\Delta^\vee}) = \sum_{j \ge 0} b_{j,i+j}(S/I_{\Delta^\vee}).$$

Thus $\dim \widetilde{\mathrm{H}}^i(\mathcal{M}_\Delta; k) = \sum_{j \ge 0} b_{j,i+j}(S/I_{\Delta^\vee})$ as desired. The statement about the regularity of $S/I_{\Delta^\vee}$ follows immediately. □

The proof of Theorem 3.1 is based on an identification of the intersection lattice of an arrangement with the lcm-lattice of a monomial ideal, and then comparison of Theorem 2.1 with the Goresky-MacPherson Formula. The important point is that the codimension of an element in the intersection lattice equals the total degree of this element in the lcm-lattice. Such a construction can be also built for arrangements other than the real coordinate subspace arrangements.

Corollary 3.3. *The following two properties are equivalent:*
(a) $\min\{\, i \mid \widetilde{\mathrm{H}}_i(\widehat{\mathcal{V}}_\Delta; k) \neq 0 \,\} = \operatorname{depth}(S/I_\Delta)$;
(b) $\operatorname{pd}(S/I_\Delta) - 1 = \operatorname{reg}(S/I_{\Delta^\vee})$.

Property (a) is proved to hold by Björner [Bj, Theorem 11.2.1(ii)]. Property (b) is proved by Terai [Te].

Proof. On the one hand we have the following equalities:

$$\begin{aligned}\operatorname{reg}(S/I_{\Delta^\vee}) &= \max\{\, i \mid \widetilde{\mathrm{H}}^i(\mathcal{M}_\Delta; k) \neq 0 \,\} \\ &= \max\{\, i \mid \widetilde{\mathrm{H}}_{n-1-i}(\widehat{\mathcal{V}}_\Delta; k) \neq 0 \,\} \\ &= n - 1 - \min\{\, j \mid \widetilde{\mathrm{H}}_j(\widehat{\mathcal{V}}_\Delta; k) \neq 0 \,\}.\end{aligned}$$

On the other hand, the Auslander-Buchsbaum equality implies that

$$\operatorname{pd}(S/I_{\Delta^\vee}) - 1 = n - 1 - \operatorname{depth}(S/I_\Delta).$$

Therefore, (a) and (b) are equivalent. □

A particular case of Corollary 3.3 says that the following two properties are equivalent:
(a) Δ is Cohen-Macaulay if and only if $\dim \widetilde{\mathrm{H}}_i(\widehat{\mathcal{V}}_\Delta; k) = 0$ for all $i \leq \dim(\Delta)$;
(b) Δ is Cohen-Macaulay if and only if the minimal free resolution of $I_{\Delta^\vee}$ is linear.

Property (a) is proved to hold by Björner [Bj, Theorem 11.2.2]. Property (b) is proved to hold by Eagon and Reiner [ER]; it provides a topological characterization of the linearity of a monomial resolution.

Corollary 3.4. *The following two properties are equivalent:*
(a) $\max\{\, i \mid \widetilde{\mathrm{H}}_i(\widehat{\mathcal{V}}_\Delta; k) \neq 0 \,\} = \dim(S/I_\Delta)$;
(b) $\min\{\, i \mid b_{j,j+i} \neq 0$ for some $j \,\}$ *is the minimal degree of a minimal monomial generator of* $I_{\Delta^\vee}$ *minus one.*

Property (a) holds by Björner [Bj, Theorem 11.2.1(i)]. It is easy to check that Property (b) holds.

Proof. On the one hand, we have the following equalities:

$$\begin{aligned}\min\{\, i \mid b_{j,j+i} \neq 0 \text{ for some } j \,\} &= \min\{\, i \mid \widetilde{\mathrm{H}}^i(\mathcal{M}_\Delta; k) \neq 0 \,\} \\ &= \min\{\, i \mid \widetilde{\mathrm{H}}_{n-1-i}(\widehat{\mathcal{V}}_\Delta; k) \neq 0 \,\} \\ &= n - 1 - \max\{\, j \mid \widetilde{\mathrm{H}}_j(\widehat{\mathcal{V}}_\Delta; k) \neq 0 \,\}.\end{aligned}$$

On the other hand we have that

$$\begin{aligned} & \min\{\text{ degree of a minimal monomial generator of } I_{\Delta^\vee}\} - 1 \\ = \quad & n - \max\{\, |\sigma| \mid \sigma \text{ is a facet of } \Delta \} - 1 \\ = \quad & n - 1 - \dim(S/I_\Delta). \end{aligned}$$

Therefore, (a) and (b) are equivalent. □

Remark 3.5 (Complexification). Fix a standard basis $f_1, \ldots, f_n$ of $\mathbf{C}^n$. The *complexification* of $\mathcal{K}_\Delta$ is the complex coordinate subspace arrangement

$$\mathcal{K}_\Delta \otimes \mathbf{C} = \{\operatorname{span}(f_j \mid j \in \sigma) \mid \sigma \in F(\Delta)\}.$$

Denote $\mathcal{M}_\Delta \otimes \mathbf{C} = \mathbf{C}^n \setminus (\mathcal{K}_\Delta \otimes \mathbf{C})$. The algebraic analogue of this complexification is the ring $S^{\bullet 2} = k[x_1, \ldots, x_n, y_1, \ldots, y_n]$ and the ideal $I^{\bullet 2}_{\Delta^\vee} = (x_{i_1} \cdots x_{i_s} y_{i_1} \cdots y_{i_s} \mid x_{i_1} \cdots x_{i_s}$ is a minimal monomial generator of $I_{\Delta^\vee}$) Theorem 3.1 shows that

$$\widetilde{\mathrm{H}}^i(\mathcal{M}_\Delta \otimes \mathbf{C}; k) \cong \bigoplus_{j \geq 0} \operatorname{Tor}^{S^{\bullet 2}}_{j,i+j}(S^{\bullet 2}/I^{\bullet 2}_{\Delta^\vee}, k).$$

The ideal $I^{\bullet 2}_{\Delta^\vee}$ can be depolarized by setting $x_i = y_i$ for $1 \leq i \leq n$; in this way, one obtains a version of the above formula over the ring S. Motivated by our work, Babson and Chan proved the following result:

Theorem 3.6 ([BCh]). *The rings* $\widetilde{H}^*(\mathcal{M}_\Delta \otimes \mathbf{C}; k)$ *and* $\operatorname{Tor}^{S^{\bullet 2}}_{*,*}(S^{\bullet 2}/I^{\bullet 2}_{\Delta^\vee}, k)$ *are isomorphic if the characteristic of* k *is not* 2.

This theorem shows that in the case of a complex coordinate subspace arrangement, the Koszul complex computing $\operatorname{Tor}^S_{*,*}(S/I, k)$ provides a much simpler model for the cohomology ring than the models in [DP] and [Yu].

An analogue of Theorem 3.6 is not valid for the structure of the cohomology algebra of the complement of a real coordinate subspace arrangement. Despite the isomorphism of vector spaces $\widetilde{\mathrm{H}}^i(\mathcal{M}_\Delta; k) \cong \oplus_{j \geq 0} \operatorname{Tor}^S_{j,i+j}(S/I_{\Delta^\vee}, k)$ given by Theorem 3.1, in general the algebras $\mathrm{H}^*(\mathcal{M}_\Delta; k)$ and $\operatorname{Tor}^S_{*,*}(S/I_{\Delta^\vee}, k)$ are not isomorphic. This is easily seen when $I_{\Delta^\vee} = (x_1, \ldots, x_n)$; in this case $\operatorname{Tor}^S_{*,*}(S/I_{\Delta^\vee}, k)$ is an exterior algebra, while $\mathrm{H}^*(\mathcal{M}_\Delta; k)$ is an algebra generated by commuting idempotents.

Corollary 3.7. *Suppose that Δ has d facets. Then*

$$\sum_{i\geq 0} \dim \widetilde{\mathrm{H}}^i(\mathcal{M}_\Delta; k) = \sum_{i\geq 0} b_i(S/I_{\Delta^\vee}) \leq \sum_{i\geq 0} c_i(n,d),$$

where $c_i(n,d)$ is the maximum number of i-dimensional faces of an n-dimensional polytope having d vertices.

There are explicit formulas for $c_i(n,d)$ (see e.g. [Zi, Corollary 8.28]) and the cyclic n-polytope $C(n,d)$ with d vertices achieves the numbers $c_i(n,d)$.

Proof. Theorem 3.1 implies that the equality in Corollary 3.7 holds. The inequality follows from [BPS, Theorem 6.3]. □

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Generic Initial Ideals and Graded Betti Numbers

Jürgen Herzog

§ Introduction

The purpose of this article is to give the algebraic background of the shifting theory developed by Kalai [26], [27]. The reader who is interested in the combinatorial aspects of the theory should consult Kalai's survey paper [26] and his article in this volume.

In the present article we are mainly interested in the behaviour of graded Betti numbers under the operation of algebraic shifting. Algebraic shifting is intimately related to the theory of generic initial ideals. In Section 1 we recall some of the basic facts of this theory. The next section is devoted to the study of stable and strongly stable ideals since generic initial ideals are of this kind, provided the base field is of characteristic 0. In Section 3 we describe the Betti numbers of stable and squarefree stable ideals, and in Section 4 the Cartan complex which provides the graded minimal free resolution of the residue class field of the exterior algebra. For the theory of squarefree monomial ideals, which is significant for combinatorial applications, it is necessary to study graded ideals, graded modules and their resolutions over the exterior algebra. In Section 5 we explain how the graded Betti numbers of squarefree monomial ideals over the exterior and symmetric algebra are related. The following two sections are devoted to the proof of a theorem on extremal Betti numbers by Bayer, Charalambous and S. Popescu [12], as well as to the corresponding theorem in the squarefree case by Aramova and the author [4]. In Section 8 we describe various shifting operators and apply the homological theory of the previous sections. Symmetric algebraic shifting and a theorem of Björner and Kalai [15] are applied in Section 9 in order to deduce a theorem on superextremal Betti numbers. In the final section extremality properties of lexsegment ideals are briefly sketched.

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Not all proofs could be included. But in most cases an outline of the proofs or precise references to the original papers are given.

Several unsolved problems and conjectures are included. The author hopes that this survey inspires the readers to study and solve some of the open problems.

§1. Generic initial ideals

Most of the content of this section can be found in the book of Eisenbud [16] or the lecture notes by M. Green [20]. We will therefore omit almost all of the proofs.

Let K be an infinite field, and $S = K[x_1, \dots, x_n]$ the polynomial ring over K. The set of monomials of degree d in S will be denoted by M_d.

We will fix a term order $<$ satisfying $x_1 > x_2 > \dots > x_n$. Let $I \subset S$ be an ideal. Then we denote by $\mathrm{in}_<(I)$ (or simply by $\mathrm{in}(I)$) the initial ideal of I, that is, the ideal which is generated by all initial terms of I.

Let $GL(n)$ denote the general linear group with coefficients in K. Any $\varphi = (a_{ij}) \in GL(n)$ induces an automorphism of the graded K-algebra S, again denoted by φ, namely

$$\varphi(f(x_1, \dots, x_n)) = f(\sum_{i=1}^{n} a_{i1}x_i, \dots, \sum_{i=1}^{n} a_{in}x_i) \quad \text{for all} \quad f \in S.$$

One basic fact in the theory of generic initial ideals is the following

Theorem 1.1 (Galligo, Bayer and Stillman). *Let $I \subset S$ be a graded ideal. Then there is a nonempty Zariski open set $U \subseteq GL(n)$ such that $\mathrm{in}(\varphi(I))$ does not depend on $\varphi \in U$. Moreover, U meets non trivially the Borel subgroup of $GL(n)$ consisting of all upper triangular invertible matrices.*

For $\varphi \in U$ the monomial ideal $\mathrm{in}(\varphi(I))$ is called the *generic initial ideal of* I, and will be denoted $\mathrm{Gin}(I)$.

For the details of the proof of Theorem 1.1 we refer to [16, Theorem 15.18]. Each homogeneous component $\mathrm{Gin}(I)_d$ of $\mathrm{Gin}(I)$ may be computed as follows: consider a transcendental field extension L/K, where L has the transcendental basis $\{a_{ij} \colon i, j = 1, \dots, n, \quad i \leq j\}$. Let $S' = L[x_1, \dots, x_n]$, $I' = \varphi(I)S'$ where $\varphi(x_j) = \sum_{i=1}^{j} a_{ij}x_i$ for $j, \dots, n$. Choose an L-basis $f_1, \dots, f_m$ of I'_d. Each of the f_i is a linear combination of monomials $u \in M_d$ whose coefficients are (homogeneous) polynomials in $K[a_{ij} \colon i, j = 1, \dots, n]$ (of degree d), say, $f_i = \sum_{u \in M_d} c_{iu}u$. Now form the $m \times |M_d|$-matrix $C = (c_{iu})$ where the columns are ordered according to the given term order, and view C as a matrix with coefficients in L. Notice that C has rank m since the polynomials $f_1, \ldots, f_m$ are linearly independent over L. For $i = 1, \ldots, m$, let u_i be the largest monomial such that $c_{iu_i} \neq 0$; then $u_i = \text{in}(f_i)$.

After elementary row operations (which amounts to choose another L-basis of I'_d), we may assume that $u_1 > u_2 > \ldots > u_m$. Then $\text{Gin}(I)_d = Ku_1 + \ldots + Ku_m$.

We order the m-tuples $(v_1, \ldots, v_m)$ of monomials of M_d lexicographically. This means that $(v_1, \ldots, v_m) > (w_1, \ldots, w_m)$ if for some i one has $v_j = w_j$ for $j < i$, and $v_i > w_i$. Then our discussion shows that $\text{Gin}(I)_d$ is the span of largest m-tupel $(u_1, \ldots, u_m)$ of monomials such that $\det(c_{iu_i})_{i=1,\ldots,m} \neq 0$.

Another basic result on generic ideals is

Theorem 1.2 (Galligo, Bayer-Stillman). *Let $I \subset S$ be a graded ideal. Then* $\text{Gin}(I)$ *is Borel fixed, that is,* $\varphi(\text{Gin}(I)) = \text{Gin}(I)$ *for all φ which belong to the Borel group of invertible upper triangular matrices.*

Generic initial ideals behave especially well when one uses the reverse lexicographic order. We will discuss this in Section 4. Let $u, v \in M_d$, $u = x_1^{a_1} \cdots x_n^{a_n}$ and $v = x_1^{b_1} \cdots x_n^{b_n}$. Then $u > v$ in reverse lexicographic order, if $\deg u > \deg v$ or $\deg u = \deg v$ and for some i one has $a_j = b_j$ for $j > i$, and $a_i < b_i$.

The following example demonstrates the difference between the lexicographic and the reverse lexicographic order. We order the monomials in three variables of degree 2 first lexicographically, and then reverse lexicographically:

(1) $x_1^2 > x_1x_2 > x_1x_3 > x_2^2 > x_2x_3 > x_3^2$
(2) $x_1^2 > x_1x_2 > x_2^2 > x_1x_3 > x_2x_3 > x_3^2$

The nice behaviour of the reverse lexicographic order is a consequence of the easy to prove

Property 1.3. *Let $<$ be the reverse lexicographic order. If $f \in S$ is a homogeneous polynomial with* $\text{in}_<(f) \in (x_i, \ldots, x_n)$ *for some i, then $f \in (x_i, \ldots, x_n)$.*

This property immediately implies (cf. [16, Proposition 15.12])

Proposition 1.4. *Let $I \subset S$ be a graded ideal. Then with respect to the reverse lexicographic order one has*

(a) $\text{in}(I) + x_nS = \text{in}(I + x_nS)$;
(b) $\text{in}(I) : x_n = \text{in}(I : x_n)$.

A monomial $u \in S$, $u = x_1^{a_1}x_2^{a_2} \cdots x_n^{a_n}$ is called *squarefree*, if $a_i \leq 1$ for $i = 1, \ldots, n$, and a monomial ideal in S is called *squarefree* if it is generated by squarefree monomials. In combinatorial contexts, squarefree monomial ideals are interesting since they appear as the defining ideals of Stanley-Reisner rings. Unfortunately Gin(I) of a squarefree monomial ideal I is never squarefree, unless I is generated by a subset of the variables. Thus for combinatorial applications one has to find an analogue of the operation Gin which yields a squarefree monomial ideal. The most natural way to define such an operation, is to work in the exterior algebra instead of the symmetric algebra.

Let V be an n-dimensional K-vector space with basis $e_1 \dots e_n$. The exterior algebra $E = \bigwedge^{\bullet}(V)$ is a finite dimensional graded K-algebra. The ith graded component $\bigwedge^{i}(V)$ has the K-basis

$$e_{j_1} \wedge e_{j_2} \wedge \cdots \wedge e_{j_i} \quad \text{with} \quad j_1 < j_2 < \ldots < j_i.$$

Let $[n] = \{1, \dots, n\}$; for a subset $\sigma \subset [n]$, $\sigma = \{j_1 < j_2 < \ldots < j_i\}$, we set $e_\sigma = e_{j_1} \wedge e_{j_2} \wedge \cdots \wedge e_{j_i}$. The elements e_σ are called the monomials of E. Term orders, initial terms and initial ideals are defined as in the polynomial ring. For example, the lexicographic or the reverse lexicographic order is defined by restriction to squarefree monomials.

In the following example we list all monomials in 4 variables of degree 2 in the exterior algebra in lexicographic and reverse lexicographic order:

(1) $e_1 \wedge e_2 > e_1 \wedge e_3 > e_1 \wedge e_4 > e_2 \wedge e_3 > e_2 \wedge e_4 > e_3 \wedge e_4$.

(2) $e_1 \wedge e_2 > e_1 \wedge e_3 > e_2 \wedge e_3 > e_1 \wedge e_4 > e_2 \wedge e_4 > e_3 \wedge e_4$.

In the exterior algebra the generic initial ideal Gin(I) of a graded ideal $I \subset E$ is defined similarly as in the case of the polynomial ring. In other words, $\text{Gin}(I) = \text{in}(\varphi(I))$ where φ is a linear automorphism of E. Of course, Gin(I) is a monomial ideal in E (which is automatically squarefree). The analogues of the theorems of Galligo, Bayer and Stillman, as well as Proposition 1.4, hold and are proved similarly in the exterior case. We refer the reader to [6] about some general facts on Gröbner basis theory in exterior algebras.

§2. Special monomial ideals

Let p be a prime number, and k and l be non-negative integers with p-adic expansion $k = \sum_i k_i p^i$ and $l = \sum_i l_i p^i$. We set $k \leq_p l$ if $k_i \leq l_i$ for all i. In order to have a consistent notation, we also set $k \leq_0 l$ if $k \leq l$ (in the usual sense).

Definition 2.1. Let p be a prime number, or $p = 0$. A monomial ideal $I \subset S$ is p-Borel, if the following condition holds: for each monomial $u \in I$, $u = \prod_i x_i^{\mu_i}$, one has $(x_i/x_j)^\nu u \in I$ for all i, j with $1 \leq i < j \leq n$ and all $\nu \leq_p \mu_j$.

The significance of p-Borel ideals follows from

Proposition 2.2. *Suppose* $\operatorname{char} K = p \geq 0$, *and let* $I \subset S$ *be a monomial ideal. Then* I *is Borel-fixed if and only if* I *is* p*-Borel.*

For the proof of Proposition 2.2 we refer to [16, Theorem 15.23].

For $p > 0$, the p-Borel ideals have a rather complicated combinatorial structure. The reader who is interested in more details about such ideals may consult [31], [3] and [23]. In these notes we will concentrate on 0-Borel ideals, which henceforth will be called *strongly stable ideals.*

For a monomial $u \in S$ we set $m(u) = \max\{i\colon x_i \quad \text{divides} \quad u\}$.

Definition 2.3. A subset $B \subset S$ of monomials is called *strongly stable*, if $x_i(u/x_j) \in B$ for all $u \in B$, all x_j that divides u, and all $i < j$. The set B is called *stable*, if $x_i(u/x_{m(u)}) \in B$ for all $u \in B$, and all $i < m(u)$.

It follows from Definition 2.1 that a strongly stable ideal is a monomial ideal I for which the set of monomials in I is a strongly stable monomial set. If the set of monomials in I is a stable set, then I is called a *stable monomial ideal.* Stable monomial ideals were introduced by Eliahou and Kervaire [17].

Examples 2.4. (a) Let $u_1, \ldots, u_m$ be monomials. There is a unique smallest strongly stable ideal I with $u_j \in I$ for $j = 1, \ldots, m$. The monomials $u_1, \ldots, u_m$ are called *Borel generators* of I, and we write $I = \langle u_1, \ldots, u_m \rangle$. I is called *principal Borel* if $I = \langle u \rangle$ for some monomial u. For example the ideal

$$I = (x_1^2, x_1x_2, x_1x_3, x_1x_4, x_2^2, x_2x_3, x_2x_4)$$

is principal Borel with Borel generator x_2x_4.

(b) A set L of monomials is called a *lexsegment*, if for all $u \in L$ and all $v \geq_{lex} u$ with $\deg v = \deg u$, it follows that $v \in L$. An monomial ideal $I \subset S$ is called a *lexsegment ideal* if the set monomials in I a lexsegment. It is obvious that lexsegment ideals are strongly stable.

(c) Replacing in (b) everywhere the word 'lex' by 'revlex', one obtains the definition of a revlexsegment and a revlexsegment ideal. It is obvious that revlexsegment ideals are strongly stable.

Remark 2.5. (a) Let I be a monomial ideal. We denote by $G(I)$ the unique minimal set of monomial generators of I. It is easily seen that I is strongly stable, if for all monomial generators u of I one has $(x_i/x_j)u \in I$ for all x_j that divide u, and all $i < j$.

Let $N \subset M_d$. The set $\{x_iu\colon u \in N, i = 1, \ldots, n\} \subset M_d$ is called the *shadow of* N, and is denoted $\operatorname{Shad}(N)$. The simple proof of the following lemma is left to the reader.

Lemma 2.6. *Let $N \subset M_d$. If N is a (strongly) stable set (resp. a lexsegment), then* $\operatorname{Shad}(N)$ *is a (strongly) stable set (resp. a lexsegment).*

Notice that the shadow of a revlexsegment is in general not a revlexsegment. For example consider the revlexsegment $\{x_1^2\}$ in $K[x_1, x_2, x_3]$. Then $x_1^2x_3$ is in the shadow of this set, but x_2^3 is not.

Let $N \subset M_d$. Then there is a unique lexsegment, denoted N^{lex} such that $|N^{lex}| = |N|$. The following important result holds

Theorem 2.7. *For any subset $N \subset M_d$ one has $|\operatorname{Shad}(N^{lex})| \leq |\operatorname{Shad}(N)|$. In other words, lexsegments have the smallest possible shadow.*

Before we indicate the proof of Theorem 2.7 we note the following consequence

Corollary 2.8. *Let $I \subset S$ be graded ideal. Then there exists a unique lexsegment ideal, denoted $I^{lex} \subset S$, such that S/I and S/I^{lex} have the same Hilbert function.*

Proof. Since S/I and $S/\operatorname{in}(I)$ have the same Hilbert function, we may replace I by $\operatorname{in}(I)$, and hence may assume that I is a monomial ideal. Let I_d be spanned by the set of monomials N_d, and I_d^{lex} the subspace of S_d spanned by N_d^{lex}. We set $I^{lex} = \bigoplus_{d \geq 0} I_d^{lex}$, and only need to show that I^{lex} is an ideal. In other words, we have to show that $\{x_1, \dots, x_n\}I_d^{lex} \subset I_{d+1}^{lex}$ for all d. By Theorem 2.7 we have $|\operatorname{Shad}(N_d^{lex})| \leq |\operatorname{Shad}(N_d)| \leq |N_{d+1}| = |N_{d+1}^{lex}|$. Since $\operatorname{Shad}(N_d^{lex})$ and N_{d+1}^{lex} are both lexsegments, this inequality implies $\operatorname{Shad}(N_d^{lex}) \subset N_{d+1}^{lex}$, as desired. Q.E.D.

For the proof of Theorem 2.7 we have to introduce some notation: let $B \subset M_d$ be a set of monomials. We let $m_i(B)$ be the number of $u \in B$ with $m(u) = i$, and set $m_{\leq i}(B) = \sum_{j=1}^{i} m_j(B)$.

Lemma 2.9. *Let $B \subset M_d$ be a stable set of monomials. Then*

(a) $m_i(\operatorname{Shad}(B)) = m_{\leq i}(B)$;

(b) $|\operatorname{Shad}(B)| = \sum_{i=1}^{n} m_{\leq i}(B)$.

Proof. (b) is of course a consequence of (a). For the proof of (a) we note that the map

$$\varphi\colon \{u \in B\colon m(u) \leq i\} \to \{u \in \operatorname{Shad}(B)\colon m(u) = i\}, \quad u \mapsto ux_i$$

is a bijection. In fact, φ is clearly injective. To see that φ is surjective, we let $v \in \operatorname{Shad}(B)$ with $m(v) = i$. Since $v \in \operatorname{Shad}(B)$, there exists $w \in B$ with $v = x_jw$ for some $j \leq i$. It follows that $m(w) \leq i$. If $j = i$, then we are done. Otherwise, $j < i$ and $m(w) = i$. Hence, since B is stable it follows that $u = (x_j/x_i)w \in B$. The assertion follows, since $v = ux_i$. Q.E.D.

Now Theorem 2.7 follows immediately from Lemma 2.9 and the next theorem which is due to Bayer [10]. We will give below the proof of the similar theorem in the squarefree case.

Theorem 2.10. *Let $L \subset M_d$ be a lexsegment, and $B \subset M_d$ be a stable set of monomials with $|L| \leq |B|$. Then $m_{\leq i}(L) \leq m_{\leq i}(B)$ for $i = 1, \dots, n$.*

The length of the shadow of a lexsegment can be computed. Let i be a positive integer. Then $a \in \mathbb{N}$ has a unique expansion

$$a = \binom{a_i}{i} + \binom{a_{i-1}}{i-1} + \cdots + \binom{a_j}{j}$$

with $a_i > a_{i-1} > \cdots > a_j \geq j \geq 1$; see [14] or [21].

We define

$$a^{\langle i \rangle} = \binom{a_i + 1}{i+1} + \binom{a_{i-1}+1}{i} + \cdots + \binom{a_j+1}{j+1},$$

and

$$a^{(i)} = \binom{a_i}{i+1} + \binom{a_{i-1}}{i} + \cdots + \binom{a_j}{j+1}.$$

Lemma 2.11. *Let $L \subset M_d$ be a lexsegment with $a = |M_d \setminus L|$. Then*

$$|M_{d+1} \setminus \mathrm{Shad}(L)| = a^{\langle d \rangle}.$$

For the proof of this lemma we refer the reader to [14, Prop.4.2.8].

As a consequence of Corollary 2.8 and Lemma 2.11 we now obtain

Theorem 2.12 (Macaulay). *Let $h\colon \mathbb{N} \to \mathbb{N}$ be a numerical function. The following conditions are equivalent:*

(a) *h is the Hilbert function of a standard graded K-algebra;*
(b) *$h(0) = 1$, and $h(d+1) \leq h(d)^{\langle d \rangle}$ for all $d \geq 0$.*

We close this section with a discussion of the analogue theorems in the squarefree case. Let $B \subset E$ be a set of monomials in the exterior algebra. Then B is called (*strongly*) *stable* if B satisfies conditions analogue to those of Definition 2.3. Thus, for example, B is stable, if for all monomials $u \in B$, $u = e_{i_1} \wedge e_{i_2} \wedge \ldots \wedge e_{i_j}$ with $i_1 < i_2 < \ldots i_j$ it follows that $e_i \wedge e_{i_1} \wedge \ldots \wedge e_{i_{j-1}} \in B$ for all $i < i_j$ and $i \notin \{i_1, \ldots, i_j\}$.

Let $u = x_{i_1} x_{i_2} \ldots x_{i_j} \in S$ be a squarefree monomial. Then we call $e_{i_1} \wedge e_{i_2} \wedge \ldots \wedge e_{i_j}$ the monomial in E corresponding to u. Let $I \subset S$ be squarefree monomial ideal, $B \subset I$ the set of squarefree monomials in I, and B' be the corresponding set of monomials in E. Notice that the K-subspace J of E spanned by B' is an ideal in E. We call it the monomial ideal in E corresponding to I, and I is called a *squarefree (strongly) stable* monomial ideal, resp. a *squarefree lexsegment* ideal, if J is a (strongly) stable resp. lexsegment ideal in E.

Corresponding to Proposition 2.2 one has

Proposition 2.13. *A Borel-fixed ideal $J \subset E$ is strongly stable. In particular, the generic initial ideal of any graded ideal in E is strongly stable.*

For the shadow of a stable set of monomials in E one has

Lemma 2.14. *Let $B \subset E_d$ be a strongly stable set of monomials. Then* $\operatorname{Shad}(B)$ *is again stable and* $|\operatorname{Shad}(B)| = \sum_{i=1}^{n-1} m_{\leq i}(B)$.

We leave the proof of Lemma 2.14 to the reader.

We now prove the squarefree version of Bayer's Theorem 2.10.

Theorem 2.15. *Let $L \subset E_d$ be a lexsegment of monomials, and $B \subset E_d$ a stable set of monomials with $|L| \leq |B|$. Then $m_{\leq i}(L) \leq m_{\leq i}(B)$ for $i = 1, \ldots, n$.*

For the proof of the theorem we need some preparation. Let $d < n$ and write N_d for the set of all (squarefree) monomials of degree d in E. If $N \subset N_d$ we denote by $\min(N)$ the smallest monomial $u \in N$ (with respect to the lexicographic order). Furthermore we define a map $\alpha\colon N_d \to N_d$ by setting $\alpha(u) = u$, if $n \notin \operatorname{supp}(u)$, and $\alpha(u) = (e_j \wedge u)/e_n$ if $n \in \operatorname{supp}(u)$, where j is the largest integer $< n$ which does not belong to $\operatorname{supp}(u)$. Here $\operatorname{supp}(u)$ is the set of elements $i \in [n]$ such that $e_i | u$.

Lemma 2.16. *With the notation introduced we have:*

(a) *The map $\alpha\colon N_d \to N_d$ is order preserving, that is, for $u, u' \in N_d$, $u \leq_{lex} u'$, one has $\alpha(u) \leq_{lex} \alpha(u')$.*

(b) *Let $B = B' + B'' \wedge e_n$ be a strongly stable set of monomials of degree d, where B' and B'' are sets of monomials in the elements $e_1, e_2, \ldots, e_{n-1}$. Then $\alpha(\min(B)) = \min(G(B'))$.*

Proof. (a) Let u and u' be two monomials of degree d with $u \leq_{lex} u'$ and $m(u) = m(u') = n$, say $u = e_{i_1} \wedge \cdots \wedge e_{i_{d-1}} \wedge e_n$ and $u' = e_{i'_1} \wedge \cdots \wedge e_{i'_{d-1}} \wedge e_n$ with $1 \leq i_1 < i_2 < \cdots < i_{d-1} < n$ and $1 \leq i'_1 < i'_2 <$

$\cdots < i'_{d-1} < n$. Then there exists an integer t with $1 \leq t \leq d-1$ such that $i_1 = i'_1, \ldots, i_{t-1} = i'_{t-1}$ and $i_t > i'_t$. Let j be the largest integer $< d$ which is not in $\text{supp}(u)$, and define j' similarly for u'. Since $i_t > i'_t$, there is at least one 'gap' in the sequence $i'_t, \ldots, i'_{d-1}, n$. Thus $j' > i'_t$. Hence if $j \geq i_t$, then the first indices of the factors of $\alpha(u)$ and $\alpha(u')$ in which they differ are again i_t and i'_t, and the inequality is preserved. On the other hand, if $j < i_t$, then we must have

$$u = e_{i_1} \wedge \cdots \wedge e_{i_{t-1}} \wedge e_{n-d+t} \wedge e_{n-d+t+1} \wedge \cdots \wedge e_{n-1} \wedge e_n,$$

and $j = i_t - 1 = n - d + t - 1$ since $i_{t-1} = i'_{t-1} < i'_t < i_t$. That is, the factors 'after' $e_{i_{t-1}}$ have the highest possible indices. It is then obvious that $\alpha(u) \leq_{lex} \alpha(u')$ as desired. By the similar way one treats the case $m(u') < m(u) = n$, while if $m(u) < m(u') = n$ one has $\alpha(u) = u \leq_{lex} u' \leq_{lex} \alpha(u')$.

(b) It follows from the above result (a) that $\alpha(\min(B)) \leq_{lex} \alpha(\min(B')) = \min(B')$ since $\min(B) \leq_{lex} \min(B')$. On the other hand, since B is strongly stable, $\alpha(\min(B)) \in B'$, which implies the reverse inequality. Q.E.D.

Proof of Theorem 2.15. We proceed by induction on n, the number of variables. The inequality $m_{\leq n}(L) \leq m_{\leq n}(B)$ is just our hypothesis. In order to prove it for $i < n$, we write $L = L' + L'' \wedge e_n$ and $B = B' + B'' \wedge e_n$ with L', L'', B' and B'' sets of monomials in $e_1, e_2, \ldots, e_{n-1}$. It is clear that L' is lexsegment, and that B' is strongly stable. Hence if we show that $|L'| \leq |B'|$, we may apply our induction hypothesis, and the assertion follows immediately.

It may be assumed that B' and B'' are lexsegments. In fact, let B^* (resp. B^{**}) be the lexsegments in $e_1, e_2, \ldots, e_{n-1}$ of degree d (resp. $d-1$) such that $|B^*| = |B'|$ (resp. $|B^{**}| = |B''|$) and set $\tilde{B} = B^* + B^{**} \wedge e_n$. Then it is not hard to see that $\tilde{B}$ is again strongly stable.

Now we are in the following situation: $L = L' + L'' \wedge e_n$ is lexsegment, and $B = B' + B'' \wedge e_n$ strongly stable as before, but in addition B' and B'' are lexsegments. Assuming $|L| \leq |B|$, we want to show that $|L'| \leq |B'|$. Thanks to Lemma 2.16 we have

$$\min(B') = \alpha(\min(B)) \leq_{lex} \alpha(\min(L)) = \min(L').$$

Since L' and B' are lexsegments, the required inequality follows. Q.E.D.

As a consequence one obtains similarly as in Corollary 2.8 that for any graded ideal $J \subset E$ there exists a unique lexsegment ideal $J^{lex} \subset E$ such that E/J and E/J^{lex} have the same Hilbert function. Detailed proofs of these statements can be found in [6].

Corollary 2.17. *Let $I \subset S$ be a squarefree monomial ideal. Then there exists a unique squarefree lexsegment ideal, denoted I^{sqlex} such that S/I and S/I^{sqlex} have the same Hilbert function.*

Proof. Let J be the corresponding ideal of I in E, B' the set of monomials of J^{lex}, and B the set of squarefree monomials in S corresponding to B'. The ideal $L \subset S$ spanned by B is clearly a squarefree lexsegment ideal. It follows from the next lemma that S/I and S/L have the same Hilbert function. Q.E.D.

Lemma 2.18. *Let $I \subset S$ be a squarefree monomial ideal, and $J \subset E$ the corresponding monomial ideal in E. Let $H_{E/J}(t) = \sum_{i=0}^{n} a_i t^i$ be the Hilbert function of E/J. Then the Hilbert function of S/I is given by*

$$H_{S/I}(t) = \sum_{i=0}^{n} a_i \frac{t^i}{(1-t)^i}.$$

This lemma implies in particular that the Hilbert function of E/J and that of S/I determine each other. A proof of this simple result can be found for example in [14, Theorem 5.1.7].

The exterior version of Lemma 2.11 is the following (cf. [6, Theorem 4.2])

Lemma 2.19. *Let $L \subset EM_d$ be a lexsegment of monomials, where EM_d denotes the set of monomials of degree d in E. Suppose that $a = |EM_d \setminus L|$. Then*

$$|EM_d \setminus \mathrm{Shad}(L)| = a^{(d)}.$$

As in the case of the polynomial rings one now deduces (cf. [6, Theorem 4.1])

Theorem 2.20 (Kruskal-Katona). *Let $(h_1, \dots, h_n)$ be a sequence of integers. Then the following conditions are equivalent:*

(a) $1 + \sum_{d=1}^{n} h_d t^d$ *is the Hilbert series of a graded K-algebra E/J;*
(b) $0 \leq h_{d+1} \leq h_d^{(d)}$ *for all i with $0 \leq d < n$.*

§3. Graded Betti numbers of initial ideals

Let M be a finitely generated graded S-module. Then M has a graded free S-resolution of the form

$$\dots \to \bigoplus_j S(-j)^{\beta_{ij}} \to \dots \to \bigoplus_j S(-j)^{\beta_{1j}} \to \bigoplus_j S(-j)^{\beta_{0j}} \to M \to 0.$$

The numbers β_{ij} are called the *graded Betti numbers of* M. Note that the Tor-groups $\mathrm{Tor}_i(K, M)$ are finitely generated, graded K-vector spaces, and that

$$\beta_{ij}(M) = \dim_K \mathrm{Tor}_i(K, M)_j \quad \text{for all} \quad i, j.$$

The following basic result holds:

Theorem 3.1. *Let $I \subset S$ be a graded ideal. Then for any term order $<$ one has*

$$\beta_{ij}(S/I) \leq \beta_{ij}(S/\operatorname{in}_<(I)) \quad \textit{for all} \quad i, j.$$

Proof. Let $\tilde{S}$ be the $K[t]$-algebra $S[t]$, where t is an indeterminate of degree 0. By [16, Theorem 15.17] there exists a graded ideal $\tilde{I} \subset \tilde{S}$ such that the $K[t]$-algebra $\tilde{S}/\tilde{I}$ is free $K[t]$-module (and thus flat over $K[t]$), and such that

$$(\tilde{S}/\tilde{I})/t(\tilde{S}/\tilde{I}) \cong S/\operatorname{in}(I), \tag{1}$$

and

$$(\tilde{S}/\tilde{I})_t \cong (S/I) \otimes_K K[t, t^{-1}], \tag{2}$$

as graded K-algebras.

Let $F_{\bullet}$ be the minimal graded free $\tilde{S}$-resolution of $\tilde{S}/\tilde{I}$. Then (1) implies that $F_{\bullet}/tF_{\bullet}$ is a graded minimal free S-resolution of S/I, so that $\beta_{ij}(\tilde{S}/\tilde{I}) = \beta_{ij}(S/\operatorname{in}(I))$ for all i and j, and (2) implies that $(F_{\bullet})_t$ is a graded (not necessarily minimal) free $S \otimes_K K[t, t^{-1}]$ resolution of $(S/I) \otimes_K K[t, t^{-1}]$. Thus, $\beta_{ij}(S/I) = \beta_{ij}((S/I) \otimes_K K[t, t^{-1}]) \leq \beta_{ij}(\tilde{S}/\tilde{I})$, as desired. Q.E.D.

Let M be a finitely generated graded S-module. The *regularity* of M is defined to be the number $\operatorname{reg}(M) = \max\{j - i \colon \beta_{ij}(M) \neq 0\}$. As an immediate consequence of Theorem 3.1 we have

Corollary 3.2. *Let $I \subset S$ be a graded ideal. Then for any term order $<$ one has*:

(a) $\operatorname{proj\,dim} S/I \leq \operatorname{proj\,dim} S/\operatorname{in}_<(I)$.
(b) $\operatorname{depth} S/I \geq \operatorname{depth} S/\operatorname{in}_<(I)$.
(c) *If $S/\operatorname{in}_<(I)$ is Cohen-Macaulay (Gorenstein), then so is S/I.*
(d) $\operatorname{reg} S/I \leq \operatorname{reg} S/\operatorname{in}_<(I)$.

We shall see in the next section that all inequalities of Corollary 3.2 become equalities, if $\operatorname{in}_<(I)$ is replaced by $\operatorname{Gin}(I)$ with respect to the reverse lexicographic order. Since by Proposition 2.2, at least in characteristic 0, the generic initial ideal is strongly stable, it is of interest to compute the graded Betti numbers of stable ideals. Eliahou and Kervaire described explicitly the resolution of such ideals. Here we are only interested in its graded Betti numbers, so that we only need to compute the graded K-vector spaces $\mathrm{Tor}_i(K, S/I)$.

Let $K_{\bullet}(x; S/I)$ be the Koszul complex of S/I with respect to $x_1, ...,$ x_n. We denote by $H_{\bullet}(x; S/I)$ the Koszul homology. Since there is a graded isomorphism $\mathrm{Tor}_{\bullet}(K, S/I) \cong H_{\bullet}(x; S/I)$, we may as well compute $H_{\bullet}(x; S/I)$ in order to determine the graded Betti numbers. Recall that $K_i(x_1, ..., x_n) = K_{\bullet}(x; S/I)$ is a free S/I- module with basis e_σ, $\sigma \subset \{1, \dots, n\}$, $|\sigma| = i$, where $e_\sigma = e_{j_1} \wedge e_{j_2} \wedge \dots \wedge e_{j_i}$ for $\sigma = \{j_1, \dots, j_i\}$, $j_1 < j_2 < \dots < j_i$. The differential ∂ of $K_{\bullet}$ is given by $\partial(e_\sigma) = \sum_{t \in \sigma} (-1)^{\alpha(\sigma,t)} x_t e_{\sigma \setminus t}$. Here $\alpha(\sigma, t) = |\{r \in \sigma \colon r < t\}|$.

For a monomial ideal I we denote by $G(I)$ the unique set of monomial generators of I. We let $\varepsilon \colon S \to S/I$ be the canonical epimorphism, and set $u' = u/x_{m(u)}$ for all $u \in G(I)$.

Theorem 3.3. *Let $I \subset S$ be a stable ideal. For all $j = 1, \dots, n$ and $i > 0$, the Koszul homology $H_i(x_j, \dots, x_n)$ is annihilated by $\mathfrak{m} = (x_1, \dots, x_n)$. In other words, all these homology modules are K-vector spaces. A basis of $H_i(x_j, \dots, x_n)$ is given by the homology classes of the cycles*

$$\varepsilon(u')e_\sigma \wedge e_{m(u)}, u \in G(I), |\sigma| = i - 1, j \le \min(\sigma), \max(\sigma) < m(u).$$

Proof. We proceed by induction on $n - j$. For $j = n$, we only have to consider $H_1(x_n)$ which is obviously minimally generated by the homology classes of the elements $\varepsilon(u')e_n$ with $u \in G(I)$ such that $m(u) = n$. Since by the definition of stable ideals $x_i u' \in I$ for all i, we see that $H_1(x_n)$ is a k-vector space.

Now assume that $j < n$, and that the assertion is proved for $j + 1$. Then $x_j H_i(x_{j+1}, \dots, x_n) = 0$ for all $i > 0$, so that the long exact sequence (cf. [14, Cor.1.6.13])

$$\cdots \xrightarrow{x_j} H_i(x_{j+1}, \dots, x_n) \longrightarrow H_i(x_j, \dots, x_n) \longrightarrow H_{i-1}(x_{j+1}, \dots, x_n)$$
$$\xrightarrow{x_j} H_{i-1}(x_{j+1}, \dots, x_n) \longrightarrow H_{i-1}(x_j, \dots, x_n) \longrightarrow \cdots$$

splits into the exact sequences

$$(3) \quad 0 \longrightarrow H_1(x_{j+1}, \dots, x_n) \longrightarrow H_1(x_j \dots, x_n) \longrightarrow S_j/I_j \xrightarrow{x_j} S_j/I_j$$

and

$$(4) \quad 0 \longrightarrow H_i(x_{j+1}, ..., x_n) \longrightarrow H_i(x_j, ..., x_n)$$
$$\longrightarrow H_{i-1}(x_{j+1}, ..., x_n) \longrightarrow 0.$$

for $i > 0$. Here S_j is the polynomial ring $K[x_1, \dots, x_j]$, I_j the ideal in S_j generated by the monomials $u \in G(I)$ which are not divisible by any x_i with $i > j$, in other words, $I_j = I \cap S_j$.

In sequence (3), $\operatorname{Ker} x_j$ is minimally generated by the residues of the monomials u' with $u \in G(I)$ and $m(u) = j$. Note that the sets $\{u \in G(I)\colon m(u) = j\}$ and $\{u \in G(I_j)\colon m(u) = j\}$ are equal, and that I_j is a stable ideal in S_j. Therefore $\operatorname{Ker} x_j$ is a K-vector space.

We now consider the short exact sequence

$$(5) \quad 0 \longrightarrow H_1(x_{j+1}, \dots, x_n) \longrightarrow H_1(x_j, \dots, x_n) \longrightarrow \operatorname{Ker} x_j \longrightarrow 0.$$

It is clear that the elements $\varepsilon(u')e_j$, $u' \in G(I)$, $m(u) = j$ are cycles in $K_1(x_j, \dots, x_n)$ such that $\delta([\varepsilon(u')e_j]) = u' + I_j$. Therefore, by (5) and our induction hypothesis, it follows that the set $\mathcal{S} = \{[\varepsilon(u')e_i]\colon u \in G(I), m(u) = i \geq j\}$ generates $H_1(x_j, \dots, x_n)$. Since I is a stable ideal we see that $x_j[\varepsilon(u')e_i] = 0$ for all $j = 1, \dots, n$ and all $[\varepsilon(u')e_i] \in \mathcal{S}$. In other words, $H_1(x_j, \dots, x_n)$ is a K-vector space. Finally, since the number of elements of $\mathcal{S}$ equals $\dim_k H_1(x_{j+1}, \dots, x_n) + \dim \operatorname{Ker} x_j$, we conclude that $\mathcal{S}$ is a basis of $H_1(x_j, \dots, x_n)$.

In order to prove our assertion for $i > 1$ we consider the exact sequences (4). By induction hypothesis the homology module $H_{i-1}(x_{j+1}, \dots, x_n)$ is a K-vector space with basis

$$[\varepsilon(u')e_\sigma \wedge e_{m(u)}], u \in G(I), |\sigma| = i-2, j+1 \leq \min(\sigma), \max(\sigma) < m(u).$$

Given such a homology class, consider the element $\varepsilon(u')e_j \wedge e_\sigma \wedge e_{m(u)}$. It is clear that this element is a cycle in $K_i(x_j, \dots, x_n)$, and that

$$\delta([\varepsilon(u')e_j \wedge e_\sigma \wedge e_{m(u)}] = \pm[\varepsilon(u')e_\sigma \wedge e_{m(u)}].$$

Thus from the exact sequence (4) and our induction hypothesis it follows that the homology classes of the cycles described in the theorem generate $H_i(x_j, \dots, x_n)$. Again the stability of the ideal I implies that $\mathfrak{m}$ annihilates all these homology classes, so that $H_i(x_j, \dots, x_n)$ is a K-vector space. Finally, just as for $i = 1$, a dimension argument shows that these homology classes form a basis of $H_i(x_j, \dots, x_n)$. Q.E.D.

Let I be a monomial ideal. We denote by $G(I)_j$ the set of monomial generators of degree j. The following result of Eliahou and Kervaire [17] follows immediately from Theorem 3.3.

Corollary 3.4. *Let $I \subset S$ be a stable ideal. Then*
(a) $\beta_{ii+j}(I) = \sum_{u \in G(I)_j} \binom{m(u)-1}{i}$;
(b) $\operatorname{proj\,dim} S/I = \max\{m(u)\colon u \in G(I)\}$;
(c) $\operatorname{reg}(I) = \max\{\deg(u)\colon u \in G(I)\}$.

With similar methods one can compute the graded Betti numbers of a squarefree stable ideal. For a monomial $u \in S$ we let $\operatorname{supp}(u) = \{i\colon x_i \text{ divides } u\}$.

Theorem 3.5. *Let $I \subset S$ be a squarefree stable ideal. Then for every $i > 0$, a basis of the homology classes of $H_i(x_1, x_2, \dots, x_n)$ is given by the homology classes of the cycles*

$$u' e_\sigma \wedge e_{m(u)}, u \in G(I), |\sigma| = i-1, \max(\sigma) < m(u), \sigma \cap \operatorname{supp}(u) = \emptyset.$$

Proof. A minimal free S-resolution of S/I is multigraded; in other words, the differentials are homogeneous homomorphisms and, for each i, we have $F_i = \bigoplus_j S(-a_{ij})$ with $a_{ij} \in \mathbb{Z}^n$. Moreover, by virtue of [24, Theorem (5.1)], all shifts a_{ij} are squarefree, i.e., $a_{ij} \in \mathbb{Z}^n$ is of the form $\sum_{t \in \tau} \varepsilon_t$, where τ is a subset of $\{1, 2, \dots, n\}$, and where $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n$ is the canonical basis of $\mathbb{Z}^n$. Thus it follows that $H_i(x_1, x_2, \dots, x_n)$ is multigraded K-vector space with $H_i(x_1, x_2, \dots, x_n)_a = 0$, if $a \in \mathbb{Z}^n$ is not squarefree. Hence, if we want to compute the homology module $H_i(x_1, x_2, \dots, x_n)$, it suffices to consider its squarefree multigraded components.

For each $0 < j < n$, there is an exact sequence whose graded part for each $a \in \mathbb{Z}^n$ yields the long exact sequence of vector spaces

$$\cdots \xrightarrow{x_j} H_i(x_{j+1}, \dots, x_n)_a \longrightarrow H_i(x_j, \dots, x_n)_a \longrightarrow H_{i-1}(x_{j+1}, \dots, x_n)_{a-\varepsilon_j}$$
$$\xrightarrow{x_j} H_{i-1}(x_{j+1}, \dots, x_n)_a \longrightarrow H_{i-1}(x_j, \dots, x_n)_a \longrightarrow \cdots.$$

We now show the following more precise result: For all $i > 0$, all $0 < j \leq n$ and all squarefree $a \in \mathbb{Z}^n$, $H_i(x_j, \dots, x_n)_a$ is generated by the homology classes of the cycles

$$u' e_\sigma \wedge e_{m(u)}, \dots u \in G(I), \dots |\sigma| = i-1$$

with

$$j \leq \min(\sigma), \max(\sigma) < m(u), \sigma \cap \operatorname{supp}(u) = \emptyset \text{ and } \sigma \cup \operatorname{supp}(u) = a.$$

The proof is achieved by induction on $n - j$. The assertion is obvious for $j = n$. We now suppose that $j < n$. For such j, but $i = 1$, the assertion is again obvious. Hence we assume in addition that $i > 1$. We first claim that

$$H_{i-1}(x_{j+1}, \dots, x_n)_{a-\varepsilon_j} \xrightarrow{x_j} H_{i-1}(x_{j+1}, \dots, x_n)_a$$

is the zero map. Since $a \in \mathbb{Z}^n$ is squarefree, the components of a are either 0 or 1. If the j-th component of a is 0, then $a - \varepsilon_j$ has a negative component; hence $H_{i-1}(x_{j+1},\dots,x_n)_{a-\varepsilon_j}=0$. Thus we may assume the j-th component of a is 1. Then $a-\varepsilon_j$ is squarefree and, by induction hypothesis, $H_{i-1}(x_{j+1},\dots,x_n)_{a-\varepsilon_j}$ is generated by the homology classes of cycles of the form $u'e_\sigma\wedge e_{m(u)}$ with $j\notin \operatorname{supp}(u)$. Such an element is mapped to the homology class of $u'x_je_\sigma\wedge e_{m(u)}$ in $H_{i-1}(x_{j+1},\dots,x_n)_a$. However, since I is stable, we have $u'x_j=0$ as desired.

From these observations we deduce that we have short exact sequences

$$0\longrightarrow H_i(x_{j+1},\dots,x_n)_a\longrightarrow H_i(x_j,\dots,x_n)_a$$
$$\longrightarrow H_{i-1}(x_{j+1},\dots,x_n)_{a-\varepsilon_j}\longrightarrow 0$$

for all $i>1$. The first map $H_i(x_{j+1},\dots,x_n)_a\rightarrow H_i(x_j,\dots,x_n)_a$ of the above exact sequence is simply induced by the natural inclusion map of the corresponding Koszul complexes, while the second map $H_i(x_j,\dots,x_n)_a\rightarrow H_{i-1}(x_{j+1},\dots,x_n)_{a-\varepsilon_j}$ is a connecting homomorphism. Given the homology class of a cycle $z=u'e_\sigma\wedge e_{m(u)}$ in $H_{i-1}(x_{j+1},\dots,x_n)_{a-\varepsilon_j}$, it is easy to see that, up to a sign, the homology class of the cycle $u'e_j\wedge e_\sigma\wedge e_{m(u)}$ in $H_i(x_j,\dots,x_n)_a$ is mapped to $[z]$. This guarantees all of our assertions as required. Q.E.D.

Corollary 3.6. *Let $I\subset S$ be a squarefree stable ideal. Then*
(a) $\beta_{ii+j}(I)=\sum_{u\in G(I)_j}\binom{m(u)-j}{i}$;
(b) $\operatorname{proj}\dim S/I=\max\{m(u)-\deg(u)+1\colon u\in G(I)\}$;
(c) $\operatorname{reg}(I)=\max\{\deg(u)\colon u\in G(I)\}$.

Remark 3.7. It follows immediately from Corollary 3.4(a) and Corollary 3.6(a) that a (squarefree) stable ideal which is generated in one degree, has a linear resolution. Very recently Römer has shown (unpublished) that among all (squarefree) ideals with linear resolution the ideals generated by (squarefree) revlexsegments have minimal Betti numbers.

§4. The Cartan complex

Let $\mathcal{M}_l$ (resp. $\mathcal{M}_r$) denote the category of finitely generated graded left (right) E-modules, and $\mathcal{M}$ the category of finitely generated graded left and right E-modules, satisfying $ax=(-1)^{\deg a\deg x}xa$ for all homogeneous elements $a\in E$ and $x\in M$. For example, any graded ideal $I\subset E$ belongs to $\mathcal{M}$.

A module $M\in\mathcal{M}_l$ has a minimal, graded free E-resolution (as a left E-module), which is always infinite, unless M is free. The ijth Betti number $\beta_{ij}(M)$ is the K-dimension of $\operatorname{Tor}_i^E(K,M)_j$. These dimensions may be computed by using the graded free E-resolution of the residue class field K. This resolution is called the Cartan complex. We will briefly describe this complex.

Let $\mathbf{v} = v_1, \cdots, v_m$ be a sequence of elements of degree 1 in E. The Cartan complex $C_{\bullet}(\mathbf{v}; E)$ of the sequence $\mathbf{v}$ with values in E is defined as the complex whose i-chains $C_i(\mathbf{v}; E)$ are the elements of degree i of the free divided power algebra $C_{\bullet}(\mathbf{v}; E) = E\langle x_1, \dots, x_m\rangle$. Recall that $C_{\bullet}(\mathbf{v}; E)$ is the polynomial ring over E in the set of variables

$$x_i^{(j)}, \quad i = 1, \dots, m, \quad j = 1, 2, \dots$$

modulo the relations

$$x_i^{(j)} x_i^{(k)} = \frac{(j+k)!}{j!k!} x_i^{(j+k)}.$$

We set $x_i^{(0)} = 1$ and $x_i^{(1)} = x_i$ for $i = 1, \dots, m$. The algebra $C_{\bullet}(\mathbf{v}; E)$ is a free E-module with basis $x^{(a)} = x_1^{(a_1)} x_2^{(a_2)} \dots x_m^{(a_m)}$, $a \in \mathbb{N}^m$. We say that $x^{(a)}$ has degree i if $|a| = i$ where $|a| = a_1 + \dots + a_m$. Thus $C_i(\mathbf{v}; E) = \bigoplus_{|a|=i} E x^{(a)}$.

The E-linear differential on $C_{\bullet}(\mathbf{v}; E)$ is defined as follows: for $x^{(a)} = x_1^{(a_1)} \cdots x_m^{(a_m)}$ we set

$$\partial(x^{(a)}) = \sum_{a_i > 0} v_i x_1^{(a_1)} \cdots x_i^{(a_i - 1)} \cdots x_m^{(a_m)}.$$

It is easily verified that $\partial \circ \partial = 0$, so that $(C_{\bullet}(\mathbf{v}; E), \partial)$ is indeed a complex. Moreover, ∂ is an E-derivation, that is, ∂ is E-linear and

$$\partial(g_1 g_2) = g_1 \partial(g_2) + \partial(g_1) g_2$$

for any two homogeneous elements g_1 and g_2 in $C_{\bullet}(\mathbf{v}; E)$.

These rules imply that the cycles $Z_{\bullet}(\mathbf{v}; E)$ of $C_{\bullet}(\mathbf{v}; E)$ form a divided power algebra, and that the boundaries $B_{\bullet}(\mathbf{v}; E)$ form an ideal in $Z_{\bullet}(\mathbf{v}; E)$, so that the homology $H_{\bullet}(\mathbf{v}; E)$ of $C_{\bullet}(\mathbf{v}; E)$ inherits a natural structure of a divided power algebra. Let M be left E-module; then $C_{\bullet}(\mathbf{v}; M) = C_{\bullet}(\mathbf{v}; E) \otimes_E M$ is called the *Cartan complex of M with respect to the sequence* $\mathbf{v}$. The homology of $C_{\bullet}(\mathbf{v}; M)$ will be denoted by $H_{\bullet}(\mathbf{v}; M)$. Note that $H_{\bullet}(\mathbf{v}; M)$ has a natural left $H_{\bullet}(\mathbf{v}; E)$-module structure.

For each $j = 1, \dots, m-1$ there exists an exact sequence of complexes

$0 \to C_{\bullet}(v_1, \dots, v_j; M) \xrightarrow{\iota} C_{\bullet}(v_1, \dots, v_{j+1}; M)$

$$\xrightarrow{\tau} C_\bullet(v_1, \dots, v_{j+1}; M)(-1) \to 0,$$

where ι is a natural inclusion map, and where τ is given by

$$\tau(g_0 + g_1 x_{j+1} + \dots + g_k x_{j+1}^{(k)}) = g_1 + g_2 x_{j+1} + \dots + g_k x_{j+1}^{(k-1)},$$

with $g_i \in C_{k-i}(v_1, \dots, v_j; M)$.

From this exact sequence one obtains immediately the following long exact sequences for the Cartan homology.

Proposition 4.1. *Let $M \in \mathcal{M}_l$; then for all $j = 1, \dots, m-1$ there exists a long exact sequence of graded left E-modules*

$$\begin{aligned} \cdots &\longrightarrow H_i(v_1, \dots, v_j; M) \xrightarrow{\alpha_i} H_i(v_1, \dots, v_{j+1}; M) \\ &\xrightarrow{\beta_i} H_{i-1}(v_1, \dots, v_{j+1}; M)(-1) \xrightarrow{\delta_{i-1}} H_{i-1}(v_1, \dots, v_j; M) \\ &\longrightarrow H_{i-1}(v_1, \dots, v_{j+1}; M) \longrightarrow \cdots . \end{aligned}$$

Here α_i is induced by the inclusion map ι, β_i by τ, and δ_{i-1} is the connecting homomorphism, which acts as follows: if $z = g_0 + g_1 x_{j+1} + \dots + g_{i-1} x_{j+1}^{(i-1)}$ is a cycle in $C_{i-1}(l_1, \dots, l_{j+1}; M)$, then $\delta_{i-1}([z]) = [g_0 v_{j+1}]$.

Let $e_1, \dots, e_n$ be a K-basis of E_1. Using Proposition 4.1 it follows easily by induction on i that $C_\bullet(e_1, \dots, e_i; E)$ is acyclic for $i = 1, \dots, n$. In particular, $C_\bullet(e_1, \dots, e_n; E)$ is a minimal, graded free E-resolution of K.

Corollary 4.2. *Let $M \in \mathcal{M}_l$. Then*

(a) *for all $i \geq 0$ there are graded isomorphisms $\operatorname{Tor}_i^E(K, M) \cong H_i(e_1, \dots, e_n; M)$ of K-vector spaces;*
(b) *for all $i \geq 0$ one has $\beta_{ii}(K) = \binom{n-1+i}{i}$ and $\beta_{ij}(K) = 0$ for $j \neq i$;*
(c) $\operatorname{reg}(M) \leq \max\{j \colon M_j \neq 0\}$.

Proof. The statements (a) and (b) are clear by the discussions preceding this corollary. Since $C_i(e_1, \dots, e_n; E) \cong \bigoplus E(-i)$, it follows from (a) that $\operatorname{Tor}_i^E(K, M)$ is a subquotient of $\bigoplus M(-i)$. This implies (c). Q.E.D.

For any finitely generated left E-module M, the Cartan cohomology with respect to the sequence $\mathbf{v} = v_1, \dots, v_m$ is defined to be the homology of the cocomplex $C^\bullet(\mathbf{v}; M) = \operatorname{Hom}_E(C_\bullet(\mathbf{v}; E), M)$. Explicitly, we have

$$C^\bullet(\mathbf{v}; M) : 0 \xrightarrow{\partial^0} C^0(M) \xrightarrow{\partial^1} C^1(M) \longrightarrow \dots ,$$

where the cochains $C^\bullet(\mathbf{v}; M)$ and the cochain maps $\partial^\bullet$ can be described as follows: the elements of $C^i(\mathbf{v}; M)$ may be identified with all homogeneous polynomials $\sum_a m_a y^a$ of degree i in the variables $y_1, \dots, y_m$ with coefficients $m_a \in M$, and where as usual for $a \in \mathbb{N}^n$, y^a denotes the monomial $y_1^{a_1} y_2^{a_2} \dots y_n^{a_n}$. The element $m_a y^a \in C^\bullet(\mathbf{v}; M)$ is defined by the mapping property

$$m_a y^a(x^{(b)}) = \begin{cases} m_a & \text{for} \quad b = a, \\ 0 & \text{for} \quad b \neq a. \end{cases}$$

After this identification the cochain maps are simply multiplication by the element $y_{\mathbf{v}} = \sum_{i=1}^n v_i y_i$. In other words, we have

$$\partial^i \colon C^i(\mathbf{v}; M) \longrightarrow C^{i+1}(\mathbf{v}; M), \quad f \mapsto y_{\mathbf{v}} f.$$

In particular we see that $C^\bullet(\mathbf{v}; E)$ may be identified with the polynomial ring $E[y_1, \dots, y_m]$, and that $C^\bullet(\mathbf{v}; M)$ is a finitely generated $C^\bullet(\mathbf{v}; E)$-module. It is obvious that cocycles and coboundaries of $C^\bullet(\mathbf{v}; M)$ are $E[y_1, \dots, y_m]$-submodules of $C^\bullet(\mathbf{v}; M)$. As $E[y_1, \dots, y_m]$ is Noetherian, it follows that the Cartan cohomology $H^\bullet(\mathbf{v}; M)$ of M is a finitely generated (graded) $E[y_1, \dots, y_m]$-module.

We set $M^* = \operatorname{Hom}_E(M, E)$. Cartan homology and cohomology are related as follows:

Proposition 4.3. *Let $M \in \mathcal{M}$. Then*

$$H_i(\mathbf{v}; M)^* \cong H^i(\mathbf{v}; M^*) \quad \textit{for all} \quad i.$$

Proof. Since E is injective, the functor $(-)^*$ commutes with homology and we obtain

$$\begin{aligned} &H_i(\mathbf{v}; M)^* \cong H^i(\operatorname{Hom}_E(C_i(\mathbf{v}; M), E)) \\ &\cong H^i(\operatorname{Hom}_E(C_i(\mathbf{v}; E), M^*) \cong H^i(\mathbf{v}; M^*). \end{aligned}$$

Q.E.D.

Proposition 4.4. *Let $M \in \mathcal{M}_l$. Then for all $j = 1, \dots, m-1$ there exists a long exact sequence of graded left E-modules*

$$\begin{aligned} &\cdots \longrightarrow H^{i-1}(v_1, \dots, v_{j+1}; M) \longrightarrow H^{i-1}(v_1, \dots, v_j; M) \\ &\longrightarrow H^{i-1}(v_1, \dots, v_{j+1}; M)(-1) \xrightarrow{y_{j+1}} H^i(v_1, \dots, v_{j+1}; M) \\ &\longrightarrow H^i(v_1, \dots, v_j; M) \longrightarrow \cdots. \end{aligned}$$

Proof. It is immediate that such a sequence exists. We only show that the map

$$H^{i-1}(v_1, \dots, v_{j+1}; M)(-1) \to H^i(v_1, \dots, v_{j+1}; M)$$

is indeed multiplication by y_{j+1}. We show this on the level of cochains. In order to simplify notation we set $C_i = C_i(v_1, \dots, v_{j+1}; E)$ for all i, and let

$$\gamma\colon \operatorname{Hom}_E(C_{i-1}, M) \to \operatorname{Hom}_E(C_i, M)$$

be the map induced by $\tau\colon C_i \to C_{i-1}$, where

$$\tau(x^{(b)}) = \begin{cases} x_1^{(b_1)} \cdots x_{j+1}^{(b_{j+1}-1)} & \text{if} \quad b_{j+1} > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Our assertion is that γ is multiplication by y_{j+1}.

For all $x^{(b)} \in C_i$ we have $\gamma(my^a)(x^{(b)}) = my^a(\tau(x^{(b)}))$. This implies that

$$\gamma(my^a)(x^{(b)}) = \begin{cases} m & \text{if} \quad (b_1, \dots, b_{j+1}) = (a_1, \dots, a_{j+1} + 1), \\ 0 & \text{otherwise.} \end{cases}$$

Hence we see that $\gamma(my^a) = my^a y_{j+1}$, as desired. Q.E.D.

§5. Simplicial cohomology

Besides Cartan cohomology, there is another natural cohomology attached to any graded E-module: let $v \in E$ be a homogeneous element of degree 1, and let $M \in \mathcal{M}_l$. Since $v^2 = 0$, we obtain a finite complex of finitely generated K-vector spaces

$$(M, v)\colon \qquad \cdots \longrightarrow M_{i-1} \xrightarrow{l_v} M_i \xrightarrow{l_v} M_{i+1} \longrightarrow \cdots$$

where l_v denotes left multiplication by v. We denote the ith cohomology of this complex by $H^i(M, v)$. Notice that $H^{\bullet}(M, v) = \bigoplus_i H^i(M, v)$ is again an object in $\mathcal{M}_l$. Indeed,

$$H^{\bullet}(M, v) = \frac{0 :_M v}{vM},$$

where $0 :_M v = \{a \in M\colon va = 0\}$.

It is clear that a short exact sequence

$$0 \longrightarrow U \longrightarrow M \longrightarrow N \longrightarrow 0,$$

of finitely generated graded E-modules induces the long exact cohomology sequence

$$\cdots \to H^i(U, v) \to H^i(M, v) \to H^i(N, v) \to H^{i+1}(U, v) \to \cdots .$$

Definition 5.1. Let $M \in \mathcal{M}_l$. An element $v \in E_1$ is called *generic for* M if $\dim_K H^i(M,v) \leq \dim_K H^i(M,u)$ for all i and all $u \in E_1$.

The property of being generic for M is an open condition, that is, there exists a non-empty Zariski open subset $G \subset E_1$, such that $v \in E_1$ is generic if and only if $v \in G$.

Let Δ be simplicial complex on the vertex set $[n] = \{1, \dots, n\}$. One denotes by $I_\Delta \subset S$ the squarefree monomial ideal generated by all monomials $x_{i_1} \cdots x_{i_k}$ such that $\{i_1, \dots, i_k\} \notin \Delta$. The K-algebra $K[\Delta] = S/I_\Delta$ is called the *Stanley-Reisner ring of* Δ. Detailed information about this well studied ring can be found in [33], [14] and [21].

We denote by J_Δ the monomial ideal in E corresponding to I_Δ. The K-algebra $K\{\Delta\} = E/J_\Delta$ is called the *exterior face ring of* Δ. This algebra was first studied in a systematic way by Kalai [26] in connection with algebraic shifting. Notice that the Hilbert series of $K\{\Delta\}$ is given by

$$H_{K\{\Delta\}}(t) = \sum_{i \geq 0} f_{i-1} t^i,$$

where f_i is the number of i-dimensional faces of Δ.

Lemma 5.2. *Let Δ be a simplicial complex, and $v \in E_1$ a generic element for $K\{\Delta\}$. Then for all i we have*

$$H^i(K\{\Delta\}, v) \cong \tilde{H}^{i-1}(\Delta; K).$$

Proof. Let $e = \sum_i^n e_i$. It follows immediately from the definition of simplicial cohomology that $\tilde{H}^{i-1}(\Delta; K) \cong H^i(K\{\Delta\}, e)$. Thus it remains to be shown that e is generic for $K\{\Delta\}$. Let $\bar{K}$ be an algebraic closure of K. Then $H^i(K\{\Delta\}, v) \otimes_K \bar{K} \cong H^i(\bar{K}\{\Delta\})$, and $\tilde{H}^{i-1}(\Delta; K) \otimes_K \bar{K} \cong \tilde{H}^{i-1}(\Delta; \bar{K})$. Thus we may as well assume that K is algebraically closed. The set L of elements $v = \sum_i^n a_i e_i \in E_1$ with $\prod_i^n a_i \neq 0$ is open. Moreover, the complexes $(K\{\Delta\}, v)$ and $(K\{\Delta\}, e)$ are isomorphic for all $v \in L$. In fact, the isomorphism of complexes is induced by the algebra automorphism $\varphi\colon K\{\Delta\} \to K\{\Delta\}$ with $\varphi(e_i) = a_i e_i$ for $i = 1, \dots, n$. Let $G \subset E_1$ be the subset of generic elements for $K\{\Delta\}$. Since K is algebraically closed and since L and G are non-empty open subsets of the irreducible space E_1, their intersection is non-empty. Let v be an element of this intersection. Then v is general, and $\dim_K H^i(K\{\Delta\}, v) = \dim_K H^i(K\{\Delta\}, e)$ for all i. This proves the assertion. Q.E.D.

For the rest of the section we discuss the following question: Let $I \subset S$ be a squarefree monomial ideal, $J \subset E$ the corresponding monomial ideal in the exterior algebra. Is there a relation between the S-resolution of I and the E-resolution of J? We will show that this is indeed the case. In order to describe this relation it is convenient to consider the attached simplicial complexes.

Both algebras, $K[\Delta]$ as well as $K\{\Delta\}$, are $\mathbb{Z}^n$-graded, and hence have $\mathbb{Z}^n$-graded resolutions. A formula for the $\mathbb{Z}^n$-graded Betti numbers $\beta_{ia}(K[\Delta])$ is given by Hochster [24] in terms of reduced simplicial homology.

Let $a \in \mathbb{Z}^n$. We set $\operatorname{supp}(a) = \{i \in [n] \colon a_i \neq 0\}$ and $|a| = \sum_i a_i$. The element $a \in \mathbb{Z}^n$ is called *squarefree*, if $a_i \in \{0,1\}$ for $i = 1, \ldots, n$.

Let $\sigma \subset [n]$. The *restriction of* Δ *to* σ is the simplicial complex $\Delta_\sigma = \{\tau \in \Delta \colon \tau \subset \sigma\}$.

Theorem 5.3. *Let Δ be a simplicial complex with vertex set $[n]$, and $a \in \mathbb{N}^n$. Then for all $i \geq 0$, we have*

(a) $\beta^S_{ia}(K[\Delta]) = 0$, *if a is not squarefree*;
(b) $\beta^S_{ia}(K[\Delta]) = \dim_K \tilde{H}_{|a|-i-1}(\Delta_{\operatorname{supp}(a)}; K)$, *if a is squarefree.*

For the proof we refer to Hochster's original paper [24], or to [14].

There is a similar kind of formula for the $\mathbb{Z}^n$-graded Betti numbers of $K\{\Delta\}$ given in [6].

Theorem 5.4. *Let Δ be a simplicial complex with vertex set $[n]$, and $a \in \mathbb{Z}^n$. Then for all $i \geq 0$, we have*

$$\beta^E_{ia}(K\{\Delta\}) = \dim \tilde{H}^{|a|-i-1}(\Delta_{\operatorname{supp}(a)}; K).$$

Proof. Set $\alpha = \operatorname{supp}(a)$, and let $\tilde{\mathcal{C}}(\Delta_\alpha)$ be the augmented oriented chain complex of Δ_α. The module $\mathcal{C}_i$ of i-chains of $\tilde{\mathcal{C}}(\Delta_\alpha)$ is the free $\mathbb{Z}$-module with basis $\sigma \in \Delta_\alpha$, $|\sigma| = i+1$. Thus the module of i-cochains $\mathcal{C}^i(K)$ is a K-vector space with basis σ^*, $\sigma \in \Delta_\alpha$, $|\sigma| = i+1$ where $\sigma^* \colon \mathcal{C}_i \to K$ is the $\mathbb{Z}$-linear map with $\sigma^*(\tau) = 0$ for $\tau \neq \sigma$ and $\sigma^*(\tau) = 1$ for $\tau = \sigma$.

On the other hand, $\operatorname{Tor}^E(K\{\Delta\}, K)_a$ may be identified with the homology of the ath graded piece $C(e_1, \ldots, e_n; K\{\Delta\})_a$ of the Cartan complex. In degree i this complex has the following K-basis

$$e_\sigma x^{(a_\sigma)}, \qquad \sigma \in \Delta_\alpha, \qquad |a_\sigma| = i.$$

Here $a_\sigma = (a'_1, \ldots, a'_n)$ where $a'_j = a_j$ for $j \notin \sigma$ and $a'_j = a_j - 1$ for $j \in \sigma$.

We define a K-linear map

$$\varphi_i\colon C_i(e_1,\dots,e_n;K\{\Delta\})_a \longrightarrow \mathcal{C}^{d-i-1}(K)$$

by setting $\varphi_i(e_\sigma x^{(a_\sigma)}) = \sigma^*$. One easily checks that φ is an isomorphism of complexes. Q.E.D.

A comparison of the formulas in Theorem 5.3 and Theorem 5.4 leads to

Corollary 5.5. *Let Δ be a simplicial complex with vertex set $[n]$. Then*

$$\sum_i \sum_{a\in\mathbb{N}^n} \beta^E_{ia}(K\{\Delta\})t^i s^a = \sum_i \sum_{a\in\mathbb{N}^n} \beta^S_{ia}(K[\Delta])\frac{t^i s^a}{\prod_{j\in\operatorname{supp} a}(1-ts_j)}.$$

There is actually an explicit construction for the E-resolution of $K\{\Delta\}$ in terms of the S-resolution of $K[\Delta]$. This construction is described in [9].

§6. Regularity and extremal Betti numbers

In this section we present the theorem of Bayer, Charalambous and S. Popescu [12] which asserts that the extremal Betti numbers of a graded ideal and its generic ideal coincide.

Throughout this section we assume that the base field K is infinite. Let M be a finitely generated graded S-module. A Betti number of M is called *extremal* if $\beta_{ii+j} = 0$ for all $(i,j) \neq (k,m)$ with $i \geq k$ and $j \geq m$. The corollary of the next theorem provides a characterization of extremal Betti numbers in terms of annihilators of almost regular sequences.

An element $x \in S_1$ is called *almost M-regular*, if the colon module $0 :_M x = \{c \in M\colon xc = 0\}$ is of finite length. The set of almost M-regular elements is a nonempty open subset of S_1. Indeed, $M/H^0_{\mathfrak{m}}(M)$ is a module of positive depth, so that the Zariski open set $\mathcal{S} \subset S_1$ of regular elements of $M/H^0_{\mathfrak{m}}(M)$ in S_1 is not empty. For any element $x \in \mathcal{S}$ we have that $0 :_M x$ is a finite length module.

Let $\mathbf{l} = l_1,\dots,l_m$ be a sequence of linear forms in $\mathcal{S}$. In order to simplify notation we set $M\langle j\rangle = M/(l_1,\dots,l_j)M$, and for $i \geq 1$ we let $H_i(j)$ be the ith Koszul homology $H_i(l_1,\dots,l_j;M)$ of M with respect to the sequence $l_1,\dots,l_j$. We further set $H_i(0) = 0$ for $i > 0$ and for $j \geq 1$ we let $H_0(j-1)$ be the colon ideal $0 :_{M\langle j-1\rangle} l_j$. Observe that, in our notation, $H_0(j)$ is *not* the 0th Koszul homology.

The sequence $\mathbf{l} = l_1, \dots, l_m$ is called an *almost regular M-sequence* if for all $j = 1, \dots, m$, the linear form l_j is almost $M_{\langle j-1\rangle}$-regular. If all permutations of the sequence $\mathbf{l}$ are almost M-regular, then we call $\mathbf{l}$ an *unconditioned almost regular M-sequence* .

Suppose $\mathbf{l} = l_1, \dots, l_m$ is almost M-regular, then all $H_i(j)$ are modules of finite length and since M is a graded S-module, all $H_i(j)$ are naturally graded. Now suppose in addition that $\mathbf{l}$ is a basis of S_1. Then there are graded isomorphisms $H_i(n)_j \cong \operatorname{Tor}_i(K, M)_j$ for all i and j. In particular, the graded ijth Betti numbers β_{ij} of M coincide with $\dim_K H_i(n)_j$.

Let N be an Artinian graded module. We set $s(N) = \max\{s \colon N_s \neq 0\}$ if $N \neq 0$ and $s(0) = -\infty$. Now we introduce the following numbers attached to M and the basis $\mathbf{l} = l_1, \dots, l_n$. We set

$$r_j = \max\{s(H_i(j)) - i \colon i \geq 1\} \text{ and } s_j = s(H_0(j-1)) \text{ for } j = 1, \dots, n,$$

and put $r_0 = 0$. We observe that $\operatorname{reg}(M) = \max\{r_n, s(M/\mathfrak{m}M)\}$.

Theorem 6.1. *Suppose that the basis $\mathbf{l} = l_1, \dots, l_n$ of S_1 is an almost regular M-sequence. Then*

(a) $r_j = \max\{s_1, \dots, s_j\}$ *for $j = 1, \dots, n$. In particular,* $r_1 \leq r_2 \leq \dots \leq r_n$.

(b) *Let $\mathcal{J} = \{j_1, \dots, j_l\}$, $1 \leq j_1 < j_2 < \dots < j_l \leq n$, be the set of elements $j \in [n]$ such that $r_j - r_{j-1} \neq 0$. Then for all t with $1 \leq t \leq l$ and all j with $j_t \leq j$ we have*
   - (i) $H_i(j)_{i+s} = 0$ *for $s > r_{j_{t-1}}$ and $i > j - j_t + 1$;*
   - (ii) $H_{j-j_t+1}(j)_{j-j_t+1+r_{j_t}} \cong H_0(j_t - 1)_{r_{j_t}}$;
   - (iii) $H_{j-j_t+1}(j)_{j-j_t+1+s}$ *is isomorphic to a submodule of* $H_0(j_t - 1)_s$ *for all $s > r_{j_{t-1}}$;*
   - (iv) $H_0(j-i)_{r_{j_t}}$ *is isomorphic to a factor module of $H_i(j)_{i+r_{j_t}}$ for all i with $i > j - j_{t+1} + 1$.*

For the proof of this theorem we refer to [4].

Corollary 6.2. *Let the numbers j_t be defined as in Theorem 6.1, and set $k_t = n - j_t + 1$ and $m_t = r_{j_t}$. Then*

(a) *the Betti number β_{ii+j} of M is extremal if and only if*

$$(i, j) \in \{(k_t, m_t) \colon t = 1, \dots, l\}.$$

Moreover, $\beta_{k_t, k_t+m_t} = \dim_K(0 : l_{j_t})_{s_{j_t}}$ *for $t = 1, \dots, l$,*

(b) *for all $t = 1, \dots, l$ we have*
   - (1) $\beta_{k_t, k_t+s} \leq \dim_K(0 : l_{j_t})_s$ *for all $s > m_{t-1}$,*
   - (2) $\beta_{i, i+m_t} \geq \dim_K(0 : l_{n-i+1})_{m_t}$ *for all $i > k_{t+1}$.*

Now we are ready to prove the main theorem of this section.

Theorem 6.3 (Bayer-Charalambous-S. Popescu). *Let $I \subset S$ be a graded ideal, and let* $\operatorname{Gin}(I)$ *be the generic initial ideal of I with respect to the reverse lexicographic order. Then for any two integers $i, j \in \mathbb{N}$ one has*

(a) *the ijth Betti number of S/I is extremal if and only if the ijth Betti number of $S/\operatorname{Gin}(I)$ is extremal;*
(b) *the corresponding extremal Betti numbers of S/I and $S/\operatorname{Gin}(I)$ are equal.*

Proof. After a generic choice of coordinates we may assume that $\operatorname{Gin}(I) = \operatorname{in}(I)$, and since the condition of being an almost regular sequence is an open condition, we may as well assume that $x_n, \dots, x_1$ is an almost regular S/I-sequence. Since $\operatorname{Gin}(I)$ is Borel fixed it follows for example from [16, Corollary 15.25] that $x_n, \dots, x_1$ is an almost regular $S/\operatorname{Gin}(I)$-sequence, too. Set $R\langle j\rangle = (S/I)/(x_n, \dots, x_j)(S/I)$ and $\bar{R}\langle j\rangle = (S/\operatorname{Gin}(I))/(x_n, \dots, x_j)(S/\operatorname{Gin}(I))$, then it follows that $0 :_{R\langle n-i+1\rangle} x_{n-i}$ as well as $0 :_{\bar{R}\langle n-i+1\rangle} x_{n-i}$ have finite length for all i. Now since the chosen term order is reverse lexicographic it follows from Proposition 1.4 that $0 :_{R\langle n-i+1\rangle} x_{n-i}$ and $0 :_{\bar{R}\langle n-i+1\rangle} x_{n-i}$ have the same Hilbert function. In particular,

$$s(0 :_{R\langle n-i+1\rangle} x_{n-i}) = s(0 :_{\bar{R}\langle n-i+1\rangle} x_{n-i}) \quad \text{for all} \quad i.$$

Thus Corollary 6.2(a) concludes the proof. Q.E.D.

Corollary 6.4. *Let $I \subset S$ be a graded ideal,* $\operatorname{Gin}(I)$ *the generic initial ideal of I with respect to the reverse lexicographic order. Then*

(a) (Bayer-Stillman) $\operatorname{reg}(I) = \operatorname{reg}(\operatorname{Gin}(I))$;
(b) $\operatorname{proj\,dim} S/I = \operatorname{proj\,dim} S/\operatorname{Gin}(I)$;
(c) *S/I is Cohen-Macaulay, if and only if $S/\operatorname{Gin}(I)$ is Cohen-Macaulay.*

§7. Extremal Betti numbers for squarefree monomial ideals

Let Δ be a simplicial complex, and $J_\Delta \subset E$ the Stanley-Reisner ideal of Δ in the exterior algebra. The *exterior algebraic shifted complex* of Δ is the simplicial complex Δ^e with

$$J_{\Delta^e} = \operatorname{Gin}(J_\Delta).$$

We will study algebraic shifting more systematically in the next section. Here we are interested in the comparison of the graded Betti numbers of $I_\Delta \subset S$ with those of I_{Δ^e}. Though it is not yet known whether or not the graded Betti numbers increase when passing from I_Δ to I_{Δ^e}, it can be shown (see [4]) that I_Δ and I_{Δ^e} have the same extremal Betti numbers. In fact one has

Theorem 7.1. *Let Δ be a simplicial complex. Then for any two integers $i, j \in \mathbb{N}$ one has*

(a) *the ijth Betti number of S/I_Δ is extremal if and only if the ijth Betti number of S/I_{Δ^e} is extremal;*
(b) *the corresponding extremal Betti numbers of S/I_Δ and S/I_{Δ^e} are equal.*

We will describe the main steps of the proof. For simplicity we set $J = J_\Delta$ $I = I_\Delta$. Set $P_j(t) = \sum_{i\geq 0} \beta^E_{ii+j}(E/J)t^i$, then Corollary 5.5 yields

$$P_j(t) = \sum_{i\geq 0}(\sum_{k=0}^{i}\binom{i+j-1}{j+k-1}\beta^S_{kk+j})t^i.$$

Setting $k(j) = \max\{k\colon \beta^S_{kk+j}(S/I) \neq 0\}$, we see that

$$P_j(t) = \frac{\sum_{k=0}^{k(j)} \beta^S_{kk+j}(S/I)t^k(1-t)^{k(j)-k} + R(t)(1-t)^{k(j)+j}}{(1-t)^{k(j)+j}},$$

with a certain polynomial $R(t)$.

We set $d_j(E/J) = k(j) + j$ and $e_j(E/J) = \beta^S_{k(j),k(j)+j}(S/I)$.

Corollary 7.2. *The following conditions are equivalent:*

(a) *$\beta^S_{ii+j}(S/I)$ is an extremal Betti number of S/I;*
(b) *$i = k(j)$, and $d_{j'}(E/J) - d_j(E/J) < j' - j$ for all $j' > j$.*

For the further discussion we need (see [4, Corollary 4.6]) a different interpretation of the numbers d_j and e_j.

Proposition 7.3. *Let $M \in \mathcal{M}$, and let $v_1, \dots, v_n$ be a generic basis of E_1. Then the natural maps*

$$H_i(v_1, \dots, v_{j+1}; M) \xrightarrow{\beta_i} H_i(v_1, \dots, v_{j+1}; M)$$

of Cartan homology attached with the sequence $v_1, \dots, v_n$ (cf. Proposition 4.1) are surjective for all $j = 0, \dots, n-1$ and all $j \gg 0$.

We now fix $M \in \mathcal{M}$ and a sequence $\mathbf{v} = v_1, \dots, v_n$ in E_1. Similarly as in Section 1 we set $M\langle j-1\rangle = M/(v_1, \dots, v_{j-1})M$ and put $H_i(j) = H_i(v_1, \dots, v_j; M)$ for $i > 0$ and $H_0(j) = H^{\bullet}(M\langle j-1\rangle, v_j)$ for

$j = 1, \ldots, n$. Furthermore we set $H_i(0) = 0$ for all i. Notice that $H_0(j)$ is *not* the 0th Cartan homology of M with respect to $v_1, \ldots, v_j$, but is the cohomology of $M\langle j-1\rangle$ with respect to v_j as defined in Section 5. From Proposition 4.1 we obtain immediately the following long exact sequence of graded E-modules

$$(6) \quad \begin{array}{l} H_2(j) \to H_1(j)(-1) \to H_1(j-1) \to H_1(j) \to H_0(j)(-1) \to 0 \\ \cdots \to H_i(j-1) \to H_i(j) \to H_{i-1}(j)(-1) \to H_{i-1}(j-1) \to \cdots. \end{array}$$

We fix an integer j. By Proposition 7.3 there exists an integer i_0 such that for all $i \geq i_0$ and all $k = 1, \ldots, n$ the sequences

$$(7) \quad 0 \longrightarrow H_{i+1}(k-1)_{(i+1)+j} \longrightarrow H_{i+1}(k)_{(i+1)+j} \longrightarrow H_i(k)_{i+j} \longrightarrow 0$$

are exact.

Set $h_i^k = \dim_K H_i(k)_{i+j}$, and $c_k = h_{i_0}^k$ for $k = 1, \ldots, n$. The exact sequences (7) yield the equations

$$(8) \qquad h_{i+1}^k = h_{i+1}^{k-1} + h_i^k$$

for all $i \geq i_0$, and $k = 1, \ldots, n$. It follows from (8) that

$$h_{i_0+i}^n = \binom{i+n-2}{n-1} c_1 + \binom{i}{n-3} c_2 + \cdots + \binom{i}{1} c_{n-1} + c_n \text{for all} i \geq 0.$$

Since $\beta_{ii+j}^E(M) = h_i^n$ for all i, we see that

$$\sum_{i \geq 0} \beta_{ii+j}^E(M) = t^{i_0+1} \sum_{i=1}^{n} \frac{c_i}{(1-t)^{n-i+1}} + Q(t),$$

where $Q(t)$ is a polynomial. Thus we obtain:

Proposition 7.4. *Let d_j and e_j be defined as above. Then*

$$d_j(E/J) = n + 1 - \min\{i \colon c_i \neq 0\} \quad \textit{and} \quad e_j(E/J) = c_{n-d_j+1}.$$

In order to relate the invariants d_j and e_j to the generalized simplicial homology modules $H_0(k)$ we need the following

Lemma 7.5. *Let $1 \leq l \leq n$ and j be integers. The following conditions are equivalent:*

(a) (1) $H_0(k)_j = 0$ *for* $k < l$, *and* $H_0(l)_j \neq 0$
  (2) $H_0(k)_{j'} = 0$ *for all* $j' > j$ *and all* $k \leq l + j - j'$.

(b) *For all* $i \geq 0$ *we have*
  (1) $H_i(k)_{i+j} = 0$ *for* $k < l$, *and* $H_i(l)_{i+j} \neq 0$
  (2) $H_i(k)_{i+j'} = 0$ *for all* $j' > j$ *and all* $k \leq l + j - j'$.

(c) *Condition* (b) *is satisfied for some* i.

Moreover, if the equivalent conditions hold, then $H_i(l)_{i+j} \cong H_0(l)_j$ *for all* $i \geq 0$.

Proof. In our proof we will use the following exact sequence

(9) $$H_i(k-1)_{i+j'} \to H_i(k)_{i+j'} \to H_{i-1}(k)_{(i-1)+j'} \to H_{i-1}(k-1)_{(i-1)+(j'+1)}$$

(a) $\Rightarrow$ (b): We prove (b) by induction on i. For $i = 0$, there is nothing to show. So now let $i > 0$ and assume that (1) and (2) hold for $i - 1$. By (9) we have the exact sequence

$$H_i(l)_{i+j} \longrightarrow H_{i-1}(l)_{(i-1)+j} \longrightarrow H_{i-1}(l-1)_{(i-1)+(j+1)}.$$

Since $l - 1 \leq l + j - (j + 1)$, we have $H_{i-1}(l-1)_{(i-1)+(j+1)} = 0$ by induction hypothesis. Also by induction hypothesis, $H_{i-1}(l)_{(i-1)+j} \neq 0$; therefore, $H_i(l)_{i+j} \neq 0$.

Now let $k < l$. Then (9) yields the exact sequence

$$H_i(k-1)_{i+j} \longrightarrow H_i(k)_{i+j} \longrightarrow H_{i-1}(k)_{(i-1)+j}.$$

By induction hypothesis we have $H_{i-1}(k)_{(i-1)+j} = 0$. Now by induction on k we may assume that $H_i(k-1)_{i+j} = 0$. Therefore, $H_i(k)_{i+j} = 0$, and this shows (1).

In order to prove (b)(2), we let $j' > j$ and $k \leq l + (j - j')$, and consider the exact sequence

$$H_i(k-1)_{i+j'} \longrightarrow H_i(k)_{i+j'} \longrightarrow H_{i-1}(k-1)_{(i-1)+j'},$$

from which the assertion follows by induction on i and k.

(c) $\Rightarrow$ (a): We show that if the conditions (1) and (2) hold for $i > 0$, then they also hold for $i - 1$. Therefore backwards induction yields the desired conclusion.

We begin with the proof of (2) for $i - 1$ by induction on k. For $k = 0$, there is nothing to show. Now let $j' > j$, and $0 < k \leq l + (j - j')$, and consider the exact sequence

$$H_i(k)_{i+j'} \longrightarrow H_{i-1}(k)_{(i-1)+j'} \longrightarrow H_{i-1}(k-1)_{(i-1)+(j'+1)}.$$

Since $k - 1 \leq l + j - (j' + 1)$ it follows by our induction hypothesis that $H_{i-1}(k-1)_{(i-1)+(j'+1)} = 0$. On the other hand, by assumption we have $H_i(k)_{i+j'} = 0$, and hence $H_{i-1}(k)_{(i-1)+j'} = 0$.

In order to prove (1) for $i - 1$ we consider the exact sequence

$$H_i(l-1)_{i+j} \longrightarrow H_i(l)_{i+j} \longrightarrow H_{i-1}(l)_{(i-1)+j} \longrightarrow H_{i-1}(l-1)_{(i-1)+(j+1)}.$$

Since $l-1 \leq l+j-(j+1)$, we know from (2) (which we have already shown for $i-1$) that $H_{i-1}(l-1)_{(i-1)+(j+1)} = 0$. By our assumption we have $H_i(l-1)_{i+j} = 0$, and hence

$$H_{i-1}(l)_{(i-1)+j} \cong H_i(l)_{i+j} \neq 0.$$

That $H_{i-1}(k)_{(i-1)+j} = 0$ for $k < l$ is proved similarly. This concludes the proof of the implication (c) $\Rightarrow$ (a).

In the proof of this implication we have just seen that $H_i(l)_{i+j} \cong H_{i-1}(l)_{(i-1)+j}$. By induction hypothesis we may assume that $H_{i-1}(l)_{(i-1)+j} \cong H_0(l)_j$, and hence $H_i(l)_{i+j} \cong H_0(l)_j$, as desired.

Q.E.D.

A pair of numbers (l, j) satisfying the equivalent conditions of Lemma 7.5 will be a called a *distinguished pair* (*for* M).

Now we may characterize the extremal Betti numbers of S/I as follows:

Corollary 7.6. *The Betti number $\beta_{ii+j}(S/I)$ is extremal if and only if $(n+1-i-j, j)$ is a distinguished pair. Moreover, if the equivalent conditions of Lemma* 7.5 *hold, then $\beta_{ii+j}(S/I) = \dim_K H_0(n+1-i-j)_j$.*

Proof. We know from Corollary 7.2 that $\beta_{ii+j}(S/I)$ is an extremal Betti number if and only if $d_{j'}(E/J) - d_j(E/J) < j' - j$ for all $j' > j$. By Proposition 7.4 this condition is equivalent to

$$\min\{k\colon H_{i_0}(k)_{i_0+j'} \neq 0\} > l + (j - j'),$$

where $l = \min\{k\colon H_{i_0}(k)_{i_0+j} \neq 0\}$. This in turn is equivalent to

$$H_{i_0}(k)_{i_0+j'} = 0 \quad \text{for} \quad k \leq l + (j - j'),$$

which means that (l, j) is a distinguished pair.

From Corollary 7.2 and Proposition 7.4 it follows that $l = n+1-i-j$. Finally, Corollary 7.2, Proposition 7.4 and Lemma 7.5 imply that

$$\beta_{ii+j}(S/I) = e_j(S/I) = c_l = \dim_K H_0(l)_j.$$

Q.E.D.

We we are ready for

Proof of Theorem 7.1. After a generic change of bases we may assume, that $\mathrm{in}(J) = \mathrm{Gin}(J)$, and that $e_n, \ldots, e_1$ is a generic basis for E/J, and since $\mathrm{in}(J)$ is Borel fixed it follows easily that $e_n, \ldots, e_1$ is a generic basis for $E/\operatorname{in}(J)$, too. We let $H_0(k) = H((E/J)/(e_n, \dots, e_{n-k+1}, e_{n-k})(E/J), e_{n-k})$. The corresponding homology modules for $E/\operatorname{in}(J)$ will be denoted by $H_0^e(k)$. It follows from the exterior analogue of Proposition 1.4 that for all $k = 1, \dots, n$ the homology modules $H_0(k)$ and $H_0^e(k)$ have the same Hilbert function. Since the Hilbert functions of these modules determine uniquely the distinguished pairs (l, j), all assertions of the theorem follow from Corollary 7.6. Q.E.D.

§8. Shifting operations

In this section we study shifting operations. They assign to each simplicial complex a shifted simplicial complex which shares basic properties with the original simplicial complex but is combinatorially simpler.

Let Δ be a simplicial complex on the vertex set $[n]$.

Definition 8.1. The simplicial complex Δ is *shifted*, if I_Δ is strongly stable. In other words, Δ is shifted if it satisfies the following property: if $\sigma \in \Delta$, $i \in \sigma$ and $j > i$, then $(\sigma \setminus \{i\}) \cup \{j\} \in \Delta$.

Following Kalai [27] we define a shifting operation by list of properties.

Definition 8.2. A map which assigns to each simplicial complex Δ on the vertex set $[n]$ a simplicial complex $\operatorname{Shift}(\Delta)$ on the same vertex set $[n]$ is called a *shifting operation*, if it satisfies the following conditions:

(S_1) $\operatorname{Shift}(\Delta)$ is shifted;
(S_2) $\operatorname{Shift}(\Delta) = \Delta$, if Δ itself is shifted;
(S_3) the simplicial complexes Δ and $\operatorname{Shift}(\Delta)$ have the same f-vector;
(S_4) if Γ is a subcomplex of Δ, then $\operatorname{Shift}(\Gamma) \subset \operatorname{Shift}(\Delta)$.

Shifting operations were first considered by Erdös, Ko, and Rado (see [1]), while algebraic shifting was introduced by Kalai [26], [27]. In this section we will present and compare the most important shifting operations.

Let us begin with

Combinatorial shifting: In the combinatorics of finite sets one considers the following operation (cf. [1]): Let $\mathcal{A}$ be a collection of subsets of $[n]$. For given integers $1 \leq i < j \leq n$, and for all $\sigma \in \mathcal{A}$ one defines:

$$S_{ij}(\sigma) = \begin{cases} (\sigma \setminus \{j\}) \cup \{i\}, & \text{if} \quad j \in \sigma, \quad i \notin \sigma, \quad (\sigma \setminus \{j\}) \cup \{i\} \notin \mathcal{A}, \\ \sigma, & \text{otherwise.} \end{cases}$$

For $1 \leq i < j \leq n$ and $a \in K$ we define an elementary automorphism $\varphi_{ij}^a \colon V \to V$ as follows: $\varphi_{ij}^a(e_k) = e_k$ if $k \neq j$, and $\varphi_{ij}^a(e_j) = ae_i + e_j$.

The following fact is easily checked

Lemma 8.3. *Let $J \subset E$ be a monomial ideal, and let $a \in K$, $a \neq 0$. Then $\mathrm{in}(\varphi_{ij}^a(J))$ has the K-basis $\{e_{S_{ij}(\sigma)} : \sigma \in \mathcal{A}\}$, where $\mathcal{A} = \{\sigma \subset [n] : e_\sigma \in J\}$.*

It follows in particular that the ideal $\mathrm{in}(\varphi_{ij}^a(J))$ does not depend on the choice of a. If Δ is a simplicial complex then $\mathrm{Shift}_{ij}(\Delta)$ is the simplicial complex defined by

$$J_{\mathrm{Shift}_{ij}(\Delta)} = \mathrm{in}(\varphi_{ij}^a(J_\Delta)).$$

Lemma 8.4. *The operator* Shift_{ij} *satisfies the conditions* (S_2), (S_3) *and* (S_4).

Proof. Suppose Δ is shifted, then I_Δ squarefree strongly stable, and so $J_\Delta \subset E$ is strongly stable. By Lemma 8.3, $\mathrm{in}(\varphi_{ij}^a(J_\Delta))$ has the K-basis $\{e_{S_{ij}(\sigma)} : \sigma \in \mathcal{A}\}$. As J_Δ is strongly stable it follows that $e_{S_{ij}(\sigma)} \in J_\Delta$ for all $e_\sigma \in J_\Delta$. This proves (S_2).

For the proof of (S_3) we note that J_Δ and $\mathrm{in}(\varphi_{ij}^a(J_\Delta))$ have the same Hilbert function. Condition (S_4) follows from Lemma 8.3. Q.E.D.

Simple examples show that (S_1) is in general not satisfied for Shift_{ij}. We will see however that a suitable sequence of these operators yield a shifted simplicial complex.

For a monomial $u \in E$ of degree d, $u = e_{j_1} \wedge \cdots \wedge e_{j_d}$, we set $c_d(u) = \sum_{k=1}^d j_k$. Moreover, if $J \subset E$ is a monomial ideal, we set $c_d(J) = \sum c_d(u)$ where the sum is taken over all monomials of degree d in J.

The following result was shown in [5]

Proposition 8.5. *Let Δ be a simplicial complex. Then*

(a) *$c_d(J_{\mathrm{Shift}_{ij}(\Delta)}) \leq c_d(J_\Delta)$ for all d;*
(b) *if Δ is not shifted, then there exist i and j with $i < j$ such that $c_d(J_{\mathrm{Shift}_{ij}(\Delta)}) < c_d(J_\Delta)$ for some d.*

Proof. Assertion (a) follows from the fact that $c_d(e_{S_{ij}(\sigma)}) \leq c_d(e_\sigma)$ for all monomials of degree d.

Suppose now that J_Δ is not strongly stable. Then there exists a squarefree monomial $e_\sigma \in J_\Delta$ (of degree d) and integers i and j with $i < j$ such that $e_{S_{ij}(\sigma)} \notin J_\Delta$. Since $c_d(e_{S_{ij}(\sigma)}) < c_d(e_\sigma)$, it follows that $c_d(J_{\mathrm{Shift}_{ij}(\Delta)}) < c_d(J_\Delta)$, as desired. Q.E.D.

Corollary 8.6. *Let Δ be a simplicial complex. Then there exists a sequence of pairs of integers $(i_1, j_1), \ldots, (i_r, j_r)$ with $i_k < j_k$ for $k =$*

$1, \ldots, r$ *such that*

$$\mathrm{Shift}_{i_r j_r}(\mathrm{Shift}_{i_{r-1} j_{r-1}}(\ldots(\mathrm{Shift}_{i_1 j_1}(\Delta)\ldots)))$$

is shifted.

Any simplicial complex which is obtained from Δ by a sequence of operations as described in Corollary 8.6 will be denoted by Δ^c. It follows from our discussions that $\Delta \mapsto \Delta^c$ is a shifting operator. We call this operator *combinatorial shifting.* Combinatorial shifting is not very natural. In fact, Δ is not even uniquely defined. The only advantage of this operator is that it is easily computable.

Conjecture 8.7. *For all simplicial complexes Δ on the vertex set $[n]$ and all integer k and l with $1 \leq k < l \leq n$ one has $\beta_{ij}(I_\Delta) \leq \beta_{ij}(I_{\mathrm{Shift}_{kl}(\Delta)})$ for all i and j. In particular, $\beta_{ij}(I_\Delta) \leq \beta_{ij}(I_{\Delta^c})$ for all i and j.*

It is only known that $\beta_{0j}(I_\Delta) \leq \beta_{0j}(I_{\mathrm{Shift}_{kl}(\Delta)})$ for all j.

Exterior algebraic shifting: Let Δ be simpicial complex, $J_\Delta \subset E$ its Stanley-Reisner ideal in the exterior algebra. Recall from Section 7 that the exterior algebraic shifted complex Δ^e of Δ is defined by the equation $J_{\Delta^e} = \mathrm{Gin}(J_\Delta)$.

Proposition 8.8. *Exterior algebraic shifting is in fact a shifting operator, that is, it satisfies the conditions (S_1) - (S_4).*

Proof. Condition (S_1) follows from Proposition 2.13, and (S_3) and (S_4) follow as for combinatorial shifting. In order to prove (S_2) we notice that for any strongly stable ideal $J \subset E$ and any invertible upper triangular matrix φ one has $\varphi(J) = J$. The assertion is clear for elementary upper triangular matrices, as well as for invertible diagonal matrices. Since these matrices generate all invertible upper triangular matrices, we get the desired conclusion. Therefore, if $J \subset E$ is strongly stable, then $\mathrm{Gin}(J) = \mathrm{in}(\varphi(J)) = \mathrm{in}(J) = J$. Q.E.D.

Conjecture 8.9. *Let Δ be simplicial complex. Then*

$$\beta_{ij}(I_\Delta) \leq \beta_{ij}(I_{\Delta^e}).$$

Note that a result similar to Theorem 3.1 holds for ideals in the exterior algebra, so that in particular one has $\beta_{ij}(J) \leq \beta_{ij}(\mathrm{Gin}^E(J))$ for all i and j. Unfortunately this does not imply the conjecture, even if one uses Corollary 5.5.

As a consequence of the fact that I_Δ and I_{Δ^e} have the same extremal Betti numbers we now derive further properties of exterior algebraic shifting which were discovered first (with different methods) by Kalai [26].

Proposition 8.10 (Kalai). *For all i one has*

$$\tilde{H}^i(\Delta; K) \cong \tilde{H}^i(\Delta^e; K).$$

Proof. Hochster's formulas (cf. Theorem 5.3) imply that

$$\beta_{in}(K[\Delta]) = \dim_K \tilde{H}_{n-i-1}(\Delta; K) \quad \text{for all } i, \tag{10}$$

and $\beta_{ij}(K[\Delta]) = 0$ for all i and all $j > n$. In particular we see that the Betti numbers $\beta_{in}(K[\Delta])$ are extremal. Thus $\beta_{in}(K[\Delta]) = \beta_{in}(K[\Delta^e])$, by Theorem 7.1. Since K is a field it follows that $\tilde{H}_{n-i-1}(\Delta; K) = \tilde{H}^{n-i-1}(\Delta; K)$ for all i, and the assertion follows. Q.E.D.

Remark 8.11. Let $J \subset E$ be a graded ideal. Using the exterior version of Proposition 1.4 one easily shows that $\dim_K H^i(E/J) = \dim_K H^i(E/\operatorname{Gin}(J))$ for all i, where $\operatorname{Gin}(J)$ is the generic initial ideal of J with respect to the reverse lexicographic order, and where $H^\bullet(M)$ denotes generalized cohomology of a graded E-module, as defined in Section 5. Note that this observation yields another proof of Proposition 8.10.

The *Alexander dual* of the simplicial complex Δ (on the vertex set $[n]$) is the simplicial complex

$$\Delta^* = \{\sigma \subset [n] \colon [n] \setminus \sigma \notin \Delta\}.$$

We shall need the following result ([18])

Theorem 8.12 (Eagon-Reiner). *Let Δ be a simplicial complex. Then the following conditions are equivalent:*

(a) *I_Δ has a linear resolution;*
(b) *the dual simplicial complex Δ^* is Cohen-Macaulay over K.*

Theorem 8.13 (Kalai). *The following conditions are equivalent:*

(a) *Δ is Cohen-Macaulay over K;*
(b) *Δ^e is Cohen-Macaulay over K;*
(c) *Δ^e is pure.*

Proof. We observe the simple fact (see for example [22, Lemma 1.1]) that $(\Delta^*)^e = (\Delta^e)^*$.

(a) $\Longleftrightarrow$ (b): By Theorem 8.12, Δ is Cohen-Macaulay over K if and only if I_{Δ^*} has linear resolution. Since, by Theorem 7.1, the regularity of $K[\Delta^*]$ and $K[(\Delta^*)^e]$ is the same, it follows that $K[\Delta^*]$ has a linear resolution if and only if $K[(\Delta^*)^e] = K[(\Delta^e)^*]$ has a linear resolution.

This in turn, again by Theorem 8.12, is the case if and only if Δ^e is Cohen-Macaulay over K.

The implication (b) $\Rightarrow$ (c) is true for any simplicial complex.

(c) $\Rightarrow$ (b): Since the maximal faces of Δ^e correspond to the minimal non-faces of $(\Delta^e)^*$, the purity of Δ^e implies that the minimal generators of the defining ideal of $(\Delta^e)^*$ all have the same degree. As $(\Delta^e)^* = (\Delta^*)^e$, we see that $(\Delta^e)^*$ is shifted, and hence its defining ideal is strongly stable. The resolution of a strongly stable ideal which is generated in one degree is linear, as follows from Corollary 3.4. This concludes the proof. Q.E.D.

Theorem 7.1 which says that extremal Betti numbers are preserved under exterior algebraic shifting can be translated into a theorem about the behaviour of links under shifting. Recall that the *link* of a face $\sigma \in \Delta$ is the simplicial complex

$$\mathrm{lk}_\Delta(\sigma) = \{\tau \in \Delta \colon \tau \cap \sigma = \emptyset \quad \text{and} \quad \tau \cup \sigma \in \Delta\}.$$

For the translation one needs the formula (see [18])

$$\tilde{H}_{i-2}(\mathrm{lk}\,\Delta^*(\sigma); K) \cong \tilde{H}^{|T|-i-1}(\Delta_T; K) \quad \text{for all } i \text{ and } \sigma \in \Delta^*,$$

where $T = [n] \setminus \sigma$. This is a slight generalization of the so-called Alexander duality formula:

$$\tilde{H}_{i-2}(\Delta^*; K) \cong \tilde{H}^{n-i-1}(\Delta; K) \quad \text{for all } i.$$

The numbers $\tilde{\beta}_i(\Gamma) = \dim_K \tilde{H}_i(\Gamma; K)$ are called the *reduced Betti numbers* (*with values in* K) of the simplicial complex Γ. Now we get

Theorem 8.14. *Let i and j be non-negative integers. Suppose $\tilde{\beta}_l(\mathrm{lk}_\Delta(\sigma)) = 0$ for all faces σ with $|\sigma| < j$, and all l with $i \leq l \leq i+t$, where $t = j - |\sigma|$. Then $\sum_{\sigma, |\sigma|=j} \tilde{\beta}_i(\mathrm{lk}_\Delta \sigma)$ is preserved under exterior algebraic shifting.*

Symmetric algebraic shifting: Let $I \subset S = K[x_1, \dots, x_n]$ be a squarefree ideal, where K is field of characteristic 0. We let $\mathrm{Gin}(I)$ be the generic initial ideal of I with respect to the reverse lexicographic term order. We know from Proposition 2.2 that $\mathrm{Gin}(I)$ is a strongly stable ideal in S. But of course it is no longer squarefree.

We will transform $\mathrm{Gin}(I)$ into a squarefree monomial ideal by applying a certain operator: for a monomial $u \in S$, $u = x_{i_1}x_{i_2}\cdots x_{i_j}\cdots x_{i_d}$ with $i_1 \leq i_2 \leq \cdots \leq i_j \leq \cdots \leq i_d$, we set

$$u^\sigma = x_{i_1}x_{i_2+1}\cdots x_{i_j+(j-1)}\cdots x_{i_d+(d-1)}.$$

It then follows immediately

$$m(u^\sigma) - \deg u^\sigma = m(u) - 1. \tag{11}$$

If L is a monomial ideal with $G(L) = \{u_1, \dots, u_s\}$, then we write L^σ for the squarefree monomial ideal generated by $u_1^\sigma, \dots, u_s^\sigma$ in $K[x_1, \dots, x_m]$, where $m = \max\{m(u) + \deg u - 1 \colon u \in G(L)\}$.

Let Δ be a simplicial complex on the vertex set $[n]$. The *symmetric algebraic shifted complex* of Δ is defined to be the simplicial complex Δ^s with

$$I_{\Delta^s} = (\mathrm{Gin}(I_\Delta))^\sigma.$$

The definition of symmetric algebraic shifting presented here is formally different from that of Kalai [27]. However it is an easy exercise to see that both notions coincide.

A priori it is not clear from the definition of symmetric algebraic shifting that Δ^s has the same vertex set $[n]$. The next lemma shows that this indeed is the case.

Lemma 8.15. *If I is a squarefree monomial ideal of $S = K[x_1, \dots, x_n]$, then $m(u) + \deg u \leq n + 1$ for all $u \in G(\mathrm{Gin}(I))$.*

Proof. Recall from Examples 2.4 that the graded Betti numbers of a strongly stable ideal I are given by Eliahou–Kervaire :

$$\beta_{i,i+j}(I) = \textstyle\sum_{u \in G(I)_j} \binom{m(u)-1}{i} \tag{12}$$

for all i and j.

Since $\mathrm{Gin}(I)$ is strongly stable, formula (12) implies that $\max\{m(u)+\deg u - 1 \colon u \in G(\mathrm{Gin}(I))\}$ is the highest shift in the resolution of $\mathrm{Gin}(I)$. The monomial ideal I being squarefree, Hochster's formula (see Theorem 5.3) guarantees that the highest shift in the resolution of I is less than or equal to n. Since the Betti numbers with highest shift in the resolution of I are extremal it follows from Theorem 6.3 that the highest shift in the resolution of I and that of $\mathrm{Gin}(I)$ coincide (see also [22]). This yields the desired inequalities. Q.E.D.

We want to point out that we defined symmetric algebraic shifting only in a polynomial rings whose base field is of characteristic 0, because otherwise we do not know if $\mathrm{Gin}(I)$ is strongly stable. It may be possible that symmetric algebraic shifting can be defined in any characteristic, provided the following question can be answered affirmatively.

Problem 8.16. Let $I \subset S = K[x_1, \dots, x_n]$ be a squarefree monomial ideal. Is it true that $\mathrm{Gin}(I)$ is strongly stable in any characteristic?

More generally one may even ask whether $\mathrm{Gin}(I)$ of a monomial ideal is strongly stable, if the characteristic of the field is larger than all exponents appearing in the monomial generators of I.

On the other hand, if I is squarefree and $\mathrm{char}\, K > n$, then $\mathrm{Gin}(I)$ is strongly stable. In fact, the highest degree of a generator of $\mathrm{Gin}(I)$ is less than or equal to $\mathrm{reg}(\mathrm{Gin}(I))$, and $\mathrm{reg}\,\mathrm{Gin}(I) = \mathrm{reg}\, I$ by Corollary 6.4. Since $\mathrm{reg}(I) \leq n$ by Theorem 5.3, we conclude that the highest degree of a generator of $\mathrm{Gin}(I)$ is $\leq n$. Hence the assertion follows from Proposition 2.2.

Note that condition (S_1) is satisfied since we have

Lemma 8.17. *Let I be a strongly stable ideal with $G(I) = \{u_1, \ldots, u_s\}$. Then the squarefree monomial ideal I^σ is squarefree strongly stable with $G(I^\sigma) = \{u_1^\sigma, \ldots, u_s^\sigma\}$.*

Proof. Suppose that, for some $u \in G(I)$, we have $u^\sigma \notin G(I^\sigma)$. Let $u = x_{i_1} \cdots x_{i_d}$ with $i_1 \leq \cdots \leq i_d$. Then, for some proper subset N of $\{1, 2, \ldots, d\}$ and for some $1 \leq q \leq s$, we have $u_q^\sigma = \prod_{j \in N} x_{i_j + (j-1)}$. Hence $u_q = \prod_{j \in N} x_{i_j + h_j}$, where h_j is the number of integers $1 \leq k < j$ with $k \notin N$. Since I is strongly stable, $\prod_{j \in N} x_{i_j}$ must belong to I. This contradicts $u \in G(I)$. Thus we have $G(I^\sigma) = \{u_1^\sigma, \ldots, u_s^\sigma\}$.

Next, to see why I^σ is squarefree strongly stable, let $u = x_{i_1} \cdots x_{i_d} \in G(I)$ and consider the monomial $(x_b u^\sigma)/x_{i_a + (a-1)}$ with $b \notin \mathrm{supp}(u^\sigma)$ and $b < i_a + (a-1)$. Let $i_p + (p-1) < b < i_{p+1} + p$ for some $p < a$ and set

$$v = (\prod_{j=1}^{p} x_{i_j}) x_{b-p} (\prod_{j=p+1}^{a-1} x_{i_j - 1})(\prod_{j=a+1}^{d} x_{i_j}).$$

Then, since $b - p < i_{p+1} \leq i_a$ and since I is strongly stable, the monomial v belongs to I. Note that $v^\sigma = (x_b u^\sigma)/x_{i_a + (a-1)}$. Say, $v = x_{\ell_1} \cdots x_{\ell_d}$ with $\ell_1 \leq \cdots \leq \ell_d$. Again, since I is strongly stable, it follows that $w = x_{\ell_1} \cdots x_{\ell_c} \in G(I)$ for some $c \leq d$. Since w^σ divides v^σ, we have $(x_b u^\sigma)/x_{i_a + (a-1)} \in I^\sigma$, as desired. Q.E.D.

The operator $I \mapsto I^\sigma$ behaves well with respect to graded Betti numbers.

Lemma 8.18. *If I is a strongly stable monomial ideal, then $\beta_{ii+j}(I) = \beta_{ii+j}(I^\sigma)$ for all i and j.*

Proof. The result follows from (11), Corollary 3.4(a) and Corollary 3.6(a). Q.E.D.

Next we indicate the proof of condition (S_2) for symmetric algebraic shifting.

Theorem 8.19. *Let $I \subset S$ be a squarefree strongly stable ideal of S. Then $I^s = I$.*

For the proof we introduce the operation τ which is inverse to σ: For a squarefree monomial $u = x_{i_1}x_{i_2}\cdots x_{i_j}\cdots x_{i_d}$ with $i_1 < i_2 < \cdots < i_j < \cdots < i_d$, we set

$$u^\tau = x_{i_1}x_{i_2-1}\cdots x_{i_j-(j-1)}\cdots x_{i_d-(d-1)}.$$

If $I \subset S$ is a squarefree monomial ideal with $G(I) = \{u_1, \dots, u_s\}$, then we write I^τ for the monomial ideal generated by $u_1^\tau, \dots, u_s^\tau$ in S.

Similarly to Lemma 8.17, we show:

Lemma 8.20. *Let I be a squarefree strongly stable ideal with $G(I) = \{u_1, \dots, u_s\}$. Then the ideal I^τ is strongly stable with $G(I^\tau) = \{u_1^\tau, \dots, u_s^\tau\}$.*

Proof. Assume that for some $u \in G(I)$, we have $u^\tau \notin G(I^\tau)$. Let $u = x_{i_1}\cdots x_{i_d}$ with $i_1 < \cdots < i_d$. Then for some proper subset $\{j_1, \dots, j_t\}$ of $\{1, 2, \dots, d\}$, where $j_1 < \cdots < j_t$, and for some $1 \leq q \leq s$, we have $u_q^\tau = \prod_{k=1}^t x_{i_{j_k}-(j_k-1)}$. Hence $u_q = \prod_{k=1}^t x_{i_{j_k}-(j_k-k)}$. Since $i_k \leq i_{j_k} - (j_k - k)$ for $1 \leq k \leq t$ and I is squarefree strongly stable, we get $x_{i_1}\cdots x_{i_t} \in I$ which contradicts $u \in G(I)$.

Now, we show that I^τ is strongly stable. Let $u = x_{i_1}\cdots x_{i_d} \in G(I)$ with $i_1 < \cdots < i_d$, and consider the monomial $v = (x_b u^\tau)/x_{i_k-(k-1)}$ with $b < i_k - (k-1)$. Let $i_p - (p-1) \leq b < i_{p+1} - p$ for some $p < k$. Then

$$v^\sigma = (\prod_{j=1}^p x_{i_j})x_{b+p}(\prod_{j=p+1}^{k-1} x_{i_j+1})(\prod_{j=k+1}^d x_{i_j}).$$

Since $b + p < i_{p+1}$ and $i_j + 1 \leq i_{j+1}$ for $p+1 \leq j \leq k-1$, and since I is squarefree strongly stable, we obtain that $v^\sigma \in I$. Say, $v^\sigma = x_{\ell_1}\cdots x_{\ell_d}$ with $\ell_1 < \cdots < \ell_d$. Again, since I is squarefree strongly stable, it follows that $w = x_{\ell_1}\cdots x_{\ell_c} \in G(I)$ for some $c \leq d$. Since w^τ divides $(v^\sigma)^\tau = v$, we have $v \in I^\tau$. Q.E.D.

The proof of Theorem 8.19 is based on the following lemma. We refer the reader to the original paper [7] for the somewhat tedious proof of the lemma.

Lemma 8.21. *Let $I \subset S$ be a squarefree strongly stable ideal generated in degree d. Let $G(I) = \{u_1, \dots, u_s\}$ where $u_1 > u_2 > \dots > u_s$.*

Let $g = (a_{ij})_{1\le i,j\le n}$ be a generic upper triangular matrix acting on S by $g(x_i) = \sum_{j=1}^{i} a_{ji}x_j$ for $1 \le i \le n$. Let c_{kj} denote the coefficient of u_j^τ in the polynomial $g(u_k)$ for $1 \le k, j \le s$. Then the determinant of the matrix $(c_{kj})_{1\le k,j\le s}$ is different from zero.

Proof of Theorem 8.19. Since the ideal I is squarefree strongly stable, I is componentwise linear [5]. Therefore by [8, Theorem 1.1], for the graded Betti numbers of I and $\mathrm{Gin}(I)$ it holds: $\beta_{ii+j}(I) = \beta_{ii+j}(\mathrm{Gin}(I))$ for all i and j. On the other hand, the ideal $\mathrm{Gin}(I)$ being strongly stable, it follows from Lemma 8.18 that $\beta_{ii+j}(\mathrm{Gin}(I)) = \beta_{ii+j}((\mathrm{Gin}(I))^\sigma)$ Thus, we obtain the equalities

$$(13) \qquad \beta_{i,i+j}(I) = \beta_{i,i+j}((\mathrm{Gin}(I))^\sigma) \quad \text{for all} \quad i, j,$$

which imply that I and $(\mathrm{Gin}(I))^\sigma$ have the same Hilbert function. Hence it is enough to prove that $I \subseteq (\mathrm{Gin}(I))^\sigma$. By Lemma 8.17 and Lemma 8.20 this inclusion is equivalent to $I^\tau \subseteq \mathrm{Gin}(I)$. So, we will show that $u^\tau \in \mathrm{Gin}(I)$ for every $u \in G(I)$.

We denote by $\langle u\rangle$ the smallest squarefree strongly stable ideal containing u. Since $I = \sum_{u\in G(I)}\langle u\rangle$, and $\mathrm{Gin}(\langle u\rangle) \subseteq \mathrm{Gin}(I)$ for every $u \in G(I)$, it is enough to show that the claim is true for squarefree Borel principal ideals. So, we may assume that $I = \langle u\rangle$. Set $d = \deg u$.

Let $G(I) = \{u_1, \ldots , u_s\}$ where $u_1 > u_2 > \cdots > u_s$. Then $u_s = u$. We may assume that the claim is true for all u_k, $1 \le k \le s-1$. Then $(u_1^\tau, u_2^\tau, \ldots , u_{s-1}^\tau) \subset \mathrm{Gin}(I)$, and since I^τ and $\mathrm{Gin}(I)$ have the same number of minimal monomial generators, one has $G(\mathrm{Gin}(I)) = \{u_1^\tau, u_2^\tau, \ldots , u_{s-1}^\tau, v\}$, where v is a monomial of degree d. We have to prove that $v = u^\tau$.

Assume $v > u^\tau$. We will see that this is impossible. First, we show that $m(v) = m(u^\tau)$. It follows from formula Corollary 3.6(a) that

$$\beta_{ii+d}((\mathrm{Gin}(I))^\sigma) = \sum_{j=1}^{s-1}\binom{m(u_j)-d}{i} + \binom{m(v^\sigma)-d}{i};$$

$$\beta_{ii+d}(I) = \sum_{j=1}^{s-1}\binom{m(u_j)-d}{i} + \binom{m(u)-d}{i}.$$

Therefore, according to (13), we obtain $\binom{m(v^\sigma)-d}{i} = \binom{m(u)-d}{i}$ which implies $m(v^\sigma) = m(u)$, so that $m(v) = m(u^\tau)$.

We fix the following notation: $u = x_{s_1}\cdots x_{s_d}$ where $s_1 < \cdots < s_d$, and $v = x_{j_1}\cdots x_{j_d}$ where $j_1 \le \cdots \le j_d$. Since $v > u^\tau$, there exits a k such that $j_i = s_i - (i-1)$ for $k+1 \le i \le d$ and $j_k < s_k - (k-1)$. As $j_d = m(v) = m(u^\tau) = s_d - (d-1)$, one has $k < d$. If $j_i + (i-1) \le s_i$

for $1 \leq i \leq k$, then $I = \langle u \rangle$ being squarefree strongly stable, one obtains that $v^\sigma \in I$ which implies $v^\sigma = u_t$ for some $1 \leq t \leq s-1$ and the contradiction $v = u_t^\tau$. Thus, there exits an ℓ, $1 \leq \ell < k$, such that $j_\ell + (\ell - 1) > s_\ell$. Then $j_\ell \leq j_k < s_k - (k-1) \leq s_d - (d-1) = m(v)$, therefore $x_{j_\ell} v / x_{m(v)} \in \mathrm{Gin}(I)$, because $\mathrm{Gin}(I)$ is strongly stable. Since $x_{j_\ell} v / x_{m(v)} > v$, we get $x_{j_\ell} v / x_{m(v)} = u_t^\tau$ for some $1 \leq t \leq s-1$. Say $u_t = x_{t_1} \cdots x_{t_d}$ where $t_1 < \cdots < t_d$. As $I = \langle u \rangle$ is a squarefree Borel principal ideal, we have $t_i \leq s_i$ for $1 \leq i \leq d$, therefore $t_i - (i-1) \leq s_i - (i-1)$ for $1 \leq i \leq d$. This contradicts $j_\ell > s_\ell - (\ell - 1)$.

Hence, $v \leq u^\tau$. Now, we apply Lemma 8.21 using same notation. We have $\mathrm{Gin}(I) = \mathrm{in}(g(I))$ and $u_j^\tau \in \mathrm{Gin}(I)$ for $1 \leq j \leq s-1$. Since the rank of the matrix $(c_{kj})_{1 \leq k,j \leq s}$ is maximal, it follows that $v \geq u^\tau$, and so $v = u^\tau$. Q.E.D.

For symmetric algebraic shifting we can prove the inequality of graded Betti numbers which we conjecture for exterior algebraic shifting.

Theorem 8.22. *Let Δ be a simplicial complex. Then*

$$\beta_{ii+j}(I_\Delta) \leq \beta_{ii+j}(I_{\Delta^s}) \quad \text{for all} \quad i \quad \text{and} \quad j.$$

Proof. The desired inequalities follow from Theorem 3.1 and Lemma 8.18. Q.E.D.

Theorem 8.22 leads us to conjecture the following inequalities:

Conjecture 8.23. *Let Δ be a simplicial complex. Then for all i and j one has*

$$\beta_{ii+j}(I_{\Delta^s}) \leq \beta_{ii+j}(I_{\Delta^e}) \leq \beta_{ii+j}(I_{\Delta^c}).$$

In virtue of Theorem 8.22 the conjecture implies the yet open inequalities

$$\beta_{ii+j}(I_\Delta) \leq \beta_{ii+j}(I_{\Delta^e})$$

for all i and j. One should expect that there is direct proof of this inequality, avoiding a comparison with the symmmetric shifted ideal. The next result shows that the extremal Betti numbers of the symmetric algebraic shifted ideals behave as expected.

Theorem 8.24. *Let Δ be a simplicial complex. Then for all i and j*

(a) *the following conditions are equivalent:*
  (i) *the ijth Betti number of I_Δ is extremal,*
  (ii) *the ijth Betti number of I_{Δ^s} is extremal.*

(b) *the corresponding extremal Betti numbers of* I_Δ *and* I_{Δ^s} *are equal.*

Proof. The corresponding statements for I and $\mathrm{Gin}^S(I)$ are proved in Theorem 6.3. Hence, since $\beta_{ij}(\mathrm{Gin}^S(I_\Delta)) = \beta_{ij}(I_{\Delta^s})$ by Lemma 8.18, we obtain the assertions for I_Δ and I_{Δ^s}, too. Q.E.D.

The invariance of the extremal Betti numbers for combinatorial shifting is unknown. To prove it, it would suffice to show that I_Δ and $I_{\mathrm{Shift}_{ij}(\Delta)}$ have the same extremal Betti numbers.

As in the case of exterior algebraic shifting we get from Theorem 8.24

Corollary 8.25. *Let K be a field of characteristic* 0. *Then*

$$\tilde{H}_i(\Delta; K) \cong \tilde{H}_i(\Delta^s; K) \quad \textit{for all} \quad i.$$

The usefulness of Proposition 8.10 and Corollary 8.25 is partially explained by the fact that $\tilde{H}_\bullet(\Delta^e; K)$ and $\tilde{H}_\bullet(\Delta^s; K)$ can be computed combinatorially in a simple way. In fact, as noted in [27] (in a different terminology), one has

Lemma 8.26. *Let Δ be a simplicial complex on the vertex set $[n]$ such that I_Δ is squarefree strongly stable. Then*

$$\begin{aligned} \dim_K \tilde{H}_i(\Delta; K) &= |\{u \in G(I_\Delta)_{i+2} \colon m(u) = n\}| \\ &= |\{\sigma \in \Delta \colon \dim \sigma = i, \quad \sigma \cup \{n\} \notin \Delta\}|. \end{aligned}$$

Proof. The first equation follows from (10) and Corollary 3.6, while the second equation follows trivially from the definitions. Q.E.D.

§9. Superextremal Betti numbers

As an application of Theorem 8.24 we prove a non-squarefree version of a theorem of Björner and Kalai [15]. We first give a more algebraic proof of their theorem, which applies to any graded ideal in the exterior algebra, and not just to monomial ideals, but nevertheless follows closely the arguments of the original proof of Björner and Kalai.

So let $J \subset E$ be a graded ideal. We set $f_{i-1} = \dim_K(E/J)_i$ for all $i \geq 0$, and call $f = (f_0, f_1, \dots)$ the f-vector of E/J. As in Section we denote by $H^i(E/J)$ the generalized simplicial cohomology of E/J. We let $\beta_{i-1} = \dim_K H^i((E/J))$, and call $\beta = (\beta_{-1}, \beta_0, \beta_1 \dots)$ the Betti sequence of E/J. Jn case $J = J_\Delta$ for some simplicial complex Δ, the β_i are the ordinary Betti numbers of Δ.

A pair of sequences $(f, \beta) \in \mathbb{N}_0^\infty$ is called *compatible* if there exists a graded K-algebra E/J such that f is the f-sequence and β the Betti sequence of E/J.

Theorem 9.1 (Björner and Kalai). *Let K be a field. The following conditions are equivalent:*

(a) *The pair of sequences (f,β) is compatible.*

(b) *Set $\chi_i = (-1)^i \sum_{j=-1}^{i}(-1)^j(f_j-\beta_j)$ for all i. Then*

  (i) $\chi_{-1}=1$ *and* $\chi_i \geq 0$ *for all* i,

  (ii) $\beta_i \leq \chi_{i-1}^{(i)} - \chi_i$ *for all* i.

Proof. (a) $\Rightarrow$ (b): The f-vectors of E/J and $E/\operatorname{Gin}(J)$ coincide, since they have the same Hilbert function. By Remark 8.11 we have $H^i(E/J) \cong H^i(E/\operatorname{Gin}(J))$ for all i. Hence also the Betti sequences of E/J and $E/\operatorname{Gin}(J)$ coincide. Thus we may replace J by $\operatorname{Gin}(J)$, and hence may as well assume that J is strongly stable.

Let J' be the ideal generated by all $u \in G(J)$ with $m(u) < n$ and all monomials $u \in E$ such that $u \wedge e_n \in G(J)$. Then J' is again strongly stable and $E_1 J' \subset J$. By Lemma 8.26, the last property implies that

$$\dim_K(J'/J)_i = |\{u \in G(J)_{i+1}\colon m(u)=n\}| = \beta_{i-1}(E/J).$$

It follows that $\dim_K(E/J')_i = f_{i-1} - \beta_{i-1}$ for all i. Now we notice that e_n is regular on E/J', in the sense that the complex

$$E/J' \xrightarrow{e_n} E/J' \xrightarrow{e_n} E/J'$$

is exact. Therefore, for each i we obtain an exact sequence of K-vector spaces

$$(14) \quad \to (E/J')_{i-1} \to (E/J')_i \to (E/J')_{i+1} \to (E/(J'+e_nE))_{i+1} \to 0,$$

and hence $\chi_i = \dim_K(E/(J'+e_nE))_{i+1}$.

Next we observe that $J'/J \cong (J'+e_nE)/(J+e_nE)$ and $E_1(J'+e_nE) \subset J+e_nE$, so that together with the Kruskal-Katona theorem (cf. Section 10) we obtain

$$\begin{aligned}\chi_i+\beta_i &= \dim_K E_{i+1} - \dim_K(J+e_nE)_{i+1}\\ &\leq \dim_K E_{i+1} - \dim_K E_1(J'+e_nE)_i \leq \chi_{i-1}^{(i)},\end{aligned}$$

as required.

(b) $\Rightarrow$ (a): The hypotheses imply that $\chi_i \leq \chi_{i-1}^{(i)}$ and $\chi_i + \beta_i \leq (\chi_{i-1}+\beta_{i-1})^{(i)}$. Thus the Kruskal-Katona theorem yields an integer m, and lexsegment ideals $L \subset N$ in the exterior algebra $E' = K\langle e_1,\dots,e_{m-1}\rangle$ such that $\dim_K(E/N)_{i+1} = \chi_i$ and $\dim_K(E/L)_{i+1} = \chi_i + \beta_i$ that for all i.

Now let $J \subset E = K\langle e_1,\dots,e_m\rangle$ be the ideal generated by the elements in $G(L)$ and all elements $u \wedge e_m$ with $u \in G(N)$. Moreover we set $J' = NE$. Then $J'/J \cong N/L$, and so

(15) $\dim_K(E/J)_{i+1} = \dim_K(N/L)_{i+1} + \dim_K(E/J')_{i+1}$
$= \beta_i + \dim_K(E/J')_{i+1}$.

On the other hand, e_m is regular on E/J', and so (14) yields

(16) $\dim_K(E/(J' + e_mE))_{i+1} = (-1)^{i+1} \sum_{j=0}^{i+1}(-1)^j \dim_K(E/J')_j$

for all i. Thus, since $E/(J' + e_mE) \cong E'/N$, it follows from (16) that

$\dim_K(E/J')_{i+1} = \dim_K(E'/N)_{i+1} + \dim_K(E'/N)_i = \chi_i + \chi_{i-1}$
$= f_i - \beta_i$.

This together with (15) implies that $\dim_K(E/J)_{i+1} = f_i$.

Finally it is clear from the construction of J that $|\{u \in G(J)_{i+2}\colon m(u) = m\}|$ equals $\dim_K(N/L)_{i+1}$ which is β_i. Thus, by Lemma 8.26, the assertion follows. Q.E.D.

The Björner-Kalai Theorem can be translated into a theorem on super extremal Betti numbers. Let $I \subset S$ be a graded ideal. We let m be the maximal integer j such that $\beta_{ij}(S/I) \neq 0$ for some i. In other words, m is the largest shift in the graded minimal free S-resolution of S/I. It is clear that $\beta_{im}(S/I)$ is an extremal Betti number for all i with $\beta_{im}(S/I) \neq 0$, and that there is at least one such i. These Betti numbers are distinguished by the fact that they are positioned on the diagonal $\{(i, m-i)\colon i = 0, \dots, m\}$ on the Betti diagram, and that all Betti numbers on the right lower side of the diagonal are zero. The ring S/I may of course have other extremal Betti numbers, not sitting on this diagonal. We call the Betti numbers β_{im}, $i = 0, \dots, m$, *super extremal*, regardless whether they are zero or not, and ask the question which sequences of numbers $(b_0, b_1, \dots, b_m)$ appear as sequences of super extremal Betti numbers for graded rings with given Hilbert function.

Before answering the question we have to encode the Hilbert function $H_{S/I}(t)$ of S/I in a suitable way. Using the additivity of the Hilbert function, the graded minimal free resolution of S/I yields the following formula:

$$H_{S/I}(t) = \frac{a_0 + a_1t + a_2t^2 + \cdots + a_mt^m}{(1-t)^n}$$

with $a_i \in \mathbb{Z}$; see for example [14]. It follows that

$$(1-t)^{n-m}H_{S/I}(t) = \frac{a_0 + a_1t + a_2t^2 + \cdots + a_mt^m}{(1-t)^m}.$$

Notice that $n-m$ may take positive or negative values. At any rate, the rational function $(1-t)^{n-m}H_{S/I}(t)$ has degree ≤ 0. One easily verifies that there is a unique expansion

$$(1-t)^{n-m}H_{S/I}(t)=\sum_{i=0}^{m} f_{i-1}\frac{t^i}{(1-t)^i}$$

with $f_i \in \mathbb{Z}$. It is clear that $f_{-1}=1$, and we shall see later that all $f_i \geq 0$. We call $f=(f_{-1},f_0,f_1,\ldots,f_{m-1})$ the *f-vector of S/I*. Given the highest shift in the resolution, the f-vector of S/I determines the Hilbert function of S/I, and vice versa.

We set $b_i=\beta_{m-i-1,m}$ and call $b=(b_{-1},\ldots,b_{m-1})$ the *super extremal sequence of S/I*. Finally we set $\chi_i=(-1)^i\sum_{j=-1}^{i}(-1)^j(f_j-b_j)$ for $i=-1,0.\ldots,m-1$. The Björner-Kalai theorem has the following counterpart.

Theorem 9.2. *Let K be a field of characteristic* 0. *Let $f=(f_{-1},f_0,\ldots,f_{m-1})$ and $b=(b_{-1},b_0,\ldots,b_{m-1})$ be sequences of nonnegative integers. The following conditions are equivalent:*

(a) *there exists a homogeneous K-algebra S/I such that f is the f-vector, and b the super extremal sequence of S/I;*

(b) (i) *$\chi_{-1}=1$ and $\chi_i\geq 0$ for all i,*
(ii) *$b_i\leq \chi_{i-1}^{(i)}-\chi_i$ for all i.*

Proof. (a) $\Rightarrow$ (b) Since the extremal Betti numbers are preserved when we pass from I to $\operatorname{Gin}^S(I)$, it follows that I and $\operatorname{Gin}^S(I)$ have the same highest shift m, and hence the same b-vector. Since S/I and $S/\operatorname{Gin}^S(I)$ have the same Hilbert function, it also follows that the f-vectors of S/I and $S/\operatorname{Gin}^S(I)$ coincide. Thus, since $\operatorname{char}(K)=0$, we may assume that I is a strongly stable monomial ideal.

The ideal I^σ is defined in $S'=K[x_1,\ldots,x_m]$ and $\beta_{ii+j}(I)=\beta_{ii+j}(I^\sigma)$ by 8.18. This implies that

$$H_{S'/I^\sigma}(t)=(1-t)^{n-m}H_{S/I}(t).$$

Hence, if we let Δ be the simplicial complex with $I_\Delta=I^\sigma$, then Δ and S/I have the same f-vector, and one has $b_i=\dim_K\tilde{H}_i(\Delta;K)$; see (5.3). Therefore, the conclusion follows from Björner-Kalai Theorem.

(b) $\Rightarrow$ (a): Given an f- and b-sequence satisfying conditions (b), there exists by 9.1 an integer m and a simplicial complex Δ on the vertex set $[m]$ whose f-vector is f and whose β-sequence is b. Then $K[x_1,\ldots,x_m]/I_\Delta$ is a homogeneous K-algebra satisfying (a). Q.E.D.

§10. Extremality properties of Lexsegment ideals

Let $I \subset S$ be a graded ideal. Then $\beta_{ij}(I) \leq \beta_{ij}(\mathrm{Gin}(I))$ for all i and j, by Theorem 3.1. Moreover it follows from Theorem 2.7 that $\beta_{0j}(\mathrm{Gin}(I)) \leq \beta_{0j}(\mathrm{Gin}(I)^{lex})$ for all j. Since $I^{lex} = \mathrm{Gin}(I)^{lex}$ we conclude that

$$\beta_{0j}(I) \leq \beta_{0j}(I^{lex}) \quad \text{for all } j.$$

Similar reasonings show that for all graded ideals $J \subset E$ one has

$$\beta_{0j}(J) \leq \beta_{0j}(J^{lex}) \quad \text{for all } j.$$

The question is whether such inequalities are valid also for the higher graded Betti numbers. In case of the polynomial ring this is known.

Theorem 10.1 (Bigatti, Hulett, Pardue). *Let $I \subset S$ be a graded ideal. Then*

$$\beta_{ij}(I) \leq \beta_{ij}(I^{lex}) \quad \textit{for all } i \textit{ and } j.$$

Bigatti [13] and Hulett [25] proved this theorem independently for base fields of characteristic 0. A proof in arbitrary characteristic was later given by Pardue [31] using some polarization trick.

In the exterior case we have (cf. [6, Theorem 4.4])

Theorem 10.2. *Let $J \subset E$ be a graded ideal. Then*

$$\beta_{ij}(J) \leq \beta_{ij}(J^{lex}) \quad \textit{for all } i \textit{ and } j.$$

Conjecture 10.3. *Let $I \subset S$ be a squarefree monomial ideal. Then*

$$\beta_{ij}(I) \leq \beta_{ij}(I^{sqlex}) \quad \textit{for all } i \textit{ and } j.$$

Theorem 10.4. *Conjecture* 10.3 *is true if* $\operatorname{char} K = 0$.

Proof. By Theorem 8.22 we have $\beta_{ij}(I) \leq \beta_{ij}(\mathrm{Gin}(I)^{\sigma})$. Now we use the result (see [5, Theorem 4.4]) that for any squarefree strongly stable ideal L one has $\beta_{ij}(L) \leq \beta_{ij}(L^{lex})$. Applying this result to $L = \mathrm{Gin}(I)^{\sigma}$ and observing that $I^{lex} = (\mathrm{Gin}(I))^{\sigma})^{lex}$, we get the desired inequalities. Q.E.D.

Theorem 10.4 was used by E. Sbarra to prove in his thesis [32] part (a) of the following theorem, while for part (b) he uses the polarization argument of Pardue. Let M be a graded S-module, and $\mathfrak{m}$ the graded maximal ideal of S. Then $H^{\bullet}_{\mathfrak{m}}(M)$ denotes the local cohomology of M. Recall that $H^{\bullet}_{\mathfrak{m}}(M)$ is naturally graded.

Theorem 10.5. *Let $S = K[x_1, \ldots, x_n]$.*

(a) *If $I \subset S$ is a squarefree monomial ideal, and* $\operatorname{char} K = 0$, *then*

$$\dim_K H^i_{\mathfrak{m}}(S/I)_j \leq \dim_K H^i_{\mathfrak{m}}(S/I^{sqlex})_j \quad \text{for all } i \text{ and } j.$$

(b) *If $I \subset S$ is a graded ideal, then*

$$\dim_K H^i_{\mathfrak{m}}(S/I)_j \leq \dim_K H^i_{\mathfrak{m}}(S/I^{lex})_j \quad \text{for all } i \text{ and } j.$$

in any characteristic.

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Algebraic Shifting

Gil Kalai

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Certain Moduli of Algebraic G-vector Bundles over Affine G-varieties

Kayo Masuda

Abstract.

Let G be a reductive complex algebraic group and P a complex G-module with algebraic quotient of dimension ≥ 1. We construct a map from a certain moduli space of algebraic G-vector bundles over P to a $\mathbb{C}$-module possibly of infinite dimension, which is an isomorphism under some conditions. We also show non-triviality of moduli of algebraic G-vector bundles over a G-stable affine hypersurface of some type. In particular, we show that the moduli space of algebraic G-vector bundles over a G-stable affine quadric with fixpoints and one-dimensional quotient contains $\mathbb{C}^p$.

§ Introduction and results

Let G be a reductive algebraic group defined over the ground field $\mathbb{C}$ of complex numbers. One of the most important problems in the theory of algebraic group action is to understand algebraic G-actions on affine space $\mathbb{A}^n$. The following problem is fundamental;

Linearization Problem

Is every action of G on $\mathbb{A}^n$ linearizable, i.e., conjugate to a linear action under polynomial automorphisms of $\mathbb{A}^n$?

In 1989, Schwarz [23] presented the first examples of *non-linearizable* actions on affine space. In fact, he first showed that there exist *non-trivial* algebraic G-vector bundles over G-modules, and the non-lineari zable actions appear on the total spaces of non-trivial algebraic G-vector bundles he found. An algebraic G-vector bundle E over an affine G-variety X is an algebraic vector bundle $p : E \to X$ together with a G-action on E such that p is G-equivariant and the action on the fibers is linear. By definition, every fiber over the fixpoint locus X^G is a G-module. An algebraic G-vector bundle is called trivial if it is isomorphic

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to a G-vector bundle of the form $X \times Q \to X$ for a G-module Q. When the base space is a G-module, if forgetting the G-action, the total space E is an affine space by the affirmative solution to the Serre Conjecture by Quillen [22] and Suslin [25]. So, the G-action on the total space of a non-trivial G-vector bundle over a G-module is a candidate for a non-linearizable action on affine space. In fact, there are some criteria for the G-action on E being non-linearizable ([1], [7], [18]). So far, all known examples of non-linearizable action are obtained from non-trivial algebraic G-vector bundles. For an abelian G, at this point, there are no counterexamples to the Linearization Problem; for, by Masuda-Moser-Petrie [19], every G-vector bundle over a G-module is trivial when G is abelian. The key point of their proof is to show that one can reduce triviality of a G-vector bundle over a G-module P to triviality of a vector bundle over the algebraic quotient space $P//G$ (=the spectrum of the ring of G-invariant polynomials on P). Since G is abelian, $P//G$ is a normal affine toric variety, and triviality of a vector bundle over a normal affine toric variety was obtained by Gubeladze [5]. We refer to Kraft [10] for recent topics in affine algebraic geometry and algebraic group action related to the Linearization Problem.

In this article, we study algebraic G-vector bundles over affine G-varieties X, especially in the case that X is a G-module. Throughout this article, we assume that X is irreducible and smooth and that X^G is non-empty. We denote by $\mathrm{VEC}_G(X, Q)$ the set of equivariant isomorphism classes of algebraic G-vector bundles over X such that every fiber over X^G is isomorphic to a G-module Q. The isomorphism class of a G-vector bundle $E \to X$ is denoted by $[E]$. Suppose that the base space is a G-module P. In this case, we have some information on $\mathrm{VEC}_G(P, Q)$ ([1], [2], [23], [11], [6], [18], [20]). By Bass-Haboush [1], every G-vector bundle over P is stably trivial, i.e., there exists a G-module S such that a Whitney sum $E \oplus (P \times S)$ is trivial. For an abelian G, $\mathrm{VEC}_G(P, Q)$ is trivial, i.e., a trivial set consisting of the trivial class $[P \times Q]$ by Masuda-Moser-Petrie [19]. For a non-abelian G, if the dimension of $P//G$ is at most one, $\mathrm{VEC}_G(P, Q)$ is well-understood. When $\dim P//G = 0$, $\mathrm{VEC}_G(P, Q)$ is trivial ([2], [12]). When $\dim P//G = 1$, however, $\mathrm{VEC}_G(P, Q)$ can be non-trivial. Schwarz ([23], cf. Kraft-Schwarz [11]) showed that $\mathrm{VEC}_G(P, Q)$ is isomorphic to an additive group $\mathbb{C}^p$ for a nonnegative integer p, and the non-trivial G-vector bundles found by Schwarz led to the first examples of non-linearizable actions on affine space, as is already mensioned above. The result of Schwarz extends to the case where the base space is a (not necessarily irreducible) G-stable affine cone X with one-dimensional quotient, namely, it holds that $\mathrm{VEC}_G(X, Q) \cong \mathbb{C}^p$ for some p ([21], [15]). However, when $\dim P//G \geq 2$, $\mathrm{VEC}_G(P,Q)$ is not finite-dimensional any more. In fact, $\mathrm{VEC}_G(P \oplus \mathbb{C}^m, Q)$ for a G-module P with one-dimensional quotient and a trivial G-module $\mathbb{C}^m$ is isomorphic to the p times direct product of a polynomial ring $\mathbb{C}[y_1, \cdots, y_m]$ where p is a nonnegative integer such that $\mathrm{VEC}_G(P,Q) \cong \mathbb{C}^p$ [16]. Furthermore, Mederer [21] presented examples of $\mathrm{VEC}_G(P,Q)$ which contains an uncountably-infinite dimensional space for a finite group G. Using Mederer's result, it is shown that $\mathrm{VEC}_G(P,Q)$ can contain an uncountably-infinite dimensional space also for a connected group G [17]. However, $\mathrm{VEC}_G(X,Q)$ is not yet classified even when X is a G-module P with $\dim P//G \geq 2$ except some special cases ([6], cf. [20]) and the cases mensioned above.

We denote by $\mathcal{O}(X)$ the $\mathbb{C}$-algebra of regular functions on X and by $\mathcal{O}(X)^G$ the subalgebra of G-invariants of $\mathcal{O}(X)$. By the finiteness theorem of Hilbert, $\mathcal{O}(X)^G$ is finitely generated and the algebraic quotient space $X//G$ is defined to be $\operatorname{Spec} \mathcal{O}(X)^G$. Let $\pi_X : X \to X//G$ be the algebraic quotient map, that is, the morphism induced by the inclusion $\mathcal{O}(X)^G \hookrightarrow \mathcal{O}(X)$. Since X is irreducible, $X//G$ is an irreducible affine variety (cf. [8]). By Luna's slice theorem [12], there is a finite stratification of $X//G = \cup_i V_i$ into locally closed subvarieties V_i such that $\pi_X|_{\pi_X^{-1}(V_i)} : \pi_X^{-1}(V_i) \to V_i$ is a G-fiber bundle (in the étale topology) and the isotropy groups of closed orbits in $\pi_X^{-1}(V_i)$ are all conjugate to a fixed reductive subgroup H_i. The unique open dense stratum of $X//G$, which we denote by U, is called the principal stratum and the corresponding isotropy group, which we denote by H, is called a principal isotropy group. The principal isotropy group H is the minimal group among H_i up to conjugation. Suppose that $\dim X//G \geq 1$. We denote by $\mathrm{VEC}_G(X,Q)_0$ the subset of $\mathrm{VEC}_G(X,Q)$ consisting of elements which are trivial over $\pi_X^{-1}(U)$ and $\pi_X^{-1}(V)$ with fiber Q where $V := X//G - U$. Though we do not know how to compute $\mathrm{VEC}_G(X,Q)$, it is not difficult to analyse $\mathrm{VEC}_G(X,Q)_0$ since every $[E] \in \mathrm{VEC}_G(X,Q)_0$ is determined by a transition function with respect to two trivializations of E. In the case that X is a (not necessarily irreducible) G-stable affine cone with $\dim X//G = 1$, in particular, a G-module with one-dimensional quotient, $\mathrm{VEC}_G(X \times \mathbb{A}^m, Q)$ and $\mathrm{VEC}_G(X \times \mathbb{A}^m, Q)_0$ coincide and we can compute $\mathrm{VEC}_G(X \times \mathbb{A}^m, Q)_0$ by analysing transition functions ([11], [16]). We assume that the ideal of V is principal; for, if $[E] \in \mathrm{VEC}_G(X,Q)$ is trivial over $\pi_X^{-1}(U)$ such that $\pi_X^{-1}(V)$ is of codimension ≥ 2, then E is trivial. Our first result is a classification of $\mathrm{VEC}_G(P,Q)_0$ for a G-module P with $\dim P//G \geq 2$.

Theorem 1. *Let P be a G-module such that $\dim P//G \geq 2$ and the ideal of the complement of the principal stratum in $P//G$ is principal.*

Let Q be a G-module. Then there exists a map

$$\Psi_{P,Q} : \mathrm{VEC}_G(P,Q)_0 \to C_P(Q).$$

Here $C_P(Q)$ is a $\mathbb{C}$-module possibly of infinite dimension (cf. Lemma 2.3). If Q is multiplicity free with respect to a principal isotropy group of P and if P has generically closed orbits, then $\Psi_{P,Q}$ is an isomorphism.

Here, a G-module Q is called "multiplicity free with respect to a reductive subgroup H" if every irreducible H-module appears in Q, viewed as an H-module, with multiplicity at most one, and we say "P has generically closed orbits" if every fiber of the quotient map π_P over the principal stratum consists of a closed orbit.

For any G-module P with one-dimensional quotient and any Q, $\Psi_{P\oplus\mathbb{C}^m,Q}$ in Theorem 1 is an isomorphism onto $C_{P\oplus\mathbb{C}^m}(Q) \cong (\mathbb{C}[y_1,\cdots, y_m])^p$, which coincides with the isomorphism obtained in [16].

Next, we investigate $\mathrm{VEC}_G(X,Q)_0$ for an affine quadric X. An affine quadric of dimension N is an affine hypersurface $X := \{(x_0,\cdots, x_N) \in \mathbb{A}^{N+1} \mid \sum_{i=0}^N x_i^2 = 1\}$. We suppose that G is connected and acts on an affine quadric X in such a way that the kernel of the action is finite. Suppose also that X^G is not empty and $\dim X//G = 1$. Then by Doebeli ([3], [4]), X is G-isomorphic to an affine quadric $X_P := \{(x,v) \in P \oplus \mathbb{C} \mid u(x) + v^2 = 1\}$, where P is an orthogonal G-module with $P//G \cong \mathbb{A}^1$ and $u(x) \in \mathcal{O}(P)^G$ is an invariant quadratic form generating $\mathcal{O}(P)^G$. The G-action on X_P is the one induced by the linear action on P. This time, however, the situation is rather different from that in the case of G-modules. The fixpoint locus X_P^G consists of two points $\{(O,\pm 1)\}$ where O is the origin of P, whereas the fixpoint locus of a G-module is an affine space, hence connected. Though $X_P//G$ is isomorphic to $\mathbb{A}^1 = \mathrm{Spec}\ \mathbb{C}[v]$, V of $X_P//G$ consists of two points $\{v = \pm 1\}$, hence V of $X_P//G$ is disconnected. For a G-module, V is connected since V is defined by invariant homogeneous polynomials. Thus we cannot apply methods in case of G-modules directly to a case of an affine quadric. While, note that X is viewed as a $G \times (\mathbb{Z}/2\mathbb{Z})$-variety, where $\mathbb{Z}/2\mathbb{Z}$ acts on $X \cong X_P \subset P \oplus \mathbb{C}$ via a (non-trivial) linear action on $\mathbb{C}$. Then $X/(\mathbb{Z}/2\mathbb{Z}) \cong P$ as a G-variety. It is easy to see that the quotient map $\pi_{\mathbb{Z}_2} : X \to X/(\mathbb{Z}/2\mathbb{Z}) \cong P$ induces an injection $\pi_{\mathbb{Z}_2}^* : \mathrm{VEC}_G(P,Q) \to \mathrm{VEC}_G(X,Q)$ (cf. [9]). Since $\mathrm{VEC}_G(P,Q) \cong \mathbb{C}^p$ by the result of Schwarz, $\mathrm{VEC}_G(X,Q)$ contains a space isomorphic to $\mathbb{C}^p$. We generalize this and obtain the following result.

Theorem 2. *Let P be a G-module with $\dim P//G \geq 1$. For $f \in \mathcal{O}(P)^G$ and an integer $d \geq 2$, let $X_P(f,d)$ be a G-stable hypersurface $\{(x,v) \in P \oplus \mathbb{C} \mid f(x) + v^d = 1\}$. Then, the quotient map*

$\pi_{\mathbb{Z}_d} : X_P(f,d) \to X_P(f,d)/(\mathbb{Z}/d\mathbb{Z}) \cong P$ *induces an injection for any* G*-module* Q

$$\pi^*_{\mathbb{Z}_d} : \mathrm{VEC}_G(P,Q) \to \mathrm{VEC}_G(X_P(f,d),Q).$$

Hence, if $\Psi_{P,Q}$ *in Theorem* 1 *is a surjection onto a non-trivial* $C_P(Q)$*, then* $\mathrm{VEC}_G(X_P(f,d),Q)$ *is non-trivial, too.*

This article consists of three parts. In section 1, we investigate $\mathrm{VEC}_G(X,Q)_0$ for an irreducible smooth affine G-variety X by analysing transition functions of G-vector bundles. We have in mind as an X a G-module. Our technique is based on the one established by Kraft-Schwarz [11]. Using the results obtained in section 1, we prove Theorem 1 in section 2. We compute $\mathrm{VEC}_G(P,Q)_0$ explicitly in examples. In section 3, we ingestigate $\mathrm{VEC}_G(X,Q)_0$ in the case where V is not connected, in particular, in the case where X is a G-stable affine hypersurface represented by an affine quadric with fixpoints and one-dimensional quotient.

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§1. General results

Let G be a reductive algebraic group and X an irreducible smooth affine G-variety. We assume that the dimension of $Y := X//G$ is greater than 0 and the ideal of $V = Y - U$ is principal, where U is the principal stratum of Y. Let $f \in \mathcal{O}(Y) = \mathcal{O}(X)^G$ be a generator of the ideal of V. We assume also that X^G is non-empty, connected and X^H is irreducible where H is a principal isotropy group of X. The object we have in mind as an X is a G-module. We will investigate $\mathrm{VEC}_G(X,Q)_0$ for a G-module Q.

Lemma 1.1. *Let* $[E] \in \mathrm{VEC}_G(X,Q)_0$*. Then* E *is trivial over* $X_h := \{x \in X \mid h(x) \neq 0\}$ *where* h *is an element of* $\mathcal{O}(Y)$ *such that* $h-1$ *is contained in the ideal* (f).

Proof. Since $E|_{\pi_X^{-1}(V)}$ is, by the assumption, isomorphic to a trivial bundle, it follows from the Equivariant Nakayama Lemma [2] that the trivialization $E|_{\pi_X^{-1}(V)} \to \pi_X^{-1}(V) \times Q$ extends to a trivialization over a G-stable open neighborhood $\tilde{U}$ of $\pi_X^{-1}(V)$. Let $\tilde{V}$ be the complement of $\tilde{U}$ in X. Since $\tilde{V}$ is a G-stable closed set, $\pi_X(\tilde{V})$ is closed in Y [8]. Note that $V \cap \pi_X(\tilde{V}) = \emptyset$ since $\pi_X^{-1}(V) \cap \tilde{V} = \emptyset$. Let $\mathfrak{I} \subset \mathcal{O}(Y)$ be the ideal which defines $\pi_X(\tilde{V})$. Then $(f) + \mathfrak{I} \ni 1$ since $V \cap \pi_X(\tilde{V}) = \emptyset$. Hence there exists an $h \in \mathfrak{I}$ such that $h-1 \in (f)$. Since $Y_h \subset Y - \pi_X(\tilde{V})$, $X_h = \pi_X^{-1}(Y_h) \subset \pi_X^{-1}(Y - \pi_X(\tilde{V})) \subset \tilde{U}$. Thus E is trivial over X_h. Q.E.D.

We define an affine scheme $\tilde{Y} = \operatorname{Spec} \tilde{A}$ by

$$\tilde{A} = \{h_1/h_2 | h_1, h_2 \in \mathcal{O}(Y), h_2 - 1 \in (f)\}.$$

Set $\tilde{Y}_f := Y_f \times_Y \tilde{Y}$, $\tilde{X} := \tilde{Y} \times_Y X$ and $\tilde{X}_f := \tilde{Y}_f \times_Y X$. The group of morphisms from X to $M := \mathrm{GL}(Q)$ is denoted by $\mathrm{Mor}\,(X, M)$ or $M(X)$. The group G acts on M by conjugation via the representation $\rho : G \to \mathrm{GL}(Q)$. The action of G on $M(X)$ is defined by $(g \cdot \mu)(x) = \rho(g)\mu(g^{-1}x)\rho(g)^{-1}$ for $g \in G$, $x \in X$, $\mu \in M(X)$. We denote the group of G-invariants of $M(X)$ by $\mathrm{Mor}\,(X, M)^G$ or $M(X)^G$. Let $[E] \in \mathrm{VEC}_G(X, Q)_0$. Then by the definition of $\mathrm{VEC}_G(X, Q)_0$, E has a trivialization over $\pi_X^{-1}(U) = X_f$, and by Lemma 1.1 E has a trivialization also over an open neighborhood of $\pi_X^{-1}(V)$, i.e., X_h for some $h \in \mathcal{O}(Y)$ such that $h - 1 \in (f)$. Hence, assigning to $[E]$ the transition function with respect to the trivializations $E|_{X_f} \cong X_f \times Q$ and $E|_{X_h} \cong X_h \times Q$, we have a bijection to a double coset (cf. [15, 3.4])

$$\mathrm{VEC}_G(X, Q)_0 \cong M(X_f)^G \backslash M(\tilde{X}_f)^G / M(\tilde{X})^G.$$

Since X^H is irreducible, the inclusion $X^H \hookrightarrow X$ induces an isomorphism $X^H /\!/ N(H) \xrightarrow{\sim} X/\!/G$ where $N(H)$ is the normalizer of H in G [14]. Set $W := N(H)/H$. When we consider X^H as a W-variety, we denote it by B. Note that the principal isotropy group of B is trivial. Let $\beta : M(X)^G \to L(B)^W$ be the restriction map where $L := \mathrm{GL}(Q)^H$. We say X *has generically closed orbits* if $\pi_X^{-1}(\xi)$ for any $\xi \in Y_f$ consists of a closed orbit, i.e. $\pi_X^{-1}(\xi) \cong G/H$. When X has generically closed orbits, $GX_f^H = X_f$. Hence $M(X_f)^G = \mathrm{Mor}\,(GX_f^H, \mathrm{GL}(Q))^G \cong L(B_f)^W$, i.e., β is an isomorphsim over Y_f. The group homomorphism β induces a map

$$\begin{aligned} \mathrm{VEC}_G(X, Q)_0 &\cong M(X_f)^G \backslash M(\tilde{X}_f)^G / M(\tilde{X})^G \\ &\to L(B_f)^W \backslash L(\tilde{B}_f)^W / \beta(M(\tilde{X})^G), \end{aligned} \tag{1}$$

which is an isomorphism when X has generically closed orbits.

We decompose Q as an H-module

$$Q \cong \oplus_{i=1}^q n_i Q_i$$

where Q_i are pairwise non-isomorphic irreducible H-modules and n_i is the multiplicity of Q_i. We call Q *multiplicity free* with respect to H if

$n_i = 1$ for all i. It follows from Schur's lemma that

$$L = \mathrm{GL}(Q)^H \cong \prod_{i=1}^{q} GL_{n_i}.$$

Let T be the center of L. Then T is W-stable and $T \cong (\mathbb{C}^*)^q$. When Q is multiplicity free with respect to H, $L = T$. Look at the action of W on T. Note that $g \in N(H)$ permutes the H-isotypic components n_iQ_i $(i = 1, \cdots, q)$. Since $w \in W$ acts on L by conjugation by $\rho(g)$ where $g \in N(H)$ is a representative of w, W acts on $T \cong (\mathbb{C}^*)^q$ by permuting $\mathbb{C}^*$s. Hence W acts on T as a subgroup of the symmetric group S_q via a continuous homomorphism from W to S_q. Thus the connected component W_0 of W containing the identity acts trivially on T and the action of W on T reduces to the action of W/W_0. The determinant map on each factor GL_{n_i} of L induces a homomorphism of groups; $\tau : L(B)^W \to T(B)^W$. The homomorphism τ induces a map

$$(2)\quad L(B_f)^W \backslash L(\tilde{B}_f)^W / \beta(M(\tilde{X})^G) \to T(\tilde{B}_f)^W / (T(B_f)^W (\tau \circ \beta) M(\tilde{X})^G).$$

By (1) and (2), we have

Lemma 1.2. *There exists a map*

$$\psi_{X,Q} : \mathrm{VEC}_G(X, Q)_0 \to T(\tilde{B}_f)^W / (T(B_f)^W (\tau \circ \beta) M(\tilde{X})^G).$$

If Q is multiplicity free with respect to H and X has generically closed orbits, then $\psi_{X,Q}$ is an isomorphism.

Remarks. 1. For $t \in \mathcal{O}(Y)$, let $\mathrm{VEC}_G(X, Q; t)$ be the subset of $\mathrm{VEC}_G(X, Q)$ consisting of elements $[E]$ such that E is trivial over $\pi_X^{-1}(Y_t)$ and its complement. Then one obtains, in a similar way to the above, a map from $\mathrm{VEC}_G(X, Q; t)$ to a quotient group.

2. When H is trivial, $M = L$ and the target residue group in Lemma 1.2 is $\mathcal{O}(\tilde{Y}_f)^* / \mathcal{O}(Y_f)^* \tau(M(\tilde{X})^G)$, where $\mathcal{O}(\tilde{Y}_f)^*$ (resp. $\mathcal{O}(Y_f)^*$) denotes the group of invertible elements in $\mathcal{O}(\tilde{Y}_f)$ (resp. $\mathcal{O}(Y_f)$). If Q contains a trivial G-module, then $\tau = \det : M(\tilde{X})^G \to \mathcal{O}(\tilde{Y})^*$ is surjective. Furthermore, if $\mathrm{Pic}\, Y = (0)$, then the residue group $\mathcal{O}(\tilde{Y}_f)^* / \mathcal{O}(Y_f)^* \tau(M(\tilde{X})^G)$ becomes trivial (cf. proof of Lemma 1.3). Thus when H is trivial and $\mathrm{Pic}\, Y = (0)$ (e.g. X is a G-module with a trivial principal isotropy group), $\psi_{X,Q}$ becomes trivial if Q contains a trivial G-module.

We will analyse the target residue group in Lemma 1.2. We pose the following conditions:

(I) V is connected and $\mathcal{O}(\pi_B^{-1}(V))^* = \mathbb{C}^*$.

(II) The restriction $\mathcal{O}(\pi_B^{-1}(V))^* \to \mathcal{O}(X^G)^*$ is an isomorphism.

It follows from the conditions (I) and (II) that the restriction of $\pi_B^{-1}(V)$ onto X^G induces an isomorphism $T(\pi_B^{-1}(V))^W \cong T^W(X^G) \cong T^W$. Set

$$\begin{aligned} T(\tilde{B})_1 &:= \{\mu \in T(\tilde{B}) \mid \mu|_{\pi_B^{-1}(V)} = I\} \\ T(\tilde{B})_1^W &:= T(\tilde{B})_1 \cap T(\tilde{B})^W \end{aligned}$$

where I is the constant map to the identity element of T. Note that $T(\tilde{B}) = T(\tilde{B})_1 T(\pi_B^{-1}(V)) = T(\tilde{B})_1 T$.

Lemma 1.3. *Suppose that the conditions* (I) *and* (II) *are satisfied. If* $\operatorname{Pic} B = (0)$ *and* $\mathcal{O}(B)^* = \mathbb{C}^*$, *then*

$$T(\tilde{B}_f)^W = T(B_f)^W T(\tilde{B})_1^W.$$

Proof. We first claim that $T(\tilde{B}_f) = T(B_f)T(\tilde{B})_1$. Since $T(\tilde{B}) = T(\tilde{B})_1 T$, it suffices to prove $T(\tilde{B}_f) = T(B_f)T(\tilde{B})$. Note that every element of $T(\tilde{B}_f)$ is considered as a transition function of a Whitney sum of line bundles over B with respect to trivializations over B_f and an open neighborhood of $\pi_B^{-1}(V)$. Since $\operatorname{Pic} B = (0)$, every line bundle over B is trivial. This implies that $T(\tilde{B}_f) = T(B_f)T(\tilde{B})$. Let $\mu \in T(\tilde{B}_f)^W$. Write $\mu = \dot{\mu}\tilde{\mu}$ with $\dot{\mu} \in T(B_f)$ and $\tilde{\mu} \in T(\tilde{B})_1$. Note that $T(B_f) \cap T(\tilde{B})_1 = T(B)_1 = \{I\}$ since $\mathcal{O}(B)^* = \mathbb{C}^*$. Since μ is W-invariant, we have $\dot{\mu}^{-1}(w \cdot \dot{\mu}) = \tilde{\mu}(w \cdot \tilde{\mu})^{-1} \in T(B_f) \cap T(\tilde{B})_1 = \{I\}$ for every $w \in W$. Hence $\dot{\mu}$ and $\tilde{\mu}$ are W-invariant, and the assertion is thus verified. Q.E.D.

Set

$$M(\tilde{X})_1^G := \{\mu \in M(\tilde{X})^G \mid \mu|_{X^G} = I\}.$$

Note that $(\tau \circ \beta)(M(\tilde{X})_1^G) \subset T(\tilde{B})_1^W$ under the conditions (I) and (II).

Lemma 1.4. *Suppose that the assumptions in Lemma* 1.3 *are satisfied. If there exists a G-equivariant morphism $r : X \to X^G$ such that $r \circ i = id$ where $i : X^G \hookrightarrow X$ is the inclusion, then there exists an isomorphism*

$$T(\tilde{B}_f)^W/(T(B_f)^W(\tau \circ \beta)M(\tilde{X})^G) \cong T(\tilde{B})_1^W/(\tau \circ \beta)(M(\tilde{X})_1^G).$$

Proof. We claim that $(\tau \circ \beta)M(\tilde{X})^G \subset (\tau \circ \beta)(M(\tilde{X})_1^G)T^W$. In fact, let $\mu \in M(\tilde{X})^G$ and $\mu_0 := \mu|_{X^G} \in M^G(X^G)$. Then $(\tau \circ \beta)\mu_0 \in T^W(X^G) \cong T^W$. Let $p : M^G(X^G) \to M(X)^G$ be the group homomorphism induced by r. Then $\tilde{\mu} := p(\mu_0) \in M(X)^G$ satisfies $\tilde{\mu}|_{X^G} = \mu_0$.

Since $\mathcal{O}(B)^* = \mathbb{C}^*$, $(\tau\circ\beta)\tilde{\mu} \in T^W$. The claim follows from that $\mu = \mu_1\tilde{\mu}$ where $\mu_1 = \mu\tilde{\mu}^{-1} \in M(\tilde{X})_1^G$. Since $(\tau\circ\beta)M(\tilde{X})^G \subset (\tau\circ\beta)(M(\tilde{X})_1^G)T^W$ and $T(B_f)^W \cap T(\tilde{B})_1^W = T(B)_1^W = \{I\}$, we obtain by Lemma 1.3 the desired isomorphism. Q.E.D.

We proceed to analyse the residue group $T(\tilde{B})_1^W/(\tau\circ\beta)(M(\tilde{X})_1^G)$.

Let $\hat{Y}$ be the completion of Y along V and let $\hat{B} = \hat{Y}\times_Y B$ and $\hat{X} = \hat{Y}\times_Y X$. Note that an element of $M(\hat{X})^G$ (resp. $T(\hat{B})^W$) is considered as an invertible matrix (resp. an invertible diagonal matrix) with entries in $\mathcal{O}(\hat{X})$ (resp. $\mathcal{O}(\hat{B})$) invariant under the G-action (resp. the W-action). For $r \geq 1$, we define

$$\begin{aligned} T(\hat{B})_r^W &:= \{\mu \in T(\hat{B})^W \mid \mu = I \mod \mathfrak{b}^r\mathcal{O}(\hat{B})\} \\ M(\hat{X})_r^G &:= \{\mu \in M(\hat{X})^G \mid \mu = I \mod \mathfrak{a}^r\mathcal{O}(\hat{X})\}, \end{aligned}$$

where $\mathfrak{a} \subset \mathcal{O}(X)$ denotes the ideal of $X^G \subset X$ and $\mathfrak{b} \subset \mathcal{O}(B)$ denotes the ideal of $\pi_B^{-1}(V)$, i.e. $\mathfrak{b} = \sqrt{(f)}$. We define $L(\hat{B})_r^W$, similarly. Then there exists a canonical map

$$T(\tilde{B})_1^W/(\tau\circ\beta)(M(\tilde{X})_1^G) \to T(\hat{B})_1^W/(\tau\circ\beta)(M(\hat{X})_1^G).$$

We will show that this canonical map is a surjection when X has generically closed orbits. First, we prove

Lemma 1.5. *For every $r \geq 1$,*

$$T(\hat{B})_1^W = T(\tilde{B})_1^W T(\hat{B})_r^W.$$

Proof. It is clear that $T(\hat{B})_1^W \supset T(\tilde{B})_1^W T(\hat{B})_r^W$. We show the opposite inclusion. Let $\mu = (\mu_1(x), \ldots, \mu_q(x)) \in T(\hat{B})_1^W$ where $\mu_i(x) \in \mathcal{O}(\hat{B})$ and $\mu_i = 1 \mod \mathfrak{b}\mathcal{O}(\hat{B})$. Recall that W acts on $T \cong (\mathbb{C}^*)^q$ by permuting $\mathbb{C}^*$s. Since the identity component W_0 acts trivially on T, $\mu_i(x) \in \mathcal{O}(\hat{B})^{W_0}$ for $1 \leq i \leq q$. Let $\bar{\mu}_i(x) \in \mathcal{O}(B)^{W_0}$ be a function such that $\mu_i(x) = \bar{\mu}_i(x) \mod \mathfrak{b}^r\mathcal{O}(\hat{B})$. Since $\mu_i = 1 \mod \mathfrak{b}\mathcal{O}(\hat{B})$, $\bar{\mu}_i = 1 \mod \mathfrak{b}$. Define $\bar{\mu} := (\bar{\mu}_1(x), \ldots, \bar{\mu}_q(x))$ and $\tilde{\mu} := \prod_{w\in W/W_0} w\cdot\bar{\mu}$. Then $\tilde{\mu} \in T(\tilde{B})_1^W$ and $\tilde{\mu}^{-1}\mu \in T(\hat{B})_r^W$. Q.E.D.

Let $\mathfrak{m}$, $\mathfrak{l}$ and $\mathfrak{t}$ be the Lie algebras of M, L and T, respectively. Then $\mathfrak{m} = \operatorname{End} Q$, $\mathfrak{l} = \operatorname{End}(Q)^H \cong \oplus_{i=1}^q M_{n_i}$ and $\mathfrak{t} \cong \mathbb{C}^q$ where M_{n_i} denotes an $(n_i \times n_i)$-matrix. Let $\beta_* : \mathfrak{m}(X)^G \to \mathfrak{l}(B)^W$ be the homomorphism of $\mathcal{O}(Y)$-modules induced by the restriction of X onto B. Similarly, let $\tau_* : \mathfrak{l}(B)^W \to \mathfrak{t}(B)^W$ be the homomorphism induced by the trace map on each M_{n_i} of $\mathfrak{l} \cong \oplus_{i=1}^q M_{n_i}$. Note that

$$\mathfrak{t}(B)^W \cong (\mathcal{O}(B) \otimes_{\mathbb{C}} \mathfrak{t})^W, \qquad \mathfrak{l}(B)^W \cong (\mathcal{O}(B) \otimes_{\mathbb{C}} \mathfrak{l})^W$$
$$\text{and} \quad \mathfrak{m}(X)^G \cong (\mathcal{O}(X) \otimes_{\mathbb{C}} \mathfrak{m})^G,$$

which are all finitely generated modules over $\mathcal{O}(Y)$ (cf. [8, II, 3.2]). For a positive integer r, we define

$$\begin{aligned} \mathfrak{t}(B)_r^W &:= (\mathfrak{b}^r \otimes_{\mathbb{C}} \mathfrak{t})^W, \\ \mathfrak{l}(B)_r^W &:= (\mathfrak{b}^r \otimes_{\mathbb{C}} \mathfrak{l})^W, \\ \mathfrak{m}(X)_r^G &:= (\mathfrak{a}^r \otimes_{\mathbb{C}} \mathfrak{m})^G, \end{aligned}$$

which are also finitely generated modules over $\mathcal{O}(Y)$. We define $\mathfrak{t}(\hat{B})_r^W$, $\mathfrak{l}(\hat{B})_r^W$ and $\mathfrak{m}(\hat{X})_r^G$, similarly. The exponentials $\exp : \mathfrak{l} \to L$ and $\exp : \mathfrak{m} \to M$ induce isomorphisms (with inverse log) $\mathfrak{l}(\hat{B})_r^W \xrightarrow{\sim} L(\hat{B})_r^W$ and $\mathfrak{m}(\hat{X})_r^G \xrightarrow{\sim} M(\hat{X})_r^G$ (Here, the latter exponential series converges in the $\mathfrak{a}$-adic topology).

Lemma 1.6. *Suppose that X has generically closed orbits. Then there exists an integer r_0 such that $\beta_*\mathfrak{m}(X)_1^G \supset \mathfrak{l}(B)_r^W$ and $\beta(M(\hat{X})_1^G) \supset L(\hat{B})_r^W$ for all $r \geq r_0$.*

Proof. Let $\{C_i\}$ and $\{A_j\}$ be generating systems of $\mathfrak{l}(B)_1^W$ and $\mathfrak{m}(X)_1^G$ over $\mathcal{O}(Y)$, respectively. Since X has generically closed orbits, $\beta_* : \mathfrak{m}(X_f)^G \to \mathfrak{l}(B_f)^W$ is an isomorphism. Thus C_i is written as $C_i = \beta_*(\sum_j c_{ij}A_j)$ where $c_{ij} \in \mathcal{O}(Y)_f$. Let $e_{ij} \geq 0$ be the minimal integer such that $f^{e_{ij}}c_{ij} \in \mathcal{O}(Y)$ and d be the minimal integer such that $\mathfrak{b}^d \subseteq (f)$. Put $e := \max_{i,j}\{e_{ij}\}$ and $r_0 := de + 1$. Then for $r \geq r_0$, any element of $\mathfrak{l}(B)_r^W$ is of the form f^eC where $C \in \mathfrak{l}(B)_1^W$. Since $C = \sum_i c_iC_i$ for $c_i \in \mathcal{O}(Y)$ and $f^ec_{ij} \in \mathcal{O}(Y)$ for every i, j, so $f^eC \in \beta_*\mathfrak{m}(X)_1^G$. Hence $\beta_*\mathfrak{m}(X)_1^G \supset \mathfrak{l}(B)_r^W$. The second inclusion follows from $\beta_*\mathfrak{m}(\hat{X})_1^G \supset \mathfrak{l}(\hat{B})_r^W$ via the exponential maps. Q.E.D.

Remark. In order to prove Lemma 1.6, it is sufficient to hold that $\beta_* : \mathfrak{m}(X_f)^G \to \mathfrak{l}(B_f)^W$ is surjective.

Since $\tau_* : \mathfrak{l}(\hat{B})_r^W \to \mathfrak{t}(\hat{B})_r^W$ is the trace map, τ_* is surjective. Hence, via the exponential maps, $T(\hat{B})_r^W = \tau(L(\hat{B})_r^W)$. Under the assumption in Lemma 1.6, $T(\hat{B})_r^W = \tau(L(\hat{B})_r^W) \subset (\tau \circ \beta)(M(\hat{X})_1^G)$ for a sufficiently large r. By this together with Lemma 1.5, we obtain

Lemma 1.7. *Suppose that X has generically closed orbits. Then the canonical map*

$$T(\tilde{B})_1^W/(\tau \circ \beta)(M(\tilde{X})_1^G) \to T(\hat{B})_1^W/(\tau \circ \beta)(M(\hat{X})_1^G)$$

is a surjection. Furthermore, if Q is multiplicity free with respect to H, then $L(\tilde{B})_1^W/\beta(M(\tilde{X})_1^G) \to L(\hat{B})_1^W/\beta(M(\hat{X})_1^G)$ is an isomorphism.

Proof. The first assertion is clear from the above statement. As for the second assertion, it suffices to show that the canonical map is injective. We will show that $\beta(M(\hat{X})_1^G) \cap L(\tilde{B})_1^W \subset \beta(M(\tilde{X})_1^G)$. Let $\hat{D} \in M(\hat{X})_1^G$ and $\beta(\hat{D}) \in \beta(M(\hat{X})_1^G) \cap L(\tilde{B})_1^W$. We regard $\hat{D}$ as an element of $\mathfrak{m}(\hat{X})^G$ and show that $\hat{D} \in \mathfrak{m}(\tilde{X})^G$. Since $\beta(\hat{D}) \in L(\tilde{B})_1^W$ is translated as $\beta_*(\hat{D}) \in \mathfrak{l}(\tilde{B})^W$, it follows from Lemma 1.6 that $f^r\beta_*(\hat{D}) = \beta_*(\tilde{D})$ for a sufficiently large r and $\tilde{D} \in \mathfrak{m}(\tilde{X})^G$. Since X has generically closed orbits, $\beta_* : \mathfrak{m}(X_f)^G \to \mathfrak{l}(B_f)^W$ is an isomorphism, so, $\beta_* : \mathfrak{m}(X)^G \to \mathfrak{l}(B)^W$ is an injection. Hence $\beta_* : \mathfrak{m}(\hat{X})^G \to \mathfrak{l}(\hat{B})^W$ is also an injection. Thus $f^r\hat{D} = \tilde{D}$. This implies that $\hat{D} \in \mathfrak{m}(\tilde{X})^G$. Hence $\hat{D} \in M(\tilde{X})_1^G$ and the assertion follows. Q.E.D.

The logarithmic map induces an isomorphism

$$T(\hat{B})_1^W/(\tau \circ \beta)(M(\hat{X})_1^G) \cong \mathfrak{t}(\hat{B})_1^W/\tau_*\beta_*\mathfrak{m}(\hat{X})_1^G.$$

We set

$$C_X(Q) := \mathfrak{t}(\hat{B})_1^W/\tau_*\beta_*\mathfrak{m}(\hat{X})_1^G.$$

When Q is multiplicity free with respect to H, $C_X(Q) = \mathfrak{l}(\hat{B})_1^W/\beta_*\mathfrak{m}(\hat{X})_1^G$.

By the results obtained so far, we have

Theorem 1.8. *There exists a map*

$$T(\tilde{B})_1^W/(\tau \circ \beta)(M(\tilde{X})_1^G) \to \mathfrak{t}(\hat{B})_1^W/\tau_*\beta_*\mathfrak{m}(\hat{X})_1^G = C_X(Q),$$

which is an isomorphism when Q is multiplicity free with respect to H and X has generically closed orbits.

§2. G-vector bundles over G-modules

In this section, we consider the case where the base space X is a G-module P and give a proof of Theorem 1 in the introduction. Let P be a G-module such that $Y = P//G$ is of dimension ≥ 1 and the ideal of $V = Y - U$ is principal. Note that the ideal of V is generated by an invariant homogeneous polynomial $f \in \mathcal{O}(P)^G$ and that V is connected. Let H be a principal isotropy group of P and let $B = P^H$.

Lemma 2.1. (1) $\operatorname{Pic} B = (0)$ *and* $\mathcal{O}(B)^* = \mathcal{O}(P^G)^* = \mathbb{C}^*$.

(2) $\pi_B^{-1}(V)$ *is a connected affine cone and* $\mathcal{O}(\pi_B^{-1}(V))^* = \mathbb{C}^*$.

Proof. (1) The assertion follows from the fact that B and P^G are affine spaces.

(2) One easily sees that $\pi_B^{-1}(V)$ is a connected affine cone. Indeed, $\pi_B^{-1}(V)$ is a union of irreducible reduced affine cones $\cup_j \operatorname{Spec} R^{(j)}$ passing through the origin. Each affine cone $\operatorname{Spec} R^{(j)}$ has a positively graded integral domain $R^{(j)} = \oplus_{k \geq 0} R_k^{(j)}$ as the coordinate ring such that $R_0^{(j)} = \mathbb{C}$. Since $(R^{(j)})^* = \mathbb{C}^*$ for each j, the standard argument in commutative algebras shows that $\mathcal{O}(\pi_B^{-1}(V))^* = \mathbb{C}^*$. Q.E.D.

The projection $p : P \to P^G$ is G-equivariant and has the property $p \circ i = id$ for the inclusion $i : P^G \hookrightarrow P$. By this fact and the results obtained so far, we obtain a map $\Psi_{P,Q}$ for a G-module Q;

$$\begin{aligned} \mathrm{VEC}_G(P,Q)_0 &\overset{\psi_{P,Q}}{\to} T(\tilde{B}_f)^W/(T(B_f)^W(\tau \circ \beta)M(\tilde{P})^G) \quad \text{(Lemma 1.2)} \\ &\cong T(\tilde{B})_1^W/(\tau \circ \beta)(M(\tilde{P})_1^G) \quad \text{(Lemmas 1.4, 2.1)} \\ &\to \mathfrak{t}(\hat{B})_1^W/\tau_*\beta_*\mathfrak{m}(\hat{P})_1^G = C_P(Q) \quad \text{(Theorem 1.8).} \end{aligned}$$

Hence we have

Theorem 2.2. *Let P be a G-module as above and let Q be a G-module. There is a map*

$$\Psi_{P,Q} : \mathrm{VEC}_G(P,Q)_0 \to C_P(Q)$$

which is an isomorphism when Q is multiplicity free with respect to H and P has generically closed orbits.

Remarks. 1. Let P be any G-module and let t be a G-invariant homogeneous polynomial on P. We use the notation in the remark of Lemma 1.2. By the construction similar to the above, one obtains a map

$$\Psi_{P,Q}(t) : \mathrm{VEC}_G(P,Q;t) \to \mathfrak{t}(\hat{B})_1^W/\tau_*\beta_*\mathfrak{m}(\hat{P})_1^G =: C_{P,t}(Q)$$

where the completion is (t)-adic completion. One can show that $\Psi_{P,Q}(t)$ is surjective for any G-module Q if one takes $t \in \mathcal{O}(Y)$ so that Y_t is contained in the principal stratum of $\overline{GP^H}$ (cf. [15, 1.1], [2, 6.5]).

2. When H is trivial and Q contains a trivial G-module, $\psi_{P,Q}$ is trivial (remark of Lemma 1.2), hence $\Psi_{P,Q}$ is also trivial.

This completes the proof of Theorem 1 in the introduction except the statement on $C_P(Q)$. Note that Theorem 1 holds also in the case $\dim P//G = 1$. When $\dim P//G = 1$, it is known that $P//G \cong \mathbb{A}^1$ and $\mathrm{VEC}_G(P \oplus \mathbb{C}^m, Q) = \mathrm{VEC}_G(P \oplus \mathbb{C}^m, Q)_0$ for $m \geq 0$ ([11], [16]). Suppose that $\dim P//G = 1$. Then $C_P(Q)$ is a finite $\mathbb{C}$-module by the formula (3) below (cf. Lemma 2.3) and $C_{P\oplus\mathbb{C}^m}(Q) \cong (\mathbb{C}[y_1, \cdots, y_m])^p$ by easy calculation. By comparing $\Psi_{P\oplus\mathbb{C}^m,Q}$ with the isomorphism $\mathrm{VEC}_G(P \oplus \mathbb{C}^m, Q) \xrightarrow{\sim} (\mathbb{C}[y_1, \cdots, y_m])^p$ given in [16] (cf. [11]), one sees that $\Psi_{P\oplus\mathbb{C}^m,Q}$ for $m \geq 0$ is an isomorphism for any P and Q.

Now, we look at $C_P(Q)$ more closely. A G-module P is called *cofree* if $\mathcal{O}(P)$ is a free module over $\mathcal{O}(P)^G$. It is known that cofree modules are coregular, i.e., $P//G$ is isomorphic to affine space (cf. [24]). Furthermore, if P^H is a cofree $N(H)$-module, then P is a cofree G-module [24]. We suppose that B is a cofree W-module and make some obeservation on $C_P(Q)$. Then, $\mathcal{O}(Y)$ is isomorphic to a polynomial ring and $\mathfrak{m}(P)^G$ and $\mathfrak{t}(B)^W$ are finite free modules over $\mathcal{O}(Y)$. Since $\mathfrak{b}$ is principal, $\mathfrak{t}(B)^W_1$ is also a finite free module over $\mathcal{O}(Y)$. The rank of $\mathfrak{t}(B)^W_1$ is the same as the rank of $\mathfrak{t}(B)^W$, which is equal to $q = \dim \mathfrak{t}$ [24]. Note that $\mathcal{O}(Y)$, $\mathfrak{m}(P)^G$ and $\mathfrak{t}(B)^W$ inherit a grading on $\mathcal{O}(P)$. Since $\mathfrak{a}$ and $\mathfrak{b}$ are homogeneous ideals, $\mathfrak{m}(P)^G_1$ and $\mathfrak{t}(B)^W_1$ are also graded. Let $\{A_i; 1 \leq i \leq \ell\}$ be a homogeneous generating system of $\mathfrak{m}(P)^G_1$ over $\mathcal{O}(Y)$ and let $\{C_i; 1 \leq i \leq q\}$ be a homogeneous basis of $\mathfrak{t}(B)^W_1$ over $\mathcal{O}(Y)$. Then

$$\tau_*\beta_*A_i = \sum_{j=1}^{q} a_{ij}C_j \quad \text{for} \quad a_{ij} \in \mathcal{O}(Y).$$

Noting that $\mathfrak{t}(\hat{B})^W_1 = \mathfrak{t}(B)^W_1 \otimes_{\mathcal{O}(Y)} \mathcal{O}(\hat{Y})$ and $\mathfrak{m}(\hat{P})^G_1 = \mathfrak{m}(P)^G_1 \otimes_{\mathcal{O}(Y)} \mathcal{O}(\hat{Y})$,

$$C_P(Q) \cong \oplus_{j=1}^{q} \mathcal{O}(\hat{Y})/\widehat{\mathfrak{a}_j} \tag{3}$$

where $\widehat{\mathfrak{a}_j} = \mathfrak{a}_j\mathcal{O}(\hat{Y})$ and $\mathfrak{a}_j$ is the ideal in $\mathcal{O}(Y)$ generated by $\{a_{ij}; 1 \leq i \leq \ell\}$. Let $e_j = \deg C_j$ and $a_i = \deg A_i$. Since τ_* and β_* preserve the grading, $\deg a_{ij} = a_i - e_j$ if $a_{ij} \neq 0$. The following is easily proved.

Lemma 2.3. *Suppose that B is cofree. If there is some j such that $a_i > e_j$ for any i, then $C_P(Q)$ is non-trivial. If there exists some j such that* $\mathrm{ht}\,\mathfrak{a}_j < \dim Y$*, then $C_P(Q)$ is an infinite dimensional $\mathbb{C}$-module.*

Remark. The module $C_P(Q)$ can be of infinite dimension, but of countably-infinite dimension.

This completes the proof of Theorem 1. By Theorem 2.2 and Lemma 2.3, we have

Corollary 2.4. *Suppose that $\Psi_{P,Q}$ in Theorem* 2.2 *is surjective and B is cofree. If $a_i > e_j$ for some j and any i, then* $\mathrm{VEC}_G(P,Q)_0$ *is non-trivial. If there exists some j such that* $\mathrm{ht}\,\mathfrak{a}_j < \dim Y$*, then* $\mathrm{VEC}_G(P,Q)_0$ *contains an infinite dimensional space.*

We give a couple of examples.

Example 2.1. Let $G = SL_n$ $(n \geq 2)$ and let P be the Lie algebra $\mathfrak{sl}_n$ with adjoint action. We denote a maximal torus of G by T_n and its Lie algebra by $\mathfrak{t}_n$. Then the principal isotropy group of $\mathfrak{sl}_n$ is T_n and $B = (\mathfrak{sl}_n)^{T_n} = \mathfrak{t}_n$. $W = N(T_n)/T_n$ is the Weyl group which is isomorphic to S_n. The algebraic quotient space Y is $\mathfrak{sl}_n//G \cong \mathfrak{t}_n//W \cong \mathbb{A}^{n-1}$ and V is of codimension one. Hence the ideal of V is generated by a single homogeneous polynomial $f \in \mathcal{O}(Y) \cong \mathbb{C}[t_1, \cdots, t_{n-1}]$. Since the general fiber of the quotient map of $\mathfrak{sl}_n$ is isomorphic to G/T_n, $\mathfrak{sl}_n$ has generically closed orbits. Let φ_1 be the standard representation space of G and φ_1^m $(m \geq 1)$ be the symmetric tensor product $S^m(\varphi_1)$. Let $Q = \varphi_1^m$. Then Q is multiplicity free with respect to T_n. Hence $L = T \cong (\mathbb{C}^*)^q$ for $q = \dim Q = \binom{n+m-1}{m}$.

Consider the case $n = 2$. Then $G = SL_2$ and the quotient map is given by the determinant map $t : P = \mathfrak{sl}_2 \to \mathfrak{sl}_2//G \cong \mathbb{A}^1$. Hence $\mathcal{O}(Y) = \mathbb{C}[t]$ and t is, as an element of $\mathcal{O}(B)^W$, written as $t = x^2$ with a coordinate x on $B = \mathfrak{t}_2 \cong \mathbb{C}$. Note that $T_2 \cong \mathbb{C}^*$ and $W \cong \mathbb{Z}/2\mathbb{Z}$. The stratification of $\mathfrak{sl}_2//G = \mathbb{A}^1$ consists of two strata, $\{0\}$ and $\mathbb{A}^1 - \{0\}$. Hence $V = \{0\}$ and $f = t$. Let R_m be the SL_2-module of binary forms of degree m. Then $P = \mathfrak{sl}_2 \cong R_2$ and $Q \cong R_m$. As a $T_2 = \mathbb{C}^*$-module, $Q = \oplus_{l=0}^m Q_{m-2l}$ where Q_{m-2l} is an irreducible T_2-module with weight $m-2l$. As a G-module, $\mathfrak{m} = \operatorname{End} R_m \cong (R_m)^* \otimes R_m \cong \oplus_{l=0}^m R_{2l}$. Hence,

$$\mathfrak{m}(\mathfrak{sl}_2)^G \cong \oplus_{l=0}^m (\mathcal{O}(R_2) \otimes R_{2l})^G = \oplus_{l=0}^m M_l$$

and

$$\mathfrak{l}(\mathfrak{t}_2)^W \cong \oplus_{l=0}^m (\mathcal{O}(\mathfrak{t}_2) \otimes R_{2l}^{T_2})^W = \oplus_{l=0}^m N_l$$

where $M_l := (\mathcal{O}(R_2) \otimes R_{2l})^G$ and $N_l := (\mathcal{O}(\mathfrak{t}_2) \otimes R_{2l}^{T_2})^W$. The modules M_l and N_l are free over $\mathcal{O}(Y) = \mathbb{C}[t]$ of rank one. In fact, since $M_l \cong \operatorname{Mor}(R_2, R_{2l})^G$, the homogeneous generator A_l of M_l is given by the l-th power map and the homogeneous generator C_l of $N_l = (\mathbb{C}[x] \otimes R_{2l}^{T_2})^W$ is given by $1 \otimes e_l$ for l even, $x \otimes e_l$ for l odd, where e_l is a base of $R_{2l}^{T_2} \cong \mathbb{C}$. Hence $\mathfrak{m}(\mathfrak{sl}_2)^G$ and $\mathfrak{l}(\mathfrak{t}_2)^W$ are free modules over $\mathbb{C}[t]$ of rank $m+1$. Note that $\deg A_l = l$ and $\deg C_l$ is 0 for l even, 1 for l odd. Since $\mathbb{C}[t]$ is a principal ideal domain, $\mathfrak{m}(\mathfrak{sl}_2)_1^G$ is also free over $\mathbb{C}[t]$. A homogeneous basis of $\mathfrak{m}(\mathfrak{sl}_2)_1^G$ over $\mathbb{C}[t]$ is $\{tA_0, A_l; l = 1, 2, \cdots, m\}$ since ${\mathfrak{sl}_2}^G = \{O\}$. Since $\mathfrak{b} = \sqrt{(t)} = (x)$, a homogeneous basis of $\mathfrak{l}(\mathfrak{t}_2)_1^W$ over $\mathbb{C}[t]$ is $\{tC_0, tC_{2l}, C_{2l-1}; l = 1, \cdots, m/2\}$ for m even, $\{tC_{2l}, C_{2l+1}; l =$

$0, 1, \cdots, [m/2]\}$ for m odd. Here, $[a]$ denotes the largest integer not-exceeding a. Since $\beta_*(A_l) = t^{[l/2]}C_l$,

$$C_{\mathfrak{sl}_2}(\varphi_1^m) \cong \mathfrak{l}(\mathfrak{t}_2)_1^W / \beta_* \mathfrak{m}(\mathfrak{sl}_2)_1^G \cong \mathbb{C}^p$$

where $p = \sum_{l=1}^m [(l-1)/2] = [(m-1)^2/4]$. Since it follows from $\mathfrak{sl}_2//G \cong \mathbb{A}^1$ that $\mathrm{VEC}_G(\mathfrak{sl}_2, \varphi_1^m) = \mathrm{VEC}_G(\mathfrak{sl}_2, \varphi_1^m)_0$, we have by Theorem 2.2

Proposition 2.5 ([23]). *Let $G = SL_2$. Then*

$$\mathrm{VEC}_G(\mathfrak{sl}_2, \varphi_1^m) \cong \mathbb{C}^p \qquad \textit{for} \quad p = [(m-1)^2/4].$$

Next, consider the case that $n \geq 3$. As a G-module,

$$\mathfrak{m} = \mathrm{End}\, \varphi_1^m \cong (\varphi_1^m)^* \otimes \varphi_1^m \cong \oplus_{l=0}^m \mathfrak{sl}_n^l$$

where $\mathfrak{sl}_n^l$ is the irreducible component of the highest weight in $S^l(\mathfrak{sl}_n)$. Hence

$$\mathfrak{m}(\mathfrak{sl}_n)^G \cong \oplus_{l=0}^m (\mathcal{O}(\mathfrak{sl}_n) \otimes \mathfrak{sl}_n^l)^G = \oplus_{l=0}^m M_l$$

where $M_l := (\mathcal{O}(\mathfrak{sl}_n) \otimes \mathfrak{sl}_n^l)^G$. Similarly,

$$\mathfrak{l}(\mathfrak{t}_n)^W \cong \oplus_{l=0}^m (\mathcal{O}(\mathfrak{t}_n) \otimes (\mathfrak{sl}_n^l)^{T_n})^W = \oplus_{l=0}^m N_l$$

where $N_l := (\mathcal{O}(\mathfrak{t}_n) \otimes (\mathfrak{sl}_n^l)^{T_n})^W$. It is known that $\mathfrak{t}_n$ is cofree (cf. [24]). Thus M_l and N_l, hence $\mathfrak{m}(\mathfrak{sl}_n)^G$ and $\mathfrak{l}(\mathfrak{t}_n)^W$, are finite free modules over $\mathcal{O}(Y)$. Since $\mathcal{O}(\mathfrak{sl}_n) \cong \oplus_{d \geq 0} S^d(\mathfrak{sl}_n)$, $M_l \cong \oplus_{d \geq 0}(S^d(\mathfrak{sl}_n) \otimes \mathfrak{sl}_n^l)^G$. Hence every homogeneous generator of M_l has degree $\geq l$. The homomorphism $\beta_* : \mathfrak{m}(\mathfrak{sl}_n)^G \to \mathfrak{l}(\mathfrak{t}_n)^W$ maps M_l to N_l. Set $M(1)_l := (\mathfrak{a} \otimes \mathfrak{sl}_n^l)^G$ and $N(1)_l := (\mathfrak{b} \otimes (\mathfrak{sl}_n^l)^{T_n})^W$. Then $\mathfrak{m}(\mathfrak{sl}_n)_1^G = \oplus_{l=0}^m M(1)_l$ and $\mathfrak{l}(\mathfrak{t}_n)_1^W = \oplus_{l=0}^m N(1)_l$. The homomorphism β_* maps $M(1)_l$ to $N(1)_l$. Let $\{A_i\}$ be a homogeneous generating system of $M(1)_m$ over $\mathcal{O}(Y)$ and $\{C_i\}$ be a homogenous basis of $N(1)_m$ over $\mathcal{O}(Y)$. Then $\beta_*(A_i) = \sum_j a_{ij} C_j$ for $a_{ij} \in \mathcal{O}(Y)$. Since $\deg A_i \geq m$ for all i and $\deg C_j < |W| + \deg f$ [8, II, 3.6], $\deg a_{ij} > 0$ if m is sufficiently large. Hence $N(1)_m / \beta_*(M(1)_m)$ is non-trivial for $m \gg 0$. We have by Theorem 2.2;

Proposition 2.6 (cf. [6]). *Let $n \geq 3$ and $G = SL_n$. For $m \geq 1$, $\mathrm{VEC}_G(\mathfrak{sl}_n, \varphi_1^m)_0 \cong C_{\mathfrak{sl}_n}(\varphi_1^m)$. In particular, $\mathrm{VEC}_G(\mathfrak{sl}_n, \varphi_1^m)_0$ is non-trivial for a sufficiently large m.*

Remark. In order to show that $C_{\mathfrak{sl}_n}(\varphi_1^m)$ contains an infinite dimensional module for $n \geq 3$, we need to prove that the height of the ideal $\mathfrak{a}_j$ generated by $a_{ij} \in \mathcal{O}(Y)$ (cf. Lemma 2.3) is smaller than $n - 1$.

However, to calculate generators of $N(1)_l$ and $M(1)_l$ by hand is a hard job.

Next is a new example of $\mathrm{VEC}_G(P,Q)_0$ containg an infinite dimensional space.

Example 2.2. Let $P = P_1 \oplus P_2$ and $G = G_1 \times G_2$ where P_i is a G_i-module with one-dimensional quotient for $i = 1, 2$. Then P is a G-module with trivial G_i-actions on P_j for $i \neq j$ and $P//G \cong \mathbb{A}^2$. A principal isotropy group H of P is $H_1 \times H_2$ where H_i is a principal isotropy group of P_i. The complement of the principal stratum in $P//G \cong \mathbb{A}^2$ is a union of two lines. Let Q_i $(i = 1, 2)$ be a G_i-module. By the statement below Theorem 2.2, there are isomorphisms $\mathrm{VEC}_{G_i}(P_i, Q_i) \cong C_{P_i}(Q_i) \cong \mathbb{C}^{p_i}$ for $i = 1, 2$. Let $Q = Q_1 \oplus Q_2$. Then Q is multiplicity free with respect to H when Q_i is multiplicity free with respect to H_i for $i = 1, 2$ and $\dim(Q_1^{H_1} \oplus Q_2^{H_2}) \leq 1$. In this case, $C_P(Q)$ is easily computed and isomorphic to $\mathbb{C}[u_1]^{p_2} \oplus \mathbb{C}[u_2]^{p_1}$ where $\mathcal{O}(P_1)^{G_1} = \mathbb{C}[u_1]$ and $\mathcal{O}(P_2)^{G_2} = \mathbb{C}[u_2]$. By Theorem 2.2, we have with the above notation

Theorem 2.7. *Suppose that Q_i is multiplicity free with respect to H_i for $i = 1, 2$ and $\dim(Q_1^{H_1} \oplus Q_2^{H_2}) \leq 1$. Then there is a map*

$$\mathrm{VEC}_G(P_1 \oplus P_2, Q_1 \oplus Q_2)_0 \to \mathbb{C}[u_1]^{p_2} \oplus \mathbb{C}[u_2]^{p_1},$$

which is an isomorphism when P_i has generically closed orbits for $i = 1, 2$.

Remark. One can show that the map in Theorem 2.7 is surjective for any Q and any P_i by using the fact that $Z_f = GP_f^H$ for $Z := \overline{GP^H}$ when $\mathcal{O}(P)^G = \mathbb{C}[f]$ (cf. [15, 1.1], the remark of Theorem 2.2).

Apply Theorem 2.7 to the case where $G = SL_2 \times SL_2$, $P = \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$, and $Q = \varphi_1^m \oplus \varphi_1^n$. Since $\mathfrak{sl}_2$ has generically closed orbits and φ_1^m is multiplicity free with respect to a principal isotropy group of $\mathfrak{sl}_2$ for $m \geq 1$, we have

Theorem 2.8. *Let $G = SL_2 \times SL_2$. Then*

$$\mathrm{VEC}_G(\mathfrak{sl}_2 \oplus \mathfrak{sl}_2, \varphi_1^m \oplus \varphi_1^n)_0 \cong \mathbb{C}[u_1]^{p(n)} \oplus \mathbb{C}[u_2]^{p(m)}.$$

Here $p(n) = [(n-1)^2/4]$ and either m or n is odd.

§3. G-vector bundles over $G \times (\mathbb{Z}/d\mathbb{Z})$-varieties

In this section, we consider in the case that V is not connected. Such a case occurs when X is a G-stable affine quadric with fixpoints and one-dimensional quotient. As is remarked in the introduction, when G is connected, such an affine quadric X is G-isomorphic to an affine quadric

$$X_P = \{(x, v) \in P \oplus \mathbb{C} \mid u(x) + v^2 = 1\}$$

where P is an orthogonal G-module with $P//G \cong \mathbb{A}^1$ and $u(x)$ is an invariant quadratic form on P such that $\mathcal{O}(P)^G = \mathbb{C}[u]$. Recall that X_P is viewed as a $G \times (\mathbb{Z}/2\mathbb{Z})$-variety. We generalize this situation. Let P be anew a G-module as in section 2, i.e., P is a G-module such that $\dim P//G \geq 1$ and the ideal of the complement of the principal stratum in $P//G$ is generated by a homogeneous polynomial $f \in \mathcal{O}(P)^G$. For $d \geq 2$, define a G-stable hypersurface $X_P(d)$ as follows;

$$X_P(d) := \{(x, v) \in P \oplus \mathbb{C} \mid f(x) + v^d = 1\}.$$

Then the fixpoint locus $X_P(d)^G$ consists of d connected components. The complement V of the principal stratum in $X_P(d)//G$ has d connected components and each connected component of $\pi_{X_P(d)}^{-1}(V)$ contains one connected component of $X_P(d)^G$. A principal isotropy group H of $X_P(d)$ is a principal isotropy group of P. As in the case of affine quadrics, $X_P(d)$ has a $\mathbb{Z}/d\mathbb{Z}$-action induced by a (non-trivial) linear action of $\mathbb{Z}/d\mathbb{Z}$ on $\mathbb{C}$. Hence $X_P(d)$ is viewed as a $G \times (\mathbb{Z}/d\mathbb{Z})$-variety. Then $X_P(d)/(\mathbb{Z}/d\mathbb{Z})$ is G-isomorphic to P. Let $\pi_{\mathbb{Z}_d} : X_P(d) \to X_P(d)/(\mathbb{Z}/d\mathbb{Z}) \cong P$ be the quotient by $\mathbb{Z}/d\mathbb{Z}$. Let $[E] \in \mathrm{VEC}_G(P, Q)$ for a G-module Q. Then $\pi_{\mathbb{Z}_d}^* E$ is a $G \times (\mathbb{Z}/d\mathbb{Z})$-vector bundle over $X_P(d)$. Viewing $\pi_{\mathbb{Z}_d}^* E$ as a G-vector bundle, we obtain a map

$$\pi_{\mathbb{Z}_d}^* : \mathrm{VEC}_G(P, Q) \to \mathrm{VEC}_G(X_P(d), Q).$$

Since $E \cong \pi_{\mathbb{Z}_d}^* E/(\mathbb{Z}/d\mathbb{Z})$ [9], we have

Lemma 3.1. *The map $\pi_{\mathbb{Z}_d}^*$ is injective.*

Note that $\pi_{\mathbb{Z}_d}^*$ maps $\mathrm{VEC}_G(P, Q)_0$ to $\mathrm{VEC}_G(X_P(d), Q)_0$. By Lemma 3.1 and Theorem 2.2, we obtain

Theorem 3.2. *The map $\pi_{\mathbb{Z}_d}^*$ induces an injection*

$$\mathrm{VEC}_G(P, Q)_0 \to \mathrm{VEC}_G(X_P(d), Q)_0.$$

Hence, if $\Psi_{P,Q}$ in Theorem 2.2 is a surjection onto a non-trivial $C_P(Q)$, then $\mathrm{VEC}_G(X_P(d), Q)_0$ is non-trivial.

If we take as an f in the definition of $X_P(d)$ any G-invariant polynomial on P, then we obtain Theorem 2 in the introduction.

Remark. Theorem 3.2 is generalized as follows. Let P_i ($i = 1, 2$) be a G_i-module such that $\dim P_1//G_1 \geq 1$ and $\dim P_2//G_2 = 1$. Let t be a homogeneous generator of $\mathcal{O}(P_2)^{G_2}$. For $f \in \mathcal{O}(P_1)^{G_1}$, define a $G_1 \times G_2$-stable hypersurface $X(f)$ as follows:

$$X(f) := \{(x_1, x_2) \in P_1 \oplus P_2 \mid f(x_1) + t(x_2) = 1\}.$$

Then the quotient map $\pi_{G_2} : X(f) \to X(f)//G_2 \cong P_1$ induces an injection for a G_1-module Q

$$\pi_{G_2}^* : \mathrm{VEC}_{G_1}(P_1, Q) \to \mathrm{VEC}_{G_1}(X(f), Q).$$

Recall that $\mathrm{VEC}_G(P, Q)_0 = \mathrm{VEC}_G(P, Q) \cong \mathbb{C}^p$ when P has one-dimensional quotient. Hence we have by Theorem 3.2

Corollary 3.3. *Suppose that X_P is a G-stable affine quadric defined as above. Then $\mathrm{VEC}_G(X_P, Q)_0$ contains a space isomorphic to $\mathbb{C}^p$ where p is a nonnegative integer such that $\mathrm{VEC}_G(P, Q) \cong \mathbb{C}^p$.*

We give a couple of examples.

Example 3.1. Let $G = SL_2$. We use the same notation as in Example 2.1. Let $P = \mathfrak{sl}_2$ and $Q = \varphi_1^m$ for $m \geq 1$. Then $\mathcal{O}(\mathfrak{sl}_2)^G = \mathbb{C}[t]$ with an invariant polynomial t of degree 2 and $\mathrm{VEC}_G(\mathfrak{sl}_2, \varphi_1^m) \cong \mathbb{C}^p$ for $p = [\frac{(m-1)^2}{4}]$. Let X be a G-stable affine quadric $\{(x, v) \in \mathfrak{sl}_2 \oplus \mathbb{C} \mid t + v^2 = 1\}$. Then by Corollary 3.3,

Proposition 3.4. *With the above notation, $\mathrm{VEC}_G(X, \varphi_1^m)_0$ contains $\mathbb{C}^p$ for $p = [\frac{(m-1)^2}{4}]$.*

Remark. It is known that $\mathrm{VEC}_G(\mathfrak{sl}_2 \oplus \mathbb{C}, \varphi_1^m)_0 \cong \mathbb{C}[v]^p$ by [16].

Example 3.2. Let $G = G_1 \times G_2$, $P = P_1 \oplus P_2$, and $Q = Q_1 \oplus Q_2$ as in Example 2.2. Let $\mathcal{O}(P_1)^{G_1} = \mathbb{C}[u_1]$ and $\mathcal{O}(P_2)^{G_2} = \mathbb{C}[u_2]$ where u_i is a G_i-invariant homogeneous polynomial on P_i. Then $P//G \cong \mathbb{A}^2 = \mathrm{Spec}\, \mathbb{C}[u_1, u_2]$ and the complement of the principal stratum is defined by $u_1 u_2 = 0$. We define for $d \geq 2$

$$X_d := \{(x_1, x_2, v) \in P_1 \oplus P_2 \oplus \mathbb{C} \mid u_1(x_1) u_2(x_2) + v^d = 1\}.$$

Then by the remark of Theorem 2.7 and Theorem 3.2,

Proposition 3.5. *Under the notation and the assumptions in Theorem* 2.7*,* $\mathrm{VEC}_G(X_d, Q_1 \oplus Q_2)_0$ *contains an infinite dimensional space if* $p_1 + p_2 > 0$.

Example 3.3. Let $G = SL_3$ and $P = \mathfrak{sl}_3$ with adjoint action. Then $P//G \cong \mathbb{A}^2$ and the complement of the principal stratum in $P//G$ is defined by an invariant homogeneous polynomial f of degree 6. For $d \geq 2$, define

$$X_d = \{(x, v) \in \mathfrak{sl}_3 \oplus \mathbb{C} \mid f + v^d = 1\}.$$

It is known that $\mathrm{VEC}_G(\mathfrak{sl}_3, \mathfrak{sl}_3)_0$ contains a space isomorphic to $\Omega^1_{\mathbb{C}}$ which is the module of Kähler differentials of $\mathbb{C}$ over $\mathbb{Q}$ [17]. Hence we have by Theorem 3.2

Proposition 3.6. $\mathrm{VEC}_G(X_d, \mathfrak{sl}_3)_0$ *contains an uncountably-infinite dimensional space.*

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Completely Parametrized $\mathbf{A}^1_*$-fibrations on the Affine Plane

Masayoshi Miyanishi

§0. Introduction

Let k be an algebraically closed field of characteristic zero, which we fix as the ground field. In the present article we consider $\mathbf{A}^1_*$-fibrations on the affine plane $\mathbf{A}^2$, where $\mathbf{A}^1_*$ denotes the affine line $\mathbf{A}^1$ with one point deleted. Let X be a smooth affine surface with Pic $(X) = (0)$ and $\Gamma(X, \mathcal{O}_X)^* = k^*$. Let $\rho : X \to B$ be an $\mathbf{A}^1_*$-fibration, where B is a smooth algebraic curve. Then ρ is untwisted because Pic $(X) = (0)$ and B is isomorphic to $\mathbf{A}^1$ or $\mathbf{P}^1$ because $\Gamma(X, \mathcal{O}_X)^* = k^*$. We call ρ a *completely* (resp. *incompletely*) *parametrized* $\mathbf{A}^1_*$-fibration if B is isomorphic to $\mathbf{P}^1$ (resp. $\mathbf{A}^1$). See [6], [8] for the definitions and relevant results. If X is the affine plane and ρ is incompletely parametrized, then there exists an irreducible polynomial $f \in \Gamma(X, \mathcal{O}_X)$ such that the fibration ρ is given as $\{F_\lambda\}_{\lambda \in k}$, where F_λ is a curve defined by $f = \lambda$. Hence f is a generically rational polynomial with two places at infinity, and such polynomials are classified by H. Saito [10] (see [7]). On the other hand, there exist no references where the completely parametrized $\mathbf{A}^1_*$-fibrations on $\mathbf{A}^2$ are explicitly classified. The fibers of the given $\mathbf{A}^1_*$-fibration form a pencil of affine plane curves parametrized by $\mathbf{P}^1$. So, the classification is made by giving the defining equation of a general member of the pencil.

For this purpose, we make use of a description of $\mathbf{A}^2$ as a homology plane with $\mathbf{A}^1_*$-fibration over $\mathbf{P}^1$ as given in [6], [8]. Our results show that the pencil is given in the form

$$\Lambda = \left\{ \left(yx^{r+1} - p(x)\right)^{\mu_1} + \lambda x^{\mu_0} = 0; \lambda \in \mathbf{P}^1 \right\},$$

where $p(x) \in k[x], \deg p(x) \leq r$ and $p(0) \neq 0$.

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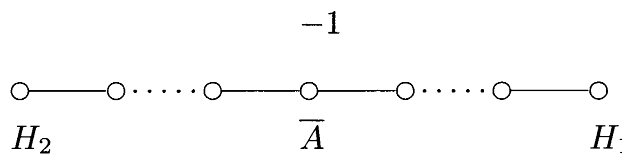
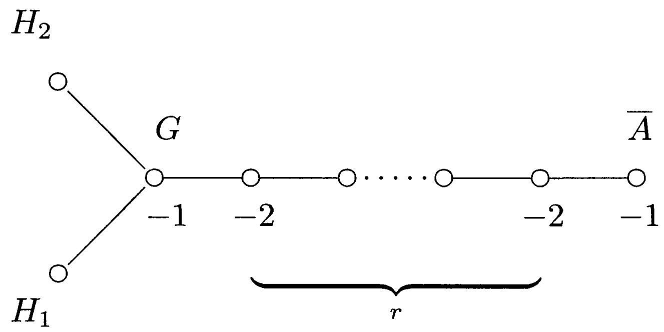
§1. $\mathbf{A}^1_*$-fibrations

Let X be a $\mathbf{Q}$-homology plane with an untwisted $\mathbf{A}^1_*$-fibration $\rho : X \to B$, where B is isomorphic to $\mathbf{P}^1$. Then every fiber but one is isomorphic to $\mathbf{A}^1_*$ if taken with the reduced structure and the excepted fiber is isomorphic to $\mathbf{A}^1$. There exists a smooth projective surface V with a $\mathbf{P}^1$-fibration $p : V \to B$ such that X is a Zariski open set of V, the boundary divisor $D := V - X$ is a divisor with simple normal crossings and p gives rise to the $\mathbf{A}^1_*$-fibration if restricted onto X. Since ρ is untwisted, there exist two cross-sections H_1 and H_2 of p, which are the loci of two points of the general fibers of ρ lying at infinity. Since the boundary divisor D has a tree as the dual graph, H_1 and H_2 meet each other at most in one point. If H_1 and H_2 meet each other, we blow up the point of intersection and its infinitely near points so that the proper transforms of H_1 and H_2 get separated from each other. Furthermore, if we assume that the embedding $X \hookrightarrow V$ is minimal in the sense that D contains no (-1) curves which are the fiber components of the $\mathbf{P}^1$-fibration p and that any contraction of such a (-1) curve makes the images of H_1 and H_2 meet each other, then it is known (cf. [6], [8]) that $\rho : X \to B$ is obtained in the following fashion.

There exists a Hirzebruch surface F_a with a minimal section M_1 and a section M_2 with $(M_1 \cdot M_2) = 0$, and there exists a sequence of blowing-ups $\sigma : V \to F_a$ such that H_1 and H_2 are the proper transforms of M_1 and M_2, respectively, and that $({H_1}^2) = ({M_1}^2) = -a$. Hence the blowing-ups σ starts with the blowing-ups of the points lying on M_2 and no points of M_1 are blown-up. The fibration $p : V \to B$ is obtained from the $\mathbf{P}^1$-fibration on F_a. Let μA be a fiber of ρ with $A \cong \mathbf{A}^1_*$ and possibly $\mu > 1$ and let $\overline{A}$ be the closure of A in V. Then the fiber of p containing $\overline{A}$ has a linear chain as the dual graph:



On the other hand, if μA is a fiber of ρ with $A \cong \mathbf{A}^1$, the dual graph of the fiber containing $\overline{A}, H_1$ and H_2 looks like



Let μA be a singular fiber of ρ, i.e., either $\mu > 1$ or $A \cong \mathbf{A}^1$. Let $\overline{A}$ be the closure of A in V. Then μ is the multiplicity of $\overline{A}$ in the fiber $p^{-1}(\rho(A))$. Let δ be the contribution of $\overline{A}$ in the total transform $\sigma^*(M_2)$. It is known (cf. [6], [8]) that $0 \le \delta < \mu$ and $\delta > 0$ if $A \cong \mathbf{A}^1_*$. We begin with recalling the following structure theorem (cf. [6], [8]).

Lemma 1.1. *Let X be a $\mathbf{Q}$-homology plane with an $\mathbf{A}^1_*$-fibration $\rho : X \to B$. Suppose $B \cong \mathbf{P}^1$ and ρ is untwisted. Let $\mu_0 A_0, ..., \mu_n A_n$ be all singular fibers with respective multiplicities $\mu_0, ..., \mu_n$, where $A_0 \cong \mathbf{A}^1$ and $A_i \cong \mathbf{A}^1_*$ for $1 \le i \le n$. Then we have the following assertions:*

(1) $\overline{\kappa}(X) = 1, 0$ *or* $-\infty$ *if and only if*

$$(n-1) - \sum_{i=1}^{n} \frac{1}{\mu_i} > 0, = 0 \text{ or } < 0, \text{ respectively.}$$

(2) $H_1(X; \mathbf{Z})$ *is a torsion group of order equal to*

$$\left| \mu_0 \cdots \mu_n a - \sum_{i=0}^{n} \mu_0 \cdots \widehat{\mu_i} \cdots \mu_n \delta_i \right| .$$

(3) *There are no homology planes X with $\overline{\kappa}(X) = 0$ and an untwisted $\mathbf{A}^1_*$-fibration $\rho : X \to B \cong \mathbf{P}^1$.*

When X is isomorphic to $\mathbf{A}^2$ in Lemma 1.1, we can specify the data more precisely.

Lemma 1.2. *With the notations of Lemma* 1.1, *the following assertions hold:*

(1) *A smooth affine surface X is isomorphic to $\mathbf{A}^2$ if and only if $\overline{\kappa}(X) = -\infty, \mathrm{Pic}\,(X) = (0)$ and $\Gamma(X, \mathcal{O}_X) = k^*$. In particular, a $\mathbf{Q}$-homology plane X is isomorphic to $\mathbf{A}^2$ if and only if $\overline{\kappa}(X) = -\infty$ and $H_1(X; \mathbf{Z}) = (0)$.*

(2) $n = 0$ *or* 1.

(3) *If* $n=0$ *then either* $a=1, \mu_0=\delta_0+1$ *or* $a=0, \delta_0=1$.
(4) *If* $n=1$ *then either*

$$a=1, \quad \mu_0\mu_1-\mu_1\delta_0-\mu_0\delta_1=\pm 1$$

or

$$a=0, \quad \mu_0=\delta_1=1, \quad \delta_0=0.$$

(5) *If* $a=n=1$ *and* $\mu_0\mu_1-\mu_1\delta_0-\mu_0\delta_1=\pm 1$, *the pair* (δ_0,δ_1) *is uniquely determined by the pair* (μ_0,μ_1). *Furthermore, if* $\mu_0\mu_1-\mu_1\delta_0-\mu_0\delta_1=1$, *then the pair* (δ_0',δ_1') *with* $\delta_i'=\mu_i-\delta_i$ $(i=0,1)$ *satisfies* $\mu_0\mu_1-\mu_1\delta_0'-\mu_0\delta_1'=-1$, *and vice versa.*

Proof. (1) We refer to [6].

(2) Note that $\mu_0\geq 1$ and $\mu_i\geq 2$ for $1\leq i\leq n$. Since $\overline{\kappa}(X)=-\infty$, it follows that

$$n-1-\frac{n}{2}\leq (n-1)-\sum_{i=1}^{n}\frac{1}{\mu_i}<0.$$

Hence $n=0$ or 1.

(3) Since $H_1(X;\mathbf{Z})=0$, we have

$$|H_1(X;\mathbf{Z})|=\left|\mu_0\cdots\mu_n a-\sum_{i=0}^{n}\mu_0\cdots\widehat{\mu_i}\cdots\mu_n\delta_i\right|=1.$$

If $n=0$ then this formula reads $\mu_0 a-\delta_0=\pm 1$, where $\mu_0>\delta_0$. Suppose $a\geq 2$. Then we have

$$(a-2)\mu_0+(\mu_0-\delta_0)+\mu_0\neq\pm 1.$$

Hence $a=0$ or 1. If $a=1$ then $\mu_0=\delta_0+1$. If $a=0$ then $\delta_0=1$.

(4) If $n=1$ then

$$a\mu_0\mu_1-\mu_1\delta_0-\mu_0\delta_1=\pm 1.$$

Suppose $a\geq 2$. Then we have

$$(a-2)\mu_0\mu_1+\mu_1(\mu_0-\delta_0)+\mu_0(\mu_1-\delta_1)\neq\pm 1.$$

Hence $a=0$ or 1. If $a=1$ then we have

$$\mu_0\mu_1-\mu_1\delta_0-\mu_0\delta_1=\pm 1.$$

If $a=0$ then $\mu_1\delta_0+\mu_0\delta_1=1$. Since $\mu_1\geq 2$, it follows that $\delta_0=0$. Then $\mu_0=\delta_1=1$.

(5) Suppose that $\mu_0\mu_1 - \mu_1\delta_0 - \mu_0\delta_1 = 1$ and $\mu_0\mu_1 - \mu_1\gamma_0 - \mu_0\gamma_1 = 1$ for the pairs (γ_0, γ_1) and (δ_0, δ_1) with $\mu_i > \gamma_i, \mu_i > \delta_i$ $(i = 0, 1)$. Then

$$\mu_1(\gamma_0 - \delta_0) = \mu_0(\delta_1 - \gamma_1).$$

Since $\gcd(\mu_0, \mu_1) = 1$, it follows that $\gamma_0 = \delta_0 + m\mu_0$ and $\delta_1 = \gamma_1 + m\mu_0$ for some integer m. If $m > 0$, then $\gamma_0 \geq \mu_0$, which is a contradiction. If $m < 0$ we obtain a contradiction in a similar fashion. So, $m = 0$. The rest is straightforward. Q.E.D.

Given a pair (μ, δ) of positive integers μ, δ with $\mu > \delta$ and $\gcd(\mu, \delta) = 1$, we define integers $\alpha_1, \alpha_2, \ldots, \alpha_s$ by expanding μ/δ in a form of continued fraction

$$\frac{\mu}{\delta} = \alpha_1 - \cfrac{1}{\alpha_1 - \cfrac{1}{\alpha_3 - \cfrac{1}{\ddots - \cfrac{1}{\alpha_s}}}}$$

where $\alpha_i \geq 2$ for $1 \leq i \leq s$. We denote this fractional expansion by $\mu/\delta = [\alpha_1, \ldots, \alpha_s]$.

Given such a pair (μ, δ), the geometric meaning of fractional expansion of μ/δ in the setting leading to Lemma 1.1 is given in th following Lemma 1.3 which is well-known (cf. [9] and [4, pp. 75–78]).

Lemma 1.3. *Let (μ, δ) be a pair of positive integers such that $\mu > \delta$ and $\gcd(\mu, \delta) = 1$. Let μA be a multiple fiber of $\rho : X \to B$ with the contribution δ of $\overline{A}$ in $\sigma^*(M_2)$. Let $\mu/\delta = [\alpha_1, \ldots, \alpha_s]$ and $\mu/(\mu - \delta) = [\alpha'_1, \ldots, \alpha'_{s'}]$ be the fractional expansions. Then the fiber $p^*(\rho(A))$ has the following dual graph:*

$$\begin{array}{ccccccccccccccccc}
-1 & & -\alpha'_1 & & & & -\alpha'_{s'} & & -1 & & -\alpha_s & & & & -\alpha_1 & & -1 \\
\circ & \text{---} & \circ & \text{---} & \cdots & \text{---} & \circ & \text{---} & \circ & \text{---} & \circ & \text{---} & \cdots & \text{---} & \circ & \text{---} & \circ \\
H_2 & & A'_1 & & & & A'_{s'} & & \overline{A} & & A_a & & & & A_1 & & H_1
\end{array}$$

where $({H_1}^2) = ({H_2}^2) = -1$ if $n = a = 1$.

The next result will clarify the geometric meaning of the condition $\mu_0\mu_1 - \mu_1\delta_0 - \mu_0\delta_1 = \pm 1$.

Lemma 1.4. *Let (μ_0, δ_0) and (μ_1, δ_1) be pairs as in Lemma 1.2 satisfying the condition $\mu_0\mu_1 - \mu_1\delta_0 - \mu_0\delta_1 = \pm 1$. Suppose that $\delta_0 > 0$ and $\delta_1 > 0$. Let $\mu_1/\delta_1 = [\alpha_1, \ldots, \alpha_s]$ and $\mu_0/\delta_0 = [\beta_1, \ldots, \beta_t]$ be the* *fractional expansions. Let E be a union of smooth rational curves with simple normal crossings on a smooth projective surface whose dual graph is given as below:*

$$
\begin{array}{ccccccccccc}
-\beta_t & & -\beta_2 & & -\beta_1 & & -1 & & -\alpha_1 & & -\alpha_2 & & -\alpha_s \\
\circ & \cdots\cdots & \circ & — & \circ & — & \circ & — & \circ & — & \circ & \cdots\cdots & \circ \\
B_t & & B_2 & & B_1 & & H_1 & & A_1 & & A_2 & & A_s
\end{array}
$$

Then the following assertions hold.

(1) *Suppose $\mu_0\mu_1 - \mu_1\delta_0 - \mu_0\delta_1 = 1$. Then E is contractible to a smooth point.*

(2) *Suppose $\mu_0\mu_1 - \mu_1\delta_0 - \mu_0\delta_1 = -1$. Then E contracts to a union of two smooth rational curves with one of the following dual graphes:*

$$
\begin{array}{ccccc}
 & 0 & -\delta & & \\
\circ & — \circ — & \circ & \cdots & (1), \\
G' & & & &
\end{array}
\quad \textit{or} \quad
\begin{array}{ccccc}
 & -\delta & 0 & & \\
\circ & — \circ — & \circ & \cdots & (2) \\
G' & & & &
\end{array}
$$

where G' denotes the proper transform of the component G in the fiber $p^*(\rho(\mu_0 A_0))$ and $(G'^2) = \delta - 1$ (resp. $(G'^2) = -1$) in the case (1) (resp. (2)).

Proof. First of all, we shall show that either $\alpha_1 = 2$ or $\beta_1 = 2$. Write the condition $\mu_0\mu_1 - \mu_1\delta_0 - \mu_0\delta_1 = \pm 1$ as

$$\left(\frac{\mu_0}{\delta_0} - 1\right)\left(\frac{\mu_1}{\delta_1} - 1\right) = 1 \pm \frac{1}{\delta_0\delta_1}.$$

Suppose $\alpha_1 \geq 3$ and $\beta_1 \geq 3$. Write $\mu_1 = \alpha_1\delta_1 - \delta_1'$ and $\mu_0 = \beta_1\delta_0 - \delta_0'$ with $0 \leq \delta_1' < \delta_1$ and $0 \leq \delta_0' < \delta_0$. Then we have

$$
\begin{aligned}
\left(\frac{\mu_0}{\delta_0} - 1\right)\left(\frac{\mu_1}{\delta_1} - 1\right) &= \left(\beta_1 - 1 - \frac{\delta_0'}{\delta_0}\right)\left(\alpha_1 - 1 - \frac{\delta_1'}{\delta_1}\right) \\
&\geq \left(\beta_1 - 2 + \frac{1}{\delta_0}\right)\left(\alpha_1 - 2 + \frac{1}{\delta_1}\right) \\
&\geq \left(1 + \frac{1}{\delta_0}\right)\left(1 + \frac{1}{\delta_1}\right) > \left(1 + \frac{1}{\delta_0\delta_1}\right)
\end{aligned}
$$

which is a contradiction.

(1) We shall prove the first assertion. Suppose $\beta_1 = 2$. Write $\mu_0 = 2\delta_0 - \delta_0'$ with $0 \le \delta_0' < \delta_0$. Suppose further that $t = 1$, i.e., $\mu_0 = 2, \delta_0 = 1, \delta_0' = 0$. Then $\mu_1 = 2\delta_1 + 1$ and the dual graph becomes

$$\begin{array}{ccccccccc}
-2 & & -1 & & -3 & & -2 & & -2 \\
\circ & \text{——} & \circ & \text{——} & \circ & \text{——} & \circ & \text{——}\cdots\cdots\text{——} & \circ \\
B_1 & & H_1 & & A_1 & & A_2 & & A_{\delta_1 - 1}
\end{array}$$

Hence it contracts to a smooth point. Suppose that $t \ge 2$. Let $\mu_0' = \delta_0, \mu_1' = \mu_1 - \delta_1$ and $\delta_1' = \delta_1$. Then the pairs (μ_0', δ_0') and (μ_1', δ_1') satisfy

$$\mu_0'\mu_1' - \mu_1'\delta_0' - \mu_0'\delta_1' = 1.$$

If $\alpha_1 = 2$ we can argue in a similar fashion. Hence we are done by induction. The first assertion is verified.

(2) Next we shall verify the second assertion. Suppose $\beta_1 = 2$ and $t = 1$. Then $\mu_1 = 2\delta_1 - 1$ and $\mu_1/\delta_1 = [2, \delta_1]$. Hence E contracts to a union of smooth rational curves with the dual graph:

$$\begin{array}{ccccc}
 & & 0 & & -\delta_1 \\
\circ & \text{——} & \circ & \text{——} & \circ \\
G & & & &
\end{array}$$

where $\delta_1 \ge 2$. Note that $\delta_1 \ne 1$. If $\alpha_1 = 2$ and $s = 1$, we have a similar conclusion as above with the second dual graph in the statement. Suppose that $\alpha_1 = \beta_1 = 2, s \ge 2$ and $t \ge 2$. We shall show that this case does not occur. Write $\mu_i = 2\delta_i - \delta_i'$ with $\delta_i' \ge 1$ for $i = 0, 1$. Then the condition $\mu_0\mu_1 - \mu_1\delta_0 - \mu_0\delta_1 = -1$ reads as $\delta_1\delta_0' + \delta_0\delta_1' = \delta_0'\delta_1' + 1$. This is a contradiction since $\delta_0 > \delta_0'$ and $\delta_1 > \delta_1'$. So, $\alpha_1 \ge 3$ if $\beta_1 = 2, s \ge 2$ and $t \ge 2$. As in the proof of the assertion (1), let $\mu_0' = \delta_0, \mu_1' = \mu_1 - \delta_1$ and $\delta_1' = \delta_1$. Then the pairs (μ_0', δ_0') and (μ_1', δ_1') satisfy

$$\mu_0'\mu_1' - \mu_1'\delta_0' - \mu_0'\delta_1' = -1.$$

Hence we are done by induction.

In the graph, call the component with self-intersection number 0 (resp. $-\delta$) L (resp. S). In view of Lemma 1.2, if E contracts to a union of two rational curves $L+S$, the linear chain E' contracts to a smooth point, where E' has the following dual graph with $\mu_0/(\mu_0 - \delta_0) = [\beta_1', \dots, \beta_{t'}']$ and $\mu_1/(\mu_1 - \delta_1) = [\alpha_1', \dots, \alpha_{s'}']$.

$$\begin{array}{ccccccccc} -\beta'_{t'} & & -\beta'_2 & -\beta'_1 & -1 & -\alpha'_1 & -\alpha'_2 & & -\alpha'_{s'} \\ \circ & \cdots & \circ & \circ & \circ & \circ & \circ & \cdots & \circ \\ B'_{t'} & & B'_2 & B'_1 & H_2 & A'_1 & A'_2 & & A'_{s'} \end{array}$$

Let W be the surface obtained from V by the contractions of E and E' as described above. Then W has a $\mathbf{P}^1$-fibration $p' : W \to \mathbf{P}^1$ given by the pencil $|L|$ and S is a cross-section of p'. In the first case, the count of the Picard number of W shows that G' is a cross-section of p' with $({G'}^2) = \delta - 1$. In the second case, the count of the Picard number shows again that $({G'}^2) = -1$ and p' has a unique singular fiber which contains G' and $\overline{A}$ as the terminal (-1) components and the (-2) components in between (see the dual graph of the fiber $p^{-1}(\rho(\mu_0 A_0))$). Q.E.D.

Consider the case where $\mu_0 = 1$ and $\delta_0 = 0$.

Lemma 1.5. *Suppose $\mu_0 = 1$ and $\delta_0 = 0$. Then $\delta_1 = 1$ if $a = 0$ and $\mu_1 = \delta_1 + 1$ if $a = 1$. Let $\mu_1/\delta_1 = [\alpha_1, \ldots, \alpha_s]$ be the fractional expansion. Let E be a union of smooth rational curves on a smooth projective surface V with the dual graph:*

$$\begin{array}{ccccc} -1 & -\alpha_1 & -\alpha_2 & & -\alpha_s \\ \circ & \circ & \circ & \cdots & \circ \\ H_1 & A_1 & A_2 & & A_s \end{array}$$

Then either E contracts to a smooth point (case $a = 1$) or E is a union of two smooth rational curves with the dual graph (case $a = 0$) :

$$\begin{array}{cc} 0 & -\mu_1 \\ \circ & \circ \end{array}$$

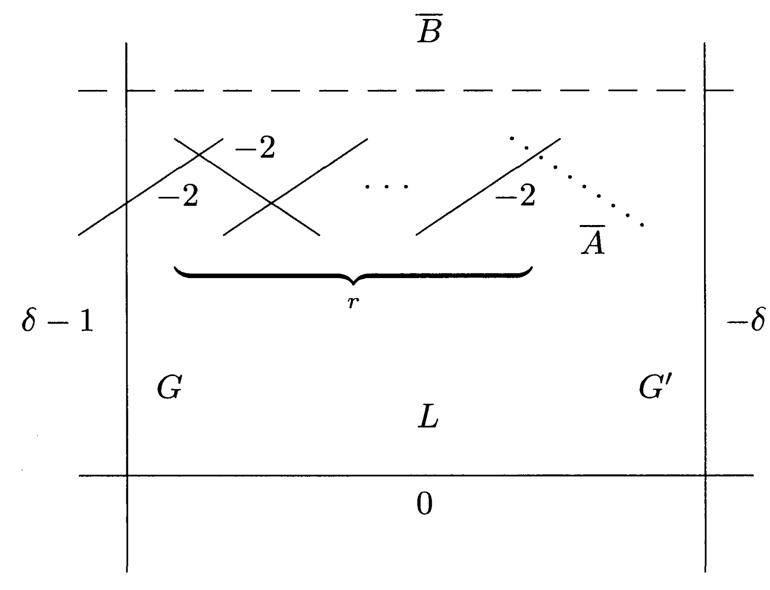
Proof. If $a = 0$ then $({H_1}^2) = 0, s = 1$ and $({A_1}^2) = -\mu_1$. If $a = 1$, then $[\alpha_1, \ldots, \alpha_s] = [2, \ldots, 2]$. It is clear that E contracts to a smooth point. Q.E.D.

§2. Explicit equations

First of all, consider the case $n = 1$. We only consider the case $a = 1$ and $\delta_0 \neq 0$. The case $a = 0$ and $\delta_0 = 0$ can be treated in a similar fashion. Furthermore, we assume that $\mu_0\mu_1 - \mu_1\delta_0 - \mu_0\delta_1 = -1$. The $\mathbf{P}^1$-fibration $p : V \to \mathbf{P}^1$, which extends the given $\mathbf{A}^1_*$-fibration $\rho : X \to \mathbf{P}^1$, has two degenerate fibers S_0 and S_1 and two sections H_1 and H_2. We assume that $S_0 \cap X = \mu_0 A$ and $S_1 \cap X = \mu_1 B$, where $A \cong \mathbf{A}^1$ and $B \cong \mathbf{A}^1_*$. Let E (resp. E') be the connected component of

$$D - G \cup \{\text{the side linear chain between } G \text{ and } \overline{A}\}$$

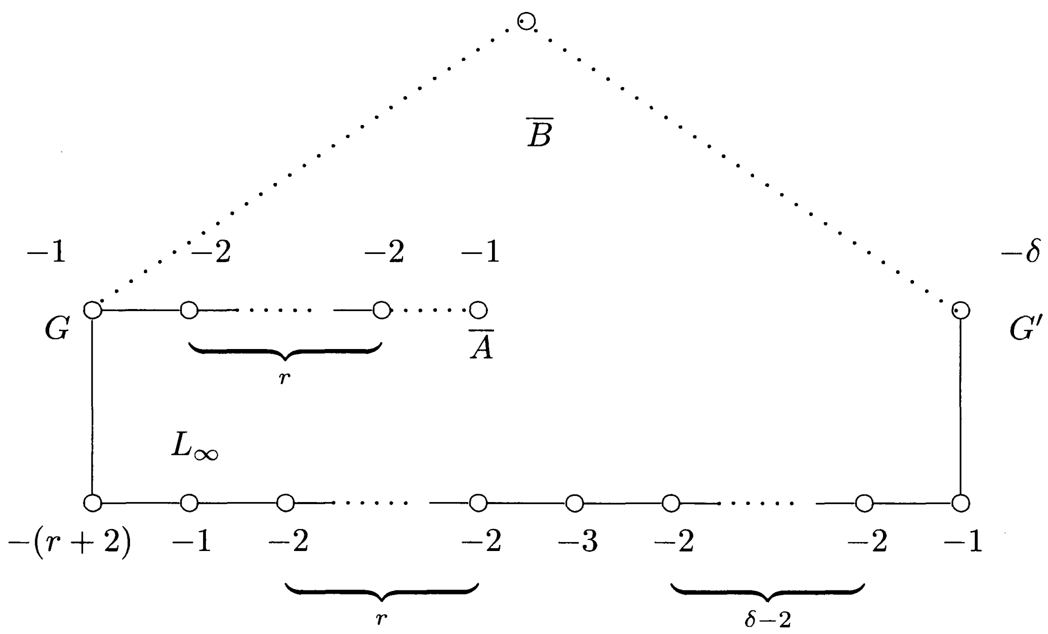
which contains H_1 (resp. H_2) (see the notations at the beginning of the section 1). By Lemma 1.4, E (resp. E') contracts to a union of two curves of the form (1) or (2) (resp. a smooth point). Suppose first that E contracts to a union of two curves of the form (1). By the contaction of E and E', we obtain a smooth projective surface W with the boundary divisor Δ such that $W - \Delta$ is isomorphic to X and Δ has the following configuration (Figure 1):



(Figure 1.)

where $\overline{A}$ (resp. $\overline{B}$) denotes, by abuse of notations, the image of $\overline{A}$ (resp. $\overline{B}$) under the contraction.

We blow up the intersection point $G \cap L$ and its infinitely near points to produce a configuration with the following dual graph (Figure 2):



(Figure 2.)

In the configuration, all curves but $\overline{A}, \overline{B}$ and L_∞ are contracted to two points, say P and Q, on the image of L_∞ (which we denote by the same symbol L_∞). In fact, the obtained surface is the projective plane $\mathbf{P}^2$ and $\mathbf{P}^2 - L_\infty$ is isomorphic to X. The image $\widetilde{B}$ of $\overline{B}$ is a curve of degree $r+2$ having a cuspidal singularity at P of multiplicity $r+1$ and passing through Q smoothly, and the image $\widetilde{A}$ of $\overline{A}$ is a line meeting $\widetilde{B}$ at P with order of contact $r+2$.

Choose a system of homogeneous coordinates (X, Y, Z) on $\mathbf{P}^2$ so that L_∞ and $\widetilde{A}$ are defined by $Z = 0$ and $X = 0$, respectively. Then $\widetilde{B}$ is defined by an equation

$$YX^{r+1} - P(X, Z) = 0,$$

where

$$P(X, Z) = a_1 X^{r+1} Z + a_2 X^r Z^2 + \cdots + a_{r+2} Z^{r+2}$$

with $a_{r+2} \neq 0$. We may assume $a_1 = 0$ by replacing Y by $Y - a_1 Z$. Let Λ be the pencil on $\mathbf{P}^2$ consisting of the closures of fibers of the given $\mathbf{A}^1_*$-fibration $\rho : X \to \mathbf{P}^1$. Since $\mu_1 B$ is a multiple fiber, we have

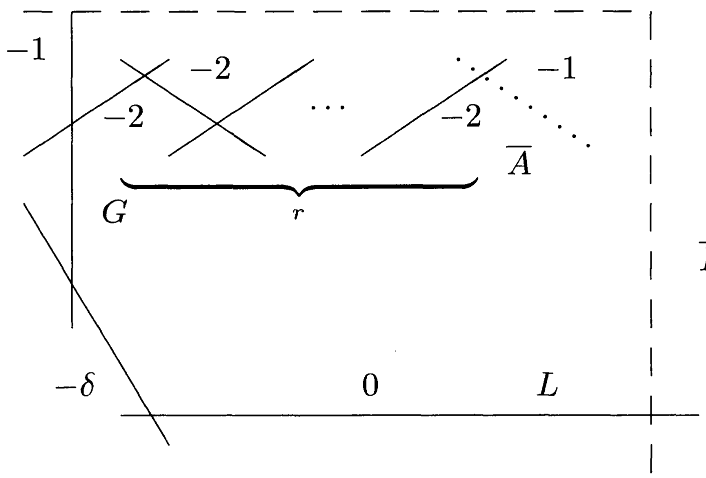
$$\Lambda = \{(YX^{r+1} - P(X,Z))^{\mu_1} + \lambda X^{\mu_0} Z^{\mu_1(r+1)+\mu_1-\mu_0} = 0;\ \lambda \in \mathbf{P}^1\} \cdots (1)$$

where we consider

$$(YX^{r+1} - P(X,Z))^{\mu_1} Z^{\mu_0 - \mu_1(r+2)} + \lambda X^{\mu_0} = 0$$

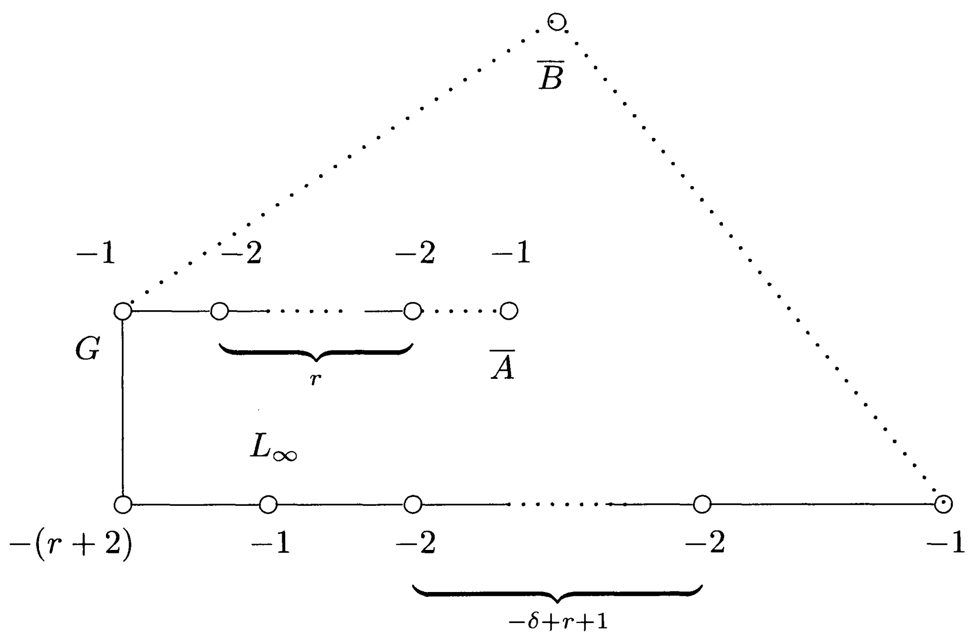
instead of the given equation if $\mu_0 > \mu_1(r+2)$.

Suppose next that E contracts to a union of two curves of the form (2). Then, with the above notation, Δ has the following configuration (Figure 3):



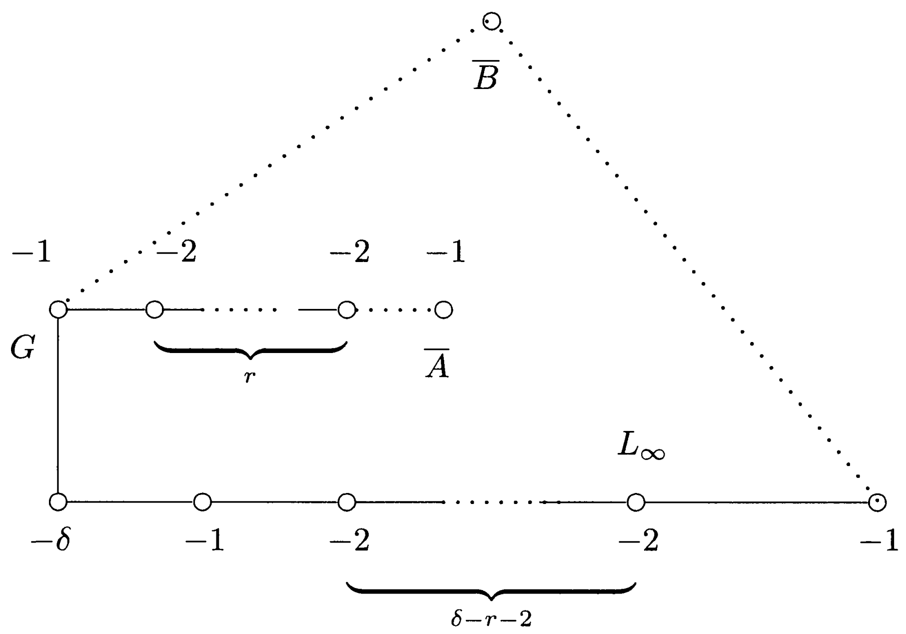
(Figure 3.)

We consider two cases according as $-\delta + r + 1 \geq 0$ or $-\delta + r + 1 < 0$. Suppose first $-\delta + r + 1 \geq 0$. Then we obtain the following dual graph after a suitable blowing-up of the above configuration (Figure 4):



(Figure 4.)

Again, all curves but $\overline{A}, \overline{B}$ and L_∞ are contracted to two points, say P and Q, on the image of L_∞. The surface obtained by this contraction is $\mathbf{P}^2$ and L_∞ is the line at infinity, i.e., $\mathbf{P}^2 - L_\infty \cong X$. The image $\widetilde{B}$ of $\overline{B}$ is a curve of degree $r+2$ having a cuspidal singurarity at P of multiplicity $r+1$ and passing through Q, and the image $\widetilde{A}$ of $\overline{A}$ is a line meeting $\widetilde{B}$ at P with order of contact $r+2$. Then we reach to the expression (1) of the pencil Λ. Consider next the case $-\delta + r + 1 < 0$. Then we blow up the intersection point $L \cap \overline{B}$ and its $(\delta - r - 2)$ infinitely near points lying on the curve $\overline{B}$ (Figure 4):



(Figure 5.)

Then all curves but $\overline{A}, \overline{B}$ and L_∞ are contracted to two points on the image of L_∞, and the surface obtained by this contraction is $\mathbf{P}^2$ with L_∞ as a line at infinity. The same argument as in the previous cases gives the expression (1) of the pencil Λ.

Consider the case $\mu_0 = 1$ and $\delta_0 = 0$. Turning the configuration upside down if necessary, we have only to consider the case $a = 0, \mu_0 = \delta_1 = 1$ and $\delta_0 = 0$. Then one can easily show that we have the same configuration as in Figure 1 with $\delta = \mu_1$ after a suitable contraction of the components of D. So, we have the same expression of Λ as given in (1).

Consider finally the case $n = 0$. The case $a = 1$ and $\mu_0 = \delta_0 + 1$ is obtained from the case $a = 0$ and $\delta = 1$ by turning the graph upside down, i.e., changing the roles of H_1 and H_2. So, we treat only the case $a = 0$ and $\delta_0 = 1$. Then we have the form (2) in the case $n = 1$. So, the argument is a complete repetition in the case $n = 1$ with the form (2). We have thus the same expression as (1) with $\mu_1 = 1$.

Hence we obtain the following result.

Theorem 2.1. *Let $\rho : X \to \mathbf{P}^1$ be an $\mathbf{A}^1_*$-fibration parametrized by $\mathbf{P}^1$. Then, with the above notations, the pencil associated to ρ is given as follows:*

$$\Lambda = \left\{ \left(yx^{r+1} - p(x)\right)^{\mu_1} + \lambda x^{\mu_0} = 0; \quad \lambda \in \mathbf{P}^1 \right\},$$

where $p(x) \in k[x], \deg p(x) \le r$ and $p(0) \ne 0$. Furthermore, we understand that $\mu_1 = 1$ when there is no multiple fiber whose reduced form is isomorphic to $\mathbf{A}^1_*$.

§3. Complements to the previous results

(I) Let C be an irreducible curve of $\mathbf{A}^2$ and let X be anew the complement $\mathbf{A}^2 - C$. In Aoki [1], it is observed whether or not X has an étale non-finite endomorphism which is not an automorphism. In the case where X has an $\mathbf{A}^1_*$-fibration $\rho : X \to B$ and ρ extends to an $\mathbf{A}^1_*$-fibration $\widetilde{\rho} : \mathbf{A}^2 \to \widetilde{B}$, i.e., a general fiber of ρ is closed in $\mathbf{A}^2$, the case $\widetilde{B} \cong \mathbf{P}^1$ is missing in the observation. We shall consider here this case by applying Theorem 2.1. Note then that C is a fiber of $\widetilde{\rho}$ taken with the reduced structure. We consider the following three cases separately:

(1) C is a multiple fiber $\mu_0 A_0$, where $A_0 \cong \mathbf{A}^1$.
(2) C is a multiple fiber $\mu_1 A_1$, where $A_1 \cong \mathbf{A}^1_*$.
(3) C is a general fiber of ρ.

In the case (1), X has logarithmic Kodaira dimension $\overline{\kappa}(X) = -\infty$ and this case is treated in [1]. In the case (2), it follows from Theorem 2.1 and the arguments leading to its proof that C is defined by an equation of the form $yx^{r+1} - p(x) = 0$, where $p(x) \in k[x], \deg p(x) \le r$ and $p(0) \ne 0$. The polynomial $yx^{r+1} - p(x)$ is then a generically rational polynomial, and this case is also treated in [1]. So, consider the case (3). By the arguments in [6] to prove the first assertion of Lemma 1.1, we know that

$$\overline{\kappa}(X) = 1 \quad \text{(resp. } 0\text{)} \ \text{ if and only if } \ n - \sum_{i=1}^{n} \frac{1}{n_i} > 0 \quad \text{(resp. } = 0\text{)},$$

where $n = 0, 1$. If $n = 1$ (resp. 0) then $\overline{\kappa}(X) = 1$ (resp. 0). If $n = 0$ (hence $\mu_1 = 1$) then the general fiber C is defined by $f = 0$ with $f = yx^{r+1} - p(x) + x^{\mu_0}$, and f is a generically rational polynomial. So we may assume that $n = 1$. Hence $\overline{\kappa}(X) = 1$.

Let $\alpha : X_1 \to X_2$ be an étale endomorphism, where we denote the source (resp. target) X by X_1 (resp. X_2). Accordingly, we denote by $\rho_i : X_i \to B_i$ $(i = 1, 2)$ the same $\mathbf{A}^1_*$-fibration $\rho : X \to B$, where $B_1 \cong B_2 \cong \mathbf{A}^1$. By [1, Lemma 3.2], there exists an endomorphism $\beta : B_1 \to B_2$ such that $\rho_2 \cdot \alpha = \beta \cdot \rho_2$.

We shall show that β is the identity automorphism. In fact, β extends to an endomorphism $\widetilde{\beta} : \widetilde{B}_1 \to \widetilde{B}_2$, where $\widetilde{B}_i \cong \mathbf{P}^1$ and $\widetilde{B}_i = B_i \cup \{P\}$ for $i = 1, 2$ with $P := \widetilde{\rho}(C)$. It is clear that $\widetilde{\beta}^{-1}(P) = P$. Let $P_i := \widetilde{\rho}(A_i)$ for $i = 0, 1$. By [3, Lemma 3.1], it follows that $\widetilde{\beta}(P_i) = P_i$ for $i = 0, 1$ because $\gcd(\mu_0, \mu_1) = 1$. Note that $\widetilde{\beta}$ is unramified at P_0 and P_1. By the same lemma, it follows that if $\widetilde{\beta}(Q) = P_i$ $(i = 0, 1)$ for $Q \neq P_i$, then the ramification index of $\widetilde{\beta}$ at Q equals to μ_i. Let $d := \deg \widetilde{\beta}$. Suppose that r (resp. s) points of $\widetilde{B}_1$ other than P_1 (resp. P_0) are mapped to P_1 (resp. P_0) under $\widetilde{\beta}$. By the Riemann-Hurwitz theorem, we have

$$\begin{aligned} -2 &= -2d + (d-1) + r(\mu_1 - 1) + s(\mu_0 - 1) \\ &= d - r - s - 3 \end{aligned}$$

where $d = \mu_1 r + 1 = \mu_0 s + 1$. Hence we obtain

$$d = r + s + 1 = \mu_1 r + 1 = \mu_0 s + 1. \tag{1}$$

If $d \neq 1$ then $r > 0$ and $s > 0$. It is then easy to derive a contradiction from (1) because $\gcd(\mu_0, \mu_1) = 1$. Hence $d = 1$. Since β is an automorphism of $\mathbf{P}^1$ fixing three points P, P_0, P_1, it follows that β is the identity automorphism.

Since α satisfies now $\rho \cdot \alpha = \rho$, the étale endomorphism α induces an endomorphism $\alpha_K : X_{1,K} \to X_{2,K}$ of the generic fiber X_K of ρ, where K is the function field of B. Since ρ is an untwisted $\mathbf{A}^1_*$-fibration, we know that $X_K = \operatorname{Spec} K[u, u^{-1}]$. Hence $\alpha_K^*(u) = au^{\pm n}$ with $a \in K^*$ and $n = \deg \alpha$. Let G be the group of the n-th roots of unity in k. Then G acts on $X_{1,K}$ and $X_{2,K}$ is the quotient curve $X_{1,K}/G$. Hence the function field $k(X_1)$ is a Galois extension of $k(X_2)$ with Galois group G. Let $\widetilde{X}_2$ (resp. W) be the normalization of X_2 (resp. $\mathbf{A}^2$) in $k(X_1)$, where X_2 is the open set $\mathbf{A}^2 - C$ of $\mathbf{A}^2$, and let $\nu : \widetilde{X}_2 \to X_2$ (resp. $\widehat{\nu} : W \to \mathbf{A}^2$) be the normalization morphism. By [5, Lemma 5], $\nu : \widetilde{X}_2 \to X_2$ is an étale Galois covering with group G with $\widetilde{X}_2$ containing X_1 as an open set, the composite $\rho_2 \cdot \nu : \widetilde{X}_2 \to B$ is an $\mathbf{A}^1_*$-fibration such that $\rho_2 \cdot \nu|_{X_1} = \rho_1$, and $(\rho_2 \cdot \nu)^{-1}(P_0)$ with $P_0 = \rho(A_0)$ is a disjoint union of n copies of the affine lines ${}^gA_0$ $(g \in G)$ so that $\widetilde{X}_2 - X_1 = \coprod_{g \in G, g \neq 1} {}^gA_0$, where

$A_0 \cong \mathbf{A}^1$. The surface W is a normal affine surface with a G-action, and $\mathbf{A}^2$ is the quotient surface W/G. Furthermore, $\widetilde{X}_2$ is a Zariski open set of W. Note that $\widetilde{\rho}\cdot\widehat{\nu} : W \to \widetilde{B}$ is an $\mathbf{A}^1_*$-fibration. Let $Z = (\widetilde{\rho}\cdot\widehat{\nu})^{-1}(P)$, where $P = \widetilde{\rho}(C)$. Then $\widehat{\nu}$ induces a finite morphism $\overline{\nu} : Z \to C$. Since the $\mathbf{A}^1_*$-fibration $\widetilde{\rho}\cdot\widehat{\nu} : W \to \widetilde{B}$ is extended to a $\mathbf{P}^1$-fibration with two cross-sections at infinity and since every irreducible component of Z has at least two places at infinity (for otherwise it cannot dominate C which is isomorphic to $\mathbf{A}^1_*$), it follows that

(1) Z is irreducible,
(2) W has no singular points along Z,
(3) Z is isomorphic to $\mathbf{A}^1_*$.

In fact, let V be a completion of W such that V is smooth along $V - W$, the complement $V - W$ supports a divisor with simple normal crossings and the $\mathbf{A}^1_*$-fibration $\widetilde{\rho}\cdot\widehat{\nu}$ extends to a $\mathbf{P}^1$-fibration $q : V \to \widetilde{B}$. If Z is reducible, the fiber $q^{-1}(P)$ must contain a loop of the irreducible components because each irreducible component of Z has at least two places at infinity. So, Z is irreducible. We may assume that $q^{-1}(P)$ contains no (-1) curves lying in $V - W$. If W has singular points on Z, the proper transform $\widehat{Z}$ of Z by a minimal resolution of singularities of W is a unique (-1) curve in the fiber meeting three or more components of the fiber. This is a contradiction. So, W is smooth along W. Now it is clear that Z is isomorphic to $\mathbf{A}^1_*$. This implies that $\widehat{\nu} : W \to \mathbf{A}^2$ is an étale finite Galois covering. Hence $\widehat{\nu}$ is an isomorphism. In particular, $\alpha : X_1 \to X_2$ is an automorphism. Thus we obtain the following:

Theorem 3.1. *Let C be an irreducible curve in $\mathbf{A}^2 := \mathrm{Spec}\, k[x,y]$ defined by*

$$\left(yx^{r+1} - p(x)\right)^{\mu_1} + \lambda x^{\mu_0} = 0,$$

where $\mu_0 \geq 1, \mu_1 > 1$ and $\lambda \neq 0$ and let $X := \mathbf{A}^2 - C$. Then $\overline{\kappa}(X) = 1$ and every étale endomorphism of X is an automorphism.

(II) In [2], we considered an automorphism of infinite order of $\mathbf{A}^2$ which stabilizes an irreducible curve C. In [2, Lemma 1.4], the case where the curve C has a defining equation

$$f := \left(yx^{r+1} - p(x)\right)^{\mu_1} + \lambda x^{\mu_0} = 0,$$

is missing. We shall complete the result by treating here the missing case. If $\mu_1 = 1$, i.e., $n = 0$, then f is a generically rational polynomial, and this case is treated in [2]. So, we assume that $\mu_1 > 1$. As in the proof of Theorem 3.1, $\overline{\kappa}(X) = 1$ and any automorphism α of X preserves the

$\mathbf{A}^1_*$-fibration ρ, i.e., $\rho \cdot \alpha = \rho$. Then $\alpha^{-1}(A_0) = A_0$ and $\alpha^{-1}(A_1) = A_1$. Namely, we have $\alpha(x) = cx$ and $\alpha(yx^{r+1} + p(x)) = d\left(yx^{r+1} + p(x)\right)$ with $c, d \in k^*$. Here note that A_0 (resp. A_1) is defined by $x = 0$ (resp. $yx^{r+1} + p(x) = 0$). Since $p(0) \neq 0$, it follows that $d = 1$. Then we have

$$\alpha(y) = c^{-(r+1)}y + \frac{p(x) - p(cx)}{c^{r+1}x^{r+1}}.$$

Hence $p(x) = p(cx)$, and c is an m-th root of unity for some m with $0 < m < r+1$ because $\deg p(x) \leq r$. So, we obtain the following:

Theorem 3.2. *Let C be an irreducible curve in $\mathbf{A}^2 := \operatorname{Spec} k[x, y]$ defined by*

$$\left(yx^{r+1} - p(x)\right)^{\mu_1} + \lambda x^{\mu_0} = 0,$$

where $\mu_0 \geq 1, \mu_1 > 1$ and $\lambda \neq 0$ and let $X := \mathbf{A}^2 - C$. Then every automorphism of $\mathbf{A}^2$ which stabilizes the curve C is of finite order.

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The Zarankiewicz Problem via Chow Forms

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The well-known *Zarankiewicz problem* [Za] is to determine the least positive integer $Z(m,n,r,s)$ such that each $m \times n$ 0-1 matrix containing $Z(m,n,r,s)$ ones has an $r \times s$ submatrix consisting entirely of ones. In graph-theoretic language, this is equivalent to finding the least positive integer $Z(m,n,r,s)$ such that each bipartite graph on m black vertices and n white vertices with $Z(m,n,r,s)$ edges has a complete bipartite subgraph on r black vertices and s white vertices.

A complete solution of the Zarankiewicz problem has not been given. While exact values of $Z(m,n,r,s)$ are known for certain infinite subsets of m,n,r and s, only asymptotic bounds are known in the general case; for example, see Čulík [Č], Füredi [F], Guy [G], Hartmann, Mycielski and Ryll-Nardzewski [HMR], Hyltén-Cavallius [HC], Irving [I], Kövári, Sós and Turán [KST], Mörs [M], Reiman [Re], Roman [Ro], Znám [Zn]. Even the case $r = s = 2$ has not been answered in general. Here we quote some known facts about this case: Hartmann, Mycielski and Ryll-Nardzewski [HMR] proved the asymptotic bounds

$$c_1 n^{4/3} < Z(n,n,2,2) < c_2 n^{3/2}$$

for some constants $c_1 \cong \frac{3}{4}$ and $c_2 \cong 2$. Kövári, Sós and Turán [KST] proved that

$$Z(n,n,2,2) \leq 2n + n^{3/2}, \lim_{n\to\infty} n^{-3/2} Z(m,n,2,2) = 1.$$

Moreover, when p is a prime integer, [KST] proved that $Z(p^2+p, p^2, 2, 2) = p^2(p+1)+1$. Hyltén-Cavallius [HC] proved that $Z(m,n,2,2) \leq \frac{n}{2} + \sqrt{nm(m-1) + \frac{n^2}{4}} + 1$. Čulík [Č] proved that for $n \geq \binom{m}{2}$,

$$Z(m,n,2,2) = \binom{m}{2} + n + 1.$$

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Reiman [Re] showed the same equality for infinitely many other m and n, and he also established a connection between finding $Z(m, n, 2, 2)$ and the existence of projective planes of given orders. This last existence question is still wide open, and hence Reiman's work provides convincing evidence that finding $Z(m, n, 2, 2)$ for all m and n is a highly non-trivial problem. Guy [G] calculated $Z(m, n, 2, 2)$ for many small values of m and n. Further asymptotic and exact values were established in [Ro], [F].

This paper is an analysis of the $r = s = 2$ case of Zarankiewicz problem from the point of view of commutative algebra. Our motivation came from the complexity theory of permanental ideals of generic matrices. This brought forth a new connection between combinatorics, computational algebra, commutative algebra, and algebraic geometry involving not only permanental ideals, but also complexity of parameters and Chow forms. We describe these connections in the first two sections of this paper. In the final section, we also exhibit a connection with hypergraphs and three dimensional matrices. However, with these new connections we have not been able to shed any new light on the Zarankiewicz problem; we have simply found several reformulations.

§1. Permanental ideals and balanced matrices

We begin by introducing permanental ideals, parameters, and complexity of parameters via Chow forms. We present the Chow form for the permanental ideals, and rephrase the question of computing $Z(m, n, 2, 2)$ in terms of the complexity of parameter ideals and their Chow forms.

Let F be a field, and let X_{ij} be indeterminates over F, where $i = 1, \dots, m$ and $j = 1, \dots, n$, with $m, n \geq 2$. Let X be the $m \times n$ matrix whose ij-th entry is X_{ij}. The matrix X is the so-called generic $m \times n$ matrix. Let P be the ideal in the polynomial ring $F[X_{ij}|i, j]$ generated by all 2×2 subpermanents of X. Specifically,

$$P = (X_{ij}X_{i'j'} + X_{i'j}X_{ij'}|i < i' \leq m, j < j' \leq n).$$

Note that the permanent is like the determinant but with all minus signs replaced by plus signs. The ideal P is called the 2×2 permanental ideal of X.

Permanental ideals have not been studied a great deal. This is partly because that they do not seem to describe geometric properties, and partly because permanents are very difficult to compute. One can calculate the determinant of an $n \times n$ matrix in $O(n^3)$ steps, but for a permanent, many more steps are needed. Calculating the permanent is in fact a #P-complete problem (see for example [V, Valiant], [B, Bürgisser]).

One measure of the complexity of an ideal is the sparsity or non-sparsity of a system of parameters modulo it. Not surprisingly, the parameters modulo the 2×2 permanental ideal are not sparse. The goal is to determine this complexity more precisely.

Definition 1. For an element $\sum_{ij} c_{ij} X_{ij}$ to be a *parameter modulo an ideal* I, it is necessary and sufficient that it avoids all the minimal primes of I. A *system of parameters modulo* I is a sequence of elements $a_1, \ldots, a_d$, where d is the dimension of the ring modulo I, such that for all $i = 1, \ldots, d$, a_i is a parameter modulo the ideal $I + (a_1, \ldots, a_{i-1})$.

A parameter is said to be *sparse* if most of the c_{ij} are zero. The *complexity of a parameter* is defined to be the number of nonzero c_{ij}. The *complexity of* I is then defined to be the smallest possible sum of all the complexities of the parameters in a system of parameters, as we vary the systems.

When $m = n = 2$, the permanental 2×2 ideal P is a prime ideal, so that any one of the four X_{ij} variables is a parameter. In this case the complexity of a single parameter is 1.

When $2 = m < n$, Laubenbacher and Swanson [LS] showed that an element $\sum_{ij} c_{ij} X_{ij}$ is a parameter modulo the permanental ideal P exactly when for each row i, at least one c_{ij} is nonzero, and for each 2×2 submatrix of X, at least one of the corresponding c_{ij} is nonzero. Thus, one can see easily that the complexity of a parameter modulo the 2×2 permanental ideal of a $2 \times n$ generic matrix is exactly $n - 1$.

Furthermore, when $m, n \geq 3$, again according to [LS], for $\sum_{ij} c_{ij} X_{ij}$ to be a parameter modulo the permanental ideal P, it is necessary that for each i, some c_{ij} is nonzero, similarly that for each j, some c_{ij} is nonzero, and lastly that for all $i < i' \leq m$, $j < j' \leq n$, at least one of $c_{ij}, c_{ij'}, c_{i'j}, c_{i'j'}$ is nonzero.

We say that a 0-1 matrix is *balanced* if every 2×2 submatrix (not necessarily contiguous) contains at least one unit element, and we let $f(m, n)$ be the minimal number of ones in a balanced $m \times n$ matrix. Note that

$$f(m, n) = mn - Z(m, n, 2, 2) + 1$$

where $Z(m, n, r, s)$ is the Zarankiewicz number.

It turns out that $f(m, n)$ equals the smallest possible complexity of a single parameter, as we prove below. It is clear from above that in the case $2 = m \leq n$, both of these numbers are $n - 1$, and similarly when

$2 = n \leq m$, both of these numbers are $m - 1$. In the sequel, we will assume (without loss of generality) that $3 \leq m, n$.

We first need a lemma:

Lemma 1. *Let $m, n \geq 3$, and let A be a balanced $m \times n$ matrix with f_A ones. Then there exists a balanced $m \times n$ matrix B with exactly f_A ones such that every row and every column of B contains at least one nonzero entry.*

Proof. Suppose that one of the rows or columns of A is zero. Without loss of generality we may assume that the first row of A is zero. As A is balanced, each of the rows $2, 3, \ldots, m$ must have at least $n - 1$ ones. Thus after possibly permuting the rows and columns of A, the first three rows are of the form

$$\begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 1 & 1 \\ * & 1 & \cdots & 1 & ** \end{bmatrix}.$$

Here, $*, **$ are either 1 or 0, but they are not both 0 since A is balanced. Let B be obtained from A by switching the $(1,1)$ and $(2,2)$ entries. Then B is still balanced with f_A ones, and every row, every column of B has at least one 1. Q.E.D.

Now we can show the connection between the complexity of parameters and $f(m, n)$:

Proposition 1. *Whenever $2 \leq m, n$, $f(m, n)$ equals the minimal possible complexity of a parameter modulo P.*

Proof. By the earlier discussion, we may take $3 \leq m \leq n$ without loss of generality. For each parameter $\sum_{ij} c_{ij} X_{ij}$, we form the balanced matrix A whose (i, j) entry equals 0 if $c_{ij} = 0$ and equals 1 otherwise. Notice that the balanced matrix A constructed in this way has the additional property that every row and every column of A contains a nonzero entry. Conversely, given a balanced matrix A, we first use Lemma 1 to convert it (non-uniquely) into a balanced matrix B such that every row and every column of B contains a nonzero entry, we then construct a parameter $\sum c_{ij} X_{ij}$ modulo the 2×2 permanental ideal by setting c_{ij} to be the ij entry of B. This element is indeed a parameter by [LS]. Q.E.D.

Thus finding $f(m, n)$, the minimal number of ones in A, is the same as finding the sparsest possible parameter for the polynomial ring modulo the permanental ideal. Hence all the values and bounds on $Z(m, n, 2, 2)$ listed at the beginning of the paper apply also for $mn + 1$ minus the smallest possible complexity of a parameter. Clearly, no parameter is sparse.

§2. Chow forms

There is another way to analyze the complexity of ideals, due to Eisenbud and Sturmfels [ES]:

Theorem 1 ([ES, Theorem 2.7]). *The complexity of an ideal I equals the least number of variables appearing in any initial monomial of the Chow form of the ring modulo I (under any monomial ordering).*

Some helpful references for Chow forms are [Sh], [ES], [GS].

Of course, calculating the complexity of an ideal is much more than calculating the complexity of a single parameter. However, the complexity of the ideal divided by the number of parameters gives an upper bound on the complexity of a parameter, and so by the previous section this is a step towards computing $Z(m, n, 2, 2)$. Thus, the problem is first to calculate the Chow form of the ring modulo P, and secondly, to find a monomial ordering on the variables under which the initial monomial of the Chow form involves the fewest number of variables.

In general, the computation of Chow forms is difficult, in technical terms even NP-hard (see [ES] for discussion). Even in the case of determinantal ideals, which tend to be much better behaved than permanental ideals, the Chow forms are difficult to compute. Glassbrenner and Smith [GS] analyzed the complexity of determinantal ideals by using the theorem of Eisenbud and Sturmfels quoted above. For the ideal of 2×2 minors of a generic $m \times n$ matrix, Glassbrenner and Smith [GS] determined that the parameter complexity is exactly mn. As the number of parameters in a parameter system for this ideal is $m + n - 1$, this implies that we can choose the first parameter with at most $\frac{mn}{m+n-1}$ non-zero coefficients c_{ij}. However, as the determinantal ideal is prime, we may choose the first parameter to be any one of the variables, and hence the smallest possible complexity of a parameter is exactly 1. In contrast, the results on the Zarankiewicz problem quoted earlier show that for the 2×2 permanental ideal the complexity of a parameter is much larger.

Even though the complexity of the permanental Chow form and the complexity of permanental parameters are much larger than the corresponding complexities for determinants, here is at least one algebro-geometric problem that turns out to be easier for permanents than for determinants: namely, the computation of Chow forms.

The Chow form of an ideal is the product of the Chow forms of its minimal primes. In addition, the Chow form of the ideal I plus an ideal generated by variables is simply the Chow form of I. Putting these facts together, we get:

Proposition 2 (see [Sh]). *The Chow form of the permanental ideal P is the product of the Chow forms of all the ideals $J_{ii'jj'}$, with $i < i' \leq m, j < j' \leq n$, where $J_{ii'jj'} = (X_{ij}X_{i'j'} + X_{i'j}X_{ij'})$ (generated by a 2×2 permanent of X).*

The calculation of the Chow form of a principal ideal is straightforward (see for example [Sh]). In particular, to define the Chow form of $X_{ij}X_{i'j'} + X_{i'j}X_{ij'}$, we first introduce 12 new variables C_{lkp}, l varying from 1 to the dimension of the polynomial ring in the four given variables modulo the quadric (which is 3), and kp varying over the subscripts of the variables X above. Let $M_{ii'jj'}$ be the 3×4 generic matrix with indeterminates C_{lkp} each of whose rows contains the variables with the same first subscript and whose columns have the matching rest of the subscripts. Explicitly,

$$M_{ii'jj'} = \begin{bmatrix} C_{1ij} & C_{1i'j} & C_{1ij'} & C_{1i'j'} \\ C_{2ij} & C_{2i'j} & C_{2ij'} & C_{2i'j'} \\ C_{3ij} & C_{3i'j} & C_{3ij'} & C_{3i'j'} \end{bmatrix}.$$

The Chow form of $X_{ij}X_{i'j'} + X_{i'j}X_{ij'}$ is given by

$$R_{ii'jj'} = \Delta_{ij}\Delta_{i'j'} + \Delta_{i'j}\Delta_{ij'},$$

where Δ_{kp} is the determinant of the submatrix of $M_{ii'jj'}$ after removing the column corresponding to kp. We thus have:

Theorem 2. *The Chow form of P is $\prod_{i,i',j,j'} R_{ii'jj'}$.*

One can verify that each $R_{ii'jj'}$ is a linear combination of 66 distinct monomials of degree 6. Hence the Chow form is the product of $\binom{m}{2}\binom{n}{2}$ factors, each of which is a linear combination of 66 monomials of degree 6. Thus while the Chow form is relatively easy to get at, its expansion is far from computationally trivial.

By the Eisenbud-Sturmfels result (Theorem 1), we now have:

Theorem 3. *The parameter complexity of the ideal P equals the minimal number of distinct variables C_{lkp} appearing in any monomial in the expansion of the Chow form of P.*

This new formulation raises more questions than answers:

Question 1. *What monomials appear in the expansion of the Chou form of P? Is there a combinatorial representation of these monomials?*

Question 2. *What is the smallest possible number of distinct variables C_{lkp} such that a monomial appearing in the Chow form of P is a* *power product of exactly these variables? Also, what is the smallest possible number of distinct variables* C_{1kp} *such that a monomial appearing in the Chow form of* P *is a power product of these variables and variables* $C_{2k'p'}, C_{3k'p'}$*?*

By Theorem 3) the answer to the first part of Question 2 is exactly the complexity of the ideal P. Furthermore, this number divided by 3 is an upper bound on the complexity of one parameter modulo P, and hence also an upper bound on $f(m,n)$.

A further question is then:

Question 3. *Does this upper bound on* $mn + 1 - f(m,n)$ *give a new lower bound on the Zarankiewicz number* $Z(m,n,2,2)$*?*

§3. Zarankiewicz problem in three dimensions and hypergraphs

It turns out that the monomials appearing in the Chow form of P are related to a certain *three-dimensional* Zarankiewicz problem. We now discuss this connection.

We will call a $3 \times m \times n$ 0-1 matrix *balanced* if (a) it contains no zero submatrix of size $2 \times 2 \times 1$, $2 \times 1 \times 2$, or $1 \times 2 \times 2$, and (b) none of the mn columns $\{A_{1,i,j}, A_{2,i,j}, A_{3,i,j}\}$ consists entirely of zeros. We define $g(m,n)$ to be the minimum number of ones in any balanced matrix of size $3 \times m \times n$.

Just as the (2-dimensional) Zarankiewicz problem can be phrased in the language of graph theory, so the above condition (a) can be expressed in terms of hypergraphs. Here we are looking for the minimum number of edges in the complete tripartite 3-graph $K_{3,m,n}$ with the property that the complement does not contain the tripartite 3-graph $K_{2,2,1}$. Condition (b) seems perhaps a little less natural. It is interesting to note, however, that condition (b) is similar to the extra condition that arose in our combinatorial interpretation of the (2-dimensional) Zarankiewicz problem: namely, that each column of the matrix should contain at least one 1. Lemma 1 showed that this extra condition was, in fact, redundant. However, this does not appear to follow easily in the 3-dimensional case.

For any monomial γ in the variables C_{lkp} ($l = 1\ldots 3$, $k = 1\ldots m$, $p = 1\ldots n$), we can form a $3 \times m \times n$ 0-1 matrix, where a 1 in position (l,k,p) indicates that γ is divisible by C_{lkp}. We then have the following:

Theorem 4. *Any monomial that appears in the expansion of the Chow form of* P *determines a balanced* $3 \times m \times n$ *submatrix. Hence, the parameter complexity of the ideal* P *is greater than or equal to* $g(m,n)$*.*

Manifestly, a $3\times m\times n$ matrix is balanced if and only if every $3\times 2\times 2$ submatrix of it is balanced. Thus, to prove the theorem, it is enough to prove the following lemma.

Lemma 2. *Any monomial appearing in the expansion of* $R_{ii'jj'} = \Delta_{ij}\Delta_{i'j'} + \Delta_{i'j}\Delta_{ij'}$ *determines a* $3\times 2\times 2$ *balanced submatrix of the matrix* (C_{lkp}).

Proof. For condition (a), there are three things to check. First consider the case of a $1\times 2\times 2$ submatrix. Without loss of generality, such a matrix corresponds to the four monomials $C_{1ij}, C_{1i'j}, C_{1ij'}, C_{1i'j'}$. Clearly, each of the four determinants Δ above will involve one of these monomials. Now consider a $2\times 1\times 2$ submatrix. Without loss of generality, such a matrix corresponds to the four monomials $C_{1ij}, C_{2ij}, C_{1ij'}$, $C_{2ij'}$. Any term in $\Delta_{i'j'}$ or in $\Delta_{i'j}$ contains one of these four monomials. Thus any term of $R_{ii'jj'}$ will contain one of these four, as well. The case of $2\times 2\times 1$ submatrices is similar.

To verify condition (b), we consider the monomials $C_{1ij}, C_{2ij}, C_{3ij}$. Clearly, any term in $\Delta_{i'j}$, $\Delta_{ij'}$, or $\Delta_{i'j'}$ contains one of these three monomials. Thus any term of $R_{ii'jj'}$ will also contain at least one of the three monomials above. Hence, condition (b) is satisfied. Q.E.D.

Given the inequality of Theorem 4, one might wonder if the complexity of ideal P is actually *equal* to $g(m,n)$. Indeed, the following converse of Lemma 2 is true. Given any $3\times 2\times 2$ balanced matrix M, there is a monomial occurring in the expansion of $\Delta_{ij}\Delta_{i'j'}+\Delta_{i'j}\Delta_{ij'}$ all of whose variables correspond to 1's in M. (One can check this, for example, by a tedious examination of cases.) Suppose that A is a balanced $3\times m\times n$ matrix. It would follow from the converse of Lemma 2 that one of the $66^{\binom{m}{2}\binom{n}{2}}$ terms in the expansion of the Chow form of P consists entirely of variables corresponding to 1's in A. Thus if no monomial in this expansion cancels out entirely, then the parameter complexity of P is exactly $g(m,n)$. Unfortunately, it is not clear to us whether or not such cancelation can occur.

In any case, each of the three $1\times m\times n$ submatrices of the $3\times m\times n$ balanced matrix is a balanced $m\times n$ matrix. Thus $g(m,n)\geq 3f(m,n)$. This finally expands on the last question:

Question 4. *Is* $mn+1-\frac{1}{3}g(m,n)$ *a new lower bound on the Zarankiewicz number* $Z(m,n,2,2)$*?*

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Notes on the Topology of Hyperplane Arrangements and Braid Groups

Claudio Procesi

Introduction.

We will be concerned with the following problem. Let V be an $n-$dimensional vector space over $\mathbf{R}$. Denote its complexification $V_{\mathbf{C}} = V + iV$.

Consider furthermore a finite family $\mathcal{H} := \mathcal{H}_I := \{H_i\}_{i\in I}$ of real hyperplanes in V which for simplicity we assume all passing through the origin. The set of given hyperplanes and all their intersections form a finite set of subspaces of V partially ordered by inclusion.

We shall restrict to the case in which $\cap_i H_i = 0$ (such an arrangement is called *essential*) in fact this is not a serious restriction.

We shall denote by

$$L(\mathcal{H}) := \{\cap_{i\in T} H_i | T \subset I\}.$$

this finite set of subspaces (closed under intersection), which will be referred to as *the real arrangement.*

The complexification of all these subspaces is the corresponding complex arrangement in $V_{\mathbf{C}}$. Our main concern will be the study of the topology of the complement in $V_{\mathbf{C}}$ of the union $\cup_{i\in I}(H_i)_{\mathbf{C}}$.

Let us denote by $\mathcal{A} := V_{\mathbf{C}} - \cup_{i\in I}(H_i)_{\mathbf{C}}$ this open set.

Of particular interest is the case in which V is a Euclidean space and the H_i are the reflection hyperplanes of a finite reflection group [Bou].

These groups have been classified by Coxeter, the finite reflection group W acts freely on $\mathcal{A}$ and we can form the covering

$$\mathcal{A} \to \mathcal{A}/W.$$

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Among these reflection groups there is the type A_n wich is the group S_{n+1} of permutations of the coordinates of $\mathbf{R^{n+1}}$ (the reflection hyperplanes are the ones of equations $x_i - x_j = 0$). In this case $\mathcal{A}/S_{n+1}$ can be identified to the space of monic polinomials of degree n with distinct roots. The homotopy groups of $\mathcal{A}$, $\mathcal{A}/W$ have been determined by Brieskorn [Br] and in the case A_n we have the classical Artin braid group B_n. Moreover it has been proved by Deligne [De] that $\mathcal{A}$, $\mathcal{A}/W$ are both $K(\pi, 1)$ spaces.

Salvetti [S1] has described a very explicit finite CW complex homotopically equivalent to $\mathcal{A}$ resp. $\mathcal{A}/W$ and, with the use of this complex many cohomology computations for these groups can be performed (cf. also [B-Z]).

De Concini and Salvetti have used these methods also to compute the cohomology of finite reflection groups. In these notes we explain some of these topics.

These notes are a first draft of a project which may never see the light and I make them available in the hope that they may be useful. Nothing new is here just maybe some improvements in the notations and presentation.

A the moment, even if the Salvetti complex is very explicit there is no real simplification available in the proof of Deligne and this topic is not included. The main open problems are related to the genus of the fibration given by the action of the reflection group on the regular part and we refer to [DS2] for details.

Note added in proof. The following paper in fact is very relevant: C. C. Squier, The homological algebra of Artin groups, *Math. Scand.*, 75 (1995), 5–43.

§1. Real arrangements

We start our analysis from real arrangements, we give some basic definitions.

With the notations of the introduction we fix a finite family $\mathcal{H} := \mathcal{H}_I := \{H_i\}_{i\in I}$ of real hyperplanes in V and denote by $L(\mathcal{H}) := \{\cap_{i\in T} H_i | T \subset I\}$ the associated *real arrangement* (i.e. the set of all possible intersections of the H_i).

Definition. The connected components of $V - \cup_i H_i$ are called chambers of the arrangement.

Clearly the chambers are connected convex open sets of V.

Given any subspace $W \in L(\mathcal{H})$ of the arrangement the set of hyperplanes in the arrangement which do not contain W cuts on W a family $\mathcal{H}|_W$ of hyperplanes and the induced arrangent in W is a subset of $L(\mathcal{H})$. The chambers of all the induced arrangements in all the subspaces in $L(\mathcal{H})$ are called *faces*, [1] the set of all faces will be denoted by $F(\mathcal{H})$.

Lemma 1.1. *The faces form a partition of V.*

The proof is by easy induction.

Let us choose for each $i \in I$ an explicit linear equation $\alpha_i = 0$ for the hyperplane H_i.

Given a chamber F, by connectedness each α_i has a definite sign ($+$ or $-$) on the points of F and conversely if on 2 points p, q in $\mathcal{A} = V - \cup_i H_i$ the funcions α_i have the same sign then this happens on the entire segment $tp + (1-t)q,\ 0 \le t \le 1$ which connects p, q in $\mathcal{A}$.

Thus a chamber determines and is determined by a sequence of signs (of course not all sequences occur).

For a face in general some of the α_i are also 0 and thus we see that more generally a face determines and it is determined by a sequence of signs $+, -, 0$ indexed by I.

This remark has an immediate implication. If we consider the arrangement $L(H_J)$ associated to a subset $J \subset I$ of the given set of hyperplanes we have:

Proposition 1.2. *Each face of the arrangement $L(H_J)$ is a union of faces of the arrangement $L(H)$.*

Lemma 1.3. *The closure of a face F is a union of faces.*

Proof. We prove this statement by induction on the dimension of the face and thus we may assume that the face is a chamber.

If $p \in \overline{F}$ is a point then $\alpha_i(p)$ is either 0 or it has the same sign of $\alpha_i(q)$ for $q \in F$.

In particular we see that the half closed segment $tp + (1-t)q,\ 0 \le t < 1$ is entirely contained in the chamber F.

Let F_1 be the face in which p is contained and $r \in F_1$ since the sequence of signs for r coincides with that of p we see that also the half closed segment $tr + (1-t)q,\ 0 \le t < 1$ is entirely contained in the chamber F and thus $r \in \overline{F}$. □

[1] In the french literature one distinguishes between *faces* as the codimension 1 faces and *facettes* for the others.

We have thus defined a partial order on the set of faces and we shall denote by $\mathcal{F}$ the partially ordered set of faces, the usual convention is $F_1 \leq F_2$ if and only if $F_2 \subset \overline{F}_1$. Thus the chambers are the minimal faces.

§2. Fans

The fundamental combinatorial object is the *nerve* of the poset $\mathcal{F}$ i.e. the simplicial complex whose vertices are in 1-1 correspondence with the faces and whose simplices correspond to totally ordered subsets of faces.

Let us axiomatize this construction. Let us call *a cone* any subset $A \subset V$ such that $v \in A,\ a > 0,\ \Longrightarrow\ av \in A$.

Definition. A polyhedral fan[2] $\mathcal{F} := \{F_i\}_{i \in I}$ is a finite family of convex cones, called the *strata* such that:
1) 0 is a stratum.
2) The closure of a stratum is a union of strata.
3) $V = \cup_{i \in I} F_i$ is a decomposition (i.e. disjoint union) of V.

By definition then the set of strata is a poset by setting $F_1 \leq F_2$ if and only if $F_2 \subset \overline{F}_1$ (F_2 is contained in the closure $\overline{F}_1$ of F_1.)

Thus the set of faces of a hyperplane arrangement is a polyhedral fan, we will see another important example when we treat complex arrangements. Let us fix a polyhedral fan, before proceding let us remark some simple facts.

a) If we intersect a line l with the strata of a fan, it becomes decomposed as disjoint union of convex strata, such that the closure of a stratum is a union of strata. Then these strata are open segments (possibly infinite) and their extremal points.

b) If $W \subset V$ is a subspace the family $W \cap F_i$ of non empty intersections is a polyhedral fan in W.

c) A polyhedral fan in the line $\mathbf{R}$ is necessarily the decomposition $\mathbf{R}^-,\ \mathbf{0}, \mathbf{R}^+$.

d) A polyhedral fan in $\mathbf{R}^2$ is given by a finite set of half lines r_i and the connected components of their complement. Notice that such components are convex if and only if the angle between two successive lines il $\leq \pi$.

[2]the definition we use is slightly more general that the one usually introduced in the theory of torus embeddings.

Part d) needs a proof. Consider a stratum S which is not a half line. S is a convex cone containing two linearly independent vectors a, b. Consider the intersection of S with the line through a, b it is a convex set A in this line which by the previous codiscussion is open, then it is easy to conclude the proof.

The main construction is a geometric realization of this poset in V but in fact this is a consequence of the construction of a *simplicial fan,* which is a pseudobaricentric subdivision of the given fan.

For this let us select in each stratum F, different from the stratum reduced to 0, a vector v_F.

There is a totally elementary but essential Lemma associated to this construction.

Lemma 2.1. *Given a vector $v \in F$ in a stratum F there exists, a unique vector $w \in \partial F$ in the boundary of F, and a unique positive numeber $a > 0$ such that:*

$$v = av_F + w.$$

Proof. If v is a multiple av_F of v_F then $a > 0$ and $w = 0$. Otherwise we work in the 2-dimensional plane π spanned by v, v_F in which the intersection $F \cap \pi$ appears as an open *convex angle* limited by two half lines which are in ∂F, then in this 2 dimensional picture the statement is clear. □

Theorem 2.2. 1) *Given a simplex $S := F_1 < F_2 < \cdots < F_k < 0$ the vectors $v_1 := v_{F_1}, v_2 := v_{F_2}, \dots v_k := v_{F_k}$ are linearly independent.*

2) *Let $C_S := \{\sum a_i v_i,\ a_i > 0\}$ the corresponding open simplicial cone. Then $V - 0$ is the disjoint union on the cones C_S.*

3) *Each stratum F is the union of the cones C_S where the first element of S is F.*

proof. We claim that all these statements are immediatete consequences of the previous Lemma. In fact let us take a vector $v \in V - 0$ then $v \in F_1$ where F_1 is a non 0 stratum uniquely determined.

By the previous Lemma $v = a_1 v_{F_1} + w_1$. If $w_1 = 0$ we stop otherwise $w_1 \in F_2 \neq 0$, $w_1 = a_2 v_{F_2} + w_2$, $a_2 > 0$ with $F_1 < F_2$. Continuing in this way we see that each point has a unque expression of the form

$$v = \sum a_i v_{F_i},\ a_i > 0,\ F_1 < F_2 < \cdots < F_k < 0.$$

□

Let us now consider, for each combinatorial simplex $S := F_1 < F_2 < \cdots < F_k$ the geometrical simplex

$$|S| := v_{F_1} * v_{F_2} * \cdots * v_{F_k}$$

convex envelope (or join) of the (independent[3]) vertices v_{F_i} (we now allow also the stratum 0).

Corollary 2.3. *The simpleces $|S|$ form a simplicial subdivision on a combinatorial ball $B_{\mathcal{F}}$ with boundary Π the union of simpleces $|S|$, $S := F_1 < F_2 < \cdots < F_k < 0$ not containing the vertex 0.*
The map $j : \mathbf{R}^+ \times \Pi \to \mathbf{V} - \mathbf{0}$, $\mathbf{j(a, v)} := \mathbf{av}$ is a homeomorphism.

Proof. We have seen that the cones C_S decompose $V - 0$ on the other hand clearly the closure $\overline{C}_S$ of the cone C_S is the union:

$$\overline{C}_S = \cup_{T \subset S} C_T$$

this implies that the simplices of Π form a simplicial complex.

For the second part it is clearly enough to show that j is bijective, for this we construct the inverse. Given a point $v \in V - 0$ we have that v is uniquely of the form:

$$v = \{\sum a_i v_i,\ a_i > 0\} \in C_S$$

we set $a = \sum_i a_i$ and $w := \frac{v}{a}$ then $w \in \Pi$ and $j^{-1}(v) = (a, w)$. □

§3. Subspace arrangements

Let us consider again a polyhedral fan $\mathcal{F} = \{F_i\}_{i \in I}$ and consider a closed subset $X \subset V$ with $X = \cup_{i \in J} F_i$ a union of strata. Let $A := V - X = \cup_{i \notin J} F_i$ also a union of strata.

Denote ass before by Π the simplicial realization of the complex of non 0 strata in $\mathcal{F}$ and let Π_X, $\Pi_X^{\perp}$ be the two full subcomplexes of Π with the vertices in X and in $\mathcal{A}$ respectively.

From the last corollary it follows that the homeomorphism j^{-1} : $V - 0 \to \mathbf{R}^+ \times \Pi$ maps $X - 0$, $\mathcal{A}$ respectively to $\mathbf{R}^+ \times \Pi_{\mathbf{X}}$, $\mathbf{R}^+ \times (\Pi - \Pi_{\mathbf{X}})$.

By standard facts $\Pi_X^{\perp}$ is a deformation retract of $\Pi - \Pi_X$ and thus we obtain:

Theorem 3.1. *The open set $\mathcal{A} = V - X$ has the same homotopy type as $\Pi_X^{\perp}$.*

[3] in the sense of affine geometry

Let us see the implication of this discussion to the topology of subspace arrangements. If we consider an arrangement of subspaces $W := \{W_j\}$ contained in $L(\mathcal{H})$ we have that:

(1) The union $V_W := \cup W_j$ of the subspaces W_j, is a union of faces.

(2) The intersection of V_W with Π is the full subcomplex Π_W with vertices the vertices v_F, $F \subset V_W$ or $v_F \in V_W$.

(3) Under the homeomorohism j the open set $V - V_W$ corresponds to

$$\mathbf{R}^+ \times (\Pi - \Pi_{\mathbf{W}}).$$

Thus consider the *orthogonal* subcomplex to Π_W i.e. the full subcomplex $\Pi_W^\perp$ having the vertices $v_F \notin V_W$.

We obtain:

Corollary 3.2. *The open set $V - V_W$ has the same homotopy type as $\Pi_W^\perp$.*

Since we will need it in a moment let us see what happens for non essential arrangements. Assume thus that the intersection $\cap H_i = A$ is a linear subspace of codimension m.

Fix a linear complement B to A so that $V = A \oplus B$ then the hyperplanes H_i intersect B in an essential arrangement $L_B(\mathcal{H})$. The faces of $L(\mathcal{H})$ can be identified with $A \times G$ with G face of $L_B(\mathcal{H})$.

Then the open set $V - \cup_i H_i$ is homeomorphic to $A \times (B - \cup_i (B \cap H_i)$.

Thus again $V - \cup_i H_i$ has the same homotopy type as the polyhedron Π associated to the induced arrangement on B.

Proposition 3.3. *If $A = \cap_i H_i$ is a subspace of codimension m the geometric realization of the poset of faces of the arrangement is a combinatorial $m-$ball.*

Before passing to complex arrangements it is useful to analyze a cellular structure of the polyhedrons $\Pi, \Pi_W, \Pi_W^\perp$.

For this we need a little more notations. Given a face F let us define by $\langle F \rangle$ the linear span of F (we know that $\langle F \rangle \in L(\mathcal{H})$ and that F is a chamber of $\langle F \rangle$).

Consider furthermore the set of indeces $J_F : \{i \in I | F \subset H_i\}$.

$$\mathcal{H}_{J_F} := \{H_i | F \subset H_i\}.$$

This is typically a non essential arrangement and $\langle F \rangle = \cap_{i \in J_F} H_i$.

We have seen that Π is a combinatorial sphere and its join with 0, $B_{\mathcal{F}} = \Pi * 0$ a ball. More generally if F is a face consider the poset $\mathcal{L}_F$ of all faces G such that $F \geq G$ i.e. such that $F \subset \overline{G}$.

We claim that:

Lemma 3.4. *As a poset $\mathcal{L}_F$ is isomorphic to the poset of faces of the configuration $\mathcal{H}_{J_F}$ of hyperplanes containing F.*

Proof. Take a face $G \in \mathcal{L}_F$, from Proposition 1 we know that it is contained in a unique face of the subarrangement $L(\mathcal{H}_{J_F})$.

Conversely take one such face G which we know (always by the same proposition) is a union of faces in $F(\mathcal{H})$.

These faces differ only for the signs of the equations α_i which do not vanish on F. Since $F \subset \overline{G}$ we must have that $F \subset \bar{F}'$ where $F' \subset G$ is a face in $F(\mathcal{H})$. This face is unique since on this face the signs of the equations α_i which do not vanish on F must have the same sign as on F. $\square$

From the previous proposition we get:

Corollary 3.5. *The nerve of the poset $\mathcal{L}_F$ is a triangulation of a combinatorial ball B_F of dimension the codimension of F.*

This fact has an important implication:

Theorem 3.6. *The boundary of B_F is the union of the B_G with $G < F$.*

$$\partial B_F = \cup_{G<F} B_G.$$

The balls B_F as F varies on all faces of the hyperplane arrangement give a cellular decomposition of the ball $B_{\mathcal{H}}$.

For any given subspace arrangement W (of the hyperplane arrangement) the polyhedron $\Pi_W^{\perp}$ is a sub cell complex given by the balls B_F as F varies on the faces F of the arrangement which are not contained in the union of the subspaces.

We will refer to B_F as the *cell dual to F*.

Product of arrangements. Before we pass to complex arrangements let us treat briefly a simple general construction. Civen two vector spaces V_1, V_2 and in each an arrangement of hyperplanes $\mathcal{H}^1, \mathcal{H}^2$ we can define the product arrangement $\mathcal{H}^1 \times \mathcal{H}^2$ in $V_1 \times V_2$ in the obvious way.

$$\mathcal{H}^1 \times \mathcal{H}^2 := \{H \times V_2,\ V_1 \times K | H \in \mathcal{H}_1,\ K \in \mathcal{H}_2\}.$$

One easily sees that the faces of this arrangemens are just products:

$$F(\mathcal{H}^1 \times \mathcal{H}^2) = \{F_1 \times F_2 | F_1 \in F(\mathcal{H}^1),\ F_2 \in F(\mathcal{H}^2)\}$$

as poset we have that $F(\mathcal{H}^1 \times \mathcal{H}^2)$ is the product $F(\mathcal{H}^1), \times F(\mathcal{H}^2)$ of the two posets with the product order $(a, b) \leq (c, d)$ if and only if $a \leq c,\ b \leq d$.

§4. Complex arrangements

It is now the time to look at complex arrangements, i.e. arrangements of hyperplanes given by real equations in complex space, or the complexification of a real arrangement $\mathcal{H}$ in V.

Of course the idea is to treat such arrangements as subspace arrangements in a real space. More precisely in $V_{\mathbf{C}} = V + iV = V \times V$ the complex hyperplane of equation $\alpha_k(v + iw) = 0$ is the real codimension 2 subspace $\tilde{H}_k := H_k + iH_k$ (where $H_k = \{v \in V | \alpha_k(v) = 0\}$).

Therefore the subspaces $\tilde{H}_k$ are part of the hyperplane arrangement associated to the real hyperplanes $H_k + iV,\ V + iH_j$, in the notations of the previous paragraph this is in fact $\mathcal{H} \times \mathcal{H}$. One can therefore apply the previous theory to this hyperplane arrangement. There is on the other hand a much more efficient way to procede due to Salvetti and we describe this.

Given a face A of the hyperplane arrangement $\mathcal{H}$ consider the hyperplane arrangement $\mathcal{H}_A$ generated by the hyperplanes containing A we consider the set

$$CF(\mathcal{H}) := \{(A, B) |\ A \in F(\mathcal{H}),\ B \in F(\mathcal{H}_A)\}$$

of pairs (A, B) where A is a face in the original hyperplane arrangement $\mathcal{H}$ while B is a face of the subarrangement $\mathcal{H}_A$.

Proposition 4.1. 1) *The sets $A \times B = A + iB$, $(A, B) \in CF(\mathcal{H})$ decompose $V_{\mathbf{C}} = V + iV$.*

2) *The closure $\overline{A + iB}$ is a union of strata $A_k + iB_k$, $(A_k, B_k) \in CF(\mathcal{H})$.*

Proof. 1) We have a decomposition $V + iV = \cup_{A \in F(\mathcal{H})} A + iV$ and then a decomposition $A + iV = \cup_{B \in F(\mathcal{H}_A)} A + iB$.

2) We have for the closure $\overline{A + iB} = \overline{A} + i\overline{B} = \overline{A} \times \overline{B}$ and $\overline{A} = \cup A_k$ is a union of faces in $F(\mathcal{H})$ while $\overline{B} = \cup B_h$ is a union of faces in $F(\mathcal{H}_A)$.

Thus $\overline{A} \times \overline{B} = \cup_{k,h} A_k \times B_h$ now the decomposition of V into faces for $F(\mathcal{H}_{A_k})$ is a refinement of the decomposition of V into faces for $F(\mathcal{H}_A)$, since A_k is in the closure of A and so the set of hyperplanes containing A_k contains the set of hyperplanes containing A. $\square$

Therefore in a natural way the set of pairs $CF(\mathcal{H})$ is also a partially ordered set and we are going (as in §1) to represent its nerve as a simplicial complex.

Remark that also the strata $A \times B$, $(A, B) \in CF(\mathcal{H})$ are convex cones (open in their closure). Thus

Theorem 4.2. *The set of strata $A \times B$, $(A, B) \in CF(\mathcal{H})$ is a polyhedral fan.*

We have to understand now how the open set $\mathcal{A}$ complement of the complex hyperplane arrangement, appears in this picture.

Proposition 4.3. *$\mathcal{A}$ is the union of the faces $A + iB$, $(A, B) \in CF(\mathcal{H})$ with B open.*

Proof. A vector $a + ib$ is in $\mathcal{A}$ if and only if b is not contained in any of the hyperplanes of $\mathcal{H}$ in which a is contained. This describes exactly the union of the strata in $CF(\mathcal{H})$ described by the proposition. $\square$

Let us thus set

$$\mathcal{F}_{\mathbf{C}} := \{A + iB \mid (A, B) \in CF(\mathcal{H}) \quad \text{with} \quad B \quad \text{open}\}.$$

This is a poset and, if we fix a vertex in each stratum of $\mathcal{F}_{\mathbf{C}}$ and construct the corresponding simplicial complex $\Pi_{\mathbf{C}}$ we have, by Theorem 3.1.

Theorem 4.4. *The complement $\mathcal{A}$ of the complex hyperplane arrangement has the same homotopy type as that of the simplicial complex $\Pi_{\mathbf{C}}$ geometric realization of $\mathcal{F}_{\mathbf{C}}$.*

We want to describe now the natural cellular structure of the poset $\mathcal{F}_{\mathbf{C}}$.

Fix a face $(A, B) \in \mathcal{F}_{\mathbf{C}}$. We want to consider the poset of all faces $(C, D) \leq (A, B)$.

By definition $(C, D) \leq (A, B)$ means $A \subset \overline{C}$, $B \subset \overline{D}$. Since B, D are open sets this condition is in fact equivalent to:

$$A \subset \overline{C}, \ B \subset D$$

thus D is the unique chamber of the configuration of hyperplanes through C which contains B. In other words, given $(A, B) \in \mathcal{F}_{\mathbf{C}}$, the subposet

$$\mathcal{F}(A, B) := \{(C, D) \in \mathcal{F}_{\mathbf{C}} | (C, D) \leq (A, B)\}$$

of $\mathcal{F}_{\mathbf{C}}$ formed by all faces $(C, D) \leq (A, B)$ is isomorphic to the poset L_A of all faces C of the hyperplane arrangement with $C \leq A$. By Corollary 3.2 the nerve of the poset $\mathcal{F}(A, B)$ is a triangulation of a combinatorial disk $\Delta(A, B)$ of dimension the codimension of A.

By construction the boundary of this ball is also a union of balls relative to pairs $(C, D) < (A, B)$ and thus:

Corollary 4.5. *We have a cell complex structure on the polyhedron* Π *in which the cells* $\Delta(A, B)$ *of dimension* k *are indexed by elements* $(A, B) \in \mathcal{F}_{\mathbf{C}}$ *with* A *of codimension* k.

The boundary of $\Delta(A, B)$ *is*

$$\partial(\Delta(A, B)) = \cup_{(A', B') < (A, B)} \Delta(A', B').$$

§5. Reflection arrangements

We consider now an $n-$dimensional Euclidean space V and the arrangement of reflection hyperplanes of a finite Coxeter group W. By this we mean that W is a finite group generated bty reflections with respect to some hyperplanes H_i and the arrangement is formed by these H_i and also all their transforms under the group W.

We plan to describe the various polyhedra considered, for real and complex arrangements, in this case and in a W equivariant way.

We start from the real polyhedron.

We assume that the only fixed vector is 0.

Fix for every H_i in the arrangement an orthogonal vector α_i so that $H_i := \{v \in V | (\alpha_i, v) = 0\}$.

The elements $\pm\alpha_i$ play the same role as the roots of a root system. Fixing a vector v outside all hyperplanes H_i determines *positive roots* and a *fundamental chamber.*

From the theory of these groups one can choose $n-$independent reflection hyperplanes $H_i, i = 1, \ldots, n$ which are the walls of a chamber C which conventionally we will call the *fundamental chamber.*

$H_i := \{v \in V | (\alpha_i, v) = 0\}$ the elements α_i correspond for root systemes to simple roots.

Thus the chamber C is a simplicial cone

$$C := \{\sum_{i=1}^{n} a_i u_i | a_i > 0\}$$

$(\alpha_i, u_j) = \delta_i^j$.

The wall H_i is spanned by the u_j, $j \neq i$, the group W is generated by the reflections s_i relative to the walls H_i.

The closure $\overline{C} := \{\sum_{i=1}^{n} a_i u_i | a_i \geq 0\}$ of C is a fundamental domain for the action of W.

The stabilizer of a face F of C acts trivially on the face and it is generated by the simple reflections s_i relative to the walls H_i with $F \subset H_i$.

F is determined by a subset $J \subset I := \{1, \dots, n\}$ we will denote it by F_J and we denote by W_J the subgroup generated by the s_i, $i \in J$. W_J is also a reflection group which may also be realized as a reflection group on the subspace $\langle F \rangle^{\perp}$ orthogonal to the span of the face F. The fixed vectors of W_J form the span $\langle F \rangle$ of the face F.

Consider now a vector $v_0 \in C$ in the open chamber. By what we have said the orbit Wv_0 gives rise to a point in each chambers and it is in 1-1 correspondence with W.

Denote by $v_w := wv_0$, $w \in W$. Let Δ be the convex hull of the points v_w. Then it is also true that the v_w span V and hence Δ is a convex polyhedral ball of dimension n. Clearly Δ is stable under W and since its extremal points are among the points Wv_0 it follows that all these points are extremal.

For any face F_J of C let us set

$$v_J := \frac{1}{|W_J|} \sum_{w \in W_J} wv_0$$

the baricenter of the orbit of v_0 under W_J.

Lemma 5.1. *We have that $v_J \in F$ and v_J is the orthogonal projection of v_0 to the span $\langle F \rangle$ of the face F.*

Proof. v_J is fixed by W_J hence it is in $\langle F \rangle$, if we decompose $v_0 = u + z$, $u \in \langle F \rangle$, $z \in \langle F \rangle^{\perp}$ we have that $\sum_{w \in W_J} wz = 0$ and hence the claim $u = v_J$.

We still have to prove that $v_J \in F$. By induction it is enough to do it when F is a codimension 1 face of C. If H_i is the wall through F and

s_i the corresponding simple reflection $v_F = 1/2(v_0 + s_i v_0)$ and H_i is the only wall separating the two chambers $C, s_i(C)$ thus the signs (α_j, w) for $j \neq i$ do not change crossing this wall and we see that $(\alpha_j, v_F) > 0$ for $j \neq i$ and so $v_F \in F$. □

Every other face F' is uniquely W equivalent to a face F_J and if $F' = wF_J$ the element w lies in a coset wW_J and so

$$v_{F'} := wv_J$$

is well defined.

We have thus defined, for all faces F of the refection arrangement a vector v_F characterized by the following properties:

1) If $F = wG$ then $v_F = wv_G$.

2) If $F \subset \overline{G}$ then v_F is the orthogonal projection of v_G to $\langle F \rangle$.

We can now consider the simplicial complex Π associated to the vertices v_F and simpleces induced from the poset structure of the faces. We have that:

Theorem 5.2. *Π is a triangulation of the ball Δ convex hull of the points v_w.*

Proof. By construction all the vertices of this polyhedron are contained in Δ and so Π triangulates some polyhedron contained in Δ but now the faces of Δ are balls of the same type for smaller reflection systems for which the coincidence is by induction and this proves the claim. □

Remark 5.3. With the notations of §3 notice that, the cell dual to a face F is the convex envelope of the orbit under the reflection group generated by the hyperplanes through F of a point v_w in a chamber of which F is a face. Let us pass now to the complexified picture and to the open set $\mathcal{A}$.

From §4 we know that this is stratified by the set

$$\mathcal{F}_{\mathbf{C}} := \{A + iB \mid (A, B) \in CF(\mathcal{H}) \quad \text{with} \quad B \quad \text{open}\}.$$

Here A is a face of the reflection arrangement while B by the description of §4 is a chamber of the reflection arrangement generated by the hyperplanes containg A.

Proposition 5.4. *There exists a unique $J \subset I$ and a unique $w \in W$ such that*

$$w(A, B) := (wA, wB) = (F_J, wB), \; C \subset wB.$$

Proof. Since $\overline{C}$ is a fundamental domain there exists a $J \subset I$ and a $w \in W$ such that $w(A) = F_J$, the set of elements $\{w' \in W | w'(A) = F_J\}$ is the coset $W_J w$.

The chamber wB is one of the chambers of the reflection arrangement generated by the hyperplanes containg F_J and W_J acts simply transitively on these chambers, of which one and only one contains C the statement follows. $\square$

We have now to choose judiciously the points $v_{(A,B)}$, $(A, B) \in CF(\mathcal{H})$ so that the resulting polyhedron is W stable. Since there is a unique $w \in W$ with $wA = F_J$, $wB \supset C$ we define

$$v_{(A,B)} := w(v_J + iv_0).$$

We obtain that:

Theorem 5.5. *The simplicial complex $\Pi_{\mathbf{C}}$ with vertices $v_{(A,B)} := w(v_J + iv_0)$ and simplices induced by the poset structure of $CF(\mathcal{H})$ is W stable moreover the homotopy equivalence between A and $\Pi_{\mathbf{C}}$ is W equivariant.*

Proof. The homeomorphism j is clearly W equivariant, but if we have a polyhedro Π a full subpolyhedron Π_X and its orthogonal $\Pi_X^\perp$ the deformation from $\Pi - \Pi_X$ to $\Pi_X^\perp$ is canonical along the rays joining a point in Π_X and in $\Pi_X^\perp$ so if we have a simplicial action of a group preserving these two polyhedra also the deformation is equivariant. $\square$

We can finally use all this to analyze the homotopy type of $\mathcal{A}/W$. From what we have seen this is homotopically equivalent to $\Pi_{\mathbf{C}}/W$.

We have seen (last corollary of §4) that $\Pi_{\mathbf{C}}$ has a cellular structure in which the cells $\Delta(A, B)$ of dimension k are indexed by elements $(A, B) \in \mathcal{F}_{\mathbf{C}}$ with A of codimension k.

Given a set $J \subset I$ with k elements we have in particular the k cell

$$C_J : \Delta(F_J, B),\ C \subset B.$$

By the previous Proposition each cell is W equivalent to one and only one of the cells C_J. Therefore we deduce that the space $\Pi_{\mathbf{C}}/W$ is obtained in some way attaching these cells.

The simplest way to describe these attachments is the following.

Consider the n cell $\Delta(0, C)$ which is the simplicial complex with vertices $v_F + iv_0$ as F runs through the faces of the real arrangement and is isomorphic (also as simplicial complex) to the ball Δ of the real picture by projection to the real part. The cells C_J are contained in $\Delta(0, C)$ and thus under projection

$$\pi : \Delta(0, C) \to \Pi_{\mathbf{C}}/W$$

is surjective. A face $\Delta(F, D)$, $C \subset D$ is identifyed to a unique face C_J by the element $w \in W$ with $wF = F_J$, $C \subset wD$.

$D_J := wD$ is the unique face of the arrangement generated by F_J and containg C. Since we have already that $C \subset D$ we must have also $wC \subset D_J$.

Lemma 5.6. *The unique element $w_0 \in W$ such that $w_0F = F_J$, $w_0D = D_J$ where D is the unique face of the arrangement generated by F and containg C is the shortest element in the coset $W_J\,w$.*

Proof. The set of elements $w|wF = F_J$ is the coset W_Jw_0. We claim that the shortest element on the coset is characterized by the fact that $l(s_iw_0) = l(w_0) + 1$ for all $i \in J$ and this in turn is equivalent to $w_0^{-1}(\alpha_i) > 0$ for all the roots α_i associated to the hyprplanes H_i, $i \in J$. Now $C := \{v|(\alpha_i, v) = \alpha_i(v) > 0, \ \forall i \in I\}$ while $D_J := \{v|\alpha_i(v) > 0, \ \forall i \in J\}$ and thus since $w_0C \subset D_J$ we have for $i \in J$ that:

$$v \in C, \ (w_0^{-1}\alpha_i, v) = (\alpha_i, w_0v) > 0.$$

□

So we have the

Theorem 5.7. *The space $\Pi_{\mathbf{C}}/W$ which is of homotopy type of $\mathcal{A}/W$ is obtained brom the ball Δ identifying each face F with the face C_J in its W orbit, using the shortest element w in the coset $W_J\,w$ for which $W_J\,wF = C_J$.*

Let us draw some interesting consequence of this.

First of all we deduce immediately Brieskorn presentation by generators and relations of the generalized braid group.

The homotopy group of $\Pi_{\mathbf{C}}/W$ is computed by just considering the 1 and 2 cells. the 1 cells give a bouquet of circles, corresponding to the 1 faces joining v_0 to s_iv_0, we denote by T_i the corresponding loop oriented from v_0 to s_iv_0. Thus the T_i are generators for the homotopy group. The 2 cells give the relations. Given 2 nodes i, j of the Dynkin diagram we deduce a relation between T_i, T_j and it easily seen to be:

$$T_iT_j = T_jT_i, \ T_iT_jT_i = T_jT_iT_j, \ T_iT_jT_iT_j = T_jT_iT_jT_i,$$
$$T_iT_jT_iT_jT_iT_j = T_jT_iT_jT_iT_jT_i,$$

according if the two nodes are joined by $0, 1, 2, 3$ edges.

First of all let us look at the 1-dimensional cells which are of the ones of vertices ws_iv_0, wv_0. If $l(ws_i) = l(w) + 1$ then w^{-1} is the element of shortest length identifying the 1-cell with s_iv_0, v_0. The generator T_i is by definition the loop associated to the oriented edge v_0, s_iv_0. Thus the lift of T_i from the point wv_0 goes to the point ws_iv_0 along this edge.

Next consider the universal covering space $\pi : \tilde{\Pi} \to \Pi \to \Pi/W$ of $\Pi_{\mathbf{C}}/W$ and of Π. Lifting the cellular structure of Π we have a paving of $\tilde{\Pi}$ by cells which are permuted by the group of deck transformations.

We fix a cell C of amaximal dimension mapping to $\Delta(0, C)$ and a base point p_0 in C mapping to v_0. Thus we identify the group of deck transformations with the generalized braid group B using this base point.

Under the homeomorphism of C to $\Delta(0, C)$ the vertices wv_0 are in the orbit of p_0 under the group of deck transformations

$$wv_0 = \pi(T_w p_0)$$

and this defines a canonical lift T_w of w.

If $w = s_{i_1}s_{i_2}\dots s_{i_k}$ is a reduced expression the we claim that

$$T_w = T_{i_1}T_{i_2}\dots T_{i_k}.$$

In fact there is a path from v_0 to wv_0 given by the edges $[s_{i_k}v_0, v_0]$, $[s_{i_1}s_{i_2}\dots s_{i_k}v_0, s_{i_2}\dots s_{i_k}v_0]$ which maps in Π/W to a path giving the element $T_{i_1}T_{i_2}\dots T_{i_k}$ of the homotopy group.

Next we identify in C the copies of the C_J which we denote by the same symbols.

We have to fix an orientation for the cells C_J this can be done by ordering the vertices and then orienting the cells C_J so that if $K \subset J$, $|K| = k - 1$ is obtained removing the h^{th} element of J the oriented cell C_K appears in the boundary of C_J with the sign $\epsilon_{K,J} := (-1)^h$.

We have thus:

Theorem 5.8. 1) *The cells in $\tilde{\Pi}$ are simply transitive orbits of the cells C_J.*

2) *Denoting by $C_k(\tilde{\Pi})$ the group of $k-$dimensional cells, under the action of B this is a free $\mathbf{Z}[\mathbf{B}]$ module with basis the cells C_J, $|J| = k$.*

3) *The boundary of the cell C_J is the sum*

$$\sum_{K \subset J,\ |K|=k-1} \epsilon_{K,J}\Big(\sum_{w \in W_J/W_K} (-1)^{l(w)} T_w\Big) C_K.$$

Where T_w denotes the canonical lift of the element of shortest length w in the coset.

Proof. The statements 1), 2) follow from the construction as for 0) and 3) we have to note that each cell F which in $\Delta(0, C)$ is in the orbit of C_J under W in $\tilde{\Pi}$ is exactly $F = T_w C_J$ (under the group of deck transformations) this is easily verified by considering the minimal path from wv_0 to v_0 followed by the two segments joining wv_0, v_0 to the centers of the respective cells. The sign $(-1)^{l(w)}$ depends of the fact that the reflections s_i reverse the orientation of the fundamental cell. □

§6. Reflection groups

In [DS2] the authors generalize the previous analysis as follows. Start from the real reflection representation V and consider instead of the complexification, the space V^m for all m. On V^m the reflection group W acts and it acts freely on the open subspace U^m obtained by removing the subspaces H^m for each reflection hyperplane.

One has naturally a set of inclusions $U^m \subset U^{m+1} \dots$ and a space U^∞ which by a simple dimension argument is contractible and hence $B_W := U^\infty / W$ is a classifying space for W.

The same method used for the complexification allows to stratify in a W equivariant way the space V^m by products $F_1 \times F_2 \times \cdots \times F_m$ where inductively:

F_1 is a face of the reflection arrangement and $Fi + 1$ is a face of the subarrangement generated by the hyperplanes which contain F_i. In this way one has a fan and U^m is a union of the strata $F_1 \times F_2 \times \cdots \times F_m$ with F_m open. Then a similar analysis gives a cellular structure on B_W. We refer to the original paper for details.

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Some Geometric Methods in Commutative Algebra

Vasudevan Srinivas

In this series of lectures, I will discuss three examples of techniques of algebraic geometry which have applications to commutative algebra. The examples chosen are those which pertain most closely to my own research interests.

§1. Hilbert functions, regularity and Uniform Artin-Rees

In this Section we describe results giving explicit, effective bounds for the Hilbert functions and postulation number of a Cohen-Macaulay module M of given dimension d and multiplicity e over a Noetherian local ring $(A, \mathfrak{m})$, with respect to a given $\mathfrak{m}$-primary ideal I. We also discuss related results bounding the Castelnuovo-Mumford regularity of the associated graded module of M with respect to I in terms of Hilbert coefficients, assuming only that M has positive depth; this leads to a new proof of the Uniform Artin-Rees theorem of Duncan and O'Carroll, and other results. The geometric technique used here is the cohomological study of the blow up of the ideal I, using in particular Grothendieck's formal function theorem.

1.1. The finiteness theorem for Hilbert functions

Recall that if $(A, \mathfrak{m})$ is a Noetherian local ring, M a finite A-module and $I \subset \mathfrak{m}$ an ideal of A such that M/IM has finite length, the *Hilbert function* (or more properly, the Hilbert-Samuel function) of M with respect to I is the numerical function

$$H_I(M)(n) = \ell(M/I^nM), \ \ \forall\, n \geq 0,$$

where we use the symbol ℓ to denote the length of a module (which has a finite composition series). Then there exists a corresponding *Hilbert polynomial*

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$$P_I(M)(x) = e_0(I,M)\binom{x+d-1}{d} + e_1(I,M)\binom{x+d-2}{d-1} + \cdots + e_d(I,M),$$

where $e_0(I,M) > 0$ is the *multiplicity* of M with respect to I, $e_j(I,M) \in \mathbb{Z}$ for all $0 \leq i \leq d = \dim M$, and such that for some non-negative integer $n_0(I,M)$,

$$H_I(M)(n) = P_I(M)(n) \;\; \forall\, n \geq n_0(I,M).$$

The integers $e_j(I,M)$ are called the *Hilbert coefficients*, and $n_0(I,M)$ is called a *postulation number*, of M with respect to I.

The first result we state is the following *Finiteness Theorem*, taken from the paper [44] of V. Trivedi. The special case when $M = A$ is a Cohen-Macaulay local ring, and $I = \mathfrak{m}$ is the maximal ideal, was treated earlier in a paper of Trivedi and myself [39].

Theorem 1.1 (Finiteness Theorem). *Let $(A,\mathfrak{m})$, I, M be as above, where M is a Cohen-Macaulay module. Let $\mathbf{e} = e_0(I,M)$. Then*

(i) $|e_j(I,M)| \leq (9\mathbf{e}^5)^{j!}$ *for $j \geq 1$, and*
(ii) $n_0(I,M) = 3^{d!-1}\mathbf{e}^{3(d-1)!-1}$ *is a postulation number for M with respect to I.*

Corollary 1.2. *For fixed positive integers d, e there are only finitely many numerical functions which can arise as the Hilbert function $H_I(M)(n)$ of a Cohen-Macaulay module M of dimension d over a Noetherian local ring, which has multiplicity e with respect to some appropriate ideal I of that local ring. In particular, only finitely many numerical functions $H : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ can arise as the Hilbert function of a Cohen-Macaulay local ring of a given dimension and multiplicity.*

Remark 1.3. [44] contains some references to the earlier results in the direction of Theorem 1.1, which essentially pertain to very special situations, or to the coefficients $e_1(I,M)$ and $e_2(I,M)$. See [45] for an extension of Theorem 1.1 to the case of generalized Cohen-Macaulay modules.

Remark 1.4. Kleiman has shown (see [11, Exp.XIII]) that only finitely many polynomials can occur as the Hilbert polynomial of an integral projective scheme, of a given dimension, embedding dimension and degree, over a field. He gives an example to show that this is false for Cohen-Macaulay schemes. On the other hand, in [41], examples are given of infinitely many polynomials which occur as Hilbert polynomials of complete, local integral domains, of a fixed dimension and multiplicity, which are quotients of a fixed regular local ring. The examples given are of quotients of dimension 2 and multiplicity 4 of a power series ring in 6 variables over a field.

We now outline a proof of the Finiteness Theorem in the special case $M = A$, $I = \mathfrak{m}$, which contains most of the features of the general case. To simplify notation, we write $e_j(M)$, $n_0(M)$, etc. to mean $e_j(\mathfrak{m}, M)$, $n_0(\mathfrak{m}, M)$, etc., for any finite A-module M.

One input is a cohomological formula for the difference between the Hilbert function and Hilbert polynomial, obtained by Johnston and Verma (see [17], and [44], Theorem 1), generalizing "classical" results of Serre. This can be expressed in two equivalent ways, involving graded local cohomology of the Rees algebra, or coherent sheaf cohomology on a blow-up. We state their result in the latter form, since that is what is useful for us later.

Theorem 1.5. *Let $(A, \mathfrak{m})$ be Noetherian local of dimension d and depth > 0, and let*

$$R = \bigoplus_{n \geq 0} \mathfrak{m}^n, \quad R_+ = \bigoplus_{n > 0} \mathfrak{m}^n$$

be the $\mathfrak{m}$-adic (graded) Rees algebra of A and its irrelevant graded ideal, respectively. Let $\pi : X = \operatorname{Proj} R \to \operatorname{Spec} A$ be the blow-up of the maximal ideal of A.

(a) *If $d \geq 2$ and* $\operatorname{depth} A \geq 2$, *then*

$$P_{\mathfrak{m}}(A)(n) - H_{\mathfrak{m}}(A)(n) = \sum_{i=1}^{d-1} (-1)^{i-1} \ell(H^i(X, \mathcal{O}_X(n))).$$

In particular, taking $n = 0$, we get

$$e_d(A) = \sum_{i=1}^{d-1} (-1)^{i-1} \ell(H^i(X, \mathcal{O}_X)). \tag{1.1}$$

(b) (Northcott) *If $d = 1$, then*

$$e_1(A) = -\ell\left(\frac{H^0(X, \mathcal{O}_X)}{A}\right).$$

Remark 1.6. Note that if $\pi : X \to \operatorname{Spec} A$ is the blow-up of the maximal ideal, then π induces an isomorphism $X - \pi^{-1}(\mathfrak{m}) \to \operatorname{Spec} A - \{\mathfrak{m}\}$. Hence for any coherent sheaf $\mathcal{F}$ on X and any $j > 0$, the cohomology A-module $H^j(X, \mathcal{F})$, which is a finite A-module, has support contained in $\{\mathfrak{m}\}$; in particular, it has finite length. Hence the formulas in (a) of the above Theorem are meaningful. A similar comment applies to (b).

Remark 1.7. [17] contains a somewhat more general result, valid for any $\mathfrak{m}$-primary ideal, formulated in terms of graded local cohomology of the Rees algebra $R(I) = \oplus_{n\geq 0} I^n$ with respect to its irrelevant graded ideal $R_+(I)$, where the depth hypotheses on A are not needed. The above formulation results from the standard relation between sheaf cohomology on X and graded local cohomology of the Rees algebra. This is further generalized in [44] to the case of Hilbert functions of A-modules, and is also expressed cohomologically as above on the blow-up $X = \operatorname{Proj} R(I)$. The formula in [17] has a sign error, corrected in [44].

Next, we recall the notion of *Castelnuovo-Mumford regularity* of a coherent sheaf $\mathcal{F}$ on the projective space $\mathbb{P}^N_A$ over a Noetherian ring A: we say that $\mathcal{F}$ is *m-regular* if $H^i(\mathbb{P}^N_A, \mathcal{F}(m-i)) = 0$ for all $i > 0$. This is closely related to the notion of Castelnuovo-Mumford regularity, in the sense of commutative algebra (see [7]), for the graded module $\oplus_{n\in\mathbb{Z}} H^0(\mathbb{P}^N_A, \mathcal{F}(n))$ over the polynomial algebra $A[X_0, \ldots, X_N]$ (the homogeneous coordinate ring of $\mathbb{P}^N_A$). We now recall some of the standard properties of regularity (see [27]).

Proposition 1.8. (i) *Let A be a Noetherian ring, and let $\mathcal{F}$ be an m-regular coherent sheaf on $\mathbb{P}^N_A = \operatorname{Proj} A[X_0, \ldots, X_N]$. Then:*

(a) *the graded $A[X_0, \ldots, X_N]$-module*

$$\Gamma_*(\mathcal{F}) = \bigoplus_{n\in\mathbb{Z}} H^0(\mathbb{P}^N_A, \mathcal{F}(n))$$

is generated by its homogeneous elements of degrees $\leq m$, and the sheaf $\mathcal{F}(n)$ is generated (as an $\mathcal{O}_{\mathbb{P}^N_A}$-module) by its global sections, for all $n \geq m$,

(b) *$H^i(\mathbb{P}^N_A, \mathcal{F}(j)) = 0$ for all $i > 0$ and $i + j \geq m$.*

(ii) *If $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is an exact sequence of coherent sheaves on $\mathbb{P}^N_A$, for a Noetherian ring A, then:*

(a) *if $\mathcal{F}'$ and $\mathcal{F}''$ are m-regular, then $\mathcal{F}$ is m-regular,*

(b) *if $\mathcal{F}$ and $\mathcal{F}''$ are m-regular, then $\mathcal{F}'$ is also m-regular precisely when $H^0(\mathbb{P}^N_A, \mathcal{F}(m-1)) \to H^0(\mathbb{P}^N_A, \mathcal{F}''(m-1))$ is surjective.*

We also need a technical lemma of Mumford (see [27]). Recall that

$$\chi(\mathcal{F}) = \sum_{i\geq 0} (-1)^i \ell(H^i(X, \mathcal{F}))$$

denotes the Euler characteristic of a coherent sheaf $\mathcal{F}$ on a Noetherian scheme X, such that all the cohomology modules of $\mathcal{F}$ have finite length, and only finitely many are non-zero (for example, this holds for any coherent $\mathcal{F}$ if X is proper over an Artinian ring). Recall also that if $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is an exact sequence of such coherent sheaves, then the long exact sequence of cohomology implies easily that $\chi(\mathcal{F}) = \chi(\mathcal{F}') + \chi(\mathcal{F}'')$. Hence, if $\mathcal{F}$ has a finite filtration $\{F^i\mathcal{F}\}$ by coherent subsheaves, then

$$\chi(\mathcal{F}) = \chi(\mathrm{gr}_F\mathcal{F}) = \sum_i \chi(F^i\mathcal{F}/F^{i+1}\mathcal{F}).$$

Lemma 1.9 (Mumford). *Let k be a field, and let $E \subset \mathbb{P}^N_k$ be a closed subscheme of dimension > 0, with Hilbert polynomial P_E (thus $P_E(n) = \chi(\mathcal{O}_E(n))$ for all $n \in \mathbb{Z}$). Let $F = E \cap H$ be the intersection with a hyperplane $H \subset \mathbb{P}^N_k$ such that the associated complex of sheaves*

$$0 \to \mathcal{O}_E(-1) \to \mathcal{O}_E \to \mathcal{O}_F \to 0$$

is exact (i.e., the linear polynomial defining H is a non zero-divisor on $\mathcal{O}_E$). Suppose that the ideal sheaf $\mathcal{I}_F$ of F in $H = \mathbb{P}^{N-1}_k$ is m'-regular, where $m' \geq 1$. Then the following properties hold:

(i) $P_E(m'-1) = \chi(\mathcal{O}_E(m'-1)) \geq 0$,
(ii) *the ideal sheaf $\mathcal{I}_E$ of E in $\mathbb{P}^N_k$ is m-regular, with $m = m' + P_E(m'-1)$,*
(iii) *$\mathcal{O}_E$ is $(m'-1)$-regular.*

Now suppose $(A, \mathfrak{m})$ is a Cohen-Macaulay local ring of dimension $d \geq 1$, with infinite residue field k (which we may assume without loss of generality). We outline the procedure which leads to an inductive proof of the Finitness Theorem.

Let

$$X = \operatorname{Proj} R = \operatorname{Proj} \bigoplus_{n \geq 0} \mathfrak{m}^n$$

be the blow-up of the maximal ideal, and

$$E = \operatorname{Proj} \bigoplus_{n \geq 0} \mathfrak{m}^n/\mathfrak{m}^{n+1}$$

the exceptional divisor. If $\ell(\mathfrak{m}/\mathfrak{m}^2) = N + 1$, then we can realize the Rees A-algebra R as a graded quotient of a polynomial algebra

$A[X_0, \ldots, X_N]$. Hence X is realized as a closed subscheme of $\mathbb{P}^N_A$, such that we have a commutative diagram of closed immersions of A-schemes

$$\begin{array}{ccc} X & \hookrightarrow & \mathbb{P}^N_A \\ \uparrow & & \uparrow \\ X \cap \mathbb{P}^N_k = E & \hookrightarrow & \mathbb{P}^N_k \end{array}$$

(here $\mathbb{P}^N_k$ is regarded as the fibre over the maximal ideal of $\pi : \mathbb{P}^N_A \to \operatorname{Spec} A$). The tautological line bundle (invertible sheaf) $\mathcal{O}_{\mathbb{P}^N_A}(1)$ restricts on X to $\mathcal{O}_X(1)$, which is naturally identified with the ideal sheaf $\mathcal{I}_{E,X}$ of E in X:

$$\mathcal{O}_{\mathbb{P}^N_A}(1) \otimes \mathcal{O}_X = \mathcal{O}_X(1) = \mathfrak{m}\mathcal{O}_X = \mathcal{I}_{E,X} \subset \mathcal{O}_X.$$

The Rees ring $R = \oplus \mathfrak{m}^n$ is identified with the homogeneous coordinate ring of $X \subset \mathbb{P}^N_A$, and the natural restriction map $H^0(\mathbb{P}^N_A, \mathcal{O}_{\mathbb{P}^N_A}(n)) \to H^0(X, \mathcal{O}_X(n))$ induces an inclusion $\mathfrak{m}^n \hookrightarrow H^0(X, \mathcal{O}_X(n))$ for all $n \geq 0$, which is an isomorphism provided $H^1(\mathbb{P}^N_A, \mathcal{I}_X(n)) = 0$, where $\mathcal{I}_X$ is the ideal sheaf of X in $\mathbb{P}^N_A$.

Let $m = m(\mathcal{I}_E)$ be the smallest integer ≥ 1 such that the ideal sheaf $\mathcal{I}_E \subset \mathcal{O}_{\mathbb{P}^N_k}$ of E in $\mathbb{P}^N_A$ is m-regular. Our next goal will be to bound (in terms of $e_0(A)$ and d) the following quantities:

(i) $|e_j(A)|$ for $1 \leq j \leq d-1$,
(ii) m,
(iii) $\dim_k H^j(\mathcal{O}_E(r))$ for $j \geq 1$, $r \geq 0$,
(the bound sought is to be independent of r),
(iv) $\dim_k H^0(\mathcal{O}_E(r))$ for $r \geq 0$,
(with a bound allowed to depend on r),
(v) $|e_d(A)|$.

We have written these expressions in the above order because the proof of each bound will use the existence of the preceeding bounds. The bounds in (i) and (v) combine to give the finiteness theorem for Hilbert polynomials. We will later show (lemma 1.12 below) how the bound in (ii) yields a bound for the postulation number as well.

First consider the case $d = 1$. The bound (i) is vacuous. For (ii), one proves directly that $m \leq e - 1$, with $e = e_0(A)$. This is just the statement (applied to the scheme $E \subset \mathbb{P}^N_k$) that the ideal sheaf of a set of e points (or more generally, of a 0-dimensional subscheme of length e) in projective space over a field is e-regular. By Proposition 1.8(ii)(b), this is equivalent to the assertion that the natural restriction map

$$f : H^0(\mathbb{P}^N_k, \mathcal{O}_{\mathbb{P}^N_k}(e-1)) \to H^0(E, \mathcal{O}_E(e-1))$$

is surjective, which can be easily proved. Since $\dim E = 0$, the bound (iii) is also vacuous. For (iv), note that $\dim H^0(E, \mathcal{O}_E(r)) = e = e_0(A)$ for all $r \in \mathbb{Z}$, again since E is 0-dimensional. For (v), the above surjectivity of f, together with Nakayama's lemma, implies that there is a surjection (and hence an isomorphism) of A-modules $\mathfrak{m}^{e-1} \to H^0(X, \mathcal{O}_X(e-1)) = \mathfrak{m}^{e-1}H^0(X, \mathcal{O}_X)$. This bounds the length of the quotient A-module $H^0(X, \mathcal{O}_X)/A$, which bounds $|e_1(A)|$ by Theorem 1.5(b) (Northcott's formula). Note that, for the case $d = 1$, we have simultaneously bounded the postulation number by $e - 1$.

Henceforth we assume $d = \dim A > 1$, and establish bounds (i)-(v) by induction on d.

Choose a "general" element $x \in \mathfrak{m} - \mathfrak{m}^2$. Then x is a non zero-divisor on A, and its image $\overline{x} \in \mathfrak{m}/\mathfrak{m}^2$ is a homogeneous superficial element in the graded ring $\oplus_{n\geq 0}\mathfrak{m}^n/\mathfrak{m}^{n+1}$ (*i.e.*, $\overline{x}$ is a non zero-divisor in high enough degrees). Let $\overline{A} = A/xA$, $\overline{\mathfrak{m}} = \mathfrak{m}\overline{A}$, so that $\overline{A}$ is a Cohen-Macaulay local ring of dimension $d-1$. Let $Y = \operatorname{Proj}\overline{R}$ with $\overline{R} = \oplus\overline{\mathfrak{m}}^n$ the corresponding blow up, and with exceptional divisor $F \subset Y$. Then $F \subset E$ is a hyperplane intersection of $E \subset \mathbb{P}^N_k$ defined by the homogenous linear equation $\overline{x} = 0$.

One has the following easy lemma, proved via Artin-Rees, using that x is a non zero-divisor on A as well as a superficial element.

Lemma 1.10. *The Hilbert polynomial of $\overline{A}$ is $P_{\overline{A}}(t) = P_A(t) - P_A(t-1)$. In particular, $e_j(\overline{A}) = e_j(A)$ for $j < d$.*

By induction on $d = \dim A$, we may thus assume given bounds (in terms of $e = e_0(A)$ and d) on the following quantities: $|e_1(A)|, \ldots,$ $|e_{d-1}(A)|$, $m' = m(\mathcal{I}_F)$ (the smallest integer $m' \geq 1$ such that $\mathcal{I}_F$ is m'-regular on $H = \mathbb{P}^{N-1}_k$), $\dim_k H^j(\mathcal{O}_F(r))$ for $j > 0$, $r \geq 0$ (bound independent of r), and $\dim_k H^0(\mathcal{O}_F(r))$ for $r \geq 0$ (bound depending on r). In particular, we are already given the desired bound (i) for the ring A.

We next observe that the Hilbert polynomial of $\mathcal{O}_E$ satisfies

$$P_E(n) = \chi(\mathcal{O}_E(n)) = P_A(n+1) - P_A(n) = \sum_{j=0}^{d-1} e_j(A)\binom{n+d-j-1}{d-j-1},$$

since we have the corresponding formula for the underlying Hilbert functions, at least for all sufficiently large n (the homogeneous coordinate ring of $E \subset \mathbb{P}^N_k$ coincides, in large enough degrees, with the graded ring $\oplus_{n\geq 0}\mathfrak{m}^n/\mathfrak{m}^{n+1}$). Now Mumford's lemma (lemma 1.9), combined with the known bound (i) above, and the bound (ii) for the ring $\overline{A}$, implies that $\mathcal{I}_E$ is m-regular for some quantity $m \geq 1$ bounded in terms of $e_0(A)$ and d. This gives the desired bound (ii) for the ring A.

The exact sequences of sheaves for all $r \geq 0$

$$0 \to \mathcal{O}_E(r) \to \mathcal{O}_E(r+1) \to \mathcal{O}_F(r+1) \to 0 \tag{1.2}$$

yield exact sequences of finte dimensional k-vector spaces

$$\cdots \to H^{j-1}(F, \mathcal{O}_F(r+1)) \to H^j(E, \mathcal{O}_E(r)) \to H^j(E, \mathcal{O}_E(r+1)) \to \cdots$$

which imply, for $j \geq 1$, $r \geq 0$,

$$\begin{aligned}&\dim_k H^j(E, \mathcal{O}_E(r)) \\ &\leq \dim_k H^{j-1}(F, \mathcal{O}_F(1)) + \cdots + \dim_k H^j(F, \mathcal{O}_F(m' - j - 1))\end{aligned}$$

since $\mathcal{O}_E$ is $m' - 1$-regular, by lemma 1.9(iii) (as $\mathcal{I}_F$ is m'-regular), and since $H^j(E, \mathcal{O}_E(r)) = 0$ for all $j > 0$ for sufficiently large r, by Serre vanishing (see [13], III Thm. 5.2). This gives the desired bound (iii) for the ring A.

For the bound (iv), first note that by the definition of Euler characteristic of a sheaf, we have

$$\dim_k H^0(E, \mathcal{O}_E) \leq \chi(\mathcal{O}_E) + \sum_{0 \leq j \leq (d-2)/2} \dim_k H^{2j+1}(E, \mathcal{O}_E).$$

But $\chi(\mathcal{O}_E) = P_A(1) - P_A(0) = e_0(A) + \cdots + e_{d-1}(A)$, so the bound (iv) for $r = 0$ (*i.e.*, for $\dim_k H^0(E, \mathcal{O}_E)$) is deduced from the bound (iii) already obtained. For $r \geq 1$, we use the exact sequences (obtained from the sheaf exact sequence (1.2))

$$0 \to H^0(E, \mathcal{O}_E(r-1)) \to H^0(E, \mathcal{O}_E(r)) \to H^0(F, \mathcal{O}_F(r)) \to \cdots$$

which implies the inequality

$$\dim_k H^0(E, \mathcal{O}_E(r)) \leq \dim_k H^0(E, \mathcal{O}_E) + \sum_{j=1}^{r} \dim_k H^0(F, \mathcal{O}_F(j)).$$

The right side is bounded, by the bound (iv) for the ring $\overline{A}$, and the bound for $r = 0$ already obtained.

Finally, we are left with the bound (v), for the absolute value of the constant term $e_d(A)$ of the Hilbert polynomial. This is really the heart of the matter, in a way. The following lemma gives the desired bound.

Lemma 1.11. *We have inequalities*

$$-\sum_{j=0}^{m'-3} \chi(\mathcal{O}_E(j)) \leq e_d(A) \leq \sum_{j=0}^{m'-3} \left[\dim_k H^0(E, \mathcal{O}_E(j)) - \chi(\mathcal{O}_E(j))\right].$$

Proof. From Theorem 1.5(a), we have the formula (1.1)

$$e_d(A) = \sum_{i=1}^{d-1}(-1)^{i-1}\ell(H^i(X,\mathcal{O}_X)).$$

From the Formal Function Theorem (see [13], III, §11), we have isomorphisms for $i > 0$, and any $r \in \mathbb{Z}$,

$$H^i(X,\mathcal{O}_X(r)) \cong \varprojlim_n H^i(X^{(n)},\mathcal{O}_{X^{(n)}}(r)), \tag{1.3}$$

where $X^{(n)} \subset X$ is the closed subscheme with ideal sheaf $\mathfrak{m}^n\mathcal{O}_X = \mathcal{O}_X(n)$ (in particular, $X^{(1)} = E$). We have exact sheaf sequences

$$0 \to \mathcal{O}_E(n) \to \mathcal{O}_{X^{(n+1)}} \to \mathcal{O}_{X^{(n)}} \to 0$$

which, since $\mathcal{O}_E$ is $m' - 1$-regular (by lemma 1.9), imply that

$$H^i(\mathcal{O}_{X^{(n+1)}}) \cong H^i(\mathcal{O}_{X^{(n)}}) \ \forall\, n \geq m' - i - 1.$$

Hence, taking $r = 0$ in the formula (1.3), we deduce from the formula (1.1) that

$$\begin{aligned} e_d &= \textstyle\sum_{i=1}^{d-1}(-1)^{i-1}\ell\left(H^i(X^{(n)},\mathcal{O}_{X^{(n)}})\right) \ \ \forall\, n \geq m' - 2 \\ &= \ell(H^0(X^{(n)},\mathcal{O}_{X^{(n)}})) - \chi(\mathcal{O}_{X^{(n)}}). \end{aligned}$$

But $\mathcal{O}_{X^{(n)}}$ is filtered by the sheaves $\mathcal{O}_E(j)$, $0 \leq j \leq n-1$, and so

$$\chi(\mathcal{O}_{X^{(n)}}) = \sum_{j=0}^{n-1}\chi(\mathcal{O}_E(j)),$$

and

$$0 \leq \ell(H^0(X^{(n)},\mathcal{O}_{X^{(n)}})) \leq \sum_{j=0}^{n-1}\dim_k H^0(E,\mathcal{O}_E(j)).$$

This implies the inequalities stated in the lemma. Q.E.D.

We now discuss the bound on the postulation number.

Lemma 1.12. *With the above notation, if $m \geq 1$ is an integer such that $\mathcal{I}_E \subset \mathcal{O}_{\mathbb{P}^N_k}$ is m-regular and $\mathcal{O}_E$ is $(m-1)$-regular on $\mathbb{P}^N_k$, then $m - 1$ is a postulation number for the Hilbert function of A.*

Proof. Since $\mathcal{O}_E$ is $(m-1)$-regular on $\mathbb{P}^N_k$, the Formal Function Theorem (formula (1.3)) implies that $\mathcal{O}_X$ is $(m-1)$-regular on $\mathbb{P}^N_A$. Hence the graded R-module $\oplus_{n\geq 0} H^0(X, \mathcal{O}_X(n))$ is generated by its homogeneous elements of degrees $\leq m-1$. Thus $H^0(X, \mathcal{O}_X(n+1)) = \mathfrak{m}H^0(X, \mathcal{O}_X(n))$ if $n \geq m-1$. From the exact sheaf sequence

$$0 \to \mathcal{O}_X(n+1) \to \mathcal{O}_X(n) \to \mathcal{O}_E(n) \to 0,$$

we get an exact sequence (defining the map γ_n)

$$0 \to H^0(X, \mathcal{O}_X(n+1)) \to H^0(X, \mathcal{O}_X(n)) \xrightarrow{\gamma_n} H^0(E, \mathcal{O}_E(n))$$

so that

$$\text{(1.4)} \qquad \operatorname{image} \gamma_n = H^0(X, \mathcal{O}_X(n)) \otimes A/\mathfrak{m} \ \forall\, n \geq m-1.$$

Now in the diagram (defining maps $\alpha_n, \beta_n, \rho_n$)

$$\begin{array}{ccc} H^0(\mathbb{P}^N_A, \mathcal{O}_{\mathbb{P}^N_A}(n)) & \xrightarrow{\alpha_n} & H^0(X, \mathcal{O}_X(n)) \\ \rho_n \downarrow & & \downarrow \gamma_n \\ H^0(\mathbb{P}^N_k, \mathcal{O}_{\mathbb{P}^N_k}(n)) & \xrightarrow{\beta_n} & H^0(E, \mathcal{O}_E(n)) \end{array}$$

we have that ρ_n is surjective for all $n \geq 0$, while β_n is surjective for $n \geq m-1$, since $\mathcal{I}_E \subset \mathcal{O}_{\mathbb{P}^N_k}$ is m-regular. Hence γ_n is surjective, and so by the formula (1.4) and Nakayama's lemma, α_n is surjective for $n \geq m-1$. But the image of α_n is $\mathfrak{m}^n \subset H^0(X, \mathcal{O}_X(n))$. Hence $m-1$ is a postulation number for the local ring A. Q.E.D.

1.2. The Regularity Number and applications

Next, we discuss the regularity number of a module M over $(A, \mathfrak{m})$ with respect to an ideal I such that M/IM has finite length. Let $d = \dim M$, and let $e_j(I, M)$, $0 \leq j \leq d$ denote the Hilbert coefficients of M with respect to I. Inductively define

$$\begin{aligned} m_1 = {} & e_0(I, M) \\ m_i = {} & m_{i-1} + e_0(I, m)\tbinom{m_{i-1}+i-2}{i-1} + e_1(I, M)\tbinom{m_{i-1}+i-3}{i-2} + \cdots \\ & + e_{i-1}(I, M), \qquad \text{for } 2 \leq i \leq d. \end{aligned}$$

Note that m_d is a polynomial with rational coefficients in $e_j(I, M)$, $0 \leq j \leq d-1$. In particular, it is independent of $e_d(I, M)$, the constant coefficient of the Hilbert polynomial of M (this is a crucial point in the inductive proof of Theorem 1.1). The integer $\mathbf{m}(I, M) = m_d(I, M)$ is called the *regularity number* of M with respect to I.

This terminology is introduced in [44], because of its relation to Castelnuovo-Mumford regularity of a certain sheaf; we comment more on this later. Its relevance to us is due to the following result (see [44], Theorem 2 and Corollary 2).

Theorem 1.13. (a) *Let M be a finite module over a Noetherian local ring $(A, \mathfrak{m})$, with* $\operatorname{depth}(M) > 0$, *and let $I \subset \mathfrak{m}$ be an ideal in A such that M/IM has finite length. Then $\mathbf{m}(I, M) - 1$ is a postulation number for M with repect to I, i.e., we have an equality between values of the Hilbert function and Hilbert polynomial*

$$H_I(M)(n) = P_I(M)(n)$$

for all $n \geq \mathbf{m}(I, M) - 1$.

(b) *Let $(A, \mathfrak{m})$ be a Noetherian local ring of dimension d and depth > 0. Let $I \subset \mathfrak{m}$ be an $\mathfrak{m}$-primary ideal, and $J \subset I$ a reduction ideal of I. Then the reduction number $r_J(I)$ is at most the regularity number $\mathbf{m}(I, A)$, i.e., we have the reduction formula*

$$J^n I^{\mathbf{m}} = I^{n+\mathbf{m}} \ \forall\, n \geq 0,$$

where $\mathbf{m} = \mathbf{m}(I, A)$.

Remark 1.14. We comment further on the regularity number $\mathbf{m}(I, M)$. Let $X = \operatorname{Proj} R(I)$ be the blow-up of the ideal I. Then the graded "Rees module" $M(I) = \oplus_{n \geq 0} I^n M$ yields a coherent sheaf $\mathcal{F}$ on X, and hence (by choosing a set of $N+1$ generators of I, and thus an embedding $X \hookrightarrow \mathbb{P}^N_A$) also on $\mathbb{P}^N_A$. This sheaf is generated by its global sections, since $M(I)$ is generated by its homogeneous elements of degree 0; hence we can find an exact sequence of coherent $\mathcal{O}_{\mathbb{P}^N_A}$-modules

$$0 \to \mathcal{K} \to \mathcal{O}_{\mathbb{P}^N_A}^{\oplus r} \to \mathcal{F} \to 0.$$

It is shown in [44] that $\mathcal{K}$ is $\mathbf{m}(I, M)$ regular on $\mathbb{P}^N_A$; this in turn controls the regularity of various other related sheaves. Note that this first syzygy sheaf $\mathcal{K}$ is the analogue for A-modules M of the ideal sheaf of the projective scheme X.

Remark 1.15. The fact that there is a connection between regularity and the reduction number was noted earlier by Trung [46]. That this implies a connection also with Hilbert coefficients seems to be the new observation in [44].

The last related result we discuss in this Section is the following "explicit" form of the Artin-Rees lemma, due to Trivedi, taken from

[44]. If $N \subset M$ are finitely generated modules over a Noetherian ring R, and I any ideal of R, an *Artin-Rees number* of (M, N) with respect to I is a positive integer n_0 such that we have the Artin-Rees formula

$$I^{n+n_0}M \cap N = I^n(I^{n_0}M \cap N) \ \ \forall\, n \geq 0$$

(the choice of the terminology "Artin-Rees number", introduced in [44], is self-explanatory).

Proposition 1.16. *Let $(A, \mathfrak{m})$ be Noetherian local, $N \subset M$ finite A-modules such that M/N has positive depth and dimension d. Then the regularity number $\mathbf{m}(\mathfrak{m}, M/N)$ is also an Artin-Rees number for (M, N) with respect to the maximal ideal $\mathfrak{m}$.*

Remark 1.17. We do not discuss the proof here; however, one of the points made by the latter part of [44] is that there is a relationship between the Artin-Rees number and Castelnuovo-Mumford regularity. The "standard" proof of the Artin-Rees lemma reduces to the statement that n_0 is an Artin-Rees number if a certain graded module is generated by its elements of degrees $\leq n_0$. On the other hand, Proposition 1.8 yields a statement that a certain graded module is generated by its homogeneous elements of bounded degree. In a sense, this is the explanation for the above mentioned relationship.

Using Proposition 1.16, and the technique of normally flat stratifications, a new proof (see [44], Theorem 4) is obtained of the following Uniform Artin-Rees Theorem of Duncan and O'Carroll (see [6] for the original proof).

Theorem 1.18 (Uniform Artin-Rees). *Let A be an excellent (or even J2) ring, and $N \subset M$ finitely generated A-modules. Then there is an $n_0 \geq 0$ such that for* any *maximal ideal $\mathfrak{m} \subset A$, we have*

$$\mathfrak{m}^{n+n_0}M \cap N = \mathfrak{m}^n(\mathfrak{m}^{n_0}M \cap N) \ \ \forall\, n \geq 0.$$

The idea of the new proof is the following: results from the theory of normal flatness imply that, since A is a J2 ring, there are only finitely many numerical functions occuring as Hilbert functions of localizations $(M/N)_\wp$ with respect to prime ideals $\wp$ of A. Hence there is a uniform bound on all the Hilbert coefficients of these localizations $(M/N)_\wp$, and hence also of the corresponding regularity numbers. Now Proposition 1.16 implies that there is a uniform bound on Artin-Rees numbers for the localizations $(M_\wp, N_\wp)$ with respect to the corresponding maximal ideals $\wp A_\wp$. For maximal ideals $\mathfrak{m}$, an Artin-Rees number for $(M_\mathfrak{m}, N_\mathfrak{m})$ with respect to $\mathfrak{m}A_\mathfrak{m}$ is automatically also an Artin-Rees number for (M, N) with respect to $\mathfrak{m}$.

If M/N is Cohen-Macaulay, then combining the Finiteness Theorem 1.1 with Proposition 1.16 yields the following result.

Corollary 1.19. *Let A be Noetherian, $N \subset M$ finitely generated modules such that $(M/N)_{\mathfrak{m}}$ is Cohen-Macaulay for each maximal ideal $\mathfrak{m} \subset A$. Then*

$$\sup_{\mathfrak{m}}\{3^{d!-1}e(\mathfrak{m},(M/N)_{\mathfrak{m}})^{3(d-1)!-1}\}$$

is an Artin-Rees number for (M, N) with respect to all maximal ideals $\mathfrak{m}$.

This has the following application to graded rings (see [44], Corollary 5):

Corollary 1.20. *Let $(A, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension d and $I \subset A$ an ideal generated by an A-sequence $f_1, \dots, f_r$ of length r. Let $\mathbf{e} = e_0(\mathfrak{m}, A/I)$ be the $\mathfrak{m}$-adic multiplicity of A/I, and set*

$$N = \max\{3^{(d-r)!}\mathbf{e}^{3(d-r-1)!-1}, (d-r)!\mathbf{e}+2\}.$$

Let J be the ideal generated by $f_1 + g_1, \dots, f_r + g_r$, where $g_1, \dots, g_r \in \mathfrak{m}^N$ are arbitrarily chosen elements. Then the $\mathfrak{m}$-adic associated graded rings (and hence the Hilbert functions) of A/I and A/J are canonically isomorphic.

§2. Chern Classes and Zero Cycles

In this Section, we discuss how the theory of Chern classes with values in the Chow ring, combined with results on 0-cycles, yields several counterexamples which are of interest in algebra. For simplicity, we will work only with varieties over an algebraically closed field k, and eventually restrict to the case $k = \mathbb{C}$, the complex number field. However, unlike the convention in [13], we will use the term "variety" even when referring to unions of irreducible varieties, and explicitly mention irreducibility when needed.

2.1. The Chow ring and Chern classes

First, we recall the definition of the graded *Chow ring* $CH^*(X) = \bigoplus_{p\geq 0} CH^p(X)$ of a non-singular variety X over an algebraically closed field k (see Fulton's book [8] for more details; see also Bloch [1]). The graded components $CH^p(X)$ generalize the more familiar notion of the *divisor class group*, which is just the group $CH^1(X)$.

If $Z \subset X$ is irreducible, let $\mathcal{O}_{Z,X}$ be the local ring of Z on X (*i.e.*, the local ring of the generic point of Z, in the terminology of Hartshorne's book [13]). The codimension of Z in X, denoted $\operatorname{codim}_X Z$, is the dimension of the local ring $\mathcal{O}_{Z,X}$. Now let

$$\begin{aligned} Z^p(X) &= \text{Free abelian group on irreducible subvarieties of } X \text{ of codi-} \\ &\qquad \text{mension } p \\ &= \text{Group of algebraic cycles on } X \text{ of codimension } p. \end{aligned}$$

For an irreducible subvariety $Z \subset X$, let $[Z]$ denote its class in $Z^p(X)$ (where $p = \operatorname{codim}_X Z$).

Let $Y \subset X$ be irreducible of codimension $p-1$, and let $k(Y)^*$ denote the multiplicative group of non-zero rational functions on Y ($k(Y)$, which is the field of rational functions on Y, is the residue field of $\mathcal{O}_{Y,X}$). For each irreducible divisor $Z \subset Y$, we have a homomorphism $\operatorname{ord}_Z : k(Y)^* \to \mathbb{Z}$, given by

$$\operatorname{ord}_Z(f) = \ell(\mathcal{O}_{Z,Y}/a\mathcal{O}_{Z,Y}) - \ell(\mathcal{O}_{Z,Y}/b\mathcal{O}_{Z,Y}),$$

for any expression of f as a ratio $f = a/b$ with $a, b \in \mathcal{O}_{Z,Y} \setminus \{0\}$. Here $\ell(M)$ denotes the length of an Artinian module M.

For $f \in k(Y)^*$, let $(f)_Y$ denote the divisor of f on Y, defined by

$$(f)_Y = \sum_{Z \subset Y} \operatorname{ord}_Z(f) \cdot [Z],$$

where Z runs over all irreducible divisors in Y; the sum has only finitely many non-zero terms, and is hence well-defined. Clearly we may also view $(f)_Y$ as an element of $Z^p(X)$.

Let $R^p(X) \subset Z^p(X)$ be the subgroup generated by cycles $(f)_Y$ as (Y, f) ranges over all irreducible subvarieties Y of X of codimension $p-1$, and all $f \in k(Y)^*$. We refer to elements of $R^p(X)$ as cycles *rationally equivalent to 0* on X. The p-th Chow group of X is defined to be

$$\begin{aligned} CH^p(X) &= \frac{Z^p(X)}{R^p(X)} \\ &= \text{group of rational equivalence classes of codimension } p\text{-cycles on } X. \end{aligned}$$

We will abuse notation and also use $[Z]$ to denote the class of an irreducible subvariety Z in $CH^p(X)$.

The graded abelian group

$$CH^*(X) = \bigoplus_{0 \le p \le \dim X} CH^p(X)$$

can be given the structure of a commutative (graded) ring via the *intersection product*. This product is characterized by the following property — if $Y \subset X$, $Z \subset X$ are irreducible of codimensions p, q respectively, and $Y \cap Z = \cup_i W_i$, where each $W_i \subset X$ is irreducible of codimension $p+q$ (we then say Y and Z *intersect properly* in X), then the intersection product of the classes $[Y]$ and $[Z]$ is

$$[Y] \cdot [Z] = \sum_i I(Y, Z; W_i)[W_i]$$

where $I(Y, Z; W_i)$ is the *intersection multiplicity* of Y and Z along W_i, defined by Serre's formula

$$I(Y, Z; W_i) = \sum_{j \geq 0} (-1)^j \ell \left(\operatorname{Tor}_j^{\mathcal{O}_{W_j,X}} (\mathcal{O}_{W_j,Y}, \mathcal{O}_{W_j,Z}) \right).$$

One of the important results proved in the book [8] is that the above procedure does give rise to a well-defined ring structure on $CH^*(X)$.

The Chow ring is an algebraic analogue for the even cohomology ring

$$\bigoplus_{i=0}^{n} H^{2i}(X, \mathbb{Z})$$

defined in algebraic topology. To illustrate this, we note the following 'cohomology-like' properties, proved in Fulton's book [8]. Here, we follow the convention of [13], and use the term "vector bundle on X" to mean "(coherent) locally free sheaf of $\mathcal{O}_X$-modules", and use the term "geometric vector bundle on X", as in [13] II Ex. 5.18, to mean a Zariski locally trivial algebraic fiber bundle $V \to X$ whose fibres are affine spaces, with linear transition functions. With this convention, we can also identify vector bundles on an affine variety $X = \operatorname{Spec} A$ with finitely generated projective A-modules; as in [13], we use the notation $\widetilde{M}$ to denote the coherent sheaf corresponding to a finitely generated A-module M.

Theorem 2.1 (Properties of the Chow ring and Chern classes).

(1) $X \mapsto \bigoplus_p CH^p(X)$ *is a contravariant functor from the category of smooth varieties over* k *to graded rings. If* $X = \coprod_i X_i$, *where* X_i *are the irreducible* (= *connected*) *components, then* $CH^*(X) = \prod_i CH^*(X_i)$. *If* X *is irreducible, then* $CH^0(X) = \mathbb{Z}$ *generated by the class* $[X]$.

(2) *If* X *is irreducible and projective* (*or more generally, proper*) *over* k *and* $d = \dim X$, *there is a well defined* degree homomorphism $\deg : CH^d(X) \to \mathbb{Z}$ *given by* $\deg(\sum_i n_i[x_i]) = \sum_i n_i$. *This allows one to define* intersection numbers *of cycles of complementary dimension, in a purely algebraic way, which agree with those defined via topology when* $k = \mathbb{C}$ (*see* (7) *below*).

(3) *If* $f : X \to Y$ *is a proper morphism of smooth varieties, there are "Gysin"* (*or "push-forward"*) *maps* $f_* : CH^p(X) \to CH^{p+r}(Y)$ *for all* p, *where* $r = \dim Y - \dim X$; *here if* $p+r < 0$, *we define* f_* *to be* 0; *the induced map* $CH^*(X) \to CH^*(Y)$ *is* $CH^*(Y)$*-linear* (projection formula), *where* $CH^*(X)$ *is regarded as a* $CH^*(Y)$*-module via the* (*contravariant*) *ring homomorphism* $f^* : CH^*(Y) \to CH^*(X)$. *If* $f : X \hookrightarrow Y$ *is the inclusion of a closed subvariety, then* f_* *is induced by the natural inclusions* $Z^p(X) \hookrightarrow Z^{p+r}(Y)$.

(4) $f^* : CH^*(X) \xrightarrow{\cong} CH^*(V)$ *for any geometric vector bundle* $f : V \to X$ (homotopy invariance). *In particular,* $CH^*(X \times \mathbb{A}^n) = CH^*(X)$, *and* $CH^*(\mathbb{A}^n) = \mathbb{Z}$.

(5) *If* V *is a vector bundle* (*i.e., locally free sheaf*) *of rank* r *on* X, *then there are* Chern classes $c_p(V) \in CH^p(X)$, *such that*

  (a) $c_0(V) = 1$,

  (b) $c_p(V) = 0$ *for* $p > r$, *and*

  (c) *for any exact sequence of vector bundles*

$$0 \to V_1 \to V_2 \to V_3 \to 0$$

  we have $c(V_2) = c(V_1)c(V_3)$, *where* $c(V_i) = \sum_p c_p(V_i)$ *are the corresponding* total Chern classes,

  (d) $c_p(V^\vee) = (-1)^p c_p(V)$, *where* $V^\vee$ *is the dual vector bundle.*

Moreover, we also have the following properties.

(6) *If* $f : \mathbb{P}(V) = \mathbf{Proj}\, S(V) \to X$ *is the projective bundle associated to a vector bundle of rank* r (*where* $S(V)$ *is the symmetric algebra of the sheaf* V *over* $\mathcal{O}_X$), *then* $CH^*(\mathbb{P}(V))$ *is a* $CH^*(X)$*-algebra generated by* $\xi = c_1(\mathcal{O}_{\mathbb{P}(V)}(1))$, *the first Chern class of the tautological line bundle, which is subject to the relation*

$$\xi^r - c_1(V)\xi^{r-1} + \cdots + (-1)^r c_n(V) = 0;$$

in particular, $CH^*(\mathbb{P}(V))$ is a free $CH^*(X)$-module with basis $1, \xi, \xi^2, \dots, \xi^{r-1}$.

(7) *If $k = \mathbb{C}$, there are cycle class homomorphisms $CH^p(X) \to H^{2p}(X, \mathbb{Z})$ such that the intersection product corresponds to the cup product in cohomology, and for a vector bundle E, the cycle class of $c_p(E)$ is the topological p-th Chern class of E.*

(8) *The first Chern class determines an isomorphism $c_1 : \mathrm{Pic} X \to CH^1(X)$ from the Picard group of line bundles on X to the first Chow group (i.e., the divisor class group) of X, such that $c_1(\mathcal{O}_X(D)) = [D] \in CH^1(X)$ for any divisor D on X. For an arbitrary vector bundle V, of rank n, we have $c_1(V) = c_1(\det V)$, where $\det V = \bigwedge^n V$.*

(9) *If $f : X \to Y$ is a morphism between non-singular varieties, V a vector bundle on Y, then the Chern classes of the pull-back vector bundle f^*V on X are given by $c(f^*V) = f^*c(V)$, where on the right, f^* is the ring homomorphism $CH^*(Y) \to CH^*(X)$* (functoriality of Chern classes). *In particular, taking $Y =$ point, we see that $c(\mathcal{O}_X) = 1 \in CH^*(X)$.*

(10) *If $i : Y \hookrightarrow X$ is the inclusion of an irreducible smooth subvariety of codimension r in a smooth variety, with normal bundle $\mathcal{N} = (\mathcal{I}_Y/\mathcal{I}_Y^2)^\vee$ (where $\mathcal{I}_Y \subset \mathcal{O}_X$ is the ideal sheaf of Y in X), then $\mathcal{N}$ is a vector bundle on Y of rank r with top Chern class*

$$c_r(\mathcal{N}) = i^* \circ i_*[Y],$$

where $[Y] \in CH^0(Y) = \mathbb{Z}$ is the generator (self-intersection formula).

Remark 2.2. If $X = \operatorname{Spec} A$ is affine, we will also sometimes write $CH^*(A)$ in place of $CH^*(X)$; similarly, by the Chern classes $c_i(P)$ of a finitely generated projective A-module P, we mean $c_i(\widetilde{P})$ where $\widetilde{P}$ is the associated vector bundle on X.

We remark that the total Chern class of a vector bundle on a smooth variety X is a *unit* in the Chow ring $CH^*(X)$, since it is of the form $1 +$ (nilpotent element). Thus the assignment $V \mapsto c(V)$ gives a homomorphism of groups from the Grothendieck group $K_0(X)$ of vector bundles (locally free sheaves) on X to the multiplicative group of those units in the graded ring $CH^*(X)$, which are expressible as $1 +$ (higher degree terms).

On a non-singular variety X, every coherent sheaf has a resolution by locally free sheaves (vector bundles) of finite rank, and the Grothendieck group $K_0(X)$ of vector bundles coincides with the Grothendieck group of coherent sheaves. There is a finite decreasing filtration $\{F^pK_0(X)\}_{p\geq 0}$

on $K_0(X)$, where $F^pK_0(X)$ is the subgroup generated by classes of sheaves supported in codimension $\geq p$. Further, $F^pK_0(X)/F^{p+1}K_0(X)$ is generated, as an abelian group, by the classes $\mathcal{O}_Z$ for irreducible subvarieties $Z \subset X$ of codimension p – for example, if $X = \operatorname{Spec} A$ is affine, we can see this using the fact that any finitely generated A-module M has a finite filtration whose quotients are of the form $A/\wp$ for prime ideals $\wp$, such that the minimal primes in $\operatorname{supp}(M)$ all occur, and their multiplicities in the filtration are independent of the choice of filtration. Thus, we have a natural surjection $Z^p(X) \to F^pK_0(X)/F^{p+1}K_0(X)$.

Now we can state the following result, sometimes called the *Riemann-Roch theorem without denominators* (see the book [8] for a proof).

Theorem 2.3. *Let X be a non-singular variety.*

(a) *If $x \in F^pK_0(X)$, then $c_i(x) = 0$ for $i < p$, and $c_p : F^pK_0(X) \to CH^p(X)$ is a group homomorphism.*
Let $\overline{c_p} : F^pK_0(X)/F^{p+1}K_0(X) \to CH^p(X)$ be the induced homomorphism.

(b) *The natural surjection $Z^p(X) \twoheadrightarrow F^pK_0(X)/F^{p+1}K_0(X)$ factors through rational equivalence, yielding a map $\psi_p : CH^p(X) \twoheadrightarrow F^pK_0(X)/F^{p+1}K_0(X)$.*

(c) *The compositions $\overline{c_p} \circ \psi_p$ and $\psi_p \circ \overline{c_p}$ both equal multiplication by the integer $(-1)^{p-1}(p-1)!$. In particular, both $\overline{c_p}$ and ψ_p are isomorphisms $\otimes\mathbb{Q}$.*

In particular, if $Z \subset X$ is an irreducible subvariety of codimension p, then $c_i([\mathcal{O}_Z]) = 0$ for $i < p$, and $c_p([\mathcal{O}_Z]) = (-1)^{p-1}(p-1)![Z] \in CH^p(X)$.

Remark 2.4. If $X = \operatorname{Spec} A$ is affine, any element $\alpha \in K_0(X)$ can be expressed as a difference $\alpha = [P] - [A^{\oplus m}]$ for some finitely generated projective A-module P and some positive integer m. Hence the total Chern class $c(\alpha)$ coincides with $c(P)$. The above theorem now implies that for any element $a \in CH^p(X)$, there is a finitely generated projective A module P with $c_p(P) = (p-1)!a$. By the Bass stability theorem, which implies that any projective A-module of rank $> d = \dim A$ has a free direct summand of positive rank, we can find a projective A-module P with $\operatorname{rank} P \leq d$ and $c_p(P) = (p-1)!a$.

Incidentally, this statement cannot be improved, in general: for any $p > 2$, there are examples of affine non-singular varieties X and elements $a \in CH^p(X)$ such that $ma \in \operatorname{image} c_p$ for some integer $m \iff (p-1)!|m$. For examples of Mohan Kumar and Nori, see [42], §17.

2.2. An example of a graded ring

We now discuss our first application of these constructions to commutative algebra, due to N. Mohan Kumar (unpublished). Let $k = \overline{k}$. We give an example of a 3-dimensional graded integral domain $A = \bigoplus_{n\geq 0} A_n$, with the following properties:

1. A is generated by A_1 as an A_0-algebra, where A_0 is a regular affine k-algebra of dimension 1,
2. the "irrelevant graded prime ideal" $P = \bigoplus_{n>0} A_n$ is the radical of an ideal generated by 2 elements,
3. P cannot be expressed as the radical of an ideal generated by 2 *homogeneous* elements.

For the example, take A_0 to be affine coordinate ring of a non-singular curve $C \subset \mathbb{A}^3_k$ such that the canonical module $\omega_{A_0} = \Omega_{A_0/k}$ is a non-torsion element of the divisor class group of A_0 (this implies k is not the algebraic closure of a finite field). In fact, if we choose A_0 to be a non-singular affine k-algebra of dimension 1 such that ω_{A_0} is non-torsion in the class group, then $C = \operatorname{Spec} A_0$ can be realized as a curve embedded in $\mathbb{A}^3_k$, by more or less standard arguments (see [13], IV, or [36], for example).

Let $R = k[x, y, z]$ denote the polynomial algebra, and let $\varphi : R \to A_0$ be the surjection corresponding to $C \hookrightarrow \mathbb{A}^3_k$. Let $I = \ker \varphi$ be the ideal of C. Then I/I^2 is a projective A_0-module of rank 2; we let

$$A = S(I/I^2) = \bigoplus_{n\geq 0} S^n(I/I^2)$$

be its symmetric algebra over A_0. We claim this graded ring A has the properties stated above.

Consider the exact sequence of projective A_0-modules

$$0 \to I/I^2 \xrightarrow{\psi} \Omega_{R/k} \otimes A_0 \xrightarrow{\overline{\varphi}} \omega_{A_0} \to 0 \tag{2.1}$$

with $\overline{\varphi}$ induced by φ, and ψ by the derivation $d : R \to \Omega_{R/k}$. Let $h : \Omega_{R/k} \otimes A_0 \to I/I^2$ be a splitting of ψ. Use h to define a homomorphism of k-algebras

$$\Phi : R \to A,$$

by setting

$$\Phi(t) = \phi(t) + h(dt) \in A_0 \oplus A_1 = A_0 \oplus I/I^2$$

for $t = x, y, z$; this uniquely specifies a k-algebra homomorphism Φ defined on the polynomial algebra R.

Clearly $\Phi(I) \subset P = \bigoplus_{n>0} A_n$, the irrelevant graded ideal, and one verifies that Φ induces isomorphisms $R/I \to A/P$ and $I/I^2 \to P/P^2$, and in fact an isomorphism between the I-adic completion of R and the P-adic completion of A.

Since $C \subset \mathbb{A}^3_k$ is a non-singular curve, it is a set-theoretic complete intersection, from a result of Ferrand and Szpiro (see [43], for example). If $a, b \in I$ with $\sqrt{(a,b)} = I$, then clearly we have $\sqrt{(\Phi(a), \Phi(b))} = P \cap Q$, for some (radical) ideal Q with $P + Q = A$. We can correspondingly write $(\Phi(a), \Phi(b)) = J \cap J'$ with $\sqrt{J} = P$, $\sqrt{J'} = Q$. Then $J/J^2 \cong (A/J)^{\oplus 2}$. This implies (by an old argument of Serre) that $\operatorname{Ext}^1_A(J, A) \cong \operatorname{Ext}^2_A(A/J, A) \cong A/J$ is free of rank 1, and any generator determines an extension

$$0 \to A \to V \to J \to 0$$

where V is a projective A-module of rank 2, and such that the induced surjection $V \otimes A/J \to J/J^2 \cong (A/J)^{\oplus 2}$ is an isomorphism.

We claim the projective module V is necessarily of the form $V = V_0 \otimes_{A_0} A$; this implies $V_0 = V \otimes_A A/P \cong J/PJ \cong (A/P)^{\oplus 2}$ is free, so that V is a free A-module, and J is generated by 2 elements. To prove the claim, note that I/I^2 is a direct summand of a free $A/I = A_0$-module of finite rank; hence there is an affine A-algebra $A' \cong A_0[x_1, \dots, x_n]$, which is a polynomial algebra over A_0, such that A is an algebra retract of A'. Now it suffices to observe that any finitely generated projective A'-module is of the form $M \otimes_{A_0} A'$, for some projective A_0-module M; this is the main result of [19] (see also [20]).

On the other hand, we claim that it is impossible to find two *homogeneous* elements $x, y \in P$ with $\sqrt{(x,y)} = P$. Indeed, let $X = \operatorname{Proj} A$, and $\pi : X \to C = \operatorname{Spec} A_0$ be the natural morphism. Then $X = \mathbb{P}(V)$ is the $\mathbb{P}^1$-bundle over C associated to the locally free sheaf $V = \widetilde{I/I^2}$ (the sheaf determined by the projective A_0-module I/I^2). Let $\xi = c_1(\mathcal{O}_X(1)) \in CH^1(X)$ be the 1st Chern class of the tautological line bundle $\mathcal{O}_X(1)$. Then by Theorem 2.1 and (2.1) above, $CH^*(X)$ is a free $CH^*(C)$-module with basis $1, \xi$, and ξ satisfies the monic relation

$$\xi^2 - c_1(V)\xi + c_2(V) = 0.$$

Since $\dim C = 1$, $CH^i(C) = 0$ for $i > 1$, and so this relation reduces to

$$\xi^2 = c_1(V)\xi.$$

From the exact sequence (2.1), we have a relation in $CH^*(C)$

$$1 = c(\mathcal{O}_C)^3 = c(\mathcal{O}_C^{\oplus 3}) = c(\Omega_{\mathbb{A}^3/k} \otimes \mathcal{O}_C) = c(V) \cdot c(\omega_C).$$

Hence $c_1(V) = -c_1(\omega_C)$, which by the choice of C is a non-torsion element of $CH^1(C)$ (which is the divisor class group of A_0). Thus $\xi^2 \in CH^2(X)$ is a non-torsion element of $CH^2(X)$.

If homogeneous elements $x, y \in P$ exist, say of degrees r and s respectively, such that $\sqrt{(x,y)} = P$, then we may regard x, y as determining global sections of the sheaves $\mathcal{O}_X(r)$ and $\mathcal{O}_X(s)$ respectively, *which have no common zeroes on X*. Let $D_x \subset X$, $D_y \subset X$ be the divisors of zeroes of $x \in \Gamma(X, \mathcal{O}_X(r))$ and $y \in \Gamma(X, \mathcal{O}_X(s))$ respectively. Then we have equations in $CH^1(X)$

$$[D_x] = c_1(\mathcal{O}_X(r)) = rc_1(\mathcal{O}_X(1)) = r\xi,$$
$$[D_y] = c_1(\mathcal{O}_X(s)) = sc_1(\mathcal{O}_X(1)) = s\xi.$$

But $D_x \cap D_y = \emptyset$. Hence in $CH^2(X)$, we have a relation

$$0 = [D_x] \cdot [D_y] = rs\xi^2,$$

contradicting that $\xi^2 \in CH^2(X)$ is a non-torsion element.

Remark 2.5. The construction of the homomorphism from the polynomial ring R to the graded ring A is an algebraic analogue of the *exponential map* in Riemannian geometry, which identifies a tubular neighbourhood of a smooth submanifold of a Riemannian manifold with the normal bundle of the submanifold (see [24, Theorem 11.1], for example). The exponential map is usually constructed using geodesics on the ambient manifold; here we use the global structure of affine space, where "geodesics" are lines, to make a similar construction algebraically. This idea appears in a paper[3] of Boratynski, who uses it to argue that a smooth subvariety of $\mathbb{A}^n$ is a set-theoretic complete intersection if and only if the zero section of its normal bundle is a set-theoretic complete intersection in the total space of the normal bundle.

2.3. Zero cycles on non-singular proper and affine varieties

In this section, we discuss results of Mumford and Roitman, which give criteria for the non-triviality of $CH^d(X)$ where X is a non-singular variety over $\mathbb{C}$ of dimension $d \geq 2$, which is either proper, or affine.

If X is non-singular and irreducible, and $\dim X = d$, then $Z^d(X)$ is just the free abelian group on the (closed) points of X. Elements of $Z^d(X)$ are called *zero cycles* on X (since they are linear combinations of irreducible subvarieties of dimension 0). In the presentation $CH^d(X) = Z^d(X)/R^d(X)$, the group $R^d(X)$ of relations is generated by divisors of rational functions on irreducible curves in X.

The main non-triviality result for zero cycles is the following result, called the *infinite dimensionality theorem for* 0*-cycles.* It was originally proved (without $\otimes\mathbb{Q}$) by Mumford [26], for surfaces, and extended to higher dimensions by Roitman [32]; the statement with $\otimes\mathbb{Q}$ follows from [33].

Theorem 2.6 (Mumford, Roitman). *Let X be an irreducible, proper, non-singular variety of dimension d over $\mathbb{C}$. Suppose X supports a non-zero regular q-form (i.e., $\Gamma(X, \Omega^q_{X/\mathbb{C}}) \neq 0$), for some $q > 0$. Then for any closed algebraic subvariety $Y \subset X$ with $\dim Y < q$, we have $CH^d(X \setminus Y) \otimes \mathbb{Q} \neq 0$.*

Corollary 2.7. *Let X be an irreducible, proper, non-singular variety of dimension d over $\mathbb{C}$, such that $\Gamma(X, \omega_X) \neq 0$. Then for any affine open subset $V \subset X$, we have $CH^d(V) \otimes \mathbb{Q} \neq 0$.*

The corollary results from the identification of ω_X with the sheaf $\Omega^d_{X/\mathbb{C}}$ of d-forms.

Bloch [1] gave another proof of the above result, using the action of algebraic correspondences on the étale cohomology, and generalized the result to arbitrary characteristics. In [37] and [38], Bloch's argument (for the case of characteristic 0) is recast in the language of differentials, extending it as well to certain singular varieties. One way of stating the infinite dimensionality results of [37] and [38], in the smooth case, is the following. The statement is technical, but it will be needed below when discussing M. Nori's construction of indecomposable projective modules.

We recall the notion of a k-generic point of an irreducible variety; we do this in a generality sufficient for our purposes. If X_0 is an irreducible k-variety, where $k \subset \mathbb{C}$ is a countable algebraically closed subfield, a point $x \in X = (X_0)_{\mathbb{C}}$ determines an irreducible subvariety $Z \subset X$, called the k-closure of X, which is the smallest subvariety of X which is defined over k (*i.e.*, of the form $(Z_0)_{\mathbb{C}}$ for some subvariety $Z_0 \subset X_0$) and contains the chosen point x. We call x a *k-generic point* if its k-closure is X itself.

In the case X_0 (and thus also X) is affine, say $X_0 = \operatorname{Spec} A$, and $X = \operatorname{Spec} A_{\mathbb{C}}$ with $A_{\mathbb{C}} = A \otimes_k \mathbb{C}$, then a point $x \in X$ corresponds to a maximal ideal $\mathfrak{m}_x \subset A_{\mathbb{C}}$. Let $\wp_x = A \cap \mathfrak{m}_x$, which is a prime ideal of A, not necessarily maximal. Then, in the earlier notation, $\wp_x$ determines an irreducible subvariety $Z_0 \subset X_0$. The k-closure $Z \subset X$ of x is the subvariety determined by the prime ideal $\wp_x A_{\mathbb{C}}$ (since k is algebraically closed, $\wp_x A_{\mathbb{C}}$ is a prime ideal). In particular, x is a k-generic point $\iff \wp_x = 0$. In this case, x determines an inclusion $A \hookrightarrow A_{\mathbb{C}}/\mathfrak{m}_x = \mathbb{C}(x) \cong \mathbb{C}$. This in turn gives an inclusion $i_x : K \hookrightarrow \mathbb{C}$ of the quotient field K of A (*i.e.*, of the function field $k(X_0)$) into the complex numbers.

In general, even if X is not affine, if we are given a k-generic point $x \in X$, we can replace X by any affine open subset defined over k, which will (because x is k-generic) automatically contain x; one verifies easily that the corresponding inclusion $K \hookrightarrow \mathbb{C}$ does not depend on the choice of this open subset. Thus we obtain an inclusion $i_x : K \hookrightarrow \mathbb{C}$ of the function field $K = k(X_0)$ into $\mathbb{C}$, associated to any k-generic point of X.

It is easy to see that the procedure is reversible: any inclusion of k-algebras $i : K \hookrightarrow \mathbb{C}$ determines a unique k-generic point of X. Indeed, choose an affine open subset $\operatorname{Spec} A = U_0 \subset X_0$, so that K is the quotient field of A. The induced inclusion $A \hookrightarrow \mathbb{C}$ induces a *surjection* of $\mathbb{C}$-algebras $A_{\mathbb{C}} \to \mathbb{C}$, whose kernel is a maximal ideal, giving the desired k-generic point.

Suppose now that X_0 is proper over k, and so X is proper over $\mathbb{C}$ (e.g., X is projective). Let $\dim X_0 = \dim X = d$. Then by the Serre duality theorem, the sheaf cohomology group $H^d(X, \mathcal{O}_X)$ is the dual $\mathbb{C}$-vector space to

$$\Gamma(X, \Omega^d_{X/\mathbb{C}}) = \Gamma(X, \omega_X) = \Gamma(X_0, \omega_{X_0}) \otimes_k \mathbb{C}.$$

Hence we may identify $H^d(X, \mathcal{O}_X) \otimes_{\mathbb{C}} \Omega^d_{\mathbb{C}/k}$ with

$$\operatorname{Hom}_{\mathbb{C}}(\Gamma(X, \omega_X), \Omega^d_{\mathbb{C}/k}) = \operatorname{Hom}_k(\Gamma(X_0, \omega_{X_0}), \Omega^d_{\mathbb{C}/k}).$$

Note that a k-generic point x determines, via the inclusion $i_x : K \hookrightarrow \mathbb{C}$, a k-linear inclusion $\Omega^n_{K/k} \hookrightarrow \Omega^n_{\mathbb{C}/k}$, and hence, via the obvious inclusion

$$\Gamma(X_0, \omega_{X_0}) = \Gamma(X_0, \Omega^n_{X_0/k}) \hookrightarrow \Omega^n_{K/k},$$

a canonical element

$$di_x \in \operatorname{Hom}_k(\Gamma(X_0, \omega_{X_0}), \Omega^d_{\mathbb{C}/k}) = H^d(X, \mathcal{O}_X) \otimes_{\mathbb{C}} \Omega^d_{\mathbb{C}/k}.$$

Theorem 2.8. *Let $k \subset \mathbb{C}$ be a countable algebraically closed subfield, and X_0 an irreducible non-singular proper k-variety of dimension d, with $\Gamma(X_0, \omega_{X_0}) \neq 0$. Let $U_0 \subset X_0$ be any Zariski open subset. Let $X = (X_0)_{\mathbb{C}}$, $U = (U_0)_{\mathbb{C}}$ be the corresponding complex varieties. Then there is a homomorphism of graded rings*

$$CH^*(U) \to \bigoplus_{p \geq 0} H^p(X, \mathcal{O}_X) \otimes_{\mathbb{C}} \Omega^p_{\mathbb{C}/k},$$

with the following properties.

(i) *If $x \in U$ is a point, which is not k-generic, then the image in $H^d(X, \mathcal{O}_X) \otimes \Omega^d_{\mathbb{C}/k}$ of $[x] \in CH^d(U)$ is zero.*

(ii) *If $x \in U$ is a k-generic point, then the image in $H^d(X, \mathcal{O}_X) \otimes \Omega^d_{\mathbb{C}/k}$ of $[x] \in CH^d(U)$ is (up to sign) the canonical element di_x described above.*

As stated earlier, the above more explicit form of the infinite dimensionality theorem follows from results proved in [37] and [38].

2.4. Some computations with Chern classes

We now study the following two problems, which turn out to have some similarities. We will show how, in each case, the problem reduces to finding an example for which the Chern classes of the cotangent bundle (*i.e.*, the sheaf of Kähler differentials) have appropriate properties. We will then see, in Example 2.12, how to construct examples with these properties. The discussion is based on the article [2] of Bloch, Murthy and Szpiro.

Problem 2.9. Find examples of n-dimensional, non-singular affine algebras A over (say) the complex number field $\mathbb{C}$, for each $n \geq 1$, such that A cannot be generated by $2n$ elements as a $\mathbb{C}$-algebra, or such that the module of Kähler differentials cannot be generated by $2n-1$ elements $da_1, \dots, da_{2n-1}$ (in contrast, it is a "classical" result that such an algebra A can always be generated by $2n+1$ elements, and its Kähler differentials can always be generated by $2n$ exact 1-forms; see, for example, [36]).

Problem 2.10. Find examples of prime ideals I of height $< N$ in a polynomial ring $\mathbb{C}[x_1, \dots, x_N]$ such that $\mathbb{C}[x_1, \dots, x_N]/I$ is regular, but I cannot be generated by $N-1$ elements (the Eisenbud-Evans conjectures, proved by Sathaye [34] and Mohan Kumar [25], imply that such an ideal I can always be generated by N elements).

First we discuss Problem 2.9. Suppose A is an affine smooth $\mathbb{C}$-algebra which is an integral domain of dimension n. Assume $X = \operatorname{Spec} A$ can be generated by $2n$ elements, *i.e.*, that there is a surjection $f : \mathbb{C}[x_1, \dots, x_{2n}] \to A$ from a polynomial ring. Let $I = \ker f$. If $i : X \hookrightarrow \mathbb{A}^{2n}_{\mathbb{C}}$ is the embedding corresponding to the surjection f, then the normal bundle to i is the sheaf $V^\vee$, where $V = \widetilde{I/I^2}$.

From the self-intersection formula, and the formula for the Chern class of the dual of a vector bundle, we see that

$$(-1)^n c_n(V) = c_n(V^\vee) = i^* i_*[X] = 0, \tag{2.2}$$

since $CH^n(\mathbb{A}^{2n}_{\mathbb{C}}) = 0$.

On the other hand, suppose $j : X \hookrightarrow Y$ is any embedding as a closed subvariety of a non-singular affine variety Y whose cotangent bundle (*i.e.*, sheaf of Kähler differentials) $\Omega_{Y/\mathbb{C}}$ is a trivial bundle. For example, we could take $Y = \mathbb{A}^{2n}_{\mathbb{C}}$, and $j = i$, but below we will consider a different example as well.

Let W be the conormal bundle of X in Y (if $Y = \operatorname{Spec} B$, and $J = \ker j^* : B \twoheadrightarrow A$, then $W = \widetilde{J/J^2}$). We then have an exact sequence of vector bundles on X

$$0 \to W \to j^*\Omega_{Y/\mathbb{C}} \to \Omega^1_{X/\mathbb{C}} \to 0.$$

Since $\Omega_{Y/\mathbb{C}}$ is a trivial vector bundle, we get that

$$c(W) = c(\Omega_{X/\mathbb{C}})^{-1} \in CH^*(X). \tag{2.3}$$

Note that this expression for $c(W)$, and hence the resulting formula for $c_n(W)$ as a polynomial in the Chern classes of $\Omega_{X/\mathbb{C}}$, is in fact *independent* of the embedding j. In particular, from (2.2), we see that $c_n(W) = 0$ for *any* such embedding $j : X \hookrightarrow Y$.

Remark 2.11. In fact, the stability and cancellation theorems of Bass imply that in the above situation, the vector bundle W itself is, up to isomorphism, independent of j, and is thus an invariant of the variety X. We call it the *stable normal bundle* of X; this is similar to the case of embeddings of smooth manifolds into Euclidean spaces. We will not need this fact in our computations below.

Returning to our discussion, we see that to find a $\mathbb{C}$-algebra A with $\dim A = n$, and which cannot be generated by $2n$ elements as a $\mathbb{C}$-algebra, it suffices to produce an embedding $j : X \hookrightarrow Y$ of $X = \operatorname{Spec} A$ into a smooth variety Y of dimension $2n$, such that

(i) $\Omega_{Y/\mathbb{C}}$ is a trivial bundle, and
(ii) if W is the conormal bundle of j, then $c_n(W) \neq 0$; in fact it suffices to produce such an embedding such that $j_*c_n(W) \in CH^{2n}(Y)$ is non-zero.

We see easily that the same example $X = \operatorname{Spec} A$ will have the property that $\Omega_{A/\mathbb{C}}$ is not generated by $2n-1$ elements; in fact if $P = \ker(f : A^{\oplus 2n-1} \twoheadrightarrow \Omega_{A/\mathbb{C}})$ for some surjection f, then $\widetilde{P}$ is a vector bundle of rank $n-1$, so that $c_n(\widetilde{P}) = 0$, while on the other hand, the exact sequence

$$0 \to P \to A^{\oplus 2n-1} \xrightarrow{f} \Omega_{A/\mathbb{C}} \to 0$$

implies that

$$c(\widetilde{P}) = c(\Omega_{X/\mathbb{C}})^{-1},$$

so that we would have

$$0 = c_n(\widetilde{P}) = c_n(W) \neq 0,$$

a contradiction.

Next we discuss the Problem 2.10 of finding an example of a "non-trivial" prime ideal $I \subset \mathbb{C}[x_1, \ldots, x_N]$ in a polynomial ring such that the quotient ring $A = \mathbb{C}[x_1, \ldots, x_N]/I$ is smooth of dimension > 0, while I cannot be generated by $N-1$ elements (by the Eisenbud-Evans conjectures, proved by Sathaye and Mohan Kumar, I can always be generated by N elements).

Suppose I can be generated by $N-1$ elements, and $\dim A/I = n > 0$. Then $I/I^2 \oplus Q = A^{N-1}$ for some projective A-module Q of rank $n-1$; hence

$$(I/I^2 \oplus Q \oplus A) \cong A^{\oplus N} \cong (I/I^2 \oplus \Omega_{A/\mathbb{C}}).$$

Hence we have an equality between total Chern classes

$$c(\Omega_{X/\mathbb{C}}) = c(\widetilde{Q}),$$

and in particular, $c_n(\Omega_{X/\mathbb{C}}) = 0$.

So if $X = \operatorname{Spec} A$ is such that $c_n(\Omega_{X/\mathbb{C}}) \in CH^n(X)$ is non-zero, then for any embedding $X \hookrightarrow \mathbb{A}^N_{\mathbb{C}}$, the corresponding prime ideal I cannot be generated by $N-1$ elements.

Example 2.12. We now show how to construct an example of an n-dimensional affine variety $X = \operatorname{Spec} A$ over $\mathbb{C}$, for any $n \geq 1$, such that, for some embedding $X \hookrightarrow Y = \operatorname{Spec} B$ with $\dim Y = 2n$, and ideal $I \subset B$, the projective module $P = I/I^2$ has the following properties:

(i) $c_n(P) \neq 0$ in $CH^n(X) \otimes \mathbb{Q}$,
(ii) if $c(P) \in CH^*(X)$ is the total Chern class, then $c(P)^{-1}$ has a non-torsion component in $CH^n(X) \otimes \mathbb{Q}$.

Then, by the discussion earlier, the affine ring A will have the properties that

(a) A cannot be generated by $2n$ elements as a $\mathbb{C}$-algebra,
(b) $\Omega_{A/\mathbb{C}}$ is not generated by $2n-1$ elements,
(c) for any way of writing $A = \mathbb{C}[x_1, \ldots, x_N]/J$ as a quotient of a polynomial ring (with N necessarily at least $2n+1$), the ideal J requires N generators (use the formula (2.3)).

The technique is that given in [2]. Let E be an elliptic curve (*i.e.*, a non-singular, projective plane cubic curve over $\mathbb{C}$), for example,

$$E = \operatorname{Proj} \mathbb{C}[x, y, z]/(x^3 + y^3 + z^3).$$

Let $E^{2n} = E \times \cdots \times E$, the product of $2n$ copies of E. Let $Y = \operatorname{Spec} B \subset E^{2n}$ be any affine open subset. By the Mumford-Roitman infinite dimensionality theorem (Theorem 2.6 above), $CH^{2n}(Y)\otimes\mathbb{Q} \neq 0$. Also, since $Y \subset E^{2n}$, clearly the $2n$-fold intersection product

$$CH^1(Y)^{\otimes 2n} \to CH^{2n}(Y)$$

is surjective. Hence we can find an element $\alpha \in CH^1(Y)$ with $\alpha^{2n} \neq 0$ in $CH^{2n}(Y)\otimes\mathbb{Q}$. Let P be the projective B-module of rank 1 corresponding to α. Since Y is affine, by Bertini's theorem, we can find elements $a_1, \ldots, a_n \in P$ such that the corresponding divisors $H_i = \{a_i = 0\} \subset Y$ are non-singular, and intersect transversally; take $X = H_1 \cap \cdots \cap H_n$. Then $X = \operatorname{Spec} A$ is non-singular of dimension n, and the ideal $I \subset B$ of $X \subset Y$ is such that $I/I^2 \cong (P \otimes_B A)^{\oplus n}$. Thus, if $j : X \hookrightarrow Y$ is the inclusion, then we have a formula between total Chern classes

$$c(I/I^2) = j^*c(P)^n = (1 + j^*c_1(P))^n = (1 + j^*\alpha)^n.$$

Hence

$$c_n(I/I^2) = j^*(\alpha)^n,$$

and so by the projection formula,

$$j_*c_n(I/I^2) = j_*(1)\alpha^n = \alpha^{2n},$$

since

$$j_*(1) = [V] = [H_1] \cdot [H_2] \cdot \cdots \cdots [H_n] = \alpha^n \in CH^n(Y),$$

as X is the complete intersection of divisors H_i, each corresponding to the class $\alpha \in CH^1(Y)$. By construction, $j_*c_n(I/I^2) \neq 0$ in $CH^{2n}(Y) \otimes \mathbb{Q}$, and so we have that $c_n(I/I^2) \neq 0$ in $CH^n(X) \otimes \mathbb{Q}$, as desired.

Similarly

$$c(I/I^2)^{-1} = (1 + j^*\alpha)^{-n}$$

has a non-zero component of degree n, which is a non-zero integral multiple of $j^*\alpha^n$.

Remark 2.13. The existence of n-dimensional non-singular affine varieties X which do not admit closed embeddings into affine $2n$-space is in contrast to the situation of differentiable manifolds — the "hard embedding theorem" of Whitney states that any smooth n-manifold has a smooth embedding in the Euclidean space $\mathbb{R}^{2n}$.

2.5. Indecomposable projective modules

Now we discuss M. Nori's (unpublished) construction of indecomposable projective modules of rank d over any affine $\mathbb{C}$-algebra $A_{\mathbb{C}}$ of dimension d, such that $U = \operatorname{Spec} A_{\mathbb{C}}$ is an open subset of a non-singular projective (or proper) $\mathbb{C}$-variety X with $H^0(X, \omega_X) = H^0(X, \Omega^d_{X/\mathbb{C}}) \neq 0$.

The idea is as follows. Fix a countable, algebraically closed subfield $k \subset \mathbb{C}$ such that X and U are defined over k; in particular, we are given an affine k-subalgebra $A \subset A_{\mathbb{C}}$ such that $A_{\mathbb{C}} = A \otimes_k \mathbb{C}$. We also have a k-variety X_0 containing $U_0 = \operatorname{Spec} A$ as an affine open subset, such that $X = (X_0)_{\mathbb{C}}$.

Let K_n be the function field of $X_0^n = X_0 \times_k \cdots \times_k X_0$ (equivalently, K_n is the quotient field of $A^{\otimes n} = A \otimes_k \cdots \otimes_k A$). We have n induced embeddings $\varphi_i : K \hookrightarrow K_n$, where $K = K_1$ is the quotient field of A, given by $\varphi_i(a) = 1 \otimes \cdots \otimes 1 \otimes a \otimes 1 \otimes \cdots \otimes 1$ with a in the i-th position.

Choose an embedding $K_n \hookrightarrow \mathbb{C}$ as a k-subalgebra. The inclusions φ_i then determine n inclusions $K \hookrightarrow \mathbb{C}$, or equivalently, k-generic points $x_1, \ldots, x_n \in X$ (in algebraic geometry, these are called "n independent generic points of X"). Let $\mathfrak{m}_i$ be the maximal ideal of $A_{\mathbb{C}}$ determined by x_i, and let $I = \cap_{i=1}^n \mathfrak{m}_i$. Clearly I is a local complete intersection ideal of height d in the d-dimensional regular ring $A_{\mathbb{C}}$. Thus we can find a projective resolution of I

$$0 \to P \to F_{d-1} \to \cdots \to F_1 \to I \to 0,$$

where F_i are free. By construction, $c(P) = c(A/I)^{(-1)^d}$. By theorem 2.3, we have

$$c(A/I) = 1 + (-1)^{d-1}(d-1)!(\sum_{i=1}^n [x_i]) \in CH^*(X).$$

Hence $c_i(P) = 0$ for $i < d$, while $c_d(P)$ is a non-zero integral multiple of the class $\sum_i [x_i] \in CH^d(U)$. This class is non-zero, from theorem 2.8 (we will get a stronger conclusion below). Hence $\operatorname{rank} P \geq d$.

By Bass' stability theorem, if $\operatorname{rank} P = d + r$, we may write $P = Q \oplus A^{\oplus r}$, where Q is projective of rank d. Then P and Q have the same Chern classes. So we can find a projective module Q of rank d with $c(Q) = 1 + m(\sum_i [x_i]) \in CH^*(U)$, for some non-zero integer m.

Suppose $Q = Q_1 \oplus Q_2$ with $\operatorname{rank} Q_1 = p$, $\operatorname{rank} Q_2 = d - p$, and $1 \leq p < d$ (thus $d > 1$). Then in $CH^*(U) \otimes \mathbb{Q}$, the class $\sum_i [x_i]$ is expressible as

$$\sum_i [x_i] = \alpha \cdot \beta, \quad \alpha \in CH^p(U) \otimes \mathbb{Q}, \ \beta \in CH^{d-p}(U) \otimes \mathbb{Q}.$$

Using the homomorphism of graded rings of Theorem 2.8,

$$CH^*(U) \otimes \mathbb{Q} \to \bigoplus_{j \geq 0} H^j(X, \mathcal{O}_X) \otimes_{\mathbb{C}} \Omega^j_{\mathbb{C}/k},$$

we see that the element

$$\xi = \sum_{i=1}^{n} di_{x_i} \in H^d(X, \mathcal{O}_X) \otimes \Omega^d_{\mathbb{C}/k}$$

is expressible as a product

$$\begin{gathered}\xi = \textstyle\sum_{i=1}^{n} di_{x_i} = \alpha \cdot \beta, \quad \alpha \in H^p(X, \mathcal{O}_X) \otimes_{\mathbb{C}} \Omega^p_{\mathbb{C}/k}, \\ \beta \in H^{d-p}(X, \mathcal{O}_X) \otimes_{\mathbb{C}} \Omega^{d-p}_{\mathbb{C}/k}.\end{gathered}$$

Let L be the algebraic closure of K_n in $\mathbb{C}$. The graded ring

$$\bigoplus_{j=0}^{d} H^j(X, \mathcal{O}_X) \otimes_{\mathbb{C}} \Omega^j_{\mathbb{C}/k} = \bigoplus_{j=0}^{d} H^j(X_0, \mathcal{O}_{X_0}) \otimes_k \Omega^j_{\mathbb{C}/k}$$

has a graded subring

$$\bigoplus_{j=0}^{d} H^j(X_0, \mathcal{O}_{X_0}) \otimes_k \Omega^j_{L/k}$$

which contains the above element ξ. We claim that ξ is then expressible as a product $\alpha \cdot \beta$ of homogeneous elements of degrees $p, d-p$ with α, β lying in this subring. Indeed, since $\mathbb{C}$ is the direct limit of its subrings B which are finitely generated L-subalgebras, we can find such a subring B, and homogeneous elements $\widetilde{\alpha}$, $\widetilde{\beta}$ of degrees p, $d-p$ in $\bigoplus_{j=0}^{d} H^j(X_0, \mathcal{O}_{X_0}) \otimes_k \Omega^j_{B/k}$ such that $\xi = \widetilde{\alpha} \cdot \widetilde{\beta}$. Choosing a maximal ideal in B, we can find an L-algebra homomorphism $B \twoheadrightarrow L$, giving rise to a graded ring homomorphism

$$f : \bigoplus_{j=0}^{d} H^j(X_0, \mathcal{O}_{X_0}) \otimes_k \Omega^j_{B/k} \to \bigoplus_{j=0}^{d} H^j(X_0, \mathcal{O}_{X_0}) \otimes_k \Omega^j_{L/k}.$$

Then $\xi = f(\widetilde{\alpha}) \cdot f(\widetilde{\beta})$ holds in $\bigoplus_{j=0}^{d} H^j(X_0, \mathcal{O}_{X_0}) \otimes_k \Omega^j_{L/k}$ itself.

Now

$$\Omega^1_{L/k} = \Omega^1_{K_n/k} \otimes_{K_n} L = \bigoplus_{j=1}^{n} \Omega^1_{K/k} \otimes_K L,$$

where the j-th summand corresponds to the j-th inclusion $K \hookrightarrow K_n$. We may write this as

$$\Omega^1_{L/k} = \Omega^1_{K/k} \otimes_K W,$$

where $W \cong L^{\oplus n}$ is an n-dimensional L-vector space with a distinguished basis. Then there are natural surjections

$$\Omega^r_{L/k} = \bigwedge_L^r (\Omega^1_{K/k} \otimes_K W) \twoheadrightarrow \Omega^r_{K/k} \otimes_K S^r(W),$$

where $S^r(W)$ is the r-th symmetric power of W as an L-vector space. In particular, since $\Omega^d_{K/k}$ is 1-dimensional over K, we get a surjection $\Omega^d_{L/k} \twoheadrightarrow S^d(W)$. This determines the component of degree d of a graded ring homomorphism

$$\Phi : \bigoplus_{j=0}^{d} H^j(X_0, \mathcal{O}_{X_0}) \otimes_k \Omega^j_{L/k} \to \bigoplus_{j=0}^{d} H^j(X_0, \mathcal{O}_{X_0}) \otimes_k \Omega^j_{K/k} \otimes_K S^j(W).$$

As in the discussion preceeding Theorem 2.8, by Serre duality on X_0, the natural inclusion $H^0(X_0, \Omega^d_{X_0/k}) \hookrightarrow \Omega^d_{K/k}$ determines a canonical element $\theta \in H^d(X_0, \mathcal{O}_{X_0}) \otimes_k \Omega^d_{K/k}$. Identifying the symmetric algebra $S^\bullet(W) = S^\bullet(L^{\oplus n})$ with the polynomial algebra $L[t_1, \ldots, t_n]$, we have that $\Phi(\xi) = \theta \cdot (t_1^d + \cdots + t_n^d)$. Hence, in the graded ring

$$\bigoplus_{j=0}^{d} H^j(X_0, \mathcal{O}_{X_0}) \otimes_k \Omega^j_{K/k} \otimes_K S^j(W),$$

the element $\theta \cdot (t_1^d + \cdots + t_n^d)$ is expressible as a product of homogeneous elements α, β of degrees p and $d-p$. Hence, by expressing

$$\alpha \in H^p(X_0, \mathcal{O}_{X_0}) \otimes_k \Omega^p_{K/k} \otimes_K S^p(W),$$
$$\beta \in H^{d-p}(X_0, \mathcal{O}_{X_0}) \otimes_k \Omega^{d-p}_{K/k} \otimes_K S^{d-p}(W)$$

in terms of K-bases of $H^p(X_0, \mathcal{O}_{X_0}) \otimes_k \Omega^p_{K/k}$ and $H^{d-p}(X_0, \mathcal{O}_{X_0}) \otimes_k \Omega^{d-p}_{K/k}$, we deduce that in the polynomial ring $S^\bullet(W) = L[t_1, \ldots, t_n]$, the "Fermat polynomial" $t_1^d + \cdots + t_n^d$ is expressible as a sum of pairwise products of homogeneous polynomials

$$t_1^d + \cdots + t_n^d = \sum_{m=1}^{N} a_m(t_1, \ldots, t_n) b_m(t_1, \ldots, t_n)$$

with

$$N = \binom{d}{p}\binom{d}{d-p}(\dim_k H^p(X_0, \mathcal{O}_{X_0}))(\dim_k H^{d-p}(X_0, \mathcal{O}_{X_0})).$$

If $n > 2N$, the system of homogeneous polynomial equations $a_1 = b_1 = \cdots = a_N = b_N = 0$ defines a non-empty subset of the projective variety $t_1^d + \cdots + t_n^d = 0$ in $\mathbb{P}_L^{n-1}$, along which this Fermat hypersurface is clearly singular — and this is a contradiction!

§3. Variants of the Noether-Lefschetz Theorem

We discuss techniques from topology and Hodge theory, namely the monodromy theory of Lefschetz pencils, and Deligne's mixed Hodge structures, which lead to the construction of unique factorization domains, and to the construction of algebraic local rings over $\mathbb{C}$ of dimension 2 with a prescribed normal singularity, and minimal divisor class group (a cyclic group, generated by the class of the canonical module). As an example, we mention the following result: any ring of the form $\mathbb{C}[x,y,z]/(z^2 + xy + f(x,y,z))$, with f a "general" polynomial of degree ≥ 5 and vanishing at $(0,0,0)$ to order 4, is a UFD.

3.1. Background and results

If A is a (Noetherian) normal local ring, $\widehat{A}$ its completion, then the map on divisor class groups $C\ell\,(A) \to C\ell\,(\widehat{A})$ is injective. This leads to a natural question.

Question 3.1. *Given $\widehat{A}$, what are the possibilities for the subgroup $C\ell\,(A) \hookrightarrow C\ell\,(\widehat{A})$?*

We will restrict attention here to the case when A has the coefficient field $\mathbb{C}$, the complex numbers; *we will assume henceforth that all local rings under consideration have coefficient field* $\mathbb{C}$, unless explicitly noted otherwise.

There is one case when the question is trivially answered: when $C\ell\,(\widehat{A}) = 0$, or equivalently, $\widehat{A}$ is a UFD. We recall some "classical" results along these lines (see [4], [22], [10], [14]).

Theorem 3.2 (Brieskorn, Lipman). *Let $\widehat{A}$ be a complete non-regular UFD with* $\dim \widehat{A} = 2$. *Then* $\widehat{A} \cong \mathbb{C}[[x,y,z]]/(x^2 + y^3 + z^5)$.

Theorem 3.3 (Grothendieck). *Let $\widehat{A} = R/(x_1, \dots, x_n)$ where R is a power series ring over $\mathbb{C}$, such that* (i) $\dim A \geq 4$, (ii) $x_1, \dots, x_n$ *is a regular sequence in R,* (iii) *$\widehat{A}_\wp$ is a UFD for all primes $\wp$ of $\widehat{A}$ of height ≤ 3. Then $\widehat{A}$ is a UFD.*

Theorem 3.4 (Hartshorne, Ogus). *Let $(\widehat{A}, \mathfrak{m})$ be a complete local ring with* $\dim \widehat{A} = d \geq 3$, *such that* (i) $\widehat{A}$ *has an isolated singularity,* (ii) $\dim_{\mathbb{C}} \mathfrak{m}/\mathfrak{m}^2 \leq 2d-3$, (iii) $\operatorname{depth} \widehat{A} \geq 3$. *Then $\widehat{A}$ is a UFD.*

In a related vein, one can ask the following.

Question 3.5. *Given $\widehat{A}$, when does there exist a Noetherian local ring A with completion $\widehat{A}$, such that A is a UFD?*

One has the following result in this direction [15] (see also [16]).

Theorem 3.6 (R. C. Heitmann). *Let R be a complete local ring over $\mathbb{C}$ of depth ≥ 2. Then there exists a local UFD A with completion $\widehat{A} = R$.*

However, the UFD constructed by Heitmann is very far from being "geometric". For example, suppose $\dim R = 2$, and R is normal but not Gorenstein. Then the corresponding ring A cannot have a dualizing module, from an old result of Murthy [28], which states that a Cohen-Macaulay UFD with a dualizing module is Gorenstein. Thus Heitmann's ring A is not a quotient of a regular local ring, for example.

So we will restrict attention, in Question 3.1, to local rings A which are essentially of finite type over $\mathbb{C}$ (*i.e.*, are localizations of finitely generated $\mathbb{C}$-algebras). We will refer to such local rings A as *geometric.* By Murthy's theorem [28] mentioned above, Question 3.5 can have a positive answer for geometric A, in dimension 2, only if $\widehat{A}$ is Gorenstein. So we are finally led to the following modification of Question 3.5.

Question 3.7. *Let R be a normal local ring. Does there exist a geometric local ring A with completion $\widehat{A} \cong R$, such that $C\ell(A)$ is the cyclic group generated by ω_A?*

Remark 3.8. Here ω_A is the dualizing module for A in the sense of [13], III,§7 — if X is any irreducible projective variety with a point $x \in X$ such that $A = \mathcal{O}_{X,x}$, then ω_A is the stalk at x of the dualizing sheaf ω_X, as defined in [13]; ω_A is in fact independent of the choice of X. As in the definition of ω_X in [13], one can characterize ω_A by a suitable dualizing property (phrased in terms of local cohomology), or as a suitable cohomology module of a dualizing complex for A.

The discussion above has been centered around making the divisor class group as small as possible. We give an example addressing the question as to how large the class group can be. This suggests that Question 3.7 is probably the only reasonable "general" question one can ask in the direction of Question 3.1.

Example 3.9. If $\widehat{A} = \mathbb{C}[[x,y,z]]/(x^2+y^3+z^7)$, then image $(C\ell\,(A) \hookrightarrow C\ell\,(\widehat{A}))$ is always a finitely generated subgroup, while $C\ell\,(\widehat{A}) = \mathbb{C}$. Without going into details, the reason for this is as follows. The *local Picard variety* of the singularity defined by $\operatorname{Spec}\widehat{A}$ (*i.e.*, the Picard variety of a minimal resolution of singularities of $\operatorname{Spec}\widehat{A}$) has connected component of the identity equal to the additive group $\mathbb{C}$ $(= \mathbb{G}_a)$, which is affine; on the other hand, if $A = \mathcal{O}_{X,x}$ is the local ring of a point $x \in X$ on a projective complex algebraic variety X, and if $Y \to X$ is a resolution of singularities, then $C\ell\,(A)$ can be realized as a quotient of the Picard group $\operatorname{Pic}(Y)$ of Y. The subgroup $\operatorname{Pic}^0(Y) \subset \operatorname{Pic}(Y)$ is an abelian variety, with finitely generated quotient group $NS(Y) = \operatorname{Pic}(Y)/\operatorname{Pic}^0(Y)$ (the *Neron-Severi group* of the projective non-singular surface Y). Now we note that any homomorphism from an abelian variety to $\mathbb{G}_a$ is zero. Presumably the finitely generated subgroup $C\ell\,(A) \subset C\ell\,(\widehat{A})$ can be of arbitrary rank, as we vary the geometric subrings A. Incidentally, $\mathbb{C}[x,y,z]/(x^2+y^3+z^7)$ is a UFD, from results of Samuel.

We now state two results in the direction of Question 3.7 which are the main focus of our discussion. These are taken from the papers [29] and [30], respectively.

Theorem 3.10 (Parameswaran + Srinivas). *Question* 3.5 (*and hence also Quesion* 3.7) *has a positive answer for isolated complete intersection singularities.*

Theorem 3.11 (Parameswaran + van Straten). *Question* 3.7 *has a positive answer for an arbitrary normal surface singularity.*

The proofs of Theorems 3.10 (in [29]) and 3.11 (in [30]) are motivated by the "classical" proof, essentially due to Lefschetz, of the Noether-Lefschetz Theorem. We next recall this statement, in two equivalent forms.

Theorem 3.12 (Noether-Lefschetz Theorem).

(a) (Algebraic Version): *Let* $F(x,y,z,w) \in \mathbb{C}[x,y,z,w]$ *be a "general" homogeneous polynomial of degree* ≥ 4. *Then* $\mathbb{C}[x,y,z,w]/(F)$ *is a UFD.*

(b) (Geometric Version): *Let* $F(x,y,z,w) \in \mathbb{C}[x,y,z,w]$ *be a "general" homogeneous polynomial of degree* ≥ 4, *and let* $X \subset \mathbb{P}^3_{\mathbb{C}}$ *be the corresponding surface. Then* $\operatorname{Pic}(X) = \mathbb{Z}$ *generated by the class of the tautological line bundle* $\mathcal{O}_X(1)$. *Equivalently, the restriction map* $\operatorname{Pic}(\mathbb{P}^3_{\mathbb{C}}) \to \operatorname{Pic}(X)$ *is an isomorphism.*

The equivalence of the two versions follows from the projective normality of X, and the formula for the divisor class group of the affine cone over a projectively normal variety (see [13] II Ex. 6.3).

3.2. Outline of the proof of the Noether-Lefschetz Theorem

The idea of the proof of the Noether-Lefschetz Theorem is to view the surface X as a general member in a 1-parameter family of such hypersurfaces in $\mathbb{P}^3_{\mathbb{C}}$. Now one uses the *monodromy theory of Lefschetz pencils* to show that $\mathrm{Pic}(\mathbb{P}^3_{\mathbb{C}}) \to \mathrm{Pic}(X)$ is an isomorphism, when $d = \deg X \geq 4$.

We outline this below, suppressing many technical details, but trying to indicate the main points of the argument (see [18], [21] and [12] for more details).

Remark 3.13. Later, we will comment on how this proof can be modified to prove Theorem 3.10 for hypersurface singularities of dimension 2. The proof of the general case of Theorem 3.10 is similar in dimension 2, but with some additional technical difficulties in dealing with complete intersections instead of hypersurfaces; the higher dimensional case of Theorem 3.10 in fact turns out to be simpler than the 2-dimensional case (for example, in dimensions ≥ 4, it follows at once from Theorem 3.3). The proof of Theorem 3.11 has the ingredients of the proof for hypersurface case, together with additional inputs from singularity theory, like a finite determinacy theorem of Pellikaan [31] and classification results of Siersma [35] on "line singularities" (singularities with 1-dimensional singular locus).

First, since the polynomial F is "general", Bertini's theorem implies that $X \subset \mathbb{P}^3_{\mathbb{C}}$ is a non-singular hypersurface. From the exact sequence of cohomology associated to the exact sheaf sequence

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-d) \xrightarrow{\cdot F} \mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_X \to 0.$$

we get $H^1(X, \mathcal{O}_X) = 0$ and (since $d \geq 4$) $H^2(X, \mathcal{O}_X) \neq 0$.

Now on an arbitrary proper $\mathbb{C}$-variety T, with associated analytic space T_{an} (which is a compact complex analytic space), the exponential sheaf sequence (with $\exp(f) = e^{2\pi i f}$)

$$0 \to \mathbb{Z}_{T_{an}} \to \mathcal{O}_{T_{an}} \xrightarrow{\exp} \mathcal{O}^*_{T_{an}} \to 0$$

gives an exact sequence in cohomology

$$0 \to \frac{H^1(T_{an}, \mathcal{O}_{T_{an}})}{H^1(T_{an}, \mathbb{Z})} \to \mathrm{Pic}(T_{an}) \to H^2(T_{an}, \mathbb{Z}) \to H^2(T_{an}, \mathcal{O}_{T_{an}})$$

where $\mathrm{Pic}(T_{an})$ is the group of isomorphism classes of analytic line bundles on T_{an}. By Serre's GAGA, the canonical map $\mathrm{Pic}(T) \to \mathrm{Pic}(T_{an})$, obtained by regarding algebraic line bundles as analytic ones, is an isomorphism. GAGA also implies that for any coherent algebraic sheaf $\mathcal{F}$ on T, with associated analytic sheaf $\mathcal{F}_{an}$, the canonical maps $H^i(T, \mathcal{F}) \to H^i(T_{an}, \mathcal{F}_{an})$ are isomorphisms of $\mathbb{C}$-vector spaces. Hence, if $H^1(T, \mathcal{O}_T) = 0$, we get an induced exact sequence

$$0 \to \mathrm{Pic}(T) \to H^2(T_{an}, \mathbb{Z}) \to H^2(T, \mathcal{O}_T).$$

This sequence is clearly also functorial in T.

Applying these remarks to the inclusion $i : X \hookrightarrow \mathbb{P}^3_{\mathbb{C}}$, since $H^i(\mathbb{P}^3_{\mathbb{C}}, \mathcal{O}_{\mathbb{P}^3_{\mathbb{C}}}) = 0$, $i = 1, 2$ we obtain a commutative diagram with exact bottom row

$$\begin{array}{ccccc} & \mathrm{Pic}(\mathbb{P}^3_{\mathbb{C}}) & \xrightarrow{\cong} & H^2((\mathbb{P}^3_{\mathbb{C}})_{an}, \mathbb{Z}) & \\ & \downarrow & & \downarrow & \\ 0 \to & \mathrm{Pic}(X) & \to & H^2(X_{an}, \mathbb{Z}) & \to H^2(X, \mathcal{O}_X). \end{array}$$

So we are reduced to proving that

$$H^2((\mathbb{P}^3_{\mathbb{C}})_{an}, \mathbb{Z}) \to H^2(X_{an}, \mathbb{Z}) \to H^2(X, \mathcal{O}_X)$$

is exact.

Since X is non-singular, Hodge Theory (in fact, the *Hodge decomposition* $H^2(X_{an}, \mathbb{C}) = H^{2,0} \oplus H^{1,1} \oplus H^{0,2}$) implies that the natural map

$$H^2(X_{an}, \mathbb{Z}) \otimes \mathbb{C} = H^2(X_{an}, \mathbb{C}) \to H^2(X_{an}, \mathcal{O}_{X_{an}}) = H^2(X, \mathcal{O}_X)$$

is surjective (it is the projection onto the summand $H^{0,2}$). Hence

$$\mathrm{Pic}(X) = \ker \left(H^2(X_{an}, \mathbb{Z}) \to H^2(X, \mathcal{O}_X) \right)$$

is a proper subgroup of $H^2(X_{an}, \mathbb{Z})$, with torsion-free quotient.

To simplify notation, we now omit the subscript *an*. We can consider a general 1-parameter family of such hypersurfaces $\{X_t\}_{t \in \mathbb{P}^1_{\mathbb{C}}}$, corresponding to a 2-dimensional $\mathbb{C}$-vector subspace V_0 of the vector space V_d of homogeneous polynomials of degree d. Let $B \subset \mathbb{P}^3_{\mathbb{C}}$ be the non-singular complete intersection curve defined by $F_1 = F_2 = 0$, for any basis $\{F_1, F_2\}$ of V_0 (by Bertini's theorem, B is a non-singular complete intersection, since $V_0 \subset V_d$ is general). (The family of hypersurfaces $\{X_t\}_{t \in \mathbb{P}^1_{\mathbb{C}}}$ is usually called a *pencil*, and $B = \bigcap_{t \in \mathbb{P}^1_{\mathbb{C}}} X_t$ is called the *base locus*).

One shows that, since the subspace $V_0 \subset V_d$ is general, $\{X_t\}_{t \in \mathbb{P}^1_{\mathbb{C}}}$ forms a *Lefschetz pencil*, which means the following.

(a) If $\widetilde{\mathbb{P}} = \{(x,t) \in \mathbb{P}^3_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}} \mid x \in X_t\}$, then $\widetilde{\mathbb{P}} \to \mathbb{P}^3_{\mathbb{C}}$ is a birational proper morphism between non-singular projective varieties, which is the blow-up of $\mathbb{P}^3_{\mathbb{C}}$ along the base locus (the smooth curve B).
(b) Let $f : \widetilde{\mathbb{P}} \to \mathbb{P}^1_{\mathbb{C}}$ be induced by the projection $\mathbb{P}^3_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}$. Then all fibers of f are irreducible, and for a finite set of (closed) points $\Delta \subset \mathbb{P}^1_{\mathbb{C}}$, we have that
    (i) X_t is non-singular for $t \notin \Delta$, and
    (ii) for any $t \in \Delta$, there is a unique singular point $x_t \in X_t$, at which X_t has an ordinary double point singularity (*i.e.*, the complete local ring $\widehat{\mathcal{O}}_{X_t,x_t}$ is isomorphic to $\mathbb{C}[[x,y,z]]/(z^2-xy)$).

Remark 3.14. A Lefschetz pencil is the complex algebraic analogue of a *Morse function* in the theory of compact differentiable manifolds.

Now $f : \widetilde{\mathbb{P}} \setminus f^{-1}(\Delta) \to \mathbb{P}^1_{\mathbb{C}} \setminus \Delta$ is a smooth proper morphism. This implies the underlying map of C^∞ manifolds is a locally trivial C^∞ fiber bundle (Ehresmann fibration theorem; see, for example, [18] for further discussion of this point). This fiber bundle structure implies that all fibers of f over $\mathbb{P}^1_{\mathbb{C}} \setminus \Delta$ are diffeomorphic, and have isomorphic singular cohomology groups. In fact, if t_1, t_2 are any two points of $\mathbb{P}^1_{\mathbb{C}} \setminus \Delta$, the choice of a path (continuous image of the unit interval $[0,1] \subset \mathbb{R}$) joining t_1 and t_2 determines an isomorphism $H^*(X_{t_1},\mathbb{Z}) \to H^*(X_{t_2},\mathbb{Z})$ between singular cohomologies; further, this isomorphism in fact depends only on the homotopy class of this path (keeping the end points fixed). In particular, taking $t_2 = t_1 = t_0$, for a chosen base point $t_0 \in \mathbb{P}^1_{\mathbb{C}} \setminus \Delta$, we obtain the *monodromy representation*

$$\rho : \pi_1(\mathbb{P}^1_{\mathbb{C}} \setminus \Delta, t_0) \to \mathrm{Aut}\,(H^*(X_{t_0},\mathbb{Z}))$$

of the fundamental group of $\mathbb{P}^1_{\mathbb{C}} \setminus \Delta$ (based at t_0) into the group of graded ring automorphisms of the cohomology of the fiber.

The Leray spectral sequence for $f : \widetilde{\mathbb{P}} \to \mathbb{P}^1$, together with the fact that the fibers X_t over points $t \in \Delta$ have only ordinary double points, is used to show that

$$\text{image}\,\left(H^2(\widetilde{\mathbb{P}},\mathbb{Z}) \to H^2(X_{t_0},\mathbb{Z})\right) = \{\text{elements of } H^2(X_{t_0},\mathbb{Z}) \text{ fixed under the monodromy action}\}$$

(this is the "easy" part of what is often called *local Lefschetz theory*). A similar spectral sequence argument, applied to cohomology with finite coefficients, also implies that coker $(H^2(\widetilde{\mathbb{P}}, \mathbb{Z}) \to H^2(X_{t_0}, \mathbb{Z}))$ is torsion-free — in fact one shows that image $H^2(\widetilde{\mathbb{P}}, \mathbb{Z}) \to H^2(X_{t_0}, \mathbb{Z}))$ is a direct summand of $H^2(X_{t_0}, \mathbb{Z})$; but the universal coefficient theorem in topology implies that $H^2(X_{t_0}, \mathbb{Z})$ is a torsion-free abelian group, since X_{t_0} is a smooth hypersurface in $\mathbb{P}^3_{\mathbb{C}}$. hence simply connected.

On the other hand, one has

$$\operatorname{Pic}(\widetilde{\mathbb{P}}) \cong \operatorname{Pic}(\mathbb{P}^3_{\mathbb{C}}) \oplus \mathbb{Z}[E] \cong H^2(\widetilde{\mathbb{P}}, \mathbb{Z}),$$

where E is the exceptional divisor, and $E \cap X_{t_0} = B$, with $\mathcal{O}_X(B) = \mathcal{O}_X(d)$. Hence

$$\begin{aligned} \text{image } \left(H^2(\widetilde{\mathbb{P}}, \mathbb{Z}) \to H^2(X_{t_0}, \mathbb{Z})\right) &= \text{image } \left(H^2(\mathbb{P}^3_{\mathbb{C}}, \mathbb{Z}) \to H^2(X_{t_0}, \mathbb{Z})\right) \\ &= \mathbb{Z}[\mathcal{O}_{X_{t_0}}(1)], \end{aligned}$$

and we are reduced to proving that

$$\begin{aligned} &\ker \left(H^2(X_{t_0}, \mathbb{Z}) \to H^2(X_{t_0}, \mathcal{O}_{X_{t_0}})\right) = \\ &\qquad \{\text{elements of } H^2(X_{t_0}, \mathbb{Z}) \text{ fixed under the monodromy action}\}. \end{aligned}$$

Since coker $(H^2(\widetilde{\mathbb{P}}, \mathbb{Z}) \to H^2(X_{t_0}, \mathbb{Z}))$ is torsion-free, we further reduce to proving that

$$\begin{aligned} &\ker \left(H^2(X_{t_0}, \mathbb{Q}) \to H^2(X_{t_0}, \mathcal{O}_{X_{t_0}})\right) = \\ &\qquad \{\text{elements of } H^2(X_{t_0}, \mathbb{Q}) \text{ fixed under the monodromy action}\}. \end{aligned}$$

Suppose we allow the base point $t_0 \in \mathbb{P}^1_{\mathbb{C}} \setminus \Delta$ to vary, and consider the corresponding variations in $H^2(X_{t_0}, \mathbb{C})$ and $H^2(X_{t_0}, \mathcal{O}_{X_{t_0}})$. In other words, we consider the sheaves $R^2 f_* \mathbb{Z}$ and $R^2 f_* \mathcal{O}_{\widetilde{\mathbb{P}}}$ restricted to $\mathbb{P}^1_{\mathbb{C}} \setminus \Delta$. These sheaves are respectively a local system, and a holomorphic vector bundle. Hence, if we fix an element of $H^2(X_{t_0}, \mathbb{Z})$, *i.e.*, an element of the stalk $(R^2 f_* \mathbb{Z})_{t_0}$, it determines a well-defined section of $R^2 f_* \mathbb{Z}$ in any open disc D around t_0 in the Riemann surface $\mathbb{P}^1_{\mathbb{C}} \setminus \Delta$; the image of this section in $R^2 f_* \mathcal{O}_{\widetilde{\mathbb{P}}} \mid_D$ is a section of a holomorphic vector bundle on D, *i.e.*, after choosing a local trivialization of this vector bundle, the section becomes a vector-valued holomorphic function on D. Using the fact that a holomorphic function on a domain in $\mathbb{C}$ has a discrete set of zeroes, one sees that the kernel of the sheaf map $R^2 f_* \mathbb{Z} \to R^2 f_* \mathcal{O}_{\widetilde{\mathbb{P}}}$ on $\mathbb{P}^1_{\mathbb{C}} \setminus \Delta$ is a *local sub-system* of $R^2 f_* \mathbb{Z} \mid_{\mathbb{P}^1_{\mathbb{C}} \setminus \Delta}$.

In more concrete language, this means that for a sufficiently general choice of base point t_0, the subspace

$$\ker \left(H^2(X_{t_0}, \mathbb{Q}) \to H^2(X_{t_0}, \mathcal{O}_{X_{t_0}})\right)$$

is at least a $\pi_1(\mathbb{P}^1_{\mathbb{C}} \setminus \Delta, t_0)$-submodule (here "general" means "in the complement of a countable subset"). This submodule clearly contains the image of $H^2(\mathbb{P}^3_{\mathbb{C}}, \mathbb{Q})$. Hence it suffices to show that the quotient

$$\frac{H^2(X_{t_0}, \mathbb{Q})}{\operatorname{image} H^2(\mathbb{P}^3_{\mathbb{C}}, \mathbb{Q})}$$

is a *simple* $\pi_1(\mathbb{P}^1_{\mathbb{C}} \setminus \Delta, t_0)$-module. This is a purely topological statement, which is proved by more carefully analyzing the monodromy representation.

Since $(\mathbb{P}^1_{\mathbb{C}})_{an}$ is the Riemann sphere S^2, it is simply connected. We may assume after reindexing that $\Delta \subset \mathbb{C} = \mathbb{P}^1_{\mathbb{C}} \setminus \{\infty\}$. The fundamental group $\pi_1(\mathbb{P}^1_{\mathbb{C}} \setminus \Delta, t_0)$ is then generated by the classes of suitably chosen loops γ_t based at t_0, indexed by elements $t \in \Delta$, which are pairwise non-intersecting (except at t_0). Here γ_t is a simple closed loop in $\mathbb{C}$, going once around t, and with winding number 0 with respect to the other points of Δ. To simplify notation, we will henceforth denote $\pi_1(\mathbb{P}^1_{\mathbb{C}} \setminus \Delta, t_0)$ by just π_1.

Recall that the topological *intersection number* (a, b) of two elements $a, b \in H^2(X_{t_0}, \mathbb{Z})$ is defined using the non-degenerate, symmetric bilinear *intersection pairing* (Poincaré duality pairing)

$$H^2(X_{t_0}, \mathbb{Z}) \otimes H^2(X_{t_0}, \mathbb{Z}) \to H^4(X_{t_0}, \mathbb{Z}) = \mathbb{Z}.$$

We thus have intersection quadratic forms on $H^2(X_{t_0}, \mathbb{Q})$ and $H^2(X_{t_0}, \mathbb{R})$. The *Hodge index theorem* implies that, since X_t is a non-singular projective surface, the intersection form on $H^2(X_{t_0}, \mathbb{R})$ has signature $(1, -1, -1, \ldots, -1)$. This means, for example, that on the orthogonal complement of $\mathbb{R}[\mathcal{O}_{X_{t_0}}(1)]$ for the intersection product, the intersection form is *negative definite*, since $[\mathcal{O}_{X_{t_0}}(1)]$ has positive self-intersection d (equal to the degree of X_{t_0} in $\mathbb{P}^3_{\mathbb{C}}$). In fact the Hodge theoretic proof of the index theorem amounts to directly proving this negative definiteness statement (a particular case of the *Hodge-Riemann bilinear relations*; see [9]).

Using the condition that the singular fibers X_t of f are irreducible with 1 ordinary double point, Lefschetz associates to each γ_t an element $\delta_t \in H^2(X_{t_0}, \mathbb{Z})$, called a *vanishing cycle*, and describes the monodromy action of the corresponding element $\rho(\gamma_t)$ on $H^2(X_{t_0}, \mathbb{Q})$ via the *Picard-Lefschetz formula*

$$\rho(\gamma_t)(a) = a + (a, \delta_t)\delta_t \;\; \forall\, a \in H^2(X_{t_0}, \mathbb{Q}),$$

where $(a, \delta_t) \in \mathbb{Z}$ is the intersection number of a with δ_t (see [12], [18], [21] for details). Further, one has a self-intersection formula

$$(\delta_t, \delta_t) = -2.$$

Another step in the proof of the Theorem is a lemma ("conjugacy of the vanishing cycles") that all the classes $\rho(\gamma_t)$, $t \in \Delta$, are contained in the same conjugacy class in the monodromy image group $\rho(\pi_1) \subset \mathrm{Aut}(H^2(X_{t_0}, \mathbb{Z}))$. This is deduced using standard homotopy arguments from a "global" geometric fact, that in the (dual) projective space parametrizing the set of all hypersurfaces of degree d in $\mathbb{P}^3_{\mathbb{C}}$, the subvariety parametrizing singular hypersurfaces (called the *discriminant locus*) is irreducible (see [18] for more details of this argument).

Next, one observes that the subspace of $H^2(X_{t_0}, \mathbb{Q})$ of elements invariant under $\rho(\pi_1)$ is the orthogonal complement of the span of the vanishing cycles δ_t, for the non-degenerate intersection pairing. This is clear, because any element of $H^2(X_{t_0}, \mathbb{Q})$ which is orthogonal to all the δ_t must, by the Picard-Lefschetz formula, be invariant under all the $\rho(\gamma_t)$, and hence under all of π_1.

Note that since $[\mathcal{O}_{X_{t_0}}(1)] \in$ image $H^2(\mathbb{P}^3_{\mathbb{C}}, \mathbb{Q})$ is π_1-invariant, it is orthogonal to all δ_t. Hence the intersection form is *negative definite* on the span of the δ_t. This implies that $H^2(X_{t_0}, \mathbb{Q})$ is the orthogonal direct sum of its π_1-submodules

$$\text{image } H^2(\mathbb{P}^3_{\mathbb{C}}, \mathbb{Q}) = H^2(X_{t_0}, \mathbb{Q})^{\pi_1}$$

(the subspace of π_1-invariants) and

$$\mathbf{V} = \sum_{t \in \Delta} \mathbb{Q}\delta_t$$

(this direct sum decomposition is the only "easy" case of the so-called "Hard Lefschetz theorem"). Since image $H^2(\mathbb{P}^3_{\mathbb{C}}, \mathbb{Q}) \neq 0$ is a proper subspace of $H^2(X_{t_0}, \mathbb{Q})$ as noted earlier, $\mathbf{V} \neq 0$.

In the light of this, the proof has been reduced to the following assertion.

Claim 3.15. $\mathbf{V} \subset H^2(X_{t_0}, \mathbb{Q})$ *is a non-trivial* simple π_1*-submodule of* $H^2(X_{t_0}, \mathbb{Q})$.

Indeed, since any two elements $\rho(\gamma_{t_1}), \rho(\gamma_{t_2}) \in \rho(\pi_1)$ are conjugate, we deduce (using the Picard-Lefschetz formula) that the corresponding vanishing cycles δ_{t_1}, δ_{t_2} are in the same $\rho(\pi_1)$-orbit. Hence any π_1-submodule of $\mathbf{V}$ containing some δ_t must be all of $\mathbf{V}$. Now if $a \in \mathbf{V}$ is a non-zero element, then negative definiteness of the intersection form yields $(a, a) < 0$, which implies $(a, \delta_t) \neq 0$ for some $t \in \Delta$. This in turn implies $a - \rho(\gamma_t)(a)$ is a non-zero multiple of δ_t, *i.e.*, the π_1-submodule generated by $\mathbb{Q}a$ contains δ_t.

3.3. Some elements of the proofs of Theorem 3.10 and Theorem 3.11

We now discuss how the Lefschetz pencil technique can be used to obtain Theorem 3.10 (in dimension 2) and 3.11.

Theorem 3.10, in dimension 2, is proved in the following stronger form (there is a similar strengthening in dimension 3, and in dimensions$\geq$ 4, it follows from Grothendieck's Theorem 3.3).

Theorem 3.16. *Let $f_i(x_1, \ldots, x_n) \in \mathbb{C}[x_1, \ldots, x_n]$, $1 \leq i \leq n-2$ be polynomials vanishing at the origin $P = (0, 0, \ldots, 0)$ such that*

$$\mathbb{C}[[x_1, \ldots, x_n]]/(f_1, \ldots, f_{n-2})$$

has an isolated complete intersection singularity. Then there exist integers $d_0 > r_0 > 0$ such that if

$$r \geq r_0, \quad d \geq \sup\{d_0, r+1, \deg f_j \ \forall\, j\}$$

and

$$V_{r,d} = \{\textit{polynomials of degree} \leq d \textit{ vanishing to order} \geq r \textit{ at } P\},$$

then for "general" $g_1, \ldots, g_{n-2} \in V_{r,d}$, we have:

(i) *$A = \mathbb{C}[x_1, \ldots, x_n]/(f_1 + g_1, \ldots, f_{n-2} + g_{n-2})$ is a UFD, with*

$$\widehat{A} \cong \mathbb{C}[[x_1, \ldots, x_n]]/(f_1, \ldots, f_{n-2}),$$

(ii) *if F_i, G_i are homogenous of degree d in $X_0, \ldots, X_n$ such that*

$$F_i(1, x_1, \ldots, x_n) = f_i, \quad G_i(1, x_1, \ldots, x_n) = g_i, \quad 1 \leq i \leq n-2,$$

then

$$\mathbb{C}[X_0, \ldots, X_n]/(F_1 + G_1, \ldots, F_{n-2} + G_{n-2})$$

is a 3-dimensional graded UFD,

(iii) *with notation as in* (ii), *the projective variety*

$$X = \operatorname{Proj} \mathbb{C}[X_0, \ldots, X_n]/(F_1 + G_1, \ldots, F_{n-2} + G_{n-2})$$

satisfies a Noether-Lefschetz theorem: the natural map $C\ell\,(\mathbb{P}^n_{\mathbb{C}}) \to C\ell\,(X)$ is an isomorphism.

Remark 3.17. In the above result, "general" means "in the complement of a countable union of hypersurfaces" (in the affine space $(V_{r,d})^{n-2}$).

Remark 3.18. To summarize the theorem in a less technical way, if we perturb the polynomials f_i defining an isolated complete intersection singularity, by adding "general" polynomials g_i vanishing to high enough order (depending on the singularity), then we obtain an algebraic local UFD from the perturbed equations $f_1+g_1 = \cdots = f_{n-2}+g_{n-2} = 0$, which has the same completion, *i.e.*, has the same singularity.

In order to prove this, one considers $S = (V_{r,d})^{n-2} = V_{r,d}\times\cdots\times V_{r,d}$, the affine space parametrizing ordered $(n-2)$-tuples $(g_1, \ldots, g_{n-2})$. For $s \in S$, let

$$X_s = \{F_1 + G_1 = \cdots = F_{n-2} + G_{n-2} = 0\} \subset \mathbb{P}^n_{\mathbb{C}}$$

be the corresponding projective subscheme. Let $L \subset S$ be a "general" line (*i.e.*, $L \cong \mathbb{A}^1_{\mathbb{C}}$ is an affine linear 1-dimensional subspace of S), and set

$$X_L = \{(x, s) \in \mathbb{P}^n_{\mathbb{C}} \times L \mid x \in X_s\}.$$

We would like to "do Lefschetz theory" for the "pencil" $f : X_L \to L$.

This requires several modifications of the earlier argument. First of all, the general subscheme X_s, $s \in L$, will be an irreducible, complete intersection surface, which has $\{P\}$ as its singular locus; further, one can ensure that every X_s is irreducible, and has at most one other singular point which is an ordinary double point. This resembles the conditions of a Lefschetz pencil, but $\{P\} \times L \subset X_L$ is still part of the singular locus.

Next, we need to construct a "simultaneous resolution of singularities" for the family $X_L \to L$ along the above curve $\{P\} \times L$ of singularities. Using a Hilbert scheme argument, this is reduced in [29] to the following problem on Hilbert functions in local algebra:

> given a complete intersection quotient $A = \mathbb{C}[[x_1, \ldots, x_n]]/(f_1, \ldots, f_{n-2})$ of a power series ring $R = \mathbb{C}[[x_1, \ldots, x_n]]$, with an isolated singularity, and an $\mathfrak{m}$-primary ideal $J \subset \mathfrak{m} = (x_1, \ldots, x_n) \subset R$, show that there exists $r > 0$ such that is an equality between Hilbert functions
>
> $$\ell(R/J^m + (f_1, \ldots, f_{n-2})) = \ell(R/J^m + (f_1 + g_1, \ldots, f_{n-2} + g_{n-2})) \qquad \forall\, m \geq 0,$$
>
> for an arbitrary choice of $g_1, \ldots, g_{n-2} \in \mathfrak{m}^r$.

In fact there is a result of Mather [23] which implies that, for some $r > 0$, and for each choice of $g_j \in \mathfrak{m}^r$, $1 \leq j \leq n-2$, there is an automorphism σ of the power series ring R such that $\sigma \equiv$ identity (mod J), and there is an equality between ideals $\sigma(f_1,\dots,f_r)=(f_1+g_1,\dots,f_r+g_r)$. This implies a positve answer to above problem.

Remark 3.19. The above question on Hilbert functions led naturally to the papers [40], [39], and thus ultimately to Theorem 1.1!

Let $Y_L\to X_L$ be the resulting "simultaneous resolution of singularities", so that $Y_L\to L$ does look more like a Lefschetz pencil (all fibers are irreducible, all but finitely many non-singular, and singular fibers have only 1 ordinary double point).

Now much of the argument is similar to the case of the Noether-Lefschetz theorem. One first uses the fact that if $s\in L$ is general, so that X_s is a surface with a unique singularity P, then

$$C\ell\,(X_s)=\mathrm{Pic}(X_s\setminus\{P\})=\mathrm{Pic}(Y_s\setminus E),$$

where $Y_s\to X_s$ is the above resolution of singularities (obtained from $Y_L\to X_L$), with exceptional set E. Using Mather's result cited above, one can arrange that the exceptional set $E\subset Y_s$ is in fact independent of $s\in L$, in the following sense — Y_L is obtained as a closed subvariety of $\widetilde{\mathbb{P}^n_{\mathbb{C}}}\times L$ for some blow-up $\widetilde{\mathbb{P}^n_{\mathbb{C}}}\to\mathbb{P}^n_{\mathbb{C}}$, such that the exceptional set for $Y_L\to X_L$ has the form $E\times L$ for some subvariety $E\subset\widetilde{\mathbb{P}^n_{\mathbb{C}}}$.

If $E_1,\dots,E_r$ are the irreducible components of E, then since Y_s is a non-singular surface, one computes that

$$\mathrm{Pic}(Y_s\setminus E)=\mathrm{Pic}(Y_s)/\{\text{subgroup generated by the classes }[\mathcal{O}_{Y_s}(E_i)]\}.$$

Thus the Theorem is equivalent to that statement that $\mathrm{Pic}(Y_s)$ is generated by the classes of $\mathcal{O}_{Y_s}(E_i)$ and the pull-back of $\mathcal{O}_{\mathbb{P}^n}(1)$.

Now suppose, for simplicity, that $n=3$, *i.e.*, we are still dealing with hypersurfaces in $\mathbb{P}^3_{\mathbb{C}}$. One can then realize Y_L as an open subset of the blow-up $\overline{Y_L}\to\mathbb{P}^3_{\mathbb{C}}$ along the union of a suitable complete intersection curve B, and a subscheme of finite length supported at P (Y_L itself is not compact, since it is proper over $L=\mathbb{A}^1_{\mathbb{C}}$). Hodge theory and the exponential sheaf sequence reduce one to proving, after some further analysis of the geometry, that

$$\mathrm{image}\left(H^2(\overline{Y_L},\mathbb{Q})\to H^2(Y_s,\mathbb{Q})\right)=\ker\left(H^2(Y_s,\mathbb{Q})\to H^2(Y_s,\mathcal{O}_{Y_s})\right).$$

Using the theory of vanishing cycles, etc. one ends up showing that

$$\mathrm{image}\left(H^2(Y_L,\mathbb{Q})\to H^2(Y_s,\mathbb{Q})\right)=\ker\left(H^2(Y_s,\mathbb{Q})\to H^2(Y_s,\mathcal{O}_{Y_s})\right).$$

To conclude, one appeals to the following result of Deligne, proved in [5] using his theory of *mixed Hodge structures*.

Theorem 3.20 (Deligne). *Let Z be a non-singular proper $\mathbb{C}$-variety, $Y \subset Z$ a non-singular closed subvariety, and $U \subset Z$ a Zariski open subset containing Y. Then*

$$\text{image}\,(H^*(Z,\mathbb{Q}) \to H^*(Y,\mathbb{Q})) = \text{image}\,(H^*(U,\mathbb{Q}) \to H^*(Y,\mathbb{Q})).$$

Remark 3.21. The application of Deligne's result, in "monodromy" situations as above, is usually called the *theorem of the fixed part* (see [5] for more discussion) – it says that if $f : U \to L$ is a smooth proper morphism between non-singular varieties, Z a non-singular complete variety containing U as a Zariski open set, and Y a fiber of f, then the subspace of $\pi_1(L)$-invariant elements of $H^*(Y,\mathbb{Q})$ is the image of the composite restriction map $H^*(Z,\mathbb{Q}) \to H^*(U,\mathbb{Q}) \to H^*(Y,\mathbb{Q})$. The special case of this result, when $Z = U$ and L are proper, is due to Griffiths.

We now make some further remarks on the proofs of the general case of Theorem 3.16, and of Theorem 3.11.

For Theorem 3.16, if $n \geq 4$, then in our above set-up, the variety Y_L, which has dimension 3, cannot be realized as an open subset of a blow-up of the ambient projective space $\mathbb{P}^n_{\mathbb{C}}$. So, in addition to the "pencil" $Y_L \to L$, one needs to also consider the "total family" of varieties parametrized by S itself, and (for example) compare the Leray spectral sequences for these two, etc. There is a "bad subvariety" $\Delta(S) \subset S$, parametrizing the fibers with additional singularities (apart from the chosen one P), such that the singular fibers of $Y_L \to L$ lie over the points of $\Delta = L \cap \Delta(S)$. Additional results needed at this stage are that $\Delta(S)$ is an *irreducible divisor*, and that (since $L \subset S$ is general) the natural map on fundamental groups $\pi_1(L \setminus \Delta) \to \pi_1(S \setminus \Delta(S))$ is *surjective* (a result of Zariski).

Finally, as before, one will end up proving that the subspace of $\pi_1(L \setminus \Delta)$-invariant elements in $H^2(Y_t,\mathbb{Z})$ coincides with the subgroup generated by the cohomology classes of the exceptional divisors E_i, and the pull-back of $\mathcal{O}_{\mathbb{P}^n_{\mathbb{C}}}(1)$. This will give the desired conclusion.

The strategy in proving Theorem 3.11 is a bit different. One first chooses some algebraic "model" for the surface singularity, *i.e.*, one finds an irreducible normal projective surface $X \subset \mathbb{P}^n_{\mathbb{C}}$ together with a point $x \in X$ such that $\mathcal{O}_{X,x}$ has the given completion, and $X \setminus \{x\}$ is non-singular. Then choose a generic linear projection $\mathbb{P}^n_{\mathbb{C}} \setminus H \to \mathbb{P}^3_{\mathbb{C}}$, which restricts to a finite, birational morphism $X \to Y$ onto a non-normal surface $Y \in \mathbb{P}^3_{\mathbb{C}}$, with X as its normalization. Finally, one analyzes the (usually 1-dimensional) singular locus $Z = Y_{sing}$ of Y; for example, one shows that at a "general" point of Z, the surface Y has complete local rings isomorphic to $\mathbb{C}[[x,y,z]]/(xy)$ (*i.e.*, Y has an "ordinary double curve singularity" at such points).

Applying the singularity theory results of Pellikaan and Siersma cited earlier, one considers deformations $\{(Y_t, Z_t)\}_t$ in $\mathbb{P}^3_{\mathbb{C}}$ of the *pair* (Y, Z). Taking "simultaneous normalizations" gives rise to corresponding deformations $\{(X_t, x_t)\}_t$ of (X, x), such that each of the complete local rings $\widehat{\mathcal{O}}_{X_t,x_t}$ is isomorphic to $\widehat{\mathcal{O}}_{X,x}$ (this is a consequence of the singularity theory inputs). Again one arrives at a sort of Noether-Lefschetz situation, with one difference: the inverse image in X_t under the normalization map $X_t \to Y_t$ of the singular locus $Z_t \subset Y_t$ (which is only a Weil divisor on Y_t) is a "new" divisor class, which does not come from a line bundle on $\mathbb{P}^3_{\mathbb{C}}$. In fact, *adjunction theory* (*i.e.*, "Grothendieck duality for the finite morphism $X_t \to Y_t$"; see for example [13]III Ex. 6.10 and Ex. 7.2) implies that this divisor represents the canonical (Weil) divisor class of X_t, upto a twist by some $\mathcal{O}_{X_t}(n)$. This gives the generator of the cyclic class group $C\ell\,(\mathcal{O}_{X_t,x_t})$ for a sufficiently general choice of the parameter t.

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