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Characterizations of Projective Space and Applications to Complex Symplectic Manifolds

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Dedicated to Professor Tetsuji Shioda on his 60th birthday

Abstract.

We obtain new criteria for a normal projective variety to be projective n-space. Our main result asserts that a normal projective variety which carries a closed, doubly-dominant, unsplitting family of rational curves is isomorphic to projective space. An immediate consequence of this is the solution of a long standing conjecture of Mori and Mukai that a smooth projective n-fold X is isomorphic to $\mathbb{P}^n$ if and only if $(C, -\mathrm{K}_X) \geq n+1$ for every curve C on X. As applications of the criteria, we study fibre space structures and birational contractions of compact complex symplectic manifolds.

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Introduction

Projective n-space $\mathbb{P}^n$ is the simplest n-dimensional algebraic variety and can accordingly be characterized in various ways. The main objectives of the present paper are:

A. To establish new characterizations of projective n-space in such a way that all the known characterizations are thereby systematically explained;

B. To apply our characterizations to morphisms from complex symplectic manifolds;

and, as prerequisites to the above two,

C. To provide a self-contained exposition of basic theory of families of rational curves, which is important for understanding detailed structure of rationally connected varieties.

Let X be a projective variety and Chow(X) the Chow scheme (see Section 1 below). Let $S \subset \mathrm{Chow}(X)$ be an irreducible subvariety and $\mathrm{pr}_S : F \to S$ the associated universal family.[2]

We say that F is a *closed family of rational curves* if S is proper and the fibre $F_s = \mathrm{pr}^{-1}(s) \subset \{s\} \times X \simeq X$ over a general point $s \in S$ is an irreducible, reduced rational curve as an effective 1-cycle.[3] Any special fibre of a family of rational curves is a 1-cycle supported by a union of rational curves. A closed family of rational curves $F \to S$ is called *maximal* if F is a union of irreducible components of F' for any family of rational curves $F' \supset F$. When every fibre F_s is irreducible and reduced (as 1-cycles), we say that F is *unsplitting*. A family of rational curves F is *dominant* if the natural projection $\mathrm{pr}_X : F \to X$ is surjective. F is *doubly dominant* if $\mathrm{pr}^{(2)}_{X\times X} : F \times_S F \to X \times X$ is surjective.

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[2] $F \subset S \times X$ is a Zariski closed subset with equidimensional fibres over S. The family F is proper over S, but not necessarily flat. Although we give a brief overview on Chow schemes and the associated universal families in Section 1, we refer the reader to [Kol] for full exposition and discussion.

[3] However, the scheme theoretic fibre F_s can contain 0-dimensional embedded components.

Main Theorem 0.1. *Let X be a normal projective variety defined over the complex number field $\mathbb{C}$ (or over an algebraically closed field of characteristic zero). If X carries a closed, maximal, unsplitting, doubly dominant family $\mathrm{pr}_S : F \to S$ of rational curves, then X is isomorphic to projective n-space $\mathbb{P}^n$, and F is the family of the lines on X parameterized by the Grassmann variety $S = \mathrm{Grass}(\mathbb{P}^n, 1)$.*

Roughly speaking, this theorem means that X is a projective space if and only if its two general points can be joined by a single rational curve of minimum degree (i.e., a line) with respect to a polarization of X. If we impose a slightly weaker condition than in Theorem 0.1, we have the following result.

Theorem 0.2. *Let X be a normal projective variety of dimension n over $\mathbb{C}$ and x a prescribed general point on it. Let $\mathrm{pr}_S : F \to S$ be a closed, maximal, doubly-dominant family of rational curves on X, and write $F\langle x\rangle \to S\langle x\rangle$ for the closed subfamily consisting of curves passing through x. If $F\langle x\rangle$ is unsplitting, then X is a quotient of $\mathbb{P}^n$ by a finite group action without fixed point locus of codimension one. In particular, X is $\mathbb{P}^n$ if it is smooth.*

A smooth projective variety X is said to be a *Fano manifold* if its anticanonical divisor $-\mathrm{K}_X$ is ample. Our Main Theorem 0.1 yields a simple numerical criterion for a Fano manifold to be projective space in terms of the *length* $l(\cdot)$ of rational curves:

Corollary 0.3 (Conjecture of Mori and Mukai). *Let X be a smooth complex Fano n-fold. Put*

$$l(X) = \min\{(C, -\mathrm{K}_X); C \subset X \textit{is a rational curve}\}.$$

Then X is isomorphic to $\mathbb{P}^n$ if and only if $l(X) \geq n+1$.

Our criterion (Theorem 0.1), stated in terms of geometry of rational curves, is strong enough to yield a whole series of characterizations of projective n-space expressed in very different languages:

Corollary 0.4. *Let X be a complex projective manifold of dimension n and $x_0 \in X$ a general point. Then the following fourteen conditions are equivalent:*

1. $X \simeq \mathbb{P}^n$;
2. Hirzebruch-Kodaira-Yau condition [HK]: *X is homotopic to $\mathbb{P}^n$*;
3. Kobayashi-Ochiai condition [KO]: *X is Fano and $c_1(X)$ is divisible by $n+1$ in $H^2(X, \mathbb{Z})$*;
4. Frankel-Siu-Yau condition [SY]: *X carries a Kähler metric of positive holomorphic bisectional curvature*;

5. Hartshorne-Mori condition [Mo1]: *The tangent bundle* Θ_X *of* X *is ample*;
6. Mori condition [Mo1]: X *is uniruled and* $\Theta_X|_C$ *is ample for an arbitrary rational curve* C *on* X;
7. Doubly transitive group action: *The action of* $\mathrm{Aut}(X)$ *on* X *is doubly transitive*;
8. Remmert-Van de Ven-Lazarsfeld condition [La]: *There exists a surjective morphism from a suitable projective space onto* X;
9. Length condition: $(C, -\mathrm{K}_X) \geq n+1$ *for every curve* C *on* X;
10. Length condition on rational curves: X *is uniruled and* $(C, -\mathrm{K}_X) \geq n+1$ *for every rational curve* C *on* X;
11. Length condition on rational curves with base point: X *is uniruled and* $(C, -\mathrm{K}_X) \geq n+1$ *for every rational curve* C *containing the prescribed general point* x_0;
12. Double dominance condition on rational curves: X *is uniruled and every reduced irreducible rational curve on* X *is a member of a doubly dominant family of rational curves*;
13. Double dominance condition on rational curves of minimum degree: X *is uniruled and a rational curve of minimum degree* (*with respect to an arbitrary fixed polarization*) *on* X *is a member of a doubly dominant family of rational curves*;
14. Dominance condition on rational curves with base point: *Every rational curve* C *passing through* x_0 *is a member of a dominant family* $F = \{C_t\}$ *of rational curves* $\{C_t\}$ *passing through the base point* x_0.

Although our result (Theorem 0.1) is far stronger than the results known before, we are not completely independent of the preceding works. Our basic strategy is in fact very similar to the argument used in [Mo1]. Given a closed, unsplitting, doubly dominant family $F \to S$ of rational curves, consider the subfamily $F\langle x\rangle \to S\langle x\rangle$. We prove that the projection $\mathrm{pr}_X : F\langle x\rangle \to X$ is actually the blow-up $\mathrm{Bl}_x(X)$ of X at x, the base variety $S\langle x\rangle$ being isomorphic to the associated exceptional divisor $E_x \simeq \mathbb{P}^{n-1}$.

If one knows that *every* point of $S\langle x\rangle$ represents a curve smooth at x, then the birationality of pr_X follows from an elementary argument (see Proposition 2.7 below). But it is by no means obvious that this smoothness condition is always satisfied. On the contrary, when $S\langle x\rangle$ happens to contain a point which represents a curve singular at x, then $F\langle x\rangle$ is never birational to X. Thus we need to rule out the existence of such bad points in S, which is done with the aid of a theorem of Kebekus

(Theorem 3.10) saying that no point of S can represent a curve which has a cuspidal singularity at the base point x.

Our characterization of projective n-space (Theorem 0.1) provides intriguing information on complex symplectic manifolds. Given a compact complex symplectic manifold Y of dimension $2n$ and an arbitrary non-constant morphism $f : \mathbb{P}^1 \rightarrow Y$, one can show that $\dim_{[f]} \mathrm{Hom}(\mathbb{P}^1, Y) \geq 2n+1$. If one knows that $f_t(\mathbb{P}^1)$ stays in a fixed n-dimensional subvariety $X \subset Y$ for any (small) deformation f_t of f, then the Main Theorem 0.1 implies that the normalization of X is necessarily $\mathbb{P}^n$. This is indeed the case in some important situations, imposing very restrictive constraints on fibre space structure of, or birational contractions from, complex symplectic manifolds. Specifically, we completely understand the symplectic resolutions of a normal projective variety with only isolated singularities. For precise statements, see Theorems 7.2, 8.3 and 9.1 below.

This paper is organized as follows:

Part I, consisting of three sections, is a review of general theory concerning families of rational curves on projective varieties. This theory is expected to be a useful tool to analyse the structure of uniruled varieties. We need here nothing very special; almost every result derives from well known geometry of ruled surfaces modulo general theory of Chow schemes and deformation.

In Section 1, we recall basic concepts and facts necessary for, and/or closely related to, the family of rational curves. Most results there are more or less known to experts, yet they are included for the coherence of the account and for the convenience of the reader.

Section 2 discusses unsplitting families of rational curves. The unsplitting condition is a very strict constraint on the family, and we obtain various estimates of the dimension of the parameter space S.

Section 3 is the survey of a recent result by Kebekus [Ke1] and [Ke2] on unsplitting families of singular rational curves. It asserts among other things that, if $F \rightarrow S$ is an unsplitting family of rational curves on a projective variety X, then no member C of S has a cuspidal singularity at a general fixed point $x \in X$.

Part II (Sections 4 and 5) treats characterizations of projective n-space. The Main Theorem 0.1 as well as Theorem 0.2 is proved in Section 4. Given a closed, doubly-dominant family of rational curves $F \rightarrow S$ which is unsplitting at a general point x (i.e., we assume that the subfamily $F\langle x\rangle \rightarrow S\langle x\rangle$ is unsplitting), we argue that the normalization of $F\langle x\rangle$ is isomorphic to a one-point blow up of $\mathbb{P}^n$.

The relationship between the various conditions in Corollary 0.4 is discussed in Section 5.

Part III (Section 6 through 9) contains applications of the Main Theorem to compact complex symplectic manifolds.

In Section 6, we review generalities on compact complex symplectic manifolds. One of the key observations is that any holomorphic map from a rational curve to a $2n$-dimensional symplectic manifold moves in a family with at least $2n+1$ independent parameters.

The first application of our Main Theorem is to fibre space structure of primitive complex symplectic manifolds. Matsushita [Mats] showed that, if such a manifold has a nontrivial fibre space structure, it must be a Lagrangian torus fibration over a $\mathbb{Q}$-Fano variety. In Section 7, we see that the base space is necessarily a projective space, provided the fibration admits a global cross section.

The second application is to birational contractions. Let Z be a compact complex symplectic manifold of dimension $2n$ and $f : Z \to \hat{Z}$ a birational contraction to a normal variety. Let $E_i \subset Z$ be an irreducible component of the exceptional locus and $B_i \subset \hat{Z}$ its image. Then we verify that the base variety B_i is again a symplectic variety of dimension $2(n-a_i)$ (possibly with singularities) and a general fibre X of the projection $E_i \to B_i$ is a union of copies of projective a_i-space. In case $a_i > 1$, X is indeed a single smooth $\mathbb{P}^{a_i}$, and the local analytic structure of $f : Z \to \hat{Z}$ is uniquely determined on a small open neighbourhood of X in Z. In order to simplify the argument, we first deal with isolated singularities (Section 8) and then general singularities (Section 9). One of the key results (unramifiedness of the normalization) is proved in Section 10.

Throughout the article, all schemes are assumed to be separated. Schemes and varieties are usually defined over $\mathbb{C}$, or, more generally, over an algebraically closed field k of characteristic zero. The assumption on the characteristic is made because we use Sard's theorem in an essential way. As far as the authors know, it is still an open problem if our results (Theorems 0.1 and 0.2) stay true in positive characteristics.

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Notation

In the present work, standard notation in algebraic geometry is freely used. For instance, $\mathrm{Hom}(Y, X)$, $\mathrm{Chow}(X)$ and $\mathrm{Hilb}(X)$ stand for the Hom scheme, the Chow scheme and the Hilbert scheme parameterizing the morphisms, the effective cycles and the subschemes, respectively. A curve $C \subset X$ or a morphism $f : Y \to X$ is denoted by $[C]$ or $[f]$ when viewed as an element of $\mathrm{Chow}(X)$ or $\mathrm{Hom}(Y, X)$.

We list below some of the *ad hoc* symbols which frequently appear in the article.

- $\overline{\bullet}$ and $\bullet^c$ generally stand for the normalization and the closure of $\bullet$.
- $\nu_\bullet : \overline{\bullet} \to \bullet$ will denote the normalization map.
- S : a closed subset of $\mathrm{Chow}(X)$ consisting of (unions of) rational curves.
- $S\langle x\rangle$: the closed subset of S consisting of members which pass through a prescribed closed point $x \in X$.
- F, $F\langle x\rangle$: the family of rational curves on X parameterized by S and $S\langle x\rangle$.
- $\mathrm{pr}_S : F \to S$, $\mathrm{pr}_X : F \to X$: the natural projections.
- $\overline{F}$, $\overline{S}$, $\overline{F}\langle x\rangle$, $\overline{S}\langle x\rangle$: the normalizations of F, S, $F\langle x\rangle$, $S\langle x\rangle$, with natural projections $\overline{\mathrm{pr}}_{\overline{S}} : \overline{F} \to \overline{S}$, $\overline{\mathrm{pr}}_{\overline{S}\langle x\rangle} : \overline{F}\langle x\rangle \to \overline{S}\langle x\rangle$, $\overline{\mathrm{pr}}_X : \overline{F}, \overline{F}\langle x\rangle \to X$.
- $\mathrm{Bl}_Y(X)$: the blowup of X along a closed subscheme Y.
- E_Y: the exceptional divisor on $\mathrm{Bl}_Y(X)$.
- $\mathrm{pr}_{\tilde{X}} : F\langle x\rangle \dashrightarrow \tilde{X} = \mathrm{Bl}_x(X)$: the natural rational map induced by the X-projection $\mathrm{pr}_X : F\langle x\rangle \to X$.

PART I. Families of Rational Curves on Projective Varieties

1. Review of basic concepts and results

In this section, we recall basic concepts and results such as deformation theory, Chow schemes and Mori's bend and break technique. Nothing very special or new appears here and experts can skip the whole section.

A. Cotangent sheaves, Zariski tangent spaces and infinitesimal deformation of morphisms

Let A be a scheme and X an A-scheme (assumed to be separated as usual). The diagonal $\Delta = \Delta_{X/A} \subset X \times_A X$ is a closed subscheme defined by the ideal sheaf $\mathfrak{I}_\Delta$. We regard $\mathcal{O}_{X\times_A X} = \mathcal{O}_X \otimes_{\mathcal{O}_A} \mathcal{O}_X$ as a left $\mathcal{O}_X$-module via the multiplication to the first factor. $\mathfrak{I}_\Delta^k$, $k = 0, 1, 2, \dots$ are naturally $\mathcal{O}_X$-modules. We have a canonical direct sum decomposition of $\mathcal{O}_{X\times_A X} = \mathfrak{I}_\Delta^0$ into the direct sum $\mathcal{O}_X \oplus \mathfrak{I}_\Delta$ as a left $\mathcal{O}_X$-module by virtue of the two canonical homomorphisms $\mathrm{pr}_1^* : \mathcal{O}_X \to \mathcal{O}_{X\times_A X}$, $\mathrm{rest}_\Delta : \mathcal{O}_{X\times_A X} \to \mathcal{O}_\Delta \simeq \mathcal{O}_X$.[4]

The coherent sheaf $\mathfrak{I}_\Delta/\mathfrak{I}_\Delta^2$ is called the *sheaf of Kähler differentials* or the *sheaf of relative* 1*-forms*, and denoted by $\Omega^1_{X/A}$. On a flat A-scheme X, $\Omega^1_{X/A}$ is locally free if and only if X is smooth over A, and in this case $\Omega^1_{X/A}$ is often called the *relative cotangent bundle* over A. When A is the spectrum of an algebraically closed field, we usually abbreviate $\Omega^1_{X/A}$ to Ω^1_X.

Explicit local description of $\Omega^1_{X/A}$ is as follows. Let $t_1, \dots, t_N$ be generators of the $\mathcal{O}_A$-algebra $\mathcal{O}_X$, with relations (or defining ideal) J. As an $\mathcal{O}_A$-module, $\mathcal{O}_X \otimes_{\mathcal{O}_A} \mathcal{O}_X$ is generated by the $t_i \otimes 1$ and $1 \otimes t_i$, with obvious relations $u \otimes 1 = 1 \otimes u = 0$ for $u \in J$ and $a \otimes 1 = 1 \otimes a$ for $a \in \mathcal{O}_A$. The ideal $\mathfrak{I}_\Delta$ is generated by $v \otimes 1 - 1 \otimes v$, $v \in \mathcal{O}_X$, and so is $\mathfrak{I}_\Delta^2$ by

$$\begin{aligned}(v \otimes 1 - 1 \otimes v)&(w \otimes 1 - 1 \otimes w)\\ &= vw \otimes 1 - v \otimes w - w \otimes v + 1 \otimes vw\\ &= v(w \otimes 1 - 1 \otimes w) + w(v \otimes 1 - 1 \otimes v) - (vw \otimes 1 - 1 \otimes vw).\end{aligned}$$

Given $v \in \mathcal{O}_X$, let $\mathrm{d}v$ denote the equivalence class of $v \otimes 1 - 1 \otimes v$ modulo $\mathfrak{I}_\Delta^2$. Then the $\mathcal{O}_X$-module $\Omega^1_{X/A}$ is generated by $\mathrm{d}v$, $v \in \mathcal{O}_X$ with relations $\mathrm{d}(vw) = v\mathrm{d}w + w\mathrm{d}v$, $v, w \in \mathcal{O}_X$ and $\mathrm{d}a = 0$ for $a \in \mathcal{O}_A$. Eventually we conclude that the $\mathcal{O}_X$-module $\Omega^1_{X/A}$ is generated

[4]The direct sum decomposition of $a \otimes b$ is given by $(ab, a \otimes b - ab \otimes 1)$.

by $\mathrm{d}t_1, \ldots, \mathrm{d}t_N$ as an $\mathcal{O}_X$-module, with two relations $\mathrm{d}(t_i t_j) = t_i \mathrm{d}t_j + t_j \mathrm{d}t_i$, $t_i, t_j \in \mathcal{O}_X$ and $\mathrm{d}u = 0, u \in \mathcal{O}_A$.

Let $f : Y \to X$ be an A-morphism between A-schemes. Then we have a commutative diagram

$$\begin{array}{ccc} \Delta_{Y/A} & \longrightarrow & \Delta_{X/A} \\ \downarrow & & \downarrow \\ Y \times_A Y & \longrightarrow & X \times_A X, \end{array}$$

which induces a natural $\mathcal{O}_X$-homomorphism $\Omega^1_{X/A} \to f_* \Omega^1_{Y/A}$ or, equivalently, an $\mathcal{O}_Y$-homomorphism $\mathrm{d}f^* : f^* \Omega^1_{X/A} \to \Omega^1_{Y/A}$, called the *differential* of f.

For an arbitrary closed embedding $f : Y \hookrightarrow X$, the differential $\mathrm{d}f^*$ is a surjection by the above description of Ω^1.

When A is the spectrum of an algebraically closed field k of characteristic zero and f is a dominant morphism between smooth k-varieties, we have the following

Theorem 1.1 (Sard's theorem). *Let k be an algebraically closed field of characteristic zero. Let X and Y be smooth k-varieties and $f : Y \to X$ a dominant morphism. Then there exists a non-empty open subset $U \subset X$ such that $\mathrm{d}f^* : f^*\Omega^1_X \to \Omega^1_Y$ is everywhere injective on $f^{-1}(U)$. Put in another way, a general fibre of f is smooth.*

Let k be an algebraically closed field and let A be Spec k; thus X is a k-scheme. Let x be a k-valued point defined by an maximal ideal $\mathfrak{M}$. Choose a generator $t_1, \ldots, t_N$ of $\mathcal{O}_{X,x}$ such that they form a k-basis of $\mathfrak{M}/\mathfrak{M}^2$. Then $(\mathcal{O}_X/\mathfrak{M}) \otimes_{\mathcal{O}_X} \Omega^1_X$ is precisely $k\mathrm{d}t_1 \oplus \cdots \oplus k\mathrm{d}t_N \simeq \mathfrak{M}/\mathfrak{M}^2$ as a k-vector space.

Given a k-valued point x of the k-scheme X, the *Zariski tangent space* $\Theta_{X,x}$ of X at x is, by definition, the set of the k-morphisms

$$f : \big(\mathrm{Spec}\ k[\varepsilon]/(\varepsilon)^2, \mathrm{Spec}\ k[\varepsilon]/(\varepsilon)\big) \to (X, x)$$

between the pointed k-schemes or, equivalently, the set of the k-algebra homomorphisms $v : \mathcal{O}_X \to k[\varepsilon]/(\varepsilon^2)$ such that $v \mod (\varepsilon)$ is equal to the evaluation map $v_x : \mathcal{O}_X \to k$ at x. By the correspondence $v \mapsto (v - v_x) : \mathcal{O}_X \to \varepsilon k$, $\Theta_{X,x}$ is naturally identified with

$$\mathrm{Hom}_k(\mathfrak{M}_x/\mathfrak{M}_x^2, k) \simeq \mathrm{Hom}_k(k(x) \otimes \Omega^1_X, k(x)).$$

(In particular, if $Y \subset X$ is a closed subscheme, there is a natural injection $\Theta_{Y,x} \subset \Theta_{X,x}$ for a k-valued point x on Y.) Grothendieck [Gro] generalized this standard fact as follows.

Proposition 1.2. *Let X and Y be k-schemes and let $\tilde{f}$: Spec $k[\varepsilon]/(\varepsilon^2) \times Y \to X$ be a k-morphism. There is a natural correspondence $\tilde{f} \mapsto \partial_\varepsilon \tilde{f} \in \mathrm{Hom}_{\mathcal{O}_Y}(f^*\Omega_X^1, \mathcal{O}_Y)$, where $f : Y \to X$ is the restriction of $\tilde{f}$ to $Y =$ Spec $k[\varepsilon]/(\varepsilon) \times Y$. Given a k-morphism $f : Y \to X$, the above gives a one-to-one correspondence between the set of the liftings of f to morphisms $\tilde{f}$: Spec $k[\varepsilon]/(\varepsilon^2) \times Y \to X$ and the k-vector space $\mathrm{Hom}_{\mathcal{O}_Y}(f^*\Omega_X^1, \mathcal{O}_Y)$.*

Proof. The topological space Spec $k[\varepsilon]/(\varepsilon^2) \times Y$ is identical with Y, and hence a k-morphism $\tilde{f}$ is uniquely determined by the continuous map f and a k-algebra homomorphism $\tilde{f}^* : \mathcal{O}_X \to \big(k[\varepsilon]/(\varepsilon^2)\big) \otimes \mathcal{O}_Y$ such that $\tilde{f}^* \mod (\varepsilon) = f^*$. By the standard embedding $k \to k[\varepsilon]/(\varepsilon^2)$, we regard f^* as a ring homomorphism to $k[\varepsilon]/(\varepsilon^2) \otimes \mathcal{O}_Y$.

Consider the natural two k-morphisms

$$(f, \tilde{f}), (f, f) : \mathrm{Spec}\ k[\varepsilon]/(\varepsilon)^2 \times Y \to X \times X$$

and the associated k-algebra homomorphisms

$$f^* \cdot f^*,\ f^* \cdot \tilde{f}^* : \mathcal{O}_X \otimes_k \mathcal{O}_X \to \big(k[\varepsilon]/(\varepsilon^2)\big) \otimes_k \mathcal{O}_Y,$$
$$a \otimes b \mapsto f^*af^*b,\ f^*a\tilde{f}^*b.$$

By construction, they satisfy

$$\begin{aligned} (f^* \cdot f^*)(\mathfrak{I}_\Delta) &= 0, \\ (f^* \cdot \tilde{f}^*)(\mathfrak{I}_\Delta) &\subset \varepsilon\mathcal{O}_Y, \\ (f^* \cdot f^* - f^* \cdot \tilde{f}^*)(\mathcal{O}_X \otimes 1) &= 0. \end{aligned}$$

Hence $f^* \cdot f^* - f^* \cdot \tilde{f}^*$ is an $\mathcal{O}_X$-linear map from $\Omega_X^1 \subset (\mathcal{O}_X \otimes \mathcal{O}_X)/\mathfrak{I}_\Delta^2$ to $\varepsilon\mathcal{O}_Y$. Then $\partial_\varepsilon \tilde{f}$ is defined to be $\varepsilon^{-1}(f^* \cdot f^* - f^* \cdot \tilde{f}^*)$. Given $\mathrm{d}\alpha = \alpha \otimes 1 - 1 \otimes \alpha \in \Omega_X^1$, $\alpha \in \mathcal{O}_X$, we have the explicit formula

$$\begin{aligned} \varepsilon\partial_\varepsilon \tilde{f}(\mathrm{d}\alpha) &= (f^* \cdot \tilde{f}^* - f^* \cdot f^*)(\alpha \otimes 1 - 1 \otimes \alpha) \\ &= f^*(\alpha)f^*(1) - f^*(1)f^*(\alpha) - f^*(\alpha)\tilde{f}^*(1) + f^*(1)\tilde{f}^*(\alpha) \\ &= \tilde{f}^*(\alpha) - f^*(\alpha). \end{aligned}$$

Conversely, given $\partial_\varepsilon \tilde{f} : \Omega_X^1 \to \mathcal{O}_Y$ and $\alpha \in \mathcal{O}_X$, we define $\tilde{f}^*(\alpha) = f^*(\alpha) + \varepsilon\partial_\varepsilon \tilde{f}(\mathrm{d}\alpha)$. $\tilde{f}^*$ is in fact a ring homomorphism because

$$\begin{aligned} \tilde{f}^*(\alpha\beta) &= f^*(\alpha)f^*(\beta) + \varepsilon f^*(\alpha)\partial_\varepsilon \tilde{f}(\mathrm{d}\beta) + \varepsilon f^*(\beta)\partial_\varepsilon \tilde{f}(\mathrm{d}\alpha) \\ &\equiv \big(f^*(\alpha) + \varepsilon\partial_\varepsilon \tilde{f}(\mathrm{d}\alpha)\big)\big(f^*(\beta) + \varepsilon\partial_\varepsilon \tilde{f}(\mathrm{d}\beta)\big) \\ &= \tilde{f}^*(\alpha)\tilde{f}^*(\beta) \mod (\varepsilon^2). \end{aligned}$$

Thus $\partial_\epsilon \tilde{f}$ uniquely determines the morphism $\tilde{f}$. Q.E.D.

The k-vector space $\Theta_{X,x}$ can be viewed as an $\mathcal{O}_X$-module through the natural surjection $\mathcal{O}_X \to k(x) \simeq k$, and hence there is a natural $\mathcal{O}_X$-homomorphism $\mathcal{T}_X^0 := \mathcal{H}om_{\mathcal{O}_X}(\Omega_X^1, \mathcal{O}_X) \to \Theta_{X,x}$. This map is, however, not a surjection in general. When X is smooth, we write Θ_X instead of $\mathcal{T}_X^0$ and call it the *tangent sheaf* of X. In this special case, the Zariski tangent space $\Theta_{X,x}$ is naturally identified with $k(x) \otimes_{\mathcal{O}_X} \Theta_X$.

Corollary 1.3. *Let X, Y and T be k-schemes and $\tilde{f} : T \times Y \to X$ a k-morphism. Given a k-valued point $t \in T$, let $f_t : Y \to X$ denote the restriction of $\tilde{f}$ to $\{t\} \times Y$. Then there exists a natural k-linear map* (Kodaira-Spencer map)

$$\partial_{T,t}\tilde{f} : \Theta_{T,t} \longrightarrow \mathrm{Hom}_{\mathcal{O}_Y}(f_t^*\Omega_X^1, \mathcal{O}_Y).$$

If T is smooth, then the Kodaira-Spencer maps define a natural $\mathcal{O}_T$-linear map

$$\partial_T\tilde{f} : \Theta_T \to (\mathrm{pr}_T)_*\mathcal{H}om(\tilde{f}^*\Omega_X^1, \mathcal{O}_{T\times Y}).$$

B. Hilbert schemes, Chow schemes and Hom schemes

Let A be a connected scheme. Let X be a flat, projective A-scheme with an A-ample line bundle L. Consider an arbitrary A-scheme T and a T-flat closed subscheme $Y \subset T \times_A X$. For each integer m and for each point $t \in T$, the Euler characteristic $\chi(Y_t, \mathcal{O}_{Y_t}(mL))$ of the fibre Y_t over t is well-defined. If m is fixed, then it is a locally constant function on T. If we fix a connected component T_0 of T, it is a polynomial function in m of degree $\dim Y/T_0$ with coefficients in the rational numbers, and is called the *Hilbert polynomial* (of Y over the connected component).

Take a polynomial $h = h(m)$ with rational coefficients. The correspondence

$$\begin{aligned} T \mapsto\ & \mathcal{H}ilb^h_{X/A}(T) \\ & := \big\{T\text{-flat closed subscheme} \subset T \times_A X \text{ with Hilbert function } h(m)\big\} \end{aligned}$$

defines a contravariant functor from the category of A-schemes to the category of sets. Indeed, if $T_1 \to T_2$ is an A-morphism and $Y \subset T_2 \times_A X$ is a T_2-flat closed subscheme with Hilbert polynomial h, then the pull-back of the family or, equivalently, the base change, $T_1 \times_{T_2} Y$ is a T_1-flat closed subscheme of $T_1 \times_A X$ with the same Hilbert polynomial. A fundamental result of Grothendieck [FGA] is that this contravariant functor is representable by the *Hilbert scheme* $\mathrm{Hilb}^h(X/A)$:

Theorem 1.4. *Let the notation be as above. For a given polynomial $h = h(m) \in \mathbb{Q}[m]$, there exist a projective A-scheme* $\mathrm{Hilb}^h(X/A)$

and a $\mathrm{Hilb}^h(X/A)$*-flat closed subscheme* $\mathcal{U}^h(X/A) \subset \mathrm{Hilb}^h(X/A) \times_A X$ *which has the following universal property:*

(Univ) *Given an arbitrary A-scheme T and arbitrary* $Y \in \mathcal{H}ilb^h_{X/A}(T)$, *there exists a unique A-morphism* $T \to \mathrm{Hilb}^h(X/A)$ *such that* $Y = T \times_{\mathrm{Hilb}^h(X/A)} \mathcal{U}^h(X/A)$.

In particular, $\mathrm{Hilb}(X/A) = \coprod_h \mathrm{Hilb}^h(X/A)$ is a countable union of projective A-schemes parameterizing the closed A-subschemes of X.

In the following, we take A to be Spec k, k being an algebraically closed field. In this specific case, we simply write $\mathrm{Hilb}(X)$ instead of $\mathrm{Hilb}(X/\mathrm{Spec}\, k)$. $\mathrm{Hilb}^h(X)(k)$ is the set of closed k-subschemes of X with Hilbert polynomial h.

If Y is a closed subscheme of (pure) dimension m, we can view Y as an effective m-cycle by forgetting the scheme structure of Y. As for the space of effective cycles, we have also the universal family parameterized by the *Chow scheme.*

Theorem 1.5. *Let* (X, L) *be a polarized projective variety (i.e., a pair of a projective variety and an ample line bundle on it). The set of effective m-cycles of degree d on X is uniquely parameterized by the k-points of the Chow scheme* $\mathrm{Chow}^d_m(X)$, *a reduced, seminormal,*[5] *projective k-scheme. The product* $\mathrm{Chow}^d_m(X) \times X$ *carries the effective cycle* $\mathrm{Univ}^d_m(X)$ *which has the following universal property.*[6] *If there is a family of effective m-cycles of degree d parameterized by a seminormal k-scheme S (i.e., if there is a closed subset* $F \subset S \times X$ *such that every closed fibre* F_s *is an m-cycle of degree d in X), then there exists a unique k-morphism* $S \to \mathrm{Chow}^d_m(X)$ *such that the family is the pullback of* $\mathrm{Univ}^d_m(X)$.

For the proof, see Kollár [Kol, Theorem 3.21].

The relationship between $\mathrm{Hilb}(X)$ and $\mathrm{Chow}(X)$ is messy in general. However, if a closed subscheme $Y \subset X$ is integral, we have a natural set theoretic one-to-one correspondence between the two schemes near $[Y]$ (which can be viewed also as an effective cycle). Hence, if in addition $\mathrm{Hilb}(X)$ is smooth at $[Y]$, $\mathrm{Hilb}(X)$ and $\mathrm{Chow}(X)$ are canonically isomorphic near $[Y]$ by seminormality of $\mathrm{Chow}(X)$.[7]

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[5]A k-scheme S is *seminormal* iff any birational, bijective morphism $S' \to S$ is an isomorphism. For instance, an ordinary node is seminormal, while a cusp is not.

[6]N.B.: $\mathrm{Univ}^d_m(X)$ is usually not flat over $\mathrm{Chow}^d_m(X)$.

[7]Notice that there is a uniquely determined morphism $\mathrm{Hilb}(X) \to \mathrm{Chow}(X)$ near $[Y]$ thanks to the universal property of Chow.

A useful sufficient condition for $\mathrm{Hilb}(X)$ to be smooth at $[Y]$ is the following

Theorem 1.6 (Grothendieck [FGA]). *Let X be a k-variety and Y a closed subscheme defined by an ideal $\mathfrak{I}_Y \subset \mathcal{O}_X$. Assume that the conormal sheaf $\mathfrak{I}_Y/\mathfrak{I}_Y^2$ is a locally free $\mathcal{O}_Y$-module*[8] *(i.e., Y is locally complete intersection in X). Then the Zariski tangent space of* $\mathrm{Hilb}(X)$ *at the point $[Y]$ is canonically isomorphic to* $\mathrm{Hom}_{\mathcal{O}_Y}(\mathfrak{I}_Y/\mathfrak{I}_Y^2, \mathcal{O}_Y) \simeq \mathrm{H}^0(Y, (\mathfrak{I}_Y/\mathfrak{I}_Y^2)^*)$. $\mathrm{Hilb}(X)$ *is smooth at $[Y]$ if the obstruction space* $\mathrm{H}^1(Y, (\mathfrak{I}_Y/\mathfrak{I}_Y^2)^*)$ *vanishes.*

While the subschemes and the effective cycles are parametrized by the Hilbert scheme and the Chow scheme, the morphisms are parametrized by the *Hom scheme.* Let X and Y be projective k-schemes. The graph of a k-morphism $f : Y \to X$ determines a closed subscheme $\Gamma_f \subset Y \times X$ isomorphic to Y via the first projection, and *vice versa.* Thus the set of morphisms $\mathrm{Hom}(Y, X) = \{f : Y \to X\}$ is a locally closed subset of $\mathrm{Hilb}(Y \times X)$ in an obvious way, and as such a countable union of quasiprojective schemes. Given specified base points $x \in X, y \in Y$, we denote by $\mathrm{Hom}(Y, X; y \mapsto x)$ the closed subset $\{[f]; f(y) = x\} \subset \mathrm{Hom}(X, Y)$. The basic properties of the Hom scheme $\mathrm{Hom}(Y, X)$ are summarized as follows:

Theorem 1.7. *Let X and Y be projective k-schemes.*

(1) *Let (S, o) be a k-scheme with a specified k-valued point and $f : Y \to X$ a k-morphism. Let $\tilde{f} : S \times Y \to X$ be a morphism such that $\tilde{f}|_{\{o\}\times Y}$ is identical with f (i.e., $\tilde{f}$ is a deformation of f parameterized by S). Then there is a natural linear map $\kappa : \Theta_{S,o} \to$* $\mathrm{Hom}_{\mathcal{O}_Y}(f^*\Omega_X^1, \mathcal{O}_Y)$, *called the* Kodaira-Spencer map. *If $\tilde{f}$ is a deformation with a base point y (i.e., if $\tilde{f}(S \times \{y\})$ is a single point $x \in X$), then $\kappa(\Theta_{S,o}) \subset$* $\mathrm{Hom}_{\mathcal{O}_Y}(f^*\Omega_X^1, \mathfrak{I}_y)$, *where $\mathfrak{I}_y$ is the ideal sheaf defining $y \in Y$. When S is smooth, then there is a natural $\mathcal{O}_S$-homomorphism $\kappa : \Theta_S \to \mathrm{pr}_{S*}\mathcal{H}om_{\mathcal{O}_{S\times Y}}(\tilde{f}^*\Omega_X^1, \mathcal{O}_{S\times Y})$.*

(2) *When X is smooth, the Kodaira-Spencer map κ gives natural identifications*

$$\Theta_{\mathrm{Hom}(Y,X),[f]} \simeq \mathrm{H}^0(Y, f^*\Theta_X),$$
$$\Theta_{\mathrm{Hom}(Y,X;y\mapsto x),[f]} \simeq \mathrm{H}^0(Y, \mathfrak{I}_y f^*\Theta_X).$$

[8]When the conormal sheaf $\mathfrak{I}_Y/\mathfrak{I}_Y^2$ is a locally free $\mathcal{O}_Y$-module, the dual locally free sheaf $(\mathfrak{I}_Y/\mathfrak{I}_Y^2)^*$ is called the *normal bundle* of Y in X and is denoted by $\mathcal{N}_{Y/X}$.

If $\mathrm{H}^1(Y, f^*\Theta_X)$ [*resp.* $\mathrm{H}^1(Y, \mathfrak{I}_y f^*\Theta_X)$] *vanishes, then the* k*-scheme* $\mathrm{Hom}(Y, X)$ [*resp.* $\mathrm{Hom}(Y, X; y \mapsto x)$] *is smooth at* $[f]$.

(3) *The formal neighbourhood of* $[f]$ *in* $\mathrm{Hom}(Y, X)$ [*resp.* $\mathrm{Hom}(Y, X; y \mapsto x)$] *is isomorphic to a subvariety in the vector space* $\mathrm{H}^0(Y, f^*\Theta_X)$ [*resp.* $\mathrm{H}^0(Y, \mathfrak{I}_y f^*\Theta_X)$] *defined by at most* $\dim \mathrm{H}^1(Y, f^*\Theta_X)$ [*resp.* $\dim \mathrm{H}^1(Y, \mathfrak{I}_y f^*\Theta_X)$] *equations. In particular, we have the following estimate of the local dimensions of Hom schemes*:

$$\dim_{[f]} \mathrm{Hom}(Y, X) \geq \dim \mathrm{H}^0(Y, f^*\Theta_X) - \dim \mathrm{H}^1(Y, f^*\Theta_X),$$
$$\dim_{[f]} \mathrm{Hom}(Y, X; y \mapsto x) \geq \dim \mathrm{H}^0(Y, \mathfrak{I}_y f^*\Theta_X) - \dim \mathrm{H}^1(Y, \mathfrak{I}_y f^*\Theta_X).$$

More precisely, the dimension of each irreducible component has the estimate as above at $[f]$.

(4) *When* Y *is a complete curve, we have*

$$\dim_{[f]} \mathrm{Hom}(Y, X) \geq -\deg f^*\mathrm{K}_X + (\dim X)\chi(Y, \mathcal{O}_Y),$$
$$\dim_{[f]} \mathrm{Hom}(Y, X; y \mapsto x) \geq -\deg f^*\mathrm{K}_X + (\dim X)\big(\chi(Y, \mathcal{O}_Y) - 1\big).$$

The statements (1) and (2) essentially follow from Corollary 1.3 together with the definition of the Hom schemes. For details of the proof, see [Gro, Exposé III, (5.6)]. The statement (3) is derived from the analysis of the obstruction spaces $\mathrm{H}^1(Y, f^*\Theta_X)$, $\mathrm{H}^1(Y, \mathfrak{I}_y f^*\Theta_X)$ (see [Mo1, Section 1]). Once (3) is established, Riemann-Roch for vector bundles on curves yields (4).

C. Bend and Break

In this subsection, everything is defined over an algebraically closed field k of arbitrary characteristic.

Let C be a smooth projective curve, X a projective variety, and Δ a smooth curve with a smooth compactification Δ^c. Let $p_1, p_2 \in C$ be two distinct k-valued points.

Let $\tilde{f} : \Delta \times C \to X$ be a morphism and let $f_s : C \to X$ stand for the restriction of $\tilde{f}$ to $\{s\} \times C$, where s is a k-valued point of Δ. Given $s \in \Delta(k)$, let $(f_s)_*(C)$ denote the naturally defined 1-cycle on X.

Theorem 1.8 (Mori's Bend and Break [Mo1]). *In the notation as above, assume that* $\dim \tilde{f}(\Delta \times C) = 2$ *and that* $\tilde{f}(\Delta \times \{p_1\})$ *is a single point. Then we can find a boundary point* $s_\infty \in \Delta^c \setminus \Delta$ *such that the limiting cycle*

$$\lim_{s \to s_\infty} (f_s)_*(C) \in \mathrm{Chow}(X)$$

contains a rational curve as an irreducible component. Assume in addition that C *is* $\mathbb{P}^1$ *and that* $\tilde{f}(\Delta \times \{p_2\})$ *is also a point. Then some limit cycle is either reducible or nonreduced.*

Proof. The morphism $\tilde{f} : \Delta \times C \to X$ does not lift to a morphism from $\Delta^c \times C$ to X. Indeed, if it did, the extended morphism $\tilde{f}^c : \Delta^c \times C \to X$ would map the curve $D_1 = \Delta^c \times \{p_1\}$ to a single point. Let H be an ample divisor on X with support away from $\tilde{f}^c(D_1)$. Since $\tilde{f}^c$ has two-dimensional image, $(\tilde{f}^{c*}H)^2 > 0$, while $(\tilde{f}^{c*}H, D_1) = 0$. Then the Hodge index theorem says that D_1^2 is negative, which is obviously not the case (D_1 is the product $\Delta^c \times \{p_1\}$).

Thus we have to replace $\Delta^c \times C$ by a suitable blown up surface Y in order to extend $\tilde{f}$ to a morphism $\tilde{f}^c : Y \to X$. Pick up $(s_\infty, p) \in \Delta^c \times C$ at which the surface Y is blown up. By construction, Y has a fibration $\pi : Y \to \Delta^c$, the fibre over s_∞ is a union of C and several copies of $\mathbb{P}^1$, and at least one of the copies is mapped onto a rational curve on X. This proves the first statement.

Let us prove the second statement. In the above notation, put $\pi^*(s) = \sum m_{s,i} Y_{s,i}$, which is an effective Cartier divisor on the smooth blown up surface Y. If t is general in Δ^c, then $\pi^{-1}(t) = C \simeq \mathbb{P}^1$, so that $\sum m_{s,i}(f_s)_* Y_{s,i}$ is a limit cycle of $(f_t)_*(C)$. Suppose that $\sum m_{s,i}(f_s)_* Y_{s,i}$ is an irreducible reduced cycle for every $s \in \Delta^c$. Then, for each $s \in \Delta^c$, there is a unique irreducible component $Y_{s,i}$ with non-constant map to X, and its coefficient $m_{s,i}$ must be one. Renumber the indices so that this unique component is $Y_{s,0}$. Since the extra components $Y_{s,i}$, $i > 0$ are mapped to single points, we may blow down these components to keep $\tilde{f}^c$ still well defined on the blown down surface Y'.

The condition $m_{s,0} = 1$ guarantees that the resulting Y' is smooth. Indeed, for the reducible fibre $\sum m_{s,i} Y_{m,i}$, we see:

(1) $K_Y Y_{s,i} < 0$ if and only if $Y_{s,i}$ is (-1)-curve; and
(2) $K_Y(\sum m_{s,i} Y_{s,i}) = -2$.

From this observation it follows that $\bigcup Y_{s,m}$ contains a (-1)-curve $\neq Y_{s,0}$, and we can smoothly contract this extra curve. Repeating this process, we eventually arrive at the smooth geometric ruled surface Y' which preserves the distinguished component $Y_{s,0}$.

On the surface Y' thus obtained, the closure of $\Delta \times \{p_i\}$ gives a section D_i', which is projected to a single point on X. Similarly as before, the pullback of an ample divisor $\tilde{f}^{c*}H$, viewed as a divisor on Y', has positive self-intersection and is disjoint from the D_i'. Thus the two distinct sections D_i' on the complete geometric ruled surface $Y' \to \Delta^c$ have negative self-intersection. It is well known, however, that a geometric ruled surface carries at most one effective curve of negative self-intersection (see e.g. [BaPeVV]). This contradiction shows the second statement. Q.E.D.

2. Unsplitting family of rational curves

Definition 2.1. Let X be a projective variety.

A *rational curve* on X is by definition the image $C = f(\mathbb{P}^1)$ of a generically one-to-one morphism $f : \mathbb{P}^1 \to X$. Since the morphism f is naturally recovered from the image C by identifying the normalization of C with $\mathbb{P}^1$, the morphism f itself is sometimes called a rational curve.

Let S be an irreducible closed subvariety of $\mathrm{Chow}(X)$ and F the associated universal cycle parameterized by S. The family $F \subset S \times X$ (or the parameter space S when there is no danger of confusion) is said to be a *dominant family* if the natural projection $\mathrm{pr}_X : F \to X$ is surjective. F is a *family of rational curves* parameterized by S if a general point $s \in S$ represents an irreducible, reduced rational curve. Any closed point s of a family of rational curves represents a cycle supported by a union of finitely many rational curves. Let $S\langle x\rangle \subset S$ denote the closed subset parameterizing the cycles passing through a point $x \in X$. A family of rational curves $F \to S$ is said to be *unsplitting at* x if every point of $s \in S\langle x\rangle$ represents an irreducible, reduced rational curve. Given an open subset $U \subset X$, we say that F (or S by abuse of terminology) is *unsplitting on* U if F is unsplitting at every $x \in U$. When F is unsplitting at every point $x \in X$ or, equivalently, when every $s \in S$ represents an irreducible, reduced rational curve, F (or S) is simply called an *unsplitting family.*

If a rational curve C has the minimum degree (with respect to an arbitrary fixed polarization)

$$\min\deg = \min\{\deg \Gamma; \Gamma \subset X \text{ is a rational curve}\},$$

then an arbitrary family of rational curves $F \subset S \times X \to S$ which contains C as a closed fibre is necessarily unsplitting. If $C \ni x$ has the minimum degree among the rational curves passing through x, i.e., if

$$\deg C = \min\deg\langle x\rangle = \min\{\deg \Gamma\,;\, \Gamma \subset X \text{ is a rational curve through } x\},$$

then F is unsplitting at x.[9]

Theorem 1.8 asserts that any nontrivial family of irreducible rational curves splits at x_i whenever the family has two base points $x_1, x_2 \in X$.

An unsplitting family of rational curves has extremely simple structure after taking the normalization:

[9] Actually, if $\deg C < 2\min\deg$ or $\deg C < 2\min\deg\langle x\rangle$, then F is unsplitting globally or at x.

Theorem 2.2 (Kollár [Kol, Theorem II. 2.8]). *Let $\mathrm{pr}_S : F \to S$ be a locally projective family of irreducible, reduced rational curves. Let $\overline{F}$ and $\overline{S}$ denote the normalizations of F and S, respectively, with a naturally induced projection $\overline{\mathrm{pr}}_{\overline{S}} : \overline{F} \to \overline{S}$. Assume that a general fibre of $\overline{\mathrm{pr}}_{\overline{S}}$ is smooth.*[10] *Then $\overline{\mathrm{pr}}_{\overline{S}}$ is a smooth morphism with every fibre isomorphic to $\mathbb{P}^1$. More precisely, $\overline{F}$ is an étale $\mathbb{P}^1$-bundle over $\overline{S}$.*

Proof. Let $s_0 \in \overline{S}$ be an arbitrary closed point and we check that $\overline{\mathrm{pr}}_{\overline{S}}$ is smooth over s_0. Since the statement is of local nature, we may replace $\overline{S}$ by an arbitrary étale neighbourhood of s_0. Let $s_0' \in S$ be the image of s_0. By our hypothesis that every fibre of pr_S is equidimensional and reduced, the projections pr_S and hence $\overline{\mathrm{pr}}_{\overline{S}}$ are smooth at the generic points of the closed fibres $\mathrm{pr}_S^{-1}(s_0')$, $\overline{\mathrm{pr}}_{\overline{S}}^{-1}(s_0)$ ([Ko, I.6.5]), and we may further assume that $\overline{\mathrm{pr}}_{\overline{S}}$ admits three disjoint sections σ_i over $\overline{S}$. Because a general fibre of $\overline{\mathrm{pr}}_{\overline{S}}$ is smooth $\mathbb{P}^1$, these three sections give rise to a (uniquely determined) rational map $f : \overline{F} \dashrightarrow \mathbb{P}^1$ which sends σ_i to $p_i \in \mathbb{P}^1$, $(p_1, p_2, p_3) = (0, 1, \infty)$. For a general point $s \in \overline{S}$, we have $\overline{F}_s \simeq \mathbb{P}^1$ and the graph Γ_s of $f|_{\overline{F}_s}$ is a well defined 1-cycle, thereby determining a rational map $\phi : \overline{S} \dashrightarrow \mathrm{Chow}(\overline{F} \times \mathbb{P}^1)$. By taking a suitable birational modification $S^\star \to \overline{S}$, we obtain a morphism $\phi^\star : S^\star \to \mathrm{Chow}(\overline{F} \times \mathbb{P}^1)$ and hence a family $\Gamma^\star \to S^\star$ of effective 1-cycles on $\overline{F} \times \mathbb{P}^1$. Let $\sigma_i^\star : S^\star \to F^\star = S^\star \times_{\overline{S}} \overline{F}$ be the disjoint three sections induced by σ_i. By construction, $\Gamma^\star$ contains the sections $\{(\sigma_i^\star(s^\star), p_i)\}_{s^\star \in S^\star}$.

The two projections $\Gamma^\star_{s^\star} \to F^\star_{s^\star}$ and $\Gamma^\star_{s^\star} \to \mathbb{P}^1$ are both of mapping degree one for each $s^\star \in S^\star$, and hence $\Gamma^\star_{s^\star} \subset F^\star_{s^\star} \times \mathbb{P}^1$ is either (a) a union of fibres of the two projections or (b) a graph of a birational morphism $g_{s^\star} : \mathbb{P}^1 \to F^\star_{s^\star} = \overline{F}_s$, where $s \in \overline{S}$ is the image of $s^\star \in S^\star$.

The subset $\Gamma^\star_{s^\star} \subset \overline{F}_s \times \mathbb{P}^1$ cannot contain all three sections $(\sigma_i^\star(s^\star), p_i)$ in the former case (a), and only the second case (b) occurs. Furthermore $g_{s^\star}$ depends only on $s \in \overline{S}$. Indeed, the source $\mathbb{P}^1$ and the target $\overline{F}_s$ depends only on s, while $g_{s^\star}(p_i) = \sigma_i^\star(s^\star) = \overline{\sigma}_i(s)$, $i = 1, 2, 3$. Hence $g_{s_1^\star} = g_{s_2^\star}$ if $s_1^\star$ and $s_2^\star \in S^\star$ lie over the same point $s \in S$. Thus the morphism $S^\star \to \mathrm{Chow}(\overline{F} \times \mathbb{P}^1)$ descends to a morphism $\overline{S} \to \mathrm{Chow}(\overline{F} \times \mathbb{P}^1)$ by Zariski's Main Theorem [Ha2, Corollary 11.4], and the associated relative 1-cycle Γ over $\overline{S}$ determines a birational morphism $\overline{S} \times \mathbb{P}^1 \to \overline{F}$, which is obviously finite. Since $\overline{F}$ is normal, we apply Zariski's Main Theorem once more to conclude that $\overline{F}$ and $\overline{S} \times \mathbb{P}^1$ are mutually isomorphic (over an étale neighbourhood of s_0). Q.E.D.

[10]This condition is automatic if the characteristic of the ground field is zero.

This in particular means the following

Corollary 2.3. *Assume that* char $k = 0$. *Let* $S \subset \mathrm{Chow}(X)$ *be a locally closed subset such that the associated family* $F \to S$ *consists of irreducible, reduced rational curves on* X. *Let* $\overline{F}$, $\overline{S}$ *and* $\overline{\mathrm{Hom}}(\mathbb{P}^1, X)$ *be the normalizations of* F, S *and* $\mathrm{Hom}(\mathbb{P}^1, X)$. *Then there exist a normal scheme* $\overline{M}$, *a finite surjective birational morphism* $\nu : \overline{M} \to M$ *onto a locally closed subset* $M \subset \mathrm{Hom}(\mathbb{P}^1, X)$ *and a commutative diagram*

$$\begin{array}{ccc} \overline{M} \times \mathbb{P}^1 & \xrightarrow[\Psi]{} & \overline{F} \\ \downarrow{\scriptstyle \mathrm{pr}_{\overline{M}}} & & \downarrow{\scriptstyle \mathrm{pr}_{\overline{S}}} \\ \overline{M} & \xrightarrow[\psi]{} & \overline{S} \end{array}$$

which makes $\overline{M} \times \mathbb{P}^1 \to \overline{M}$ *equivariant* $\mathrm{Aut}(\mathbb{P}^1)$*-torsors (equivariant principal* $\mathrm{Aut}(\mathbb{P}^1)$*-bundles, in other words) over* $\overline{F} \to \overline{S}$.

Proof. In general, given a normal variety Z and a morphism $h : Z \to \mathrm{Hom}(\mathbb{P}^1, X)$, there is a natural morphism $\psi : Z \to \mathrm{Chow}(X)$ by the correspondence $z \mapsto f_*([\mathbb{P}^1])$, where $[f] = h(z)$.

The algebraic group $\mathrm{Aut}(\mathbb{P}^1)$ naturally acts on $\mathrm{Hom}(\mathbb{P}^1, X)$ from the left by $g([f]) = [f \cdot g^{-1}]$, $f \in \mathrm{Hom}(\mathbb{P}^1, X)$, $g \in \mathrm{Aut}(\mathbb{P}^1)$. If $M \subset \mathrm{Hom}(\mathbb{P}^1, X)$ is an $\mathrm{Aut}(\mathbb{P}^1)$-stable subset, then $\mathrm{Aut}(\mathbb{P}^1)$ acts also on $\overline{M}$, the normalization of M. It is clear that $\psi(g([f])) = \psi([f])$.

Given a family $F \to S$ as above, the normalization $\overline{F}$ is an étale $\mathbb{P}^1$-bundle over $\overline{S}$, so that the universal property of the Hom scheme tells us that there is an étale local morphism $\sigma_\iota : \overline{S} \to \mathrm{Hom}(\mathbb{P}^1, X)$ if we fix an étale local trivialization $\iota : \overline{F} \xrightarrow{\sim} \overline{S} \times \mathbb{P}^1$. If ι and ι' are two local trivializations, then there exists an $\overline{S}$-automorphism j of $\overline{S} \times \mathbb{P}^1$ such that $\iota' = \iota \circ j$, so that $\mathrm{Aut}_{\overline{S}}(\overline{S} \times \mathbb{P}^1)\overline{\sigma}_\iota$ is independent of the choice of local trivializations. Thus, by defining M to be the orbit $\mathrm{Aut}_{\overline{S}}(\overline{S} \times \mathbb{P}^1)(\sigma_\iota(\overline{S})) \subset \mathrm{Hom}(\mathbb{P}^1, X)$, we have a surjection $\psi : \overline{M} \to \overline{S}$, closed fibres being isomorphic to $\mathrm{Aut}(\mathbb{P}^1)$. Q.E.D.

From the observation above and Mori's Bend and Break (Theorem 1.8), we derive the following dimension estimate for unsplitting families:

Proposition 2.4. *We assume that* char $k = 0$. *Let* X *be a projective variety with a closed point* x *and a closed subset* $Z \subset X$ *off* x. *Let* $\mathrm{pr}_S : F \to S \subset \mathrm{Chow}(X)$ *be a closed family of rational curves parameterized by an irreducible variety* S, *and* pr_X *the natural second projection. Let* $F\langle x\rangle \to S\langle x\rangle$ *[resp.* $F\langle x, Z\rangle \to S\langle x, Z\rangle$*] denote the closed subfamily* *consisting of curves passing through x [resp. both x and Z]. Assume that $S\langle x\rangle$ is non-empty. Then:*

(1) *We have a general dimension estimate*

$$\dim S \leq \dim\ S\langle x\rangle + \dim\ \mathrm{pr}_X(F) - 1,$$
$$\dim S\langle x\rangle \leq \dim\ S\langle x, Z\rangle + \dim\ \mathrm{pr}_X(F\langle x\rangle) - \dim\big(Z \cap \mathrm{pr}_X(F\langle x\rangle)\big) - 1,$$

Moreover, there are open subsets $U \subset \mathrm{pr}_X(F)$, $U' \subset \mathrm{pr}_X(F\langle x\rangle)$ such that, if $x \in U$ and $Z \cap \mathrm{pr}_X F\langle x\rangle \subset U'$, the above inequalities become eualities.

(2) *If $F \to S$ is unsplitting at x, then we have*

$$\dim\ S\langle x, Z\rangle \ \leq\ \dim\big(Z \cap \mathrm{pr}_X(F\langle x\rangle)\big),$$
$$\dim S \leq \dim \mathrm{pr}_X(F) + \dim \mathrm{pr}_X(F\langle x\rangle) - 2 \leq 2\dim X - 2.$$

Furthermore the projection $\mathrm{pr}_X : F\langle x\rangle \to X$ is finite over $X \backslash \{x\}$ (under the condition that F is unsplitting at x).

Proof. Take a closed point $x' \in Z$. By definition,

$$S\langle x\rangle = \mathrm{pr}_S(\mathrm{pr}_X^{-1}(x)),$$
$$S\langle x, x'\rangle = \mathrm{pr}_S\big((\mathrm{pr}_X|_{F\langle x\rangle})^{-1}(x')\big),$$

and the inequalities in (1) follow from standard dimension count. Under the unsplitting condition, the Bend-and-Break theorem (Theorem 1.8) shows that $\dim S\langle x, x'\rangle \leq 0$ for each $x' \in Z$, whence follows the first inequality in (2). When Z is a point x', substitute $\dim S\langle x, Z\rangle$ by 0 in the inequalities in (1), and we get the second inequality in (2). The fibre of $\mathrm{pr}_X|_{F\langle x\rangle}$ over x' is essentially $S\langle x, x'\rangle$ and hence finite. Q.E.D.

A family $F \to S$ of rational curves parameterized by an irreducible closed variety $S \subset \mathrm{Chow}(X)$ is said to be *maximal* if there is no family $F' \to S'$ of rational curves such that $S' \subset \mathrm{Chow}(X)$ is irreducible and that $S' \supsetneqq S$. In view of Corollary 2.3, $F \to S$ is maximal if and only if there is an irreducible component $M \subset \mathrm{Hom}(\mathbb{P}^1, X)$ such that S is the natural image of $\overline{M}$.

Corollary 2.5 (Fujita, Ionescu [Io, Theorem 0.4], Wiśniewski [W, Theorem 1.1]). *Let C be an irreducible, reduced rational curve on an n-dimensional smooth projective variety X over a field of characteristic zero. Take a maximal family of rational curves $F \to S$ which contains C as a fibre. If $(C, -\mathrm{K}_X) > \dim \mathrm{pr}_X(F) + \dim \mathrm{pr}_X(F\langle x\rangle) - n + 1$, then $F \to S$ is a splitting family.*

Proof. Let $f : \mathbb{P}^1 \to X$ be the composite of the normalization $\mathbb{P}^1 \to C$ and the embedding $C \hookrightarrow X$. The maximality condition on

S means that S is the image of an irreducible component $M \ni [f]$ of $\mathrm{Hom}(\mathbb{P}^1, X)$ and, in particular,

$$\dim S = \dim_{[f]} M - 3.$$

Then the estimate (Theorem 1.7) tells us that

$$\dim_{[f]} M \geq \dim \mathrm{pr}_X(F) + \dim \mathrm{pr}_X(F\langle x\rangle) + 2,$$

so that

$$\dim_{[C]} S = \dim_{[f]} M - 3 \geq \dim \mathrm{pr}_X(F) + \dim \mathrm{pr}_X(F\langle x\rangle) - 1.$$

But Proposition 2.4(2) asserts that F must split at x under this condition. Q.E.D.

Example 2.6 (Dimension estimate of the exceptional loci of extremal contractions[11] of smooth varieties). Suppose that C is an extremal rational curve on a smooth projective variety X and that the associated extremal contraction $\mathrm{cont}_{[C]}$ is a birational morphism: $X \to Z$, with exceptional set $E \subset X$ mapped to $B \subset Z$. We may assume that $C \subset E_b = \mathrm{cont}_{[C]}^{-1}(b)$ for a general point $b \in B$ and C is a rational curve of minimum degree in E_b. Let $F \to S$ be the maximal family of rational curves on E which contains C as a closed fibre. Then, for an arbitrary closed point $x \in E_b$, the closed subset $S\langle x\rangle \subset S$ consists of rational curves on E_b, and hence $F\langle x\rangle$ is unsplitting (by the condition that $\deg C$ is minimum). Furthermore we have

$$\begin{gathered}\mathrm{pr}_X(F) \subset E,\\ \dim \mathrm{pr}(F\langle x\rangle) \leq \dim E_b = \dim E - \dim B\end{gathered}$$

for the general point x in E. Hence

$$\begin{gathered}0 < (C, -\mathrm{K}_X) \leq 2\dim E - \dim B - n + 1,\\ \dim E \geq \frac{n + \dim B}{2},\end{gathered}$$

[11] A projective, surjective, birational morphism $X \to Z$ is called an *extremal contraction* if the relative Picard number $\rho(X/Y)$ is one with the anticanonical bundle $-K_X$ relatively ample over Z. Given such a contraction, we can find a curve $C \subset X$ such that an effective 1-cycle C' on X collapes to a point on Z if and only if $\mathbb{R}[C] = \mathbb{R}[C']$ in $N_1(X)$, the numerical equivalence classes of the 1-cycles. There is an one-to-one correspondence between the *extremal ray* $\mathbb{R}_{\geq 0}[C] \subset N_1(X)$ and the extremal contraction $\mathrm{cont}_{[C]}$. It is known that an extremal ray is generated by a rational curve (*extremal rational curve*). We refer the reader to [Mo2], [Mo3] and [KM] for the theory of extremal contractions.

thus ruling out birational extremal contractions with small exceptional loci.[12]

Thus the unsplitting condition is already a tight constraint for a closed family of rational curves $F \to S$. If we impose an additional condition, however, we can say much more about the projection $\mathrm{pr}_X : F\langle x\rangle \to X$. Indeed, we have the following proposition, which is a naive prototype of the more complex argument in Section 4 below.

Proposition 2.7 (char $k = 0$). *Let X be a projective variety and x a closed point on the smooth locus of X. Let $S \subset \mathrm{Chow}(X)$ be a closed irreducible subvariety and $F \to S$ the associated universal family of cycles. Assume that $F \to S$ satisfies the following three conditions:*

(a) (unsplitting) *Each fibre F_s is a reduced irreducible rational curve on X.*

(b) (base point) *Each F_s passes through the base point $x \in X$ (i.e., $S = S\langle x\rangle$).*

(c) (smoothness at the base point) *Each F_s is smooth at x.*

Then the natural projection $\mathrm{pr}_X : F \to X$ is birational onto the image.

Proof. Take the normalization $\overline{\mathrm{pr}}_{\overline{S}} : \overline{F} \to \overline{S}$ of the fibre space $\mathrm{pr}_S : F \to S$. Let $\overline{\mathrm{pr}}_X : \overline{F} \to X$ be the natural projection. Then, by condition (c), the inverse image $\sigma_x = \overline{\mathrm{pr}}_X^{-1}(x)$ is a single nonsingular point on each fibre $\overline{F}_s$ over $s \in \overline{S}$. Thus σ_x is a well-defined section of the $\mathbb{P}^1$-bundle morphism $\overline{\mathrm{pr}}_{\overline{S}}$, and in particular defines a Cartier divisor on $\overline{F}$. By this property, $\overline{\mathrm{pr}}_X : \overline{F} \to X$ naturally lifts to a morphism $\overline{\mathrm{pr}}_{\tilde{X}} : \overline{F} \to \tilde{X} = \mathrm{Bl}_x(X)$. Let $\tilde{F}_s \subset \tilde{X}$ be the strict transform of $F_s \subset X$. It is clear that $\overline{\mathrm{pr}}_{\tilde{X}}(\overline{F}_s) = \tilde{F}_s$.

Put $Y = \overline{\mathrm{pr}}_{\tilde{X}}(\overline{F}) \subset \tilde{X}$. Let $Y^\circ \subset Y$ and $\tilde{F}_s^\circ \subset \tilde{F}_s$ be the smooth loci. Then we prove the inclusion relation

$$\overline{\mathrm{pr}}_{\tilde{X}}^{-1}\big(\tilde{F}_s^\circ \cap (Y^\circ \setminus \{x\})\big) \subset \{\overline{s}\} \times \overline{F}_s$$

where $\overline{s} \in \overline{S}$ lies over $s \in S$. Once this is proven, the assertion is more or less clear.

Note that $\tilde{F}_s^\circ \cap Y^\circ$ is locally complete intersection in Y°, so that its inverse image in $\overline{F}$ does not contain a zero-dimensional component. Thus it suffices to show that any complete curve Γ contained in $\overline{\mathrm{pr}}_{\tilde{X}}^{-1}(\tilde{F}_s)$ is of the form $\{\overline{s}\} \times \overline{F}_s$. If Γ is one of the fibre of $\overline{\mathrm{pr}}_{\overline{S}}$, then the universal

[12]In view of Theorem 0.1, we can show that the normalization of a general fibre E_b of the projection $E \to B$ is a disjoint union of finite quotients of projective $\frac{n-\dim B}{2}$-space if $\dim E$ attains the minimum possible value $\frac{n+\dim B}{2}$.

property of Chow schemes implies that the image of Γ on F must be $\{s\} \times F_s$, and the assertion readily follows.

We derive a contradiction from the hypothesis that Γ is not a fibre. Let B be the normalization of $\overline{\mathrm{pr}}_{\overline{S}}(\Gamma) \subset \overline{S}$. Consider the fibre product $\overline{F}_B = B \times_{\overline{S}} \overline{F}$, which is a geometric ruled surface over B with a distinguished section $\sigma_x : B \to \overline{F}_B$ and a projection $\phi : \overline{F}_B \to \tilde{X}$. Let $H \in \mathrm{Pic}(\tilde{X})$ be the total transform of an ample divisor on X and $E_x \subset \tilde{X}$ the exceptional divisor. Then $\phi^* E_x = \sigma_x$ by construction. It is well-known that the Néron-Severi group $\mathrm{NS}(\overline{F}_B)$ is freely generated by the section σ_x and a fibre $\mathfrak{f}$ of the ruling. If $d > 0$ denotes the mapping degree of the surjection $\Gamma \to \tilde{F}_s$, then we compute the intersection numbers on the ruled surface $\overline{F}_B$:

$$\begin{aligned}
(\mathfrak{f}, \phi^* H) &= (\overline{F}_s, \overline{\mathrm{pr}}_{\tilde{X}}^* H) = (\tilde{F}_s, H) = \deg F_s > 0, \\
(\sigma_x, \phi^* H) &= \deg(\sigma_x / E_x)(E_x, H) = 0, \\
(\Gamma, \sigma_x) &= (\Gamma, \phi^* E_x) = d(\tilde{F}_s, E_x) = d = d(\mathfrak{f}, \sigma_x) > 0, \\
(\Gamma, \phi^* H) &= d(\tilde{F}_s, H) = d \deg(F_s) = d(\mathfrak{f}, \phi^* H) > 0.
\end{aligned}$$

The first two equations show that the two divisors H and σ_x are independent in $\mathrm{NS}(\overline{F}_B) \otimes_{\mathbb{Z}} \mathbb{Q} \simeq \mathbb{Q}^{\oplus 2}$, and thus the last two equalities yield the numerical equivalence $\Gamma \approx d\mathfrak{f}$. However, an irreducible effective divisor numerically equivalent to a multiple of a fibre $\mathfrak{f}$ is necessarily a fibre, contradicting our assumption.[13] Q.E.D.

Corollary 2.5 bounds the intersection number $(C, -\mathrm{K}_X)$ from above under the condition that C cannot be deformed to split off extra components. The dominance condition on F gives a bound in the opposite direction.

Theorem 2.8 (char $k = 0$). *Let* $\mathrm{pr}_S : F \to S$ *be a family of rational curves on a projective variety* X *and* C *an irreducible, reduced member of the family. Fix a general closed point* x *on* X.

(1) *Assume that* F *is dominant and that* $\mathrm{pr}_X(C)$ *lies on the smooth locus of* X. *Then we have the inequality*

$$(C, -\mathrm{K}_X) \geq \dim \mathrm{pr}_X(F\langle x \rangle) + 1.$$

[13] We have actually proved something a little stronger than the statement of Proposition 2.7: $\overline{\mathrm{pr}}_{\tilde{X}} : \overline{F} \to \tilde{Y} = \overline{\mathrm{pr}}_{\tilde{X}}(\overline{F}) \subset \tilde{X} = \mathrm{Bl}_x(X)$ is an isomorphism over the smooth locus $\tilde{U}^\circ$ of an open neighbourhood $U \subset \tilde{Y}$ of the exceptional divisor $E_x \cap \tilde{Y}$.

If, in addition, $F \to S$ *is closed, maximal and unsplitting at* x, *we have the equality*

$$(C, -\mathrm{K}_X) = \dim \mathrm{pr}_X(F\langle x\rangle) + 1 = \dim F\langle x\rangle + 1.$$

(2) *Let* $F \to S$ *be a closed, maximal, dominant family of rational curves on* X. *Assume that*

a) F *is unsplitting at* x, *that*
b) C *is a general fibre of* $F \to S$ *and that*
c) $\mathrm{pr}_X(C)$ *lies on the smooth locus of* X *and passes through* x.

Let $f : \mathbb{P}^1 \to X$ *be the composite of the normalization morphism* $\mathbb{P}^1 \to C$ *and the projection* $\mathrm{pr}_X : C \to X$. *Then*

$$f^*\Theta_X \simeq \mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus e-1} \oplus \mathcal{O}^{\oplus n-e},$$

where $e = \dim F\langle x\rangle = \dim S\langle x\rangle + 1$.

Proof. Since the intersection number is invariant under the flat deformation, we may assume that C is a general member of the family in order to compute $(C, -K_X)$. Consider the normalization $\overline{\mathrm{pr}}_{\overline{S}} : \overline{F} \to \overline{S}$ of the fibration with the second projection $\overline{\mathrm{pr}}_X : \overline{F} \to X$. Let $\overline{C} \simeq \mathbb{P}^1$ be the normalization of C. We view $\overline{C}$ as a general fibre over $\overline{S}$. Then $\overline{F}$ is étale locally a product $\overline{S} \times \mathbb{P}^1$. Hence we have a natural homomorphism between tangent spaces $\Theta_{\overline{C}\times\overline{S}} \to \overline{\mathrm{pr}}_X^*\Theta_X$, of maximal rank at a general point of $\overline{C} = \{[\overline{C}]\} \times \overline{C}$. The sheaf $\Theta_{\overline{C}\times\overline{S}}$ restricted to $\overline{C}$ is isomorphic to $\mathcal{O}(2) \oplus \mathcal{O}^{\dim S}$. In particular, if F is dominant, then $\overline{\mathrm{pr}}_X^*\Theta_X$ is semipositive on $\overline{C}$ by Sard's theorem.

Next consider the subfamily $F\langle x\rangle \to S\langle x\rangle$ and its normalization $\overline{F}\langle x\rangle \to \overline{S}\langle x\rangle$, a $\mathbb{P}^1$-bundle. The inverse image $\overline{\mathrm{pr}}_X^{-1}(x)$ in $\overline{F}\langle x\rangle$ contains a component σ which is finite and dominating over $\overline{S}\langle x\rangle$. Hence, after a suitable finite base change $W \to \overline{S}\langle x\rangle$, we get a section $\sigma_0 : W \to \overline{F}_W = W \times_{\overline{S}} \overline{F}$ which covers the multisection $\sigma \subset \overline{F}\langle x\rangle$. Then, étale locally, $\overline{F}_W = W \times \overline{C}_0$, $\sigma_0 = W \times \{\infty\}$, where $\overline{C}_0$ is the fibre of a general point of $\overline{S}\langle x\rangle$. Hence we have a natural morphism $W \to \mathrm{Hom}(\mathbb{P}^1, X; \infty \mapsto x)$. The differential of the associated morphism $W \times \mathbb{P}^1 \to X$ has rank equal to $\dim \mathrm{pr}_X(F\langle x\rangle)$ at a general point of C_0 and fits into the commutative diagram

$$\begin{array}{ccc} \mathrm{pr}_W^*\Theta_W|_{C_0} & \longrightarrow & \overline{\mathrm{pr}}_X^*\Theta_X(-\sigma_0)|_{C_0} \\ \downarrow & & \downarrow \\ \Theta_{W\times\overline{C}_0}|_{C_0} \simeq \mathrm{pr}_W^*\Theta_W|_{C_0} \oplus \Theta_{C_0} & \longrightarrow & \overline{\mathrm{pr}}_X^*\Theta_X|_{C_0}, \end{array}$$

where the vertical arrows stand for natural inclusion maps. Since F is dominant over X, we may assume that x is general and that $C = C_0$. Thus we conclude that the vector bundle $\mathcal{E} = \overline{\mathrm{pr}}_X^* \Theta_X|_{\overline{C}}$ on $\overline{C} \simeq \mathbb{P}^1$ satisfies:

(1) $\mathcal{E}$ is a direct sum of line bundles $L_1, \dots, L_n$;
(2) Each summand L_i has nonnegative degree;
(3) There are are at least $e = \dim \mathrm{pr}_X(F\langle x \rangle)$ summand of positive degree;
(4) There is a summand of degree ≥ 2.

Hence $(C, -\mathrm{K}_X) = \deg \Theta_X|_C = \deg \mathcal{E} \geq \dim \mathrm{pr}_X(F\langle x \rangle) + 1$. If F is unsplitting at x, we have the inequality of the converse direction by Corollary 2.5, so that $L_1 = \mathcal{O}(2)$, $L_2, \dots, L_e = \mathcal{O}(1)$, $L_{e+1}, \dots, L_n = \mathcal{O}$. Q.E.D.

3. Unsplitting families of singular rational curves and a theorem of Kebekus

In this section, every scheme or morphism is defined over an algebraically closed field k of characteristic zero.

Definition 3.1. A singular rational curve C is said to be *nodal* if the normalization $\nu : \mathbb{P}^1 \to C$ is set theoretically not bijective.[14] Any nodal rational curve is (non-canonically) a birational image of the standard nodal curve obtained by identifying two distinct points (say 0 and ∞) of $\mathbb{P}^1$. The standard nodal curve is isomorphic to the plane cubic $y^2 = x^2(x-1)$.

Similarly, C is said to have a *cusp* at $x \in C$ if there is a point $p \in \overline{C} \simeq \mathbb{P}^1$ (say ∞) such that $\mathfrak{I}_x \mathcal{O}_{\mathbb{P}^1} \subset \mathfrak{I}_p^2$, or, equivalently, if a sufficiently small analytic (or formal) neighbourhood of x in C has an irreducible branch which has multiplicity ≥ 2 at x. (Here $\mathfrak{I}_\bullet$ denotes the defining ideal of $\bullet$.) Any cuspidal curve is an image of the standard cuspidal cubic $y^2 = x^3$ by a birational morphism.

Note that a singular curve C is nodal or cuspidal at some point $x \in C$, but these two properties are not mutually exclusive.

An unsplitting family of rational curves $F \to S$ is said to be a *family of singular* [resp. *nodal, cuspidal*] rational curves if a general member (= general fibre) is a singular [resp. nodal, cuspidal] rational curve. Every member of a family of singular [resp. cuspidal] rational curves is singular [resp. cuspidal], while special members of a family of nodal curves may not be nodal.

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[14]Our definition of nodal curves is not quite standard. We do not require that the normalization map $\mathbb{P}^1 \to C$ is unramified. The curve C can be nodal and simultaneously cuspidal at a given point x.

Lemma 3.2. *Assume that* char $k = 0$.

(1) *Let $F \to S$ be an unsplitting family of nodal rational curves. Then, after a surjective finite base change $W \to S$ by a normal variety W, we can find two distinct sections $\sigma_1, \sigma_2 : W \to \overline{F}_W$, into the normalization of $F_W = W \times_S F$, such that the natural projection $\overline{F}_W \to F_W$ identifies the two sections σ_1 and σ_2.*

(2) *If $F \to S$ is an unsplitting family of cuspidal rational curves, then after a surjective finite base change $W \to S$, there exists a section $\sigma : W \to \overline{F}_W$ such that $\mathcal{O}_{F_W} \subset \mathcal{O}_W + \nu_* \mathcal{O}_{\overline{F}_W}(-2\sigma)$ via the projection $\nu : \overline{F}_W \to F_W$.*

Proof. Let Sing(F/S) denote the closed subset

$$\bigcup_{[C] \in S} \{[C]\} \times \mathrm{Sing}(C) \subset F.$$

When F is a family of singular rational curves, Sing(F/S) is finite and dominant over S. Choose an irreducible component $V \subset \mathrm{Sing}(F/S)$ which surjects onto S, and let W be the normal Galois closure of the projection $V \to S$. Then the inverse image $\Sigma \subset \overline{F}_W$ of V is a union of sections. If a general member of F is nodal, we can choose V so that Σ contains at least two distinct sections. If every member of F is cuspidal, we can choose V such that $\mathfrak{I}_V \mathcal{O}_{\overline{F}_W}$ is contained in $\mathfrak{I}_\sigma^2$ for a suitable irreducible component σ of Σ. Q.E.D.

Corollary 3.3 (Existence of singular cubic models for unsplitting family of singular rational curves [Ke1] and [Ke2]). *Let the notation be as in Lemma* 3.2. *If $F \to S$ is an unsplitting family of nodal* [resp. *cuspidal*] *rational curves, then after a finite surjective base change $W \to S$, there exists a family $\mathcal{C} \to W$ such that*

(1) *each fibre $\mathcal{C}_w$ is an irreducible singular plane cubic and a general fibre is a nodal plane cubic* [resp. *each fibre $\mathcal{C}_w$ is a cuspidal plane cubic*] *and that*

(2) *the natural projection $\overline{F}_W \to F_W$ factors through $\mathcal{C}$.*

The Bend and Break (Theorem 1.8) gives a constraint for an unsplitting family of nodal rational curves.

Proposition 3.4. *Let X be a projective variety and $S \subset \mathrm{Chow}(X)$ a closed subvariety which parameterizes an unsplitting family F of nodal rational curves on X (i.e., a general member of F is a nodal curve). Let* Node(F/S) *be the locally closed subset of nodal loci of the fibres (i.e.,* Node(F/S) *is* Sing(F/S) *minus the purely cuspidal locus). Then the natural projection* Node$(F/S) \to X$ *via* $\mathrm{pr}_X : F \to X$ *is quasi-finite. In particular,* $\dim S = \dim \mathrm{Node}(F/S) \le \dim X$.

Proof. By definition, Node(F/S) is dominant and quasi-finite over S. Assume that there is a (not necessarily complete) curve B in Node(F/S) which is contracted to a single point $x \in X$ via pr_X. Take the closure B^c of B in $F\langle x\rangle$. Its image $V \subset S$ via pr_S is a non-trivial complete curve. By the base change $V \to S$, we get a one-parameter family of nodal curves. Take a suitable finite, smooth base change $W \to V$ such that the inverse image of B in the normalization $\overline{F}_W$ of $F_W = W \times_S F$ is a union of two or more sections. Since $\overline{F}_W$ is a $\mathbb{P}^1$-bundle over the smooth curve W, it is a ruled surface with two distinct sections which are contracted to the point x. This means $\overline{F}_W$ contains two distinct curves with negative self-intersection number, which is impossible by elementary theory of ruled surfaces. Q.E.D.

Definition 3.5. Let Σ be a reduced scheme. By a *family of irreducible singular plane cubics* $\mathcal{C}$ over Σ, we mean a proper (not necessarily projective) flat morphism $\pi : \mathcal{C} \to \Sigma$ whose fibres are isomorphic to irreducible singular plane cubics. Note that, in our definition, the structure of plane cubics may change from fibre to fibre, and therefore π is not necessarily (globally) projective. A generic example of such families is indeed non-projective.

Corollary 3.3 asserts that an unsplitting family of singular curves $F \to S$ is dominated by $\mathcal{C} \to \Sigma$, a family of irreducible singular plane cubics.

In general, a one-parameter family of singular cubic curves has nodal generic fibre and several cuspidal special fibres. Only very special families have fibres of constant type. In such an exceptional case, the projectivity condition almost completely determines the global structure of the family.

Lemma 3.6 (Kebekus [Ke1] and [Ke2]). *Let* $\pi : \mathcal{C} = \{(s, C_s)\}_{s\in\Sigma} \to \Sigma$ *be a family of irreducible singular cubic curves over a smooth projective curve* Σ*. Suppose that one of the following mutually exclusive conditions is satisfied:*

(**C**) *A general member* C_s *is a cuspidal cubic, or*

(**N**) *Every member* C_s *is a nodal cubic.*

If $\mathcal{C}$ *is furthermore projective*[15] *over* Σ*, then after replacing the base curve* Σ *by a suitable finite étale cover, we can find a section* $\sigma : \Sigma \to \mathcal{C}$ *such that* $\sigma(\Sigma)$ *lies on the smooth locus of* $\mathcal{C}$*. In Case* (**N**) (*i.e., every*

[15]There are several ways to define projectivity. Our convention here is that an S-scheme Y is projective over S if there exists a line bundle L globally defined on Y which is relatively ample.

C_s is nodal), the normalization of $\mathcal{C}$ is a trivial bundle $\Sigma \times \mathbb{P}^1$ after an étale base change.

The proof of Lemma 3.6 relies on the theory of relative Picard schemes.

Given a family of irreducible singular cubic curves $\mathcal{C}$ parameterized by Σ, let

$$\mathrm{Pic}(\mathcal{C}/\Sigma) = \coprod_{d \in \mathbb{Z}} \mathrm{Pic}^d(\mathcal{C}/\Sigma)$$

denote the *relative Picard scheme*, which is a Σ-scheme-functor defined as follows. For an arbitrary Σ-scheme T, the set of the T-valued points $\mathrm{Pic}^d(\mathcal{C}/\Sigma)(T)$ consists of the equivalence classes of line bundles on $T \times_\Sigma \mathcal{C}$ of which restriction to each fibre over T has degree d. Here two line bundles L_1 and L_2 are said to be equivalent if and only if there exists a line bundle M on T such that L_2 and $\mathrm{pr}_T^* M \otimes L_1$ are mutually isomorphic as $\mathcal{O}_{T \times_\Sigma \mathcal{C}}$-modules.

The relative Picard scheme $\varpi : \mathrm{Pic}(\mathcal{C}/\Sigma) \to \Sigma$ and its open and closed subset (degree-zero part) $\varpi^0 : \mathrm{Pic}^0(\mathcal{C}/\Sigma) \to \Sigma$ are commutative group schemes over Σ, the multiplication being the tensor product. (In the following we adopt multiplicative notation for the group law of the relative Picard scheme.)

When $\mathcal{C} \to \Sigma$ is a family of irreducible singular plane cubics, the fibre $(\varpi^0)^{-1}(s)$ over a closed point $s \in \Sigma$ is either the algebraic torus $\mathbb{G}_{\mathrm{m}}$ ($= \mathbb{C}^\times$ if the ground field is $\mathbb{C}$) or the additive group scheme $\mathbb{G}_{\mathrm{a}}$ ($= \mathbb{C}$) according as C_s is nodal or cuspidal. $\mathrm{Pic}^0(\mathcal{C}/\Sigma)$ has a canonical global section $[\mathcal{O}_\mathcal{C}]$, which is the unity section with respect to the group law.

Since each fibre C_s over $s \in \Sigma$ is reduced, the fibre space $\pi : \mathcal{C} \to \Sigma$ admits an analytic local section σ_s defined on a neighbourhood of s (alternatively, a local section in étale topology). An arbitrary local section σ_s determines

(1) a local identification (in analytic or étale topology)

$$\mathrm{Pic}^d(\mathcal{C}/\Sigma) \xrightarrow{\sim} \mathrm{Pic}^0(\mathcal{C}/\Sigma)$$
$$[L] \mapsto [L \otimes \mathcal{O}(-d\sigma_s)],$$

and hence gives

(2) a natural $\mathrm{Pic}^0(\mathcal{C}/\Sigma)$-torsor structure on $\mathrm{Pic}^d(\mathcal{C}/\Sigma)$.

Let $\mathcal{U} = \{U_i\}$ be an open covering of Σ (in analytic or étale topology) and $\sigma_i : U_i \to \mathcal{C}$ a local section. It is straightforward to check that the cohomology class $\{\sigma_i^d \sigma_j^{-d}\} \in \mathrm{H}^1(\mathcal{U}, \mathrm{Pic}^0(\mathcal{C}/\Sigma))$ does not depend on the choice of the open covering $\mathcal{U}$ or on the choice of the σ_i, and thus determines an invariant (the characteristic class)

$$\begin{aligned} c(\mathrm{Pic}^d(\mathcal{C}/\Sigma)) &= c(\mathrm{Pic}^1(\mathcal{C}/\Sigma))^d \\ &\in \mathrm{H}^1_{\mathrm{ann}}(\Sigma, \mathrm{Pic}^0(\mathcal{C}/\Sigma))\ , \text{ or } \in \mathrm{H}^1_{\text{ét}}(\Sigma, \mathrm{Pic}^0(\mathcal{C}/\Sigma)) \end{aligned}$$

of the torsor. The characteristic class vanishes if and only if $\mathrm{Pic}^d(\mathcal{C}/\Sigma)$ admits a global section over Σ.

General nonsense tells us that the structure of $\mathrm{Pic}^d(\mathcal{C}/\Sigma)$ as a $\mathrm{Pic}^0(\mathcal{C}/\Sigma)$-torsor is completely determined by the characteristic class. In particular, $\mathrm{Pic}^d(\mathcal{C}/\Sigma)$ is isomorphic to $\mathrm{Pic}^0(\mathcal{C}/\Sigma)$ (i.e., a trivial $\mathrm{Pic}^0(\mathcal{C}/\Sigma)$-torsor) if and only if $\mathrm{Pic}^d(\mathcal{C}/\Sigma)$ admits a global section over Σ.

These general remarks being said, we note the following

Lemma 3.7. *Let $\pi : \mathcal{C} \to \Sigma$ be a family of irreducible singular cubic curves as above. There is a natural Σ-isomorphism between $\mathrm{Pic}^1(\mathcal{C}/\Sigma)$ and the open subset $\mathcal{C}^\circ$ of $\mathcal{C}$ consisting of the smooth points of the fibres. Given integers d and m, the natural morphism $[m]$: $\mathrm{Pic}^d(\mathcal{C}/\Sigma) \to \mathrm{Pic}^{md}(\mathcal{C}/\Sigma)$ defined by $[L] \mapsto [L^{\otimes m}]$ is surjective. Let U be an open subset of Σ. If every geometric fibre is cuspidal or every fibre is nodal over U and if $\tau : U \to \mathrm{Pic}^{dm}(\mathcal{C}/\Sigma)$ is a section, then $[m]^{-1}(\tau(U)) \subset \mathrm{Pic}^d(\mathcal{C}/\Sigma)$ is étale and finite over U.*

Proof. For a given Σ-scheme T and a given T-valued point $\sigma : T \to \mathcal{C}^\circ_T = T \times_\Sigma \mathcal{C}^\circ$, the correspondence $\sigma \mapsto [\mathcal{O}_{\mathcal{C}_T}(\sigma(T))]$ defines a natural morphism $\mathcal{C}^\circ(T) \to \mathrm{Pic}^1(\mathcal{C}/\Sigma)(T)$.

Conversely, given an line bundle L of relative degree one on $\mathcal{C}_T$, Riemann-Roch for the curve C_t of arithmetic genus one tells us that the linear system $|L|_{C_t}|$ consists of a unique effective member $\sigma(t)$, a single smooth point. This correspondence induces the inverse morphism: $\mathrm{Pic}^1(\mathcal{C}/\Sigma)(T) \to \mathcal{C}^\circ(T)$, $[L] \mapsto \sigma(t)$. Thus we have a natural isomorphism $\mathcal{C}^\circ \simeq \mathrm{Pic}^1(\mathcal{C}/\Sigma)$.

The endomorphism $[m] : \mathrm{Pic}^0(\mathcal{C}/\Sigma) \to \mathrm{Pic}^0(\mathcal{C}/\Sigma)$ is surjective. Indeed, this map is fibrewise given by $z \mapsto z^m$ on $\mathbb{G}_{\mathrm{m}}$ and by $z \mapsto mz$ on $\mathbb{G}_{\mathrm{a}}$. Then the natural $\mathrm{Pic}^0(\mathcal{C}/\Sigma)$-torsor structure on $\mathrm{Pic}(\mathcal{C}/\Sigma)$ yields the surjectivity of $[m] : \mathrm{Pic}^d(\mathcal{C}/\Sigma) \to \mathrm{Pic}^{md}(\mathcal{C}/\Sigma)$.

In order to check the final statements, we notice that there is a local isomorphism $\mathcal{C}|_V \simeq V \times C_{s_0}$ over a small analytic (or étale) open subset $V \subset U$. Then it is obvious that $[m] : \mathrm{Pic}^0(\mathcal{C}/\Sigma)|_V \to \mathrm{Pic}^0(\mathcal{C}/\Sigma)|_V$ is surjective with kernel isomorphic to μ_m (the group of the m-th roots of unity) or $\{0\}$ according as the fibres are multiplicative or additive.

Q.E.D.

Remark 3.8. If the type of the fibre of $\pi : \mathcal{C} \to \Sigma$ jumps at a closed point $s \in \Sigma$, then the kernel of $[m] : \mathrm{Pic}^0(\mathcal{C}/\Sigma) \to \mathrm{Pic}^0(\mathcal{C}/\Sigma)$ is not flat, not proper over Σ.

Assume that $\mathcal{C}_s$ is cuspidal with other fibres being nodal. Identify $\mathrm{Pic}^0(\mathcal{C}/\Sigma)$ locally with $\mathrm{Pic}^1(\mathcal{C}/\Sigma) = \mathcal{C}^\circ$. The normalization $\overline{\mathcal{C}}$ is a natural compactification of $\mathcal{C}^\circ$. The complement $D = \overline{\mathcal{C}} \setminus \mathcal{C}^\circ$ is a union of two sections meeting each other at $x \in \mathcal{C}$ over s. Then the closure of $\mathrm{Ker}[m]$ is a union of section $\sigma_0 = [\mathcal{O}_{\mathcal{C}}]$ and $m-1$ sections $\sigma_1, \dots, \sigma_{m-1} : \Sigma \to \overline{\mathcal{C}}$ which meet D at x.

Proof of Lemma 3.6. Since $\pi : \mathcal{C} \to \Sigma$ is assumed to be projective, there exists a global ample line bundle L on $\mathcal{C}$. If $\deg L|_{\mathcal{C}_s} = d$, then $[L]$ is a global section of $\mathrm{Pic}^d(\mathcal{C}/\Sigma)$. Consider the surjective morphism $[d] : \mathrm{Pic}^1(\mathcal{C}/\Sigma) \to \mathrm{Pic}^d(\mathcal{C}/\Sigma)$. The inverse image $\tilde{\Sigma} = [d]^{-1}([L]) \subset \mathrm{Pic}^1(\mathcal{C}/\Sigma)$ is finite étale over Σ by Lemma 3.7. Hence by the étale base change $\mathcal{C}_{\tilde{\Sigma}} = \tilde{\Sigma} \times_\Sigma \mathcal{C}$, we get either a single section σ or d disjoint sections $\sigma_i : \tilde{\Sigma} \to \mathrm{Pic}^1(\mathcal{C}_{\tilde{\Sigma}}/\tilde{\Sigma})$ according to the case (**C**) or (**N**). Noting that $\mathrm{Pic}^1(\mathcal{C}_{\tilde{\Sigma}}/\tilde{\Sigma})$ is naturally identified with $\mathcal{C}^\circ_{\tilde{\Sigma}}$, we are done. Q.E.D.

Lemma 3.6 has intriguing applications to the geometry of rational curves on uniruled varieties.

Corollary 3.9. *Let X be a projective variety and $x \in X$ a closed point. Let $S \subset \mathrm{Chow}(X)$ be a closed subvariety such that $S = S\langle x \rangle$ (i.e., every $s \in S$ represents an effective cycle passing through x). Assume that S is a family of unsplitting rational curves (at x) and that each element s represents a singular rational curve.*

(1) *If every $s \in S$ represents a curve $C_s \subset X$ with at least one cuspidal singularity, then each irreducible component of $\mathrm{pr}_X(\mathrm{Cusp}(F/S))$, the locus of the cuspidal singularities of the C_s in X, is either identical with the one-point set $\{x\}$ or disjoint from x.*

(2) *If no $s \in S$ corresponds to a curve C_s with cuspidal singularities, then S is a finite set.*

Proof. Assume that S is a curve. Take a suitable smooth projective curve Σ which dominates S. In case (1) or (2), we can find a family of singular cubics $\mathcal{C} \to \Sigma$, with every fibre being accordingly cuspidal or nodal, such that $\mathcal{C} \to \Sigma$ dominates the universal family $F \to S$. Let $\tilde{\mathrm{pr}}_X : \mathcal{C} \to X$ be the projection naturally induced by $\mathrm{pr}_X : F \to S$. The normalization $\tilde{\mathcal{C}} \to \Sigma$ of the family $\mathcal{C} \to \Sigma$ is a $\mathbb{P}^1$-bundle. By the unsplitting property (at x) of $F \to S$, it follows that pr_X and $\tilde{\mathrm{pr}}_X$ are finite over $X \setminus \{x\}$. Hence the one-dimensional component of $\tilde{\mathrm{pr}}_X^{-1}(x)$ is a curve with negative self-intersection and hence the unique minimal section σ_x of the geometric ruled surface $\tilde{\mathcal{C}} \to \Sigma$.

When $\mathcal{C}$ is a family of nodal cubics, Lemma 3.6 says that $\mathcal{C}$ is essentially (namely after a base change) a trivial bundle without any negative section, meaning the assertion (2).

If $\mathcal{C}$ is a family of cuspidal cubics, let $\gamma \subset \tilde{\mathcal{C}}$ be the section obtained as the inverse image of the cuspidal locus. Then, by Lemma 3.6, there exists a section $\sigma : \Sigma \to \tilde{\mathcal{C}}$ which does not meet γ, implying that the unique negative section σ_x must coincide either with σ or with γ. Consequently, γ is either away from, or identical with, σ_x, yielding (1) when $\dim S = 1$. If S has dimension two or more, we can verify the assertion by taking all curves in it. Q.E.D.

Theorem 3.10 (Kebekus [Ke1] and [Ke2]). *Let X be a projective variety of dimension n and $S \subset \mathrm{Chow}(X)$ a closed, dominant family of rational curves on X. Assume that S is unsplitting on an open subset $U \subset X$ and fix a general point $x \in U$. Then*

(1) *there is no member of S which has a cuspidal singularity at x;*

(2) *there exist only finitely many members of S which have singularity at x (the singular point x of such a member C is necessarily nodal by (1)); and*

(3) *if C is a member of S which is singular at x, then there are an n-dimensional locally closed subset $\Sigma \subset S$ and a one-dimensional locally closed subset $\Sigma\langle x\rangle \subset S\langle x\rangle$ such that $[C] \in \Sigma\langle x\rangle \subset \Sigma$ and that Σ consists of nodal curves.*

Proof. We start with the proof of (1). Suppose that for every $x \in X$ there is a member of S with a cuspidal singularity at x. Then, by Corollary 3.3, it follows that there exists a family of cuspidal cubics $\pi : \mathcal{C} \to T$ with cuspidal locus $\mathrm{Cusp}(\mathcal{C}/T)$ and morphisms $\Phi : \mathcal{C} \to F$, $\varphi : T \to S$ such that

a) the diagram

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow[\Phi]{} & F \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \mathrm{pr}_S} \\ T & \xrightarrow[\varphi]{} & S \end{array}$$

is commutative, that

b) the closed fibre $\mathcal{C}_t$ over $t \in T$ is a partial normalization of $F_{\varphi(t)}$ and that

c) the restsriction of $\Psi = \mathrm{pr}_X \Phi : \mathcal{C} \to X$ to $\mathrm{Cusp}(\mathcal{C}/T)$ is a surjective morphism onto X.

In particular, by simple dimension count, we have

$$\dim \pi(\Psi^{-1}(x)) = \dim \Psi^{-1}(x)$$
$$= \dim(\Psi^{-1}(x) \cap \mathrm{Cusp}(\mathcal{C}/T)) + 1 = \dim \pi(\Psi^{-1}(x) \cap \mathrm{Cusp}(\mathcal{C}/T)) + 1,$$

provided $x \in X$ is general. Hence we can find a pointed smooth complete curve (Σ, o) and a non-constant morphism $f : \Sigma \to \pi(\Psi^{-1}(x)) \subset T$ such that

$$f(o) \in \pi(\Psi^{-1}(x) \cap \mathrm{Cusp}(\mathcal{C}/T)),$$
$$f(\Sigma) \not\subset \pi(\Psi^{-1}(x) \cap \mathrm{Cusp}(\mathcal{C}/T)).$$

Let $g : \mathcal{C}_\Sigma \to X$ be the naturally induced morphism from the one-parameter family of plane cubics $\mathcal{C}_\Sigma = \Sigma \times_T \mathcal{C}$ to X. By construction, $g^{-1}(x)$ contains a rational section over Σ but $\pi_\Sigma(g^{-1}(x) \cap \mathrm{Cusp}(\mathcal{C}_\Sigma/\Sigma))$ is a finite set containing o, contradicting Corollary 3.9(1). This completes the proof of (1).

The assertion (2) follows from Proposition 3.4.

In order to prove (3), let T be an irreducible component of the closed subset $\subset S$, which parameterizes the singular rational curves. If there is a member $C \in S$ which is singular at a general point x, then there exists a $T \subset S$ and the associated family $G \to T$, with $\mathrm{Sing}(G/T)$ dominating X. By (1), a general point of T represents a curve without cusps on an open subset $U \subset X$. Hence the dominant morphism $\mathrm{Sing}(G/T) \to X$ is generically finite over U, so that

$$\dim T = \dim \mathrm{Sing}(G/T) = \dim U = \dim X = n,$$
$$\dim G = \dim T + 1 = n + 1,$$
$$\dim T\langle x \rangle = \dim G - \dim X = 1,$$

whence follows the statement (3). Q.E.D.

PART II. Characterizations of Projective n-Space

4. A characterization of projective n-space by the existence of unsplitting, doubly-dominant family of rational curves

In this section, we prove that a normal variety which carries an unsplitting, doubly dominant family of rational curves F parametrized by an irreducible variety S is necessarily projective space.

Given a closed point $x \in X$ and a closed family $F \to S$ of rational curves in X, put

$$S\langle x\rangle = \mathrm{pr}_S(\mathrm{pr}_X^{-1}(x)),$$
$$F\langle x\rangle = S\langle x\rangle \times_S F.$$

The fibre space $F\langle x\rangle \to S\langle x\rangle$ is nothing but the closed subfamily consisting of curves passing through x.

Lemma 4.1. (1) *A family F of rational curves on X is doubly dominant if and only if the natural projection $\mathrm{pr}_X : F\langle x\rangle \to X$ is surjective for each closed point $x \in X$. If F is doubly dominant and x is general, then the restriction of pr_X to each irreducible component of $F\langle x\rangle$ is a morphism onto X.*

(2) *If F is doubly dominant and unsplitting at a general point $x \in X$, then $\mathrm{pr}_X : F\langle x\rangle \to X$ is finite over $X \setminus \{x\}$.*

Proof. The statement (1) is a verbal rephrasing of our definition, whereas (2) was proved by Proposition 2.4(2). Q.E.D.

Our goal in this section is the following

Theorem 4.2. *Let X be a normal projective variety defined over an algebraically closed field of characteristic zero. If X carries a closed, irreducible, maximal, doubly-dominant family of rational curves $F \to S$ which is unsplitting on an open subset U (i.e., every point of $S\langle x\rangle$ represents an irreducible and reduced curve for $x \in U$), then X is a finite quotient of $\mathbb{P}^n$ by $\pi_1(X \setminus \mathrm{Sing}(X))$. If $F \to S$ is everywhere unsplitting, then X is isomorphic to a projective space $\mathbb{P}^n$.*

The proof of Theorem 4.2 consists of ten steps. From Step 1 through Step 9, we require that the doubly-dominant family $F \to S$ is unsplitting on an open subset, while in Step 10 we assume that F is everywhere unsplitting. A rough plan of our proof is as follows:

Take the normalization $\overline{F}\langle x\rangle \to \overline{S}\langle x\rangle$ of the subfamily $F\langle x\rangle \to S\langle x\rangle$ (more precisely, an irreducible component of this subfamily). This $\mathbb{P}^1$-bundle carries a distinguished section $\sigma : \overline{S}\langle x\rangle \to \overline{F}\langle x\rangle$, whose image in X is the base point x. Note that $\sigma = \sigma(\overline{S}\langle x\rangle)$ is a Cartier divisor on $\overline{F}\langle x\rangle$. The inverse image of x in $\overline{F}\langle x\rangle$ is the disjoint union of σ and a finite closed subscheme, say Δ. The monoidal transformation at Δ gives a morphism $\widetilde{\mathrm{pr}}_{\tilde{X}} : \tilde{F}\langle x\rangle \to \tilde{X} = \mathrm{Bl}_x(X)$. In Step 3, we show that $\widetilde{\mathrm{pr}}_{\tilde{X}}$ maps $\sigma \simeq \overline{S}\langle x\rangle$ birationally onto the exceptional divisor $E_x \subset \tilde{X}$. In Step 4, the projection $\overline{F}\langle x\rangle \to X$ turns out to be unramified in codimension one. From Step 5 through 8, we observe that $\overline{F}\langle x\rangle$ is a $\mathbb{P}^1$-bundle over $\mathbb{P}^{n-1}$, from which we conclude in Step 9 that $\overline{F}\langle x\rangle$ is a one-point blow up of $\mathbb{P}^n$, with X a quotient of $\mathbb{P}^n$ by a finite group. Finally in Step 10, we check that F cannot be globally unsplitting if X is a non-trivial quotient of $\mathbb{P}^n$.

Before going into the proof, we fix notation.

By $\overline{*}$, we denote the normalization of $*$. If F is unsplitting at x, then $\overline{F}\langle x\rangle$ is a $\mathbb{P}^1$-bundle over $\overline{S}\langle x\rangle$. $\overline{\mathrm{pr}}_X$ and $\overline{\mathrm{pr}}_{\overline{S}\langle x\rangle}$ stand for the natural projections from $\overline{F}\langle x\rangle$ to X and to $\overline{S}\langle x\rangle$.

Assume that F is unsplitting at a general point $x \in X$. Then by Kebekus' Theorem 3.10, almost every member C of $S\langle x\rangle$ is smooth at x, except for possibly finitely many C_i's which have nodal singularities at x. Thus the pullback of the closed subscheme $x \in X$ on the normalization of C_i is a disjoint union of reduced points. It follows that $\overline{\mathrm{pr}}_X^{-1}(x) \subset \overline{F}\langle x\rangle$ is a union of a divisor σ which is a (single-valued) section of the fibration $\overline{F}\langle x\rangle \to \overline{S}\langle x\rangle$ plus a closed subscheme supported on finitely many points on $\overline{F}\langle x\rangle$ away from σ. In what follows, the section σ is referred as the *distinguished section* (with respect to the fibration $F\langle x\rangle \to S\langle x\rangle$).

In general, the closed subset $S\langle x\rangle$ of the irreducible projective variety S could be reducible. We denote by $S_0\langle x\rangle$ an (arbitrary) irreducible component of $S\langle x\rangle$.

When a smooth curve C lies in the smooth locus of X, the normal bundle of C is denoted by $\mathcal{N}_{C/X}$.

Given a closed smooth point z of a variety Z, let $\mu_z : \mathrm{Bl}_z(Z) \to Z$ be the blowing-up at z and let $E_z \subset \mathrm{Bl}_z(Z)$ stand for the associated exceptional divisor. $\mathrm{Bl}_x(X)$ is usually denoted by $\tilde{X}$ in order to simplify the notation.

Let us begin the proof of Theorem 4.2.

Step 1. *Let C be a general member of $S\langle x\rangle$. Then C is smooth and lies on the smooth locus of X with normal bundle $\mathcal{N}_{C/X}$ isomorphic to $\mathcal{O}(1)^{n-1}$. $S\langle x\rangle$ is smooth at $[C]$ and there exists an open (in Zariski topology) neighbourhood $V \subset S\langle x\rangle$ of $[C]$, such that the restriction of $\mathrm{pr}_X : F \to X$ to $V \times_S F$ naturally lifts to an étale morphism to* $\mathrm{Bl}_x(X)$.

Proof. By Theorem 3.10, almost every $C \in S\langle x\rangle$ is smooth at x. By Proposition 2.4(2), we have $\dim S\langle x, \mathrm{Sing}(X)\rangle \leq \dim \mathrm{Sing}(X)$. Since X is normal of dimension n, the singular locus of X has dimension $\leq n-2$. Hence $S\langle x, \mathrm{Sing}(X)\rangle$ has dimension $\leq n-2$, while the dominant family $S\langle x\rangle$ has dimension $n-1$. Thus a general member C is smooth at x and is off $\mathrm{Sing}(X)$. In particular, the inverse image of $x \in X$ via the projection $\overline{\mathrm{pr}}_X : \overline{F}\langle x\rangle \to X$ is, near the general fibre $\overline{C} \subset \overline{F}\langle x\rangle$, exactly the distinguished section σ, which is a Cartier divisor. Hence, around $\overline{C}$, we can naturally lift $\overline{\mathrm{pr}}_X$ to a morphism $\overline{\mathrm{pr}}_{\tilde{X}}$ to $\tilde{X} = \mathrm{Bl}_x(X)$.

Since $x \in X$ is general and $F\langle x\rangle$ is dominant, Theorem 2.8(2) applies to show that $\overline{\mathrm{pr}}_X^* \Theta_X|_{\overline{C}} \simeq \mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus n-1}$. Noticing that C is smooth at x and that $\overline{\mathrm{pr}}_X$ is of maximal rank at a general point of $\overline{C}$, we infer that

$$\overline{\mathrm{pr}}_{\tilde{X}}^* \Theta_{\tilde{X}}|_{\overline{C}} \simeq \mathcal{O}(2) \oplus \mathcal{O}^{\oplus n-1} = \Theta_{\overline{F\langle x\rangle}}|_{\overline{C}}.$$

This means that $\overline{\mathrm{pr}}_{\tilde{X}}$ is an analytic isomorphism near $\overline{C}$ and therefore that C is smooth outside x as well as at x. Q.E.D.

Step 2. *Let $T \to \overline{S}\langle x\rangle$ be a non-constant morphism from a smooth curve. Put $\overline{F}_T = T \times_{\overline{S}\langle x\rangle} \overline{F}\langle x\rangle$ and let $\Phi : \overline{F}_T \to X$ and $\overline{\mathrm{pr}}_T : \overline{F}_T \to T$ be the natural projections. Then we have*

$$J_x = \mathfrak{I}_x \mathcal{O}_{\overline{F}_T} = \mathfrak{I}_{\sigma_T} \prod_i \mathfrak{I}_i,$$

where σ_T is a section $T \to \overline{F}_T$ and $\mathfrak{I}_i$ is the defining ideal of a zero-dimensional closed subscheme supported by a point p_i away from σ_T.

Furthermore, around p_i, there exists a local coordinate system (z_1, z_2) such that $\mathfrak{I}_i = (z_1, z_2^{m_i})$, $m_i \in \mathbb{Z}_{>0}$. In particular, the monoidal transformation $\nu : \tilde{F}_T \to \overline{F}_T$ with respect to the ideal $J_x \subset \mathcal{O}_{\overline{F}_T}$ gives a normal variety with at worst rational double points of type A_{m_i-1}. The ideal $\mathfrak{I}_i$ defines an exceptional Cartier divisor $m_i\tau_i$ over $p_i \in \overline{F}_T$, where $\tau_i \subset \tilde{F}_T$ is a Weil divisor isomorphic to $\mathbb{P}^1$.

Assume that a closed subscheme $Z \subsetneqq \Phi(\overline{F}_T)$ satisfies the following two conditions

(1) *$Z \ni x$,*
(2) *Z transversally meets all the smooth analytic branches of C_j at x whenever C_j in $S\langle x\rangle$ has a singularity at x.*

Then we have the inclusion relations

$$\mathcal{O}_{\tilde{F}_T}(-m_i\tau_i) \supset \mathfrak{I}_Z \mathcal{O}_{\tilde{F}_T} \not\subset \mathcal{O}_{\tilde{F}_T}((-m_i - 1)\tau_i).$$

Proof. Since each fibre of $F\langle x\rangle \to S\langle x\rangle$ passes through x and has no cuspidal singularity at x, the closed subset $\Phi^{-1}(x) \subset \overline{F}_T$ cuts out a non-empty, reduced closed subset from each fibre of $\overline{F}_T \to T$. Furthermore, there are only finitely many fibres $\subset F\langle x\rangle$ that have nodal singularities at x. This implies that the subset defined by J_x is a disjoint union of a section σ_T plus a 0-dimensional subscheme away from σ_T. Let p_i be a closed point which supports a zero-dimensional connected component, with $t_i = \overline{\mathrm{pr}}_T(p_i)$ its image in T. Let (z_1, z_2) be a local coordinate system of the smooth ruled surface $\overline{F}_T$ around p_i such that the fibre $\overline{\mathrm{pr}}_T^{-1}(t_i)$ is defined by $z_2 = 0$. Then we have

$$\mathfrak{I}_i/\mathfrak{I}_i \cap (z_2) = (z_1)/(z_1 z_2) \subset \mathcal{O}_{\overline{F}_T, p_i}/(z_2)$$

because $\mathfrak{I}_i$ defines the reduced point p_i on the fibre. Hence $\mathfrak{I}_{i,p_i}$ is of the form $(z_1, z_2^{m_i})$, an ideal generated by two generators. Near p_i, the monoidal transformation ν is thus a subvariety in $\overline{F}_T \times \mathbb{P}^1$ defined by

$$z_2^{m_i} = u z_1 \text{ or } z_1 = v z_2^{m_i}$$

on two affine subsets $\simeq \overline{F}_T \times \mathbb{A}^1$. If $m_i > 1$, then $\tilde{F}_T$ has a unique singularity $z_1 = z_2 = u = 0$ of type $A_{m_i - 1}$ over p_i. The ideal $\mathfrak{I}_i = (z_1, z_2^{m_i})$ is a principal ideal generated by z_1 or $z_2^{m_i}$ on the open subsets, while the Weil divisor τ_i is defined by $z_1 = z_2 = 0$.

Put $\overline{C}_i = \overline{\mathrm{pr}}_T^{-1}(\overline{\mathrm{pr}}_T(p_i)) \subset \overline{F}_T$. Since $\Phi(p_i) = \Phi(\sigma_T \cap \overline{C}_i) = x$, the image $\Phi(C_i)$ has a nodal singularity at x. Therefore the two conditions (1) and (2) on the closed subscheme Z mean that

(1*) $\mathfrak{I}_i \supset \mathfrak{I}_Z \mathcal{O}_{\overline{F}_T}$ and that

(2*) $\mathfrak{I}_Z \mathcal{O}_{\overline{C}_i} = (z_1)$ in terms of the local coordinate as above.

These two properties amount to saying that $\mathfrak{I}_Z \mathcal{O}_{\overline{F}_T, p_i}$ contains an element of the form $(\text{unit}) z_1 + g z_2^{m_i}$, $g \in \mathcal{O}_{\overline{F}_T, p_i}$. The pullback of this element generates $z_1 \mathcal{O}_{\tilde{F}_T} = \mathcal{O}(-m_i \tau_i) \subset \mathcal{O}_{\tilde{F}_T}$ on the first affine open subset given by $z_2^{m_i} = u z_1$. Q.E.D.

Step 3. *Let $x \in X$ be a general closed point and $\tilde{X} = \mathrm{Bl}_x(X)$ the one-point blowup at x, with the exceptional divisor $E_x \subset \tilde{X}$. Fix an arbitrary irreducible component $F_0\langle x\rangle \to S_0\langle x\rangle$ of the fibre space $F\langle x\rangle \to S\langle x\rangle$. Let $\overline{\mathrm{pr}}_{\tilde{X}} : \overline{F}_0\langle x\rangle \dashrightarrow \tilde{X}$ be the dominant rational map induced by the projection $\overline{\mathrm{pr}}_X : \overline{F}_0\langle x\rangle \to X$. Blow up $\overline{F}\langle x\rangle$ along a zero-dimensional subscheme away from σ to eliminate the indeterminacy of $\overline{\mathrm{pr}}_{\tilde{X}}$ and we get a morphism $\tilde{\mathrm{pr}}_{\tilde{X}} : \tilde{F}\langle x\rangle \to \tilde{X} = \mathrm{Bl}_x(X)$. Let $s \in \overline{S}_0\langle x\rangle$ be a general closed point such that*

(1) *$\overline{C}_s = \overline{\mathrm{pr}}_{\overline{S}}^{-1}(s) \subset \overline{F}_0\langle x\rangle$ is mapped onto a smooth curve C_s on $X \setminus \mathrm{Sing}(X)$ with normal bundle $\mathcal{O}(1)^{\oplus n-1}$ and that*

(2) *C_s is transversal with every member C' of $S\langle x\rangle$ which is singular at x.*

Let $\tilde{C}_s \subset \tilde{X}$ be the strict transform of $C_s \subset X$. Then $\tilde{\mathrm{pr}}_{\tilde{X}}^{-1}(\tilde{C}_s)$ is scheme-theoretically a union of $\overline{C}_s$ and a closed subscheme which does not meet σ. The restriction of $\tilde{\mathrm{pr}}_{\tilde{X}}$ to σ gives a birational morphism $\overline{S}_0\langle x\rangle \simeq \sigma \to E_x \simeq \mathbb{P}^{n-1}$. (Here $\overline{C}_s$ and σ are viewed as subschemes of

$\tilde{F}_0\langle x\rangle$ since the monoidal transformation $\tilde{F}_0\langle x\rangle \to \overline{F}_0\langle x\rangle$ does not affect the neighbourhoods of $\overline{C}_s$ and σ.)

Proof. For simplicity of the notation, assume that $S\langle x\rangle$ is irreducible. For general case, one has only to put the subscript $_0$ to everything relevant.

Fix a general point $s \in \overline{S}\langle x\rangle$ and let $\overline{C} \subset \overline{F}\langle x\rangle$ be the fibre over s. $C = \overline{\mathrm{pr}}_X(\overline{C}) \subset X$ is an everywhere smooth rational curve through x and off $\mathrm{Sing}(X)$. It is easy to check that C satisfies the conditions (1) and (2) above.

The smooth curve C is locally complete intersection on X, so that $\overline{\mathrm{pr}}_X^{-1}(C)$ is a union of the distinguished section σ and purely one-dimensional components (because $\overline{\mathrm{pr}}_X$ is finite over $X \setminus \{x\}$). One of the one-dimensional components is the trivial one $\overline{C} = B_0$. Let B_j, $j = 1, 2, \ldots$ be the extra one-dimensional components. Then we have the following claim

Claim. *The extra components $B_j \subset \overline{F}\langle x\rangle$ $(j \geq 1)$ do not meet the distinguished section σ.*

Assume for a while that this claim is true. Blow up $\overline{F}\langle x\rangle$ at the finitely many points p_i so that we have a well-defined morphism $\tilde{\mathrm{pr}}_{\tilde{X}} : \tilde{F}\langle x\rangle \to \tilde{X} = \mathrm{Bl}_x(X)$, a process which does not affect open neighbourhoods of σ and of $\overline{C}$ in $\overline{F}\langle x\rangle$.

Let $\tilde{C} \subset \tilde{X}$ denote the strict transform of $C \subset X$. Then, by construction, $\tilde{\mathrm{pr}}_{\tilde{X}}^{-1}(\tilde{C})$ is the union of $\tilde{C}$ and the strict transforms $\tilde{B}_j$ of B_j, $j = 1, 2, \ldots$. In particular, the inverse image of $E_x \cap \tilde{C}$ in $\sigma \simeq \tilde{\sigma} \subset \tilde{F}\langle x\rangle$ is

$$\tilde{\sigma} \cap \left(\tilde{C} \cup \tilde{B}_1 \cup \tilde{B}_2 \cup \cdots\right) = \tilde{\sigma} \cap \overline{C},$$

a single point. Thus the projection $\tilde{\mathrm{pr}}_{\tilde{X}} : \sigma = \tilde{\sigma} \to E_x$ is generically one-to-one.

This shows that the assertion we want to prove follows from Claim above.

Proof of Claim. Fix an arbitrary component B_j and denote it by B for the sake of simplicity of notation. Let Γ be the normalization of $\overline{\mathrm{pr}}_{\overline{S}\langle x\rangle}(B) \subset \overline{S}\langle x\rangle$, and $\overline{F}_\Gamma$ the fibre product $\Gamma \times_{\overline{S}\langle x\rangle} \overline{F}\langle x\rangle$. $\overline{F}_\Gamma$ is a $\mathbb{P}^1$-bundle over the smooth curve Γ. The inverse image of $B \subset \overline{F}\langle x\rangle$ in $\overline{F}_\Gamma$ contains a unique one-dimensional irreducible component $\overline{B}_\Gamma$, dominating $C \subset X$ via the natural projection $\Phi : \overline{F}_\Gamma \to X$.

Let $\nu : \tilde{F}_\Gamma \to \overline{F}_\Gamma$ be the monoidal transformation with respect to the ideal $\mathfrak{I}_x\mathcal{O}_{\overline{F}_\Gamma}$. The naturally induced morphism $\tilde{\Phi} : \tilde{F}_\Gamma \to \tilde{X} = \mathrm{Bl}_x(X)$ is finite when restricted to $\tilde{F}_\Gamma \setminus \sigma_\Gamma$, and we have $\tilde{\Phi}^* E_x = \sigma_\Gamma + \sum m_i \tau_i$.

(Since ν is an isomorphism near the section σ_Γ, we refer by σ_Γ also its inverse image in $\tilde{F}_\Gamma$.)

Consider the commutative diagram

$$\begin{array}{ccccc} \tilde{F}_\Gamma & \xrightarrow[\tilde{\Phi}]{} & \tilde{X} = \mathrm{Bl}_x(X) & \longleftarrow & \tilde{C} \\ \downarrow{\scriptstyle \nu} & & \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \simeq} \\ \overline{F}_\Gamma & \xrightarrow[\Phi]{} & X & \longleftarrow & C. \end{array}$$

By Step 2, we have

$$(*) \qquad \mathcal{O}_{\tilde{F}_\Gamma}(-\sigma_\Gamma - \sum m_i\tau_i) \supset \mathfrak{I}_C\mathcal{O}_{\tilde{F}_\Gamma} \not\subset \mathcal{O}_{\tilde{F}_\Gamma}(-\sigma_\Gamma - \sum m_i\tau_i - \tau_{i_0})$$

for any irreducible component τ_{i_0} of the exceptional divisor.

The ideal sheaf $\mathfrak{I}_C\mathcal{O}_{\overline{F}_\Gamma} \subset \mathcal{O}_{\overline{F}_\Gamma}$ is of the form $\mathfrak{q}(-D - \sigma_\Gamma)$, where D is an effective Cartier divisor on the smooth surface $\overline{F}_\Gamma$ and $\mathfrak{q} \subset \mathcal{O}_{\overline{F}_\Gamma}$ is an ideal defining a zero-dimensional closed subscheme. Thus the Cartier divisor ν^*D is necessarily of the form $\tilde{D} + \sum a_i\tau_i$, where a_i is a non-negative integer $\le m_i$ (this is true because the Weil divisor τ_i is irreducible and reduced). By construction, $\overline{B}_\Gamma \subset \overline{F}_\Gamma$ is contained in D and so is its strict transform $\tilde{B}_\Gamma$ in $\tilde{D}$.

Let d be the mapping degree of the finite cover $\tilde{\Phi}|_{\tilde{D}} : \tilde{D} \to \tilde{C}$.

Since $(\tilde{C}, E_x) = 1$ by the smoothness of C at x, we have

$$d = (\tilde{D}, \tilde{\Phi}^*E_x) = (\tilde{D}, \sigma_\Gamma) + \sum(\tilde{D}, m_i\tau_i)$$

or, equivalently,

$$0 \le (\tilde{D}, \sigma_\Gamma) = d - \delta, \quad \delta = \sum(\tilde{D}, m_i\tau_i) = -\sum a_i m_i \tau_i^2.$$

(The intersection number $\tau_i^2 \in \mathbb{Q}$ is well defined because $m_i\tau_i$ is Cartier.) Noting that $\tilde{D} = \nu^*D - \sum a_i\tau_i$ and that the exceptional divisors τ_i are disjoint from σ_Γ, we obtain

$$(D, \sigma_\Gamma) = (\tilde{D}, \sigma_\Gamma) = d - \delta,$$

which yields the numerical equivalence

$$(**) \qquad D \approx d\mathfrak{f} + \frac{\delta}{-\sigma_\Gamma^2}\sigma_\Gamma,$$

where $\mathfrak{f}$ denotes a fibre of the ruling $\mathrm{pr}_\Gamma : \overline{F}_\Gamma \to \Gamma$.

Indeed, this is a direct consequence of (a) the fact that the Néron-Severi group of $\overline{F}_\Gamma$ is freely generated by $\mathfrak{f}$ and σ_Γ, together with (b) the following table of intersection numbers:

$$\begin{aligned}
(\mathfrak{f}^2) = 0, \quad (\mathfrak{f}, \sigma_\Gamma) = 1, \quad (\sigma_\Gamma^2) < 0, \\
(\mathfrak{f}, \Phi^* H) = (C, H) > 0, \\
(\sigma_\Gamma, \Phi^* H) = \deg(\sigma_\Gamma \to \Phi(\sigma_\Gamma))(\Phi(\sigma_\Gamma), H) = 0, \\
(D, \sigma_\Gamma) = d - \delta = d(\mathfrak{f}, \sigma_\Gamma) - \delta, \\
(D, \Phi^* H) = d(C, H) = d(\mathfrak{f}, \Phi^* H) > 0.
\end{aligned}$$

Here H denotes an ample divisor on X.

From the numerical equivalence (**) we deduce

$$\begin{aligned}
\frac{\delta}{-\sigma_\Gamma^2} \geq 1 \in \mathbb{Z}, \\
D^2 = \frac{2d\delta - \delta^2}{-\sigma_\Gamma^2}.
\end{aligned}$$

Then we get

$$\tilde{D}^2 = D^2 + (\sum a_i \tau_i)^2 \geq D^2 - \delta \geq D^2 - \delta \frac{\delta}{-\sigma_\Gamma^2} = 2(d - \delta) \frac{\delta}{-\sigma_\Gamma^2} \geq 0.$$

(Here we used the fact that $\tau_i^2 < 0, a_i^2 \leq a_i m_i$, $\tau_i \tau_j = \delta_{ij} \tau_i^2$.) The curve $B_\Gamma \subset \overline{F}_\Gamma$, which is an irreducible component of D, is away from σ_Γ if $d - \delta = (D, \sigma_\Gamma) = 0$. Hence Claim above reduces to the inequality $\tilde{D}^2 \leq 0$, which we derive from the following observation.[16]

Since C is general, the normal bundle $\mathcal{N}_{C/X}$ is of the form $\mathcal{O}(1)^{\oplus n-1}$ and so $\mathcal{N}_{\tilde{C}/\tilde{X}}$ is trivial. Let $\hat{X} \to \tilde{X}$ be the blow-up along $\tilde{C}$, with exceptional divisor $E_{\tilde{C}} \subset \hat{X}$. The triviality of the normal bundle means that the divisor $E_{\tilde{C}}|_{E_{\tilde{C}}}$ on $E_{\tilde{C}}$ is seminegative. Perform blowing-ups

[16]What we use below is the seminegativity of $\mathcal{N}_{\tilde{C}/\tilde{X}}$ rather than its triviality. If we start from a dominant family of rational curves unsplitting at a general point $x \in X$, then $\mathcal{N}_{\tilde{C}/\tilde{X}}$ is always seminegative by virtue of Theorem 2.8(2). Thus the above Claim as well as the statement of this step applies to a fairly wide class of families of rational curves.

$\hat{F}_\Gamma \to \tilde{F}_\Gamma$ to have a commutative diagram

$$\begin{array}{ccc} \hat{F}_\Gamma & \xrightarrow[\hat{\Phi}]{} & \hat{X} \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} \\ \tilde{F}_\Gamma & \xrightarrow[\tilde{\Phi}]{} & \tilde{X}. \end{array}$$

Then $\hat{\Phi}^* E_{\tilde{C}} = \alpha^* \tilde{D} + A$, where A is an effective divisor lying over finitely many points on $\overline{F}_\Gamma$. Thus

$$\tilde{D}^2 = (\alpha^* \tilde{D})^2 \le (\alpha^* \tilde{D}, \hat{\Phi}^* E_{\tilde{C}}) \le 0,$$

because $\hat{\Phi}^* E_{\tilde{C}}$ restricted to $\alpha^* \tilde{D}$ is semi-negative.

This completes the proofs of Claim and of Step 3 as well. Q.E.D.

Step 4. *The projection* $\overline{\mathrm{pr}}_X : \overline{F}_0\langle x\rangle \to X$ *is unramified over* $X \setminus (\mathrm{Sing}(X) \cup \{x\})$. *In particular, there exists a proper finite morphism* $Y \to X$, *étale over* $X \setminus \mathrm{Sing}(X)$ *and a birational morphism* $\overline{F}_0\langle x\rangle \to Y$ *which factor the morphism* $\overline{\mathrm{pr}}_X$.

Proof. In order to prove the assertion, take a (unique) normal birational modification $\lambda : F_0^\sharp\langle x\rangle \dashrightarrow \overline{F}_0\langle x\rangle$ such that $\mathrm{pr}_X : \overline{F}_0\langle x\rangle \to X$ lifts to a finite morphism

$$\mathrm{pr}^\sharp_{\tilde{X}} : F_0^\sharp\langle x\rangle \to \tilde{X} = \mathrm{Bl}_x(X).$$

(Such a modification is constructed by first blowing up $\overline{F}_0\langle x\rangle$ and then taking the Stein factorization with respect to the projection onto $\tilde{X}$.) Since $\overline{F}_0\langle x\rangle \to X$ is finite over $X \setminus \{x\}$, we have the identity $F_0^\sharp\langle x\rangle = \overline{F}_0\langle x\rangle$ outside the inverse images of $x \in X$.

The strict transform $\sigma^\sharp$ of the distinguished section σ is a subvariety in $F_0^\sharp\langle x\rangle$ which is finite and birational over E_x, and hence isomorphic to $E_x \simeq \mathbb{P}^{n-1}$ by Zariski's Main Theorem.

Recall that $\sigma^\sharp$ is a connected component of $(\mathrm{pr}^\sharp_{\tilde{X}})^{-1} E_x$. Furthermore, the normal variety $F_0^\sharp\langle x\rangle$ is smooth in codimension one and so is it at a general point of $\sigma^\sharp$. In particular, the Cartier divisor $\mathrm{pr}^{\sharp *}_{\tilde{X}} E_x$ is of the form $a\sigma^\sharp$ locally near a general point of $\sigma^\sharp$ and hence globally on a neighbourhood of $\sigma^\sharp$. On the other hand, the strict transform $C^\sharp \subset F_0^\sharp\langle x\rangle$ of a general fibre $\overline{C} \subset \overline{F}_0\langle x\rangle$ satisfies

$$(C^\sharp, \sigma^\sharp) = (\overline{C}, \sigma) = 1,$$
$$(C^\sharp, \mathrm{pr}^{\sharp *}_{\tilde{X}} E_x) = (\mathrm{pr}^\sharp_{\tilde{X}}(\tilde{C}), E_x) = 1.$$

Thus $a = 1$ and we have $\mathrm{pr}_{\tilde{X}}^{\sharp *} E_x = \sigma^\sharp$ near $\sigma^\sharp$, showing that $\mathrm{pr}_{\tilde{X}}^{\sharp *}$ is unramified near $\sigma^\sharp$.

By Step 1, $\overline{\mathrm{pr}}_X$ is unramified on $\big(U \cup \overline{\mathrm{pr}}_{\overline{S}_0\langle x\rangle}^{-1}(W)\big) \setminus \sigma$, where $W \subset \overline{S}_0\langle x\rangle$ is an open dense subset. This implies that $\overline{\mathrm{pr}}_X|_{(\overline{F}_0\langle x\rangle\setminus\sigma)}$ is unramified in codimension one, or equivalently, $\overline{\mathrm{pr}}_X$, finite over $X \setminus \{x\}$, is unramified over $X \setminus$ (subset of codimension ≥ 2), and we have the assertion by the purity of the branch loci. Q.E.D.

Step 5. *$\overline{S}_0\langle x\rangle$ is a smooth variety birational to $\mathbb{P}^{n-1}$. In particular, the $\mathbb{P}^1$-bundle $\overline{F}_0\langle y\rangle$ and Y defined in Step 4 are both smooth and simply connected.*

Proof. Let $U \subset \overline{F_0}\langle x\rangle$ be a small open neighbourhood of σ. $U \setminus \sigma$ is unramified over $X \setminus \{x\}$ and hence smooth. Furthermore, for each $s \in \overline{S}_0\langle x\rangle$, the fibre $\overline{C}_s = \overline{\mathrm{pr}}_S^{-1}(s)$ is a smooth $\mathbb{P}^1$. Therefore, for each point $p \in \overline{C}_s \cap (U \setminus \sigma_0)$, there is a smooth $(n-1)$-dimensional analytic (or étale) slice $\overline{S}_p^\dagger$ which cuts out p from $\overline{C}_s$, inducing an analytic local isomorphism $(\overline{S}_0\langle x\rangle, s) \simeq \overline{S}_p^\dagger$. Thus $\overline{S}_0\langle x\rangle$ is everywhere smooth.

In the proof of Step 3, we have checked that $\overline{S}_0\langle x\rangle$ was birational to $\mathbb{P}^{n-1}$, and hence $\pi_1(\overline{S}_0\langle x\rangle) \simeq \pi_1(\mathbb{P}^{n-1}) = (1)$ because of the birational invariance of the fundamental group of smooth projective varieties. Y is birational to the smooth variety $\overline{F}_0\langle x\rangle$. Furthermore, $Y -$ (the finitely many smooth points over $x \in X$) is isomorphic to an open subset of $\overline{F}_0\langle x\rangle$, so that Y is smooth and hence simply connected. Q.E.D.

Step 6. *Let Y be the smooth variety constructed above via $\overline{F}_0\langle x\rangle$. Then Y is a compactification of the universal cover of $X \setminus \mathrm{Sing}(X)$ by finitely many varieties and hence independent of the choice of the base point $x \in X$. Thus a point y lying over a general point x is again a general point of Y, and $\overline{F}\langle x\rangle \to \overline{S}\langle x\rangle$ defines a dominant unsplitting family of rational curves on Y through a general closed point $y \in Y$.*

Proof. For each fibre $\overline{C} \simeq \mathbb{P}^1$, the morphism $\overline{\mathrm{pr}}_Y|_{\overline{C}}$ is generically one-to-one because so is $\overline{\mathrm{pr}}_X|_{\overline{C}}$. Hence $\{C_Y\} = \{\overline{\mathrm{pr}}_Y(\overline{C})\}$ is a closed, dominant unsplitting family of rational curves (perhaps non-effectively) parameterized by $\overline{S}\langle x\rangle$. The image of σ in Y is a single point y because it is irreducible and finite over x. This implies the assertion. Q.E.D.

Step 7. *Put*

$$G_0 = \{(s, \overline{\mathrm{pr}}_Y(\overline{C}_s); s \in \overline{S}_0\langle x\rangle\} \subset \overline{S}_0\langle x\rangle \times Y.$$

Then we have $\overline{F}_0\langle x\rangle \simeq G_0$; i.e., every $\overline{\mathrm{pr}}_Y(\overline{C_s})$ is smooth.

Proof. The projection $\overline{\mathrm{pr}}_Y : \overline{F}_0\langle x\rangle \to Y$ factors through G_0 by construction. Since $\overline{\mathrm{pr}}_Y$ is birational and finite over $Y \setminus \{y\}$, we have $\overline{F}_0\langle x\rangle \simeq G \simeq Y$ over $Y \setminus \{y\}$. Thus every $\overline{\mathrm{pr}}_Y(\overline{C}_s) \subset Y$ must be smooth off the base point y.

If, however, some $\overline{\mathrm{pr}}_Y(\overline{C}_s)$ has a singularity at a general point y, then Theorem 3.10(3) asserts that there is a one-parameter subfamily of nodal curves $\{\overline{\mathrm{pr}}_Y(\overline{C}_t)\}$ with moving nodal locus, which contradicts what we have just seen. Hence $\overline{\mathrm{pr}}_Y(\overline{C}_s)$ is smooth also at y. Q.E.D.

Step 8. *Let $\overline{\mathrm{pr}}_Y : \overline{F}_0\langle x\rangle \to Y$ be the birational morphism as above. Then $\overline{S}_0\langle x\rangle \simeq \overline{\mathrm{pr}}_Y^{-1}(y) \subset \overline{F}_0\langle x\rangle$ is isomorphic to $\mathbb{P}^{n-1}$ and $\overline{F}_0\langle x\rangle \simeq$ $\mathrm{Bl}_y(Y)$.*

Proof. Since every fibre $\overline{C} = \overline{C}_s$ is isomorphically mapped onto a curve through $y = \overline{\mathrm{pr}}_Y(\sigma)$ in Y, we have a natural lift $\overline{\mathrm{pr}}_{\tilde{Y}} : \overline{F}\langle x\rangle \to \tilde{Y} = \mathrm{Bl}_y(Y)$. This birational morphism induces a birational morphism $\sigma \to E_y$ $(\simeq E_x)$ and the equality $\overline{\mathrm{pr}}_{\tilde{Y}}^* E_y = \sigma$. Hence $\overline{\mathrm{pr}}_{\tilde{Y}}$ is unramified at a general point of σ, and in the same time unramified on $U \setminus \sigma$, where U is an open neighbourhood of σ. This shows that $\overline{\mathrm{pr}}_{\tilde{Y}}$ is unramified in U by the purity of the ramification locus on smooth varieties. Thus we have a isomorphism between U and an open neighbourhood of E_y in Y, inducing $\sigma \simeq E_y$. In particular, $\overline{F}\langle x\rangle$ is birational, finite over $\mathrm{Bl}_y(Y)$, and hence isomorphic to $\mathrm{Bl}_y(Y)$ by Zariski's Main Theorem. Q.E.D.

Step 9. *$Y \simeq \mathbb{P}^n$ and $\overline{\mathrm{pr}}_Y(\overline{C})$ is a line. In other words, X is the finite quotient $\mathbb{P}^n/G$ and the unsplitting rational curve $C \subset X$ is the image of a line $\subset \mathbb{P}^n$, where $G = \pi_1(X \setminus \mathrm{Sing}(X))$.*

Proof. $\overline{F}_0\langle x\rangle \simeq \mathrm{Bl}_y(Y)$ is a $\mathbb{P}^1$-bundle over $\overline{S}_0\langle x\rangle \simeq \mathbb{P}^{n-1}$. Since the distinguished section σ is the exceptional divisor $E_y \simeq \mathbb{P}^{n-1}$ via the isomorphism $\overline{F}_0\langle x\rangle \simeq \mathrm{Bl}_y(Y)$, we have $\mathcal{O}_\sigma(\sigma) \simeq \mathcal{O}(-1)$. Thus the natural exact sequence

$$0 \to \mathcal{O}_{\overline{F}_0\langle x\rangle} \to \mathcal{O}_{\overline{F}_0\langle x\rangle}(\sigma) \to \mathcal{O}_\sigma(\sigma) \to 0$$

and the pushforward by $\mathrm{pr}_{\overline{S}_0\langle x\rangle}$ induce the exact sequence

$$0 \to \mathcal{O}_{\overline{S}_0\langle x\rangle} \to \mathrm{pr}_{\overline{S}_0\langle x\rangle\,*}\mathcal{O}_{\overline{F}_0\langle x\rangle}(\sigma) \to \mathcal{O}_{\overline{S}_0\langle x\rangle}(-1) \to 0.$$

This means that $\mathrm{Bl}_y(Y) \simeq \overline{F}_0\langle x\rangle$ is the projective bundle $\mathbb{P}_{\mathbb{P}^{n-1}}(\mathcal{O} \oplus \mathcal{O}(-1))$ with σ being $\mathbb{P}_{\mathbb{P}^{n-1}}(\mathcal{O}(-1))$, implying that Y is isomorphic to

$\mathbb{P}^n$.[17] Q.E.D.

We have thus completed the proof of Theorem 0.2, while Theorem 0.1 follows from

Step 10. *$X = Y \simeq \mathbb{P}^n$ if $F \to S$ is everywhere unsplitting.*

Proof. Since $Y \setminus$ (finite set) is the universal cover of $X \setminus \mathrm{Sing}(X)$, the fundamental group $G = \pi_1(X \setminus \mathrm{Sing}(X))$ naturally acts on $Y \simeq \mathbb{P}^n$ and the normal variety X is the finite quotient under this action.

Suppose that G is non-trivial. Take an arbitrary element $g \neq 1 \in G \subset \mathrm{Aut}(\mathbb{P}^n) = \mathbf{PGL}(n+1,k)$. Then $g \in \mathbf{PGL}(n+1,k)$, an element of finite order, can be diagonalized with at least two distinct eigenvalues λ_1, λ_2. Choose eigenvectors $v_1, v_2 \in \mathbb{C}^{n+1}$ corresponding to the eigenvalues. The two dimensional space $\mathbb{C}v_1 + \mathbb{C}v_2$ defines a $\langle g \rangle$-stable line C_0 in $\mathbb{P}^n \simeq Y$, on which g acts non-trivially.

Recall that $F\langle x \rangle$ was a family of lines in $Y = \mathbb{P}^n$ passing through y lying over $x \in X$. It follows, then, that each point of S is the image of a line $\subset Y$ in X. In particular, S can be viewed as a closed subset of the quotient of the Grassmann variety $\mathrm{Grass}(\mathbb{P}^n, 1)$ by the action of G. Since S is a doubly dominant family of curves, we have

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[17] A more elementary and direct proof is the following: By pulling back a hyperplane h on $\overline{S}_0\langle x \rangle$, we get a base-point-free effective divisor $\tilde{H}_x$ on $\overline{F}_0\langle x \rangle \simeq \tilde{Y} = \mathrm{Bl}_y(Y)$. $\tilde{H}_x|_\sigma$ is a hyperplane in $E_x \simeq \sigma \simeq \overline{S}_0\langle x \rangle$, so that $\tilde{H}_x \simeq \mu_Y^* H_x - E_y$, where H_x is an effective divisor on Y. The obvious equality $\tilde{H}_x^n = h^n = 0$ on $\overline{F}_0\langle x \rangle = \mathrm{Bl}_y(Y)$ then gives $H_x^n = 1$. The free $(n-1)$-dimensional linear system $|\tilde{H}_x|$ on $\tilde{Y}$ can be viewed as a linear subsystem with a unique base point $y \in Y$ of the complete linear system $|H_x|$ on Y. Viewed as a linear system on Y, the base locus of the bigger linear system $|H_x|$ is contained in $y \in Y$ lying over $x \in X$. By moving around the prescribed base point $x \in X$, we get a new linear subsystem $|\tilde{H}_{x'}|$ of $|H_y|$ on Y with a single base point y' over $x' \neq x$. The two divisors H_x and $H_{x'}$ are obviously algebraically equivalent and hence linearly equivalent by the vanishing of the irregularity $q(Y)$ (it is birational to a $\mathbb{P}^1$-bundle over $\mathbb{P}^{n-1}$). Thus $|H_x| = |H_{x'}| = |H|$. Take a general member $D \in |\tilde{H}_{x'}|$ and consider the linear system $\Lambda \subset |H|$ spanned by D and $|\tilde{H}_x|$. Λ is an n-dimensional linear system free from base points and hence gives rise to a morphism $\phi_\Lambda : Y \to \mathbb{P}^n$ of mapping degree $H^n = 1$. This birational morphism ϕ_Λ is finite and hence an isomorphism by Zariski's Main Theorem. To see this, it suffices to check that $(\Gamma, H) > 0$ for an arbitrary effective curve Γ on Y. The strict transform $\tilde{\Gamma} \subset \tilde{X} \simeq \overline{F}_0\langle x \rangle$ is a curve not contained in $E_y = \sigma$. Hence $(\Gamma, H) = (\tilde{\Gamma}, \tilde{H} + \sigma) \geq (\tilde{\Gamma}, \tilde{H})$, the third term being positive unless $\tilde{\Gamma}$ is a fibre $\overline{C}$ over a point in $\overline{S}_0\langle x \rangle$. For a fibre $\tilde{\Gamma} = \overline{C}$, we have $(\overline{\mathrm{pr}}_Y(\overline{C}), H) = (\overline{C}, \sigma) = 1$, so that the image of $\overline{\mathrm{pr}}_Y(\overline{C}) \subset Y$ in $\mathbb{P}^n$ is a line.

$\dim S \geq 2n - 2 = \dim \mathrm{Grass}(1, \mathbb{P}^n)$, so that $S = \mathrm{Grass}(\mathbb{P}^n, 1)/G$, a $2(n-1)$-dimensional irreducible variety. Consequently the image of C_0 is represented by a point $\in S$, while $\mathrm{pr}_X(([C_0], C_0)) \subset X$ factors through the quotient $C_0/\langle g \rangle$. Hence $\mathrm{pr}_X|_{C_0} : C_0 \to X$ is not generically one-to-one, contradicting the global unsplitting property of the family. Q.E.D.

Example 4.3. The everywhere unsplitting condition on $F \to S$ is really necessary in order to characterize projective spaces. The easiest example is constructed in dimension two as follows.

The cyclic group $G = \mathbb{Z}/(p)$ of order p, an odd prime number, effectively acts on $Y = \mathbb{P}^2$ via the diagonal action $\mathrm{diag}(1, \exp \frac{2\pi\sqrt{-1}}{p}, \exp \frac{-2\pi\sqrt{-1}}{p})$ of a generator. If $y \in Y$ is general, then the orbit $G(y) \subset Y$ is not collinear, or, equivalently, no line passing through y is G-stable. This means that the images of the lines on Y in $X = Y/G$ form a closed, doubly-dominant family $F \to S$ of rational curves on X which is unsplitting at a general point. The line connecting y and $g(y)$ is mapped to a nodal curve on X, so that there are exactly $\frac{p-1}{2}$ points of S which represent curves with nodes at a fixed general point x.

Around the fixed point $(1:0:0) \in \mathbb{P}^2$, the quotient morphism $\mathbb{P}^2 \to X$ is given by

$$(s,t) \mapsto (st, s^p, t^p) \in X = \{(u,v,w); vw = u^p\}$$

in terms of affine coordinates. Hence a general line $\{(1 : s : as)\}_{s \in \mathbb{C} \cup \infty}$ passing through $(1,0,0)$ is mapped to the curve $\{(as^2, s^p, a^p s^p)\}$ with a $(2,p)$-cusp. Thus there are finitely many cuspidal curves $\in S$ passing through a given general base point x.

This construction easily carries over to higher dimension (for instance, consider $X = \mathbb{P}^n/G$, $G = \mathbb{Z}/(p)$, where p is a prime number $\geq n+1$), showing that

a) There are lot of singular quotients of $\mathbb{P}^n$ which carry doubly dominant families of rational curves unsplitting at a general point, and also that
b) The dimension estimates (Theorem 3.10(1) – (3)) in Kebekus' theorem are optimal in general.

5. Various characterizations of projective spaces

In this section, we derive from Main Theorem 0.1 various characterizations of projective spaces given in Corollary 0.4.

For the proof of Corollary 0.4, let us begin with trivial implications:

(a) The condition $X \simeq \mathbb{P}^n$ implies all the other conditions;

(b) (Length condition) $\Rightarrow$ (Length condition for rational curves) $\Rightarrow$ (Length condition for rational curves with base point); and
(c) (double dominance condition for rational curves) $\Leftrightarrow$ (dominance condition for rational curves with base point) $\Rightarrow$ (double dominance condition for rational curves of minimum degree).

Furthermore we know the implication relations

(d) (Frankel-Siu-Yau condition) $\Rightarrow$ (Hartshorne-Mori condition) $\Rightarrow$ (Mori condition).

Indeed, as is well known, the positivity of the holomorphic bisectional curvature yields the ampleness of the tangent bundle, while from the ampleness of tangent bundle Θ_X (or from a weaker condition that $-\mathrm{K}_X$ is ample) follows the uniruledness of X.

Our Main Theorem asserts that the double dominance of an unsplitting family implies $X \simeq \mathbb{P}^n$, while

(e) a family of curves of minimum degree is always unsplitting,

and hence (double dominance condition of rational curves of minimum degree) implies $X \simeq \mathbb{P}^n$.

Thus only the following implications remain to be checked:

(f) (Hirzebruch-Kodaira-Yau condition) $\Rightarrow$ (Kobayashi-Ochiai condition) $\Rightarrow$ (Length condition);
(g) (Mori condition) $\Rightarrow$ (Length condition for rational curves);
(h) (doubly transitive group action) $\Rightarrow$ (Mori condition);
(i) (Remmert-Van de Ven-Lazarsfeld condition) $\Rightarrow$ (Length condition for rational curves with base point) $\Rightarrow$ (double dominance condition for rational curves of minimum degree).

In what follows, we check the implications above one by one. Almost everything is an easy exercise except for the proof of the implication (Hirzebruch-Kodaira-Yau) $\Rightarrow$ (Kobayashi-Ochiai), where we need the topological invariance of some Chern numbers plus the characterization of ball quotients due to S.-T. Yau.

Proof of (Hirzebruch-Kodaira-Yau) $\Rightarrow$ (Kobayashi-Ochiai): Assume that X is homotopic to $\mathbb{P}^n$. Noting that X is simply connected and complex projective, we have $\mathrm{Pic}(X) \simeq H^2(X,\mathbb{Z}) \simeq H^2(\mathbb{P}^n,\mathbb{Z}) \simeq \mathbb{Z}$. The first Chern class $c_1(X)$ can be written as mh, where m is an integer and h is the positive generator of $\mathrm{Pic}(X)$. The Chern number $c_1^n(X) = m^n$ is a homotopy invariant up to sign [Hi], so that $m = \pm(n+1)$. The Kobayashi-Ochiai condition is thus satisfied modulo the positivity of m. Suppose that m were negative. Then K_X would be ample and hence X would carry a Kähler Einstein metric [Y1], [Y2] and [Au]. The Chern number $c_1^{n-2}(2(n+1)c_2 - nc_1^2)$ is again a homotopy invariant (up to sign) and hence zero because $X \approx \mathbb{P}^n$. By Chen-Ogiue-Yau's result [CO], [Y1] and [Y2], this would imply that the universal cover of X is the open unit ball $B_n = \mathbf{SU}(1,n)/\mathbf{S}(\mathbf{U}(1) \times \mathbf{U}(n))$, contradicting the assumption that the compact manifold X is simply connected. Q.E.D.

Proof of (Kobayashi-Ochiai) $\Rightarrow$ (Length): Since $-\mathrm{K}_X$ is ample and divisible by $n+1$, it follows that $(C, -\mathrm{K}_X) \geq n+1$ for any effective curve on X. Q.E.D.

Proof of (Doubly transitive group action) $\Rightarrow$ (Mori): A complex Lie group which holomorphically acts on X with a fixed point is necessarily a linear algebraic group, a rational variety (a non-trivial action of a complex torus cannot have fixed points). Hence the orbit space X is uniruled (actually unirational). Let $C \subset X$ be an irreducible curve and $x \in C$ a smooth point. A doubly transitive action of $\mathrm{Aut}(X)$ gives rise to vector fields on X with zero at a given point x. This shows that $\Theta_X \otimes \mathcal{O}_C(-x)$ is generated by global sections. Hence $\Theta_X|_C$ is ample. Q.E.D.

Proof of (Mori) $\Rightarrow$ (Length of rational curves): Let $f : \mathbb{P}^1 \to X$ be a morphism, which is birational onto its image. If $\Theta_X|_{f(\mathbb{P}^1)}$ is ample, then so is $f^*\Theta_X$. Thus $f^*\Theta_X = \mathcal{O}(d_1) \oplus \cdots \oplus \mathcal{O}(d_n), d_i \geq 1$. Since there is a non-zero natural homomorphism $\mathcal{O}(2) \simeq \Theta_{\mathbb{P}^1} \to f^*\Theta_X$, some d_i must be at least 2. Hence $(C, -\mathrm{K}_X) = \deg f^*\Theta_X = \sum d_i \geq n+1$. Q.E.D.

Proof of (Remmert-Van de Ven-Lazarsfeld) $\Rightarrow$ (Length of rational curves with base point): Let $g : \mathbb{P}^N \to X$ be a surjective morphism. Clearly $N \geq n = \dim X$. Take an ample divisor H on X. If $n = 0$, then there is nothing to prove. If n is positive, then g^*H cannot be trivial so that $(g^*H)^N = g^*(H^N) > 0$. This shows that $N = n$. In particular by Sard's theorem, there is a non-empty open subset $U \subset X$ such that g is unramified over U. Pick up $x_0 \in X$ from U. Let $C \subset X$ be an irreducible curve through x_0 and $(\tilde{C}, \tilde{x_0}) \subset \mathbb{P}^n$ a pointed irreducible curve such that $g(\tilde{C}) = C, g(\tilde{x}_0) = x_0$. We have a natural homomorphism $\Theta_{\mathbb{P}^n}|_{\tilde{C}} \to (g|_{\tilde{C}})^*\Theta_X$, which is an isomorphism at $\tilde{x}_0$ and hence globally injective. Since $\Theta_{\mathbb{P}^n}$ is ample, so is $(g|_{\tilde{C}})^*\Theta_X$. Thus $\Theta_X|_C$ is ample and, if C is rational, $\deg \Theta_X|_C = \deg \nu^*\Theta_X \geq n+1$, where $\nu : \mathbb{P}^1 \to C \hookrightarrow X$ is the normalization. Q.E.D.

Proof of (Length of rational curves with base point) $\Rightarrow$ (double dominance of rational curves of minimum degree): Let C be a rational curve passing through x_0. Assume that C has minimal degree among such curves. Let $f : \mathbb{P}^1 \to X$ be the morphism induced by the normalization. Then, if $(C, -\mathrm{K}_X) \geq n+1$, then the

$\mathrm{Hom}(\mathbb{P}^1, X; \infty \mapsto x_0)$ has dimension at least $n+1$ at $[f]$, giving rise to a family of morphisms as desired. Q.E.D.

Remark 5.1. For a singular projective variety X, the conditions (Remmert-Van de Ven-Lazarsfeld), (Double dominance of rational curves of minimum degree) and (Dominance of rational curves with base point) in Corollary 0.4 still make sense. When X is normal, Theorem 0.2 asserts that

$$\begin{aligned}(X \simeq \mathbb{P}^n) &\Leftrightarrow \text{(Double dominance of rational curves of minimum degree)}\\ &\Rightarrow \text{(Dominance of rational curves with base point)}\\ &\Rightarrow \text{(Remmert-Van de Ven-Lazarsfeld).}\end{aligned}$$

If we drop the normality condition, the third still implies the fourth because they are both of birational nature. However, the first two conditions are not mutually equivalent any more. The most trivial example is this: let $x_1, x_2 \in \mathbb{P}^n$ be two distinct point and let X be $\mathbb{P}^n$ with these two points x_1, x_2 identified (it is easy to check that X is projective).

Remark 5.2. Let us have a quick glance at the history of characterizations of projective spaces.

The Hirzebruch-Kodaira characterization [HK] (1957) was obtained (under the condition that the first Chern class c_1 is positive) as a beautiful application of the two milestones of the age: Hirzebruch's Riemann-Roch [Hi] (1953) and Kodaira vanishing [Kod] (1953). The solution of Calabi's conjecture by S.-T. Yau [Y1], [Y2] and T. E. Aubin [Au] (1978) enabled us to drop the positivity assumption on c_1. S. Kobayashi and T. Ochiai [KO] (1973) found that one can relax the diffeomorphism condition to the divisibility condition on c_1; in fact, this condition together with Kodaira vanishing completely determines[18] the Hilbert polynomial $h(t) = \chi(X, \mathcal{O}(tH))$, where H is an ample divisor such that $[H]$ is a positive generator of $\mathrm{H}^2(X, \mathbb{Z}) \simeq \mathbb{Z}$. These two earlier characterizations are topological, or rather cohomological, in nature.

In 1960s, conjectures on more geometric characterizations were proposed by T. Frankel [Fr], R. Hartshorne [Ha1], R. Remmert and A. Van de Ven [RV]. The first two conjectures were formulated in terms

[18]Indeed, we have $h(-1) = \cdots = h(-n) = 0$, $h(0) = 1$ for the polynomial h of degree n, so that $h(t) = (1/n!)(t+1)\cdots(t+n)$. In particular, we have $H^n = 1$, $\dim \mathrm{H}^0(X, \mathcal{O}(H)) = h(1) = n+1$. Similarly, if X is an n-dimensional Fano with $-K_X = nH$, then $h(t) = (-1)^n h(-n-t)$ (Serre duality), $h(0) = 1$, $h(-1) = \cdots = h(-n+1) = 0$, and hence $h(t) = (2/n!)(t+(n/2))(t+1)\cdots(t+n-1)$, $H^n = 2$, $\dim \mathrm{H}^0(X, \mathcal{O}(H)) = h(1) = n+2$.

of the positivity of tangent bundles, while the third is concerned with holomorphic images of projective spaces.

It was perceived wisdom among experts that the key to the Frankel/Hartshorne conjecture was the existence of lines, i.e., rational curves C on the variety X with $(C, -\mathrm{K}_X) = n + 1$. Y.-T. Siu and S.-T. Yau [SY] (1980) solved the Frankel conjecture by realizing lines as energy-minimizing C^∞ images of the Riemann sphere S^2 with the aid of an imposing work of Sacks-Uhlenbeck [SaU] on harmonic maps. In their proof, the positivity of the curvature tensor is essential to ensure global convergence of energy minimizing sequences, and so their method does not work under weaker conditions like the positivity of the Ricci tensor. Shortly before their work, however, S. Mori [Mo1] (1978) discovered a revolutionary technique (Bend and Break plus modulo p reduction) to produce rational curves under much milder conditions: if the canonical bundle is not nef, then we can always find rational curves [MiMo]. Once sufficiently many rational curves were found, the Hartshorne conjecture was not hard to prove any more.[19] R. Lazarsfeld [La] (1983) applied Mori's result to solve the conjecture of Remmert-Van de Ven.

The characterization in terms of the length was proposed by S. Mori and S. Mukai at Taniguchi Conference, Katata 1987, where the participants jointly compiled a list of open problems in algebraic geometry. The formulation of Theorem 0.1 in terms of the existence of a doubly-dominant, unsplitting family is presumably new. One of the cornerstones of our result is Theorem 3.10 due to S. Kebekus, who obtained the result in the spring of 2000 during his stay at RIMS, Kyoto University.

Remark 5.3. Various results on projective n-space have been extended to n-dimensional smooth hyperquadric $\mathrm{Q}_n \subset \mathbb{P}^{n+1}$, the second simplest n-fold. For instance, the following six conditions on a smooth complex Fano n-fold X $(n \geq 2)$ are known to be equivalent:

- $X \simeq \mathrm{Q}_n$;
- Brieskorn condition [Br]: X is diffeomorphic to Q_n;
- Kobayashi-Ochiai condition [KO]: $c_1(X)$ is divisible by n in $\mathrm{H}^2(X, \mathbb{Z})$;
- Siu condition [Si]: X carries a Kähler metric of semi-positive holomorphic bisectional curvature with certain non-degeneracy condition;
- Cho-Sato condition I [CS1]: X is a holomorphic image of Q_n and is not isomorphic to $\mathbb{P}^n$;

[19]Step 4 through Step 9 in the previous section essentially reproduce the proof.

- Cho-Sato condition II [CS2]: $\wedge^2\Theta_X$ is ample but Θ_X is not ample.

These are of course the counterparts of (Hirzebruch-Kodaira), (Kobayashi-Ochiai), (Frankel-Siu-Yau), (Remmert-Van de Ven-Lazarsfeld) and (Hartshorne-Mori) for projective spaces, respectively. In view of the apparent parallelism above, it is quite natural to ask if our method (possibly after some minor modifications) applies to hyperquadrics. To be more specific, we conjecture that the following two conditions on X are also equivalent to the above six:

- Length condition: $\min\{(C, -\mathrm{K}_X); C \subset X \text{ is a curve}\} = n$;
- Subdouble dominance of rational curves: Let $F \to S$ be an arbitrary maximal family of rational curves on X. Then, F is a dominant family and, for general $x \in X$, the projection $\mathrm{pr}_X : F\langle x\rangle \to X$ has image of dimension $\geq n-1$.[20]

Thanks to the classification of Del Pezzo surfaces and Fano threefolds [Is1], [Is2] and [MoMu], the conjecture is verified up to dimension three.

PART III. Applications to Complex Symplectic Manifolds

6. Complex symplectic manifolds: generalities

This section is a concise review of the theory of complex symplectic manifolds. For proofs and further discussion, we refer the reader for example to Beauville [Bea1] and Fujiki [Fu].

Let Y be a Kähler manifold and η a d-closed[21] holomorphic 2-form on it. η defines a skew-symmetric, $\mathcal{O}_X$-bilinear pairing: $\Theta_Y \times \Theta_Y \to \mathcal{O}_Y$. When this pairing is everywhere non-degenerate, we call the pair (Y, η), or simply Y itself, a *complex symplectic manifold*, and η is said to be a *complex symplectic form* or a *complex symplectic structure*[22] on Y.

A (complex) symplectic manifold is necessarily of even dimension $2n$. The non-degeneracy condition of η is equivalent to saying that $\wedge^n\eta$ is a nowhere vanishing $2n$-form. In particular, the canonical bundle K_Y of a (complex) symplectic manifold Y is trivial. The symplectic form η gives a standard isomorphism $\Theta_Y \simeq \Omega^1_Y$.

[20] A normal (singular) hyperquadric $\subset \mathbb{P}^{n+1}$ also satisfies this condition. One could ask if finite quotients of hyperquadrics are characterized as normal, projective, uniruled varieties which satisfy the above subdouble dominance condition.

[21] As is well known, the d-closedness of η is automatic if Y is compact.

[22] We often drop the adjective "complex" when there is no danger of confusion.

Let $Z \subset Y$ be an analytic subset (closed, open, locally closed, or whatever). A subbundle[23] $\mathcal{E} \subset \Theta_Y|_Z$ is said to be *isotropic* if η identically vanishes on $\mathcal{E} \times \mathcal{E}$. The rank of an isotropic subbundle $\mathcal{E}$ does not exceed $n = (1/2)\text{rank}\, \Theta_Y$. An isotropic subbundle of rank n is called *Lagrangian.* If $\mathcal{E} \subset \Theta_Y|_Z$ is Lagrangian, η gives a non-degenerate pairing between $\mathcal{E}$ and $(\Theta_Y|_Z)/\mathcal{E}$, and thereby a natural isomorphism between $(\Theta_Y|_Z)/\mathcal{E}$ and the dual of $\mathcal{E}$.

In general, a symplectic structure η is not unique even modulo the equivalence via non-zero constant scalar multiples. For instance, if η' is any (possibly degenerate) d-closed holomorphic 2-form, $\eta + t\eta'$ is again a symplectic form if the parameter t is sufficiently close to zero. However, when $H^0(Y, \Omega_Y^2)$ is one-dimensional, there is essentially only one symplectic structure, and we call Y a *primitive symplectic manifold* in this case.

The importance of primitive complex symplectic manifolds in the framework of the classification theory of Kähler manifolds is illustrated by the following

Theorem 6.1 ("Bogomolov decomposition" due to Berger [Ber]-Yau [Y2]-Bogomolov [Bo]-Beauville [Beal]). *Let W be a compact Kähler manifold whose canonical class $-c_1(W) \in \mathrm{H}^2(W, \mathbb{Q})$ is zero. Then there exist a finite étale cover $\tilde{W} \to W$, a Ricci-flat Kähler metric g on $\tilde{W}$ and a Riemannian decomposition $\tilde{W} \simeq A \times \prod Y_i \times \prod Z_j$ such that*

(a) *A is a flat complex torus (with trivial holonomy); that*
(b) *Y_i is a simply connected, primitive symplectic manifold of dimension $2d_i$ with holonomy group $\mathbf{Sp}(2d_i)$; and that*
(c) *Z_j is an n_j-dimensional simply connected manifold whose holonomy group is $\mathbf{SU}(n_j)$, $n_j \geq 3$ (i.e., Z_i is an $\mathbf{SU}$-manifold).*

Each class appearing in the above decomposition has trivial canonical class. The three classes are separated from each other by simple birational invariants:

Complex torus $\iff \mathrm{H}^0(\Omega^1) \neq 0,$
Symplectic manifold $\iff \mathrm{H}^0(\Omega^1) = 0,\ \mathrm{H}^0(\Omega^2) \neq 0,$
$\mathbf{SU}$-*manifold* $\iff \mathrm{H}^0(\Omega^1) = \mathrm{H}^0(\Omega^2) = 0.$

[23] A coherent subsheaf $\mathcal{E}$ of a vector bundle (= locally free sheaf) $\mathcal{F}$ is called a *subbundle* if $\mathcal{F}/\mathcal{E}$ is locally free. A subbundle is always locally free.

In essence, this theorem asserts that the classification of compact Kähler manifolds with trivial canonical classes reduces to that of compact primitive symplectic manifolds and **SU**-manifolds.[24]

The (analytic) local structure of a symplectic manifold is extremely rigid. Indeed we have the following

Theorem 6.2 (Darboux). *Let (Y, η) be a complex symplectic manifold. Then, around each point of Y, we can find a local analytic coordinate system, called a Darboux coordinate system, $(x_1, \dots, x_n; y_1, \dots, y_n)$ such that*

$$\eta = dx_1 \wedge dy_1 + dx_2 \wedge dy_2 + \cdots + dx_n \wedge dy_n.$$

Despite the rigidity of the local structure, the global structure of compact symplectic manifolds is rich and admits abundant non-trivial deformation of complex structure.

Theorem 6.3 (Bogomolov [Bo]-Beauville [Bea1]; for the generalization to **SU**-manifolds, see Tian [Ti] and Todorov [To]). *Let Y be a compact complex symplectic manifold. Then the deformation space* (the Kuranishi space) *of the complex structure of Y is smooth and its tangent space at $[M]$ is exactly $\mathrm{H}^1(Y, \Theta_Y) \simeq \mathrm{H}^1(Y, \Omega^1_Y)$. The Kuranishi space is a Kähler manifold via the Weil-Petersson metric defined by the canonical Hodge pairing on $\mathrm{H}^1(Y, \Theta_M)$ induced by a prescribed Ricci-flat Kähler metric on M. In particular, given a homology class $\alpha \in \mathrm{H}_2(Y, \mathbb{Q}) \cap (\mathrm{H}^{2,0}(Y))^{\perp}$ (which is represented by a rational algebraic 1-cycle by a theorem of Lefschetz), there is a one-parameter deformation $\mathcal{Y} = \{Y_t\}_{t \in T}$ such that the flat lifting $\alpha_t \in \mathrm{H}_2(Y_t, \mathbb{Q})$ of α is no more an algebraic cycle for general t.*

If we deform the complex structure of given compact Y in a general direction, then Y will not contain any compact analytic curve. In fact, given an algebraic cycle $\alpha \in Y$, the algebraicity of α is preserved in a $\mathbb{Q}$-rational hyperplane $\alpha^{\perp}$ in $\mathrm{H}^1(Y, \Omega^1) \subset \big(\mathrm{H}_2(Y, \mathbb{C})\big)^*$, which is analytically locally identified with the Kuranishi space. More generally, the set of the deformations with Picard number $\geq \rho$ locally forms a countable union of linear affine subspaces of codimension ρ in the Kuranishi space.

An immediate consequence of this observation is

Proposition 6.4. *Let Y be a compact complex symplectic manifold of dimension $2n$ and $f : \mathbb{P}^1 \to Y$ a non-constant morphism. Then $\mathrm{Hom}(\mathbb{P}^1, Y)$ is of dimension $\geq 2n + 1$ at $[f]$.*

[24]Manifolds within these two classes are often referred as "Calabi-Yau manifolds".

Proof. We can construct a one-parameter deformation $\mathcal{Y} = \{Y_t\}$, $Y = Y_0$ such that the algebraicity of the cycle $f(\mathbb{P}^1)$ is destroyed on general Y_t. Hence (the underlying reduced structure of) $\mathrm{Hom}(\mathbb{P}^1, \mathcal{Y})$ is locally identical with $\mathrm{Hom}(\mathbb{P}^1, Y)$ around $[f]$. On the other hand, since the canonical bundle $\mathrm{K}_{\mathcal{Y}}$ is trivial, we have

$$\dim_{[f]} \mathrm{Hom}(\mathbb{P}^1, Y) = \dim_{[f]} \mathrm{Hom}(\mathbb{P}^1, \mathcal{Y}) \geq \chi(\mathbb{P}^1, f^*\Omega^1_{\mathcal{Y}}) = 2n + 1.$$

Q.E.D.

Definition 6.5. Let (Y, η) be a complex symplectic manifold of dimension $2n$. A subvariety $Z \subset Y$ is said to be *isotropic* if $\Theta_{Z^\circ} \subset \Theta_Y|_{Z^\circ}$ is isotropic (equivalently: if $\eta|_Z \in H^0(Z, \Omega^2_Z/(\text{torsion}))$ identically vanishes). Here the symbol Z° stands for the smooth locus of Z. Any isotropic subvariety Z has dimension $\leq n$. If $\dim Z$ attains the maximum n, Z is called a *Lagrangian* subvariety.

If a subvariety Z contains sufficiently many rational curves, then it is necessarily isotropic (Lagrangian provided $\dim Z = n$). Indeed, we have:

Proposition 6.6. *Let Z be a closed r-dimensional subvariety of a $2n$-dimensional compact complex symplectic manifold (Y, η). If there is a family of rational curves $\tilde{f} : T \times \mathbb{P}^1 \to Z \subset Y$, such that*

(1) *$\tilde{f}(T \times \{\infty\})$ is a single closed point and that*

(2) *$\tilde{f}(T \times \mathbb{P}^1)$ contains an open dense subset of Z,*

then Z is an isotropic (Lagrangian, if $r = n$) subvariety.

Proof. If necessary, by changing the parameter space T by a suitable resolution, we may assume that T is a complex manifold. We have a natural generically injective homomorphism

$$\mathrm{pr}_T^*\Theta_T \oplus \mathrm{pr}_{\mathbb{P}^1}^*\Theta_{\mathbb{P}^1}(-\infty) \to \mathcal{H}om(\tilde{f}^*\Omega^1_Z, \mathcal{O}_{T\times\mathbb{P}^1}(-\infty)) \subset (\tilde{f}^*\Theta_Y)(-\infty).$$

Hence $\mathcal{H}om(\tilde{f}^*\Omega^1_Z, \mathcal{O}_{T\times\mathbb{P}^1})$ is ample when restricted to general $\{t\} \times \mathbb{P}^1$. The restriction of η to Z induces a bilinear pairing on $\mathcal{H}om(\Omega^1_Z, \mathcal{O}_Z)$ with values in the trivial line bundle $\mathcal{O}_Z$. Since the former vector bundle is ample on general $f_t(\mathbb{P}^1)$, this pairing identically vanishes on $f_t(\mathbb{P}^1)$. In view of the condition (2), this proves the assertion. Q.E.D.

Remark 6.7. In Proposition 6.6, if the closed point $\tilde{f}(T \times \{\infty\})$ is a non-singlular point of Z, it follows that Z is rationally connected.[25]

[25] A projective variety X is *rationally connected* if its two general points can be joined by an irreducible rational curve on X.

It is a general fact that a rationally connected variety has no nonzero global holomorphic r-forms, $r > 0$ ([KMM]).

7. Fibre space structure of primitive complex symplectic manifolds

Let Y be a projective, primitive complex symplectic manifold of dimension $2n$. Yau's theorem [Y1] and [Y2] asserts that if we are given a Kähler class η_0, Y carries a unique Ricci flat Kähler metric g whose Kähler form η is cohomologous to the given η_0. The Ricci-flat Kähler metric with holonomy $\mathbf{Sp}(2n)$ furnishes Y with a natural hyperkähler structure, which governs the Hodge-Lefschetz decomposition of the cohomology ring $H^{\bullet,\bullet}(Y, \mathbb{C})$. In view of this special structure of the cohomology, Beauville [Beal] defined a symmetric bilinear form (*Beauville quadratic form*[26]) $Q(\cdot,\cdot)$ on the Néron-Severi group $\mathrm{NS}(Y)$ such that $D^{2n} = cQ(D,D)^n$, $D \in \mathrm{NS}(Y)$, where c is a constant independent of D.

The existence of the Beauville quadratic form yields non-trivial information on the cone of divisors. If D is a nef divisor $\not\approx 0$ with $D^{2n} = 0$ and H is ample, then we have

$$\sum \binom{2n}{k} t^k D^{2n-k} H^k = (D+tH)^{2n} = cQ(D+tH, D+tH)^n$$
$$= c\big(Q(D,D) + 2tQ(D,H) + t^2 Q(H,H)\big)^n.$$

If we compare the coefficients of t^k in the left-hand side and in the right-hand side, we easily get $Q(D,D) = 0$, $Q(D,H) > 0$, $D^nH^n > 0$ and $D^{n+i}H^{n-i} = 0$, $i > 0$. Namely, a non-zero nef divisor on a primitive complex symplectic manifold Y of dimension $2n$ is either big or looks like a pullback of an ample divisor on an n-dimensional variety.

Starting from this observation, D. Matsushita discovered that any non-trivial fibre space structure of Y must be of very restricted type.

Theorem 7.1 (Matsushita [Mats]). *Let Y be a projective, primitive complex symplectic manifold and let $\pi : Y \to X$ be a morphism onto a normal projective variety X with $0 < \dim X < 2n = \dim Y$ and with $\pi_* \mathcal{O}_Y = \mathcal{O}_X$. Then:*

(1) *X is of dimension n and $\mathbb{Q}$-factorial, i.e., any Weil divisor on X is a Cartier divisor if multiplied by a suitable positive integer. The Picard number of X is one.*

(2) *X has only log-terminal singularities.*

---

[26] For a systematic account on the Hodge-Lefschetz decomposition on hyperk̈hler manifolds and the Beauville quadratic form Q, see Fujiki [Fu].

(3) *The anticanonical divisor* $-K_X$ *is ample (as a* $\mathbb{Q}$*-Cartier divisor). In other words,* X *is a* $\mathbb{Q}$*-Fano variety.*

(4) *Every fibre* Y_x *of* π *is of pure dimension* n *and each of its components is Lagrangian.*

The property (1) specifically means that a nonzero effective Weil divisor D and an effective curve C on X necessarily meet each other.[27] The property (4) implies that a smooth fibre A of π is an abelian variety because Θ_A and the trivial normal bundle $\mathcal{N}_{A/Y} \simeq \mathcal{O}_A^{\oplus n}$ are mutually duals via the symplectic pairing.[28]

Under a certain technical condition, Main Theorem 0.1 shows that the base variety X of a non-trivial fibration of a primitive symplectic manifold Y is a projective space. Namely:

Theorem 7.2. *Let* $\pi : Y \to X$ *be as in Theorem* 7.1 *and assume in addition that* π *admits a section* $\sigma : X \to Y$*. Then* X *is isomorphic to* $\mathbb{P}^n$*.*

The proof of Theorem 7.2 consists of several steps. We start with easy observations.

Lemma 7.3. X *is a smooth uniruled variety. Let* $x \in X$ *be a general point and* $C \subset X$ *a rational curve of minimum degree passing through* x*. If* C *is general, then* C *is smooth and*

$$\Theta_X|_C \simeq \mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus e-1} \oplus \mathcal{O}^{\oplus n-e}.$$

Proof. The projection π and the section σ give a canonical injection $\mathcal{O}_X \hookrightarrow \mathcal{O}_Y$ and a natural surjection $\mathcal{O}_Y \twoheadrightarrow \mathcal{O}_X$, thereby inducing an $\mathcal{O}_X$-linear map $\Omega^1_X \to \Omega^1_Y$ and an $\mathcal{O}_Y$-linear surjection $\Omega^1_Y \to \Omega^1_X$. Therefore Ω^1_X is a direct summand of the $\mathcal{O}_X$-module $\Omega^1_Y|_{\sigma(X)}$. Since $\Omega^1_Y|_{\sigma(X)}$ is locally free by the smoothness of Y, we conclude that Ω^1_X is also locally free; that is, X is smooth. In particular, X is a Fano manifold by Matsushita's theorem. Any Fano manifold (more generally, any $\mathbb{Q}$-Fano variety) is uniruled by [MiMo]. The proof of the second and third statements are given in Theorem 2.8. Q.E.D.

From now on, we fix the notation as follows.

Let x be a general point on X, and $C \subset X$ a general (smooth) rational curve of minimum degree passing through x. By the embedding

[27] We need this fact to prove Theorem 7.2 below.

[28] An alternative proof is by the famous theorem of Liouville on completely integrable Hamiltonian systems (see Arnold [Ar, p.272]).

$\sigma : X \to Y$, we view C and X as closed subschemes of Y. Define $f : \mathbb{P}^1 \to Y$ by fixing an isomorphism $\mathbb{P}^1 \simeq C \subset \sigma(X) \subset Y$.

Lemma 7.4. *We have an isomorphism*

$$f^*\Theta_Y \simeq \mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus e-1} \oplus \mathcal{O}^{\oplus 2n-2e} \oplus \mathcal{O}(-1)^{\oplus e-1} \oplus \mathcal{O}(-2).$$

Proof. Since X is a Fano manifold, there is no non-zero global 2-form on X.[29] Hence X is a Lagrangian submanifold in Y. Then the symplectic form η defines an isomorphism between

$$f^*\Theta_X \simeq \mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus e-1} \oplus \mathcal{O}^{\oplus n-e}$$

and the dual of $f^*\Theta_Y/f^*\Theta_X$. Q.E.D.

Corollary 7.5. $\mathrm{Hom}(\mathbb{P}^1, Y)$ *is smooth at* $[f]$. *If* $[h] \in \mathrm{Hom}(\mathbb{P}^1, Y)$ *is sufficiently close to* $[f]$, *then* $h^*\Theta_Y$ *has the same decomposition type as* $f^*\Theta_Y$.

Proof. By Lemma 7.4, we have $\dim \mathrm{H}^0(\mathbb{P}^1, f^*\Theta_Y) = 2n+1$, whilst we have the estimate $\dim_{[f]} \mathrm{Hom}(\mathbb{P}^1, Y) \geq 2n+1$ by Definition 6.5. This shows that $\mathrm{Hom}(\mathbb{P}^1, Y)$ is smooth at $[f]$ and the differential of the universal morphism $\mathrm{Hom}(\mathbb{P}^1, Y) \times \mathbb{P}^1 \to Y$ has rank $2n-e$ at $([f], p)$, where $p \in \mathbb{P}^1$ is general. Therefore, if $[h] \in \mathrm{Hom}(\mathbb{P}^1, Y)$ is sufficiently close to $[f]$, we have

$$\dim \mathrm{H}^0(\mathbb{P}^1, h^*\Theta_Y) = \dim_{[h]} \mathrm{Hom}(\mathbb{P}^1, Y) = \dim_{[f]} \mathrm{Hom}(\mathbb{P}^1, Y) = 2n+1$$

and $\mathrm{H}^0(\mathbb{P}^1, h^*\Theta_Y)$ generates a subsheaf $\mathcal{E} \subset h^*\Theta_Y$ of rank $2n-e$. The quotient $\mathcal{E}/\Theta_{\mathbb{P}^1}$ is semi-positive, of rank $2n-e-1$, of degree $e-1$, and is isomorphic to $\mathcal{O}(1)^{\oplus e-1} \oplus \mathcal{O}^{\oplus 2n-2e}$ when $h = f$. This decomposition type is obviously stable under small deformation, and hence

$$\mathcal{E} \simeq \mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus e-1} \mathcal{O}^{\oplus 2n-2e}.$$

Then by the existence of non-degenerate pairing on $h^*\Theta_Y \supset \mathcal{E}$, we get the assertion. Q.E.D.

Fix a general point $x \in \sigma(X) \simeq X$ and a general rational curve $C \subset \sigma(X)$ of minimum degree through x. Recall that $C \simeq \mathbb{P}^1$ lies on the smooth locus of $\sigma(X)$. Let $f : \mathbb{P}^1 \simeq C \to \sigma(X) \subset Y$ be the embedding given above, M a sufficiently small Zariski open neighbourhood of $[f]$ in $\mathrm{Hom}(\mathbb{P}^1, Y)$, and $\tilde{f} : M \times \mathbb{P}^1 \to Y$ the restriction of the universal

[29] By Kodaira vanishing, we have $\mathrm{H}^2(X, \mathcal{O}_X) = \mathrm{H}^0(X, \Omega^2_X) = 0$.

morphism. Pick up a general point $z = h(p) = \tilde{f}([h], p)$ of $\tilde{f}(M \times \mathbb{P}^1) \subset Y$ and put $M_z = M \cap \mathrm{Hom}(\mathbb{P}^1, Y; p \mapsto z)$. M and M_z are smooth because

$$\dim \mathrm{H}^0(\mathbb{P}^1, h^*\Theta_Y) = \dim M = 2n+1,$$
$$\dim \mathrm{H}^0(\mathbb{P}^1, h^*\Theta_Y(-p)) = \dim M_z = n+1.$$

Let $\tilde{f}_z : M_z \times \mathbb{P}^1 \mapsto Y$ denote the restriction of $\tilde{f}$.

By Corollary 7.5, the universal morphism $\tilde{f} : M \times \mathbb{P}^1 \to Y$ is everywhere of constant rank $2n - e$, so that $\tilde{f}(M \times \mathbb{P}^1) \subset Y$ is an immersed locally closed submanifold of dimension $2n - e$ (after shrinking M to a smaller open subset if necessary). In particular, the normalization Z of $\tilde{f}(M \times \mathbb{P}^1)$ is smooth. The morphism $\tilde{f} : M \times \mathbb{P}^1 \to Y$ factors through $\tilde{g} : M \times \mathbb{P}^1 \to Z$ and the natural immersion $Z \to Y$. Notice that $\sigma(X) \subset Y$ is contained in the closure of $\tilde{f}(M \times \mathbb{P}^1)$ because $\Theta_{\sigma(X)}|_C$ is semipositive.

Let Z_z denote the normalization of $\tilde{f}_z(M_z \times \mathbb{P}^1) \subset \tilde{f}(M \times \mathbb{P}^1)$. Let $\tilde{g}_z : M_z \times \mathbb{P}^1 \to Z_z$ and $j : Z_z \to Z$ be the morphisms defined in an obvious manner. Our morphism $h : \mathbb{P}^1 \to \tilde{f}(M_z \times \mathbb{P}^1) \subset \tilde{f}(M \times \mathbb{P}^1) \subset X$ naturally defines morphisms $g_z : \mathbb{P}^1 \to Z_z$, $g = jg_z : \mathbb{P}^1 \to Z$.

Then an immediate consequence of Lemmas 7.3 and 7.4 is the following

Lemma 7.6. *We have*

$$g^*\Theta_Z = \mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus e-1} \oplus \mathcal{O}^{\oplus 2n-2e},$$
$$g_z^*\Theta_{Z_z} \subset \mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus e-1} \subset g_z^*\Theta_{Z_z}(p).$$

The composite natural projection $Z \to Y \xrightarrow{\pi} X$ is everywhere of rank n near the curve $g(\mathbb{P}^1) \subset Z$.

Lemma 7.7. *Let the notation be as above and assume that M is sufficiently small. Then*

(1) *The 2-form η defines a degenerate bilinear form of constant rank $2n - 2e$ on Θ_Z in a natural way.*

(2) *Z_z is smooth and $j : Z_z \to Z$ is an embedding.*

(3) *The subsheaf $\Theta_{Z_z} \subset j^*\Theta_Z$ is determined by the following null-space property:*

$$v \in \Theta_{Z_z} \iff \eta(v, w) = 0 \text{ for arbitrary } w \in j^*\Theta_Z.$$

Thus the subset $Z_z \subset Z$, attached to a general point $z \in Z$, is indeed an integral submanifold of a foliation of rank e on the smooth, locally closed variety Z.

Proof. The rank of Θ_Z is everywhere $2n-e$, while Θ_Y is of rank $2n$. Elementary linear algebra then shows that the bilinear form $\eta|_{\Theta_Z}$ has pointwise rank $\geq 2n-2e$, and the associated null space

$$\mathrm{Null}_\eta(\Theta_{Z,p}) = \{v \in \Theta_{Z,p}; \eta(v, \Theta_{Z,p}) = 0\}$$

has complex dimension $\leq e$. As a function in $p \in Z$, $\dim \mathrm{Null}_\eta(\Theta_{Z,p})$ is uppersemicontinuous (equivalently, rank $\eta|_{\Theta_{Z,p}}$ is lower semicontinuous). Hence the statement (1) follows if we check that $\dim \mathrm{Null}_\eta(\Theta_{Z,p}) = e$ for general $p \in Z$.

On $\mathbb{P}^1$, the maximal positive subbundle $\mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus e-1} \subset g^*\Theta_Z$ is of course ample and $g^*\Theta_Z = g_z^* j^*\Theta_Z$ is semipositive, so that any pairing $g_z^*\Theta_{Z_z} \times g_z^* j^*\Theta_Z \to \mathcal{O}$ identically vanishes. This means that the pairing

$$\eta : \Theta_{Z_z} \times j^*\Theta_Z \to \mathcal{O}_{Z_z}$$

vanishes on $g_z(\mathbb{P}^1)$. By deforming h and thereby $g_z(\mathbb{P}^1) \subset Z_z$, we conclude that the pairing $\Theta_{Z_z} \times j^*\Theta_Z \to \mathcal{O}_{Z_z}$ must identically vanish on Z_z. Hence, on an open dense subset of Z_z, the vector bundle Θ_{Z_z} of rank e is pointwise identical with $\mathrm{Null}_\eta(\Theta_{Z,p})$, showing that $\dim \mathrm{Null}_\eta(\Theta_{Z,p})$ is constant on Z_z. Noticing that the family $\{Z_z\}_{z \in Z}$ sweeps out Z, we conclude that the subspaces $\mathrm{Null}_\eta(\Theta_{Z,p})$ have constant dimension e on Z and gives rise to a subbundle $\mathrm{Null}_\eta(\Theta_Z) \subset \Theta_Z$.

The Lie bracket $[\ ,\ ]$ defined on Θ_Z induces an $\mathcal{O}_Z$-homomorphism $\wedge^2 \mathrm{Null}_\eta(\Theta_Z) \to \Theta_Z/\mathrm{Null}_\eta(\Theta_Z)$. If we restrict things to a small deformation of $g_z(\mathbb{P}^1)$, the target is a trivial vectorbundle while the source is ample, and hence any $\mathcal{O}_Z$-homomorphism $\wedge^2 \mathrm{Null}_\eta(\Theta_Z) \to \Theta_Z/\mathrm{Null}_\eta(\Theta_Z)$ is a zero map. This means that the subbundle $\mathrm{Null}_\eta(\Theta_Z)$ is involutive, i.e., $[\mathrm{Null}_\eta(\Theta_Z), \mathrm{Null}_\eta(\Theta_Z)] \subset \mathrm{Null}_\eta(\Theta_Z)$. Thus $\mathrm{Null}_\eta(\Theta_Z)$ defines a foliation on Z.

Furthermore, we have seen that Θ_{Z_z} is identical with the subbundle $\mathrm{Null}_\eta(\Theta_Z)$ (on an open dense subset of Z_z). In other words, Z_z is an integral submanifold of the foliation given by $\mathrm{Null}_\eta(\Theta_Z)$, whence follows (2) and (3). Q.E.D.

Corollary 7.8. *There exist a variety B and a dominant morphism $Z \to B$ of which a general fibre is a manifold of the form $(Z_z)^c \cap Z$, $z \in Z$. (Here $\bullet^c$ denotes the Zariski closure.)*

Proof. Given a point z on Z, the integral variety of a non-singular foliation passing through z is uniquely determined and, in our situation, is a constructible set of the form $Z_{z'}$ for some $z' \in Z$. Hence we have a well-defined morphism $Z \to \mathrm{Chow}(Z^c)$, $z \mapsto [(Z_{z'})^c]$. Q.E.D.

Recall that the closure of $\tilde{f}(M \times \mathbb{P}^1)$ contains $\sigma(X)$. More precisely, $\tilde{f}(M \times \mathbb{P}^1)$ (as well as its normalization Z) contains a smooth small open neighbourhood $U \subset \sigma(X)$ of $C = f(\mathbb{P}^1)$. Hence the natural rational map $\sigma(X) \dashrightarrow Z$ and the fibration $Z \to \mathrm{Chow}(Z^c)$ induce a dominant rational map $\phi : \sigma(X) \dashrightarrow B' \subset \mathrm{Chow}(Z^c)$, well defined as a morphism on U.

Lemma 7.9. $\left(\tilde{f}(M_x \times \mathbb{P}^1)\right)^c = \sigma(X)$*, or equivalently* $e = n$*, the superscript* ^{c} *denoting the closure. In particular,* $\dim_{[f]} \mathrm{Hom}(\mathbb{P}^1, \sigma(X)) = 2n + 1$.

Proof. Suppose otherwise. The projective variety B' is then of positive dimension. Take a divisor D' on B' away from the single point

$$\phi(C) = \phi(f(\mathbb{P}^1)) = \phi(\sigma(X) \cap Z_z) \in B',$$

and let D be its inverse image on $\sigma(X)$. (More precisely, blow up $\sigma(X)$ along centres away from U to resolve the indeterminacy of ϕ, and we get the diagram

$$\begin{array}{ccc} \sigma(X)' & \xrightarrow[\phi']{} & B' \subset \mathrm{Chow}(Z) \\ \downarrow{\scriptstyle \mu} & & \\ \sigma(X). & & \end{array}$$

Then we define D to be $\mu(\phi'^* D')$.) D is well-defined on U (i.e., independent of the choice of $\sigma(X)'$) and is away from $\sigma(X) \cap Z_z \supset C$. In other words, there exists an effective Weil divisor D on $\sigma(X)$ which does not meet the effective curve C. This contradicts Matsushita's result that $\sigma(X) \simeq X$ has Picard number one. Q.E.D.

Theorem 7.2 immediately follows from this lemma and our Main Theorem 0.1.

Example 7.10. The following example of non-trivial fibration of primitive complex symplectic manifolds is due to Fujiki ($r = 2$) and Beauville [Bea1] (r general).

Let $S \to \mathbb{P}^1$ be an elliptic K3 surface with everywhere non-vanishing 2-form η. $\mathrm{Hilb}^r(S)$, the Hilbert scheme of 0-dimensional closed subschemes of degree r on S, is known to be smooth by a result of Fogarty [Fo]. There is a natural birational morphism from $\mathrm{Hilb}^r(S)$ to the symmetric product $\mathrm{Sym}^r(S) = S \times \cdots \times S/\mathfrak{S}_r$, which is identified with the Chow scheme of the effective 0-cycles of degree r.[30] Thus we have natural

[30] By virture of the normality of the symmetric product and the universal property of the Chow scheme.

morphisms $\mathrm{Hilb}^r(S) \to \mathrm{Sym}^r(S) \to \mathrm{Sym}^r(\mathbb{P}^1) \simeq \mathbb{P}^r$, defining an abelian fibration structure of $\mathrm{Hilb}^r(S)$ over $\mathbb{P}^r$. The $\mathfrak{S}_r$-invariant 2-form $\sum \mathrm{pr}_i^* \eta$ naturally lifts to a symplectic form on $\mathrm{Hilb}^r(S)$, nonzero holomorphic 2-form unique up to non-zero factor.

Examples of this type form a 19-dimensional family because the deformation of the elliptic fibre space structure of S is parameterized by a 19-dimensional space thanks to the global Torelli for K3 surfaces [BaPeVV, VII.11.1].

We can check that the second Betti cohomology group of $\mathrm{Hilb}^r(S)$ is generated by $\mathrm{H}^2(S, \mathbb{C})$ and the exceptional divisor E over the diagonal

$$\Delta = \{(x_1, \ldots, x_r); x_i = x_j \text{ for some } (i,j), i \neq j\}/\mathfrak{S}_r \subset \mathrm{Sym}^r(S),$$

so that

$$\mathrm{H}^1(\mathrm{Hilb}^r(S), \Theta_{\mathrm{Hilb}^r(S)}) \simeq \mathrm{H}^1(\mathrm{Hilb}^r(S), \Omega^1) \simeq \mathrm{H}^1(S, \Omega^1) \oplus \mathbb{C}[E] \simeq \mathbb{C}^{21}.$$

The Kuranishi space T of Hilb^r is thus 21-dimensional, and the subspace T_0 which preserves the fibre space structure is of dimension 20. That is to say a general element $t \in T_0$ is not represented as a Hilbert scheme of a K3 surface any more.

Remark 7.11. Let $\pi : Y \to X$ be a fibre space structure of a projective primitive complex symplectic manifold Y over a normal projective variety X. Assume that every scheme theoretic fibre of π has a reduced irreducible component or, equivalently, that π admits analytic local sections at every point $x \in X$. The dual abelian fibration $\pi^\dagger : \mathrm{Pic}^0(Y/X) \to X$ is naturally a (non-proper) group scheme over X, and so is the double dual $\pi^\ddagger : \mathrm{Pic}^0\big(\mathrm{Pic}^0(Y/X)/X\big) \to X$. By choosing a local analytic section $\sigma_U : U \to Y|_U$ defined on a small Stein open subset $U \subset X$, we have a natural identification $\mathrm{Pic}^0\big(\mathrm{Pic}^0(Y/X)/X\big)|_U \simeq Y^\circ|_U$, where $Y^\circ \subset Y$ is the open subset consisting of the non-critical points of π. This isomorphism provides Y° with a $\mathrm{Pic}^0\big(\mathrm{Pic}^0(Y/X)/X\big)$-torsor structure over X, or a natural action of $\mathrm{Pic}^0\big(\mathrm{Pic}^0(Y/X)/X\big)$ on Y°. The global structure of Y° is recovered from $\mathrm{Pic}^0\big(\mathrm{Pic}^0(Y/X)/X\big)$ and the patching data $\eta \in \mathrm{H}^1_{\mathrm{ann}}(X, \mathrm{Pic}^0(\mathrm{Pic}^0(Y/X)/X))$.

Suppose that this $\mathrm{Pic}^0\big(\mathrm{Pic}^0(Y/X)/X\big)$-action on Y° extends to an action on the compactification Y. (This is indeed the case when the degenerations are semistable.) Under this additional assumption, we can naturally construct a smooth compactification $\pi^\ddagger : Y^\ddagger \to X$ of the group scheme $\mathrm{Pic}^0\big(\mathrm{Pic}^0(Y/X)/X\big)$ by identifying $Y^\ddagger|_U$ with $Y|_U$ and patching these together via the given data η. On a smooth fibre, the patchig is defined by a suitable translation, which does not affect the relative cotangent sheaf $\Omega^1_{Y/X}$. Therefore we have a canonical isomorphism $\pi_*\Omega^1_{Y/X} \simeq \pi^\ddagger_*\Omega^1_{Y^\ddagger/X}$, implying that $Y^\ddagger$ is again a primitive complex symplectic manifold. Applying Theorem 7.2 to $Y^\ddagger$ instead of Y, we conclude that the base variety X is $\mathbb{P}^n$. In this sense, Theorem 7.2 states something stronger than it sounds.

Theorem 7.2 and Remark 7.11 in mind, we ask the following questions:

Problem 7.12. (1) *Is it possible to completely classify symplectic n-dimensional complex torus fibrations free from multiple fibres over $\mathbb{P}^n$?*

(2) *Is there a fibration $\pi : Y \to X$ from a primitive compact symplectic manifold onto a normal variety which is not $\mathbb{P}^n$?* (If there is any, its projection π will have non-semistable, perhaps multiple, fibres.)

Most primitive symplectic manifolds known so far carry non-trivial fibrations if we suitably deform its complex structure, although we have no idea if that is always the case. For instance, if there is a symplectic manifold with $\mathrm{h}^{1,1} = 1$, it does not allow any non-trivial fibration.

Problem 7.13. (1) Let Y be a compact primitive complex symplectic manifold with $\dim \mathrm{H}^1(Y, \Omega^1_Y) \geq 2$ (or, equivalently, $\dim \mathrm{H}^2(Y, \mathbb{C}) \geq 4$). By suitably deforming the complex structure, we can assume that the Picard number is exactly $\mathrm{h}^{1,1}$. Assume that there is a divisor D with $D^{2n} = 0$. Then, *is it possible to find D such that the linear system $|D|$ defines a non-trivial fibration of Y?*

(2) Since the holonomy group of a primitive symplectic manifold Y is the full symplectic group $\mathbf{Sp}(2n)$, it follows that $\mathrm{H}^0(Y, \Omega^p_Y)$ is $\mathbb{C}$ or 0 according to the parity of $p \leq 2n$. However, the higher cohomology $\mathrm{H}^q(Y, \Omega^p_Y)$ could be highly non-trivial. *Are there a priori dimension estimates (from above and/or from below) for the Betti numbers of primitive symplectic manifolds?*

8. Symplectic resolutions of an isolated singularity

Let $\hat{Z}$ be s normal variety of even dimension $2n$ with a single isolated singularity, and $\pi : Z \to \hat{Z}$ a symplectic resolution. Namely, π is a projective bimeromorphic morphism from a complex manifold Z, which carries an everywhere non-degenerate closed holomorphic 2-form η. Let $E = \bigcup E_i$ denote the exceptional locus of π, each E_i being an irreducible component of E.

Recall that a pure dimensional closed subvariety $W \subset Z$ is called *Lagrangian* if $\dim W = n$ and $\eta|_W$ is identically zero.

Example 8.1. Let $Z = \operatorname{Spec} \operatorname{Sym}\Theta_X$ be the (total space of the) cotangent bundle Θ^*_X of a smooth projective variety X, $\mathrm{pr}_X : Z \to X$

the standard projection, and $0_X \subset Z$ the zero-section. Θ_Z is naturally isomorphic to $\mathrm{pr}_X^*\Theta_X \oplus \mathrm{pr}_X^*\Theta_X^*$, so that it carries a standard symplectic form η defined by

$$\eta\big((\alpha_1,\beta_1),(\alpha_2,\beta_2)\big) = \langle\alpha_1|\beta_2\rangle - \langle\alpha_2|\beta_1\rangle,$$

where $\alpha_i \in \Theta_X$, $\beta_i \in \Theta_X^*$, and $\langle\cdot|\cdot\rangle$ stands for the canonical pairing between the duals.[31]

The normal bundle of $0_X \simeq X$ is of course isomorphic to Θ_X^*. By Hartshorne-Mori, it is negative if and only if X is $\mathbb{P}^n$. In this case, thanks to a theorem of Grauert [Gra], we can contract 0_X to a point o to get a symplectic resolution $\pi : Z \to \hat{Z}$.

The symplectic resolution $\pi : Z \to \hat{Z}$ described in Example 8.1 is said to be *standard*. The following fact (the existence of "standard flops") is an important feature of standard symplectic resolutions.

Lemma 8.2. *Let $X \subset Z$ be a Lagrangian submanifold isomorphic to $\mathbb{P}^n$ in a complex symplectic manifold. Then a small analytic neighbourhood of X in Z is isomorphic to a standard symplectic resolution described in Example* 8.1 *of an isolated singularity. If $n \geq 2$ (i.e., if $\dim Z \geq 4$), let $\mu : \tilde{Z} = \mathrm{Bl}_X(Z) \to Z$ denote the blowing-up along X. Then the exceptional divisor $\tilde{X} = E_X \subset \tilde{Z}$ is a $\mathbb{P}^{n-1}$-bundle over X and admits another $\mathbb{P}^{n-1}$-fibration over $X' \simeq \mathbb{P}^n$, and accordingly $\tilde{Z}$ has another blowing-down $\mu' : \tilde{Z} \to Z'$ onto a new symplectic manifold.*[32] *Given $p \in X$, the closed subset $\mu'\big(\mu^{-1}(p)\big)$ is a hyperplane in X'.*

Proof. Because of the symplectic paring η, we easily deduce that the normal bundle $\mathcal{N}_{X/Z}$ is naturally isomorphic to the negative vector bundle Θ_X^*. Then we have

$$\mathrm{H}^1(X, \mathrm{Sym}^r\mathcal{N}_{X/Z}^*) = \mathrm{H}^1(X, \Theta_Z \otimes \mathrm{Sym}^r\mathcal{N}_{X/Z}^*) = 0, \quad r > 0,$$

and a theorem of Grauert [Gra, Sect. 4, Satz 7] applies to prove that Z is locally biholomorphic to the standard resolution around X.

[31] While $\Theta_X^* = \mathrm{Spec}\,\mathrm{Sym}\Theta_X$ is a symplectic manifold, its projectivization $W = \mathbb{P}(\Theta_X) = \mathrm{Proj}\,\mathrm{Sym}\Theta_X$ is a *complex contact manifold*, an odd-dimensional analogue of a symplectic manifold (see [Bea2]). Namely there is a subbundle $\mathcal{F} \subset \Theta_W$ of corank one together with a non-degenerate skew symmetric pairing $\mathcal{F} \times \mathcal{F} \to \Theta_W/\mathcal{F} \simeq \mathcal{O}_W(\mathbf{1})$, where $\mathcal{O}_W(\mathbf{1})$ stands for the tautological line bundle on the projective bundle.

[32] The birational map $Z \dashrightarrow Z'$ is a typical flop. We refer the reader to [KM] for flips and flops.

The blown up variety $\tilde{Z}$ is Proj $\bigoplus \mathfrak{I}_X^m$ and the exceptional divisor is given by Proj $\bigoplus (I_X^m/I_X^{m+1})$ = Proj $\bigoplus \mathrm{Sym}^m \Theta_X$. We have a standard Euler exact sequence

$$0 \to \mathcal{O} \to \mathcal{O}(1)^{\oplus n+1} \to \Theta_X \to 0,$$

which means that the exceptional divisor $\tilde{X} = E_X$ is a hypersurface in $\mathbb{P}^n \times X \simeq \mathbb{P}^n \times \mathbb{P}^n$ of bidegree $(1,1)$. We can easily check that the first projection $\tilde{X} \to \mathbb{P}^n$ gives another $\mathbb{P}^{n-1}$-fibre space structure, each fibre F of which satisfying $\mathcal{O}_{\tilde{Z}}(\tilde{X})|_F \simeq \mathcal{O}_F(-1)$. Therefore $(\tilde{Z}, \tilde{X})$ can be blown down to (Z', X'), a pair of smooth varieties. The symplectic form η can be viewed as a non-degenerate form on $Z' \setminus X'$. Recalling that the codimension of $X' \subset Z'$ is two or more, the form η naturally extends to a non-degenerate holomorphic 2-form on Z'. Thus we get the assertion. Q.E.D.

The main result of this section is the following

Theorem 8.3. *Let $\hat{Z}$ be a normal projective variety of dimension $2n$ with a single isolated singularity. Assume that there exists a symplectic resolution $\pi : Z \to \hat{Z}$; in other words, π is a birational morphism from the smooth, projective, complex symplectic variety Z onto $\hat{Z}$. Then:*

(1) The exceptional locus $E \subset \hat{Z}$ of π is a union of Lagrangian submanifolds isomorphic to $\mathbb{P}^n$.

(2) When $n = 1$, the exceptional locus E is a tree of smooth $\mathbb{P}^1$'s with configuration of one of the ADE*-singularities.*

(3) If $n \geq 2$, then E consists of a single smooth $\mathbb{P}^n$ and $\pi : Z \to \hat{Z}$ is analytically-locally a standard resolution.

For the proof, we need several easy results.

Lemma 8.4. *Every component E_i of the exceptional locus is uniruled. Furthermore* $\mathrm{R}^i\pi_*\mathcal{O}_Z = 0$, $i > 0$.

Proof. The first statement is a special case of Theorem 1 of [Ka]. (Essentially the adjunction plus Miyaoka-Mori criterion for uniruledness [MiMo].) In order to show the second statement, notice that $\wedge^n \eta$ defines a nowhere vanishing $2n$-form, so that the dualizing sheaf ω_Z is isomorphic to $\mathcal{O}_Z$. Then the Grauert-Riemenschneider vanishing [GraR] yields $\mathrm{R}^i\pi_*\mathcal{O} \simeq \mathrm{R}^i\pi_*\omega = 0$, $i > 0$. Q.E.D.

Corollary 8.5. *Let $\tilde{E}_i$ be a non-singular model of E_i, an irreducible component of the exceptional locus E. Then the pullback of a holomorphic 2-form η on Z to $\tilde{E}_i$ is identically zero. In particular, E_i is isotropic with respect to the symplectic form η, and hence* $\dim E_i \leq n$.

Proof. Take an embedded resolution $\mu : \tilde{Z} \to Z$ of $E \subset Z$. Thus the inverse image $\tilde{E} \subset \tilde{Z}$ of $E \subset Z$ is a divisor of simple normal crossings. Let $D \subset \tilde{E}$ be an arbitrary irreducible component. In order to prove the assertion, it suffices to show that the natural restriction map $\mathrm{H}^0(\tilde{Z}, \Omega^2_{\tilde{Z}}) \to \mathrm{H}^0(D, \Omega^2_D)$ identically vanishes. By Hodge theory, this map is the complex conjugate of the restriction map $\mathrm{H}^2(\tilde{Z}, \mathcal{O}_{\tilde{Z}}) \to \mathrm{H}^2(D, \mathcal{O}_D)$. Take an effective, sufficiently ample divisor $\hat{H}$ on $\hat{Z}$ such that $\mathrm{H}^i\big(\hat{Z}, (\mathrm{R}^j(\pi\mu)_*\mathcal{O}_{\tilde{Z}})(\hat{H})\big) = 0$ for $i > 0, j \geq 0$. Thus, by Leray spectral sequence,

$$\mathrm{H}^2(\tilde{Z}, \mathcal{O}_{\tilde{Z}}(\mu^*\pi^*\hat{H})) \simeq \mathrm{H}^0(\hat{Z}, \mathrm{R}^2(\pi\mu)_*\mathcal{O}_{\tilde{Z}}(\hat{H})).$$

By construction, $\mu^*\pi^*\hat{H}$ is trivial on D. In the meantime, since $\mathrm{R}^j\pi_*\mathcal{O}_{\hat{Z}} = 0$ and Z is non-singular, we have $\mathrm{R}^j(\pi\mu)_*\mathcal{O} = 0$, $j > 0$. Looking at the commutative diagram

$$\begin{array}{ccc} \mathrm{H}^2(\tilde{Z}, \mathcal{O}) & \longrightarrow & \mathrm{H}^2(D, \mathcal{O}) \\ \downarrow & & \simeq\downarrow \\ 0 = \mathrm{H}^2(\tilde{Z}, \mathcal{O}(\mu^*\pi^*\hat{H})) & \longrightarrow & \mathrm{H}^2(D, \mathcal{O}(\mu^*\pi^*\hat{H})) \end{array}$$

we conclude that the restriction map in question is zero. Q.E.D.

Lemma 8.6. *Let $p_i \in E_i$ be a point not contained in $\bigcup_{j\neq i} E_j$ and take a rational curve $C_i \subset E_i \subset Z$ such that C_i passes through p_i. Let $g_i : \mathbb{P}^1 \to E_i$ be a standard morphism obtained by normalizing C_i. Then we have:*

$$\dim_{[g_i]} \mathrm{Hom}(\mathbb{P}^1, E_i) \geq 2n + 1.$$

Furthermore, given any rational curve $C \subset E = \bigcup E_i$, there is some component $E_0 \supset C$ such that $\dim_{[g]} \mathrm{Hom}(\mathbb{P}^1, E_0) \geq 2n + 1$, where $g : \mathbb{P}^1 \to E_0 \subset Z$ is induced by the normalization of C.

Proof. Since $\pi(g_i(\mathbb{P}^1)) = \pi(g(\mathbb{P}^1)) = o$ and $\hat{Z}$ is Kähler, every deformation of g_i or g in $\mathrm{Hom}(\mathbb{P}^1, Z)$ maps $\mathbb{P}^1$ to E. Hence

$$\dim_{[g_i]} \mathrm{Hom}(\mathbb{P}^1, E_i) = \dim_{[g_i]} \mathrm{Hom}(\mathbb{P}^1, E) = \dim_{[g_i]} \mathrm{Hom}(\mathbb{P}^1, Z) \geq 2n+1,$$
$$\dim_{[g]} \mathrm{Hom}(\mathbb{P}^1, E) = \dim_{[g]} \mathrm{Hom}(\mathbb{P}^1, Z) \geq 2n + 1$$

by Definition 6.5. On the other hand, we have a set theoretical equality $\mathrm{Hom}(\mathbb{P}^1, E) = \bigcup \mathrm{Hom}(\mathbb{P}^1, E_i)$. Q.E.D.

If $C_i \subset E_i$ is chosen to have minimum degree among the rational curves passing through p_i, then, locally around $[g_i]$, the scheme

$\operatorname{Hom}(\mathbb{P}^1, E_i)$ is identical with $\bigcup \operatorname{Hom}(\mathbb{P}^1, E_i)$ and has dimension $\leq 2\dim E + 1 \leq 2n+1$ by Corollaries 8.5 and 2.5. If C is a rational curve of minimum degree in E, then $\dim_{[g]} \operatorname{Hom}(\mathbb{P}^1, E) = \dim_{[g]} \operatorname{Hom}(\mathbb{P}^1, E_0) \leq 2n + 1$ for some $E_0 \supset C$.

Comparing these with Lemma 8.6 and Theorem 0.1, we have proved the following

Corollary 8.7. *$E_i \subset Z$ is Lagrangian, i.e., $\dim E_i = n$. Its normalization $\overline{E}_i$ is isomorphic to a finite quotient of $\mathbb{P}^n$ and there is at least one component, say E_0, such that $\overline{E}_0$ is $\mathbb{P}^n$.*

When $n = 1$, the symplectic manifold Z is a K3 surface (it cannot be an abelian surface because of the existence of the exceptional divisor E) and E is an effective divisor with negative definite intersection matrix. Each E_i is (-2)-curve, while $\mathrm{R}^1\pi_*\mathcal{O}_Z = 0$. Hence the singularity of $\hat{Z}$ is a rational double point and E is a chain of $\mathbb{P}^1$'s of which the dual graph is one of the Dynkin diagrams of type ADE. Thus, in order to complete the proof of Theorem 8.3, we may assume that n is at least two.

For a while, we fix an irreducible component E_0 whose normalization $\overline{E}_0$ is isomorphic to $\mathbb{P}^n$.

Lemma 8.8. *Let $f : C \to \overline{E}_0 \simeq \mathbb{P}^n$ be a morphism from a smooth complete (not necessarily rational) curve. Assume that the normalization morphism $\nu_0 : \overline{E}_0 \to E_0 \subset Z$ is unramified. Then there is a natural exact sequence*

$$0 \to f^*\Theta_{\mathbb{P}^n} \to f^*\nu_0^*\Theta_Z \to f^*\Omega^1_{\mathbb{P}^n} \to 0.$$

If f is non-constant, then we have $\mathrm{H}^0(C, f^*\nu_0^*\Theta_Z) = \mathrm{H}^0(C, f^*\Theta_{\mathbb{P}^n})$. If, in addition, $\mathrm{H}^1(C, f^*\Theta_{\mathbb{P}^n}) = 0$, then $\operatorname{Hom}(C, Z)$ is smooth at $[\nu_0 f]$ and is locally (in Zariski topology) identified with $\operatorname{Hom}(C, \overline{E}_0)$.

Proof. Trivial. Q.E.D.

Let $\nu : \tilde{E} = \coprod \overline{E}_i \to E = \bigcup E_i$ be the normalization morphism, with $\nu_i : \overline{E}_i \to E_i$ being the normalization of each irreducible component.

Corollary 8.9. *Assume that the normalization morphism $\nu_0 : \overline{E}_0 \simeq \mathbb{P}^n \to E_0$ is unramified. Then the singular locus $\operatorname{Sing}(E_0)$ of E_0 and the intersection $E_0 \cap (E \setminus E_0)$ are both zero-dimensional.*

Proof. Consider an irreducible curve $\hat{C}_0$ on $E_0 \subset E$, with the normalization C_0. Let $C_{i\alpha} \subset \overline{E}_i$ be the normalization of a one-dimensional irreducible component[33] of $\nu_i^{-1}(\hat{C}_0)$, $\alpha = 1, \ldots, m(i)$. Because of the

[33] Of course $\nu_i^{-1}(\hat{C}_0)$ could contain extra zero-dimensional components.

unramifiedness condition, we see that $\hat{C}_0$ sits in the singular locus of E if and only if $\coprod_{i,\alpha} C_{i\alpha}$ is not isomorphic to C_0.

Take an arbitrary irreducible component C' of the fibre product of the $C_{i\alpha}$ over C_0. Let $\tilde{C}$ be the Galois closure of the finite covering $C' \to C_0$ and let $\tilde{f}_{i\alpha} : \tilde{C} \to C_{i\alpha}$ denote the canonical projection. In short, we take a Galois cover $\tilde{C} \to C_0$ which gives a commutative diagram

$$\begin{array}{ccc} \tilde{C} & \xrightarrow[\tilde{f}_{i\alpha}]{} & C_{i\alpha} \\ \downarrow{\scriptstyle \tilde{f}_{j\beta}} & & \downarrow{\scriptstyle \nu_i} \\ C_{j,\beta} & \xrightarrow[\nu_j]{} & C_0 \end{array}$$

for any quadruple $(i, j; \alpha, \beta)$. Since $\tilde{C}$ is Galois over C_0 with Galois group Γ, we can write $C_0 = \tilde{C}/\Gamma$, $C_{i\alpha} = \tilde{C}/\Gamma_{i\alpha}$, where $\Gamma_{i\alpha} \subset \Gamma$ is a subgroup.

We consider the modulo p reductions of Z, C_0, $\tilde{C}$ etc., where p is a sufficiently large prime number. Let $\Phi : C \to \tilde{C}$ be the geometric Frobenius and put $f_{i\alpha} = \tilde{f}_{i\alpha}\Phi$. Since $\Theta_{\overline{E}_0}$ is ample, $\mathrm{H}^1(C, f_{0\alpha}\Theta_{\overline{E}_0})$ vanishes provided the prime number p is sufficiently large. Thus Lemma 8.8 applies to show that $\mathrm{Hom}(C, Z)$ is locally irreducible at $\nu_i f_{i\alpha} = \nu_j f_{j\beta}$, and locally identified with the germ $\big(\mathrm{Hom}(C, \overline{E}_0), [\nu_0 f_{0\alpha}]\big)$. The morphism $f = \nu_i f_{i\alpha} : C \to Z$ does not depend on the choice of the indices (i, α) corresponding to $f_{i\alpha}$. This shows that $\mathrm{Hom}(\mathbb{P}^1, \overline{E}_i)$, which is a subset of $\mathrm{Hom}(\mathbb{P}^1, Z)$, is identical with $\mathrm{Hom}(\mathbb{P}^1, \overline{E}_{i,0})$ at $[f_{i\alpha}]$ for $i \neq 0$, where $\overline{E}_{i,0} \subset \overline{E}_i$ is the inverse image of $E_i \cap E_0$.

Recall that $\overline{E}_i$ is a finite quotient $\mathbb{P}^n/G_i$, $G_i \subset \mathrm{Aut}(\mathbb{P}^n)$. If the characteristic p is sufficiently large (for example $p \gg |G_i|, |\Gamma|$), we can construct a finite cover $C^* \to C \simeq C^*/G_i'$, $G_i' \subset G_i$, which completes the commutative diagram

$$\begin{array}{ccccc} C^* & \longrightarrow & \overline{C \times_{\overline{E}_i} \mathbb{P}^n} & \longrightarrow & \mathbb{P}^n \\ & & \downarrow & & \downarrow \\ & & C & \longrightarrow & \overline{E}_i. \end{array}$$

(Namely, C^* is an irreducible component of the normalization of $C \times_{\overline{E}_i} \mathbb{P}^n$, and $G_i \subset G \subset \mathrm{Aut}(\overline{C \times_{\overline{E}_i} \mathbb{P}^n})$ is the stabilizer of C^*.) Then we have a natural morphism $\mathrm{Hom}(C^*, \mathbb{P}^n)^{G_i} \to \mathrm{Hom}(C, \overline{E}_i)$. Noting the ampleness of $\Theta_{\mathbb{P}^n}$, it turns out there are lot of G_i-invariant (more adequately, G_i-equivariant) morphisms $C^* \to \mathbb{P}^n$ or, equivalently, lot of deformation of $f_{i\alpha} : C \to \overline{E}_i$, when p is sufficiently large. This contradicts $\mathrm{Hom}(C, \overline{E}_i) = \mathrm{Hom}(C, \overline{E}_{i,0})$, the existence of $C_{i\alpha}$, $\tilde{f}_{i\alpha}$, $i \neq 0$ assumed. Thus any curve $\hat{C}_0 \subset E_0$ is not contained in other components E_i, i.e., $E_0 \cap (E \setminus E_0)$ is zero-dimensional.

We check next that E_0 has no self-intersection of positive dimension. Since $\overline{E}_0 \simeq \mathbb{P}^n \to E_0$ is unramified, an irreducible component $C_{0\alpha}$ of $\nu_0^{-1}(\hat{C}_0)$ is an étale cover of C_0. We prove first that $C_{0\alpha}$ is isomorphic to C_0. Assume that $\Gamma = \mathrm{Gal}(\tilde{C}/C_0) \neq \Gamma_{0\alpha} = \mathrm{Gal}(\tilde{C}/C_{0\alpha})$. Then we can find $\tilde{\gamma} \in \Gamma \setminus \Gamma_{0\alpha}$. The automorphism $\tilde{\gamma}$ of $\tilde{C}$ lifts to an automorphism γ of C in a unique manner (recall that $C \to \tilde{C}$ was Frobenius). Thus $f_{0\alpha}\gamma \neq f_{0\alpha}$, while $\nu_0 f_{0\alpha}\gamma = \nu_0 f_{0,\alpha}$ by construction. This implies that a unique morphism $f = \nu_0 f_{0\alpha} \in \mathrm{Hom}(C, Z)$ corresponds to two different morphisms $\in \mathrm{Hom}(\mathbb{P}^1, \overline{E}_0)$, contradicting the local birational isomorphism $\mathrm{Hom}(C, Z) \simeq \mathrm{Hom}(C, \overline{E}_0)$ between smooth schemes. Hence each $C_{0\alpha}$ is isomorphic to C_0.

Finally, if there were two components $C_{0\alpha}$, $C_{0\beta}$, then the same argument shows that $f = \nu_0 f_{0\alpha} = \nu_0 f_{0\beta}$ would correspond to two different element of $\mathrm{Hom}(C, \overline{E}_0)$, another contradiction.

Summing up things together, we conclude that the inverse image of $\hat{C}_0 \subset E$ in the normalization $\coprod \overline{E}_i$ is a single curve $\hat{C}_{01}$ birational to C_0 in E_0, plus (possibly) zero-dimensional components. This shows the assertion. Q.E.D.

Lemma 8.10. *Let the notation be as above. Assume that the normalization map $\nu_i : \overline{E}_i \to E_i$ is unramified whenever $\overline{E}_i$ is isomorphic to $\mathbb{P}^n$. Then every E_i is smooth and isomorphic to $\mathbb{P}^n$. Two components E_i, E_j meet each other transversally at finitely many points and the associated dual graph*[34] *is a tree.*

Proof. First we prove that $\overline{E}_i \simeq \mathbb{P}^n$ for each i.

The index set I of the irreducible components E_i is a disjoint sum $I^+ \cup I^-$, where $\overline{E}_i \simeq \mathbb{P}^n$ if and only if $i \in I^+$. It suffices to derive a contradiction from the hypothesis $I^- \neq \emptyset$. Put $E^+ = \bigcup_{i \in I^+} E_i$, $E^- = \bigcup_{i \in I^-} E_i$. Lemma 8.9 asserts that $E^+ \cap E^-$ is a finite set. Assume that a rational curve $C \subset E^-$ has the minimum degree among all the rational curves in E^-. Let $f : \mathbb{P}^1 \to E^- \subset E \subset Z$ be the morphism obtained by the normalization of C. Since C is not contained in E^+, we have a local (set theoretical) i dentity around $[f]$:

$$\mathrm{Hom}(\mathbb{P}^1, Z) = \mathrm{Hom}(\mathbb{P}^1, E^-) = \mathrm{Hom}(\mathbb{P}^1, E_{i_0}) \text{ for some } i_0 \in I^-.$$

[34]The vertices are the irreducible components E_i and two vertices E_i and E_j are joined by $\sharp(E_i \cap E_j)$ edges.

Thus the deformations of f give an unsplitting, doubly dominant family of rational curves on E_{i_0}, so that $\overline{E}_{i_0} \simeq \mathbb{P}^n$, contradicting the definition of I^-. Thus every $\overline{E}_i$ is isomorphic to $\mathbb{P}^n$ and hence $\mathrm{Sing}(E)$ is zero-dimensional by virtue of Corollary 8.9.

Next we show the second assertion. Let $\mathfrak{I}_E \subset \mathcal{O}_Z$ be the defining ideal of E and t a large integer. The natural short exact sequence

$$0 \to \mathfrak{I}_E/\mathfrak{I}_E^t \to \mathcal{O}_Z/\mathfrak{I}_E^t \to \mathcal{O}_E \to 0$$

induces the cohomology exact sequence

$$\mathrm{H}^1(Z, \mathcal{O}_Z/\mathfrak{I}_E^t) \to \mathrm{H}^1(E, \mathcal{O}_E) \to \mathrm{H}^2(Z, \mathfrak{I}_E/\mathfrak{I}_E^t).$$

Since $(\hat{Z}, o)$ is a rational singularity, the image the first term $\mathrm{H}^1(Z, \mathcal{O}_Z/\mathfrak{I}_E^t)$ in $\mathrm{H}^1(E, \mathcal{O}_E)$ vanishes for sufficiently large t. The sheaf $\mathfrak{I}_E/\mathfrak{I}_E^t$ is a successive extension of the $\mathfrak{I}_E^s/\mathfrak{I}_E^{s+1}$, $s = 1, \dots, t-1$, which are identical with the $\mathrm{Sym}^s\Theta_{\mathbb{P}^n}$ outside finitely many points. Because of the well known vanishing of $\mathrm{H}^q(\mathbb{P}^n, \mathrm{Sym}^s\Theta_{\mathbb{P}^n})$, $q > 0$, the third term $\mathrm{H}^2(Z, \mathfrak{I}_E/\mathfrak{I}_E^t)$ also vanishes.[35] Thus the middle term $\mathrm{H}^1(E, \mathcal{O}_E)$ is zero.

On the other hand, the normalization $\nu : \coprod \overline{E}_i \to E$, $\overline{E}_i \simeq \mathbb{P}^n$ induces the exact sequence

$$0 \to \mathcal{O}_E \to \bigoplus_i \nu_{i*}\mathcal{O}_{\overline{E}_i} \to \mathcal{S} \to 0,$$

where $\mathcal{S}$ is a skyscraper sheaf supported by the finitely many singular points of E. The equality $\mathrm{H}^1(E, \mathcal{O}_E) = 0$ holds if and only if $\bigoplus \mathrm{H}^0(\overline{E}_i, \mathcal{O}_{\overline{E}_i})/\mathrm{H}^0(E, \mathcal{O}_E) \simeq \mathbb{C}^{N-1} \to \mathcal{S}$ is a surjection, where N is the number of the irreducible components. Now it is an easy exercise to check that this condition is satisfied only if each component E_i is smooth, meeting other components transversally, and the associated dual graph is a tree. Q.E.D.

Lemma 8.11. *Assume that* $n \geq 2$ *and that* $\nu : \overline{E} = \coprod \overline{E}_i \to E$ *is unramified whenever* $\overline{E}_i \simeq \mathbb{P}^n$. *Then* E *is a single* $\mathbb{P}^n$.

Proof. Since $\hat{Z}$ is normal, the exceptional locus E is connected by Zariski's main theorem. Therefore, if E is reducible, then there exist irreducible components E_1 and E_2 which mutually meet at an isolated point p. Consider the blowing-up $\mu : \tilde{Z} \to Z$ along E_1. By Lemma 8.2, we can blow down $\tilde{Z}$ in another direction to get a new symplectic manifold Z'. The strict transform $\mu_{\mathrm{strict}}^{-1}(E_2) \subset \tilde{Z}$ contains a

[35] N.B.: We cannot conclude the vanishing of $\mathrm{H}^1(Z, \mathfrak{I}_E/\mathfrak{I}_E^t)$ because of the difference between $\mathfrak{I}_E^s/\mathfrak{I}_E^{s+1}$ and $\mathrm{Sym}^s\Theta_{\mathbb{P}^n}$ in dimension zero.

divisor $\mu_{\text{strict}}^{-1}(E_2) \cap \mu^{-1}(E_1)$. Then the blowing-down $\mu' : \tilde{Z} \to Z'$ keeps the $\mu^{-1}(p) \simeq \mathbb{P}^{n-1}$ untouched, so that the strict transform $E'_2 \subset Z'$ of $E_2 \subset Z$ is a blown up $\mathbb{P}^n$, a smooth variety not isomorphic to $\mathbb{P}^n$ (note that $n > 1$) nor to a finite quotient of $\mathbb{P}^n$. The naturally induced birational map $Z' \dashrightarrow \hat{Z}$ is well defined as a set theoretic map, and indeed a morphism by the normality of $\hat{Z}$. Then we can apply Corollary 8.7 to Z' to infer that $\overline{E}_2 \simeq E'_2$ must be $\mathbb{P}^n$ or a finite quotient of $\mathbb{P}^n$, a contradiction. Q.E.D.

The proof of Theorem 8.3 is now reduced to the following assertion:

Theorem 8.12. *The normalization map $\nu : \mathbb{P}^n \to X$ is unramified; in other words, the generically injective map $\nu : \mathbb{P}^n \to Z$ is an immersion.*

The proof of this theorem is given in Section 10 below.

Remark 8.13. It would be natural to ask if Theorem 8.3 can be localized. Most steps of our proof in fact apply also to local situations (germs of isolated singularities, say). The compactness of the symplectic manifold Z enters simply to ensure that $\dim_{[f]} \mathrm{Hom}(\mathbb{P}^1, Z) \geq 2n + 1$ for every non-constant morphism $f : \mathbb{P}^1 \to Z$.

Let $(Z, E) \to (\hat{Z}, o)$ be a projective symplectic resolution of the germ of an isolated singularity of dimension $2n$. The condition of trivial canonical bundle guarantees that $\dim_{[f]} \mathrm{Hom}(\mathbb{P}^1, Z) \geq 2n$ for arbitrary non-constant f, from which we deduce:

a) The exceptional locus E has pure dimension n and each component of it is rationally connected;
b) E is Lagrangian in Z.

These property strongly suggest that the normalization of E is a disjoint union of finite quotients of projective space $\mathbb{P}^n$ or of (possibly singular) hyperquadrics (see Remark 5.3).

9. Symplectic resolution of non-isolated singularities

Let Z be a compact, Kähler, complex symplectic manifold with symplectic form η and let $\pi : Z \to \hat{Z}$ be a projective bimeromorphic morphism onto a normal, Kähler complex space $\hat{Z}$. Let $E_0 \subset Z$ be an irreducible component of the exceptional locus E and B_0 its image in $\hat{Z}$.

Let us fix the notation as follows:

q : a general point $\in B_0$.
X : the general fibre $(\pi|_{E_0})^{-1}(q)$.
p : a general point on X (and hence general in E_0).
$a := \dim X$.
$b := \dim B_0$.

In the notation above, we prove:

Theorem 9.1. *We have $b = 2n - 2a$. An open dense subset U of the smooth locus of B_0 carries a natural symplectic structure induced by η. A general fibre X of the fibration $E_0 \to B_0$ is a union of copies of smooth $\mathbb{P}^a$. If $a = 1$, X is a tree of smooth $\mathbb{P}^1$'s with configuration of type* ADE. *If $a \geq 2$, then $X \simeq \mathbb{P}^a$, so that $E_0|_U$ is an étale $\mathbb{P}^a$-bundle over U.*

Roughly speaking, the exceptional set of a projective bimeromorphic morphism from a complex symplectic manifold is essentially a contraction of projective space and the symplectic structure of the source manifold is inherited by the resulting singular loci.[36]

Most part of the proof of Theorem 9.1 is exactly the same as that of Theorem 8.3 with some flavour of Section 7, and so the proof of the subsequent lemmas will be a little sketchy.

Lemma 9.2. *The exceptional set E_0 is uniruled. $\mathrm{R}^i\pi_*\mathcal{O}_Z = 0$ for $i > 0$ and the 2-form η is identically zero as a 2-form on $X \subset E_0$. (More precisely, the pullback of η to a non-singular model of X identically vanishes.)*

The proof is exactly the same as in Lemma 8.4 and Corollary 8.5, and left to the reader.

Let $C \subset X$ be a rational curve passing through a general smooth point $p \in X$, and assume that the degree of C attains minimum among such rational curves. Let $f : \mathbb{P}^1 \to X \subset E_0 \subset Z$ be the morphism induced by the normalization of C.

Lemma 9.3. *Let $\overline{E}_0$ be the normalization of the irreducible variety E_0 and $\overline{f} : \mathbb{P}^1 \to \overline{E}_0$ the map which $f : \mathbb{P}^1 \to E_0$ naturally induces (recall that C, which contains p, does not lie in the singular locus of E_0). If C or, equivalently, $f : \mathbb{P}^1 \to E_0$ is generally chosen, then*

$$\overline{f}^*\Omega^1_{\overline{X}} \simeq \mathcal{O}(-2) \oplus \mathcal{O}(-1)^{\oplus e-1} \oplus \mathcal{O}^{a-e},$$
$$\overline{f}^*\Omega^1_{\overline{E}_0} \simeq \mathcal{O}(-2) \oplus \mathcal{O}(-1)^{\oplus e-1} \oplus \mathcal{O}^{a+b-e},$$

where $1 \leq e \leq a$.

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[36](After a comment of D. Barlet) The compact variety B_0 is a *symplectic variety* in the following sense: (a) η can be viewed as a (meromorphic) d-closed 2-form on B_0; (b) If $\pi : E_0 \to B_0$ is flat over $V \subset (B_0)_{\mathrm{smooth}}$, then $\eta|_V$ is a holomorphic symplectic form; and (c) η is holomorphic on B_0, i.e., for any compact 2-chain $\gamma \subset B_0$, we have $|\int_\gamma \eta| < +\infty$. These properties can be easily verified from the subsequent proof of Theorem 9.1.

Proof. Any small deformation f_t of the morphism $f : \mathbb{P}^1 \to Z$ has image inside E_0. In fact, $\pi f(\mathbb{P}^1)$ is a single point, and so is $\pi f_t(\mathbb{P}^1)$ thanks to the Kähler condition on $\hat{Z}$, meaning that a curve $f_t(\mathbb{P}^1) \subset Z$ must stay in E_0 (actually in some closed fibre over a point $\in B_0$).

If we impose the condition that $f_t(\mathbb{P}^1)$ contains $p \in X$, then $f_t(\mathbb{P}^1)$ necessarily sits in X. Since $\overline{f}$ is general, its image $\overline{f}(\mathbb{P}^1)$ does not meet the singular locus of $\overline{E}_0$. Hence Theorem 2.8 applies to get the direct sum decomposition of $\overline{f}^*\Omega^1_{\overline{E}_0}$ as above. In particular, the deformation of f with base condition $\infty \mapsto p$ is unobstructed, and $\mathrm{H}^0(\mathbb{P}^1, \overline{f}^*\Theta_{\overline{E}_0}(-(\infty)))$ generates a subspace of $\overline{f}^*\Theta_X$, meaning that $e \leq a$. Q.E.D.

Lemma 9.4. *The integer e being as in Lemma 9.3, we have the equality $e = 2n-a-b$ or, equivalently, $\dim E_0 = 2n-e$. The kernel $\mathcal{K}$ of the natural homomorphism $f^*\Omega^1_Z \to \overline{f}^*\Omega^1_{\overline{E}_0}$ is ample $\simeq \mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus e-1}$. We have an isomorphism*

$$f^*\Omega^1_Z \simeq \mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus e-1} \oplus \mathcal{O}^{\oplus 2n-2e} \oplus \mathcal{O}(-1)^{\oplus e-1} \oplus \mathcal{O}(-2).$$

Proof. Since a small deformation of C stays in E_0, we have

$$a+b+e+1 = \dim_{[\overline{f}]} \mathrm{Hom}(\mathbb{P}^1, \overline{E}_0) = \dim_{[f]} \mathrm{Hom}(\mathbb{P}^1, Z) \geq 2n+1,$$

so that

$$\operatorname{rank} \mathcal{K} = 2n - a - b \leq e.$$

Then the existence of the symplectic form, or the self-duality of $f^*\Omega^1_Z$, implies the assertion. Q.E.D.

Corollary 9.5. *$f^*\eta$, viewed as a bilinear form on $\overline{f}^*\Theta_{\overline{E}_0}$, is a degenerate form of rank $2n-e$, and determines a non-degenerate bilinear form on $\overline{f}^*\Theta_{\overline{E}}/\mathcal{P}$, where $\mathcal{P} \simeq \mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus e-1}$ is the maximal ample subbundle.*

Proof. An easy exercise. Q.E.D.

Since $\eta_0 = \eta|_{\Theta_{\overline{E}_0} \times \Theta_{\overline{E}_0}}$ is a well-defined skew-symmetric bilinear form, this means

Corollary 9.6. *η_0 is a degenerate skew-symmetric bilinear form of rank $2n-2e$ on an open subset $U \supset \overline{f}(\mathbb{P}^1)$ of $\overline{E}_0$, and the subsheaf*

$$\mathcal{P} = \{v \in \Theta_{\overline{E}_0}; \eta_0(v, *) \equiv 0\} \subset \Theta_{\overline{E}_0}$$

is locally free of rank e on U.

Corollary 9.7. *Let $S\langle p\rangle \subset \mathrm{Chow}(\tilde{E}_0)$ be the closed subset parameterizing the rational curves of minimum degree through a general point*

$p \in \tilde{E}_0$, and $F\langle p\rangle \to S\langle p\rangle$ the associated family. (For simplicity we assume $S\langle p\rangle$ is irreducible.) Let $U \subset F\langle p\rangle$ denote a small open neighbourhood of $\{[C]\}\times C$. Then the natural image $\overline{X}_p$ of U on $\overline{X}$ is smooth at p as well as at a general point $p' \in \overline{X}_p$, with tangent spaces exactly $\mathcal{P}\otimes\mathbb{C}(p)$ and $\mathcal{P}\otimes\mathbb{C}(p')$, respectively.

The dimension of $\overline{X}_p$ is clearly e and it is doubly covered by rational curves of minimum degree. Hence by Theorem 0.2, the normalization of $\overline{X}_p$ is a finite quotient of $\mathbb{P}^e$.

Lemma 9.8. *$e = a$, so that the Zariski closure of $\overline{X}_p$ is $\overline{X}$ and* $\mathrm{pr}_Z(F) = E_0$.

Proof. As before, we denote the normalization by putting overlines. Recall that η induces a non-degenerate pairing on $\Theta_{\overline{E}_0}/\mathcal{P}$, a sheaf of rank $2n-2e$. By Lemma 8.2 above, η identically vanishes on $\Theta_{\overline{X}}\times\Theta_{\overline{X}}$. If $\mathcal{P}$ is strictly smaller than $\Theta_{\overline{X}}$, then η induces a non-zero pairing

$$\Theta_{\overline{X}}\times(\Theta_{\overline{E}_0}|_{\overline{X}}/\Theta_{\overline{X}})\to\mathcal{O}_{\overline{X}}.$$

Our fibre space structure $\overline{E}_0\to\overline{B}_0$ gives an isomorphism

$$\mathcal{Q} := \Theta_{\overline{E}_0}|_{\overline{X}}/\Theta_{\overline{X}}\simeq\mathcal{O}^{\oplus b}$$

in an obvious manner. Then there exists a global section θ of $\mathcal{Q}$ such that $\eta(\cdot,\theta)$, viewed as a linear form on $\Theta_{\tilde{X}}$, is not identically zero; in other words, $\eta(\cdot,\theta)$ is a global d-closed 1-form on $\overline{X}$ or, more precisely, on a smooth model $\tilde{X}$ of $\overline{X}$. This is ruled out by the property $\mathrm{R}^1\pi_*\mathcal{O}_Z = 0$ in view of Lemma 9.9 below. This contradiction comes from the hypothesis $\mathcal{P}\neq\Theta_{\overline{X}}$. Q.E.D.

Lemma 9.9. *Let $\pi : W\to\hat{W}$ be a bimeromorphic projective morphism from a manifold W to a normal complex space. Let $T\subset W$ be the inverse image of a point $o\in\hat{W}$ (we equip T with reduced structure). If $\mathrm{R}^1\pi_*\mathcal{O}_W = 0$, then there is no non-zero* d*-closed holomorphic* 1*-form on T.*[37]

Proof. Let $U\subset W$ be a sufficiently small open neighbourhood of T. Consider the truncated De Rham exact sequence

$$0\to\mathbb{C}\to\mathcal{O}\to\mathrm{d}\mathcal{O}\to 0$$

[37]More precisely, if $\{U_i\}$ is a collection of open subsets of W which covers T and ζ_i is a d-closed holomorphic 1-form on U_i such that $\zeta_i=\zeta_j$ on $T\cap U_i\cap U_j$, then the pullback of ζ_i to a resolution $\tilde{T}$ of any component of T is identically zero.

and the associated commutative diagram of cohomology groups

$$\begin{array}{ccccccccc}
\mathrm{H}^0(\mathbb{C}_U) & \longrightarrow & \mathrm{H}^0(\mathcal{O}_U) & \longrightarrow & \mathrm{H}^0(\mathrm{d}\mathcal{O}_U) & \longrightarrow & \mathrm{H}^1(\mathbb{C}_U) & \longrightarrow & \mathrm{H}^1(\mathcal{O}_U) \\
\simeq\downarrow & & \downarrow & & \downarrow & & \simeq\downarrow & & \downarrow \\
\mathrm{H}^0(\mathbb{C}_T) & \longrightarrow & \mathrm{H}^0(\mathcal{O}_T) & \longrightarrow & \mathrm{H}^0(\mathrm{d}\mathcal{O}_T) & \longrightarrow & \mathrm{H}^1(\mathbb{C}_T) & \longrightarrow & \mathrm{H}^1(\mathcal{O}_T)
\end{array}$$

of which the rows are exact. By our hypothesis, $\mathrm{H}^1(U, \mathcal{O}_U) = 0$, so that the edge homomorphism $\mathrm{H}^0(T, d\mathcal{O}_T) \to \mathrm{H}^1(T, \mathbb{C}_T)$ is surjective. It is injective as well because of the isomorphisms $\mathrm{H}^0(\mathbb{C}_T) \simeq \mathrm{H}^0(\mathcal{O}_T) \simeq \mathbb{C}$. Thus we have $\mathrm{H}^0(\mathrm{d}\mathcal{O}_T) \simeq \mathrm{H}^1(\mathbb{C}_T)$.

Suppose that there is a non-zero element $\zeta \in \mathrm{H}^0(T, \mathrm{d}\mathcal{O}_T)$. Then its complex conjugate $\overline{\zeta}$ is of course d-closed, defining a cohomology class $\in \mathrm{H}^1(T, \mathbb{C}_T)$. If $\mathrm{H}^0(T, d\mathcal{O}_T) \neq 0$, then there is a resolution V of some component of T such that the pullbacks of $\mathrm{H}^0(T, d\mathcal{O}_T)$ and $\mathrm{H}^0(T, d\overline{\mathcal{O}}_T)$ are non-zero. Since V is projective, these two spaces are independent in $\mathrm{H}^1(V, \mathbb{C})$, and so are in $\mathrm{H}^1(T, \mathbb{C})$, contracting $\mathrm{H}^1(\mathbb{C}_T) \simeq \mathrm{H}^0(\mathrm{d}\mathcal{O}_T)$. Q.E.D.

Now we are in the position to photocopy the arguments in the previous section to get the following:

Lemma 9.10. *If $a = 1$, X is a tree of $\mathbb{P}^1$'s with configuration of ADE singularities. If $a \geq 2$, then X is a single $\mathbb{P}^a$.*

Proof. Let us check what kind of results were used in the proof of similar statements in the previous section. They were:

(1) The vanishing of $\mathrm{R}^i\pi_*\mathcal{O}_Z$, the counterpart of which was established in Lemma 9.2;
(2) Deformation argument on rational curves, which perfectly works also in this new context;
(3) The existence of a flop Lemma 8.2; and finally
(4) The unramifiedness of the normalization map $\overline{X}_i \simeq \mathbb{P}^a \to X_i$.

The unramifiedness will be proved in the next section. The existence of a flop is still valid by the following lemma. Q.E.D.

Lemma 9.11. *Let (Z, η) be a complex symplectic manifold of dimension $2n$. Suppose that Z contains $E_0 = \mathbb{P}^a \times \Delta^{2n-2a}$ as a closed subset, Δ^{2n-2a} denoting a small polydisc of dimension $2n-2a$, and that $\Theta_{\mathbb{P}^a}$ is exactly the null-space of $(\Theta_{E_0}|_{\mathbb{P}^a}, \eta)$. Assume that $a \geq 2$ and let $\mu : \tilde{Z} \to Z$ be the blow up along E_0. Then $\tilde{Z}$ has a blow down in another direction.*

Analytically locally Z looks like a symplectic product $W \times \Delta^{2n-2a}$ around E_0, and the proof is completely parallel to Lemma 8.2.

Remark 9.12. The vanishing of $\mathrm{R}^i\pi_*\mathcal{O}_Z$ is a fairly restrictive condition if combined with our result $\overline{X} = \mathbb{P}^a/G$. For example, we can easily show that if the normalization map $\nu : \overline{X} \to X$ is unramified outside finitely many points, then ν is everywhere unramified.

Example 9.13. We take up once more Hilbert schemes of a K3 surface.

Let S be a K3 surface and consider the Hilbert scheme $Z = \mathrm{Hilb}^r(S)$ parameterizing the closed subschemes with constant Hilbert polynomial $h(t) = r$. An element of $[z] \in \mathrm{Hilb}^r(S)$ is a zero-dimensional scheme defined by an ideal sheaf I_z such that $\dim_{\mathbb{C}}(\mathcal{O}/I_z) = r$. As was mentioned in Example 7.10, Z is a complex symplectic manifold.

By forgetting the scheme structure of z and viewing it as an effective 0-cycle of degree r, we get a natural morphism $\pi_r : Z \to \hat{Z} = \mathrm{Sym}^r(S) \simeq \mathrm{Chow}_0^r(S)$,

Let us study the structure of the birational morphism π_r for small values of r.

Case $r = 2$. If a 0-cycle $p_1 + p_2$ is supported by two distinct points, the corresponding scheme is uniquely defined by the ideal $\mathfrak{M}_{p_1}\mathfrak{M}_{p_2}$, so that π is an isomorphism over $\mathrm{Sym}^2S \setminus S$.[38] When $p_1 = p_2 = p$, then a closed subscheme of degree two supported by p is determined by an ideal I such that $\mathfrak{M}_p^2 \subset I \subset \mathfrak{M}_p$ and that $\dim_{\mathbb{C}} \mathfrak{M}_p/I = 1$. It follows that $\pi_2^{-1}([2p])$ is naturally isomorphic to $\mathbb{P}(\mathfrak{M}_p/\mathfrak{M}_p^2) \simeq \mathbb{P}^1$, and π_2 has a $\mathbb{P}^1$-bundle structure over the diagonal $S \subset \mathrm{Sym}^2S$. Thus the exceptional locus E of the birational contraction π_2 is a smooth $\mathbb{P}^1$-bundle over S.

Suppose that S contains a (-2)-curve. Then we can contract this curve to get a normal surface S'. The symmetric product Sym^2S' is a normal variety with a unique point q (of course singular) such that the projection $\mathrm{Sym}^2S \to \mathrm{Sym}^2S'$ has $\mathbb{P}^2$ as a fibre over q. Looking at the construction closely, we find that the exceptional locus of the birational contraction $\pi_2' : \mathrm{Hilb}^2(S) \to \mathrm{Sym}^2S'$ consists of two irreducible components, one of which being a $\mathbb{P}^1$-bundle over S and the other being $\mathbb{P}^2$ over q. The two components meet each other along a smooth quadric in $\mathbb{P}^2$.

Case $r = 3$. Let us look at the fibre of π_3 over a cycle $\gamma = p_1 + p_2 + p_3$ on S.

If the three points are mutually distinct, then the fibre is a single point corresponding the ideal $I_y = \mathfrak{M}_{p_1}\mathfrak{M}_{p_2}\mathfrak{M}_{p_3}$.

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[38]Here the image of the diagonal $\Delta_S \subset S \times S$ in Sym^2S is identified with S.

When $p = p_1 = p_2 \neq p_3$, the fibre $\pi^{-1}(\gamma)$ is a scheme defined by $I\mathfrak{M}_{p_3}$, where $\mathfrak{M}_p^2 \subset I \subset \mathfrak{M}_p$ and $\dim \mathcal{O}/I = \dim \mathfrak{M}_p/I + 1 = 2$. Hence there is a one-to-one correspondence between $\pi^{-1}(\gamma)$ and the set of one-dimensional subspace in $\mathfrak{M}_p/\mathfrak{M}_p^2 \simeq \mathbb{C}^2$, which is nothing but $\mathbb{P}^1$.

In the most degenerate case $p = p_1 = p_2 = p_3$, the fibre $E_0 = \pi_3^{-1}(\gamma)$ consists of the ideals I such that $\mathfrak{M}_p^3 \subset I \subset \mathfrak{M}_p$ and that $\dim \mathfrak{M}_p/I = 2$. Hence we have an injection from E_0 into the Grassmann variety $\mathrm{Grass}(\mathfrak{M}_p/\mathfrak{M}_p^3, 2)$. There is a naturally marked closed point $[\mathfrak{M}_p^2]$ in E_0. This point represents a (unique) closed subscheme of length three which is supported by p but not locally complete intersection on S.

Assume that the ideal I is not contained in $\mathfrak{M}_p^2$ or, equivalently, that $I/(I \cap \mathfrak{M}_p^2) \neq 0$. By Nakayama's lemma,

$$\dim_{\mathbb{C}}(I + \mathfrak{M}_p^2)/\mathfrak{M}_p^2) = \dim_{\mathbb{C}} \mathfrak{M}_p/\mathfrak{M}_p^2 = 2$$

would mean the absurd equality $I = \mathfrak{M}_p$, and so $I + \mathfrak{M}_p^2/\mathfrak{M}_p^2$ must be one-dimensional. Consequently $(I \cap \mathfrak{M}_p^2)/\mathfrak{M}_p^3$ is necessarily two-dimensional. Let (x, y) be a local coordinate system. Let

$$w = ax + by + q(x, y) \in \mathfrak{M}_p/\mathfrak{M}_p^3$$

be an element of $I/\mathfrak{M}_p^3$ such that

(1) a, b are constants $\in \mathbb{C}$ and $(a, b) \neq (0, 0)$,

(2) $q(x, y)$ is a homogeneous quadratic polynomial in (x, y).

Then this element w uniquely determines

$$I = \mathcal{O}_S w + \mathfrak{M}_p^3 = \big(w, (\bar{b}x - \bar{a}x)^3\big),$$

where $\bar{a}, \bar{b}$ are the complex conjugates of a, b. In fact, the three-dimensional vector space $I/\mathfrak{M}_p^3$ has basis $(ax + by + q(x, y), (ax + by)x, (ax+by)y)$. Given I, the choice of the linear part (a, b) is unique up to non-zero factor. The choice of the quadratic part $q(x, y)$ involves, however, indeterminacy. Indeed, we can change it by adding a quadric of the form $(ax+by)(tx+uy)$, where t, u are arbitrary constants. Thus we have a natural projection $(E_0 \setminus [\mathfrak{M}_p^2]) \to \mathbb{P}^1$, $[I] \mapsto (ax + by) \in \mathbb{P}(\mathfrak{M}_p/\mathfrak{M}_p^2)^*$, of which the fibre is isomorphic to the one-dimensional vector space $\mathfrak{M}_p^2/\big((ax + by)\mathfrak{M}_p + \mathfrak{M}_p^3\big) \simeq \mathbb{C}$. The closed subset $E_0 \subset \mathrm{Hilb}^3(S)$ is therefore a compactification of a $\mathbb{C}$-bundle over $\mathbb{P}^1$ by the single point $[\mathfrak{M}_p^2]$, implying that $E_0 \simeq \mathbb{P}^2$.

In order to have a global picture of the exceptional locus of π_3, we have to check what happens to the point $[I_p\mathfrak{M}_{p_3}]$ $(\mathfrak{M}_p^2 \subsetneqq I_p \subsetneqq \mathfrak{M}_p)$ when p_3 tends to p.

As a reference point $\in \pi_3^{-1}(3p) \subset \mathrm{Hilb}^3(S)$, we take the ideal I generated by (x, y^3) (i.e., $a = 1, b = q(x,y) = 0$), which is a local complete intersection. Hence its universal infinitesimal deformation[39]is given by $\mathrm{Hom}(I/I^2, \mathcal{O}/I) \simeq \mathbb{C}^6$. Recalling the construction of the universal deformation, we have the following explicit description of the family of ideals I_t parameterized by six parameters $t_1, \ldots, t_6$:

$$I_t = (x + t_1 y^2 + t_2 y + t_3, y^3 + t_4 y^2 + t_5 y + t_6).$$

The condition that the scheme is supported by one or two points is thus described by the vanishing of the discriminant $t_4^2 t_5^2 + 18 t_4 t_5 t_6 - 4t_5^3 - 4t_4^3 t_6 - 27 t_6^2$ of the second cubic polynomial in y, thereby defining an locally irreducible hypersurface on the parameter space $\mathbb{C}^6$.

The above observations show that

(1) the exceptional locus of the projection $\pi_3 : \mathrm{Hilb}^3(S) \to S^{(3)}$ consists of a single irreducible component E; that
(2) E is stratified into the disjoint sum of a $\mathbb{P}^1$-bundle over the symplectic 4-fold $S^{(2)} \setminus S$ (the bigger diagonal in $S^{(3)}$) and a $\mathbb{P}^2$-bundle over S (the smaller diagonal); and that
(3) the $\mathbb{P}^a$ bundle structure of the exceptional locus E is not globally defined.

10. Unramifiedness of the normalization map

In this section, we give the deferred proof of the unramifiedness of the normalization map attached to the exceptional locus of a symplectic resolution.

Let $\pi\ Z \to \hat{Z}$ be a projective symplectic resolution of a projective normal variety of dimension $2n$, and let $E \subset Z$ be the exceptional locus with image B in $\hat{Z}$. Take a general point $b \in B$. Then $X = \pi^{-1}(b)$ is a union of a-dimensional Lagrangian subvarieties X_i in a $2a$-dimensional symplectic submanifold $\subset Z$. Their normalizations $\overline{X}_i$ are isomorphic to finite quotients of $\mathbb{P}^a$ and there is at least one component, say X_0, such that $\overline{X}_0 \simeq \mathbb{P}^a$. If $\overline{X}_i$ is not $\mathbb{P}^a$, then X_i necessarily meets another component X_j along a positive dimensional subset.

In the above notation, we show the following

Theorem 10.1. *The normalization map*

$$\nu_0 : \overline{X}_0 \to X_0 \subset X = \pi^{-1}(b) \subset Z$$

is unramified if $b \in B$ is general and if $\overline{X}_0 \simeq \mathbb{P}^a$.

[39]By Theorem 1.6, $\mathrm{Hilb}^r(X)$ is smooth at $[\gamma]$ provided the finite scheme γ is locally complete intersection.

When X has only isolated singularities, this theorem is easy to prove:

Lemma 10.2. *If X has only isolated singularities, then X is a union of smooth $\mathbb{P}^a$.*

Proof. Note that $\overline{X}_i \simeq \mathbb{P}^a$ because the intersection $X_i \cap X_j$ is a finite set for all $j \neq i$.

Consider the short exact sequence $0 \to \mathfrak{I}_X \to \mathcal{O}_Z \to \mathcal{O}_X \to 0$ and the associated long exact sequence

$$0 = \mathrm{R}^1\pi_*\mathcal{O}_Z \to \mathrm{H}^1(X, \mathcal{O}_X) \to \mathrm{R}^2\pi_*\mathcal{I}_Z.$$

Outside the finitely many singular points, all the components X_i are isomorphic to $\mathbb{P}^a$ and mutually disjoint. Therefore

$$\mathrm{H}^2(X, \mathfrak{I}_X^i/\mathfrak{I}_X^{i+1}) \simeq \mathrm{H}^2(\mathbb{P}^a, \mathrm{Sym}^i\Theta_{\mathbb{P}^a})^{\oplus s} = 0, \quad i = 1, 2, \ldots,$$

where s is the number of the irreducible components X_i. In particular, $\mathrm{R}^2\pi_*\mathfrak{I}_X/\mathfrak{I}_X^k$ vanishes for k very large, so that $\mathrm{R}^2\pi_*\mathfrak{I}_X = 0$. This shows that $\mathrm{H}^1(X, \mathcal{O}_X) = 0$.

On the other hand, the normalization map gives the exact sequence

$$0 \to \mathcal{O}_X \to \nu_*(\mathcal{O}_{\mathbb{P}^a})^{\oplus s} \to \mathcal{S} \to 0,$$

where $\mathcal{S}$ is a skyscraper sheaf. Taking cohomology, we have the exact sequence

$$0 \to \mathbb{C}^{s-1} \to \mathrm{H}^0(X, \mathcal{S}) \to \mathrm{H}^1(X, \mathcal{O}_X) = 0,$$

which shows that the length of $\mathcal{S} = \nu_*\mathcal{O}_{\overline{X}}/\mathcal{O}_X$ is equal to $s-1$, s being the number of the irreducible components of X. Since X is connected, this is possible only if each X_i is smooth and any two components are disjoint or meet transversally at a single point. Q.E.D.

The key to the argument above is the simple fact that $\hat{Z}$ has only rational singularities. However, we are not sure if this rationality property directly entails Theorem 10.1 when X has singularities of positive dimension, and we indeed take a completely different approach in this case. While the proof of Lemma 10.2 was done without detailed information on the local structure of X, the core of our proof of Theorem 10.1 below is the local analysis of the singular locus of X. The results Theorem 10.3 through Proposition 10.6 deal with local reductions of singular Lagrangian subvarieties in general. With the aide of this reduction and the spannedness result Lemma 10.7 – Corollary 10.9 for defining ideals of exceptional loci, we show that the non-immersed points of $\nu_0 : \mathbb{P}^n \to Z$ have so special properties described in Lemma 10.10 – Lemma 10.16 that the existence of such points is eventually ruled out in Lemma 10.17.

For simplicity of the notation, we assume that $\hat{Z}$ has a single isolated singularity in what follows. In particular, X is a union of Lagrangian subvarieties in Z. The general case is easily reduced to this special case.

In order to analyze the local property of $X \subset Z$ along the singular locus of X, we use the following *local primitive decomposition theorem,* or the *Lagrangian reduction theorem* for Lagrangian subvarieties, which asserts that a Lagrangian subvariety with smooth normalization is locally a Lagrangian product of a smooth submanifold and a singular Lagrangian subvariety of maximal embedding dimension:

Theorem 10.3. *Let (M, η) be a complex symplectic manifold of dimension $2n$ and L a Lagrangian subvariety in M. Assume that the normalization $\overline{L}$ of L is smooth. Let $o \in L$ be a singular point of L and let e denote the embedding dimension of L at o. If we replace M by a small neibourhood of o, then there exist a symplectic manifold (M', η') of dimension $2n' = 2(e - n)$, a Lagrangian subvariety $L' \subset M'$, an e-dimensional submanifold $N \supset L$ of M and a projection $\sigma : N \to M'$ such that*

(1) *L' has embedding dimension $2n'$ at o'; that*
(2) *σ is a smooth fibration with fibres of dimension $2n - e$; and that*
(3) *$L = \sigma^{-1}(L') \subset N$.*

Proof. Our proof is the induction on $2n - e$. If $2n - e = 0$, we simply let $M' = M$, $L' = L$.

Assume that $e < 2n$. Then we have a smooth hypersurface $D \subset M$ which contains L. The restriction of η to the tangent bundle $\Theta_D \subset \Theta_M|_D$ is everywhere of rank $2n - 1$ and the null space $\mathrm{Null}_\eta \subset \Theta_D$ is everywhere one-dimensional, giving rise to an integrable foliation and a smooth fibration $\sigma_1 : D \to M_1$ with integral submanifolds of Null_η as fibres. Furthermore, the Lagrangian condition on L implies the inclusion relation $\mathrm{Null}_\eta|_L \subset \Theta_L$, and a fibre of σ is either contained in L or away from L. Hence L is the inverse image $\sigma^{-1}(L_1)$ of $L_1 = \sigma_1(L) \subset M_1$. The tangent space of M_1 is isomorphic to $\Theta_N/\mathrm{Null}_\eta$, on which η induces a non-degenerate pairing η_1. It is easy to show that η_1 is d-closed, so that M_1 is a symplectic manifold with Lagrangian subvariety L_1 of embedding dimension $e - 1$. L is locally a product of L_1 and $\mathbb{C}$. Then the induction hypothesis shows the assertions (1) through (3). Q.E.D.

Similarly we can prove the following

Lemma 10.4. *Let (M, o) be a complex symplectic manifold of dimension $2n$ and L a Lagrangian subvariety whose normalization $\overline{L}$ is smooth. Let $(R, o) \subset \overline{L}$ be a smooth subvariety of dimension r and assume that $\nu|_R : R \to M$ is an isomorphism. Take a general smooth* *complete intersection* $T_R \subset M$ *of codimension* r *transversally intersecting* $\nu(R)$ *at* o. *Then, analytically locally around* o, *there exists a smooth fibration* $\phi_R : T_R \to M^\dagger$ *over a symplectic manifold of dimension* $2n^\dagger = 2n - 2r$ *such that*

(1) $L^\dagger = \phi_R(L \cap T_R) \subset M^\dagger$ *is a Lagrangian subvariety whose normalization* $\nu^\dagger : \overline{L}^\dagger \to L^\dagger$ *is a morphism from a smooth manifold and that*

(2) $L \cap T_R$ *is locally isomorphic to* $L^\dagger$ *and hence has embedding dimension* $\leq 2n - 2r$.

The proof is completely parallel to that of Theorem 10.3 and left to the reader.

Corollary 10.5. *In Lemma* 10.4, *let* R *be either an irreducible component of the discriminant locus* $\nu_L^{-1}(\mathrm{Sing}(L)) \subset \overline{L}$ *or an irreducible component of the ramification locus*

$$\mathrm{Ram}(\overline{L}) = \{p \in \overline{L};\ \ \nu_L : \overline{L} \to Z \ \ \textit{is not an immersion at } p\}$$

and o *a general point of* R. *Then the resulting Lagrangian subvariety* $L^\dagger$ *has embedding dimension* $2n^\dagger$.

Proof. The normalization of $L \cap T_R$ is a general complete intersection in $\overline{L}$ and hence smooth. The singular locus or branch locus of $L \cap T_R$ is the finite set $R \cap T_R$. If the embedding dimension e of the reduction $L^\dagger \simeq L \cap T_R$ is less than $2n^\dagger$, apply Theorem 10.3 to $L^\dagger$ to conclude that $L^\dagger$ is locally a product $\mathbb{C} \times$ (Lagrangian subvariety) and hence the discriminant locus or the ramification locus of $\overline{L}^\dagger \to L^\dagger$ cannot be zero-dimensional. Q.E.D.

Proposition 10.6. *In Lemma* 10.4, *assume that* $R \subset \overline{L}$ *is an irreducible component either of the ramification locus* $\mathrm{Ram}(L)$ *or of the discriminant locus* $\nu^{-1}(\mathrm{Sing}(L))$, *and let* o *be a general point in it. Then the resulting Lagrangian subvariety* $L^\dagger \subset M^\dagger$ *is isomorphic with the Lagrangian reduction* $L' \subset M'$ *obtained in Theorem* 10.3. *If* R *and* o *are an irreducible component of* $\mathrm{Ram}(L)$ *and a general point on it, the tangent map* $\mathrm{d}\nu_{L'} : \Theta_{\overline{L}'} \to \nu_{L'}^*\Theta_{M'}$ *is the zero map at the inverse image of the isolated branch points* $\in L'$.

Proof. $L \cap T_R$ is isomorphic to $L^\dagger$ and its embedding dimension is $2n^\dagger$. Since T_R is a smooth complete intersection defined by r equations $f_1 = \cdots = f_r = 0$, we have $\Omega^1_{L\cap T} = \mathcal{O}_{L\cap T} \otimes (\Omega^1_L/(\mathrm{d}f_1, \ldots, \mathrm{d}f_r)$ so that the embedding dimension of L is $2n^\dagger + r = \dim T_R < 2n$. Hence L has non-trivial Lagrangian reduction and $L \cap T_R$ is (locally) a section of the reduction, thus giving local isomorphisms $L' \simeq L \cap T_R \simeq L^\dagger$.

Let R be a component of $\mathrm{Ram}(L)$. If $d\nu_{L'}$ does not vanish at o, we can find a smooth curve $\Gamma \subset \overline{L'} \ni o$ such that $\nu|_\Gamma$ is an isomorphism into M'. Then apply Lemma 10.4 to (M', L', Γ) and we conclude that a general hypersurface of L' has embedding dimension $\leq \dim M' - 2$ or, equivalently, that the embedding dimension of L' is $\leq \dim M' - 1$, a contradiction. Q.E.D.

Let us return to our original situation as in Theorem 10.1. Take an irreducible component $X_0 \subset X$ such that $\overline{X_0} \simeq \mathbb{P}^n$. Let $\mathfrak{M} \subset \mathcal{O}_{\hat{Z}}$ be the maximal ideal defining the isolated singular point $o \in \hat{Z}$. Let $l+1$ be the number of the irreducible components X_i of X and let $\mathbf{a} = (a_0, \dots, a_l)$ be an $(l+1)$-tuple of positive integers. We write $\mathbf{b} \geq \mathbf{a}$ or $\mathbf{b} \gg \mathbf{a}$ if $b_i \geq a_i$ or $b_i \gg a_i$ for all i. For simplicity of the notation, we adopt the convention $s = (s, \dots, s)$ for a positive integer s. Let $\mathfrak{I}_X^{(\mathbf{a})} \subset \mathcal{O}_Z$ denote the $\mathbf{a}$-th *symbolic power* of the ideal $\mathfrak{I}_X$ (see [Matm, p.56]); *i.e.*, $\mathfrak{I}_X^{(\mathbf{a})} = \bigcap_i \mathfrak{I}_{X_i}^{(a_i)}$ and $\mathfrak{I}_{X_i}^{(a_i)} = \mathfrak{I}_{X_i}^a \mathcal{O}_{Z, \mathfrak{I}_{X_i}} \cap \mathcal{O}_Z$.

Then we have the following easy

Lemma 10.7. *Let $\hat{U} \subset \hat{Z}$ be an affine (or Stein) neibourhood of the singular point o and $U \subset Z$ its inverse image. Then*

(1) *Given positive integers $b \geq a$, the $\mathcal{O}_{\hat{U}}$-module $\mathfrak{M}^a/\mathfrak{M}^b$ and the $\mathcal{O}_U$-module $\pi^*(\mathfrak{M}^a/\mathfrak{M}^b)$ are generated by global sections.*

(2) *If $\mathbf{c} \gg b \gg \mathbf{a}$, then $\mathfrak{I}_X^{(\mathbf{c})} \subset \pi^*\mathfrak{M}^b \subset \mathfrak{I}_X^{(\mathbf{a})} \subset \pi^*\mathfrak{M} \subset \mathfrak{I}_X$.*

(3) *There exists $\mathbf{a} \gg 0$ such that $\mathcal{O}_X$-module $\mathfrak{I}_X^{(\mathbf{a})}/\mathfrak{I}_X^{(\mathbf{a}+\mathbf{e}_0))}$ is generated by global sections, where $\mathbf{e}_0 = (1, 0, \dots, 0)$.*

Proof. (1) and (2) are trivial. The statement (3) easily derives from the following three observations:

(1) $\pi^*\mathfrak{M}/\pi^*\mathfrak{M}^c$ and its quotient $\pi^*\mathfrak{M}/\mathfrak{I}_X^{(\mathbf{b})}$ are generated by global sections;
(2) The $\mathcal{O}_Z$-module $\pi^*\mathfrak{M}/\mathfrak{I}_X^{(\mathbf{b})}$ has filtrations with associated graded module $\bigoplus(\pi^*\mathfrak{M} \cap \mathfrak{I}_X^{(\mathbf{b}_j)})/(\pi^*\mathfrak{M} \cap \mathfrak{I}_X^{(\mathbf{b}_{j+1})})$, where $\mathbf{b}_0 = 1 \leq \mathbf{b}_1 \leq \cdots \leq \mathbf{b}_m = \mathbf{b}$ is an increasing sequence;
(3) Standard multiplications of the ring $\mathcal{O}_Z$ induce a natural $\mathcal{O}_X$-homomorphism $(\mathfrak{I}_X^{(\mathbf{j})}/\mathfrak{I}_X^{(\mathbf{j}+\mathbf{e})}) \otimes (\mathfrak{I}_X^{(\mathbf{k})}/\mathfrak{I}_X^{(\mathbf{k}+\mathbf{e})}) \to \mathfrak{I}_X^{(\mathbf{j}+\mathbf{k})}/\mathfrak{I}_X^{(\mathbf{j}+\mathbf{k}+\mathbf{e})}$, where $0 < \mathbf{e} \leq 1$.

The detail is left to the reader. Q.E.D.

Let $\nu_0 : \mathbb{P}^n \to X_0 \subset X \subset Z$ be the normalization map of an irreducible component $X_0 \subset X$. *From now on until Lemma* 10.16, *we assume that ν_0 is not an immersion so that the ramification locus* $\mathrm{Ram}(X_0) \subset \mathbb{P}^n$ *is non-empty.*

Recall that $\nu_0^*(\mathfrak{I}^{(\mathbf{a})}/\mathfrak{I}^{(\mathbf{a}+\mathbf{e}_0)})/(\text{torsion})$ is a subsheaf of $\mathrm{Sym}^a\Theta_{\mathbb{P}^n}$, where $a = a_0$, $\mathbf{a} = (a_0, a_1, \dots)$. Choose a general point p of an irreducible component R of $\mathrm{Ram}(X_0) \subset \mathbb{P}^n$. Let $(T_0 : T_1 : \cdots : T_n)$ be homogeneous linear coordinates of $\mathbb{P}^n$ such that $p = (1 : 0 : \cdots : 0)$. The linear functions $t_i = T_i/T_0$, $i = 1, \dots, n$, define affine coordinates at p, while t_0 is the constant function 1. The global vector fields on $\mathbb{P}^n$ are generated by

$$\theta_{ij} = T_i \frac{\partial}{\partial T_j} = \begin{cases} t_i \frac{\partial}{\partial t_i} & \text{if } j \neq 0 \\ -t_i(t_1 \frac{\partial}{\partial t_1} + \cdots + t_n \frac{\partial}{\partial t_n}) & \text{if } j = 0. \end{cases}$$

This means that if a global vector field θ on $\mathbb{P}^n$ vanishes at p to the second order, then θ is locally contained in the line bundle $\mathcal{L}$ generated by the Euler vector field $-\theta_{00} = t_1 \frac{\partial}{\partial t_1} + \cdots + t_n \frac{\partial}{\partial t_n} \in (t_1, \dots, t_n)\Theta_{\mathbb{P}^n}$.

Corollary 10.8. *There exists* $\mathbf{a} = (a, a_1, \dots) \geq 1$ *such that*

$$\nu_0^*(\mathfrak{I}_X^{(\mathbf{a})}/\mathfrak{I}_X^{(\mathbf{a}+\mathbf{e}_0)})/(\text{torsion}) + (t_1, \dots, t_n)^{a-1}\mathcal{L} \cdot \mathrm{Sym}^{a-1}\Theta_{\mathbb{P}^n} \supset (t_1, \dots, t_n)^a \mathrm{Sym}^a\Theta_{\mathbb{P}^n}$$

locally around p.

Proof. The $\mathcal{O}_{\mathbb{P}^n}$-module $\nu^*(\mathfrak{I}_X^{(\mathbf{a})}/\mathfrak{I}_X^{(\mathbf{a}+\mathbf{e}_0)})/(\text{torsion}) \subset \mathrm{Sym}^a\Theta_{\mathbb{P}^n}$ is generated by global sections and, at general points, coincides with the total space $\mathrm{Sym}^a\Theta_{\mathbb{P}^n}$. It is well known that a global section ξ of $\mathrm{Sym}^a\Theta_{\mathbb{P}^n}$ is a sum of the products of θ_{ij}, which have zero of order $\leq a$ at p modulo $\mathcal{L} \cdot \mathrm{Sym}^{a-1}\Theta_{\mathbb{P}^n}$. Q.E.D.

Let R denote the ramification locus of $\nu_0 : \mathbb{P}^n \to Z$ and r its dimension. Choose a general point p of an r-dimensional component of R. By Theorem 10.3 – Proposition 10.6, we find a small neighbourhood $N \subset Z$ of p and a Lagrangian reduction $\phi : (N, X_0, p) \to (M, L, o)$, where $L \subset M$ is a Lagrangian subvariety of dimension $m = n - r$ with a single isolated branch point $o = \phi(p)$. (Here we write X_0 instead of $X_0 \cap N$ by abuse of notation.)

Since X_0 is locally a product, we have local decompositions

$$\Theta_{\mathbb{P}^n} = \mathcal{F} \oplus \phi^*\Theta_{\overline{L}},$$
$$\mathfrak{I}_{X_0}/\mathfrak{I}_{X_0}^2 = \mathcal{F} \oplus \phi^*(\mathfrak{I}_L/\mathfrak{I}_L^2),$$

where $\mathcal{F}$ is a subbundle of $\Theta_{\mathbb{P}^n}$ such that $\mathcal{F}|_R = \Theta_R$. In particular, we have a locally defined sujection $\mathfrak{I}_X^{(a)}/\mathfrak{I}_X^{(a+1)} \to \phi^*(\mathfrak{I}_L^{(a)}/\mathfrak{I}_L^{(a+1)})$. If we let $\nu_L : \overline{L} \to L$ stand for the normalization, there is a natural inclusion

$$\nu_L^*(\mathfrak{I}_L^{(a)}/\mathfrak{I}_L^{(a+1)}) \subset (t_1, \dots, t_{n-r})\mathrm{Sym}^a\Theta_{\overline{L}}$$

and a local surjection $\Theta_{\mathbb{P}^n}|_{\nu_0^{-1}(W)} \to \Theta_{\overline{L}}$, where $t_1, \dots, t_{n-r}$ are local coordinates of $\overline{L}$ at $\phi(p)$ and $W \subset Z$ is a general smooth complete intersection of codimension r. (In what follows, we write p instead of $\phi(p)$ for simplicity of the notation.) The image of the Euler vector field $-\theta_{00}$ in $\Theta_{\overline{L}}$ is of the form

$$\sum_{i=1}^{n-r} t_i \frac{\partial}{\partial t_i}$$

modulo $(t_1, \dots, t_{n-r})^2 \Theta_{\overline{L}}$, thus generating a line bundle $\mathcal{L}_L \subset (t_1, \dots, t_{n-r})\Theta_{\overline{L}}$.

Corollary 10.9. *Let a be a large, sufficiently divisible integer. Then, locally around p, the $\mathcal{O}_{\overline{L}}$-module $(t_1, \dots, t_{n-r})^a \mathrm{Sym}^a \Theta_{\overline{L}}$ is generated by the submodule $(t_1, \dots, t_{n-r})^{a-1} \mathcal{L}_L \cdot \mathrm{Sym}^{a-1} \Theta_{\overline{L}}$ and the pullbacks of local sections of the $\mathcal{O}_L$-module $\mathfrak{I}_L^{(a)}/\mathfrak{I}_L^{(a+1)}$.*

Let $z_1, \dots, z_{2(n-r)}$ be local coordinates of $M \supset L$ at p. There is an integer $d \geq 2$ such that

$$(t_1, \dots, t_{n-r})^d \supset (z_1, \dots, z_{2(n-r)})\mathcal{O}_{\overline{L}} \not\subset (t_1, \dots, t_{n-r})^{d+1}.$$

After performing a linear coordinate change of $(z_1, \dots, z_{2(n-r)})$ if necessary, we may assume that the equivalence classes $\overline{z}_1, \dots, \overline{z}_s$ of $z_1, \dots, z_s$ are linearly independent in $(t_1, \dots, t_{n-r})^d/(t_1, \dots, t_{n-r})^{d+1} \simeq \mathbb{C}^{\binom{n-r+d-1}{d}}$ and that $\overline{z}_{s+1} = \cdots = \overline{z}_{n-r} = 0$.

Lemma 10.10. *Let the notation be as above. Then:*

(1) *The congruence relation*

$$\sum_{i=1}^{n-r} t_i \frac{\partial}{\partial t_i} \equiv d \sum_{j=1}^{2(n-r)} z_j \frac{\partial}{\partial z_j} \quad \mod (t_1, \dots, t_{n-r})^{d+1} \nu_0^* \Theta_M$$

holds.

(2) *The sheaf $(t_1, \dots, t_{n-r})^a \mathrm{Sym}^a \Theta_{\overline{L}}$, which is viewed as an $\mathcal{O}_M$-module, is generated by*

$$\mathfrak{I}_L^{(a)}/\mathfrak{I}_L^{(a+1)} + (t_1, \dots, t_{n-r})^{ad} \mathcal{L} \cdot \mathrm{Sym}^{a-1} \Theta_{\overline{L}} + (t_1, \dots, t_{n-r})^{ad+1} \mathrm{Sym}^a \Theta_{\overline{L}}.$$

(3) *The $\mathbb{C}$-vector space spanned by the homogeneous polynomials of degree a in $\overline{z}_1, \dots, \overline{z}_s$ is identical with $(t_1, \dots, t_{n-r})^{ad}/(t_1, \dots, t_{n-r})^{ad+1}$.*

Proof. Noting that

$$z_{s+1} \equiv \cdots \equiv z_{2(n-r)} \equiv 0 \quad \mod (t_1, \dots, t_{n-r})^{d+1},$$

we derive (1) from the Euler identity

$$\sum_i t_i \frac{\partial z_j}{\partial t_i} \equiv dz_j \qquad \mathrm{mod}\ (t_1, \ldots, t_{n-r})^{d+1},$$

and, in view of this, (2) readily follows from Corollary 10.9.

The identity (2) specifically implies that

$$(*) \qquad \sum_{\sigma \in \mathfrak{S}_a} \frac{\partial z_{j_{\sigma 1}}}{\partial t_{i_1}} \cdots \frac{\partial z_{j_{\sigma a}}}{\partial t_{i_a}} \in (z_1, \ldots, z_{2(n-a)})^a + (t_1, \ldots, t_{n-r})^{ad+1}.$$

The vector space $\big((t_1, \ldots, t_{n-r})/(t_1, \ldots, t_{n-r})^2\big) \otimes \Theta_{\overline{L}}$, naturally identified with the Lie algebra $\mathfrak{gl}(n-r)$, canonically acts on the vector space

$$(t_1, \ldots, t_{n-r})/(t_1, \ldots, t_{n-r})^2 \simeq \mathbb{C}(p) \otimes \Omega_{\overline{L}},$$

and so does

$$\mathrm{Sym}^a \mathfrak{gl}(n-r) = \big((t_1, \ldots, t_{n-r})^a/(t_1, \ldots, t_{n-r})^{a+1}\big) \otimes \mathrm{Sym}^a \Theta_{\overline{L}}$$

on the vector space

$$(t_1, \ldots, t_{n-r})^{ad}/(t_1, \ldots, t_{n-r})^{ad+1} \simeq \mathbb{C}(p) \otimes \mathrm{Sym}^a \Omega^1_{\overline{L}}.$$

It is easy to check that $(t_1, \ldots, t_{n-r})^{ad}/(t_1, \ldots, t_{n-r})^{ad+1}$ contains no nontrivial subspace stable under this action. On the other hand, the relation $(*)$ is nothing but the stabiliy of the subspace $(\overline{z}_1, \ldots, \overline{z}_{2(n-r)})^a$, whence follows (3). Q.E.D.

Corollary 10.11. *Let $\mu_L : \widetilde{L} \to \overline{L}$ and $\mu_M : \widetilde{M} \to M$ be the blowups at $p = (0, \ldots, 0)$ and $\nu_L(p)$. Then $\nu_L : \overline{L} \to M$ naturally lifts to a morphism $\tilde{\nu}_L : \widetilde{L} \to \widetilde{M}$. The exceptional divisor $E_L \simeq \mathbb{P}^{n-r-1} \subset \widetilde{L}$ is isomorphically mapped to a subvariety of $E_M \simeq \mathbb{P}^{2(n-r)-1} \subset \widetilde{M}$ defined by quadratic homogeneous polynomials.*

Proof. By Lemma 10.10(3), we have

$$(z_1, \ldots, z_{2(n-r)})^a \mathcal{O}_{\widetilde{L}} = (t_1, \ldots, t_{n-r})^{ad} \mathcal{O}_{\widetilde{L}} = \mathcal{O}_{\widetilde{L}}(-adE_L),$$

so that $(z_1, \ldots, z_{2(n-r)})\mathcal{O}_{\widetilde{L}} = \mathcal{O}_{\widetilde{L}}(-dE_L)$, meaning that ν_L lifts to $\tilde{\nu}_L$ by the universal property of monoidal transformations.

We can view $\overline{z}_j$ as a homogeneous polynomial of degree d. Then Lemma 10.10(3) means that the system of polynomials $\overline{z}_1, \ldots, \overline{z}_{2(n-r)}$, as a free linear subsystem of $|\mathcal{O}_{\mathbb{P}^{n-r-1}}(d)|$, defines an embedding $E_L \simeq$

$\mathbb{P}^{n-r-1} \hookrightarrow E_M \simeq \mathbb{P}^{2(n-r)-1}$. In particular, the defining ideal $\overline{\mathfrak{J}}_L$ of E_L in E_M is locally free and satisfies $\overline{\mathfrak{J}}_L^{(a)} = \overline{\mathfrak{J}}_L^a$.

Via the inclusion $\Theta_{\overline{L}} \subset \nu_L^* \Theta_M \simeq \nu_L^* \Omega_M^1$ and Lemma 10.10(3), we have

$$\prod_{\alpha=1}^{a} t_{i_\alpha} \frac{\partial}{\partial t_{j_\alpha}} = \prod_{\alpha=1}^{a} \Big(\sum_{k=1}^{2(n-r)} t_{i_\alpha} \frac{\partial z_k}{\partial t_{j_\alpha}} \frac{\partial}{\partial z_k} \Big) \subset \big(t_i \frac{\partial z_k}{\partial t_j}\big)^a \nu_L^* \Omega_M^1$$
$$\equiv (t_1, \dots, t_{n-r})^{ad} \nu_L^* \Omega_M^1 = (z_1, \dots, z_s)^a \nu_L^* \Omega_M^1 \mod (t_1, \dots, t_{n-r})^{ad+1}.$$

This means that a section of $\mathfrak{J}_L^{(a)} / \mathfrak{J}_L^{(a+1)}$ has image of the form $\sum_{IJ} a_{IJ} z^I \mathrm{d}z^J$ in $\mathrm{Sym}^a \Omega_M^1 / (z_1, \dots, z_{2(n-r)})^{a+1} \mathrm{Sym}^a \Omega_M^1$, where I, J are multi-indices with $|I| = |J| = a$. We note that, if the section does not vanish at $\nu_L(p)$, at least one of the holomorphic functions $a_{IJ}(z)$ does not vanish at $(0, \dots, 0)$. Indeed, a non-zero local section determines a non-zero element of $(t_1, \dots, t_{n-r})^a \mathrm{Sym}^a \Theta_{\overline{L}}$, which in turn defines a non-zero linear transformation of

$$(z_1, \dots, z_{2(n-r)})^a + (t_1, \dots, t_{n-r})^{ad+1} / (t_1, \dots, t_{n-r})^{ad+1}$$
$$= (t_1, \dots, t_{n-r})^{ad} / (t_1, \dots, t_{n-r})^{ad+1}.$$

Thus the sheaf $\mathfrak{J}_L^{(a)} / (z_1, \dots, z_n) \mathfrak{J}_L^{(a)}$ is generated by homogeneous polynomials precisely of degree $2a$ (degree a from the part $(z_1, \dots, z_{2(n-r)})^a$ and degree a from $\mathrm{Sym}^a \Omega_M^1$). Of course this sheaf is the a-th power $\overline{\mathfrak{J}}_L^a$ of the ideal sheaf of E_L, and E_L is necessarily defined by quadratic equations in E_M. Q.E.D.

The embedding $\mathbb{P}^{n-r-1} = E_L \hookrightarrow E_M = \mathbb{P}^{2(n-r)-1}$ is a linear projection from the d-th Veronese embedding

$$\mathbb{P}^{n-r-1} \hookrightarrow \mathbb{P}^{\binom{d+n-r-1}{n-r-1}-1}$$

to a linear subsystem. Then a classical lemma of Terracini (see [Zak], p.2) yields the following

Lemma 10.12. *Assume that* $n - r \geq 3$ *or* $d \geq 3$. *Then* $\overline{z}_1, \dots, \overline{z}_{2(n-r)}$ *are linearly independent in* $(t_1, \dots, t_{n-r})^d / (t_1, \dots, t_{n-r})^{d+1}$.

Proof. The embedding $E_L \simeq \mathbb{P}^{n-r-1} \hookrightarrow \mathbb{P}^{2(n-r)-1}$ is a linear projection of the d-th Veronese embedding. The Terracini lemma states that the dimension of the secant variety SX of the subvariety $E_L \subset \mathbb{P}^{\binom{d+n-r-1}{n-r-1}}$ is given by the dimension of the linear span

$$\langle T_{E_L,x}, T_{E_L,y} \rangle \subset \mathbb{P}^{\binom{d+n-r-1}{n-r-1}}$$

of the embedded tangent spaces $T_{E_L,x}, T_{E_L,y} \subset \mathbb{P}^{\binom{d+n-r-1}{n-r-1}}$ of E_L at general points $x, y \in E_L$. In case $d > 2$, this shows that $\dim SX = 2(n-r)-1$ and we cannot isomorphically project E_L to $\mathbb{P}^{2(n-r-1)}$, meaning the linear independence of $z_1, \dots, z_{2(n-r)}$.

Suppose that $d = 2$ and that $z_1, \dots, z_{2(n-r)}$ are linearly dependent. Then we have an embedding $\mathbb{P}^{n-r-1} \subset \mathbb{P}^{2(n-r-1)}$, defining equations of which are hyperquadrics by Corollary 10.11. It follows that $E_L = \mathbb{P}^{n-r-1}$ is a complete intersection of quadrics. Indeed, the degree of E_L is 2^{n-r-1} and there are at least $n-r-1$ independent quadrics that contains E_L. In view of the adunction formula, the subvariety $\mathbb{P}^{n-r-1} \subset \mathbb{P}^{2(n-r-1)}$ is a complete intersection of quadrics if and only if $n-r-1 = 1$. Q.E.D.

Lemma 10.13. *If $n-r \geq 2, d \geq 3$, then $d = 3, n-r = 2$.*

Proof. By construction, $E_L = \mathbb{P}^{n-r-1} \subset \mathbb{P}^{2(n-r)-1}$ has degree d^{n-r-1} and is defined as an intersection of quadrics. Hence we have a trivial inequality $2^{n-r} \geq d^{n-r-1}$. If the equality holds, then $d = 4$, $n-r = 2$ and $\mathbb{P}^1 \subset \mathbb{P}^3$ would be a complete intersection of two quadrics, which would be a curve of arithmetic genus one. Thus we have only two possibilities: $d = 2$ and $d = 3, n-r = 2$. Q.E.D.

Lemma 10.14. *The case $n-r = 2, d = 3$ does not occur.*

Proof. Let (x, y) be a local coordinate system on $\overline{L}$. Under the assumption of the lemma, we can assume that $z_1 \equiv x^3, z_2 \equiv x^2y, z_3 \equiv xy^2, z_4 \equiv y^3$ modulo $(x,y)^4$. Blow up $\overline{L}$ and M at p and $\nu_L(p)$. Then, by taking a suitable holomorphic function u on $\tilde{L}$, we get a local coordinate system (x, u) on $\tilde{L}$ such that $z_1 = x^3, z_2 = x^3u, z_3 = x^3u^2 + x^4g, z_3 = x^3u^3 + x^4h$, where g, h are holomorphic. Let us prove that z_3 and z_4 are functions in (x^3, u) and hence $\nu_L : \overline{L} \to M$ cannot be bimeromorphic.

Recall that E_L is defined by three quadrics and $\mathfrak{I}_L^{(a)}$ is generated by elements congruent to the products of these quadrics modulo terms of degree $\geq 2a+1$. We have therefore relations of functions in (x, u):

$$(z_iz_j - z_kz_l)^a \equiv 0 \mod (z_1, \dots, z_4)^{2a+1} \text{ if } i+j = k+l.$$

Assume that z_3 or z_4 contains a term $x^{3(m+1)+c}$, where m is a non-negative integer and $0 < c < 3$. Let $3(m+1)+c$ be the minimum of such exponents and assume that the minimum is attained by a term in z_3. Then $(z_1z_3 - z_2^2)^a$ contains the non-zero x^{3a+m+c}-term, while the terms of degree $\geq 2a+1$ terms do not. This means that $(z_1z_3 - z_2^2)^a +$ (terms of higher order) cannot vanish on L, contradicting our assumption. In case z_4 contains the term x^{3m+c} and z_3 does not, then look at a second equation $(z_2z_4 - z_3^2)^a + (\text{terms of order } 2a+1) = 0$ and we similarly get the absurd conclusion that z_4 is a function in (x^3, u). Q.E.D.

Lemma 10.15. *If $d = 2$, then $n - r \leq 2$.*

Proof. Let $\mathcal{N}^*$ be the conormal bundle of $E_L = \mathbb{P}^{n-r-1} \subset E_M = \mathbb{P}^{2(n-r)-1}$. A quadric relation defines an injection

$$\tilde{\nu}_L^* \mathcal{O}_{E_M}(-2) = \mathcal{O}_{E_L}(-2d) = \mathcal{O}_{E_L}(-4) \subset \mathcal{N}^*$$

and hence an element of $\mathrm{H}^0(\mathcal{N}^*(4))$. The assumption that $E_L = \mathbb{P}^{n-r-1}$ is defined by quadratic equations in $\mathbb{P}^{2(n-r)-1}$ is rephrased as the global generation of $\mathcal{N}^*(4)$, meaning that $\mathcal{N}^*(4)$ is a direct sum of trivial line bundles $\mathcal{O}$ and a nef big vector bundle $\mathcal{P}$. By Serre duality, $\mathrm{H}^1(\mathbb{P}^{n-r-1}, \mathcal{P}) \simeq \mathrm{H}^1(\mathbb{P}(\mathcal{P}), \mathcal{O}(\mathbf{1}))$ is the dual of

$$\mathrm{H}^{\operatorname{rank}\mathcal{P}+2(n-r)-3}(\mathbb{P}(\mathcal{P}), \mathcal{O}(-(n-r+1)\mathbf{1}))$$

and vanishes by the Kawamata-Viehwg vanishing theorem. Consequently we have $\mathrm{H}^1(\mathbb{P}^{n-r-1}, \mathcal{N}^*(4)) = 0$.

In view of the standard dual Euler exact sequences

$$0 \to \tilde{\nu}_L^* \Omega^1_{E_M} \to \mathcal{O}(-2)^{\oplus 2(n-r)} \to \mathcal{O} \to 0,$$
$$0 \to \Omega^1_{E_L} \to \mathcal{O}(-1)^{\oplus n-r} \to \mathcal{O} \to 0,$$

the conormal bundle $\mathcal{N}^*$ is explicitly described by the exact sequence

$$0 \to \mathcal{N}^* \to \mathcal{O}(-2)^{\oplus 2(n-r)} \to \mathcal{O}(-1)^{\oplus n-r} \to 0.$$

In view of the vanishing of $\mathrm{H}^1(\mathbb{P}^{n-r-1}, \mathcal{N}^*(4))$, we infer that

$$2(n-r)\dim \mathrm{H}^0(\mathcal{O}(2)) - (n-r)\dim \mathrm{H}^0(\mathcal{O}(3)) = \dim \mathrm{H}^0(\mathcal{N}^*(4)).$$

On the other hand, $\dim \mathrm{H}^0(\mathcal{N}^*(4))$ is nothing but the number of quadratic equations of $E_L \subset E_M$, which was at least

$$\operatorname{length}(t_1, \ldots, t_{n-r})\Theta_{\overline{L}}/(\mathcal{L}_L + (t_1, \ldots, t_{n-r})^2\Theta_{\overline{L}}) = (n-r)^2 - 1.$$

Hence

$$(n-r)^2\{(n-r+1) - \frac{1}{6}(n-r+2)(n-r+1) - 1\} \geq -1,$$

yielding the inequality $n - r \leq 2$. Q.E.D.

Lemma 10.16. $n - r = 1$.

Proof. It suffices to exclude the case $d = n - r = 2$. In this case, we may well assume that $z_1 \equiv x^2, z_2 \equiv xy, z_3 \equiv y^2, z_4 \equiv 0$ modulo $(x, y)^3$. Let η_{ij} be the $\mathrm{d}z_i \wedge \mathrm{d}z_j$-coefficient of the symplectic form η on M. Then, since $L \subset M$ is Lagrangian,

$$0 = \nu_L^* \eta \equiv (2x^2\eta_{12} + 4xy\eta_{13} + 2y^2\eta_{23})\mathrm{d}x \wedge \mathrm{d}y \qquad \mathrm{mod}\ (x, y)^3,$$

and hence

$$\eta_{12} \equiv \eta_{13} \equiv \eta_{23} \equiv 0 \quad \mathrm{mod}\ (z_1, z_2, z_3, z_4),$$

meaning that η at $(0,0,0,0)$ has the three-dimensional isotropic subspace spanned by $\frac{\partial}{\partial z_i}$, $i = 1, 2, 3$, thus contradicting the nondegeneracy of η. Q.E.D.

Lemma 10.17. *The case $n - r = 1$ is impossible.*

Proof. In this case, $\mathfrak{I}_L \subset \mathcal{O}_M$ is an invertible sheaf with $\mathfrak{I}_L^{(a)} = \mathfrak{I}_L^a$. Thus it suffices to show that $\nu_L^* \mathfrak{I}_L/\mathfrak{I}_L^2 \neq t\Theta_{\overline{L}}$, where t is a local parameter of the curve $\overline{L}$. If d is the multiplicity of L at o, then there exists an integer $e > d$ such that $z_1 = t^d$, $z_2 = (\mathrm{unit})t^e$. The image of $t(\partial/\partial t)$ is

$$\begin{aligned} t\frac{\mathrm{d}z_1}{\mathrm{d}t}\frac{\partial}{\partial z_1} + t\frac{\mathrm{d}z_2}{\mathrm{d}t}\frac{\partial}{\partial z_2} &= \eta_{12}t(\frac{\mathrm{d}z_1}{\mathrm{d}t}\mathrm{d}z_2 - \frac{\mathrm{d}z_2}{\mathrm{d}t}\mathrm{d}z_1) \\ &\equiv \eta_{12}(0)(dz_1\mathrm{d}z_2 - ez_2\mathrm{d}z_1) \quad \mathrm{mod}\ t^{d+1}\mathcal{O}_{\overline{L}}\mathrm{d}z_2 + t^{e+1}\mathcal{O}_{\overline{L}}\mathrm{d}z_1. \end{aligned}$$

If this comes from $(\mathfrak{I}_L/\mathfrak{I}_L^2)$, then the defining equation of L contains an element $\in \eta_{12}(d-e)z_1z_2 + (z_2^2, z_1^3, z_1^2z_2)$, which we can easily rule out by comparing the degrees in t. Q.E.D.

Thus we have eventually excluded the possibility that $\nu_0 : \mathbb{P}^n \to X$ is not an immersion, completing the proof of Theorem 10.1.

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The Geometry of Siegel Modular Varieties

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Contents

Introduction

Siegel modular varieties are interesting because they arise as moduli spaces for abelian varieties with a polarization and a level structure, and also because of their concrete analytic realization as locally symmetric varieties. Even in the early days of modern algebraic geometry the study of quartic surfaces led to some specific examples of these moduli spaces being studied in the context of projective geometry. Later advances in complex analytic and algebraic geometry and in number theory have given us many very effective tools for studying these varieties and their various compactifications, and in the last ten years a considerable amount of progress has been made in understanding the general picture. In this survey we intend to give a reasonably thorough account of the more recent work, though mostly without detailed proofs, and to describe sufficiently but not exhaustively the earlier work of, among others, Satake, Igusa, Mumford and Tai that has made the recent progress possible.

We confine ourselves to working over the complex numbers. This does not mean that we can wholly ignore number theory, since much of what is known depends on interpreting differential forms on Siegel modular varieties as Siegel modular forms. It does mean, though, that we are neglecting many important, interesting and difficult questions: in particular, the work of Faltings and Chai, who extended much of the compactification theory to $\operatorname{Spec}\mathbb{Z}$, will make only a fleeting appearance. To have attempted to cover this material would have greatly increased the length of this article and would have led us beyond the areas where we can pretend to competence.

The plan of the article is as follows.

In Section I we first give a general description of Siegel modular varieties as complex analytic spaces, and then explain how to compactify them and obtain projective varieties. There are essentially two related ways to do this.

In Section II we start to understand the birational geometry of these compactified varieties. We examine the canonical divisor and explain some results which calculate the Kodaira dimension in many cases and the Chow ring in a few. We also describe the fundamental group.

In Section III we restrict ourselves to the special case of moduli of abelian surfaces (Siegel modular threefolds), which is of particular interest. We describe a rather general lifting method, due to Gritsenko in the form we use, which produces Siegel modular forms of low weight by starting from their behaviour near the boundary of the moduli space. This enables us to get more precise results about the Kodaira dimension in a few interesting special cases, due to Gritsenko and others. Then we describe some results of a more general nature, which tend to show that in most cases the compactified varieties are of general type. In the last part of this section we examine some finite covers and quotients of moduli spaces of polarized abelian surfaces, some of which can be interpreted as moduli of Kummer surfaces. The lifting method gives particularly good results for these varieties.

In Section IV we examine three cases, two of them classical, where a Siegel modular variety (or a near relative) has a particularly good projective description. These are the Segre cubic and the Burkhardt quartic, which are classical, and the Nieto quintic, which is on the contrary a surprisingly recent discovery. There is a huge body of work on the first two and we cannot do more than summarize enough of the results to enable us to highlight the similarities among the three cases.

In Section V we examine the moduli spaces of $(1, t)$-polarized abelian surfaces (sometimes with level structure) for small t. We begin with the famous Horrocks-Mumford case, $t = 5$, and then move on to the work of Manolache and Schreyer on $t = 7$ and Gross and Popescu on other cases, especially $t = 11$.

In Section VI we return to the compactification problems and describe very recent improvements brought about by Alexeev and Nakamura, who (building on earlier work by Nakamura, Namikawa, Tai and Mumford) have shed some light on the question of whether there are compactifications of the moduli space that are really compactifications of moduli, that is, support a proper universal family.

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I. Siegel modular varieties

In this section we give the basic definitions in connection with Siegel modular varieties and sketch the construction of the Satake and toroidal compactifications.

I.1. Arithmetic quotients of the Siegel upper half plane

To any point τ in the *upper half plane*

$$\mathbb{H}_1 = \{\tau \in \mathbb{C}\,; \quad \operatorname{Im} \tau > 0\}$$

one can associate a lattice

$$L_\tau = \mathbb{Z}\tau + \mathbb{Z}$$

and an *elliptic curve*

$$E_\tau = \mathbb{C}/L_\tau.$$

Since every elliptic curve arises in this way one obtains a surjective map

$$\mathbb{H}_1 \to \{\text{elliptic curves}\}/\text{ isomorphism.}$$

The group $\mathrm{SL}(2,\mathbb{Z})$ acts on $\mathbb{H}_1$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : \tau \mapsto \frac{a\tau + b}{c\tau + d}$$

and

$$E_\tau \cong E_{\tau'} \Leftrightarrow \tau \sim \tau' \bmod \mathrm{SL}(2,\mathbb{Z}).$$

Hence there is a bijection

$$X^\circ(1) = \mathrm{SL}(2,\mathbb{Z})\backslash\mathbb{H}_1 \xrightarrow{1:1} \{\text{elliptic curves}\}/\text{ isomorphism.}$$

The j-function is an $\mathrm{SL}(2,\mathbb{Z})$-invariant function on $\mathbb{H}_1$ and defines an isomorphism of Riemann surfaces

$$j : X^\circ(1) \cong \mathbb{C}.$$

An *abelian variety* (over the complex numbers $\mathbb{C}$) is a g-dimensional complex torus $\mathbb{C}^g/L$ which is a projective variety, i.e. can be embedded into some projective space $\mathbb{P}^n$. Whereas every 1-dimensional torus $\mathbb{C}/L$ is an algebraic curve, it is no longer true that every torus $X = \mathbb{C}^g/L$ of dimension $g \geq 2$ is projective. This is the case if and only if X admits a *polarization.* There are several ways to define polarizations. Perhaps the most common definition is that using Riemann forms. A *Riemann form* on $\mathbb{C}^g$ with respect to the lattice L is a hermitian form $H \geq 0$ on $\mathbb{C}^g$ whose imaginary part $H' = \mathrm{Im}(H)$ is integer-valued on L, i.e. defines an alternating bilinear form

$$H' : L \otimes L \to \mathbb{Z}.$$

The $\mathbb{R}$-linear extension of H' to $\mathbb{C}^g$ satisfies $H'(x,y) = H'(ix,iy)$ and determines H by the relation

$$H(x,y) = H'(ix,y) + iH'(x,y).$$

H is positive definite if and only if H' is non-degenerate. In this case H (or equivalently H') is called a *polarization.* By the elementary divisor theorem there exists then a basis of L with respect to which H' is given by the form

$$\Lambda = \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix} \quad , \quad E = \begin{pmatrix} e_1 & & \\ & \ddots & \\ & & e_g \end{pmatrix},$$

where the $e_1, \dots, e_g$ are positive integers such that $e_1 | e_2 \dots | e_g$. The g-tuple $(e_1, \dots, e_g)$ is uniquely determined by H and is called the *type of the polarization.* If $e_1 = \dots = e_g = 1$ one speaks of a *principal polarization.* A *(principally) polarized abelian variety* is a pair (A, H) consisting of a torus A and a (principal) polarization H.

Assume we have chosen a basis of the lattice L. If we express each basis vector of L in terms of the standard basis of $\mathbb{C}^g$ we obtain a matrix $\Omega \in M(2g \times g, \mathbb{C})$ called a *period matrix* of A. The fact that H is hermitian and positive definite is equivalent to

$${}^t\Omega\Lambda^{-1}\Omega = 0, \text{ and } i\ {}^t\Omega\Lambda^{-1}\bar{\Omega} > 0.$$

These are the *Riemann bilinear relations.* We consider vectors of $\mathbb{C}^g$ as row vectors. Using the action of $\mathrm{GL}(g, \mathbb{C})$ on row vectors by right multiplication we can transform the last g vectors of the chosen basis of L to be $(e_1, 0, \dots, 0), (0, e_2, 0, \dots, 0), \dots, (0, \dots, 0, e_g)$. Then Ω takes on the form

$$\Omega = \Omega_\tau = \begin{pmatrix} \tau \\ E \end{pmatrix}$$

and the Riemann bilinear relations translate into

$$\tau = {}^t\tau, \quad \operatorname{Im} \tau > 0.$$

In other words, the complex $(g \times g)$-matrix τ is an element of the *Siegel space of degree* g

$$\mathbb{H}_g = \{\tau \in M(g \times g, \mathbb{C}); \tau = {}^t\tau, \operatorname{Im} \tau > 0\}.$$

Conversely, given a matrix $\tau \in \mathbb{H}_g$ we can associate to it the period matrix Ω_τ and the lattice $L = L_\tau$ spanned by the rows of Ω_τ. The complex torus $A = \mathbb{C}^g / L_\tau$ carries a Riemann form given by

$$H(x, y) = x \operatorname{Im}(\tau)^{-1}\ {}^t\bar{y}.$$

This defines a polarization of type $(e_1, \ldots, e_g)$. Hence for every given type of polarization we have a surjection

$$\mathbb{H}_g \to \{(A,H); (A,H) \text{ is an } (e_1,\ldots,e_g)\text{-polarized ab.var.}\}/\text{ isom.}$$

To describe the set of these isomorphism classes we have to see what happens when we change the basis of L. Consider the *symplectic group*

$$\mathrm{Sp}(\Lambda,\mathbb{Z}) = \{h \in \mathrm{GL}(2g,\mathbb{Z});\ h\Lambda{}^t h = \Lambda\}.$$

As usual we write elements $h \in \mathrm{Sp}(\Lambda,\mathbb{Z})$ in the form

$$h = \begin{pmatrix} A & B \\ C & D \end{pmatrix}; \quad A,\ldots,D \in M(g\times g,\mathbb{Z}).$$

It is useful to work with the "right projective space P of $\mathrm{GL}(g,\mathbb{C})$" i.e. the set of all $(2g\times g)$-matrices of rank g divided out by the equivalence relation

$$\begin{pmatrix} M_1 \\ M_2 \end{pmatrix} \sim \begin{pmatrix} M_1 M \\ M_2 M \end{pmatrix} \text{ for any } M \in \mathrm{GL}(g,\mathbb{C}).$$

Clearly P is isomorphic to the Grassmannian $G = \mathrm{Gr}(g,\mathbb{C}^{2g})$. The group $\mathrm{Sp}(\Lambda,\mathbb{Z})$ acts on P by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{bmatrix} M_1 \\ M_2 \end{bmatrix} = \begin{bmatrix} AM_1 + BM_2 \\ CM_1 + DM_2 \end{bmatrix}$$

where [ ] denotes equivalence classes in P. One can embed $\mathbb{H}_g$ into P by $\tau \mapsto \begin{bmatrix} \tau \\ E \end{bmatrix}$. Then the action of $\mathrm{Sp}(\Lambda,\mathbb{Z})$ restricts to an action on the image of $\mathbb{H}_g$ and is given by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{bmatrix} \tau \\ E \end{bmatrix} = \begin{bmatrix} A\tau + BE \\ C\tau + DE \end{bmatrix} = \begin{bmatrix} (A\tau+BE)(C\tau+DE)^{-1}E \\ E \end{bmatrix}.$$

In other words, $\mathrm{Sp}(\Lambda,\mathbb{Z})$ acts on $\mathbb{H}_g$ by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} : \tau \mapsto (A\tau+BE)(C\tau+DE)^{-1}E.$$

We can then summarize our above discussion with the observation that for a given type $(e_1,\ldots,e_g)$ of a polarization the quotient

$$\mathcal{A}_{e_1,\ldots,e_g} = \mathrm{Sp}(\Lambda,\mathbb{Z})\backslash\mathbb{H}_g$$

parametrizes the isomorphism classes of $(e_1, \ldots, e_g)$-polarized abelian varieties, i.e. $\mathcal{A}_{e_1,\ldots,e_g}$ is the coarse moduli space of $(e_1, \ldots, e_g)$-polarized abelian varieties. (Note that the action of $\mathrm{Sp}(\Lambda, \mathbb{Z})$ on $\mathbb{H}_g$ depends on the type of the polarization.) If we consider principally polarized abelian varieties, then the form Λ is the standard symplectic form

$$J = \begin{pmatrix} 0 & \mathbf{1}_g \\ -\mathbf{1}_g & 0 \end{pmatrix}$$

and $\mathrm{Sp}(\Lambda, \mathbb{Z}) = \mathrm{Sp}(2g, \mathbb{Z})$ is the standard symplectic integer group. In this case we use the notation

$$\mathcal{A}_g = \mathcal{A}_{1,\ldots,1} = \mathrm{Sp}(2g, \mathbb{Z}) \backslash \mathbb{H}_g.$$

This clearly generalizes the situation which we encountered with elliptic curves. The space $\mathbb{H}_1$ is just the ordinary upper half plane and $\mathrm{Sp}(2, \mathbb{Z}) = \mathrm{SL}(2, \mathbb{Z})$. We also observe that multiplying the type of a polarization by a common factor does not change the moduli space. Instead of the group $\mathrm{Sp}(\Lambda, \mathbb{Z})$ one can also use a suitable conjugate which is a subgroup of $\mathrm{Sp}(J, \mathbb{Q})$. One can then work with the standard symplectic form and the usual action of the symplectic group on Siegel space, but the elements of the conjugate group will in general have rational and no longer just integer entries.

One is often interested in polarized abelian varieties with extra structures, the so-called *level structures.* If L is a lattice equipped with a non-degenerate form Λ the *dual lattice* $L^\vee$ of L is defined by

$$L^\vee = \{y \in L \otimes \mathbb{Q};\ \Lambda(x, y) \in \mathbb{Z} \text{ for all } x \in L\}.$$

Then $L^\vee / L$ is non-canonically isomorphic to $(\mathbb{Z}_{e_1} \times \ldots \times \mathbb{Z}_{e_g})^2$. The group $L^\vee / L$ carries a skew form induced by Λ and the group $(\mathbb{Z}_{e_1} \times \ldots \times \mathbb{Z}_{e_g})^2$ has a $\mathbb{Q}/\mathbb{Z}$-valued skew form which with respect to the canonical generators is given by

$$\begin{pmatrix} 0 & E^{-1} \\ -E^{-1} & 0 \end{pmatrix}.$$

If (A, H) is a polarized abelian variety, then a *canonical level structure* on (A, H) is a symplectic isomorphism

$$\alpha : L^\vee / L \to (\mathbb{Z}_{e_1} \times \ldots \times \mathbb{Z}_{e_g})^2$$

where the two groups are equipped with the forms described above. Given Λ we can define the group.

$$\mathrm{Sp}^{\mathrm{lev}}(\Lambda, \mathbb{Z}) := \{h \in \mathrm{Sp}(\Lambda, \mathbb{Z});\ h|_{L^\vee/L} = \mathrm{id}_{\ L^\vee/L}\}.$$

The quotient space

$$\mathcal{A}^{\mathrm{lev}}_{e_1,\dots,e_g} := \mathrm{Sp}^{\mathrm{lev}}(\Lambda,\mathbb{Z})\backslash\mathbb{H}_g$$

has the interpretation

$$\mathcal{A}^{\mathrm{lev}}_{e_1,\dots,e_g} = \{(A,H,\alpha);\ (A,H) \text{ is an } (e_1,\dots,e_g)\text{-polarized abelian variety}, \alpha \text{ is a canonical level structure}\}/\text{ isom.}$$

If Λ is a multiple nJ of the standard symplectic form then $\mathrm{Sp}(nJ,\mathbb{Z}) = \mathrm{Sp}(J,\mathbb{Z})$ but

$$\Gamma_g(n) := \mathrm{Sp}^{\mathrm{lev}}(nJ,\mathbb{Z}) = \{h \in \mathrm{Sp}(J,\mathbb{Z});\ h \equiv \mathbf{1} \bmod n\}.$$

This group is called the *principal congruence subgroup* of level n. A *level-n structure* on a principally polarized abelian variety (A,H) is a canonical level structure in the above sense for the polarization nH. The space

$$\mathcal{A}_g(n) := \Gamma_g(n)\backslash\mathbb{H}_g$$

is the moduli space of principally polarized abelian varieties with a level-n structure.

The groups $\mathrm{Sp}(\Lambda,\mathbb{Z})$ act properly discontinuously on the Siegel space $\mathbb{H}_g$. If $e_1 \geq 3$ then $\mathrm{Sp}^{\mathrm{lev}}(\Lambda,\mathbb{Z})$ acts freely and consequently the spaces $\mathcal{A}^{\mathrm{lev}}_{e_1,\dots,e_g}$ are smooth in this case. The finite group $\mathrm{Sp}(\Lambda,\mathbb{Z})/\mathrm{Sp}^{\mathrm{lev}}(\Lambda,\mathbb{Z})$ acts on $\mathcal{A}^{\mathrm{lev}}_{e_1,\dots,e_g}$ with quotient $\mathcal{A}_{e_1,\dots,e_g}$. In particular, these spaces have at most finite quotient singularities.

A torus $A = \mathbb{C}^g/L$ is projective if and only if there exists an ample line bundle $\mathcal{L}$ on it. By the Lefschetz theorem the first Chern class defines an isomorphism

$$c_1 : \mathrm{NS}(A) \cong H^2(A,\mathbb{Z}) \cap H^{1,1}(A,\mathbb{C}).$$

The natural identification $H_1(A,\mathbb{Z}) \cong L$ induces isomorphisms

$$H^2(A,\mathbb{Z}) \cong \mathrm{Hom}(\textstyle\bigwedge^2 H_1(A,\mathbb{Z}),\mathbb{Z}) \cong \mathrm{Hom}(\textstyle\bigwedge^2 L,\mathbb{Z}).$$

Hence given a line bundle $\mathcal{L}$ the first Chern class $c_1(\mathcal{L})$ can be interpreted as a skew form on the lattice L. Let $H' := -c_1(\mathcal{L}) \in \mathrm{Hom}(\bigwedge^2 L,\mathbb{Z})$. Since $c_1(\mathcal{L})$ is a $(1,1)$-form it follows that $H'(x,y) = H'(ix,iy)$ and hence the associated form H is hermitian. The ampleness of $\mathcal{L}$ is equivalent to positive definiteness of H. In this way an ample line bundle defines, via its first Chern class, a hermitian form H. Reversing this process one can also associate to a Riemann form an element in $H^2(A,\mathbb{Z})$ which is the first Chern class of an ample line bundle $\mathcal{L}$. The line bundle $\mathcal{L}$ itself is only defined up to translation. One can also view level structures from this point of view. Consider an ample line bundle $\mathcal{L}$ representing a polarization H. This defines a map

$$\begin{array}{rccl} \lambda: & A & \longrightarrow & \hat{A} = \operatorname{Pic}^0 A \\ & x & \longmapsto & t_x^* \mathcal{L} \otimes \mathcal{L}^{-1} \end{array}$$

where t_x is translation by x. The map λ depends only on the polarization, not on the choice of the line bundle $\mathcal{L}$. If we write $A = \mathbb{C}^g/L$ then we have $\operatorname{Ker}\lambda \cong L^\vee/L$ and this defines a skew form on $\operatorname{Ker}\lambda$, the *Weil pairing.* This also shows that $\operatorname{Ker}\lambda$ and the group $(\mathbb{Z}_{e_1} \times \ldots \times \mathbb{Z}_{e_g})^2$ are (non-canonically) isomorphic. We have already equipped the latter group with a skew form. From this point of view a canonical level structure is then nothing but a symplectic isomorphism

$$\alpha : \operatorname{Ker}\lambda \cong (\mathbb{Z}_{e_1} \times \ldots \times \mathbb{Z}_{e_g})^2.$$

I.2. Compactifications of Siegel modular varieties

We have already observed that the j-function defines an isomorphism of Riemann surfaces

$$j : X^\circ(1) = \mathrm{SL}(2,\mathbb{Z})\backslash\mathbb{H}_1 \cong \mathbb{C}.$$

Clearly this can be compactified to $X(1) = \mathbb{P}^1 = \mathbb{C}\cup\{\infty\}$. It is, however, important to understand this compactification more systematically. The action of the group $\mathrm{SL}(2,\mathbb{Z})$ extends to an action on

$$\overline{\mathbb{H}}_1 = \mathbb{H}_1 \cup \mathbb{Q} \cup \{i\infty\}.$$

The extra points $\mathbb{Q} \cup \{i\infty\}$ form one orbit under this action and we can set

$$X(1) = \mathrm{SL}(2,\mathbb{Z})\backslash\overline{\mathbb{H}}_1.$$

To understand the structure of $X(1)$ as a Riemann surface we have to consider the stabilizer

$$P(i\infty) = \left\{ \pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} ; n \in \mathbb{Z} \right\}$$

of the point $i\infty$. It acts on $\mathbb{H}_1$ by $\tau \mapsto \tau + n$. Taking the quotient by $P(i\infty)$ we obtain the map

$$\begin{array}{ccl} \mathbb{H}_1 & \longrightarrow & D_1^* = \{z \in \mathbb{C};\quad 0 < |z| < 1\} \\ \tau & \longmapsto & t = e^{2\pi i \tau}. \end{array}$$

Adding the origin gives us the "partial compactification" D_1 of D_1^*. For ε sufficiently small no two points in the punctured disc D_ε^* of radius ε are identified under the map from D_1^* to the quotient $\mathrm{SL}(2,\mathbb{Z})\backslash\mathbb{H}_1$. Hence we obtain $X(1)$ by

$$X(1) = X^\circ(1) \cup_{D_\varepsilon^*} D_\varepsilon.$$

This process is known as "adding the cusp $i\infty$". If we take an arbitrary arithmetic subgroup $\Gamma \subset \mathrm{SL}(2,\mathbb{Z})$ then $\mathbb{Q} \cup \{i\infty\}$ will in general have several, but finitely many, orbits. However, given a representative of such an orbit we can always find an element in $\mathrm{SL}(2,\mathbb{Z})$ which maps this representative to $i\infty$. We can then perform the above construction once more, the only difference being that we will, in general, have to work with a subgroup of $P(i\infty)$. Using this process we can always compactify the quotient $X^\circ(\Gamma) = \Gamma\backslash\mathbb{H}_1$, by adding a finite number of cusps, to a compact Riemann surface $X(\Gamma)$.

The situation is considerably more complicated for higher genus g where it is no longer the case that there is a unique compactification of a quotient $\mathcal{A}(\Gamma) = \Gamma\backslash\mathbb{H}_g$. There have been many attempts to construct suitable compactifications of $\mathcal{A}(\Gamma)$. The first solution was given by Satake ([Sa]) in the case of $\mathcal{A}_g$. Satake's compactification $\bar{\mathcal{A}}_g$ is in some sense minimal. The boundary $\bar{\mathcal{A}}_g\backslash\mathcal{A}_g$ is set-theoretically the union of the spaces $\mathcal{A}_i, i \leq g-1$. The projective variety $\bar{\mathcal{A}}_g$ is normal but highly singular along the boundary. Satake's compactification was later generalized by Baily and Borel to arbitrary quotients of symmetric domains by arithmetic groups. By blowing up along the boundary, Igusa ([I3]) constructed a partial desingularization of Satake's compactification. The boundary of Igusa's compactification has codimension 1. The ideas of Igusa together with work of Hirzebruch on Hilbert modular surfaces were the starting point for Mumford's general theory of toroidal compactifications of quotients of bounded symmetric domains ([Mu3]). A detailed description of this theory can be found in [AMRT]. Namikawa showed in [Nam2] that Igusa's compactification is a toroidal compactification in Mumford's sense. Toroidal compactifications depend on the choice of cone decompositions and are, therefore, not unique. The disadvantage of this is that this makes it difficult to give a good modular interpretation for these compactifications. Recently, however, Alexeev and Nakamura ([AN], [Ale1]) partly improving work of Nakamura and Namikawa ([Nak1], [Nam1]) have made progress by showing that the toroidal compactification $\mathcal{A}_g^*$ which is given by the second Voronoi decomposition represents a good functor. We shall return to this topic in chapter VI of our survey article.

This survey article is clearly not the right place to give a complete exposition of the construction of compactifications of Siegel modular varieties. Nevertheless we want to sketch the basic ideas behind the construction of the Satake compactification and of toroidal compactifications. We shall start with the *Satake compactification.* For this we consider an arithmetic subgroup Γ of $\mathrm{Sp}(2g,\mathbb{Q})$ for some $g \geq 2$. (This is no restriction since the groups $\mathrm{Sp}(\Lambda,\mathbb{Z})$ which arise for non-principal polarizations are conjugate to subgroups of $\mathrm{Sp}(2g,\mathbb{Q})$). A *modular form* of *weight* k with respect to the group Γ is a holomorphic function

$$F : \mathbb{H}_g \longrightarrow \mathbb{C}$$

with the following transformation behaviour with respect to the group Γ:

$$F(M\tau) = \det(C\tau + D)^k F(\tau) \quad \text{for all } M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma.$$

(For $g = 1$ one has to add the condition that F is holomorpic at the cusps, but this is automatic for $g \geq 2$). If Γ acts freely then the automorphy factor $\det(C\tau + D)^k$ defines a line bundle L^k on the quotient $\Gamma\backslash\mathbb{H}_g$. In general some elements in Γ will have fixed points, but every such element is torsion and the order of all torsion elements in Γ is bounded (see e.g. [LB, p.120]). Hence, even if Γ does not act freely, the modular forms of weight nk_0 for some suitable integer k_0 and $n \geq 1$ are sections of a line bundle L^{nk_0}. The space $M_k(\Gamma)$ of modular forms of fixed weight k with respect to Γ is a finite-dimensional vector space and the elements of $M_{nk_0}(\Gamma)$ define a rational map to some projective space $\mathbb{P}^N$. If n is sufficiently large it turns out that this map is actually an immersion and the Satake compactification $\mathcal{A}(\Gamma)$ can be defined as the projective closure of the image of this map.

There is another way of describing the Satake compactification which also leads us to toroidal compactifications. The *Cayley transformation*

$$\begin{array}{rcl} \Phi : \mathbb{H}_g & \to & \mathrm{Sym}(g,\mathbb{C}) \\ \tau & \mapsto & (\tau - i\mathbf{1})(\tau + i\mathbf{1})^{-1} \end{array}$$

realizes $\mathbb{H}_g$ as the symmetric domain

$$\mathcal{D}_g = \{Z \in \mathrm{Sym}(g,\mathbb{C});\ \mathbf{1} - Z\bar{Z} > 0\}.$$

Let $\bar{\mathcal{D}}_g$ be the topological closure of $\mathcal{D}_g$ in $\mathrm{Sym}(g,\mathbb{C})$. The action of $\mathrm{Sp}(2g,\mathbb{R})$ on $\mathbb{H}_g$ defines, via the Cayley transformation, an action on $\mathcal{D}_g$ which extends to $\bar{\mathcal{D}}_g$. Two points in $\bar{\mathcal{D}}_g$ are called equivalent if they can be connected by finitely many holomorphic curves. Under this equivalence relation all points in $\mathcal{D}_g$ are equivalent. The equivalence classes of $\bar{\mathcal{D}}_g\backslash\mathcal{D}_g$ are called the proper boundary components of $\bar{\mathcal{D}}_g$. Given any point $Z \in \bar{\mathcal{D}}_g$ one can associate to it the real subspace $U(Z) = \operatorname{Ker}\psi(Z)$ of $\mathbb{R}^{2g}$ where

$$\psi(Z) : \mathbb{R}^{2g} \to \mathbb{C}^g, \nu \mapsto \nu \begin{pmatrix} i(\mathbf{1}+Z) \\ \mathbf{1}-Z \end{pmatrix}.$$

Then $U(Z)$ is an isotropic subspace of $\mathbb{R}^{2g}$ equipped with the standard symplectic form J. Moreover $U(Z) \neq 0$ if and only if $Z \in \bar{\mathcal{D}}_g\backslash\mathcal{D}_g$ and $U(Z_1) = U(Z_2)$ if and only if Z_1 and Z_2 are equivalent. This defines a bijection between the proper boundary components of $\bar{\mathcal{D}}_g$ and the non-trivial isotropic subspaces of $\mathbb{R}^{2g}$.

For any boundary component F we can define its stabilizer in $\operatorname{Sp}(2g, \mathbb{R})$ by

$$\mathcal{P}(F) = \{h \in \operatorname{Sp}(2g, \mathbb{R});\ h(F) = F\}.$$

If $U = U(F)$ is the associated isotropic subspace, then

$$\mathcal{P}(F) = \mathcal{P}(U) = \{h \in \operatorname{Sp}(2g, \mathbb{R});\ Uh^{-1} = U\}.$$

A boundary component F is called *rational* if $\mathcal{P}(F)$ is defined over the rationals or, equivalently, if $U(F)$ is a rational subspace, i.e. can be generated by rational vectors. Adding the rational boundary components to $\mathcal{D}_g$ one obtains the *rational closure* $\mathcal{D}_g^{\mathrm{rat}}$ of $\mathcal{D}_g$. This can be equipped with either the Satake topology or the cylindrical topology. The Satake compactification, as a topological space, is then the quotient $\Gamma\backslash\mathcal{D}_g^{\mathrm{rat}}$. (The Satake topology and the cylindrical topology are actually different, but the quotients turn out to be homeomorphic.) For $g = 1$ the above procedure is easily understood: the Cayley transformation ψ maps the upper half plane $\mathbb{H}_1$ to the unit disc D_1. Under this transformation the rational boundary points $\mathbb{Q}\cup\{i\infty\}$ of $\mathbb{H}_1$ are mapped to the rational boundary points of D_1. The relevant topology is the image under ψ of the *horocyclic topology* on $\overline{\mathbb{H}}_1 = \mathbb{H}_1 \cup \mathbb{Q} \cup \{i\infty\}$.

Given two boundary components F and F' with $F \neq F'$ we say that F is *adjacent* to F' (denoted by $F' \succ F$) if $F \subset \overline{F'}$. This is the case if and only if $U(F') \subsetneqq U(F)$. In this way we obtain two partially ordered sets, namely

$$\begin{aligned} (X_1, <) &= (\{\text{proper rational boundary components } F \text{ of } \mathcal{D}_g\}, \succ) \\ (X_2, <) &= (\{\text{non-trivial isotropic subspaces } U \text{ of } \mathbb{Q}^g\}, \subsetneqq). \end{aligned}$$

The group $\operatorname{Sp}(2g, \mathbb{Q})$ acts on both partially ordered sets as a group of automorphisms and the map $f : X_1 \to X_2$ which associates to each

F the isotropic subspace $U(F)$ is an $\mathrm{Sp}(2g,\mathbb{Q})$-equivariant isomorphism of partially ordered sets. To every partially ordered set $(X,<)$ one can associate its *simplicial realization* $\mathrm{SR}(X)$ which is the simplicial complex consisting of all simplices $(x_0,\ldots,x_n)$ where $x_0,\ldots,x_n \in X$ and $x_0 < x_1 < \ldots < x_n$. The *Tits building* $\mathcal{T}$ of $\mathrm{Sp}(2g,\mathbb{Q})$ is the simplicial complex $\mathcal{T} = \mathrm{SR}(X_1) = \mathrm{SR}(X_2)$. If Γ is an arithmetic subgroup of $\mathrm{Sp}(2g,\mathbb{Q})$, then the Tits building of Γ is the quotient $\mathcal{T}(\Gamma) = \Gamma\backslash\mathcal{T}$.

The group $\mathcal{P}(F)$ is a maximal parabolic subgroup of $\mathrm{Sp}(2g,\mathbb{R})$. More generally, given any flag $U_1 \subsetneqq \ldots \subsetneqq U_l$ of isotropic subspaces, its stabilizer is a parabolic subgroup of $\mathrm{Sp}(2g,\mathbb{R})$. Conversely any parabolic subgroup is the stabilizer of some isotropic flag. The maximal length of an isotropic flag in $\mathbb{R}^{2g}$ is g and the corresponding subgroups are the minimal parabolic subgroups or *Borel subgroups* of $\mathrm{Sp}(2g,\mathbb{R})$. We have already remarked that a boundary component F is rational if and only if the stabilizer $\mathcal{P}(F)$ is defined over the rationals, which happens if and only if $U(F)$ is a rational subspace. More generally an isotropic flag is rational if and only if its stabilizer is defined over $\mathbb{Q}$. This explains how the Tits building $\mathcal{T}$ of $\mathrm{Sp}(2g,\mathbb{Q})$ can be defined using parabolic subgroups of $\mathrm{Sp}(2g,\mathbb{R})$ which are defined over $\mathbb{Q}$. The Tits building of an arithmetic subgroup Γ of $\mathrm{Sp}(2g,\mathbb{Q})$ can, therefore, also be defined in terms of conjugacy classes of groups $\Gamma \cap \mathcal{P}(F)$.

As an example we consider the integer symplectic group $\mathrm{Sp}(2g,\mathbb{Z})$. There exists exactly one maximal isotropic flag modulo the action of $\mathrm{Sp}(2g,\mathbb{Z})$, namely

$$\{0\} \subsetneqq U_1 \subsetneqq U_2 \subsetneqq \ldots \subsetneqq U_g; \quad U_i = \mathrm{span}(e_1,\ldots,e_i).$$

Hence the Tits building $\mathcal{T}(\mathrm{Sp}(2g,\mathbb{Z}))$ is a $(g-1)$-simplex whose vertices correspond to the space U_i. This corresponds to the fact that set-theoretically

$$\bar{\mathcal{A}}_g = \mathcal{A}_g \amalg \mathcal{A}_{g-1} \amalg \ldots \amalg \mathcal{A}_1 \amalg \mathcal{A}_0.$$

With these preparations we can now sketch the construction of a toroidal compactification of a quotient $\mathcal{A}(\Gamma) = \Gamma\backslash\mathbb{H}_g$ where Γ is an arithmetic subgroup of $\mathrm{Sp}(2g,\mathbb{Q})$. We have to compactify $\mathcal{A}(\Gamma)$ in the direction of the cusps, which are in 1-to-1 correspondence with the vertices of the Tits building $\mathcal{T}(\Gamma)$. We shall first fix one cusp and consider the associated boundary component F, resp. the isotropic subspace $U = U(F)$. Let $\mathcal{P}(F)$ be the stabilizer of F in $\mathrm{Sp}(2g,\mathbb{R})$. Then there is an exact sequence of Lie groups

$$1 \to \mathcal{P}'(F) \to \mathcal{P}(F) \to \mathcal{P}''(F) \to 1$$

where $\mathcal{P}'(F)$ is the centre of the unipotent radical $R_u(\mathcal{P}(F))$ of $\mathcal{P}(F)$. Here $\mathcal{P}'(F)$ is a real vector space isomorphic to $\mathrm{Sym}(g', \mathbb{R})$ where $g' = \dim U(F)$. Let $P(F) = \mathcal{P}(F) \cap \Gamma, P'(F) = \mathcal{P}'(F) \cap \Gamma$ and $P''(F) = P(F)/P'(F)$. The group $P'(F)$ is a lattice of maximal rank in $\mathcal{P}'(F)$. To F one can now associate a torus bundle $\mathcal{X}(F)$ with fibre $T = P'(F) \otimes_{\mathbb{Z}} \mathbb{C}/P'(F) \cong (\mathbb{C}^*)^{g'}$ over the base $S = F \times V(F)$ where $V(F) = R_u(\mathcal{P}(F))/\mathcal{P}'(F)$ is an affine abelian Lie group and hence a vector space. To construct a partial compactification of $\mathcal{A}(\Gamma)$ in the direction of the cusp corresponding to F, one then proceeds as follows:

(1) Consider the partial quotient $X(F) = P'(F)\backslash\mathbb{H}_g$. This is a torus bundle with fibre $(\mathbb{C}^*)^{\frac{1}{2}g'(g'+1)}$ over some open subset of $\mathbb{C}^{\frac{1}{2}(g(g+1)-g'(g'+1))}$ and can be regarded as an open subset of the torus bundle $\mathcal{X}(F)$.
(2) Choose a fan Σ in the real vector space $\mathcal{P}'(F) \cong \mathrm{Sym}(g', \mathbb{R})$ and construct a trivial bundle $\mathcal{X}_\Sigma(F)$ whose fibres are torus embeddings.
(3) If Σ is chosen compatible with the action of $P''(F)$, then the action of $P''(F)$ on $\mathcal{X}(F)$ extends to an action of $P''(F)$ on $\mathcal{X}_\Sigma(F)$.
(4) Denote by $X_\Sigma(F)$ the interior of the closure of $X(F)$ in $\mathcal{X}_\Sigma(F)$. Define the partial compactification of $\mathcal{A}(\Gamma)$ in the direction of F as the quotient space $Y_\Sigma(F) = P''(F)\backslash X_\Sigma(F)$.

To be able to carry out this programme we may not choose the fan Σ arbitrarily, but we must restrict ourselves to *admissible* fans Σ (for a precise definition see [Nam2, Definition 7.3]). In particular Σ must define a cone decomposition of the cone $\mathrm{Sym}_+(g', \mathbb{R})$ of positive definite symmetric $(g' \times g')$-matrices. The space $Y_\Sigma(F)$ is called the partial compactification in the direction F.

The above procedure describes how to compactify $\mathcal{A}(\Gamma)$ in the direction of one cusp F. This programme then has to be carried out for each cusp in such a way that the partial compactifications glue together and give the desired toroidal compactification. For this purpose we have to consider a collection $\tilde{\Sigma} = \{\Sigma(F)\}$ of fans $\Sigma(F) \subset \mathcal{P}'(F)$. Such a collection is called an *admissible collection of fans* if

(1) Every fan $\Sigma(F) \subset \mathcal{P}'(F)$ is an admissible fan.
(2) If $F = g(F')$ for some $g \in \Gamma$, then $\Sigma(F) = g(\Sigma(F'))$ as fans in the space $\mathcal{P}'(F) = g(\mathcal{P}'(F'))$.
(3) If $F' \succ F$ is a pair of adjacent rational boundary components, then equality $\Sigma(F') = \Sigma(F) \cap \mathcal{P}'(F')$ holds as fans in $\mathcal{P}'(F') \subset \mathcal{P}'(F)$.

The conditions (2) and (3) ensure that the compactifications in the direction of the various cusps are compatible and can be glued together. More precisely we obtain the following:

(2′) If $g \in \Gamma$ with $F = g(F')$, then there exists a natural isomorphism $\tilde{g} : X_{\Sigma(F')}(F') \to X_{\Sigma(F)}(F)$.

(3′) Suppose $F' \succ F$ is a pair of adjacent rational boundary components. Then $P'(F') \subset P'(F)$ and there exists a natural quotient map $\pi_0(F', F) : X(F') \to X(F)$. Because of (3) this extends to an étale map: $\pi(F', F) : X_{\Sigma(F')}(F') \to X_{\Sigma(F)}(F)$.

We can now consider the disjoint union

$$X = \coprod_F X_{\Sigma(F)}(F)$$

over all rational boundary components F. One can define an equivalence relation on X as follows: if $x \in X_{\Sigma(F)}(F)$ and $x' \in X_{\Sigma(F')}(F')$, then

(a) $x \sim x'$ if there exists $g \in \Gamma$ such that $F = g(F')$ and $x = \tilde{g}(x')$.

(b) $x \sim x'$ if $F' \succ F$ and $\pi(F', F)(x') = x$.

The *toroidal compactification* of $\mathcal{A}(\Gamma)$ defined by the admissible collection of fans $\tilde{\Sigma}$ is then the space

$$\mathcal{A}(\Gamma)^* = X/\sim .$$

Clearly $\mathcal{A}(\Gamma)^*$ depends on $\tilde{\Sigma}$. We could also have described $\mathcal{A}(\Gamma)^*$ as $Y/\sim$ where $Y = \amalg\, Y_{\Sigma(F)}(F)$ and the equivalence relation $\sim$ on Y is induced from that on X. There is a notion of a *projective* admissible collection of fans (see [Nam2, Definition 7.22]) which ensures that the space $\mathcal{A}(\Gamma)^*$ is projective.

For every toroidal compactification there is a natural map $\pi : \mathcal{A}(\Gamma)^* \to \bar{\mathcal{A}}(\Gamma)$ to the Satake compactification. Tai, in [AMRT], showed that if $\mathcal{A}(\Gamma)^*$ is defined by a projective admissible collection of fans, then π is the normalization of the blow-up of some ideal sheaf supported on the boundary of $\bar{\mathcal{A}}(\Gamma)$.

There are several well known cone decompositions for $\mathrm{Sym}_+(g', \mathbb{R})$: see e.g. [Nam2, section 8]. The *central cone decomposition* was used by Igusa ([I1]) and leads to the *Igusa compactification.* The most important decomposition for our purposes is the *second Voronoi decomposition.* The corresponding compactification is simply called the *Voronoi compactification.* The Voronoi compactification $\mathcal{A}(\Gamma)^* = \mathcal{A}_g^*$ for $\Gamma = \mathrm{Sp}(2g, \mathbb{Z})$ is a projective variety ([Ale1]). For $g = 2$ all standard known cone decompositions coincide with the *Legendre decomposition.*

II. Classification theory

Here we discuss known results about the Kodaira dimension of Siegel modular varieties and about canonical and minimal models. We also report on some work on the fundamental group of Siegel modular varieties.

II.1. The canonical divisor

If one wants to prove results about the Kodaira dimension of Siegel modular varieties, one first has to understand the canonical divisor. For an element $\tau \in \mathbb{H}_g$ we write

$$\tau = \left(\begin{array}{ccc|c} \tau_{11} & \cdots & \tau_{1,g-1} & \tau_{1g} \\ \vdots & & & \vdots \\ \tau_{1,g-1} & \cdots & \tau_{g-1,g-1} & \tau_{g-1,g} \\ \hline \tau_{1g} & \cdots & \tau_{g-1,g} & \tau_{gg} \end{array}\right) = \left(\begin{array}{c|c} \tau' & {}^t z \\ \hline z & \tau_{gg} \end{array}\right).$$

Let

$$d\tau = d\tau_{11} \wedge d\tau_{12} \wedge \ldots \wedge d\tau_{gg}.$$

If F is a modular form of weight $g+1$ with respect to an arithmetic group Γ, then it is easy to check that the form $\omega = F d\tau$ is Γ-invariant. Hence, if Γ acts *freely*, then

$$K_{\mathcal{A}(\Gamma)} = (g+1)L$$

where L is the line bundle of modular forms, i.e. the line bundle given by the automorphy factor $\det(C\tau + D)$. If Γ does not act freely, let ${}^\circ\!\mathcal{A}(\Gamma) = \mathcal{A}(\Gamma) \backslash R$ where R is the branch locus of the quotient map $\mathbb{H}_g \to \mathcal{A}(\Gamma)$. Then by the above reasoning it is still true that

$$K_{{}^\circ\!\mathcal{A}(\Gamma)} = (g+1)L|_{{}^\circ\!\mathcal{A}(\Gamma)}.$$

In order to describe the canonical bundle on a toroidal compactification $\mathcal{A}(\Gamma)^*$ we have to understand the behaviour of the differential form ω at the boundary. To simplify the exposition, we shall first consider the case $\Gamma_g = \mathrm{Sp}(2g, \mathbb{Z})$. Then there exists, up to the action of Γ, exactly one maximal boundary component F. We can assume that $U(F) = U = \mathrm{span}(e_g)$. The stabilizer $P(F) = P(U)$ of U in Γ_g is generated by elements of the form

$$g_1 = \begin{pmatrix} A & 0 & B & 0 \\ 0 & 1 & 0 & 0 \\ C & 0 & D & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad g_2 = \begin{pmatrix} \mathbf{1}_{g-1} & 0 & 0 & 0 \\ 0 & \pm 1 & 0 & 0 \\ 0 & 0 & \mathbf{1}_{g-1} & 0 \\ 0 & 0 & 0 & \pm 1 \end{pmatrix},$$

$$g_3 = \begin{pmatrix} \mathbf{1}_{g-1} & 0 & 0 & {}^tN \\ M & 1 & N & 0 \\ 0 & 0 & \mathbf{1}_{g-1} & -{}^tM \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad g_4 = \begin{pmatrix} \mathbf{1}_{g-1} & 0 & 0 & 0 \\ 0 & 1 & 0 & S \\ 0 & 0 & \mathbf{1}_{g-1} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_{g-1}$, $M, N \in \mathbb{Z}^{g-1}$ and $S \in \mathbb{Z}$.

The group $P'(F)$ is the rank 1 lattice generated by g_4, and the partial quotient with respect to $P'(F)$ is given by

$$\begin{array}{rcl} e(F) : \mathbb{H}_g & \longrightarrow & \mathbb{H}_{g-1} \times \mathbb{C}^{g-1} \times \mathbb{C}^* \\ \tau & \longmapsto & (\tau', z, t = e^{2\pi i \tau_{gg}}). \end{array}$$

Here $\mathbb{H}_{g-1} \times \mathbb{C}^{g-1} \times \mathbb{C}^*$ is a rank 1 torus bundle over $\mathbb{H}_{g-1} \times \mathbb{C}^{g-1} = F \times V(F)$. Partial compactification in the direction of F consists of adding $\mathbb{H}_{g-1} \times \mathbb{C}^{g-1} \times \{0\}$ and then taking the quotient with respect to $P''(F)$. Since $d\tau_{gg} = (2\pi i)^{-1} dt/t$ it follows that

$$\omega = (2\pi i)^{-1} F \frac{d\tau_{11} \wedge \ldots \wedge d\tau_{g-1,g} \wedge dt}{t}$$

has a pole of order 1 along the boundary, unless F vanishes there. Moreover, since $F(g_4(\tau)) = F(\tau)$ it follows that F has a Fourier expansion

$$F(\tau) = \sum_{n \geq 0} F_n(\tau', z) t^n.$$

A modular form F is a *cusp form* if $F_0(\tau', z) = 0$, i.e. if F vanishes along the boundary. (If Γ is an arbitrary arithmetic subgroup of $\mathrm{Sp}(2g, \mathbb{Q})$ we have in general several boundary components and then we require vanishing of F along each of these boundary components.) The above discussion can be interpreted as follows. First assume that Γ is neat (i.e. the subgroup of $\mathbb{C}^*$ generated by the eigenvalues of all elements of Γ is torsion free) and that $\mathcal{A}(\Gamma)^*$ is a smooth compactification with the following property: for every point in the boundary there exists a representative $x \in X_{\Sigma(F)}(F)$ for some boundary component such that $X_{\Sigma(F)}(F)$ is smooth at x and $P''(F)$ acts freely at x. (Such a toroidal compactification always exists if Γ is neat.) Let D be the boundary divisor of $\mathcal{A}(\Gamma)^*$. Then

$$K_{\mathcal{A}(\Gamma)^*} = (g+1)L - D.$$

Here L is the extension of the line bundle on modular forms on $\mathcal{A}(\Gamma)$ to $\mathcal{A}(\Gamma)^*$. This makes sense since by construction the line bundle extends to the Satake compactification $\bar{\mathcal{A}}(\Gamma)$ and since there is a natural map

$\pi : \mathcal{A}(\Gamma)^* \to \bar{\mathcal{A}}(\Gamma)$. We use the same notation for L and $\pi^* L$. If Γ does not act freely we can define the open set $^\circ\!\mathcal{A}(\Gamma)^*$ consisting of $^\circ\!\mathcal{A}(\Gamma)$ and those points in the boundary which have a representative $x \in X_{\Sigma(F)}(F)$ where $P''(F)$ acts freely at x. In this case we still have

$$K_{^\circ\!\mathcal{A}(\Gamma)^*} = ((g+1)L - D)|_{^\circ\!\mathcal{A}(\Gamma)^*}.$$

This shows in particular that every cusp form F of weight $g+1$ with respect to Γ defines via $\omega = F d\tau$ a differential N-form on $^\circ\!\mathcal{A}(\Gamma)^*$ where $N = \frac{g(g+1)}{2}$ is the dimension of $\mathcal{A}(\Gamma)$. It is a non-trivial result of Freitag that every such form can be extended to any smooth projective model of $\mathcal{A}(\Gamma)$. If we denote by $S_k(\Gamma)$ the space of cusp forms of weight k with respect to Γ, then we can formulate Freitag's result as follows.

Theorem II.1.1 ([F]). *Let $\tilde{\mathcal{A}}(\Gamma)$ be a smooth projective model of $\mathcal{A}(\Gamma)$. Then every cusp form F of weight $g+1$ with respect to Γ defines a differential form $\omega = F d\tau$ which extends to $\tilde{\mathcal{A}}(\Gamma)$. In particular, there is a natural isomorphism*

$$\Gamma(\tilde{\mathcal{A}}(\Gamma), \omega_{\tilde{\mathcal{A}}(\Gamma)}) \cong S_{g+1}(\Gamma)$$

and hence $p_g(\tilde{\mathcal{A}}(\Gamma)) = \dim S_{g+1}(\Gamma)$.

Proof. See [F, Satz III.2.6] and the remark following this. Q.E.D.

Similarly a form of weight $k(g+1)$ which vanishes of order k along the boundary defines a k-fold differential form on $^\circ\!\mathcal{A}(\Gamma)^*$. In general, however, such a form does not extend to a smooth model $\tilde{\mathcal{A}}(\Gamma)$ of $\mathcal{A}(\Gamma)$.

II.2. The Kodaira dimension of $\mathcal{A}_g(n)$

By the *Kodaira dimension* of a Siegel modular variety $\mathcal{A}(\Gamma)$ we mean the Kodaira dimension of a smooth projective model of $\mathcal{A}(\Gamma)$. Such a model always exists and the Kodaira dimension is independent of the specific model chosen. It is a well known result that $\mathcal{A}_g$ is of general type for $g \geq 7$. This was first proved by Tai for $g \geq 9$ ([T1]) and then improved to $g \geq 8$ by Freitag ([F]) and to $g \geq 7$ by Mumford ([Mu4]). In this section we want to discuss the proof of the following result.

Theorem II.2.1 ([T1], [F], [Mu4] and [H2]). *$\mathcal{A}_g(n)$ is of general type for the following values of g and $n \geq n_0$:*

g	2	3	4	5	6	≥ 7
n_0	4	3	2	2	2	1

We have already seen that the construction of differential forms is closely related to the existence of cusp forms. Using Mumford's extension of

Hirzebruch proportionality to the non-compact case and the Atiyah-Bott fixed point theorem it is not difficult to show that the dimension of the space of cusp forms of weight k grows as follows:

$$\dim S_k(\Gamma_g) \sim 2^{-N-g} k^N V_g \pi^{-N}$$

where

$$N = \frac{g(g+1)}{2} = \dim \mathcal{A}_g(n)$$

and V_g is Siegel's symplectic volume

$$V_g = 2^{g^2+1}\pi^N \prod_{j=1}^{g} \frac{(j-1)!}{2j!} B_j.$$

Here B_j are the Bernoulli numbers.

Every form of weight $k(g+1)$ gives rise to a k-fold differential form on $^\circ\!\mathcal{A}_g(n)$. If $k = 1$, we have already seen that these forms extend by Freitag's extension theorem to every smooth model of $\mathcal{A}_g(n)$. This is no longer automatically the case if $k \geq 2$. Then one encounters two types of obstructions: one is extension to the boundary (since we need higher vanishing order along D), the other type of obstruction comes from the singularities, or more precisely from those points where $\Gamma_g(n)$ does not act freely. These can be points on $\mathcal{A}_g(n)$ or on the boundary. If $n \geq 3$, then $\Gamma_g(n)$ is neat and in particular it acts freely. Moreover we can choose a suitable cone decomposition such that the corresponding toroidal compactification is smooth. In this case there are no obstructions from points where $\Gamma_g(n)$ does not act freely. If $n = 1$ or 2 we shall, however, always have such points. It is one of the main results of Tai ([T1, Section 5]) that for $g \geq 5$ all resulting singularities are *canonical*, i.e. give no obstructions to extending k-fold differential forms to a smooth model. The remainder of the proof of Tai then consists of a careful analysis of the obstructions to the extension of k-forms to the boundary. These obstructions lie in a vector space which can be interpreted as a space of Jacobi forms on $\mathbb{H}_{g-1} \times \mathbb{C}^{g-1}$. Tai gives an estimate of this space in [T1, Section 2] and compares it with the dimension formula for $S_k(\Gamma_g)$.

The approach developed by Mumford in [Mu4] is more geometric in nature. First recall that

$$(1) \qquad K|_{^\circ\!\mathcal{A}_g^*(n)} = ((g+1)L - D)|_{^\circ\!\mathcal{A}_g^*(n)}.$$

Let $\bar{\Theta}_{\mathrm{null}}$ be the closure of the locus of pairs (A, Θ) where A is an abelian variety and Θ is a symmetric divisor representing a principal polarization such that Θ has a singularity at a point of order 2. Then one can show that for the class of $\bar{\Theta}_{\mathrm{null}}$ on $\mathcal{A}_g^*(n)$:

$$\text{(2)} \qquad [\bar{\Theta}_{\mathrm{null}}] = 2^{g-2}(2^g+1)L - 2^{2g-5}D.$$

One can now use (2) to eliminate the boundary D in (1). Since the natural quotient $\mathcal{A}_g^*(n) \to \mathcal{A}_g^*$ is branched of order n along D one finds the following formula for K:

$$\text{(3)} \qquad K|_{{}^\circ\mathcal{A}_g^*(n)} = \left((g+1) - \frac{2^{g-2}(2^g+1)}{n2^{2g-5}}\right)L + \frac{1}{n2^{2g-5}}[\bar{\Theta}_{\mathrm{null}}].$$

In view of Tai's result on the singularities of $\mathcal{A}_g^*(n)$ this gives general type whenever the factor in front of L is positive and $n \geq 3$ or $g \geq 5$. This gives all cases in the list with two exceptions, namely $(g,n) = (4,2)$ and $(7,1)$. In the first case the factor in front of L is still positive, but one cannot immediately invoke Tai's result on canonical singularities. As Salvetti Manni has pointed out, one can, however, argue as follows. An easy calculation shows that for every element $\sigma \in \Gamma_g(2)$ the square $\sigma^2 \in \Gamma_g(4)$. Hence if σ has a fixed point then $\sigma^2 = 1$ since $\Gamma_g(4)$ acts freely. But now one can again use Tai's extension result (see [T1, Remark after Lemma 4.5] and [T1, Remark after Lemma 5.2]).

This leaves the case $(g,n) = (7,1)$ which is the main result of [Mu4]. Mumford considers the locus

$$N_0 = \{(A,\Theta)\,;\ \operatorname{Sing}\Theta \neq \emptyset\}$$

in $\mathcal{A}_g$. Clearly this contains Θ_{null}, but is bigger than Θ_{null} if $g \geq 4$. Mumford shows that the class of the closure $\bar{N}_0$ on $\mathcal{A}_g^*$ is

$$\text{(4)} \qquad [\bar{N}_0] = \left(\frac{(g+1)!}{2} + g!\right)L - \frac{(g+1)!}{12}D$$

and hence one finds for the canonical divisor:

$$K|_{{}^\circ\mathcal{A}_g^*(n)} = \frac{12(g^2-4g-17)}{g+1}L + \frac{12}{(g+1)!}[\bar{N}_0].$$

Since the factor in front of L is positive for $g = 7$ one can once more use Tai's extension result to prove the theorem for $(g,n) = (7,1)$.

The classification of the varieties $\mathcal{A}_g(n)$ with respect to the Kodaira dimension is therefore now complete with the exception of one important case:

Problem. Determine the Kodaira dimension of $\mathcal{A}_6$.

All other varieties $\mathcal{A}_g(n)$ which do not appear in the above list are known to be either rational or unirational. Unirationality of $\mathcal{A}_5$ was proved by Donagi ([D]) and independently by Mori and Mukai ([MM]) and Verra ([V]). Unirationality of $\mathcal{A}_4$ was shown by Clemens ([Cl]) and unirationality of $\mathcal{A}_g, g \leq 3$ is easy. For $g = 3$ there exists a dominant map from the space of plane quartics to $\mathcal{M}_3$ which in turn is birational to $\mathcal{A}_3$. For $g = 2$ one can use the fact that $\mathcal{M}_2$ is birational to $\mathcal{A}_2$ and that every genus 2 curve is a 2:1 cover of $\mathbb{P}^1$ branched in 6 points. Rationality of these spaces is a more difficult question. Igusa ([I1]) showed that $\mathcal{A}_2$ is rational. The rationality of $\mathcal{M}_3$, and hence also of $\mathcal{A}_3$, was proved by Katsylo ([K]). The space $\mathcal{A}_3(2)$ is rational by the work of van Geemen ([vG]) and Dolgachev and Ortland ([DO]). The variety $\mathcal{A}_2(3)$ is birational to the Burkhardt quartic in $\mathbb{P}^4$ and hence also rational. This was proved by Todd in 1936 ([To]) and Baker in 1942 (see [Ba2]), but see also the thesis of Finkelnberg ([Fi]). The variety $\mathcal{A}_2(2)$ is birational to the Segre cubic (cf. [vdG1]) in $\mathbb{P}^4$ and hence also rational. The latter two cases are examples of Siegel modular varieties which have very interesting projective models. We will come back to this more systematically in chapter IV. It should also be noted that Yamazaki ([Ya]) was the first to prove that $\mathcal{A}_2(n)$ is of general type for $n \geq 4$.

All the results discussed above concern the case of principal polarization. The case of non-principal polarizations of type $(e_1, \ldots, e_g)$ was also studied by Tai.

Theorem II.2.2 ([T2]). *The moduli space $\mathcal{A}_{e_1,\ldots,e_g}$ of abelian varieties with a polarization of type $(e_1, \ldots, e_g)$ is of general type if either $g \geq 16$ or $g \geq 8$ and all e_i are odd and sums of two squares.*

The essential point in the proof is the construction of sufficiently many cusp forms with high vanishing order along the boundary. These modular forms are obtained as pullbacks of theta series on Hermitian or quaternionic upper half spaces.

More detailed results are known in the case of abelian surfaces ($g = 2$). We will discuss this separately in chapters III and V.

By a different method, namely using symmetrization of modular forms, Gritsenko has shown the following:

Theorem II.2.3 ([Gr1]). *For every integer t there is an integer $g(t)$ such that the moduli space $\mathcal{A}_{1,\ldots,1,t}$ is of general type for $g \geq g(t)$. In particular $\mathcal{A}_{1,\ldots,1,2}$ is of general type for $g \geq 13$.*

Proof. See [Gr1, Satz 1.1.10], where an explicit bound for $g(t)$ is given. Q.E.D.

Once one has determined that a variety is of general type it is natural to ask for a minimal or canonical model. For a given model this means asking whether the canonical divisor is nef or ample. In fact one can ask more generally what the nef cone is. The Picard group of $\mathcal{A}_g^*, g = 2, 3$ is generated (modulo torsion) by two elements, namely the ($\mathbb{Q}$-) line bundle L given by modular forms of weight 1 and the boundary D.

In [H2] one of us computed the nef cone of $\mathcal{A}_g^*, g = 2, 3$. The result is given by the theorem below. As we shall see one can give a quick proof of this using known results about $\overline{\mathcal{M}}_g$ and the Torelli map. However this approach cannot be generalized to higher genus since the Torelli map is then no longer surjective, nor to other than principal polarizations. For this reason an alternative proof was given in [H2] making essential use of a result of Weissauer ([We]) on the existence of cusp forms of small slope which do not vanish on a given point in Siegel space.

Theorem II.2.4. *Let $g = 2$ or 3. Then a divisor $aL - bD$ on $\mathcal{A}_g^*$ is nef if and only if $b \geq 0$ and $a - 12b \geq 0$.*

Proof. First note that the two conditions are necessary. In fact let C be a curve which is contracted under the natural map $\pi : \mathcal{A}_g^* \to \bar{\mathcal{A}}_g$ onto the Satake compactification. The divisor $-D$ is π-ample (cf. also [Mu4]) and L is the pull-back of a line bundle on $\bar{\mathcal{A}}_g$. Hence $(aL - bD).C \geq 0$ implies $b \geq 0$. Let C be the closure of the locus given by split abelian varieties $E \times A'$ where E is an arbitrary elliptic curve and A' is a fixed abelian variety of dimension $g - 1$. Then C is a rational curve with $D.C = 1$ and $L.C = 1/12$. This shows that $a - 12b \geq 0$ for every nef divisor D.

To prove that the conditions stated are sufficient we consider the Torelli map $t : \mathcal{M}_g \to \mathcal{A}_g$ which extends to a map $\bar{t} : \overline{\mathcal{M}}_g \to \mathcal{A}_g^*$. This map is surjective for $g = 2, 3$. Here $\overline{\mathcal{M}}_g$ denotes the compactification of $\mathcal{M}_g$ by stable curves. It follows that for every curve C in $\mathcal{A}_g^*$ there exists a curve C' in $\overline{\mathcal{M}}_g$ which is finite over C. Hence a divisor on $\mathcal{A}_g^*, g = 2, 3$ is nef if and only if this is true for its pull-back to $\overline{\mathcal{M}}_g$. We can now use Faber's paper ([Fa]). Then $\bar{t}^*L = \lambda$ where λ is the Hodge bundle and $\bar{t}^*D = \delta_0$. Here δ_0 is the boundary ($g = 2$), resp. the closure of the locus of genus 2 curves with one node ($g = 3$). The result now follows from [Fa] since $a\lambda - b\delta_0$ is nef on $\overline{\mathcal{M}}_g$, $g = 2, 3$ if $b \geq 0$ and $a - 12b \geq 0$. Q.E.D.

Corollary II.2.5. *The canonical divisor on $\mathcal{A}_2^*(n)$ is nef but not ample for $n = 4$ and ample for $n \geq 5$. In particular $\mathcal{A}_2^*(4)$ is a minimal model and $\mathcal{A}_2^*(n)$ is a canonical model for $n \geq 5$.*

This was first observed, though not proved in detail, by Borisov in an early version of [Bori].

Corollary II.2.6. *The canonical divisor on $\mathcal{A}_3^*(n)$ is nef but not ample for $n = 3$ and ample for $n \geq 4$. In particular $\mathcal{A}_3^*(3)$ is a minimal model and $\mathcal{A}_3^*(n)$ is a canonical model for $n \geq 4$.*

Proof of the corollaries. Nefness or ampleness of K follows immediately from Theorem II.2.4 since

$$(g+1) - \frac{12}{n} \geq 0 \Leftrightarrow \begin{cases} n \geq 4 & \text{if} \quad g = 2 \\ n \geq 3 & \text{if} \quad g = 3. \end{cases}$$

To see that K is not ample on $\mathcal{A}_2^*(4)$ nor on $\mathcal{A}_3^*(3)$ we can again use the curves C coming from products $E \times A'$ where A' is a fixed abelian variety of dimension $g - 1$. For these curves $K.C = 0$. □

For $g \geq 4$ it is, contrary to what was said in [H2], no longer true that the Picard group is generated by L and D. Here we simply state the

Problem. Describe the nef cone of $\mathcal{A}_g^*$.

In [H3] the methods of [H2] were used to prove ampleness of K in the case of $(1,p)$-polarized abelian surfaces with a canonical level structure and a level-n structure, for p prime and $n \geq 5$, provided p does not divide n.

Finally we want to mention some results concerning the Chow ring of $\mathcal{A}_g^*$. The Chow groups considered here are defined as the invariant part of the Chow ring of $\mathcal{A}_g^*(n)$. The Chow ring of $\overline{\mathcal{M}}_2$ was computed by Mumford [Mu5]. This gives also the Chow ring of $\mathcal{A}_2^*$, which was also calculated by a different method by van der Geer in [vdG3].

Theorem II.2.7 ([Mu5] and [vdG3]). *Let $\lambda_1 = \lambda$ and λ_2 be the tautological classes on $\mathcal{A}_2^*$. Let σ_1 be the class of the boundary. Then*

$$\mathrm{CH}_{\mathbb{Q}}(\mathcal{A}_2^*) \cong \mathbb{Q}[\lambda_1, \lambda_2, \sigma_1]/I$$

where I is the ideal generated by the relations

$$(1 + \lambda_1 + \lambda_2)(1 - \lambda_1 + \lambda_2) = 1,$$
$$\lambda_2 \sigma_1 = 0,$$
$$\sigma_1^2 = 22\sigma_1\lambda_1 - 120\lambda_1^2.$$

The ranks of the Chow groups are $1, 2, 2, 1$.

Van der Geer also computed the Chow ring of $\mathcal{A}_3^*$.

Theorem II.2.8 ([vdG3]). *Let $\lambda_1, \lambda_2, \lambda_3$ be the tautological classes in $\mathcal{A}_3^*$ and σ_1, σ_2 be the first and second symmetric functions in the boundary divisors (viewed as an invariant class on $\mathcal{A}_g^*(n)$). Then*

$$\mathrm{CH}_{\mathbb{Q}}(\mathcal{A}_3^*) \cong \mathbb{Q}[\lambda_1, \lambda_2, \lambda_3, \sigma_1, \sigma_2]/J$$

where J is the ideal generated by the relations

$$\begin{aligned} &(1+\lambda_1+\lambda_2+\lambda_3)(1-\lambda_1+\lambda_2-\lambda_3) = 1,\\ &\lambda_3\sigma_1 \;= \lambda_3\sigma_2 = \lambda_1^2\sigma_2 = 0,\\ &\sigma_1^3 \quad = 2016\lambda_3 - 4\lambda_1^2\sigma_1 - 24\lambda_1\sigma_2 + \tfrac{11}{3}\sigma_2\sigma_1,\\ &\sigma_2^2 \quad = 360\lambda_1^3\sigma_1 - 45\lambda_1^2\sigma_1^2 + 15\lambda_1\sigma_2\sigma_1. \end{aligned}$$

The ranks of the Chow groups are $1, 2, 4, 6, 4, 2, 1$.

Proof. See [vdG3]. The proof uses in an essential way the description of the Voronoi compactification $\mathcal{A}_3^*$ given by Nakamura ([Nak1]) and Tsushima ([Ts]). Q.E.D.

II.3. Fundamental groups

The fundamental group of a smooth projective model $\widetilde{\mathcal{A}}(\Gamma)$ of $\mathcal{A}(\Gamma)$ is independent of the specific model chosen. We assume in this section that $g \geq 2$, so that the dimension of $\mathcal{A}(\Gamma)$ is at least 3.

The first results about the fundamental group of $\widetilde{\mathcal{A}}(\Gamma)$ were obtained by Heidrich and Knöller ([HK], [Kn]) and concern the principal congruence subgroups $\Gamma(n) \subset \mathrm{Sp}(2g, \mathbb{Z})$. They proved the following result.

Theorem II.3.1 ([HK],[Kn]). *If $n \geq 3$ or if $n = g = 2$ then $\widetilde{\mathcal{A}}_g(n)$ is simply-connected.*

As an immediate corollary (first explicitly pointed out by Heidrich-Riske) one has

Corollary II.3.2 ([H–R]). *If Γ is an arithmetic subgroup of $\mathrm{Sp}(2g, \mathbb{Q})$, then the fundamental group of $\widetilde{\mathcal{A}}(\Gamma)$ is finite.*

Corollary II.3.2 follows from Theorem II.3.1 because any subgroup of $\mathrm{Sp}(2g, \mathbb{Z})$ of finite index contains a principal congruence subgroup of some level.

Proof. The proof of Theorem II.3.1 uses the fact that there is, up to the action of the group $\mathrm{Sp}(2g, \mathbb{Z}_n)$, only one codimension 1 boundary component F in the Igusa compactification $\mathcal{A}_g^*(n)$. Suppose for simplicity that $n \geq 4$, so that $\Gamma(n)$ is neat. A small loop passing around this component can be identified with a loop in the fibre $\mathbb{C}^*$ of $\mathcal{X}(F)$ and hence with a generator u_F of the 1-dimensional lattice $P'(F)$. This loop determines an element γ_F, usually non-trivial, of $\pi_1(\mathcal{A}_g(n))$ (which is simply $\Gamma(n)$, since $\Gamma(n)$ is torsion-free and hence acts freely on $\mathbb{H}_g$). The element γ_F is in the kernel of the map $\pi_1(\mathcal{A}_g(n)) \to \pi_1(\widetilde{\mathcal{A}}_g(n))$, so u_F is in the kernel of $\Gamma(n) \to \pi_1(\widetilde{\mathcal{A}}_g(n))$. But it turns out that the normalizer of $P'(F)$ in $\Gamma(n)$ is the whole of $\Gamma(n)$, as was shown by Mennicke ([Me]) by a direct calculation. Q.E.D.

We (the authors of the present article) applied this method in [HS2] to the case of $\mathcal{A}_{1,p}^{\text{lev}}$ for $p \geq 5$ prime, where there are many codimension 1 boundary components. A minor extra complication is the presence of some singularities in $\Gamma\backslash\mathbb{H}_2$, but they are easily dealt with. In [S1] one of us also considered the case of $\mathcal{A}_{1,p}$. We found the following simple result.

Theorem II.3.3 ([HS2] and [S1]). *If $p \geq 5$ is prime then $\tilde{\mathcal{A}}_{1,p}^{\text{lev}}$ and $\tilde{\mathcal{A}}_{1,p}$ are both simply-connected.*

In some other cases one knows that $\widetilde{\mathcal{A}}(\Gamma)$ is rational and hence simply-connected. In all these cases, as F. Campana pointed out, it follows that the Satake compactification, and any other normal model, is also simply-connected.

By a more systematic use of these ideas, one of us [S1] gave a more general result, valid in fact for all locally symmetric varieties over $\mathbb{C}$. From it several results about Siegel modular varieties can be easily deduced, of which Theorem II.3.4 below is the most striking.

Theorem II.3.4 ([S1]). *For any finite group G there exists a $g \geq 2$ and an arithmetic subgroup $\Gamma \subset \mathrm{Sp}(2g, \mathbb{Z})$ such that $\pi_1(\widetilde{\mathcal{A}}(\Gamma)) \cong G$.*

Proof. We choose an $l \geq 4$ and a faithful representation $\rho : G \to \mathrm{Sp}(2g, \mathbb{F}_p)$ for some prime p not dividing $2l|G|$. The reduction mod p map $\phi_p : \Gamma(l) \to \mathrm{Sp}(2g, \mathbb{F}_p)$ is surjective and we take $\Gamma = \phi_p^{-1}(\rho(G))$. As this is a subgroup of $\Gamma(l)$ it is neat, and under these circumstances the fundamental group of the corresponding smooth compactification of $\mathcal{A}(\Gamma)$ is Γ/Υ, where Υ is a certain subgroup of Γ generated by unipotent elements (each unipotent element corresponds to a loop around a boundary component). From this it follows that $\Upsilon \subset \operatorname{Ker}\phi_p = \Gamma(pl)$. Then from Theorem II.3.1 applied to level pl it follows that $\Upsilon = \Gamma(pl)$ and hence that the fundamental group is $\Gamma/\Gamma(pl) \cong G$. Q.E.D.

For $G = D_8$ we may take $g = 2$; in particular, the fundamental group of a smooth projective model of a Siegel modular threefold need not be abelian. Apart from the slightly artificial examples which constitute Theorem II.3.4, it is also shown in [S1] that a smooth model of the double cover $\tilde{\mathcal{N}}_5$ of Nieto's threefold $\mathcal{N}_5$ has fundamental group $\mathbb{Z}_2 \times \mathbb{Z}_2$. The space $\tilde{\mathcal{N}}_5$ will be discussed in Section IV below: it is birational with the moduli space of abelian surfaces with a polarization of type $(1,3)$ and a level-2 structure.

III. Abelian surfaces

In the case of abelian surfaces the moduli spaces $\mathcal{A}_{1,t}$ and $\mathcal{A}_{1,t}^{\text{lev}}$ of abelian surfaces with a $(1,t)$-polarization, resp. with a $(1,t)$-polarization and a canonical level structure were investigated by a number of authors. One of the starting points for this development was the paper by Horrocks and Mumford ([HM]) which established a connection between the Horrocks-Mumford bundle on $\mathbb{P}^4$ and the moduli space $\mathcal{A}_{1,5}^{\text{lev}}$.

III.1. The lifting method

Using a version of Maaß lifting Gritsenko has proved the existence of a weight 3 cusp forms for almost all values of t. Before we can describe his lifting result recall the *paramodular group* $\mathrm{Sp}(\Lambda,\mathbb{Z})$ where

$$\Lambda = \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix}, E = \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix}$$

for some integer $t \geq 1$, with respect to a basis (e_1,e_2,e_3,e_4). This group is conjugate to the (rational) paramodular group

$$\Gamma_{1,t} = R^{-1}\,\mathrm{Sp}(\Lambda,\mathbb{Z})R, \qquad R = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & t \end{pmatrix}.$$

It is straightforward to check that

$$\Gamma_{1,t} = \left\{ g \in \mathrm{Sp}(4,\mathbb{Q});\ g \in \begin{pmatrix} \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & t\mathbb{Z} \\ t\mathbb{Z} & \mathbb{Z} & t\mathbb{Z} & t\mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & t\mathbb{Z} \\ \mathbb{Z} & t^{-1}\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \end{pmatrix} \right\}.$$

Then $\mathcal{A}_{1,t} = \Gamma_{1,t} \backslash \mathbb{H}_2$ is the moduli space of $(1,t)$-polarized abelian surfaces. In this chapter we shall denote the elements of $\mathbb{H}_2$ by

$$\tau = \begin{pmatrix} \tau_1 & \tau_2 \\ \tau_2 & \tau_3 \end{pmatrix} \in \mathbb{H}_2.$$

The Tits building of $\Gamma_{1,t}$, and hence the combinatorial structure of the boundary components of the Satake or the Voronoi (Igusa) compactification of $\mathcal{A}_{1,t}$ are known, at least if t is square free: see [FrS], where Tits buildings for some other groups are also calculated. There are exactly $\mu(t)$ corank 1 boundary components, where $\mu(t)$ denotes the number of prime divisors of t ([Gr1, Folgerung 2.4]). If t is square free, then there exists exactly one corank 2 boundary component ([Fr, Satz 4.7]). In particular, if $t > 1$ is a prime number then there exist two corank 1 boundary components and one corank 2 boundary component. These boundary components belong to the isotropic subspaces spanned by e_3 and e_4, resp. by $e_3 \wedge e_4$. In terms of the Siegel space the two corank 1 boundary components correspond to $\tau_1 \to i\infty$ and $\tau_3 \to i\infty$. For $t = 1$ these two components are equivalent under the group $\Gamma_{1,1} = \mathrm{Sp}(4, \mathbb{Z})$.

Gritsenko's construction of cusp forms uses a version of Maaß lifting. In order to explain this, we first have to recall the definition of *Jacobi forms.* Here we restrict ourselves to the case of $\Gamma_{1,1} = \mathrm{Sp}(4, \mathbb{Z})$. The stabilizer of $\mathbb{Q}e_4$ in $\mathrm{Sp}(4, \mathbb{Z})$ has the structure

$$P(e_4) \cong \mathrm{SL}(2, \mathbb{Z}) \ltimes H(\mathbb{Z})$$

where $\mathrm{SL}(2, \mathbb{Z})$ is identified with

$$\left\{ \begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} ; \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z}) \right\}$$

and

$$H(\mathbb{Z}) = \left\{ \begin{pmatrix} 1 & 0 & 0 & \mu \\ \lambda & 1 & \mu & r \\ 0 & 0 & 1 & -\lambda \\ 0 & 0 & 0 & 1 \end{pmatrix} ; \lambda, \mu, r \in \mathbb{Z} \right\}$$

is the *integral Heisenberg group.*

Every modular form $F \in M_k(\mathrm{Sp}(4, \mathbb{Z}))$ of weight k with respect to $\mathrm{Sp}(4, \mathbb{Z})$ has a Fourier extension with respect to τ_3 which is of the following form

$$F(\tau) = \sum_{m \geq 0} f_m(\tau_1, \tau_2) e^{2\pi i m \tau_3}.$$

The same is true for modular forms with respect to $\Gamma_{1,t}$, the only difference is that the factor $\exp(2\pi i m \tau_3)$ has to be replaced by $\exp(2\pi i m t \tau_3)$. The coefficients $f_m(\tau_1, \tau_2)$ are examples of *Jacobi forms.* Formally Jacobi forms are defined as follows:

Definition. A *Jacobi form* of *index* m and *weight* k is a holomorphic function

$$\Phi = \Phi(\tau, z) : \mathbb{H}_1 \times \mathbb{C} \to \mathbb{C}$$

which has the following properties:

(1) It has the *transformation behaviour*

(a) $\Phi\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\right) = (c\tau+d)^k e^{\frac{2\pi i c m z^2}{c\tau+d}} \Phi(\tau, z),$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})$

(b) $\Phi(\tau, z+\lambda\tau+\mu) = e^{-2\pi i m(\lambda^2\tau+2\lambda z)}\Phi(\tau, z), \quad \lambda, \mu \in \mathbb{Z}.$

(2) It has a Fourier expansion

$$\Phi(\tau, z) = \sum_{\substack{n,l\in\mathbb{Z}, n\geq 0 \\ 4nm\geq l^2}} f(n,l)e^{2\pi i(n\tau+lz)}.$$

A Jacobi form is called a *cusp form* if one has strict inequality $4nm > l^2$ in the Fourier expansion.

Note that for $z = 0$ the transformation behaviour described by (1)(a) is exactly that of a modular form. For fixed τ the transformation law (1)(b) is, up to a factor 2 in the exponent, the transformation law for theta functions. One can also summarize (1)(a) and (1)(b) by saying that $\Phi = \Phi(\tau, z)$ is a modular form with respect to the *Jacobi group* $\mathrm{SL}(2,\mathbb{Z}) \ltimes H(\mathbb{Z})$. (Very roughly, Jacobi forms can be thought of as sections of a suitable $\mathbb{Q}$-line bundle over the universal elliptic curve, which doesn't actually exist.) The Jacobi forms of weight k and index m form a vector space $J_{k,m}$ of finite dimension. The standard reference for Jacobi forms is the book by Eichler and Zagier ([EZ]).

As we have said before, Jacobi forms arise naturally as coefficients in the Fourier expansion of modular forms. These coefficients are functions, or more precisely sections of a suitable line bundle, on a boundary component of the Siegel modular threefold. The idea of lifting is to reverse this process. Starting with a Jacobi form one wants to construct a Siegel modular form where this Jacobi form appears as a Fourier coefficient. This idea goes back to Maaß ([Ma2]) and has in recent years been refined in several ways by Gritsenko, Borcherds and others: see e.g.[Gr1], [Gr3], [GrN] and [Borc]. The following lifting result is due to Gritsenko.

Theorem III.1.1 ([Gr1]). *There is a lifting, i.e. an embedding*

$$\mathrm{Lift} : J_{k,t} \longrightarrow M_k(\Gamma_{1,t})$$

of the space of Jacobi forms of weight k and index t into the space of modular forms of weight k with respect to the paramodular group $\Gamma_{1,t}$. The lifting of a Jacobi cusp form is again a cusp form.

Proof. For details see [Gr1, Hauptsatz 2.1] or [Gr2, Theorem 3]. For a Jacobi form $\Phi = \Phi(\tau z)$ with Fourier expansion

$$\Phi(\tau, z) = \sum_{\substack{n,l \in \mathbb{Z} \\ 4nt \geq l^2}} f(n,l) e^{2\pi i (n\tau + lz)}$$

the lift can be written down explicitly as

$$\operatorname{Lift} \Phi(\tau) = \sum_{4tmn \geq l^2} \sum_{a|(n,l,m)} a^{k-1} f\left(\frac{nm}{a^2}, \frac{l}{a}\right) e^{2\pi i (n\tau_1 + l\tau_2 + mt\tau_3)}.$$

Q.E.D.

Since one knows dimension formulae for Jacobi cusp forms one obtains in this way lower bounds for the dimension of the space of modular forms and cusp forms with respect to the paramodular group. Using this together with Freitag's extension theorem it is then easy to obtain the following corollaries.

Corollary III.1.2. *Let $p_g(t)$ be the geometric genus of a smooth projective model of the moduli space $\mathcal{A}_{1,t}$ of $(1,t)$-polarized abelian surfaces. Then*

$$p_g(t) \geq \sum_{j=1}^{t-1} \left(\{2j+2\}_{12} - \left\lfloor \frac{j^2}{12} \right\rfloor \right)$$

where

$$\{m\}_{12} = \begin{cases} \lfloor \frac{m}{12} \rfloor & \text{if } m \not\equiv 2 \mod 12 \\ \lfloor \frac{m}{12} \rfloor - 1 & \text{if } m \equiv 2 \mod 12 \end{cases}$$

and $\lfloor x \rfloor$ denotes the integer part of x.

This corollary also implies that $p_g(t)$ goes to infinity as t goes to infinity.

Corollary III.1.3. *The Kodaira dimension of $\mathcal{A}_{1,t}$ is non-negative if $t \geq 13$ and $t \neq 14, 15, 16, 18, 20, 24, 30,$ or 36. In particular these spaces are not unirational.*

Corollary III.1.4. *The Kodaira dimension of $\mathcal{A}_{1,t}$ is positive if $t \geq 29$ and $t \neq 30, 32, 35, 36, 40, 42, 48,$ or 60.*

On the other hand one knows that $\mathcal{A}_{1,t}$ is rational or unirational for small values of t. We have already mentioned that Igusa proved rationality of $\mathcal{A}_{1,1} = \mathcal{A}_2$ in [I1]. Rationality of $\mathcal{A}_{1,2}$ and $\mathcal{A}_{1,3}$ was proved by Birkenhake and Lange ([BL]). Birkenhake, Lange and van Straten

([BLvS]) also showed that $\mathcal{A}_{1,4}$ is unirational. It is a consequence of the work of Horrocks and Mumford ([HM]) that $\mathcal{A}_{1,5}^{\text{lev}}$ is rational. The variety $\mathcal{A}_{1,7}^{\text{lev}}$ is birational to a Fano variety of type V_{22} ([MS]) and hence also rational. The following result of Gross and Popescu was stated in [GP1] and is proved in the series of papers [GP1]–[GP4].

Theorem III.1.5 ([GP1], [GP2], [GP3] and [GP4]). *$\mathcal{A}_{1,t}^{\text{lev}}$ is rational for $6 \leq t \leq 10$ and $t = 12$ and unirational, but not rational, for $t = 11$. Moreover the variety $\mathcal{A}_{1,t}$ is unirational for $t = 14, 16, 18$ and* 20.

We shall return to some of the projective models of the modular varieties $\mathcal{A}_{1,t}$ in chapter V. Altogether this gives a fairly complete picture as regards the question which of the spaces $\mathcal{A}_{1,t}$ can be rational or unirational. In fact there are only very few open cases.

Problem. Determine whether the spaces $\mathcal{A}_{1,t}$ for $t = 15, 24, 30$, or 36 are unirational.

III.2. General type results for moduli spaces of abelian surfaces

In the case of moduli spaces of abelian surfaces there are a number of concrete bounds which guarantee that the moduli spaces $\mathcal{A}_{1,t}$, resp. $\mathcal{A}_{1,t}^{\text{lev}}$ are of general type. Here we collect the known results and comment on the different approaches which enable one to prove these theorems.

Theorem III.2.1 ([HS1] and [GrH1]). *Let p be a prime number. The moduli spaces $\mathcal{A}_{1,p}^{\text{lev}}$ are of general type if $p \geq 37$.*

Proof. This theorem was first proved in [HS1] for $p \geq 41$ and was improved in [GrH1] to $p = 37$. The two methods of proof differ in one important point. In [HS1] we first estimate how the dimension of the space of cusp forms grows with the weight k and find that

$$\dim S_{3k}\left(\Gamma_{1,p}^{\text{lev}}\right) = \frac{p(p^4-1)}{640}k^3 + O(k^2). \tag{5}$$

These cusp forms give rise to k-fold differential forms on ${}^{\circ}\mathcal{A}_{1,p}^{\text{lev}}$ and we have two types of obstruction to extending them to a smooth projective model of $\mathcal{A}_{1,p}^{\text{lev}}$: one comes from the boundary and the other arises from the elliptic fixed points. To calculate the number of obstructions from the boundary we used the description of the boundary of the Igusa compactification (which is equal to the Voronoi decomposition) given in [HKW2]. We found that the number of obstructions to extending k-fold differentials is bounded by

$$H_B(p,k) = \frac{(p^2-1)}{144}(9p^2+2p+11)k^3 + O(k^2). \tag{6}$$

The singularities of the moduli spaces $\mathcal{A}_{1,p}^{\text{lev}}$ and of the Igusa compactification were computed in [HKW1]. This allowed us to calculate the obstructions arising from the fixed points of the action of the group $\Gamma_{1,p}^{\text{lev}}$. The result is that the number of these obstructions is bounded by

$$H_S(p,k) = \frac{1}{12}(p^2-1)\left(\frac{7}{18}p - 1\right)k^3 + O(k^2). \tag{7}$$

The result then follows from comparing the leading terms of (6) and (7) with that of (5).

The approach in [GrH1] is different. The crucial point is to use Gritsenko's lifting result to produce non-zero cusp forms of weight 2. The first prime where this works is $p = 37$, but it also works for all primes $p > 71$. Let G be a non-trivial modular form of weight 2 with respect to $\Gamma_{1,37}$. Then we can consider the subspace

$$V_k = G^k M_k\left(\Gamma_{1,37}^{\text{lev}}\right) \subset M_{3k}\left(\Gamma_{1,37}^{\text{lev}}\right).$$

The crucial point is that the elements of V_k vanish by construction to order k on the boundary. This ensures that the extension to the boundary imposes no further conditions. The only possible obstructions are those coming from the elliptic fixed points. These obstructions were computed above. A comparison of the leading terms again gives the result. Q.E.D.

The second method described above was also used in the proof of the following two results.

Theorem III.2.2 ([OG] and [GrS]). *The moduli space $\mathcal{A}_{1,p^2}^{\text{lev}}$ is of general type for every prime $p \geq 11$.*

This was proved in [GrS] and improves a result of O'Grady ([OG]) who had shown this for $p \geq 17$. The crucial point in [GrS] is that, because of the square p^2, there is a covering $\mathcal{A}_{1,p^2} \to \mathcal{A}_{1,1}$. The proof in [GrS] then also uses the existence of a weight 2 cusp form with respect to the group Γ_{1,p^2} for $p \geq 11$. The only obstructions which have to be computed explicitly are those coming from the elliptic fixed points. The essential ingredient in O'Grady's proof is the existence of a map from a partial desingularization of a toroidal compactification to the space $\overline{\mathcal{M}}_2$ of semi-stable genus 2 curves.

A further result in this direction is

Theorem III.2.3 ([S2]). *The moduli spaces $\mathcal{A}_{1,p}$ are of general type for all primes $p \geq 173$.*

It is important to remark that in this case there is no natural map from $\mathcal{A}_{1,p}$ to the moduli space $\mathcal{A}_{1,1} = \mathcal{A}_2$ of principally polarized abelian surfaces. A crucial ingredient in the proof of the above theorem is the calculation of the singularities of the spaces $\mathcal{A}_{1,p}$ which was achieved by Brasch ([Br]). Another recent result is

Theorem III.2.4 ([H3]). *The moduli spaces of $(1,d)$-polarized abelian surfaces with a full level-n structure are of general type for all pairs (d,n) with $(d,n) = 1$ and $n \geq 4$.*

A general result due to L. Borisov is

Theorem III.2.5 ([Bori]). *There are only finitely many subgroups H of $\mathrm{Sp}(4,\mathbb{Z})$ such that $\mathcal{A}(H)$ is not of general type.*

Note that this result applies to the groups $\Gamma_{1,p}^{\mathrm{lev}}$ and Γ_{1,p^2} which are both conjugate to subgroups of $\mathrm{Sp}(4,\mathbb{Z})$, but does not apply to the groups $\Gamma_{1,p}$, which are not. (At least for $p \geq 7$: the subgroup of $\mathbb{C}^*$ generated by the eigenvalues of non-torsion elements of $\Gamma_{1,p}$ contains pth roots of unity, as was shown by Brasch in [Br], but the corresponding group for $\mathrm{Sp}(4,\mathbb{Z})$ has only 2- and 3-torsion.)

We shall give a rough outline of the proof of this result. For details the reader is referred to [Bori]. We shall mostly comment on the geometric aspects of the proof. Every subgroup H in $\mathrm{Sp}(4,\mathbb{Z})$ contains a principal congruence subgroup $\Gamma(n)$. The first reduction is the observation that it is sufficient to consider only subgroups H which contain a principal congruence subgroup $\Gamma(p^t)$ for some prime p. This is essentially a group theoretic argument using the fact that the finite group $\mathrm{Sp}(4,\mathbb{Z}_p)$ is simple for all primes $p \geq 3$. Let us now assume that H contains $\Gamma(n)$ (we assume $n \geq 5$). This implies that there is a finite morphism $\mathcal{A}_2(n) \to \mathcal{A}(H)$. The idea is to show that for almost all groups H there are sufficiently many pluricanonical forms on the Igusa (Voronoi) compactification $X = \mathcal{A}_2^*(n)$ which descend to a smooth projective model of $\mathcal{A}(H)$. For this it is crucial to get a hold on the possible singularities of the quotient Y. We have already observed in Corollary II.2.6 that the canonical divisor on X is ample for $n \geq 5$. The finite group $\bar{H} = \Gamma_2(n)/H$ acts on X and the quotient $Y = \bar{H}\backslash X$ is a (in general singular) projective model of $\mathcal{A}(H)$. Since X is smooth and $\bar{H}$ is finite, the variety Y is normal and has log-terminal singularities, i.e. if $\pi : Z \to Y$ is a desingularization whose exceptional divisor $E = \sum_i E_i$

has simple normal crossings, then

$$K_Z = \pi^* K_Y + \sum_i (-1+\delta_i)E_i \quad \text{with } \delta_i > 0.$$

Choose $\delta > 0$ such that $-1+\delta$ is the minimal discrepancy. By L_X, resp. L_Y we denote the $\mathbb{Q}$-line bundle whose sections are modular forms of weight 1. Then $L_X = \mu^* L_Y$ where $\mu : X \to Y$ is the quotient map.

The next reduction is that it suffices to construct a non-trivial section $s \in H^0(m(K_Y - L_Y))$ such that $s_y \in \mathcal{O}_Y\left(m(K_Y - L_Y)\mathfrak{m}_y^{m(1-\delta)}\right)$ for all $y \in Y$ where Y has a non-canonical singularity. This is enough because $\pi^*(sH^0(mL_Y)) \subset H^0(mK_Z)$ and the dimension of the space $H^0(mL_Y)$ grows as m^3.

The idea is to construct s as a suitable $\bar{H}$-invariant section

$$s \in H^0\big(\mu^*(m(K_Y - L_Y))\big)^{\bar{H}}$$

satisfying vanishing conditions at the branch locus of the finite map $\mu : X \to Y$. For this one has to understand the geometry of the quotient map μ. First of all one has branching along the boundary $D = \sum D_i$ of X. We also have to look at the Humbert surfaces

$$\mathcal{H}_1 = \left\{\tau = \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_3 \end{pmatrix}; \tau_1, \tau_3 \in \mathbb{H}_1\right\} = \text{Fix}\begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}$$

and

$$\mathcal{H}_4 = \left\{\tau = \begin{pmatrix} \tau_1 & \tau_2 \\ \tau_2 & \tau_3 \end{pmatrix}; \tau_1 = \tau_3\right\} = \text{Fix}\begin{pmatrix} 0 & 1 & & \\ 1 & 0 & & \\ & & 0 & 1 \\ & & 1 & 0 \end{pmatrix}.$$

Let

$$\mathcal{F} = \bigcup_{g \in \mathrm{Sp}(4,\mathbb{Z})} g(\mathcal{H}_1) \quad , \quad \mathcal{G} = \bigcup_{g \in \mathrm{Sp}(4,\mathbb{Z})} g(\mathcal{H}_2)$$

and let

$$F = \overline{\pi(\mathcal{F})}, \quad G = \overline{\pi(\mathcal{G})}$$

where $\pi : \mathbb{H}_2 \to \Gamma(n)\backslash\mathbb{H}_2 \subset X$ is the quotient map. One can then show that the branching divisor of the map $\mathcal{A}(\Gamma_2(n)) \to \mathcal{A}(H)$ is contained in $F \cup G$ and that all singularities in $\mathcal{A}(H)$ which lie outside $\mu(F \cup G)$ are canonical. Moreover the stabilizer subgroups in $\mathrm{Sp}(4,\mathbb{Z})$ of points in $\mathcal{F} \cup \mathcal{G}$ are solvable groups of bounded order. Let $F = \sum F_i$ and

$G = \sum G_i$ be the decomposition of the surfaces F and G into irreducible components. We denote by d_i, f_i and g_i the ramification order of the quotient map $\mu : X \to Y$ along D_i, F_i and G_i. The numbers f_i and g_i are equal to 1 or 2. One has

$$\mu^*(m(K_Y - L_Y)) = m(K_X - L_X) - \sum_i m(d_i - 1)D_i - \sum_i m(f_i - 1)F_i - \sum_i m(g_i - 1)G_i.$$

Recall that the finite group $\bar{H}$ is a subgroup of the group $\bar{G} = \Gamma/\Gamma(n) = \mathrm{Sp}(4, \mathbb{Z}_n)$. The crucial point in Borisov's argument is to show, roughly speaking, that the index $[\bar{G} : \bar{H}]$ can be bounded from above in terms of the singularities of Y. There are several such types of bounds depending on whether one considers points on the branch locus or on one or more boundary components. We first use this bound for the points on X which lie on 3 boundary divisors. Using this and the fact that Y has only finite quotient singularities one obtains the following further reduction: if R is the ramification divisor of the map $\mu : X \to Y$, then it is enough to construct a non-zero section in $H^0(m(K_X - L_X - R))$ for some $m > 0$ which lies in $\mathfrak{m}_x^{mk(\mathrm{Stab}^H x)}$ for all points x in X which lie over non-canonical points of Y and which are not on the intersection of 3 boundary divisors. Here $k(\mathrm{Stab}^H x)$ is defined as follows. First note that $\mathrm{Stab}^H x$ is solvable and consider a series

$$\{0\} = H_0 \lhd H_1 \lhd \ldots \lhd H_t = \mathrm{Stab}^H x$$

with H_i/H_{i-1} abelian of exponent k_i. Take $k' = k_1 \cdot \ldots \cdot k_t$. Then $k(\mathrm{Stab}^H x)$ is the minimum over all k' which are obtained in this way. To obtain an invariant section one can then take the product with respect to the action of the finite group $\bar{H}$. Now recall that all non-canonical points on $\mathcal{A}(H)$ lie in $\mu(F \cup G)$. The subgroup $Z\,\mathrm{Stab}^H D_i$ of $\mathrm{Stab}^H D_i$ which acts trivially on D_i is cyclic of order d_i. Moreover if x lies on exactly one boundary divisor of X then the order of the group $\mathrm{Stab}^H x / Z\,\mathrm{Stab}^H D_i$ is bounded by 6 and if x lies on exactly 2 boundary divisors, then the order of this group is bounded by 4. Using this one can show that there is a constant c (independent of H) such that it is sufficient to construct a non-zero section in $m(K_X - L_X - cR)$ for some positive m. By results of Yamazaki [Ya] the divisor $mK_X - 2mL_X$ is effective. It is, therefore, sufficient to prove the existence of a non-zero section in $m(K_X - 2cR)$. The latter equals

$$mK_X - 2c\sum_i m(d_i - 1)D_i - 2c\sum_i m(f_i - 1)F_i - 2c\sum_i m(g_i - 1)G_i.$$

We shall now restrict ourselves to obstructions coming from components F_i; the obstructions coming from G_i, D_i can be treated similarly. Since $h^0(mK_X) > c_1 n^{10} m^3$ for some $c_1 > 0$, $m \gg 0$ one has to prove the following result: let $\varepsilon > 0$; then for all but finitely many subgroups H one has

$$\sum_{f_i=2} (h^0(mK_X) - h^0(mK_X - 2cmf_iF_i)) \leq \varepsilon n^{10} m^3 \quad \text{for} \quad m \gg 0$$

and all n. This can finally be derived from the following boundedness result. Let $\varepsilon > 0$ and assume that

$$\frac{\#\{F_i\ ;\ f_i = 2\}}{\#\{F_i\}} \geq \varepsilon,$$

then the index $[\bar{G} : \bar{H}]$ is bounded by an (explicitly known) constant depending only on ε. The proof of this statement is group theoretic and the idea is as follows. Assume the above inequality holds: then H contains many involutions and these generate a subgroup of $\mathrm{Sp}(4, \mathbb{Z})$ whose index is bounded in terms of ε.

III.3. Left and right neighbours

The paramodular group $\Gamma_{1,t} \subset \mathrm{Sp}(4, \mathbb{Q})$ is (for $t > 1$) not a maximal discrete subgroup of the group of analytic automorphisms of $\mathbb{H}_2$. For every divisor $d \| t$ (i.e. $d | t$ and $(d, t/d) = 1$) one can choose integers x and y such that

$$xd - yt_d = 1, \quad \text{where } t_d = t/d.$$

The matrix

$$V_d = \frac{1}{\sqrt{d}} \begin{pmatrix} dx & -1 & 0 & 0 \\ -yt & d & 0 & 0 \\ 0 & 0 & d & yt \\ 0 & 0 & 1 & dx \end{pmatrix}$$

is an element of $\mathrm{Sp}(4, \mathbb{R})$ and one easily checks that

$$V_d^2 \in \Gamma_{1,t}, \qquad V_d \Gamma_{1,t} V_d^{-1} = \Gamma_{1,t}.$$

The group generated by $\Gamma_{1,t}$ and the elements V_d, i.e.

$$\Gamma_{1,t}^{\ddagger} = \langle \Gamma_{1,t},\ V_d;\ d \| t \rangle$$

does not depend on the choice of the integers x, y. It is a normal extension of $\Gamma_{1,t}$ with

$$\Gamma_{1,t}^{\ddagger} / \Gamma_{1,t} \cong (\mathbb{Z}_2)^{\mu(t)}$$

where $\mu(t)$ is the number of prime divisors of t. If t is square-free, it is known that $\Gamma_{1,t}^{\ddagger}$ is a maximal discrete subgroup of $\mathrm{Sp}(4,\mathbb{R})$ (see [Al],[Gu]). The coset $\Gamma_{1,t}V_t$ equals $\Gamma_{1,t}V_t'$ where

$$V_d = \begin{pmatrix} 0 & \sqrt{t}^{-1} & 0 & 0 \\ \sqrt{t} & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{t} \\ 0 & 0 & \sqrt{t}^{-1} & 0 \end{pmatrix}.$$

This generalizes the Fricke involution known from the theory of elliptic curves. The geometric meaning of the involution $\bar{V}_t : \mathcal{A}_{1,t} \to \mathcal{A}_{1,t}$ induced by V_t is that it maps a polarized abelian surface (A, H) to its dual. A similar geometric interpretation can also be given for the involutions V_d (see [GrH2, Proposition 1.6] and also [Br, Satz (1.11)] for the case $d = t$). We also consider the degree 2 extension

$$\Gamma_{1,t}^{+} = \langle \Gamma_{1,t}, V_t \rangle$$

of $\Gamma_{1,t}$. If $t = p^n$ for a prime number p, then $\Gamma_{1,t}^{+} = \Gamma_{1,t}^{\ddagger}$. The groups $\Gamma_{1,t}^{\ddagger}$ and $\Gamma_{1,t}^{+}$ define Siegel modular threefolds

$$\mathcal{A}_{1,t}^{\ddagger} = \Gamma_{1,t}^{\ddagger} \setminus \mathbb{H}_2, \qquad \mathcal{A}_{1,t}^{+} = \Gamma_{1,t}^{+} \setminus \mathbb{H}_2.$$

Since $\Gamma_{1,t}^{\ddagger}$ is a maximal discrete subgroup for t square free the space $\mathcal{A}_{1,t}^{\ddagger}$ was called a *minimal Siegel modular threefold.* This should not be confused with minimal models in the sense of Mori theory.

The paper [GrH2] contains an interpretation of the varieties $\mathcal{A}_{1,t}^{\ddagger}$ and $\mathcal{A}_{1,t}^{+}$ as moduli spaces. We start with the spaces $\mathcal{A}_{1,t}^{\ddagger}$.

Theorem III.3.1 ([GrH2]).

(i) *Let A, A' be two $(1,t)$-polarized abelian surfaces which define the same point in $\mathcal{A}_{1,t}^{\ddagger}$. Then their (smooth) Kummer surfaces X, X' are isomorphic.*

(ii) *Assume that the Néron-Severi group of A and A' is generated by the polarization. Then the converse is also true: if A and A' have isomorphic Kummer surfaces, then A and A' define the same point in $\mathcal{A}_{1,t}^{\ddagger}$.*

The proof of this theorem is given in [GrH2, Theorem 1.5]. The crucial ingredient is the Torelli theorem for K3 surfaces. The above theorem says in particular that an abelian surface and its dual have isomorphic Kummer surfaces. This implies a negative answer to a problem posed by

Shioda, who asked whether it was true that two abelian surfaces whose Kummer surfaces are isomorphic are necessarily isomorphic themselves. In view of the above result, a general $(1,t)$-polarized surface with $t>1$ gives a counterexample: the surface A and its dual $\hat{A}$ have isomorphic Kummer surfaces, but A and $\hat{A}$ are not isomorphic as polarized abelian surfaces. If the polarization generates the Néron-Severi group this implies that A and $\hat{A}$ are not isomorphic as algebraic surfaces. In view of the above theorem one can interpret $\mathcal{A}^{\ddagger}_{1,t}$ as the space of Kummer surfaces associated to $(1,t)$-polarized abelian surfaces.

The space $\mathcal{A}^{+}_{1,t}$ can be interpreted as a space of lattice-polarized $K3$-surfaces in the sense of [N3] and [Dol]. As usual let E_8 be the even, unimodular, positive definite lattice of rank 8. By $E_8(-1)$ we denote the lattice which arises from E_8 by multiplying the form with -1. Let $\langle n\rangle$ be the rank 1 lattice $\mathbb{Z}l$ with the form given by $l^2=n$.

Theorem III.3.2 ([GrH2]). *The moduli space $\mathcal{A}^{+}_{1,t}$ is isomorphic to the moduli space of lattice polarized $K3$-surfaces with a polarization of type $\langle 2t\rangle\oplus 2E_8(-1)$.*

For a proof see [GrH2, Proposition 1.4]. If

$$L=\mathbb{Z}e_1\oplus\mathbb{Z}e_2\oplus\mathbb{Z}e_3\oplus\mathbb{Z}e_4,$$

then $\bigwedge^2 L$ carries a symmetric bilinear form $(\ ,\ )$ given by

$$x\wedge y=(x,y)e_1\wedge e_2\wedge e_3\wedge e_4\in\bigwedge^4 L.$$

If $w_t=e_1\wedge e_3+te_2\wedge e_4$, then the group

$$\tilde{\Gamma}_{1,t}=\{g: L\to L;\ \bigwedge^2 g(w_t)=w_t\}$$

is isomorphic to the paramodular group $\Gamma_{1,t}$. The lattice $L_t=w_t^{\perp}$ has rank 5 and the form $(\ ,\ )$ induces a quadratic form of signature (3, 2) on L_t. If $\mathrm{O}(L_t)$ is the orthogonal group of isometries of the lattice L_t, then there is a natural homomorphism

$$\bigwedge^2:\quad \Gamma_{1,t}\cong\tilde{\Gamma}_{1,t}\longrightarrow \mathrm{O}(L_t).$$

This homomorphism can be extended to $\Gamma^{\ddagger}_{1,t}$ and

$$\Gamma^{\ddagger}_{1,t}/\Gamma_{1,t}\cong \mathrm{O}(L_t^{\vee}/L_t)\cong(\mathbb{Z}_2)^{\mu(t)}$$

where $L_t^{\vee}$ is the dual lattice of L_t. This, together with Nikulin's theory ([N2], [N3]) is the crucial ingredient in the proof of the above theorems.

The varieties $\mathcal{A}^{+}_{1,t}$ and $\mathcal{A}^{\ddagger}_{1,t}$ are quotients of the moduli space $\mathcal{A}_{1,t}$ of $(1,t)$-polarized abelian surfaces. In [GrH3] there is an investigation into an interesting class of Galois coverings of the spaces $\mathcal{A}_{1,t}$. These coverings are called *left neighbours*, and the quotients are called *right neighbours*. To explain the coverings of $\mathcal{A}_{1,t}$ which were considered in [GrH3], we have to recall a well known result about the commutator subgroup $\mathrm{Sp}(2g,\mathbb{Z})'$ of the symplectic group $\mathrm{Sp}(2g,\mathbb{Z})$. Reiner [Re] and Maaß [Ma1] proved that

$$\mathrm{Sp}(2g,\mathbb{Z})/\mathrm{Sp}(2g,\mathbb{Z})' = \begin{cases} \mathbb{Z}_{12} & \text{for } g=1 \\ \mathbb{Z}_2 & \text{for } g=2 \\ 1 & \text{for } g \geq 3 \end{cases}.$$

The existence of a character of order 12 of $\mathrm{Sp}(2,\mathbb{Z}) = \mathrm{SL}(2,\mathbb{Z})$ follows from the Dedekind η-function

$$\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n), \qquad q = e^{2\pi i\tau}.$$

This function is a modular form of weight 1/2 with a multiplier system of order 24. Its square η^2 has weight 1 and is a modular form with respect to a character v_η of order 12. For $g=2$ the product

$$\Delta_5(\tau) = \prod_{(m,m')\ \text{even}} \Theta_{mm'}(\tau,0)$$

of the 10 even theta characteristics is a modular form for $\mathrm{Sp}(4,\mathbb{Z})$ of weight 5 with respect to a character of order 2.

In [GrH3] the commutator subgroups of the groups $\Gamma_{1,t}$ and $\Gamma^{+}_{1,t}$ were computed. For $t \geq 1$ we put

$$t_1 = (t,12), \qquad t_2 = (2t,12).$$

Theorem III.3.3 ([GrH3]). *For the commutator subgroups* $\Gamma'_{1,t}$ *of* $\Gamma_{1,t}$ *and* $(\Gamma^{+}_{1,t})'$ *of* $\Gamma^{+}_{1,t}$ *one obtains*

(i) $\Gamma_{1,t}/\Gamma'_{1,t} \cong \mathbb{Z}_{t_1} \times \mathbb{Z}_{t_2}$

(ii) $\Gamma^{+}_{1,t}/(\Gamma^{+}_{1,t})' \cong \mathbb{Z}_2 \times \mathbb{Z}_{t_2}$.

This was shown in [GrH3, Theorem 2.1].

In [Mu1] Mumford pointed out an interesting application of the computation of $\mathrm{Sp}(2,\mathbb{Z})'$ to the Picard group of the moduli stack $\underline{\mathcal{A}}_1$. He showed that

$$\mathrm{Pic}(\underline{\mathcal{A}}_1) \cong \mathbb{Z}_{12}.$$

In the same way the above theorem implies that

$$\operatorname{Pic}(\underline{\mathcal{A}}_2) = \operatorname{Pic}(\underline{\mathcal{A}}_{1,1}) \cong \mathbb{Z} \times \mathbb{Z}_2$$

and

$$\operatorname{Tors}\operatorname{Pic}(\underline{\mathcal{A}}_{1,t}) = \mathbb{Z}_{t_1} \times \mathbb{Z}_{t_2}.$$

The difference between the cases $\underline{\mathcal{A}}_{1,1}$ and $\underline{\mathcal{A}}_{1,t}$, $t > 1$ is that one knows that the rank of the Picard group of $\underline{\mathcal{A}}_2 = \underline{\mathcal{A}}_{1,1}$ is 1, whereas the rank of the Picard group of $\underline{\mathcal{A}}_{1,t}$, $t > 1$ is unknown. One only knows that it is positive. This is true for all moduli stacks of abelian varieties of dimension $g \geq 2$, since the bundle L of modular forms of weight 1 is non-trivial. The difference from the genus 1 case lies in the fact that there the boundary of the Satake compactification is a divisor.

Problem. Determine the rank of the Picard group $\operatorname{Pic}(\underline{\mathcal{A}}_{1,t})$.

We have already discussed Gritsenko's result which gives the existence of weight 3 cusp forms for $\Gamma_{1,t}$ for all but finitely many values of t. We call these values

$$t = 1, 2, \ldots, 12, 14, 15, 16, 18, 20, 24, 30, 36$$

the *exceptional* polarizations. In many cases the results of Gross and Popescu show that weight 3 cusp forms indeed cannot exist. The best possible one can hope for is the existence of weight 3 cusp forms with a character of a small order. The following result is such an existence theorem.

Theorem III.3.4 ([GrH3]). *Let t be exceptional.*

(i) *If $t \neq 1, 2, 4, 5, 8, 16$ then there exists a weight 3 cusp form with respect to $\Gamma_{1,t}$ with a character of order 2.*
(ii) *For $t = 8, 16$ there exists a weight 3 cusp form with a character of order 4.*
(iii) *For $t \equiv 0 \mod 3, t \neq 3, 9$ there exists a weight 3 cusp form with a character of order 3.*

To every character $\chi : \Gamma_{1,t} \to \mathbb{C}^*$ one can associate a Siegel modular variety

$$\mathcal{A}(\chi) = \operatorname{Ker}\chi \setminus \mathbb{H}_2.$$

The existence of a non-trivial cusp form of weight 3 with a character χ then implies by Freitag's theorem the existence of a differential form on a smooth projective model $\tilde{\mathcal{A}}(\chi)$ of $\mathcal{A}(\chi)$. In particular the above result proves the existence of abelian covers $\mathcal{A}(\chi) \to \mathcal{A}_{1,t}$ of small degree with $p_g(\tilde{\mathcal{A}}(\chi)) > 0$.

The proof is again an application of Gritsenko's lifting techniques. To give the reader an idea we shall discuss the case $t = 11$ which is particularly interesting since by the result of Gross and Popescu $\mathcal{A}_{1,11}$ is unirational, but not rational. In this case $\Gamma_{1,11}$ has exactly one character χ_2. This character has order 2. By the above theorem there is a degree 2 cover $\mathcal{A}(\chi_2) \to \mathcal{A}_{1,11}$ with positive geometric genus. In this case the lifting procedure gives us a map

$$\text{Lift}\colon J^{\text{cusp}}_{3,\frac{11}{2}}(v_\eta^{12} \times v_H) \to S_3(\Gamma_{1,11}, \chi_2).$$

Here v_η is the multiplier system of the Dedekind η-function and v_η^{12} is a character of order 2. The character v_H is a character of order 2 of the integer Heisenberg group $H = H(\mathbb{Z})$. By $J^{\text{cusp}}_{3,\frac{11}{2}}(v_\eta^2 \times v_H)$ we denote the Jacobi cusp forms of weight 3 and index 11/2 with a character $v_\eta^{12} \times v_H$. Similarly $S_3(\Gamma_{1,11}, \chi_2)$ is the space of weight 3 cusp form with respect to the group $\Gamma_{1,11}$ and the character χ_2. Recall the Jacobi theta series

$$\vartheta(\tau, z) = \sum_{m\in\mathbb{Z}} \left(-\frac{4}{m}\right) q^{m^2/8} r^{m/2} \qquad (q = \mathrm{e}^{2\pi i \tau},\ r = \mathrm{e}^{2\pi i z})$$

where

$$\left(-\frac{4}{m}\right) = \begin{cases} \pm 1 & \text{if } m \equiv \pm 1 \mod 4 \\ 0 & \text{if } m \equiv 0 \mod 2. \end{cases}$$

This is a Jacobi form of weight 1/2, index 3/2 and multiplier system $v_\eta^3 \times v_H$. For an integer a we can consider the Jacobi form

$$\vartheta_a(\tau, z) = \vartheta(\tau, az) \in J_{\frac{1}{2},\frac{1}{2}a^2}(v_\eta^3 \times v_H^a).$$

One then obtains the desired Siegel cusp form by taking

$$F = \text{Lift}(\eta^3 \vartheta^2 \vartheta_3) \in S_3(\Gamma_{1,11}, \chi_2).$$

Finally we want to consider the maximal abelian covering of $\mathcal{A}_{1,t}$, namely the Siegel modular threefold

$$\mathcal{A}^{\text{com}}_{1,t} = \Gamma'_{1,t} \setminus \mathbb{H}_2.$$

By $\tilde{\mathcal{A}}^{\text{com}}_{1,t}$ we denote a smooth projective model of $\mathcal{A}^{\text{com}}_{1,t}$.

Theorem III.3.5 ([GrH3]).

(i) *The geometric genus of* $\tilde{\mathcal{A}}^{\text{com}}_{1,t}$ *is* 0 *if and only if* $t = 1, 2, 4, 5$.
(ii) *The geometric genus of* $\mathcal{A}^{\text{com}}_{1,3}$ *and* $\mathcal{A}^{\text{com}}_{1,7}$ *is* 1.

The proof can be found as part of the proof of [GrH3, Theorem 3.1].

At this point we should like to remark that all known construction methods fail when one wants to construct modular forms of small weight with respect to the groups $\Gamma^+_{1,t}$ or $\Gamma^\ddagger_{1,t}$. We therefore pose the

Problem. Construct modular forms of small weight with respect to the groups $\Gamma^+_{1,t}$ and $\Gamma^\ddagger_{1,t}$.

IV. Projective models

In this section we describe some cases in which a Siegel modular variety is or is closely related to an interesting projective variety. Many of the results are very old.

IV.1. The Segre cubic

Segre's cubic primal, or the *Segre cubic*, is the subvariety $\mathcal{S}_3$ of $\mathbb{P}^5$ given by the equations

$$\sum_{i=0}^{5} x_i = \sum_{i=0}^{5} x_i^3 = 0$$

in homogeneous coordinates $(x_0 : \ldots : x_5)$ on $\mathbb{P}^5$. Since it lies in the hyperplane $\left(\sum x_i = 0\right) \subset \mathbb{P}^5$ it may be thought of as a cubic hypersurface in $\mathbb{P}^4$, but the equations as given here have the advantage of showing that there is an action of the symmetric group $\mathrm{Sym}(6)$ on $\mathcal{S}_3$.

These are the equations of $\mathcal{S}_3$ as they are most often given in the literature but there is another equally elegant formulation: $\mathcal{S}_3$ is given by the equations

$$\sigma_1(x_i) = \sigma_3(x_i) = 0$$

where $\sigma_k(x_i)$ is the kth elementary symmetric polynomial in the x_i,

$$\sigma_k(x_i) = \sum_{\#I=k} \prod_{i\in I} x_i.$$

To check that these equations do indeed define $\mathcal{S}_3$ it is enough to notice that

$$3\sigma_3(x_i) = \left(\sum x_i\right)^3 - 3\left(\sum x_i\right)\left(\sum x_i^2\right) - \sum x_i^3.$$

Lemma IV.1.1. *$\mathcal{S}_3$ is invariant under the action of* $\mathrm{Sym}(6)$ *and has ten nodes, at the points equivalent to* $(1:1:1:-1:-1:-1)$ *under the* $\mathrm{Sym}(6)$*-action. This is the maximum possible for a cubic hypersurface in* $\mathbb{P}^4$*, and any cubic hypersurface with ten nodes is projectively equivalent to* $\mathcal{S}_3$*.*

Many other beautiful properties of the Segre cubic and related varieties were discovered in the nineteenth century.

The dual variety of the Segre cubic is a quartic hypersurface $\mathcal{I}_4 \subset \mathbb{P}^4$, the *Igusa quartic.* If we take homogeneous coordinates $(y_0 : \ldots : y_5)$ on $\mathbb{P}^5$ then it was shown by Baker ([Ba1]) that $\mathcal{I}_4$ is given by

$$\sum_{i=0}^{5} y_i = a^2 + b^2 + c^2 - 2(ab + bc + ca) = 0$$

where

$$a = (y_1 - y_5)(y_4 - y_2), \quad b = (y_2 - y_3)(y_5 - y_0) \text{ and } c = (y_0 - y_4)(y_3 - y_1).$$

This can also be written in terms of symmetric functions in suitable variables as

$$\sigma_1(x_i) = 4\sigma_4(x_i) - \sigma_2(x_i)^2 = 0.$$

This quartic is singular along $\binom{6}{2} = 15$ lines ℓ_{ij}, $0 \le i < j \le 5$, and $\ell_{ij} \cap \ell_{mn} = \emptyset$ if and only if $\{i,j\} \cap \{m,n\} \neq \emptyset$. There are $\frac{1}{2}\binom{6}{3} = 10$ smooth quadric surfaces Q_{ijk} in $\mathcal{I}_4$, such that, for instance, ℓ_{01}, ℓ_{12} and ℓ_{20} lie in one ruling of $Q_{012} = Q_{345}$ and ℓ_{34}, ℓ_{45} and ℓ_{53} lie in the other ruling. The birational map $\mathcal{I}_4 \dashrightarrow \mathcal{S}_3$ given by the duality blows up the 15 lines ℓ_{ij}, which resolves the singularities of $\mathcal{I}_4$, and blows down the proper transform of each Q_{ijk} (still a smooth quadric) to give the ten nodes of $\mathcal{S}_3$.

It has long been known that if $H \subset \mathbb{P}^4 = \left(\sum_{i=0}^{5} y_i\right)$ is a hyperplane which is tangent to $\mathcal{I}_4$ then $H \cap \mathcal{I}_4$ is a Kummer quartic surface. This fact provides a connection with abelian surfaces and their moduli. The Igusa quartic can be seen as a moduli space of Kummer surfaces. In this case, because the polarization is principal, two abelian surfaces giving the same Kummer surface are isomorphic and the (coarse) moduli space of abelian surfaces is the same as the moduli space of Kummer surfaces. This will fail in the non-principally polarized case, in IV.3, below.

Theorem IV.1.2. *$\mathcal{S}_3$ is birationally equivalent to a compactification of the moduli space $\mathcal{A}_2(2)$ of principally polarized abelian surfaces with a level-2 structure.*

The Segre cubic is rational. An explicit birational map $\mathbb{P}^3 \dashrightarrow \mathcal{S}_3$ was given by Baker ([Ba1]) and is presented in more modern language in [Hun].

Corollary IV.1.3. *$\mathcal{A}_2^*(2)$ is rational.*

A much more precise description of the relation between $\mathcal{S}_3$ and $\mathcal{A}_2(2)$ is given by this theorem of Igusa.

Theorem IV.1.4 ([I2]). *The Igusa compactification $\mathcal{A}_2^*(2)$ of the moduli space of principally polarized abelian surfaces with a level-2 structure is isomorphic to the the blow-up $\tilde{\mathcal{S}}_3$ of $\mathcal{S}_3$ in the ten nodes. The Satake compactification $\bar{\mathcal{A}}_2(2)$ is isomorphic to $\mathcal{I}_4$, which is obtained from $\tilde{\mathcal{S}}_3$ by contracting* 15 *rational surfaces to lines.*

Proof. The Satake compactification is $\operatorname{Proj} \mathcal{M}\big(\Gamma_2(2)\big)$, where $\mathcal{M}(\Gamma)$ is the ring of modular forms for the group Γ. The ten even theta characteristics determine ten theta constants $\theta_{m_0}(\tau), \ldots, \theta_{m_9}(\tau)$ of weight $\frac{1}{2}$ for $\Gamma_2(2)$, and $\theta^4_{m_i}(\tau)$ is a modular form of weight 2 for $\Gamma_2(2)$. These modular forms determine a map $f : \mathcal{A}_2(2) \to \mathbb{P}^9$ whose image actually lies in a certain $\mathbb{P}^4 \subset \mathbb{P}^9$. The integral closure of the subring of $\mathcal{M}\big(\Gamma_2(2)\big)$ generated by the $\theta^4_{m_i}$ is the whole of $\mathcal{M}\big(\Gamma_2(2)\big)$ and there is a quartic relation among the $\theta^4_{m_i}$ (as well as five linear relations defining $\mathbb{P}^4 \subset \mathbb{P}^9$) which, with a suitable choice of basis, is the quartic $a^2 + b^2 + c^2 - 2(ab + bc + ca) = 0$. Furthermore, f is an embedding and the closure of its image is normal, so it is the Satake compactification. Q.E.D.

The Igusa compactification is, in this context, the blow-up of the Satake compactification along the boundary, which here consists of the fifteen lines ℓ_{ij}. The birational map $\mathcal{I}_4 \dashrightarrow \mathcal{S}_3$ does this blow-up and also blows down the ten quadrics Q_{ijk} to the ten nodes of $\mathcal{S}_3$.

For full details of the proof see [I2]; for a more extended sketch than we have given here and some further facts, see [Hun]. We mention that the surfaces Q_{ijk}, considered as surfaces in $\mathcal{A}_2(2)$, correspond to principally polarized abelian surfaces which are products of two elliptic curves.

Without going into details, we mention also that $\mathcal{I}_4$ may be thought of as the natural compactification of the moduli of ordered 6-tuples of distinct points on a conic in $\mathbb{P}^2$. Such a 6-tuple determines 6 lines in $\check{\mathbb{P}}^2$ which are all tangent to some conic, and the Kummer surface is the double cover of $\check{\mathbb{P}}^2$ branched along the six lines. The order gives the level-2 structure (note that $\Gamma_2/\Gamma_2(2) \cong \mathrm{Sp}(4, \mathbb{Z}_2) \cong \mathrm{Sym}(6)$.) The abelian surface is the Jacobian of the double cover of the conic branched at the six points. On the other hand, $\mathcal{S}_3$ may be thought of as the natural compactification of the moduli of ordered 6-tuples of points on a line: for this, see [DO].

The topology of the Segre cubic and related spaces has been studied by van der Geer ([vdG1]) and by Lee and Weintraub ([LW1], [LW2]).

The method in [LW1] is to show that the isomorphism between the open parts of $\mathcal{S}_3$ and $\mathcal{A}_2(2)$ is defined over a suitable number field and use the Weil conjectures.

Theorem IV.1.5 ([LW1] and [vdG1]). *The homology of the Igusa compactification of $\mathcal{A}_2(2)$ is torsion-free. The Hodge numbers are $h^{0,0} = h^{3,3} = 1$, $h^{1,1} = h^{2,2} = 16$ and $h^{p,q} = 0$ otherwise.*

By using the covering $\mathcal{A}_2(4) \to \mathcal{A}_2(2)$, Lee and Weintraub [LW3] also prove a similar result for $\mathcal{A}_2(4)$.

IV.2. The Burkhardt quartic

The *Burkhardt quartic* is the subvariety $\mathcal{B}_4$ of $\mathbb{P}^4$ given by the equation

$$y_0^4 - y_0(y_1^3 + y_2^3 + y_3^3 + y_4^3) + 3y_1y_2y_3y_4 = 0.$$

This form of degree 4 was found by Burkhardt ([Bu]) in 1888. It is the invariant of smallest degree of a certain action of the finite simple group $\mathrm{PSp}(4, \mathbb{Z}_3)$ of order 25920 on $\mathbb{P}^4$, which arises in the study of the 27 lines on a cubic surface. In fact this group is a subgroup of index 2 in the Weyl group $W(E_6)$ of E_6, which is the automorphism group of the configuration of the 27 lines. The 27 lines themselves can be recovered by solving an equation whose Galois group is $W(E_6)$ or, after adjoining a square root of the discriminant, $\mathrm{PSp}(4, \mathbb{Z}_3)$.

Lemma IV.2.1. *$\mathcal{B}_4$ has forty-five nodes. Fifteen of them are equivalent to $(1 : -1 : 0 : 0 : 0 : 0)$ under the action of $\mathrm{Sym}(6)$ and the other thirty are equivalent to $(1 : 1 : \xi_3 : \xi_3 : \xi_3^2 : \xi_3^2)$, where $\xi_3 = e^{2\pi i/3}$. This is the greatest number of nodes that a quartic hypersurface in $\mathbb{P}^4$ can have and any quartic hypersurface in $\mathbb{P}^4$ with 45 nodes is projectively equivalent to $\mathcal{B}_4$.*

This lemma is an assemblage of results of Baker ([Ba2]) and de Jong, Shepherd-Barron and Van de Ven ([JSV]): the bound on the number of double points is the Varchenko (or spectral) bound [Va], which in this case is sharp.

We denote by $\theta_{\alpha\beta}(\tau)$, $\alpha, \beta \in \mathbb{Z}_3$, the theta constants

$$\theta_{\alpha\beta}(\tau) = \theta \begin{bmatrix} 0 & 0 \\ \alpha & \beta \end{bmatrix} (\tau, 0) = \sum_{n \in \mathbb{Z}^2} \exp\{\pi i\, {}^t n \tau n + 2\pi i(\alpha n_1 + \beta n_2)\}$$

where $\tau \in \mathbb{H}_2$. Here we identify $\alpha \in \mathbb{Z}_3$ with $\alpha/3 \in \mathbb{Q}$. The action of $\Gamma_2(1) = \mathrm{Sp}(4, \mathbb{Z})$ on $\mathbb{H}_2$ induces a linear action on the space spanned by these $\theta_{\alpha\beta}$, and $\Gamma_2(3)$ acts trivially on the corresponding projective space. Since $-1 \in \Gamma_2(1)$ acts trivially on $\mathbb{H}_2$, this gives an action of

$\mathrm{PSp}(4,\mathbb{Z})/\Gamma_2(3) \cong \mathrm{PSp}(4,\mathbb{Z}_3)$ on $\mathbb{P}^8$. The subspace spanned by the $y_{\alpha\beta} = \frac{1}{2}(\theta_{\alpha\beta} + \theta_{-\alpha,-\beta})$ is invariant. Burkhardt studied the ring of invariants of this action. We put $y_0 = -y_{00}$, $y_1 = 2y_{10}$, $y_2 = 2y_{01}$, $y_3 = 2y_{11}$ and $y_4 = 2y_{1,-1}$.

Theorem IV.2.2 ([Bu] and [vdG2]). *The quartic form* $y_0^4 - y_0(y_1^3 + y_2^3 + y_3^3 + y_4^3) + 3y_1y_2y_3y_4$ *is an invariant, of lowest degree, for this action. The map*

$$\tau \longmapsto (y_0 : y_1 : y_2 : y_3 : y_4)$$

defines a map $\mathbb{H}_2/\Gamma_2(3) \rightarrow \mathcal{B}_4$ *which extends to a birational map* $\mathcal{A}_2^*(3) \dashrightarrow \mathcal{B}_4$.

This much is fairly easy to prove, but far more is true: van der Geer, in [vdG2], gives a short modern proof as well as providing more detail. The projective geometry of $\mathcal{B}_4$ is better understood by embedding it in $\mathbb{P}^5$, as we did for $\mathcal{S}_3$. Baker [Ba2] gives explicit linear functions $x_0, \dots, x_5$ of $y_0, \dots, y_4$ such that $\mathcal{B}_4 \subset \mathbb{P}^5$ is given by

$$\sigma_1(x_i) = \sigma_4(x_i) = 0.$$

The details are reproduced in [Hun].

Theorem IV.2.3 ([To] and [Ba2]). $\mathcal{B}_4$ *is rational: consequently* $\mathcal{A}_2^*(3)$ *is rational.*

This was first proved by Todd ([To]); later Baker ([Ba2]) gave an explicit birational map from $\mathbb{P}^3$ to $\mathcal{B}_4$.

To prove Theorem IV.2.2 we need to say how to recover a principally polarized abelian surface and a level-3 structure from a general point of $\mathcal{B}_4$. The linear system on a principally polarized abelian surface given by three times the polarization is very ample, so the theta functions $\theta_{\alpha\beta}(\tau, z)$ determine an embedding of $A_\tau = \mathbb{C}^2/\mathbb{Z}^2 + \mathbb{Z}^2\tau$ ($\tau \in \mathbb{H}_2$) into $\mathbb{P}^8$. Moreover the extended Heisenberg group G_3 acts on the linear space spanned by the $\theta_{\alpha\beta}$. The Heisenberg group of level 3 is a central extension

$$0 \longrightarrow \mu_3 \longrightarrow H_3 \longrightarrow \mathbb{Z}_3^2 \longrightarrow 0$$

and G_3 is an extension of this by an involution ι. The involution acts by $z \mapsto -z$ and $\mathbb{Z}_3^2$ acts by translation by 3-torsion points. The space spanned by the $y_{\alpha\beta}$ is invariant under the normalizer of the Heisenberg group in $\mathrm{PGL}(4,\mathbb{C})$, which is isomorphic to $\mathrm{PSp}(4,\mathbb{Z}_3)$, so we get an action of this group on $\mathbb{P}^4$ and on $\mathcal{B}_4 \subset \mathbb{P}^4$.

For a general point $p \in \mathcal{B}_4$ the hyperplane in $\mathbb{P}^4$ tangent to $\mathcal{B}_4$ at p meets $\mathcal{B}_4$ in a quartic surface with six nodes, of a type known as a

Weddle surface. Such a surface is birational to a unique Kummer surface (Hudson ([Hud]) and Jessop ([Je]) both give constructions) and this is the Kummer surface of A_τ.

It is not straightforward to see the level-3 structure in this picture. One method is to start with a principally polarized abelian surface (A, Θ) and embed it in $\mathbb{P}^8$ by $|3\Theta|$. Then there is a projection $\mathbb{P}^8 \to \mathbb{P}^3$ under which the image of A is the Weddle surface, so one identifies this $\mathbb{P}^3$ with the tangent hyperplane to $\mathcal{B}_4$. The Heisenberg group acts on $\mathbb{P}^8$ and on $H^0\big(\mathbb{P}^8, \mathcal{O}_{\mathbb{P}^8}(2)\big)$, which has dimension 45. In $\mathbb{P}^8$, A is cut out by nine quadrics in $\mathbb{P}^8$. The span of these nine quadrics is determined by five coefficients $\alpha_0, \ldots, \alpha_4$ which satisfy a homogeneous Heisenberg-invariant relation of degree 4. As the Heisenberg group acting on $\mathbb{P}^4$ has only one such relation this relation must again be the one that defines $\mathcal{B}_4$. Thus the linear space spanned by nine quadrics, and hence A with its polarization and Heisenberg action, are determined by a point of $\mathcal{B}_4$. The fact that the two degree 4 relations coincide is equivalent to saying that $\mathcal{B}_4$ has an unusual projective property, namely it is self-Steinerian.

It is quite complicated to say what the level-3 structure means for the Kummer surface. It is not enough to look at the Weddle surface: one also has to consider the image of A in another projection $\mathbb{P}^8 \to \mathbb{P}^4$, which is again a birational model of the Kummer surface, this time as a complete intersection of type $(2,3)$ with ten nodes. More details can be found in [Hun].

The details of this proof were carried out by Coble ([Cob]), who also proved much more about the geometry of $\mathcal{B}_4$ and the embedded surface $A_\tau \subset \mathbb{P}^8$. The next theorem is a consequence of Coble's results.

Theorem IV.2.4 ([Cob]). *Let $\pi : \tilde{\mathcal{B}}_4 \to \mathcal{B}_4$ be the blow-up of $\mathcal{B}_4$ in the 45 nodes. Then $\tilde{\mathcal{B}}_4 \cong \mathcal{A}_2^*(3)$; the exceptional surfaces in $\tilde{\mathcal{B}}_4$ correspond to the Humbert surfaces that parametrize product abelian surfaces. The Satake compactification is obtained by contracting the preimages of 40 planes in $\mathcal{B}_4$, each of which contains 9 of the nodes.*

One should compare the birational map $\mathcal{A}_2^*(3) \dashrightarrow \mathcal{B}_4$ with the birational map $\mathcal{I}_4 \dashrightarrow \mathcal{S}_3$ of the previous section.

By computing the zeta function of $\tilde{\mathcal{B}}_4$ over $\mathbb{F}_q$ for $q \equiv 1 \pmod 3$, Hoffman and Weintraub ([HoW]) calculated the cohomology of $\mathcal{A}_2^*(3)$.

Theorem IV.2.5 ([HoW]). *$H^i(\mathcal{A}_2^*(3), \mathbb{Z})$ is free: the odd Betti numbers are zero and $b_2 = b_4 = 61$.*

In fact [HoW] gives much more detail, describing the mixed Hodge structures, the intersection cohomology of the Satake compactification, the $\mathrm{PSp}(4, \mathbb{Z}_3)$-module structure of the cohomology and some of the cohomology of the group $\Gamma_2(3)$. The cohomology of $\Gamma_2(3)$ was also partly computed, by another method, by MacPherson and McConnell ([McMc]), but neither result contains the other.

IV.3. The Nieto quintic

The *Nieto quintic* $\mathcal{N}_5$ is the subvariety of $\mathbb{P}^5$ given in homogeneous coordinates $x_0, \ldots, x_5$ by

$$\sigma_1(x_i) = \sigma_5(x_i) = 0.$$

This is conveniently written as $\sum x_i = \sum \frac{1}{x_i} = 0$. As in the cases of $\mathcal{S}_3$ and $\mathcal{B}_4$, this form of the equation displays the action of Sym(6) and is preferable for most purposes to a single quintic equation in $\mathbb{P}^4$. Unlike $\mathcal{S}_3$ and $\mathcal{B}_4$, which were extensively studied in the nineteenth century, $\mathcal{N}_5$ and its relation to abelian surfaces was first studied only in the 1989 Ph.D. thesis of Nieto ([Ni]) and the paper of Barth and Nieto ([BN]).

We begin with a result of van Straten ([vS]).

Theorem IV.3.1 ([vS]). *$\mathcal{N}_5$ has ten nodes but (unlike $\mathcal{S}_3$ and $\mathcal{B}_4$) it also has some non-isolated singularities. However the quintic hypersurface in $\mathbb{P}^4$ given as a subvariety of $\mathbb{P}^5$ by*

$$\sigma_1(x_i) = \sigma_5(x_i) + \sigma_2(x_i)\sigma_3(x_i) = 0.$$

has 130 *nodes and no other singularities.*

This threefold and the Nieto quintic are both special elements of the pencil

$$\sigma_1(x_i) = \alpha\sigma_5(x_i) + \beta\sigma_2(x_i)\sigma_3(x_i) = 0$$

and the general element of this pencil has 100 nodes. Van der Geer ([vdG2]) has analysed in a similar way the pencil

$$\sigma_1(x_i) = \alpha\sigma_4(x_i) + \beta\sigma_2(x_i)^2 = 0$$

which contains $\mathcal{B}_4$ (45 nodes) and $\mathcal{I}_4$ (15 singular lines) among the special fibres, the general fibre having 30 nodes.

No example of a quintic 3-fold with more than 130 nodes is known, though the Varchenko bound in this case is 135.

$\mathcal{N}_5$, like $\mathcal{S}_3$ and $\mathcal{B}_4$, is related to abelian surfaces via Kummer surfaces. The Heisenberg group $H_{2,2}$, which is a central extension

$$0 \to \mu_2 \to H_{2,2} \to \mathbb{Z}_2^4 \to 0$$

acts on $\mathbb{P}^3$ via the Schrödinger representation on $\mathbb{C}^4$. This is fundamental for the relation between $\mathcal{N}_5$ and Kummer surfaces.

Theorem IV.3.2 ([BN]). *The space of $H_{2,2}$-invariant quartic surfaces in $\mathbb{P}^3$ is 5-dimensional. The subvariety of this $\mathbb{P}^5$ which consists of those $H_{2,2}$-invariant quartic surfaces that contain a line is three-dimensional and its closure is projectively equivalent to $\mathcal{N}_5$. There is a double cover $\tilde{\mathcal{N}}_5 \to \mathcal{N}_5$ such that $\tilde{\mathcal{N}}_5$ is birationally equivalent to $\mathcal{A}^*_{1,3}(2)$.*

Proof. A general $H_{2,2}$-invariant quartic surface X containing a line ℓ will contain 16 skew lines (namely the $H_{2,2}$-orbit of ℓ). By a theorem of Nikulin ([N1]) this means that X is the minimal desingularization of the Kummer surface of some abelian surface A. The $H_{2,2}$-action on X gives rise to a level-2 structure on A, but the natural polarization on A is of type $(1,3)$. There is a second $H_{2,2}$-orbit of lines on X and they give rise to a second realization of X as the desingularized Kummer surface of another (in general non-isomorphic) abelian surface $\hat{A}$, which is in fact the dual of A. The moduli points of A and $\hat{A}$ (with their respective polarizations, but without level structures) in $\mathcal{A}_{1,3}$ are related by $V_3(A) = \hat{A}$, where V_3 is the Gritsenko involution described in III.3, above.

Conversely, given a general abelian surface A with a $(1,3)$-polarization and a level-2 structure, let $\widetilde{\mathrm{Km}}\,A$ be the desingularized Kummer surface and $\mathcal{L}$ a symmetric line bundle on A in the polarization class. Then the linear system $|\mathcal{L}^{\otimes 2}|^-$ of anti-invariant sections embeds $\widetilde{\mathrm{Km}}\,A$ as an $H_{2,2}$-invariant quartic surface and the exceptional curves become lines in this embedding. This gives the connection between $\mathcal{N}_5$ and $\mathcal{A}_{1,3}(2)$. Q.E.D.

The double cover $\tilde{\mathcal{N}}_5 \to \mathcal{N}_5$ is the inverse image of $\mathcal{N}_5$ under the double cover of $\mathbb{P}^5$ branched along the coordinate hyperplanes.

$\mathcal{N}_5$ is not very singular and therefore resembles a smooth quintic threefold in some respects. Barth and Nieto prove much more.

Theorem IV.3.3 ([BN]). *Both $\mathcal{N}_5$ and $\tilde{\mathcal{N}}_5$ are birationally equivalent to (different) Calabi-Yau threefolds. In particular, the Kodaira dimension of $\mathcal{A}^*_{1,3}(2)$ is zero.*

The fundamental group of a smooth projective model of $\mathcal{A}^*_{1,3}(2)$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ (see [S1] and II.3 above). Hence, as R. Livné has pointed out, there are four unramified covers of such a model which are also Calabi-Yau threefolds. In all other cases where the Kodaira dimension of a Siegel modular variety (of dimension > 1) is known, the variety is either of general type or uniruled.

It is a consequence of the above theorem that the modular group $\Gamma_{1,3}(2)$ which defines the moduli space $\mathcal{A}_{1,3}(2)$ has a unique weight-3 cusp form (up to a scalar). This cusp form was determined in [GrH4].

Recall that there is a weight-3 cusp form Δ_1 for the group $\Gamma_{1,3}$ with a character of order 6. The form Δ_1 has several interesting properties, in particular it admits an infinite product expansion and determines a generalized Lorentzian Kac-Moody superalgebra of Borcherds type (see [GrN]).

Theorem IV.3.4 ([GrH4]). *The modular form Δ_1 is the unique weight-3 cusp form of the group $\Gamma_{1,3}(2)$.*

Using this, it is possible to give an explicit construction of a Calabi-Yau model of $\mathcal{A}_{1,3}(2)$ which does not use the projective geometry of [BN].

Nieto and the authors of the present survey have investigated the relation between $\tilde{\mathcal{N}}_5$ and $\mathcal{A}^*_{1,3}(2)$ in more detail. $\mathcal{N}_5$ contains 30 planes which fall naturally into two sets of 15, the so-called S- and V-planes.

Theorem IV.3.5 ([HNS1]). *The rational map $\mathcal{A}^*_{1,3}(2) \dashrightarrow \mathcal{N}_5$ (which is generically 2-to-1) contracts the locus of product surfaces to the 10 nodes. The locus of bielliptic surfaces is mapped to the V-planes and the boundary of $\mathcal{A}^*_{1,3}(2)$ is mapped to the S-planes. Thus by first blowing up the singular points and then contracting the surfaces in $\tilde{\mathcal{N}}_5$ that live over the S-planes to curves one obtains the Satake compactification.*

In [HNS2] we gave a description of some of the degenerations that occur over the S-planes.

One of the open problems here is to give a projective description of the branch locus of this map. The projective geometry associated with the Nieto quintic is much less worked out than in the classical cases of the Segre cubic and the Burkhardt quartic.

Theorem IV.3.6 ([HSGS]). *The varieties $\mathcal{N}_5$ and $\tilde{\mathcal{N}}_5$ have rigid Calabi-Yau models. Both Calabi-Yaus are modular: more precisely, their L-function is equal (up to the Euler factors at bad primes) to the Mellin transform of the normalised weight 4 cusp form of level 6.*

V. Non-principal polarizations

We have encountered non-principal polarizations and some of the properties of the associated moduli spaces already. For abelian surfaces, a few of these moduli spaces have good descriptions in terms of projective geometry, and we will describe some of these results for abelian surfaces below. We begin with the most famous case, historically the starting point for much of the recent work on the whole subject.

V.1. Type $(1,5)$ and the Horrocks-Mumford bundle

In this section we shall briefly describe the relation between the Horrocks-Mumford bundle and abelian surfaces. Since this material has been covered extensively in another survey article (see [H1] and the references quoted there) we shall be very brief here.

The existence of the Horrocks-Mumford bundle is closely related to abelian surfaces embedded in $\mathbb{P}^4$. Indeed, let $A \subset \mathbb{P}^4$ be a smooth abelian surface. Since $\omega_A = \mathcal{O}_A$ it follows that the determinant of the normal bundle of A in $\mathbb{P}^4$ is $\det N_{A/\mathbb{P}^4} = \mathcal{O}_A(5) = \mathcal{O}_{\mathbb{P}^4}(5)_{|A}$, i.e. it can be extended to $\mathbb{P}^4$. It then follows from the Serre construction (see e.g. [OSS, Theorem 5.1.1]) that the normal bundle $N_{A/\mathbb{P}^4}$ itself can be extended to a rank 2 bundle on $\mathbb{P}^4$. On the other hand the double point formula shows immediately that a smooth abelian surface in $\mathbb{P}^4$ can only have degree 10, so the hyperplane section is a polarization of type $(1,5)$. Using Reider's criterion (see e.g. [LB, chapter 10, §4]) one can nowadays check immediately that a polarization of type $(1,n), n \geq 5$ on an abelian surface with Picard number $\rho(A) = 1$ is very ample. The history of this subject is, however, quite intricate. Comessatti proved in 1916 that certain abelian surfaces could be embedded in $\mathbb{P}^4$. He considered a 2-dimensional family of abelian surfaces, namely those which have real multiplication in $\mathbb{Q}(\sqrt{5})$. His main tool was theta functions. His paper ([Com]) was later forgotten outside the Italian school of algebraic geometers. A modern account of Comessatti's results using, however, a different language and modern methods was later given by Lange ([L]) in 1986. Before that Ramanan ([R]) had proved a criterion for a $(1,n)$-polarization to be very ample. This criterion applies to all $(1,n)$-polarized abelian surfaces (A,H) which are cyclic n-fold covers of a Jacobian. In particular this also gives the existence of abelian surfaces in $\mathbb{P}^4$. The remaining cases not covered by Ramanan's paper were treated in [HL].

With the exception of Comessatti's essentially forgotten paper, none of this was available when Horrocks and Mumford investigated the existence of indecomposable rank 2 bundles on $\mathbb{P}^4$. Although they also convinced themselves of the existence of smooth abelian surfaces in $\mathbb{P}^4$ they then presented a construction of their bundle F in [HM] in cohomological terms, i.e. they constructed F by means of a *monad.* A monad is a complex

$$(M) \qquad A \xrightarrow{p} B \xrightarrow{q} C$$

where A, B and C are vector bundles, p is injective as a map of vector bundles, q is surjective and $q \circ p = 0$. The cohomology of (M) is

$$F = \operatorname{Ker} q / \operatorname{Im} p$$

which is clearly a vector bundle. The Horrocks-Mumford bundle can be given by a monad of the form

$$V \otimes \mathcal{O}_{\mathbb{P}^4}(2) \xrightarrow{p} 2 \bigwedge^2 T_{\mathbb{P}^4} \xrightarrow{q} V^* \otimes \mathcal{O}_{\mathbb{P}^4}(3)$$

where $V = \mathbb{C}^5$ and $\mathbb{P}^4 = \mathbb{P}(V)$. The difficulty is to write down the maps p and q. The crucial ingredient here is the maps

$$\begin{array}{rcllcl} f^+ & : & V \longrightarrow \bigwedge^2 V, & f^+(\sum v_i e_i) & = & \sum v_i e_{i+2} \wedge e_{i+3} \\ f^- & : & V \longrightarrow \bigwedge^2 V, & f^-(\sum v_i e_i) & = & \sum v_i e_{i+1} \wedge e_{i+4} \end{array}$$

where $(e_i)_{i \in \mathbb{Z}_5}$ is the standard basis of $V = \mathbb{C}^5$ and indices have to be read cyclically. The second ingredient is the Koszul complex on $\mathbb{P}^4$, especially its middle part

$$\begin{array}{ccccc} \bigwedge^2 V \otimes \mathcal{O}_{\mathbb{P}^4}(1) & & \xrightarrow{\quad \wedge s \quad} & & \bigwedge^3 V \otimes \mathcal{O}_{\mathbb{P}^4}(2) \\ & \searrow^{p_0} & & \nearrow^{q_0} & \\ & & \bigwedge^2 T_{\mathbb{P}^4}(-1) & & \end{array}$$

where $s : \mathcal{O}_{\mathbb{P}^4}(-1) \to V \otimes \mathcal{O}_{\mathbb{P}^4}$ is the tautological bundle map. The maps p and q are then given by

$$\begin{array}{lclcl} p : V \otimes \mathcal{O}_{\mathbb{P}^4}(2) & \xrightarrow{(f^+, f^-)} & 2 \bigwedge^2 V \otimes \mathcal{O}_{\mathbb{P}^4}(2) & \xrightarrow{2p_0(1)} & 2 \bigwedge^2 T_{\mathbb{P}^4} \\ q : 2 \bigwedge^2 T_{\mathbb{P}^4} & \xrightarrow{2q_0(1)} & 2 \bigwedge^3 V \otimes \mathcal{O}_{\mathbb{P}^4}(3) & \xrightarrow{(-{}^t f^-, {}^t f^+)} & V^* \otimes \mathcal{O}_{\mathbb{P}^4}(3). \end{array}$$

Once one has come up with these maps it is not difficult to check that p and q define a monad. Clearly the cohomology F of this monad is a rank 2 bundle and it is straightforward to calculate its Chern classes to be

$$c(F) = 1 + 5h + 10h^2$$

where h denotes the hyperplane section. Since this polynomial is irreducible over the integers it follows that F is indecomposable.

One of the remarkable features of the bundle F is its symmetry group. The *Heisenberg group* of level n is the subgroup H_n of $\mathrm{SL}(n, \mathbb{C})$ generated by the automorphisms

$$\sigma : \quad e_i \mapsto e_{i-1}, \quad \tau : e_i \mapsto \varepsilon^i e_i \quad (\varepsilon = e^{2\pi i/n}).$$

Since $[\sigma, \tau] = \varepsilon \cdot \mathrm{id}_V$ the group H_n is a central extension

$$0 \to \mu_n \to H_n \to \mathbb{Z}_n \times \mathbb{Z}_n \to 0.$$

Let N_5 be the normalizer of the Heisenberg group H_5 in $\mathrm{SL}(5,\mathbb{C})$. Then $N_5/H_5 \cong \mathrm{SL}(2,\mathbb{Z}_5)$ and N_5 is in fact a semi-direct product

$$N_5 \cong H_5 \rtimes \mathrm{SL}(2,\mathbb{Z}_5).$$

Its order is $|N_5| = |H_5| \cdot |\mathrm{SL}(2,\mathbb{Z}_5)| = 125 \cdot 120 = 15{,}000$. One can show that N_5 acts on the bundle F and that it is indeed its full symmetry group ([De]).

The Horrocks-Mumford bundle is *stable*. This follows since $F(-1) = F \otimes \mathcal{O}_{\mathbb{P}^4}(-1)$ has $c_1(F(-1)) = 3$ and $h^0(F(-1)) = 0$. Indeed F is the unique stable rank 2 bundle with $c_1 = 5$ and $c_2 = 10$ ([DS]). The connection with abelian surfaces is given via sections of F. Since $F(-1)$ has no sections every section $0 \neq s \in H^0(F)$ vanishes on a surface whose degree is $c_2(F) = 10$.

Proposition V.1.1. *For a general section $s \in H^0(F)$ the zero-set $X_s = \{s = 0\}$ is a smooth abelian surface of degree* 10.

Proof. [HM, Theorem 5.1]. The crucial point is to prove that X_s is smooth. The vector bundle F is globally generated outside 25 lines L_{ij} in $\mathbb{P}^4$. It therefore follows from Bertini that X_s is smooth outside these lines. A calculation in local coordinates then shows that for general s the surface X_s is also smooth where it meets the lines L_{ij}. It is then an easy consequence of surface classification to show that X_s is abelian. Q.E.D.

In order to establish the connection with moduli spaces it is useful to study the space of sections $H^0(F)$ as an N_5-module. One can show that this space is 4-dimensional and that the Heisenberg group H_5 acts trivially on $H^0(F)$. Hence $H^0(F)$ is an $\mathrm{SL}(2,\mathbb{Z}_5)$-module. It turns out that the action of $\mathrm{SL}(2,\mathbb{Z}_5)$ on $H^0(F)$ factors through an action of $\mathrm{PSL}(2,\mathbb{Z}_5) \cong A_5$ and that as an A_5-module $H^0(F)$ is irreducible. Let $U \subset \mathbb{P}^3 = \mathbb{P}(H^0(F))$ be the open set parametrising *smooth* Horrocks-Mumford surfaces X_s. Then X_s is an abelian surface which is fixed under the Heisenberg group H_5. The action of H_5 on X_s defines a canonical level-5 structure on X_5. Let $\mathcal{A}_{1,5}^{\mathrm{lev}}$ be the moduli space of triples (A, H, α) where (A, H) is a $(1,5)$-polarized abelian surface and α a canonical level structure and denote by ${}^{\circ}\mathcal{A}_{1,5}^{\mathrm{lev}}$ the open part where the polarization H is very ample. Then the above discussion leads to

Theorem V.1.2 ([HM]). *The map which associates to a section s the Horrocks-Mumford surface $X_s = \{s = 0\}$ induces an isomorphism of U with ${}^{\circ}\mathcal{A}_{1,5}^{\mathrm{lev}}$. Under this isomorphism the action of* $\mathrm{PSL}(2,\mathbb{Z}_5) = A_5$ *on U is identified with the action of* $\mathrm{PSL}(2,\mathbb{Z}_5)$ *on $\mathcal{A}_{1,5}^{\mathrm{lev}}$ which permutes the canonical level structures on a $(1,5)$-polarized abelian surface. In particular $\mathcal{A}_{1,5}^{\mathrm{lev}}$ is a rational variety.*

Proof. [HM, Theorem 5.2]. Q.E.D.

The inverse morphism

$$\varphi : {}^{\circ}\mathcal{A}_{1,5}^{\mathrm{lev}} \to U \subset \mathbb{P}(H^0(F)) = \mathbb{P}^3$$

can be extended to a morphism

$$\tilde{\varphi} : (\mathcal{A}_{1,5}^{\mathrm{lev}})^* \to \mathbb{P}(H^0(F))$$

where $(\mathcal{A}_{1,5}^{\mathrm{lev}})^*$ denotes the Igusa (=Voronoi) compactification of $\mathcal{A}_{1,5}^{\mathrm{lev}}$. This extension can also be understood in terms of degenerations of abelian surfaces. Details can be found in [HKW2].

V.2. Type (1, 7)

The case of type $(1,7)$ was studied by Manolache and Schreyer ([MS]) in 1993. We are grateful to them for making some private notes and a draft version of [MS] available to us and answering our questions. Some of their results have also been found by Gross and Popescu ([GP1] and [GP3]) and by Ranestad: see also [S-BT].

Theorem V.2.1 ([MS]). *$\mathcal{A}_{1,7}^{\mathrm{lev}}$ is rational, because it is birationally equivalent to a Fano variety of type V_{22}.*

Proof. We can give only a sketch of the proof here. For a general abelian surface A with a polarization of type $(1,7)$ the polarization is very ample and embeds A in $\mathbb{P}^6$. In the presence of a canonical level structure the $\mathbb{P}^6$ may be thought of as $\mathbb{P}(V)$ where V is the Schrödinger representation of the Heisenberg group H_7. We also introduce, for $j \in \mathbb{Z}_7$, the representation V_j, which is the Schrödinger representation composed with the automorphism of H_7 given by $e^{2\pi i/7} \mapsto e^{6\pi ij/7}$. These can also be thought of as representations of the extended Heisenberg group G_7, the extension of H_7 by an extra involution coming from -1 on A. The representation S of G_7 is the character given by this involution (so S is trivial on H_7).

It is easy to see that $A \subset \mathbb{P}^6$ is not contained in any quadric, that is $H^0\big(\mathcal{I}_A(2)\big) = 0$, and from this it follows that there is an H_7-invariant resolution

$$\begin{aligned} 0 \leftarrow \mathcal{I}_A \leftarrow 3V_4 \otimes \mathcal{O}(-3) \leftarrow 7V_1 \otimes \mathcal{O}(-4) \leftarrow 6V_2 \otimes \mathcal{O}(-5) \\ \leftarrow 2V \otimes \mathcal{O}(-6) \oplus \mathcal{O}(-7) \leftarrow 2\mathcal{O}(-7) \leftarrow 0. \end{aligned}$$

By using this and the Koszul complex one obtains a symmetric resolution

$$0 \leftarrow \mathcal{O}_A \leftarrow \mathcal{O} \stackrel{\beta}{\leftarrow} 3V_4 \otimes \mathcal{O}(-3) \stackrel{\alpha}{\leftarrow} 2S \otimes \Omega^3 \stackrel{\alpha'}{\leftarrow} 3V_1 \otimes \mathcal{O}(-4) \stackrel{\beta'}{\leftarrow} \mathcal{O}(-7) \leftarrow 0.$$

This resolution is G_7-invariant. Because of the G_7-symmetry, α can be described by a 3×2 matrix X whose entries lie in a certain 4-dimensional space U, which is a module for $\mathrm{SL}(2, \mathbb{Z}_7)$. The symmetry of the resolution above amounts to saying that α' is given by the matrix $X' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} {}^tX$, and the complex tells us that $\alpha\alpha' = 0$. The three 2×2 minors of X cut out a twisted cubic curve C_A in $\mathbb{P}(U^\vee)$ and because of the conditions on α the ideal I_A of this cubic is annihilated by the differential operators

$$\Delta_1 = \frac{\partial^2}{\partial u_0 \partial u_1} - \frac{1}{2}\frac{\partial^2}{\partial u_2^2},$$
$$\Delta_2 = \frac{\partial^2}{\partial u_0 \partial u_2} - \frac{1}{2}\frac{\partial^2}{\partial u_3^2},$$
$$\Delta_3 = \frac{\partial^2}{\partial u_0 \partial u_3} - \frac{1}{2}\frac{\partial^2}{\partial u_1^2}$$

where the u_i are coordinates on U.

This enables one to recover the abelian surface A from C_A. If we write $R = \mathbb{C}[u_0, u_1, u_2, u_3]$ then we have a complex (the Hilbert-Burch complex)

$$0 \longleftarrow R/I_A \longleftarrow R \longleftarrow R(-2)^{\oplus 3} \xleftarrow{X} R(-3)^{\oplus 2} \longleftarrow 0.$$

It is exact, because otherwise one can easily calculate the syzygies of I_A and see that they cannot be the syzygies of any ideal annihilated by the three Δ_i. So I_A determines α (up to conjugation) and the symmetric resolution of $\mathcal{O}_A$ can be reconstructed from α.

Let H_1 be the component of the Hilbert scheme parametrising twisted cubic curves. For a general net of quadrics $\delta \subset \mathbb{P}(U^\vee)$ the subspace $H(\delta) \subset H_1$ consisting of those cubics annihilated by δ is, by a result of Mukai ([Muk]), a smooth rational Fano 3-fold of genus 12, of the type known as V_{22}. To check that this is so in a particular case it is enough to show that $H(\delta)$ is smooth. We must do so for $\delta = \Delta = \mathrm{Span}(\Delta_1, \Delta_2\Delta_3)$. Manolache and Schreyer show that $H(\Delta)$ is isomorphic to the space $\mathrm{VSP}\left(\bar{X}(7), 6\right)$ of polar hexagons to the Klein quartic curve (the modular curve $\bar{X}(7)$):

$$\mathrm{VSP}\left(\bar{X}(7), 6\right) = \left\{\{l_1, \ldots, l_6\} \subset \mathrm{Hilb}^6(\check{\mathbb{P}}^2) \mid \sum l_i^4 = x_0^3x_1 + x_1^3x_2 + x_2^3x_0\right\}.$$

(To be precise we first consider all 6-tuples $(l_1, \dots, l_6)$ where the l_i are pairwise different with the above property and then take the Zariski-closure in the Hilbert scheme.) It is known that VSP $\big(\bar{X}(7), 6\big)$ is smooth, so we are done. Q.E.D.

Manolache and Schreyer also give an explicit rational parametrization of VSP $\big(\bar{X}(7), 6\big)$ by writing down equations for the abelian surfaces. They make the interesting observation that this rational parametrization is actually defined over the rational numbers.

V.3. Type $(1, 11)$

The spaces $\mathcal{A}_{1,d}^{\mathrm{lev}}$ for small d are studied by Gross and Popescu, ([GP1], [GP2], [GP3] and [GP4]). In particular, in [GP2], they obtain a description of $\mathcal{A}_{1,11}^{\mathrm{lev}}$.

Theorem V.3.1 ([GP2]). *There is a rational map* $\Theta_{11} : \mathcal{A}_{1,11}^{\mathrm{lev}} \dashrightarrow \mathrm{Gr}(2,6)$ *which is birational onto its image. The closure of* $\mathrm{Im}\,\Theta_{11}$ *is a smooth linear section of* $\mathrm{Gr}(2,6)$ *in the Plücker embedding and is birational to the Klein cubic in* $\mathbb{P}^4$*. In particular* $\mathcal{A}_{1,11}^{\mathrm{lev}}$ *is unirational but not rational.*

The Klein cubic is the cubic hypersurface in $\mathbb{P}^4$ with the equation

$$\sum_{i=0}^{4} x_i^2 x_{i+1} = 0$$

with homogeneous coordinates x_i, $i \in \mathbb{Z}_5$. It is smooth, and all smooth cubic hypersurfaces are unirational but not rational [CG] and [IM].

The rational map Θ_{11} arises in the following way. For a general abelian surface A in $\mathcal{A}_{1,11}^{\mathrm{lev}}$, the polarization (which is very ample) and the level structure determine an H_{11}-invariant embedding of A into $\mathbb{P}^{10}$. The action of -1 on A lifts to $\mathbb{P}^{10} = \mathbb{P}(H^0(\mathcal{L}))$ and the (-1)-eigenspace of this action on $H^0(\mathcal{L})$ (where $\mathcal{L}$ is a symmetric bundle in the polarizing class) determines a $\mathbb{P}^4$, called $\mathbb{P}^- \subset \mathbb{P}^{10}$. We choose coordinates $x_0, \dots, x_{10}$ on $\mathbb{P}^{10}$ with indices in $\mathbb{Z}_{11}$ such that $x_1, \dots, x_5$ are coordinates on $\mathbb{P}^-$, so that on $\mathbb{P}^-$ we have $x_0 = 0$, $x_i = -x_{-i}$. The matrix T is defined to be the restriction of R to $\mathbb{P}^-$, where

$$R_{ij} = x_{j+i} x_{j-i}, \qquad 0 \le i, j \le 5.$$

(This is part of a larger matrix which describes the action on $H^0\big(\mathcal{O}_{\mathbb{P}^{10}}(2)\big)$ of H_{11}.) The matrix T is skew-symmetric and non-degenerate at a general point of $\mathbb{P}^-$. However, it turns out that for a general $A \in \mathcal{A}_{1,11}^{\mathrm{lev}}$ the rank of T at a general point $x \in A \cap \mathbb{P}^-$ is 4.

For a fixed A, the kernel of T is independent of the choice of x (except where the dimension of the kernel jumps), and this kernel is the point $\Theta_{11}(A) \in \mathrm{Gr}(2,6)$.

From the explicit matrix R, finally, Gross and Popescu obtain the description of the closure of $\mathrm{Im}\,\Theta_{11}$ as being the intersection of $\mathrm{Gr}(2,6)$ with five hyperplanes in Plücker coordinates. The equation of the Klein cubic emerges directly (as a 6×6 Pfaffian), but it is a theorem of Adler ([AR]) that the Klein cubic is the only degree 3 invariant of $\mathrm{PSL}(2,\mathbb{Z}_{11})$ in $\mathbb{P}^4$.

V.4. Other type $(1,t)$ cases

The results of Gross and Popescu for $t = 11$ described above are part of their more general results about $\mathcal{A}_{1,t}^{\mathrm{lev}}$ and $\mathcal{A}_{1,t}$ for $t \geq 5$. In the series of papers [GP1]–[GP4] they prove the following (already stated above as Theorem III.1.5).

Theorem V.4.1 ([GP1], [GP2], [GP3] and [GP4]). *$\mathcal{A}_{1,t}^{\mathrm{lev}}$ is rational for $6 \leq t \leq 10$ and $t = 12$ and unirational, but not rational, for $t = 11$. Moreover the variety $\mathcal{A}_{1,t}$ is unirational for $t = 14, 16, 18$ and 20.*

The cases have a different flavour depending on whether t is even or odd. For odd $t = 2d+1$ the situation is essentially as described for $t = 11$ above: there is a rational map $\Theta_{2d+1} : \mathcal{A}_{1,t}^{\mathrm{lev}} \dashrightarrow \mathrm{Gr}(d-3, d+1)$, which can be described in terms of matrices or by saying that A maps to the H_t-subrepresentation $H^0(\mathcal{I}_A(2))$ of $H^0(\mathcal{O}_A(2))$. In other words, one embeds A in $\mathbb{P}^{t-1}$ and selects the H_t-space of quadrics vanishing along A.

Theorem V.4.2 ([GP1]). *If $t = 2d+1 \geq 11$ is odd then the homogeneous ideal of a general H_t-invariant abelian surface in $\mathbb{P}^{t-1}$ is generated by quadrics; consequently Θ_{2d+1} is birational onto its image.*

For $t = 7$ and $t = 9$ this is not true: however, a detailed analysis is still possible and is carried out in [GP3] for $t = 7$ and in [GP2] for $t = 9$. For $t \geq 13$ it is a good description of the image of Θ_t that is lacking. Even for $t = 13$ the moduli space is not unirational and for large t it is of general type (at least for t prime or a prime square).

For even $t = 2d$ the surface $A \subset \mathbb{P}^{t-1}$ meets $\mathbb{P}^- = \mathbb{P}^{d-2}$ in four distinct points (this is true even for many degenerate abelian surfaces). Because of the H_t-invariance these points form a $\mathbb{Z}_2 \times \mathbb{Z}_2$-orbit and there is therefore a rational map $\Theta_{2d} : \mathcal{A}_{1,t}^{\mathrm{lev}} \dashrightarrow \mathbb{P}^-/(\mathbb{Z}_2 \times \mathbb{Z}_2)$.

Theorem V.4.3 ([GP1]). *If $t = 2d \geq 10$ is even then the homogeneous ideal of a general H_t-invariant abelian surface in $\mathbb{P}^{t-1}$ is generated by quadrics (certain Pfaffians) and Θ_{2d} is birational onto its image.*

To deduce Theorem V.4.1 from Theorem V.4.2 and Theorem V.4.3 a careful analysis of each case is necessary: for $t = 6, 8$ it is again the case that A is not cut out by quadrics in $\mathbb{P}^{t-1}$. In those cases when rationality or unirationality can be proved, the point is often that there are pencils of abelian surfaces in suitable Calabi-Yau 3-folds and these give rise to rational curves in the moduli spaces. Gross and Popescu use these methods in [GP2] ($t = 9, 11$), [GP3] ($t = 6, 7, 8$ and 10), and [GP4] ($t = 12$) to obtain detailed information about the moduli spaces $\mathcal{A}_{1,t}^{\mathrm{lev}}$. In [GP4] they also consider the spaces $\mathcal{A}_{1,t}$ for $t = 14, 16, 18$ and 20.

VI. Degenerations

The procedure of toroidal compactification described in [AMRT] involves making many choices. Occasionally there is an obvious choice. For moduli of abelian surfaces this is usually the case, or nearly so, since one has the Igusa compactification (which is the blow-up of the Satake compactification along the boundary) and all known cone decompositions essentially agree with this one. But generally toroidal compactifications are not so simple. One has to make further modifications in order to obtain acceptably mild singularities at the boundary. Ideally one would like to do this in a way which is meaningful for moduli, so as to obtain a space which represents a functor described in terms of abelian varieties and well-understood degenerations. The model, of course, is the Deligne-Mumford compactification of the moduli space of curves.

VI.1. Local degenerations

The first systematic approach to the local problem of constructing degenerations of polarized abelian varieties is Mumford's paper [Mu2] (conveniently reprinted as an appendix to [FC]). Mumford specifies degeneration data which determine a family G of semi-abelian varieties over the spectrum S of a complete normal ring R. Faltings and Chai ([FC]) generalized this and also showed how to recover the degeneration data from such a family. This semi-abelian family can then be compactified: in fact, Mumford's construction actually produced the compactification first and the semi-abelian family as a subscheme. However, although G is uniquely determined, the compactification is non-canonical. We may as well assume that R is a DVR and that G_η, the generic fibre, is an abelian scheme: the compactification then amounts to compactifying the central fibre G_0 in some way.

Namikawa (see for instance [Nam3] for a concise account) and Nakamura ([Nak1]) used toroidal methods to construct natural compactifications in the complex-analytic category, together with proper degenerating families of so-called stable quasi-abelian varieties. Various difficulties, including non-reduced fibres, remained, but more recently Alexeev and Nakamura ([Ale1] and [AN]) have produced a more satisfactory and simpler theory. We describe their results below, beginning with their simplified version of the constructions of Mumford and of Faltings and Chai. See [FC], [Mu2], or [AN] itself for more.

R is a complete DVR with maximal ideal I, residue field $k = R/I$ and field of fractions K. We take a split torus $\tilde{G}$ over $S = \operatorname{Spec} R$ with character group X and let $\tilde{G}(K) \cong (K^*)^g$ be the group of K-valued points of $\tilde{G}$. A set of periods is simply a subgroup $Y \subset \tilde{G}(K)$ which is isomorphic to $\mathbb{Z}^g$. One can define a polarization to be an injective map $\phi : Y \to X$ with suitable properties.

Theorem VI.1.1 ([Mu2] and [FC]). *There is a quotient $G = \tilde{G}/Y$ which is a semi-abelian scheme over S : the generic fibre G_η is an abelian scheme over* $\operatorname{Spec} K$ *with a polarization (given by a line bundle $\mathcal{L}_\eta$ induced by ϕ).*

Mumford's proof also provides a projective degeneration, in fact a wide choice of projective degenerations, each containing G as an open subscheme.

Theorem VI.1.2 ([Mu2], [Ch], [FC] and [AN]). *There is an integral scheme $\tilde{P}$, locally of finite type over S, containing $\tilde{G}$ as an open subscheme, with an ample line bundle $\tilde{\mathcal{L}}$ and an action of Y on $(\tilde{P}, \tilde{\mathcal{L}})$. There is an S-scheme $P = \tilde{P}/Y$, projective over S, with $P_\eta \cong G_\eta$ as polarized varieties, and G can be identified with an open subscheme of P.*

Many technical details have been omitted here. $\tilde{P}$ has to satisfy certain compatibility and completeness conditions: of these, the most complicated is a completeness condition which is used in [FC] to prove that each component of the central fibre P_0 is proper over k. Alexeev and Nakamura make a special choice of $\tilde{P}$ which, among other merits, enables them to dispense with this condition because the properness is automatic.

Mumford proved this result in the case of maximal degeneration, when G_0 is a torus over k. That condition which was dropped in [FC] and also in [AN] where $\tilde{G}$ is allowed to have an abelian part. Then $\tilde{G}$ and G_0 are Raynaud extensions, that is, extensions of abelian schemes by tori, over R and k respectively. The extra work entailed by this is carried out in [FC] but the results, though a little more complicated to state, are essentially the same as in the case of maximal degeneration.

In practice one often starts with the generic fibre G_η. According to the semistable reduction theorem there is always a semi-abelian family $G \to S$ with generic fibre G_η, and the aim is to construct a uniformization $G = \tilde{G}/Y$. It was proved in [FC] that this is always possible.

The proof of VI.1.1, in the version given by Chai ([Ch]) involves implicitly writing down theta functions on $\tilde{G}(K)$ in order to check that the generic fibre is the abelian scheme G_η. These theta functions can be written (analogously with the complex-analytic case) as Fourier power series convergent in the I-adic topology, by taking coordinates $w_1, \ldots, w_g$ on $\tilde{G}(K)$ and setting

$$\theta = \sum_{x \in X} \sigma_x(\theta) w^x$$

with $\sigma_x(\theta) \in K$. In particular theta functions representing elements of $H^0(G_\eta, \mathcal{L}_\eta)$ can be written this way and the coefficients obey the transformation formula

$$\sigma_{x+\phi(y)}(\theta) = a(y)b(y,x)\sigma_x(\theta)$$

for suitable functions $a : Y \to K^*$ and $b : Y \times X \to K^*$.

For simplicity we shall assume for the moment that the polarization is principal: this allows us to identify Y with X via ϕ and also means that there is only one theta function, ϑ. The general case is only slightly more complicated.

These power series have K-coefficients and converge in the I-adic topology but their behaviour is entirely analogous to the familiar complex-analytic theta functions. Thus there are cocycle conditions on a and b and it turns out that b is a symmetric bilinear form on $X \times X$ and a is an inhomogeneous quadratic form. Composing a and b with the valuation yields functions $A : X \to \mathbb{Z}$, $B : X \times X \to \mathbb{Z}$, and they are related by

$$A(x) = \frac{1}{2}B(x,x) + \frac{rx}{2}$$

for some $r \in \mathbb{N}$. We fix a parameter $s \in R$, so $I = sR$.

Theorem VI.1.3 ([AN]). *The normalization of the scheme* $\operatorname{Proj} R[s^{A(x)} w^x \theta; x \in X]$ *is a relatively complete model* $\tilde{P}$ *for the maximal degeneration of principally polarized abelian varieties associated with* G_η.

Similar results hold in general. The definition of $\tilde{P}$ has to be modified slightly if G_0 has an abelian part. If the polarization is non-principal it may be necessary to make a ramified base change first, since otherwise there may not be a suitable extension of $A : Y \to \mathbb{Z}$ to $A : X \to \mathbb{Z}$. Even for principal polarization it may be necessary to make a base change if we want the central fibre P_0 to have no non-reduced components.

The proof of Theorem VI.1.3 depends on the observation that the ring

$$R[s^{A(x)}w^x\theta; x \in X]$$

is generated by monomials. Consequently $\tilde{P}$ can be described in terms of toric geometry. The quadratic form B defines a *Delaunay decomposition* of $X \otimes \mathbb{R} = X_{\mathbb{R}}$. One of the many ways of describing this is to consider the paraboloid in $\mathbb{R}e_0 \oplus X_{\mathbb{R}}$ given by

$$x_0 = A(x) = \frac{1}{2}B(x,x) + \frac{rx}{2},$$

and the lattice $M = \mathbb{Z}e_0 \oplus X$. The convex hull of the points of the paraboloid with $x \in X$ consists of countably many facets and the projections of these facets on $X_{\mathbb{R}}$ form the Delaunay decomposition. This decomposition determines $\tilde{P}$. It is convenient to express this in terms of the *Voronoi decomposition* Vor_B of $X_{\mathbb{R}}$ which is dual to the Delaunay decomposition in the sense that there is a 1-to-1 inclusion-reversing correspondence between (closed) Delaunay and Voronoi cells. We introduce the map $dA : X_{\mathbb{R}} \to X^*_{\mathbb{R}}$ given by

$$dA(\xi)(x) = B(\xi, x) + \frac{rx}{2}.$$

Theorem VI.1.4 ([AN]). *$\tilde{P}$ is the torus embedding over R given by the lattice $N = M^* \subset \mathbb{R}e_0^* \oplus X^*_{\mathbb{R}}$ and the fan Δ consisting of $\{0\}$ and the cones on the polyhedral cells making up $(1, -dA(\mathrm{Vor}_B))$.*

Using this description, Alexeev and Nakamura check the required properties of $\tilde{P}$ and prove Theorem VI.1.3. They also obtain a precise description of the central fibres $\tilde{P}_0$ (which has no non-reduced components if we have made a suitable base change) and P_0 (which is projective). The polarized fibres $(P_0, \mathcal{L}_0)$ that arise are called *stable quasi-abelian varieties*, as in [Nak1]. In the principally polarized case P_0 comes with a Cartier divisor Θ_0 and (P_0, Θ_0) is called a *stable quasi-abelian pair.* We refer to [AN] for a precise intrinsic definition, which does not depend on first knowing a degeneration that gives rise to the stable quasi-abelian variety. For our purposes all that matters is that such a characterization exists.

VI.2. Global degenerations and compactification

Alexeev, in [Ale1], uses the infinitesimal degenerations that we have just been considering to tackle the problem of canonical global moduli. For simplicity we shall describe results of [Ale1] only in the principally polarized case.

We define a *semi-abelic variety* to be a normal variety P with an action of a semi-abelian variety G having only finitely many orbits, such that the stabilizer of the generic point of P is a connected reduced subgroup of the torus part of G. If $G = A$ is actually an abelian variety then Alexeev refers to P as an *abelic variety*: this is the same thing as a torsor for the abelian variety A. If we relax the conditions by allowing P to be semi-normal then P is called a *stable semi-abelic variety* or SSAV.

A *stable semi-abelic pair* (P, Θ) is a projective SSAV together with an effective ample Cartier divisor Θ on P such that Θ does not contain any G-orbit. The degree of the corresponding polarization is $g!h^0(\mathcal{O}_P(\Theta))$, and P is said to be principally polarized if the degree of the polarization is $g!$ If P is an abelic variety then (P, Θ) is called an *abelic pair.*

Theorem VI.2.1 ([Ale1]). *The categories $\underline{A}_g$ of g-dimensional principally polarized abelian varieties and $\underline{AP}_g$ of principally polarized abelic pairs are naturally equivalent. The corresponding coarse moduli spaces $\mathcal{A}_g$ and $\mathcal{AP}_g$ exist as separated schemes and are naturally isomorphic to each other.*

Because of this we may as well compactify $\mathcal{AP}_g$ instead of $\mathcal{A}_g$ if that is easier. Alexeev carries out this program in [Ale1]. In this way, he obtains a proper algebraic space $\overline{\mathcal{AP}}_g$ which is a coarse moduli space for stable semi-abelic pairs.

Theorem VI.2.2 ([Ale1]). *The main irreducible component of $\overline{\mathcal{AP}}_g$ (the component that contains $\mathcal{AP}_g = A_g$) is isomorphic to the Voronoi compactification $\mathcal{A}_g^*$ of $\mathcal{A}_g$. Moreover, the Voronoi compactification in this case is projective.*

The first part of Theorem VI.2.2 results from a careful comparison of the respective moduli stacks. The projectivity, however, is proved by elementary toric methods which, in view of the results of [FC], work over $\operatorname{Spec} \mathbb{Z}$.

In general $\overline{\mathcal{AP}}_g$ has other components, possibly of very large dimension. Alexeev has examined these components and the SSAVs that they parametrize in [Ale2].

Namikawa, in [Nam1], already showed how to attach a stable quasi-abelian variety to a point of the Voronoi compactification. Namikawa's families, however, have non-reduced fibres and require the presence of a level structure: a minor technical alteration (a base change and normalization) has to be made before the construction works satisfactorily. See [AN] for this and also for an alternative construction using explicit local families that were first written down by Chai ([Ch]). The use of abelic rather than abelian varieties also seems to be essential in order to obtain a good family: this is rather more apparent over a non-algebraically closed field, when the difference between an abelian variety (which has a point) and an abelic variety is considerable.

Nakamura, in [Nak2], takes a different approach. He considers degenerating families of abelian varieties with certain types of level structure. In his case the boundary points correspond to *projectively stable quasi-abelian schemes* in the sense of GIT. His construction works over $\operatorname{Spec}\mathbb{Z}[\zeta_N, 1/N]$ for a suitable N. At the time of writing it is not clear whether Nakamura's compactification also leads to the second Voronoi compactification.

References

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On Semistable Extremal Neighborhoods

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Dedicated to Professor Heisuke HIRONAKA on the occasion of his seventieth birthday

Abstract.

We give an explicit description of an extremal nbd $X \supset C \simeq \mathbb{P}^1$ of type $k2A$. We also give a criterion for X to be a flipping contraction and an explicit description of the contraction and the flip.

§1. Introduction

In the three dimensional minimal model program, flips and divisorial contractions are the fundamental birational maps. Among them, flips are proved to exist [8]. This paper is concerned with the classification of flips. We give a brief background.

Let $f : X \to Y$ be a projective birational morphism from a threefold X with only terminal singularities to a normal threefold Y and $Q \in Y$ such that $C = f^{-1}(Q)$ is a curve and $-K_X$ is f-ample.

We note that, in the context of the minimal model program, we often assume that X is $\mathbb{Q}$-factorial and put the condition $\rho(X/Y) = 1$ on the relative Picard number. In this paper, we do not assume these conditions, because they are not preserved when we work on the associated formal scheme.

For an arbitrarily small open set $U \ni Q$, we call $f^{-1}(U) \supset C \to U \ni Q$ *an extremal neighborhood* (or, *an extremal nbd*, for short). It is said to be *flipping* (resp. *divisorial*) if the exceptional set is a curve (resp. a divisor). An extremal nbd is said to be *irreducible* if C is irreducible.

In [5], the irreducible extremal nbds $X \supset C \to Y \ni Q$ are studied as follows. A general member D of $|-K_X|$ is proved to have only Du Val singularities, and the irreducible extremal nbds are classified

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into 6 types $k1A$, $k2A$, $cD/3$, IIA, IC, kAD [5, (2.2)] according to the singularities of D. The first two (resp. the last four) cases are said to be *semistable* (resp. *exceptional*). For the exceptional irreducible flipping extremal nbds $X \supset C$, the singularities of the general member H of $|\mathcal{O}_X|$ containing C are computed in [5, Chapters 6–9] and the irreducible flipping $X \supset C$ is reconstructed as an essentially arbitrary one-parameter deformation space of H [5, Theorems 13.9–13.12] and the flip is described [5, Theorems 13.17 and 13.18].

However if we start with H of an irreducible semistable extremal nbd X, whether or not X is flipping depends not only on H but also on the individual one-parameter deformation, which is quite different from the exceptional cases.

In this paper, we treat the case of $k2A$. (The case of $k1A$ will be treated elsewhere.) In Section 2, we give an expression of an extremal nbd $X \supset C$ of type $k2A$ in terms of coordinates (Theorem 2.2) and graded rings (Definition 2.8, Theorem 2.9).

Section 3 is the core algorithm section of this paper, where we introduce a sequence $d(n)$ (Definitions 3.2 and 3.11) and present a series of divisions (Theorems 3.10–3.13) starting with the "graded equations" in Theorem 2.9.

Section 4 is the main section for applications, where we give a necessary and sufficient condition (Corollary 4.1) for $X \supset C$ to be flipping in terms of $d(n)$. Furthermore, the extremal contraction (Theorem 4.3) and the flip are explicitly constructed (Theorem 4.7).

In Section 5, we present the division in the case of a multi-parameter deformation space of H and comment on some of the further directions.

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§2. Good coordinates

2.1. Let $f : X \supset C\ (\simeq \mathbb{P}^1) \to Y \ni Q$ be an extremal nbd of type $k2A$ [5] with two terminal singular points P_1, P_2 of indices $m_1, m_2 > 1$ and axial multiplicities $\alpha_1, \alpha_2 \geq 1$, respectively.

Let $D \in |-K_X|$ be a Du Val member, whose minimal resolution has the dual configuration

$$\underbrace{\circ - \cdots - \circ}_{m_1\alpha_1 - 1} - C' - \underbrace{\circ - \cdots - \circ}_{m_2\alpha_2 - 1},$$

where C' is the proper transform of C and $\circ$ denotes an exceptional curve and all these curves are (-2)-curves [5, 2.2.4]. By adding two (non-compact) curves ℓ'_i at both ends

$$\ell'_1 - \underbrace{\circ - \cdots - \circ}_{m_1\alpha_1 - 1} - C' - \underbrace{\circ - \cdots - \circ}_{m_2\alpha_2 - 1} - \ell'_2,$$

we obtain a reduced curve on the minimal resolution such that the intersection numbers with C' and $\circ$'s are zero. Since $H^1(X, \mathcal{O}_X) = 0$ [8], we see that $\ell_1 + C + \ell_2 \sim 0$ on D, where $\ell_i \subset D$ denotes the image of ℓ'_i. Let

$$H_D := \ell_1 + C + \ell_2 \ (\sim 0 \text{ on } D).$$

By the exact sequence

$$0 \to \mathcal{O}_X(K_X) \to \mathcal{O}_X \to \mathcal{O}_D \to 0$$

and the Grauert Riemenschneider vanishing $H^1(X, \mathcal{O}_X(K_X)) = 0$ [8], we obtain a trivial Cartier divisor $H = (u = 0)$ on X, which is normal and induces H_D on D.

Theorem 2.2. *Let U_i be the $\mathbb{Z}_{m_i}$-quotient of a "hypersurface" of $\mathbb{C}^4$,*

$$U_i := (\xi_i, \eta_i, \zeta_i, u; \xi_i\eta_i = g_i(\zeta_i^{m_i}, u))/\mathbb{Z}_{m_i}(1, -1, a_i, 0),$$

where a_i is an integer $\in [1, m_i]$ prime to m_i and $g_i(T, u) \in \mathbb{C}[[u]][T]$ is a monic polynomial in T of degree, say ρ_i such that $g_i(\zeta_i^{m_i}, u)$ is square-free. Let $P_i := 0$ and $C_i := \xi_i$-axis$/\mathbb{Z}_{m_i}$. U_i is defined to be a formal scheme along $C_i \simeq \mathbb{C}^1$ with only terminal singularities.

For a suitable choice of a_i and $g_i(T, u)$, we have

1. *these C_1 and C_2 are patched together to form $C \simeq \mathbb{P}^1$ and U_1 and U_2 are patched together to form the completion $\hat{X}$ of X along C by the identification on $U_1 \cap U_2$:*

$$\xi_1^{m_1} = (\xi_2^{m_2})^{-1},$$
$$\frac{\zeta_1}{\xi_1^{a_1}} = \xi_2^{m_2}\frac{\zeta_2}{\xi_2^{a_2}},$$

2. *$D = (\zeta_1 = 0)/\mathbb{Z}_{m_1} \cup (\zeta_2 = 0)/\mathbb{Z}_{m_2}$ and $H = (u = 0)$ under the identification.*

Remark 2.3. The assertions of Theorem 2.2 modulo the equation u of H, that is the corresponding assertion for H is easily seen as follows.

By the construction, $f(H_D)$ has an ordinary double point at Q. Since $K_H + H_D \sim 0$, we see that $(f(H), f(H_D))$ is lc ([6, (5.58)] or [12]). Since $f(H_D)$ has two analytic branches, this means that $(f(H), f(H_D))$ is the quotient of (xy-plane, $(xy = 0)$) by a diagonal action of some cyclic group [1] or [3]. In particular, it is toric.

Since C is a log crepant divisor of $(f(H), f(H_D))$ (that is, $K_H + H_D \sim 0$ and, and H_D is a reduced curve containing C), (H, H_D) is toric as well as $(f(H), f(H_D))$.

The index-1 cover of $(H \setminus \ell_i, H_D \setminus \ell_i)$ is toric. By the descrition of the terminal singularities P_i [7, 11], we obtain the isomorphisms

$$H \setminus \ell_{3-i} \simeq (\xi_i, \eta_i, \zeta_i; \xi_i\eta_i = \zeta_i^{m_i\rho_i})/\mathbb{Z}_{m_i}(1, -1, a_i)$$

with the properties

1. $H_D = (\zeta_i = 0)/\mathbb{Z}_{m_i}, m_i\rho_i\ell_i = (\xi_i = 0)/\mathbb{Z}_{m_i}$ and $C \setminus \ell_{3-i} = (\zeta_i = \eta_i = 0)/\mathbb{Z}_{m_i}$ on $H \setminus \ell_{3-i}$ under the identification, and
2. $\xi_1^{m_1} = \xi_2^{-m_2}$ on $H \setminus (\ell_1 \cup \ell_2)$,

for some $\rho_i \in \mathbb{Z}_{>0}$ and $a_i \in [1, m_i]$ such that $(a_i, m_i) = 1$. Once these properties are checked, it is easy to see the following.

3. $K_H \sim (m_1 - a_1) \cdot (\xi_1 = 0)/\mathbb{Z}_{m_i} - a_2 \cdot (\xi_2 = 0)/\mathbb{Z}_{m_2}$
4. $\xi_1^{-a_1}\zeta_1 = \xi_2^{m_2-a_2}\zeta_2$ on $H \setminus (\ell_1 \cup \ell_2)$

Indeed, the property 3 follows from $gr_X^0\omega \simeq \mathcal{O}_C(-1)$ [8, (1.14.(i))] and the assertion that

$$x_i^{m_i-a_i} \operatorname{Res} \frac{d\xi_i \wedge d\eta_i \wedge d\zeta_i}{\xi_i\eta_i - \zeta_i^{m_i\rho_i}} = -x_i^{m_i-a_i}(d\xi_i/\xi_i) \wedge d\zeta_i$$

is a generator of $gr_X^0\omega|_H$ on $H \setminus \ell_{3-i}$. The property 4 follows from the property 3 because $m_1 d\xi_1/\xi_1 = -m_2 d\xi_2/\xi_2$.

Proof of Theorem 2.2. We note that Theorem 2.2 is proved modulo (u), the equation of H (Remark 2.3). On the completion $\hat{X}$ of X along C, let U_i be the complement of P_{3-i}. Assume that Theorem 2.2 is proved modulo $(u)^N$ for some $N > 0$. We attach subscript N to the coordinates and the equations that are chosen to work for $(u)^N$.

From the $\mathbb{Z}_{m_i}$-invariant relation

$$\xi_{N,i}\eta_{N,i} = g_{N,i}(\zeta_{N,i}^{m_i}, u) \mod (u)^N,$$

we have

$$\xi_{N,i}\eta_{N,i} = g_i'(\zeta_{N,i}^{m_i}, u) + u^N\xi_{N,i}\alpha_i + u^N\eta_{N,i}\beta_i \mod (u)^{N+1}$$

for some $\alpha_i, \beta_i \in \mathbb{C}[\xi_{N,i}][[u, \eta_{N,i}, \zeta_{N,i}]]$ and $g_i'(T, u) \in \mathbb{C}[[u, T]]$ such that $g_i' \equiv g_{N,i} \mod (u)^N$. Then

$$(\xi_{N,i} - u^N\alpha_i)(\eta_{N,i} - u^N\beta_i) = g_i''(\zeta_{N,i}^{m_i}, u) \mod (u)^{N+1}$$

for some $g_i'' \in \mathbb{C}[[u, T]]$ such that $g_i'' \equiv g_{N,i} \mod (u)^N$.

The Cartier divisor

$$\Phi_i = (\xi_{N,i} - u^N\alpha_i)/\xi_i) = (\xi_{N,i} = u^N\alpha_i)/\mathbb{Z}_{m_i} - (\xi_{N,i} = 0)/\mathbb{Z}_{m_i}$$

on a neighborhood of P_i extends to a principal divisor on X, because Φ_i intersects properly with C and $(\Phi_i \cdot C) = 0$. By the exact sequence

$$0 \to \mathcal{O}_{\hat{X}}(\Phi_i) \xrightarrow{u^N} \mathcal{O}_{\hat{X}}(\Phi_i) \to \mathcal{O}_{N\hat{H}} \to 0$$

and $H^1(\hat{X}, \mathcal{O}_{\hat{X}}) = 0$, there is a rational function φ_i on $\hat{X}$ such that $(\varphi_i) = \Phi_i$ and $\varphi_i|_{N\hat{H}} = 1$. We note that φ_i is invertible on U_{3-i}.

Let

$$\xi_{N+1,i} = \xi_{N,i} \cdot \varphi_i \cdot \varphi_{3-i}^{-m_{3-i}/m_i}.$$

Then we have

$$\begin{aligned} \xi_{N+1,i} &\equiv \xi_{N,i} \mod (u)^N, \\ \xi_{N+1,1}^{m_1} &= \xi_{N+1,2}^{-m_2} \end{aligned}$$

and $\xi_{N+1,i} = (\xi_{N,i} - u^N\alpha_i) \cdot (\text{unit on } U_i)$. Let

$$\zeta_{N+1,i} = \begin{cases} \zeta_{N,1}\varphi_2^{m_2-a_2}\varphi_2^{-a_1m_2/m_1} & i = 1, \\ \zeta_{N,2}\varphi_1^{-a_1}\varphi_1^{-(m_2-a_2)m_1/m_2} & i = 2. \end{cases}$$

We note that $\zeta_{N+1,i}/\zeta_{N,i}$ is a unit on U_i and that

$$\begin{aligned} \xi_{N+1,1}^{-a_1}\zeta_{N+1,1} &= \xi_{N+1,2}^{m_2-a_2}\zeta_{N+1,2}, \\ \zeta_{N+1,i} &\equiv \zeta_{N,i} \mod (u)^N. \end{aligned}$$

By the Weierstrass preparation theorem, there exist a unit $\gamma_i \in \mathbb{C}[[u,T]]$ such that $\gamma_i \equiv 1 \mod (u)^N$ and $g_{N+1,i} := g_i''\gamma_i$ is a monic polynomial $\in \mathbb{C}[[u]][T]$. We then define $\eta_{N+1,i}$ such that $\eta_{N+1,i} = (\eta_{N,i} - u^N\beta_i) \cdot (\text{unit on } U_i)$ by the relation on U_i:

$$\xi_{N+1,i}\eta_{N+1,i} = (\xi_{N,i} - u^N\alpha_i)(\eta_{N,i} - u^N\beta_i)(\zeta_{N+1,i}/\zeta_{N,i})^{m_i\rho_i}\gamma_i.$$

By $g_{N+1,i}(T, 0) = T^{\rho_i}$ and $\zeta_{N+1,i}/\zeta_{N,i} \equiv 1 \mod (u)^N$, we have

$$g_{N+1,i}(\zeta_{N+1,i}^{m_i}, u) \equiv g_{N+1,i}(\zeta_{N,i}^{m_i}, u) \cdot (\zeta_{N+1,i}/\zeta_{N,i})^{m_i\rho_i} \mod (u)^{N+1},$$

and hence

$$\xi_{N+1,i}\eta_{N+1,i} \equiv g_{N+1,i}(\zeta_{N+1,i}^{m_i}, u) \mod (u)^{N+1}.$$

Thus the theorem is proved modulo $(u)^{N+1}$. We can let $n \to \infty$.

Finally $g_i(\zeta_i^{m_i}, u)$ is square-free, because otherwise U_i has a non-isolated singularity. Q.E.D.

Remark 2.4. The numbers a_i and ρ_i in Theorem 2.2 can be easily read off from the information on $X \supset C$, D and H. Furthermore, α_i is related to g_i.

1. The numbers a_i are uniquely determined by $X \supset C \ni P_i$ because the $\mathbb{Z}_{m_i}$-action is normalized by its action on the ξ_i-axis.
2. By
$$H = (\xi_i \eta_i = g_i(\zeta_i^{m_i}, 0))/\mathbb{Z}_{m_i}(1, -1, a_i),$$
the index-one cover of H at P_i is an $A_{m_i\rho_i}$-point $(\xi_i\eta_i = g_i(\zeta_i^{m_i}, 0))$. Thus ρ_i is uniquely determined by H.
3. Similarly by
$$\begin{aligned} D &= (\xi_i \eta_i = g_i(0, u))/\mathbb{Z}_{m_i}(1, -1, 0), \\ &\simeq (xy = g_i(0, u)^{m_i}), \end{aligned}$$
we have $(g_i(0, u)) = (u)^{\alpha_i}$ in $\mathbb{C}[[u]]$.

Remark 2.5. Under the notation of Theorem 2.2, let $S_i = (\xi_i = 0)/\mathbb{Z}_{m_i}$. Then

1. $m_1 S_1 \sim m_2 S_2$, $S_1 \cap S_2 = \emptyset$,
2. $K_X \sim (m_1 - a_1)S_1 - a_2 S_2$, and
3. $-m_i K_X \sim \delta S_{3-i}$, where $\delta := a_1 m_2 + a_2 m_1 - m_1 m_2 > 0$.

Let
$$z \in \Gamma(\hat{X}, \mathcal{O}(a_1 S_1 + (a_2 - m_2)S_2)), u \in \Gamma(\hat{X}, \mathcal{O}), x_i \in \Gamma(\hat{X}, \mathcal{O}(S_i))$$
be the sections defining D, H and S_i.

Let $G_i(T_1, T_2) = g_i(T_1/T_2, u)T_2^{\rho_i} \in \mathbb{C}[[u]][T_1, T_2]$, which is a homogeneous polynomial in T_1, T_2 of degree ρ_i. Since $z^{m_i}, x_{3-i}^{\delta}$ are both sections of $\mathcal{O}(\delta S_i)$, we can consider the section $G_i(z^{m_i}, x_{3-i}^{\delta})$ of $\mathcal{O}(\rho_i \delta S_{3-i})$. The section is divisible by x_i and the quotient y_i satisfies the condition
$$y_i \in \Gamma(\hat{X}, \mathcal{O}(\rho_i \delta S_{3-i} - S_i)),$$
which follows immediately from the local equation $\xi_i \eta_i = g_i(\zeta^{m_i}, u)$. We have thus two equations:
$$x_1 y_1 - G_1(z^{m_1}, x_2^{\delta}) = 0, \quad x_2 y_2 - G_2(z^{m_2}, x_1^{\delta}) = 0,$$
where $G_1(z^{m_1}, x_2^{\delta})$ and $G_1(z^{m_1}, x_2^{\delta})$ are square-free (Theorem 2.2).

The contractibility of C implies the following positivity result.

Proposition 2.6. *Under the notation and the assumptions of Theorem 2.2, we have*

$$\Delta := \rho_1 m_1^2 - \delta\rho_1\rho_2 m_1 m_2 + \rho_2 m_2^2 > 0.$$

Proof. By the property 1 of Remark 2.3, we have

$$(\ell_i \cdot C)_H = \frac{1}{m_i\rho_i}(S_i \cdot C) = \frac{1}{m_i^2\rho_i}.$$

Since $H \cap D = \ell_1 + C + \ell_2$, we have the following by Remark 2.5.

$$\begin{aligned}(\ell_1 + C + \ell_2 \cdot C)_H = (D \cdot C) &= \frac{1}{m_1}(-m_1 K_X \cdot C) \\ &= \frac{1}{m_1}(\delta S_2 \cdot C) = \frac{\delta}{m_1 m_2}.\end{aligned}$$

Thus we have

$$(C^2)_H = \frac{-\Delta}{\rho_1\rho_2 m_1^2 m_2^2}.$$

Since C is an exceptional curve on H, we have $(C^2)_H < 0$. Q.E.D.

Remark 2.7. The properties that

$$\begin{aligned} z &\in \Gamma(\hat{X}, \mathcal{O}(a_1 S_1 + (a_2 - m_2)S_2)), \\ u &\in \Gamma(\hat{X}, \mathcal{O}), \\ x_i &\in \Gamma(\hat{X}, \mathcal{O}(S_i)), \\ y_i &\in \Gamma(\hat{X}, \mathcal{O}(\rho_i\delta S_{3-i} - S_i)) \end{aligned}$$

in Remark 2.5 can be rephrased as follows. Let the group

$$\Gamma := \{(\gamma_1, \gamma_2) \in (\mathbb{C}^*)^2 | \gamma_1^{m_1} = \gamma_2^{m_2}\}$$

act on $H^0(\hat{X}, \mathcal{O}(\lambda_1 S_1 + \lambda_2 S_2))$ via the multiplication by $\gamma_1^{\lambda_1}\gamma_2^{\lambda_2}$. Then we have

$$\gamma(x_i, z, y_i, u) = (\gamma_i x_i, \gamma_1^{a_1}\gamma_2^{a_2 - m_2} z, \gamma_{3-i}^{\rho_i\delta}\gamma_i^{-1} y_i, u),$$

and $x_i y_i - G_i(z^{m_i}, x_{3-i}^{\delta})$ is semi-invariant under the Γ-action. The scheme $\hat{X}$ has an alternate description in terms of these data as follows.

Definition 2.8. Let $a_i, m_i, \alpha_i, \rho_i$ be positive integers (cf. Remark 2.10) and $G_i(T_1, T_2) \in \mathbb{C}[[u]][T_1, T_2]$ a homogeneous polynomial in T_1 and T_2 of degree ρ_i $(i = 1, 2)$ such that

1. $a_i \le m_i$ and $(a_i, m_i) = 1$,
2. $\delta = a_1 m_2 + a_2 m_1 - m_1 m_2 > 0$,
3. $G_i(1,0) = 1$, $G_i(0,1)\mathbb{C}[[u]] = u^{\alpha_i}\mathbb{C}[[u]]$,
4. $G_i(T_1^{m_i}, 1)$ is reduced, and
5. $\Delta = \rho_1 m_1^2 - \delta\rho_1\rho_2 m_1 m_2 + \rho_2 m_2^2 > 0$.

Let $R := \mathbb{C}[[u]][x_1, y_1, x_2, y_2, z]$ be the $\mathbb{C}[[u]]$-algebra with the Γ-action in Remark 2.7, and let $W = \operatorname{Spec} R/I$ be the scheme with the Γ-action, where I is the ideal given by

$$I := (x_1 y_1 - G_1(z^{m_1}, x_2^\delta), x_2 y_2 - G_2(z^{m_2}, x_1^\delta)).$$

Set

$$X := (W \setminus V(x_1, x_2))/\Gamma \supset C := V(y_1, y_2, z)/\Gamma \simeq \mathbb{P}^1$$

and $\{P_i\} = V(x_i, y_1, y_2, z, u)$, where $V(I)$ denots the closed subset defined by all the equations in I.

Theorem 2.9. *With the above notation and the assumptions, we have the following.*

1. *X is a normal scheme of dimension 3 such that $X \setminus \{P_1, P_2\}$ is smooth and*

   $$P_i \in X \simeq (\xi_i, \eta_i, \zeta_i, u; \xi_i\eta_i = G_i(\zeta_i^{m_i}, 1))/\mathbb{Z}_{m_i}(1, -1, a_i, 0)$$

   is a terminal singularity with index m_i and $P_i \in C = \xi_i$-axis$/\mathbb{Z}_{m_i}$ under the identification,
2. *$S_i := (x_i = 0)/\Gamma$ is a $\mathbb{Q}$-Cartier Weil divisor on X, and a rational function ϕ on W such that $\phi/x_1^{-b_1}x_2^{-b_2}$ is Γ-invariant defines a $\mathbb{Q}$-Cartier Weil divisor $(\phi = 0)/\Gamma \sim b_1 S_1 + b_2 S_2$. In particular, $D := (z = 0)/\Gamma \in |-K_X|$ and $H = (u = 0)$ are as in 2.1,*
3. *the completion of X along C is isomorphic to $\hat{X}$ given in Theorem 2.2.*

Proof. On $U_i = \{x_{3-i} \neq 0\}$, we normalize $x_{3-i} = 1$ and set $\xi_i := x_i$, $\eta_i := y_i$ and $\zeta_i := z$ with the relation $\xi_i\eta_i = G_i(\zeta_i, 1)$. Note that η_{3-i} is not needed because $y_{3-i} = G_{3-i}(\zeta_i^{m_{3-i}}, \xi_i^\delta)$. The stabilizer $\Gamma_i \simeq \mathbb{Z}_{m_i}$ of x_{3-i} acts on $(\xi_i, \eta_i, \zeta_i, u)$ via the grading $(1, -1, a_i, 0) \mod (m_i)$, and the quotient is isomorphic to U_i. The rest of the assertion 1 follows from [7, 11]. The patching of the coordinates is obtained by

$$\gamma(\xi_1, 1, \zeta_1) = (1, \xi_2, \zeta_2).$$

Indeed we obtain $\gamma_1 = \xi_1^{-1}$, $\gamma_2 = \xi_2$ (whence $\xi_1^{m_1} = \xi_2^{-m_2}$) and the relation for ζ_i's: $\xi_1^{-a_1}\xi_2^{a_2 - m_2}\zeta_1 = \zeta_2$. This proves the assertion 3, and the rest is obvious (cf. Remark 2.3). Q.E.D.

Remark 2.10. In 2.1, if we assume $m_1, m_2 \geq 1$ and that there is a Du Val member $C \subset D \in |-K_X|$ whose minimal resolution has the dual configuration

$$\circ - \cdots - \circ - C' - \circ - \cdots - \circ,$$

then Theorem 2.2 still holds. In this case, the axial multiplicity α_i is undefined and Remark 2.4.3 is irrelevant for i such that $m_i = 1$, and most importantly a general member of $|-K_X|$ does not contain C. That is, $X \supset C$ is an easy case of $k1A$.

In Definition 2.8, we assume $m_1, m_2 \geq 1$. This allows us to treat $k2A$ and some easy case of $k1A$ with no changes in our treatment.

§3. A division algorithm

3.1. We maintain the notation and the assumptions of Definition 2.8. We note that if $G_i(0,1) = u^{\alpha_i} v_i^{\rho_i \delta}$ for some unit $v_i \in \mathbb{C}[[u]]$ then replacing x_i, y_i by $x_i v_{3-i}^{-1}, y_i v_{3-i}$, we may assume $G_i(0,1) = u^{\alpha_i}$. In other words, we may further assume

$$G_i(T_1, T_2) = T_1^{\rho_i} + \cdots + u^{\alpha_i} T_2^{\rho_i}$$

without loss of generality.

We will study when $X \supset C \simeq \mathbb{P}^1$ is a flipping nbd.

Definition 3.2. In addition to the above $G_1(T_1, T_2)$, $G_2(T_1, T_2)$, we introduce $G_i(T_1, T_2)$ $(i = 3, 4)$ as follows:

$$G_i(T_1, T_2) := G_{i-2}(u^{\alpha_{i-2}} T_2, T_1)/u^{\alpha_{i-2}} \in \mathbb{C}[[u]][T_1, T_2] \quad (i = 3, 4).$$

We note that $G_i(T_1, T_2)$ is homogeneous of degree $\rho_i = \rho_{i-2}$ and is of the form

$$G_i(T_1, T_2) = T_1^{\rho_i} + \cdots + u^{\alpha_i} T_2^{\rho_i},$$

where $\alpha_i = \alpha_{i-2}(\rho_{i-2} - 1)$. We remark that $G_i \not\equiv T_1^{\rho_i} \mod (u)$ if $\rho_{i-2} = 1$.

For a positive integer a and an integer x, let x umod a be the integer $y \in [1, a]$ such that $y \equiv x \mod a$. For arbitrary $i \in \mathbb{Z}$, we use the following notation:

$$G_i := G_{i \text{ umod } 4}, \ \rho_i := \rho_{i \text{ umod } 4}, \ \alpha_i := \alpha_{i \text{ umod } 4}, \ \alpha_{i,2} := \alpha_{i \text{ umod } 2}.$$

We note then the obvious $\rho_i = \rho_{i-2}$ and the following formula

$$G_i(T_1, T_2) = G_{i-2}(u^{\alpha_{i-2,2}} T_2, T_1)/u^{\alpha_{i-2}} \quad (\forall i). \tag{3.1}$$

Let $d(n) \in \mathbb{Z}$ $(n \in \mathbb{Z})$ be a sequence determined by

$$d(1) = m_1,\ d(2) = m_2,\ d(n+1) + d(n-1) = \delta\rho_n d(n) \quad (\forall n).$$

Let $e(n) \in \mathbb{Z}$ $(n \in \mathbb{Z})$ be another sequence determined by

$$e(0) = 0, e(1) = -\alpha_1, e(2) = -\alpha_2,\ e(3) = 0,$$

$$e(n+1) + e(n-1) = \delta\rho_n e(n) + \delta\alpha_{n-2} - \alpha_{n-1,2}\ (n \neq 1, 2).$$

Let $\varepsilon := (\rho_1\rho_2\delta)^2 - 4\rho_1\rho_2$, the discriminant of the quadratic form $q(x_1, x_2) := \rho_1 x_1^2 - \rho_1\rho_2\delta x_1 x_2 + \rho_2 x_2^2$.

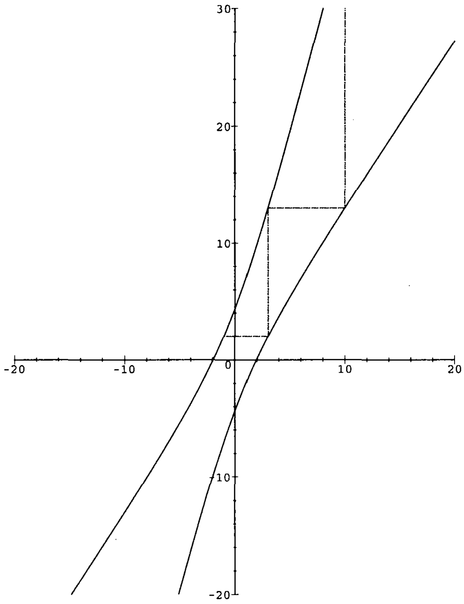
Lemma 3.3. *Let δ, ρ_n be as above and let $\varepsilon := (\rho_1\rho_2\delta)^2 - 4\rho_1\rho_2$. Let $x(n) \in \mathbb{Q}$ be a sequence for $n \in \mathbb{Z}$ such that*

$$x(n) = \delta\rho_{n-1}x(n-1) - x(n-2).$$

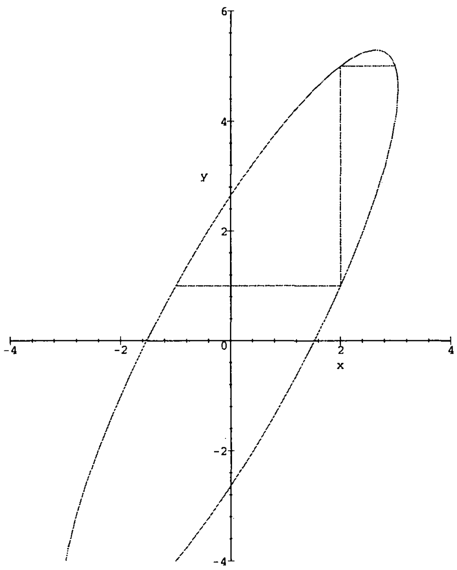
Then we have the following.

1. *If $x(n_0 - 1) > x(n_0 + 1)$ for some n_0 such that $0 \leq x(n_0)$, then $x(n-1) > x(n+1)$ for every $n \geq n_0$ such that $0 \leq x(n_0), \cdots, x(n)$.*
2. *If $x(n_0-1) = x(n_0+1)$ for some n_0, then $x(n_0-n) = x(n_0+n)$ for every n. If furthermore $x(n_0) = x(n_0+2)$ and $(x(n_0), x(n_0+1)) \neq (0,0)$, then $\varepsilon = 0$ and $q(x(1), x(2)) = 0$.*
3. *Assume that $\varepsilon \geq 0$. If $x(n_0 - 1) < x(n_0 + 1)$ (resp. $x(n_0 - 1) > x(n_0+1)$) for some n_0, then $x(n-1) < x(n+1)$ (resp. $x(n-1) > x(n+1)$) for every n.*

Proof. Draw the graph of the conic $C := \{(x_1, x_2) \mid q(x_1, x_2) = A\}$, with some constant A so that $(x(2i-1), x(2i)) \in C$ for some i.



$\varepsilon \geq 0$



$\varepsilon < 0$

The induction formula for $x(n)$ implies that

$$(x(2i+1), x(2i)) \in C \Leftrightarrow (x(2i-1), x(2i)) \in C \quad (\forall i),$$
$$(x(2i+1), x(2i)) \in C \Leftrightarrow (x(2i+1), x(2i+2)) \in C \quad (\forall i).$$

So $(x(2i+1), x(2i)), (x(2i-1), x(2i))$ all lie on C. Except for the second half of the assertion (2), the assertions are obvious from the geometric considerations.

For the second half of the assertion (2), assume that $x(n_0 - 1) = x(n_0 + 1), x(n_0) = x(n_0 + 2)$ and $(x(n_0), x(n_0 + 1)) \neq (0, 0)$. By the first half of the assertion (2), we have $x(n) = x(n+2)$ for all n. By $x(i-1) = x(i+1)$, we see that the line $x_i = x(i)$ is tangent to the conic at the point $P = (x(1), x(2)) \neq (0, 0)$ for $i = 1, 2$. This means that P is a singular point of C, whence C is a double line. Hence $\varepsilon = 0$ and $A = 0$. Q.E.D.

Corollary 3.4. *If we switch* $(a_1, m_1, \alpha_1, \rho_1, x_1, y_1, G_1)$ *and* $(a_2, m_2, \alpha_2, \rho_2, x_2, y_2, G_2)$*, then* $(\alpha_n, d(n), e(n))$ *and* $(\alpha_{3-n}, d(3-n), e(3-n))$ *are switched for all* n*. Modulo this switching, we may assume that* $d(1) > d(3)$*.*

Proof. The first assertion is obvious. We note that $d(1) = m_1 > 0$ and $d(2) = m_2 > 0$. Thus we are also done if $d(1) > d(3)$ or if $d(0) > d(2)$ by Lemma 3.3.1. So we may assume that $d(1) \leq d(3)$ and $d(0) \leq d(2)$.

By $\Delta > 0$, we have $(d(0), d(1)) \neq (d(2), d(3))$ by Lemma 3.3.2. Hence we have either $d(1) < d(3)$ or $d(0) < d(2)$.

If we switch the two sets as above, we have either $d(2) < d(0)$ or $d(3) < d(1)$ after the switch. Thus $d(1) > d(3)$ by Lemma 3.3.1. Q.E.D.

Lemma 3.5. *Assume that* $d(1) > d(3)$ (*cf. Corollary* 3.4) *and that* $\varepsilon < 0$. *Then* $d(k) < 0$ *for some* $k \leq 5$.

Proof. By $\varepsilon = \rho_1\rho_2(\delta^2\rho_1\rho_2 - 4) < 0$, we have $\delta = 1$ and $\rho_1\rho_2 \leq 3$.

Assume first that $\rho_1 = 1$ and $\rho_2 \leq 3$. By $d(3) = \rho_2 m_2 - m_1 < m_1$, we have $\rho_2 m_2 < 2m_1$. Thus the lemma follows from

$$\begin{aligned} d(5) &= \rho_2(\rho_2 - 2)m_2 - (\rho_2 - 1)m_1 \\ &< (\rho_2 - 2)2m_1 - (\rho_2 - 1)m_1 \\ &= (\rho_2 - 3)m_1 \leq 0. \end{aligned}$$

Assume next that $\rho_2 = 1$ and $\rho_1 \leq 3$. By $d(3) = m_2 - m_1 < m_1$, we have $m_2 < 2m_1$, and we are done by

$$\begin{aligned} d(5) &= (\rho_1 - 2)m_2 - (\rho_1 - 1)m_1 \\ &< (\rho_1 - 2)2m_1 - (\rho_1 - 1)m_1 \\ &= (\rho_1 - 3)m_1 \leq 0. \end{aligned}$$

Q.E.D.

Remark 3.6.

1. We note that $e(4) = \delta\alpha_1 > 0$, $e(5) = (\delta^2\rho_2 - 1)\alpha_1 + \delta\alpha_2 > 0$ and

$$e(6) = (\delta^2\rho_2 + \rho_1 - 3)\delta\rho_1\alpha_1 + (\delta^2\rho_1 - 1)\alpha_2.$$

   In particular, $e(6) \leq 0$ implies that $\delta = 1, \rho_1 = 1$ and $\rho_2 = 1, 2$.
2. If we set

$$e_0(n) := -\alpha_{n-2}/\rho_n \ (\forall n),$$

   then $e_0(n)$'s satisfy the conditions for $e(n)$ except for the values of $e(0), \cdots, e(3)$. Therefore if we put $e_1(n) = e(n) - e_0(n)$, then

$$e_1(0) = \alpha_2/\rho_2, e_1(1) = -\alpha_1/\rho_1, e_1(2) = -\alpha_2/\rho_2, e_1(3) = \alpha_1/\rho_1$$

   and the following induction formula holds.

$$e_1(n+1) + e_1(n-1) = \delta\rho_n e_1(n) \quad (n \neq 1, 2).$$

Corollary 3.7. *Assume that* $\varepsilon \geq 0$. *Then* $e_1(n) > e_1(n-2)$ *for all* n; $e(n) > 0$ *for all* $n \geq 4$; $e(n) \geq \alpha_1 + \alpha_2$ *for all* $n \geq 7$.

Proof. We have $e_1(2) = -\alpha_2/\rho_2$ and $e_1(4) = \delta\alpha_1 + \alpha_2/\rho_2$. If we temporarily mean the coefficient of α_i with the subscript α_i, then we have $e_1(4)_{\alpha_i} > e_1(2)_{\alpha_i}$ for $i = 1, 2$. By $\varepsilon \geq 0$, we can apply Lemma 3.3 to $e_1(n)_{\alpha_i}$ and obtain $e_1(n)_{\alpha_i} > e_1(n-2)_{\alpha_i}$ for all $n \geq 4$ and $i = 1, 2$. Since $e_0(n)$ depends only on $n \mod (4)$, we have $e(n) \geq e(n-4) + \alpha_1 + \alpha_2$ for all $n \geq 6$. Also by $e_0(n) \in \alpha_{n,2} \cdot [-1, 0]$, we have $e(n)_{\alpha_i} - e(n-2)_{\alpha_i} > 0$ (resp. ≥ 0) for $i \not\equiv n$ (resp. $i \equiv n$) $\mod (2)$ if $n \geq 4$. In other words, we have $e(n) \geq e(n-2) + \alpha_{n+1,2}$ for all $n \geq 4$.

By $e(3) = 0$ and $e(4) = \delta\alpha_1$, we have $e(5) \geq \alpha_2$, $e(6) \geq \alpha_1$ and $e(n) \geq \alpha_1 + \alpha_2$ $(n \geq 7)$. Q.E.D.

Corollary 3.8. *Assume that* $d(1) > d(3)$ (*cf. Corollary* 3.4). *Let* k *be the smallest integer* ≥ 3 *such that* $d(k) \leq 0$. (*The integer* k *exists by Lemma* 3.3.) *Then* $e(n) > 0$ *if* $4 \leq n \leq k + 1$.

Proof. We note that $e(4), e(5) > 0$ by Remark 3.6.1. Thus we are done if $k \leq 4$. If $\varepsilon \geq 0$, then $e(n) > 0$ for all $n \geq 4$ by Corollary 3.7. Thus we are also done if $\varepsilon \geq 0$.

Thus we may assume that $\varepsilon < 0$ and $d(4) > 0$. Hence $k = 5$ by Lemma 3.5. It is enough to derive a contradiction assuming $e(6) \leq 0$. By Remark 3.6.1, we have $\delta = \rho_1 = 1$ and $\rho_2 = 1, 2$. We have

$$m_1 > d(3) = \rho_2 m_2 - m_1 > 0,$$

$$d(4) = d(3) - m_2 = (\rho_2 - 1)m_2 - m_1 > 0.$$

From the second equation, we have $\rho_2 = 2$. We have $m_1 > m_2$ from the first and $m_2 > m_1$ from the second. This is a contradiciton. Q.E.D.

Definition 3.9. Let $S_i = (x_i = 0)/\Gamma$ and $D = (z = 0)/\Gamma$ be $\mathbb{Q}$-Cartier Weil divisors on X by Theorem 2.9.2, then we have $-a_1 S_1 + (m_2 - a_2)S_2 + D \sim 0$, $m_1 D \sim \delta S_2$ and $m_2 D \sim \delta S_1$.

Then we introduce the following sections and divisors.

$$\begin{aligned}
F_0 &:= y_1 \in H^0(X, \mathcal{O}(L_0)), \text{where} \quad L_0 := \delta\rho_1 L_1 - L_2,\\
F_1 &:= x_2 \in H^0(X, \mathcal{O}(L_1)), \text{where} \quad L_1 := S_2,\\
F_2 &:= x_1 \in H^0(X, \mathcal{O}(L_2)), \text{where} \quad L_2 := S_1,\\
F_3 &:= y_2 \in H^0(X, \mathcal{O}(L_3)), \text{where} \quad L_3 := \delta\rho_2 L_2 - L_1.
\end{aligned}$$

We note that the formulas

$$F_{n-1}F_{n+1} = G_n(z^{d(n)}, F_n^\delta) \qquad (n = 1, 2)$$

can be rewritten in the form

$$F_{n-1}F_{n+1} = G_{n-2}(F_n^\delta, z^{d(n)}u^{e(n)})u^{\alpha_n} \qquad (n = 1, 2).$$

by the formula (3.1).

These L_i and F_i are extended as follows.

Theorem 3.10. *Let $n_0, n_1 \in \mathbb{Z}$ be such that $n_0 \le 1$, $2 \le n_1$ and*

$$\begin{aligned} d(n) > 0 &\quad \text{if } n \in [n_0, n_1], \\ e(n) > 0 &\quad \text{if } n \in [n_0, n_1] \setminus [0,3]. \end{aligned}$$

Then $L_0, \cdots, L_3$ and $F_0, \cdots, F_3$ can be extended to divisors L_n and $F_n \in H^0(X, \mathcal{O}(L_n))$ for $n \in [n_0 - 1, n_1 + 1]$ such that the following hold.

0_n. *$L_{n-1} + L_{n+1} = \delta\rho_n L_n$, if $n \in [n_0, n_1]$.*

1_n. *F_n, F_{n-1} are relatively prime on X (that is, $\{F_n = F_{n-1} = 0\}$ contains no divisors on X), if $n \in [n_0, n_1 + 1]$.*

2_n. *F_n, zu are relatively prime on X, if $n \in [n_0 - 1, n_1 + 1]$.*

3_n. $$F_{n-1}F_{n+1} = \begin{cases} G_n(z^{d(n)}, F_n^\delta) = G_{n-2}(F_n^\delta, z^{d(n)}u^{e(n)})u^{\alpha_n} \\ (n = 1, 2), \\ G_{n-2}(F_n^\delta, z^{d(n)}u^{e(n)}) \\ (n \ne 1, 2), \end{cases}$$

if $n \in [n_0, n_1]$.

Proof. By Corollary 3.4, we only need to consider $n \ge 2$. Thus we set $n_0 = 1$ and use induction on n_1.

The theorem is obvious if $n_1 = 2$. Assume that $n_1 \ge 3$ and let $n = n_1 \ge 3$. By the induction hypotheses, it is enough to define L_{n+1} by the assertion 0_n, construct F_{n+1} satisfying the assertion 3_n and prove the assertions 1_{n+1} and 2_{n+1}.

We will construct F_{n+1} satisfying 3_n using 1_{n-1}, 2_{n-1}, 3_{n-2} and 3_{n-1}. During this proof, $\equiv$ denotes the congruence modulo the ideal $F_{n-1}\mathbb{C}[[u]][F_{n-3}, F_{n-2}, F_{n-1}, F_n, z]$ unless otherwise mentioned. We first claim that

$$F_{n-2}^\delta F_n^\delta \equiv z^{d(n-2)}u^{e(n-2)+\alpha_{n-2,2}} \cdot z^{d(n)}u^{e(n)}.$$

We note that the claim is obvious for $n = 3$ by $F_1^\delta F_3^\delta \equiv z^{\delta\rho_2 d(2)}$. If $n \ge 4$, the claim follows from

$$\begin{aligned} F_{n-2}^\delta F_n^\delta &= G_{n-3}(F_{n-1}^\delta, z^{d(n-1)}u^{e(n-1)})^\delta \\ &\equiv z^{\delta\rho_{n-3}d(n-1)}u^{\delta\rho_{n-3}e(n-1)+\delta\alpha_{n-3}} \\ &= z^{d(n-2)}u^{e(n-2)+\alpha_{n-2,2}} \cdot z^{d(n)}u^{e(n)}. \end{aligned}$$

In the following, we use a temporary notation that # denotes any sufficiently large integer. Let

$$M := F_{n-2}^{\delta\rho_{n-2}} G_{n-2}(F_n^\delta, z^{d(n)}u^{e(n)})z^{\#}u^{\#}.$$

Then by the above claim, we have the following.

$$\begin{aligned} M &= G_{n-2}(F_{n-2}^{\delta}F_n^{\delta}, F_{n-2}^{\delta}z^{d(n)}u^{e(n)})z^{\#}u^{\#} \\ &\equiv G_{n-2}(z^{d(n-2)}u^{e(n-2)+\alpha_{n-2,2}}, F_{n-2}^{\delta})z^{\#}u^{\#} \\ &= G_{n-4}(F_{n-2}^{\delta}, z^{d(n-2)}u^{e(n-2)})z^{\#}u^{\#} \\ &= F_{n-1}F_{n-3}z^{\#}u^{\#} \\ &\equiv 0. \end{aligned}$$

Thus $G_{n-2}(F_n^{\delta}, z^{d(n)}u^{e(n)})$ (or $G_n(z^{d(n)}, F_n^{\delta})$ if $n = 1, 2$) vanishes on the divisor $(F_{n-1} = 0)$ by 1_{n-1} and 2_{n-1}. Hence we obtain $F_{n+1} \in H^0(X, \mathcal{O}(L_{n+1}))$ as claimed.

We will prove 1_{n+1} and 2_{n+1} using 2_n and 3_n. We see that $F_{n+1} = zu = 0$ implies $F_n = 0$ by the formula 3_n. Indeed one can use $d(n), e(n) > 0$ if $n \geq 4$ and $d(3) > 0$ and $G_1(T_1, T_2) \equiv T_1^{\rho_1} \mod (u)$ if $n = 3$. Thus by 2_n, F_{n+1} and zu are relatively prime on X, which is 2_{n+1}. $F_n = F_{n+1} = 0$ implies $zu = 0$. So by 2_n, F_n, F_{n+1} are relatively prime on X, which is 1_{n+1}. Q.E.D.

Definition 3.11. By Corollary 3.4, we will assume that $d(1) > d(3)$. Let $k \geq 3$ be the smallest integer such that $d(k) \leq 0$, which exists by Corollary 3.3.1. Then we have

$$\begin{aligned} d(1), d(2), \cdots, d(k-1) &> 0, \\ e(4), \cdots, e(k+1) &> 0, \end{aligned}$$

by Corollary 3.8.

By Theorem 3.10, $\mathbb{Q}$-Cartier Weil divisors L_i and sections $F_i \in H^0(X, \mathcal{O}_X(L_i))$ $(i = 0, \cdots, k)$ satisfy the following.

0. $L_{n-1} + L_{n+1} = \delta\rho_n L_n$, if $1 \leq n \leq k-1$.
1. F_n, F_{n-1} are relatively prime on X if $1 \leq n \leq k$.
2. F_n, zu are relatively prime on X, if $0 \leq n \leq k$.
3. $F_{n-1}F_{n+1} = \begin{cases} G_n(z^{d(n)}, F_n^{\delta}) = G_{n-2}(F_n^{\delta}, z^{d(n)}u^{e(n)})u^{\alpha_n} \\ (n = 1, 2), \\ G_{n-2}(F_n^{\delta}, z^{d(n)}u^{e(n)}) \\ (n \neq 1, 2), \end{cases}$
   if $1 \leq n \leq k-1$.

We then introduce the modified sequence $d^*(n)$ for the uniform treatment of F_n as follows.

$$d^*(n) = \begin{cases} d(n) & (n \leq k), \\ -d(n-2) & (n \geq k+1). \end{cases}$$

The following is one of the key results that the exceptional locus C of X is a set-theoretic complete intersection of two divisors: $F_k = F_{k+1} = 0$ (cf. [4, 20.11]).

Theorem 3.12. *Under the notation and the assumptions of Definition* 3.11, *we have*

$$F_{k+1} := \frac{G_{k-2}(F_k^\delta z^{-d(k)}, u^{e(k)})}{F_{k-1}} = \frac{G_{k-2}(F_k^\delta, z^{d(k)} u^{e(k)}) z^{-\rho_{k-2} d(k)}}{F_{k-1}}$$

belongs to $H^0(X, \mathcal{O}(L_{k+1}))$, *where* $L_{k+1} := -L_{k-1}$.

Furthermore, F_k *and* F_{k+1} *satisfy the following.*

1. $C = \{F_{k+1} = F_k = 0\}$ *as a set.*
2. $C = \{F_{k+1} = u = 0\}$ *as a set.*

Proof. The proof that F_{k+1} is a regular section of $\mathcal{O}(L_{k+1})$ is similar to the one for F_{n+1} in Theorem 3.10, and we omit it.

The assertion 1 is immediately reduced to 2. Indeed $F_{k+1} = F_k = 0$ implies $u = 0$ by the definition of F_{k+1} (note that $e(k) > 0$ if $k \geq 4$). It remains to prove the assertion 2.

Let $F_n|_H$ denote the restriction of F_n to H and $(F_n|_H)$ the divisor defined by $F_n|_H = 0$. We note

$$\begin{array}{rcllll} D \cap H & = & \ell_1 & +\ell_2 & +C \\ (F_1|_H) & = & & m_2\rho_2\ell_2 & \\ (F_2|_H) & = & m_1\rho_1\ell_1 & & \\ (F_3|_H) & = & m_2\rho_2\ell_1 & & +C. \end{array}$$

We claim

$$(3.2) \qquad (F_n|_H) \equiv \rho_{n-1} d(n-1)\ell_1 \mod \mathbb{Z}C \quad \text{for } n \in [2, k].$$

We prove the claim by induction on n, where the cases $n = 2, 3$ are checked. Assume that the claim is proved up to n ($\leq k-1$). By Definition 3.11, we have

$$\begin{aligned} (F_{n+1}|_H) &\equiv (\delta\rho_{n-2}\rho_{n-1} d(n-1) - \rho_{n-2} d(n-2))\ell, \\ &\equiv \rho_n d(n)\ell_1 \mod \mathbb{Z}C. \end{aligned}$$

Thus the claim is proved. We then have

$$\begin{aligned} (F_{k+1}|_H) &\equiv \rho_{k-2}(\delta \cdot (F_k|_H) - d(k)D) - (F_{k-1}|_H) \\ &\equiv 0 \mod \mathbb{Z}C. \end{aligned}$$

Hence $C = \{F_{k+1} = u = 0\}$, and we are done. Q.E.D.

There is another important division.

Theorem 3.13. *Under the notation and the assumptions of Theorem* 3.12, *we have*

$$\begin{aligned} F_{k+2} &:= \frac{G_{k-1}(F_{k+1}^{\delta} z^{d(k-1)}, u^{e(k+1)})}{F_k} \\ &= \frac{G_{k-1}(F_{k+1}^{\delta}, z^{d^*(k+1)} u^{e(k+1)}) z^{-\rho_{k-1} d^*(k+1)}}{F_k} \end{aligned}$$

belongs to $H^0(X, \mathcal{O}(L_{k+2}))$, *where* $L_{k+2} := -L_k$.

Furthermore, $C = \{F_{k-1} = F_{k+2} = u = 0\}$.

Proof. We will closely follow the proof for Theorem 3.10. In this proof, $\equiv$ denotes the congruence modulo $F_k\mathbb{C}[[u]][F_{k-2}, F_{k-1}, F_k, F_{k+1}, z]$.

By the induction formula of $e(n)$, we have:

$$\begin{aligned} F_{k-1}^{\delta} F_{k+1}^{\delta} z^{d(k-1)} &= G_{k-2}(F_k^{\delta} z^{-d(k)}, u^{e(k)})^{\delta} z^{d(k-1)} \\ &\equiv z^{d(k-1)} u^{\delta\rho_{k-2} e(k) + \delta\alpha_{k-2}} \\ &= u^{e(k+1)} \cdot z^{d(k-1)} u^{\alpha_{k-1,2} + e(k-1)}. \end{aligned}$$

Thus for

$$M := F_{k-1}^{\delta\rho_{k-1}} G_{k-1}(F_{k+1}^{\delta} z^{d(k-1)}, u^{e(k+1)}) u^{\#},$$

we have the following.

$$\begin{aligned} M &= G_{k-1}(F_{k-1}^{\delta} F_{k+1}^{\delta} z^{d(k-1)}, F_{k-1}^{\delta} u^{e(k+1)}) u^{\#} \\ &\equiv G_{k-1}(z^{d(k-1)} u^{\alpha_{k-1,2} + e(k-1)}, F_{k-1}^{\delta}) u^{\#} \\ &= G_{k-3}(F_{k-1}^{\delta}, z^{d(k-1)} u^{e(k-1)}) u^{\#} \\ &= F_k F_{k-2} u^{\#} \\ &\equiv 0. \end{aligned}$$

Since $F_{k-1}u, F_k$ are relatively prime on X by Theorem 3.10, we see that $G_{k-1}(F_{k+1}^{\delta} z^{d(k-1)}, u^{e(k+1)})$ is divisible by F_k and that F_{k+2} is a regular section of L_{k+2}.

For the last assertion, we borrow the notation and the argument in the proof of the previous theorem 3.12. We saw $(F_{k-1}|_H) \subset C \cup \ell_1$ there. By the formula (3.2), we have

$$\begin{aligned} (F_{k+2}|_H) &\equiv \rho_{k-1}(\delta(F_{k+1}|_H) + d(k-1)(D \cap H)) - (F_k|_H) \\ &\equiv \rho_{k-1} d(k-1) - \rho_{k-1} d(k-1) \equiv 0 \quad \operatorname{mod} \mathbb{Z}C. \end{aligned}$$

This means $(F_{k+2}|_H) \subset C \cup \ell_2$, which proves the claim. Q.E.D.

The following are elementary properties which are immediate to check.

Proposition 3.14. *We have*

1. $d^*(n)L_{n+1} - d^*(n+1)L_n \sim 0$ *is a generating relation for* L_n, L_{n+1} *in* Pic X *if* $1 \leq n \leq k+1$,
2. $(L_n \cdot C) = d^*(n)/(m_1 m_2)$ *if* $1 \leq n \leq k+2$.

Proof. The case $n = 1$ of the first assertion is on $m_1 S_1 \sim m_2 S_2$. If $iS_1 - jS_2 \sim 0$ then it is Cartier at P_1. Hence $m_1 | i$ and $iS_1 - jS_2$ is a multiple of $m_1 S_1 - m_2 S_2$. If $2 \leq n \leq k-1$, then we are done by the change of the basis, $L_{n+1} \sim \delta\rho_n L_n - L_{n-1}$:

$$\begin{aligned} d(n)L_{n+1} - d(n+1)L_n &\sim d(n)(\delta\rho_n L_n - l_{n-1}) - d(n+1)L_n \\ &\sim d(n-1)L_n - d(n)L_n. \end{aligned}$$

The cases $n = k, k+1$ follow from the case $n = k-1$. The first assertion is thus proved. The second assertion follows from the first because $(L_1 \cdot C) = 1/m_2$ and $(L_2 \cdot C) = 1/m_1$. Q.E.D.

Proposition 3.15. *Let* $c(i)$ $(i \in [1, k+2])$ *be a sequence determined by*

i. $c(1) = a_1$, $c(2) = m_2 - a_2$,
ii. $c(n+1) = \delta\rho_n c(n) - c(n-1)$ *for* $n \in [2, k-1]$,
iii. $c(k+1) = -c(k-1)$, $c(k+2) = -c(k)$.

Then we have

1. $-K_X \sim c(n)L_{n+1} - c(n+1)L_n$ *if* $1 \leq n \leq k+1$,
2. $c(n)d^*(n+1) - c(n+1)d^*(n) = \delta$ *for all* $n \in [1, k+1]$,
3. $c(n)$ *and* $d^*(n)$ *are relatively prime for all* $n \in [1, k+2]$, *and*
4. $-d^*(n)K_X \sim \delta L_n$ *for all* $n \in [1, k+1]$.

Proof. The case $n = 1$ of the assertion 1 follows from $-K_X \sim a_1 L_2 - (m_2 - a_2)L_1$. One can check the case $n \in [2, k-1]$ inductively by using $L_{n+1} \sim \delta\rho_n L_n - L_{n-1}$ $(n \in [2, k-1])$ as follows:

$$\begin{aligned} c(n)L_{n+1} - c(n+1)L_n &\sim c(n)(\delta\rho_n L_n - L_{n-1}) - c(n+1)L_n \\ &\sim c(n-1)L_n - c(n)L_{n-1} \sim -K_X. \end{aligned}$$

The cases $n = k, k+1$ are equivalent to the case $n = k-1$ because $L_{k+1} \sim -L_{k-1}$ and $L_{k+2} \sim -L_k$. This proves the assertion 1.

By the induction formula, we immediately see that the value

$$c(n)d(n+1) - c(n+1)d(n) = c(n-1)d(n) - c(n)d(n-1)$$

does not depend on n and hence equal to $\delta = c(1)d(2) - c(2)d(1)$, which proves the assertion 2.

Let $\gcd(n) = (c(n), d(n))$, then $\gcd(n)$ divides δ. By the induction formula, we see that $d(i) \equiv -d(i-2)$ and $c(i) \equiv -c(i-2)$ modulo $\gcd(n)$ for all i. This implies that $\gcd(n)$ divides $\gcd(1)$ or $\gcd(2)$ depending on the parity of n. Since $\gcd(1) = (a_1, m_1) = 1$ and $\gcd(2) = (a_2, m_2) = 1$, we get $\gcd(n) = 1$, the assertion 3.

The assertion 4 follows from Proposition 3.14.

$$\begin{aligned} -d^*(n)K_X &\sim c(n)d^*(n)L_{n+1} - c(n+1)d^*(n)L_n \\ &\sim (c(n)d^*(n+1) - c(n+1)d^*(n))L_n = \delta L_n. \end{aligned}$$

Q.E.D.

§4. Contractions and flips

In this section, we give an explicit description of the contractions and the flips using the divisions in Section 3.

Although we work on the specific model X, our description can treat arbitrary extremal nbd of type $k2A$ by passing to the formal completion or the associated analytic space by Theorems 2.2 and 2.9.

First by Theorem 3.12 alone (without the further division), we can decide exactly when $X \supset C$ is a flipping nbd as follows.

Corollary 4.1. *Let $X \supset C \simeq \mathbb{P}^1$ be the scheme introduced in* 3.1. *Under the notation and the assumptions of Theorem* 3.12, *we have*

1. *If $d(k) < 0$ then the formal completion $\hat{X}$ and the associated algebraic space of $X \supset C$ are flipping nbds.*
2. *If $d(k) = 0$ then $\hat{X} \supset C$ is not a flipping nbd. Indeed, C is a fiber of a divisorial contraction of $\hat{X}$ (or the algebraic space X) and $\{F_{k+1} = 0\}$ is the exceptional divisor for the contraction.*

Proof. By Theorem 3.12, C is a set-theoretic complete intersection of two Cartier divisors $N_1 := a(F_k = 0) \sim aL_k$ and $N_2 := a(F_{k+1} = 0) \sim aL_{k+1}$ for some integer $a > 0$.

Assume that $d(k) < 0$. Then $-N_1$ and $-N_2$ are ample on C by $(L_k \cdot C), (L_{k+1} \cdot C) < 0$. Then the defining ideal J of $N_1 \cap N_2$ has the property that J/J^2 is ample on $C = \operatorname{Supp}(\mathcal{O}_X/J)$. Thus $C \subset \hat{X}$ can be contracted by [2, 6.2] and the associated algebraic space can be contracted by [2, 3.1]. Since $(K_X \cdot C) < 0$, these are flipping contractions.

Assume thet $d(k) = 0$. Then $aL_k \sim 0$ and $(L_{k+1} \cdot C) < 0$. Then $F_k^a : X \to \mathbb{A}^1$ induces, on the divisor N_2, a morphism $g : N_2 \to \mathbb{A}^1$ such that $C = g^{-1}(0)$ as a set. We note that $\mathcal{O}_{N_2}(-N_2)$ is g-ample by

$(L_{k+1} \cdot C) < 0$. Thus we can similarly see that N_2 can be contracted to a curve such that C is one of its set-theoretic fiber by a birational contraction of the formal completion (and also the associated algebraic space) of $X \supset C$. We note that $-K_X$ is relatively ample and N_2 is exceptional with respect to the contraction. Q.E.D.

The extremal contraction of $\hat{X}$ is expressed as Spec $H^0(\hat{X}, \mathcal{O}_{\hat{X}})$ (or its formal scheme version). We give here an explicit construction using the further division Theorems 3.12 and 3.13.

Definition 4.2. Let

$$\begin{array}{lll} y_1' & := F_{k+2} & \in H^0(X, \mathcal{O}(-L_k)), \\ x_2' & := F_{k+1} & \in H^0(X, \mathcal{O}(L_{k+1})), \\ x_1' & := F_k & \in H^0(X, \mathcal{O}(L_k)), \\ y_2' & := F_{k-1} & \in H^0(X, \mathcal{O}(-L_{k+1})), \\ z & & \in H^0(X, \mathcal{O}(c(k)L_{k+1} + c(k-1)L_k)), \end{array}$$

on which we have the Γ-action defined in Remark 2.7. We rewrite the action as follows.

By $\mathbb{Z}L_1 + \mathbb{Z}L_2 = \mathbb{Z}L_k + \mathbb{Z}L_{k+1} \subset \text{Pic } X$ (Proposition 3.14.1), we set

$$\Gamma' := \text{Hom } (\mathbb{Z}L_k + \mathbb{Z}L_{k+1}, \mathbb{C}^*) = \text{Hom } (\mathbb{Z}L_1 + \mathbb{Z}L_2, \mathbb{C}^*) = \Gamma,$$

and $\gamma' \in \Gamma'$ acts on $H^0(X, \mathcal{O}(L_i))$ as the multiplication by $\gamma'(L_i) \in \mathbb{C}^*$. Let $m_1' := d(k-1) > 0$, $m_2' := -d(k) \geq 0$. Then we have $m_2' L_{k+1} \sim m_1' L_k$ (Proposition 3.14) and hence

$$\Gamma' = \{\gamma' = (\gamma_1', \gamma_2') \in (\mathbb{C}^*)^2 \mid (\gamma_1')^{m_1'} = (\gamma_2')^{m_2'}\}.$$

The Γ-action is equivalent to the Γ'-action given by

$$\gamma'(x_i', z, y_i', u) = ((\gamma_i')x_i', (\gamma_1')^{c(k-1)}(\gamma_2')^{c(k)}z, (\gamma_i')^{-1}y_i', u).$$

Γ' acts on the ring $R' := \mathbb{C}[[u]][x_1', y_1', x_2', y_2', z]$, the ideal

$$I := (x_1'y_1' - G_{k-1}((x_2')^{\delta} z^{m_1'}, u^{e(k+1)}), x_2'y_2' - G_{k-2}((x_1')^{\delta} z^{m_2'}, u^{e(k)}))$$

and the scheme $W' :=$ Spec R'/I'. We note that it is easy to check that W' is a complete intersection and is an integral domain by Proposition 4.8 and that W' is normal by the Jacobian criterion. Let

$$(R'/I')^{\Gamma'} := \{r \in R'/I' \mid \gamma' r = r\}$$

and $Y :=$ Spec $(R'/I')^{\Gamma'}$ with the origin 0. Because of the construction, we have a natural morphism $\pi : X \to Y$.

Theorem 4.3. *There is an open subset $U \ni 0$ of Y such that $\pi : \pi^{-1}(U) \to U$ is either a flipping contraction with C the only flipping curve (the case $m'_2 > 0$), or a divisorial contraction with $(F_{k+1} = 0)$ the only exceptional divisor (the case $m'_2 = 0$).*

Proof. First of all, W (cf. Definition 2.8) and W' are birationally equivalent because of the inductive formulas in Defintion 3.11 and Theorems 3.12 and 3.13. The birational map is Γ-equivariant as explained in Definition 4.2. Because of these, it is easy to see that π is birational.

Next, we claim that $\pi^{-1}(0) = C$ as a set, and prove it in two cases.

Case 1 $(m'_2 > 0)$. For arbitrary i, j, we have $u, (x'_i)^a(y'_j)^b \in (R'/I')^{\Gamma'}$ for some positive integers a, b depending on x'_i, y'_j. Thus

$$C \subset \pi^{-1}(0) \subset \{x'_1 = x'_2 = 0\} \cup \{y'_1 = y'_2 = u = 0\}.$$

Thus by Theorems 3.12 and 3.13, we have $\pi^{-1}(0) = C$ as a set.

Case 2 $(m'_2 = 0)$. We have $(y'_1)^a, (x'_1)^a, x'_2 y'_2, u \in (R'/I')^{\Gamma'}$ for some positive integer a. Thus

$$C \subset \pi^{-1}(0) \subset \{x'_1 = x'_2 = 0\} \cup \{y'_1 = y'_2 = u = 0\}.$$

The rest is the same as Case 1. This proves the claim.

By [9], π can be extended to a proper birational morphism $\bar{\pi} : \bar{X} \to Y$. Then we have $\pi^{-1}(0) = C$. Indeed, by the normality of Y, $\pi^{-1}(0)$ is a connected set containing C as a connected component.

Thus it is enough to set $U = Y \setminus \bar{\pi}(\bar{X} \setminus X)$ to make π proper above it. Shrink U further so that L_{k-1} is π-ample over U.

Assume that $m'_2 > 0$. Then $C = \pi^{-1}(0)$ is a set-theoretic complete intersection of two π-negative divisors $(F_k = 0)$ and $(F_{k+1} = 0)$. Every π-exceptional curve $\subset \pi^{-1}(U)$ is contained in these divisors. Thus C is the only π-exceptional curve $\subset \pi^{-1}(U)$.

Assume next that $m'_2 = 0$. In this case, the arguments are similar to those in the proof of Corolary 4.1.2. $C = (F_k = 0) \cap (F_{k+1} = 0)$ being π-exceptional and $F_k \sim 0$ imply that F_{k+1} is contracted by π. Then, $-F_{k+1}$ being π-ample implies that F_{k+1} contains all the curves contracted by π. Q.E.D.

We will closely study the divisorial contraction or the flip as follows.

Definition 4.4. Let $a'_1 := c(k-1)$ umod (m'_1) (cf. Definition 3.2), and if $m'_2 > 0$ then we also let $a'_2 := c(k)$ umod (m'_2). Since $(c(i), d(i)) = 1$ by Proposition 3.15, we have $(m'_i, a'_i) = 1$ and $0 < a'_i \leq m'_i$ if $m'_i > 0$.

Theorem 4.5. *With the above notation and assumptions, assume further that* $d(k) = 0$. *Then* $m'_1 = d(k-1) = \delta = \gcd(m_1, m_2)$, $c(k) = -1$ *and we have a terminal singularity of index* m'_1,

$$0 \in Y \simeq (\xi, \eta, \zeta, u; \xi\eta - G_{k-1}(\zeta^{m'_1}, u^{e(k+1)}))/\mathbb{Z}_{m'_1}(1, -1, a'_1, 0),$$

where the π*-fundamental set is the curve* $\{\zeta = G_{k-2}(\xi^{m'_1}, u^{e(k)}) = 0\}/\mathbb{Z}_{m'_1}$ *under the identification.*

Proof. By the induction formula, we see $\gcd(m_1, m_2) = \gcd(d(i), d(i+1))$ for all i. In particular, we have $d(k-1) = \gcd(m_1, m_2)$. By Proposition 3.15, we have $c(k) = \pm 1$ and $-c(k)d(k-1) = \delta$. Thus $c(k) = -1$ and $m'_1 = d(k-1) = \delta$.

By $d(k) = 0$, we have $\Gamma' = \mathbb{Z}_{m'_1} \times \mathbb{C}^*$ and can obtain the isomorphism by taking the invariants in two steps. We note that $\xi = x'_1$, $\eta = y'_1$ and $\zeta = x'_1 z$. Since the fundamental set on Y is defined by $x'_2 y'_2 = x'_2 z = 0$, we are done by $x'_2 y'_2 = G_{k-2}(\xi^{m'_1}, u^{e(k)})$. Q.E.D.

Definition 4.6. Let

$$X' := (W' \setminus \{x'_1 = x'_2 = 0\})/\Gamma' \supset C' := \{y'_1 = y'_2 = z = u = 0\}/\Gamma'$$

and P'_i the point, $x'_i = y'_1 = y'_2 = z = u = 0$. We note that $C' \simeq \mathbb{P}^1$.

Let $\pi' : X' \to Y$ be the induced morphism.

Theorem 4.7. *With the above notation and the assumptions, assume further that* $d(k) < 0$. *Then we have*

1. X' *is a normal scheme of dimension* 3 *such that* $X' \setminus \{P'_i, P'_2\}$ *is smooth and the germ*

   $$P'_i \in X' \simeq (\xi'_i, \eta'_i, \zeta'_i, u; \xi'_i\eta'_i = G'_i(\zeta'^{m'_i}_i, 1))/\mathbb{Z}_{m'_i}(1, -1, a'_i, 0)$$

   is a terminal singularity of index m'_i *and* $P_i \in C' = \xi'_i$*-axis*$/\mathbb{Z}_{m'_i}$ *under the identification, where*

   $$G'_i(T_1, T_2) := G_{k-i}(T_1, u^{e(k+2-i)}T_2) \quad (i = 1, 2),$$

2. X' *is proper and is the flip of* X *over some open set* $\ni 0$ *of* Y.

Proof. The proof of the first assertion is similar to the one for Theorem 2.9, and we omit it.

As in the proof of Theorem 4.3, we see that π' is a birational morphism. Although X' is proper over X, we only claim it over an open set $\ni 0$. For this we only need to show $(\pi')^{-1}(0) = C'$ as in the proof of Theorem 4.3.

For arbitrary i, j, we have $(x_i')^a(y_j')^b, (x_i')^a z^c \in (R'/I')^{\Gamma'}$ for some positive integers a, b, c depending on x_i', y_j', z. Thus

$$(\pi')^{-1}(0) \subset \{x_1' = x_2' = 0\} \cup \{y_1' = y_2' = z = u = 0\} = C',$$

and the properness is settled.

By the construction of W' and by $C = \{F_k = F_{k+1} = 0\}$ (Theorem 3.12), we have a natural birational morphism $X \setminus C \to X'$. By Theorem 4.3, $X \setminus C \simeq X' \setminus C'$ over an open set $U \ni 0$ of Y. It only remains to show that $(K_{X'} \cdot C') > 0$.

Let $S_1' := (x_1' = 0)/\Gamma'$ be the $\mathbb{Q}$-Cartier divisor on X'. Then $(S_1' \cdot C') > 0$. Since $X \simeq X'$ in codimension 1, we can pull back S_1' and $K_{X'}$ on X' to $L_k \sim (F_k = 0)$ and K_X on X. By $-d(k)K_X \sim \delta L_k$ on X (Proposition 3.15), we have $-d(k)K_{X'} \sim \delta S_1'$. Hence $(K_{X'} \cdot C') > 0$ as required. Q.E.D.

We used the following elementary result in this section. We give a proof for the readers' convenience.

Proposition 4.8. *Let A be an integral domain, and x_1, x_2, u_1, $u_2 \in A$. Assume that x_1, x_2 are prime elements (that is, x_1A, x_2A are non-zero prime ideals) and that (x_1, x_2) is a prime ideal $\neq x_1A, x_2A$. Then*

1. *If $u_1 \notin x_1A$, then $A[y_1]/(x_1y_1 - u_1)$ is an integral domain.*
2. *If $u_1 \notin (x_1, x_2)$ and $u_2 \notin x_2A$, then $A[y_1, y_2]/(x_1y_1 - u_1, x_2y_2 - u_2)$ is an integral domain.*

Proof. Let $P := \{f(y_1) \in A[y_1] \mid f(u_1/x_1) = 0\}$.

We claim that if $f(y_1) = a_ny_1^n + \cdots + a_0 \in P \setminus \{0\}$, then $n > 0$ and $a_n \in x_1A$. Indeed $n \geq \deg f \geq 1$ is obvious, and from

$$a_nu_1^n + x_1(a_{n-1}u_1^{n-1} \cdots + a_0x_1^{n-1}) = 0,$$

we get $a_n \in x_1A$ by $u_1 \notin x_1A$. This proves the claim.

For the assertion 1, it is enough to prove that $P = (x_1y_1 - u_1)$.

Let $f \in P \setminus \{0\}$. By the claim, we can lower $\deg f$ modulo $(x_1y_1 - u_1)$. Hence the assertion 1 is proved by induction on $\deg f$.

For the assertion 2, let $S = A[y_1]/(x_1y_1 - u_1)$, which is an integral domain. We note that $S/x_2S \simeq (A/x_2A)[y_1]/(x_1y_1 - u_1)$ is an integral domain by $u_1 \notin (x_1, x_2)$. We claim that $u_2 \mod x_2S \neq 0$. Indeed $u_2 \mod x_2A$ is a non-zero constant of the integral domain $(A/x_2A)[y_1]$. Hence $u_2 \mod x_2A \notin (A/x_2A)[y_1](x_1y_1 - u_1)$, which proves the claim. Finally applying the assertion 1 on S, we obtain the assertion 2. Q.E.D.

§5. Further discussions

In this section, we consider the case of a base ring which is more general than $\mathbb{C}[[u]]$ in Definition 3.2. We note that, over Spec $\mathbb{Z}$, our finite group action $\mathbb{Z}_m$ is actually the finite multiplicative group scheme action $\mu_n \subset \mathbb{G}_m$, which is linearly reductive over Spec $\mathbb{Z}$. Hence no changes are needed for characteristic ≥ 0.

Definition 5.1. Let (Λ, m_Λ) be a regular local ring and let $u_1, u_2 \in m_\Lambda$ be non-zero elements. Let α_i, m_i, ρ_i be positive integers and $G_i(T_1, T_2) \in \Lambda[T_1, T_2]$ a homogeneous polynomial in T_1 and T_2 of degree ρ_i $(i = 1, 2)$ such that

1. $a_i \leq m_i$ and $(a_i, m_i) = 1$,
2. $\delta := a_1 m_2 + a_2 m_1 - m_1 m_2 > 0$,
3. the coefficient of $T_1^{\rho_i}$ (resp. $T_2^{\rho_i}$) in G_i is 1 (resp. u_1),
4. $\Delta := \rho_1 m_1^2 - \delta \rho_1 \rho_2 m_1 m_2 + \rho_2 m_2^2 > 0$.

By formally writing $\alpha_i = log_u u_i$ (or $u^{\alpha_i} = u_i$) for $i = 1, 2$, Definition 3.2 applies to our case. By Corollary 3.4, we may assume that

5. $d(1) > d(3)$.

Corollary 3.8 implies that

6. $u^{e(n)} \in (u_1, u_2)\Lambda$ if $4 \leq n \leq k + 1$.

Let $R := \Lambda[x_1, y_1, x_2, y_2, z]$ be the Λ-algebra with the Γ-action in Remark 2.7, and let $W =$ Spec R/I be the scheme with the Γ-action, where I is the ideal given by

$$I := (x_1 y_1 - G_1(z^{m_1}, x_2^\delta), x_2 y_2 - G_2(z^{m_1}, x_1^\delta)).$$

As in Definition 2.8, we set

$$X := (W \setminus V(x_1, x_2))/\Gamma \supset C := V(y_1, y_2, z, m_\Lambda)/\Gamma \simeq \mathbb{P}^1_{\text{Spec } \Lambda/m_\Lambda}$$

and $P_i = V(x_i, y_1, y_2, z, m_\Lambda)/\Gamma \simeq$ Spec Λ/m_Λ. Let L_i be the $\mathbb{Q}$-Cartier divisor classes and $F_i \in H^0(X, \mathcal{O}(L_i))$ be the sections as in Definition 3.9.

Theorem 5.2. *$L_0, \cdots, L_3$ and $F_0, \cdots, F_3$ can be extended to $\mathbb{Q}$-Cartier divisor classes L_i and sections $F_i \in H^0(X, \mathcal{O}(L_i))$ for $i \in [0, k+2]$ such that the following hold.*

0_n. $L_{n-1} + L_{n+1} = \begin{cases} \delta \rho_n L_n & (n \leq k-1) \\ 0 & (n = k, k+1), \end{cases}$ *if* $n \in [1, k+1]$.

1_n. *F_n, F_{n-1} are relatively prime on X if $n \in [1, k+2]$.*

2_n. *$F_n, z u_1 u_2$ are relatively prime on X, if $n \in [0, k+2]$.*

$$3_n.\ F_{n-1}F_{n+1} = \begin{cases} G_{n-2}(F_n^\delta z^{-d^*(n)}, u^{e(n)}) \\ = G_{n-2}(F_n^\delta, z^{d^*(n)}u^{e(n)})z^{-\rho_n d^*(n)} & (n = k, k+1), \\ G_{n-2}(F_n^\delta, z^{d^*(n)}u^{e(n)}) & (2 < n < k), \\ G_n(z^{d(n)}, F_n^\delta) \\ = G_{n-2}(F_n^\delta, z^{d(n)}u^{e(n)})u^{\alpha_n} & (n = 1, 2), \end{cases}$$

if $n \in [1, k+1]$.

Our argument here is slightly stronger than those for Theorems 3.10, 3.12 and 3.13, since we introduce the intermediate schemes X^i and study them closely.

Lemma 5.3. *Let the notation and the assumptions be as in Theorem* 5.2. *For* $i \in [1, k]$, *let* $R^i := \Lambda[F_{i-1}, \cdots, F_{i+2}, z]$ *be the polynomial ring with* 5 *variables and* $I^i \subset R^i$ *the ideal generated by the relations* 3_i *and* 3_{i+1}.

As in Definition 2.8, *let* $X^i := (\mathrm{Spec}\ R^i/I^i \setminus \{F_i = F_{i+1} = 0\})/\Gamma$, *and let* L_j^i $(j \in [i-1, i+2])$ *be the* $\mathbb{Q}$*-Cartier divisor class on* X^i *induced by* F_j. *By the condition corresponding to* 0_n, *we define* L_j^i *for all* $j \in [0, k+2]$. *Let* B_1^i (*resp.* B_2^i) *be the closed subset of* X^i *defined by* $F_{i-1} = F_i = 0$ (*resp.* $F_{i+1} = F_{i+2} = 0$).

Then for every $i \in [1, k]$, *we have the following.*

1. R^i/I^i *is a normal domain of complete intersection,*
2. $\mathrm{codim}_{X^i}(B_j^i) \geq 2$ *for every* $j = 1, 2$.

By the relations $3_1, \cdots, 3_n$, $R^1/I^1, \cdots, R^{n-1}/I^{n-1}$ *are all birational to each other. For every* $i \in [2, k]$, *the birational map* $X^{i-1} \dashrightarrow X^i$ *induces*

3. *an isomorphism* $X^{i-1} \setminus B_2^{i-1} \simeq X^i \setminus B_1^i$,
4. *the identification* $L_j^{i-1} = L_j^i$, *which is simply denoted by* L_j, *and*
5. $H^0(X^{i-1}, \mathcal{O}(L_j)) = H^0(X^i, \mathcal{O}(L_j))$ *for all* j.

Proof. It is easy to check that R^i/I^i is a complete intersection integral domain by Proposition 4.8 and $u_1u_2 \neq 0 \in \Lambda$. The normality can be checked by the Jacobian criterion at codimension 1 points, which is the assertion 1. Again using $u_1u_2 \neq 0 \in \Lambda$, one can easily check the assertion 2.

We now regard F_{i+2} as a rational function in $F_{i-2}, \cdots, F_{i+1}, z$. We can see that the regular section

$$F_{i+2}F_i \in H^0(X^{i-1}, \mathcal{O}(L_{i+2}^{i-1} + L_i^{i-1}))$$

satisfies the condition

$$F_{i+2}F_i(F_{i-1}zu_1u_2)^{\#} \in F_i\Lambda[F_{i-2}, F_{i-1}, F_i, F_{i+1}, z]$$

from the relations 3_{i+1}, 3_i and 3_{i-1} by pure computation, where # denotes an arbitrarily large positive integer. Indeed the computation was carried out in the proof of Theorem 3.10 with $n = i+1 \le k-1$. Since the computation is similar in other cases, we omit the computation. This means that the regular section $F_{i+2}F_i$ vanishes on the divisor $(F_i = 0) \sim L_i^{i-1}$, whence $F_{i+1} \in H^0(X^{i-1}, \mathcal{O}(L_{i+1}^{i-1}))$. Hence $(F_{i-1}, F_i, F_{i+1}, F_{i+2}, z)$ induce a morphism $X^{i-1} \setminus B_2^{i-1} \to X^i \setminus B_1^i$.

The inverse $X^i \setminus B_1^i \to X^{i-1} \setminus B_2^{i-1}$ can be constructed similarly from the assertion:

$$F_{i-2}F_i(F_{i+1}zu_1u_2)^{\#} \in F_i\Lambda[F_{i-1}, F_i, F_{i+1}, F_{i+2}, z].$$

Indeed, we can prove this using 3_{i-1}, 3_i and 3_{i+1} by the computation similar to the above. The rest are obvious. Q.E.D.

Proof of Theorem 5.2. By 0_n, we define L_j's. By Lemma 5.3, we have the extension F_j ($j \in [0, k+2]$) satisfying 3_n ($n \in [1, k+1]$) by $F_j \in H^0(X, \mathcal{O}(L_j)) = H^0(X^i, \mathcal{O}(L_j))$ for some $i \in [1, k]$ such that $j \in [i-1, i+2]$.

By Lemma 5.3, the assertions 1_n and 2_m can be examined on X^i such that $n, m \in [i-1, i+2]$.

On X^i, we know that $B_1^i = \{F_{i-1} = F_i = 0\}$, $\emptyset = \{F_i = F_{i+1} = 0\}$, $B_2^i = \{F_{i+1} = F_{i+2} = 0\}$ are of codimension ≥ 2 on X^i. This proves 1_n. The computation of 2_n can be done through a simple but tedious computation, which we omit. Q.E.D.

Remark 5.4. For the family of surfaces $\pi : X \to \operatorname{Spec} \Lambda$, we have a divisorial contraction or a flipping contraction depending on whether $d(k) = 0$ or not (Corollary 4.1). It is not difficult to obtain an analogue of Theorem 2.2 for a multi-parameter analytic deformation space of H and analogues of Theorems 3.12 and 3.13 for Λ, and furthermore to carry out a detailed computation as in Sections 2 and 4.

For instance, C need not be the only contractible curve over $[m_\Lambda]$ because we do not assume $G_i \equiv T_1^{\rho_i} \bmod m_\Lambda$ in Definition 5.1.3. The contractible curves over $[m_\Lambda]$ are contained in $F_4 = 0$, which follows from $F_k = F_{k+1} = u_1 = u_2 = 0$ through the relations in Theorem 5.2.3.

Using such G_i, we can systematically construct reducible flipping curves.

Remark 5.5. An interesting problem in order to understand flips is to find the generators of the graded ring $\oplus_{\nu \in Z} H^0(X, \mathcal{O}(\nu K_X))$ or some of its variants. We note that our z, $F_0, \cdots, F_{k+2}$ are a part of the key generators.

It is possible to carry out further divisions to get F_i for $i < 0$ and $i > k+2$. The former case was treated in Theorem 3.10. The latter case corresponds to the case $i < 0$ for X' in Theorem 4.7, or we can continue the division imitating the arguments in Theorem 3.10.

However this immediate generalization does not give the right homogeneous elements as pointed out by M. Reid. He has been proposing a more general division [10] using pfaffians.

Our standpoint is that, with our easier divisions, we can determine many of the structures of the flips.

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Local Structure of an Elliptic Fibration

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Abstract.

We classify all the projective elliptic fibrations defined over a unit polydisc whose discriminant loci are contained in a union of coordinate hyperplanes, up to the bimeromorphic equivalence relation. If the monodromies are unipotent and if general singular fibers are not of multiple type, then we can construct relative minimal models.

Contents

§ Introduction

The aim of this paper is to describe the local structure of a projective elliptic fibration over a complex analytic manifold which is a smooth morphism outside a normal crossing divisor of the base manifold. An elliptic fibration is a proper surjective morphism $f\colon X \to S$ of complex analytic varieties whose general fibers are nonsingular elliptic curves. It is not necessarily a flat morphism. We consider the case S is a unit polydisc

$$\Delta^d := \{(t_1, t_2, \ldots, t_d) \in \mathbb{C}^d \mid |t_i| < 1 \text{ for all } i\}$$

and suppose further that f is smooth over $S^\star = S \setminus D$, where D is the normal crossing divisor $D := \{t_1 t_2 \cdots t_l = 0\}$ for some $1 \leq l \leq d$. We are interested in what kind of such elliptic fibrations exist, up to the bimeromorphic equivalence relation over S. For the purpose, it is important to understand the notion of period mappings and monodromies. A smooth fiber is an elliptic curve isomorphic to a torus $\mathbb{C}/(\mathbb{Z}\omega + \mathbb{Z})$, where the period $\omega \in \mathbb{H} := \{\omega \in \mathbb{Z} \mid \operatorname{Im}\omega > 0\}$ is determined up to the action of $\mathrm{SL}(2,\mathbb{Z})$. By considering the ambiguity, we have a period mapping (function) $\omega\colon U \to \mathbb{H}$ from the universal covering space $U \simeq \mathbb{H}^l \times \Delta^{d-l}$ of $S^\star$ into the upper half plane $\mathbb{H}$, and a monodromy representation $\rho\colon \pi_1(S^\star) \to \mathrm{SL}(2,\mathbb{Z})$ such that for $\gamma \in \pi_1(S^\star)$ and $z \in U$,

$$\omega(\gamma z) = \frac{a_\gamma \omega(z) + b_\gamma}{c_\gamma \omega(z) + d_\gamma}, \quad \text{where} \quad \rho(\gamma) = \begin{pmatrix} a_\gamma & b_\gamma \\ c_\gamma & d_\gamma \end{pmatrix}.$$

The period function and the monodromy representation define a polarized variation of Hodge structures of rank two and weight one [G] (cf. §1). Thus f induces a variation of Hodge structures $H(f)$ on $S^\star$. We classify all the variations of Hodge structures over $S^\star$ in §2 and §3. After fixing a variation of Hodge structures H, we shall classify elliptic fibrations by determining the following set $\mathcal{E}^+(S,D,H)$: Let $(f\colon X \to S, \phi)$ be a pair of a projective elliptic fibration f smooth over $S^\star$ and an isomorphism $\phi\colon H \simeq H(f)$ as variations of Hodge structures. Two such pairs $(f_1\colon X_1 \to S, \phi_1)$ and $(f_2\colon X_2 \to S, \phi_2)$ are called bimeromorphically equivalent over S, if there is a bimeromorphic mapping $\varphi\colon X_1 \cdots\to X_2$ over S such that the induced isomorphism $\varphi^*\colon H(f_2) \simeq H(f_1)$ satisfies $\phi_1 = \varphi^* \circ \phi_2$. The set $\mathcal{E}^+(S,D,H)$ is defined to be the set of bimeromorphic equivalence classes of all such pairs. For any variation of Hodge structures H on $S^\star$, we have a projective elliptic fibration $p\colon B(H) \to S$ with $H \simeq H(p)$ which admits a section $S \to B(H)$. This is uniquely determined up to the bimeromorphic equivalence relation over S and is called the *basic elliptic fibration* associated with H. It determines a distinguished element of $\mathcal{E}^+(S,D,H)$, and thus it is also called the *basic member*. For the study of other elements of $\mathcal{E}^+(S,D,H)$, we first consider a special case where the following two conditions are satisfied:

- The monodromy matrices $\rho(\gamma)$ are all unipotent;
- The fibration f admits a meromorphic section over a neighborhood of any point of $S \setminus Z$ for a Zariski closed subset Z of codimension greater than one.

Under the situation, our Theorems 4.3.1 and 4.3.2 state that f is bimeromorphically equivalent to a basic elliptic fibration. Further the basic elliptic fibration is a smooth elliptic fibration or a *toric model* which is constructed by the method of toroidal embedding theory ([KKMS]).

These are minimal elliptic fibrations. Next, for a general elliptic fibration $f\colon X \to S$, we have a finite ramified covering of the form

$$\tau\colon T = \Delta^l \times \Delta^{d-l} \longrightarrow S = \Delta^l \times \Delta^{d-l}$$
$$(\theta_1, \theta_2, \dots, \theta_l, t_{l+1}, t_{l+2}, \dots, t_d) \mapsto (\theta_1^{m_1}, \theta_2^{m_2}, \dots, \theta_l^{m_l}, t_{l+1}, t_{l+2}, \dots, t_d)$$

for some $m_i \geq 1$ such that $f_T\colon X \times_S T \to T$ satisfies the above two conditions. Hence f_T is bimeromorphically equivalent to $p_T\colon B(H) \times_S T \to T$. Therefore the bimeromorphic equivalence class of f is determined by a meromorphic action of the Galois group $\mathrm{Gal}(\tau)$ on p_T. The basic elliptic fibration p is a group object over $S^\star$. Hence the sheaf $\mathfrak{S}_{H/S}$ of germs of meromorphic sections of p is a sheaf of abelian groups. Since we always fix a marking of variation of Hodge structures, the action of an element of the Galois group is written as the translation by a meromorphic section of p_T. Therefore, $\mathcal{E}^+(S, D, H)$ is identified with the inductive limit of Galois cohomology groups

$$\varinjlim H^1(\mathrm{Gal}(\tau), H^0(T, \mathfrak{S}_{H_T/T})),$$

where H_T is the pullback of H on $T^\star := \tau^{-1}(S^\star)$ and the limit is taken over all such coverings τ described as above. We shall describe the limits and the actions of Galois groups on basic elliptic fibrations in §§5–7.

Background

The study of elliptic fibration was developed by Kodaira's work on elliptic surfaces, i.e., elliptic fibrations over curves, in [Kd1] and [Kd2].

In the work, first the classification of singular fibers of minimal elliptic fibrations is obtained by a calculation of intersection numbers of irreducible components. The following types of singular fibers are listed (cf. Figure 3 and Figure 4): ${}_m\mathrm{I}_a$, I_a^*, II, II*, III, III*, IV, IV*, where $a \geq 0$ and $m \geq 1$.

Next, the basic elliptic fibration is constructed from the data of period function and monodromy representation defined on a Zariski-open subset of the base curve, which were essentially called *functional* and *homological* invariants, respectively. The construction is natural over the Zariski-open subset. To obtain an extension of the basic fibration to the whole base curve, we may assume that the curve is a unit disc Δ and the Zariski-open subset is the punctured disc $\Delta^\star = \Delta \setminus \{0\}$. If the monodromy matrix is trivial, then the period mapping is single-valued and thus the smooth basic fibration is naturally extended. In the case the monodromy is unipotent of infinite order, i.e., I_a, $(a > 0)$, Kodaira made a technical construction of the basic fibration. But now it can be replaced by the method of toroidal embedding theory ([KKMS]). For other cases of monodromy matrices, a branched covering $\Delta \to \Delta$ reduces to the study of actions of the Galois group on the above basic fibrations. The quotient spaces, their desingularizations and further their (relative) minimal models are obtained by careful calculations. The result corresponds to the classification of non-multiple singular fibers. Kodaira proved that every elliptic surface without multiple fibers is a twist of the basic member by the translation by local sections. Thus the set of such fibrations is essentially identified with the cohomology group $H^1(S, \mathfrak{S})$, where S is the base curve and $\mathfrak{S}$ is the sheaf of germs of sections of the basic fibration.

For general elliptic fibrations, Kodaira showed in [Kd2] that every multiple fibers are obtained from an elliptic surface without multiple fibers by *logarithmic transformations.*

His theory contained not only local but also global properties of elliptic fibrations. This was generalized to the study of degenerations of abelian varieties, where a particular open subset of the basic elliptic fibration is considered as the *Néron model.* In the purely algebraic situation, birational equivalence classes are determined only by smooth parts, or more strictly, by generic fibers. Hence the study of multiple fibers is replaced by that of Galois cohomology groups $H^1(\mathrm{Gal}(L/K), E(L))$, where E is an elliptic curve with origin (hence is fixed a group structure) over a field K, L/K is a Galois extension, $E(L)$ is the group of L-valued points.

In the analytic situation, Kawai ([Kwi]) succeeded in generalizing the construction of basic members to the case of elliptic fibrations over surfaces, where the resulting ambient spaces were not necessarily nonsingular. Ueno ([U]) obtained their desingularizations, which however are not distinguished models in their bimeromorphic equivalence classes. To obtain a good model, we had to wait the development of the minimal model theory.

Their basic members were also determined by functional and homological invariants. Now we know that giving these invariants is equivalent to giving a polarized variation of Hodge structures of weight one and rank two (cf. [G]). This is also equivalent to giving a *Weierstrass model* [Ny4]. It was proved that every elliptic fibration admitting a section is bimeromorphically equivalent to a Weierstrass model. Before [Ny4], Miranda ([Mi]) studied the desingularizations of Weierstrass models over surfaces, where he obtained flat minimal models after changing the base surface by blow-ups.

Compared with the progress in the study of elliptic fibration admitting a global section, few results were known for general elliptic fibrations. For example, some interesting examples are found in the case multiple fibers appear. Especially, Fujimoto ([Fm]) constructed them by a generalization of logarithmic transformation. Some of them induce examples of deformations of complex manifolds under which the plurigenera are not invariant.

The minimal model theory of projective varieties (cf. [KMM]) together with its generalization to complex analytic varieties [Ny3, §4] allows us to study the minimality of elliptic fibrations. For the classification of elliptic fibrations, it is essential to determine the relative minimal models. Since Mori ([Mo]) has proved the three-dimensional flip conjecture, there exist relative minimal models for a given projective elliptic fibration over a surface. These minimal models usually have terminal singularities and are not uniquely determined in their bimeromorphic equivalence classes. However, every two bimeromorphically equivalent minimal models are connected by a sequence of flops [Kw4] and [Kl2]. We have studied elliptic fibrations over surfaces by applying the minimal model theory in [Ny5], whose Main Theorem corresponds to Theorems 4.3.1 and 4.3.2.

Previous version

The author intended to write this paper as "Elliptic fibrations over surfaces II," that is a continuation of [Ny5]. He considered the cases of non-unipotent monodromies and of multiple fibers, by taking a suitable finite Kummer covering $\Delta^2 \to \Delta^2$. The study was reduced to that of Galois actions on special basic fibrations. The classification of the actions was to be the contents of "Part II." But a few months later, the author obtained a generalization of Main Theorem of [Ny5] to the higher dimensional case. The three-dimensional flip theorem ([Mo]) was essential in the proof in [Ny5]. He found a new idea to prove it without using the flip theorem. By the progress, the classification of actions of covering groups is also extended to higher dimensional case. This is essentially reduced to calculating Galois cohomology groups. The first version [Ny7] appeared in a preprint series of Department of Mathematics, Faculty of Science, University of Tokyo in 1991.

The construction of the first version is as follows: §§1–3 are devoted to some basics on elliptic fibrations. The basic properties on period functions for smooth elliptic fibrations are explained in §1. Especially, variations of Hodge structures, basic elliptic fibrations and their torsors are discussed. In §2, monodromy representations over a product of punctured discs are studied. The types of monodromies are classified into: I_0, $\mathrm{I}_0^{(*)}$, $\mathrm{II}^{(*)}$, $\mathrm{III}^{(*)}$, $\mathrm{IV}_+^{(*)}$, $\mathrm{IV}_-^{(*)}$, $\mathrm{I}_{(+)}$, $\mathrm{I}_{(+)}^{(*)}$ (cf. Table 2). The sets of smooth elliptic fibrations over the base with a fixed variation of Hodge structures are calculated in each type. Thus all the smooth elliptic fibrations over the product of punctured discs are described. However, the calculation of Galois cohomology groups contains some errors. The canonical extension of the variation of Hodge structures to $S = \Delta^d$ is explained in §3. We have some results on locally projective or Kähler elliptic fibrations from fundamental isomorphisms Corollary 3.2.1 for direct image sheaves of canonical sheaves and from torsion free theorems for the higher direct image sheaves. Examples of non-Kähler elliptic fibrations are given. In §4, toric models are constructed, which are basic elliptic fibrations corresponding to variations of Hodge structures with non-trivial but only unipotent monodromies. These are given by the method of toroidal embedding theory. Similar constructions appeared in the study of degeneration of abelian varieties (cf. [Nk], [Nm]). The last part of §4 is devoted to proving the main results Theorems 4.3.1 and 4.3.2, which are generalizations of Main Theorem in [Ny5]. In §5, elliptic fibrations over curves are studied from a viewpoint of toric models. In §6, the case of finite monodromies is studied and possible elliptic fibrations are described as the quotient space of basic smooth fibration by an action of Galois group. In §7, the case of infinite monodromies are treated. However, the calculation of some Galois cohomology groups in the case of $\mathrm{I}^{(*)}_{(+)}$ is not clearly mentioned. It had two appendices, where elliptic fibrations over surfaces are studied by the method of minimal model theory. In Appendix A, the study of elliptic fibrations over a surface is shown to be reduced, in some sense, to that of *standard* elliptic fibrations. They are relative minimal fibrations with only equi-dimensional fibers and satisfy more conditions. In Appendix B, the good minimal model conjecture is proved for compact Kähler threefolds admitting elliptic fibrations. This is a generalization of the unpublished paper [Ny6].

Present version

The author left the first version untouched about five years. In the period, he received a paper [DG] of Dolgachev and Gross, where elliptic fibrations over surfaces are studied in the purely algebraic situation. For the use of étale cohomology theory, they looked carefully at the models obtained by Miranda ([Mi]) and calculated similar Galois cohomology groups. In our first version, the author did not understand the importance of describing the groups. In the study of the groups, he found a kind of generalization of étale cohomology theory, by which we can consider global structures of elliptic fibrations in the analytic situation. This is named the ∂-étale cohomology theory and is written in [Ny8] in 1996. The results on Galois cohomology groups in this paper are also derived from [Ny8], since the structure of $\mathcal{E}^+(S, D, H)$ is studied in more general case. Under the influence of [Ny8], the author decided to write a new version of this paper. The preparation however has been slow.

The major difference between previous and present versions is as follows: §0 is added. Here an elementary descent theory, G-linearization, and torsors are discussed. §§0–3 are still preliminary sections. In §2, we change the base space $S^\star$ to be a product of punctured discs and polydiscs, i.e., $S^\star = (\Delta^\star)^l \times \Delta^{d-l}$. We divide the case $\mathrm{I}^{(*)}_{(+)}$ into three subcases (cf. Table 3). By a similar method to [Ny8], we calculate the related group cohomologies in each type. Similarly to the previous version, all the smooth elliptic fibrations over $S^\star$ are described. In §3, we explain more on the canonical extension of a variation of Hodge structures of rank two, weight one defined on $S^\star$ to $S := \Delta^d$. In particular, we determine possible period functions. In §4, we add a discussion on a kind of generalization of torsors in §4.1, which are torsors in a sense of bimeromorphic geometry. It is important, since our basic fibrations are not group objects, but have group structures in meromorphic sense. We also give an extension Theorem 4.1.1 of smooth projective elliptic fibrations. The description of toric models in §§4.2 and 4.3 are essentially same as before, except the following two things:

- The proof of Proposition 4.2.12 is replaced. Original argument is combinatorial and the new one is an application of the theory of elliptic surfaces.
- Another proof of Corollary 4.3.3 is added, in which the toric models are not used. This is based on an argument of Viehweg in [V, 9.10].

§§5–7 are devoted to the calculation of the set $\mathcal{E}^+(S, D, H)$ and the description of any projective elliptic fibrations over $S = \Delta^d$ with discriminant locus D. In §5, we consider not only the case S is a curve but the case $l = 1$, i.e., the discriminant locus D is a smooth divisor. We have unique minimal models in this case. We treat the case H has a finite monodromy group in §6 and the remaining case in §7. The calculation is much simpler than that in the previous version. In Appendix B, B.8 is corrected.

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Notation

We use the same notation as in [Ny3], [Ny4], and [KMM], and need the following in addition.

Complex analytic space: We treat only complex analytic spaces which are Hausdorff and have countable open bases. A complex analytic variety means an irreducible and reduced complex analytic space. A complex analytic manifold means a nonsingular complex analytic variety. Every complex analytic manifolds should be connected.

Polydisc: Let Δ^d be the d-dimensional unit polydisc

$$\{(t_1, t_2, \dots, t_d) \in \mathbb{C}^d \mid |t_i| < 1 \text{ for } 1 \le i \le d\}$$

with respect to a coordinate system $(t_1, t_2, \dots, t_d)$. The coordinate hyperplane $\{t_i = 0\}$ is often denoted by D_i. We denote by $\Delta^\star$ the punctured disc $\Delta \setminus \{0\}$. Thus $(\Delta^\star)^l \times \Delta^{d-l} \simeq \Delta^d \setminus \bigcup_{i=1}^l D_i$.

Exponential mapping: We denote the function $\exp(2\pi\sqrt{-1}z)$ by $e(z)$ for $z \in \mathbb{C}$. The universal covering space of the punctured disc $\Delta^\star$ is isomorphic to the upper half plane $\mathbb{H} := \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$. The function $z \mapsto e(z)$ induces a universal covering mapping $\mathbb{H} \to \Delta^\star$.

Pullback of open subsets: Let $f\colon V \to W$ be a morphism of complex analytic spaces. For an open subset $U \subset W$, we shall denote the pullback $f^{-1}(U)$ by $V_{|U}$.

Morphisms over a fixed base space: Let $f\colon X \to S$ and $g\colon Y \to S$ be morphisms of complex analytic spaces. A morphism

$h: X \to Y$ is called a morphism over S, if $f = g \circ h$. A complex analytic space over S is a morphism $f: X \to S$ from a complex analytic space.

Duals: Dual objects are indicated by $\vee$. For example, we denote by $\mathcal{F}^\vee$ the dual $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$ for an $\mathcal{O}_X$-module $\mathcal{F}$ of a complex analytic space X.

Elimination: For a sequence of letters $a_1, a_2, \ldots, a_n$ and for $1 \leq i \leq n$, if we eliminate a_i, then we denote the remaining sequence by $a_1, a_2, \ldots, \widehat{a_i}, \ldots, a_n$.

Special complex numbers: We write $\boldsymbol{\omega} := e(2\pi\sqrt{-1}/3)$ and $\boldsymbol{i} := \sqrt{-1}$.

Fibrations and Projective morphisms: A proper surjective morphism $f: X \to S$ of complex analytic varieties is called a *fibration* if X and S are normal and if any fibers of f are connected. A proper morphism f is called a *projective* morphism if there exists an f-ample line bundle (invertible sheaf) on X (cf. [Ny3]). $f: X \to S$ is called a *locally projective morphism*, if there is an open covering $\bigcup S_\lambda = S$ such that $X_{|S_\lambda} \to S_\lambda$ is a projective morphism for any λ. Note that the composite of two projective morphisms is not always projective. This is only a locally projective morphism. The composite of two locally projective morphisms is not always locally projective.

Minimal models: A fibration $f: X \to S$ is said to be a *minimal* fibration (or a *minimal model*) over a point $P \in S$, if the following conditions are all satisfied:

(1) f is a (locally) projective morphism;
(2) X has only terminal singularities;
(3) X is $\mathbb{Q}$-factorial over P (cf. [Ny3, §4]);
(4) the canonical divisor (class) K_X of X is f-nef over P, i.e., the intersection numbers $(K_X \cdot C) \geq 0$ for any irreducible curves C such that $f(C) = P$.

Although, sometimes, a fibration $f: X \to S$ is said to be a minimal fibration even if it does not satisfy the condition (3).

Elliptic fibrations: A fibration $f: X \to S$ is called an *elliptic fibration* if general fibers are nonsingular elliptic curves. In this paper, we shall treat mainly the projective elliptic fibrations (cf. §3.3). If $\dim S = 1$, every elliptic fibration is a locally projective morphism. But there is an elliptic fibration over Δ^2 whose central fiber is a Hopf surface (cf. Examples 3.3.4 and 3.3.5).

Sections: Let $f: X \to Y$ be a proper surjective morphism between complex analytic varieties. A closed subvariety $\Sigma \subset X$ is called a *section* of f if f induces an isomorphism $\Sigma \simeq Y$. If $\Sigma \to Y$ is a bimeromorphic morphism, then Σ is called by a *meromorphic section.* Furthermore a morphism $\sigma\colon Y \to X$ such that $f \circ \sigma = \mathrm{id}_Y$ is also called a section of f.

Variation of Hodge structures: (cf. §1.1.) Since we consider elliptic fibrations, we treat variations of Hodge structures only of rank two and of weight one. Further we always assume such variation of Hodge structures admits a polarization defined over $\mathbb{Z}$.

§0. Descent theory

0.1. General situation

Let X be a complex analytic space and let $\sigma\colon G \times X \to X$ be a left action of a discrete group G. Suppose that the action is properly discontinuous. For the quotient morphism $\tau\colon X \to Y := G\backslash X$, there is a canonical morphism $G \times X \ni (g, x) \mapsto (gx, x) \in X \times_Y X$. It is an isomorphism if the action is fixed point free. For a complex analytic spaces Z, let $F(Z)$ be one of the following categories:

(1) The category of sheaves of abelian groups over Z;
(2) The category of complex analytic spaces over Z.

Then we have pullback functors $\tau^*\colon F(Y) \to F(X)$, $\sigma^*, p_2^*\colon F(X) \to F(G \times X)$, and

$$p_{31}^*,\, p_{32}^*,\, p_{21}^*\colon F(G \times X) \to F(G \times G \times X),$$

where $p_2\colon G \times X \to X$ denotes the second projection and p_{31}, p_{32}, p_{21} the morphisms $G \times G \times X \to G \times X$ defined by

$$p_{31}\colon (g, h, x) \mapsto (gh, x),\ p_{32}\colon (g, h, x) \mapsto (h, x),\ p_{21}\colon (g, h, x) \mapsto (g, hx).$$

Suppose that there is an isomorphism $\psi\colon \xi \simeq \tau^*\eta$ for objects $\xi \in F(X)$ and $\eta \in F(Y)$. Then we have a natural isomorphism

$$\phi := \phi(\eta, \psi) := p_2^*(\psi)^{-1} \circ \sigma^*(\psi)\colon \sigma^*\xi \to \sigma^*\tau^*\eta = p_2^*\tau^*\eta \to p_2^*\xi,$$

which satisfies the following cocycle condition:

$$p_{31}^*(\phi) = p_{32}^*(\phi) \circ p_{21}^*(\phi). \tag{0.1}$$

Definition 0.1.1. A pair (ξ, ϕ) consisting of an object $\xi \in F(X)$ and an isomorphism $\phi\colon \sigma^*\xi \to p_2^*\xi$ satisfying the cocycle condition (0.1), is called a *G-equivariant object* of $F(X)$. A morphism $f\colon (\xi_1, \phi_1) \to (\xi_2, \phi_2)$ is defined to be a morphism $f\colon \xi_1 \to \xi_2$ in $F(X)$ such that $\phi_1 \circ p_2^*(f) = \phi_2 \circ \sigma^*(f)$. We denote by $F(X, G)$ the category of G-equivariant objects of $F(X)$.

Let us denote by $L_g\colon X \to X$ the action of $g \in G$. We can identify L_g with the composite $X = \{g\} \times X \subset G \times X \xrightarrow{\sigma} X$. For an isomorphism $\phi\colon \sigma^*\xi \to p_2^*\xi$ in $F(X)$ and for an element $g \in G$, let ϕ_g be the morphism $\phi_g = \phi_{|\{g\}\times X}\colon L_g^*\xi \to \xi$. Then the cocycle condition (0.1) is equivalent to

$$\phi_{gh} = \phi_h \circ L_h^*(\phi_g) \tag{0.2}$$

for any $g, h \in G$. Thus ϕ of a G-equivariant object (ξ, ϕ) is determined by the collection $\{\phi_g\}$ satisfying (0.2).

The natural functor $\tau^*\colon F(Y) \to F(X)$ factors through $F(X, G) \to F(X)$. As in the usual descent theory, we have the following:

Lemma 0.1.2. *Suppose that the action of G on X is free. Then the natural functor $\tau^*\colon F(Y) \to F(X, G)$ gives an equivalence of categories. That is, for a G-equivariant object $(\xi, \phi) \in F(X, G)$, there exist an object $\eta \in F(Y)$ and an isomorphism $\psi\colon \xi \simeq \tau^*\eta$ such that $\phi = \phi(\eta, \psi)$, and furthermore, the pair (η, ψ) is uniquely determined up to the following equivalence relation: $(\eta, \psi) \sim (\eta', \psi')$ if and only if there is an isomorphism $\theta\colon \eta \to \eta'$ such that $\psi' = \tau^*(\theta) \circ \psi$.*

For two G-equivariant objects (ξ_1, ϕ_1) and (ξ_2, ϕ_2) of $F(X)$, the set $\mathrm{Hom}_{F(X)}(\xi_1, \xi_2)$ of morphisms admits a right action of G as follows: For $g \in G$ and a morphism $f\colon \xi_1 \to \xi_2$,

$$f^g := \phi_{2,g} \circ L_g^*(f) \circ \phi_{1,g}^{-1}\colon \xi_1 \xrightarrow{\phi_{1,g}^{-1}} L_g^*\xi_1 \xrightarrow{L_g^*(f)} L_g^*\xi_2 \xrightarrow{\phi_{2,g}} \xi_2.$$

Similarly, for a G-equivariant object (ξ, ϕ), we have a right action of G on the automorphism group $\mathrm{Aut}_{F(X)}(\xi)$.

Lemma 0.1.3. *Let (ξ, ϕ) be a G-equivariant object of $F(X)$. Then the set of isomorphism classes of G-equivariant objects of the form (ξ, ϕ') is identified with the cohomology set $H^1(G, \mathrm{Aut}_{F(X)}(\xi))$, where the action of G on $\mathrm{Aut}_{F(X)}(\xi)$ is determined by ϕ.*

Proof. Let $\phi'_g\colon L_g^*\xi \to \xi$ be the restriction of ϕ' to $\{g\} \times X$ and set $\rho(g) := \phi'_g \circ \phi_g^{-1} \in \mathrm{Aut}_{F(X)}(\xi)$. Then for $g, h \in G$, we have

$$\rho(gh) = \rho(h) \circ \phi_h \circ L_h^*(\rho(g)) \circ \phi_h^{-1} = \rho(h) \circ \rho(g)^h.$$

Thus $\{\rho(g)\}$ defines a cocycle in $Z^1(G, \mathrm{Aut}_{F(X)}(\xi))$. Conversely, for a cocycle $\{\rho(g)\}$, the collection $\{\phi'_g := \rho(g) \circ \phi_g\}$ defines an isomorphism ϕ' satisfying the cocycle condition (0.1). Suppose that two cocycles $\{\rho_1(g)\}$ and $\{\rho_2(g)\}$ define two isomorphisms $\phi_1, \phi_2\colon \sigma^*\xi \to p_2^*\xi$, respectively.

Then (ξ, ϕ_1) is isomorphic to (ξ, ϕ_2) in $F(X, G)$ if and only if $\{\rho_1(g)\}$ and $\{\rho_2(g)\}$ are cohomologous. Q.E.D.

Corollary 0.1.4. *Suppose that the action of G on X is free. Let η be an object of $F(Y)$. Then the set of isomorphism classes of $\eta' \in F(Y)$ admitting an isomorphism $\tau^*\eta \simeq \tau^*\eta'$ is identified with the cohomology set $H^1(G, \mathrm{Aut}(\tau^*\eta))$.*

0.2. G-linearization

Let X, Y, G be same as before. We shall recall the notion of G-linearization (cf. [Mu1]). For a sheaf $\mathcal{F}$ of abelian groups on X, a G-linearization is an isomorphism $\phi\colon \sigma^{-1}\mathcal{F} \to p_2^{-1}\mathcal{F}$ satisfying the cocycle condition $p_{31}^*(\phi) = p_{32}^*(\phi) \circ p_{21}^*(\phi)$. Therefore this is the case $F(Z)$ is the category of sheaves of abelian groups on Z. For two G-linearized sheaves $(\mathcal{F}_1, \phi_1)$ and $(\mathcal{F}_2, \phi_2)$, the tensor product $\mathcal{F}_1 \otimes \mathcal{F}_2$ has a G-linearization $\phi_1 \otimes \phi_2$. A G-linearization ϕ on the sheaf $\mathcal{H}om(\mathcal{F}_1, \mathcal{F}_2)$ is given by

$$\phi\colon \mathcal{H}om(\sigma^{-1}\mathcal{F}_1, \sigma^{-1}\mathcal{F}_2) \ni \alpha \mapsto \phi_2 \circ \alpha \circ \phi_1^{-1} \in \mathcal{H}om(p_2^{-1}\mathcal{F}_1, p_2^{-1}\mathcal{F}_2).$$

As in §0.1, from a G-linearization on $\mathcal{F}$, we have a right action of G on the set $H^0(X, \mathcal{F}) = \mathrm{Hom}_X(\mathbb{Z}_X, \mathcal{F})$. This is called the dual action of G in [Mu1]. Therefore, the direct image sheaf $\tau_*\mathcal{F}$ also admits the right action of G. Let $\mathcal{G}$ be the G-invariant part of $\tau_*\mathcal{F}$, i.e., $\mathcal{G} := \mathcal{H}om_{\mathbb{Z}_Y[G]}(\mathbb{Z}_Y, \tau_*\mathcal{F})$. If the action of G is free, then there is an isomorphism $\mathcal{F} \simeq \tau^{-1}\mathcal{G}$ by Lemma 0.1.2. The set of isomorphism classes of G-linearizations of $\mathcal{F}$ is identified with the cohomology set $H^1(G, \mathrm{Aut}(\mathcal{F}))$ by Lemma 0.1.3. The cohomology groups $H^p(X, \mathcal{F}) \simeq H^p(Y, \tau_*\mathcal{F})$ have also right G-module structures, since so does $\tau_*\mathcal{F}$. Here we recall the following:

Lemma 0.2.1 (Hochschild–Serre spectral sequence). *Suppose that the action of G on X is free. Let $\mathcal{G}$ be a sheaf of abelian groups on Y. Then there is a spectral sequence*:

$$E_2^{p,q} = H^p(G, H^q(X, \tau^{-1}(\mathcal{G}))) \Longrightarrow H^{p+q}(Y, \mathcal{G}).$$

In particular, if $H^i(X, \tau^{-1}(\mathcal{G})) = 0$ for any $i > 0$, then, for all p, we have an isomorphism

$$H^p(G, H^0(X, \tau^{-1}(\mathcal{G}))) \simeq H^p(Y, \mathcal{G}).$$

Next we shall consider the case $F(Z)$ is the category of sheaves of $\mathcal{O}_Z$-modules in §0.1, where $\mathcal{O}_Z$ denotes the structure sheaf. The $\mathcal{O}_X$

has a natural G-linearization which is explicitly written as follows: The isomorphisms $\phi_g\colon L_g^{-1}\mathcal{O}_X \simeq \mathcal{O}_X$ are given by

$$H^0(gU, \mathcal{O}_X) \ni f \mapsto f^g \in H^0(U, \mathcal{O}_X),$$

where $U \subset X$ is an open subset and $f^g(z) := f(gz)$ for $z \in U$. A G-linearization of an $\mathcal{O}_X$-module $\mathcal{F}$ is called $\mathcal{O}_X$-linear if the multiplication $\mathcal{O}_X \times \mathcal{F} \to \mathcal{F}$ is compatible with G-linearizations. Then we have the category of G-linearized $\mathcal{O}_X$-modules. This is identified with the category of $\mathcal{O}_Y$-modules when G acts on X freely. For a G-linearized $\mathcal{O}_X$-module $\mathcal{F}$, the set of isomorphism classes of $\mathcal{O}_X$-linear G-linearizations is identified with the cohomology set $H^1(G, \mathrm{Aut}_{\mathcal{O}_X}(\mathcal{F}))$ by Lemma 0.1.3.

Next we shall consider a special case. Let M be an abelian group. Suppose that X is connected and that there is a G-linearization ϕ on the constant sheaf $M_X := M \otimes \mathbb{Z}_X$ which is different from the trivial G-linearization induced from $M_Y := M \otimes \mathbb{Z}_Y$. The ϕ corresponds to an element of $H^1(G, \mathrm{Aut}(M_X))$ by Lemma 0.1.3. Since X is connected, we have $\mathrm{Aut}(M_X) = \mathrm{Aut}(M)$ and thus G acts trivially on $\mathrm{Aut}(M)$. Therefore, the cohomology set $H^1(G, \mathrm{Aut}(M_X))$ is identified with the set $\mathrm{Hom}_{\mathrm{anti}}(G, \mathrm{Aut}(M))$ of anti-group homomorphisms from G to $\mathrm{Aut}(M)$. Thus M has a right G-module structure, which is nothing but the right module structure of $M = H^0(X, M_X)$ induced from ϕ. If M has a right G-module structure, then there is uniquely a group homomorphism $\rho\colon G \to \mathrm{Aut}(M)$ such that $x^g = \rho(g^{-1})(x)$ for any $x \in M$. The tensor product $M \otimes_{\mathbb{Z}} \mathcal{O}_X$ has a natural G-linearization induced from $M_X \simeq \tau^{-1}M_Y$ and from the natural G-linearization of $\mathcal{O}_X$. We have another G-linearization of $M \otimes_{\mathbb{Z}} \mathcal{O}_X$ from ϕ above. Thus we have:

Lemma 0.2.2. *Suppose that X is connected and let M be an abelian group. Then the set of G-linearizations of the constant sheaf M_X is identified with the set* $\mathrm{Hom}(G, \mathrm{Aut}(M))$ *of group homomorphisms. For a homomorphism $\rho\colon G \to \mathrm{Aut}(M)$, the corresponding G-linearizations of M_X and $M \otimes_{\mathbb{Z}} \mathcal{O}_X$, respectively, are given in the following way*:

$$H^0(gU, M_X) = M \ni x \mapsto \rho(g^{-1})(x) \in M = H^0(U, M_X),$$
$$H^0(gU, M \otimes \mathcal{O}_X) \ni v \mapsto v^g \in H^0(U, M \otimes \mathcal{O}_X),$$

where $U \subset X$ is a connected open subset, $g \in G$ and for $z \in U$,

$$v^g(z) := \rho(g^{-1})v(gz).$$

Suppose further that Y is a connected analytic space and let H be a *locally constant sheaf* with fiber M, i.e., H is isomorphic to the constant sheaf M_Y locally on Y, and that $\tau\colon X \to Y$ is the universal covering space. Then G is isomorphic to the fundamental group $\pi_1(Y, y)$ for a point $y \in Y$ and acts on X freely. Thus there exist an isomorphism $\psi \colon M_X \simeq \tau^{-1}H$ and also a G-linearization $\phi = \phi(H, \psi)$ on M_X. Hence we have a group homomorphism $\rho \colon G = \pi_1(Y, y) \to \mathrm{Aut}(M)$, which is called the *monodromy representation* of H.

0.3. Torsors

Still let X, Y, G be same as in §0.1. We shall consider the case $F(Z)$ is the category of complex analytic spaces over Z. Let $f \colon W \to X$ be a morphism of complex analytic spaces. Suppose that there is an isomorphism $\phi \colon \sigma^*(W) := (G \times X) \times_X W \to p_2^*(W) := (G \times X) \times_X W$ over $G \times X$ satisfying the cocycle condition (0.1). Then the restriction of ϕ to $\{g\} \times X$ defines an isomorphism $\phi_g \colon L_g^* W \simeq W$. These ϕ_g satisfy the cocycle condition (0.2). From ϕ_g, we have the following commutative diagram:

$$\begin{array}{ccccc} W & \xrightarrow{\phi_g^{-1}} & L_g^* W & \longrightarrow & W \\ & & {\scriptstyle L_g^*(f)}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ & & X & \xrightarrow[L_g]{} & X. \end{array}$$

Let $\varphi(g) \colon W \to W$ be the composite of the morphisms appearing at the top sequence in the diagram above. Then we have $\varphi(gh) = \varphi(g) \circ \varphi(h)$. Therefore G acts holomorphically on W from the left and the action is compatible with $W \to X$. Therefore we have the quotient space $V = G \backslash W$ over Y. If G acts on X freely, then so on W. Hence $W \to V$ is étale and $W \simeq V \times_Y X$, in the case.

Next we shall consider a special case. Suppose that the action of G on X is free. Let $B \to Y$ be an analytic space over Y admitting a group structure, i.e., the functor $Z \mapsto \mathrm{Hom}_Y(Z, B)$ from the category of complex analytic spaces over Y to the category of sets factors through the category of groups. Thus the set $B(X/Y) := \mathrm{Hom}_Y(X, B)$ is considered as the group of sections of $B_X := B \times_Y X \to X$. We have a right action of G on the group $B(X/Y) = \mathrm{Hom}_X(Y_X, B_X)$ by §0.1. There is an injection $B(X/Y) \ni \sigma \mapsto \mathrm{tr}(\sigma) \in \mathrm{Aut}_X(B_X)$, where $\mathrm{tr}(\sigma)$ is the left multiplication mapping

$$B_X = B \times_Y X \ni (b, x) \mapsto (\sigma(x)b, x) \in B_X.$$

This injection is a G-linear group homomorphism, i.e, $\mathrm{tr}(\sigma^g) = \mathrm{tr}(\sigma)^g$ for $g \in G$ and $\mathrm{tr}(\sigma_1\sigma_2) = \mathrm{tr}(\sigma_1) \circ \mathrm{tr}(\sigma_2)$. A cocycle $\{\sigma_g\}$ in $Z^1(G, B(X/Y))$

defines an element of $H^1(G, \mathrm{Aut}_X(B_X))$ and determines a smooth morphism $V \to Y$ from the quotient space $V := G\backslash B_X$ by the action:

$$B_X \ni (b, x) \mapsto (\sigma_g(x)b, gx) \in B_X.$$

Then $B \to Y$ acts on $V \to Y$ from right by:

$$V \times_Y B \ni ([b, x], b') \mapsto [bb', x] \in V,$$

where $[b, x]$ denotes the image of $(b, x) \in B_X = B \times_Y X$ under the quotient morphism $B_X \to V$. Furthermore we have a B_X-linear isomorphism $B_X \simeq V_X$.

Definition 0.3.1. A smooth morphism $V \to Y$ is called a *torsor* of $B \to Y$ if $B \to Y$ acts on $V \to Y$ from the right and there exist an open covering $\{Y_\lambda\}$ of Y and B-linear isomorphisms $B_{|Y_\lambda} \simeq V_{|Y_\lambda}$.

The set of isomorphism classes of torsors of $B \to Y$ whose pullbacks to X are trivialized is identified with the cohomology set $H^1(G, B(X/Y))$ by Lemma 0.1.3. The set of isomorphism classes of torsors of $B \to Y$ itself is identified with the cohomology set $H^1(Y, \mathcal{O}(B/Y))$, where $\mathcal{O}(B/Y)$ is the sheaf of germs of sections of $B \to Y$, i.e., $H^0(U, \mathcal{O}(B/Y)) = B(U/Y)$ for open subsets $U \subset Y$. Therefore we have an injection

$$H^1(G, B(X/Y)) \hookrightarrow H^1(Y, \mathcal{O}(B/Y)).$$

As an analogy of Lemma 0.2.1, we see that the injection is extended to a sequence:

$$H^1(G, B(X/Y)) \to H^1(Y, \mathcal{O}(B/Y)) \to H^1(X, \mathcal{O}(B_X/X)),$$

which is exact in the following sense: If an element of $H^1(Y, \mathcal{O}(B/Y))$ goes to the trivial element in $H^1(X, \mathcal{O}(B_X/X))$, then it comes from $H^1(G, B(X/Y))$.

We can also consider similar things in the case the action of G is not necessarily free. But for the resulting quotient space V, the induced morphism $V \to Y$ is not necessarily a smooth morphism. We can also consider the case that $B \to Y$ has only a meromorphic group structure and the group G is finite. By replacing $B(X/Y)$ by a group of meromorphic sections of $B_X \to X$, we obtain a meromorphic action of G on B_X from an element of $H^1(G, B(X/Y))$. Since G is finite, we have a meromorphic quotient V (up to the bimeromorphic equivalence relation) of B_X by the action.

§1. Smooth elliptic fibrations

1.1. Variation of Hodge structures of rank two and weight one

An elliptic curve C is isomorphic to a complex torus $\mathbb{C}/L$, where $L = L_\omega = \mathbb{Z} + \mathbb{Z}\omega$ for some $\omega \in \mathbb{H}$. Under a natural isomorphism $\pi_1(C) \simeq H_1(C,\mathbb{Z}) \simeq L$, we have the following two loops γ_1 and γ_0 of C corresponding to ω and 1 in L, respectively:

$$\gamma_1\colon [0,1] \ni t \mapsto t\omega \in \mathbb{C}, \quad \gamma_0\colon [0,1] \ni t \mapsto t \in \mathbb{C}.$$

For the coordinate z of $\mathbb{C}$, $\mathrm{d}z$ defines a holomorphic 1-form on C. Further $H^1(C,\mathbb{C})$ is spanned by the cohomology classes of $\mathrm{d}z$ and $\mathrm{d}\bar{z}$. The Hodge decomposition $H^1(C,\mathbb{C}) = H^{1,0} \oplus H^{0,1}$ is given by $H^{1,0} = \mathbb{C}\,\mathrm{d}z$ and $H^{0,1} = \mathbb{C}\,\mathrm{d}\bar{z}$. Let (e_1, e_0) be the dual base of $H^1(C,\mathbb{Z})$ to (γ_1,γ_0). Then $\mathrm{d}z = e_0 + \omega e_1$ in $H^1(C,\mathbb{C})$, since

$$\int_{\gamma_0} \mathrm{d}z = 1 \quad \text{and} \quad \int_{\gamma_1} \mathrm{d}z = \omega.$$

Let $\bigwedge^2 H^1(C,\mathbb{Z}) \simeq H^2(C,\mathbb{Z}) \to \mathbb{Z}$ be an isomorphism sending $e_0 \wedge e_1$ to 1. Let $Q\colon H^1(C,\mathbb{Z}) \times H^1(C,\mathbb{Z}) \to \mathbb{Z}$ be the induced skew symmetric bilinear form. Then

$$\int_C \frac{\sqrt{-1}}{2}\,\mathrm{d}z \wedge \mathrm{d}\bar{z} = \frac{\sqrt{-1}}{2} Q(\mathrm{d}z, \mathrm{d}\bar{z}) = \operatorname{Im}\omega.$$

Let $H_1(C,\mathbb{Z}) \to (H^{1,0})^\vee = \operatorname{Hom}(H^{1,0},\mathbb{C}) \simeq \mathbb{C}$ be the homomorphism given by the integral

$$\gamma \mapsto \int_\gamma \mathrm{d}z.$$

We see that the induced homomorphism $H_1(C,\mathbb{C}) \to (H^{1,0})^\vee$ is dual to the injection $H^{1,0} \to H^1(C,\mathbb{C})$. Moreover we have a commutative diagram of exact sequences:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^{1,0} & \longrightarrow & H^1(C,\mathbb{C}) & \longrightarrow & H^{0,1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle q} & & \downarrow & & \\
0 & \longrightarrow & (H^{0,1})^\vee & \longrightarrow & H_1(C,\mathbb{C}) & \longrightarrow & (H^{1,0})^\vee & \longrightarrow & 0,
\end{array}$$

where $q\colon H^1(C,\mathbb{C}) \to H_1(C,\mathbb{C})$ is the isomorphism by Poincaré duality determined by Q, explicitly by $q(e_0) = \gamma_1$ and $q(e_1) = -\gamma_0$.

A polarized Hodge structure $H = (H, Q, F^\bullet)$ of rank two and of weight one is defined to be the following data (cf. [D], [G]):

(1) A free $\mathbb{Z}$-module H of rank two;
(2) A skew symmetric bilinear form $Q\colon H \times H \to \mathbb{Z}$ inducing an isomorphism $\bigwedge^2 H \simeq \mathbb{Z}$;
(3) A descending filtration of vector subspaces of $H_{\mathbb{C}} := H \otimes \mathbb{C}$:

$$0 = F^2(H_{\mathbb{C}}) \subset F^1(H_{\mathbb{C}}) \subset F^0(H_{\mathbb{C}}) = H_{\mathbb{C}}$$

satisfying the following conditions:
(a) $\dim F^1(H_{\mathbb{C}}) = 1$;
(b) $F^1(H_{\mathbb{C}}) \oplus \overline{F^1(H_{\mathbb{C}})} = H_{\mathbb{C}}$, where $\overline{F^1(H_{\mathbb{C}})}$ denotes the complex conjugate;
(c) $\sqrt{-1}Q(x, \overline{x}) > 0$ for any nonzero element x of $F^1(H_{\mathbb{C}})$.

The Q is called the polarization of H and $\{F^p\}$ is called the Hodge filtration. The condition (c) is called the Hodge–Riemann bilinear relation. For the elliptic curve C above, the data $(H^1(C, \mathbb{Z}), Q, F^1 = H^{1,0})$ form a polarized Hodge structure of rank two and of weight one. Conversely, any polarized Hodge structure of rank two and of weight one defines an elliptic curve inducing the same Hodge structure.

Let S be a complex analytic variety. A polarized variation of Hodge structures $H = (H, Q, \mathcal{F}^\bullet)$ of rank two and weight one over S is defined to be the following data (cf. [D], [G]):

(1) A locally constant sheaf H with fiber $\mathbb{Z}^{\oplus 2}$;
(2) A skew symmetric bilinear form $Q\colon H \times H \to \mathbb{Z}_S$ inducing an isomorphism $\bigwedge^2 H \simeq \mathbb{Z}_S$;
(3) A descending sequence of holomorphic subbundles:

$$0 = \mathcal{F}^2(\mathcal{H}) \subset \mathcal{F}^1(\mathcal{H}) \subset \mathcal{F}^0(\mathcal{H}) = \mathcal{H} := H \otimes \mathcal{O}_S,$$

where the restriction $(H_s, Q_s, \mathcal{F}^\bullet \otimes \mathbb{C}(s))$ to the fiber over any point $s \in S$ forms a polarized Hodge structure of rank two and of weight one.

Note that the Griffiths transversality condition is satisfied automatically in this case.

Example 1.1.1. Let $f\colon X \to S$ be a smooth elliptic fibration, i.e., a smooth proper surjective morphism with elliptic curves as fibers. Then $H := R^1 f_* \mathbb{Z}_X$ is a locally constant sheaf with fiber $\mathbb{Z}^{\oplus 2}$. The cup product $R^1 f_* \mathbb{Z}_X \times R^1 f_* \mathbb{Z}_X \to R^2 f_* \mathbb{Z}_X$ and the trace map $R^2 f_* \mathbb{Z}_X \simeq \mathbb{Z}_S$ define a skew symmetric bilinear form Q on H. Let

$$0 \to f^{-1}\mathcal{O}_S \to \mathcal{O}_X \xrightarrow{\mathrm{d}_{X/S}} \Omega^1_{X/S} \to 0$$

be the relative Poincaré exact sequence. By taking higher direct images, we have an exact sequence:

$$0 \to f_*\Omega^1_{X/S} \to R^1 f_* f^{-1}\mathcal{O}_S \simeq H \otimes \mathcal{O}_S \to R^1 f_* \mathcal{O}_X \to 0.$$

Let $\mathcal{F}^1(\mathcal{H})$ be the subbundle $f_*\Omega^1_{X/S}$ of $\mathcal{H} := H \otimes \mathcal{O}_S$. Then the conditions (a), (b), (c) above are satisfied on each fiber. Thus we have a variation of Hodge structures of weight one and rank two from a smooth elliptic fibration.

Let H be a variation of Hodge structures of rank two and weight one whose local constant system H is trivial. Then we can choose a base (e_0, e_1) of $H^0(S, H)$ so that $Q(e_0, e_1) = 1$. Denoting $\mathcal{L}_H := \mathcal{H}/\mathcal{F}^1(\mathcal{H})$, we have a surjection $r\colon \mathcal{O}_S^{\oplus 2} \simeq \mathcal{H} \to \mathcal{L}_H$. The sections $r(e_0)$ and $r(e_1)$ of $\mathcal{L}_H$ are nowhere vanishing. We then define a function by

$$\omega(z) := -\frac{r(e_0)}{r(e_1)}$$

for $z \in S$. The Hodge subbundle $\mathcal{F}^1(\mathcal{H})$ is generated by $\omega(z)e_1 + e_0$. Hence the Hodge–Riemann bilinear relation implies that $\operatorname{Im}\omega(z) > 0$, i.e., ω is a mapping into the upper half plane $\mathbb{H}$. Let $(e_0^\sharp, e_1^\sharp)$ be another base of $H^0(S, H)$ with $Q(e_0^\sharp, e_1^\sharp) = 1$. Then

$$(e_1, e_0) = (e_1^\sharp, e_0^\sharp)\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

for a matrix in $\mathrm{SL}(2, \mathbb{Z})$. Let $\omega^\sharp(z) = -r(e_0^\sharp)/r(e_1^\sharp)$ be the similarly defined function. Since $\omega^\sharp(z)e_1^\sharp + e_0^\sharp$ is also a generator of $\mathcal{F}^1(\mathcal{H})$, there is a nowhere vanishing holomorphic function $u(z)$ such that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \omega(z) \\ 1 \end{pmatrix} = u(z)\begin{pmatrix} \omega^\sharp(z) \\ 1 \end{pmatrix}.$$

Thus $u(z) = c\omega(z) + d$ and $\omega^\sharp(z) = (a\omega(z) + b)/(c\omega(z) + d)$.

Definition 1.1.2. The $\omega(z)$ is called the period function.

Next suppose further that there is a properly discontinuous action of a discrete group Γ on S and that the variation of Hodge structures H admits a Γ-linearization. This means that the locally constant system H and Hodge filtrations $\mathcal{F}^\bullet(\mathcal{H})$ admit compatible Γ-linearizations which preserve the polarization Q. For the right Γ-module structure of $H^0(S, H)$, we have a group homomorphism $\rho\colon \Gamma \to \operatorname{Aut}(H^0(S, H))$ such that $x^\gamma = \rho(\gamma)^{-1}x$ for $x \in H^0(S, H)$. Since Q is preserved, we have a matrix

$$\rho(\gamma) = \begin{pmatrix} a_\gamma & b_\gamma \\ c_\gamma & d_\gamma \end{pmatrix}$$

in $\mathrm{SL}(2, \mathbb{Z})$ such that $(e_1^\gamma, e_0^\gamma) = (e_1, e_0)\rho(\gamma)^{-1}$. We shall write an element of $H^0(S, H) = \mathbb{Z}e_1 \oplus \mathbb{Z}e_0$ by a column vector ${}^{\mathrm{t}}(x, y)$ consisting of integers which corresponds to $xe_1 + ye_0$. Let $H^\vee$ be the dual locally constant system $\mathcal{H}om(H, \mathbb{Z}_S)$ and let $(e_0^\vee, e_1^\vee)$ be the dual base to (e_0, e_1). We identify a row vector (m, n) consisting of integers with $me_1^\vee + ne_0^\vee$ in $H^0(S, H^\vee)$. Since $(m, n) \cdot {}^{\mathrm{t}}(x, y)^\gamma = (m, n)\rho(\gamma^{-1})\,{}^{\mathrm{t}}(x, y)$, the right Γ-module structure of $H^0(S, H^\vee)$ is described by $(m, n)^\gamma = (m, n)\rho(\gamma)$. Let $q\colon H \to H^\vee$ be the isomorphism defined by $q(x)(y) = Q(x, y)$ for $x, y \in H$. Then we have $q(e_0) = e_1^\vee$, and $q(e_1) = -e_0^\vee$. More explicitly, we have

$$q\begin{pmatrix} x \\ y \end{pmatrix} = (x, y)\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Thus q is compatible with Γ-linearizations, since

$${}^{\mathrm{t}}\rho(\gamma^{-1}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\rho(\gamma)\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^{-1}.$$

We shall also write an element of $H^0(S, \mathcal{H})$ by a column vector

$$\boldsymbol{v}(z) = \begin{pmatrix} f(z) \\ g(z) \end{pmatrix}$$

consisting of global holomorphic functions which corresponds to $f(z)e_1 + g(z)e_0$. Then the right Γ-module structure of $H^0(S, \mathcal{H})$ is given by $\boldsymbol{v}(z)^\gamma = \rho(\gamma)^{-1}\boldsymbol{v}(\gamma z)$. Since $\mathcal{F}^1(\mathcal{H})$ is generated by $\omega(z)e_1 + e_0$, for each $\gamma \in \Gamma$, we have

$$\rho(\gamma)\begin{pmatrix} \omega(z) \\ 1 \end{pmatrix} = (c_\gamma\omega(z) + d_\gamma)\begin{pmatrix} \omega(\gamma z) \\ 1 \end{pmatrix}.$$

In particular, we have

$$\omega(\gamma z) = \frac{a_\gamma\omega(z) + b_\gamma}{c_\gamma\omega(z) + d_\gamma}. \tag{1.1}$$

Now we have the following commutative diagram of exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{F}^1(\mathcal{H}) & \longrightarrow & \mathcal{H} & \longrightarrow & \mathcal{L}_H & \longrightarrow & 0 \\ & & \| & & \| & & \| & & \\ 0 & \longrightarrow & \mathcal{O}_S & \xrightarrow{\ i\ } & \mathcal{O}_S^{\oplus 2} & \xrightarrow{\ p\ } & \mathcal{O}_S & \longrightarrow & 0, \end{array}$$

where i and p are defined by:

$$i\colon 1 \mapsto \begin{pmatrix} \omega(z) \\ 1 \end{pmatrix} \quad \text{and} \quad p\colon \begin{pmatrix} \alpha(z) \\ \beta(z) \end{pmatrix} \mapsto \alpha(z) - \omega(z)\beta(z).$$

There is a Γ-linearization on $\mathcal{O}_S \simeq \mathcal{L}_H$. The induced right action of Γ on $H^0(S, \mathcal{O}_S)$ is described as follows: For a holomorphic function $f(z)$ on S and for $\gamma \in \Gamma$,

$$f^{\gamma}(z) := (c_{\gamma}\omega(z) + d_{\gamma})f(\gamma z).$$

The homomorphism $\mathcal{H} \to \mathcal{L}_H$ is isomorphic to the dual of $\mathcal{F}^1(\mathcal{H}) \to \mathcal{H}$. Thus the composite

$$H^{\vee} \xrightarrow{q^{-1}} H \otimes \mathcal{O}_S \to \mathcal{L}_H$$

is induced from the following Γ-linearized homomorphism:

$$H^{\vee} \simeq \mathbb{Z}_S^{\oplus 2} \ni (m, n) \mapsto m\omega(z) + n \in \mathcal{O}_S.$$

Next, we consider a polarized variation of Hodge structures H of rank two and of weight one on a complex analytic variety S whose local constant system is not necessarily constant. Let $\tau\colon U \to S$ be the universal covering mapping. Then $\tau^{-1}H = (\tau^{-1}H, Q, \tau^*\mathcal{F}^{\bullet}(\mathcal{H}))$ is a variation of Hodge structures with a trivial locally constant system. We have an action of the fundamental group $\Gamma = \pi_1(S, s)$ for a point $s \in S$ on U and a Γ-linearization on the variation of Hodge structures $\tau^{-1}H$. Thus by the previous argument, we have a period function $\omega(z)$ for $z \in U$ and a monodromy representation $\rho\colon \Gamma \to \mathrm{SL}(2, \mathbb{Z})$ satisfying (1.1). Let $\mathcal{L}_H$ denote the quotient $\mathcal{H}/\mathcal{F}^1(\mathcal{H})$. Then the homomorphism $\tau^{-1}H \to \tau^*\mathcal{L}_H$ is isomorphic to

$$\mathbb{Z}_U^{\oplus 2} \ni (m, n) \mapsto m\omega(z) + n \in \mathcal{O}_U.$$

Here the right actions of $\gamma \in \Gamma$ on $H^0(U, \mathbb{Z}_U^{\oplus 2})$ and $H^0(U, \mathcal{O}_U)$ are given by:

$$(m, n) \mapsto (m, n)\rho(\gamma) \quad \text{and} \quad f(z) \mapsto f^{\gamma}(z) := (c_{\gamma}w(z) + d_{\gamma})f(\gamma z).$$

Therefore a polarized variation of Hodge structures of rank two and of weight one is determined by a monodromy representation $\rho\colon \pi_1(S, s) \to \mathrm{SL}(2, \mathbb{Z})$ and a period function $\omega\colon U \to \mathbb{H}$ satisfying the condition (1.1).

1.2. Smooth basic elliptic fibrations

We shall define the *basic elliptic fibration* associated with the variation of Hodge structures H on S. Let $\mathbb{V}(H) := \mathbb{V}(\mathcal{L}_H) \to S$ be the holomorphic line bundle associated with the invertible sheaf $\mathcal{L}_H$. For the injection $H \hookrightarrow \mathcal{L}_H$, we have a corresponding subspace $\mathbb{L}(H) \subset \mathbb{V}(H)$ étale over S. Locally on S, $\mathbb{V}(H) \simeq S \times \mathbb{C}$ and $\mathbb{L}(H) \simeq S \times \mathbb{Z}^2$. Since $\mathbb{L}(H)$ is a discrete subgroup of $\mathbb{V}(H)$ over S, we can define the quotient $B(H) := \mathbb{V}(H)/\mathbb{L}(H)$. This is also described in the following way: Let $\omega(z)$ for $z \in U$ and $\rho(\gamma)$ for $\gamma \in \Gamma = \pi_1(S, s)$, respectively, be the period function and the monodromy representation defined as before. For $\gamma \in \Gamma$ and $(m, n) \in \mathbb{Z}^{\oplus 2}$, we define an automorphism $\Phi(\gamma, (m, n))$ of $U \times \mathbb{C}$ by:

$$U \times \mathbb{C} \ni (z, \zeta) \mapsto \left(\gamma z, \frac{\zeta + m\omega(z) + n}{c_\gamma \omega(z) + d_\gamma}\right).$$

Then for any $\gamma_1, \gamma_2 \in \Gamma$ and $(m_1, n_1), (m_2, n_2) \in \mathbb{Z}^{\oplus 2}$, we obtain

$$\begin{aligned} \Phi(\gamma_1, (m_1, n_1)) \circ \Phi(\gamma_2, (m_2, n_2)) &= \Phi(\gamma_1\gamma_2, (m_3, n_3)), \\ \text{where} \quad (m_3, n_3) &= (m_2, n_2) + (m_1, n_1)\rho(\gamma_2). \end{aligned}$$

Thus the semi-direct product $\Gamma \ltimes \mathbb{Z}^{\oplus 2}$ acts on $U \times \mathbb{C}$ from the left. Since this action is properly discontinuous and fixed point free, we have the quotient variety $B(H)$ smooth over S. By the argument of §1.1, the quotient of $U \times \mathbb{C}$ by the subgroup $\Gamma \ltimes 0$ is isomorphic to $\mathbb{V}(H)$. Therefore, we have an elliptic fibration $p\colon B(H) \to S$ canonically from H. The zero section of $\mathbb{V}(H) \to S$ defines a section $\sigma_0\colon S \to B(H)$. Note that $R^1p_*\mathbb{Z}_{B(H)} \simeq H$ as variations of Hodge structures. By the construction, $p\colon B(H) \to S$ has a group structure whose zero section is σ_0.

Definition 1.2.1 (cf. [Kd1]). The elliptic fibration $p\colon B(H) \to S$ is said to be the *smooth basic elliptic fibration* associated with the polarized variation of Hodge structures H.

Let $\sigma\colon S \to B(H)$ be another section of p. Since $p\colon B(H) \to S$ has a group structure, we have the translation morphism $\mathrm{tr}(\sigma)\colon B(H) \to B(H)$ over S. Then $\mathrm{tr}(\sigma)$ preserves the variation of Hodge structures H, i.e.,

$$\mathrm{tr}(\sigma)^*\colon R^1p_*\mathbb{Z}_{B(H)} \to R^1p_*\mathbb{Z}_{B(H)}$$

is the identity mapping. Conversely, we have the following:

Lemma 1.2.2. *Let $\varphi\colon B(H) \to B(H)$ be an automorphism over S which induces the identity on $H = R^1p_*\mathbb{Z}_{B(H)}$. Then $\varphi = \mathrm{tr}(\sigma)$ for a section $\sigma\colon S \to B(H)$.*

Proof. Let $\sigma\colon S \to B$ be the composite of the zero section $\sigma_0\colon S \to B$ and $\varphi\colon B \to B$. Then the composite of φ and the inverse of the translation $\mathrm{tr}(\sigma)$ also induces the identity on $R^1p_*\mathbb{Z}_B$. Thus it is enough to prove that φ is the identity morphism provided that φ preserves the zero section. We see that this should be an identity on any fiber, by a property of automorphisms of elliptic curves. Q.E.D.

Some properties on morphisms of elliptic curves are generalized to:

Lemma 1.2.3.

(1) *Let H_1 and H_2 be two variations of Hodge structures of weight one and rank two over S and let $\varphi\colon B(H_1) \to B(H_2)$ be a morphism over S. Then $\varphi = \mathrm{tr}(\sigma) \circ \phi$ for the translation morphism $\mathrm{tr}(\sigma)$ by a section $\sigma\colon S \to B(H_2)$ and a group homomorphism $\phi\colon B(H_1) \to B(H_2)$ over S.*

(2) *Let $\phi\colon B(H) \to B(H)$ be an automorphism over S preserving the zero section. Then the order of ϕ is finite and is one of $\{1, 2, 3, 4, 6\}$.*

Proposition 1.2.4 (cf. [Kd1]). *Let $f\colon X \to S$ be a smooth elliptic fibration of complex analytic varieties such that $H \simeq R^1f_*\mathbb{Z}_X$ as variations of Hodge structures. Assume that f admits a section $\sigma\colon S \to X$. Then there exists an isomorphism $h\colon X \to B(H)$ over S such that $h \circ \sigma = \sigma_0$.*

Proof. Let $\Delta_X \subset X \times_S X$ be the diagonal locus, $\Sigma := \sigma(S) \subset X$, p_1, p_2 the first and the second projections, respectively, and let $\Sigma_X := p_2^{-1}(\Sigma) \subset X \times_S X$. We consider the invertible sheaf

$$\mathcal{N} := \mathcal{O}_{X \times_S X}(\Delta_X - \Sigma_X).$$

Then for any $x \in X$, we have an isomorphism

$$\mathcal{N}_{|p_1^{-1}(x)} \simeq \mathcal{O}_{f^{-1}(f(x))}([x] - [\sigma(f(x))]),$$

where $[x]$ denotes the prime divisor supported at x on the elliptic curve $f^{-1}(f(x))$. Let c be the image of $\mathcal{N}$ under the natural homomorphism

$$H^1(X \times_S X, \mathcal{O}^{\star}_{X \times_S X}) \to H^0(X, R^1p_{1*}\mathcal{O}^{\star}_{X \times_S X}).$$

We shall also consider the following exact sequence induced from the exponential sequence on $X \times_S X$:

$$0 \to R^1p_{1*}\mathbb{Z}_{X \times_S X} \to R^1p_{1*}\mathcal{O}_{X \times_S X} \to R^1p_{1*}\mathcal{O}^{\star}_{X \times_S X} \to \\ \to R^2p_{1*}\mathbb{Z}_{X \times_S X} \simeq \mathbb{Z}_X.$$

We infer that $R^1p_{1*}\mathbb{Z}_{X\times_S X} \simeq f^{-1}H$, $R^1p_{1*}\mathcal{O}_{X\times_S X} \simeq f^*\mathcal{L}_H$, and that the natural inclusion $H \hookrightarrow \mathcal{L}_H$ determined by the variation of Hodge structures induces the injection $R^1p_{1*}\mathbb{Z}_{X\times_S X} \to R^1p_{1*}\mathcal{O}_{X\times_S X}$ above. Let $\mathfrak{S}$ be the cokernel of $f^{-1}H \hookrightarrow f^*\mathcal{L}_H$. Then $c \in H^0(X, \mathfrak{S})$. Since $f^*\mathcal{L}_H \to \mathfrak{S}$ is surjective, we have an open covering $\{X_\lambda\}$ of X and sections $\alpha_\lambda \in H^0(X_\lambda, f^*\mathcal{L}_H)$ such that $c_{|X_\lambda}$ is the image of α_λ. Then

$$\alpha_{\lambda|X_\lambda\cap X_\mu} - \alpha_{\mu|X_\lambda\cap X_\mu} \in H^0(X_\lambda \cap X_\mu, f^{-1}H).$$

The α_λ defines a morphism $h_\lambda \colon X_\lambda \to \mathbb{V}(H)$ over S. Further $h_\lambda(x) - h_\mu(x) \in \mathbb{L}(H)$ for $x \in X_\lambda \cap X_\mu$. Therefore we have a global morphism $h \colon X \to B(H) = \mathbb{V}(H)/\mathbb{L}(H)$ over S. By construction, h does not depend on the choices of open coverings $\{X_\lambda\}$ and sections $\{\alpha_\lambda\}$.

We shall show that $h(\Sigma)$ coincides with the zero section of $B(H) \to S$. By considering the restrictions to $\Sigma \simeq S$ of $f^{-1}H$, $f^*\mathcal{L}_H$, $\mathfrak{S}$, and $R^1p_{1*}\mathcal{O}^\star_{X\times_S X}$, we have the following commutative diagram:

$$\begin{array}{ccc} H^0(X, \mathfrak{S}) & \longrightarrow & H^0(X, R^1p_{1*}\mathcal{O}^\star_{X\times_S X}) \\ \downarrow & & \downarrow \\ H^0(S, \mathcal{L}_H/H) & \longrightarrow & H^0(S, R^1f_*\mathcal{O}^\star_X). \end{array}$$

The both horizontal homomorphisms are injective. From an isomorphism $\mathcal{N}_{|\Sigma\times_S X} \simeq \mathcal{O}_X$ and the commutative diagram

$$\begin{array}{ccc} H^1(X\times_S X, \mathcal{O}^\star_{X\times_S X}) & \longrightarrow & H^1(\Sigma\times_S X, \mathcal{O}^\star_{\Sigma\times_S X}) \\ \downarrow & & \downarrow \\ H^0(X, R^1p_{1*}\mathcal{O}^\star_{X\times_S X}) & \longrightarrow & H^0(S, R^1f_*\mathcal{O}^\star_X), \end{array}$$

we infer that the image of c in $H^0(S, \mathcal{L}_H/H)$ is zero. Thus $h(\Sigma)$ coincides with the zero section.

Finally, we shall prove that h is an isomorphism. We have only to check it on each fiber of $X \to S$. The restriction of h to a fiber $E := f^{-1}(P)$ is essentially isomorphic to:

$$E \ni x \mapsto \mathcal{O}([x] - [\sigma(P)]) \in \mathrm{Pic}^0(E).$$

Therefore this is an isomorphism. Q.E.D.

We thus obtain a one to one correspondence between the set of isomorphism classes of smooth basic elliptic fibrations and that of polarized variations of Hodge structures of rank two, weight one over S. Next, we shall relate them with Weierstrass models [MS], [Ny4]. Let

$(\mathcal{L}, \alpha, \beta)$ be a triplet consisting of an invertible sheaf $\mathcal{L}$ on S and sections $\alpha \in H^0(S, \mathcal{L}^{\otimes(-4)})$, $\beta \in H^0(S, \mathcal{L}^{\otimes(-6)})$ such that $0 \neq 4\alpha^3 + 27\beta^2 \in H^0(S, \mathcal{L}^{\otimes(-12)})$. For the $\mathbb{P}^2$-bundle $p\colon \mathbb{P} := \mathbb{P}(\mathcal{O}_S \oplus \mathcal{L}^{\otimes 2} \oplus \mathcal{L}^{\otimes 3}) \to S$, let $\mathcal{O}(1)$ be the tautological line bundle. According to the natural inclusions

$$\mathcal{O}_S \hookrightarrow \mathcal{O}_S \oplus \mathcal{L}^{\otimes 2} \oplus \mathcal{L}^{\otimes 3}, \ \mathcal{L}^{\otimes 2} \hookrightarrow \mathcal{O}_S \oplus \mathcal{L}^{\otimes 2} \oplus \mathcal{L}^{\otimes 3}, \ \mathcal{L}^{\otimes 3} \hookrightarrow \mathcal{O}_S \oplus \mathcal{L}^{\otimes 2} \oplus \mathcal{L}^{\otimes 3},$$

we have sections $Z \in H^0(\mathbb{P}, \mathcal{O}(1))$, $X \in H^0(\mathbb{P}, \mathcal{O}(1) \otimes p^*(\mathcal{L}^{\otimes(-2)}))$, and $Y \in H^0(\mathbb{P}, \mathcal{O}(1) \otimes p^*(\mathcal{L}^{\otimes(-3)}))$. Then $Y^2 Z - (X^3 + \alpha X Z^2 + \beta Z^3)$ is a global section of $\mathcal{O}(3) \otimes p^*\mathcal{L}^{\otimes(-12)}$. The zero locus of the section is called the Weierstrass model and denoted by $W_S(\mathcal{L}, \alpha, \beta)$. The section $\{X = Z = 0\}$ of $\mathbb{P} \to S$ is contained in $W_S(\mathcal{L}, \alpha, \beta)$, which is called the canonical section.

Fact 1.2.5 ([MS]). Let $f\colon X \to S$ be a smooth elliptic fibration admitting a section $\sigma\colon S \to X$. Then there exist a triplet $(\mathcal{L}, \alpha, \beta)$ and an isomorphism $\mu\colon X \to W_S(\mathcal{L}, \alpha, \beta)$ over S such that $\mu \circ \sigma$ is the canonical section.

In this case, $\mathcal{L} \simeq R^1 f_* \mathcal{O}_X$ and the discriminant $4\alpha^3 + 27\beta^2$ is a nowhere vanishing section. Therefore the following three sets can be identified to each other:

- The set of isomorphism classes of variations of Hodge structures of weight one and rank two over S;
- The set of isomorphism classes of smooth basic elliptic fibrations over S;
- The set of triplets $(\mathcal{L}, \alpha, \beta)$ as above with $4\alpha^3 + 27\beta^2$ nowhere vanishing, modulo the following equivalence relation: $(\mathcal{L}, \alpha, \beta) \sim (\mathcal{L}', \alpha', \beta')$ if and only if there is a nowhere vanishing section $\varepsilon \in H^0(S, \mathcal{L}' \otimes \mathcal{L}^{\otimes(-1)})$ such that $\alpha = \varepsilon^4 \alpha'$ and $\beta = \varepsilon^6 \beta'$.

Remark 1.2.6. Let us consider the case $S = \mathbb{H} = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$ and $\omega(z) = z$ for $z \in \mathbb{H}$. Then ω defines a variation of Hodge structures and the corresponding smooth basic elliptic fibration over $\mathbb{H}$ is sometimes called the "universal" elliptic fibration. By the theory of Weierstrass' $\wp$-function, this is isomorphic to the Weierstrass model

$$W_{\mathbb{H}}(\mathcal{O}_{\mathbb{H}}, \boldsymbol{\alpha}, \boldsymbol{\beta}) = \{ ((X:Y:Z), z) \in \mathbb{P}^2 \times \mathbb{H} \mid Y^2 Z = X^3 + \boldsymbol{\alpha}(z) X Z^2 + \boldsymbol{\beta}(z) Z^3 \},$$

where $\boldsymbol{\alpha}(z) := -15 G_4(z)$, $\boldsymbol{\beta}(z) := -35 G_6(z)$, and $G_k(z)$ is the Eisenstein series

$$G_k(z) := \sum_{(0,0) \neq (m,n) \in \mathbb{Z}^{\oplus 2}} (mz + n)^{-k}$$

of weight k. The following facts are known.

(1) $4\boldsymbol{\alpha}(z)^3 + 27\boldsymbol{\beta}(z)^2 = -(1/16)\boldsymbol{\Delta}(z)$, where $\boldsymbol{\Delta}(z)$ is the cusp form of weight 12 of the modular group $\mathrm{SL}(2,\mathbb{Z})$ represented by

$$\boldsymbol{\Delta}(z) = (2\pi)^{12} q \prod_{\nu=1}^{\infty} (1 - q^{\nu})^{24}$$

for $q = \exp(2\pi\sqrt{-1}z)$.

(2) The $\mathrm{SL}(2,\mathbb{Z})$-invariant function

$$\boldsymbol{j}(z) := \frac{4\boldsymbol{\alpha}(z)^3}{4\boldsymbol{\alpha}(z)^3 + 27\boldsymbol{\beta}(z)^2}$$

is called the *elliptic modular function* and induces an isomorphism $\mathrm{SL}(2,\mathbb{Z})\backslash\mathbb{H} \simeq \mathbb{C}$.

(3) The function $\boldsymbol{j}(z) - 12^{-3}q^{-1}$ is a holomorphic function near $q = 0$.

Definition 1.2.7. Let H be a polarized variation of Hodge structures of rank two weight one. The *J-function* of H is defined to be $J(t) := \boldsymbol{j}(\omega(z))$, where $\tau(z) = t \in S$. The J-function of a smooth elliptic fibration $X \to S$ should be the J-function of the corresponding polarized variation of Hodge structures.

In particular, the J-function of a smooth Weierstrass model $W(\mathcal{L}, \alpha, \beta) \to S$ is given by $J(t) = 4\alpha^3/(4\alpha^3 + 27\beta^2)$.

In papers [Kd1] and [Kwi], the J-function is called the *functional invariant* and the monodromy representation of H (or that restricted to the Zariski-open subset $\{t \in S \colon J(t) \neq 0, 1\}$) is called the *homological invariant.* Here a period function ω is a multi-valued analytic function satisfying $\boldsymbol{j}(\omega) = J$ and the condition (1.1). A pair consisting of such a period function and a monodromy representation is called a *characteristic pair* in [U].

1.3. General smooth elliptic fibrations

Let $\mathfrak{S}_H$ be the sheaf of germs of sections of the smooth basic elliptic fibration $p \colon B := B(H) \to S$. Then this is a sheaf of abelian groups. From the surjection $\mathbb{V}(H) \to B(H)$, we have the following exponential exact sequence (cf. [Kd1]):

$$0 \to H \to \mathcal{L}_H \to \mathfrak{S}_H \to 0.$$

For $\eta \in H^1(S, \mathfrak{S}_H)$, we can define a torsor $B(H)^{\eta} \to S$ of $p \colon B \to S$. By a similar argument to [Kd1], we can prove the following:

Proposition 1.3.1 (cf. [Kd1, 10.1]). *Any smooth elliptic fibration $f\colon X \to S$ with an isomorphism $R^1 f_* \mathbb{Z}_X \simeq H$ is isomorphic to $B(H)^\eta \to S$ for some $\eta \in H^1(S, \mathfrak{S}_H)$.*

Proof. Since f is smooth, we have an open covering $\{S_\lambda\}_{\lambda \in \Lambda}$ of S and sections $S_\lambda \to X_{|U_\lambda}$. Therefore there exist isomorphisms $\phi_\lambda\colon X_{|U_\lambda} \to B_{|U_\lambda}$ by Proposition 1.2.4. Here we may assume that the induced isomorphisms $\phi_\lambda^*\colon (R^1 p_* \mathbb{Z}_B)_{|U_\lambda} \to (R^1 f_* \mathbb{Z}_X)_{|U_\lambda}$ are glued to the given isomorphism $H \simeq R^1 f_* \mathbb{Z}_X$. Let us consider the composites $\varphi_{\lambda,\mu} := (\phi_\lambda \circ \phi_\mu^{-1})_{|U_\lambda \cap U_\mu}$. Then $\varphi_{\lambda,\mu}$ induces the identity on $(R^1 p_* \mathbb{Z}_B)_{|U_\lambda \cap U_\mu}$. Thus by Lemma 1.2.2, there exists a section $\eta_{\lambda,\mu}$ such that $\varphi_{\lambda,\mu}$ is the translation morphism $\mathrm{tr}(\eta_{\lambda,\mu})$. Since $\varphi_{\lambda,\mu} \circ \varphi_{\mu,\nu} \circ \varphi_{\nu,\lambda}$ is identical over $U_\lambda \cap U_\mu \cap U_\nu$ for $\lambda, \mu, \nu \in \Lambda$, we have $\eta_{\lambda,\mu} + \eta_{\mu,\nu} + \eta_{\nu,\lambda} = 0$ over $U_\lambda \cap U_\mu \cap U_\nu$. Therefore $f\colon X \to S$ is isomorphic to $B(H)^\eta$ for $\eta = \{\eta_{\lambda,\mu}\}_{\lambda,\mu \in \Lambda}$. Q.E.D.

We shall explain more about the cohomology class η. For a smooth elliptic fibration $f\colon X \simeq B(H)^\eta \to S$, let us consider the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R^1 f_* \mathbb{Z}_X & \longrightarrow & R^1 f_* \mathcal{O}_X & \longrightarrow & R^1 f_* \mathcal{O}_X^\star & \longrightarrow & R^2 f_* \mathbb{Z}_X \\ & & \| & & \| & & & & \\ 0 & \longrightarrow & H & \longrightarrow & \mathcal{L}_H & \longrightarrow & \mathfrak{S}_H & \longrightarrow & 0. \end{array}$$

Then we have a homomorphism $\Phi_X\colon \mathfrak{S}_H \to R^1 f_* \mathcal{O}_X^\star$ such that the sequence

$$(1.2) \qquad 0 \to \mathfrak{S}_H \xrightarrow{\Phi_X} R^1 f_* \mathcal{O}_X^\star \to R^2 f_* \mathbb{Z}_X \simeq \mathbb{Z}_S \to 0$$

is exact. We have the following description of Φ_X: Let $\Sigma_0 := \sigma_0(S) \subset B(H)$ be the zero section and let Σ be a section of $p\colon B(H)_{|\mathcal{U}} \to \mathcal{U}$ for an open subset $\mathcal{U} \subset S$. We shall take an open covering $\{U_\lambda\}$ of S and isomorphisms $\phi_\lambda\colon X_{|U_\lambda} \to B(H)_{|U_\lambda}$ as in the proof of Proposition 1.3.1. Let $\mathcal{M}_\lambda$ be the invertible sheaf

$$\phi_\lambda^*(\mathcal{O}(\Sigma - \Sigma_0)_{|B(H)_{|U_\lambda \cap \mathcal{U}}}).$$

Since $\phi_\lambda \circ \phi_\mu^{-1}$ on $B(H)_{|U_\lambda \cap U_\mu}$ is the translation mapping by a section $\eta_{\lambda,\mu}$, there exist invertible sheaves $\mathcal{N}_{\lambda,\mu}$ on $U_\lambda \cap U_\mu \cap \mathcal{U}$ such that

$$(\mathcal{M}_\lambda)_{|U_\lambda \cap U_\mu \cap \mathcal{U}} \simeq (\mathcal{M}_\mu)_{|U_\lambda \cap U_\mu \cap \mathcal{U}} \otimes f^* \mathcal{N}_{\lambda,\mu}.$$

Therefore we have an element $\Phi_X(\Sigma) \in H^0(\mathcal{U}, R^1 f_* \mathcal{O}_X^\star)$, which does not depend on the choices of open coverings $\{U_\lambda\}$ and isomorphisms

ϕ_λ. This is a description of the homomorphism $\Phi_X \colon H^0(\mathcal{U}, \mathfrak{S}_H) \to H^0(\mathcal{U}, R^1 f_* \mathcal{O}_X^\star)$. Let us consider a connecting homomorphism

$$\mathbb{Z} = H^0(S, \mathbb{Z}_S) \to H^1(S, \mathfrak{S}_H) \tag{1.3}$$

of the sequence (1.2). By the description of Φ_X, we see that the image of 1 is just η. Thus we have proved:

Lemma 1.3.2 (cf. [Ny8]). *Let $f \colon X \to S$ be a smooth elliptic fibration with H as a variation of Hodge structures. Suppose that $X \simeq B(H)^\eta$ over S. Then there exists an exact sequence* (1.2) *and the image of* 1 *under the connecting homomorphism* (1.3) *is* η.

Proposition 1.3.3 (cf. [Kd1, 11.5]). *Let $f \colon X \to S$ be a smooth elliptic fibration and let $\eta \in H^1(S, \mathfrak{S}_H)$ be the corresponding cohomology class. Then the following three conditions are equivalent*:

(1) *There is a prime divisor $D \subset X$ dominating S*;
(2) *The smooth elliptic fibration $f \colon X \to S$ is a projective morphism, i.e., there is an f-ample line bundle on X*;
(3) *η is a torsion element of $H^1(S, \mathfrak{S}_H)$.*

Proof. (1) $\Longrightarrow$ (2): The invertible sheaf $\mathcal{O}_X(D)$ is f-ample.
(2) $\Longrightarrow$ (3): By (1.2), we have the following long exact sequence:

$$0 \to H^0(S, \mathfrak{S}_H) \to H^0(S, R^1 f_* \mathcal{O}_X^\star) \to H^0(S, \mathbb{Z}) \to H^1(S, \mathfrak{S}_H).$$

An f-ample invertible sheaf defines an element of $H^0(S, R^1 f_* \mathcal{O}_X^\star)$, which is mapped to a positive integer in $\mathbb{Z} = H^0(S, \mathbb{Z})$. Thus by Lemma 1.3.2, the η is a torsion element.

(3) $\Longrightarrow$ (1): Let us assume that $m\eta = 0$ for a positive integer m. We shall consider the multiplication by m:

$$m\times \colon B(H) \ni b \mapsto mb = b + \cdots + b \in B(H).$$

Then by gluing $m\times \colon B(H)_{|S_\lambda} \to B(H)_{|S_\lambda}$, we have an étale finite morphism $\mu \colon X \simeq B(H)^\eta \to B(H)^{m\eta} \simeq B(H)$. Thus an irreducible component D of $\mu^*(\Sigma)$ dominates S. Q.E.D.

By the proof, we can take a divisor $D \subset X$ in (1) to be étale over S. However in general there is a prime divisor of X which is not étale over S.

Example 1.3.4. Let us consider the ruled surface $\Sigma_1 := \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(-1)) \to \mathbb{P}^1$ and a double covering $E \to \mathbb{P}^1$ from an elliptic curve. Let V be the fiber product $\Sigma_1 \times_{\mathbb{P}^1} E$. By considering the blow-down $\Sigma_1 \to \mathbb{P}^2$

of the unique (-1)-curve, we have a morphism $h\colon V \to \Sigma_1 \times E \to \mathbb{P}^2 \times E$. The image $h(V) \subset \mathbb{P}^2 \times E$ dominates $\mathbb{P}^2$, but $h(V) \to \mathbb{P}^2$ is not étale.

In the case the mapping degree of $D \to S$ is one, we have:

Lemma 1.3.5. *Let $f\colon X \to S$ be a smooth elliptic fibration over a complex* manifold *S. If a prime divisor D of X dominates S bimeromorphically, then $D \to S$ is isomorphic.*

Proof. Suppose that $h := f_{|D}\colon D \to S$ is not an isomorphism. Then the support of a non-trivial fiber $h^{-1}(s)$ is an elliptic curve. On the other hand, we have a bimeromorphic morphism $\nu\colon M \to D$ from a manifold M such that every non-trivial fiber of $h \circ \nu\colon M \to S$ is a union of rational curves. This is a contradiction. Q.E.D.

1.4. Smooth elliptic fibrations whose pullbacks are basic

Let $f\colon X \to S$ be a smooth elliptic fibration, $\tau\colon U \to S$ the universal covering mapping, and let $\Gamma = \pi_1(S, s)$. Suppose that the pullback $f_U\colon U \times_S X \to U$ admits a global section. Let $p\colon B = B(H) \to S$ be the basic smooth elliptic fibration associated with the variation of Hodge structures H induced from f. By Proposition 1.3.1, f is considered to be a torsor of p and it corresponds to a cohomology class η in $H^1(S, \mathfrak{S}_H)$. Further by Lemma 0.1.3, η is contained in $H^1(\Gamma, H^0(U, \tau^{-1}\mathfrak{S}_H))$, where we consider the following edge sequence of the Hochschild–Serre spectral sequence Lemma 0.2.1:

$$0 \to H^1(\Gamma, H^0(U, \tau^{-1}\mathfrak{S}_H)) \to H^1(S, \mathfrak{S}_H) \to H^0(\Gamma, H^1(U, \tau^{-1}\mathfrak{S}_H)).$$

Looking at the exact sequence:

$$(1.4)\quad 0 \to \tau^{-1}H \simeq \mathbb{Z}_U^{\oplus 2} \to \tau^{-1}(\mathcal{L}_H) \simeq \mathcal{O}_U \to \tau^{-1}\mathfrak{S}_H \simeq \mathfrak{S}_{\tau^{-1}H} \to 0,$$

we have an isomorphism

$$H^0(U, \mathfrak{S}_{\tau^{-1}H}) \simeq H^0(U, \mathcal{O}_U)/(\mathbb{Z}\omega + \mathbb{Z}),$$

where $\omega\colon U \to \mathbb{H}$ is the period function, since U is simply connected. Hence an element of $H^1(\Gamma, H^0(U, \mathfrak{S}_{\tau^{-1}H}))$ is represented by a collection of global holomorphic functions $\{F_\gamma(z)\}_{\gamma\in\Gamma}$ on U satisfying the cocycle condition:

$$(1.5)\qquad F_{\gamma\delta}(z) \equiv F_\delta(z) + (c_\delta\omega(z) + d_\delta)F_\gamma(\delta z) \mod \mathbb{Z}\omega(z) + \mathbb{Z},$$

for $z \in U$ and $\gamma, \delta \in \Gamma$ (cf. §1.1). Two collections $\{F_\gamma^{(1)}(z)\}$ and $\{F_\gamma^{(2)}(z)\}$ of holomorphic functions determine the same cohomology class in $H^1(\Gamma, H^0(U, \mathfrak{S}_{\tau^{-1}H}))$ if and only if there is a global holomorphic function $H(z)$ on U such that

$$(1.6)\quad F_\gamma^{(1)}(z) - F_\gamma^{(2)}(z) \equiv H(z) - (c_\gamma \omega(z) + d_\gamma) H(\gamma z) \mod \mathbb{Z}\omega(z) + \mathbb{Z}.$$

Let $F := \{F_\gamma(z)\}$ be a collection satisfying (1.5) and let $B_U := B(\tau^{-1}H) \simeq U \times_S B(H)$. Then F defines a left action of Γ on B_U, which is described as follows: For $\gamma \in \Gamma$, let us define the following automorphism of $U \times \mathbb{C}$:

$$U \times \mathbb{C} \ni (z, \zeta) \mapsto \left(\gamma z, \frac{\zeta + F_\gamma(z)}{c_\gamma \omega(z) + d_\gamma} \right).$$

Then it induces an automorphism $\Phi_F(\gamma)$ of $B_U \simeq U \times \mathbb{C}/(\mathbb{Z}\omega + \mathbb{Z})$. Here we have $\Phi_F(\gamma) \circ \Phi_F(\delta) = \Phi_F(\gamma\delta)$ for $\gamma, \delta \in \Gamma$. Thus we have the left action by Φ_F which is compatible with the action of Γ on U. Let B^F be the quotient $\Gamma \backslash B_U$ by the action. Then we have a smooth elliptic fibration $p^F : B^F \to S$. Therefore we have:

Lemma 1.4.1. *Let $f \colon X \to S$ be a smooth elliptic fibration which induces the variation of Hodge structures H on S. Suppose that $U \times_S X \to U$ admits a global section for the universal covering mapping $U \to S$. Then there is a collection of global holomorphic functions $F = \{F_\gamma(z)\}_{\gamma \in \pi_1(S,s)}$ on U satisfying the condition* (1.5) *such that f is isomorphic to $p^F : B^F \to S$ over S.*

Remark. Since $B_U \simeq U \times \mathbb{C}/(\mathbb{Z}\omega + \mathbb{Z})$, we can describe B^F as the quotient of $U \times \mathbb{C}$ by an action of a suitable group. Let $\Phi_F(\gamma, m, n)$ be an automorphism of $U \times \mathbb{C}$ defined by

$$U \times \mathbb{C} \ni (z, \zeta) \mapsto \left(\gamma z, \frac{\zeta + F_\gamma(z) + m\omega(z) + n}{c_\gamma \omega(z) + d_\gamma} \right),$$

for $m, n \in \mathbb{Z}$. For $\gamma, \delta \in \Gamma$, let us define a pair $(A_{\gamma,\delta}, B_{\gamma,\delta})$ of integers by

$$A_{\gamma,\delta}\omega(z) + B_{\gamma,\delta} := F_\delta(z) - F_{\gamma\delta}(z) + (c_\delta \omega(z) + d_\delta) F_\gamma(\delta z).$$

Then we have $\Phi_F(\gamma, m_1, n_1) \circ \Phi_F(\delta, m_2, n_2) = \Phi_F(\gamma\delta, m_3, n_3)$, where

$$(m_3, n_3) = (m_2, n_2) + (m_1, n_1) \begin{pmatrix} a_\gamma & b_\gamma \\ c_\gamma & d_\gamma \end{pmatrix} + (A_{\gamma,\delta}, B_{\gamma,\delta}).$$

Let c be the cohomology class determined by $\{(A_{\gamma,\delta}, B_{\gamma,\delta})\}$ in $H^2(\Gamma, \mathbb{Z}^{\oplus 2})$. Then we have a corresponding group $G(c)$ which is an extension of Γ by the right Γ-module $\mathbb{Z}^{\oplus 2}$. We have a left action of $G(c)$ on $U \times \mathbb{C}$ by Φ_F. The correspondence $F \mapsto c$ induces a homomorphism $H^1(\Gamma, H^0(U, \mathfrak{S}_{\tau^{-1}H})) \to H^2(\Gamma, \mathbb{Z}^{\oplus 2})$, which is derived from the exact sequence (1.4). The B^F is isomorphic to the quotient space $G(c) \backslash (U \times \mathbb{C})$.

§2. Smooth elliptic fibrations over $(\Delta^\star)^l \times \Delta^{d-l}$

2.1. Monodromy representations

Let S be a d-dimensional unit polydisc Δ^d with a coordinate system $(t_1, t_2, \ldots, t_d)$, i.e.,

$$S = \left\{(t_1, t_2, \ldots, t_d) \in \mathbb{C}^d \mid |t_i| < 1 \text{ for any } i\right\}$$

for a positive integer d. Let D be a divisor $\{t_1 t_2 \cdots t_l = 0\}$ for a positive integer $l \leq d$, i.e., $D = \sum_{i=1}^{l} D_i$, where $D_i = \{t_i = 0\}$ is the i-th coordinate hyperplane. We denote by $S^\star$ the complement $S \setminus D$ and by $j\colon S^\star \hookrightarrow S$ the natural inclusion. Since $S^\star$ is isomorphic to $(\Delta^\star)^l \times \Delta^{d-l}$, the universal covering space U of $S^\star$ is isomorphic to $\mathbb{H}^l \times \Delta^{d-l}$, where $\mathbb{H}$ is the upper half plane $\{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$. For a coordinate system $z = (z_1, z_2, \ldots, z_l, t_{l+1}, \ldots, t_d)$ of U, the universal covering mapping $\boldsymbol{e}\colon U \to S^\star$ is given by:

$$\boldsymbol{e}(z) = (e(z_1), e(z_2), \ldots, e(z_l), t_{l+1}, \ldots, t_d),$$

where $e(z) := \exp(2\pi\sqrt{-1}z)$. For $1 \leq i \leq l$, let γ_i be the automorphism of U defined by:

$$(z_1, z_2, \ldots, z_l, t') \mapsto (z_1, z_2, \ldots, z_{i-1}, z_i + 1, z_{i+1}, \ldots, z_l, t'),$$

where $t' = (t_{l+1}, t_{l+2}, \ldots, t_d)$. Then the fundamental group $\pi_1 := \pi_1(S^\star)$ is a free abelian group of rank l generated by $\gamma_1, \gamma_2, \ldots, \gamma_l$.

In this section, we shall consider smooth elliptic fibrations defined over $S^\star$. First of all, we shall describe all the variations of Hodge structures of rank two and weight one defined over $S^\star$. Note that the monodromy matrices are quasi-unipotent by Borel's lemma (cf. [Sc, (4.5)]). We have the following classification of quasi-unipotent matrices in $\mathrm{SL}(2, \mathbb{Z})$:

Lemma 2.1.1 (cf. [Kd1]). *A quasi-unipotent matrix in* $\mathrm{SL}(2, \mathbb{Z})$ *is conjugate exactly to one of the matrices of Table* 1 *in* $\mathrm{SL}(2, \mathbb{Z})$.

Table 1. Quasi unipotent matrices in $\mathrm{SL}(2, \mathbb{Z})$.

I_a $(a \in \mathbb{Z})$	II	III	IV
$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$
I_b^* $(b \in \mathbb{Z})$	II^*	III^*	IV^*
$\begin{pmatrix} -1 & -b \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$

The monodromy matrix $\rho(\gamma_i)$ for γ_i is said to be the *monodromy matrix around the coordinate hyperplane* D_i.

Lemma 2.1.2. *Let $g(t)$ be a holomorphic function on $t \in S^\star$ such that $e(g(t))$ is a meromorphic function on S. Then $g(t)$ is holomorphic also on S.*

Proof. There exist integers a_i for $1 \leq i \leq l$ and a nowhere vanishing function $u(t)$ on S such that $e(g(t)) = u(t)\prod_{i=1}^{l} t_i^{a_i}$. We have a holomorphic function $h(t)$ such that $u(t) = e(h(t))$ on S. Then for the coordinate system $(z_1, z_2, \ldots, z_l, t')$ of U, we have

$$g(t) - h(t) - \sum_{i=1}^{l} a_i z_i \in \mathbb{Z}.$$

Since this is a constant function, $a_i = 0$ for all i. Hence $g(t)$ is holomorphic on S. Q.E.D.

Lemma 2.1.3 (cf. [Kd1, 7.3]). *Let $\rho\colon \pi_1 \to \mathrm{SL}(2,\mathbb{Z})$ be the monodromy representation associated with a variation of Hodge structures of weight one rank two defined on $S^\star$. Then the conjugation by a suitable matrix in $\mathrm{SL}(2,\mathbb{Z})$ changes all the monodromy matrices $\rho(\gamma_i)$ to be matrices listed in Table 1 at the same time. If $\rho(\gamma_i)$ corresponds to the matrix of the form I_a or I_a^*, then $a \geq 0$.*

Proof. By Lemma 2.1.1, the first assertion is derived from the commutativity of $\rho(\gamma)$'s. For the rest, we may assume that $d = l = 1$ and $\rho(\gamma_1)$ is of type I_a or I_a^*. Then the period function $\omega(z)$ satisfies $\omega(z+1) = \omega(z) + a$ by (1.1). Thus the function $e(\omega(z))$ is invariant under the action of π_1. Thus there is a holomorphic function $W(t)$ on $S^\star$ such that $W(e(z)) = e(\omega(z))$. Since $|W(t)| < 1$ for any $t \in S^\star$, $W(t)$ is still holomorphic over $0 \in S$. On the other hand, the function $\omega(z) - az$ is also invariant under the action of π_1. Thus we have a holomorphic function $g(t)$ on $S^\star$ such that $g(e(z)) = \omega(z) - az$. Then $W(t) = t^a e(g(t))$. Thus $g(t)$ is also holomorphic over $0 \in S$ by Lemma 2.1.2. Therefore $a \geq 0$. Q.E.D.

We can define the types of monodromy representations $\rho\colon \pi_1 \to \mathrm{SL}(2,\mathbb{Z})$ as in Table 2. By Lemmas 2.1.1 and 2.1.3, any monodromy representation ρ is one of types as above, up to conjugation in $\mathrm{SL}(2,\mathbb{Z})$. We call the image of ρ by the *monodromy group*. If the monodromy group is not finite, then ρ is of type $\mathrm{I}_{(+)}$ or $\mathrm{I}_{(+)}^{(*)}$. In the case $\mathrm{I}_{(+)}$, we set $\boldsymbol{a} := (a_1, a_2, \ldots, a_l) \in \mathbb{Z}^{\oplus l}$, where $\rho(\gamma_i)$ is of type I_{a_i} for $1 \leq i \leq l$. Further we put $\alpha := \gcd(\boldsymbol{a})$. In the cases $\mathrm{I}_0^{(*)}$ and $\mathrm{I}_{(+)}^{(*)}$, let c_i be one of $\{0, 1\}$

Table 2. Type of monodromy representations.

I_0	All the $\rho(\gamma_i)$ are of type I_0.
$\mathrm{I}_0^{(*)}$	One of $\rho(\gamma_i)$ is of type I_0^*. Others are of types I_0 or I_0^*.
$\mathrm{II}^{(*)}$	One of $\rho(\gamma_i)$ is of type II or II*. Others are of types I_0, I_0^*, II, II*, IV or IV*.
$\mathrm{III}^{(*)}$	One of $\rho(\gamma_i)$ is of type III or III*. Others are of types I_0, I_0^*, III or III*.
$\mathrm{IV}_+^{(*)}$	One of $\rho(\gamma_i)$ is of type IV or IV*. Others are of types I_0, IV or IV*.
$\mathrm{IV}_-^{(*)}$	One of $\rho(\gamma_i)$ is of type IV or IV* and another $\rho(\gamma_j)$ is of type I_0^*. Others are of types I_0, I_0^*, IV or IV*.
$\mathrm{I}_{(+)}$	Any $\rho(\gamma_i)$ is of type I_{a_i}, where one of a_i is positive.
$\mathrm{I}_{(+)}^{(*)}$	One of $\rho(\gamma_i)$ is of type $\mathrm{I}_{a_i}^*$. Others are of types I_{a_j} or $\mathrm{I}_{a_j}^*$, where one of a_i is positive.

Table 3. Subcases of $\mathrm{I}_{(+)}^{(*)}$.

$\mathrm{I}_{(+)}^{(*)}(0)$	$\boldsymbol{a}^* \equiv 0 \mod 2$
$\mathrm{I}_{(+)}^{(*)}(1)$	$\boldsymbol{a}^* \equiv \boldsymbol{c} \mod 2$
$\mathrm{I}_{(+)}^{(*)}(2)$	$\boldsymbol{a}^* \wedge \boldsymbol{c} \not\equiv 0 \mod 2$

such that $(-1)^{c_i}$ is the eigenvalue of $\rho(\gamma_i)$. We set $\boldsymbol{c} := (c_1, c_2, \dots, c_l)$. In the case $\mathrm{I}_{(+)}^{(*)}$, we further define $a_i^* := (-1)^{c_i} a_i$, where $\rho(\gamma_i)$ is of type I_{a_i} or $\mathrm{I}_{a_i}^*$. We also set $\boldsymbol{a}^* := (a_1^*, a_2^*, \dots, a_l^*)$. We divide the case $\mathrm{I}_{(+)}^{(*)}$ into three subcases as in Table 3.

Proposition 2.1.4. *Let $\rho\colon \pi_1 \to \mathrm{SL}(2,\mathbb{Z})$ and $\omega(z)$, respectively, be the monodromy representation and the period function associated with a variation of Hodge structures of rank two and weight one defined over $S^\star$. The following four conditions are equivalent:*

(1) *The type of the monodromy representation is either $\mathrm{I}_{(+)}$ or $\mathrm{I}_{(+)}^{(*)}$;*
(2) *There exists a holomorphic function h on S such that the period function is given by*

$$\omega(z) = \sum_{i=1}^{l} a_i z_i + h(t),$$

where one of a_i is positive;

(3) *The J-function $J(t) = \boldsymbol{j}(\omega(z))$ is not holomorphic at $\{0\}$ in S*;

(4) *The monodromy group is not a finite group.*

Proof. (1) $\Longrightarrow$ (2): For the function $g(z) := \omega(z) - \sum_{i=1}^{l} a_i z_i$, we have $g(\gamma_i z) = g(z)$ for any γ_i. Thus there is a holomorphic function $h(t)$ on $t \in S^\star$ such that $g(z) = h(t)$. As in the proof of Lemma 2.1.3, we have also a holomorphic function $W(t)$ on S such that $W(t) = \prod_{i=1}^{l} t_i^{a_i} e(h(t))$ on $S^\star$. Therefore $e(h(t))$ is meromorphic on S. Thus $h(t)$ is still holomorphic on S by Lemma 2.1.2.

(2) $\Longrightarrow$ (3): By (2), we have $e(\omega(z)) = u(t) \prod_{i=1}^{l} t_i^{a_i}$ for a nowhere vanishing function $u(t)$ on S. Thus by Remark 1.2.6, $J(t)$ is a meromorphic function with poles of order a_i on each coordinate hyperplane D_i.

(3) $\Longrightarrow$ (4): Suppose that the monodromy group is finite. Then we can take a Kummer covering

$$\tau \colon \Delta^d = \Delta^l \times \Delta^{d-l} \ni \theta = (\theta_1, \theta_2, \dots, \theta_l, t') \mapsto (\theta_1^{m_1}, \theta_2^{m_2}, \dots, \theta_l^{m_l}, t') \in S$$

such that the pullback of the variation of Hodge structures on $\tau^{-1}(S^\star)$ has a trivial monodromy group. Thus there exists a holomorphic function $H(\theta)$ on Δ^d such that $\omega(z) = H(\theta) = H(\theta', t')$ for $\theta' = (e(z_1/m_1), e(z_2/m_2), \dots, e(z_l/m_l))$. Hence $J(t) = \boldsymbol{j}(\omega(z)) = \boldsymbol{j}(H(\theta))$ is holomorphic on Δ^d. Thus $J(t)$ is also holomorphic on S.

(4) $\Longrightarrow$ (1): Trivial. Q.E.D.

Corollary 2.1.5. *The J-function $J(t) = \boldsymbol{j}(\omega(z))$ induces a holomorphic map $J \colon S \to \mathbb{P}^1$. The image contains ∞ if and only if the monodromy representation is of type $\mathrm{I}_{(+)}$ or $\mathrm{I}_{(+)}^{(*)}$.*

The classification of possible period functions $\omega(z)$ is given in Corollary 3.1.6.

2.2. Classification of smooth projective elliptic fibrations over $(\Delta^\star)^l \times \Delta^{d-l}$

Let H be a variation of Hodge structures of weight one and rank two on $S^\star = (\Delta^\star)^l \times \Delta^{d-l}$. We may assume that for the monodromy representation $\rho \colon \pi_1 \to \mathrm{SL}(2, \mathbb{Z})$, every $\rho(\gamma)$ for $\gamma \in \pi_1$ are matrices listed in Table 1. The Hodge filtrations are determined by the period function $\omega(z)$ on U such that

$$\omega(\gamma z) = \frac{a_\gamma \omega(z) + b_\gamma}{c_\gamma \omega(z) + d_\gamma}, \quad \text{where} \quad \rho(\gamma) = \begin{pmatrix} a_\gamma & b_\gamma \\ c_\gamma & d_\gamma \end{pmatrix}.$$

By Proposition 1.3.1, any smooth elliptic fibration on $S^\star$ is isomorphic to $B(H)^\eta \to S^\star$ for some $\eta \in H^1(S^\star, \mathfrak{S}_H)$. Let us consider the exact sequence

$$0 \to H \to \mathcal{L}_H \to \mathfrak{S}_H \to 0.$$

Since $H^i(U, \mathbb{Z}) = H^i(U, \mathcal{O}_U) = 0$ for $i > 0$, applying Lemma 0.2.1, we have isomorphisms:

$$H^p(S^\star, H) \simeq H^p(\pi_1, \mathbb{Z}^{\oplus 2}), \quad H^p(S^\star, \mathcal{L}_H) \simeq H^p(\pi_1, H^0(U, \mathcal{O}_U)),$$
$$H^p(S^\star, \mathfrak{S}_H) \simeq H^p(\pi_1, H^0(U, \mathfrak{S}_{\boldsymbol{e}^{-1}H})),$$
$$H^1(S^\star, \mathfrak{S}_H) \simeq H^2(S^\star, H) \simeq H^2(\pi_1, \mathbb{Z}^{\oplus 2})$$

for any p. From the vanishing $H^1(U, \boldsymbol{e}^{-1}\mathfrak{S}_H) = 0$, we see that any smooth elliptic fibration over U admits a global section. Therefore by Lemma 1.4.1, for any smooth elliptic fibration $X \to S^\star$ having H as a variation of Hodge structures, there is a collection of holomorphic functions $F := \{F_\gamma(z) \,|\, \gamma \in \pi_1\}$ on U such that F satisfies the condition (1.5) and $X \simeq B(H)^F$ over $S^\star$.

Theorem 2.2.1 ([Ny8, (3.1)]). *The group cohomology groups $H^p(\pi_1, \mathbb{Z}^{\oplus 2})$ are calculated as in Table 4.*

Proof. Let $R := \mathbb{Z}[\pi_1] = \mathbb{Z}[\gamma_1^{\pm 1}, \gamma_2^{\pm 1}, \dots, \gamma_l^{\pm 1}]$ be the group ring for $\pi_1 \simeq \mathbb{Z}^{\oplus l}$. Then we have a standard resolution

$$\cdots \to \bigwedge^{p+1}(R^{\oplus l}) \to \bigwedge^{p}(R^{\oplus l}) \to \cdots \to R \to \mathbb{Z} \to 0$$

of the trivial π_1-module $\mathbb{Z}$, where the canonical base $e_{i_0} \wedge e_{i_1} \wedge \cdots \wedge e_{i_p} \in \bigwedge^{p+1}(R^{\oplus l})$ for $1 \le i_0 < i_1 < \cdots < i_p \le l$ is mapped to

$$\sum_{j=0}^{p}(-1)^j e_{i_0} \wedge e_{i_1} \wedge \cdots \wedge \widehat{e_{i_j}} \wedge \cdots \wedge e_{i_p}(1 - \gamma_{i_j}).$$

The group cohomology $H^p(\pi_1, \mathbb{Z}^{\oplus 2})$ for the right π_1-module $\mathbb{Z}^{\oplus 2}$ is isomorphic to the p-th cohomology of the complex

$$\cdots \to \mathrm{Hom}_R(\bigwedge^{p}(R^{\oplus l}), \mathbb{Z}^{\oplus 2}) \xrightarrow{\mathrm{d}^p} \mathrm{Hom}_R(\bigwedge^{p+1}(R^{\oplus l}), \mathbb{Z}^{\oplus 2}) \to \cdots .$$

Let I be the unit matrix. Then the d^p is described as:

$$\mathrm{d}^p(x)(e_{i_0} \wedge e_{i_1} \wedge \cdots \wedge e_{i_p}) = \sum_{j=0}^{p}(-1)^j x(e_{i_0} \wedge e_{i_1} \wedge \cdots \wedge \widehat{e_{i_j}} \wedge \cdots \wedge e_{i_p})(I - \rho(\gamma_{i_j})).$$

We denote $\boldsymbol{i} := \sqrt{-1}$ and $\boldsymbol{\omega} := \exp(2\pi\sqrt{-1}/3)$. Let A be the commutative algebra defined as follows:

$$A := \begin{cases} \mathbb{Z}, & \text{in the cases } \mathrm{I}_0,\ \mathrm{I}_0^{(*)}; \\ \mathbb{Z}[\boldsymbol{\omega}], & \text{in the cases } \mathrm{II}^{(*)},\ \mathrm{IV}_+^{(*)},\ \mathrm{IV}_-^{(*)}; \\ \mathbb{Z}[\boldsymbol{i}], & \text{in the case } \mathrm{III}^{(*)}; \\ \mathbb{Z}[\varepsilon]/(\varepsilon^2), & \text{in the cases } \mathrm{I}_{(+)},\ \mathrm{I}_{(+)}^{(*)}. \end{cases}$$

Then we can consider $\mathbb{Z}^{\oplus 2}$ as an A-module by regarding the elements $\boldsymbol{i}$, $\boldsymbol{\omega}$ and ε as:

$$\boldsymbol{i} \leftrightarrow \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \boldsymbol{\omega} \leftrightarrow \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}, \quad \varepsilon \leftrightarrow \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Thus there is a natural ring homomorphism $\phi\colon R \to A$ from which the R-module structure of $\mathbb{Z}^{\oplus 2}$ is derived. More precisely, $\phi(\gamma)$ is determined according to types of the matrix $\rho(\gamma)$ as in Table 5. For all the cases except I_0, $\mathrm{I}_0^{(*)}$, we have the following isomorphism $\mathbb{Z}^{\oplus 2} \simeq A$ as

Table 4. List of cohomology groups $H^p(\pi_1, \mathbb{Z}^{\oplus 2})$.

Type	H^0	H^1	H^p $(p \geq 2)$
I_0	$\mathbb{Z}^{\oplus 2}$	$\mathbb{Z}^{\oplus 2l}$	$\mathbb{Z}^{\oplus 2\binom{l}{p}}$
$\mathrm{I}_0^{(*)}$	0	$(\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$	$(\mathbb{Z}/2\mathbb{Z})^{\oplus 2\binom{l-1}{p-1}}$
$\mathrm{II}^{(*)}$	0	0	0
$\mathrm{III}^{(*)}$	0	$\mathbb{Z}/2\mathbb{Z}$	$(\mathbb{Z}/2\mathbb{Z})^{\oplus\binom{l-1}{p-1}}$
$\mathrm{IV}_+^{(*)}$	0	$\mathbb{Z}/3\mathbb{Z}$	$(\mathbb{Z}/3\mathbb{Z})^{\oplus\binom{l-1}{p-1}}$
$\mathrm{IV}_-^{(*)}$	0	0	0
$\mathrm{I}_{(+)}$	$\mathbb{Z}$	$\mathbb{Z}^{\oplus l} \oplus \mathbb{Z}/\alpha\mathbb{Z}$	$\mathbb{Z}^{\oplus\binom{l}{p}} \oplus (\mathbb{Z}/\alpha\mathbb{Z})^{\oplus\binom{l-1}{p-1}}$
$\mathrm{I}_{(+)}^{(*)}(0)$	0	$(\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$	$(\mathbb{Z}/2\mathbb{Z})^{\oplus 2\binom{l-1}{p-1}}$
$\mathrm{I}_{(+)}^{(*)}(1)$	0	$\mathbb{Z}/4\mathbb{Z}$	$(\mathbb{Z}/4\mathbb{Z})^{\oplus\binom{l-1}{p-1}}$
$\mathrm{I}_{(+)}^{(*)}(2)$	0	$\mathbb{Z}/2\mathbb{Z}$	$(\mathbb{Z}/2\mathbb{Z})^{\oplus\binom{l-1}{p-1}}$

Table 5. Image of γ.

γ	I_0	I_0^*	II	II*	III	III*	IV	IV*	I_a	I_a^*
$\phi(\gamma)$	1	-1	$-\boldsymbol{\omega}$	$-\boldsymbol{\omega}^2$	$-\boldsymbol{i}$	$\boldsymbol{i}$	$\boldsymbol{\omega}^2$	$\boldsymbol{\omega}$	$1+a\varepsilon$	$-(1+a\varepsilon)$

A-modules:

$$\mathbb{Z}^{\oplus 2} \ni (m,n) \mapsto \begin{cases} m\boldsymbol{\omega}+n, & \text{in the cases } \mathrm{II}^{(*)},\ \mathrm{IV}_+^{(*)},\ \mathrm{IV}_-^{(*)}; \\ m\boldsymbol{i}+n, & \text{in the case } \mathrm{III}^{(*)}; \\ m+n\varepsilon, & \text{in the cases } \mathrm{I}_{(+)},\ \mathrm{I}_{(+)}^{(*)}. \end{cases}$$

We define $b_i := 1-\phi(\gamma_i) \in A$ and $\boldsymbol{b} := (b_1, b_2, \dots, b_l) \in A^{\oplus l}$. Then $H^p(\pi_1, \mathbb{Z}^{\oplus 2})$ is isomorphic to the p-th cohomology group of the following complex:

$$0 \to M \xrightarrow{\boldsymbol{b}\wedge} M \otimes_A (A^{\oplus l}) \xrightarrow{\boldsymbol{b}\wedge} M \otimes_A \bigwedge^2 (A^{\oplus l}) \xrightarrow{\boldsymbol{b}\wedge} \cdots,$$

where $M = \mathbb{Z}^{\oplus 2}$ as an A-module. Here for $\boldsymbol{x} \in M \otimes \bigwedge^p(A^{\oplus l})$, $\boldsymbol{b}\wedge\boldsymbol{x}$ is defined as follows: Let $x_{i_1,i_2,\dots,i_p}$ be the $(i_1,i_2,\dots,i_p)$-coefficient of $\boldsymbol{x}$ for $1 \le i_1 < i_2 < \cdots < i_p \le l$. Then the $(i_0,i_1,\dots,i_p)$-coefficient of $\boldsymbol{b}\wedge\boldsymbol{x}$ for $1 \le i_0 < i_1 < \cdots < i_p \le l$ is defined by:

$$\sum_{j=0}^{p} (-1)^j x_{i_0,i_1,\dots,\widehat{i_j},\dots,i_p} \cdot b_j.$$

We shall calculate the cohomology group $H^p = H^p(\pi_1, \mathbb{Z}^{\oplus 2})$ in each type of monodromy representations.

The case I_0: We have $A=\mathbb{Z}$ and $\boldsymbol{b}=0$. Thus $H^p \simeq \mathbb{Z}^{\oplus 2} \otimes \bigwedge^p(\mathbb{Z}^{\oplus l})$.

The cases $\mathrm{II}^{(*)}$ and $\mathrm{IV}_-^{(*)}$: In the case $\mathrm{II}^{(*)}$, one of b_i is $1+\boldsymbol{\omega} = -\boldsymbol{\omega}^2$ or $1+\boldsymbol{\omega}^2 = -\boldsymbol{\omega}$. Since these are units of $A=\mathbb{Z}[\boldsymbol{\omega}]$, there is a matrix $P \in \mathrm{GL}(l, A)$ such that $\boldsymbol{b} = (1,0,\dots,0)P$. Therefore for an $\boldsymbol{x} \in \bigwedge^p(A^{\oplus l})$, $\boldsymbol{b}\wedge\boldsymbol{x} = 0$ if and only if $\boldsymbol{x} = \boldsymbol{b}\wedge\boldsymbol{y}$ for some $\boldsymbol{y}$. Hence $H^p = 0$ for any p. In the case $\mathrm{IV}_-^{(*)}$, one of b_i is $1-\boldsymbol{\omega}$ or $1-\boldsymbol{\omega}^2$ and another b_j is 2. Since $(1-\boldsymbol{\omega})-2 = \boldsymbol{\omega}^2$ and $(1-\boldsymbol{\omega}^2)-2 = \boldsymbol{\omega}$ are units in A, by the same reason as above, we have $H^p = 0$ for any p.

The case $\mathrm{I}_0^{(*)}$: We have $A=\mathbb{Z}$ and $\boldsymbol{b} = 2\boldsymbol{c}$. Then we can find a matrix $P \in \mathrm{GL}(l,\mathbb{Z})$ such that $\boldsymbol{c} = (1,0,\dots,0)P$. Therefore, if $\boldsymbol{x} \in \bigwedge^p(\mathbb{Z}^{\oplus l})$ satisfies $\boldsymbol{b}\wedge\boldsymbol{x} = 2\boldsymbol{c}\wedge\boldsymbol{x} = 0$, then $\boldsymbol{x} = \boldsymbol{c}\wedge\boldsymbol{y}$ for some $\boldsymbol{y} \in \bigwedge^{p-1}(\mathbb{Z}^{\oplus l})$. Suppose that the $\boldsymbol{x} = \boldsymbol{c}\wedge\boldsymbol{y}$ is written by $\boldsymbol{b}\wedge\boldsymbol{y}'$ for some

$\boldsymbol{y}' \in \bigwedge^{p-1}(\mathbb{Z}^{\oplus l})$. Then $\boldsymbol{y} - 2\boldsymbol{y}' \in \mathrm{Ker}(\boldsymbol{c}\wedge)$. This implies

$$\boldsymbol{y} \mod 2 \in \mathrm{Im}\left(\boldsymbol{c}\wedge\colon \bigwedge^{p-2}((\mathbb{Z}/2\mathbb{Z})^{\oplus l}) \to \bigwedge^{p-1}((\mathbb{Z}/2\mathbb{Z})^{\oplus l})\right). \tag{2.1}$$

Conversely, if $\boldsymbol{y}$ satisfies the condition (2.1), then $\boldsymbol{c}\wedge\boldsymbol{y} = \boldsymbol{b}\wedge\boldsymbol{y}'$ for some $\boldsymbol{y}'$. Therefore we have

$$H^p \simeq \mathbb{Z}^{\oplus 2} \otimes \bigwedge^{p-1}((\mathbb{Z}/2\mathbb{Z})^{\oplus l}/(\mathbb{Z}/2\mathbb{Z})\boldsymbol{c}) \simeq \mathbb{Z}^{\oplus 2} \otimes \bigwedge^{p-1}((\mathbb{Z}/2\mathbb{Z})^{\oplus (l-1)}).$$

The cases $\mathrm{III}^{(*)}$, $\mathrm{IV}_+^{(*)}$, $\mathrm{I}_{(+)}^{(*)}(0)$, $\mathrm{I}_{(+)}^{(*)}(1)$: We have an element $\boldsymbol{u} \in A^{\oplus l}$ such that $\boldsymbol{u} = (1, 0, \ldots, 0)\cdot P$ for some $P \in \mathrm{GL}(l, A)$ and $\boldsymbol{b} = \delta\boldsymbol{u}$. More explicitly, we can choose

$$\delta = \begin{cases} 1 - \boldsymbol{i}, & \text{in the case } \mathrm{III}^{(*)}; \\ 1 - \boldsymbol{\omega}, & \text{in the case } \mathrm{IV}_+^{(*)}; \\ 2, & \text{in the case } \mathrm{I}_{(+)}^{(*)}(0); \\ 2 - \varepsilon, & \text{in the case } \mathrm{I}_{(+)}^{(*)}(1). \end{cases}$$

Further $\boldsymbol{u} = \boldsymbol{c} - (\varepsilon/2)\boldsymbol{a}^*$ and $\boldsymbol{u} = \boldsymbol{c} - (\varepsilon/2)(\boldsymbol{a}^* - \boldsymbol{c})$ in the cases $\mathrm{I}_{(+)}^{(*)}(0)$ and $\mathrm{I}_{(+)}^{(*)}(1)$, respectively. Therefore for an $\boldsymbol{x} \in \bigwedge^p(A^{\oplus l})$, $\boldsymbol{b}\wedge\boldsymbol{x} = 0$ if and only if $\boldsymbol{x} = \boldsymbol{u}\wedge\boldsymbol{y}$ for some $\boldsymbol{y}$. For such $\boldsymbol{y} \in \bigwedge^{p-1}(A^{\oplus l})$, the condition: $\boldsymbol{u}\wedge\boldsymbol{y} = \boldsymbol{b}\wedge\boldsymbol{y}'$ for some $\boldsymbol{y}' \in \bigwedge^{p-1}(A^{\oplus l})$ is equivalent to: $\boldsymbol{y} \mod \delta$ is contained in the image of $\boldsymbol{u}\wedge$. Thus

$$H^p \simeq \bigwedge^{p-1}((A/\delta A)^{\oplus l}/(A/\delta A)\boldsymbol{u}) \simeq \bigwedge^{p-1}((A/\delta A)^{\oplus(l-1)}).$$

We note that

$$A/\delta A \simeq \begin{cases} \mathbb{Z}/2\mathbb{Z}, & \text{in the case } \mathrm{III}^{(*)}; \\ \mathbb{Z}/3\mathbb{Z}, & \text{in the case } \mathrm{IV}_+^{(*)}; \\ (\mathbb{Z}/2\mathbb{Z})^{\oplus 2}, & \text{in the case } \mathrm{I}_{(+)}^{(*)}(0); \\ \mathbb{Z}/4\mathbb{Z}, & \text{in the case } \mathrm{I}_{(+)}^{(*)}(1). \end{cases}$$

The case $\mathrm{I}_{(+)}$: Let $\boldsymbol{u} \in \mathbb{Z}^{\oplus l}$ be the vector such that $\boldsymbol{a} = \alpha\boldsymbol{u}$. Then $\boldsymbol{u} = (1, 0, \ldots, 0)\cdot P$ for some $P \in \mathrm{GL}(l, \mathbb{Z})$. We have $\boldsymbol{b} = -\alpha\varepsilon\boldsymbol{u}$. We take an element $\boldsymbol{x} = \boldsymbol{x}_0 + \varepsilon\boldsymbol{x}_1 \in \bigwedge^p(A^{\oplus l})$, where $\boldsymbol{x}_0, \boldsymbol{x}_1 \in \bigwedge^p(\mathbb{Z}^{\oplus l})$. Then $\boldsymbol{b}\wedge \boldsymbol{x} = 0$ if and only if $\boldsymbol{x}_0 = \boldsymbol{u}\wedge\boldsymbol{y}_0$ for some $\boldsymbol{y}_0 \in \bigwedge^{p-1}(\mathbb{Z}^{\oplus l})$. Furthermore for such $\boldsymbol{y}_0$ and $\boldsymbol{x}_1$, $\boldsymbol{u}\wedge\boldsymbol{y}_0 + \varepsilon\boldsymbol{x}_1 = \boldsymbol{b}\wedge\boldsymbol{v}$ for some $\boldsymbol{v} \in \bigwedge^{p-1}(A^{\oplus l})$ if and only if $\boldsymbol{u} \wedge \boldsymbol{y}_0 = 0$ and $\boldsymbol{x}_1 = \alpha \boldsymbol{u} \wedge \boldsymbol{v}_0$ for some $\boldsymbol{v}_0 \in \bigwedge^{p-1}(\mathbb{Z}^{\oplus l})$. Therefore H^p is isomorphic to

$$\begin{aligned}
&\operatorname{Im}\left(\boldsymbol{u}\wedge\colon \bigwedge^{p-1}(\mathbb{Z}^{\oplus l}) \to \bigwedge^{p}(\mathbb{Z}^{\oplus l})\right) \oplus \operatorname{Coker}\left(\alpha\boldsymbol{u}\wedge\colon \bigwedge^{p-1}(\mathbb{Z}^{\oplus l}) \to \bigwedge^{p}(\mathbb{Z}^{\oplus l})\right) \\
&\simeq \bigwedge^{p-1}(\mathbb{Z}^{\oplus l}/\mathbb{Z}\boldsymbol{u}) \oplus \bigwedge^{p}(\mathbb{Z}^{\oplus l})\Big/(\alpha\boldsymbol{u} \wedge \bigwedge^{p-1}(\mathbb{Z}^{\oplus l})) \\
&\simeq \bigwedge^{p-1}(\mathbb{Z}^{\oplus l}/\mathbb{Z}\boldsymbol{u}) \oplus \bigwedge^{p}(\mathbb{Z}^{\oplus l}/\mathbb{Z}\boldsymbol{u}) \oplus \left(\bigwedge^{p-1}(\mathbb{Z}^{\oplus l}/\mathbb{Z}\boldsymbol{u}) \otimes \mathbb{Z}/\alpha\mathbb{Z}\right) \\
&\simeq \mathbb{Z}^{\binom{l}{p}} \oplus (\mathbb{Z}/\alpha\mathbb{Z})^{\oplus\binom{l-1}{p-1}}.
\end{aligned}$$

The case $\mathrm{I}^{(*)}_{(+)}(2)$: We have $\boldsymbol{b} = 2\boldsymbol{c} - \varepsilon \boldsymbol{a}^*$, where $\boldsymbol{c} \wedge \boldsymbol{a}^* \not\equiv 0 \mod 2$. Let us take an element $\boldsymbol{x} = \boldsymbol{x}_0 + \varepsilon \boldsymbol{x}_1 \in \bigwedge^p(A^{\oplus l})$, where $\boldsymbol{x}_0, \boldsymbol{x}_1 \in \bigwedge^p(\mathbb{Z}^{\oplus l})$. Suppose that $\boldsymbol{b} \wedge \boldsymbol{x} = 0$. Then $\boldsymbol{c} \wedge \boldsymbol{x}_0 = 0$ and $\boldsymbol{a}^* \wedge \boldsymbol{x}_0 = 2\boldsymbol{c} \wedge \boldsymbol{x}_1$. Thus there exist $\boldsymbol{y}_0, \boldsymbol{y}_1 \in \bigwedge^{p-1}(\mathbb{Z}^{\oplus l})$ such that

$$\boldsymbol{x}_0 = \boldsymbol{c} \wedge \boldsymbol{y}_0, \quad 2\boldsymbol{x}_1 = -\boldsymbol{a}^* \wedge \boldsymbol{y}_0 + \boldsymbol{c} \wedge \boldsymbol{y}_1.$$

Since $\boldsymbol{c} \wedge \boldsymbol{a}^* \not\equiv 0 \mod 2$, we have $\boldsymbol{z}_0, \boldsymbol{z}_1, \boldsymbol{z}_2 \in \bigwedge^{p-2}(\mathbb{Z}^{\oplus l})$ and $\boldsymbol{y}'_0, \boldsymbol{y}'_1 \in \bigwedge^{p-1}(\mathbb{Z}^{\oplus l})$ such that

$$\boldsymbol{y}_0 = \boldsymbol{c} \wedge \boldsymbol{z}_0 + \boldsymbol{a}^* \wedge \boldsymbol{z}_1 + 2\boldsymbol{y}'_0, \quad \boldsymbol{y}_1 = \boldsymbol{c} \wedge \boldsymbol{z}_2 - \boldsymbol{a}^* \wedge \boldsymbol{z}_0 + 2\boldsymbol{y}'_1.$$

Therefore we have

$$\boldsymbol{x}_0 = \boldsymbol{c} \wedge \boldsymbol{a}^* \wedge \boldsymbol{z}_1 + 2\boldsymbol{c} \wedge \boldsymbol{y}'_0, \quad \boldsymbol{x}_1 = -\boldsymbol{a}^* \wedge \boldsymbol{y}'_0 + \boldsymbol{c} \wedge \boldsymbol{y}'_1. \tag{2.2}$$

Conversely, if there exist $\boldsymbol{z}_1, \boldsymbol{y}'_0, \boldsymbol{y}'_1$ satisfying (2.2), then $\boldsymbol{x} = \boldsymbol{x}_0 + \varepsilon \boldsymbol{x}_1$ satisfies $\boldsymbol{b} \wedge \boldsymbol{x} = 0$. Next for such $\boldsymbol{z}_1, \boldsymbol{y}'_0, \boldsymbol{y}'_1$, suppose that $\boldsymbol{x}_0 + \varepsilon \boldsymbol{x}_1 = \boldsymbol{b} \wedge (\boldsymbol{w}_0 + \varepsilon \boldsymbol{w}_1)$ for some $\boldsymbol{w}_0, \boldsymbol{w}_1 \in \bigwedge^{p-1}(\mathbb{Z}^{\oplus l})$. Then we have

$$\boldsymbol{c} \wedge \boldsymbol{a}^* \wedge \boldsymbol{z}_1 + 2\boldsymbol{c} \wedge \boldsymbol{y}'_0 = 2\boldsymbol{c} \wedge \boldsymbol{w}_0, \quad -\boldsymbol{a}^* \wedge \boldsymbol{y}'_0 + \boldsymbol{c} \wedge \boldsymbol{y}'_1 = 2\boldsymbol{c} \wedge \boldsymbol{w}_1 - \boldsymbol{a}^* \wedge \boldsymbol{w}_0.$$

Therefore there exist $\boldsymbol{v}_0, \boldsymbol{v}_1 \in \bigwedge^{p-3}(\mathbb{Z}^{\oplus 2})$ and $\boldsymbol{z}'_1, \boldsymbol{q} \in \bigwedge^{p-2}(\mathbb{Z}^{\oplus l})$ such that

$$\begin{aligned}
&\boldsymbol{z}_1 = \boldsymbol{a}^* \wedge \boldsymbol{v}_0 + \boldsymbol{c} \wedge \boldsymbol{v}_1 + 2\boldsymbol{z}'_1, \quad \boldsymbol{w}_0 = \boldsymbol{y}'_0 + \boldsymbol{a}^* \wedge \boldsymbol{z}'_1 + \boldsymbol{c} \wedge \boldsymbol{q}, \\
&\boldsymbol{c} \wedge \boldsymbol{y}'_1 = \boldsymbol{c} \wedge (\boldsymbol{a}^* \wedge \boldsymbol{q} + 2\boldsymbol{w}_1).
\end{aligned}$$

Hence we see

$$\boldsymbol{c} \wedge \boldsymbol{a}^* \wedge \boldsymbol{z}_1 \equiv 0 \mod 2, \quad \boldsymbol{c} \wedge \boldsymbol{a}^* \wedge \boldsymbol{y}'_1 \equiv 0 \mod 2. \tag{2.3}$$

Conversely, if $\boldsymbol{z}_1$ and $\boldsymbol{y}'_1$ satisfy the condition (2.3), then $\boldsymbol{x}_0 + \varepsilon\boldsymbol{x}_1 = \boldsymbol{b}\wedge(\boldsymbol{w}_0+\varepsilon\boldsymbol{w}_1)$ for some $\boldsymbol{w}_0, \boldsymbol{w}_1 \in \bigwedge^{p-1}(\mathbb{Z}^{\oplus l})$. Therefore H^p is isomorphic to

$$\operatorname{Im}\left(\boldsymbol{c}\wedge\boldsymbol{a}^*\wedge\colon \bigwedge^{p-2}(\mathbb{Z}^{\oplus l}) \to \bigwedge^{p}(\mathbb{Z}/2\mathbb{Z})^{\oplus l}\right) \oplus \operatorname{Im}\left(\boldsymbol{c}\wedge\boldsymbol{a}^*\wedge\colon \bigwedge^{p-1}(\mathbb{Z}^{\oplus l}) \to \bigwedge^{p+1}(\mathbb{Z}/2\mathbb{Z})^{\oplus l}\right)$$
$$\simeq \bigwedge^{p-2}(\mathbb{Z}/2\mathbb{Z})^{\oplus(l-2)} \oplus \bigwedge^{p-1}(\mathbb{Z}/2\mathbb{Z})^{\oplus(l-2)} \simeq (\mathbb{Z}/2\mathbb{Z})^{\oplus\binom{l-1}{p-1}}.$$

Thus we are done. Q.E.D.

In order to obtain a collection $\{F_\gamma(z)\}$ of holomorphic functions on $U = \mathbb{H}^l \times \Delta^{d-l}$ satisfying (1.5), it is enough to have a collection of holomorphic functions $F = \{F_i(z)\}_{i=1}^l$ satisfying the condition:

$$\text{(2.4)}\quad F_j(z) - (c_{\gamma_i}\omega(z) + d_{\gamma_i})F_j(\gamma_i z) \equiv F_i(z) - (c_{\gamma_j}\omega(z) + d_{\gamma_j})F_i(\gamma_j z) \mod \mathbb{Z}\omega(z) + \mathbb{Z}$$

for all $1 \le i, j \le l$. Once we have a collection F satisfying the condition (2.4), then we have a smooth elliptic fibration $B^F \to S^\star$ as the quotient of B_U by the following action of $\gamma_i \in \pi_1 \simeq \mathbb{Z}^{\oplus l}$:

$$[z, \zeta] \mapsto \left[\gamma_i z, \frac{\zeta + F_i(z)}{c_{\gamma_i}\omega(z) + d_{\gamma_i}}\right],$$

where $[z, \zeta]$ denotes the image of $(z, \zeta) \in U \times \mathbb{C}$ under the morphism $U \times \mathbb{C} \to B_U \simeq U \times \mathbb{C}/(\mathbb{Z}\omega + \mathbb{Z})$. For two collections $F = \{F_i\}_{i=1}^l$ and $F' = \{F'_i\}_{i=1}^l$, they induce same elliptic fibration if and only if there exists a holomorphic function $H(z)$ on U such that

$$\text{(2.5)}\quad F_i(z) - F'_i(z) \equiv H(z) - (c_{\gamma_i}\omega(z) + d_{\gamma_i})H(\gamma_i z) \mod \mathbb{Z}\omega(z) + \mathbb{Z}.$$

For a collection F, let $(P_{i,j}, Q_{i,j})$ for $1 \le i < j \le l$ be pairs of integers defined by

$$P_{i,j}\omega(z) + Q_{i,j} := F_i(z) - (c_{\gamma_j}\omega(z) + d_{\gamma_j})F_i(\gamma_j z) - (F_j(z) - (c_{\gamma_i}\omega(z) + d_{\gamma_i})F_j(\gamma_i z)).$$

Then $\{(P_{i,j}, Q_{i,j})\}$ defines an element $\boldsymbol{x}$ of $M \otimes_A \bigwedge^2(A^{\oplus l})$, where $M = \mathbb{Z}^{\oplus 2}$ as an A-module (cf. Theorem 2.2.1). Here $\boldsymbol{b} \wedge \boldsymbol{x} = 0$. In order to determine all the possible smooth elliptic fibrations, we have only to find collections F of holomorphic functions which cover all the representatives $\boldsymbol{x}$ of the cohomology group $H^2(\pi_1, \mathbb{Z}^{\oplus 2})$. We shall give such collections of holomorphic functions explicitly.

The case I_0: Let $m_{i,j}$ and $n_{i,j}$ be integers for $1 \le i, j \le l$ such that $m_{j,i} = -m_{i,j}$ and $n_{j,i} = -n_{i,j}$. We have

$$F_j(z) - F_j(\gamma_i z) - (F_i(z) - F_i(\gamma_j z)) = m_{i,j}\omega(z) + n_{i,j}$$

for the functions

$$F_i(z) := \frac{1}{2}\sum_{k=1}^{l}(m_{i,k}\omega(z) + n_{i,k})z_k.$$

Thus the cohomology class in $H^2(\pi_1, \mathbb{Z}^{\oplus 2})$ induced from the collection $\{F_i(z)\}$ is essentially $\{(m_{i,j}, n_{i,j})\} \in (\bigwedge^2(\mathbb{Z}^{\oplus l}))^{\oplus 2}$.

The case $\mathrm{I}_0^{(*)}$: In the proof of Theorem 2.2.1, if $\boldsymbol{x} \in \bigwedge^2(\mathbb{Z}^{\oplus l})$ satisfies $\boldsymbol{b} \wedge \boldsymbol{x} = 0$, then $\boldsymbol{x} = \boldsymbol{c} \wedge \boldsymbol{y}$ for some $\boldsymbol{y} \in \mathbb{Z}^{\oplus l}$. Let $\boldsymbol{y}_1 := (m_1, m_2, \ldots, m_l)$ and $\boldsymbol{y}_2 := (n_1, n_2, \ldots, n_l)$ be elements of $\mathbb{Z}^{\oplus l}$. Then for any $0 \le i \ne j \le l$, the (i,j) components of vectors $\boldsymbol{c} \wedge \boldsymbol{y}_1$ and $\boldsymbol{c} \wedge \boldsymbol{y}_2$ are $c_i m_j - c_j m_i$ and $c_i n_j - c_j n_i$, respectively. We have

$$\begin{aligned} F_j(z) - (-1)^{c_i} F_j(\gamma_i z) - (F_i(z) - (-1)^{c_j} F_i(\gamma_j z)) \\ = (c_i m_j - c_j m_i)\omega(z) + (c_i n_j - c_j n_i) \end{aligned}$$

for the functions

$$F_i(z) := (m_i/2)\omega(z) + (n_i/2).$$

Therefore these collections $\{F_i\}$ induce all the cohomology classes in $H^2(\pi_1, \mathbb{Z}^{\oplus 2})$. By the proof of Theorem 2.2.1, for two collections of integers (m_i, n_i) and (m'_i, n'_i), the corresponding $\{F_i\}$'s determine the same cohomology class if and only if

$$m_i - m'_i \equiv k_1 c_i \mod 2 \quad \text{and} \quad n_i - n'_i \equiv k_2 c_i \mod 2$$

for some integers k_1, k_2.

The cases $\mathrm{II}^{(*)}$ *and* $\mathrm{IV}_{-}^{(*)}$: We have $H^2(\pi_1, \mathbb{Z}^{\oplus 2}) = 0$. Hence it is enough to set $F_i(z) = 0$ for all i.

The cases $\mathrm{III}^{(*)}$, $\mathrm{IV}_{+}^{(*)}$, $\mathrm{I}_{(+)}^{(*)}(0)$, $\mathrm{I}_{(+)}^{(*)}(1)$: In the proof of Theorem 2.2.1, we write $\boldsymbol{b} = \delta\boldsymbol{u}$ and $\boldsymbol{u} = (1, 0, \ldots, 0) \cdot P$ for some $P \in \mathrm{GL}(l, A)$. If $\boldsymbol{x} \in \bigwedge^2(A^{\oplus l})$ satisfies $\boldsymbol{b} \wedge \boldsymbol{x} = 0$, then $\boldsymbol{x} = \boldsymbol{u} \wedge \boldsymbol{y}$ for some $\boldsymbol{y} \in A^{\oplus l}$.

For $\boldsymbol{y} = (y_1, y_2, \ldots, y_l) \in A^{\oplus l}$, there are integers m_i, n_i for $1 \leq i \leq l$ such that

$$y_i = \begin{cases} m_i \boldsymbol{i} + n_i, & \text{in the case } \mathrm{III}^{(*)}; \\ m_i \boldsymbol{\omega} + n_i, & \text{in the case } \mathrm{IV}_{+}^{(*)}; \\ m_i + n_i \varepsilon & \text{in the cases } \mathrm{I}_{(+)}^{(*)}(0),\ \mathrm{I}_{(+)}^{(*)}(1). \end{cases}$$

Looking at $\boldsymbol{u} \wedge \boldsymbol{y} = \delta^{-1} \boldsymbol{b} \wedge \boldsymbol{y}$, we define rational numbers p_i, q_i for $1 \leq i \leq l$ by

$$\delta^{-1} y_i = \begin{cases} p_i \boldsymbol{i} + q_i, & \text{in the case } \mathrm{III}^{(*)}; \\ p_i \boldsymbol{\omega} + q_i & \text{in the case } \mathrm{IV}_{+}^{(*)}; \\ p_i + q_i \varepsilon & \text{in the cases } \mathrm{I}_{(+)}^{(*)}(0),\ \mathrm{I}_{(+)}^{(*)}(1). \end{cases}$$

We set $F_i(z) := p_i \omega(z) + q_i$. Then $(P_{i,j}, Q_{i,j})$ defined by $\{F_i\}$ as above is calculated by

$$(P_{i,j}, Q_{i,j}) = (p_i, q_i)(I - \rho(\gamma_j)) - (p_j, q_j)(I - \rho(\gamma_i)).$$

Thus $\boldsymbol{x}$ induced from $(P_{i,j}, Q_{i,j})$ corresponds to $\boldsymbol{u} \wedge \boldsymbol{y}$. Hence such collections $\{F_i(z)\}$ cover all the cohomology classes in $H^2(\pi_1, \mathbb{Z}^{\oplus 2})$. We have the following expression of $F_i(z)$ by means of (m_i, n_i):

$$F_i(z) = \begin{cases} \dfrac{m_i + n_i}{2}\omega(z) + \dfrac{n_i - m_i}{2}, & \text{in the case } \mathrm{III}^{(*)}; \\ \dfrac{m_i + n_i}{3}\omega(z) + \dfrac{2n_i - m_i}{3}, & \text{in the case } \mathrm{IV}_{+}^{(*)}; \\ \dfrac{m_i}{2}\omega(z) + \dfrac{n_i}{2}, & \text{in the case } \mathrm{I}_{(+)}^{(*)}(0); \\ \dfrac{m_i}{2}\omega(z) + \dfrac{m_i + 2n_i}{4}, & \text{in the case } \mathrm{I}_{(+)}^{(*)}(1). \end{cases}$$

For two collections of pairs of integers $\{(m_i, n_i)\}$ and $\{(m'_i, n'_i)\}$, they define a same cohomology class if and only if there is an integer k such that

$$k\delta^{-1} b_i = \begin{cases} (m_i - m'_i)\boldsymbol{i} + (n_i - n'_i), & \text{in the case } \mathrm{III}^{(*)}; \\ (m_i - m'_i)\boldsymbol{\omega} + (n_i - n'_i), & \text{in the case } \mathrm{IV}_{+}^{(*)}; \\ (m_i - m'_i) + (n_i - n'_i)\varepsilon, & \text{in the cases } \mathrm{I}_{(+)}^{(*)}(0),\ \mathrm{I}_{(+)}^{(*)}(1). \end{cases}$$

The case $\mathrm{I}_{(+)}$: Let $\omega(z) = \sum_{i=1}^{l} a_i z_i + h(t)$ be the period function. Let m_i and $n_{i,j}$ are integers for $1 \leq i, j \leq l$ such that $n_{j,i} = -n_{i,j}$. Let

$\alpha = \gcd(a_1, a_2, \dots, a_l)$. We set

$$F_i(z) := \frac{1}{2\alpha}(m_i\omega(z)^2 - \sum_{k=1}^{l}(m_i a_k^2 + \alpha n_{i,k})z_k).$$

Then for $1 \le i, j \le l$,

$$F_j(\gamma_i z) - F_j(z) - (F_i(\gamma_j z) - F_i(z)) = \frac{1}{\alpha}(a_i m_j - a_j m_i)\omega(z) + n_{i,j}.$$

By the proof of Theorem 2.2.1, these $\{F_i\}$ cover all the cohomology classes in $H^2(\pi_1, \mathbb{Z}^{\oplus 2})$. For two collections $\{m_i, n_{i,j}\}$ and $\{m'_i, n'_{i,j}\}$, the corresponding $\{F_i\}$'s determine same cohomology class if and only if there exists a vector $(v_1, v_2, \dots, v_l) \in \mathbb{Z}^{\oplus l}$ such that

$$a_i(m_j - m'_j) = a_j(m_i - m'_i) \quad \text{and} \quad n_{i,j} - n'_{i,j} = a_i v_j - a_j v_i.$$

The case $\mathrm{I}_{(+)}^{(*)}(2)$: By the proof of Theorem 2.2.1, if an $\boldsymbol{x} \in \bigwedge^2(A^{\oplus l})$ satisfies $\boldsymbol{b} \wedge \boldsymbol{x} = 0$, where $A = \mathbb{Z}[\varepsilon]$, $\boldsymbol{b} = 2\boldsymbol{c} - \varepsilon\boldsymbol{a}^*$, then there exist vectors $\boldsymbol{y}'_1, \boldsymbol{w}_0, \boldsymbol{w}_1 \in \mathbb{Z}^{\oplus l}$ and an integer z_1 such that

$$\boldsymbol{x} = z_1 \boldsymbol{c} \wedge \boldsymbol{a}^* + \varepsilon \boldsymbol{c} \wedge \boldsymbol{y}'_1 + \boldsymbol{b} \wedge (\boldsymbol{w}_0 + \varepsilon \boldsymbol{w}_1).$$

We denote $z_1 = m$ and $\boldsymbol{y}'_1 = (n_1, n_2, \dots, n_l)$. If we set $p_i = (m/2)a_i^*$, $q_i = n_i/2$, then

$$(2c_i - \varepsilon a_i^*)(p_j + \varepsilon q_j) - (2c_j - \varepsilon a_j^*)(p_i + \varepsilon q_i) = m(c_i a_j^* - c_j a_i^*) + \varepsilon(c_i n_j - c_j n_i).$$

Let us consider the functions:

$$F_i(z) = \frac{m a_i^*}{2}\omega(z) + \frac{n_i}{2}.$$

Then $\{F_i(z)\}$ satisfies the cocycle condition (2.4) and every element in $H^2(\pi_1, \mathbb{Z}^{\oplus 2})$ is induced from such $\{F_i(z)\}$ for some $m, n_1, n_2, \dots, n_l \in \mathbb{Z}$. For two collections of integers $\{m, n_1, n_2, \dots, n_l\}$, $\{m', n'_1, n'_2, \dots, n'_l\}$, the corresponding $\{F_i(z)\}$'s determine same cohomology class if and only if $m \equiv m' \mod 2$ and

$$(n_1, n_2, \dots, n_l) \wedge \boldsymbol{c} \wedge \boldsymbol{a}^* \equiv (n'_1, n'_2, \dots, n'_l) \wedge \boldsymbol{c} \wedge \boldsymbol{a}^* \mod 2.$$

Therefore by combining with Proposition 1.3.3, we have:

Theorem 2.2.2. *Let* $f\colon X \to S^\star \simeq (\Delta^\star)^l \times \Delta^{d-l}$ *be a smooth elliptic fibration. Then* X *is isomorphic to the quotient of the total space* B_U *of the basic fibration* $B_U \to U$ *by the following action of* $\pi_1 \simeq \mathbb{Z}^{\oplus l}$:

$$B_U \ni [z, \zeta] \mapsto \left[\gamma_i z, \frac{\zeta + F_i(z)}{c_{\gamma_i}\omega(z) + d_{\gamma_i}}\right],$$

Table 6. Collections of holomorphic functions.

Type	$F_i(z)$	Condition
I_0	$\frac{1}{2}\sum_{k=1}^{l}(m_{i,k}\omega(z)+n_{i,k})z_k.$	$m_{i,j}, n_{i,j} \in \mathbb{Z}$: $m_{j,i}=-m_{i,j}$, $n_{j,i}=-n_{i,j}$
$\mathrm{I}_0^{(*)}$	$\frac{m_i}{2}\omega(z)+\frac{n_i}{2}$	$m_i, n_i \in \mathbb{Z}$
$\mathrm{II}^{(*)}$	0	
$\mathrm{III}^{(*)}$	$\frac{m_i+n_i}{2}\omega(z)+\frac{n_i-m_i}{2}$	$m_i, n_i \in \mathbb{Z}$
$\mathrm{IV}_+^{(*)}$	$\frac{m_i+n_i}{3}\omega(z)+\frac{2n_i-m_i}{3}$	$m_i, n_i \in \mathbb{Z}$
$\mathrm{IV}_-^{(*)}$	0	
$\mathrm{I}_{(+)}$	$\frac{1}{2\alpha}\left(m_i\omega(z)^2-\sum_{k=1}^{l}(m_i a_k^2+\alpha n_{i,k})z_k\right)$	$m_i, n_{i,j} \in \mathbb{Z}$: $n_{j,i}=-n_{i,j}$
$\mathrm{I}_{(+)}^{(*)}(0)$	$\frac{m_i}{2}\omega(z)+\frac{n_i}{2}$	$m_i, n_i \in \mathbb{Z}$
$\mathrm{I}_{(+)}^{(*)}(1)$	$\frac{m_i}{2}\omega(z)+\frac{m_i+2n_i}{4}$	$m_i, n_i \in \mathbb{Z}$
$\mathrm{I}_{(+)}^{(*)}(2)$	$\frac{ma_i^*}{2}\omega(z)+\frac{n_i}{2}$	$m, n_i \in \mathbb{Z}$

where $[z,\zeta] \in B_U$ *is the image of a point* $(z,\zeta) \in U \times \mathbb{C}$, *and* $\{F_i(z)\}$ *is one of the collections of holomorphic functions listed in Table* 6. *If* H *is not of type* I_0 *nor* $\mathrm{I}_{(+)}$, *then* f *is a projective morphism. If* H *is of type* I_0 *and* f *is projective, then we can take* $F_i(z)=0$ *for any* i.

§3. Canonical extensions of variations of Hodge structures

3.1. Canonical extensions

Let H be a variation of Hodge structures of weight one, rank two on $S^\star = (\Delta^\star)^l \times \Delta^{d-l}$. As in §§1 and 2, H is determined by the monodromy representation $\rho\colon \pi_1 := \pi_1(S^\star) \simeq \mathbb{Z}^{\oplus l} \to \mathrm{SL}(2,\mathbb{Z})$ and the period function $\omega(z)$. Let $\rho(\gamma) = S(\gamma)U(\gamma)$ be a decomposition of the monodromy matrix $\rho(\gamma)$ for $\gamma \in \pi_1$ such that $S(\gamma)$ is semi-simple, $U(\gamma)$ is unipotent, and $S(\gamma)U(\gamma) = U(\gamma)S(\gamma)$. If H is one of types I_0, $\mathrm{I}_0^{(*)}$, $\mathrm{I}_{(+)}$, $\mathrm{I}_{(+)}^{(*)}$, then $S(\gamma) = \pm I$ for any $\gamma \in \pi_1$, where I denotes the unit matrix. If H is of other type, then $U(\gamma) = I$ for any $\gamma \in \pi_1$. Thus all $S(\gamma)$ and $U(\gamma)$ are uniquely determined and commute to each other. The eigenvalue of

Table 7. Order of $S(\gamma_i)$.

$\rho(\gamma_i)$	I_a	I_b^*	II	II*	III	III*	IV	IV*
m_i	1	2	6	6	4	4	3	3

$S(\gamma_i)$ is contained in $\{\pm 1, \pm\boldsymbol{\omega}^{\pm 1}, \pm\boldsymbol{i}\}$. Let m_i be its order (cf. Table 7). Now we consider the *unipotent reduction* of H. This is a Kummer covering defined by:

$$\tau\colon T = \Delta^l \times \Delta^{d-l} \ni \theta = (\theta_1, \theta_2, \dots, \theta_l, t') \longmapsto (\theta_1^{m_1}, \theta_2^{m_2}, \dots, \theta_l^{m_l}, t') \in \Delta^l \times \Delta^{d-l} = S.$$

We denote $T^\star := \tau^{-1}(S^\star)$ and let $H_T := \tau^{-1}H$ be the induced variation of Hodge structures on $T^\star$. Then all the monodromy matrices of H_T are unipotent. Let N_i, R_i^u and R_i^ℓ for $1 \le i \le l$ be the matrices satisfying the following four conditions:

(1) N_i is nilpotent and $\exp(N_i) = U(\gamma_i)$;
(2) R_i^u, R_i^ℓ are semi-simple and $\exp(R_i^u) = \exp(R_i^\ell) = S(\gamma_i)$;
(3) All the eigenvalues of R_i^u are contained in $2\pi\sqrt{-1}\,(-1, 0]$;
(4) All the eigenvalues of R_i^ℓ are contained in $2\pi\sqrt{-1}\,[0, 1)$.

Then these matrices also commute to each other. Let M_i^u and M_i^ℓ be the matrices $R_i^u + N_i$ and $R_i^\ell + N_i$, respectively, for $1 \le i \le l$. Let $\boldsymbol{e}\colon U = \mathbb{H}^l \times \Delta^{d-l} \to (\Delta^\star)^l \times \Delta^{d-l} = S^\star$ be the universal covering map defined in §2.1 and let (e_0, e_1) be the basis of $H^0(U, \boldsymbol{e}^{-1}H) \simeq \mathbb{Z}^{\oplus 2}$ defined in §1.1. The e_1 and e_0, respectively, are identified with column vectors

$$e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad e_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and they satisfy $Q(e_0, e_1) = 1$, where Q is the polarization of H. Let $\mathcal{H} = H \otimes \mathcal{O}_{S^\star}$. Then $\boldsymbol{e}^{-1}\mathcal{H} \simeq \mathcal{O}_U^{\oplus 2}$. Therefore as in §2.1, the right action of $\gamma \in \pi_1$ induced from $\mathcal{H}$ on $H^0(U, \mathcal{O}^{\oplus 2})$ is written by

$$\boldsymbol{v}^\gamma(z) := \rho(\gamma)^{-1}\boldsymbol{v}(\gamma z),$$

where we consider $\boldsymbol{v}(z) \in H^0(U, \mathcal{O}_U^{\oplus 2})$ as a column vector. The holomorphic vector $\boldsymbol{v}(z)$ is invariant under this action if and only if $\boldsymbol{v} = \boldsymbol{e}^* v$ for some $v \in H^0(S^\star, \mathcal{H})$. Therefore for holomorphic vectors

$$^u\boldsymbol{v}_1(z) := \exp(\sum_{i=1}^l z_i M_i^u) \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad ^u\boldsymbol{v}_0(z) := \exp(\sum_{i=1}^l z_i M_i^u) \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

$$^\ell\boldsymbol{v}_1(z) := \exp(\sum_{i=1}^l z_i M_i^\ell) \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad ^\ell\boldsymbol{v}_0(z) := \exp(\sum_{i=1}^l z_i M_i^\ell) \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

we have global holomorphic sections ${}^uv_1$, ${}^uv_0$, ${}^\ell v_1$, ${}^\ell v_0 \in \Gamma(S^\star, \mathcal{H})$ such that ${}^u\boldsymbol{v}_i = \boldsymbol{e}^*({}^uv_i)$ and ${}^\ell\boldsymbol{v}_i = \boldsymbol{e}^*({}^\ell v_i)$ for $i = 0, 1$. Thus $\mathcal{H} = \mathcal{O}_{S^\star}\, {}^uv_1 \oplus \mathcal{O}_{S^\star}\, {}^uv_0 = \mathcal{O}_{S^\star}\, {}^\ell v_1 \oplus \mathcal{O}_{S^\star}\, {}^\ell v_0$.

Definition 3.1.1 (cf. [Kl1], [Mw]). The *upper* and the *lower canonical extensions* ${}^u\mathcal{H}$ and ${}^\ell\mathcal{H}$ of $\mathcal{H}$ to S are defined to be the subsheaves $\mathcal{O}_S\, {}^uv_1 \oplus \mathcal{O}_S\, {}^uv_2$ and $\mathcal{O}_S\, {}^\ell v_1 \oplus \mathcal{O}_S\, {}^\ell v_2$ of $j_*\mathcal{H}$, respectively, where $j\colon S^\star \hookrightarrow S$ denotes the open immersion. We define the induced filtration by:

$$\mathcal{F}^p({}^u\mathcal{H}) := j_*\mathcal{F}^p(\mathcal{H}) \cap {}^u\mathcal{H}, \quad \mathcal{F}^p({}^\ell\mathcal{H}) := j_*\mathcal{F}^p(\mathcal{H}) \cap {}^\ell\mathcal{H}$$

and define a quotient sheaf $\mathcal{L}_{H/S} := {}^\ell\mathcal{H}/\mathcal{F}^1({}^\ell\mathcal{H})$.

Remark. (1). We have ${}^\ell\mathcal{H} \subset {}^u\mathcal{H}$. If the monodromy matrices $\rho(\gamma_i)$ are all unipotent, then ${}^u\mathcal{H} = {}^\ell\mathcal{H}$. Thus ${}^u(\tau^*\mathcal{H}) = {}^\ell(\tau^*\mathcal{H})$. We see that ${}^\ell\mathcal{H}$ is the $\mathrm{Gal}(\tau)$-invariant part of $\tau_*({}^\ell(\tau^*\mathcal{H}))$ and that ${}^\ell(\mathcal{H}^\vee) \simeq ({}^u\mathcal{H})^\vee$, where $\mathcal{F}^\vee$ denotes the dual $\mathcal{H}om(\mathcal{F}, \mathcal{O})$.

(2). Let H be a variation of Hodge structures of weight one, rank two on $M \setminus D$, where M is a complex manifold and D is a normal crossing divisor on M. Then the local canonical extensions ${}^u\mathcal{H}$ and ${}^\ell\mathcal{H}$ are patched together. Thus we can define globally the upper and the lower canonical extensions to M.

The following result is known as a part of the nilpotent orbit theorem [Sc].

Lemma 3.1.2. *$\mathcal{F}^1({}^u\mathcal{H})$ and $\mathcal{F}^1({}^\ell\mathcal{H})$ are subbundles of rank one of ${}^u\mathcal{H}$ and ${}^\ell\mathcal{H}$, respectively. In particular, $\mathcal{L}_{H/S}$ is an invertible sheaf.*

Proof. $\mathcal{F}^1({}^\ell\mathcal{H})$ is the $\mathrm{Gal}(\tau)$-invariant part of $\tau_*\mathcal{F}^1({}^\ell(\tau^*\mathcal{H}))$ for the unipotent reduction. If $\mathcal{F}^1({}^\ell(\tau^*\mathcal{H}))$ is a subbundle of ${}^\ell(\tau^*\mathcal{H})$, then $\mathcal{F}^1({}^\ell\mathcal{H})$ is also a subbundle of ${}^\ell\mathcal{H}$, $\mathcal{L}_{H/S}$ is an invertible sheaf, and $\mathcal{F}^1({}^u\mathcal{H})$ is an invertible sheaf dual to $\mathcal{L}_{H/S}$. Thus we may assume that the monodromy of H is unipotent. We consider a generator $\omega(z)e_1 + e_0$ of $\boldsymbol{e}^*\mathcal{F}^1(\mathcal{H})$ corresponding to

$$\begin{pmatrix} \omega(z) \\ 1 \end{pmatrix}.$$

Now we have $\omega(z) = \sum_{i=1}^{l} a_i z_i + h(t)$ for a holomorphic function $h(t)$ by Proposition 2.1.4 and

$$M_i^\ell = N_i = \begin{pmatrix} 0 & a_i \\ 0 & 0 \end{pmatrix},$$

where we consider $a_i = 0$ in the case H is of type I_0. Thus

$$\begin{pmatrix} \omega(z) \\ 1 \end{pmatrix} = \exp(\sum_{i=1}^{l} z_i M_i^\ell) \begin{pmatrix} h(t) \\ 1 \end{pmatrix}.$$

Therefore the generator is written by $h(t)\,{}^\ell v_1 + {}^\ell v_2$. Hence $\mathcal{F}^1({}^\ell\mathcal{H})$ is generated by $h(t)\,{}^\ell v_1 + {}^\ell v_2$ and is a subbundle of ${}^\ell\mathcal{H}$. Q.E.D.

Lemma 3.1.3. *There exist natural injections $j_*H \to {}^\ell\mathcal{H}$ and $j_*H \to {}^\ell\mathcal{H} \to \mathcal{L}_{H/S}$.*

Proof. We have only to check the image of $j_*H \to j_*\mathcal{H}$ is contained in ${}^\ell\mathcal{H}$, since $H \cap \mathcal{F}^1(\mathcal{H}) = 0$. The stalk $(j_*H)_0$ is the π_1-invariant part of $\Gamma(U, \boldsymbol{e}^{-1}H)$. If H is neither of types I_0 nor $\mathrm{I}_{(+)}$, then the stalk $(j_*H)_0$ is zero. Assume that H is of type $\mathrm{I}_{(+)}$. Then

$$M_i^\ell = \begin{pmatrix} 0 & a_i \\ 0 & 0 \end{pmatrix}$$

for any i. Hence the stalk $(j_*H)_0 \simeq \mathbb{Z}$ is generated by $e_1 \in H^0(U, \boldsymbol{e}^{-1}H)$ above and

$${}^\ell\boldsymbol{v}_1(z) = \exp(\sum_{i=1}^{l} z_i M_i^\ell) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Therefore the image of e_1 is contained in $({}^\ell\mathcal{H})_0$. Finally, assume that H is of type I_0. Then all $M_i^\ell = 0$. Thus ${}^\ell\boldsymbol{v}_1(z) = e_1$ and ${}^\ell\boldsymbol{v}_0(z) = e_0$. Hence images of e_1, e_0 are contained in $({}^\ell\mathcal{H})_0$. Thus $j_*H \subset {}^\ell\mathcal{H}$. Q.E.D.

For the period function $\omega(z)$, we have:

$$\begin{pmatrix} \omega(\gamma z) \\ 1 \end{pmatrix} = (c_\gamma \omega(z) + d_\gamma)^{-1} \rho(\gamma) \begin{pmatrix} \omega(z) \\ 1 \end{pmatrix}$$

for any $\gamma \in \pi_1$, from the formula (1.1). Let us consider the following holomorphic vectors

$${}^u\mathbf{V}(z) := \exp(-\sum_i z_i M_i^u) \begin{pmatrix} \omega(z) \\ 1 \end{pmatrix}, \quad {}^\ell\mathbf{V}(z) := \exp(-\sum_i z_i M_i^\ell) \begin{pmatrix} \omega(z) \\ 1 \end{pmatrix},$$
$$\boldsymbol{u}(z) := \exp(-\sum_i z_i N_i) \begin{pmatrix} \omega(z) \\ 1 \end{pmatrix}.$$

Then for $\gamma \in \pi_1$, we have:

$$\begin{aligned} {}^u\mathbf{V}(\gamma z) &= (c_\gamma \omega(z) + d_\gamma)^{-1}\, {}^u\mathbf{V}(z), \\ {}^\ell\mathbf{V}(\gamma z) &= (c_\gamma \omega(z) + d_\gamma)^{-1}\, {}^\ell\mathbf{V}(z), \\ \boldsymbol{u}(\gamma z) &= (c_\gamma \omega(z) + d_\gamma)^{-1} S(\gamma) \boldsymbol{u}(z). \end{aligned} \tag{3.1}$$

Since $S(\gamma) = I$ for $\gamma \in \pi_1(T^\star) = \bigoplus_{i=1}^{l} m_i \mathbb{Z} \subset \pi_1$ and since

$$N_i = \begin{pmatrix} 0 & a_i \\ 0 & 0 \end{pmatrix}$$

for some $a_i \geq 0$, there exists a holomorphic function $h(\theta)$ defined on $T^\star$ such that

$$\boldsymbol{u}(z) = \begin{pmatrix} h(\theta) \\ 1 \end{pmatrix}, \quad \omega(z) = \sum_{i=1}^{l} a_i z_i + h(\theta),$$

where we write $\theta = (\theta', t') \in (\Delta^\star)^l \times \Delta^{d-l} = T^\star$. By Proposition 2.1.4, $h(\theta)$ is holomorphic on T. Note that one of a_i is positive if and only if the monodromy group is not finite. For γ_i, let $\gamma_i\theta$ be the point

$$(\theta_1, \theta_2, \ldots, \theta_{i-1}, e(1/m_i)\theta_i, \theta_{i+1}, \ldots, \theta_l, t').$$

Then we can define $\gamma\theta$ also for $\gamma \in \pi_1$. By (3.1), we have:

$$\begin{pmatrix} h(\gamma\theta) \\ 1 \end{pmatrix} = (c_\gamma \omega(z) + d_\gamma)^{-1} S(\gamma) \begin{pmatrix} h(\theta) \\ 1 \end{pmatrix}. \tag{3.2}$$

Note that $c_\gamma \omega(z) + d_\gamma = c_\gamma h(\theta) + d_\gamma$. Therefore

$$S(\gamma) \begin{pmatrix} h(0) \\ 1 \end{pmatrix} = (c_\gamma h(0) + d_\gamma) \begin{pmatrix} h(0) \\ 1 \end{pmatrix}.$$

Thus, $c_\gamma h(0) + d_\gamma$ is an eigenvalue of $S(\gamma)$. If $S(\gamma) = \pm I$ for any γ, i.e., H is one of types I_0, $\mathrm{I}_0^{(*)}$, $\mathrm{I}_{(+)}$, $\mathrm{I}_{(+)}^{(*)}$, then $h(\theta)$ is a holomorphic function on $t \in S$. In the case H is one of types $\mathrm{II}^{(*)}$, $\mathrm{III}^{(*)}$, $\mathrm{IV}_+^{(*)}$, $\mathrm{IV}_-^{(*)}$, we define the matrix:

$$P := \begin{pmatrix} h(0) & \overline{h(0)} \\ 1 & 1 \end{pmatrix}.$$

Then we have

$$P^{-1} S(\gamma) P = \begin{pmatrix} (c_\gamma h(0) + d_\gamma) & 0 \\ 0 & (c_\gamma h(0) + d_\gamma)^{-1} \end{pmatrix}$$

for any $\gamma \in \pi_1$. In any case of H, we have integers $-m_i < p_i, q_i \leq 0$ and $0 \leq p_i', q_i' < m_i$ for $1 \leq i \leq l$ satisfying the following condition (cf. Table 8):

$$c_{\gamma_i} h(0) + d_{\gamma_i} = e\left(\frac{p_i}{m_i}\right) = e\left(\frac{p_i'}{m_i}\right) = e\left(-\frac{q_i}{m_i}\right) = e\left(-\frac{q_i'}{m_i}\right).$$

Here $-p_i = q_i'$ and $-q_i = p_i'$. We define rational numbers $\delta_i := -p_i/m_i$.

Lemma 3.1.4. *Suppose that H is one of types $\mathrm{II}^{(*)}$, $\mathrm{III}^{(*)}$, $\mathrm{IV}_+^{(*)}$, $\mathrm{IV}_-^{(*)}$. Let A be the algebra defined in the proof of Theorem* 2.2.1 *and let*

Table 8. Related numbers for monodromy matrices.

$\rho(\gamma_i)$	I_a	I_b^*	II	II*	III	III*	IV	IV*
m_i	1	2	6	6	4	4	3	3
$-p_i = q_i'$	0	1	1	5	1	3	1	2
$-q_i = p_i'$	0	1	5	1	3	1	2	1
$\delta_i := -p_i/m_i$	0	1/2	1/6	5/6	1/4	3/4	1/3	2/3

w_A be the constant defined by:

$$w_A = \begin{cases} \boldsymbol{\omega}, & \textit{in the case } A = \mathbb{Z}[\boldsymbol{\omega}]; \\ \boldsymbol{i}, & \textit{in the case } A = \mathbb{Z}[\boldsymbol{i}]. \end{cases}$$

Then $\phi(\gamma) = c_\gamma h(0) + d_\gamma$ in $A \subset \mathbb{C}$ for any γ, where $\phi(\gamma)$ is defined in Table 5. In particular, $h(0) = w_A$. Let $\psi(\theta)$ be the holomorphic function

$$\psi(\theta) := \frac{h(\theta) - w_A}{h(\theta) - \overline{w_A}}$$

defined on T. Then it satisfies the following conditions:

(1) $\psi(0) = 0$;

(2) *For any $\theta \in T$ and $\gamma \in \pi_1$, $|\psi(\theta)| < 1$ and $\psi(\gamma\theta) = \phi(\gamma)^{-2}\psi(\theta)$;*

(3) *For any $\theta \in T$, and $\gamma \in \pi_1$,*

$$h(\theta) = \frac{w_A - \overline{w_A}\psi(\theta)}{1 - \psi(\theta)} \quad \textit{and} \quad c_\gamma h(\theta) + d_\gamma = \phi(\gamma)\frac{1 - \psi(\gamma\theta)}{1 - \psi(\theta)};$$

(4) *There is a holomorphic function $\psi_0(t)$ on S such that $|\psi_0(t)| \leq 1$ for any $t \in S$ and*

$$\psi(\theta) = \psi_0(t) \prod_{i=1}^{l} \theta_i^{\langle 2\delta_i \rangle m_i}.$$

Proof. Since all $a_i = 0$ in this case, we have $\omega(z) = h(\theta)$. Thus $\operatorname{Im} h(\theta) > 0$. Since $c_\gamma h(0) + d_\gamma$ is an eigenvalue of $\rho(\gamma) = S(\gamma)$ and $\operatorname{Im} h(0) > 0$, we have $\phi(\gamma) = c_\gamma h(0) + d_\gamma$. The equality $h(0) = w_A$ is derived from Table 1. Then we have

$$P^{-1}S(\gamma)P = \begin{pmatrix} \phi(\gamma) & 0 \\ 0 & \phi(\gamma)^{-1} \end{pmatrix},$$

for any $\gamma \in \pi_1$ and for the matrix

$$P = \begin{pmatrix} h(0) & \overline{h(0)} \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} w_A & \overline{w_A} \\ 1 & 1 \end{pmatrix}.$$

Thus for the holomorphic vector

$$\begin{pmatrix} a(\theta) \\ b(\theta) \end{pmatrix} := P^{-1} \begin{pmatrix} h(\theta) \\ 1 \end{pmatrix},$$

we have

$$a(\theta) = \frac{h(\theta) - \overline{w_A}}{w_A - \overline{w_A}}, \qquad b(\theta) = \frac{-h(\theta) + w_A}{w_A - \overline{w_A}},$$
$$a(\gamma\theta) = (c_\gamma h(\theta) + d_\gamma)^{-1}\phi(\gamma)a(\theta), \quad b(\gamma\theta) = (c_\gamma h(\theta) + d_\gamma)^{-1}\phi(\gamma)^{-1}b(\theta).$$

Note that $a(\theta)$ is a nowhere vanishing function on T. Since $\psi(\theta) = -a(\theta)^{-1}b(\theta)$, we have

$$h(\theta) = \frac{w_A - \overline{w_A}\psi(\theta)}{1 - \psi(\theta)} \quad \text{and} \quad \alpha(\theta)^{-1} = 1 - \psi(\theta).$$

Thus $\psi(\theta)$ satisfies the required conditions. Q.E.D.

Corollary 3.1.5. *A variation of Hodge structures H of one of types $\mathrm{II}^{(*)}$, $\mathrm{III}^{(*)}$, $\mathrm{IV}_+^{(*)}$, $\mathrm{IV}_-^{(*)}$ on $S^\star$ is determined by a surjective group homomorphism $\phi\colon \pi_1 \twoheadrightarrow A^\star \subset \mathbb{C}^\star$ and a holomorphic function $\widetilde{\psi}(z)$ on U such that $|\widetilde{\psi}(z)| < 1$ and $\widetilde{\psi}(\gamma z) = \phi(\gamma)^{-2}\widetilde{\psi}(z)$ for any $z \in U$ and $\gamma \in \pi_1$, where A is one of subalgebras $\mathbb{Z}[\boldsymbol{\omega}]$ and $\mathbb{Z}[\boldsymbol{i}]$ of $\mathbb{C}$ and $A^\star := A \cap \mathbb{C}^\star$. Here the period function is given by*

$$\omega(z) = \frac{w_A - \overline{w_A}\widetilde{\psi}(z)}{1 - \widetilde{\psi}(z)}.$$

Corollary 3.1.6. *The period function $\omega(z)$ is written in the following form according as the type of monodromy representation:*

I_0, $\mathrm{I}_0^{(*)}$: $\quad \omega(z) = h(t)$, *where* $\operatorname{Im} h(t) > 0$;

$\mathrm{I}_{(+)}, \mathrm{I}_{(+)}^{(*)}$: $\quad \omega(z) = \sum_{i=1}^{l} a_i z_i + h(t)$, *where* $\operatorname{Im} h(t) \geq 0$;

$\mathrm{II}^{(*)}, \mathrm{IV}_+^{(*)}, \mathrm{IV}_-^{(*)}$:

$$\omega(z) = \frac{\boldsymbol{\omega} - \boldsymbol{\omega}^2\psi_0(t)\prod_{i=1}^{l}\theta_i^{\langle 2\delta_i\rangle m_i}}{1 - \psi_0(t)\prod_{i=1}^{l}\theta_i^{\langle 2\delta_i\rangle m_i}}, \qquad \textit{where } |\psi_0(t)| \leq 1;$$

$$\mathrm{III}^{(*)}: \qquad \omega(z) = \frac{\boldsymbol{i} + \boldsymbol{i}\psi_0(t)\prod_{i=1}^{l}\theta_i^{\langle 2\delta_i\rangle m_i}}{1 - \psi_0(t)\prod_{i=1}^{l}\theta_i^{\langle 2\delta_i\rangle m_i}}, \qquad \textit{where } |\psi_0(t)| \le 1.$$

We shall describe generators of $\mathcal{F}^1({}^u\mathcal{H})$ and $\mathcal{F}^1({}^\ell\mathcal{H})$, explicitly. In the case H is one of types $\mathrm{II}^{(*)}$, $\mathrm{III}^{(*)}$, $\mathrm{IV}_+^{(*)}$, $\mathrm{IV}_-^{(*)}$, we can write

$$P^{-1}\begin{pmatrix} h(\theta) \\ 1 \end{pmatrix} = \frac{1}{1-\psi(\theta)}\begin{pmatrix} 1 \\ -\psi(\theta) \end{pmatrix},$$

where the function $\psi(\theta)$ is written by

$$\psi(\theta) = \psi_0(t)\prod_{i=1}^{l}\theta_i^{\langle 2\delta_i\rangle m_i}$$

for a holomorphic function $\psi_0(t)$ defined on S. We see that $(p_i+p_i')/m_i + \langle 2\delta_i\rangle$ is 0 or 1 for any i. Let us define holomorphic functions $A(t)$ and $B(t)$ on S by:

$$\begin{pmatrix} A(t) \\ B(t) \end{pmatrix} := P\begin{pmatrix} 1 \\ -\psi_0(t)\prod_{i=1}^{l}\theta_i^{p_i+p_i'+\langle 2\delta_i\rangle m_i} \end{pmatrix}.$$

We define also a holomorphic function $a(\theta) := (1-\psi(\theta))^{-1}$ over T. In the case H is one of types I_0, $\mathrm{I}_0^{(*)}$, $\mathrm{I}_{(+)}$, $\mathrm{I}_{(+)}^{(*)}$, we set $P := I$,

$$\begin{pmatrix} A(t) \\ B(t) \end{pmatrix} := \begin{pmatrix} h(\theta) \\ 1 \end{pmatrix},$$

and $a(\theta) := 1$. Then we see that $A(t)$ and $B(t)$ have no common zeros on S in any case.

Lemma 3.1.7. *$A(t)\,{}^u v_1 + B(t)\,{}^u v_0$ and $A(t)\,{}^\ell v_1 + B(t)\,{}^\ell v_0$ are generators of $\mathcal{F}^1({}^u\mathcal{H})$ and $\mathcal{F}^1({}^\ell\mathcal{H})$, respectively.*

Proof. We can write

$${}^u\mathbf{V}(z) = \exp(-\sum_{i=1}^{l} z_i R_i^u)\begin{pmatrix} h(\theta) \\ 1 \end{pmatrix}, \quad {}^\ell\mathbf{V}(z) = \exp(-\sum_{i=1}^{l} z_i R_i^\ell)\begin{pmatrix} h(\theta) \\ 1 \end{pmatrix}.$$

By definition, we see

$$P^{-1}R_i^u P = 2\pi\sqrt{-1}\begin{pmatrix} \frac{p_i}{m_i} & 0 \\ 0 & \frac{q_i}{m_i} \end{pmatrix}, \quad PR_i^\ell P^{-1} = 2\pi\sqrt{-1}\begin{pmatrix} \frac{p_i'}{m_i} & 0 \\ 0 & \frac{q_i'}{m_i} \end{pmatrix}.$$

Therefore we have

$$(3.3)\qquad {}^u\mathbf{V}(z) = a(\theta)\prod_{i=1}^{l}\theta_i^{-p_i}\begin{pmatrix}A(t)\\B(t)\end{pmatrix},$$

$$(3.4)\qquad {}^\ell\mathbf{V}(z) = a(\theta)\prod_{i=1}^{l}\theta_i^{-p_i'}\begin{pmatrix}A(t)\\B(t)\end{pmatrix}.$$

Let $v \in \Gamma(S, {}^u\mathcal{H})$ be a section such that $v_{|S^\star} \in \Gamma(S^\star, \mathcal{F}^1(\mathcal{H}))$. Then $v = u_1(t)\,{}^uv_1 + u_0(t)\,{}^uv_0$ for some holomorphic functions $u_1(t), u_0(t)$ on S and $\boldsymbol{e}^*(v) = \widetilde{\varphi}(z)(e_0 + \omega(z)e_1)$ for a holomorphic function $\widetilde{\varphi}(z)$ on U. Now $\boldsymbol{e}^*(v)$ corresponds to the vector

$$\exp(\sum_{i=1}^{l} z_i M_i^u)\begin{pmatrix}u_1(t)\\u_0(t)\end{pmatrix}.$$

Hence

$$\begin{pmatrix}u_1(t)\\u_0(t)\end{pmatrix} = \widetilde{\varphi}(z)^u\mathbf{V}(z) = \widetilde{\varphi}(z)a(\theta)\prod_{i=1}^{l}\theta_i^{-p_i}\begin{pmatrix}A(t)\\B(t)\end{pmatrix}$$

by (3.3). Thus $\widetilde{\varphi}(z)a(\theta)\prod_{i=1}^{l}\theta_i^{-p_i}$ is a holomorphic function $\varphi(t)$ on S. Therefore $v = \varphi(t)(A(t)\,{}^uv_1 + B(t)\,{}^uv_0)$. Thus $A(t)\,{}^uv_1 + B(t)\,{}^uv_0$ is a generator of $\mathcal{F}^1({}^u\mathcal{H})$. Similarly we can prove $A(t)\,{}^\ell v_1 + B(t)\,{}^\ell v_0$ is a generator of $\mathcal{F}^1({}^\ell\mathcal{H})$ by using (3.4). Q.E.D.

Let $\boldsymbol{\Delta}(z)$ be the cusp form of weight 12 (cf. Remark 1.2.6). Let the section $\xi \in H^0(S^\star, \mathcal{F}^1(\mathcal{H})^{\otimes 12})$ correspond to $\boldsymbol{\Delta}(\omega(z))(\omega(z)e_1 + e_0)^{\otimes 12}$.

Corollary 3.1.8 (cf. [U]). *The ξ extends to a holomorphic section of $\mathcal{F}^1({}^u\mathcal{H})^{\otimes 12}$ over S. The effective divisor* $\operatorname{div}(\xi)$ *is written by* $\sum_{i=1}^{l}(a_i + 12\delta_i)D_i$.

Proof. By the argument above, $\omega(z) = \sum_{i=1}^{l} a_i z_i + h(\theta)$ for non-negative integers a_i and a holomorphic function $h(\theta)$ on T. Note that if $a_i > 0$ then monodromy matrix $\rho(\gamma_i)$ is of type I_{a_i} or $\mathrm{I}^*_{a_i}$. By Remark 1.2.6, the function $\boldsymbol{\Delta}(\omega(z))$ is written as $u(\theta)\prod_{i=1}^{l}\theta_i^{m_i a_i}$ for a nowhere vanishing function $u(\theta)$ on T. On the other hand, by (3.3), we have

$$\omega(z)e_1 + e_0 = a(\theta)(\prod_{i=1}^{l}\theta_i^{-p_i})(A(t)\,{}^uv_1 + B(t)\,{}^uv_0),$$

where $A(t)\,{}^uv_1 + B(t)\,{}^uv_0$ is a generator of $\mathcal{F}^1({}^u\mathcal{H})$ by Lemma 3.1.7. By computing the vanishing order of $\prod_i \theta_i^{-12p_i}\prod_{i=1}^{l}\theta_i^{m_i a_i}$ along each coordinate hyperplane $\{\theta_i = 0\}$, we are done. Q.E.D.

3.2. Torsion free theorems

Let $f\colon Y \to M$ be an elliptic fibration (not necessarily projective) between complex manifolds such that f is smooth outside a normal crossing divisor $D = \bigcup D_i$ on M. Then we have a variation of Hodge structures $H := (R^1 f_* \mathbb{Z}_Y)_{|M \setminus D}$. Let ${}^u\mathcal{H}$ and ${}^\ell\mathcal{H}$ be the upper and the lower canonical extensions, respectively of $\mathcal{H} = H \otimes \mathcal{O}_{M\setminus D}$ to M defined in Definition 3.1.1 (cf. [Kl1], [Mw]). Also we denote by $\mathcal{F}^p({}^u\mathcal{H})$ and $\mathcal{F}^p({}^\ell\mathcal{H})$ the induced p-th filtrations (cf. Definition 3.1.1). As a corollary of Corollary 3.1.8, we have:

Corollary 3.2.1 ([Kw2]). *Let $J\colon M \to \mathbb{P}^1$ be the J-function associated with H. Then there is an isomorphism*

$$\mathcal{F}^1({}^u\mathcal{H})^{\otimes 12} \simeq J^*\mathcal{O}(1) \otimes \mathcal{O}_M(\sum 12\delta_i D_i),$$

where the rational numbers δ_i are determined by the types of the monodromy matrices around D_i as in Table 8.

Proof. By Corollary 3.1.8, $\xi = \mathbf{\Delta}(\omega(z))(\omega(z)e_1 + e_0)^{\otimes 12}$ is a section of $H^0(M, \mathcal{F}^1({}^u\mathcal{H})^{\otimes 12})$ such that $\operatorname{div}(\xi) = \sum_i (a_i + 12\delta_i)D_i$. Here if $a_i = a > 0$, then the monodromy matrix around D_i is of type I_a or I_a^*. Thus we have the isomorphism $J^*\mathcal{O}(1) \simeq \mathcal{O}(\sum a_i D_i)$ by Proposition 2.1.4, which implies the expected isomorphism. Q.E.D.

The following theorem was proved by [Kl1] (cf. [Ny2]) for algebraic case and by [Mw] for projective morphisms. On the other hand, Saito independently proved this by using his theory of Hodge modules in [Sa1]. He also had a generalization to the case of Kähler morphisms (cf. [Sa2], [Sa3]). Takegoshi ([Ta]) also gives another proof for Kähler morphisms by an L^2-method.

Theorem 3.2.2. *Let $\pi\colon X \to W$ be a projective surjective morphism from a complex analytic manifold X onto a complex analytic variety W. Then the higher direct images $R^i\pi_*\omega_X$ are torsion free for $i \geq 0$. Moreover the following properties hold*:

(1) *Assume that W is nonsingular and π is smooth outside a normal crossing divisor D of W. Let $d = \dim X - \dim W$ and let ${}^u\mathcal{H}^{d+i}$ be the upper canonical extension of the variation of Hodge structures $(R^{d+i}\pi_*\mathbb{Z}_X)_{|W\setminus D}$ for any $i \geq 0$. Then we have*

$$R^i\pi_*\omega_{X/W} \simeq \mathcal{F}^d({}^u\mathcal{H}^{d+i}),$$

where $\mathcal{F}^d$ denotes the induced d-th filter from the Hodge filtration;

(2) *Assume that there is a projective morphism* $f\colon W \to V$ *to a complex analytic variety* V. *Then for an* f*-ample invertible sheaf* $\mathcal{A}$ *of* W *and for integers* $p > 0$ *and* $i \geq 0$, *we have*

$$R^p f_*(\mathcal{A} \otimes R^i \pi_* \omega_X) = 0.$$

We shall consider the above theorem in the case of (not necessarily projective) elliptic fibrations.

Theorem 3.2.3. *Let* $f\colon Y \to W$ *be an elliptic fibration, which is not necessarily projective, between complex manifolds. Suppose that* f *is smooth outside a normal crossing divisor on* W. *Then there exist the following isomorphisms*:

$$R^i f_* \omega_{Y/W} \simeq \begin{cases} \mathcal{F}^1({}^u\mathcal{H}), & i = 0; \\ \mathcal{O}_S, & i = 1; \\ 0, & i > 1, \end{cases} \quad R^i f_* \mathcal{O}_Y \simeq \begin{cases} \mathcal{O}_S, & i = 0; \\ Gr^0_{\mathcal{F}}({}^\ell\mathcal{H}), & i = 1; \\ 0, & i > 1. \end{cases}$$

Proof. If f is a locally projective morphism, then these are isomorphic by Theorem 3.2.2. If $\dim W = 1$, then f is a flat morphism. Thus f is a locally projective morphism by Claim 3.2.4 below. Thus even in the case $\dim W > 1$, the double duals of $R^i f_* \omega_{Y/W}$ and $R^i f_* \mathcal{O}_Y$, respectively, are isomorphic to the right hand side of the corresponding formula. Hence we have only to check the formula locally on W. We may assume that the monodromy matrices are unipotent by taking the unipotent reduction. By the flattening theorem, we have a proper bimeromorphic morphism $\mu\colon M \to W$ from a nonsingular manifold M such that the fiber product $Y \times_W M \to M$ induces a flat morphism $g\colon Z \to M$ from the main component Z of $Y \times_W M$. We may assume that g is smooth outside a normal crossing divisor D of M.

Claim 3.2.4. *g is locally a projective morphism.*

Proof. Let us consider the following exact sequence induced by g_* from an exponential sequence:

$$R^1 g_* \mathcal{O}_Z \to R^1 g_* \mathcal{O}_Z^\star \to R^2 g_* \mathbb{Z}_Z \to R^2 g_* \mathcal{O}_Z.$$

Now we have $R^2 g_* \mathcal{O}_Z = 0$. Note that the stalk $(R^2 g_* \mathbb{Z}_Z)_P$ is isomorphic to $H^2(g^{-1}(P), \mathbb{Z})$ for $P \in M$. Thus we have an invertible sheaf $\mathcal{L}$ on an open neighborhood of $g^{-1}(P)$ such that the intersection numbers $\mathcal{L} \cdot C$ are positive for any irreducible components of $C \subset g^{-1}(P)$. Hence $\mathcal{L}$ is g-ample over an open neighborhood of $\{P\}$. Q.E.D.

Proof of Theorem 3.2.3 *continued.* There is an elliptic fibration $\pi\colon X \to M$ from a complex manifold X such that π is bimeromorphically equivalent to g over M and that π and g are isomorphic to each other over $M \setminus D$. By Claim 3.2.4, g is bimeromorphically equivalent to a projective morphism locally over M. Since the canonical extension of $(R^1\pi_*\mathbb{Z}_X)_{|M\setminus D} \otimes \mathcal{O}_{M\setminus D}$ and the induced filtrations are pullbacks of the corresponding sheaves on W, we have:

$$R^i\pi_*\omega_{X/M} \simeq \begin{cases} \mu^*(\mathcal{F}^1(\mathcal{H})), & \text{if } i = 0; \\ \mathcal{O}_M, & \text{if } i = 1; \\ 0, & \text{otherwise,} \end{cases}$$

by Theorem 3.2.2. Since $R^i\mu_*\omega_M = 0$ for $i > 0$, we have:

$$R^i f_*\omega_{Y/W} \simeq R^i(\mu \circ \pi)_*\omega_{X/W} \simeq \begin{cases} \mathcal{F}^1(\mathcal{H}), & \text{if } i = 0; \\ \mathcal{O}_W, & \text{if } i = 1; \\ 0, & \text{otherwise.} \end{cases}$$

By duality, we also have the isomorphisms for $R^i f_*\mathcal{O}_Y$. Q.E.D.

Remark. By the same argument, we can show such isomorphisms exist in the case the general fibers of $Y \to S$ are curves. But if it is not an elliptic fibration, then $Y \to S$ is bimeromorphically equivalent to a projective morphism. Therefore this is already proved by Theorem 3.2.2.

Corollary 3.2.5 (cf. [U, 6.1], [Kw2, 20]). *Let $f\colon Y \to W$ be an elliptic fibration between complex manifolds such that f is smooth outside a normal crossing divisor $D = \cup D_i$. Then we have*:

$$(f_*\omega_{Y/W})^{\otimes 12} \simeq J^*\mathcal{O}(1) \otimes \mathcal{O}(\sum 12\delta_i D_i).$$

Now we shall prove the following:

Theorem 3.2.6 (Torsion free theorem). *Let $\pi\colon X \to W$ be an elliptic fibration from a complex manifold X onto a complex analytic variety W. Then the higher direct image sheaves $R^i\pi_*\omega_X$ are torsion free. Further if there exist a projective morphism $f\colon W \to V$ onto a complex analytic variety V and an f-ample invertible sheaf $\mathcal{A}$ on W, then*

$$R^p f_*(\mathcal{A} \otimes R^i\pi_*\omega_X) = 0$$

for any $p > 0$ and $i \geq 0$.

Proof. First we shall show that $R^1\pi_*\omega_X$ is torsion free and that $R^i\pi_*\omega_X = 0$ for $i \geq 2$. We may assume that there exist a complex manifold M and morphisms $g\colon X \to M$ and $\mu\colon M \to W$ such that g is smooth outside a normal crossing divisor $D = \bigcup D_i$ on M, μ is a bimeromorphic morphism, and that $\pi = \mu \circ g$. Thus by Theorem 3.2.3 and Corollary 3.2.5, we have

$$(g_*\omega_X)^{\otimes 12} \simeq \omega_M^{\otimes 12} \otimes J^*\mathcal{O}(1) \otimes \mathcal{O}(\sum 12\delta_i D_i),$$

$$R^i g_*\omega_X \simeq \begin{cases} \omega_M, & \text{for } i = 1; \\ 0, & \text{for } i > 1. \end{cases}$$

Note that $g_*\omega_{X/M} - \sum \delta_i D_i$ is μ-nef. By applying the vanishing theorem [Ny3, 3.6, 3.7], we have

$$R^k\mu_*(g_*\omega_X) = 0 \quad \text{and} \quad R^k\mu_*\omega_M = 0$$

for $k > 0$. Therefore by the Leray spectral sequence, we have

$$R^i\pi_*\omega_X \simeq \begin{cases} \mu_*(g_*\omega_X), & \text{for } i = 0; \\ \mu_*(\omega_M), & \text{for } i = 1; \\ 0, & \text{for } i > 1. \end{cases}$$

Thus $R^i\pi_*\omega_X$ are torsion free. Next we shall prove the vanishing:

$$R^p f_*(\mathcal{A} \otimes R^i\pi_*\omega_X) = 0.$$

By the argument above, we have only to consider the cases $i = 0, 1$. Since $\mu^*\mathcal{A}$ is $(f \circ \mu)$-nef-big, the $g_*\omega_{X/M} - \sum_i \delta_i D_i + \mu^*\mathcal{A}$ is also $(f \circ \mu)$-nef-big. Thus by [Ny3, 3.7], we have

$$R^p(f \circ \mu)_*(\mu^*\mathcal{A} \otimes \omega_M) = 0 \quad \text{and} \quad R^p(f \circ \mu)_*(\mu^*\mathcal{A} \otimes \pi_*\omega_X) = 0$$

for $p > 0$. Thus by the argument above, we have the desired vanishing.
Q.E.D.

3.3. Projective morphisms

We have the following criterion for a given proper surjective morphism to be locally projective.

Proposition 3.3.1. *Let $\pi \colon X \to V$ be a proper surjective morphism from a complex analytic manifold X onto a complex analytic variety V. Suppose that the stalk $(R^2\pi_*\mathcal{O}_X)_P = 0$ for a point $P \in V$. Then π is projective over $\{P\}$ if and only if there is an open neighborhood U of P in V such that $\pi^{-1}(U)$ admits a Kähler metric.*

Proof. It is enough to prove that π is projective over $\{P\}$ under the assumption: X is Kähler. Let us consider the following exact sequence induced by π_* from the exponential sequence:

$$R^1\pi_*\mathcal{O}_X \to R^1\pi_*\mathcal{O}_X^\star \to R^2\pi_*\mathbb{Z}_X \to R^2\pi_*\mathcal{O}_X = 0.$$

We note that the stalk $(R^2\pi_*\mathbb{Z}_X)_P$ is isomorphic to $H^2(\pi^{-1}(P),\mathbb{Z})$. Let ω be a Kähler form on $\pi^{-1}(U)$ for an open neighborhood U of $\{P\}$. Then its cohomology class $[\omega]$ should be an element of $H^2(\pi^{-1}(U),\mathbb{R})$. Let us denote by the same $[\omega]$ the image of $[\omega]$ under the map $H^2(\pi^{-1}(U),\mathbb{R}) \to H^2(\pi^{-1}(P),\mathbb{R})$. We define the *Kähler cone* $\mathrm{KC}(X/V;\ P)$ over $\{P\}$ to be the subset of $H^2(\pi^{-1}(P),\mathbb{R})$ consisting of all the $[\omega]$ for Kähler forms ω defined on some neighborhoods of $\pi^{-1}(P)$.

Claim 3.3.2. $\mathrm{KC}(X/V;\ P)$ *is an open subset of* $H^2(\pi^{-1}(P),\mathbb{R})$.

Proof. Note that $(R^1\pi_*\mathcal{O}_X^\star)_P \to H^2(\pi^{-1}(P),\mathbb{Z})$ is surjective. Thus for any element $\tau \in H^2(\pi^{-1}(P),\mathbb{R})$, we have a d-closed real $(1,1)$-form η on a neighborhood of $\pi^{-1}(P)$ such that its cohomology class $[\eta]$ is τ. Let ω be a Kähler form and let η_i for $1 \le i \le n$ be d-closed real $(1,1)$-forms on a neighborhood of $\pi^{-1}(P)$ such that $\{[\eta_i]\}$ is a basis of $H^2(\pi^{-1}(P),\mathbb{R})$. Since $\pi^{-1}(P)$ is a compact subset, there exists a positive number ε such that if x_i are real numbers with $|x_i| < \varepsilon$, then $\omega + \sum_{1\le i\le n} x_i\eta_i$ is also a Kähler form on a neighborhood of $\pi^{-1}(P)$. Thus the Kähler cone is open. Q.E.D.

Proof of Proposition 3.3.1 *continued.* By the above claim, we obtain an invertible sheaf $\mathcal{L}$ on a neighborhood of $\pi^{-1}(P)$ which has a positive Hermitian metric. Thus $\mathcal{L}$ is π-ample. Therefore π is a projective morphism over $\{P\}$. Q.E.D.

As a consequence of Claim 3.2.3 and Proposition 3.3.1, we have:

Theorem 3.3.3. *Let $f\colon Y \to M$ be an elliptic fibration from a complex Kähler manifold Y onto a complex manifold M such that f is smooth outside a normal crossing divisor on M. Then f is a locally projective morphism.*

In the case M is a nonsingular curve, any elliptic fibration $Y \to M$ is a locally projective morphism. But in the case $\dim M \ge 2$, there exist non-projective elliptic fibrations.

Example 3.3.4. Let Δ^2 be the two-dimensional unit disc with a coordinate system (t_1,t_2), $\mu\colon S \to \Delta^2$ the blowing-up at $0 = (0,0) \in \Delta^2$, and let D be the exceptional divisor on S. Then S is covered by open subsets U_0 and U_1 such that

(1) $U_0 = \{\,(x_0,y_0) \in \mathbb{C}^2 \,|\, |x_0| < 1, |x_0y_0| < 1\}$, $\mu^*(t_1) = x_0$ and $\mu^*(t_2) = x_0y_0$ on U_0,
(2) $U_1 = \{\,(x_1,y_1) \in \mathbb{C}^2 \,|\, |x_1| < 1, |x_1y_1| < 1\}$, $\mu^*(t_1) = x_1y_1$ and $\mu^*(t_2) = x_1$ on U_1,

(3) $U_{0,1} := \{ (x_0, y_0) \in U_0 \mid y_0 \neq 0\}$ and $U_{1,0} := \{ (x_1, y_1) \in U_1 \mid y_1 \neq 0\}$ are isomorphic to each other by

$$\begin{cases} x_1 &= x_0 y_0, \\ y_1 &= y_0^{-1}, \end{cases} \quad \text{and} \quad \begin{cases} x_0 &= x_1 y_1, \\ y_0 &= y_1^{-1}. \end{cases}$$

We take an elliptic curve E_ρ that is the quotient manifold of $\mathbb{C}^\star$ by the action:

$$\mathbb{C}^\star \ni u \mapsto u\rho$$

for $\rho \in \mathbb{C}^\star$ with $|\rho| < 1$. Let us consider the following isomorphism:

$$U_{0,1} \times \mathbb{C}^\star \ni ((x_0, y_0),\, u) \mapsto ((x_0 y_0, y_0^{-1}),\, u y_0) \in U_{1,0} \times \mathbb{C}^\star \,.$$

This induces the isomorphism $U_{0,1} \times E \simeq U_{1,0} \times E$, by which we can patch $U_0 \times E$ and $U_1 \times E$. Thus we obtain a smooth elliptic fibration $f\colon X \to S$. Note that $f^{-1}(D)$ is isomorphic to the Hopf surface H_ρ, which is defined to be the quotient manifold of $\mathbb{C}^2 \setminus \{(0,0)\}$ by the action

$$\mathbb{C}^2 \setminus \{(0,0)\} \ni (z_1, z_2) \mapsto (\rho z_1, \rho z_2)\,.$$

Thus f is a locally projective morphism but not a projective morphism. Further the composite $\mu \circ f\colon X \to \Delta^2$ is an elliptic fibration smooth outside $\{0\}$ and the central fiber $f^{-1}\mu^{-1}(0)$ is isomorphic to the Hopf surface.

Similar constructions to this example are found in [Kt], [Ts]. We have the following generalization:

Example 3.3.5. Let us consider the following three-dimensional complex manifold:

$$M := \left\{ (x, y, z_1, z_2) \in \Delta^2 \times (\mathbb{C}^2 \setminus \{(0,0)\}) \mid x z_2 = y z_1 \right\}.$$

Here we consider the following three actions:

$$(x, y, z_1, z_2) \mapsto (\mu x, y, \mu z_1, z_2), \quad (x, \mu y, z_1, \mu z_2), \quad (x, y, \rho z_1, \rho z_2),$$

where $\mu := e(1/m)$ for a positive integer m and $\rho \in \mathbb{C}^\star$ satisfies $|\rho| < 1$. Therefore $(\mathbb{Z}/m\mathbb{Z}) \times (\mathbb{Z}/m\mathbb{Z}) \times \mathbb{Z}$ acts on M properly discontinuously and freely. Thus we have the quotient manifold X with an elliptic fibration $g\colon X \to \Delta^2$ defined by $(x, y, z_1, z_2) \mapsto (t_1, t_2) = (x^m, y^m)$. Further we have an elliptic fibration $f\colon X \to S \subset \Delta^2 \times \mathbb{P}^1$ by $(x, y, z_1, z_2) \mapsto (x^m, y^m, (z_1^m : z_2^m))$, where $\nu\colon S \to \Delta^2$ is the blowing-up at $0 = (0,0) \in \Delta^2$. Here $g = \nu \circ f$. The f is smooth outside $D = \nu^{-1}(0)$ and $f^*(D) = m f^{-1}(D)$, where the central fiber $f^{-1}(D) = g^{-1}(0)$ is the Hopf surface

H_{ρ^m} (cf. Example 3.3.4) and $f^{-1}(D) \to D$ is the induced smooth elliptic fibration. Therefore we have the following canonical bundle formula:

$$K_X \sim f^*(K_S) + (m-1)f^{-1}(D) \sim g^*(K_{\Delta^2}) + (2m-1)f^{-1}(D).$$

Note that if $m = 1$, then this f is nothing but the same f as Example 3.3.4. The multi-valued map:

$$(\Delta^2)^\circ := \Delta^2 \setminus \{(0,0)\} \ni (t_1, t_2) \mapsto (t_1^{1/m}, t_2^{1/m}, t_1^{1/m}, t_2^{1/m}) \in M$$

defines a holomorphic section. Thus by Proposition 1.2.4, $X \times_{\Delta^2} (\Delta^2)^\circ \simeq E_\rho \times (\Delta^2)^\circ$, since $(\Delta^2)^\circ$ is simply connected.

§4. Toric models

4.1. Basic elliptic fibrations

An *elliptic fibration* is defined to be a proper surjective morphism of complex analytic varieties whose general fibers are elliptic curves. An elliptic fibration is said to be *basic* if there is a meromorphic section. Let S be a d-dimensional complex manifold, D a normal crossing divisor, and let $S^\star := S \setminus D$. If an elliptic fibration over S is smooth over $S^\star$, then any meromorphic section is holomorphic over $S^\star$ by Lemma 1.3.5. Let H be a variation of Hodge structures on $S^\star$, $p^\star \colon B(H)^\star \to S^\star$ the associated smooth basic elliptic fibration, and let $\sigma_0^\star \colon S^\star \to B(H)^\star$ be the zero section. Then by [Ny4, (2.5)], there exists a *minimal* triplet $(\mathcal{L}_{H/S}, \alpha, \beta)$ on S such that the Weierstrass model $W := W(\mathcal{L}_{H/S}, \alpha, \beta) \to S$ is an extension of $p^\star$ to S and the canonical section is an extension of $\sigma_0^\star$. The W has only rational singularities and the invertible sheaf $\mathcal{L}_{H/S}$ is isomorphic to (cf. Theorem 3.2.2):

$$\mathrm{Gr}^0_{\mathcal{F}}({}^\ell\mathcal{H}) = {}^\ell\mathcal{H}/\mathcal{F}^1({}^\ell\mathcal{H}).$$

Let $\nu^\star \colon H \to H$ be an automorphism as a variation of Hodge structures over $S^\star$. Then $\nu^\star$ is of finite order, which is one of $\{1, 2, 3, 4, 6\}$. By the uniqueness of the extension W [Ny4, (2.5)], we have an automorphism $\nu \colon W \to W$ over S inducing $\nu^\star$ over $S^\star$. The automorphism ν is defined by

$$W \ni (X : Y : Z) \mapsto (\varepsilon^2 X : Y : \varepsilon^3 Z)$$

for a primitive m-th root ε of 1, where m is the order of $\nu^\star$. By taking an equivariant resolution of singularities, we have an extension $p \colon B(H) \to S$ of $p^\star$ satisfying the following conditions:

(1) $B(H)$ is nonsingular;

(2) $B(H) \to S$ admits a section $\sigma_0 \colon S \to B(H)$ which is an extension of $\sigma_0^\star$;
(3) For any such automorphism $\nu \colon W \to W$ as above, the induced bimeromorphic automorphism $\nu \colon B(H) \cdots\to B(H)$ is holomorphic.

The section $\sigma_0 \colon S \to B(H)$ is also called the zero section. By the existence of such extensions and by Proposition 1.3.3, we have the following:

Theorem 4.1.1 (cf. [Ny8]). *Let $f^\star \colon X^\star \to S^\star$ be a smooth projective elliptic fibration over a complex manifold $S^\star$. Suppose that $S^\star$ is isomorphic to a Zariski-open subset of another complex manifold S. Then $f^\star$ extends to a projective elliptic fibration over S.*

Proof. A prime divisor $R^\star$ of $X^\star$ is finite étale over $S^\star$ by Proposition 1.3.3. Let $S'^\star \to S^\star$ be the Galois closure of $R^\star \to S^\star$. Then $S'^\star$ is realized as a Zariski-open subset of a complex manifold S' and the finite étale morphism $S'^\star \to S^\star$ extends to a generically finite proper morphism $S' \to S$, by a theorem of Grauert–Remmert [GR]. Here we may assume that the Galois group G of $S'^\star \to S^\star$ acts holomorphically on S'. The pullback $X^\star \times_{S^\star} S'^\star \to S'^\star$ admits a section. Thus by the previous argument, we can extend the smooth basic elliptic fibration to a basic elliptic fibration $B(H') \to S'$, where the action of the Galois group G on $X^\star \times_{S^\star} S'^\star$ induces a holomorphic action on $B(H')$. Hence we have only to take the quotient. Q.E.D.

Let $\sigma \colon S \cdots\to B(H)$ be a meromorphic section. We denote by Σ and Σ_0 the images of σ and σ_0, respectively. Let us consider the diagonal $\Delta_{B(H)} \subset B(H) \times_S B(H)$ and take a bimeromorphic morphism $\mu \colon Z \to B(H) \times_S B(H)$ from a complex manifold Z onto the main component of $B(H) \times_S B(H)$ which is isomorphic over $B(H)^\star \times_{S^\star} B(H)^\star$. Let Δ' be the proper transform of $\Delta_{B(H)}$ in Z and let $p_1, p_2 \colon Z \to B(H)$ be the first and the second projections, respectively. We consider an invertible sheaf

$$\mathcal{N} := \mathcal{O}_Z(\Delta' - p_2^*(\Sigma_0) + p_2^*(\Sigma)).$$

Then for $b \in B(H)^\star = p^{-1}(S^\star)$, we have an isomorphism:

$$\mathcal{N}_{|p_1^{-1}(b)} \simeq \mathcal{O}_{p^{-1}(p(b))}([b] - [\sigma_0(p(b))] + [\sigma(p(b))]),$$

which is an invertible sheaf of degree one on the elliptic curve $p^{-1}(p(b))$. By replacing Z by a further blowing up, we have an effective divisor $E \subset Z$ such that $p_1^* p_{1*}\mathcal{N} \simeq \mathcal{N} \otimes \mathcal{O}(-E)$. An irreducible component E_0 of E dominates $B(H)$ bimeromorphically, and the other components do not dominate $B(H)$. Let $\mathrm{tr}(\sigma) \colon B(H) \cdots\to B(H)$ be the meromorphic mapping over S associated with the graph $\mu(E_0) \subset B(H) \times_S B(H)$. Then the restriction of $\mathrm{tr}(\sigma)$ to $p^{-1}(S^\star) = B(H)^\star$ is nothing but the translation morphism by the section σ. We call $\mathrm{tr}(\sigma)\colon B(H) \cdots\to B(H)$ by the *translation mapping* by a meromorphic section σ. By the same argument, we see that $p\colon B(H) \to S$ has a meromorphic group structure, i.e., there exist a multiplication mapping $B(H) \times_S B(H) \cdots\to B(H)$ over S and an inverse $B(H) \to B(H)$ which are extensions of the same objects for $p^\star\colon B(H)^\star \to S^\star$. We also have the following generalization of Lemma 1.2.2:

Lemma 4.1.2. *Let $\varphi\colon B(H) \cdots\to B(H)$ be a bimeromorphic mapping over S inducing the identity homomorphism on $(R^1 p_* \mathbb{Z}_{B(H)})_{|S^\star}$. Then there exists a meromorphic section $\sigma\colon S \cdots\to B(H)$ such that $\varphi = \mathrm{tr}(\sigma)$.*

In particular, every bimeromorphic automorphism $\varphi\colon B(H) \cdots\to B(H)$ over S is expressed as the composite of a translation mapping and an automorphism ν of finite order explained as before. By [Ny4, (2.1)], we have:

Lemma 4.1.3. *Let $f\colon X \to S$ be an elliptic fibration smooth over $S^\star$ which induces an isomorphism $H \simeq (R^1 f_* \mathbb{Z}_X)_{|S^\star}$ as variations of Hodge structures. Suppose that f admits a meromorphic section $\sigma\colon S \cdots\to X$. Then there exists a bimeromorphic mapping $h\colon X \cdots\to B(H)$ such that $h \circ \sigma$ is the zero section σ_0.*

Let $\mathcal{U} \subset S$ be an open subset. The set of meromorphic sections $\{\sigma\colon \mathcal{U} \cdots\to B(H)_{|\mathcal{U}}\}$ forms a subgroup of $H^0(\mathcal{U} \cap S^\star, \mathfrak{S}_H)$. From the subgroups, we define a subsheaf $\mathfrak{S}_{H/S}$ of $j_* \mathfrak{S}_H$, where j denotes the inclusion $S^\star \hookrightarrow S$. We call $\mathfrak{S}_{H/S}$ by the sheaf of germs of meromorphic sections of $B(H) \to S$. Obviously, it does not depend on the choice of $B(H)$. Let $f\colon X \to S$ be an elliptic fibration smooth over $S^\star$ and let $\phi\colon H \simeq (R^1 f_* \mathbb{Z}_X)_{|S^\star}$ be an isomorphism as variations of Hodge structures. Suppose that f admits a meromorphic section locally on S, i.e., there exist an open covering $S = \bigcup_{\lambda \in \Lambda} S_\lambda$ and meromorphic sections $S_\lambda \cdots\to X_{|S_\lambda}$ for any λ. Then we have bimeromorphic mappings $\varphi_\lambda\colon X_{|S_\lambda} \cdots\to B(H)_{|S_\lambda}$ over S_λ such that these φ_λ^* induce the given isomorphism $\phi\colon H \simeq (R^1 f_* \mathbb{Z}_X)_{|S^\star}$. Then for $\lambda, \mu \in \Lambda$, the transition mapping $\varphi_\lambda \circ \varphi_\mu^{-1}\colon B(H)_{|S_\lambda \cap S_\mu} \cdots\to B(H)_{|S_\lambda \cap S_\mu}$ is the translation mapping by a meromorphic section $\eta_{\lambda,\mu}$ of $B(H) \to S$ over $S_\lambda \cap S_\mu$. Therefore we have $\eta_{\lambda,\mu} + \eta_{\mu,\nu} + \eta_{\nu,\lambda} = 0$ for $\lambda, \mu, \nu \in \Lambda$. This collection $\{\eta_{\lambda,\mu}\}$ defines a cohomology class in $H^1(S, \mathfrak{S}_{H/S})$, which is independent of the choices of an open covering $\{S_\lambda\}$ and bimeromorphic mappings $\{\varphi_\lambda\}$. Let us denote the cohomology class by $\eta(X/S, \phi)$. Let $f'\colon X' \to S$ be another elliptic fibration smooth over $S^\star$ and let $\phi'\colon H \simeq (R^1 f'_* \mathbb{Z}_{X'})_{|S^\star}$ be an isomorphism. Assume that f' also admits a meromorphic section locally over S and that $\eta(X/S, \phi) = \eta(X'/S, \phi')$. Then there exists a bimeromorphic mapping $\psi\colon X \cdots\to X'$ over S such that $\phi = \psi^* \circ \phi'$. Therefore these $\eta(X/S, \phi)$ define a natural equivalence relation for such pairs $(X/S, \phi)$. Let $(f\colon X \to S, \phi)$ be a pair as above and suppose that X is nonsingular. Then we have the following exact sequence by Theorem 3.2.3:

$$0 \to R^1 f_* \mathbb{Z}_X \to R^1 f_* \mathcal{O}_X \to R^1 f_* \mathcal{O}_X^\star \to R^2 f_* \mathbb{Z}_X \to 0.$$

The natural homomorphism $R^1 f_* \mathbb{Z}_X \to j_* H$ induces a commutative diagram (cf. Lemma 3.1.3)

$$\begin{array}{ccc} R^1 f_* \mathbb{Z}_X & \longrightarrow & R^1 f_* \mathcal{O}_X \\ \downarrow & & \| \\ j_* H & \longrightarrow & \mathcal{L}_{H/S}. \end{array}$$

Let $\mathcal{V}_X$ be the kernel of the homomorphism

$$R^1 f_* \mathcal{O}_X^\star \to j_*((R^1 f_* \mathcal{O}_X^\star)_{|S^\star}).$$

For a meromorphic section $\sigma\colon S \cdots\to B(H)$, we can attach an invertible sheaf $\mathcal{O}_{B(H)}(\Sigma - \Sigma_0)$, where $\Sigma = \sigma(S)$ and $\Sigma_0 = \sigma_0(S)$. By considering $\varphi_\lambda^* \mathcal{O}_{B(H)}(\Sigma - \Sigma_0)$, we have an element of $H^0(S, R^1 f_* \mathcal{O}_X^\star / \mathcal{V}_X)$, since the transition mappings $\varphi_\lambda \circ \varphi_\mu^{-1}$ are translations. Therefore as in §1.3, we have an injective homomorphism

$$\Phi_X : \mathfrak{S}_{H/S} \to R^1 f_* \mathcal{O}_X^\star / \mathcal{V}_X,$$

which extends to the exact sequence (cf. (1.2)):

$$0 \to \mathfrak{S}_{H/S} \overset{\Phi_X}{\to} R^1 f_* \mathcal{O}_X^\star / \mathcal{V}_X \to \mathbb{Z}_S \to 0. \tag{4.1}$$

By the similar argument to the proof of Lemma 1.3.2, we have:

Lemma 4.1.4 (cf. [Ny8]). *The cohomology class $\eta(X/S, \phi)$ is the image of* 1 *under the connecting homomorphism*

$$\mathbb{Z} = H^0(S, \mathbb{Z}_S) \to H^1(S, \mathfrak{S}_{H/S})$$

derived from (4.1).

Proposition 4.1.5.

(1) *Let* $f\colon X \to S$ *be an elliptic fibration smooth over* $S^\star$ *admitting a meromorphic section locally over* S *and let* $\phi\colon H \simeq (R^1 f_* \mathbb{Z}_X)_{|S^\star}$ *be an isomorphism. Then the cohomology class* $\eta(X/S, \phi) \in H^1(S, \mathfrak{S}_{H/S})$ *is a torsion element if and only if* f *is bimeromorphically equivalent to a projective morphism over* S.

(2) *For a torsion element* $\eta \in H^1(S, \mathfrak{S}_{H/S})$, *there exist a pair* $(X/S, \phi)$ *as above such that* $\eta = \eta(X/S, \phi)$.

Proof. (1). Suppose that f is bimeromorphically equivalent to a projective morphism and that X is nonsingular. Then we have an invertible sheaf $\mathcal{M}$ on X such that $\deg \mathcal{M}_{|f^{-1}(s)} > 0$ for a general fiber $f^{-1}(s)$. At the exact sequence (4.1), $\mathcal{M} \in H^1(X, \mathcal{O}_X^\star)$ induces a positive integer in $\mathbb{Z} = \Gamma(S, \mathbb{Z}_S)$. Hence by Lemma 4.1.4, $\eta(X/S, \phi)$ is a torsion element. Next conversely suppose $m\eta(X/S, \phi) = 0$ for a positive integer m. Then by the same argument as in the proof of Proposition 1.3.3, we have a generically finite meromorphic mapping $X \cdots\to B(H)$ over S. Therefore $f\colon X \to S$ is bimeromorphically equivalent to a projective morphism.

(2). Suppose that $m\eta = 0$ for a positive integer m. Let $\{S_\lambda\}$ be a locally finite open covering of S and $\{\eta_{\lambda,\mu}\}$ be a cocycle of meromorphic sections of $B(H)$ representing η. We may assume that there exist meromorphic sections ξ_λ over S_λ such that $m\eta_{\lambda,\mu} = \xi_\mu - \xi_\lambda$ over $S_\lambda \cap S_\mu$. As in the previous argument we have multiplication mappings

$$\psi_\lambda\colon B(H)_{|S_\lambda} \overset{m\times}{\cdots\to} B(H)_{|S_\lambda}.$$

Then $\mathrm{tr}(\xi_\mu) \circ \mathrm{tr}(\xi_\lambda)^{-1} \circ \psi_\mu = \psi_\lambda \circ \mathrm{tr}(\eta_{\lambda,\mu})$. The meromorphic sections $\eta_{\lambda,\mu}$, ξ_λ are holomorphic over $S^\star$. Therefore we have the patching $X^\star := \bigcup_\lambda B^\star(H)_{|S^\lambda}$ by $\{\eta_{\lambda,\mu}\}$ and a finite étale morphism $\psi^\star\colon X^\star \to B(H)^\star$ over $S^\star$. By construction, the elliptic fibration $f^\star\colon X^\star \to B(H)^\star \to S^\star$ induces an isomorphism $\phi\colon H \simeq R^1 f_*^\star \mathbb{Z}_{X^\star}$ as variations of Hodge structures. By a theorem of Grauert–Remmert ([GR]), there exist a generically finite morphism $\psi\colon X \to B(H)$ extending $\psi^\star\colon X^\star \to B^\star(H)$. Then the composite $f\colon X \to B(H) \to S$ is an elliptic fibration bimeromorphically equivalent to a projective morphism. By the uniqueness of the extension of finite morphisms, we have bimeromorphic mappings $\varphi_\lambda\colon X_{|S_\lambda} \cdots\to B(H)_{|S_\lambda}$ such that $\psi_{|S_\lambda} = \psi_\lambda \circ \varphi_\lambda$ and $\varphi_\lambda = \mathrm{tr}(\eta_{\lambda,\mu}) \circ \varphi_\mu$ for λ, μ. Therefore $\eta(X/S, \phi) = \eta$. Q.E.D.

Next we consider another situation. Let $f\colon X \to S$ be an elliptic fibration smooth over $S^\star$ and let $\phi\colon H \simeq (R^1 f_* \mathbb{Z}_X)_{|S^\star}$ be an isomorphism of variations of Hodge structures. Suppose that there is a finite Galois covering $\tau\colon T \to S$ with the Galois group G such that T is a complex manifold, $\tau^{-1}(D) = D_T$ is a normal crossing divisor, τ is étale over $S^\star$, and that the fiber product $X_T := T \times_S X \to T$ admits a meromorphic section. Let H_T be the the variation of Hodge structures $\tau^{-1}H$ on $T^\star := T \setminus \tau^{-1}(D)$ and let $B(H_T) \to T$ be a similar basic elliptic fibration. Then $X_T \to T$ is bimeromorphically equivalent to $B(H_T) \to T$. Then from f, we have a cohomology class in $H^1(G, H^0(T, \mathfrak{S}_{H_T/T}))$ by the same argument as in §0.3 and §1.4. Conversely, let us take an element $\eta \in H^1(G, H^0(T, \mathfrak{S}_{H_T/T}))$. Then by an argument in §0.3, η induces a left meromorphic action of G on $B(H_T)$. This is described as follows: Let $\{\eta_g\}_{g\in G}$ be a cocycle of meromorphic sections of $B(H_T)$ representing η. Let $G \times B(H_T) \cdots\to B(H_T)$ be the natural meromorphic action of G which defines $B(H)$ as the quotient. Let $\phi_g\colon B(H_T) = \{g\} \times B(H_T) \cdots\to B(H_T)$ be induced bimeromorphic automorphisms. Then the new action of G on $B(H_T)$ is defined by $\phi'_g := \phi_g \circ \mathrm{tr}(\eta_g)$. Since G is a finite group, we can consider the quotient X of $B(H_T)$ by G. Then we obtained an elliptic fibration. Let $\mathcal{E}(S, D, H, T)$ be the set of bimeromorphic equivalence classes of pairs $(f\colon X \to S, \phi)$ consisting of an elliptic fibration $f\colon X \to S$ smooth over $S^\star$ and an isomorphism $\phi\colon H \simeq (R^1 f_*\mathbb{Z}_X)_{|S^\star}$ such that $X \times_S T \to T$ admits a meromorphic section. Here two pairs $(f_1\colon X_1 \to S, \phi_1)$ and $(f_2\colon X_2 \to S, \phi_2)$ are called to be bimeromorphically equivalent if there is a bimeromorphic mapping $\varphi\colon X_1 \cdots\to X_2$ over S such that $\phi_1 = \varphi^* \circ \phi_2$. Then we have:

Lemma 4.1.6. *There is a one to one correspondence between* $H^1(G, H^0(T, \mathfrak{S}_{H_T/T}))$ *and* $\mathcal{E}(S, D, H, T)$.

4.2. Construction of toric models

We fix positive integers $1 \le l \le d$ and nonnegative integers a_i for $1 \le i \le l$, where we assume $a := \sum_{i=1}^{l} a_i > 0$.

Definition 4.2.1. A map $\sigma\colon \mathbb{Z} \to \{1, 2, \dots, l\}$ is called a *sign function* with respect to $(a_1, a_2, \dots, a_l)$ if the following two conditions are satisfied:

(1) $\sigma(m + a) = \sigma(m)$ for $m \in \mathbb{Z}$;
(2) $a_i = \sharp\{0 \le j < a \,|\, \sigma(j) = i\}$ for any i.

For a sign function σ and for given integers b_i for $1 \le i \le l$, there exist maps $I_i\colon \mathbb{Z} \to \mathbb{Z}$ with $I_i(0) = b_i$ such that for $m \in \mathbb{Z}$,

$$I_i(m+1) = \begin{cases} I_i(m) + 1, & \text{if } \sigma(m) = i; \\ I_i(m), & \text{otherwise.} \end{cases}$$

Definition 4.2.2. We call the map I_i by the *index function* at i with respect to the sign function σ and the initial value b_i.

For any integer k, let $\mathcal{C}_k \subset \mathbb{R}^{d+1}$ be the rational polyhedral cone

$$\left\{ (u_1, \ldots, u_d, y) \in \mathbb{R}^{d+1} \,\middle|\, u_i \geq 0,\ \sum_{i=1}^{l} I_i(k) u_i \leq y \leq \sum_{i=1}^{l} I_i(k+1) u_i \right\}.$$

Then the semigroup

$$\mathcal{C}_k^{\vee} \cap \mathbb{Z}^{d+1} := \left\{ (n_1, \ldots, n_{d+1}) \,\middle|\, \sum_{i=1}^{d} n_i u_i + n_{d+1} y \geq 0 \text{ for } (u_i, y) \in \mathcal{C}_k \right\}$$

is finitely generated. Let R_k be the associated semigroup ring over $\mathbb{C}$. It is easy to show the following:

Lemma 4.2.3. *Let $\mathbb{C}[t_1, t_2, \ldots, t_d, s]$ be the polynomial ring of $(d+1)$-variables. Then R_k is isomorphic to a $\mathbb{C}$-subalgebra of $\mathbb{C}[t_1^{\pm 1}, t_2^{\pm 1}, \ldots, t_d^{\pm 1}, s^{\pm 1}]$ generated by monomials*

$$t_1, t_2, \ldots, t_d, \quad s \prod_{i=1}^{l} t_i^{-I_i(k)}, \quad s^{-1} \prod_{i=1}^{l} t_i^{I_i(k+1)}.$$

By the theory of torus embeddings, $\operatorname{Spec} R_k$ are patched together and form a nonsingular scheme $\mathcal{M}_{\sigma,(b_i)}$ locally of finite type over $\operatorname{Spec} \mathbb{C}[t_1, t_2, \ldots, t_d]$. We note that $\mathcal{M}_{\sigma,(0)_{i=1}^{l}}$ is isomorphic to $\mathcal{M}_{\sigma,(b_i)}$ by the morphism:

$$(t_1, t_2, \ldots, t_d, s) \mapsto \left(t_1, t_2, \ldots, t_d, s \prod_{i=1}^{l} t_i^{b_i} \right).$$

Thus we denote $\mathcal{M}_{\sigma,(b_i)}$ simply by $\mathcal{M}_{\sigma}$. Let Δ^d be the unit polydisc defined in

$$(\operatorname{Spec} \mathbb{C}[t_1, t_2, \ldots, t_d])^{\mathrm{an}} \simeq \mathbb{C}^d.$$

Let $(\Delta^d)^{\star} := (\Delta^{\star})^l \times \Delta^{d-l}$ and let $(\Delta^d)^{\circ}$ be the complement of the following subset in Δ^d:

$$\bigcup_{1 \leq i < j \leq l} \{t_i = t_j = 0\}.$$

Then $(\Delta^d)^{\circ} = \Delta^d \setminus \operatorname{Sing} D$ and $(\Delta^d)^{\star} = \Delta^d \setminus \operatorname{Supp} D$, where $D = \sum_{i=1}^{l} D_i = \{t_1 t_2 \cdots t_l = 0\}$. We shall consider the analytic space

$$\mathcal{X}_{\sigma} := (\mathcal{M}_{\sigma})^{\mathrm{an}} \times_{(\operatorname{Spec} \mathbb{C}[t_1, t_2, \ldots, t_d])^{\mathrm{an}}} \Delta^d$$

and the projection $\pi\colon \mathcal{X}_\sigma \to \Delta^d$. We define $\mathcal{X}_\sigma^\circ := \mathcal{X}_\sigma \times_{\Delta^d} (\Delta^d)^\circ$ and $\mathcal{X}_\sigma^\star := \mathcal{X}_\sigma \times_{\Delta^d} (\Delta^d)^\star$. Since $\mathcal{X}_\sigma^\star \simeq (\Delta^d)^\star \times \mathbb{C}^\star$ does not depend on the choice of σ, we write $\mathcal{X}^\star := \mathcal{X}_\sigma^\star$. Here the variable s above is considered to be a coordinate of $\mathbb{C}^\star$. Let us define

$$\mathcal{X}_\sigma^{(k)} := (\operatorname{Spec} R_k)^{\mathrm{an}} \times_{(\operatorname{Spec}\mathbb{C}[t_1,t_2,\dots,t_d])^{\mathrm{an}}} \Delta^d.$$

Lemma 4.2.4. *$\mathcal{X}_\sigma$ is simply connected.*

Proof. By construction, $\mathcal{X}_\sigma$ contains $\mathcal{X}_\sigma^{(k)}$ as a Zariski-open subset, which is isomorphic to $\{(u,v) \in \mathbb{C}^2 |\ |uv| < 1\} \times \Delta^{d-1}$. This is simply connected. Thus $\mathcal{X}_\sigma$ is also simply connected. Q.E.D.

Lemma 4.2.5. *For any sign functions σ, $\mathcal{X}_\sigma^\circ$ are isomorphic to each other.*

Proof. For any $\mathbf{J} = (j_1, j_2, \dots, j_l) \in \mathbb{Z}^{\oplus l}$ and for $1 \le i \le l$, we consider the algebra

$${}_iR_{\mathbf{J}} := \mathbb{C}\left[t_1, t_2, \dots, t_d, s\prod_{k=1}^{l} t_k^{-j_k}, s^{-1}t_i \prod_{k=1}^{l} t_k^{j_k}\right].$$

It is enough to show that $\mathcal{X}_\sigma^\circ$ contains $(\operatorname{Spec} {}_iR_{\mathbf{J}})^{\mathrm{an}} \times_{(\operatorname{Spec}\mathbb{C}[t])^{\mathrm{an}}} (\Delta^d)^\circ$ as an open subset naturally, where $t = (t_1, t_2, \dots, t_d)$. There is an integer m such that $\sigma(m) = i$ and $I_i(m) = j_i$. Then two algebras ${}_iR_{\mathbf{J}}$ and R_m are isomorphic to each other up to the localization by $\prod_{k\neq i} t_k$. Next we fix $i' \neq i$ and compare two algebras ${}_iR_{\mathbf{J}}$ and ${}_{i'}R_{\mathbf{J}}$. Then there is an open immersion $\operatorname{Spec}({}_iR_{\mathbf{J}}[t_i^{-1}]) \to \operatorname{Spec}({}_{i'}R_{\mathbf{J}}[t_i^{-1}])$. Hence by combining with the previous argument, we have an open immersion

$$\operatorname{Spec}\left({}_iR_{\mathbf{J}}\left[\prod\nolimits_{k\neq i'} t_k^{-1}\right]\right) \to \operatorname{Spec}\left(R_{m'}\left[\prod\nolimits_{k\neq i'} t_k^{-1}\right]\right)$$

for some m'. Thus we are done. Q.E.D.

In what follows, we also denote $\mathcal{X}^\circ = \mathcal{X}_\sigma^\circ$. Note that $\mathcal{X}^\circ$ is also simply connected, since $\operatorname{codim}(\mathcal{X}_\sigma \setminus \mathcal{X}^\circ) \ge 2$. Now we shall take a period function $\omega(z)$ on $U = \mathbb{H}^l \times \Delta^{d-l}$ of the form (cf. Proposition 2.1.4):

$$\omega(z) = \sum_{i=1}^{l} a_i z_i + h(t),$$

where $h(t)$ is a holomorphic function on Δ^d such that $\operatorname{Im} h(t) \ge 0$. Then we have a variation of Hodge structures H of type $\mathrm{I}_{(+)}$ on $(\Delta^d)^\star$ whose monodromy matrix around the coordinate hyperplane D_i is of type I_{a_i}.

We have $e(\omega(z)) = e(h(t))\prod_{i=1}^{l} t_i^{a_i}$. Let us consider an automorphism ϑ of $\mathcal{X}^\star$ defined by:

$$\vartheta\colon \mathcal{X}^\star \simeq (\Delta^d)^\star \times \mathbb{C}^\star \ni (t, s) \mapsto \left(t,\, s \cdot e(h(t)) \prod_{i=1}^{l} t_i^{a_i}\right).$$

This induces also a holomorphic automorphism of $\mathcal{X}_\sigma$. In fact, we have isomorphisms $\mathcal{X}_\sigma^{(k)} \simeq \mathcal{X}_\sigma^{(k+a)}$ by ϑ. From the inequality $|e(\omega(z))| < 1$ and from a similar argument to [Nk], we have:

Lemma 4.2.6. *The action of ϑ on $\mathcal{X}_\sigma$ is properly discontinuous and fixed point free.*

Therefore we can define the quotient manifold X_σ, which has a structure of an elliptic fibration $p\colon X_\sigma \to \Delta^d$. Here we usually assume that all the initial values $b_i = 0$. By composing the morphism

$$\Delta^d \ni t \mapsto (t, 1) \in \mathcal{X}_\sigma$$

with the quotient morphism $q\colon \mathcal{X}_\sigma \to X_\sigma$, we have a section $\sigma_0\colon \Delta^d \to X_\sigma$.

Definition 4.2.7. We call $p\colon X_\sigma \to \Delta^d$ by the *toric model* of type σ and the section $\sigma_0\colon \Delta^d \to X_\sigma$ by the *zero section* of p.

Lemma 4.2.8. *We have the following properties on the toric model:*

(1) *The period function of p is of the form $\omega(z) = \sum_{i=1}^{d} a_i z_i + h(t)$ on $\mathbb{H}^l \times \Delta^{d-l}$;*
(2) *The monodromy matrix $\rho(\gamma_i)$ is of type I_{a_i};*
(3) *The fiber $p^{-1}(0)$ is isomorphic to a cycle of rational curves, the number of whose components is $a = \sum_{i=1}^{d} a_i$. In particular, p is a flat morphism;*
(4) *The canonical bundle of X_σ is trivial and hence $p\colon X_\sigma \to \Delta^d$ is a minimal elliptic fibration.*

Proof. It is enough to prove (4). Let us consider the meromorphic $(d+1)$-form

$$dt_1 \wedge dt_2 \wedge \cdots \wedge dt_d \wedge \frac{ds}{s}$$

on $\mathcal{X}_\sigma$. It is easily checked that this is holomorphic and is a nowhere vanishing section of the canonical bundle of $\mathcal{X}_\sigma$. Further this is invariant under ϑ. Thus this induces a nowhere vanishing section of the canonical bundle of X_σ. Q.E.D.

Theorem 4.2.9 (minimal model). *Let $f\colon Y \to \Delta^d$ be a minimal projective elliptic fibration which is bimeromorphically equivalent to $p\colon X_\sigma \to \Delta^d$. Then there exist a sign function σ' and a bimeromorphic morphism $X_{\sigma'} \to Y$ over Δ^d such that $X_{\sigma'}$ is a $\mathbb{Q}$-factorialization of Y.*

We divide the proof into the following 4 steps.

Step 1. Since $f\colon Y \to \Delta^d$ and $p\colon X_\sigma \to \Delta^d$ are minimal models, the bimeromorphic mapping $Y \cdots\to X_\sigma$ is an isomorphism in codimension one. Let A be a general irreducible divisor on Y which is f-ample. Then its proper transform Γ in X_σ is also an irreducible divisor. Suppose that Γ is p-nef. Then Γ is p-semi-ample by [Ny3, 4.8, 4.10]. Since Y and X_σ are isomorphic in codimension one, we have a bimeromorphic morphism $X_\sigma \to Y$ over Δ^d sending Γ to A. Therefore we are done in the case Γ is p-nef.

Next assume that Γ is not p-nef. It is enough to find a suitable sign function σ' such that the proper transform of A in $X_{\sigma'}$ is relatively nef over Δ^d.

Step 2. Now the positive integer $a = \sum_{i=1}^{l} a_i$ satisfies $a \geq 2$. Otherwise, the central fiber $p^{-1}(0)$ is irreducible, so Γ is p-nef. For $k \in \mathbb{Z}$, let $\widetilde{C}_k$ be the irreducible component of the central fiber $\pi^{-1}(0) \subset \mathcal{X}_\sigma$ which intersects $\mathcal{X}_\sigma^{(k)} \cap \mathcal{X}_\sigma^{(k+1)}$, $\widetilde{E}_k$ the component of $\pi^{-1}(D_{\sigma(k)})$ containing $\widetilde{C}_k$, and let $\widetilde{F}_k$ be the component of $\pi^{-1}(D_{\sigma(k+1)})$ containing $\widetilde{C}_k$. Also for $\kappa \in \mathbb{Z}/a\mathbb{Z}$, let C_κ be the image of $\widetilde{C}_k$ under the quotient morphism $\mathcal{X}_\sigma \to X_\sigma$, where $k \mod a = \kappa$. Further let E_κ and F_κ be the images of $\widetilde{E}_k$ and $\widetilde{F}_k$, respectively. The following lemma is easily shown:

Lemma 4.2.10.

(1) *If $\sigma(\kappa) = \sigma(\kappa+1)$, then $E_\kappa = F_\kappa \simeq \mathbb{P}^1 \times D_{\sigma(\kappa)}$ over $D_{\sigma(\kappa)}$ and C_κ is the central fiber of $E_\kappa \to D_{\sigma(\kappa)}$. In particular, the normal bundle $N_{C_\kappa/\mathcal{X}_\sigma}$ is isomorphic to $\mathcal{O}(-2) \oplus \mathcal{O}^{\oplus(d-1)}$.*

(2) *If $\sigma(\kappa) \neq \sigma(\kappa+1)$, then the complete intersection $E_\kappa \cap F_\kappa$ is isomorphic to $\mathbb{P}^1 \times \left(D_{\sigma(\kappa)} \cap D_{\sigma(\kappa+1)}\right)$ over $D_{\sigma(\kappa)} \cap D_{\sigma(\kappa+1)}$, where C_κ is the central fiber of $E_\kappa \cap F_\kappa \to D_{\sigma(\kappa)} \cap D_{\sigma(\kappa+1)}$.*

(3) *In the case $\sigma(\kappa) \neq \sigma(\kappa+1)$, the normal bundle of $E_\kappa \cap F_\kappa$ in $\mathcal{X}_\sigma$ is isomorphic to $p_1^*\mathcal{O}(-1)^{\oplus 2}$, where p_1 is the first projection*

$$E_\kappa \cap F_\kappa \simeq \mathbb{P}^1 \times \left(D_{\sigma(\kappa)} \cap D_{\sigma(\kappa+1)}\right) \to \mathbb{P}^1 \ .$$

In particular, the normal bundle $N_{C_\kappa/\mathcal{X}_\sigma}$ is isomorphic to $\mathcal{O}(-1)^{\oplus 2} \oplus \mathcal{O}^{\oplus(d-2)}$.

Step 3. Since Γ is not p-nef, there exists a curve $C_\kappa \subset p^{-1}(0)$ such that $\Gamma \cdot C_\kappa < 0$. If $\sigma(\kappa) = \sigma(\kappa+1)$, then $\Gamma \cdot \gamma < 0$ for any fiber γ of

$E_\kappa \to D_{\sigma(\kappa)}$. Thus $E_\kappa \subset \Gamma$. This is impossible, since Γ is irreducible and is dominating Δ^d. Therefore $\sigma(\kappa) \neq \sigma(\kappa+1)$. Let $X'_\sigma \to X_\sigma$ be the blowing-up along $E_\kappa \cap F_\kappa$. By Lemma 4.2.10, we can blow-down X'_σ along the other ruling of the exceptional divisor which is isomorphic to

$$\mathbb{P}^1 \times \mathbb{P}^1 \times \left(D_{\sigma(\kappa)} \cap D_{\sigma(\kappa+1)}\right).$$

Thus we obtain another manifold X''_σ. (cf. Figure 1). By considering this process on $\mathcal{X}_\sigma$ and by applying the torus embedding theory, we see that $X''_\sigma \simeq X_{\sigma'}$ for a sign function σ' determined by

$$\sigma'(j) := \begin{cases} \sigma(\kappa+1), & \text{if } j = \kappa; \\ \sigma(\kappa), & \text{if } j = \kappa+1; \\ \sigma(j), & \text{otherwise.} \end{cases}$$

Let C'_κ be the fiber over $0 \in \Delta^d$ of the image of the exceptional divisor, C'_j the proper transform of C_j for $\kappa \neq j \in \mathbb{Z}/a\mathbb{Z}$ and let Γ' be the proper transform of Γ. Then we have:

Lemma 4.2.11.

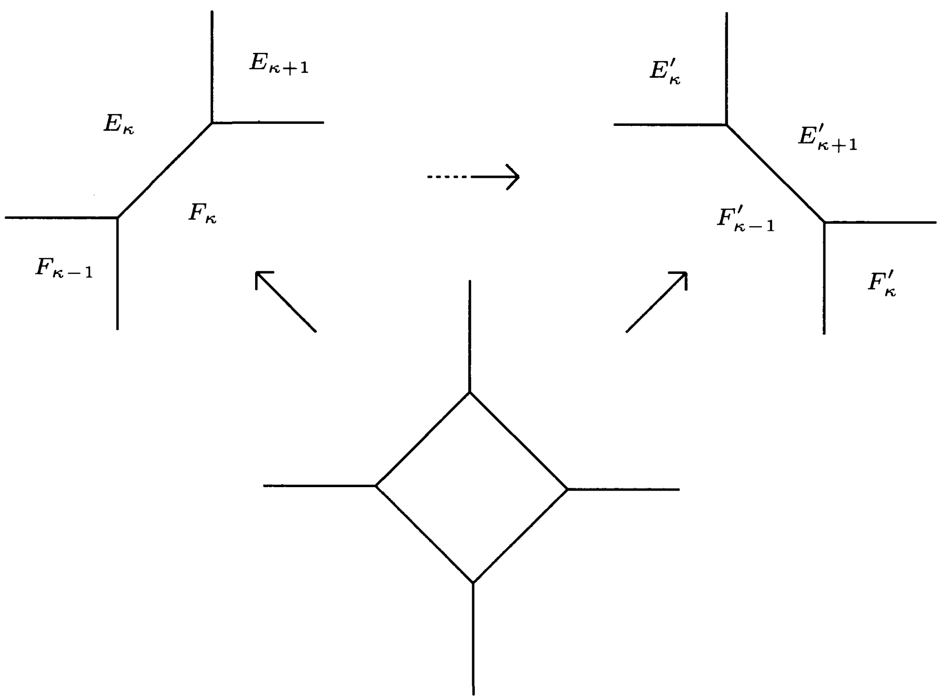

Figure 1. Flop.

(1) *If $a > 2$, then*

$$\Gamma' \cdot C'_j = \begin{cases} -\Gamma \cdot C_\kappa, & \text{if } j = \kappa; \\ \Gamma \cdot C_j + \Gamma \cdot C_\kappa, & \text{if } j = \kappa - 1 \text{ or } j = \kappa + 1; \\ \Gamma \cdot C_j, & \text{otherwise.} \end{cases}$$

(2) *If $a = 2$, then*

$$\Gamma' \cdot C'_j = \begin{cases} -\Gamma \cdot C_\kappa, & \text{if } j = \kappa; \\ \Gamma \cdot C_j + 2\Gamma \cdot C_\kappa, & \text{if } j = \kappa + 1. \end{cases}$$

Step 4. Let δ be the covering degree of $\Gamma \to \Delta^d$ and let us consider the following set of mappings:

$$S_\delta^{(a)} := \left\{ \phi \colon \mathbb{Z}/a\mathbb{Z} \to \mathbb{Z} \,\middle|\, \sum_{x \in \mathbb{Z}/a\mathbb{Z}} \phi(x) = \delta \right\}.$$

An element $\phi \in S_\delta^{(a)}$ is called *nef* if $\phi(x) \geq 0$ for all $x \in \mathbb{Z}/a\mathbb{Z}$. For $y \in \mathbb{Z}/a\mathbb{Z}$ and $\phi \in S_\delta^{(a)}$ with $\phi(y) < 0$, we define the *flop* $F_y(\phi) \in S_\delta^{(a)}$ of ϕ at y as follows:

(1) (cf. Figure 2) If $a > 2$, then

$$F_y(\phi)(x) := \begin{cases} -\phi(x), & \text{if } x = y; \\ \phi(x) + \phi(y), & \text{if } x = y - 1 \text{ or } x = y + 1; \\ \phi(x), & \text{otherwise.} \end{cases}$$

(2) If $a = 2$, then

$$F_y(\phi)(x) := \begin{cases} -\phi(x), & \text{if } x = y; \\ \phi(x) + 2\phi(y), & \text{if } x = y + 1. \end{cases}$$

By the previous argument, Theorem 4.2.9 is deduced from the following:

Proposition 4.2.12 (Termination). *Any $\phi \in S_\delta^{(a)}$ turns to be nef after a finite number of flops.*

A lot of proofs of the proposition seem to be known. The following one is an application of the theory of elliptic surfaces.

Proof. Let $p \colon X \to \Delta$ be the toric model over a one-dimensional disc Δ, determined by a positive integer $a > 1$ and a period function $\omega(z) = az$. Then X is minimal over Δ and the central fiber $p^*(0)$

is a union of smooth rational curves, the number of whose irreducible components is a. We can write:

$$p^*(0) = \sum_{i \in \mathbb{Z}/a\mathbb{Z}} C_i.$$

For $\phi \in S_\delta^{(a)}$, let L be a Cartier divisor on X with intersection numbers $L \cdot C_i = \phi(i)$ for all i. Note that such L exists since there exist divisors Γ_i for $i \in \mathbb{Z}/a\mathbb{Z}$ such that $\Gamma_i \cdot C_j = \delta_{i,j}$. Suppose that $\phi(y) < 0$ for some $y \in \mathbb{Z}/a\mathbb{Z}$. Let us consider the divisor $L' = L + \phi(y)C_y$. Then we have $L' \cdot C_j = \phi'(j)$ for $\phi' = F_y(\phi)$. We here look at the exact sequence:

$$0 \to \mathcal{O}(L') \to \mathcal{O}(L) \to \mathcal{O}(L) \otimes \mathcal{O}_B \to 0,$$

where B is the effective divisor $-\phi(y)C_y$. We have $\chi(B, \mathcal{O}(L)\otimes\mathcal{O}_B) = 0$. Thus $h^0(B, \mathcal{O}(L)\otimes\mathcal{O}_B) = h^1(B, \mathcal{O}(L)\otimes\mathcal{O}_B)$. Let l and l' be the lengths of the skyscraper sheaves $R^1p_*\mathcal{O}(L)$ and $R^1p_*\mathcal{O}(L')$, respectively. If $p_*\mathcal{O}(L') \simeq p_*\mathcal{O}(L)$, then $l = l'$. If $p_*\mathcal{O}(L') \hookrightarrow p_*\mathcal{O}(L)$ is not an isomorphism, then $l' < l$. In the former case, we have

$$\operatorname{Im}(p^*p_*\mathcal{O}(L) \to \mathcal{O}(L)) \subset \mathcal{O}(L') \subset \mathcal{O}(L).$$

Thus after a finite number of flops, we come to the second situation. However the length is a nonnegative integer. Therefore we can not perform flops infinitely. Q.E.D.

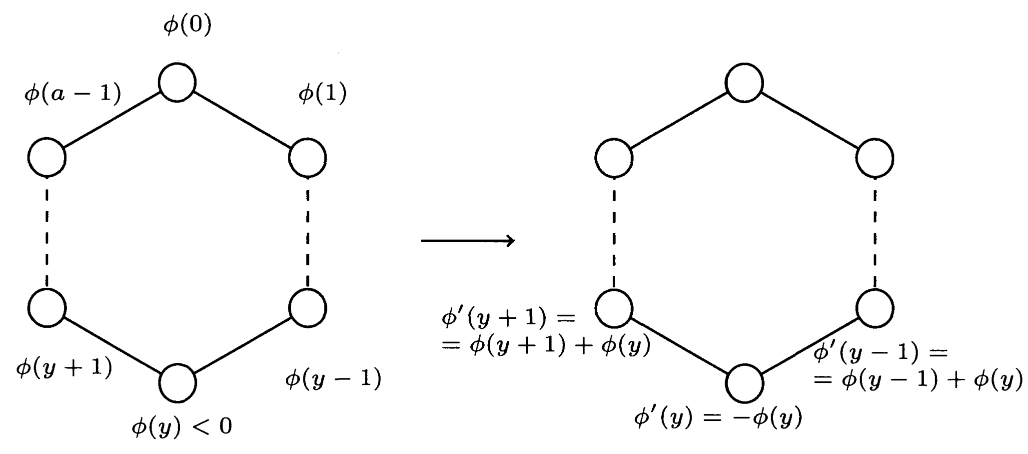

Figure 2. Rule of flops in the case $a > 2$.

Let H be the variation of Hodge structures on $(\Delta^d)^\star$ induced from the period function $\omega(z) = \sum_{i=1}^l a_i z_i + h(t)$ and monodromy matrices $\rho(\gamma_i)$ of type I_{a_i}. Then the group of meromorphic sections of the toric model $p\colon X_\sigma \to \Delta^d$ is isomorphic to $H^0(\Delta^d, \mathfrak{S}_{H/\Delta^d})$. We shall study the sheaf $\mathfrak{S}_{H/\Delta^d}$.

Lemma 4.2.13. *Let $\Delta^d \cdots\to X_\sigma$ be a meromorphic section. Then there exist a sign function σ' such that the induced mapping $\Delta^d \cdots\to X_{\sigma'}$ is holomorphic.*

Proof. Let $\Gamma \subset X_\sigma$ be the image of the meromorphic section. Suppose that $\Gamma \to \Delta^d$ is not an isomorphism. Then $\Gamma \cdot C < 0$ for an irreducible curve C contained in a fiber of $\Gamma \to \Delta^d$. Thus Γ is not relatively nef over Δ^d. By the same argument as in the proof of Proposition 4.2.12, we can find an expected sign function. Q.E.D.

We have the quotient morphism $\mathcal{X}_\sigma \to X_\sigma$ by the action of ϑ. Since this is the universal covering mapping of X_σ, every meromorphic section of $X_\sigma \to \Delta^d$ has a lift to $\mathcal{X}_\sigma$. Note that this is holomorphic over $(\Delta^d)^\circ$.

Lemma 4.2.14. *The sheaf of germs of meromorphic sections of $\pi\colon \mathcal{X}_\sigma \to \Delta^d$ is isomorphic to the sheaf $\mathcal{O}_{\Delta^d}(*D^+)^\star$ of germs of meromorphic functions whose zeros and poles are contained in the divisor $D^+ := \{\prod_{a_i>0} t_i = 0\}$.*

Proof. A holomorphic section of $\mathcal{X}_\sigma \to \Delta^d$ for some σ is given by $s = v(t)\prod_{i=1}^l t_i^{I_i(k)}$ for some holomorphic function $v(t)$ on Δ^d and an integer k, where I_i denotes the index function at i with respect to σ. By Lemma 4.2.13, we see that every meromorphic sections of π is written by

$$s = u(t)\prod_{i=1}^{l} t_i^{m_i}$$

for a nowhere vanishing function $u(t)$ on Δ^d and integers m_i, for $1 \leq i \leq l$ with $a_i > 0$. Q.E.D.

Then we have a natural surjective homomorphism $\mathcal{O}_{\Delta^d}(*D^+)^\star \twoheadrightarrow \mathfrak{S}_{H/\Delta^d}$. The kernel is isomorphic to $\mathbb{Z}$ whose generator corresponds to the meromorphic function $e(h(t))\prod_{i=1}^l t_i^{a_i}$. Therefore there exist the following two exact sequences:

$$0 \to \mathbb{Z}_{\Delta^d} \to \mathcal{O}_{\Delta^d}(*D^+)^\star \to \mathfrak{S}_{H/\Delta^d} \to 0; \tag{4.2}$$

$$0 \to \mathcal{O}^\star_{\Delta^d} \to \mathcal{O}_{\Delta^d}(*D^+)^\star \to \bigoplus_{a_i>0} \mathbb{Z}_{D_i} \to 0. \tag{4.3}$$

4.3. Smooth model and toric model theorems

Theorem 4.3.1 (Smooth model theorem). *Let $f\colon Y \to \Delta^d$ be a projective elliptic fibration such that*

(1) *f is a smooth morphism over $(\Delta^d)^\star$,*
(2) *the monodromy representation is of type I_0,*
(3) *$f^{-1}(P)$ has a reduced component for a general point P of each D_i.*

Then f is bimeromorphically equivalent to the smooth basic elliptic fibration $p\colon B(H) \to \Delta^d$, where H is the variation of Hodge structures induced from f.

Proof. We have a Zariski-open subset V of Δ^d such that Y is flat over V, $\operatorname{codim}(\Delta^d \setminus V) \geq 2$, and that $f_{|V}\colon Y_{|V} \to V$ admits a meromorphic section locally over V. Thus we obtain a cohomology class $\eta = \eta(Y_{|V}/V, \phi) \in H^1(V, \mathfrak{S}_{H/\Delta^d})$, where $\phi\colon H \simeq (R^1 f_* \mathbb{Z}_Y)_{|V}$ is an isomorphism. Note that η is a torsion element by Proposition 4.1.5. There is an exact sequence:

$$0 \to H \simeq \mathbb{Z}_{\Delta^d}^{\oplus 2} \to \mathcal{O}_{\Delta^d} \to \mathfrak{S}_{H/\Delta^d} \to 0.$$

Hence we have an isomorphism:

$$H^1(V, \mathcal{O}_V) \simeq H^1(V, \mathfrak{S}_{H/\Delta^d}),$$

since $H^i(V, \mathbb{Z}) = 0$ for $i = 1, 2$. Thus the torsion element η must be zero. This means that $Y_{|V}$ is bimeromorphically equivalent to $B(H)_{|V}$ over V. Since $B(H)\backslash p^{-1}(V)$ has codimension greater than one, the meromorphic mapping to $Y_{|V}$ extends to a meromorphic mapping $B(H) \cdots\to Y$ over Δ^d. Hence f is bimeromorphically equivalent to p. Q.E.D.

Theorem 4.3.2 (Toric model theorem). *Let $f\colon Y \to \Delta^d$ be a projective elliptic fibration such that*

(1) *f is a smooth morphism over $(\Delta^d)^\star$,*
(2) *the monodromy matrix $\rho(\gamma_i)$ around the coordinate hyperplane D_i is of type I_{a_i},*
(3) *one of $\{a_i\}$ is not zero,*
(4) *$f^{-1}(P)$ has a reduced component for a general point P of each D_i.*

Then f is bimeromorphically equivalent to a toric model $p\colon X_\sigma \to \Delta^d$.

Proof. Since the monodromy representation of f is of type $\mathrm{I}_{(+)}$, by Proposition 2.1.4, we may assume that the period function of f over $(\Delta^d)^\star$ is written as:

$$\omega(z) = \sum_{i=1}^{l} a_i z_i + h(t)$$

for a holomorphic function $h(t)$ on Δ^d. Let $p\colon X_\sigma \to \Delta^d$ be a toric model constructed from the variation of Hodge structures H. We have a Zariski-open subset $V \subset (\Delta^d)^\circ$ such that $f\colon Y \to \Delta^d$ is flat over V, $\operatorname{codim}((\Delta^d)^\circ \setminus V) \geq 2$ and that $f_{|V}\colon Y_{|V} \to V$ admits a meromorphic section locally over V. Therefore, we have a cohomology class $\eta(Y_{|V}/V, \phi) \in H^1(V, \mathfrak{S}_{H/\Delta^d})$, where $\phi\colon H \simeq (R^1 f_* \mathbb{Z}_Y)_{|Y^\star}$ is an isomorphism. By Proposition 4.1.5, $\eta(Y_{|V}/V, \phi)$ is a torsion element of $H^1(V, \mathfrak{S}_{H/\Delta^d})$. We have an isomorphism $H^1(V, \mathcal{O}_{\Delta^d}(*D^+)^\star) \simeq H^1(V, \mathfrak{S}_{H/\Delta^d})$ from (4.2). From (4.3), we have a surjection

$$H^0\left(V, \mathcal{O}_{\Delta^d}(*D^+)^\star\right) \twoheadrightarrow H^0\Big(V, \bigoplus_{a_i>0} \mathbb{Z}_{D_i}\Big) \simeq \bigoplus_{a_i>0} \mathbb{Z}$$

and an exact sequence

$$0 \to H^1(V, \mathcal{O}^\star_{\Delta^d}) \to H^1\left(V, \mathcal{O}_{\Delta^d}(*D^+)^\star\right) \to$$
$$\to H^1\Big(V, \bigoplus_{a_i>0} \mathbb{Z}_{D_i}\Big) \simeq \bigoplus_{a_i>0} H^1(D_i \cap V, \mathbb{Z}).$$

Note that $H^1(D_i \cap V, \mathbb{Z})$ are torsion free abelian groups and the exponential sequence on Δ^d induces the isomorphism $H^1(V, \mathcal{O}_{\Delta^d}) \simeq H^1(V, \mathcal{O}^\star_{\Delta^d})$. Hence $H^1(V, \mathcal{O}^\star_{\Delta^d})$ has a structure of $\mathbb{C}$-vector space. Therefore $H^1(V, \mathfrak{S}_{H/\Delta^d})$ is torsion free. Thus $\eta(Y_{|V}/V, \phi) = 0$, which means that $Y_{|V} \to V$ is bimeromorphically equivalent to the toric model $p\colon X_\sigma \to \Delta^d$ over V. Hence we have a bimeromorphic mapping $(X_\sigma)_{|V} \cdots\to Y_{|V}$ over V. Since $\operatorname{codim}(X_\sigma \setminus p^{-1}((\Delta^d)^\circ)) \geq 2$, $\operatorname{codim}(p^{-1}((\Delta^d)^\circ) \setminus p^{-1}(V)) \geq 2$, and since $f\colon Y \to \Delta^d$ is a projective morphism, the meromorphic mapping extends to a bimeromorphic mapping $X_\sigma \cdots\to Y$ over Δ^d.

Q.E.D.

By taking a unipotent reduction and a further Kummer coverings $\Delta^d \to \Delta^d$, we have the following:

Corollary 4.3.3. *Let $f\colon Y \to \Delta^d$ be a projective elliptic fibration smooth over $(\Delta^d)^\star$. Then there is a finite branched covering $T \to \Delta^d$ étale over $(\Delta^d)^\star$ such that $Y \times_{\Delta^d} T \to T$ admits a meromorphic section.*

We shall give another proof of Corollary 4.3.3, which is based on an argument of Viehweg ([V, 9.10]).

Proof. Let $A \subset Y$ be a prime divisor dominating Δ^d. By taking a normalization of A, we have a generically finite surjective morphism $V \to \Delta^d$ such that $Y \times_{\Delta^d} V \to V$ admits a meromorphic section. Let $V \to T \to \Delta^d$ be the Stein factorization. Then $\tau\colon T \to \Delta^d$ is a finite morphism. By taking its Galois closure, we may suppose that τ

is a Galois covering with a Galois group G. Let us consider a basic elliptic fibration $p\colon B(H) \to \Delta^d$ associated with the variation of Hodge structures H induced from f over $(\Delta^d)^\star$ and let B_T be the normalization of the main component of $B(H) \times_{\Delta^d} T$. We fix a bimeromorphic mapping $\varphi\colon Y \times_{\Delta^d} T \cdots\to B_T$ over T which keeps the isomorphism of variations of Hodge structures determined by f and p. Let $\mathfrak{S}_{H_T/T}$ be the sheaf of germs of meromorphic sections of $B_T \to T$. Then by the argument preceeding to Lemma 4.1.6, we have a cohomology class $\eta \in H^1(G, H^0(T, \mathfrak{S}_{H_T/T}))$. For a cocycle $\{\eta_g\}_{g\in G}$ representing η, we have a meromorphic mapping

$$\phi'(g) := \phi(g) \circ \mathrm{tr}(\eta_g)\colon B_T \cdots\to B_T \to B_T,$$

where $\phi(g)$ is induced from

$$\mathrm{id} \times_{\Delta^d} g\colon B(H) \times_{\Delta^d} T \to B(H) \times_{\Delta^d} T.$$

We can take $\{\eta_g\}$ to satisfy $\phi'(g) = \varphi \circ \phi_Y(g) \circ \varphi^{-1}$, where

$$\phi_Y(g) := (\mathrm{id}_Y \times_{\Delta^d} g)\colon Y \times_{\Delta^d} T \to Y \times_{\Delta^d} T.$$

Let n be the order of η. Then η comes from $H^1(G, H^0(T, K_n))$, where

$$K_n := \mathrm{Ker}(\mathfrak{S}_{H_T/T} \stackrel{n\times}{\to} \mathfrak{S}_{H_T/T}).$$

Thus we have a cocycle $\{\eta_g^0\}$ of $H^0(T, K_n)$ and a meromorphic section $\sigma \in H^0(T, \mathfrak{S}_{H/T})$ such that

$$\eta_g = \eta_g^0 + \sigma - \sigma^g.$$

By replacing φ by $\mathrm{tr}(\sigma) \circ \varphi$, we may assume that $\eta_g = \eta_g^0 \in H^0(T, K_n)$. For a prime divisor Γ of Δ^d with $\Gamma \cap (\Delta^d)^\star \neq \emptyset$, let Γ' be the unique irreducible component of $p^{-1}(\Gamma)$ dominating Γ. Let $R(\Gamma)$ be the ramification group for Γ, that is the subgroup of G consisting of all the elements $g \in G$ satisfying the following condition: for any prime divisor Γ_i of T dominating Γ, $g\colon T \to T$ induces the identity on Γ_i. If $g \in R(\Gamma)$, then $\phi'(g)$ also induces the identity on every prime divisor Γ_i' on B_T dominating Γ'. Therefore η_g coincides with the zero section at least over $\Gamma \cap (\Delta^d)^\star$. Since K_n is a local constant system with fiber $(\mathbb{Z}/n\mathbb{Z})^{\oplus 2}$ over $\tau^{-1}((\Delta^d)^\star)$, every $\eta_g = 0$. Let R be the subgroup of G generated by all such ramification subgroups $R(\Gamma)$. Then it is a normal subgroup and $\phi(g) = \phi'(g)$ for any $g \in R$. Therefore the quotient $R\backslash Y \to R\backslash T$ still admits a meromorphic section. Hence if we take such a Galois covering $\tau\colon T \to S$ with the degree of τ being minimal, then R must be trivial. This means that τ is unramified over $(\Delta^d)^\star$. Thus we are done. Q.E.D.

§5. Elliptic fibrations with smooth discriminant loci

Let S, D, $j\colon S^\star \hookrightarrow S$, $e\colon U \to S^\star$ be the same objects as in §2. A variation of Hodge structures H on $S^\star$ is determined by a monodromy representation $\rho\colon \pi_1 := \pi_1(S^\star) \to \mathrm{SL}(2,\mathbb{Z})$ and a period mapping $\omega\colon U \to \mathbb{H}$. They are described in Table 2 and Corollary 3.1.6.

Definition. A finite ramified covering $\tau\colon T \to S$ is called a *U-covering* if the following conditions are satisfied:

(1) $T \simeq \Delta^d = \Delta^l \times \Delta^{d-l}$ and τ is given by

$$\theta = (\theta_1, \theta_2, \dots, \theta_l, t') \mapsto (\theta_1^{m_1}, \theta_2^{m_2}, \dots, \theta_l^{m_l}, t') \in S,$$

for some positive integers $m_1, m_2, \dots, m_l$;

(2) The induced variation of Hodge structures $H_T := \tau^{-1}H$ defined on $T^\star := \tau^{-1}(S^\star)$ has only unipotent monodromies, i.e., it is of type I_0 or type $\mathrm{I}_{(+)}$.

Let $\tau\colon T \to S$ be a U-covering. Then the period mapping of H is written by

$$\omega(z) = \sum_{i=1}^{l} a_i z_i + h(\theta)$$

for nonnegative integers a_i and holomorphic function $h(\theta)$ defined over T. If the monodromy group of H is finite, then all $a_i = 0$. Otherwise, $h(\theta)$ is the pullback of a holomorphic function on S. Under this situation, we shall classify projective elliptic fibrations $f\colon Y \to S$ smooth over $S^\star$ such that the induced variations of Hodge structures are isomorphic to the given H. More precisely, we shall describe the set $\mathcal{E}^+(S,D,H)$ defined as follows: Let $(f\colon Y \to S, \phi)$ be a pair of a projective elliptic fibration $f\colon Y \to S$ smooth over $S^\star$ and an isomorphism $\phi\colon H \simeq (R^1 f_* \mathbb{Z})_{|S^\star}$ as variations of Hodge structures. Two such pairs $(f_1\colon Y_1 \to S, \phi_1)$ and $(f_2\colon Y_2 \to S, \phi_2)$ are called bimeromorphically equivalent over S if there is a bimeromorphic mapping $\varphi\colon Y_1 \cdots\to Y_2$ over S such that $\phi_1 = \varphi^* \circ \phi_2$. We define $\mathcal{E}^+(S,D,H)$ to be the set of all the equivalence classes by this relation. Let $(f\colon Y \to S, \phi)$ be an element in $\mathcal{E}^+(S,D,H)$. Then by Corollary 4.3.3, we can find a U-covering $\tau\colon T \to S$ satisfying the following condition:

(5.1) $\qquad Y \times_S T \to T$ admits a meromorphic section.

Therefore, we have

$$\mathcal{E}^+(S,D,H) = \varinjlim_{T \to S} \mathcal{E}(S,D,H,T),$$

where the inductive limit is taken over all the U-coverings $\tau\colon T \to S$. Note that the set $\mathcal{E}(S, D, H, T)$ is identified with the cohomology group $H^1(\mathrm{Gal}(\tau), H^0(T, \mathfrak{S}_{H_T/T}))$ by Lemma 4.1.6. We see further that if a U-covering $\tau\colon T \to S$ satisfies the condition (5.1) for $Y \to S$, then $Y \times_S T \to T$ is bimeromorphically equivalent to a smooth morphism or a toric model by Theorems 4.3.1 and 4.3.2. For a cohomology class in $H^1(\mathrm{Gal}(\tau), H^0(T, \mathfrak{S}_{H_T/T}))$, we have a meromorphic action of the Galois group $\mathrm{Gal}(\tau)$ on the smooth or the toric model. We shall describe all such actions. In this section, we treat the case $l = 1$. We can further construct minimal models of the quotient varieties. For $l \geq 2$, we treat the case H has only finite monodromies in §6 and the rest case in §7.

5.1. Finite monodromy case

Assume that $l = 1$. We denote $a = a_1$ and $m = m_1$ for a U-covering $\tau\colon T \to S$. Thus τ is defined by $(\theta_1, t') \mapsto (\theta_1^m, t')$. Suppose that the order of the monodromy matrix $\rho(\gamma_1)$ is finite. Then $a = 0$ and $j_{T*}H_T$ is a constant sheaf for the immersion $j_T\colon T^\star \hookrightarrow T$. We denote the constant sheaf by the same symbol H_T. The exact sequence

$$0 \to H_T \to \mathcal{L}_{H_T} \to \mathfrak{S}_{H_T} \to 0$$

defined over $T^\star$ extends to:

$$0 \to \mathbb{Z}_T^{\oplus 2} \to \mathcal{O}_T \to \mathfrak{S}_{H_T/T} \to 0.$$

Every sheaves appearing in both sequences are canonically G-linearized, where $G = \mathrm{Gal}(\tau) \simeq \mathbb{Z}/m\mathbb{Z}$. Therefore the right action of G on $\mathbb{Z}^{\oplus 2} = H^0(T, \mathbb{Z}_T^{\oplus 2})$ is induced from the right multiplication of $\rho(\gamma_1)$, and that on $H^0(T, \mathcal{O}_T)$ is described by:

$$f(\theta) \mapsto (c_{\gamma_1} h(\theta) + d_{\gamma_1}) f(\gamma_1 \theta),$$

where $\omega(z) = h(\theta)$ is the period function. The right action on $H^0(T, \mathfrak{S}_{H_T/T})$ is induced from two actions above. We have an exact sequence:

$$\begin{aligned} H^1(G, H^0(T, \mathcal{O}_T)) \to H^1(G, H^0(T, \mathfrak{S}_{H_T/T})) \to \\ \to H^2(G, \mathbb{Z}^{\oplus 2}) \to H^2(G, H^0(T, \mathcal{O}_T)). \end{aligned}$$

Here we note that $H^p(G, H^0(T, \mathcal{O}_T)) = 0$ for $p > 0$, since $H^0(T, \mathcal{O}_T)$ is a $\mathbb{C}$-vector space. Thus

$$H^1(G, H^0(T, \mathfrak{S}_{H_T/T})) \simeq H^2(G, \mathbb{Z}^{\oplus 2}).$$

Let π_1' be the fundamental group of $T^\star$. Then $\pi_1/\pi_1' \simeq G$. We have the following Hochschild–Serre spectral sequence:

$$E_2^{p,q} = H^p(G, H^q(\pi_1', \mathbb{Z}^{\oplus 2})) \Longrightarrow E^{p+q} = H^{p+q}(\pi_1, \mathbb{Z}^{\oplus 2}).$$

Suppose that H is of type I_0. Then $E^1 = \mathbb{Z}^{\oplus 2} \to E_2^{0,1} = \mathbb{Z}^{\oplus 2}$ is the multiplication map by m. Since $H^2(\pi_1, \mathbb{Z}^{\oplus 2}) = 0$, we have

$$H^2(G, \mathbb{Z}^{\oplus 2}) \simeq (\mathbb{Z}/m\mathbb{Z})^{\oplus 2}.$$

Suppose that H is not of type I_0. Then $E_2^{0,1} = H^0(G, \mathbb{Z}^{\oplus 2}) = 0$. Since $H^2(\pi_1, \mathbb{Z}^{\oplus 2}) = 0$, we have

$$H^1(G, H^0(T, \mathfrak{S}_{H_T/T})) \simeq H^2(G, \mathbb{Z}) = E_2^{2,0} = 0.$$

This means that every elliptic fibration $Y \to T$ appearing in $\mathcal{E}^+(S, D, H)$ is a basic fibration in this case. Therefore we have:

Theorem 5.1.1. *For $S = \Delta^d$, $D = \{t_1 = 0\}$ and for a variation of Hodge structures H on $S^\star = S \setminus D$ such that the order of the monodromy matrix is finite, we have the following identification*:

$$\mathcal{E}^+(S, D, H) = \begin{cases} (\mathbb{Q}/\mathbb{Z})^{\oplus 2}, & H \textit{ is of type } \mathrm{I}_0, \\ 0, & \textit{otherwise.} \end{cases}$$

5.2. Infinite monodromy case

Assume that the order of the monodromy matrix $\rho(\gamma_1)$ is infinite, i.e., $\rho(\gamma_1)$ is of type I_a or I_a^* for a positive integer a. The period function $\omega(z)$ is written by $\omega(z) = az_1 + h(t)$, where $h(t)$ is a holomorphic function on S. Let $\tau\colon T \to S$ be a U-covering determined by $\theta = (\theta_1, t') \mapsto (\theta_1^m, t')$. Then from the exact sequences (4.2), (4.3), we have exact sequences of right G-modules:

$$(5.2) \qquad 0 \to \mathbb{Z} \to H^0(T, \mathcal{O}_T(*D_T)^\star) \to H^0(T, \mathfrak{S}_{H_T/T}) \to 0,$$

$$(5.3) \qquad 0 \to H^0(T, \mathcal{O}_T^\star) \to H^0(T, \mathcal{O}_T(*D_T)^\star) \to H^0(D_T, \mathbb{Z}) \simeq \mathbb{Z} \to 0,$$

where $D_T = \tau^{-1}(D) = \{\theta_1 = 0\}$ and $1 \in \mathbb{Z}$ is mapped to the function $e(h(t))t_1^a$. Suppose first that $\rho(\gamma_1)$ is of type I_a. Then its action on $\mathbb{Z}$ is trivial and that on $H^0(T, \mathcal{O}_T(*D_T)^\star)$ is written as:

$$H^0(T, \mathcal{O}_T(*D_T)^\star) \ni v(\theta) = v(\theta_1, t') \mapsto v\left(e(1/m)\theta_1, t'\right).$$

Thus the action on the group $H^0(T, \mathcal{O}_T(*D_T)^\star) \simeq H^0(T, \mathcal{O}_T^\star) \oplus \mathbb{Z}$ is expressed by:

$$(u(\theta), n) = (u(\theta_1, t'), n) \mapsto (u(e(1/m)\theta_1, t')e(n/m), n).$$

Let $[u(\theta), n]$ be the image of $(u(\theta), n)$ under the homomorphism $H^0(T, \mathcal{O}_T(*D_T)^\star) \to H^0(T, \mathfrak{S}_{H_T/T})$. Since $G \simeq \mathbb{Z}/m\mathbb{Z}$, the cohomology group $H^1(G, H^0(T, \mathfrak{S}_{H_T/T}))$ is isomorphic to Z^1/B^1, where

$$Z^1 = \left\{ \xi \in H^0(T, \mathfrak{S}_{H_T/T}) \,\middle|\, \sum_{i=0}^{m-1} \xi^{\gamma_1^i} = 0 \right\},$$
$$B^1 = \left\{ \xi = \eta - \eta^{\gamma_1} \,\middle|\, \eta \in H^0(T, \mathfrak{S}_{H_T/T}) \right\}.$$

For an element $\xi = [u(\theta), n]$, this is contained in Z^1 if and only if there is an integer k such that

$$\left(e\Big(\frac{n(m-1)}{2}\Big) \prod_{i=0}^{m-1} u\left(e(i/m)\theta_1, t'\right), nm \right) = (e(kh(t)), kma).$$

Therefore $n = ka$ and

$$\prod_{i=0}^{m-1} u\left(e(i/m)\theta_1, t'\right) = e\Big(kh(t) - \frac{n(m-1)}{2}\Big).$$

Hence there exist a nowhere vanishing function $u_1(\theta)$ and a positive integer n_1 such that

$$u(\theta) = e\Big(\frac{n_1 + kh(t)}{m} - \frac{ka(m-1)}{2m}\Big) u_1(\theta) u_1\left(e(1/m)\theta_1, t'\right)^{-1}.$$

If $\xi = [u(\theta), n]$ is contained in B^1, then $n = k'ma$ and

$$u(\theta) = u_2(\theta) u_2\left(e(1/m)\theta_1, t'\right)^{-1} e\left(-n'/m + k'h(t)\right),$$

for integers k', n' and a nowhere vanishing function $u_2(\theta)$. Thus $k = mk'$ and $n_1 + n' \equiv k'am(m-1)/2 \mod m$. Hence $H^1(G, H^0(T, \mathfrak{S}_{H_T/T})) \simeq \mathbb{Z}/m\mathbb{Z}$ and its generator is written by

$$\xi = \xi_m = \left[e\Big(\frac{h(t)}{m} - \frac{a(m-1)}{2m}\Big), a \right].$$

Let $T' \simeq \Delta^d \to T$ be a finite ramified covering branched only over D_T defined by $\theta' = (\theta'_1, t') \mapsto (\theta_1'^{m'}, t')$. Then $\mathrm{Gal}(T'/T) \simeq \mathbb{Z}/m'\mathbb{Z}$ and $\mathrm{Gal}(T'/S) \simeq \mathbb{Z}/mm'\mathbb{Z}$. The image of ξ_m under the homomorphism

$$H^1(\mathrm{Gal}(T/S), H^0(T, \mathfrak{S}_{H_T/T})) \to H^1(\mathrm{Gal}(T'/S), H^0(T', \mathfrak{S}_{H_{T'}/T'}))$$

is $m'\xi_{mm'}$. Therefore

$$\varinjlim_{T/S} H^1(\mathrm{Gal}(T/S), H^0(T, \mathfrak{S}_{H_T/T})) \simeq \mathbb{Q}/\mathbb{Z}.$$

Next suppose that $\rho(\gamma_1)$ is of type I_a^*. At the exact sequence (5.2), its action on $\mathbb{Z}$ is the multiplication of -1 and that on $H^0(T, \mathcal{O}_T(*D_T)^\star)$ is written by:

$$H^0(T, \mathcal{O}_T(*D_T))^\star \ni v(\theta_1, t') \mapsto v\left(e(1/m)\theta_1, t'\right)^{-1}.$$

Thus the action on the group $H^0(T, \mathcal{O}_T(*D_T)^\star) \simeq H^0(T, \mathcal{O}_T^\star) \oplus \mathbb{Z}$ is expressed by:

$$(u(\theta), n) = (u(\theta_1, t'), n) \mapsto \left(u\left(e(1/m)\theta_1, t'\right)^{-1} e(-n/m), -n\right).$$

Let $[u(\theta), n]$ be the image of $(u(\theta), n)$ under the homomorphism $H^0(T, \mathcal{O}_T(*D_T)^\star) \to H^0(T, \mathfrak{S}_{H_T/T})$. Since $G \simeq \mathbb{Z}/m\mathbb{Z}$, the cohomology group $H^1(G, H^0(T, \mathfrak{S}_{H_T/T}))$ is isomorphic to Z^1/B^1, where

$$Z^1 = \left\{ \xi \in H^0(T, \mathfrak{S}_{H_T/T}) \,\middle|\, \sum_{i=0}^{m-1} \xi^{\gamma_1^i} = 0 \right\},$$
$$B^1 = \left\{ \xi = \eta - \eta^{\gamma_1} \,\middle|\, \eta \in H^0(T, \mathfrak{S}_{H_T/T}) \right\}.$$

For an element $\xi = [u(\theta), n]$, this is contained in Z^1 if and only if

$$\left((-1)^n \prod_{i=0}^{m-1} u\left(e(i/m)\theta_1, t'\right)^{(-1)^i}, 0 \right) = (1, 0).$$

By taking $\theta_1 = 0$, we see that n is even. Since $H^1(G, H^0(T, \mathcal{O}_T)) = 0$ and $H^2(G, \mathbb{Z}) = 0$, we have $H^1(G, H^0(T, \mathcal{O}_T^\star)) = 0$. Therefore there exist a nowhere vanishing function $u_1(\tau)$ such that

$$u(\theta) = u_1(\theta) u_1\left(e(1/m)\theta_1, t'\right).$$

Let $v(\theta) := e(-n/(4m))u_1(\theta)$. Then $\xi = [u(\theta), n] = \eta - \eta^{\gamma_1}$ for $\eta = [v(\theta), n/2]$. Hence ξ is contained in B^1. Therefore $H^1(G, H^0(T, \mathfrak{S}_{H_T/T})) = 0$. Thus we have:

Theorem 5.2.1. *For $S = \Delta^d$, $D = \{t_1 = 0\}$ and for a variation of Hodge structures H on $S^\star = S \setminus D$ such that the order of the monodromy matrix is infinite, we have the following identification*:

$$\mathcal{E}^+(S, D, H) = \begin{cases} \mathbb{Q}/\mathbb{Z}, & \textit{in the case } \mathrm{I}_a; \\ 0, & \textit{in the case } \mathrm{I}_a^*. \end{cases}$$

5.3. Minimal models

Suppose that $\rho(\gamma_1)$ is of type I_0. Let $(p/m, q/m)$ be a pair of rational numbers, where p, q, m are positive integers and $\gcd(m, p, q) = 1$. Giving such a pair modulo $\mathbb{Z}^{\oplus 2}$ is equivalent to giving an element of $(\mathbb{Q}/\mathbb{Z})^{\oplus 2}$ whose order is m. Let $\tau\colon T \to S$ be the cyclic covering defined as before with mapping degree m. The smooth basic fibration $B(H_T) \to T$ is described as the quotient space of $T \times \mathbb{C}$ by the following action of $(n_1, n_2) \in \mathbb{Z}^{\oplus 2}$:

$$T \times \mathbb{C} \ni (\theta, \zeta) \mapsto (\theta, \zeta + n_1 h(t) + n_2),$$

where $h(t) = \omega(z)$ is the period function. From $(p/m, q/m)$, we have the following action of the generator 1 of $G \simeq \mathbb{Z}/m\mathbb{Z}$ on $B(H_T)$:

$$B(H_T) \ni [(\theta_1, t'), \zeta] \mapsto \left[(e(1/m)\theta_1, t'), \zeta + \frac{p}{m}h(t) + \frac{q}{m}\right].$$

An elliptic fibration $f\colon Y \to S$ smooth over $S^\star$ having H as its variation of Hodge structures is bimeromorphically equivalent to the quotient space by the action above for some $(p/m, q/m)$. Note that the action is free. Therefore the quotient space X is nonsingular. Let D_X be the support of the divisor π^*D, where $\pi\colon X \to S$ is the induced elliptic fibration. Then $\pi^*D = mD_X$ and thus the singular fibers of π are elliptic curves $\mathbb{C}/(\mathbb{Z}h(0,t') + \mathbb{Z} + \mathbb{Z}((p/m)h(0,t') + (q/m)))$ with multiplicity m. We have the canonical bundle formula:

$$\omega_X \simeq \pi^*(\omega_S) \otimes \mathcal{O}_X((m-1)D_X).$$

In particular, $\pi\colon X \to S$ is the unique minimal model in the bimeromorphic equivalence class over S.

Theorem 5.3.1. *Let $f\colon Y \to S = \Delta^d$ be a projective elliptic fibration smooth outside $D = \{t_1 = 0\}$. Suppose that the induced variation of Hodge structures is of type I_0. Then f has a unique minimal model $\pi\colon X \to S$ such that X is nonsingular, π is a flat morphism, and*

$$\omega_X \simeq \pi^*\omega_S \otimes \mathcal{O}_X((m-1)D_X)$$

for some positive integer m, where $D_X \to D$ is a smooth elliptic fibration and $\pi^*D = mD_X$.

In the case $d = 1$, the singular fiber is called of type ${}_m\mathrm{I}_0$ in Kodaira ([Kd1]) (cf. Figure 3).

Next suppose that $\rho(\gamma_1)$ is of type I_a for $a > 0$. Let p and m be coprime positive integers, $T \to S$ the finite cyclic covering

$$T \ni \theta = (\theta_1, t') \mapsto (\theta_1^m, t') \in S$$

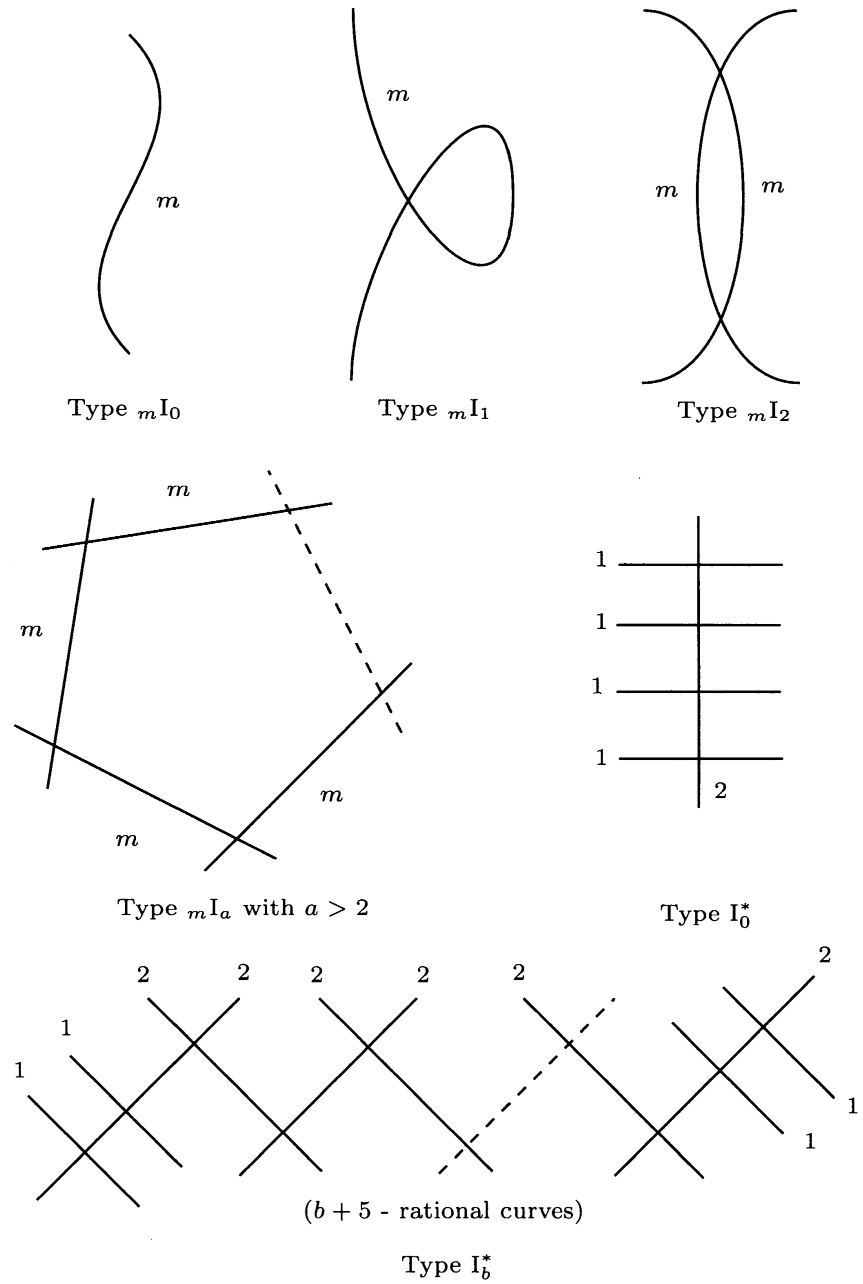

Figure 3. Singular fibers of Types $_m\mathrm{I}_a$ and I_b^*.

defined as before, and let $X \to T$ be the toric model associated with the period function $\omega(z) = az_1 + h(t)$. The X is the quotient space of $\mathcal{X} = \bigcup_{k\in\mathbb{Z}} \mathcal{X}^{(k)}$ by the action: $\mathcal{X} \ni s \mapsto s\theta_1^{ma} e(h(t))$, where s is a coordinate of $\mathbb{C}^\star$ under the isomorphism $T^\star \times_T \mathcal{X} \simeq T^\star \times \mathbb{C}^\star$. We now

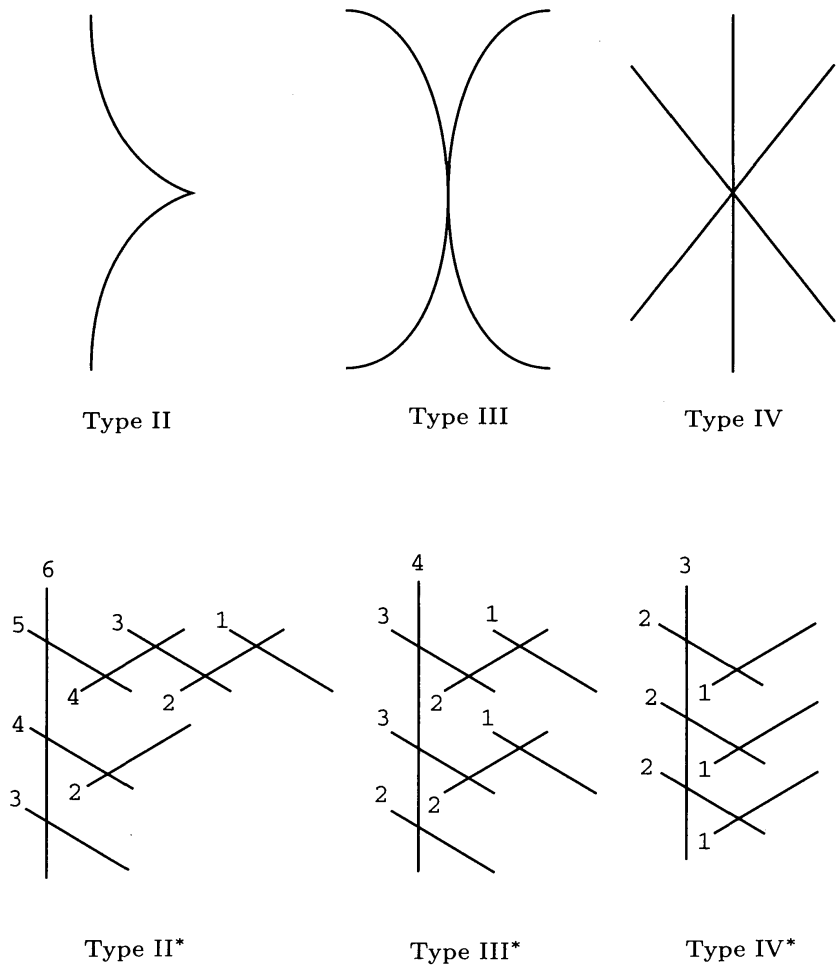

Figure 4. Singular fibers of Types II, III, IV, II*, III* and IV*.

have the following action of $\rho(\gamma_1)$ on X corresponding to $p/m \in \mathbb{Q}/\mathbb{Z}$:

$$X \ni [\theta, s] = [(\theta_1, t'), s] \\ \longmapsto \left[(e(1/m)\theta_1, t'), s \cdot e\left(\frac{p}{m} \left(h(t) - \frac{(m-1)a}{2} \right) \right) \theta_1^{pa} \right],$$

where $[\theta, s]$ denotes the image of the point $(\theta, s) \in T^\star \times \mathbb{C}^\star$. Note that the action is holomorphic and free on the whole space X. Let $\pi \colon Z \to S$ be the elliptic fibration obtained as the quotient by the action. Let D_Z be the support of the divisor $\pi^* D$. Then each fiber of $D_Z \to D$ is a cycle of rational curves, the number of whose irreducible components is a, and $\pi^*D = mD_Z$. The canonical bundle ω_Z is isomorphic to $\pi^*\omega_S \otimes \mathcal{O}((m-1)D_Z)$. In particular, $\pi\colon Z \to S$ is the unique minimal model of $f: Y \to S$. Thus we have:

Theorem 5.3.2. *Let $f\colon Y \to S = \Delta^d$ be a projective elliptic fibration which is smooth outside $D = \{t_1 = 0\}$. Suppose that the induced variation of Hodge structures is of type I_a for $a > 0$. Then f has a unique minimal model $\pi\colon Z \to S$ such that Z is nonsingular, π is a flat morphism, and*

$$\omega_Z \simeq \pi^*\omega_S \otimes \mathcal{O}_Z((m-1)D_Z)$$

for some positive integer m, where each fiber of $D_Z \to D$ is a cycle of rational curves, the number of whose irreducible components is a, and $\pi^*D = mD_Z$.

In the case $d = 1$, the singular fiber is called of type ${}_m\mathrm{I}_a$ in Kodaira ([Kd1]) (cf. Figure 3).

Finally, we shall consider the rest cases, i.e., $\rho(\gamma_1)$ is one of types I_0^*, II, II*, III, III*, IV, IV* and I_a^* for $a > 0$. We have only to give a minimal model for a basic elliptic fibration $B(H) \to S$. There are two methods. First one is starting from the description of the action of G on the smooth basic elliptic fibration or the toric model over T. After resolving the singularities of quotient spaces, we take successive contractions of the exceptional curves of the first kind. This is done in [Kd1] in the case $d = 1$. The same argument works even in the case $d > 1$, since the singularities are of similar types. In the second method, we use Weierstrass models. We may assume that $B(H) \simeq W_S(\mathcal{O}_S, \alpha, \beta)$ for some functions α, β such that $4\alpha^3 + 27\beta^2$ vanishes only over $D = \{t_1 = 0\}$. Since $\rho(\gamma_1)$ is one of such types, we see that $\alpha = \beta = 0$ on D. The singular locus of the Weierstrass model $\{Y^2Z = X^3 + \alpha(t)XZ^2 + \beta(t)Z^3\}$ is the locus $\{Y = t_1 = 0\}$. This singularity is locally isomorphic to $F \times \mathbb{C}^{d-1}$, where F is a surface singularity and is a germ of a rational double point. Therefore by taking standard resolution of singularities $Z \to W_S(\mathcal{O}_S, \alpha, \beta)$, we have a minimal elliptic fibration $\pi\colon Z \to S$. Therefore we have:

Theorem 5.3.3. *Let $f\colon Y \to S = \Delta^d$ be a projective elliptic fibration which is smooth outside $D = \{t_1 = 0\}$. Suppose that the induced variation of Hodge structures is not of type I_a $(a \geq 0)$. Then f admits a meromorphic section and has a unique minimal model $\pi\colon Z \to S$ such that Z is nonsingular, π is a flat morphism, and $\omega_Z \simeq \pi^*\omega_S$, where each fiber of $\pi^*D \to D$ is isomorphic to the singular fiber of the same type obtained in Kodaira* ([Kd1]) (cf. Figure 3 and Figure 4).

Since the minimal models are unique, we have the following result from Theorems 5.3.1, 5.3.2 and 5.3.3:

Corollary 5.3.4. *Let $f: Y \to S$ be an elliptic fibration over a complex manifold S. Assume the following conditions are satisfied:*

(1) *f is smooth outside a nonsingular divisor $D \subset S$;*
(2) *For any point $P \in S$, there is an open neighborhood $\mathcal{U}$ such that $Y_{|\mathcal{U}} \to \mathcal{U}$ is bimeromorphically equivalent to a projective morphism.*

Then there is a minimal elliptic fibration $\pi: X \to S$ for f such that X is nonsingular and π is flat.

Let $f: Y \to S$ be an elliptic fibration smooth outside a normal crossing divisor D. Let $C \subset S$ be a general smooth curve intersecting an irreducible component D_i transversely at one general point P. Over an open neighborhood $\mathcal{U}$ of P, we have the unique minimal elliptic fibration $Z_{\mathcal{U}} \to \mathcal{U}$ from $Y_{|\mathcal{U}} \to \mathcal{U}$, by above theorems. Then the singular fiber over P of the minimal elliptic surface obtained from the fiber product $Y \times_{\Delta^d} C \to C$ is isomorphic to that of $Z_{\mathcal{U}} \to \mathcal{U}$.

Definition 5.3.5. The *singular fiber type* of f over the divisor D_i is defined to be the type of the fiber over $P = C \cap D_i$ of the minimal elliptic surface obtained from the fiber product $Y \times_{\Delta^d} C \to C$ for a general curve C.

§6. Finite monodromy case

6.1. Cohomology groups

Let $S = \Delta^d$, $D = \{t_1 t_2 \cdots t_l = 0\} = \sum_{i=1}^{l} D_i$, $S^\star = S \setminus D$ be the same objects as in §2 and let H be a polarized variation of Hodge structures of rank two and weight one defined on $S^\star$. In this section, we treat the case the monodromy group $\mathrm{Im}(\pi_1 \to \mathrm{SL}(2, \mathbb{Z}))$ is a finite group. Then by §5, the singular fiber type over the coordinate hyperplane D_i is one of ${}_m\mathrm{I}_0$, I_0^*, II, II*, III, III*, IV and IV*.

Let $\tau: T \to S$ be a U-covering. Then the pullback $\tau^{-1}H$ extends trivially to a constant system $\mathbb{Z}_T^{\oplus 2}$ together with Hodge filtrations. The period function $\omega(z)$ of H is written by $\omega(z) = h(\theta)$ for a holomorphic function $h(\theta)$ on T. Let H_T be the variation of Hodge structures on T. We have an exact sequences:

$$0 \to H_T \simeq \mathbb{Z}_T^{\oplus 2} \to \mathcal{L}_{H_T} \simeq \mathcal{O}_T \to \mathfrak{S}_{H_T} \to 0.$$

Since there is a factorization $\pi_1 \to G = \mathrm{Gal}(\tau) \to \mathrm{SL}(2, \mathbb{Z})$ of the monodromy representation, the sheaves $\mathbb{Z}_T^{\oplus 2}$, $\mathcal{O}_T$, and $\mathfrak{S}_{H_T}$ are G-linearized.

By taking global sections, we have:

$$0 \to \mathbb{Z}^{\oplus 2} \to H^0(T, \mathcal{O}_T) \to H^0(T, \mathfrak{S}_{H_T}) \to 0,$$

where the right G-module structures of $\mathbb{Z}^{\oplus 2}$ and $H^0(T, \mathcal{O}_T)$ are given by:

$$\mathbb{Z}^{\oplus 2} \ni (m, n) \mapsto (m, n)\rho(\gamma) \text{ and } H^0(T, \mathcal{O}_T) \ni f(\theta) \mapsto (c_\gamma h(\theta) + d_\gamma) f(\gamma\theta),$$

respectively, for $\gamma \in \pi_1$. Since $H^0(T, \mathcal{O}_T)$ is a $\mathbb{C}$-vector space, we have $H^i(G, H^0(T, \mathcal{O}_T)) = 0$ for $i > 0$. Therefore

$$H^1(G, H^0(T, \mathfrak{S}_{H_T})) \simeq H^2(G, \mathbb{Z}^{\oplus 2}).$$

First of all, let us consider the case H is of type I_0. Then $\mathbb{Z}^{\oplus 2}$ is a trivial π_1-module. We have the following commutative diagram of exact sequences:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z}^{\oplus 2} & \longrightarrow & \mathbb{Q}^{\oplus 2} & \longrightarrow & (\mathbb{Q}/\mathbb{Z})^{\oplus 2} & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbb{Z}^{\oplus 2} & \longrightarrow & H^0(T, \mathcal{O}_T) & \longrightarrow & H^0(T, \mathfrak{S}_{H_T}) & \longrightarrow & 0.
\end{array}$$

Here the homomorphism $\mathbb{Q}^{\oplus 2} \to H^0(T, \mathcal{O}_T)$ is given by $(q_1, q_2) \mapsto q_1 h(\theta) + q_2$. Therefore we have also an isomorphism:

$$H^1(G, (\mathbb{Q}/\mathbb{Z})^{\oplus 2}) \simeq H^1(G, H^0(T, \mathfrak{S}_{H_T})).$$

Hence the set $\mathcal{E}^+(S, D, H)$ is identified with

$$\varinjlim_{\pi_1 \twoheadrightarrow G} H^1(G, (\mathbb{Q}/\mathbb{Z})^{\oplus 2}),$$

where the limit is taken over all the finite quotient groups G of π_1. By considering the isomorphism $H^1(G, (\mathbb{Q}/\mathbb{Z})^{\oplus 2}) \simeq \mathrm{Hom}(G, (\mathbb{Q}/\mathbb{Z})^{\oplus 2})$, we have

$$\mathcal{E}^+(S, D, H) = \mathrm{Hom}(\pi_1, (\mathbb{Q}/\mathbb{Z})^{\oplus 2}) = (\mathbb{Q}/\mathbb{Z})^{\oplus 2l}.$$

Next let us consider the case H is not of type I_0. For the U-covering $T \to S$, let π_1' be the fundamental group of $T^\star$. Then $G = \pi_1/\pi_1'$ and we have the Hochschild–Serre spectral sequence:

$$E_2^{p,q} = H^p(G, H^q(\pi_1', \mathbb{Z}^{\oplus 2})) \Longrightarrow H^{p+q}(\pi_1, \mathbb{Z}^{\oplus 2}).$$

Since $\mathbb{Z}^{\oplus 2}$ is a trivial π_1'-module, we have

$$H^q(\pi_1', \mathbb{Z}^{\oplus 2}) \simeq \begin{cases} \mathbb{Z}^{\oplus 2}, & q = 0; \\ \mathrm{Hom}(\bigwedge^q \pi_1', \mathbb{Z}^{\oplus 2}), & q > 0. \end{cases}$$

We know $H^0(\pi_1, \mathbb{Z}^{\oplus 2}) = 0$ by Theorem 2.2.1. Hence $E_2^{0,1} = E_2^{0,2} = 0$. Therefore we have an injection

$$E_2^{2,0} = H^2(G, \mathbb{Z}^{\oplus 2}) \hookrightarrow H^2(\pi_1, \mathbb{Z}^{\oplus 2}).$$

We shall show it is also surjective for some U-covering $T \to S$. The cohomology group $H^2(\pi_1, \mathbb{Z}^{\oplus 2})$ parameterizes all the smooth elliptic fibrations over $S^\star$ having the same variation of Hodge structures by §2.2. Since this is a finite group by Theorem 2.2.1, they are all projective morphisms. Therefore by Theorem 4.1.1, we can extend them to projective morphisms over S. Hence

$$H^2(\pi_1, \mathbb{Z}^{\oplus 2}) = \bigcup H^2(G, \mathbb{Z}^{\oplus 2}),$$

where G is taken to be the Galois group of a U-covering. Therefore

$$\mathcal{E}^+(S, D, H) = H^2(G, \mathbb{Z}^{\oplus 2}) = H^2(\pi_1, \mathbb{Z}^{\oplus 2})$$

for some U-covering $T \to S$. As a result, we have:

Theorem 6.1.1.

$$\mathcal{E}^+(S, D, H) = \begin{cases} (\mathbb{Q}/\mathbb{Z})^{\oplus 2l}, & H \text{ is of type } \mathrm{I}_0; \\ (\mathbb{Z}/2\mathbb{Z})^{\oplus 2(l-1)}, & H \text{ is of type } \mathrm{I}_0^{(*)}; \\ 0, & H \text{ is of types } \mathrm{II}^{(*)} \text{ or } \mathrm{IV}_-^{(*)}; \\ (\mathbb{Z}/2\mathbb{Z})^{\oplus (l-1)}, & H \text{ is of type } \mathrm{III}^{(*)}; \\ (\mathbb{Z}/3\mathbb{Z})^{\oplus (l-1)}, & H \text{ is of type } \mathrm{IV}_+^{(*)}. \end{cases}$$

6.2. Construction

Case I_0. A variation of Hodge structures H of this type is defined only by a single-valued holomorphic period function $\omega(t)\colon S \to \mathbb{H}$. The basic smooth elliptic fibration $B(H) \to S$ associated with H is constructed as the quotient of $S \times \mathbb{C}$ by the following action of $(m, n) \in \mathbb{Z}^{\oplus 2}$:

$$(t, \zeta) \mapsto (t, \zeta + m\omega(t) + n).$$

Let (p_i, q_i) be elements of $(\mathbb{Q}/\mathbb{Z})^{\oplus 2}$ for $1 \le i \le l$. Let m_i be the order of (p_i, q_i) in $(\mathbb{Q}/\mathbb{Z})^{\oplus 2}$ and let $\tau\colon T = \Delta^d \to S$ be the Kummer covering defined by:

$$T \ni \theta = (\theta_1, \theta_2, \ldots, \theta_l, t') \mapsto (\theta_1^{m_1}, \theta_2^{m_2}, \ldots, \theta_l^{m_l}, t') \in S.$$

The action of the Galois group $G = \mathrm{Gal}(\tau) \simeq \bigoplus_{i=1}^l \mathbb{Z}/m_i\mathbb{Z}$ on $T \times_S B(H)$ is as follows:

$$[\theta, \zeta] \mapsto [\gamma_i \theta, \zeta + p_i \omega(t) + q_i],$$

where $[\theta, \zeta]$ denotes the image of $(\theta, \zeta) \in T \times \mathbb{C}$ in $T \times_S B(H)$. Then the quotient $G \backslash (T \times_S B(H)) \to G \backslash T \simeq S$ is an elliptic fibration corresponding to the element $\{(p_i, q_i)\}_{i=1}^{l} \in \mathcal{E}^+(S, D, H)$.

Theorem 6.2.1. *If H is of type I_0, then there exists a minimal elliptic fibration for any element of $\mathcal{E}^+(S, D, H)$.*

Proof. We consider the fixed points for the action of $\gamma \in G$ on $T \times_S B(H)$. Then we see that if

$$\gamma = \gamma_1^{k_1} \gamma_2^{k_2} \cdots \gamma_l^{k_l}$$

has a fixed point, then any points over the locus $\{\theta = \gamma\theta\}$ are fixed and

$$\sum_{i=1}^{l} k_i(p_i, q_i) = 0 \quad \text{in} \quad (\mathbb{Q}/\mathbb{Z})^{\oplus 2}. \tag{6.1}$$

Let $G_0 \subset G$ be the subgroup consisting of any γ satisfying (6.1). Then $G_0 \backslash (T \times_S B(H)) \simeq (G_0 \backslash T) \times_S B(H)$. Note that the singularities of $G_0 \backslash T$ are described by means of a torus embedding theory. By [R], there is a toroidal partial resolution of singularities $V \to G_0 \backslash T$ such that V has only terminal singularities and the canonical divisor K_V is relatively nef over S. It is constructed by a decomposition of the cone associated with $G_0 \backslash T$. Since G/G_0 preserves the decomposition, we have an action of G/G_0 on V. Thus we have an action of the same group on $V \times_S B(H)$ which is bimeromorphically equivalent to that on $(G_0 \backslash T) \times_S B(H)$. We see the action on $V \times_S B(H)$ is free. Thus the quotient space $X := (G/G_0) \backslash (V \times_S B(H))$ has only terminal singularities and the canonical divisor K_X is relatively nef over S. Hence we obtain a minimal model. Q.E.D.

Example 6.2.2. Let $l = d = 2$ and take $(1/2, 1/2), (1/3, 1/3) \in (\mathbb{Q}/\mathbb{Z})^{\oplus 2}$. Then $m_1 = 2$ and $m_2 = 3$. For integers k_1, k_2, $k_1(1/2, 1/2) + k_2(1/3, 1/3) = 0$ if and only if $k_1 \equiv 0 \mod 2$ and $k_2 \equiv 0 \mod 3$. Thus the action of the Galois group G on $T \times_S B(H)$ is free. Hence the quotient space $X = G \backslash (T \times_S B(H))$ is nonsingular and the elliptic fibration $f : X \to S = \Delta^2$ is a flat morphism. The fiber over a point of $D_1 \setminus \{(0,0)\}$ (resp. $D_2 \setminus \{(0,0)\}$) is a multiple fiber with multiplicity 2 (resp. 3). The central fiber is also a non-reduced curve with multiplicity 6, whose support is a nonsingular elliptic curve. We have:

$$K_X \sim f^* K_S + D_1' + 2D_2',$$

where D_i' is the irreducible component of $f^*(D_i)$ for $i = 1, 2$.

Example 6.2.3. Let $l = d = 2$ and take $(1/2, 1/2), (1/4, 3/4) \in (\mathbb{Q}/\mathbb{Z})^{\oplus 2}$. Then $m_1 = 2$ and $m_2 = 4$. For integers k_1, k_2, $k_1(1/2, 1/2) + k_2(1/4, 3/4) = 0$ if and only if k_2 is even and $k_1 + k_2/2$ is also even. Thus G_0 is generated by $\gamma_1\gamma_2^2$, which is of order 2. The action of G_0 on $T = \Delta^2$ is written by:

$$(\theta_1, \theta_2) \mapsto (-\theta_1, -\theta_2).$$

Therefore $G_0\backslash T$ has an ordinary double point as the singularity. Let $V \to G_0\backslash T$ be the minimal desingularization. Then $G/G_0 \simeq \mathbb{Z}/4\mathbb{Z}$ acts on V and the quotient space $W := (\mathbb{Z}/4\mathbb{Z})\backslash V$ is obtained by the blowing up of $S = \Delta^2$ along the ideal (t_1^2, t_2). The W has one exceptional curve $C \simeq \mathbb{P}^1$ and proper transforms D_1', D_2' of coordinate lines $D_i = \{t_i = 0\}$. The intersection $D_1' \cap C$ is one point and it is an ordinary double point. The minimal elliptic fibration $X \to W$ is smooth outside $D_1' \sqcup D_2'$, and singular fiber type over D_1' is ${}_2\mathrm{I}_0$ and that over D_2' is ${}_4\mathrm{I}_0$.

Other Cases. Let $\tau\colon T = \Delta^d \to S$ be a U-covering. The natural extension of $\tau^{-1}H$ on $T^\star$ to T is denoted by H_T. Let $\omega(z) = h(\theta)\colon T \to \mathbb{H}$ be the period function and let $B(H_T) \to T$ be the associated smooth basic elliptic fibration. The $B(H_T)$ is isomorphic to the quotient space of $T \times \mathbb{C}$ by the following action of $(m, n) \in \mathbb{Z}^{\oplus 2}$:

$$(\theta, \zeta) \mapsto (\theta, \zeta + mh(\theta) + n).$$

Let us consider functions $F_i(z)$ listed in the Table 6. These are holomorphic function over T, since the period function $\omega(z) = h(\theta)$ is so. Similarly to Theorem 2.2.2, if we take the U-covering $\tau\colon T \to S$ in a suitable way, then we can define an action of the Galois group $G = \mathrm{Gal}(\tau)$ on $B(H_T)$ by:

$$[\theta, \zeta] \mapsto \left[\gamma_i\theta, \frac{\zeta + F_i(z)}{c_{\gamma_i}h(\theta) + d_{\gamma_i}}\right].$$

The quotient by the action induces an elliptic fibration $X \to S$, which is of course the extension of the corresponding smooth elliptic fibration over $S^\star$. Since all the cohomology classes of $H^2(\pi_1, \mathbb{Z}^{\oplus 2})$ are represented by the functions $F_i(z)$, we have all the elliptic fibrations corresponding to elements of $\mathcal{E}^+(S, D, H)$. The possible singular fiber types over coordinate hyperplanes are listed in Table 9.

§7. Infinite monodromy case

Let H be a variation of Hodge structures of type $\mathrm{I}_{(+)}$ or $\mathrm{I}_{(+)}^{(*)}$ on $S^\star$. We use the same notation as in §2. The period function $\omega(z)$ is written by

Table 9. Possible singular fiber types.

Type of monodromy	Singular fiber types ($0 \le a, m \in \mathbb{Z}$)
I_0	${}_m\mathrm{I}_0$
I_0^*	I_0, ${}_2\mathrm{I}_0$, I_0^*
$\mathrm{II}^{(*)}$	I_0, I_0^*, II, II^*, IV, IV^*
$\mathrm{III}^{(*)}$	I_0, ${}_2\mathrm{I}_0$, I_0^*, III, III^*
$\mathrm{IV}_+^{(*)}$	I_0, ${}_3\mathrm{I}_0$, IV
$\mathrm{IV}_-^{(*)}$	I_0, I_0^*, IV, IV^*
$\mathrm{I}_{(+)}$	${}_m\mathrm{I}_a$
$\mathrm{I}_{(+)}^{(*)}(0)$	I_a, ${}_2\mathrm{I}_a$, I_a^*
$\mathrm{I}_{(+)}^{(*)}(1)$	I_a, ${}_2\mathrm{I}_a$, ${}_4\mathrm{I}_a$, I_a^*
$\mathrm{I}_{(+)}^{(*)}(2)$	I_a, ${}_2\mathrm{I}_a$, I_a^*

$$\omega(z) = \sum_{i=1}^{l} a_i z_i + h(t)$$

for a holomorphic function $h(t)$ on S, where $a_i \ge 0$ and one of a_i is positive. Let $\tau\colon T \simeq \Delta^d \to S$ be a U-covering determined by $\tau^* t_i = \theta_i^{m_i}$ for $1 \le i \le l$. Then the monodromy matrix around the coordinate hyperplane $D_{T,i} = \{\theta_i = 0\}$ is of type $\mathrm{I}_{m_i a_i}$. Let G be the Galois group $\mathrm{Gal}(\tau) \simeq \bigoplus_{i=1}^{l} \mathbb{Z}/m_i\mathbb{Z}$ and let π' be the kernel of $\pi_1 \twoheadrightarrow G$, which is the fundamental group of $T^\star = \tau^{-1}(S^\star)$. We shall calculate the cohomology group $H^1(G, H^0(T, \mathfrak{S}_{H_T/T}))$. Let D_T^+ be the divisor $\{\prod_{a_i>0} \theta_i = 0\}$. Then as in (4.2) and (4.3), we have the following two exact sequences:

$$0 \to \mathbb{Z}_T \to \mathcal{O}_T(*D_T^+)^\star \to \mathfrak{S}_{H_T/T} \to 0; \tag{7.1}$$

$$0 \to \mathcal{O}_T^\star \to \mathcal{O}_T(*D_T^+)^\star \to \bigoplus_{a_i>0} \mathbb{Z}_{D_{T,i}} \to 0. \tag{7.2}$$

Note that the $\bigoplus_{a_i>0} \mathbb{Z}_{D_{T,i}}$ is considered to be a submodule of $R^1 j_{T*}\mathbb{Z}_{T^\star} \simeq \bigoplus_{i=1}^{l} \mathbb{Z}_{D_{T,i}}$, where j_T is an immersion $T^\star \hookrightarrow T$. By taking global sections, we have the following two exact sequences of G-modules:

$$0 \to \mathbb{Z} \to H^0(T, \mathcal{O}_T(*D_T^+)^\star) \to H^0(T, \mathfrak{S}_{H_T/T}) \to 0; \tag{7.3}$$

(7.4)

$$0 \to H^0(T, \mathcal{O}_T^\star) \to H^0(T, \mathcal{O}_T(*D_T^+)^\star) \to H^0(T, \bigoplus_{a_i>0} \mathbb{Z}_{D_{T,i}}) =: L_T^+ \to 0.$$

Here L_T^+ is considered to be a submodule of $H^0(T, R^1 j_{T*}\mathbb{Z}_{T^\star}) \simeq \mathrm{Hom}(\pi', \mathbb{Z})$. Since we fix generators γ_i of π_1, we have a natural isomorphism $\mathrm{Hom}(\pi_1, \mathbb{Z}) \simeq \mathbb{Z}^{\oplus l}$. By using this isomorphism, we identify L_T^+ with $\bigoplus_{a_i>0}(1/m_i)\mathbb{Z}$, i.e., we shall write an element of L_T^+ by $(q_1, q_2, \dots, q_l) \in \mathbb{Q}^{\oplus l}$, where $q_j = 0$ for $a_j = 0$ and $m_j q_j \in \mathbb{Z}$ for all j. By the sequence (7.4), we see that $H^0(T, \mathcal{O}_T(*D_T^+)^\star)$ is isomorphic to the direct sum $H^0(T, \mathcal{O}_T^\star) \oplus L_T^+$ as an abelian group. The isomorphism is described by:

$$(u(\theta), (q_i)) \mapsto u(\theta) \prod_{a_i>0} \theta_i^{m_i q_i}.$$

Therefore the induced action of $\gamma_j \in \pi_1$ (more precisely, the image of γ_j in G) on the direct sum is written by:

$$(u(\theta), (q_i)) \mapsto \left((u(\gamma_j\theta)e(q_j))^{(-1)^{c_j}}, (-1)^{c_j}(q_i) \right).$$

By (7.1) and (7.2), we have a homomorphism $\mathcal{O}_T^\star \to \mathfrak{S}_{H_T/T}$, from which the following exact sequence of G-modules is derived:

$$(7.5) \qquad 0 \to H^0(T, \mathcal{O}_T^\star) \to H^0(T, \mathfrak{S}_{H_T/T}) \to L_T^+/\mathbb{Z}\boldsymbol{a} \to 0,$$

where $\boldsymbol{a} := (a_1, a_2, \dots, a_l) \in \mathbb{Z}^{\oplus l} = \mathrm{Hom}(\pi_1, \mathbb{Z})$.

7.1. Case $\mathrm{I}_{(+)}$

Suppose that H is of type $\mathrm{I}_{(+)}$. Then every $c_i = 0$. Thus $\mathbb{Z}$ in the sequence (7.3) and L_T^+ are trivial G-modules. For $(u(\theta), \boldsymbol{q}) \in H^0(T, \mathcal{O}_T^\star) \oplus L_T^+$, let $[u(\theta), \boldsymbol{q}]$ be the image under the homomorphism

$$H^0(T, \mathcal{O}_T(*D_T^+)^\star) \to H^0(T, \mathfrak{S}_{H_T/T}),$$

where $\boldsymbol{q} = (q_1, q_2, \dots, q_l) \in L_T^+$. Since G is isomorphic to $\bigoplus_{i=1}^l \mathbb{Z}/m_i\mathbb{Z}$, the cohomology group $H^1(G, H^0(T, \mathfrak{S}_{H_T/T}))$ is isomorphic to Z^1/B^1, where Z^1 and B^1 is defined by:

$$Z^1 := \left\{ (\xi_i)_{i=1}^l \in H^0(T, \mathfrak{S}_{H_T/T})^{\oplus l} \,\middle|\, \sum_{r=0}^{m_i-1} \xi_i^{\gamma_i^r} = 0,\ \xi_i - \xi_i^{\gamma_j} = \xi_j - \xi_j^{\gamma_i} \right\};$$

$$B^1 := \left\{ (\eta - \eta^{\gamma_i})_{i=1}^l \,\middle|\, \eta \in H^0(T, \mathfrak{S}_{H_T/T}) \right\}.$$

Lemma 7.1.1. *For $\xi_i = [u_i(\theta), \boldsymbol{q}^i]$, the collection $(\xi_i)_{i=1}^l$ is contained in Z^1 if and only if there exist integers n_i, rational numbers λ_i for $1 \le i \le l$, and a nowhere vanishing function $v(\theta)$ defined on T satisfying the following conditions for any $1 \le i, j \le l$:*

(1) $m_i\lambda_i \in \mathbb{Z}$;
(2) $(n_i/m_i)a_j \equiv (n_j/m_j)a_i \mod \mathbb{Z}$;
(3) $\boldsymbol{q}^i = (n_i/m_i)\boldsymbol{a}$;
(4)

$$u_i(\theta) = e\left(\lambda_i - \frac{(m_i-1)n_i a_i}{2m_i} + \frac{h(t)n_i}{m_i}\right) v(\theta)v(\gamma_i\theta)^{-1}.$$

Proof. We see the condition is equivalent to the following condition by simple calculation: there exist integers n_i for $1 \leq i \leq l$ such that

(1)

$$\prod_{r=0}^{m_i-1} u_i(\gamma_i^r\theta) = e\left(-\frac{q_i^i m_i(m_i-1)}{2} + n_i h(t)\right),$$

(2) $m_i\boldsymbol{q}^i = n_i\boldsymbol{a}$,
(3) $u_i(\theta)u_i(\gamma_j\theta)^{-1}e(-q_j^i) = u_j(\theta)u_j(\gamma_i\theta)^{-1}e(-q_i^j)$,

for any i, j, where $\boldsymbol{q}^i = (q_1^i, q_2^i, \ldots, q_l^i) \in L_T^+$. By taking $\theta = 0$ in (3), we see $q_j^i - q_i^j \in \mathbb{Z}$. Further by (2), we have $q_j^i = (n_i/m_i)a_j$ for any i, j. Therefore $(n_i/m_i)a_j \equiv (n_j/m_j)a_i \mod \mathbb{Z}$ for any i, j. Let us define

$$v_i(\theta) := u_i(\theta)e\left(\frac{(m_i-1)n_i a_i}{2m_i} - \frac{h(t)n_i}{m_i}\right).$$

Then we have $\prod_{r=0}^{m_i-1} v_i(\gamma_i^r\theta) = 1$ and $v_i(\theta)v_i(\gamma_j\theta)^{-1} = v_j(\theta)v_j(\gamma_i\theta)^{-1}$ for i, j. Thus $\{v_i(\theta)\}$ defines an element of $H^1(G, H^0(T, \mathcal{O}_T^\star))$. Thus we have rational numbers λ_i for $1 \leq i \leq l$ with $m_i\lambda_i \in \mathbb{Z}$ and a nowhere vanishing function $v(\theta)$ on T such that

$$v_i(\theta) = e(\lambda_i)v(\theta)v(\gamma_i\theta)^{-1}. \qquad \text{Q.E.D.}$$

By considering the condition that the collection (ξ_i) is contained in B^1, we have:

Corollary 7.1.2. *The cohomology group $H^1(G, H^0(T, \mathfrak{S}_{H_T/T}))$ is isomorphic to*

$$\bigoplus_{a_i=0} m_i^{-1}\mathbb{Z}/\mathbb{Z}$$

$$\oplus \left\{(n_i/m_i) \in \bigoplus_{i=1}^{l} m_i^{-1}\mathbb{Z}/\mathbb{Z} \,\middle|\, (n_i/m_i)a_j \equiv (n_j/m_j)a_i \mod \mathbb{Z}\right\}.$$

Since our description of $L_T^+ \subset \mathrm{Hom}(\pi_1, \mathbb{Q})$ is compatible with any further U-coverings $T' \to T \to S$, we have:

Theorem 7.1.3. *Suppose that H is of type $\mathrm{I}_{(+)}$ and the monodromy matrix around the coordinate hyperplane D_i is of type I_{a_i}. Then the set $\mathcal{E}^+(S, D, H)$ is identified with the group*

$$\bigoplus_{a_i=0} \mathbb{Q}/\mathbb{Z} \oplus \left\{ (p_i) \in \bigoplus_{i=1}^{l} \mathbb{Q}/\mathbb{Z} \,\middle|\, p_i a_j \equiv p_j a_i \mod \mathbb{Z} \text{ for any } i, j \right\}.$$

Let k be the number of indices $1 \le i \le l$ with $a_i > 0$ and let $\alpha := \gcd a_i$. Then the group is isomorphic to:

$$(\mathbb{Q}/\mathbb{Z})^{\oplus(l-k+1)} \oplus (\mathbb{Z}/\alpha\mathbb{Z})^{\oplus(l-1)}.$$

Next, we shall construct the elliptic fibration associated with an element of $\mathcal{E}^+(S, D, H)$. By Theorem 7.1.3, every element of $\mathcal{E}^+(S, D, H)$ is determined by l-pairs of rational numbers (p_i, q_i) for $1 \le i \le l$ such that $q_i = 0$ for $a_i > 0$ and that $p_i a_j \equiv p_j a_i \mod \mathbb{Z}$ for any $1 \le i, j \le l$. Let m_i be the order of (p_i, q_i) in $(\mathbb{Q}/\mathbb{Z})^{\oplus 2}$ and let $\tau \colon T = \Delta^d \to S$ be the U-covering with $\tau^* t_i = \theta_i^{m_i}$ for $1 \le i \le l$. Let $X_\sigma \to T$ be the toric model associated with the variation of Hodge structures $\tau^{-1}H$ on $T^\star$ and with a suitable sign function σ. For the universal covering space $\mathcal{X}_\sigma$ of X_σ, $\mathcal{X}^\circ$ and $\mathcal{X}^\star$ are the open subsets $(\mathcal{X}_\sigma)_{|S^\circ}$ and $(\mathcal{X}_\sigma)_{|S^\star}$, respectively. We know that $\mathcal{X}^\star \simeq T^\star \times \mathbb{C}^\star$. For $1 \le i \le l$, we have the following isomorphism of $\mathcal{X}^\star$:

$$(\theta, s) \mapsto \left(\gamma_i \theta, s \cdot e\left(q_i - \frac{p_i a_i (m_i - 1)}{2} + p_i h(t) \right) \left(\prod_{j=1}^{l} \theta_j^{m_j a_j} \right)^{p_i} \right).$$

This extends to an isomorphism of $\mathcal{X}^\circ$. Since it is compatible with the action of

$$\vartheta \colon (\theta, s) \mapsto \left(\theta, s \cdot e(h(t)) \prod_{i=1}^{l} \theta_i^{m_i a_i} \right),$$

we have a meromorphic action of $\mathrm{Gal}(\tau)$ on X_σ. If we choose a sign function σ with respect to $(m_1 a_1/n, m_2 a_2/n, \ldots, m_l a_l/n)$ for $n = \gcd(m_1 a_1, m_2 a_2, \ldots, m_l a_l)$, then the action is holomorphic. By taking the quotient, we have an expected elliptic fibration.

7.2. Case $\mathrm{I}_{(+)}^{(*)}$

Next suppose that H is of type $\mathrm{I}_{(+)}^{(*)}$. Then γ_i acts on $\mathbb{Z}$ in the sequence (7.3) and on L_T^+ as the multiplication of $(-1)^{c_i}$. By the isomorphism as an abelian group

$$H^0(T, \mathcal{O}_T(*D_T^+)^\star) \simeq H^0(T, \mathcal{O}_T^\star) \oplus L_T^+,$$

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathbb{Z} & \longrightarrow & \mathbb{Z}^{\oplus 2} & \longrightarrow & \mathbb{Z} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & H^0(T, \mathcal{O}_T) & \longrightarrow & H^0(T, \mathcal{O}_T) \oplus L_T^+ & \longrightarrow & L_T^+ & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & H^0(T, \mathcal{O}_T^\star) & \longrightarrow & H^0(T, \mathfrak{S}_{H_T/T}) & \longrightarrow & L_T^+/\mathbb{Z}\boldsymbol{a} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

Figure 5. (cf. Lemma 7.2.1.)

we have a G-module structure on the direct sum. We can define a compatible π_1-module structure on $H^0(T, \mathcal{O}_T) \oplus L_T^+$, where the γ_j acts as

$$(f(\theta), (q_i)) \mapsto (-1)^{c_j}(f(\gamma_j \theta) + q_j, (q_i)).$$

Then we have:

Lemma 7.2.1. *We have an exact sequence of π_1-modules:*

$$0 \to \mathbb{Z}^{\oplus 2} \to H^0(T, \mathcal{O}_T) \oplus L_T^+ \to H^0(T, \mathfrak{S}_{H_T/T}) \to 0, \tag{7.6}$$

where $\mathbb{Z}^{\oplus 2}$ is the π_1-module associated with the monodromy representation $\pi_1 \to \mathrm{SL}(2, \mathbb{Z})$ and the first homomorphism is given by:

$$\mathbb{Z}^{\oplus 2} \ni (m, n) \mapsto (mh(t) + n, m\boldsymbol{a}) \in H^0(T, \mathcal{O}_T) \oplus L_T^+.$$

Further there is the commutative diagram Figure 5, *where the left vertical sequence is induced from the exponential sequence of T, and the right vertical sequence is induced from $1 \mapsto \boldsymbol{a}$.*

Thus we have a long exact sequence:

$$
\begin{aligned}
0 \to H^0(\pi_1', \mathbb{Z}^{\oplus 2}) &\to H^0(\pi_1', H^0(T, \mathcal{O}_T) \oplus L_T^+) \to H^0(T, \mathfrak{S}_{H_T/T}) \to \\
\to H^1(\pi_1', \mathbb{Z}^{\oplus 2}) &\to H^1(\pi_1', H^0(T, \mathcal{O}_T) \oplus L_T^+) \to \cdots
\end{aligned}
$$

$$
\begin{array}{ccccc}
H^0(G, L_T^+/\mathbb{Z}\boldsymbol{a}) & \longrightarrow & H^1(G, H^0(T, \mathcal{O}_T^\star)) & \longrightarrow & \\
\downarrow & & \downarrow & & \\
H^0(G, H^1(\pi_1', \mathbb{Z}^{\oplus 2})) & \longrightarrow & H^2(G, H^0(\pi_1', \mathbb{Z}^{\oplus 2})) & \longrightarrow &
\end{array}
$$

$$
\begin{array}{cccccc}
\longrightarrow & H^1(G, H^0(T, \mathfrak{S}_{H_T/T})) & \longrightarrow & H^1(G, L_T^+/\mathbb{Z}\boldsymbol{a}) & \longrightarrow \\
 & \downarrow & & \downarrow & \\
\longrightarrow & F^1(H^2(\pi_1, \mathbb{Z}^{\oplus 2})) & \longrightarrow & H^1(G, H^1(\pi_1', \mathbb{Z}^{\oplus 2})) & \longrightarrow
\end{array}
$$

$$
\begin{array}{cc}
\longrightarrow & H^2(G, H^0(T, \mathcal{O}_T^\star)) \\
 & \downarrow \\
\longrightarrow & H^3(G, H^0(\pi_1', \mathbb{Z}^{\oplus 2})).
\end{array}
$$

Figure 6. (cf. Lemma 7.2.3.)

Lemma 7.2.2.

(1) $H^0(\pi_1', H^0(T, \mathcal{O}_T) \oplus L_T^+) \simeq H^0(T, \mathcal{O}_T)$.

(2) *The image of* $H^0(T, \mathfrak{S}_{H_T/T}) \to H^1(\pi_1', \mathbb{Z}^{\oplus 2})$ *is isomorphic to* $L_T^+/\mathbb{Z}\boldsymbol{a}$.

Proof. Let $(f(\theta), (q_i))$ be a π_1'-invariant element of $H^0(T, \mathcal{O}_T) \oplus L_T^+$. Then $m_i q_i = 0$ for any i. Hence $q_i = 0$. Conversely, $(f(\theta), 0)$ is π_1'-invariant. Hence (1) is derived. Furthermore, we see that the injection

$$H^0(\pi_1', \mathbb{Z}^{\oplus 2}) \to H^0(\pi_1', H^0(T, \mathcal{O}_T) \oplus L_T^+)$$

is isomorphic to $\mathbb{Z} \to H^0(T, \mathcal{O}_T)$, which sends 1 to 1. Thus we have (2), by Lemma 7.2.1. Q.E.D.

By considering Hochschild–Serre's spectral sequence, we have:

Lemma 7.2.3. *The commutative diagram* Figure 6 *of exact sequences exists, in which the top sequence is a part of a long exact sequence induced from* (7.5) *and the bottom sequence is a part of the edge sequence of Hochschild–Serre's spectral sequence for* $\mathbb{Z}^{\oplus 2}$. *The* $F^i(H^2(\pi_1, \mathbb{Z}^{\oplus 2}))$ *is the filtration induced from the spectral sequence.*

Corollary 7.2.4. *The homomorphism*

$$H^1(G, H^0(T, \mathfrak{S}_{H_T/T})) \to H^2(\pi_1, \mathbb{Z}^{\oplus 2})$$

is an injection.

Proof. Let us consider the commutative diagram in Figure 6. The cokernel of $L_T^+/\mathbb{Z}\boldsymbol{a} \to H^1(\pi_1', \mathbb{Z}^{\oplus 2})$ is a torsion free group, where γ_i acts as the multiplication of $(-1)^{c_i}$. Thus the G-invariant part of the cokernel is zero. Therefore the first vertical homomorphism is an isomorphism and the fourth one is injective. The second homomorphism is also an isomorphism, since $H^i(G, H^0(T, \mathcal{O}_T)) = 0$ for $i > 0$. Therefore the homomorphism in question is injective. Q.E.D.

Theorem 7.2.5. *Suppose that H is of type $\mathrm{I}_{(+)}^{(*)}$. Then the set $\mathcal{E}^+(S, D, H)$ is identified with the group $H^2(\pi_1, \mathbb{Z}^{\oplus 2})$.*

Proof. We know $\mathcal{E}^+(S, D, H) = \varinjlim_{T \to S} H^1(G, H^0(T, \mathfrak{S}_{H_T/T}))$. Thus we have an injection $\mathcal{E}^+(S, D, H) \hookrightarrow H^2(\pi_1, \mathbb{Z}^{\oplus 2})$ by Corollary 7.2.4. On the other hand, $\mathcal{E}^+(S^\star, \emptyset, H)$ is identified with $H^2(\pi_1, \mathbb{Z}^{\oplus 2})$ by Theorem 2.2.2. Since this is a torsion group, every smooth elliptic fibration over $S^\star$ having H as a variation of Hodge structures extends to a projective elliptic fibration over S by Theorem 4.1.1. Thus the mapping $\mathcal{E}^+(S, D, H) \to H^2(\pi_1, \mathbb{Z}^{\oplus 2})$ is bijective. Q.E.D.

Next we shall construct the elliptic fibration associated with an element of $\mathcal{E}^+(S, D, H)$. Let H be a variation of Hodge structures of type $\mathrm{I}_{(+)}^{(*)}$. Let $\tau \colon T = \Delta^d \to S$ be a U-covering with $\tau^* t_i = \theta_i^{m_i}$ for sufficiently large m_i, e.g., $m_i = 4$. Let $X_\sigma \to T$ be a toric model associated with the same variation of Hodge structures as $\tau^{-1}H$. As in the previous case, let us consider $\mathcal{X}^\circ$ and $\mathcal{X}^\star$. Then $\mathcal{X}^\star \simeq T^\star \times \mathbb{C}^\star$. According to the types $\mathrm{I}_{(+)}^{(*)}(0)$, $\mathrm{I}_{(+)}^{(*)}(1)$ and $\mathrm{I}_{(+)}^{(*)}(2)$, let $F_i(z)$ be the function listed in Table 6. Since $\omega(z) = \sum a_i z_i + h(t)$, $e(F_i(z))$ is the multiple of unit holomorphic functions on S and monomials of θ_i for $1 \le i \le l$. Hence by the mapping:

$$\mathcal{X}^\star \ni (\theta, s) \mapsto \left(\gamma_i \theta, (s \cdot e(F_i(z)))^{(-1)^{c_i}}\right),$$

we have a holomorphic action of the Galois group $G = \mathrm{Gal}(\tau)$ on the $\mathcal{X}^\circ$ and a meromorphic action on the toric model X_σ. Although it is not necessarily a holomorphic action, by taking its 'quotient', we have an expected elliptic fibration. The possible singular fiber types over coordinate hyperplanes are listed in Table 9.

§Appendix A. Standard elliptic fibrations over surfaces

We shall study elliptic fibrations over normal surfaces. If the base surface is nonsingular and the fibration is smooth outside a normal crossing divisor, then the local bimeromorphic structures are classified in §6 and §7. But here we do not use these results but the flip theorem [Mo] and the flop theorem [Kw4] (cf. [Kl2]) for threefolds. We shall prove the following:

Theorem A.1. *Let $\pi\colon X \to S$ be a locally projective elliptic fibration over a normal complex analytic surface S. Then there exist a* standard *elliptic fibration $f\colon Y \to T$ and a bimeromorphic morphism $\mu\colon T \to S$ such that π and $\mu \circ f$ are bimeromorphically equivalent and K_Y is $\mu \circ f$-semi-ample.*

A *standard* elliptic fibration is defined as follows:

Definition A.2. Let $f\colon Y \to T$ be an elliptic fibration over a normal surface T. If the following conditions are satisfied, then f is said to be a *standard* elliptic fibration:

(1) Y has only terminal singularities;
(2) Y has only $\mathbb{Q}$*-factorial singularities*, i.e., for each point $y \in Y$ and for any Weil divisor D defined on a neighborhood of y, mD is Cartier at y for a positive integer m;
(3) f is a locally projective morphism;
(4) f is an equi-dimensional morphism, i.e., every fiber of f is one-dimensional;
(5) There exists an effective $\mathbb{Q}$-divisor Δ on T such that (T, Δ) is log-terminal and $K_Y \sim_{\mathbb{Q}} f^*(K_T + \Delta)$.

Remark A.3.

(1) $\mu \circ f$ may not to be a locally projective morphism.
(2) If T is nonsingular, a standard elliptic fibration $f\colon Y \to T$ is a flat morphism.
(3) For the definition of log-terminal pair, see [KMM], or [Ny3].

Therefore, the classification of threefolds admitting elliptic fibrations is reduced to that of standard elliptic fibrations.

For the proof, we recall the following semi-ampleness theorem. This was originally proved by [Kw3, 6.1] in the case S is a point. It is generalized to the algebraic case in [Ny1, 5] (cf. [KMM, 6-1-1]) and to the case X is a variety in class $\mathcal{C}$ and S is a point in [Ny3, 5.5]. Further [Ny3, 5.8] treats in a case of degenerations. But these proofs are essentially same and depend on the torsion free Theorem 3.2.2. Thus we have:

Theorem A.4 (Semi-ampleness theorem). *Let $\pi\colon X \to S$ be a projective surjective morphism from a normal complex variety X onto a complex variety S, Δ an effective $\mathbb{Q}$-divisor of X and H a $\mathbb{Q}$-divisor of X. Then H is π-semi-ample if the following conditions are satisfied:*

(1) *(X, Δ) is log-terminal;*
(2) *H and $H-(K_X+\Delta)$ are π-nef;*
(3) *$\nu(H-(K_X+\Delta)_{|X_s}) = \kappa(H-(K_X+\Delta)_{|X_s})$ for a general fiber X_s;*
(4) *$\kappa(aH-(K_X+\Delta)_{|X_s}) \geq 0$ and $\nu(aH-(K_X+\Delta)_{|X_s}) = \nu(H-(K_X+\Delta)_{|X_s})$ for some $a > 1$ on a general fiber X_s.*

Here $\nu(D)$ denotes the numerical D-dimension (cf. [KMM, 6-1-1]).

Proposition A.5. *Let $\pi\colon X \to S$ be a locally projective elliptic fibration over a surface S. Then there exist a locally projective elliptic fibration $g\colon Z \to R$, a bimeromorphic morphism $\nu\colon R \to S$, and an effective $\mathbb{Q}$-divisor Λ on R satisfying the following conditions:*

(1) *π and $\nu \circ g$ are bimeromorphically equivalent;*
(2) *Z has only terminal singularities;*
(3) *Z is $\mathbb{Q}$-factorial over any point of S;*
(4) *(R, Λ) is log-terminal;*
(5) *$K_Z \sim_{\mathbb{Q}} g^*(K_R + \Lambda)$;*
(6) *$K_R + \Lambda$ is ν-ample.*

Proof. Since π is locally projective, for each point $s \in S$ there is an open neighborhood $\mathcal{U}_s$ such that $\pi^{-1}(\mathcal{U}_s) \to \mathcal{U}_s$ is a projective morphism. Thus by applying minimal model theorem [Mo], [Ny3, §4] to $(\mathcal{U}_s, s)$, we have an elliptic fibration $h_s\colon \mathcal{Z}_s \to \mathcal{U}'_s$ such that

- $\mathcal{U}'_s \subset \mathcal{U}_s$ is also an open neighborhood of s,
- $\mathcal{Z}_s$ has only terminal singularities,
- $\mathcal{Z}_s$ is $\mathbb{Q}$-factorial over s,
- h_s is a projective morphism, bimeromorphic to π over $\mathcal{U}'_s$,
- $K_{\mathcal{Z}_s}$ is h_s-nef.

The $\mathcal{Z}_s$ is not uniquely determined in general, but by [Kw4], it is determined up to a sequence of flops. Thus except a discrete set of points of S, $\mathcal{Z}_s$ is uniquely determined. Therefore we can patch these $\mathcal{Z}_s$ and get a locally projective elliptic fibration $h\colon Z \to S$ such that

- Z has only terminal singularities,
- Z is $\mathbb{Q}$-factorial over any point of S,
- h is a locally projective morphism, bimeromorphic to π,
- K_Z is h-nef.

By Theorem A.4, we see that K_Z is h-semi-ample. Therefore there exist a bimeromorphic morphism $\nu\colon R \to S$, a $\mathbb{Q}$-Cartier divisor L on R, and an elliptic fibration $g\colon Z \to R$ such that $h = \nu \circ g$, $K_Z \sim_{\mathbb{Q}} g^*L$, and L is ν-ample. By [Ny4, 0.4], we have an effective $\mathbb{Q}$-divisor Λ on R such that (R, Λ) is log-terminal and $K_Z \sim_{\mathbb{Q}} g^*(K_R + \Lambda)$. Q.E.D.

Proposition A.6. *Let $\pi\colon X \to S$, $g\colon Z \to R$, and $\nu\colon R \to S$ be as in Proposition A.5. Then there exist an equi-dimensional elliptic fibration $g'\colon Z' \to T$ and a bimeromorphic morphism $\delta\colon T \to R$ satisfying the following conditions:*

(1) *$\delta \circ g'$ and g are bimeromorphically equivalent;*
(2) *$\mu := \nu \circ \delta$ and $\mu \circ g'$ are locally projective morphisms;*
(3) *Z' has only terminal singularities and is $\mathbb{Q}$-factorial over any point of S;*
(4) *$K_{Z'}$ is $\mathbb{Q}$-linearly equivalent to the pullback of $K_R + \Lambda$.*

Proof. We may assume that g is not equi-dimensional. In general, g is equi-dimensional over a neighborhood of $\nu^{-1}(s)$ for $s \in S$ except a discrete set of points. Thus we can consider locally on S. Let us take such exceptional point $P \in S$ and look at the vector spaces $N^1(Z/S; P)$, $N^1(Z/R; \nu^{-1}(P))$, $N_1(Z/R; \nu^{-1}(P))$ (cf. [Ny3, §4]), etc.

Step 1. By the assumption, there is a prime divisor E on Z such that $g(E)$ is a point and $\nu \circ g(E) = P$. Then we can take an effective divisor D on Z such that $D + kE$ is the pullback of an effective Cartier divisor on R for some integer $k > 0$ and D does not contain E. We consider the minimal model program for the log-terminal pair $(Z, \varepsilon D)$ for $0 < \varepsilon \ll 1$ in $N_1(Z/R; \nu^{-1}(P))$. Note that K_Z is $\mathbb{Q}$-linearly equivalent to the pullback of a $\mathbb{Q}$-divisor of R. If $-E$ is not g-nef, then there exist an extremal ray and its contraction morphism over R. Since extremal curves are contained in D, E can not to be contracted. By continuing such contractions and flops over $(R, \nu^{-1}(P))$, we have an elliptic fibration $q_1\colon V_1 \to R$ such that

(1) q_1 is bimeromorphically equivalent to g,
(2) V_1 has only canonical singularities,
(3) $K_{V_1} \sim_{\mathbb{Q}} q_1^*(K_R + \Lambda)$,
(4) $-E'$ is q_1-nef,

where E' is the strict transform of E in V_1. By Theorem A.4, $-E'$ is q_1-semi-ample. Therefore there exist an elliptic fibration $V_1 \to R_1$ and a bimeromorphic morphism $\delta_1\colon R_1 \to R$ such that q_1 is the composition of these morphisms and $-E'$ is the pullback of a δ_1-ample $\mathbb{Q}$-divisor on R_1. Thus δ_1 is not an isomorphism. Here we note that δ_1 and q_1 are projective morphisms over a neighborhood of $\nu^{-1}(P)$. Hence $\rho(R_1/R; \nu^{-1}(P)) > 0$. Since V_1 has only canonical singularities, we can take a crepant morphism $Z_1 \to V_1$ such that Z_1 has only terminal singularities, is $\mathbb{Q}$-factorial over P, and is projective over a neighborhood of $\nu^{-1}(P)$. Hence Z_1 and Z are isomorphic to each other in codimension one and $\rho(Z/R;\nu^{-1}(P)) = \rho(Z_1/R;\nu^{-1}(P))$.

Step 2. Further assume that the induced morphism $f_1\colon Z_1 \to R_1$ is not equi-dimensional over $\delta_1^{-1}\nu^{-1}(P)$. Then by the same argument in *Step* 1, we have morphisms $f_2\colon Z_2 \to R_2$ and $\delta_2\colon R_2 \to R_1$ such that Z_2 has only terminal singularities, is $\mathbb{Q}$-factorial and projective over P, and $\rho(R_2/R;\nu^{-1}(P)) > \rho(R_1/R;\nu^{-1}(P))$. Therefore

$$\rho(Z/R;\nu^{-1}(P)) = \rho(Z_2/R;\nu^{-1}(P)) > \\ > \rho(R_2/R;\nu^{-1}(P)) > \rho(R_1/R;\nu^{-1}(P)).$$

If f_2 is not equi-dimensional, we can continue this process. After a finite number of steps, f_m should be equi-dimensional, since $\rho(R_i/R;\nu^{-1}(P))$ are bounded. Thus we obtain the desired $Z' := Z_m$ and $T := R_m$ over P. Q.E.D.

Remark A.7. For the equi-dimensional morphism $Z' \to T$, T is uniquely determined. Because if $Z'' \to T'$ satisfies the same conditions, then Z'' and Z are isomorphic in codimension one. Thus for every prime divisor Γ on T, its proper transform in T' must be a prime divisor. Thus $T' \simeq T$. Note that $Z' \to S$ is a locally projective morphism.

Definition A.8. The morphism $Z' \to T$ in Proposition A.6 is said to be an *equi-dimensional model* of $\pi\colon X \to S$.

Lemma A.9. *Let $f\colon Y \to T$ be a minimal elliptic fibration over a surface T such that Y is $\mathbb{Q}$-factorial over any point of T and f is equi-dimensional. Then Y has only $\mathbb{Q}$-factorial singularities.*

Proof. Let $\nu\colon Y' \to Y$ be a bimeromorphic morphism whose exceptional locus is a union of discrete curves. Then Y' has only terminal singularities and ν is crepant, i.e., $K_{Y'} \sim_{\mathbb{Q}} \nu^* K_Y$. Since $f\circ\nu\colon Y' \to T$ is also equi-dimensional, $f\circ\nu$ is a locally projective morphism by the same argument as in Claim 3.2.4. Since Y is $\mathbb{Q}$-factorial over any point of T, ν must be an isomorphism. Thus by the existence of $\mathbb{Q}$-factorialization [Kw4], we are done. Q.E.D.

Proof of Theorem A.1. Let $f\colon Y \to T$ be a minimal model of an equi-dimensional model $g'\colon Z' \to T$ such that Y is $\mathbb{Q}$-factorial over any point of T. Since Y and Z' are having only terminal singularities, they are isomorphic in codimension one. Thus by flops, we can take Y to be a partial resolution of Z'. Therefore f is also equi-dimensional. Thus by Lemma A.9, f is a standard elliptic fibration. Q.E.D.

§Appendix B. Minimal models for elliptic threefolds

Minimal model theory is not yet developed for compact Kähler manifolds. But we have the following theorem in [Ny6]:

Theorem B.1. *Let X be a compact Kähler threefold of algebraic dimension two. Then X is uniruled or there exists a good minimal model of X.*

Here we say that X is *uniruled* if there exists a dominant meromorphic mapping $Y \times \mathbb{P}^1 \cdots\to X$ such that $\dim Y = \dim X - 1$. A *good minimal model* of X is defined to be a complex normal variety V satisfying the following conditions:

(1) V is bimeromorphically equivalent to X;
(2) V has only terminal singularities;
(3) The canonical divisor K_V is semi-ample.

We shall generalize to the following:

Theorem B.2. *Let $\pi\colon X \to B$ be a proper surjective morphism from a complex Kähler threefold X onto a complex variety B. Suppose that there exists an elliptic fibration $f\colon X \to S$ and a proper surjective morphism $g\colon S \to B$ such that $\pi = g \circ f$. Then the general fiber of $\pi\colon X \to B$ is uniruled or X admits a relative good minimal model over B.*

Here, a *relative good minimal model* over B is defined to be a proper surjective morphism $V \to B$ such that V has only terminal singularities and the canonical divisor K_V is relatively semi-ample over B.

We note the following lemma which is derived from Theorem 3.2.2 and from the similar argument of [Ny3, 3.12]:

Lemma B.3. *Let $f\colon X \to Y$ and $g\colon Y \to Z$ be projective morphisms of complex varieties where X is nonsingular. Let D be a $\mathbb{Q}$-divisor on X whose fractional part $\langle D\rangle$ is supported in a normal crossing divisor. Assume that there exists a g-nef-big $\mathbb{Q}$-Cartier divisor L such that $D \sim_{\mathbb{Q}} f^*(L)$. Then*

$$R^p g_*(R^i f_* \mathcal{O}_X(K_X + \ulcorner D \urcorner)) = 0$$

for $i \geq 0$ and $p > 0$, where $\ulcorner D \urcorner$ denotes the round-up *of D.*

In the case of elliptic fibrations, we have the following generalization:

Proposition B.4. *Let $f\colon X \to Y$ be an elliptic fibration from a complex manifold X onto a complex variety Y, $g\colon Y \to Z$ a projective morphism onto a complex variety Z, and let D be a $\mathbb{Q}$-divisor on X*

whose fractional part $\langle D\rangle$ is supported in a normal crossing divisor. Assume that there exists a g-nef-big $\mathbb{Q}$-Cartier divisor L such that $D \sim_{\mathbb{Q}} f^*L$. Then $R^i f_*\mathcal{O}_X(K_X + \ulcorner D \urcorner)$ is torsion free and $R^p g_*(R^i f_*\mathcal{O}_X(K_X + \ulcorner D \urcorner)) = 0$ for $i \geq 0$ and $p > 0$.

Proof. Since the statement is local on Z, we may assume that Z is a Stein space. By Lemma B.3, we may assume further that f is not bimeromorphically equivalent to a projective morphism. As in [Ny3, 3.12], we may assume that there exist a bimeromorphic morphism $\nu\colon Y' \to Y$ such that

(1) Y' is nonsingular,
(2) $g \circ \nu\colon Y' \to Z$ is a projective morphism,
(3) there is an elliptic fibration $f'\colon X \to Y'$ with $\nu \circ f' = f$,
(4) f' is smooth outside a normal crossing divisor on Y',
(5) there is an effective $\mathbb{Q}$-divisor Δ on Y' with $\nu^*L - \delta\Delta$ being $g \circ \nu$-ample for $0 < \delta \ll 1$,
(6) $\operatorname{Supp}\langle D\rangle \cup \operatorname{Supp} f'^*(\Delta)$ is a normal crossing divisor.

Since $\ulcorner D - \delta f'^*(\Delta) \urcorner = \ulcorner D \urcorner$, by Leray's spectral sequence, we can reduce to the situation such that $Y = Y'$ and L is g-ample. Then by the proof of [Ny3, 3.9], we may assume further that there exists a commutative diagram:

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\tilde{f}} & \widetilde{Y} \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\lambda} \\ X & \xrightarrow{f} & Y, \end{array}$$

where

(1) $\widetilde{X}$ and $\widetilde{Y}$ are nonsingular,
(2) φ is generically finite, λ is projective, and $\tilde{f}$ is an elliptic fibration,
(3) $\operatorname{Supp}\varphi^*\langle D\rangle$ is a normal crossing divisor and $\varphi^*(D)$ is a Cartier divisor,
(4) $\mathcal{O}_X(K_X + \ulcorner D \urcorner)$ is a direct summand of $\varphi_*\mathcal{O}_{\widetilde{X}}(K_{\widetilde{X}} + \varphi^*(D))$.

Therefore by replacing X and Y by $\widetilde{X}$ and $\widetilde{Y}$, respectively, we can reduce to the case where the following conditions are satisfied:

(1) Y is nonsingular;
(2) $f\colon X \to Y$ is smooth outside a normal crossing divisor on Y;
(3) L is a g-ample Cartier divisor;
(4) D is a Cartier divisor with $D \sim f^*(L)$.

Then by Theorem 3.2.3, $\mathcal{F}^i := R^i f_*\mathcal{O}_X(K_X)$ are locally free sheaves. Thus $\mathcal{F}^i = 0$ for $i \geq 2$, $\mathcal{F}^1 \simeq \mathcal{O}_Y(K_Y)$ and $(\mathcal{F}^0)^{\otimes 12} \simeq \mathcal{O}_Y(12(K_Y + \Delta))$

for some effective $\mathbb{Q}$-divisor Δ whose support is a normal crossing divisor and whose *round-down* $\llcorner \Delta \lrcorner = 0$. Since

$$\mathcal{F}^0 - \Delta + L - K_Y$$

is g-ample, we are done by applying [Ny3, 3.5]. Q.E.D.

Corollary B.5. *Let $f: X \to Y$ be an elliptic fibration, $g: Y \to Z$ a projective morphism for normal complex varieties X, Y and Z. Then $R^2(g \circ f)_* \mathcal{O}_X = 0$ if the following conditions are satisfied:*

(1) *$(X, 0)$ is log-terminal;*
(2) *There is a $\mathbb{Q}$-divisor L on Y such that $-K_X \sim_{\mathbb{Q}} f^*(L)$;*
(3) *L is g-ample.*

Proof. Let $\mu: M \to X$ be a modification such that μ-exceptional locus is a normal crossing divisor $\bigcup E_i$. Then we have $K_M \sim_{\mathbb{Q}} \mu^*(K_X) + \sum_i a_i E_i$ for $a_i > -1$. Then for $D = \sum_i a_i E_i - K_M$, we have

$$R^p g_* (R^i (f \circ \mu)_* \mathcal{O}_M (K_M + \ulcorner D \urcorner)) = 0$$

for $p > 0$ by Proposition B.4. Since $R^i \mu_* \mathcal{O}_M (K_M + \ulcorner D \urcorner) = 0$ for $i > 0$ (cf. [Ny3, 3.6]) and $\mu_* \mathcal{O}_M (K_M + \ulcorner D \urcorner) \simeq \mathcal{O}_X$, we have $R^p g_* R^i f_* \mathcal{O}_X \simeq 0$ for $p > 0$. Since $R^i f_* \mathcal{O}_X = 0$ for $i > 1$ by Proposition B.4, $R^2(g \circ f)_* \mathcal{O}_X = 0$. Q.E.D.

Proposition B.6 (cf. [Ft1]). *Let T be a normal compact complex surface in class $\mathcal{C}$. Suppose that there is an effective $\mathbb{Q}$-divisor Δ on T such that (T, Δ) is log-terminal and $(K_T + \Delta) \cdot C \geq 0$ for any irreducible curve C on T. Then $K_T + \Delta$ is semi-ample.*

Proof. This is proved by [Ft1] in the case $a(T) = 2$. We thus assume that $a(T) < 2$. Therefore $p_g(T) > 0$ by Proposition 3.3.1. There exists an effective $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on T which is $\mathbb{Q}$-linearly equivalent to $K_T + \Delta$. Hence $(K_T + \Delta)^2 \geq 0$. Since $a(T) < 2$, we have $(K_T + \Delta)^2 = 0$.

Step 1. Reduction to the case T is nonsingular.
Let $\mu: M \to T$ be the minimal resolution of singularities of T. Then we have

$$K_M \sim_{\mathbb{Q}} \mu^*(K_T + \Delta) + \sum a_i E_i$$

for $a_i > -1$, where E_i is a μ-exceptional curve or the proper transform of a component of $\operatorname{Supp} \Delta$. Since μ is minimal, $a_i \leq 0$ for all i. Let $\Delta' := -\sum_i a_i E_i$. Then $K_M + \Delta' \sim_{\mathbb{Q}} \mu^*(K_T + \Delta)$. By definition, (M, Δ') is log-terminal. Thus we may assume that T is nonsingular.

Step 2. Reduction to the case T is relatively minimal.

Let $\nu\colon T \to T_1$ be the contraction of a (-1)-curve E on T and let $\Delta_1 := \nu_*(\Delta)$. Here (-1)-curve means an exceptional curve of the first kind. Then we have $K_T + \Delta \sim_{\mathbb{Q}} \rho^*(K_{T_1} + \Delta_1) - bE$ for some $b \geq 0$, since $(K_T + \Delta) \cdot E \geq 0$. Thus $(K_{T_1} + \Delta_1) \cdot \Gamma \geq 0$ for any irreducible curve Γ on T_1 and

$$0 = (K_{T_1} + \Delta_1)^2 \geq (K_T + \Delta)^2 = 0 .$$

Hence $b = 0$. Therefore by continuing the contractions of (-1)-curves, we may assume that T is a relatively minimal model.

Step 3. Case $a(T) = 0$.
Assume that $a(T) = 0$. Then by the classification of surfaces, T is a complex torus or a K3 surface. If T is a complex torus, then $\Delta = 0$, since T has no curves. Therefore $K_T + \Delta \sim_{\mathbb{Q}} 0$. Assume that T is a K3 surface. Then by the Riemann–Roch formula,

$$h^0(m\Delta) + h^0(-m\Delta) \geq 2,$$

for any m with $m\Delta$ is Cartier. Since $a(T) = 0$, we have also $\Delta = 0$. Thus $K_T + \Delta \sim_{\mathbb{Q}} 0$.

Step 4. Case $a(T) = 1$.
Assume that $a(T) = 1$. Then there exist a minimal elliptic fibration $f\colon T \to C$ over a smooth curve C. By the canonical bundle formula, we see that $K_T \sim_{\mathbb{Q}} f^*(K_C + B)$ for an effective $\mathbb{Q}$-divisor B on C with $\llcorner B \lrcorner = 0$. Since $a(T) = 1$, no curves Γ of T dominate C. Therefore every component of Δ is contained in fibers of f. Now $K_T + \Delta$ is f-nef. Thus Δ is also f-nef. Therefore there is another effective $\mathbb{Q}$-divisor B' on C such that $\Delta \sim_{\mathbb{Q}} f^*(B')$. Therefore $K_T + \Delta \sim_{\mathbb{Q}} f^*(K_C + B + B')$. Hence $K_T + \Delta$ is semi-ample. Q.E.D.

Lemma B.7. *Let $f\colon X \to M$ be a fibration between complex manifolds whose general fiber is $\mathbb{P}^1$. Suppose that there exist two prime divisors $D_1 \neq D_2$ on X and a Cartier divisor E on D_1 such that*

(1) *D_1 and D_2 dominate M bimeromorphically,*

(2) $\mathcal{O}_{D_1}(D_1) \simeq \mathcal{O}_{D_1}(E)$.

Then f is bimeromorphically equivalent to the first projection $M \times \mathbb{P}^1 \to M$.

Proof. By the generically surjective homomorphism $f^*f_*\mathcal{O}_X(D_1) \to \mathcal{O}_X(D_1)$, we may assume that X is isomorphic to $\mathbb{P}_M(\mathcal{E})$ for a locally free sheaf $\mathcal{E}$ of rank two and that D_1 and D_2 are sections of f. Then there exist two exact sequences:

$$0 \to \mathcal{O}_M \to \mathcal{E} \to \mathcal{L} \to 0, \tag{B.1}$$

$$\text{(B.2)} \qquad 0 \to \mathcal{M} \to \mathcal{E} \to \mathcal{L} \otimes \mathcal{M}^{-1} \to 0,$$

where $\mathcal{L}$ and $\mathcal{M}$ are invertible sheaves on M. Here we consider that the section D_1 corresponds to the exact sequence (B.1). Then $\mathcal{O}_{D_1}(D_2)$ is isomorphic to $\mathcal{L} \otimes \mathcal{M}^{-1}$. As in the elementary transformations, we blow-up along $D_1 \cap D_2$ and contract the proper transform of $f^{-1}(f(D_1 \cap D_2))$. Then we can make $D_1 \cap D_2 = \emptyset$. Therefore we may assume that $\mathcal{E} \simeq \mathcal{O} \oplus \mathcal{L}$. By the assumption, there is a Cartier divisor L on M such that $\mathcal{L} \simeq \mathcal{O}_M(L)$. Therefore X is bimeromorphic to $M \times \mathbb{P}^1$. Q.E.D.

Proof of Theorem B.2. By taking the Stein factorization, we may assume that $\pi\colon X \to B$ is a fibration.

Step 1. Case $\pi\colon X \to B$ is a locally projective morphism.
If $\dim B = 0$, then X is projective and theorem is true by [Mo], [KMM], and [Kw3]. If $\dim B \geq 1$ and if the general fiber of π is not uniruled, then for any point $b \in B$, we have an open neighborhood $\mathcal{U}_b \subset B$ and a relative good minimal model $Z_b \to \mathcal{U}_b$ which is bimeromorphically equivalent to $\pi^{-1}(\mathcal{U}_b) \to \mathcal{U}_b$ by [Ny3, §4], [Mo], and A.4. Further except a discrete set of points $\{b_i\}$, $Z_b \to \mathcal{U}_b$ is the unique minimal model of $\pi^{-1}(\mathcal{U}_b) \to \mathcal{U}_b$. Thus we can glue these $Z_b \to \mathcal{U}_b$ and obtain a relative good minimal model $Z \to B$ of π.

In what follows, we assume that π is not a locally projective morphism.

Step 2. Inductive step.
We may assume that X and S are nonsingular and there exists a normal crossing divisor $D = \bigcup D_i$ of S such that f is smooth outside D. Then by Theorem 3.3.3, f is a locally projective morphism. Thus by applying Proposition A.5, we have an elliptic fibration $h\colon Y \to T$ between normal varieties and an effective $\mathbb{Q}$-divisor Δ_T on T such that

(1) there is a bimeromorphic morphism $\mu\colon T \to S$,
(2) $h\colon Y \to T$ is bimeromorphically equivalent to $f\colon X \to S$,
(3) Y has only terminal singularities,
(4) Y is $\mathbb{Q}$-factorial over any points of S,
(5) (T, Δ_T) is log-terminal,
(6) $K_Y \sim_{\mathbb{Q}} h^*(K_T + \Delta_T)$.

Suppose that $(K_T + \Delta_T) \cdot C < 0$ for an irreducible curve C contained in a fiber of $T \to B$ satisfying $C^2 < 0$. Then we have a contraction $\delta\colon T \to T'$ of C. Since $(K_T + \Delta_T) \cdot C < 0$, $-(K_T + \Delta_T)$ is δ-ample. Thus for $\Delta_{T'} := \delta_* \Delta_T$, $(T', \Delta_{T'})$ is also log-terminal and

$$\delta_* \mathcal{O}_T(\llcorner m(K_T + \Delta_T) \lrcorner) \simeq \mathcal{O}_{T'}(\llcorner m(K_{T'} + \Delta_{T'}) \lrcorner)$$

for any $m \geq 0$. By applying Corollary B.5 to $Y \to T \to T'$, we have $R^2(\delta \circ h)_* \mathcal{O}_Y = 0$. Therefore by Proposition 3.3.1, $\delta \circ h$ is bimeromorphic to a locally projective morphism. Hence by Proposition A.5, there is a minimal model $h'\colon Y' \to T'$ such that h' is bimeromorphically equivalent to $\delta \circ h$ and $K_{Y'}$ is h'-semi-ample. Since

$$h'_*\mathcal{O}_{Y'}(mK_{Y'}) \simeq \mathcal{O}_{T'}(m(K_{T'} + \Delta_{T'}))$$

for infinitely many m, we see that $K_{Y'} \sim_{\mathbb{Q}} h'^*(K_{T'}+\Delta_{T'})$. By continuing this process and by Theorem A.1, we may assume that the following conditions are satisfied:

(1) $h\colon Y \to T$ is bimeromorphically equivalent to $f\colon X \to S$;
(2) h is a standard elliptic fibration;
(3) $K_Y \sim_{\mathbb{Q}} h^*(K_T + \Delta_T)$, where (T, Δ_T) is log-terminal;
(4) There is no irreducible curve C on T such that $C^2 < 0$, $(K_T + \Delta_T) \cdot C < 0$, and that C is contained in a fiber of $q\colon T \to B$.

Step 3. Case $1 \leq \dim B \leq 2$.
In this case, we have $R^2 g_*\mathcal{O}_S = 0$. Thus $g\colon S \to B$ is a locally projective morphism by Proposition 3.3.1. Therefore $q\colon T \to S$ is also a locally projective morphism. Suppose that the genus $p_g(F) = 0$ for the general fibers F of $\pi\colon X \to B$. Then $\dim B = 1$, otherwise, the general fibers of π are elliptic curves. Hence the general fibers are Kähler surfaces with $p_g = 0$, so we have $R^i\pi_*\mathcal{O}_X = 0$ for $i = 1$, 2 by [St]. Using Proposition 3.3.1, we see that π is a locally projective morphism. This is a contradiction. Therefore $\pi_*\omega_X \neq 0$. Hence for any point $P \in B$, $K_T + \Delta_T$ is $\mathbb{Q}$-linearly equivalent to an effective $\mathbb{Q}$-divisor over P. Thus $K_T+\Delta_T$ is q-nef. By Theorem A.4, $K_T+\Delta_T$ is q-semi-ample. Therefore $Y \to B$ is a good minimal model in this case.

Step 4. Case $\dim B = 0$ and T is a projective surface. (cf. [Ny6])
If $K_T + \Delta_T$ is nef on T, then $K_T + \Delta_T$ is semi-ample by [Ft1]. Thus Y is a good minimal model. Next assume that $K_T + \Delta_T$ is not nef. Then by *Step* 3 and the cone theorem for (T, Δ_T), there exists a contraction morphism $\sigma\colon T \to C$ such that $\dim C < 2$. Then by Corollary B.5, we see that $R^2(\sigma \circ h)_*\mathcal{O}_Y = 0$. Therefore $\sigma \circ h$ is bimeromorphically equivalent to a locally projective morphism by Proposition 3.3.1. Thus C is a smooth curve. Let F be a general fiber of $\sigma \circ h$. Then F is a ruled surface such that $-K_F$ is semi-ample and $K_F^2 = 0$. Suppose that the irregularity $q(F) = 0$. Then we have $R^1(\sigma \circ h)_*\mathcal{O}_Y = 0$ by [St]. Thus $H^2(Y, \mathcal{O}_Y) = 0$, so Y is Moishezon by Proposition 3.3.1. This is a contradiction. Hence $q(F) = 1$ and F is a minimal ruled surface over an elliptic curve E. Thus by applying the relative minimal model theory to $\sigma \circ h\colon Y \to C$, we have a meromorphic map $\zeta\colon Y \cdots\to N$ over C, where N is a normal nonprojective surface and $N \to C$ is an elliptic fibration. For the general fiber F, ζ induces the projection $F \to E$.

Let $\widetilde{Y} \to Y$ be a modification such that $\widetilde{Y} \to N$ is a morphism and let H_1 and H_2 be general ample divisors on T. Then $D_1' := h^*(H_1)$ and $D_2' := h^*(H_2)$ dominate N by ζ. Thus we can take a finite covering $N' \to N$, a modification Y' of the fiber product $\widetilde{Y} \times_N N'$ and prime divisors D_1 and D_2 such that $Y' \to N'$ and D_1, D_2 satisfy the condition of Lemma B.7. Therefore Y is dominated by $N' \times \mathbb{P}^1$.

Step 5. Case $\dim B = 0$ and T is not projective.
In this case $K_T + \Delta_T$ is semi-ample by Proposition B.6. Thus we are done. Q.E.D.

Corollary B.8. *Let X be a compact Kähler threefold admitting an elliptic fibration. Then X is uniruled or there is a good minimal model of X. In each case, there exist a normal compact complex surface T, an effective $\mathbb{Q}$-divisor Δ_T, and a standard elliptic fibration $h\colon Y \to T$ such that (T, Δ_T) is log-terminal, Y is bimeromorphically equivalent to X, $K_Y \sim_{\mathbb{Q}} h^*(K_T + \Delta_T)$. If X is uniruled, then T must be projective, so the algebraic dimension $a(X) \geq 2$. If X is not uniruled, then we can take T so that $K_T + \Delta_T$ is semi-ample. If $a(X) \leq 1$ and $\kappa(X) = 0$, then there is a finite covering $\widetilde{Y} \to Y$ such that*

(1) *the covering is étale outside the non-Gorenstein locus of Y,*
(2) *$\widetilde{Y}$ is a three-dimensional complex torus or the product of an elliptic curve and a $K3$ surface.*

Proof. We have only to prove the last statement. First assume that $a(X) \leq 1$, $\kappa(X) = 0$, and $p_g(X) = 1$. Then $K_Y \sim 0$. Since T is not ruled, we see that $\Delta_T = 0$, T has only rational double points as singularities, and $K_T \sim_{\mathbb{Q}} 0$. The inequality $a(T) \leq a(X) \leq 1$ implies that T is a two-dimensional complex torus or its minimal desingularization is a K3 surface. Therefore, the elliptic fibration $h\colon Y \to T$ is smooth outside the singular locus of T by Theorem 4.3.1. For a singular point $P \in T$, there exist an open neighborhood $\mathcal{U} \subset T$ and a finite Galois covering $\mathcal{V} \to \mathcal{U}$ from a nonsingular surface étale outside P such that the normalization $\mathcal{Y}$ of $Y \times_T \mathcal{V}$ induces a smooth elliptic fibration $\mathcal{Y} \to \mathcal{V}$. Here $\mathcal{Y} \to Y$ is an étale morphism since Y has only Gorenstein terminal singularities (cf. [Kw4, 5.1]). In particular, the fiber of $h\colon Y \to T$ over P is an elliptic curve. Now we have isomorphisms $R^1h_*\mathcal{O}_Y \simeq R^1h_*\omega_Y \simeq \omega_T \simeq \mathcal{O}_T$. Since X is compact and Kähler, the natural homomorphism $H^1(Y, \mathcal{O}_Y) \to H^0(T, R^1h_*\mathcal{O}_T)$ is surjective. We infer that $q(Y) = q(T) + 1$. Thus $q(Y) = 1$ or $q(Y) = 3$ according as T is bimeromorphic to a K3 surface or T is a complex torus. Let $Y \to A$ be the Albanese mapping, which is a fiber space by [Kw1]. If $q(Y) = 3$, then Y is isomorphic to a complex torus and $h\colon Y \to T$ is a fiber bundle. Suppose that $q(Y) = 1$. Then the induced morphism

$Y \to A \times T$ is surjective, since general fibers of h dominate the elliptic curve A. In particular, every smooth fibers of h are isomorphic to each other. Let $T' \subset Y$ be a general fiber of $Y \to A$. Then T' is nonsingular and dominates T. Hence T' is a complex torus of dimension two or a K3 surface. Further $T' \to T$ is a finite morphism étale outside the singular locus of T, since every fibers of $Y \to T$ are elliptic curves and since possible exceptional curves for $T' \to T$ should be rational. We infer that the normalization $\widetilde{Y}$ of the fiber product $Y \times_T T'$ is isomorphic to the product of T' and a fiber. Since $\widetilde{Y} \to Y$ is étale outside the singular locus of Y and since Y has only Gorenstein terminal singularities, Y is nonsingular and $\widetilde{Y} \to Y$ is an étale covering.

Next, we treat the general case $\kappa(X) = 0$ and $a(X) \leq 1$. Since $K_Y \sim_{\mathbb{Q}} 0$, there is a finite covering $Y' \to Y$ such that Y' has only Gorenstein terminal singularities, the covering is étale outside the non-Gorenstein locus of Y, and $K_{Y'} \sim 0$. Let $Y' \to T' \to T$ be the Stein factorization. Then $Y' \to T'$ is also an equi-dimensional elliptic fibration. Thus Y' admits a finite étale covering $\widetilde{Y} \to Y'$ from a complex torus or the product of an elliptic curve and a K3 surface. Q.E.D.

Finally, we note that the good minimal model conjecture for non-Kähler threefolds is not true in general. For example, we have the following:

Proposition B.9. *There exists a compact complex threefold X with $\kappa(X) = 2$ such that K_Y is not semi-ample for any normal variety Y with only terminal singularities bimeromorphically equivalent to X.*

Proof. Let T be a nonsingular minimal projective surface of general type and let $\mu\colon S \to T$ be the blowing-up at a point $P \in T$. Then by Example 3.3.5, we have an elliptic fibration $f\colon X \to S$ smooth outside $D := \mu^{-1}(P)$ such that $f^*(D) = mf^{-1}(D)$ for some positive integer m, where $f^{-1}(D)$ is isomorphic to a Hopf surface. Then by the canonical bundle formula, we see that

$$K_X \sim f^*(K_S) + (m-1)f^{-1}(D) \sim f^*\mu^*(K_T) + (2m-1)f^{-1}(D).$$

Therefore $H^0(X, nK_X) \simeq H^0(T, nK_T)$ for any $n \geq 0$. Suppose that there exists a normal complex threefold Y with only terminal singularities such that it is bimeromorphically equivalent to X and K_Y is semi-ample. Then we have a projective bimeromorphic morphism $\lambda\colon Z \to Y$ and a bimeromorphic morphism $\nu\colon Z \to X$ from a complex manifold Z. By construction, we see that $\lambda^*(K_Y) \sim_{\mathbb{Q}} \nu^* f^* \mu^*(K_T)$. Therefore the proper transform of the Hopf surface $f^{-1}(D)$ must be a λ-exceptional divisor of Z. Since λ is a projective morphism, this is a contradiction. Q.E.D.

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