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Global and Local Properties of Pencils of Algebraic Curves

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§0. Introduction

Let S be a non-singular projective surface over $\mathbb{C}$, and let $f : S \to B$ be a relatively minimal fibration of curves of genus g over a non-singular projective curve B of genus b. In this article, we discuss some recent developments in the area where its global and local properties interact each other, with comments on several interesting open questions.

From the global point of view, our motivation comes from the study of minimal surfaces of general type. In the birational sense, any algebraic surface has a fibration over a curve, because it has an algebraic function. For surfaces of small Kodaira dimension, we can choose among various pencils a "preassigned" pencil such as Mori fibrations or Iitaka fibrations. On the other hand, there seems to be no canonical way in finding a pencil which reflects well the structure of a surface of general type. However, we often see that a pencil structure appears naturally for them as well. In the series of papers [42], Horikawa showed that most surfaces which are geographically close to the Noether line, $K_S^2 = 2\chi(\mathcal{O}_S) - 6$, have a pencil of curves of genus 2 which is induced on S from a ruling of its canonical image via the canonical map. Similar phenomena can be observed for canonical surfaces close to the Castelnuovo line $K_S^2 = 3\chi(\mathcal{O}_S) - 10$ ([22], [10]). In this case, the quadric hull of the canonical image is a threefold of small degree and its ruling usually induces on S a pencil of non-hyperelliptic curves of genus 3 (see [52], [44]). One can find a lot of such observations in literatures (e.g., Xiao's works) indicating the importance of a systematic study of fibered surfaces in the study of surfaces of general type.

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In §1, we describe the zone of existence for fibrations of genus g with respect to two relative numerical invariants χ_f and $K^2_{S/B}$, where we put $\chi_f = \chi(\mathcal{O}_S) - (g-1)(b-1)$ and $K_{S/B} = K_S - f^*K_B$ for a fibration $f : S \to B$. Recall that we have the fundamental inequality in surface geography

$$K^2_{S/B} \geq \frac{4(g-1)}{g}\chi_f$$

called the *slope inequality*. It was first discovered by Horikawa and Persson for hyperelliptic pencils (Part V of [42], [81]) and proved by Xiao [96] in general. In §1.2, we give two different proofs due respectively to Xiao [96] and Moriwaki [73]. A general philosophy in geography of fibered surfaces is that, if we impose a certain condition on a general fiber, then the zone of existence of such pencils is restricted further, and, as a result, a sharper slope inequality holds. For example, consider the case $g = 3$. If a general fiber is non-hyperelliptic, then we have $K^2_{S/B} \geq 3\chi_f$ ([53]), while the bound of genus 3 implied by the slope inequality is $8/3$.

Now, as an ideal model, we recall here Horikawa's works [41] and [43]. Let $f : S \to B$ be a fibration of genus 2, and let $F_1, \cdots, F_l$ be singular fibers of f. Then:

(i) One can define a nonnegative integer $\mathrm{Ind}(F_i)$ for each fiber germ of F_i so that the equality

$$K^2_{S/B} = 2\chi_f + \sum_{i=1}^{l} \mathrm{Ind}(F_i)$$

holds. We call $\mathrm{Ind}(F_i)$ the Horikawa index of F_i.

(ii) One can classify the fiber germs with positive Horikawa index.

(iii) A germ F_i with $\mathrm{Ind}(F_i) = k > 0$ has a local splitting deformation to k fibers with $\mathrm{Ind} = 1$ modulo fibers with $\mathrm{Ind} = 0$.

How to generalize (i), (ii) and (iii) is the motivation of the following discussions of the slope equality, classification of degenerations and Morsification, respectively.

Our first aim is to generalize the equality in (i). Let $f : S \to B$ be a pencil with a certain condition on its general fiber. If there exists a rational number λ, a finite set of fibers $F_1, \cdots, F_l$ and well-defined non-negative rational numbers $\mathrm{Ind}(F_i)$ satisfying

$$K^2_{S/B} = \lambda\chi_f + \sum_{i=1}^{l} \mathrm{Ind}(F_i),$$

we call it the slope equality which, once obtained, has nice geographic applications. We describe two known examples, hyperelliptic fibrations and Clifford general fibrations of odd genus, in §2.2 and §2.3, respectively. Note that F_i may be a smooth fiber in general. Indeed, if f is a non-hyperelliptic pencil of genus 3, then the generic fiber with $\mathrm{Ind} = 1$ is a smooth hyperelliptic curve (e.g., [52]).

The essence of the slope equality is the local concentration of the global invariants of the surface on a finite number of fiber germs. In §2.1, we define the local signature $\sigma(F_i)$ for fiber germs by using the Horikawa index and the Euler contribution, which enables us to restate the slope equality as the "local concentration formula" of the global signature:

$$\mathrm{Sign}(S) = \sum_{i=1}^{l} \sigma(F_i).$$

This itself has a certain topological meaning. For example, consider hyperelliptic fibrations. The Horikawa index, and therefore the local signature, is explicitly calculated in terms of the datum of the singularities of the branch curve of the double covering as in §2.2. On the other hand, Endo [31] defined the topological local signature $\sigma_{top}(F_i)$ by using the Meyer cocycle [68]. Then Terasoma [89] showed that our $\sigma(F_i)$ coincides with $\sigma_{top}(F_i)$. Furuta [33] defined the topological local signature in more general situations. It is interesting to establish relations with the Horikawa index.

The second aim is to develop the classification theory of singular fiber germs. Kodaira [51] and Namikawa-Ueno [76], [77] studied genus one and two cases, respectively. We recall the classification of genus three case obtained in [9], and discuss the method in §3. Here our central tool is Matsumoto-Montesinos' theorem ([64], [65]) which characterizes the monodromy in the mapping class group Γ_g of genus g. To be more precise, let $f : S \to \Delta$ be a degeneration of curves of genus $g \geq 2$ over a 1-dimensional unit disk Δ with a unique singular fiber $F = f^{-1}(0)$. If we fix $t_0 \in \Delta^*$, then the monodromy action of $\pi_1(\Delta^*, t_0) \simeq \mathbb{Z}$ induces modulo isotopy an orientation-preserving homeomorphism of the fiber $\phi_f : F_0 \to F_0$, where $F_0 = f^{-1}(t_0)$. Since the change of the base point t_0 corresponds to the conjugation in Γ_g, f gives us a uniquely determined element $[\phi_f]$ in the set $\widehat{\Gamma}_g$ of all conjugacy classes of Γ_g. We call $[\phi_f]$ the *topological monodromy* of f. Matsumoto-Montesinos' theorem [64] states that an element of $\widehat{\Gamma}_g$ is realizable as the topological monodromy of a degeneration if and only if it is the class of a pseudo-periodic map of negative twist. Moreover this class is completely determined by certain invariants called valencies, screw numbers and the action on the

extended partition graph ([78], [79], [65]). Therefore, the classification of degenerations is reduced to determining these invariants. In §3, we describe the method to carry it out. By this method, for any genus in theory, we can classify singular fibers, topological monodromies and the topological structure of the stable curve corresponding to the moduli point of the given degeneration at the same time.

The last aim is to consider Morsification of degenerations [83]. Let $f : S \to \Delta$ be a degeneration with a unique singular fiber $F = f^{-1}(0)$. If their exists a relative deformation $\{f_u \colon S_u \to \Delta\}_{u \in \Delta'}$ with $f_0 = f$ such that f_u has $l \geq 2$ singular fibers $F_{u,1}, \cdots, F_{u,l}$, then we say that F splits into $F_{u,1}, \cdots, F_{u,l}$. Starting from a given germ $F_{u,i}$ $(1 \leq i \leq l)$, we seek for its splittings successively. Such a reduction will terminate after a finite number of steps. Then we say that F splits into atomic fibers via several splitting families. One of the central problems is Xiao Gang's Morsification conjecture [83] that any atomic fiber has a simple description (see §4.1 for the precise statement). There are two steps to be considered for that: The first is to construct splitting families of a given germ F. The second is to determine all the atomic fibers in a certain category.

In §4.2, we recall the construction of hyperelliptic splitting families via the splitting deformation of singularities of the branch curve of the double cover ([2], [3]). As a result, any hyperelliptic singular fiber is reduced to very simple classes of fiber germs via several splitting families. Moreover we can give the list of hyperelliptic atomic fibers of genus 3 [3]. Since hyperelliptic splitting families satisfy the conservation law of the Horikawa index, it is also considered as the algebraic Morsification defined in §4.1.

In §4.3, we give and discuss seven open questions concerning the Morsification of degenerations.

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§1. Geography of fibered surfaces

1.1. Geography

Though our main interest is in surfaces fibered over a curve, we would like to start from recalling some results about surfaces of general type [82].

Since Bogomolov-Miyaoka-Yau's inequality was established, the geography of surfaces of general type, introduced by Ulf Persson [81], has been one of the main subjects in the surface theory. Recall that the numerical characters of a minimal surface S of general type satisfy

$$\chi(\mathcal{O}_S) > 0, \quad K_S^2 > 0, \quad 2\chi(\mathcal{O}_S) - 6 \leq K_S^2 \leq 9\chi(\mathcal{O}_S).$$

These inequalities enable us to plot the pair $(\chi(\mathcal{O}_S), K_S^2)$ as a lattice point in the area defined by them in the xy- plane. The upper bound $y = 9x$ is the Bogomolov-Miyaoka-Yau line, and it is famous that surfaces (whose numerical characters are) on this line can be obtained as a quotient of the complex ball. The lower bound $y = 2x - 6$ is the Noether line, and regular surfaces near this line were studied in detail by Horikawa [42]. In the intermediate area, we can also find some important lines. We would like to recall, among others, the Castelnuovo line $y = 3x - 10$. Castelnuovo's second inequality [15] says that $K_S^2 \ geq 3\chi(\mathcal{O}_S) - 10$ holds for canonical surfaces S, where a minimal surface of general type is called canonical if its canonical map is birational onto the image. Furthermore, it is known that the canonical map of surfaces in the region $2x - 6 \leq y < 3x - 10$ gives a double covering of a ruled surface (cf. [15]). In other words, those who live in this area are of hyperelliptic type. Almost all regular canonical surfaces on $y = 3x - 10$, $3x - 9$ have a pencil of non-hyperelliptic curves of genus three (see [22], [10], [52], [44]). Therefore, surfaces fibered over curves appear quite naturally through the canonical map. This is one of the main reasons why we are interested in pencils of curves. Note also that Persson [81] and Chen [24] constructed fibered surfaces whose invariants can almost fill the zone of existence $2x - 6 \leq y \leq 9x$ (see also [7]).

Let $f : S \to B$ be a surjective morphism of a non-singular projective surface S onto a non-singular projective curve B of genus b with connected fibers. We say that f is a *relatively minimal* fibration if there are no (-1)-curves (i.e., a non-singular rational curve with self-intersection number -1) contained in fibers. We denote by g the genus of a general fiber F of f. If $g = 0$, then f is a $\mathbb{P}^1$-bundle and there exists a vector bundle $\mathcal{E}$ of rank two with $S \simeq \mathbb{P}_B(\mathcal{E})$. If $g = 1$, then it is so called an elliptic surface whose structure was studied by Kodaira [51] extensively.

In what follows, a fibration means a relatively minimal fibration of genus $g \geq 2$, unless otherwise stated explicitly.

Let $K_{S/B}$ be the relative canonical bundle $K_S - f^*K_B$. By Arakelov's theorem (see [16]), it is a nef line bundle,

that is, the intersection number $K_{S/B}C$ is non-negative for any irreducible curve C on S. Furthermore, it is known that $K_{S/B}C = 0$ if and only if C is a (-2)-curve contained in a fiber. Hence one can regard the relative canonical bundle as a candidate of the canonical bundle of minimal surfaces of general type. Now, we can introduce three numerical invariants associated to f.

1. $K^2_{S/B}$
2. $\chi_f := \deg f_*\omega_{S/B} = \chi(\mathcal{O}_S) - (g-1)(b-1)$
3. $e_f := e(S) - e(F)e(B)$, where F is a general fiber of f, and $e(X)$ denotes the topological Euler number of the space X.

Note that e_f has the following "localization"

$$e_f = \sum_{P \in B} e_f(f^{-1}P), \tag{1.1.1}$$

where $e_f(f^{-1}P) := e(f^{-1}P) + 2g - 2$ (Euler contribution).

The three invariants are non-negative integers related by Noether's formula:

$$K^2_{S/B} + e_f = 12\chi_f \tag{1.1.2}$$

Hence one can choose any two of them as basic invariants. We choose here $K^2_{S/B}$ and χ_f as basics, and will consider relations among them, usually assuming that f is not a locally trivial fibration (i.e., not an analytic fiber bundle). This condition is equivalent to assuming that $\chi_f > 0$. In such a case, the ratio

$$\lambda_f := K^2_{S/B}/\chi_f \tag{1.1.3}$$

is called the *slope* of the fibration [96].

On one hand, it is easy to get the upper bound for $K^2_{S/B}$ in terms of χ_f: Since $e_f \geq 0$, we get $K^2_{S/B} \leq 12\chi_f$. The equality holds here if and only if f is a *Kodaira fibration*, that is, f has no singular fibers but with variable moduli. On the other hand, the lower bound is non-trivial, and it is called the *slope inequality*:

$$K^2_{S/B} \geq \frac{4(g-1)}{g}\chi_f \tag{1.1.4}$$

shown by Xiao [96] (see also [28] for semi-stable fibrations).

All the above enable us to plot $(\chi_f, K^2_{S/B})$ on the plane, quite similarly as in the case of surfaces of general type. The zone of existence is given by

$$x > 0, \quad y > 0, \quad \left(4 - \frac{4}{g}\right) x \leq y \leq 12x$$

Fibrations on the lower bound $y = (4 - 4/g)x$ are of hyperelliptic type with only "simple" singular fibers (see §2.2). Fibrations on the upper bound $y = 12x$ are of non-hyperelliptic type also with only beautiful (non-singular !) fibers. Then one may ask:

- What happens in the intermediate area ? How about singular fibers ?
- Are there any important lines like the Castelnuovo line ?

Remark. For hyperelliptic fibrations, Xiao [97] showed

$$\lambda_f \leq \begin{cases} 12 - \dfrac{8g+4}{g^2} & \text{if } g \text{ is even,} \\ 12 - \dfrac{8g+4}{g^2-1} & \text{if } g \text{ is odd} \end{cases}$$

See also [66].

1.2. Proofs of the Slope Inequality

Here we outline two proofs of the slope inequality both of which involve an interesting vector bundle argument.

(A) The first proof is due to Xiao [96]. Let $\mathcal{E}$ be a vector bundle (or a locally free sheaf) on a non-singular irreducible curve. We denote by $\mathrm{rk}\mathcal{E}$ the rank of $\mathcal{E}$ and by $\deg \mathcal{E}$ the degree of the determinant line bundle of $\mathcal{E}$. The ratio $\mu(\mathcal{E}) := \deg \mathcal{E}/\mathrm{rk}\mathcal{E}$ is called the slope of $\mathcal{E}$. Recall that $\mathcal{E}$ is said to be semi-stable if $\mu(\mathcal{V}) \leq \mu(\mathcal{E})$ holds for any subbundle $\mathcal{V}$ of $\mathcal{E}$. Even if $\mathcal{E}$ is not semi-stable, we have a uniquely determined filtration, the Harder-Narashimhan filtration [35],

$$0 =: \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_n = \mathcal{E} \tag{1.2.1}$$

satisfying the following properties

(1) Each $\mathcal{E}_i/\mathcal{E}_{i-1}$ is semi-stable

(2) $\mu_1 > \cdots > \mu_n$, where $\mu_i := \mu(\mathcal{E}_i/\mathcal{E}_{i-1})$

If we put $r_i = \mathrm{rk}\mathcal{E}_i$, then

$$\deg \mathcal{E} = \sum_{i=1}^{n} r_i(\mu_i - \mu_{i+1}), \qquad (\text{where } \mu_{n+1} = 0). \tag{1.2.2}$$

We consider $f_*\omega_{S/B}$ and its Harder-Narashimhan filtration. Since $f_*\omega_{S/B}$ is nef by Fujita's theorem [32], μ_n is non-negative. The invariant χ_f can be calculated by the formula (1.2.2).

For each i, the composite of the natural sheaf homomorphisms

$$f^*\mathcal{E}_i \hookrightarrow f^*f_*\omega_{S/B} \to \omega_{S/B}$$

induces a rational map $S \to \mathbb{P}_B(\mathcal{E}_i)$. Let $\rho : \tilde{S} \to S$ be a minimal succession of blowing-ups such that the above map becomes a morphism for every i. We denote by M_i the pull-back to $\tilde{S}$ of the tautological line bundle H_i on $\mathbb{P}_B(\mathcal{E}_i)$. Put $M_{n+1} = \rho^*K_{S/B}$. For simplicity, we denote a general fiber of $\tilde{S} \to B$ also by F.

By the construction, we have effective divisors Z_i such that

$$\rho^*K_{S/B} \equiv M_i + Z_i, \quad Z_1 \geq \cdots \geq Z_n \geq Z_{n+1} = 0$$

where the symbol $\equiv$ means numerical equivalence of divisors. According to a theorem of Miyaoka [69], the $\mathbb{Q}$-divisors $H_i - \mu_i\Gamma$ are nef, where Γ denotes a fiber of $\mathbb{P}_B(\mathcal{E}_i) \to B$. It follows that $N_i := M_i - \mu_iF$ is also nef being the pull-back of a nef divisor.

Put $d_i = N_iF = M_iF$. Since $M_i|_F$ is a special divisor on F which induces a map into $\mathbb{P}^{r_i-1}$ (a fiber of $\mathbb{P}_B(\mathcal{E}_i) \to B$), we have $d_i \geq 2r_i - 2$ by Clifford's theorem. Note that we have $d_n = d_{n+1} = 2g - 2$.

For $i > j$, we have $N_i = N_j + (\mu_i - \mu_j)F + (Z_j - Z_i)$ and

$$\begin{aligned} N_i^2 &= N_iN_j + d_i(\mu_i - \mu_j) + N_i(Z_j - Z_i) \\ &= N_j^2 + (d_i + d_j)(\mu_i - \mu_j) + (N_i + N_j)(Z_j - Z_i) \\ &\geq N_j^2 + (d_i + d_j)(\mu_i - \mu_j) \end{aligned}$$

by the nefness of the N_i's. Hence we have:

Lemma 1.1 ([96]). *Let $\{i_1, \cdots, i_m\}$ be a sequence of indices with $1 \leq i_1 < \cdots < i_m \leq n$. Then*

$$K_{S/B}^2 \geq \sum_{p=1}^{m}(d_{i_p} + d_{i_{p+1}})(\mu_{i_p} - \mu_{i_{p+1}})$$

where $i_{m+1} = n + 1$.

We can now show the slope inequality. First, suppose that $f_*\omega_{S/B}$ is semi-stable. We apply Lemma 1.1 to the sequence $\{1\}$ to get

$$K^2_{S/B} \geq (4g-4)\mu_1 = (4-4/g)\chi_f$$

which is what we want. We can assume that $f_*\omega_{S/B}$ is not semi-stable. Applying Lemma 1.1 to $\{1, 2, \cdots, n\}$ and to $\{1, n\}$, we respectively get

$$\begin{aligned} K^2_{S/B} &\geq \textstyle\sum_{i=1}^{n}(d_i + d_{i+1})(\mu_i - \mu_{i+1}) \\ &\geq \textstyle\sum(2r_i + 2r_{i+1} - 4)(\mu_i - \mu_{i+1}) \\ &\geq \textstyle\sum_{i=1}^{n-1}(4r_i - 2)(\mu_i - \mu_{i+1}) + (4g-4)\mu_n \\ &= 4\chi_f - 2(\mu_1 + \mu_n) \end{aligned}$$

and

$$K^2_{S/B} \geq (0 + 2g - 2)(\mu_1 - \mu_n) + (4g-4)\mu_n = 2(g-1)(\mu_1 + \mu_n)$$

From these two, we get (1.1.4) by eliminating $\mu_1 + \mu_n$.

(B) The second proof is taken from Moriwaki [73]. We need the following two results:

Lemma 1.2 ([80]). *For a general fiber F, the kernel of the evaluation map $H^0(F, K_F) \otimes \mathcal{O}_F \to \mathcal{O}_F(K_F)$ is a semi-stable vector bundle.*

Lemma 1.3 (Bogomolov instability theorem [20]). *Let $\mathcal{F}$ be a torsion free sheaf on a non-singular projective surface S and put*

$$\delta(\mathcal{F}) = 2\mathrm{rk}(\mathcal{F})c_2(\mathcal{F}) - (\mathrm{rk}(\mathcal{F}) - 1)c_1^2(\mathcal{F}).$$

If $\delta(\mathcal{F}) < 0$, then there exists a non-zero saturated subsheaf $\mathcal{G}$ of $\mathcal{F}$ such that

$$D := \mathrm{rk}(\mathcal{F})c_1(\mathcal{G}) - \mathrm{rk}(\mathcal{G})c_1(\mathcal{F})$$

is in the positive cone, that is, $D^2 > 0$ and $DH > 0$ for any ample divisor H.

We let $\mathcal{F}$ be the kernel of the natural sheaf homomorphism $\phi : f^*f_*\omega_{S/B} \to \omega_{S/B}$. Since a general fiber F is a non-singular curve of genus $g \geq 2$, K_F is generated by its global sections. Hence ϕ is generically surjective. Its image L is torsion free and $\mathcal{F}$ is locally free, since they are a first and a second syzygy, respectively, on a non-singular surface.

We can find an effective divisor Z which is vertical with respect to f and satisfies $c_1(L) = K_{S/B} - Z$. It is the fixed part of the linear system

$|K_{S/B} + f^*\mathcal{L}|$ for any sufficiently ample divisor $\mathcal{L}$ on B; we have $Z = Z_n$ in the notation of (A). Note also that $c_2(L)$ is nothing but the length of the isolated base points of the variable part (with natural scheme structure).

Lemma 1.4. *$\delta(\mathcal{F})$ is non-negative.*

Proof. We assume that $\delta(\mathcal{F}) < 0$ and show that this leads us to a contradiction. Let $\mathcal{G}$ and D be as in Lemma 1.3. Since F is nef, we always have $FD \geq 0$. Since $D^2 > 0$ and $F^2 = 0$, Hodge's index theorem implies that $FD > 0$. We have

$$FD = \mathrm{rk}(\mathcal{F}) \deg(\mathcal{G}|_F) - \mathrm{rk}(\mathcal{G}) \deg(\mathcal{F}|_F).$$

Thus $FD > 0$ means that $\mathcal{F}|_F$ has a destabilizing subsheaf $\mathcal{G}|_F$. But it is impossible by Lemma 1.2. Therefore, $\delta(\mathcal{F}) \geq 0$. Q.E.D.

We calculate $\delta(\mathcal{F})$. From the exact sequence

$$0 \to \mathcal{F} \to f^* f_* \omega_{S/B} \to L \to 0$$

we get

$$c_1(\mathcal{F}) = f^* \det(f_* \omega_{S/B}) - c_1(L)$$

and $\chi(\mathcal{F}) = \chi(f^* f_* \omega_{S/B}) - \chi(L)$. Since the Riemann-Roch theorem gives us

$$\chi(\mathcal{F}) = \frac{c_1(\mathcal{F})(c_1(\mathcal{F}) - K_S)}{2} - c_2(\mathcal{F}) + \mathrm{rk}(\mathcal{F})\chi(\mathcal{O}_S),$$

we get

$$c_2(\mathcal{F}) = c_1(L)^2 - 2(g-1)\chi_f - c_2(L).$$

Hence, using $c_1 = c_1(L) = K_{S/B} - Z$, we get

$$\begin{aligned} & \delta(\mathcal{F}) \\ = \; & 2(g-1)(c_1^2 - 2(g-1)\chi_f - c_2(L)) - (g-2)(c_1^2 - 2\chi_f c_1 F) \\ = \; & gK_{S/B}^2 - 4(g-1)\chi_f - g(2K_{S/B} - Z)Z - 2(g-1)c_2(L). \end{aligned}$$

Now, $\delta(\mathcal{F}) \geq 0$ is equivalent to

$$K_{S/B}^2 \geq \frac{4(g-1)}{g}\chi_f + (2K_{S/B} - Z)Z + \frac{2(g-1)}{g}c_2(L).$$

Here, $K_{S/B}$ is nef, $Z^2 \leq 0$ (since Z is vertical), and $c_2(L) \geq 0$. Therefore, we get the slope inequality (1.1.4) as desired.

1.3. Further remarks

(1) It is known that if $\lambda_f = 4 - 4/g$, then f is necessarily a hyperelliptic fibration (see [96], [54]). Hence the lower bound of the slope of non-hyperelliptic fibrations is bigger. Unfortunately, the precise bound is known only for small genus:

- $g = 3$: $\lambda_f \geq 3$ ([53], [23], [83], [44], [45])
- $g = 4$: $\lambda_f \geq 24/7$, and $\lambda_f \geq 7/2$ if a general fiber has two distinct g^1_3's ([25], [54])
- $g = 5$: $\lambda_f \geq 40/11$ for f trigonal[1] and $\lambda_f \geq 4$ for f non-trigonal ([54])

For trigonal fibrations, we gave in [55] a very rough bound $14(g-1)/(3g+1)$, while it goes up to $24(g-1)/(5g+1)$ when semi-stable [86]. For bielliptic fibrations of genus $g \geq 5$, Barja [12] has shown $\lambda_f \geq 4$. It would be very interesting to find the accurate lower bound of the slope of non-hyperelliptic fibrations. See §2.3 below for an attempt.

(2) For a fibration $f : S \to B$, we put $q_f = q(S) - b$. When $q_f > 0$, we call f an irregular fibration. In this case, we can construct an unramified covering of S not coming from those of B, inducing fibrations over B with slope equals the original λ_f and of an arbitrary large genus. Hence the slope inequality (formally letting $g \to \infty$) gives us:

Theorem (Xiao [96]). *If $q_f > 0$, then $K^2_{S/B} \geq 4\chi_f$.*

It is known that the inequality is sharp when $q_f = 1$, but may not when $q_f \geq 2$. It would be interesting to have a sharper lower bound for the slope of irregular fibrations including g and q_f. An attempt along this line can be found in a recent paper by Barja and Zucconi [14] (see also [13]). For hyperelliptic fibrations, Xiao gave further interesting results in [98] and [99].

§2. Slope equality and the local invariants

2.1. Horikawa index and local signature

Fiber germ. Let Δ be a small open neighborhood of the origin $0 \in \mathbb{C}$, usually a disk around 0, and let $f_\Delta : S_\Delta \to \Delta$ be a relatively minimal fibration of genus g. We assume that the critical value of f_Δ, if exists, is 0. Let $f_{\Delta'} : S_{\Delta'} \to \Delta'$ be another such fibration. We say that f_Δ and $f_{\Delta'}$ are (analytically) equivalent if there exists a biholomorphic

[1] This may not be sharp.

map $\phi : S_\Delta|_{\Delta\cap\Delta'} \to S_{\Delta'}|_{\Delta\cap\Delta'}$ such that $f_\Delta|_{\Delta\cap\Delta'} = f'_\Delta|_{\Delta\cap\Delta'} \circ \phi$. This equivalence relation leads us to the notion of the (singular) *fiber germ* at 0. A representative $f_\Delta : S_\Delta \to \Delta$ of a fiber germ will be simply called a degeneration.

Similarly, we can introduce the notion of topological equivalence by considering an orientation preserving homeomorphism ϕ.

Slope equality. Let us consider and fix a certain property (*) for curves of genus g, and consider the set $\mathcal{C}(*)$ of all the fiber germs of genus g whose general fiber satisfies (*). A function $\mathrm{Ind} : \mathcal{C}(*) \to \mathbb{Q}$ is called a *Horikawa index* with respect to (*) if the following are satisfied for any global fibration $f : S \to B$ of genus g whose general fiber satisfies (*):

1. $\mathrm{Ind}(f^{-1}P)$ is a non-negative rational number for any $P \in B$ and is equal to zero if the fiber over P satisfies (*).
2. there exists a rational number λ, which depends only on g and (*), satisfying $4 - 4/g \leq \lambda \leq 12$ and

$$K^2_{S/B} = \lambda\chi_f + \sum_{P\in B} \mathrm{Ind}(f^{-1}P). \tag{2.1.1}$$

Suppose that there exists a global fibration $f : S \to B$ of genus g whose general fiber satisfies (*) and $\mathrm{Ind}(f^{-1}P) = 0$ holds for any $P \in B$. When invariants of such fibrations can fill (almost) all the lattice points on the line $y = \lambda x$, the equality (2.1.1) will be called a *slope equality*. Once it is established, λ gives us the slope of the line which is the lower bound of the zone of the existence of fibrations with property (*), and the Horikawa index measures how far it is from the lower bound.

The equality of type (2.1.1) was first obtained by Horikawa in [41] for genus 2 fibrations, and it was applied successfully to surfaces of general type near the Noether line [42]. This is the reason why we call Ind by his name.

Definition 2.1. Suppose that a Horikawa index with respect to (*) is given. Let $f_\Delta : S_\Delta \to \Delta$ be a degeneration which represents an element in $\mathcal{C}(*)$. We call such a degeneration a (*)-degeneration for simplicity. The central fiber $f^{-1}(0)$ is called a *critical* fiber if $\mathrm{Ind}(f^{-1}(0)) > 0$. As usual, it is called a *singular* fiber if $e_f(f^{-1}(0)) > 0$.

Local signature. Since a projective surface S is an orientable 4-dimensional differentiable manifold, the signature $\mathrm{Sign}(S)$ of the intersection form on H^2 is one of the most important topological invariants.

A function $\sigma : \mathcal{C}(*) \to \mathbb{Q}$ is called a *local signature* with respect to (*) if the following are satisfied for any global fibration $f : S \to B$ of genus g whose general fiber satisfies (*):

1. $\sigma(f^{-1}P) = 0$ if the fiber over $P \in B$ is non-singular and satisfies (*), and
2. the signature can be expressed as

$$\text{Sign}(S) = \sum_{P \in B} \sigma(f^{-1}P) \tag{2.1.2}$$

Suppose that (2.1.1) is obtained with $\lambda < 12$. Since the signature is given by

$$\text{Sign}(S) = \frac{1}{3}(c_1^2(S) - 2c_2(S)) = K_{S/B}^2 - 8\chi_f$$

if we put

$$\sigma(f^{-1}P) := \frac{4}{12-\lambda}\text{Ind}(f^{-1}P) - \frac{8-\lambda}{12-\lambda}e_f(f^{-1}P) \tag{2.1.3}$$

then (1.1.1) and (2.1.1) give us

$$\text{Sign}(S) = \sum_{P \in B} \sigma(f^{-1}P).$$

Hence, σ is a local signature with respect to (*).

Conversely, suppose that we are given a local signature and that we know all values $\sigma(f^{-1}P)$. If there exists a rational number μ with $-4 - 4/g \leq \mu \leq 4$ such that

1. the inequality

$$\mu(\sigma(f^{-1}P) + e_f(f^{-1}P)) \leq 4\sigma(f^{-1}P) \tag{2.1.4}$$

holds for all fiber germs $f^{-1}P \in \mathcal{C}(*)$, and

2. the equality holds in (2.1.4) as long as the fiber $f^{-1}P$ satisfies (*),

then putting $\lambda = \mu + 8$, we will be able to define $\text{Ind}(f^{-1}P)$ by using (2.1.3) and get an equality of type (2.1.1).

Remark. Though we naturally expect that Horikawa index and Local signature are unique if exist, the above definitions can say nothing about that. The local invariants will have desired properties if we can find a nice condition (*).

In the following two subsections, we shall describe two cases where the equality of type (2.1.1) is known.

2.2. Hyperelliptic fibrations—double coverings

The slope inequality was first proved for hyperelliptic fibrations by Persson [81] and Horikawa [42], Part V. They used the double covering method which has been one of the important tools in the surface theory.

Though our concern is in hyperelliptic fibrations, we start from a more general setting: Let $f : S \to B$ be a relatively minimal fibration of genus $g \geq 2$ and assume that a general fiber F has an involution which extends to an involution ι of the whole S. We are in such a situation when f is a hyperelliptic fibration: Since the relative canonical map is a generically finite rational map of degree two, S has an involution which restricts to the hyperelliptic involution on F.

Let $\rho : \hat{S} \to S$ be the blowing-ups at all the isolated fixed points of $\langle\iota\rangle$-action, and let $\hat{\iota}$ be the induced involution on $\hat{S}$. Since the fixed locus of $\hat{\iota}$ is of codimension 1, the quotient space $\hat{W} := \hat{S}/\langle\hat{\iota}\rangle$ is non-singular, and the quotient map $\hat{S} \to \hat{W}$ is a finite double covering. Then the branch locus $\hat{R}$ is a non-singular divisor on $\hat{W}$, and we can find a line bundle $\hat{L}$ on $\hat{W}$ satisfying $[\hat{R}] = 2\hat{L}$ and such that $\hat{S}$ is isomorphic to the double covering of $\hat{W}$ constructed in the total space of $\hat{L}$ in the usual way.

$$\begin{array}{ccc} \hat{S} & \longrightarrow & \hat{W} \\ \rho\downarrow & & \downarrow \\ S & & W \\ & f\searrow \quad \swarrow\pi & \\ & B & \end{array}$$

Since $\hat{W}$ has a natural fibration over B, say of genus h, we can take a relatively minimal model $\pi : W \to B$. Such W is unique when $h \geq 1$. On the other hand, if $h = 0$, that is, $\hat{W}$ is a ruled surface over B, a relatively minimal model is not unique, and we can move from a model to another via elementary transformations (that is, blow up a fiber at a point and then blow down the proper transform).

We let R be the direct image of $\hat{R}$ in W as divisor. Then R is a reduced divisor with a natural line bundle L satisfying $2L = [R]$. Hence we can construct in the total space of L a normal surface S' birational to S, in the usual way. Note that, by Hurwitz formula, we have

$$2g - 2 = 2(2h - 2) + R\Gamma, \tag{2.2.1}$$

where Γ is a general fiber of π.

Since S' is a divisor in a non-singular threefold, the dualizing sheaf $\omega_{S'}$ is an invertible sheaf. More concretely, it is induced by $K_W + L$.

We can calculate the numerical invariants of $f' : S' \to B$ as follows:

$$\omega^2_{S'/B} = 2(K_{W/B} + L)^2 = 2K^2_{W/B} + 2K_{W/B}R + R^2/2$$

Since

$$\begin{aligned} \chi(\mathcal{O}_{S'}) &= \chi(\mathcal{O}_W) + \chi(\mathcal{O}_W(-L)) \\ &= 2\chi(\mathcal{O}_W) + R^2/8 + K_W R/4 \end{aligned}$$

we get

$$\begin{aligned} \chi_{f'} &= 2\chi(\mathcal{O}_W) + R^2/8 + K_W R/4 - (g-1)(b-1) \\ &= 2\chi_\pi + R^2/8 + K_W R/4 - (b-1)R\Gamma/2 \\ &= 2\chi_\pi + R^2/8 + K_{W/B}R/4. \end{aligned}$$

From these, it follows

$$\omega^2_{S'/B} - 4\chi_{f'} = 2(K^2_{W/B} - 4\chi_\pi) + K_{W/B}R. \tag{2.2.2}$$

The singularities of S' can be resolved in a natural way by the *canonical resolution* (or even resolution). Let P be a singular point of R of multiplicity m_1. Let $\sigma_1 : W_1 \to W$ be the blowing-up at P, $E = \sigma_1^{-1}(P)$ the exceptional (-1)-curve. Then $R_1 = \sigma_1^* R - 2[m_1/2]E$ is a reduced divisor and $[R_1] = 2L_1$, where $L_1 = \sigma_1^* L - [m_1/2][E]$. Furthermore, the double covering of W_1 branched along R_1 (constructed in L_1) has a birational morphism onto S'. Continuing this process, we get a sequence

$$\begin{array}{ccccccc} S_n & \to & S_{n-1} & \to \cdots \to & S_1 & \to & S' \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ W_n & \to & W_{n-1} & \to \cdots \to & W_1 & \to & W \end{array} \tag{2.2.3}$$

such that the branch locus R_n of $S_n \to W_n$ is non-singular. If we choose such a sequence of minimal length, then it can be shown that S_n is isomorphic to $\hat{S}$. Let $m_1, \ldots, m_n$ denote the multiplicity sequence. Then a calculation shows that the differences of invariants are given by

$$\chi_{f'} - \chi_{f_n} = \frac{1}{2}\sum_{i=1}^{n} \left[\frac{m_i}{2}\right]\left(\left[\frac{m_i}{2}\right] - 1\right), \tag{2.2.4}$$

$$\omega^2_{S'/B} - K^2_{S_n/B} = 2\sum_{i=1}^{n} \left(\left[\frac{m_i}{2}\right] - 1\right)^2 \tag{2.2.5}$$

Therefore, letting ϵ denote the number of blowing-downs to obtain S from $\hat{S}$, we get

$$K^2_{S/B} - 4\chi_f = 2(K^2_{W/B} - 4\chi_\pi) + K_{W/B}R + \sum_{i=1}^{n}\left(\left[\frac{m_i}{2}\right] - 1\right) + \epsilon \tag{2.2.6}$$

We assume now that f is a hyperelliptic fibration. By (2.2.1), we have $R\Gamma = 2g+2$. Since we can write $R \equiv -(g+1)K_{W/B} + \alpha\Gamma$ with some integer α, it follows from (2.2.2) that

$$\omega^2_{S'/B} = \frac{4(g-1)}{g}\chi_{f'}$$

because $K^2_{W/B} = \chi_\pi = 0$. (N.B. Curiously enough, the equality always holds in the slope inequality for the singular model $f' : S' \to B$.) Then, taking the contribution of the resolution into account, we get

$$K^2_{S/B} = \frac{4(g-1)}{g}\chi_f + \frac{2}{g}\sum_i \left(\left[\frac{m_i}{2}\right] - 1\right)\left(g - \left[\frac{m_i}{2}\right]\right) + \epsilon. \tag{2.2.7}$$

We can take W so that all the m_i satisfy $m_i \leq g+2$ (e.g., [99]). It follows that all the summands in the correction term are non- negative, and we get the slope inequality. It may be clear that the correction term can be localized to give us the Horikawa index. Needless to say, the property (*) in §2.1 is "hyperelliptic" here.

Endo [31] defined topologically the local signature for hyperelliptic fibrations. In analytic case, it coincides with ours obtained as in §2.1 using the Horikawa index (see [2] and [31]).

Remark. When $h = 1$, it follows from (2.2.6) that $K^2_{S/B} \geq 4\chi_f$ as shown by Barja in [12].

2.3. Fibrations of general Clifford index —relative canonical algebra

The second method to get a slope equality is the use of the relative canonical algebra ([23], [83], [56]).

For any non-negative integer n, put $\mathcal{R}_n = f_*(\omega^{\otimes n}_{S/B})$. These are nef locally free sheaves satisfying

$$\mathrm{rk}\mathcal{R}_n = \begin{cases} 1 & \text{if } n = 0, \\ g & \text{if } n = 1, \\ (2n-1)(g-1) & \text{if } n \geq 2 \end{cases}$$

and

$$\deg \mathcal{R}_n = \begin{cases} 0 & \text{if } n = 0, \\ \chi_f & \text{if } n = 1, \\ \frac{1}{2}n(n-1)K^2_{S/B} + \chi_f & \text{if } n \geq 2. \end{cases}$$

The relative canonical algebra for f is the $\mathcal{O}_B$-algebra

$$\mathcal{R}(f) = \bigoplus_{n \geq 0} \mathcal{R}_n.$$

It is attractive for a naive reason that its Proj gives the relative canonical model.

Let X be a non-singular projective curve of genus $g \geq 2$. The Clifford index of X is defined as

$$\mathrm{Cliff}(X) = \min\{\deg L - 2\dim |L| \mid L \in \mathrm{Pic}(X),\ h^0(L) > 1,\ h^1(L) > 1\}$$

when $g \geq 4$. If $g = 2$, we put $\mathrm{Cliff}(X) = 0$. If $g = 3$, we put $\mathrm{Cliff}(X) = 0$ or 1 according to whether X is hyperelliptic or not. See [27] for further properties of the Clifford index. The Clifford index of a fibration $f : S \to B$, which we denote by $\mathrm{Cliff}(f)$, is defined as that of a general fiber F. Then we have $0 \leq \mathrm{Cliff}(f) \leq (g-1)/2$.

Let $f : S \to B$ be a fibration with $\mathrm{Cliff}(f) > 0$. The contraction with the identity in $H^0(B, End(\mathcal{R}_1))$ and the multiplication define an $\mathcal{O}_B$-linear map $d_{i,j} : \bigwedge^i \mathcal{R}_1 \otimes \mathcal{R}_j \to \bigwedge^{i-1} \mathcal{R}_1 \otimes \mathcal{R}_{j+1}$ satisfying $\mathrm{Im}(d_{i,j}) \subset \mathrm{Ker}(d_{i-1,j+1})$. Hence, as in [34], we can consider a Koszul complex

$$0 \to \wedge^{c+1}\mathcal{R}_1 \to \cdots \to \wedge^i \mathcal{R}_1 \otimes \mathcal{R}_{c+1-i} \xrightarrow{d_{i,c+1-i}} \wedge^{i-1}\mathcal{R}_1 \otimes \mathcal{R}_{c+2-i} \to$$

$$\cdots \to \mathcal{R}_1 \otimes \mathcal{R}_c \to \mathcal{R}_{c+1} \to 0$$

for $c = \mathrm{Cliff}(f)$. We put

$$\mathcal{K}_{i,j} = \mathrm{Ker}(d_{i,j})/\mathrm{Im}(d_{i+1,j-1}).$$

Then $\mathcal{K}_{i,c+1-i}$ is a torsion sheaf for $0 \leq i \leq c-2$, because the Koszul complex

$$\begin{aligned} &\textstyle\bigwedge^{i+1} H^0(K_F) \otimes H^0((c-i)K_F) \to \\ &\textstyle\quad \bigwedge^{i} H^0(K_F) \otimes H^0((c-i+1)K_F) \to \\ &\textstyle\qquad \bigwedge^{i-1} H^0(K_F) \otimes H^0((c-i+2)K_F) \end{aligned} \tag{2.3.1}$$

is exact at the middle term for any general fiber F by [34]. The differential $d_{c+1,0} : \bigwedge^{c+1} \mathcal{R}_1 \to \bigwedge^c \mathcal{R}_1 \otimes \mathcal{R}_1$ is clearly injective and the quotient is locally free. We have

$$\mathcal{K}_{c,1} \simeq \mathrm{Ker}\left(\frac{\bigwedge^c \mathcal{R}_1 \otimes \mathcal{R}_1}{\bigwedge^{c+1} \mathcal{R}_1} \xrightarrow{\bar{d}_{c,1}} \bigwedge^{c-1} \mathcal{R}_1 \otimes \mathcal{R}_2\right)$$

where $\bar{d}_{c,1}$ denote the natural map induced from $d_{c,1}$. It is locally free being a subsheaf of a locally free sheaf on a smooth curve. We have

$$\sum_{i=0}^{c+1}(-1)^i \deg\left(\bigwedge^{c+1-i} \mathcal{R}_1 \otimes \mathcal{R}_i\right) = \sum_{i=0}^{c+1}(-1)^i \deg(\mathcal{K}_{c+1-i,i}).$$

By using formulae giving rank and degree of $\mathcal{R}_n$, we get

$$\begin{aligned} K_{S/B}^2 = & \frac{(g-1)(g+2-2c)}{g-c}\chi_f \\ & -\binom{g-3}{c-1}^{-1} \sum_{i=0}^{c+1}(-1)^{i+1} \deg(\mathcal{K}_{c+1-i,i}). \end{aligned} \tag{2.3.2}$$

Note that, if Green's conjecture [34] on syzygies of canonical curves holds, $\mathcal{K}_{c-1,2}$ is a torsion sheaf for $c = \mathrm{Cliff}(f)$. If $\mathcal{K}_{c-1,2}$ is torsion, then the rank of $\mathcal{K}_{c,1}$ is given by

$$\mathrm{rk}(\mathcal{K}_{c,1}) = \binom{g-1}{c-1} \frac{(g-1-c)(g-1-2c)}{c+1}. \tag{2.3.3}$$

We now assume that g is odd and $\mathrm{Cliff}(f) = c = (g-1)/2$. Then Green's conjecture is true ([40] and [88]) and we see that $\mathcal{K}_{(g-3)/2,2}$ is torsion. Furthermore, we have $\mathcal{K}_{(g-1)/2,1} = 0$ because its rank is zero by (2.3.3). Hence (2.3.2) gives us:

$$K_{S/B}^2 = \frac{6(g-1)}{g+1}\chi_f + \sum_{P\in B} \mathrm{Ind}(f^{-1}P), \tag{2.3.4}$$

where

$$\mathrm{Ind}(f^{-1}P) := \binom{g-3}{(g-3)/2}^{-1} \sum_{i=2}^{(g+1)/2} (-1)^i \mathrm{length}(\mathcal{K}_{(g+1)/2-i,i})_P.$$

We can show ([56]) that $\mathrm{Ind}(f^{-1}P)$ is non-negative, and it is zero only if $|K_F|$ has no base points and (2.3.1) with $i = (g-3)/2$ is exact at the middle term for $F = f^{-1}P$. Therefore, we get the slope equality (2.3.4) with respect to the condition (*): $\mathrm{Cliff}(F) = (g-1)/2$.

We can define the local signature as described in 2.1. But, this time, it is not known whether our local signature coincides with the topological one introduced by Furuta [33].

Remarks. (1) We emphasize here that a critical fiber is not necessarily a singular fiber and vice versa. For example, let $f : S \to B$ be

a Kodaira fibration of genus three. Then $K^2_{S/B} = 12\chi_f$. On the other hand, since $\mathrm{Cliff}(f) = 1$, we have

$$K^2_{S/B} = 3\chi_f + \sum_{P \in B} \mathrm{Ind}(f^{-1}P)$$

by (2.3.4). Hence

$$\sum_{P \in B} \mathrm{Ind}(f^{-1}P) = 9\chi_f.$$

Recall that any fiber of f is non-singular. It follows that f has exactly $9\chi_f$ non-singular hyperelliptic fibers (counting infinitely near ones) which are critical.

(2) Beside its importance, the structure of the relative canonical algebra is not well understood. Miles Reid conjectured that it is generated in degrees lesser than or equal to three, and related in degrees lesser than or equal to six (1-2-3 conjecture [83]). It is shown in [58] that the relative canonical algebra is generated in degrees lesser than or equal to four, and that the 1-2-3 conjecture breaks down in one case that the fiber is a multiple fiber whose canonical linear system contains a (-1) elliptic cycle as a fixed part. The annoying exception actually occurs [59].

(3) To prove the uniqueness of the local signature, Nariya Kawazumi kindly suggests us to argue as follows. Let $\mathcal{M}_g$ be the moduli space of curves of odd genus g $geq3$, and let D be the locus on which the Clifford index drops, that is, D is the "k-gonal locus" in the sense of [38]. Recall that $H^2(\mathcal{M}_g, \mathbb{Q}) \simeq \mathbb{Q}$ by Harer's theorem [36] and is generated by the 1st Morita-Mumford class ([74], [71]). Let τ denote Meyer's signature cocycle [68]. Then it is known the class $-3[\tau] \in H^2(\mathcal{M}_g)$ is the 1st Morita-Mumford class [72]. What we saw above is that $[\tau]$ goes to 0 via the natural map $H^2(\mathcal{M}_g) \to H^2(\mathcal{M}_g \setminus D)$. Since it factors as

$$H^2(\mathcal{M}_g) \to H^2(\pi_1(\mathcal{M}_g \setminus D)) \hookrightarrow H^2(\mathcal{M}_g \setminus D)$$

we see that $[\tau]$ also goes to 0 in $H^2(\pi_1(\mathcal{M}_g \setminus D))$. It follows that there exists a function $\eta : \pi_1(\mathcal{M}_g\ setminusD) \to \mathbb{Q}$ such that η goes to τ. Then to show the uniqueness, we only have to show that $H^1(\mathcal{M}_g \setminus D) = 0$.

Consider the exact sequence of cohomology groups with coefficients in $\mathbb{Q}$:

$$H^1(\mathcal{M}_g) \to H^1(\mathcal{M}_g \setminus D) \to H^2(\mathcal{M}_g, \mathcal{M}_g \setminus D) \to H^2(\mathcal{M}_g) \to H^2(\mathcal{M}_g \setminus D)$$

Since we have $H^1(\mathcal{M}_g) = 0$ by [36], it is also sufficient to see that $H^2(\mathcal{M}_g, \mathcal{M}_g \setminus D) \simeq \mathbb{Q}$.

For $g = 3$, the argument works, because $\mathcal{M}_3 \setminus D$ is nothing but the moduli of smooth plane quartics.

(4) We imagined that the Clifford index gives us a good condition (*) to get a slope equality. But the above computation implies that the answer seems to be negative. For example, consider a fibration of even genus with general Clifford index $g/2 - 1$. Then (2.3.2) gives us

$$K^2_{S/B} = \frac{8(g-1)}{g+2}\chi_f - \begin{pmatrix} g-3 \\ g/2-2 \end{pmatrix} \sum_{i=1}^{g/2} (-1)^{i+1} \deg(\mathcal{K}_{g/2-i,i}).$$

However, it contains the part involving $\mathcal{K}_{g/2-1,1}$ which has not been localized yet. So, we will need some additional (or even, completely different) assumptions.

Problem. Find a nicer condition (*) giving us a slope equality.

Recently, Yoshikawa [100] defined the local signature for general Lefschetz fibrations. His generality condition (*) is the vanishing of all even theta characteristics. However, it seems hard to induce the slope equality from his local signature, since we must compute all possible values $\sigma(F)$ to get λ as in (2.1.1). We hope that his or one of the conditions given by Furuta [33] is the right answer to the above question.

§3. Topological classification of degenerations

3.1. Moduli point and topological monodromy

Here we study singular fiber germs, especially their topological nature.

Let $f : S \to \Delta$ be the normally minimal model of a degeneration of genus g, with a unique singular fiber $F = f^{-1}(0)$. That is, it is the model such that the singularities of the reduced scheme of F are at most normal crossing and any (-1) curve in the components of F meets other components of F at more than two points. Note that such a model uniquely exists among the birational equivalence class of f.

Recall that, in the study of degenerations of elliptic curves by Kodaira [51], the notion of the limit of the period and the monodromy in $\mathrm{SL}(2, \mathbb{Z})$ are essential. We introduce their candidates in our context: the moduli point and topological monodromy.

Moduli point. Let $\mathcal{M}_g$ be the moduli space of smooth curves of genus g. We can associate to our normally minimal model $f : S \to \Delta$ the moduli map $\alpha_f : \Delta^* \to \mathcal{M}_g$, where $\Delta^* = \Delta \setminus \{0\}$. Since Δ is one-dimensional, α_f extends to a morphism

$$\overline{\alpha}_f : \Delta \longrightarrow \overline{\mathcal{M}}_g$$

where $\overline{\mathcal{M}}_g$ is the moduli space of stable curves of genus g [29]. We call $\overline{\alpha}_f(0)$ the *moduli point* of f. Let $\widetilde{f} : \widetilde{S} \to \Delta$ be the stable reduction of f. (Namely, let $\Delta \to \Delta$ be the cyclic covering between disks branching only at the origin so that the desingularization of the pull back of f by this map induces a semi-stable family $S' \to \Delta$. We obtain $\widetilde{S}$ from S' by contracting chains of (-2) curves. Thus $\widetilde{S}$ has at most rational double points of type A as its singularity.) Then the stable curve which corresponds to the moduli point of f coincides with the fiber $\widetilde{f}^{-1}(0)$.

Topological monodromy. Let Γ_g be the mapping class group of genus g. Choose $t_0 \in \Delta^*$ and put $F_0 = f^{-1}(t_0)$. The monodromy action of $\pi_1(\Delta^*, t_0) \simeq \mathbb{Z}$ induces modulo isotopy an orientation-preserving homeomorphism $\phi_f : F_0 \to F_0$. Since the change of the base point t_0 corresponds to the conjugation in Γ_g, f determines an element $[\phi_f]$ of the set $\widehat{\Gamma}_g$ of all conjugacy classes of Γ_g. We call $[\phi_f]$ the *topological monodromy* of f.

Obviously, one can associate another monodromy: By choosing a suitable symplectic 1-homology basis of F_0, we can define the homomorphism $\eta : \Gamma_g \to \mathrm{Sp}(2g, \mathbb{Z})$. We call the equivalence class of $\eta(\phi_f)$ in the conjugacy class of $\mathrm{Sp}(2g, \mathbb{Z})$ the *homological monodromy* of f.

An advantage of considering the topological monodromy is that it can distinguish degenerations whose ϕ_f's are contained in the kernel of the map η, i.e. the Torelli group (see also [6]). For instance, we consider the degeneration of genus 2 whose singular fiber is a stable curve consisting of two nonsingular components with one node. Then the topological monodromy is the integral Dehn twist along a *separated curve*, i.e. a simple closed real curve on the Riemann surface whose complement is disconnected. Since this curve does not intersect the usual symplectic homology basis, the homological monodromy is trivial. (See also [76], p. 350.)

From now on, monodromy means topological monodromy, and we sometimes identify it with the representative homeomorphism.

3.2. Namikawa-Ueno's example

Contrary to the case of degenerations of elliptic curves [51], the singular fiber F (as a one-dimensional non-reduced analytic space) in general can determine neither the analytic structure nor the topological structure of the degeneration itself when $g \geq 2$. In some sense, this phenomenon is analogous to the fact that the exceptional curve of the good resolution of an isolated surface singularity does not necessarily determine the germ of the original singularity itself.

Namikawa-Ueno found the series of examples implying the above fact. Among them, the simplest ones are of type $2I_{0-0}$ in [77], p.159, and of type II^*_{1-0} in [77], p.172. Namely, there exist two degenerations $f_i : S_i \to \Delta$ $(i = 1, 2)$ of genus 2 such that

(i) $F_i = f_i^{-1}(0)$ $(i = 1, 2)$ are mutually isomorphic and each of them is written as $2C_1 + 2C_2 + C_3 + C_4$ where C_1 is a nonsingular elliptic curve, C_j $(2 \leq j \leq 4)$ are nonsingular rational curves, $C_1^2 = C_2^2 = -1$, $C_3^2 = C_4^2 = -2$ and $C_1C_2 = C_2C_3 = C_2C_4 = 1$,

(ii) $\overline{\alpha}_{f_1}(0)$ is a stable curve consisting of two nonsingular components with one node, while $\overline{\alpha}_{f_2}(0)$ is an irreducible stable curve with one node,

(iii) ϕ_{f_1} (resp. ϕ_{f_2}) is of order 2, and is the rotation of angle π around the diameter of a separated curve (resp. a non-separated curve). These homeomorphisms are not conjugate each other in Γ_2 (see also [64], §7).

Here we reconstruct these examples. The method is nothing but the "converse process of stable reduction", and is a special case of [87], Part II.

We first construct f_1. Let E be an elliptic curve. We fix a point P_0 on E. Let U be a small open disk neighborhood of P_0 in E. Let E_i $(i = 1, 2)$ be two copies of E, and we fix the identification maps $\tau_i : E \to E_i$. Put $P_i = \tau_i(P_0)$ and $U_i = \tau_i(U)$. Let z_i be the local coordinate on U_i such that $P_i = \{z_i = 0\}$ and $U_i = \{|z_i| < 1\}$. Let Δ_i $(i = 1, 2)$ be two copies of a unit disk defined by $\{t_i \in \mathbb{C} | |t_i| < 1\}$. We put

$$V_i = \{(z_i, t_i) \in U_i \times \Delta_i \mid 0 \leq |z_i| \leq |t_i|^2\}.$$

Let W be an analytic subset in a three-dimensional polydisk defined by

$$W = \{(x_1, x_2, t) \in \Delta^3 \mid x_1x_2 = t^2,\ |x_1| < 1, |x_2| < 1, |t| < 1\}.$$

Then W has an A_1-singularity at $Q = \{(x_1, x_2, t) = (0, 0, 0)\}$. We obtain an analytic surface $\widetilde{S}_1$ by patching up three pieces $E_1 \times \Delta_1 \setminus V_1$, $E_2 \times \Delta_2 \setminus V_2$ and W in the following way: Patch $W \setminus \{Q\}$ and $U_1 \times \Delta_1 \setminus V_1$

by $x_1 = z_1$, $x_2 = t_1^2/z_1$ and $t = t_1$. Patch $W \setminus \{Q\}$ and $U_2 \times \Delta_2 \setminus V_2$ by $x_2 = z_2$, $x_1 = t_2^2/z_2$ and $t = t_2$. Then we have a natural holomorphic map $\widetilde{f}_1 : \widetilde{S}_1 \to \Delta$. The general fiber of $\widetilde{f}_1$ is a smooth curve of genus 2, since it is a connected sum of two tori. The singular fiber $\widetilde{f}_1^{-1}(0)$ is a stable curve of genus 2 consisting of two smooth components with one node, and $\widetilde{S}_1$ has an A_1-singularity at Q.

Now we define three maps $E_1 \times \Delta_1 \setminus V_1 \to E_2 \times \Delta_2 \setminus V_2$, $E_2 \times \Delta_2 \setminus V_2 \to E_1 \times \Delta_1 \setminus V_1$ and $W \to W$ by $(P, t_1) \mapsto (\tau_2 \circ \tau_1^{-1}(P), -t_2)$, $(P', t_2) \mapsto (\tau_1 \circ \tau_2^{-1}(P'), -t_1)$ and $(x_1, x_2, t) \mapsto (x_2, x_1, -t)$, respectively. These maps are well-patched and define an involution $\widetilde{\tau} : \widetilde{S}_1 \to \widetilde{S}_1$.

Let $f_1^\sharp : S_1^\sharp := \widetilde{S}_1/\langle\widetilde{\tau}\rangle \longrightarrow \Delta$ be the quotient of $\widetilde{f}_1$ by $\widetilde{\tau}$ where the base disk is the quotient space of the original disk by the involution $t \to -t$. Then it is easy to see that the resolution space S_1 of $S_1^\sharp$ induces the desired degeneration $f_1 : S_1 \to \Delta$.

Next we construct f_2. Let E' be another elliptic curve so that there exists a free action $\rho : E' \to E'$ of order 2 such that the quotient of E' by ρ coincides with E. We fix two points Q_i $(i = 1, 2)$ on E' which satisfy $\rho(Q_1) = Q_2$ so that Q_i goes to P_0 by the quotient map $E' \to E$. We fix an open disk neighborhood U_i' of Q_i which satisfy $\rho(U_1') = U_2'$ and $U_1' \cap U_2' = \emptyset$. We define the closed region V_i' of $U_i' \times \Delta$ similarly as in the previous example. We obtain an analytic surface $\widetilde{S}_2$ by patching two pieces $(E' \times \Delta) \setminus (V_1' \cup V_2')$ and W along the locus $(U_1' \times \Delta \setminus V_1') \coprod (U_2' \times \Delta \setminus V_2')$ and $W \setminus \{Q\}$ similarly. We obtain the family $\widetilde{f}_2 : \widetilde{S}_2 \to \Delta$ where $\widetilde{f}_2^{-1}(0)$ is an irreducible stable curve of genus 2 with one node. $\widetilde{S}_2$ has a natural involution so that the resolution space of the quotient space of $\widetilde{S}_2$ by this involution induces the desired degeneration.

3.3. Matsumoto-Montesinos' theorem and classification

In order to study the monodromy of degenerations, we first review the theorem of Thurston [92] about the classification of elements in Γ_g.

Let Σ be a Riemann surface of genus g with k disk boundaries with negative Euler number, i.e. $g + k - 2 > 0$. Let $\phi : \Sigma \to \Sigma$ be a homeomorphism. (Homeomorphism is always assumed to be orientation-preserving.) We call ϕ is *reducible* by an *admissible system of cut curves* Γ if;

(a) Γ is a disjoint union of simple closed curves on Σ,
(b) any connected component of the complement of Γ has negative Euler number,
(c) ϕ preserves Γ as a set.

Thurston's theorem *Any homeomorphism ϕ of Σ is isotopic to a diffeomorphism ϕ' of Σ such that, either*

(i) *ϕ' is periodic, i.e. there exists a positive integer n such that $(\phi')^n$ is the identity map, or*

(ii) *ϕ' is pseudo-Anosov, i.e. there exists a real number $\lambda > 1$ and a pair of transverse measured foliations $\mathcal{F}^s$ and $\mathcal{F}^u$ such that $\phi'(\mathcal{F}^s) = 1/\lambda\mathcal{F}^s$ and $\phi'(\mathcal{F}^u) = \lambda\mathcal{F}^u$, or*

(iii) *ϕ' is reducible by a certain admissible system Γ of cut curves. Γ has a ϕ'-invariant annular neighborhood $\mathcal{A}(\Gamma)$ so that the complement $\Sigma \setminus \mathcal{A}(\Gamma)$ has a decomposition $\coprod_j \Sigma^{(j)}$ ($\Sigma^{(j)}$ may be disconnected) which satisfies*;

(iiia) *ϕ' preserves each component $\Sigma^{(j)}$,*

(iiib) *each component map $\phi'|_{\Sigma^{(j)}} : \Sigma^{(j)} \to \Sigma^{(j)}$ satisfies (i) or (ii).*

If all the component maps of ϕ' satisfy (i), we call ϕ *pseudo-periodic.*

Back to our situation, we consider (local 1-parameter) degenerations of curves of genus $g \geq 2$. It is known that the monodromy of degeneration is pseudo-periodic (cf. [26], [46]). Moreover this map is of negative twist, i.e. has a property of one-sided chirality ([85], [30]). More precisely, the power of this map which stabilize any curve of the admissible system induces on its annular neighborhood an integral Dehn twist of right-hand direction.

Let $\mathcal{S}_g$ be the set of topological equivalence classes of degenerations in 2.1, and let $\widehat{\mathcal{P}}_g^-$ be the subset of $\widehat{\Gamma}_g$ whose elements consist of the conjugacy classes of pseudo-periodic maps of negative twist. By taking the monodromy, we have a natural map

$$\rho : \mathcal{S}_g \longrightarrow \widehat{\mathcal{P}}_g^- .$$

Then:

Matsumoto-Montesinos' theorem *The map ρ is bijective.*

We summarize the key points of their proof [64]. For any $\phi \in \widehat{\mathcal{P}}_g^-$, they construct the degeneration whose monodromy coincides with ϕ in the following way. Among the isotopy class of ϕ, they choose an extremal element which is in some sense "the nearest" to a locally holomorphic map, and is called the superstandard form of ϕ. According to the nature of the superstandard form, they construct a certain topological covering map

$$\pi_\phi : \Sigma_g \longrightarrow V_\phi$$

form a Riemann surface Σ_g of genus g. V_ϕ is the topological underlying space of a Riemann surface with nodes, and a multiplicity is attached to each component of it. The map π_ϕ is generically finite-to-one, but some simple closed curves collapse to points. They call V_ϕ the *generalized quotient space* of ϕ, and call π_ϕ the generalized quotient map of ϕ.

They construct a fiber bundle of Riemann surfaces $f_\epsilon : \mathcal{V}_\epsilon \to S^1_\epsilon = \{t \in \mathbb{C} \mid |t| = \epsilon\}$ such that the monodromy of f_ϵ coincides with the superstandard form of ϕ and the "diameter of the vanishing cycle" tend to 0 if $\epsilon \mapsto 0$. They put the generalized quotient space V_ϕ over the origin, and associate the fiber space

$$f : S = \coprod_{0<\epsilon<1} \mathcal{V}_\epsilon \coprod V_\phi \longrightarrow \Delta = \coprod_{0<\epsilon<1} S^1_\epsilon \coprod \{0\}.$$

f naturally admit a complex structure and is the desired degeneration. Especially V_ϕ is nothing but the singular fiber of f (see also [18], [94]).

We discuss more closely around this theorem. First we assume that ϕ is a periodic map of order n. Let P be a point in Σ_g. There is a positive integer $\alpha(P)$ such that the points $P, \phi(P), \cdots, \phi^{\alpha(P)-1}(P)$ are mutually distinct and $\phi^{\alpha(P)}(P) = P$. We have $\alpha(P) = n$ for a generic point P. While if $\alpha(P) < n$, we call P a *multiple point* of ϕ.

Let C be an oriented simple closed curve on Σ_g. (We write it as $\vec{C}$ when we want to emphasize its orientation.) Let $m = m(C)$ be the smallest positive integer such that $\phi^m(\vec{C}) = \vec{C}$, i.e. $\phi^m(C) = C$ as a set and ϕ^m preserves the orientation of C. The restriction $\phi^m|_C$ is a periodic map of order, say, $\lambda \geq 1$. Note that $n = m\lambda$. Let Q be any point on C, and suppose that the images of Q under the iteration of ϕ^m are ordered as $(Q, \phi^{m\sigma}(Q), \phi^{2m\sigma}(Q), \ldots, \phi^{(\lambda-1)m\sigma}(Q))$ when we view in the direction of $\vec{C}$, where σ is an integer with $0 \leq \sigma \leq \lambda - 1$ and g.c.d$(\sigma, \lambda) = 1$. Let δ be the integer which satisfies

$$\sigma\delta \equiv 1 \pmod{\lambda}, \quad 0 \leq \delta \leq \lambda - 1.$$

Then the action of ϕ^m on $\vec{C}$ is topologically equivalent to the rotation of angle $2\pi\delta/\lambda$ with a suitable parameterization of $\vec{C}$ as an oriented circle. Nielsen [78] called the triple (m, λ, σ) the *valency* of $\vec{C}$ with respect to ϕ. The *valency of a multiple point* P is defined to be the valency of the boundary curve ∂D_P, oriented from the outside of an invariant disk neighborhood D_P of P.

Nielsen's theorem ([78], §11) says that the conjugacy class of a periodic map is completely determined by the order n and the *total valency*, i.e. the set of valencies of multiple points.

In order to determine the total valency, the following method is useful (cf. [9], §1). By Kerckhoff's theorem [50], we may assume that ϕ is an analytic automorphism under a certain complex structure. Let $\Pi : \Sigma_g \to \Sigma_{g'}$ be the quotient n-fold cyclic covering of ϕ, where g' is the genus of the base. We denote by $\lambda_1, \ldots, \lambda_l$ the ramification indices of Π and let $(n/\lambda_i, \lambda_i, \sigma_i)$ $(1 \leq i \leq l)$ be the valencies of ϕ of the ramification points. By combining the following facts, we can determine the total valency:

(i) (Hurwitz formula) $2(g-1)/n = 2(g'-1) + \Sigma_{i=1}^{l}(1 - 1/\lambda_i)$.
(ii) (Nielsen's integral condition [78], (4.6)) $\Sigma_{i=1}^{l}\sigma_i/\lambda_i$ is an integer.
(iii) (Wiman [95]) $n \leq 4g + 2$.
(iv) (Harvey [39]) Set $M = \text{l.c.m}(\lambda_1, \ldots, \lambda_l)$. Then;
(iv-a) $\text{l.c.m}(\lambda_1, \ldots, \widehat{\lambda_i}, \ldots, \lambda_l) = M$ for all i, where $\widehat{\lambda_i}$ means the omission of λ_i.
(iv-b) M divides n, and if $g' = 0$, $n = M$.
(iv-c) $l \neq 1$, and $l \geq 3$ if $g' = 0$.
(iv-d) If $2|M$, the number of $\lambda_1, \ldots, \lambda_l$ which are divisible by the maximal power of 2 dividing M is even.

Example 3.1. The classification of non-identical conjugacy classes of periodic maps of genus 1 and 2 is as follows. The result is classical for $g = 1$. We apply the above criterion for $g = 2$. For simplicity, if we have the total valency $(n/\lambda_i, \lambda_i, \sigma_i)$ $(1 \leq i \leq l)$, we symbolically write it as $\sigma_1/\lambda_1 + \ldots + \sigma_l/\lambda_l$. (We do not specify g' when $g' = 0$.)

(i) $g = 1$:

1. $n = 6; 1/6 + 1/3 + 1/2.\ 5/6 + 2/3 + 1/2.$
2. $n = 4; 1/4 + 1/4 + 1/2.\ 3/4 + 3/4 + 1/2.$
3. $n = 3; 1/3 + 1/3 + 1/3.\ 2/3 + 2/3 + 2/3.$
4. $n = 2; 1/2 + 1/2 + 1/2 + 1/2.$
5. $g' = 1$, n is arbitrary and Π is an unramified covering.

(ii) $g = 2$:

1. $n = 10; 1/10 + 2/5 + 1/2.\ 3/10 + 1/5 + 1/2.\ 7/10 + 4/5 + 1/2.\ 9/10 + 3/5 + 1/2.$
2. $n = 8; 1/8 + 3/8 + 1/2.\ 5/8 + 7/8 + 1/2.$
3. $n = 6; 1/6 + 1/6 + 2/3.\ 5/6 + 5/6 + 1/3.\ 1/3 + 2/3 + 1/2 + 1/2.$
4. $n = 5; 1/5 + 1/5 + 3/5.\ 1/5 + 2/5 + 2/5.\ 2/5 + 4/5 + 4/5.\ 3/5 + 3/5 + 4/5.$
5. $n = 4, 1/4 + 3/4 + 1/2 + 1/2.$
6. $n = 3, 1/3 + 1/3 + 2/3 + 2/3.$
7. $n = 2, 1/2 + 1/2 + 1/2 + 1/2 + 1/2 + 1/2.$
8. $g' = 1$, $n = 2$ and $1/2 + 1/2$.

According to [64], we associate to the valency (m, λ, σ) a sequence of integers $a_0 > a_1 > \cdots > a_l = 1$ such that

$$a_0 = \lambda, a_1 = \sigma, a_{j+1} + a_{j-1} \equiv 0 \pmod{a_j} \quad (0 \leq j \leq l-1).$$

We set $m_j = ma_j$, and call $m_1, \cdots, m_l$ the *multiplicity sequence* of the valency. Then the "tail part" of the generalized quotient space is the open disk with multiplicity m_0 followed by a tree of $\mathbb{P}^1$'s with multiplicity $m_1, \cdots, m_l$ meeting transversally (see [65], p.72, Figure 1).

In the case of periodic maps, we can construct the generalized quotient space V_ϕ by patching all the tail parts of the total valency to the "body part" $\Sigma_{g'}$.

By this method, we can easily construct all the degenerations of genus 1 and 2 with periodic monodromy ¿from the datum in Example 3.1. For instance, if $g = 1, n = 6$ and the total valency is $5/6 + 2/3 + 1/2$, then the multiplicity sequences are $(6,5,4,3,2,1)$, $(6,4,2)$ and $(6,3)$. We connect the three trees to a rational component of multiplicity 6. This induces the degeneration of type II^* in [51]. The case $g = 1, n = 6$ and $1/6 + 1/3 + 1/2$ corresponds to the normally minimal model of type II in [51]. The case $g = 2, n = 8$ and $5/8 + 7/8 + 1/2$ corresponds to type VII^* in [76], p.340, etc. The moduli points of these degenerations are smooth curves. (However we cannot determine the limit of the period matrix by this method, while it is possible by [76] for $g = 2$.)

Next we assume that ϕ is a reducible pseudo-periodic map. We identify ϕ with its representative ϕ' in Thurston's theorem. Let Γ_i be a simple closed curve of the admissible system Γ of ϕ. Let β_i the smallest positive integer such that $\phi^{\beta_i}(\vec{\Gamma}_i) = \vec{\Gamma}_i$, and let L_i be the smallest positive integer such that ϕ^{L_i} is an integral Dehn twist (say e_i-times). According to [78], we define the *screw number* of Γ_i by $s(\Gamma_i) = e_i\beta_i/\ L_i$. If β_i is even and $\phi^{\beta_i/2}(\vec{\Gamma}_i) = -\vec{\Gamma}_i$ (i.e. $\phi^{\beta_i/2}$ preserves Γ_i as a set but reverses its orientation), we call Γ_i *amphidrome*. Otherwise, we call it *non-amphidrome*.

Let B_k be a connected component of $\Sigma_g \setminus \Gamma$. Let b_k be the smallest positive integer such that ϕ^{b_k} stabilizes B_k. We call the periodic map $\phi^{b_k}|_{B_k}$ the *stabilized component map* of ϕ for B_k. Let $G(\Gamma)$ be the *extended partition graph* ([64]), i.e. the vertex v_{B_k} of $G(\Gamma)$ corresponds to a component B_k and the edge e_{Γ_j} corresponds to a curve Γ_j , and e_{Γ_j} is connected to v_{B_k} if Γ_j is a boundary of B_k. We apriori give an orientation on e_{Γ_j} in a suitable way. Note that, if Γ_j is amphidrome, then the action of $\phi^{\beta_i/2}$ to e_{Γ_j} changes its orientation.

Nielsen and Matsumoto-Montesinos' theorem ([79], [64]) says that the conjugacy class of a pseudo-periodic map is completely determined by

(i) the set of stabilized component maps,
(ii) the set of screw numbers,
(iii) the action of the map to the extended partition graph.

The generalized quotient space V_ϕ is constructed as follows: We first construct the "parts" of V_ϕ by the data (i), and then combine these parts by the datum (ii) and (iii).

We explain this process explicitly by the following simple examples (compare with [77], pp.158–169): Let Γ be a separated simple closed curve on a Riemann surface Σ_2 of genus 2. We write

$$\Sigma_2 = \Sigma^{(1)} \cup \mathcal{A}(\Gamma) \cup \Sigma^{(2)}$$

where $\Sigma^{(i)}$ $(i = 1, 2)$ is a torus with one disk boundary $\partial\Sigma^{(i)}$ and $\mathcal{A}(\Gamma)$ is an annulus around Γ. The graph $G(\Gamma)$ consists of two vertices $v_{\Sigma^{(1)}}$ and $v_{\Sigma^{(2)}}$ which are combined by an oriented edge e_Γ. The action of ϕ to $G(\Gamma)$ is either (a) the identity, or (b) of order 2 and is generated by the permutation $(v_{\Sigma^{(1)}}, v_{\Sigma^{(2)}})$ reversing the orientation of e_Γ.

Assume (a). We have a closed torus $\overline{\Sigma^{(i)}}$ by patching a disk U on $\Sigma^{(i)}$ along the boundary and have a periodic homeomorphism $\overline{\phi}^{(i)} : \overline{\Sigma^{(i)}} \longrightarrow \overline{\Sigma^{(i)}}$ which is an extension of the component map $\phi|_{\Sigma^{(i)}}$ $(i = 1, 2)$ so that $\overline{\phi}^{(i)}|_U$ has a unique multiple point. The total valency of $\overline{\phi}^{(i)}$ is one of the table in Example 3.1. For example, we assume that the total valency of $\overline{\phi}^{(1)}$ (resp. $\overline{\phi}^{(2)}$) is $5/6 + 2/3 + 1/2$ (resp. $2/3 + 2/3 + 2/3$). Then the valency of the curve $\partial\Sigma^{(1)} = \partial U$ (resp. $\partial\Sigma^{(2)} = \partial U$) is automatically $5/6$ (resp. $2/3$). The two parts of V_ϕ are degenerate elliptic curves of type II^* and of type IV^*. We patch them as follows: Put

$$K = -5/6 - 2/3 - s(\Gamma).$$

Note that the screw number $s(\Gamma)$ is non-positive since ϕ is of negative twist. Then K is an integer which satisfies $K \geq -1$ ([64]). If $K > 0$, we have the sequence $(6, 5, 4, 3, 2, 1, 1, \cdots, 1, 1, 2, 3)$ by adding the the multiplicity sequences of $5/6$ and $2/3$ to $(K - 1)$ times of 1. If $K = 0$, the sequence is $(6, 5, 4, 3, 2, 1, 2, 3)$. If $K = -1$, we shorten the original two sequences and get $(6, 5, 4, 3, 2, 3)$. We obtain V_ϕ by combining the above two parts along the tree of smooth rational curves which have the above multiplicity sequence. Therefore we have degenerations of type II^*-IV^*-m and of type II^*-IV^*-α in [77], p.164, 165.

Assume (b). Then $(\phi^2)_{\Sigma(i)}$ $(i = 1, 2)$ are the stabilized component maps. The total valencies of these maps coincide with each other. Suppose it is $3/4 + 3/4 + 1/2$, for instance. Then the valency of each boundary curve is automatically $3/4$. Since Γ is amphidrome, we use the "tail of type D", i.e. whose dual graph is a Dynkin diagram of type D (see [65], p.73, Figure 2). We obtain V_ϕ from the double multiple of the degenerate elliptic fiber of type III^* by adding the tail of type D. We consequently get the degeneration of type $2III^*$-m in [77], p.168. The other cases are similar.

Lemma 3.2. *Matsumoto-Montesinos' theorem induces the stable reduction theorem.*

Proof. (cf. [9], §4) Let ϕ_f be the monodromy of a given degeneration $f : S \to \Delta$. Since ϕ_f is pseudo-periodic, there exists a positive integer N such that ϕ_f^N is an integral Dehn twist along the annular neighborhood of the admissible system Γ_ϕ of cut curves of ϕ. Since all the component maps of ϕ_f^N are trivial, the generalized quotient space of ϕ_f^N is a semi-stable curve. Let $\widetilde{f} : \widetilde{S} \to \Delta$ be the desingularization of the pull back of f by the cyclic covering $\Delta \to \Delta$ given by $t \mapsto t^N$. Since the monodromy of $\widetilde{f}$ coincides with ϕ_f^N, $\widetilde{f}$ is a semi-stable family. Therefore we obtain the stable family by contracting chains of (-2) curves on $\widetilde{S}$. Q.E.D.

The proof especially says that the topological type of the stable curve which is the moduli point of f can be obtained by shrinking Γ_ϕ to ordinary double points. In other words, the dual graph of the stable curve of the moduli point coincides with $G(\Gamma)$.

We can classify degenerations of curves of genus g topologically, according to the following steps:

Step 1: Classify admissible systems Γ of cut curves on Σ_g, i.e. classify stable curves of genus g.

Step 2: Classify cyclic actions on $G(\Gamma)$, i.e. classify cyclic actions on the dual graph of the stable curve.

Step 3: Classify periodic maps of Riemann surfaces of genus $\leq g$ with disk boundaries, i.e. classify cyclic analytic automorphisms of pointed curves of genus $\leq g$.

Step 4: Classify pseudo-periodic maps, i.e. classify cyclic automorphisms of stable curves.

However, we must spend a lot of time in our life for the explicit calculations for large g. In the case of $g = 3$, we have:

Proposition 3.3 ([9]). *We can explicitly classify the singular fibers, the monodromies and the open strata which contain the moduli points (i.e. the topological structure of the stable curves corresponding to the moduli points) at the same time for $g = 3$.*

For the proof, we apply the above four steps to each stable curve of genus 3. We have 42 types of such curves in total.

Remarks. (1) If we restrict the map $\rho : \mathcal{S}_g \longrightarrow \widehat{\mathcal{P}}_g^-$ to a certain class of degenerations, what happens ? For instance, let $\mathcal{SH}_g$ be the set of topological equivalence classes of hyperelliptic degenerations (i.e. degenerations whose general fibers are hyperelliptic curves). Let $\mathcal{H}_g$ be the conjugacy classes of the hyperelliptic mapping class group of genus g, i.e. consisting of elements of Γ_g which commute with the hyperelliptic involution. Then we have a natural map

$$\rho' : \mathcal{SH}_g \longrightarrow \mathcal{H}_g \cap \widehat{\mathcal{P}}_g^-.$$

Endo's question asks whether ρ' is bijective. Since Ishizaka [48] has classified hyperelliptic degenerations of genus 3, we can expect an answer at least in this case.

(2) Historically Nielsen [79] first studied pseudo-periodic maps under the name "surface transformation class of algebraically finite type". From the reconstruction of Thurston's theorem by Bers [17], function theorists usually call a non-periodic pseudo-periodic map (resp. periodic map) a map of parabolic type (resp. map of elliptic type).

Contrary to the local monodromy case, if we consider the *global* monodromy of a 1-parameter degeneration, there really appear various classes in the above classification of Γ_g ([47], [84]). The systematic study in this field seems to be untouched and will be interesting.

§4. Morsification of degenerations

4.1. Definition of Morsifications

(A) Classical Morsification. Let Δ, Δ' be sufficiently small open disks. Let $f\colon S \longrightarrow \Delta$ be a degeneration of curves of genus g with a unique singular fiber $F = f^{-1}(0)$. Assume that there exist a smooth threefold M and a holomorphic map $\mathbf{f}\colon M \longrightarrow \Delta \times \Delta'$ such that the restriction $\mathbf{f}_u\colon M_u \longrightarrow \Delta \times \{u\}$ of $\mathbf{f}$ over $u \in \Delta$ satisfies the following:

(i) $\mathbf{f}_0$ coincides with f,
(ii) $\mathbf{f}_u$ $(u \neq 0)$ has $l \geq 1$ singular fiber germs $F_{u,1}, \ldots, F_{u,l}$. l is independent on u.

Then we call $\mathbf{f}$ a *splitting family* of the germ F, and we symbolically write it as

$$F \longrightarrow F_{u,1} + \cdots + F_{u,l}.$$

Moreover, if $l \geq 2$, we call $\mathbf{f}$ a *proper* splitting family of F. If $l = 1$, we call $\mathbf{f}$ an *equisingular deformation* of F. If F has no proper splitting families even after equisingular deformations, then we call F an *atomic fiber*. We say that the germ F is *morsified* to the set of germs $\mathcal{S}_F = \{F^{(1)}, \ldots, F^{(k)}\}$ if:

(a) each $F^{(i)}$ $(1 \leq i \leq k)$ is an atomic fiber,
(b) F is decomposed into the members of $\mathcal{S}_F$ via several splitting families.

Namely, if F is not atomic, F has a splitting family $F \longrightarrow F_{u,1} + \cdots + F_{u,l}$. If one of the elements of $\{F_{u,1}, \ldots, F_{u,l}\}$ does not belong to $\mathcal{S}_F$, say $F_{u,i_0} \notin \mathcal{S}_F$, then F_{u,i_0} has a splitting family $F_{u,i_0} \longrightarrow F'_{v,1} + \cdots + F'_{v,l'}$ (It is possible that F_{u,i_0} has an equisingular deformation $F_{u,i_0} \longrightarrow \widetilde{F_{u,i_0}}$ and then has a proper splitting family $\widetilde{F_{u,i_0}} \longrightarrow F'_{v,1} + \cdots + F'_{v,l'}$). If F'_{v,j_0} $(1 \leq j_0 \leq l')$ does not belong to $\mathcal{S}_F$, then F'_{v,j_0} has a splitting family. Continue this process. After finite steps, all the terminated fiber germs belong to $\mathcal{S}_F$.

If F is morsified to $\mathcal{S}_F$ via only one splitting family, we say that F is *directly morsified* to $\mathcal{S}_F$. The set of atomic fibers $\mathcal{S} = \{\widetilde{F}^{(1)}, \ldots, \widetilde{F}^{(\tilde{l})}\}$ is called the *complete system of atomic fibers of genus* g if any fiber germ of genus g is morsified to a subset of $\mathcal{S}$.

Now, Xiao Gang's morsification conjecture [83] can be stated as:

Morsification Conjecture. Any atomic fiber is either a fiber germ with only one Morse critical point or a multiple of a smooth curve.

(B) Algebraic Morsification. We consider a certain condition (*) for smooth curves as in §2.1 with respect to which the Horikawa index is well-defined. Let $f\colon S \longrightarrow \Delta$ be a (*)-degeneration with the unique critical or singular fiber $F = f^{-1}(0)$. Assume that there exist a smooth threefold M and a holomorphic map $\mathbf{f}\colon M \longrightarrow \Delta \times \Delta'$ such that

(i) $\mathbf{f}_0$ coincides with f,
(ii) $\mathbf{f}_u$ $(u \neq 0)$ is a (*)-degeneration and has $l \geq 1$ critical or singular fibers $F_{u,1}, \ldots, F_{u,l}$, where l is independent on u,
(iii) (conservation of index) $\mathrm{Ind}(F) = \sum_{i=1}^{l} \mathrm{Ind}(F_{u,i})$.

Then we call **f** a (*)-splitting family of F. In a similar way, we define (*)-atomic fiber, complete system of (*)-atomic fibers and so on.

Question. Does the condition (iii) follow from (i) and (ii) ?

(C) Topological Morsification. Let $\overline{f}\colon \overline{S} \longrightarrow \overline{\Delta}$ be a holomorphic fibration between a complex surface $\overline{S}$ with boundary and a closed disk $\overline{\Delta} = \{t \in \mathbf{C} |\ |t| \leq \varepsilon\}$, i.e., $\overline{f}$ is the restriction of a certain holomorphic fibration $f^{(0)}\colon S^{(0)} \longrightarrow \Delta^{(0)} = \{t \in \mathbf{C} |\ |t| < \varepsilon + \varepsilon'\}$ to $\overline{\Delta} \subset \Delta^{(0)}$. Assume that $F = f^{-1}(0)$ is the unique singular fiber.

Suppose that there exists another holomorphic fibration $\overline{f'}\colon \overline{S'} \longrightarrow \overline{\Delta}$ with singular fibers $F'_1, \ldots, F'_l$ which satisfies the following: There exists an orientation-preserving diffeomorphism $\mathcal{H}\colon \overline{S} \longrightarrow \overline{S'}$ which commutes with $\overline{f}$ and $\overline{f'}$ on the boundary, i.e., $(\overline{f'}|_{\partial \overline{S'}}) \circ \mathcal{H}|_{\partial \overline{S}} = \overline{F}|_{\partial S}$. Then we say that F splits differentiably into F'_1, $ldots, F'_l$ ([63], Definition 3.4) and write $F \overset{C^\infty}{\longrightarrow} F'_1 + \cdots + F'_l$. Based on this notion, we can similarly define C^∞-atomic fibers, C^∞-complete system of atomic fibers and so on.

Lemma 4.1 ([3], §1). *A classical splitting $F \longrightarrow F_{u,1} + \cdots + F_{u,l}$ naturally induces a C^∞-splitting $F \overset{C^\infty}{\longrightarrow} F_{u,1} + \cdots + F_{u,l}$ whenever $|u|$ is sufficiently small.*

The proof is an analogue of the C^∞ local triviality of the deformation space of complex manifolds without boundary.

Historical remark. As to the classical Morsification, Moishezon [70] studied elliptic fibrations and proved that any Kodaira singular fiber is directly morsified to either:

(a) an irreducible stable curve with one node, or
(b) a multiple of a smooth elliptic curve.

For the study of elliptic fibrations over multi-dimensional base via minimal model theory, see Nakayama [75].

As to the algebraic morsification, a systematic study has been done only for hyperelliptic degenerations. For $g = 2$, Horikawa [43] proved that, modulo fibers with Ind $= 0$, any critical fiber is directly morsified to a several number of a unique atomic fiber, i. e., the stable curve with two components with one node. We remark that this can be also shown by deforming the canonical algebra, since it is a complete intersection [67]. We will discuss further developements for hyperelliptic fibrations in §4.2. For non-hyperelliptic degenerations of genus three, we showed in Appendix of [57] that, modulo fibers with Ind $= 0$, any 2-connected

critical fiber is morsified to (smooth) hyperelliptic fibers of Horikawa index one, by deforming the canonical algebra. Similar calculations may be possible by using results in [67]. But, the following conjecture still remains open:

Conjecture ([83]). *The complete system of atomic fibers of non-hyperelliptic degenerations of genus three consists of*

(a) *an irreducible stable curve with one node,* $\mathrm{Ind} = 0$,
(b) *a smooth hyperelliptic curve,* $\mathrm{Ind} = 1$,
(c) *a stable curve with two components with one node,* $\mathrm{Ind} = 2$,
(d) *a double multiple of a smooth curve of genus two,* $\mathrm{Ind} = 3$.

As to the topological morsification, topologists consider its analogue for a wider class of fibrations, that is, "locally analytic fibrations" (cf. [63]). For instance, a good torus fibration ([62]) is such a generalized object of an elliptic fibration, and Matsumoto [62] and Ue [93] studied the Morsification problem for good torus fibrations.

4.2. Hyperelliptic splitting families

Here, according to [2] and [3], we construct some hyperelliptic splitting families and determine the complete system of hyperelliptic atomic fibers of genus three.

We set the condition (*) in §2.1 to be hyperelliptic. Let $f\colon S \to \Delta$ be a hyperelliptic degeneration of genus g. Then a (*)-critical fiber of f is necessarily a singular fiber. In fact, hyperelliptic curves do not specialize to a non- hyperelliptic curve via a deformation by the closedness of hyperelliptic locus in $\mathcal{M}_g$. Moreover, the notion of algebraic splitting family is almost equivalent to the notion of classical splitting family by the following:

Lemma 4.2 ([3], §1). *If a hyperelliptic singular fiber germ F splits as $F \longrightarrow F_{u,1} + \cdots + F_{u,l}$, then*

$$\mathrm{Ind}(F) = \sum_{i=1}^{l} \mathrm{Ind}(F_{u,1}).$$

Proof. By (2.1.3) with $\lambda = 4(g-1)/g$, we have

$$\mathrm{Ind}(F) = \frac{2g+1}{g}\sigma(F) + \frac{g+1}{g}e_f(F).$$

On the other hand, Terasoma [89] shows that

$$\sigma(F) = \sigma_{top}(F),$$

where $\sigma_{top}(F)$ is the topological local signature of Endo citeEndo. It follows from Lemma 4.1 that

$$\sigma_{top}(F) = \sum_{i=1}^{l} \sigma_{top}(F_{u,i}), \quad e_f(F) = \sum_{i=1}^{l} e_f(F_{u,i}).$$

Therefore, we obtain the assertion. Q.E.D.

Let $f\colon S \to \Delta$ be a hyperelliptic degeneration of genus g with the unique singular fiber $F = f^{-1}(0)$. As we explained in §2.2, f is determined by the branch divisor R on $W = \mathbb{P}^1 \times \Delta$. Let $\pi\colon W \to \Delta$ be the natural projection. If a point $P \in R$ is a singular point of R or R has a component which is tangential to the fiber of π at P, we call P a *bad point* of R. Since f is smooth over $\Delta \setminus \{0\}$ by the assumption, bad points of R are on $\Gamma_0 = \pi^{-1}(0)$.

Now one of our main methods constructing splitting families of f is to find a reduced divisor $\mathbf{R}$ on $\mathbf{W} = \mathbb{P}^1 \times \Delta \times \Delta'$ such that the restriction $\mathbf{R}_u = \mathbf{R}|_{\mathbb{P}^1 \times \Delta \times \{u\}}$ over $u \in \Delta'$ satisfies the following:

(a) $\mathbf{R}_0$ coincides with the original R,

(b) The bad points of $\mathbf{R}_u$ $(u \neq 0)$ are on several fibers of the projection

$$\pi_u\colon \mathbb{P}^1 \times \Delta \times \{u\} \longrightarrow \Delta \times \{u\},$$

(c) Let $\phi\colon M' \to \mathbf{W}$ be the singular double cover branched along $\mathbf{R}$ in the total space of the square root bundle of $[\mathbf{R}]$. The natural map $f'\colon M' \to \Delta$ ^{prime} induces a deformation of normal surface singularities. Then f' has a simultaneous resolution (after a base change if necessary), which gives us a splitting family.

We describe a sufficient condition for (c). Let $m_1, \ldots, m_n$ be the multiplicity sequence of the even resolution (2.2.3) of R. We re-order the sequence $\{[m_i/2]\}$ according to the usual ordering $\geq$ to get a number sequence $\{h_i\}_{i=1}^n$, $h_1 \geq \cdots$ $geqh_n$. We put

$$HM(R) := \{h_i\}_{i=1}^n.$$

and call it the *half multiplicity sequence* of R. The half multiplicity sequence $HM(\mathbf{R}_u)$ of $\mathbf{R}_u$ is defined in the same way. If we have

(c′) $\quad HM(R) = HM(\mathbf{R}_u)$

for any $u \in \Delta'$, then the summation of the geometric genus of singularities on the fiber of f' is preserved via the deformation by the formula (2.2.4). Therefore the condition (c′) implies the condition (c) by Laufer's theorem [60].

In order to find $\mathbf{R}$ satisfying (a), (b) and (c′), we use the "perturbation method on the way of its resolution process". We explain it by using the following example. (See also Example 3.6 and Figure 2 in [2].)

Let (x,t) be the system of coordinates of W, where x is the inhomogeneous coordinate of $\mathbb{P}^1$. Let R be the divisor on W defined by

$$(x^3 + t^{6n})(x-1)(x-2)(x-3) = 0.$$

Then R has the unique bad point $Q = \{x = t = 0\}$, and we have $\Gamma_0 R = 6$. The $2n$-th branch divisor R_{2n} by the composition of blowups $\sigma_{2n} \circ \cdots \sigma_1 \colon W_{2n} \longrightarrow W$ with their centers are infinitely near to Q is nonsingular. We have $HM(R) = \{2, \ldots, 2, 1, \ldots, 1\}$ (1,2 appears n-times). The double cover $\widetilde{S}$ of W_{2n} branched along R_{2n} is nonsingular and the natural fibration $\widetilde{f} \colon \widetilde{S} \longrightarrow \Delta$ has n (-1)-curves on $\widetilde{f}^{-1}(0)$. By contracting them, we get a semi-stable degeneration of genus two whose singular fiber $F = f^{-1}(0)$ consists of two smooth elliptic curves and n (-2)-curves. We have $\mathrm{Ind}(F) = n$ by (2.2.7). Horikawa [41] called F a singular fiber *of type* rmI_n.

Now let $R_{2k} \subset W_{2k}$ $(1 \leq k \leq n-1)$ be the $2k$-th branch divisor on the way of its resolution process. R_{2k} has an unique singular point Q_{2k} on the exceptional curve E_{2k} by σ_{2k}, and the equation of R_{2k} at Q_{2k} is written as $(x')^3 + t^{6(n-k)} = 0$ with respect to the natural local coordinates on a neighborhood U around $Q_{2k} \in W_{2k}$.

We can define the divisor $\widetilde{\mathbf{R}}$ on $W_{2k} \times \Delta'$ such that the equation of $\widetilde{\mathbf{R}}$ in $U \times \Delta'$, with coordinates (x', t, u), is written as

$$(x')^3 + (t-u)^{6(n-k)} = 0.$$

We resolve the singularities on the singular double cover branched along $\widetilde{\mathbf{R}}$ simultaneously with respect to the base Δ' by the relative version of the canonical resolution. Geometrically, this process is essentially the same as the following:

Let $\mathbf{R}$ be the image of $\widetilde{\mathbf{R}}$ by the contraction map $W_{2k} \times \Delta' \longrightarrow W \times \Delta'$. Then the restriction $\mathbf{R}_u$ of $\mathbf{R}$ to $W \times \{u\}$ $(u \neq 0)$ has two bad points Q' and Q'' on $\pi^{-1}(0)$ and $\pi^{-1}(u)$, respectively. The typical equation of $\mathbf{R}$ is

$$(x^3 + t^{6k})(x^3 + (t-u)^{6(n-k)})(x-1)(x-2)(x-3) = 0.$$

We have

$$HM(\mathbf{R}_u) = \{2, \ldots, 2, 1, \ldots, 1\} = HM(R).$$

Note that the contributions to $HM(\mathbf{R}_u)$ $(u \neq 0)$ come from both Q' and Q'', while the contributions to $HM(R)$ come from Q only. Therefore, $\mathbf{R}$ satisfies the desired conditions (a), (b) and (c'). The simultaneous resolution space induces the splitting family of type I_n into type I_k and type I_{n-k}.

Of course, we can define $\mathbf{R}$ directly by the above equation in this special case. However, this argument, called a *fission* in [2], is useful for constructing more complicated hyperelliptic splitting families.

Remark. The idea of the perturbation method on the way of its resolution process is originally used by A'Campo [1] for studying the Morsification of plane curve singularities. It is also applied to study the signature of the Milnor fiber of a type of normal surface singularities [8].

Assume that two reduced (possibly reducible) curves C_1 and C_2 on a nonsingular surface meet at a point Q. For a positive integer n, we say that C_1 is *n-tangential* to C_2 at Q if the following condition holds: Blow-up n-times successively at infinitely near points of Q. Then there exist a locally analytic component C_1' of C_1 at Q and a locally analytic component C_2' of C_2 at Q so that the proper transforms of C_1' and C_2' by the composition of these blow-ups still meet each other. Namely, C_1' contacts C_2' at Q of order at least n. We say that C_1 is n-tangential to C_2 if there exists a point Q such that C_1 is n-tangential to C_2 at Q. Let (W, R) be a representative of the branch curve of a germ F. If R contains Γ_0, then we set $R_{hor} := R - \Gamma_0$, and otherwise we set $R_{hor} := R$. We call R_{hor} the horizontal part of R.

We call a representative (W, R) *trivial* if one of the following two conditions is satisfied:
(i) R_{hor} intersects Γ_0 at (just) one point Q_1 such that R_{hor} is not tangential to Γ_0 at Q_1, or
(ii) R_{hor} intersects Γ_0 at (just) two points Q_1 and Q_2 such that R_{hor} is smooth at Q_1 meeting transversally to Γ_0, and R_{hor} is not tangential to Γ_0 at Q_2.

Otherwise, we call (W, R) a *non-trivial* representative. A trivial representaive (W, R) of F becomes a non-trivial representative after a finite number of elementary transformations.

With this notation, we introduce some special classes of singular fiber germs.

(i) Type 0_0: R is smooth on W and meets Γ_0 transversally except at one point P where the order of contact is two. Then the corresponding

singular fiber F is an irreducible stable curve with one node. If a representative of the branch curve (W, R) has the above property, we call the fiber germ F a germ of type 0_0.

(ii) Class I: R does not contain Γ_0 and meets Γ_0 transversally except at one point P which is an ordinary singularity of R of multiplicity $2g'+2$, where g' is an integer with $1 \leq g' \leq [(g-1)/2]$. Then F is a stable curve of two components of genera g' and $g-g'-1$ with two nodes. If a representative of the branch curve (W, R) has the above property, we say that the fiber germ F belongs to class I. In fact, this class contains $[(g-1)/2]$ types.

(iii) Class II: R contains Γ_0. We can produce a minimal succession of blow-ups (2.2.3) such that any singularity of R_n is ordinary. Denote by $\mathbf{E}_j$ $(1 \leq j \leq n)$ the total transform of Γ_0 by $\sigma_j \circ \sigma_{j-1} \circ \cdots \circ \sigma_1$. Then the following conditions (a) through (e) hold:

(a) R_n contains $\mathbf{E}_n$,
(b) Any singularity of R_n has even multiplicity,
(c) For any $0 \leq i \leq n$, any singularity of R_i with even multiplicity is ordinary,
(d) If R is not 1-tangential to Γ_0 at P, then the following condition hold: Blow-up at P. Then $R_1 - \mathbf{E}_1$ is 3-tangential to $\mathbf{E}_1$,
(e) If R has an ordinary double point, then R_{hor} is 3-tangential to Γ_0.

We say that a germ F belongs to class II if at least one nontrivial representative (W, R) of the branch curve satisfies $R \supset \Gamma_0$ and moreover any nontrivial representative (W, R) of the branch curve with $R \supset \Gamma_0$ has the above conditions (a) through (e).

Proposition 4.3 ([2]). *Any hyperelliptic singular fiber germ F splits into the germs of members of type 0_0, class I and class II via several proper splitting families.*

Moreover, if $\mathrm{Ind}(F) = 0$, *then F splits into several germs of type* 0_0.

We use several variations of the method of the fission to prove it. We do not know whether class II fibers have further proper hyperelliptic splitting families or not. However, if we admit equisingular deformations, some of them really split as follows.

From now, we consider the case $g = 3$. We have the following list of fibers of type 0_0, class I and class II (class II has three types II(i), II(ii) and II(iii)):

Type 0_0: F is an irreducible stable curve with one node.

Type I: F is a stable curve with two smooth elliptic components with two nodes.

Type II(i): $R = R_{hor} + \Gamma_0$ has two bad points Q_i $(i = 1, 2)$. Q_1 (resp. Q_2) is an ordinary singularity of multiplicity four (resp. six). F is a stable curve consisting of a smooth elliptic component and a smooth genus two component meeting at a point.

Type II(ii): $R = R_{hor} + \Gamma_0$ has two bad points Q_i $(i = 1, 2)$. R_{hor} has two smooth local components at Q_i such that each of them contacts to Γ_0 to the second order. We can write F as $F = 2C_1 + 2C_2 + 2C_3$, where C_1 and C_3 are smooth elliptic curves and C_2 is a smooth rational curve with $C_1C_2 = C_2C_3 = 1$, $C_1^2 = C_3^2 = -1$ and $C_2^2 = -2$.

Type II(iii): $R = R_{hor} + \Gamma_0$ has a unique bad point Q. R_{hor} has four smooth local components at Q such that each of them contacts to Γ_0 to the second order. F is a double multiple of a smooth curve of genus two.

The Horikawa indices and the Euler contributions of these germs are:

	0_0	I	II (i)	II (ii)	II (iii)
Ind	0	2/3	5/3	10/3	8/3
e_f	1	2	1	4	2

If there were a proper splitting family of a germ in these classes, the possible case would be the following (a) or (b) by the conservation of invariants:

(a) II(ii) $\longrightarrow 0_0 + 0_0 + \text{II(i)} + \text{II(i)}$
(b) II(ii) $\longrightarrow$ I + II(iii)

We can show that the case (a) is impossible by a certain monodromy argument, while (b) actually occurs.

Lemma 4.4. *There exists a type* II(ii) *fiber germ F such that F splits into type* I *and type* II(iii).

We have two proofs in [3], one of which is as follows. We first construct a hyperelliptic stable family $\widetilde{\mathbf{f}} : \widetilde{M} \longrightarrow \Delta \times \Delta'$ of genus three (where $\widetilde{M}$ has singularities) such that $\widetilde{\mathbf{f}}_u : \widetilde{M}_u \longrightarrow \Delta \times \{u\}$ satisfies the following:

(a) $\mathbf{f}_0$ has a unique singular fiber $\mathbf{f}_0^{-1}(0)$ of type I so that two nodes of $\mathbf{f}_0^{-1}(0)$ are A_1-singularities on $\widetilde{M}$,
(b) $\mathbf{f}_u$ $(u \neq 0)$ has two singular fibers $\mathbf{f}_u^{-1}(u)$ and $\mathbf{f}_u^{-1}(-u)$ of type I so that the nodes of them are nonsingular points on $\widetilde{M}$,

(c) there exists an involution $\iota\colon \widetilde{M} \longrightarrow \widetilde{M}$ fiber-wise with respect to the base Δ' so that τ sends $\mathbf{f}_u^{-1}(u)$ isomorphically onto $\mathbf{f}_u^{-1}(-u)$ and τ acts on $\mathbf{f}_u^{-1}(0)$ as a fixed point free automorphism.

Then the resolution space of the quotient space $\widetilde{M}/\langle\iota\rangle$ induces a desired splitting family. Since we can connect any type II(ii) fiber germ and the fiber germ F in Lemma 4.4 via equisingular deformations, we get:

Proposition 4.5 ([3]). *The complete system of hyperelliptic atomic fibers of genus three are of type* 0_0, I, II(i) *and* II(iii).

4.3. Questions and remarks

Morsification of fiber germs may be a new mathematical area involving algebraic geometry, low-dimensional topology, Teichmüller theory and so on. Many natural and fundamental questions are not settled, or, even, are not under consideration. Here we pick up some of them and give comments.

1. Direct Morsification. If a germ F has a splitting $F \longrightarrow F_{u,1} + F_{u,2} + \cdots$ and each $F_{u,i}$ $(i = 1, 2, \dots)$ has a splitting $F_{u,i} \longrightarrow F'_{u,i,1} + F'_{u,i,2} +$ *cdots*. Then is there a family which realizes $F \longrightarrow F'_{u,1,1} + F'_{u,1,2} + \cdots F'_{u,2,1} + F'_{u,2,2} + \cdots$ directly ? If one has a complete system of atomic fibers in a certain category, then does a germ F in the category have a direct Morsification to atomic fibers ?

2. General construction of splitting families. As to the classical Morsification, Takamura began his pioneering work [87]. The main method of [87], Parts II and III, is as follows: He first reconstructs Matsumoto-Montesinos' families and shows that we can choose a "linear degeneration" as a representative of the topological equivalence class of a given germ F. The linear degeneration is constructed as a hypersurface of a certain explicit ambient threefold $X \supset S \xrightarrow{f} \Delta$. X is the "plumbing space" of the normal bundles of components of F, and is a generalization of Hirzebruch-Jung string in some sense. Terasoma's theorem [90] says that if two degenerations f_1 and f_2 are topologically equivalent, then we can connect f_1 and f_2 via several equisingular deformations. Therefore, the linear degeneration is also a representative of the equisingular deformation class of F.

Now he constructs explicitly relative deformations $\{f_u\colon X_u \supset S_u \longrightarrow \Delta\}_{u\in\Delta'}$ of f so that f_u $(u \neq 0)$ has several singular fibers. If we had $X_u \simeq X$, then the deformation would behave as an infinitesimal displacement in X. But X really has a jumping deformation similarly as

in the case of rational normal scrolls ([37]), and the constructions of $\{f_u\}_{u\in\Delta'}$ become delicate.

Takamura announces that he can obtain the complete systems of classical atomic fibers of genus $g \leq 5$ by this method.

As to the algebraic Morsification, our knowledge is very poor except for hyperelliptic degenerations. We expect that pioneers will come up soon. Another problem to be considered is the Morsification problem for the base locus of the canonical linear system of singular fiber germs. We imagine that $\mathrm{Bs}|K_F|$ of an atomic fiber F is simple and K_F is ample. The last assertion means that (-2)-curves will disappear along Morsification, while it is impossible under deformations of " global" surfaces.

3. Versal family. Can we describe a versal family of a given fiber germ F ? Moreover, can we prove the existence of the Morsification via the versal family ?

This question comes from an analogy of the well-known Morse theory on hypersurface singularities (e.g., [5]). For the readers' convenience, we review the argument.

We consider germs of n-dimensional isolated hypersurface singularities. Among them, a singularity with Milnor number $\mu = 1$ is called an A_1-singularity. A typical equation is $x_0^2 + x_1^2 + \cdots + x_n^2 = 0$. An A_1-singularity is sometimes called a *Morse singularity*, because:

Any isolated hypersurface singularity $(V, \mathbf{0})$ *with Milnor number* $\mu(V, \mathbf{0}) = \mu$ *is directly morsified to* μ A_1*-singularities.*

Indeed, let $x = (x_0, x_1, \ldots, x_n)$ be the system of coordinates on a domain $U \subset \mathbb{C}^{n+1}$ containing the origin $\mathbf{0}$. Let $f(x_0, x_1, \ldots, x_n) = 0$ be the equation in U of the germ $(V, \mathbf{0})$. The Kuranishi space of $(V, \mathbf{0})$ is unobstructed and coincides with

$$\mathrm{Ext}^1(\Omega^1_V, \mathcal{O}_V) \simeq J_f = \mathbb{C}\{x_0, x_1, \ldots, x_n\}/I,$$

where $I = (f, \partial f/\partial x_0, \ldots, \partial f/\partial x_n)$. Since $\dim_{\mathbb{C}} J_f = \mu(V, \mathbf{0}) = \mu$, we identify J_f with $\mathbb{C}^\mu$. Let

$$\widetilde{F} = (\widetilde{f}, \widetilde{\pi}) \colon U \times \mathbb{C}^\mu \longrightarrow \mathbb{C} \times \mathbb{C}$$

be the versal family or the universal unfolding of f ([91]). Namely, if $u = (u_1, \ldots, u_\mu)$ is the system of coordinates of $mathbbC^\mu$ and $\widetilde{f} = \widetilde{f}(x, u)$, then $\{\partial \widetilde{f}/\partial u_j|_{u=0}\}_{j=1}^{l}$ generates J_f. If we fix $u \in \mathbb{C}^\mu$, then the map $\widetilde{f}(x, u) \colon U \times \{u\} \to \mathbb{C}$ $times\{u\}$ is a $\mu : 1$ finite map for generic u. We define the bifurcation locus $\mathrm{Bif}(\widetilde{f})$ of $\widetilde{f}$ as the subset of $\mathbb{C}^\mu$ consisting of the elements u such that the map $\widetilde{f}(x, u)$ is not $\mu : 1$. By [61], $\mathrm{Bif}(\widetilde{f})$

is a proper analytic subset of $\mathbb{C}^\mu$. Let Δ be a local curve on $\mathbb{C}^\mu$ passing through the origin $\mathbf{0}$ such that $\Delta \bigcap \mathrm{Bif}(\widetilde{f}) = \{\mathbf{0}\}$. We restrict the versal family $\widetilde{F}$ on Δ. Then we have the splitting family which includes the direct Morsification of $(V, \mathbf{0})$.

4. Maximal molecular fiber. If we have a proper splitting $F' \longrightarrow F'_{u,1} + \cdots + F'^{prime}_{u,l}$, we say that F'_{u,i_0} $(1 \leq i_0 \leq l)$ is a *proper deformation* of F'. If a germ F cannot be a proper deformation of any germ F' of the same genus, we call F a maximal molecular fiber.

Can we prove the existence of maximal molecular fibers and moreover classify them ?

As an analogy in singularity theory, we consider deformations of a germ (V, P) of a rational double point. Let $G(V, P)$ be the Dynkin diagram of (V, P), that is, the dual graph of the exceptional divisor on its minimal resolution. Then it is well-known that $G(V, P)$ is one of types A_n, D_n, E_6, E_7 and E_8, and that (V, P) is a deformation of another rational double point (V', P') if and only if $G(V, P)$ is a sub-Dynkin diagram of $G(V', P')$ ([4], [21]). Therefore E_8-singularity is the only "maximal molecular singularity" in the category of rational double points.

If we determine the complete systems of atomic fibers of genus g, then we may ask for the classification of maximal molecular fibers of genus g. Especially, the versal family of a maximal molecular fiber seems to be interesting.

5. Monodromy realizable relation. Let $\mathbf{f}' \colon M' \to \Delta'' \times \Delta'''$ be a proper splitting family of genus g of a germ F, and let $\mathcal{D}$ be the discriminant locus of $\mathbf{f}'$. We choose a closed polydisk $\overline{\Delta} \times \overline{\Delta'} \subset \Delta'' \times \Delta'''$ which satisfies $(\partial\overline{\Delta} \times \overline{\Delta'}) \cap \mathcal{D} = \emptyset$, where $\partial\overline{\Delta}$ is the boundary circle of $\overline{\Delta}$. Let $\mathbf{f} \colon \overline{M} \longrightarrow \overline{\Delta} \times \overline{\Delta'}$ be the restriction of $\mathbf{f}'$ over $\overline{\Delta} \times \overline{\Delta'}$. We fix a continuous section $\lambda \colon \overline{\Delta'} \longrightarrow \partial\overline{\Delta} \times \overline{\Delta'}$ with $pr_2 \circ \lambda = id_{\overline{\Delta'}}$. We identify the circle $\partial\overline{\Delta} \times \{u\}$ $(u \in \overline{\Delta'})$ with the oriented simple closed curve $\vec{\partial}_u$ which starts from $\lambda(u)$, goes in counter-clockwise direction and ends with $\lambda(u)$.

We fix $u_0 \in \overline{\Delta'} \backslash \{0\}$. We identify the image $\lambda([0, u_0])$ with the oriented path $\vec{\lambda}$ on $\overline{\Delta} \times \overline{\Delta'}$ which starts from $\lambda(0)$ and ends with $\lambda(u)$. Then two paths $\vec{\partial}_{u_0}$ and $\vec{\lambda} \circ \vec{\partial}_0 \circ \vec{\lambda}^{-1}$ are mutually homotopic on $(\overline{\Delta} \times \overline{\Delta'}) \setminus \mathcal{D}$.

Let $\mathbf{f}_{u_0}^{-1}(d_1), \cdots, \mathbf{f}_{u_0}^{-1}(d_l)$ be the set of all singular fibers of $\mathbf{f}_{u_0} \colon M_{u_0} \to \overline{\Delta} \times \{u_0\}$, i.e. $\mathcal{D} \cap (\overline{\Delta} \times \{u_0\}) = \{d_1, \cdots, d_l\}$. Let $\overline{U}_j$ $(1 \leq j \leq l)$ be a closed connected simple region on $\overline{\Delta} \times \{u_0\}$ which satisfies

(i) $\overline{U}_j \cap \mathcal{D} = \{d_j\}$,
(ii) the boundary $\partial\overline{U}_j$ does not contain d_j,
(iii) $\lambda(u_0) \in \partial\overline{U}_j$.

We identify $\partial\overline{U}_j$ with the oriented simple closed curve $\vec{\gamma}_j$ which starts from $\lambda(u_0)$, goes in counter-clockwise direction and ends with $\lambda(u_0)$. We change the order of $\vec{\gamma}_1, \cdots, \vec{\gamma}_l$ if necessary, then two paths $\vec{\partial}_{u_0}$ and $\vec{\gamma}_l \circ \cdots \circ \vec{\gamma}_1$ are mutually homotopic on $\overline{\Delta} \times \{u_0\} \setminus (\mathcal{D} \cap (\overline{\Delta} \times \{u_0\}))$.

Let $\phi_j \colon \mathbf{f}_{u_0}^{-1}(\lambda(u_0)) \longrightarrow \mathbf{f}_{u_0}^{-1}(\lambda(u_0))$ $(1 \leq j \leq l)$ be the monodromy diffeomorphism of the closed Riemann surface $\mathbf{f}_{u_0}^{-1}(\lambda(u_0))$ of genus g according to the path $\vec{\gamma}_j$. Similarly, let $\phi_0 \colon \mathbf{f}_0^{-1}(\lambda(0)) \longrightarrow \mathbf{f}_0^{-1}(\lambda(0))$ be the diffeomorphism according to $\vec{\partial}_0$, and let $\psi_\lambda \colon \mathbf{f}_{u_0}^{-1}(\lambda(0)) \longrightarrow \mathbf{f}_{u_0}^{-1}(\lambda(u_0))$ be the diffeomorphism according to $\vec{\lambda}$. If we identify these diffeomorphisms with their isotopy classes, we have a relation

$$(**) \qquad \psi_\lambda \circ \phi_0 \circ \psi_\lambda^{-1} = \phi_l \circ \cdots \circ \phi_1$$

in Γ_g. Note that the elements $\phi_0, \phi_1, \cdots, \phi_l$ are contained in the set of pseudo-periodic maps of negative twist $\mathcal{P}_g^-$. We call (**) the *monodromy relation* of $\mathbf{f}$.

Conversely, let $a_0, a_1, \cdots, a_l$ be elements of $\mathcal{P}_g^-$ and b an element of Γ_g which satisfy

$$(***) \qquad b a_0 b^{-1} = a_l \cdots a_1$$

in Γ_g. If this relation coincides with the monodromy relation of a certain splitting family of a certain germ F, we call (***) a *monodromy realizable relation*. Our question is:

Can we characterize intrinsically monodromy realizable relations ?

For example, the Birman-Hilden relation ([19])

$$\iota = \tau_1 \cdots \tau_{2g-2} \tau_{2g-1}^2 \tau_{2g-2} \cdots \tau_1$$

is a monodromy realizable relation, where ι is the hyperelliptic involution and τ_i $(1 \leq i \leq 2g-1)$ is a right-hand full Dehn twist along the i-th curve of the Birman-Hilden base. For the proof, see Matsumoto [63], Example A, Ito [49], or [2], Examples 3.12 and 3.14. (Note that b is an element of the hyperelliptic mapping class group by the construction of [63] and [2]. Hence $b\iota b^{-1} = \iota$.)

By taking the square of both sides of the relation, we get

$$id = \tau_1 \cdots \tau_{2g-1}^2 \cdots \tau_1^2 \cdots \tau_{2g-1}^2 \cdots \tau_1.$$

We claim that this is not a monodromy realizable relation. Indeed, the degeneration with trivial topological monodromy is a topologically trivial degeneration, i.e. the central fiber is a smooth curve. The trivial degeneration has no proper splitting family by the closedness of the discriminant locus.

6. Moduli point and monodromy of a two-parameter degeneration. quad A splitting family $\mathbf{f}\colon M \to \Delta \times \Delta'$ is nothing but a two-parameter degeneration of curves. As an analogy of the case of one-parameter degenerations discussed in §3, we should consider the moduli point and the monodromy of $\mathbf{f}$.

Note that the discriminant locus $\mathcal{D}$ of $\mathbf{f}$ has a bad singularity at the origin $\mathbf{0}$ in general. Therefore the moduli map

$$\alpha_{\mathbf{f}}\colon (\Delta \times \Delta') \setminus \mathcal{D} \longrightarrow \mathcal{M}_g$$

is not necessarily extended to a *morphism* from $\Delta \times \Delta'$ to $\overline{\mathcal{M}}_g$. More precisely, if we fix $u \in \Delta'$, then the one-parameter degeneration $\mathbf{f}_u\colon M_u \longrightarrow \Delta \times \{u\}$ has the extension of the moduli map $\overline{\alpha}_{\mathbf{f}_u} : \Delta \times \{u\} \longrightarrow \overline{\mathcal{M}}_g$ as a morphism. If we define the extension $\overline{\alpha}_{\mathbf{f}}\colon \Delta \times \Delta' \longrightarrow \overline{\mathcal{M}}_g$ of $\alpha_{\mathbf{f}}$ by $\overline{\alpha}_{\mathbf{f}}(t,u) := \overline{\alpha}_{\mathbf{f}_u}(t)$, then the map fails to be continuous at $\mathbf{0}$ in general. We have many examples such that the maps $\overline{\alpha}_{\mathbf{f}}$ "jump" at $\mathbf{0}$. For instance, the family of Example 3.12 in [2] has such a property. Therefore, the moduli point is not well-defined in the usual sense.

Is the space $\overline{\mathcal{M}}_g$ too "narrow" to consider this type of problems ?

We can also consider the topological monodromy

$$\pi_1(\Delta \times \Delta' \setminus \mathcal{D}) \longrightarrow \widehat{\Gamma}_g$$

in a similar way. However, the Matsumoto-Montesinos type arguments seem to be unknown.

7. Global Morsification. We should consider the Morsification problem for global pencils $f\colon S \longrightarrow B$ discussed in §§1, 2. We guess that this may be the original motivation of Xiao-Reid and Horikawa. However, almost nothing seems known.

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Integral p-adic Hodge Theory

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Abstract.

We give an overview of some questions and results in integral p-adic Hodge theory. A few proofs are supplied.

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§0. Introduction

Fix a prime number p, an algebraic closure $\overline{\mathbf{Q}}_p$ of the field $\mathbf{Q}_p$ of p-adic numbers with ring of integers $\overline{\mathbf{Z}}_p$, and a finite extension F of $\mathbf{Q}_p$ inside $\overline{\mathbf{Q}}_p$ with ring of integers $\mathcal{O}_F$.

Roughly speaking, p-adic Hodge theory (over F) is the study of de Rham and p-adic étale cohomologies of (proper smooth) schemes over F. The research for relations between these two cohomology groups gave birth to Fontaine's theory of semi-stable (and potentially semi-stable) p-adic representations of $\mathrm{Gal}(\overline{\mathbf{Q}}_p/F)$ and, in the course of time, p-adic

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Hodge theory also included the study of these Galois representations.

Integral p-adic Hodge theory could be today defined as the study of Galois stable $\mathbf{Z}_p$-*lattices* in semi-stable p-adic representations together with their links with the various *integral* p-adic cohomologies of proper smooth schemes over F. Integral p-adic Hodge theory gives back classical p-adic Hodge theory (by inverting p), but it also gives rise to completely new characteristic p phenomena (by reducing modulo p). Thus, it is richer than p-adic Hodge theory. It is also much more complicated. Although p-adic Hodge theory is now mostly complete (by the work of many people including Tate, Raynaud, Grothendieck, Bloch, Messing, Fontaine, Colmez, Faltings, Kato, Hyodo, Tsuji...), integral p-adic Hodge theory is far from being as well understood and there remains a great deal to be found before one has a complete theory. Of course, if such a theory exists, it should also contain all the results of p-adic Hodge theory.

This text tries to "take stock" of the situation of integral p-adic Hodge theory so far, although it certainly couldn't pretend to be fully exhaustive. It has two aims: the first is to give the best possible conjectures and results up to now, the second is to prove the minor things which were not already proved in order to state these results. It is organized as follows. In section 1, we recall the basic definitions and results on semi-stable p-adic representations. The key role here is played by weakly admissible filtered (φ, N)-modules. In section 2, we give a conjectural description of Galois stable lattices in semi-stable p-adic representations with small Hodge-Tate weights and we explain the known cases of this conjecture. The idea is to define integral structures also on the filtered modules side called strongly divisible lattices (or strongly divisible modules). In section 3, we prove the case where the Hodge-Tate weights are between 0 and 1. The crystalline case is a consequence of a link between strongly divisible modules (in that case) and p-divisible groups over $\mathcal{O}_F$ (Cor. 3.2.4). To deduce the semi-stable case (§3.5), we need a technical result on Galois representations arising from finite flat group schemes that we prove via the theory of norm fields (Theorem 3.4.3). In section 4, we consider the "higher weight" cases and give a link between strongly divisible modules and some cohomology groups H^m's with $m < p-1$. Finally, in section 5, using strongly divisible lattices we compute the reduction modulo p of Galois stable $\mathbf{Z}_p$-lattices in some two dimensional semi-stable p-adic representations and show how variable this reduction can be.

We have restricted ourselves to finite extensions F of $\mathbf{Q}_p$ mainly for simplicity. All the statements of this paper, except those of section 5, should hold *verbatim* for any complete local field of characteristic 0 with perfect residue field of characteristic p.

This text is an extended version of a talk given in July 2000 at the conference "Algebraic Geometry 2000" in Azumino. I would like to thank the organizers for inviting me to this conference, and thus giving me the opportunity to come to Japan for the first time. I also thank B. Conrad and A. Mézard for their comments on an earlier version of this text.

§1. Review of semi-stable p-adic representations

Let $\overline{\mathbf{F}}_p$ be the residue field of $\overline{\mathbf{Z}}_p$ (an algebraic closure of the finite field $\mathbf{F}_p$) and $\mathbf{F} \subset \overline{\mathbf{F}}_p$ the residue field of F. Let $f := [\mathbf{F} : \mathbf{F}_p]$ and $e := [F : F_0]$ where $F_0 \subset F$ is the maximal unramified subfield of F. We write G_F for $\mathrm{Gal}(\overline{\mathbf{Q}}_p/F)$ and σ for the arithmetic Frobenius on F_0. If ℓ is any prime number, an ℓ-adic representation of G_F is, by definition, a continuous linear representation of G_F on a finite dimensional $\mathbf{Q}_\ell$-vector space V.

Definition 1.1 ([14]). A p-adic representation V of G_F is called semi-stable if:

$$\dim_{F_0}(B_{st} \otimes_{\mathbf{Q}_p} V)^{G_F} = \dim_{\mathbf{Q}_p} V.$$

Here, B_{st} is Fontaine's ring of p-adic periods defined in [13] (see also [20]). It is endowed with an action of G_F. The exponent G_F on the left hand side means we take the elements of $B_{st} \otimes_{\mathbf{Q}_p} V$ which are fixed by G_F. If V is *any* p-adic representation of G_F, one has only an inequality $\dim_{F_0}(B_{st} \otimes_{\mathbf{Q}_p} V)^{G_F} \leq \dim_{\mathbf{Q}_p} V$ ([14]).

Definition 1.1 is not very explicit. Fortunately, a recent result of Colmez and Fontaine ([10]) gives an alternative description of semi-stable p-adic representations which is very explicit and useful. Define a filtered (φ, N)-module to be a finite dimensional F_0-vector space D endowed with:

- a σ-linear injective map $\varphi : D \to D$ (the "Frobenius")
- a linear map $N : D \to D$ such that $N\varphi = p\varphi N$ (the "monodromy")
- a decreasing filtration $(\mathrm{Fil}^i D_F)_{i \in \mathbf{Z}}$ on $D_F := F \otimes_{F_0} D$ by F-vector subspaces such that $\mathrm{Fil}^i D_F = D_F$ for $i \ll 0$ and $\mathrm{Fil}^i D_F = 0$ for $i \gg 0$.

The conditions on φ and N imply that N is nilpotent. Let D be a filtered (φ, N)-module and define:

• $t_H(D) := \sum_{i \in \mathbf{Z}} i \dim_F \mathrm{gr}^i D_F$ where $\mathrm{gr}^i D_F = \mathrm{Fil}^i D_F / \mathrm{Fil}^{i+1} D_F$

• $t_N(D) := \sum_{\alpha \in \mathbf{Q}} \alpha \dim_{\overline{\mathbf{Q}}_p} \overline{D}_\alpha$ where $\overline{D}_\alpha$ is the sum of the characteristic subspaces of $\overline{\mathbf{Q}}_p \otimes_{F_0} D$ for the eigenvalues of $Id \otimes \varphi^f$ having valuation α (here the valuation is normalized so that p^f has valuation 1).

It is clear that $t_H(D) \in \mathbf{Z}$ and one can prove $t_N(D) \in \mathbf{Z}$ (see e.g. [1]). By definition a filtered (φ, N)-submodule of D is a filtered (φ, N)-module D' equipped with an injection $D' \hookrightarrow D$ that commutes with φ and N and for which $\mathrm{Fil}^i D'_F = D'_F \cap \mathrm{Fil}^i D_F$.

Definition 1.2 ([14]). A filtered (φ, N)-module D is weakly admissible if $t_H(D) = t_N(D)$ and if $t_H(D') \leq t_N(D')$ for any filtered (φ, N)-submodule D' of D.

If V is a semi-stable p-adic representation of G_F, one can prove that the F_0-vector space $D_{st}(V) := (B_{st} \otimes_{\mathbf{Q}_p} V)^{G_F}$ is a weakly admissible filtered (φ, N)-module in a natural, although not quite canonical, way (see [13]). The aforementioned result of Colmez and Fontaine is:

Theorem 1.3 ([10]). *The functor $D_{st} : V \mapsto (B_{st} \otimes_{\mathbf{Q}_p} V)^{G_F}$ establishes an equivalence of categories between the category of semi-stable p-adic representations of G_F and the category of weakly admissible filtered (φ, N)-modules.*

Note that the functor D_{st} is not canonical since it depends on a filtration on $F \otimes_{F_0} B_{st}$ (or equivalently of an embedding $F \otimes_{F_0} B_{st} \hookrightarrow B_{dR}$ since the filtration is induced via such an embedding by the filtration on B_{dR}) which itself depends on the choice of a uniformizer π in F. When $N = 0$ on $D_{st}(V)$, V is said to be crystalline and in that case $D_{st}(V)$ is independant of any choice.

In the sequel, we will instead use the contravariant functor $D^*_{st}(V) := D_{st}(V^*)$, where V^* is the dual representation of V (crystalline/semi-stable if and only if V is). The reason for this is that the Hodge-Tate weights of V are exactly the $i \in \mathbf{Z}$ such that $\mathrm{gr}^i D^*_{st}(V)_F \neq 0$ (with D_{st}, it would be the $-i$ such that $\mathrm{gr}^i D_{st}(V)_F \neq 0$, see [14]). A quasi-inverse to D^*_{st} is then given by:

$$V^*_{st}(D) := \mathrm{Hom}_{\varphi,N}(D, B_{st}) \cap \mathrm{Hom}_{\mathrm{Fil}\cdot}(D_F, F \otimes_{F_0} B_{st}),$$

that is to say the $\mathbf{Q}_p$-vector space of F_0-linear maps $f : D \to B_{st}$ being compatible with all the structures (G_F acting by $(g \cdot f)(x) := g(f(x))$).

We will use this quasi-inverse in the sequel.

To finish this section, we remind the reader that a description similar to 1.3 also exists for semi-stable ℓ-adic representations of G_F with $\ell \neq p$ (i.e. ℓ-adic representations such that the inertia acts unipotently) and that it is essentially trivial: they are described by finite dimensional $\mathbf{Q}_\ell$-vector spaces endowed with a continuous linear action of $\mathrm{Gal}(F^{nr}/F)$ (which plays the role of the Frobenius) and with a nilpotent endomorphism N (the monodromy) such that $N\varphi = p^f \varphi N$ where φ is the geometric Frobenius of $\mathrm{Gal}(F^{nr}/F)$ and F^{nr} the maximal unramified extension of F inside $\overline{\mathbf{Q}}_p$. Recall that $g = \exp(Nt_\ell(g))$ if $g \in I_F := \mathrm{Gal}(\overline{\mathbf{Q}}_p/F^{nr})$ and $t_\ell : I_F \to \mathbf{Z}_\ell(1) \simeq \mathbf{Z}_\ell$ is the tame ℓ-component of I_F.

§2. Lattices in semi-stable representations with low Hodge-Tate weights

On the side of p-adic representations of G_F, there is an obvious integral structure, namely the $\mathbf{Z}_p$-lattices that are preserved by the action of G_F (which always exist because G_F is compact). Thus, granting Theorem 1.3, one can ask whether there also exists a corresponding integral structure on the filtered module side.

2.1. Basic assumptions

Let us first examine the ℓ-adic situation. Let V be a semi-stable ℓ-adic representation of G_F and D the associated $(\mathrm{Gal}(F^{nr}/F), N)$-vector space defined at the end of the previous section. *If* $N^\ell = 0$, there are nice integral structures on D that correspond to G_F-stable lattices in V, namely the $\mathbf{Z}_\ell$-lattices in D that are preserved by $\mathrm{Gal}(F^{nr}/F)$ and N. But if $N^\ell \neq 0$, this doesn't work anymore because we cannot use the operators $\frac{N^i}{i!}$ when $i \geq \ell$ to rebuild the unipotent action of inertia on the Galois side (and in that case, one usually works directly with Galois lattices). As the p-adic side is much more involved than the ℓ-adic one, one can expect to need, *at least*, the assumption $N^p = 0$ on D .

Definition 2.1.1. (1) A weakly admissible filtered (φ, N)-module D such that $\mathrm{Fil}^0 D_F = D_F$, $\mathrm{Fil}^m D_F \neq 0$, and $\mathrm{Fil}^{m+1} D_F = 0$ (for some $m \in \mathbf{N}$) has no m-component if, inside the abelian category of weakly admissible modules, D has no non zero weakly admissible quotient $\overline{D}$ such that $\mathrm{Fil}^m \overline{D}_F = \overline{D}_F$.
(2) A semi-stable p-adic representation V of G_F with Hodge-Tate weights between 0 and m has no m-component if $D^*_{st}(V)$ has no m-component.

Note that filtered (φ, N)-modules arising from unipotent p-divisible groups over $\mathcal{O}_F$ (i.e. p-divisible groups with connected Cartier dual) have no 1-component.

It turns out that in the p-adic setting, one is naturally led to either the hypothesis:

Basic Assumption 2.1.2. *Either the Hodge-Tate weights of the semi-stable p-adic representation V are between 0 and m with $m < p-1$ or they are between 0 and $p-1$ and V has no $p-1$-component.*

or its equivalent filtered variant:

Basic Assumption 2.1.3. *Either the filtration on the weakly admissible filtered module D is such that $\mathrm{Fil}^0 D_F = D_F$ and $\mathrm{Fil}^{m+1} D_F = 0$ with $m < p-1$ or it is such that $\mathrm{Fil}^0 D_F = D_F$ and $\mathrm{Fil}^p D_F = 0$ and D has no $p-1$-component.*

Equivalently, one could just say $\mathrm{Fil}^{p-1} D_F = 0$ in the first case of 2.1.3, but it's convenient to have an integer m as in 2.1.2 and 2.1.3. Twisting by the cyclotomic character, one could also weaken Assumption 2.1.2 (resp. 2.1.3) to just require that the *difference* between the extreme Hodge-Tate weights (resp. the *length* of the filtration) is smaller than m. Without assumption on m, it is not yet known how Galois lattices can be described in general in terms of integral structures on the filtered (φ, N)-modules. The link with our ℓ-adic prelude is provided by:

Lemma 2.1.4. *Let D be a weakly admissible filtered (φ, N)-module such that $\mathrm{Fil}^0 D_F = D_F$ and $\mathrm{Fil}^p D_F = 0$. Then $N^p = 0$ on D.*

Proof. Let $P_H(D)$ (resp. $P_N(D)$) be the Hodge (resp. Newton) polygon associated to D, i.e. the convex polygon such that the part of slope $i \in \mathbf{N}$ (resp. $\alpha \in \mathbf{Q}^+$) is of length $\dim_F \mathrm{gr}^i D_F$ (resp. $\dim_{\overline{\mathbf{Q}}_p} \overline{D}_\alpha$, see §1). The weak admissibility condition implies that $P_H(D)$ lies under $P_N(D)$ and that they have the same endpoints (see [15]). From the corresponding drawing and the assumptions on D, one must have $\alpha \leq p-1$ if $\overline{D}_\alpha \neq 0$ (since $p-1$ is the highest possible slope on $P_H(D)$). But from $N\varphi = p\varphi N$, we get $N(\overline{D}_\alpha) \subset \overline{D}_{\alpha-1}$ if $\alpha \geq 1$ and $N(\overline{D}_\alpha) = 0$ otherwise. Thus, $N^p(\overline{D}_\alpha) = 0$ for all α, i.e. $N^p = 0$. Q.E.D.

Note that Assumptions 2.1.2 or 2.1.3 here are really *stronger* than just $N^p = 0$ (for instance, in the crystalline case, $N = 0$ but m can be arbitrary).

2.2. Strongly divisible modules

In this section, we define integral structures for filtered (φ, N)-modules satisfying Assumption 2.1.3 and we state the main conjecture.

From now on, we fix a uniformizer π in F and denote by $E(u)$ its minimal polynomial (an Eisenstein polynomial of degree e). Let S be the p-adic completion of $W(\mathbf{F})[u, \frac{u^{ie}}{i!}]_{i\in\mathbf{N}}$ where u is an indeterminate and endow S with the following structures:
- a continuous σ-linear Frobenius still denoted $\sigma : S \to S$ such that $\sigma(u) = u^p$
- a continuous linear derivation $N : S \to S$ such that $N(u) = -u$
- a decreasing filtration $(\mathrm{Fil}^i S)_{i\in\mathbf{N}}$ where $\mathrm{Fil}^i S$ is the p-adic completion of $\sum_{j\geq i} S\frac{E(u)^j}{j!}$ (one checks $\frac{E(u)^j}{j!} \in S$).

Note that $N\sigma = p\sigma N$, $N(\mathrm{Fil}^{i+1}S) \subset \mathrm{Fil}^i S$ for $i \in \mathbf{N}$ and $\sigma(\mathrm{Fil}^i S) \subset p^i S$ for $i \in \{0, ..., p-1\}$.

Let D be a weakly admissible filtered (φ, N)-module and assume that $\mathrm{Fil}^0 D_F = D_F$. Let:

$$\mathcal{D} := S \otimes_{W(\mathbf{F})} D$$

and define :
- $\varphi := \sigma \otimes \varphi : \mathcal{D} \to \mathcal{D}$
- $N := N \otimes Id + Id \otimes N : \mathcal{D} \to \mathcal{D}$
- $\mathrm{Fil}^0\mathcal{D} := \mathcal{D}$ and, by induction:

$$\mathrm{Fil}^{i+1}\mathcal{D} := \{x \in \mathcal{D} \mid N(x) \in \mathrm{Fil}^i\mathcal{D} \text{ and } f_\pi(x) \in \mathrm{Fil}^{i+1} D_F\}$$

where $f_\pi : \mathcal{D} \twoheadrightarrow D_F$ is defined by $s(u) \otimes x \mapsto s(\pi)x$.

One can show the map f_π induces surjections $\mathrm{Fil}^i\mathcal{D} \twoheadrightarrow \mathrm{Fil}^i D_F$ ([8]). The filtered module $\mathcal{D}$ has the advantage over the filtered module D that all of its data are defined at the same level (no need to extend scalars to F). Moreover, one can prove that the knowledge of $\mathcal{D}$ is equivalent to that of D ([8]). It turns out the integral structures will naturally live inside the $\mathcal{D}$'s. But first, we note that there is a "B_{st}-counterpart" to this construction, i.e. there is a "period S-algebra", first introduced by Kato in [22] and that the author named $\widehat{B_{st}}$, such that the couple $(B_{st}, \widehat{B_{st}})$ is somewhat analogous to the couple $(D, \mathcal{D})$. More precisely, if t denotes Fontaine's analogue of $2\pi i$ (see [13]), then $\widehat{B_{st}} = \widehat{A_{st}}[1/t]$ where $\widehat{A_{st}}$ is (non canonically) isomorphic to the p-adic completion of

$A_{cris}[X, \frac{X^i}{i!}]_{i\in\mathbf{N}}$. Here, A_{cris} is the integral version of B_{cris} ([13]), X is an indeterminate, and $A_{cris}[X, \frac{X^i}{i!}]_{i\in\mathbf{N}}$ is an S-module via the map $u \mapsto [\underline{\pi}](1+X)^{-1}$ where $[\underline{\pi}]$ is a specific element of A_{cris} made out of a compatible system of p^n-th roots of π in $\overline{\mathbf{Q}}_p$. See [3] for details, where it is also explained how to endow $A_{cris}[X, \frac{X^i}{i!}]_{i\in\mathbf{N}}$ with a continuous action of G_F (which is non trivial on X), Frobenius and monodromy maps, and a decreasing filtration, with all of these structures inducing the previous structures on S, the usual structures on A_{cris}, and ultimately only depending, up to isomorphism, on the choice of π and not on any other choice.

Now, let us go back to the initial problem of defining integral structures:

Definition 2.2.1. Let D be a weakly admissible filtered (φ, N)-module such that $\mathrm{Fil}^0 D_F = D_F$ and $\mathrm{Fil}^{m+1} D_F = 0$ with $m < p$. A strongly divisible lattice (or module) in $\mathcal{D}$ is an S-submodule $\mathcal{M}$ of $\mathcal{D}$ such that:
(1) $\mathcal{M}$ is free of finite rank over S and $\mathcal{M}[\frac{1}{p}] \xrightarrow{\sim} \mathcal{D}$
(2) $\mathcal{M}$ is stable under φ and N
(3) $\varphi(\mathrm{Fil}^m \mathcal{M}) \subset p^m \mathcal{M}$ where $\mathrm{Fil}^m \mathcal{M} := \mathcal{M} \cap \mathrm{Fil}^m \mathcal{D}$.

One can show this definition doesn't depend on m (provided of course $\mathrm{Fil}^{m+1} D_F = 0$ and $m < p$). Using the weak admissibility of D, one can also show that condition (3) in Definition 2.2.1 is actually equivalent to the apparently stronger condition that $\varphi(\mathrm{Fil}^m \mathcal{M})$ spans $p^m \mathcal{M}$ over S (see §2.1 of [3]).

Examples 2.2.2. (1) Let D be the trivial filtered module (i.e. $D = F_0$ with $\mathrm{Fil}^1 D_F = 0$, $N = 0$ and $\varphi = \sigma$). Then S is a strongly divisible lattice in $\mathcal{D} = S[\frac{1}{p}]$.

(2) Let D be as in 2.2.1 and assume $F = F_0 = W(\mathbf{F})[1/p]$ and $N = 0$. Recall ([17]) that a strongly divisible module in the sense of Fontaine and Laffaille is a $W(\mathbf{F})$-lattice M in D such that $\varphi(\mathrm{Fil}^i M) \subset p^i M$ for all $i \in \mathbf{N}$ where $\mathrm{Fil}^i M := M \cap \mathrm{Fil}^i D$. As previously, because D is weakly admissible, this is equivalent to $M = \sum_i \frac{\varphi}{p^i}(\mathrm{Fil}^i M)$. Let $\mathcal{M} := S \otimes_{W(\mathbf{F})} M \subset \mathcal{D}$, then $\mathcal{M}$ is a strongly divisible lattice in $\mathcal{D}$ in the sense of 2.2.1.

(3) Assume $F = F_0$ and let $D = F_0 e_1 \oplus F_0 e_2$ with $\varphi(e_1) = p^r e_1$, $\varphi(e_2) = p^{r-1} e_1$ ($r \in \mathbf{N}$, $2r \leq (p-1)$), $N(e_1) = e_2$, $N(e_2) = 0$, $\mathrm{Fil}^i D =$

$F_0(e_1 + \mathcal{L}e_2)$ if $1 \leq i \leq 2r-1$ ($\mathcal{L} \in W(\mathbf{F})$) and $\mathrm{Fil}^i D = 0$ if $i \geq 2r$. Then one can check that $Se_1 \oplus S(e_2/p)$ is a strongly divisible lattice in $\mathcal{D}$.

(4) Assume $F = F_0(\pi)$ with $\pi^{p-1} = -p$ and let $D = F_0e_1 \oplus F_0e_2$ with $\varphi(e_1) = pe_1$, $\varphi(e_2) = pe_2$, $N = 0$, $\mathrm{Fil}^1 D_F = \mathrm{Fil}^2 D_F = F(e_1 + \pi e_2)$, $\mathrm{Fil}^i D_F = 0$ if $i \geq 3$ and assume $p \geq 5$. Then one can check that $Se_1 \oplus S\big(e_2 + \frac{u^{p(p-2)}}{pU} e_1\big)$ is a strongly divisible lattice in $\mathcal{D}$ where $U = \frac{p-2}{p-1}\big(\frac{u^{p(p-1)}}{p} + 1\big) - 1 \in S^\times$.

For $m \in \mathbf{N}$ consider the category $\mathcal{C}_m$ of S-modules $\mathcal{M}$ endowed with a σ-linear endomorphism φ, a $W(\mathbf{F})$-linear endomorphism N satisfying $N(sx) = N(s)x + sN(x)$ ($s \in S$, $x \in \mathcal{M}$), and an S-submodule $\mathrm{Fil}^m \mathcal{M}$, with morphisms being S-linear maps that preserve Fil^m and commute with φ and N. For $m < p$, we define the category of *strongly divisible modules of weight* $\leq m$ as the full subcategory of $\mathcal{C}_m$ consisting of objects that are isomorphic to a strongly divisible module in some $S \otimes_{W(\mathbf{F})} D$ for D weakly admissible as in 2.2.1. It turns out one can directly describe this category:

Theorem 2.2.3. *The category of strongly divisible modules of weight* $\leq m$ ($m < p$) *is the full subcategory of* $\mathcal{C}_m$ *of objects* $\mathcal{M}$ *satisfying the following conditions*:
(1) $\mathcal{M}$ *is free of finite rank over* S
(2) $(\mathrm{Fil}^m S)\mathcal{M} \subset \mathrm{Fil}^m \mathcal{M}$
(3) $\mathrm{Fil}^m \mathcal{M} \cap p\mathcal{M} = p\mathrm{Fil}^m \mathcal{M}$
(4) $\varphi(\mathrm{Fil}^m \mathcal{M})$ *spans* $p^m \mathcal{M}$
(5) $N\varphi = p\varphi N$
(6) $(\mathrm{Fil}^1 S)N(\mathrm{Fil}^m \mathcal{M}) \subset \mathrm{Fil}^m \mathcal{M}$.

The point is to prove that $\mathcal{M}[1/p] \simeq S \otimes_{W(\mathbf{F})} D$ for a (unique) filtered (φ, N)-module D and that this D is weakly admissible. This is done in [8] and [3] for $m < p-1$ but the proof readily extends to the case $m < p$. Of course, when m grows, these categories are full subcategories one of the other.

Definition 2.2.4. A strongly divisible module of weight $\leq m$ has no m-component if the corresponding weakly admissible D has no m-component (cf. 2.1.1).

To a strongly divisible module $\mathcal{M}$ of weight $\leq m$ one can associate the $\mathbf{Z}_p[G_F]$-module:

$$T^*_{st}(\mathcal{M}) := \mathrm{Hom}_{S,\varphi,N,\mathrm{Fil}^m}(\mathcal{M}, \widehat{A_{st}})$$

where one considers S-linear maps from $\mathcal{M}$ to $\widehat{A_{st}}$ that commute with φ, N and preserve Fil^m (this doesnt depend on $m < p$ such that $\mathcal{M}$ is of weight $\leq m$). The group G_F acts by $(g \cdot f)(x) := g(f(x))$.

Proposition 2.2.5. *Let $\mathcal{M}$ be a strongly divisible module of weight $\leq m$ ($m < p$) and D the corresponding weakly admissible filtered (φ, N)-module. Then $T^*_{st}(\mathcal{M})$ is a Galois stable $\mathbf{Z}_p$-lattice in $V^*_{st}(D)$ (see §1 for V^*_{st}).*

Proof. We only give a sketch here and refer the reader to [3] or [8] for details. Let $\mathcal{D} := \mathcal{M}[1/p] = S \otimes_{W(\mathbf{F})} D$. As $T^*_{st}(\mathcal{M})$ is clearly a Galois stable $\mathbf{Z}_p$-lattice in $V^*_{st}(\mathcal{D}) := \mathrm{Hom}_{S,\varphi,N,\mathrm{Fil}^m}(\mathcal{D}, \widehat{A_{st}}[1/p])$, the real issue is to prove that $V^*_{st}(\mathcal{D})$ is isomorphic as a Galois representation to $V^*_{st}(D)$. Note first that an S-linear map $f : \mathcal{D} \to \widehat{A_{st}}[1/p]$ preserves Fil^m if and only if it preserves Fil^i for $0 \leq i \leq m$. There is a ring morphism commuting with G_F and compatible with the filtration $\widehat{A_{st}}[1/p] \to B_{dR}$, $X \mapsto \frac{[\underline{\pi}]}{\pi} - 1$ where $[\underline{\pi}]$ is the "specific" element of A_{cris} previously mentionned. Using that $D = \{x \in \mathcal{D} \mid N^n(x) = 0 \text{ for some } n \in \mathbf{N}\}$, one gets any $f \in V^*_{st}(\mathcal{D})$ sends D to $B^+_{cris}[\log(1+X)] \subset \widehat{A_{st}}[1/p]$. Composing with the above ring morphism and using the surjectivity of $\mathrm{Fil}^i\mathcal{D} \to \mathrm{Fil}^i D_F$, one ends up with an F_0-linear map $\overline{f} : D \to B^+_{st} \subset B_{dR}$ that commutes with φ and N, preserves the filtration after extending scalars to F and is such that the diagram:

$$\begin{array}{ccccc} \mathcal{D} & & \xrightarrow{f} & & \widehat{A_{st}}[1/p] \\ {\scriptstyle f_\pi}\downarrow & & & & \downarrow \\ D_F & \xrightarrow{1\otimes\overline{f}} & F \otimes_{F_0} B^+_{st} & \hookrightarrow & B_{dR} \end{array}$$

commutes. This gives an injective $\mathbf{Q}_p$-linear map $V^*_{st}(\mathcal{D}) \to V^*_{st}(D)$ which is easily checked to be surjective. Q.E.D.

Our main conjecture is:

Conjecture 2.2.6. (1) If $m < p-1$, the functor $\mathcal{M} \mapsto T^*_{st}(\mathcal{M})$ establishes an anti-equivalence of categories between the category of strongly divisible modules of weight $\leq m$ and the category of G_F-stable lattices in semi-stable representations of G_F with Hodge-Tate weights in $\{0, ..., m\}$.
(2) If $m = p-1$, the functor $\mathcal{M} \mapsto T^*_{st}(\mathcal{M})$ establishes an anti-equivalence of categories between the category of strongly divisible modules of weight $\leq p-1$ that have no $p-1$-component and the category of G_F-stable lattices in semi-stable representations of G_F with Hodge-Tate weights in $\{0, ..., p-1\}$ that have no $p-1$-component.

In particular, if V is a semi-stable p-adic representation of G_F with Hodge-Tate weights in $\{0, ..., m\}$ (without m-component if $m = p-1$), then Galois lattices in V should *exactly* correspond to strongly divisible modules in the associated $S \otimes_{W(\mathbf{F})} D$. The following theorem summarizes the known cases of conjecture 2.2.6:

Theorem 2.2.7. *Conjecture* 2.2.6 *is true in the following two cases*:
(1) $m < p-1$ *and* $e = 1$
(2) $m = 0$ *or* $m = 1$ *and* $p \neq 2$.

Case (1) is proven in [4] using results of [3]. The method is a generalization of that of Fontaine and Laffaille who did the subcase $m < p-1$, $e = 1$, $N = 0$ ([17]). At the time of [17], the ring S and S-modules like $\mathcal{D}$ and $\mathcal{M}$ were not yet defined, but in that case one can manage with $W(\mathbf{F})$-lattices only, namely those lattices defined in Example (2) of 2.2.2. In the other cases, one can not dispense with S, which makes the theory much more complicated, even when $e = 1$. Case (2) is proven in the next section using the theory of p-divisible groups.

There are two other partial results in the direction of 2.2.6. The first is that if $em < p-1$, then $\mathcal{D}$ at least *always* contains a strongly divisible lattice ([3]). The second is that for $m < p-1$ the restriction of T^*_{st} to the subcategory of "filtered-free" strongly divisible modules of weight $\leq m$ is *fully faithful* ([12]). Here, by filtered free, we mean there is a *basis* $(e_i)_{1 \leq i \leq d}$ of the underlying S-module $\mathcal{M}$ and integers $0 \leq r_1 \leq ... \leq r_d \leq m$ such that:

$$\mathrm{Fil}^m \mathcal{M} = \Big(\bigoplus_i E(u)^{r_i} S e_i\Big) + (\mathrm{Fil}^m S)\mathcal{M}.$$

Unfortunately, most of the strongly divisible modules are not filtered free, but they are if $m \leq 1$ ([5]) or if $e = 1$ and $N = 0$ ([17]). In particular, the full faithfulness of T^*_{st} in case (2) of 2.2.7 was thus proven in [12] (we will derive it below from Tate's full faithfulness theorem).

§3. Finite flat group schemes, p-divisible groups, and norm fields

In this section, we prove statement (2) of Theorem 2.2.7. We first deal with the case of lattices in crystalline representations using results on p-divisible groups (§3.2). Then we derive the general case using the theory of norm fields (§3.3, §3.4 and §3.5).

3.1. The case $m = 0$

This is the case of unramified p-adic representations of G_F. Define the category of *étale* $(\varphi, W(\mathbf{F}))$*-modules* as the category of free $W(\mathbf{F})$-modules of finite rank equipped with a bijective σ-linear endomorphism φ. Then it has long been known (see [16] for instance) that the functor $M \mapsto \mathrm{Hom}_{W(\mathbf{F}),\varphi}(M, W(\overline{\mathbf{F}}_p))$ establishes an anti-equivalence of categories between étale $(\varphi, W(\mathbf{F}))$-modules and G_F-stable lattices in unramified p-adic representations of G_F.

View $W(\mathbf{F})$ as an S-module by sending u and its divided powers to 0. To a strongly divisible module $\mathcal{M}$ of weight 0, one can associate $M := \mathcal{M} \otimes_S W(\mathbf{F})$ and endow it with the image of φ (the image of N being 0). It is clear M is then an étale $(\varphi, W(\mathbf{F}))$-module. Statement (2) of 2.2.7 in the case $m = 0$ comes down to:

Proposition 3.1.1. *The functor $\mathcal{M} \mapsto \mathcal{M} \otimes_S W(\mathbf{F})$ establishes an equivalence of categories between strongly divisible modules of weight 0 and étale $(\varphi, W(\mathbf{F}))$-modules.*

This is the well-known "Dwork's trick" ([21]) in a divided power context.

3.2. Classification of group schemes and consequences

From now on, we assume $m = 1$ and $p \neq 2$. We connect some of the strongly divisible modules of weight ≤ 1 to p-divisible groups over $\mathcal{O}_F$.

As with Galois lattices, it is tempting, using the alternative definition of strongly divisible modules given by Theorem 2.2.3, to reduce strongly divisible modules of weight $\leq m$ modulo arbitrary powers of p. For $m = 1$, we are led to the following category $\underline{\mathcal{M}}_0^1$.

An object of $\underline{\mathcal{M}}_0^1$ is a triple $(\mathcal{M}, \mathrm{Fil}^1\mathcal{M}, \varphi_1)$ where:
(1) $\mathcal{M}$ is an S-module of finite type isomorphic to $\oplus_{n \in \mathbf{Z}_{>0}} (S/p^n S)^{r_n}$ for integers r_n which are almost all equal to 0
(2) $\mathrm{Fil}^1\mathcal{M}$ is an S-submodule of $\mathcal{M}$ containing $(\mathrm{Fil}^1 S)\mathcal{M}$
(3) $\varphi_1 : \mathrm{Fil}^1\mathcal{M} \to \mathcal{M}$ is an additive map such that:

$$\varphi_1(sx) = \frac{\frac{\sigma}{p}(s)}{\frac{\sigma}{p}(E(u))}\varphi_1(E(u)x)$$

where $s \in \mathrm{Fil}^1 S$ and $x \in \mathcal{M}$ (note that $\frac{\sigma}{p}(E(u)) \in S^\times$) and such that $\mathcal{M}$ is generated by $\varphi_1(\mathrm{Fil}^1\mathcal{M})$ as an S-module.

A morphism between two objects of $\underline{\mathcal{M}}_0^1$ is an S-linear map sending Fil^1 to Fil^1 and commuting with φ_1. The map φ_1 has to be thought as the p-torsion version of the map $\frac{\varphi}{p}|_{\mathrm{Fil}^1}$. The condition $\mathrm{Fil}^1\mathcal{M} \cap p\mathcal{M} = p\mathrm{Fil}^1\mathcal{M}$ turns out to be automatically satisfied on an object of $\underline{\mathcal{M}}_0^1$. We could define a similar category by requiring the existence of a "monodromy map" N on the S-modules $\mathcal{M}$ (as for strongly divisible modules), but Lemma 3.2.1 below shows that the objects of $\underline{\mathcal{M}}_0^1$ are *already endowed* with a canonical N, and there will be no need here to consider more general torsion objects.

Lemma 3.2.1. *Let $\mathcal{M}$ be an object of $\underline{\mathcal{M}}_0^1$. There is a unique additive map $N : \mathcal{M} \to \mathcal{M}$ such that:*
(1) $N(sx) = N(s)x + sN(x)$ *for* $s \in S$ *and* $x \in \mathcal{M}$
(2) $\varphi_1\big(E(u)N(x)\big) = \frac{\sigma}{p}(E(u))N\big(\varphi_1(x)\big)$ *for* $x \in \mathrm{Fil}^1\mathcal{M}$
(3) $N(\mathcal{M}) \subset u\mathcal{M}$.

Proof. Assume two such N exist and let Δ be their difference. Let $x \in \mathrm{Fil}^1\mathcal{M}$, from conditions (2) and (3) we get:

$$\Delta(\varphi_1(x)) = \Big(\frac{\sigma}{p}\big(E(u)\big)\Big)^{-1}\varphi_1\big(E(u)\Delta(x)\big) \in u^p\mathcal{M}.$$

Since $\mathcal{M}$ is spanned by the image of φ_1 and Δ is S-linear, one has $\Delta(\mathcal{M}) \subset u^p\mathcal{M}$. An obvious induction then yields $\Delta = 0$. For the existence of N, there are 3 possible proofs: (1) one can (tediously) build it by pure linear algebra; (2) one can use 3.2.2 below which implies by [2] that there *must exist* a connection $\nabla : \mathcal{M} \to \mathcal{M} \otimes_S Sdu$ and N is defined by $u\nabla(x) = -N(x) \otimes du$; (3) one easily builds explicitly such an N when $\mathcal{M}$ is free over S ([5]), then, using 3.2.2 below and the fact any commutative finite flat group scheme is the kernel of an isogeny between p-divisible groups, one gets that any object of $\underline{\mathcal{M}}_0^1$ is the quotient of two strongly divisible modules and one takes the quotient N. Q.E.D.

Note that any morphism in $\underline{\mathcal{M}}_0^1$ automatically commutes with the respective N given by 3.2.1. The main purpose in defining the category $\underline{\mathcal{M}}_0^1$ lies in:

Theorem 3.2.2 ([5]). *There is an anti-equivalence of categories between $\underline{\mathcal{M}}_0^1$ and the category of commutative finite flat group schemes G over $\mathcal{O}_F$ such that $\mathrm{Ker}(p_G^n) = G$ for some $n \in \mathbf{N}$ and $\mathrm{Ker}(p_G^n)$ is flat over $\mathcal{O}_F$ for all $n \in \mathbf{N}$ (where p_G^n is multiplication by p^n on G).*

One can dispense with the last flatness assumption on the kernels $\mathrm{Ker}(p_G^n)$, but the price is that one has to consider more complicated

S-modules than just $\oplus(S/p^nS)^{r_n}$ for which I do not know the explicit structure (see [5]). This assumption is automatically satisfied if $e < p-1$.

Remark 3.2.3. More general objects than those of $\underline{\mathcal{M}}_0^1$, e.g. objects of $\underline{\mathcal{M}}_0^1$ endowed with an additive map N satisfying (1) and (2) of 3.2.1 but not (3), may correspond to "log-group schemes" (i.e. group objects in the category of log-schemes).

Taking the projective limit in 3.2.2 and using 3.2.1 yields:

Corollary 3.2.4 ([5]). *There is an anti-equivalence of categories between the category of strongly divisible modules $\mathcal{M}$ of weight ≤ 1 such that $N(\mathcal{M}) \subset u\mathcal{M}$ and the category of p-divisible groups over $\mathcal{O}_F$.*

Using this corollary, one can prove the following special case of (2), 2.2.7:

Theorem 3.2.5. *The functor $\mathcal{M} \mapsto T_{st}^*(\mathcal{M})$ establishes an anti-equivalence of categories between the category of strongly divisible modules $\mathcal{M}$ of weight ≤ 1 such that $N(\mathcal{M}) \subset u\mathcal{M}$ and the category of G_F-stable lattices in crystalline representations of G_F with Hodge-Tate weights in $\{0,1\}$.*

Proof. The full faithfulness is a well-known theorem of Tate ([27]). By [5], one knows that any crystalline V with Hodge-Tate weights in $\{0,1\}$ contains at least one lattice which is isomorphic to the Tate module of some p-divisible group over $\mathcal{O}_F$. But Raynaud's argument ([24]) then shows this must hold for *any* lattice in such a V. Using 3.2.4, this ensures the essential surjectivity. Q.E.D.

The rest of §3 will be devoted to the rest of the proof of (2), 2.2.7, i.e. the case of semi-stable *non*-crystalline representations.

3.3. Group schemes of type $(p,...,p)$ and norm fields

In this section, we state a variant of 3.2.2 for group schemes killed by p in terms of modules over the ring of integers of the norm field of an infinite wildly ramified extension of F. This variant will be used in the next section to prove a result on representations of G_F coming from group schemes. Recall that a group scheme of type $(p,...,p)$ is by definition a commutative finite flat group scheme killed by p.

Choose $(\pi_n)_{n\in\mathbf{N}} \in \overline{\mathbf{Q}}_p^{\mathbf{N}}$ such that $\pi_0 = \pi$, $\pi_{n+1}^p = \pi_n$ and let $F_n := F(\pi_n)$, $\mathcal{O}_{F_n}$ its ring of integers, $F_\infty := \cup F_n$ and $G_{F_\infty} := \mathrm{Gal}(\overline{\mathbf{Q}}_p/F_\infty)$ (in particular $F_0 = F$). It is proven in [30] that the projective limit

$\varprojlim F_n$ (resp. $\varprojlim \mathcal{O}_{F_n}$) with the norms as transition maps is in a natural way a field (resp. a ring) of characteristic p which can be identified with $\mathbf{F}((\underline{\pi}))$ (resp. $\mathbf{F}[[\underline{\pi}]]$). Here, $\underline{\pi}$ is the element $(..., \pi_n, \pi_{n-1}, ..., \pi_0) \in \varprojlim \mathcal{O}_{F_n}$. Such fields as $\varprojlim F_n$ are called *norm fields* in [30]. Let $\mathbf{F}((\underline{\pi}))^{\text{sep}}$ be a separable closure of $\mathbf{F}((\underline{\pi}))$. The main result of [30] is a canonical identification $G_{F_\infty} \simeq \text{Gal}\big(\mathbf{F}((\underline{\pi}))^{\text{sep}}/\mathbf{F}((\underline{\pi}))\big)$ which gives a surprising alternative description of the Galois group G_{F_∞}.

Let σ be the Frobenius on $\mathbf{F}((\underline{\pi}))$ and $\mathbf{F}[[\underline{\pi}]]$. We introduce two kinds of modules:

• The category of *étale* $(\varphi, \mathbf{F}((\underline{\pi})))$*-modules* is the category of finite dimensional $\mathbf{F}((\underline{\pi}))$-vector spaces $\mathfrak{D}$ endowed with a σ-linear map $\varphi : \mathfrak{D} \to \mathfrak{D}$ inducing an isomorphism (or equivalently a surjection):

$$\mathbf{F}((\underline{\pi})) \otimes_{\sigma, \mathbf{F}((\underline{\pi}))} \mathfrak{D} \xrightarrow{1 \otimes \varphi} \mathfrak{D}$$

(with obvious morphisms between objects).

• The category of $(\varphi, \mathbf{F}[[\underline{\pi}]])$*-modules of height* ≤ 1 is the category of free $\mathbf{F}[[\underline{\pi}]]$-modules of finite rank $\mathfrak{M}$ endowed with a σ-linear map $\varphi : \mathfrak{M} \to \mathfrak{M}$ such that $\underline{\pi}^e \mathfrak{M}$ is contained in the $\mathbf{F}[[\underline{\pi}]]$-submodule of $\mathfrak{M}$ generated by $\varphi(\mathfrak{M})$ (ibid.).

If $\mathfrak{M}$ is an object of the second category, then $\mathfrak{M}[1/\underline{\pi}]$ is obviously an object of the first. The following two theorems give alternative descriptions of these two categories:

Theorem 3.3.1 ([16]). *The functor*:

$$\mathfrak{D} \mapsto T^*(\mathfrak{D}) := \text{Hom}_{\mathbf{F}((\underline{\pi})), \varphi}(\mathfrak{D}, \mathbf{F}((\underline{\pi}))^{\text{sep}})$$

establishes an anti-equivalence of categories between the category of étale $(\varphi, \mathbf{F}((\underline{\pi})))$*-modules and the category of continuous representations of* $G_{F_\infty} \simeq \text{Gal}\big(\mathbf{F}((\underline{\pi}))^{\text{sep}}/\mathbf{F}((\underline{\pi}))\big)$ *on finite dimensional* $\mathbf{F}_p$*-vector spaces.*

Theorem 3.3.2. *There is an anti-equivalence of categories between the category of* $(\varphi, \mathbf{F}[[\underline{\pi}]])$*-modules of height* ≤ 1 *and the category of group schemes of type* $(p, ..., p)$ *over* $\mathcal{O}_F$*. Moreover, if* G *is such a group scheme and* $\mathfrak{M}(G)$ *is the corresponding* $\mathbf{F}[[\underline{\pi}]]$*-module, one has an* $\mathbf{F}_p[G_{F_\infty}]$*-module isomorphism:* $T^*(\mathfrak{M}[1/\underline{\pi}]) \simeq G(\overline{\mathbf{Q}}_p)|_{G_{F_\infty}}$.

Proof. Granting 3.2.2, it is enough to prove that there is an equivalence of categories between $(\varphi, \mathbf{F}[[\underline{\pi}]])$-modules of height ≤ 1 and objects of $\underline{\mathcal{M}}_0^1$ killed by p, commuting with the functors to Galois representations. Let $\mathfrak{M}$ be a $(\varphi, \mathbf{F}[[\underline{\pi}]])$-module of height ≤ 1 and view S/pS as an

$\mathbf{F}[[\underline{\pi}]]$-algebra via $\Pi : \mathbf{F}[[\underline{\pi}]] \to S/pS$, $\sum x_i \underline{\pi}^i \mapsto \sum x_i u^i$. One associates to $\mathfrak{M}$ an object $\mathcal{M}$ of $\underline{\mathcal{M}}_0^1$ as follows:
- as an S-module, $\mathcal{M} := S/pS \otimes_{\sigma \circ \Pi, \mathbf{F}[[\underline{\pi}]]} \mathfrak{M}$
- $\mathrm{Fil}^1 \mathcal{M} := \{y \in \mathcal{M} \mid (Id \otimes \varphi)(y) \in \mathrm{Fil}^1 S/pS \otimes_{\mathbf{F}[[\underline{\pi}]]} \mathfrak{M}\}$
- $\varphi_1 : \mathrm{Fil}^1 \mathcal{M} \to \mathcal{M}$ is defined as the composite:

$$\mathrm{Fil}^1 \mathcal{M} \xrightarrow{Id \otimes \varphi} \mathrm{Fil}^1 S/pS \otimes_{\mathbf{F}[[\underline{\pi}]]} \mathfrak{M} \xrightarrow{\frac{\sigma}{p} \otimes Id} S/pS \otimes_{\sigma \circ \Pi, \mathbf{F}[[\underline{\pi}]]} \mathfrak{M} \simeq \mathcal{M}.$$

Using the fact $\mathfrak{M}$ is of height ≤ 1, it is easy to see that the image of φ_1 generates everything. This process obviously defines a functor to $\underline{\mathcal{M}}_0^1$. It turns out this functor is an equivalence of categories on objects killed by p. Using [5], Lemma 2.1.2.1 and [5], Proposition 2.1.2.2, the proof is almost *verbatim* the proof of [6], Theorem 4.1.1. The only difference is that here $\mathrm{Ker}(\Pi) = (\pi^{ep})$ and $\mathrm{Ker}(\sigma \circ \Pi) = (\pi^e)$ instead of (π^p) and (π) in *loc.cit.* and this doesn't change the argument. For the Galois actions, let R be the projective limit

$$\ldots \xrightarrow{\mathrm{Frob}} \overline{\mathbf{Z}}_p/p\overline{\mathbf{Z}}_p \xrightarrow{\mathrm{Frob}} \overline{\mathbf{Z}}_p/p\overline{\mathbf{Z}}_p \xrightarrow{\mathrm{Frob}} \overline{\mathbf{Z}}_p/p\overline{\mathbf{Z}}_p$$

and R^{DP} the Divided Power envelope of R with respect to the ideal generated by the image of $\underline{\pi}^e$ i.e. by the element $(\ldots, \overline{\pi}_2^e, \overline{\pi}_1^e, \overline{\pi}_0^e) \in R$ where $\overline{\pi}_i$ is the image of π_i in $\overline{\mathbf{Z}}_p/p\overline{\mathbf{Z}}_p$ (see [29] for instance). One can endow R^{DP} with a Fil^1 and a φ_1 ([29]) and view it as an S-module via $u \mapsto \mathrm{image}(\underline{\pi})$. By [5] and [3] Lemma 2.3.1.1, the restriction to G_{F_∞} of the Galois representation associated to $\mathcal{M}$ is isomorphic to $\mathrm{Hom}_{S,\varphi_1,\mathrm{Fil}^1}(\mathcal{M}, R^{DP})$ (with left action of G_{F_∞} on R^{DP} and obvious notations). Thus, one has to compare $\mathrm{Hom}_{\mathbf{F}((\underline{\pi})),\varphi}(\mathfrak{M}[1/\underline{\pi}], \mathbf{F}((\underline{\pi}))^{\mathrm{sep}})$ and $\mathrm{Hom}_{S,\varphi_1,\mathrm{Fil}^1}(\mathcal{M}, R^{DP})$. Using [6], Lemma 2.3.3, the proof is again (almost) *verbatim* the proof of [6], Proposition 4.2.1. Q.E.D.

Remark 3.3.3. Theorem 3.3.2 implies that representations of G_{F_∞} coming from $(\varphi, \mathbf{F}[[\underline{\pi}]])$-modules of height ≤ 1 can be extended to G_F. We will see in the next section that this extension is essentially unique.

Remark 3.3.4. Let $\underline{\mathbf{Z}/p\mathbf{Z}}$ and $\underline{\mu_p}$ be the usual group schemes of rank p. Using [5] and the above proof, one can see that $\mathfrak{M}(\underline{\mathbf{Z}/p\mathbf{Z}}) = \mathbf{F}[[\underline{\pi}]]e_1$ with $\varphi(e_1) = e_1$ and that $\mathfrak{M}(\underline{\mu_p}) = \mathbf{F}[[\underline{\pi}]]e_2$ with $\varphi(e_2) = -F(0)^{-1}\underline{\pi}^e e_2$ where $F(0) = -\frac{E(0)}{p}$ (recall $E(u)$ is the minimal polynomial of π). One of the problems with the category $\underline{\mathcal{M}}_0^1$ for $p = 2$ is that there is no map corresponding to the non-trivial morphism of group schemes $\underline{\mathbf{Z}/2\mathbf{Z}} \to \underline{\mu_2}$ sending 1 to -1. However, for $p = 2$, *there is* a non-trivial map $\mathfrak{M}(\underline{\mu_2}) \to \mathfrak{M}(\underline{\mathbf{Z}/2\mathbf{Z}})$ that commutes with φ, namely:

$e_2 \mapsto F(0)^{-1}\underline{\pi}^e e_1$ (this map would give 0 in $\underline{\mathcal{M}}_0^1$ by the functor in the proof of 3.3.2). So, one can ask whether statement 3.3.2 *still holds* for $p = 2$ although statement 3.2.2 doesn't...

3.4. A full faithfulness result

Lemma 3.4.1. *Let $\mathfrak{D}' \hookrightarrow \mathfrak{D}$ (resp. $\mathfrak{D}' \twoheadrightarrow \mathfrak{D}$) be an injection (resp. a surjection) of étale $(\varphi, \mathbf{F}((\underline{\pi})))$-modules and assume $\mathfrak{D}$ (resp. $\mathfrak{D}'$) is generated by a $(\varphi, \mathbf{F}[[\underline{\pi}]])$-module $\mathfrak{M}$ (resp. $\mathfrak{M}'$) of height ≤ 1. Then $\mathfrak{M} \cap \mathfrak{D}'$ (resp. image$(\mathfrak{M}')$) is a $(\varphi, \mathbf{F}[[\underline{\pi}]])$-module of height ≤ 1.*

Proof. The surjection case is obvious. For the injection, it is clear that $\mathfrak{M}' := \mathfrak{M} \cap \mathfrak{D}'$ is stable under φ and is a direct factor of $\mathfrak{M}$. Let $(f_1, ..., f_d)$ be a basis of $\mathfrak{M}$ (over $\mathbf{F}[[\underline{\pi}]]$) such that $(f_1, ..., f_{d'})$ is a basis of $\mathfrak{M}'$ and denote by $(\overline{f}_{d'+1}, ..., \overline{f}_d)$ the image basis of $\mathfrak{M}/\mathfrak{M}'$. By assumption, there are $s_{ij} \in \mathbf{F}[[\underline{\pi}]]$ such that $\underline{\pi}^e f_i = \sum_{j=1}^{d} s_{ij}\varphi(f_j)$ for $1 \leq i \leq d$. For $d' + 1 \leq i \leq d$, this implies that $(\varphi(\overline{f}_{d'+1}), ..., \varphi(\overline{f}_d))$ is a basis of $(\mathfrak{M}/\mathfrak{M}')[1/\underline{\pi}]$ since it generates this module, and for $1 \leq i \leq d'$ this implies $0 = \sum_{j=d'+1}^{d} s_{ij}\varphi(\overline{f}_j)$; i.e. $s_{ij} = 0$ for $d' + 1 \leq j \leq d$ (and $1 \leq i \leq d'$). Hence, $\varphi(\mathfrak{M}')$ generates $\underline{\pi}^e\mathfrak{M}'$. Q.E.D.

Lemma 3.4.2. *Let G_1 and G_2 be two group schemes of type $(p, ..., p)$ over $\mathcal{O}_F$. Then any $\mathbf{F}_p[G_{F_\infty}]$-isomorphism $G_1(\overline{\mathbf{Q}}_p)|_{G_{F_\infty}} \simeq G_2(\overline{\mathbf{Q}}_p)|_{G_{F_\infty}}$ is an $\mathbf{F}_p[G_F]$-isomorphism.*

Proof. Fix such an $\mathbf{F}_p[G_{F_\infty}]$-isomorphism. Let $\mathfrak{M}_i$ be the $(\varphi, \mathbf{F}[[\underline{\pi}]])$-module of height ≤ 1 associated to G_i by 3.3.2 and let $\mathfrak{D} := \mathfrak{M}_i[1/\underline{\pi}]$, which doesn't depend on $i \in \{1, 2\}$ by assumption and 3.3.1. Then $\mathfrak{M} := \mathfrak{M}_1 + \mathfrak{M}_2 \subset \mathfrak{D}$ is obviously still a $(\varphi, \mathbf{F}[[\underline{\pi}]])$-module of height ≤ 1 and thus corresponds to a group scheme $G/\mathcal{O}_F$. The two injections $\mathfrak{M}_i \hookrightarrow \mathfrak{M}$ give morphisms of group schemes $G \to G_i$ such that $G(\overline{\mathbf{Q}}_p)|_{G_{F_\infty}} \xrightarrow{\sim} G_i(\overline{\mathbf{Q}}_p)|_{G_{F_\infty}}$ by 3.3.2. This implies $G_1(\overline{\mathbf{Q}}_p) \simeq G(\overline{\mathbf{Q}}_p) \simeq G_2(\overline{\mathbf{Q}}_p)$ and all of these isomorphisms obviously commute with G_F since they come from morphisms of group schemes. Q.E.D.

We say a representation of G_F on a finite length $\mathbf{Z}_p$-module is *finite flat* if it is isomorphic to the representation of G_F on $G(\overline{\mathbf{Q}}_p)$ for some commutative finite flat group scheme G over $\mathcal{O}_F$ killed by some power of p. The process of schematic closure ([24]) then shows this category is abelian and stable under formation of subobjects and quotients.

Theorem 3.4.3. *The fonctor "restriction to G_{F_∞}" from finite flat representations of G_F to representations of G_{F_∞} is fully faithful. Its essential image is stable under formation of subobjects and quotients.*

Proof. We first start with the full faithfulness. By a standard devissage, one is reduced to the case of representations on $\mathbf{F}_p$-vector spaces. Let G_1, G_2 be two group schemes of type $(p,...,p)$, $\mathfrak{M}_1$, $\mathfrak{M}_2$ the corresponding $(\varphi,\mathbf{F}[[\underline{\pi}]])$-modules of height ≤ 1 and $\mathfrak{D}_i := \mathfrak{M}_i[1/\underline{\pi}]$ $(i=1,2)$. Assume there is an $\mathbf{F}_p[G_{F_\infty}]$-morphism $G_2(\overline{\mathbf{Q}}_p)|_{G_{F_\infty}} \to G_1(\overline{\mathbf{Q}}_p)|_{G_{F_\infty}}$ i.e. by 3.3.1 a morphism $f : \mathfrak{D}_1 \to \mathfrak{D}_2$. By 3.4.1, $f(\mathfrak{M}_1)$ and $\mathfrak{M}_2 \cap f(\mathfrak{D}_1)$ are two $(\varphi,\mathbf{F}[[\underline{\pi}]])$-modules of height ≤ 1 that generate $f(\mathfrak{D}_1)$. They correspond to two group schemes G_1', G_2' such that $G_2'(\overline{\mathbf{Q}}_p)|_{G_{F_\infty}} \simeq G_1'(\overline{\mathbf{Q}}_p)|_{G_{F_\infty}}$ and we have morphisms of group schemes $G_1' \to G_1$ and $G_2 \to G_2'$ by 3.3.2. Hence the morphism $G_2(\overline{\mathbf{Q}}_p)|_{G_{F_\infty}} \to G_1(\overline{\mathbf{Q}}_p)|_{G_{F_\infty}}$ factorizes through:

$$G_2(\overline{\mathbf{Q}}_p)|_{G_{F_\infty}} \longrightarrow G_2'(\overline{\mathbf{Q}}_p)|_{G_{F_\infty}} \simeq G_1'(\overline{\mathbf{Q}}_p)|_{G_{F_\infty}} \longrightarrow G_1(\overline{\mathbf{Q}}_p)|_{G_{F_\infty}}.$$

By 3.4.2, $G_2'(\overline{\mathbf{Q}}_p) \simeq G_1'(\overline{\mathbf{Q}}_p)$ as $\mathbf{F}_p[G_F]$-modules from which we get that the map $G_2(\overline{\mathbf{Q}}_p) \to G_1(\overline{\mathbf{Q}}_p)$ commutes with G_F. This gives the full faithfulness. For the rest of the statement, it is enough to prove that any G_{F_∞}-subrepresentation of a finite flat G_F-representation T is preserved by G_F (and hence is finite flat). We proceed by induction on $n \in \mathbf{N}$ such that $p^nT = 0$. For $n = 1$, this is a consequence of 3.4.1 (together with 3.3.1 and 3.3.2). Assume this holds for $n-1$ and let $T' \subset T$ be a G_{F_∞}-subrepresentation with $p^nT = 0$. Then $T/(T'+pT)$ is a quotient of T/pT, hence is preserved by G_F by the case $n = 1$. By the full faithfulness, the morphism $T \to T/(T'+pT)$ commutes with G_F hence $T'+pT$ is preserved by G_F. Now $(T'+pT)/T'$ is a quotient of pT, hence is preserved by G_F by the case $n-1$. By the full faithfulness applied to $T'+pT \to (T'+pT)/T'$, T' is preserved by G_F. Q.E.D.

Corollary 3.4.4. *Let V be a crystalline representation of G_F with Hodge-Tate weights in $\{0,1\}$ and $T \subset V$ a $\mathbf{Z}_p$-lattice which is stable under G_{F_∞}. Then T is stable under G_F.*

Proof. Let T' be a $\mathbf{Z}_p[G_F]$-lattice containing T and recall that by 3.2.4 and 3.2.5, T' is the Tate module of a p-divisible group over $\mathcal{O}_F$. By 3.4.3 any $\mathbf{Z}_p[G_{F_\infty}]$-submodule of T'/p^nT' is stable under G_F for any $n \in \mathbf{N}$. Thus, $T/T \cap p^nT'$ is stable under G_F, i.e. $g(T) \subset T+p^nT'$ for any $g \in G_F$ and $n \in \mathbf{N}$, which implies $g(T) \subset \cap_n(T+p^nT') = T$. Q.E.D.

3.5. Lattices in semi-stable representations with Hodge-Tate weights in $\{0,1\}$

We finish the proof of (2), 2.2.7 using Corollary 3.4.4 above. We choose $(\pi_n)_{n\in\mathbf{N}}$ as in §3.3 and define F_∞ and G_{F_∞} in the same way.

Let D be a weakly admissible filtered (φ, N)-module such that $\mathrm{Fil}^0 D_F = D_F$ and $\mathrm{Fil}^2 D_F = 0$. Let $V := V^*_{st}(D)$ as in §1 and $\mathcal{D} := S \otimes_{W(\mathbf{F})} D$ as in §2.2. Recall we have defined $V^*_{st}(\mathcal{D})$ in the proof of Proposition 2.2.5 and shown that $V^*_{st}(\mathcal{D}) \xrightarrow{\sim} V^*_{st}(D)$. Define:

$$V^*_{cris}(D) := \mathrm{Hom}_{\varphi}(D, B_{cris}) \cap \mathrm{Hom}_{\mathrm{Fil}}(D_F, F \otimes_{F_0} B_{cris})$$

and $V^*_{cris}(\mathcal{D}) := \mathrm{Hom}_{S,\varphi,\mathrm{Fil}^1}(\mathcal{D}, B_{cris})$ where we view B_{cris} as an S-algebra by sending u to the element $[\underline{\pi}]$ corresponding to the p^n-th roots π_n (§2.2). We have ring morphisms $B_{st} \to B_{cris}$ and $\widehat{A_{st}}[1/p] \to B_{cris}$ obtained by sending $\log \frac{[\underline{\pi}]}{\pi}$ and X to 0.

Lemma 3.5.1. (1) *The map $f \mapsto f|_D$ induces an isomorphism of $\mathbf{Q}_p$-vector spaces $V^*_{cris}(\mathcal{D}) \xrightarrow{\sim} V^*_{cris}(D)$.*
(2) *The ring homomorphisms $B_{st} \to B_{cris}$ and $\widehat{A_{st}}[1/p] \to B_{cris}$ induce isomorphisms of $\mathbf{Q}_p$-vector spaces $V^*_{st}(D) \xrightarrow{\sim} V^*_{cris}(D)$ and $V^*_{st}(\mathcal{D}) \xrightarrow{\sim} V^*_{cris}(\mathcal{D})$.*
(3) *The diagram*

$$\begin{array}{ccc} V^*_{st}(\mathcal{D}) & \xrightarrow{\sim} & V^*_{cris}(\mathcal{D}) \\ \wr\downarrow & & \downarrow\wr \\ V^*_{st}(D) & \xrightarrow{\sim} & V^*_{cris}(D) \end{array}$$

is commutative.

Proof. (3) follows from the definition of the various maps. To prove (1) and (2), we first note that we can replace everywhere B_{cris} by B^+_{cris} and B_{st} by B^+_{st}. For B_{cris}, this is a direct consequence of [13], Theorem5.3.7(i). For B_{st}, one can argue as follows. Let $f \in V^*_{st}(D)$, $x \in D \setminus \{0\}$ and $r \in \mathbf{Z}_{\geq 0}$ such that $N^{r+1}(x) = 0$ but $N^r(x) \neq 0$. Then $f(\varphi^s(N^r(x))) \in \mathrm{Fil}^0 B_{cris}$ for all $s \in \mathbf{Z}$ which implies $f(N^r(x)) \in B^+_{cris}$ by [13], Theorem5.3.7(i). Hence $f(N^{r-1}(x)) \in \mathrm{Fil}^0 B_{cris} + B^+_{cris} \log \frac{[\underline{\pi}]}{\pi}$. Since $\varphi(\log \frac{[\underline{\pi}]}{\pi}) = p \log \frac{[\underline{\pi}]}{\pi}$, the same argument shows $f(N^{r-1}(x)) \in B^+_{cris} + B^+_{cris} \log \frac{[\underline{\pi}]}{\pi}$ and we deduce $f(x) \in B^+_{st}$ by induction. The isomorphism in (1) comes from the facts that $\mathrm{Fil}^1 \mathcal{D} = f_\pi^{-1}(\mathrm{Fil}^1 D_F)$ and $[\underline{\pi}] - \pi \in \mathrm{Fil}^1(F \otimes_{F_0} B^+_{cris})$ where $f_\pi : \mathcal{D} \to D_F$ is the map of §2.2. Note that it exists because there is just Fil^1. One checks the ring homomorphisms $B^+_{st} \to B^+_{cris}$ and $\widehat{A_{st}}[1/p] \to B^+_{cris}$ deduced from the ones without $+$ commute with φ and preserve Fil^1 and hence induce maps of vector spaces as in (2). Since we know that $V^*_{st}(\mathcal{D}) \xrightarrow{\sim} V^*_{st}(D)$, we only have to prove $V^*_{st}(D) \xrightarrow{\sim} V^*_{cris}(D)$ thanks to (1) and the commutativity in (3). The inverse map $V^*_{cris}(D) \to V^*_{st}(D)$ is given by $f \mapsto f + \log \frac{[\underline{\pi}]}{\pi} f \circ N$ (using $B^+_{cris} \subset B^+_{st}$). Q.E.D.

Remark 3.5.2. One can prove that the above isomorphism $V^*_{st}(\mathcal{D}) \xrightarrow{\sim} V^*_{cris}(\mathcal{D})$ does not require $\mathrm{Fil}^2 D_F = 0$. Also, all the isomorphisms in

Lemma 3.5.1 commute with G_{F_∞} although they do not commute with G_F.

Lemma 3.5.3. *Let D' be the same filtered (φ, N)-module as D but with $N = 0$. Then D' is also weakly admissible.*

Proof. With the notations of §1, we have:

$$\overline{\mathbf{Q}}_p \otimes_{F_0} N(D) = \oplus_\alpha N(\overline{D}_\alpha)$$

with $N(\overline{D}_\alpha) \subset \overline{D}_{\alpha-1}$ (since $N\varphi^f = p^f\varphi^f N$). But $\overline{D}_\alpha = 0$ if $\alpha \notin [0,1]$ (weak admissibility condition) so $N(D) \subset \overline{D}_0$ which implies $t_N(N(D)) = 0$ and also $t_H(N(D)) = 0$ since $0 \leq t_H(N(D)) \leq t_N(N(D))$. Note that $N^2 = 0$ (same proof as for 2.1.4). Let $D^0 \subset D$ be a F_0-vector subspace stable under φ but not necessarily under N with the induced filtration $\mathrm{Fil}^i D_F^0 := D_F^0 \cap \mathrm{Fil}^i D_F$ $(i = 0, 1)$. Define $D^1 := D^0 + N(D^0)$. From the exact sequence $0 \to N(D^0) \to D^1 \to D^0/(D^0 \cap N(D^0)) \to 0$, the weak admissibility condition for $D^1 \subset D$, and the additivity property of t_H and t_N, we have:

$$t_H^1(D^0/(D^0 \cap N(D^0))) \leq t_N(D^0/(D^0 \cap N(D^0)))$$

where by t_H^1 we mean the t_H computed with the filtration on $D^0/(D^0 \cap N(D^0))$ coming from the quotient filtration of D^1. From the exact sequence $0 \to D^0 \cap N(D^0) \to D^0 \to D^0/(D^0 \cap N(D^0)) \to 0$, we deduce $t_H(D^0) = t_H^0(D^0/(D^0 \cap N(D^0)))$ and $t_N(D_0) = t_N(D^0/(D^0 \cap N(D^0)))$ where by t_H^0 we mean the t_H computed with the filtration on $D^0/(D^0 \cap N(D^0))$ coming from the quotient filtration of D^0. From the inclusion $D^0 \subset D^1$ and the above inequality, we get:

$$t_H^0(D^0/(D^0 \cap N(D^0))) \leq t_H^1(D^0/(D^0 \cap N(D^0))) \leq t_N(D^0/(D^0 \cap N(D^0)))$$

hence $t_H(D^0) \leq t_N(D^0)$. This gives the desired result. Q.E.D.

Let $\mathcal{D}' := S \otimes_{W(\mathbf{F})} D'$ (with its usual structures) and note that $\mathcal{D}' \simeq \mathcal{D}$ except for the operator N. We have $V_{cris}^*(D') = V_{cris}^*(D)$ and $V_{cris}^*(\mathcal{D}') = V_{cris}^*(\mathcal{D})$ since the definition of these vector spaces do not use N. Using (2), Lemma 3.5.1, we deduce isomorphisms $V_{st}^*(D') \simeq V_{st}^*(D)$ and $V_{st}^*(\mathcal{D}') \simeq V_{st}^*(\mathcal{D})$ such that the diagram:

$$\begin{array}{ccc} V_{st}^*(\mathcal{D}) & \xrightarrow{\sim} & V_{st}^*(\mathcal{D}') \\ \wr\downarrow & & \downarrow\wr \\ V_{st}^*(D) & \xrightarrow{\sim} & V_{st}^*(D') \end{array}$$

commutes. We call V the commun underlying $\mathbf{Q}_p$-vector space and ρ', ρ the two different Galois actions $G_F \to \mathrm{Aut}(V)$ corresponding to D'

and D respectively. Let $D(-1)$ be the filtered (φ, N)-module defined by $\mathrm{Fil}^m D(-1)_F := \mathrm{Fil}^{m+1} D_F$, $\varphi_{D(-1)} := p^{-1}\varphi_D$ and $N_{D(-1)} := N_D$. The operator N induces a morphism of filtered modules $N : D(-1) \to D$ and thus a morphism of Galois representations:

$$N : (V, \rho) \to (V \otimes \mathbf{Q}_p(-1), \rho \otimes \chi^{-1})$$

where $\mathbf{Q}_p(-1)$ is the $\mathbf{Q}_p$-dual of $\mathbf{Q}_p(1) := \left(\varprojlim \mu_{p^n}(\overline{\mathbf{Q}}_p)\right) \otimes \mathbf{Q}_p$ and χ is the p-adic cyclotomic character. Working out the isomorphism $V^*_{st}(D') \simeq V^*_{st}(D)$ from the proof of 3.5.1, we easily obtain:

Lemma 3.5.4. *Let $t_p : G_F \to \varprojlim \mu_{p^n}(\overline{\mathbf{Q}}_p) = \mathbf{Z}_p(1)$ be the 1-cocycle defined by $t_p(g) := \left(g(\pi_n)/\pi_n\right)_{n \in \mathbf{N}}$. Then:*

$$\rho = (Id + t_p \otimes N) \circ \rho'.$$

From this and the results of §3.4, we obtain the following key corollary:

Corollary 3.5.5. *Let $T \subset V$ be a $\mathbf{Z}_p$-lattice which is stable under ρ. Then T is also stable under ρ', or equivalently $N(T) \subset T \otimes \mathbf{Z}_p(-1)$.*

Proof. Since $t_p(g) = 0$ if $g \in G_{F_\infty}$, it follows from 3.5.4 that T is preserved by $\rho'(G_{F_\infty}) = \rho(G_{F_\infty})$. By 3.4.4, T is stable under ρ'. Q.E.D.

Now, let $\mathcal{M}$ be a strongly divisible lattice in $\mathcal{D}$. We denote by $\mathcal{M}'$ the image of $\mathcal{M}$ in $\mathcal{D}'$ under the identification $\mathcal{D} \simeq \mathcal{D}'$. In particular, as S-modules, $\mathcal{M}$ and $\mathrm{Fil}^1\mathcal{M}$ are just the same as $\mathcal{M}'$ and $\mathrm{Fil}^1\mathcal{M}'$.

Lemma 3.5.6. (1) *The S-module $\mathcal{M}'$ is preserved N in $\mathcal{D}'$, i.e. $\mathcal{M}'$ is a strongly divisible lattice in $\mathcal{D}'$.*
(2) *Under the isomorphism $V^*_{st}(D) \simeq V^*_{st}(D')$, the lattice $T^*_{st}(\mathcal{M})$ corresponds to the lattice $T^*_{st}(\mathcal{M}')$.*

Proof. For (1), we have to prove $N(\mathcal{M}) \subset \mathcal{M}$ with N being $N \otimes 1$ on $\mathcal{D}' = S \otimes D' = S \otimes D$. By Lemma 3.2.1 or by [5] Proposition 5.1.3, there is a unique additive map $N' : \mathcal{M} \to \mathcal{M}$ such that $N'(sx) = N(s)x + sN'(x)$, $N'(\mathcal{M}) \subset u\mathcal{M}$ and $N'\varphi = p\varphi N'$. As $D = \cap_{n \in \mathbf{N}} \varphi^n(\mathcal{D})$ (this is easily checked), the last commutativity condition implies $N'(D) \subset D$ and the condition $N'(\mathcal{M}) \subset u\mathcal{M}$ implies $N'|_D = 0$. Hence, on $\mathcal{M}[1/p] = S \otimes D$, N' is exactly $N \otimes 1$. This proves (1). Recall from Lemma 3.5.1 and the foregoing that we have a commutative diagram:

$$\begin{array}{ccc} V^*_{st}(\mathcal{D}) & \xrightarrow{\sim} & V^*_{st}(\mathcal{D}') \\ \wr \downarrow & & \downarrow \wr \\ V^*_{cris}(\mathcal{D}) & = & V^*_{cris}(\mathcal{D}') \end{array}$$

where the top arrow is the identification $V^*_{st}(D) \simeq V^*_{st}(D')$. In order to prove (2), it is enough to prove that the two lattices $T^*_{st}(\mathcal{M}) \subset V^*_{st}(\mathcal{D})$ and $T^*_{st}(\mathcal{M}') \subset V^*_{st}(\mathcal{D}')$ map to the same $\mathbf{Z}_p$-module in $V^*_{cris}(\mathcal{D}) = V^*_{cris}(\mathcal{D}')$. Define $T^*_{cris}(\mathcal{M}) := \mathrm{Hom}_{S,\varphi,\mathrm{Fil}^1}(\mathcal{M}, A_{cris}) \subset V^*_{cris}(\mathcal{D})$ and likewise for $T^*_{cris}(\mathcal{M}')$. Since N is not involved, we have $T^*_{cris}(\mathcal{M}) = T^*_{cris}(\mathcal{M}')$. By [3] Lemma 2.3.1.1, $T^*_{st}(\mathcal{M})$ (resp. $T^*_{st}(\mathcal{M})$) exactly maps to $T^*_{cris}(\mathcal{M})$ (resp. $T^*_{cris}(\mathcal{M}')$) under $V^*_{st}(\mathcal{D}) \xrightarrow{\sim} V^*_{cris}(\mathcal{D})$ (resp. $V^*_{st}(\mathcal{D}') \xrightarrow{\sim} V^*_{cris}(\mathcal{D}')$). This gives (2). Q.E.D.

Corollary 3.5.7. *Statement* (2) *of* 2.2.7 *holds.*

Proof. We can assume $m = 1$. We first prove the full faithfulness. Let $\mathcal{M}_1$, $\mathcal{M}_2$ be two strongly divisible modules of weight ≤ 1, T_1, T_2 their corresponding lattices and $f : T_2 \to T_1$ a Galois morphism. Let $V_i := T_i \otimes \mathbf{Q}_p$, $D_i := D^*_{st}(V_i)$, D'_i as before and $V'_i := V^*_{st}(D'_i)$ $(i \in \{1,2\})$. Recall $V_i \simeq V'_i$ as vector spaces. The map f induces $f : V_2 \to V_1$ and $f : V'_2 \to V'_1$ which is G_F-equivariant for both actions of G_F (look at the corresponding map on D_i and D'_i). By 3.5.5, T_i is Galois stable in V'_i and thus $f : T_2 \to T_1$ commutes with this "crystalline" Galois action. By 3.5.6 and 3.2.5, it induces a morphism $\mathcal{M}_1 \to \mathcal{M}_2$. It remains to prove this morphism commutes with the original N, but this is obvious since this is so for $\mathcal{M}_1[1/p] \to \mathcal{M}_2[1/p]$. Let us now prove the essential surjectivity. Let V be a semi-stable p-adic representation with Hodge-Tate weights in $\{0,1\}$ and $T \subset V$ a Galois stable lattice. Let $D := D^*_{st}(V)$ and D', V' as before. Since T is also Galois stable in V' (Corollary 3.5.5, this is the key point), by 3.2.5 it corresponds to a strongly divisible lattice $\mathcal{M}$ in $\mathcal{D}' := S \otimes D'$. By statement (2) of 3.5.6, it remains to prove that $\mathcal{M}$ is stable under N in $\mathcal{D} := S \otimes D$. Denote by N' the S-derivation on $\mathcal{M}$ induced by $\mathcal{D}'$, by $N(V)$ the unramified quotient of V corresponding to $N(D) \subset D$ (see the proof of 3.5.3) and by $N(T)$ the image of T in $N(V)$. One has an injection of crystalline representations with Hodge-Tate weights in $\{0,1\}$, $N(V) \otimes \mathbf{Q}_p(1) \hookrightarrow V$, which induces $N(T) \otimes \mathbf{Z}_p(1) \hookrightarrow T$ by 3.5.5. If $\mathcal{M}_0$ denotes the strongly divisible lattice in $S \otimes N(D)$ corresponding to $N(T)$ (case $m = 0$) and $\mathcal{M}_0(1)$ the obvious one corresponding to $N(T) \otimes \mathbf{Z}_p(1)$, then by 3.2.5 we have morphisms $\mathcal{M} \to \mathcal{M}_0(1)$ and $\mathcal{M}_0 \to \mathcal{M}$, the composite of which is $N - N' : \mathcal{M} \to \mathcal{M}(1)$ (with obvious notation, one checks this by looking over $S[1/p]$). Forgetting the twist "(1)", this implies $N(\mathcal{M}) \subset \mathcal{M}$. Q.E.D.

§4. Integral p-adic cohomologies

In this section, we suggest a cohomological interpretation of strongly divisible modules.

We fix a proper smooth scheme X over $\mathrm{Spec}(F)$ and we assume X admits a proper semi-stable model $\mathcal{X}$ over $\mathcal{O}_F$ (i.e. étale-locally $\mathcal{X}$ is smooth over $\mathcal{O}_F[X_1,...,X_r]/(X_1\cdots X_r-\pi)$ for some r). Let $\mathcal{Y}:=\mathcal{X}\times_{\mathrm{Spec}(\mathcal{O}_F)}\mathrm{Spec}(\mathbf{F})$ and $\mathcal{X}_1:=\mathcal{X}\times_{\mathrm{Spec}(\mathcal{O}_F)}\mathrm{Spec}(\mathcal{O}_F/p\mathcal{O}_F)$. Endow $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{X}_1$ with their natural log-structure ([22]) and for $m\in\mathbf{N}$ denote by:

$$\begin{aligned} H^m_{\text{ét}}(X\times_F\overline{\mathbf{Q}}_p,\mathbf{Z}_p) &:= \varprojlim H^m((X\times_F\overline{\mathbf{Q}}_p)_{\text{ét}},\mathbf{Z}/p^n\mathbf{Z}) \\ H^m_{\text{ét}}(X\times_F\overline{\mathbf{Q}}_p,\mathbf{Q}_p) &:= H^m_{\text{ét}}(X\times_F\overline{\mathbf{Q}}_p,\mathbf{Z}_p)\otimes_{\mathbf{Z}_p}\mathbf{Q}_p \end{aligned}$$

the usual p-adic étale cohomology groups of X. By [28], $H^m_{\text{ét}}(X\times_F\overline{\mathbf{Q}}_p,\mathbf{Q}_p)$ is a semi-stable p-adic representation of G_F with Hodge-Tate weights in $\{-m,...,0\}$. Moreover, if $V^m:=H^m_{\text{ét}}(X\times_F\overline{\mathbf{Q}}_p,\mathbf{Q}_p)^*$ ($\mathbf{Q}_p$-dual) and $D^m:=D^*_{st}(V^m)$ is the associated filtered (φ,N)-module (see §1), then:

$$D^m\simeq H^m_{\log-\mathrm{cris}}(\mathcal{Y}/W(\mathbf{F}))\otimes F_0 \tag{1}$$

where:

$$H^m_{\log-\mathrm{cris}}(\mathcal{Y}/W(\mathbf{F})):=\varprojlim H^m_{\log-\mathrm{cris}}\big(\mathcal{Y}/\mathrm{Spec}(W_n(\mathbf{F}))\big)$$

is the log-crystalline cohomology of $\mathcal{Y}$ with respect to the base scheme $\mathrm{Spec}(W_n(\mathbf{F}))$ endowed with the log-structure $\big(\mathbf{N}\to W_n(\mathbf{F}),1\mapsto 0\big)$. More precisely this cohomology is naturally endowed with operators φ and N and one has an isomorphism (depending on the choice of π):

$$F\otimes_{W(\mathbf{F})}H^m_{\log-\mathrm{cris}}(\mathcal{Y}/W(\mathbf{F}))\simeq H^m_{\mathrm{dR}}(X)$$

where $H^m_{\mathrm{dR}}(X)$ is the usual de Rham cohomology of X endowed with its Hodge filtration. Then (1) is an isomorphism of *filtered (φ,N)-modules* (see [19], [22] and [28] for details).

Now, let:

$$\mathcal{D}^m:=S\otimes_{W(\mathbf{F})}D^m$$

and endow it with the same structures as in section 2.2. It is shown in [19] that there is an isomorphism of $S[1/p]$-modules:

$$\mathcal{D}^m\simeq H^m_{\log-\mathrm{cris}}(\mathcal{X}_1/S)\otimes F_0$$

where:

$$H^m_{\log-\mathrm{cris}}(\mathcal{X}_1/S) := \varprojlim H^m_{\log-\mathrm{cris}}\big(\mathcal{X}_1/\mathrm{Spec}(S/p^nS)\big)$$

is the log-crystalline cohomology of $\mathcal{X}_1$ with respect to the base scheme $\mathrm{Spec}(S/p^nS)$ endowed with the log-structure $\big(\mathbf{N} \to S/p^nS, 1 \mapsto u\big)$. Here the log-scheme $\mathcal{X}_1$ is viewed over $\mathrm{Spec}(S/p^nS)$ via the embedding $\mathrm{Spec}(\mathcal{O}_F/p\mathcal{O}_F) \hookrightarrow \mathrm{Spec}(S/p^nS)$, $u \mapsto \pi$. Assume $m < p-1$ and consider:

$$T^m := \mathbf{Z}_p\text{–dual of } \big(H^m_{\text{ét}}(X \times_F \overline{\mathbf{Q}}_p, \mathbf{Z}_p)/\text{torsion}\big).$$

Then T^m is a Galois stable lattice in V^m. Conjecture 2.2.6 predicts there should exist a corresponding strongly divisible lattice in $\mathcal{D}^m$. Consider:

$$\mathcal{M}^m := H^m_{\log-\mathrm{cris}}(\mathcal{X}_1/S)/\text{torsion}.$$

One can prove that $\mathcal{M}^m \subset \mathcal{D}^m$ and that it is stable under φ and N ([19]).

Question 4.1. *Assume* $m < p-1$.
(1) Is $\mathcal{M}^m$ a strongly divisible lattice in $\mathcal{D}^m$ in the sense of Definition 2.2.1?
(2) If this is so, is $T^*_{st}(\mathcal{M}^m)$ isomorphic to T^m?

The following theorem summarizes the known answers to these questions:

Theorem 4.2. *The answer to questions* (1) *and* (2) *of* 4.1 *is yes in the following two cases*:
(1) $e = 1$
(2) $m \leq 1$.

Case (1) is proven in [7]. The method is a generalization of that of Fontaine and Messing (syntomic cohomology) who did the subcase $e = 1$, $N = 0$ ([18]). However, the proofs are more involved because strongly divisible modules when $N \neq 0$ are much more complicated than when $N = 0$, even if $e = 1$ (see [9] for instance). Case (2) is a special case of results of Faltings and is proven in [12] using his theory of almost étale extensions.

§5. A glimpse at reduction modulo p

Integral p-adic Hodge theory has the virtue that we can form its reduction modulo p. We provide here some samples of such reductions

(Propositions 5.2, 5.3 and Theorem 5.4). More precisely we reduce modulo p lattices in some 2-dimensional (over $\overline{\mathbf{Q}}_p$) semi-stable representations of G_F for $F = \mathbf{Q}_p$. This is the simplest case, although not so simple! In the sequel, we denote by $I_{\mathbf{Q}_p}$ the inertia subgroup of $G_{\mathbf{Q}_p}$, by val the p-adic valuation normalized by $\mathrm{val}(p) = 1$ and by ified character of $G_{\mathbf{Q}_p}$ sending the arithmetic Frobenius to λ.

Let us consider semi-stable p-adic representations V of $G_{\mathbf{Q}_p}$ endowed with an embedding $E \hookrightarrow \mathrm{Aut}_{G_{\mathbf{Q}_p}}(V)$ where E is a finite (arbitrarily large) extension of $\mathbf{Q}_p$ inside $\overline{\mathbf{Q}}_p$ such that $\dim_E V = 2$. In that case, $D := D_{st}(V^*)$ is also a 2-dimensional E-vector space with E-linear φ, N and filtration. We assume moreover $\mathrm{Fil}^0 D = D$ and $\mathrm{Fil}^1 D \neq 0$, and we denote by $k \geq 2$ the smallest integer such that $\mathrm{Fil}^k D = 0$. Since $\dim_E D = 2$, we have $\mathrm{Fil}^1 D = \mathrm{Fil}^2 D = \ldots = \mathrm{Fil}^{k-1} D$. We denote by $\mathcal{O}_E$ the ring of integers of E and by $\mathfrak{m}_E$ its maximal ideal.

Examples 5.1. The following three examples exhaust all the possibilities of two dimensional semi-stable representations of $G_{\mathbf{Q}_p}$ with Hodge-Tate weights $(0, k-1)$:

$$(1) \quad \begin{cases} D & = & Ee_1 \oplus Ee_2 \\ \varphi(e_1) & = & p^{k-1}(\lambda e_1 + \mu e_2) \\ \varphi(e_2) & = & \lambda^{-1} e_2 \\ N & = & 0 \\ \mathrm{Fil}^{k-1} D & = & Ee_1 \\ (\lambda, \mu) & \in & \mathcal{O}_E^\times \times E \end{cases}$$

$$(2) \quad \begin{cases} D & = & Ee_1 \oplus Ee_2 \\ \varphi(e_1) & = & p^{k-1} e_2 \\ \varphi(e_2) & = & -e_1 + \mu e_2 \\ N & = & 0 \\ \mathrm{Fil}^{k-1} D & = & Ee_1 \\ \mu & \in & \mathfrak{m}_E \end{cases}$$

$$(3) \quad \begin{cases} \varphi(e_1) & = & p^{k/2} \lambda e_1 \\ \varphi(e_2) & = & p^{k/2-1} \lambda e_2 \\ \mathrm{Fil}^{k-1} D & = & E(e_1 - \mathcal{L} e_2) \\ N(e_1) & = & e_2 \\ N(e_2) & = & 0 \\ k & \in & 2\mathbf{Z}_{>0} \\ (\lambda, \mathcal{L}) & \in & \{\pm 1\} \times E \end{cases}$$

(The reader can check that the above filtered (φ, N)-modules are all weakly admissible.)

Following Serre ([26]), define for $n \in \mathbf{Z}_{>0}$ and $g \in I_{\mathbf{Q}_p}$:

$$\theta_{p^n-1}(g) := \frac{g(p^{1/(p^n-1)})}{p^{1/(p^n-1)}} \in \mu_{p^n-1}(\overline{\mathbf{Q}}_p) \simeq \mathbf{F}_{p^n}^\times \hookrightarrow \overline{\mathbf{F}}_p^\times.$$

This turns out to be independent of the choice of $p^{1/(p^n-1)}$ and defines a tamely ramified character $\theta_{p^n-1} : I_{\mathbf{Q}_p} \to \overline{\mathbf{F}}_p^\times$. Let $T \subset V$ be a Galois stable $\mathcal{O}_E$-lattice and $\overline{T} := T \otimes_{\mathcal{O}_E} (\mathcal{O}_E/\mathfrak{m}_E)$ its reduction "modulo p". By [26] the semi-simplification of this reduction can be described in terms of powers of the characters θ_{p^n-1}. For instance, by noticing that the Galois representations associated to the filtered modules of Example 5.1 (1) are reducible, one immediately gets:

Proposition 5.2 ([17]). *Let V be a semi-stable p-adic representation of $G_{\mathbf{Q}_p}$ such that $D^*_{st}(V)$ is as in Example* 5.1(1). *Let $T \subset V$ be a Galois stable $\mathcal{O}_E$-lattice. Then:*

$$\overline{T} \simeq \begin{pmatrix} \theta_{p-1}^{k-1}\mathrm{Frob}(\overline{\lambda}) & * \\ 0 & \mathrm{Frob}(\overline{\lambda}^{-1}) \end{pmatrix}.$$

For cases (2) and (3) of 5.1, one needs integral p-adic Hodge theory. By computing explicit strongly divisible lattices in $S \otimes D$ for D as in 5.1 (2), (3) and reducing them modulo p, one gets, assuming of course $k < p+1$ (Basic Assumption):

Proposition 5.3 ([17]). *Let V be a semi-stable p-adic representation of $G_{\mathbf{Q}_p}$ such that $D^*_{st}(V)$ is as in Example* 5.1(2) *and assume $k < p+1$. Let $T \subset V$ be a Galois stable $\mathcal{O}_E$-lattice. Then:*

$$\overline{T}|_{I_{\mathbf{Q}_p}} \otimes \overline{\mathbf{F}}_p \simeq \begin{pmatrix} \theta_{p^2-1}^{k-1} & 0 \\ 0 & \theta_{p^2-1}^{p(k-1)} \end{pmatrix}.$$

And, finally, the semi-stable non-crystalline case, which is somewhat more involved:

Theorem 5.4 ([9]). *Let V be a semi-stable p-adic representation of $G_{\mathbf{Q}_p}$ such that $D^*_{st}(V)$ is as in Example* 5.1(3) *and assume $k < p+1$. Let $T \subset V$ be a Galois stable $\mathcal{O}_E$-lattice. Define $\ell := \mathrm{val}(\mathcal{L})$, $[\ell]$ the greatest integer $\leq \ell$, and, if $\ell \in \mathbf{Z}$, $\alpha := \mathcal{L}/p^\ell$. Let $H_0 := 0$ and, for $n \in \mathbf{Z}_{>0}$, $H_n := \sum_{i=1}^n \frac{1}{i}$. Define also:*

$$a := (-1)^{\frac{k}{2}}\Big(-1 + \frac{k}{2}\Big(\frac{k}{2}-1\Big)\big(\mathcal{L} + 2H_{k/2-1}\big)\Big)$$

and if $\ell \in \{-\frac{k}{2}+2, -\frac{k}{2}+1, ..., -1\}$:

$$b := (-1)^{\frac{k}{2}-\ell}\left(\frac{k}{2}-\ell\right)\binom{\frac{k}{2}-1-\ell}{-2\ell+1}\alpha.$$

(1) *If* $\mathrm{val}(a) = 0$, *then*:

$$\overline{T} \simeq \begin{pmatrix} \theta_{p-1}^{\frac{k}{2}}\mathrm{Frob}(\overline{a}^{-1}\overline{\lambda}) & * \\ 0 & \theta_{p-1}^{\frac{k}{2}-1}\mathrm{Frob}(\overline{a\lambda}) \end{pmatrix}$$

or

$$\overline{T} \simeq \begin{pmatrix} \theta_{p-1}^{\frac{k}{2}-1}\mathrm{Frob}(\overline{a\lambda}) & * \\ 0 & \theta_{p-1}^{\frac{k}{2}}\mathrm{Frob}(\overline{a}^{-1}\overline{\lambda}) \end{pmatrix}.$$

(2) *If* $\mathrm{val}(a) > 0$, *then*:

$$\overline{T}|_{I_{\mathbf{Q}_p}} \otimes \overline{\mathbf{F}}_p \simeq \begin{pmatrix} \theta_{p^2-1}^{\frac{k}{2}-1+p\frac{k}{2}} & 0 \\ 0 & \theta_{p^2-1}^{\frac{k}{2}+p(\frac{k}{2}-1)} \end{pmatrix}.$$

(3) *If* $\mathrm{val}(a) < 0$ (*i.e.* $\ell < 0$), *then*:

- *if* $\ell < -\frac{k}{2}+2$, *then* $\overline{T}|_{I_{\mathbf{Q}_p}} \otimes \overline{\mathbf{F}}_p \simeq \begin{pmatrix} \theta_{p^2-1}^{k-1} & 0 \\ 0 & \theta_{p^2-1}^{p(k-1)} \end{pmatrix}$,
- *if* $-\frac{k}{2}+2 \leq \ell < 0$ *and* $\ell \notin \mathbf{Z}$, *then*:

$$\overline{T}|_{I_{\mathbf{Q}_p}} \otimes \overline{\mathbf{F}}_p \simeq \begin{pmatrix} \theta_{p^2-1}^{\frac{k}{2}-[\ell]+p(\frac{k}{2}+[\ell]-1)} & 0 \\ 0 & \theta_{p^2-1}^{\frac{k}{2}+[\ell]-1+p(\frac{k}{2}-[\ell])} \end{pmatrix},$$

- *if* $-\frac{k}{2}+2 \leq \ell < 0$ *and* $\ell \in \mathbf{Z}$, *then*:

$$\begin{aligned} \overline{T} &\simeq \begin{pmatrix} \theta_{p-1}^{\frac{k}{2}-\ell}\mathrm{Frob}(\overline{b}^{-1}\overline{\lambda}) & * \\ 0 & \theta_{p-1}^{\frac{k}{2}+\ell-1}\mathrm{Frob}(\overline{b\lambda}) \end{pmatrix} \\ &\text{or} \\ \overline{T} &\simeq \begin{pmatrix} \theta_{p-1}^{\frac{k}{2}+\ell-1}\mathrm{Frob}(\overline{b\lambda}) & * \\ 0 & \theta_{p-1}^{\frac{k}{2}-\ell}\mathrm{Frob}(\overline{b}^{-1}\overline{\lambda}) \end{pmatrix}. \end{aligned}$$

Remark 5.5. Proposition 5.3 and Theorem 5.4 are wrong in general for $k > p+1$.

These results can be applied to modular forms. Fix an embedding $\overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}}_p$. Let f be a cuspidal eigenform on $\Gamma_0(N)$ of weight $k \geq 2$ and

$\rho_f : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to GL_2(E_f)$ the p-adic global representation associated to f where $E_f \subset \overline{\mathbf{Q}}_p$ is a finite extension of $\mathbf{Q}_p$. Denote by $\overline{\rho}_f$ the semi-simplification modulo p of ρ_f and let $\overline{\rho}_{f,p} := \overline{\rho}_f|_{G_{\mathbf{Q}_p}}$. By [28] and [25], one easily deduces from Propositions 5.2, 5.3 and Theorem 5.4:

Corollary 5.6 (Deligne). *Let f be a cuspidal eigenform of weight k for $\Gamma_0(N)$ with $(p,N)=1$. Let a_p be the eigenvalue of the Hecke operator T_p and assume $\mathrm{val}(a_p)=0$. Then $\overline{\rho}_{f,p}$ is as in 5.2 with $\lambda \in \mathcal{O}_{E_f}^{\times}$ such that $p^{k-1}\lambda + \lambda^{-1} = a_p$.*

Corollary 5.7 (Fontaine, Serre). *Let f be a cuspidal eigenform of weight k for $\Gamma_0(N)$ with $(p,N)=1$ and $2 \leq k \leq p$. Let a_p be the eigenvalue of the Hecke operator T_p and assume $\mathrm{val}(a_p) \neq 0$. Then $\overline{\rho}_{f,p}$ is as in 5.3.*

Corollary 5.8 ([9]). *Let f be a cuspidal eigenform of weight k for $\Gamma_0(N)$ with $p \| N$ and $2 \leq k \leq p$. Assume f is new at p. Let a_p be the eigenvalue of the Hecke operator T_p and $\mathcal{L}_p(f) \in E_f$ the invariant associated to f ([23]). Then $\overline{\rho}_{f,p}$ is as in 5.4 with $\mathcal{L} = \mathcal{L}_p(f)$ and $\lambda = a_p/p^{k/2-1} \in \{\pm 1\}$.*

Remark 5.9. Corollaries 5.6 and 5.7 were originally proven in several letters (letter from Deligne to Serre (28/05/74) for the first and letters from Serre to Fontaine (27/05/79) and Fontaine to Serre (25/06/79 and 10/07/79) for the second). One can find published alternative proofs of these corollaries in [11] which don't use neither p-adic Hodge theory nor integral p-adic Hodge theory, i.e. don't use nor prove Propositions 5.2 and 5.3, but show that Corollary 5.7 also holds in weight $k = p+1$ (integral p-adic Hodge theory cannot yet deal directly with this case because of Assumption 2.1.2).

As a conclusion, let us mention the following fact. In [9], it is proven that there is a surprising link between the various cases of Theorem 5.4 and the Jordan-Hölder decomposition of the representation $\mathrm{Sym}^{k-2}\overline{\mathbf{F}}_p^2 \otimes_{\overline{\mathbf{F}}_p} \mathrm{St}$ of $\mathrm{GL}_2(\mathbf{Z}_p)$. Here St is the Steinberg representation of $\mathrm{GL}_2(\mathbf{Z}_p)$ in characteristic p, i.e. the inflation to $\mathrm{GL}_2(\mathbf{Z}_p)$ of the natural representation of $\mathrm{GL}_2(\mathbf{F}_p)$ on the space of functions $\mathbf{P}^1(\mathbf{F}_p) \to \overline{\mathbf{F}}_p$ with average value 0 (with $g \in \mathrm{GL}_2(\mathbf{F}_p)$ acting on a function through the usual action of g^{-1} on $\mathbf{P}^1(\mathbf{F}_p)$). This gives a mysterious link between integral p-adic Hodge theory and the representation theory of $\mathrm{GL}_2(\mathbf{Z}_p)$. I hope more is true in that direction.

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A Survey of the Hodge-Arakelov Theory of Elliptic Curves II

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Abstract.

The purpose of the present manuscript is to continue the survey of the *Hodge-Arakelov theory of elliptic curves* (cf. [7], [8], [9], [10], [11]) that was begun in [12]. This theory is a sort of "Hodge theory of elliptic curves" analogous to the classical *complex and p-adic* Hodge theories, but which exists in the *global arithmetic framework of Arakelov theory*. In particular, in the present manuscript, we focus on the aspects of the theory (cf. [9], [10], [11]) developed subsequent to those discussed in [12], but prior to the conference "Algebraic Geometry 2000" held in Nagano, Japan, in July 2000. These developments center around the natural connection that exists on the pair consisting of the *universal extension of an elliptic curve*, equipped with an *ample line bundle*. This connection gives rise to a natural object — which we call the *crystalline theta object* — which exhibits many interesting and unexpected properties. These properties allow one, in particular, to understand at a rigorous mathematical level the (hitherto purely "philosophical") relationship between the classical Kodaira-Spencer morphism and the Galois-theoretic *"arithmetic Kodaira-Spencer morphism"* of Hodge-Arakelov theory. They also provide a method (under certain conditions) for *"eliminating the Gaussian poles,"* which are the main obstruction to applying Hodge-Arakelov theory to diophantine geometry. Finally, these techniques allow one to give a new proof of the main result of [7] using *characteristic p methods*. It is the hope of the author to survey more recent developments (i.e., developments that occurred subsequent to "Algebraic Geometry 2000") in a sequel to the present manuscript.

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§1. General Introduction

We begin our general introduction to the topics presented in the present manuscript by reviewing the *fundamental argument from algebraic geometry* whose *arithmetic analogue* is the central goal of the Hodge-Arakelov theory of elliptic curves.

Let S be a *smooth, proper, geometrically connected algebraic curve* over an *algebraically closed field of characteristic zero* k. Let

$$E \to S$$

be a *family of one-dimensional semi-abelian varieties* whose generic fiber is *proper*. Thus, (except for a finite number of exceptions) the fibers of $E \to S$ are *elliptic curves*. Let us write

$$\Sigma \subseteq S$$

for the finite set of points over which the fiber of $E \to S$ fails to be an elliptic curve, and

$$\omega_E \stackrel{\text{def}}{=} \Omega_{E/S}|_{0_E}$$

for the restriction of the sheaf of relative differentials to the zero section of $E \to S$. Then the *height* of the family $E \to S$ is defined to be:

$$\mathrm{ht}_E \stackrel{\text{def}}{=} \deg_S(\omega_E^{\otimes 2})$$

(i.e., the degree on S of the line bundle in parentheses). The height is a measure of the *arithmetic complexity* of the family $E \to S$. For instance, the family is *isotrivial* (i.e., becomes trivial upon applying some finite flat base extension $T \to S$) if and only if $\mathrm{ht}_E = 0$.

In some sense, the most important property of the height in this context is the fact that (in the nonisotrivial case) it is *universally bounded* by invariants depending only on the pair (S, Σ). This bound — *"Szpiro's conjecture for function fields"* — is as follows:

$$\text{(1)} \qquad \mathrm{ht}_E \leq 2g_S - 2 + |\Sigma|$$

(where g_S is the genus of S, and $|\Sigma|$ is the cardinality of Σ). The proof, in the present geometric context, is the following simple argument: Write $\overline{\mathcal{M}}_{\mathrm{ell}}$ for the *compactified moduli stack of elliptic curves over* k, and $\infty \subseteq \overline{\mathcal{M}}_{\mathrm{ell}}$ for the divisor at infinity of this stack. Thus, $E \to S$ defines a *classifying morphism*

$$\kappa : S \to \overline{\mathcal{M}}_{\mathrm{ell}}$$

whose (logarithmic) *derivative*

$$d\kappa : \omega_E^{\otimes 2} \cong \kappa^* \Omega_{\overline{\mathcal{M}}_{\mathrm{ell}}/k}(\infty) \to \Omega_S(\Sigma)$$

is *nonzero* (so long as we assume that the family $E \to S$ is nonisotrivial). Thus, since $d\kappa$ is a generically nonzero morphism between line bundles on the curve S, the degree of its domain (i.e., ht_E) is $\leq$ the degree of its range (i.e., $2g_S - 2 + |\Sigma|$), so we obtain the desired inequality.

Note that in the above argument, the most essential ingredient is the *Kodaira-Spencer morphism*, i.e., the *derivative* $d\kappa$. Until recently, no analogue of such a derivative existed in the "arithmetic case" (i.e., of elliptic curves over number fields). On the other hand:

> *The Hodge-Arakelov theory of elliptic curves gives rise to a natural analogue of the Kodaira-Spencer morphism in the arithmetic context of an elliptic curve over a number field.*

A survey of the basic theory of this arithmetic Kodaira-Spencer morphism, together with a detailed explanation of the sense in which it may be regarded as being analogous to the classical geometric Kodaira-Spencer morphism, may be found in [12].

At a more technical level, in some sense the most *fundamental result* of Hodge-Arakelov theory is the following: Let E be an *elliptic curve over a field* K *of characteristic zero*. Let d be a positive integer, and $\eta \in E(K)$ a torsion point of order not dividing d. Write

$$\mathcal{L} \stackrel{\mathrm{def}}{=} \mathcal{O}_E(d \cdot [\eta])$$

for the line bundle on E corresponding to the divisor of multiplicity d with support at the point η. Write

$$E^\dagger \to E$$

for the *universal extension of the elliptic curve*, i.e., *the moduli space of pairs* $(\mathcal{M}, \nabla_{\mathcal{M}})$ *consisting of a degree zero line bundle* $\mathcal{M}$ *on* E, *equipped with a connection* $\nabla_{\mathcal{M}}$. Thus, $E^{\dagger}$ is an affine torsor on E under the module ω_E of invariant differentials on E. In particular, since $E^{\dagger}$ is (Zariski locally over E) the spectrum of a polynomial algebra in one variable with coefficients in the sheaf of functions on E, it makes sense to speak of the "relative degree over E" – which we refer to in this paper as the *torsorial degree* – of a function on $E^{\dagger}$. Note that (since we are in characteristic zero) the subscheme $E^{\dagger}[d] \subseteq E^{\dagger}$ of d-torsion points of $E^{\dagger}$ maps isomorphically to the subscheme $E[d] \subseteq E$ of d-torsion points of E. Then in its simplest form, the main theorem of [7] states the following:

Theorem 1.1. (Simple Version of the Hodge-Arakelov Comparison Isomorphism) *Let E be an elliptic curve over a field K of characteristic zero. Write $E^{\dagger} \to E$ for its universal extension. Let d be a positive integer, and $\eta \in E(K)$ a torsion point whose order does not divide d. Write $\mathcal{L} \stackrel{\text{def}}{=} \mathcal{O}_E(d \cdot [\eta])$. Then the natural map*

$$\Gamma(E^{\dagger}, \mathcal{L})^{<d} \to \mathcal{L}|_{E^{\dagger}[d]}$$

given by restricting sections of $\mathcal{L}$ over $E^{\dagger}$ whose torsorial degree is $< d$ to the d-torsion points $E^{\dagger}[d] \subseteq E^{\dagger}$ is a **bijection** *between K-vector spaces of dimension d^2.*

The remainder of the main theorem essentially consists of specifying precisely *how one must modify the integral structure of* $\Gamma(E^{\dagger}, \mathcal{L})^{<d}$ *over more general bases* in order to obtain an isomorphism at the finite and infinite primes of a number field, as well as for degenerating elliptic curves.

The relationship between Theorem 1.1 and the classical Kodaira-Spencer morphism is discussed in detail — albeit at a rather *philosophical* level — in [12], §1.3, 1.4. The analogue in the arithmetic case of the geometric argument used above to prove (1) is discussed in [12], §1.5.1. The upshot of this argument in the arithmetic case is that in order to derive diophantine equalities analogous to (1) in the arithmetic case — i.e., *Szpiro's conjecture* — it is necessary to eliminate certain *unwanted poles* — called *Gaussian poles* — that occur in the construction of the arithmetic Kodaira-Spencer morphism.

In the present manuscript, we discuss the following further developments in the theory (cf. §3, 4):

(i) a method for *eliminating the Gaussian poles* under certain conditions (cf. Corollary 3.5);
(ii) an argument which shows that (under certain conditions) the reduction in positive characteristic of the arithmetic Kodaira-Spencer morphism *coincides* with the classical geometric Kodaira-Spencer morphism (cf. Corollary 3.6);
(iii) a *alternative proof* of Theorem 1.1 using characteristic p methods (cf. Theorem 4.3).

Thus, development (i) *brings us closer to the goal of applying Hodge-Arakelov theory to proving Szpiro's conjecture* (for elliptic curves over number fields). Unfortunately, the conditions under which the argument of (i) may be carried out *do not hold* (at least in the naive sense) for elliptic curves over number fields. Nevertheless, there is substantial hope that certain new constructions will allow us to realize these conditions even for elliptic curves over number fields (cf. §3 for more on this issue). Development (ii) is significant in that it shows that the *analogy between the arithmetic and classical geometric Kodaira-Spencer morphisms* is not just philosophy, but *rigorous mathematics*! Finally, the significance of development (iii) is that it provides a much more *conceptual*, as well as *technically simpler* proof of the "fundamental theorem of Hodge-Arakelov theory" (i.e., Theorem 1.1).

Underlying all of these new developments (especially (i), (ii)) is the theory of the *crystalline theta object*, to be discussed in §2. This object is a *locally free sheaf equipped with a connection and a Hodge filtration*, hence is reminiscent of the *"$\mathcal{MF}^{\nabla}$-objects"* of [2], §2. On the other hand, many of its properties, such as *"Griffiths semi-transversality"* and *the vanishing of the (higher) p-curvatures*, are somewhat different from (and indeed, somewhat surprising from the point of view of the theory of) $\mathcal{MF}^{\nabla}$-objects. Nevertheless, these properties are of crucial importance in the arguments that underly developments (i), (ii).

§2. The Crystalline Theta Object

2.1. The Complex Analogue

We begin the discussion of this § by motivating our construction in the abstract algebraic case by first examining the complex analogue of the abstract algebraic theory.

Let E be an *elliptic curve over* $\mathbb{C}$ (the field of complex numbers). In this discussion of the complex analogue, we shall regard E as a *complex manifold* (rather than an algebraic variety). Let us write $\mathcal{O}_E$ (respectively, $\mathcal{O}_{E_{\mathbb{R}}}$) for the sheaf of complex analytic (respectively, real analytic)

complex-valued functions on E. In both the complex and real analytic categories, we have *exponential exact sequences*:

$$0 \longrightarrow 2\pi i \cdot \mathbb{Z} \longrightarrow \mathcal{O}_E \xrightarrow{\exp} \mathcal{O}_E^{\times} \longrightarrow 0$$

$$0 \longrightarrow 2\pi i \cdot \mathbb{Z} \longrightarrow \mathcal{O}_{E_{\mathbb{R}}} \xrightarrow{\exp} \mathcal{O}_{E_{\mathbb{R}}}^{\times} \longrightarrow 0$$

Since (as is well-known from analysis) $H^1(E, \mathcal{O}_{E_{\mathbb{R}}}) = H^2(E, \mathcal{O}_{E_{\mathbb{R}}}) = H^2(E, \mathcal{O}_E) = 0$, taking cohomology thus gives rise to the following exact sequences:

$$\begin{aligned} 0 \longrightarrow H^1(E, 2\pi i \cdot \mathbb{Z}) \longrightarrow \quad & H^1(E, \mathcal{O}_E) \longrightarrow H^1(E, \mathcal{O}_E^{\times}) \\ \xrightarrow{\deg} \; & H^2(E, 2\pi i \cdot \mathbb{Z}) = \mathbb{Z} \longrightarrow 0 \end{aligned}$$

$$0 \longrightarrow H^1(E, \mathcal{O}_{E_{\mathbb{R}}}^{\times}) \xrightarrow{\deg} H^2(E, 2\pi i \cdot \mathbb{Z}) = \mathbb{Z} \longrightarrow 0$$

In other words, (as is well-known) the isomorphism class of a *holomorphic line bundle* on E is not determined just by its degree (which is a topological invariant), but also has *continuous holomorphic moduli* (given by $H^1(E, \mathcal{O}_E) \neq 0$), while the isomorphism class of a *real analytic line bundle* is completely determined by its degree. Thus, in particular, *the complex analytic pair* $(E, \mathcal{L})$ *(i.e., a "polarized elliptic curve") has nontrivial moduli*, and in fact, even if the moduli of E are held fixed, $\mathcal{L}$ itself has nontrivial moduli. (Here, by "nontrivial moduli," we mean that there exist continuous families of such objects which are not locally isomorphic to the trivial family.) On the other hand, (if we write $\mathcal{L}_{\mathbb{R}} \stackrel{\text{def}}{=} \mathcal{L} \otimes_{\mathcal{O}_E} \mathcal{O}_{E_{\mathbb{R}}}$, then) *the real analytic pair* $(E_{\mathbb{R}}, \mathcal{L}_{\mathbb{R}})$ *has trivial moduli*, i.e., continuous families of such objects are always locally isomorphic to the trivial family. Put another way,

> *The real analytic pair* $(E_{\mathbb{R}}, \mathcal{L}_{\mathbb{R}})$ *is a* **topological invariant** *of the polarized elliptic curve* $(E, \mathcal{L})$.

Note that once one admits that $E_{\mathbb{R}}$ itself is a "topological invariant" of E (a fact which may be seen immediately by thinking of $E_{\mathbb{R}}$ as $H^1(E, \mathbf{S}^1)$, where $\mathbf{S}^1 \subseteq \mathbb{C}^{\times}$ is the unit circle), (one checks easily that) the fact that $\mathcal{L}_{\mathbb{R}}$ is also a topological invariant follows essentially from the fact that $H^1(E, \mathcal{O}_{E_{\mathbb{R}}}) = 0$. Note that if one thinks of the *universal extension* $E^{\dagger}$ as the *algebraic analogue* of $E_{\mathbb{R}}$, then the analogue of this fact in the algebraic context is given precisely by the (easily verified) fact:

$$H^1(E^\dagger, \mathcal{O}_{E^\dagger}) = 0$$

(that is, the first cohomology module of the algebraic coherent sheaf $\mathcal{O}_{E^\dagger}$ vanishes). In §2.3 below, we would like to exploit this fact to show that the pair $(E^\dagger, \mathcal{L}|_{E^\dagger})$ has a natural structure of *crystal* (valued in the category of "polarized varieties," or varieties equipped with an ample line bundle), and that — by placing an appropriate *integral structure* on $E^\dagger$ — one may show that this crystal exists naturally not only in characteristic 0, but also in *mixed characteristic*. Thus, in summary, the analogy that we wish to assert here is the following:

topological invariance of $(E_{\mathbb{R}}, \mathcal{L}_{\mathbb{R}})$ $\longleftrightarrow$ $(E^\dagger, \mathcal{L}|^\dagger_E)$ is a **crystal**

In fact, it is useful here to recall that in some sense:

> *The essential spirit of the "Hodge-Arakelov Theory of Elliptic Curves" may be summarized as being the Hodge theory of the pairs* $(E_{\mathbb{R}}, \mathcal{L}_{\mathbb{R}})$ *(at archimedean primes),* $(E^\dagger, \mathcal{L}|^\dagger_E)$ *(with appropriate integral structures — to be discussed in §2.3 below — at non-archimedean primes), as opposed to the "usual Hodge theory of an elliptic curve" which may be thought of as the Hodge theory of* $E_{\mathbb{R}}$ *(at archimedean primes) or* $E^\dagger$ *(at non-archimedean primes).*

In this connection, we note that the "Hodge-Arakelov theory of an elliptic curve at an archimedean prime" is discussed/reviewed in detail in [7], Chapter VII, §4.

2.2. The Case of Ordinary p-adic Elliptic Curves

Central to the theory of the mixed characteristic analogue of the ideas presented in §2.1 is the theory of the *étale integral structure on the universal extension* of an elliptic curve. In the present §, we would like to review the definition of the étale integral structure and discuss its structure in the case of an ordinary p-adic elliptic curve.

Let us work over a formal neighborhood of the point at infinity on the moduli stack of elliptic curves — i.e., say, the spectrum of a ring of power series of the form

$$S \stackrel{\text{def}}{=} \text{Spec}(\mathcal{O}[[q]])$$

where $\mathcal{O}$ is a Dedekind domain of mixed characteristic and q an indeterminate. Write $\widehat{S}$ for the completion of S with respect to the q-adic topology, and $E \to S$ for the *tautological degenerating elliptic curve* (more

precisely: one-dimensional semi-abelian scheme), with "q-parameter" equal to q. Then we have natural isomorphisms

$$E|_{\widehat{S}} \cong \mathbb{G}_m; \quad \omega_E = \mathcal{O}_S \cdot d\log(U); \quad E^\dagger|_{\widehat{S}} \cong \mathbb{G}_m \times \mathbb{A}^1$$

where we write U for the usual multiplicative coordinate on $\mathbb{G}_m$ (cf. [7], Chapter III, Theorem 2.1; the discussion of [7], Chapter III, §6, for more details). If we write "T" for the standard coordinate on this affine line, then near infinity, the *standard integral structure* on $E^\dagger$ may be described as that given by

$$\bigoplus_{r \geq 0} \mathcal{O}_{\mathbb{G}_m} \cdot T^r$$

On the other hand:

Definition 2.1. The *étale integral structure on* $E^\dagger$ (near infinity) is given by

$$\bigoplus_{r \geq 0} \mathcal{O}_{\mathbb{G}_m} \cdot \binom{T}{r}$$

where $\binom{T}{r} \stackrel{\text{def}}{=} \frac{1}{r!}T(T-1)\cdot\cdots\cdot(T-(r-1)))$.

Moreover, although the above definition of this integral structure is only valid near infinity, this integral structure *extends uniquely over the entire (compactified) moduli stack of elliptic curves* $\overline{\mathcal{M}}_{\text{ell}}$ (over $\mathbb{Z}$) — cf. [9], §1. (Note that this uniqueness is a consequence of the fact that $\overline{\mathcal{M}}_{\text{ell}}$ is a regular scheme of dimension 2.)

Thus, given a *family of elliptic curves* $E \to S$ over an arbitrary $\mathbb{Z}$-flat base S, we obtain a *naturally defined "étale integral structure"* on the universal extension $E^\dagger$, i.e., a group scheme

$$E^\dagger_{\text{et}} \to S$$

(which is not of finite type over S).

Next, we would like to examine this étale integral structure in more detail in the case of a *family of ordinary p-adic elliptic curves.* Thus, let S be a *p-adic formal scheme which is formally smooth over* $\mathbb{Z}_p$, and assume also that we are given a *family of ordinary elliptic curves*

$$E \to S$$

such that the associated classifying morphism $S \to (\mathcal{M}_{\text{ell}})_{\mathbb{Z}_p}$ is formally (i.e., relative to the p-adic topology) *étale.* (Here, by "ordinary," we

mean that the fibers of $E \to S$ over all the points of $S_{\mathbb{F}_p}$ have nonzero Hasse invariant.) For $n \geq 1$, write

$$E[p^n] \stackrel{\text{def}}{=} \text{Ker}([p^n] : E \to E)$$

for the kernel of multiplication by p^n on E. Then, as is well-known (cf., e.g., [6], p. 150), there is a unique exact sequence

$$0 \to E[p^n]^{\boldsymbol{\mu}} \to E[p^n] \to E[p^n]^{\text{et}} \to 0$$

of finite flat group schemes over E such that $E[p^n]^{\boldsymbol{\mu}}$ (respectively, $E[p^n]^{\text{et}}$) is étale locally isomorphic to $\boldsymbol{\mu}_{p^n}$ (respectively, $\mathbb{Z}/p^n\mathbb{Z}$).

Let us write:

$$E^{\text{F}^n} \stackrel{\text{def}}{=} E/E[p^n]^{\boldsymbol{\mu}}$$

Then since $E^{\text{F}^n} \to S$ is a family of elliptic curves, and the classifying morphism associated to $E \to S$ is *étale*, it follows that $E^{\text{F}^n} \to S$ defines a morphism:

$$\Phi_S^n : S \to S$$

One checks easily that $\Phi_S \stackrel{\text{def}}{=} \Phi_S^1$ is a *lifting of the Frobenius morphism in characteristic p*, and that Φ_S^n is the result of iterating Φ_S a total of n times (as the notation suggests). The morphism

$$\mathcal{V} : E^{\text{F}} \to E$$

given by forming the quotient of E^{F} by the image in E^{F} of $E[p]$ will be referred to as the the *Verschiebung morphism* associated to E. For any $n \geq 1$, the morphism

$$\mathcal{V}^n : E^{\text{F}^n} \to E$$

given by forming the quotient of E^{F^n} by the image in E^{F^n} of $E[p^n]$ is easily seen to be equal (as the notation suggests) to the "n-th iterate" (i.e., up to various appropriate base changes by iterates of Φ_S) of $\mathcal{V}$. Note that the kernel of $\mathcal{V}^n$ may be identified with $E[p^n]^{\text{et}}$. In particular, it follows that $\mathcal{V}^n$ is *étale of degree* p^n.

Thus, we obtain a tower

$$\cdots \to E^{\text{F}^n} \to E^{\text{F}^{n-1}} \to \cdots \to E^{\text{F}} \to E$$

of étale isogenies of degree p. Let us denote the p-adic completion of the inverse limit of this system of isogenies by E^{F^∞}. Thus, in particular, we have a natural morphism

$$E^{\mathrm{F}^\infty} \to E$$

In [9], §2.2, a certain canonical section

$$\kappa[p^n] : E^{\mathrm{F}^n} \otimes \mathbb{Z}/p^n\mathbb{Z} \to E^\dagger \otimes \mathbb{Z}/p^n\mathbb{Z}$$

is constructed of the universal extension of E over E^{F^n}. This section is the "ordinary analogue" of the section near infinity (cf. the discussion at the beginning of the present §2.2) given by sending $T \mapsto 0$. Letting $n \to \infty$, we thus obtain a morphism

$$\kappa[p^\infty] : E^{\mathrm{F}^\infty} \to (E^\dagger)^\wedge$$

(where "$\wedge$" denotes p-adic completion). Finally, by computing near infinity, one shows that this morphism in fact *factors uniquely through the étale integral structure*, i.e., determines a morphism:

$$\kappa^\infty_{\mathrm{et}} : E^{\mathrm{F}^\infty} \to (E^\dagger_{\mathrm{et}})^\wedge$$

Theorem 2.2. (Explicit Description of the Étale Integral Structure of an Ordinary Elliptic Curve) *The natural morphism*

$$\kappa^\infty_{\mathrm{et}} : E^{\mathrm{F}^\infty} \to (E^\dagger_{\mathrm{et}})^\wedge$$

from the p-adic completion of the "Verschiebung tower" of E to the p-adic completion of the universal extension of E equipped with the étale integral structure is an isomorphism.

We refer to [9], §2.2, for more details. The essential idea of the proof is to apply the following *well-known bijection* (due to Mahler — cf., e.g., [5], §3.2):

$$\mathrm{Comb}(\mathbb{Z}_p)^\wedge \xrightarrow{\sim} \mathrm{Cont}(\mathbb{Z}_p, \mathbb{Z}_p)$$

from the p-adic completion of the free $\mathbb{Z}_p$-module $\mathrm{Comb}(\mathbb{Z}_p)$ on the generators $\binom{T}{r}$ to the space of continuous functions $\mathbb{Z}_p \to \mathbb{Z}_p$ (given by evaluating the indeterminate T on elements of $\mathbb{Z}_p$).

Remark 2.3. One way to think of the content of Theorem 2.2 is as the assertion that:

> *The étale integral structure on $E^\dagger$ is very closely related to the p-adic Hodge theory of E.*

Since $E^{\dagger}_{\text{et}}$ is defined globally over $\mathbb{Z}$ whenever E is defined globally over $\mathbb{Z}$, Theorem 2.2 may also be taken as asserting that $E^{\dagger}_{\text{et}}$ enjoys the following remarkable interpretation:

> *The universal extension of an elliptic curve equipped with the étale integral structure is a natural globalization over $\mathbb{Z}$ of the (very local!) p-adic Hodge theory of the elliptic curve.*

This interpretation is very much in line with the general philosophy of Hodge-Arakelov theory (cf., e.g., [12], §1.3).

2.3. Integral Structures and Connections

Let

$$f : E \to S$$

be a *family of elliptic curves* over a scheme S. In order to formulate the various versions of the comparison isomorphism of Hodge-Arakelov theory (cf., e.g., Theorem 1.1) properly in mixed characteristic, it is necessary to consider not only the universal extension $E^{\dagger}$, but also certain $E^{\dagger}$-torsors — which we shall refer to as *Hodge torsors* — as follows. Let $\mathcal{L}$ be a *line bundle on E*. For simplicity, we assume that the (relative) degree (over S) of $\mathcal{L}$ is *one*. Then we define

$$E^{\mathcal{L}} \to E$$

to be the ω_E-torsor over E parametrizing (over an open $U \subseteq E$) the ($\mathcal{O}_S$-linear) *connections $\nabla_{\mathcal{L}}$ on the line bundle $\mathcal{L}$*. Often to simplify the notation (as well as to emphasize the analogy with $E^{\dagger}$), we will denote $E^{\mathcal{L}}$ by

$$E^*$$

(where we think of the "$*$" as being set equal to $\mathcal{L}$).

Then E^* admits a *natural $E^{\dagger}$-action*, defined as follows. If $\nabla_{\mathcal{L}}$ is the connection corresponding to a section of $E^* \to E$ over some open $U \subseteq E$, and ∇_α is a connection on the degree zero line bundle $\mathcal{O}_E([0_E] - [\alpha])$ for some point $\alpha \in E(S)$, then we let the point (α, ∇_α) of $E^{\dagger}(S)$ act on E^* by

$$\nabla_{\mathcal{L}} \mapsto (\mathcal{T}_\alpha^* \nabla_{\mathcal{L}}) \otimes \nabla_\alpha$$

— where we denote by $\mathcal{T}_\alpha : E \to E$ the morphism "translation by α," and we observe that

$$(\mathcal{T}_\alpha^* \mathcal{L}) \otimes \mathcal{O}_E([0_E] - [\alpha]) \xrightarrow{\sim} \mathcal{L}$$

(by the elementary theory of line bundles on elliptic curves), so $(\mathcal{T}_\alpha^* \nabla_{\mathcal{L}}) \otimes \nabla_\alpha$ defines a connection on $\mathcal{L}$ over $\mathcal{T}_\alpha^* U$, i.e., a section of $E^* \to E$ over $\mathcal{T}_\alpha^* U$, as desired. Thus, we obtain an action of $E^\dagger$ on E^* which is *compatible* (via the projections $E^\dagger \to E$, $E^* \to E$) with the usual translation action of E on itself. Moreover, it is an easy exercise to show that *the action of $E^\dagger$ on E^* defines a structure of $E^\dagger$-torsor on E^*.*

When S is *$\mathbb{Z}$-flat*, and $\mathcal{L}$ is the degree one line bundle defined by a *torsion point* $\eta \in E(S)$, then there is also a natural analogue of the *étale integral structure* on $E^\dagger$ (cf. §2.2) for E^*. For simplicity, we restrict ourselves here to the case where $* = \mathcal{O}_E([0_E])$. Then (since $(\overline{\mathcal{M}}_{\mathrm{ell}})_{\mathbb{Z}}$ is connected and regular of dimension two) the étale integral structure on E^* is uniquely determined over arbitrary S once it is determined "near infinity," i.e., in the case $S = \mathrm{Spec}(\mathbb{Z}[[q]])$. Now, as we saw at the beginning of §2.2, the *standard integral structure on $E^\dagger$* is given (near infinity) by:

$$\bigoplus_{r \geq 0} \mathcal{O}_{\mathbb{G}_m} \cdot T^r$$

Relative to this notation, the *standard integral structure on E^** is given by:

$$\bigoplus_{r \geq 0} \mathcal{O}_{\mathbb{G}_m} \cdot (T - \frac{1}{2})^r$$

Thus, just as the *étale integral structure on $E^\dagger$* is given by:

$$\bigoplus_{r \geq 0} \mathcal{O}_{\mathbb{G}_m} \cdot \binom{T}{r}$$

it should not surprise the reader that the *étale integral structure on E^** is given by:

$$\bigoplus_{r \geq 0} \mathcal{O}_{\mathbb{G}_m} \cdot \binom{T - \frac{1}{2}}{r}$$

Here, we note that in the case of a *more general torsion point* $\eta \in E(\mathcal{O}[[q^{1/N}]])$ (where $\mathcal{O}$ is the ring of integers of some number field), the "$\frac{1}{2}$" is replaced by a rational number of the form $\beta - \frac{1}{2}$, where $\beta \in \mathbb{Q}$ is any representative of the unique class in $\mathbb{Q}/\mathbb{Z}$ such that — if we think of $E(\mathcal{O}[[q^{1/N}]])$ as $\mathbb{G}_m(\mathcal{O}[[q^{1/N}]][q^{-1}])/q^{\mathbb{Z}}$ via the Schottky uniformization

of the Tate curve E — then η is equal to the point determined by $\zeta \cdot q^{\beta}$ for some root of unity ζ.

Just as in the case of $E^{\dagger}$, the étale integral structure on E^* extends over all of $(\overline{\mathcal{M}}_{\mathrm{ell}})_{\mathbb{Z}}$. At primes other than 2, this follows immediately from the case of $E^{\dagger}$. On the other hand, at the prime 2, this extension result is much more difficult to prove (cf. [9], §8.2, 8.3) and requires various complicated 2-adic computations involving higher p-curvatures (where $p = 2$). At any rate, once this extension is made, one obtains an S-scheme (which is not of finite type over S)

$$E^*_{\mathrm{et}}$$

equipped with a natural structure of $E^{\dagger}_{\mathrm{et}}$-*torsor*.

Next, let us assume that S is *smooth of finite type over* $\mathbb{Z}$. In this situation, it is proven in [9], Lemma 5.1, that the well-known connection on $E^{\dagger}$ determines natural connections on $E^{\dagger}_{\mathrm{et}}$ and E^*_{et}. Moreover, by using the fact (cf. [9], Theorem 4.3) that

$$f_*(\mathcal{O}_{E^*_{\mathrm{et}}}) = \mathcal{O}_S; \quad \mathbf{R}^1 f_*(\mathcal{O}_{E^*_{\mathrm{et}}}) = 0$$

(where, by abuse of notation, we denote all structure morphisms to S by "f") one derives (cf. [9], Theorem 5.2) the following *fundamental* (i.e., relative to the theory of [9]) *result* (cf. the discussion of §2.1):

Theorem 2.4. *Let $\epsilon \in E^*_{\mathrm{et}}(S)$ be a horizontal point of E^*_{et}. Then restriction to ϵ defines a natural bijection between connections (over $\mathbb{Z}$) on the pair $(E^*_{\mathrm{et}}, \mathcal{L}|_{E^*_{\mathrm{et}}})$ and connections on the line bundle $\mathcal{L}|_{\epsilon}$ on S.*

Remark 2.5. Typically, in the situations that we are interested in, there will be a natural choice of connection on $\mathcal{L}|_{\epsilon}$, so we shall not discuss this technical issue further in this survey. Thus, (once one chooses a connection on $\mathcal{L}|_{\epsilon}$) there is a *natural choice of connection on* $(E^*_{\mathrm{et}}, \mathcal{L}|_{E^*_{\mathrm{et}}})$. In particular, we thus obtain a natural *crystal valued in the category of schemes equipped with an ample line bundle* on the crystalline site of PD-thickenings of S over $\mathbb{Z}$.

Before proceeding, we observe that the crucial fact

$$f_*(\mathcal{O}_{E^*_{\mathrm{et}}}) = \mathcal{O}_S; \quad \mathbf{R}^1 f_*(\mathcal{O}_{E^*_{\mathrm{et}}}) = 0$$

— which is immediate in characteristic 0 — holds in mixed characteristic *only for the étale integral structure*, not for the standard integral structure on E^*. This is the *technical reason* for the appearance of the étale integral structure in Theorem 2.4. It is an interesting coincidence that this integral structure just happens to be the *same integral structure* as

the integral structure that was applied in [7] to make the comparison isomorphism (cf. Theorem 1.1) valid in mixed characteristic.

Once one has a connection on the pair $(E^*_{\text{et}}, \mathcal{L}|_{E^*_{\text{et}}})$, the next natural step is to consider the *resulting connection* $\nabla_{\mathcal{V}}$ on the locally free (albeit of infinite rank!) sheaf

$$\mathcal{V} \stackrel{\text{def}}{=} f_*(\mathcal{L}|_{E^*_{\text{et}}})$$

on S. Note that by considering the *"torsorial degree"* of sections of $\mathcal{L}$ over E^*_{et} (i.e., relative degree over E), one obtains a natural *"Hodge filtration"*

$$F^r(\mathcal{V}) \subseteq \mathcal{V}$$

whose subquotients are given by:

$$(F^{r+1}/F^r)(\mathcal{V}) = \frac{1}{r!} \cdot \tau_E^{\otimes r} \otimes_{\mathcal{O}_S} f_*(\mathcal{L})$$

(where $f_*(\mathcal{L})$ is the push-forward of the original line bundle $\mathcal{L}$ on E). The triple

$$(\mathcal{V}, F^r(\mathcal{V}), \nabla_{\mathcal{V}})$$

is referred to as the *crystalline theta object* (cf. [9], Theorem 8.1) and is the main object of study in [9].

2.4. Comparison Isomorphism at Infinity

In this §, we would like to take a closer look at the *crystalline theta object* introduced in §2.3 in a neighborhood of infinity, i.e., over the base $S = \text{Spec}(\mathbb{Z}[[q]])$. In this case, if we think of the tautological elliptic curve over $U_S \stackrel{\text{def}}{=} S[q^{-1}]$, i.e., the *Tate curve*, as the quotient

$$\mathbb{G}_m/q^{\mathbb{Z}}$$

and write U for the standard multiplicative coordinate on $\mathbb{G}_m$, then sections of $\mathcal{V} \stackrel{\text{def}}{=} f_*(\mathcal{L}|_{E^*_{\text{et}}})$ may be thought of as linear combinations of the *theta function*

$$\Theta \stackrel{\text{def}}{=} \sum_{k \in \mathbb{Z}} (-1)^k \cdot q^{\frac{1}{2}k^2 - \frac{1}{2}k} \cdot U^{k-\frac{1}{2}} \cdot \theta$$

(where "$U^{-\frac{1}{2}} \cdot \theta$" is a certain natural trivialization of $\mathcal{L}$ over $\mathbb{G}_m$, and we note that the exponent of q is always *integral*) and its *derivatives* (i.e., the result of applying polynomials (with $\mathcal{O}_S$-coefficients) in $(U \cdot \partial/\partial U)$).

In fact, in the present context, it is natural to consider certain *specific derivatives* (i.e., *"congruence canonical Schottky-Weierstrass zeta functions"* — cf. [9], §6) of the theta function, as follows. For integers $r \geq 0$, let

$$\lambda_r \stackrel{\text{def}}{=} \lfloor \frac{r}{2} - \frac{1}{2} \rfloor$$

(i.e., the greatest integer $\leq$ the number inside the "$\lfloor - \rfloor$");

$$F^r(\mathbb{Z}) \stackrel{\text{def}}{=} \{0 - \lambda_r, 1 - \lambda_r, \dots, r - 1 - \lambda_r\} \subseteq \mathbb{Z}$$

Thus, $F^{r+1}(\mathbb{Z}) \supseteq F^r(\mathbb{Z})$ is obtained from $F^r(\mathbb{Z})$ by appending one more integer "$k[r]$" directly to the left/right of $F^r(\mathbb{Z})$ (where "left/right" depends only on the parity of r). In particular, the map $\mathbb{Z}_{\geq 0} \to \mathbb{Z}$ given by

$$r \mapsto k[r]$$

is a *bijection*. Also, let us write:

$$\Psi(k) \stackrel{\text{def}}{=} \frac{1}{2}k^2 - \frac{1}{2}k$$

Then a topological (with respect to the q-adic topology) $\mathcal{O}_{\widehat{S}}$-basis of $\mathcal{V}$ is given by $\zeta_0^{\text{CG}}, \zeta_1^{\text{CG}}, \dots, \zeta_r^{\text{CG}}, \dots \in \mathcal{V}$, where we define:

$$\zeta_r^{\text{CG}} = \sum_{k \in \mathbb{Z}} \binom{k + \lambda_r}{r} \cdot (-1)^k \cdot q^{\frac{1}{2}k^2 - \frac{1}{2}k} \cdot U^{k - \frac{1}{2}} \cdot \theta$$

(cf. [7], Chapter V, Theorem 4.8).

Next, let us recall that the integral structure on E^*_{et} was defined by:

$$\bigoplus_{r \geq 0} \mathcal{O}_{\mathbb{G}_m} \cdot \binom{T - \frac{1}{2}}{r}$$

hence that sending $T \mapsto 0$ defines a q-adic section

$$\kappa_{\mathbb{G}_m} : (\mathbb{G}_m)_{\widehat{S}} \to E^*_{\text{et}}|_{\widehat{S}}$$

(at least after one inverts 2). Then pulling back via $\kappa_{\mathbb{G}_m}$ and applying the *trivialization* of $\mathcal{L}|_{\mathbb{G}_m}$ given by "$U^{-\frac{1}{2}} \cdot \theta$" yields a natural *evaluation morphism*:

$$\Xi : \mathcal{V} \to \mathcal{L}|_{(\mathbb{G}_m)_{\widehat{S}}} \stackrel{\sim}{\to} \mathcal{O}_{(\mathbb{G}_m)_{\widehat{S}}}$$

which is, in fact, *integral over* $\mathbb{Z}$ (cf. [9], Theorem 6.1). Moreover, the connection $\nabla_{\mathcal{V}}$ is *projectively compatible* — that is, compatible up to a possible "error term" consisting of a *scalar-valued* (logarithmic) differential on S (cf. [9], Theorem 6.1, for more details) — with the natural connection on $\mathcal{O}_{(\mathbb{G}_m)_{\widehat{S}}}$ (regarded as a quasi-coherent $\mathcal{O}_{\widehat{S}}$-module) arising from the fact that $(\mathbb{G}_m)_{\widehat{S}}$ arises by pulling back to $\widehat{S}$ the $\mathbb{Z}$-*scheme* $\mathbb{G}_m$ (i.e., put another way, the unique connection for which integral powers of U are *horizontal*).

Next, let us observe (cf. [7], Chapter V, Theorem 4.8) that the sections ζ_r^{CG} introduced above have the following *congruence property*:

$$\Xi(\zeta_r^{\mathrm{CG}}) \equiv 0 \pmod{q^{\Psi(k[r])}}$$

Thus, it is natural to consider the sections

$$\widetilde{\zeta}_r^{\mathrm{CG}} \stackrel{\mathrm{def}}{=} q^{-\Psi(k[r])} \cdot \zeta_r^{\mathrm{CG}} \in q^{-\infty} \cdot \mathcal{V}$$

which define a *new integral structure* on $\mathcal{V}$, which we denote by

$$\mathcal{V}^{\mathrm{GP}}$$

— where the "GP" stands for *"Gaussian poles,"* i.e., the poles arising from the $q^{-\Psi(k[r])}$ — cf. [7], Chapter VI, Theorem 4.1 and the Remarks following that theorem. In particular, the $\widetilde{\zeta}_r^{\mathrm{CG}}$ form a *topological* $\mathcal{O}_{\widehat{S}}$-*basis* for $\mathcal{V}^{\mathrm{GP}}$, and Ξ factors through $\mathcal{V}^{\mathrm{GP}}$ to form a morphism:

$$\Xi^{\mathrm{GP}} : \mathcal{V}^{\mathrm{GP}} \to \mathcal{O}_{(\mathbb{G}_m)_{\widehat{S}}}$$

Now we have the following *Schottky-theoretic analogue* (i.e., d-torsion points are replaced by "$\mathbb{G}_m$") of the original *Hodge-Arakelov comparison isomorphism* (cf., Theorem 1.1):

Theorem 2.6. (Schottky-Theoretic Hodge-Arakelov Comparison Isomorphism) *The evaluation map introduced above is an isomorphism:*

$$\Xi^{\mathrm{GP}} : \mathcal{V}^{\mathrm{GP}} \stackrel{\sim}{\to} \mathcal{O}_{(\mathbb{G}_m)_{\widehat{S}}}$$

which is **projectively horizontal** *with respect to the natural (logarithmic) connection* $\nabla_{\mathcal{V}}$ *(cf. Theorem* 2.4*) on the left and the trivial connection (i.e., for which the integral powers of* U *are horizontal) on the right.*

Proof. This is essentially a combination of [9], Theorems 6.1, 6.2. The main idea is that, since the integral powers of U form a topological $\mathcal{O}_{\widehat{S}}$-basis of $\mathcal{O}_{(\mathbb{G}_m)_{\widehat{S}}}$, it suffices to prove the result modulo q, in which case

it essentially follows from the fact that the binomial coefficient functions form a $\mathbb{Z}$-basis of the space of integer-valued polynomial functions on $\mathbb{Z}$. Q.E.D.

A number of interesting corollaries may be read off of Theorem 2.6, as follows. The first such corollary is the *computation of the monodromy* of $(\mathcal{V}, \nabla_{\mathcal{V}})$ at the point at infinity, i.e., $V(q) \subseteq S$ (cf. [9], Corollary 6.3). More important from the point of view of the further development of the theory is the *vanishing of the p-curvature* (cf. [3], §5,6, for a discussion of the notion of "p-curvature") — cf. [9], Corollary 6.4.

In fact, in the present context, in addition to the p-curvature, the *higher p-curvatures* of the pair $(\mathcal{V}, \nabla_{\mathcal{V}})$ may also be defined (cf. [9], §7.1). Then one has the following result (cf. [9], Corollary 7.6):

Corollary 2.7. *All of the higher p-curvatures of the crystalline theta object vanish (over an arbitrary $\mathbb{Z}$-smooth base S).*

Proof. Note that since these higher p-curvatures all form sections of *locally free sheaves* (at least in, say, the universal case $S = (\overline{\mathcal{M}}_{\mathrm{ell}})_{\mathbb{Z}}$) it suffices to check that they are zero in a neighborhood of infinity, i.e., for the present $S = \mathrm{Spec}(\mathbb{Z}[[q]])$. Thus, we may apply the isomorphism of Theorem 2.6, so Corollary 2.7 follows from the fact that $\mathcal{O}_{(\mathbb{G}_m)_{\widehat{S}}}$ is (topologically) generated by its *horizontal sections.* Q.E.D.

In more down-to-earth terms, Corollary 2.7 may be interpreted as follows. It follows from the general theory of integrable connections that in any *PD formal neighborhood* (where, "PD" stands for "puissances divisées," i.e., divided powers) of a point of S, sections of $\mathcal{V}$ may be *PD formally integrated* to form *horizontal sections of* $(\mathcal{V}, \nabla_{\mathcal{V}})$ in the given PD formal neighborhood. In general, such horizontal sections are not defined on a *formal neighborhood* of the given point of S, i.e., (if, for instance, t is a local coordinate on a relatively one-dimensional smooth S over $\mathbb{Z}$, then) such horizontal sections are only defined once one introduces the *divided powers*:

$$\frac{t^n}{n!}$$

(where $n \geq 0$ is an integer). To state then that *"all the higher p-curvatures vanish"* means that *such horizontal sections exist as formal power series in t with integral coefficients.*

The above discussion motivates the following point of view. Near infinity, the horizontal sections of the above discussion are simply the integral powers of U. Thus, the *classical theta expansion*

$$\Theta \stackrel{\text{def}}{=} \sum_{k \in \mathbb{Z}} (-1)^k \cdot q^{\frac{1}{2}k^2 - \frac{1}{2}k} \cdot U^{k-\frac{1}{2}} \cdot \theta$$

may be thought of (in the present context) as the *"expansion of a generator of $F^1(\mathcal{V})$ in terms of local horizontal sections of $(\mathcal{V}, \nabla_{\mathcal{V}})$."* In particular, if one takes this point of view, then it is natural to consider the expansion of a generator of $F^1(\mathcal{V})$ in terms of local horizontal sections of $(\mathcal{V}, \nabla_{\mathcal{V}})$ at points of $(\overline{\mathcal{M}}_{\text{ell}})_{\mathbb{Z}}$ other than the point at infinity. These *crystalline theta expansions* are discussion in more detail in [9], §7.2.

2.5. The Associated Kodaira-Spencer Morphism

Let

$$f : E \to S$$

be a *family of elliptic curves* over a *smooth $\mathbb{Z}$-scheme S*. We shall write $(\mathcal{V}, F^r(\mathcal{V}), \nabla_{\mathcal{V}})$ for the associated *crystalline theta object* (cf. §2.3).

In this §, we would like to consider the relationship between $\nabla_{\mathcal{V}}$ and $F^r(\mathcal{V})$. The first, most fundamental property of this relationship is that, unlike the $\mathcal{MF}^{\nabla}$-objects of [2], §2, which satisfy *Griffiths transversality*, i.e., the connection maps F^r into F^{r+1}, the crystalline theta object only satisfies the following weaker property (cf. [9], Theorem 8.1):

Theorem 2.8. (Griffiths Semi-transversality) *The crystalline theta object satisfies "Griffiths semi-transversality," i.e., $\nabla_{\mathcal{V}}$ maps $F^r(\mathcal{V})$ into $F^{r+2}(\mathcal{V})$ for all integers $r \geq 0$.*

Remark 2.9. The reason that the crystalline theta object is only "semi-transversal" is the following: The connection of Theorem 2.4 is defined in *two steps*. If one thinks in terms of isomorphisms between pull-backs to "nearby points" that differ by a square nilpotent ideal, *first* one must establish the isomorphism between the two pull-backs of E^*_{et}. This causes the Hodge filtration to jump *once*. *Then* (once one has an isomorphism between the underlying schemes, i.e., between the two pull-backs of E^*_{et}) one must establish an isomorphism between the two pull-backs of $\mathcal{L}$. This causes the Hodge filtration to jump *once more*. We refer to the discussion of [9], §8.1, for more details.

Thus, by analogy to the usual Kodaira-Spencer morphism in the Griffiths-transversal case (which measures the extent to which the connection fails to *preserve* the Hodge filtration, i.e., map F^r into F^r), it is natural to define the *Kodaira-Spencer morphism of the crystalline theta object* as follows: By Theorem 2.8, $\nabla_{\mathcal{V}}$ defines a morphism:

$$F^1(\mathcal{V}) \to F^3(\mathcal{V}) \otimes_{\mathcal{O}_S} \Omega_{S/\mathbb{Z}}$$

If we then compose with the projection $F^3 \twoheadrightarrow F^3/F^2$, then we obtain an $\mathcal{O}_S$-linear morphism:

$$\kappa_{\mathcal{V}} : F^1(\mathcal{V}) \to (F^3/F^2)(\mathcal{V}) \otimes_{\mathcal{O}_S} \Omega_{S/\mathbb{Z}} \xrightarrow{\sim} \frac{1}{2} \cdot \tau_E^{\otimes 2} \otimes_{\mathcal{O}_S} F^1(\mathcal{V}) \otimes_{\mathcal{O}_S} \Omega_{S/\mathbb{Z}}$$

Moreover, since $F^1(\mathcal{V}) = f_*\mathcal{O}_E([0_E])$ is a *line bundle*, it follows that we may regard $\kappa_{\mathcal{V}}$ as an $\mathcal{O}_S$-linear morphism:

$$\omega_E^{\otimes 2} \to \frac{1}{2} \cdot \Omega_{S/\mathbb{Z}}$$

On the other hand, we recall that the *"classical Kodaira-Spencer morphism"* of the family $E \to S$ is defined as follows: If we write

$$\mathcal{E} \stackrel{\text{def}}{=} \mathbf{R}^1 f_{\text{DR},*}\mathcal{O}_E$$

for the first relative de Rham cohomology module of $E \to S$, and $F^1(\mathcal{E}) \subseteq \mathcal{E}$ (respectively, $\nabla_{\mathcal{E}}$) for its Hodge filtration (respectively, Gauss-Manin connection), then $\nabla_{\mathcal{E}}$ defines a morphism

$$F^1(\mathcal{E}) \to \mathcal{E} \otimes_{\mathcal{O}_S} \Omega_{S/\mathbb{Z}}$$

which may be composed with the projection $\mathcal{E} \twoheadrightarrow \mathcal{E}/F^1(\mathcal{E}) \xrightarrow{\sim} F^1(\mathcal{E}) \otimes_{\mathcal{O}_S} \tau_E^{\otimes 2}$ to obtain a morphism:

$$\kappa_{\mathcal{E}} : F^1(\mathcal{E}) \to F^1(\mathcal{E}) \otimes_{\mathcal{O}_S} \tau_E^{\otimes 2} \otimes_{\mathcal{O}_S} \Omega_{S/\mathbb{Z}}$$

Since $F^1(\mathcal{E}) = \omega_E$ is a line bundle, $\kappa_{\mathcal{E}}$ may thus be regarded as a morphism:

$$\omega_E^{\otimes 2} \to \Omega_{S/\mathbb{Z}}$$

Then we have the following result (cf. [9], Theorem 8.1):

Theorem 2.10. (The Kodaira-Spencer Morphism of the Crystalline Theta Object) *The Kodaira-Spencer morphism*

$$\kappa_{\mathcal{V}} : \omega_E^{\otimes 2} \to \frac{1}{2} \cdot \Omega_{S/\mathbb{Z}}$$

associated to the crystalline theta object $(\mathcal{V}, F^r(\mathcal{V}), \nabla_{\mathcal{V}})$ *is equal to* $\frac{1}{2}$ *times the classical Kodaira-Spencer morphism*

$$\kappa_{\mathcal{E}} : \omega_E^{\otimes 2} \to \Omega_{S/\mathbb{Z}}$$

associated to the relative first de Rham cohomology module of the family $E \to S$.

Proof. It suffices to prove this result in the universal case, i.e., when $S = (\overline{\mathcal{M}}_{\mathrm{ell}})_{\mathbb{Z}}$. Moreover, the asserted equality clearly holds over all of $(\overline{\mathcal{M}}_{\mathrm{ell}})_{\mathbb{Z}}$ if and only if it holds in a neighborhood of infinity. Thus, we may assume that $S = \mathrm{Spec}(\mathbb{Z}[[q]])$. Now the main idea is to consider the theta expansion:

$$\Theta \stackrel{\mathrm{def}}{=} \sum_{k \in \mathbb{Z}} (-1)^k \cdot q^{\frac{1}{2}k^2 - \frac{1}{2}k} \cdot U^k \cdot (U^{-\frac{1}{2}} \cdot \theta)$$

By Theorem 2.6, the connection $\nabla_{\mathcal{V}}$ amounts (in the context of such expansions) to the (logarithmic) partial derivative $q \cdot \partial/\partial q$. On the other hand, generators of $F^r(\mathcal{V})$ for $r > 1$ are obtained by taking various derivatives of Θ with respect to U. Since applying $q \cdot \partial/\partial q$ to the k-th term of the expansion amounts to multiplying the k-th term by $\frac{1}{2}k^2 - \frac{1}{2}k$, it follows that (if we ignore the trivialization "$(U^{-\frac{1}{2}} \cdot \theta)$," then) applying $q \cdot \partial/\partial q$ to the k-th term of the expansion gives the same result as applying $\frac{1}{2}(U \cdot \partial/\partial U)^2 - \frac{1}{2}(U \cdot \partial/\partial U)$ to this term. That is to say, we have:

$$\{q \cdot \partial/\partial q\} \cdot \Theta = \{\frac{1}{2}(U \cdot \partial/\partial U)^2 - \frac{1}{2}(U \cdot \partial/\partial U)\} \cdot \Theta$$

On the other hand, near infinity, the classical Kodaira-Spencer morphism amounts to the identification of "$q \cdot \partial/\partial q$" with "$(U \cdot \partial/\partial U)^2$." This completes the proof of the asserted equality. Q.E.D.

Remark 2.11. The *vanishing* of the p-curvatures (cf. Corollary 2.7) in the presence of a Kodaira-Spencer morphism which is an *isomorphism* (cf. Theorem 2.10) is somewhat surprising from the point of view of the classical theory of $\mathcal{MF}^{\nabla}$-objects arising from families of varieties, in which *vanishing of the p-curvature is related to vanishing of the Kodaira-Spencer morphism* (cf. [4]).

§3. Lagrangian Galois Actions

In this §, we apply the theory of §2 to

(i) show how to *eliminate the Gaussian poles* from (a certain version of) the arithmetic Kodaira-Spencer morphism under certain conditions (cf. Corollary 3.5);

(ii) show that the reduction in positive characteristic of (a certain version of) the arithmetic Kodaira-Spencer morphism *coincides* with

the classical geometric Kodaira-Spencer morphism (cf. Corollary 3.6).

The ideas surveyed in this § are discussed in more detail in [10], §2 (in the case of odd p); [11], §3 (in the case of $p = 2$). The key idea here is that the theory of the *crystalline theta object* allows one to study the (globally defined) *discrete* arithmetic Kodaira-Spencer morphism from a (local p-adic) *continuous* point of view.

Let p be an *odd prime*. Let $K \stackrel{\text{def}}{=} \mathbb{Q}_p(\zeta_p)$, where ζ_p is a *primitive p-th root of unity*. Write $\mathcal{O}_K$ (respectively, $\mathfrak{m}$; k) for the ring of integers of K (respectively, maximal ideal of $\mathcal{O}_K$; residue field of $\mathcal{O}_K$);

$$U_S \stackrel{\text{def}}{=} S \backslash V(q) \subseteq S \stackrel{\text{def}}{=} \operatorname{Spec}(\mathcal{O}_K[[q]])$$

and

$$E \to S$$

for the tautological *degenerating elliptic curve* (more precisely: one-dimensional semi-abelian scheme) with "q-parameter" equal to q over S. Also, let us write:

$$Z \stackrel{\text{def}}{=} \operatorname{Spec}(\mathcal{O}_K[[q^{\frac{1}{p}}]])$$

Thus, over $\mathbb{Q}$, $Z_{\mathbb{Q}} \to S_{\mathbb{Q}}$ is finite, étale over $(U_S)_{\mathbb{Q}}$, and Galois, with Galois group

$$G = \mathbb{F}_p(1)$$

(where the "(1)" denotes a Tate twist). As is well-known, over U_S, the *p-torsion points* of $E \to S$ fit into a natural exact sequence:

$$0 \to \boldsymbol{\mu}_p|_{U_S} \to E[p]|_{U_S} \to \mathbb{F}_p|_{U_S} \to 0$$

which splits over $U_Z \stackrel{\text{def}}{=} U_S \times_S Z$. In the following discussion, we consider the *particular splitting*

$$H|_{U_Z} \subseteq E[p]|_{U_Z}$$

(i.e., subgroup scheme $H|_{U_Z}$ that projects isomorphically onto $\mathbb{F}_p|_{U_Z}$) of this exact sequence defined by the p-th root of q given by $q^{\frac{1}{p}}$.

Next, let us write $C \to S$ for the unique *semi-stable model* compactifying $E \to S$. Thus, C is *regular*, and the complement of its unique node is equal to E. On the other hand, $C_Z \stackrel{\text{def}}{=} C \times_S Z$ is no longer regular. Let us denote by

$$C'_Z \to Z$$

the unique *regular semi-stable model* over Z compactifying $E|_{U_Z}$. Thus, the complement

$$E'_Z \to Z$$

of the nodes of C'_Z has a natural group scheme structure whose special fiber (i.e., fiber over $V(q^{\frac{1}{p}})$) is a disjoint union of p copies of $\mathbb{G}_m$.

Next, let us observe that $H|_{U_Z}$ determines (by taking the closure in E'_Z) a *closed subgroup scheme*

$$H \subseteq E'_Z$$

of E'_Z. We may then form the *quotient*:

$$E_Z \subseteq E'_Z \twoheadrightarrow E_H \stackrel{\text{def}}{=} E'_Z/H$$

Thus, $E_H \to Z$ is a *one-dimensional semi-abelian scheme* with "q-parameter" equal to $q^{\frac{1}{p}}$. Let us write

$$\mathcal{L}_H \stackrel{\text{def}}{=} \mathcal{O}_{E_H}([0_{E_H}])$$

and $\mathcal{L}'_Z \stackrel{\text{def}}{=} \mathcal{L}_H|_{E'_Z}$; $\mathcal{L}_Z \stackrel{\text{def}}{=} \mathcal{L}'_Z|_{E_Z}$. Moreover, it may be shown (cf. [10], §2.1, for more details) that, if we denote by $\epsilon \in E_Z(Z)$ the unique section of order 2, then although *a priori*,

$$\mathcal{L}_\epsilon \stackrel{\text{def}}{=} \mathcal{L}_Z|_\epsilon$$

appears only to be defined over Z, in fact, it *admits a natural G-action* (compatible with the G-action on $\mathcal{O}_Z$), as if it arose as the pull-back to Z of a line bundle on S.

On the other hand, note (cf. the exact sequence discussed above) that $\boldsymbol{\mu}_p$ may be regarded as a *closed subgroup scheme*:

$$(\boldsymbol{\mu}_p)_Z \subseteq E_H$$

In addition, one has a natural closed subgroup scheme

$$\{\pm 1\} \hookrightarrow E_H$$

so we shall write "$(-\boldsymbol{\mu}_p)_Z \subseteq E_H$" for the *translate* of $(\boldsymbol{\mu}_p)_Z$ by the element "-1."

If we now *substitute* the pair $(E_H, \mathcal{L}_H)$ into the theory of §2.3, 2.4, we obtain the corresponding *crystalline theta object*

$$(\mathcal{V}_H, F^r(\mathcal{V}_H), \nabla_{\mathcal{V}_H})$$

over Z by considering sections of $\mathcal{L}_H$ over the *Hodge torsor equipped with its étale integral structure* $E^*_{H,\mathrm{et}}$ associated to this pair. Moreover, it follows immediately from the definition of the integral structure on $E^*_{H,\mathrm{et}}$ that the subgroup schemes $(\boldsymbol{\mu}_p)_Z, \{\pm 1\} \subseteq E_H$ *lift naturally* to closed subschemes

$$(\boldsymbol{\mu}_p)_Z, \{\pm 1\} \subseteq E^*_{H,\mathrm{et}}$$

hence that we obtain a natural lifting

$$(-\boldsymbol{\mu}_p)_Z \subseteq E^*_{H,\mathrm{et}}$$

of $(-\boldsymbol{\mu}_p)_Z \subseteq E_H$. On the other hand, by the *theory of theta groups* (cf., e.g., [7], Chapter IV, §1), we have a natural *theta trivialization*

$$\mathcal{L}_H|_{(-\boldsymbol{\mu}_p)_Z} \xrightarrow{\sim} \mathcal{L}_\epsilon \otimes_{\mathcal{O}_Z} \mathcal{O}_{(\boldsymbol{\mu}_p)_Z}$$

of the restriction of $\mathcal{L}_H$ to $(-\boldsymbol{\mu}_p)_Z$.

In particular, if we *restrict* sections of $\mathcal{L}_H$ over $E^*_{H,\mathrm{et}}$ to $(-\boldsymbol{\mu}_p)_Z$ and then apply the *theta trivialization*, we obtain a *morphism*

$$\Xi_{\mathcal{V}_H} : \mathcal{V}_H \to \mathcal{L}_\epsilon \otimes_{\mathcal{O}_Z} \mathcal{O}_{(\boldsymbol{\mu}_p)_Z}$$

whose restriction

$$\Xi_{\mathcal{H}_{\mathrm{DR}}} : \mathcal{H}_{\mathrm{DR}} \to \mathcal{L}_\epsilon \otimes_{\mathcal{O}_Z} \mathcal{O}_{(\boldsymbol{\mu}_p)_Z}$$

to $\mathcal{H}_{\mathrm{DR}} \overset{\mathrm{def}}{=} F^p(\mathcal{V}_H) \subseteq \mathcal{V}_H$ is "essentially" (i.e., up to taking H-invariants) the *restriction morphism appearing in the original Hodge-Arakelov Comparison Isomorphism* (cf. Theorem 1.1). That is to say, it follows from the main theorem of [7] that:

$$\Xi_{\mathcal{H}_{\mathrm{DR}}}|_{U_Z} \textit{ is an isomorphism.}$$

(In order to make it an isomorphism over Z, one must introduce the *"Gaussian poles"* on $\mathcal{H}_{\mathrm{DR}}$.) In particular, since the Galois group G *acts naturally on the range of* $\Xi_{\mathcal{H}_{\mathrm{DR}}}$, we get a natural action of G on the domain of $\Xi_{\mathcal{H}_{\mathrm{DR}}}$, at least over U_Z.

Definition 3.1. This action of G on $\mathcal{H}_{\mathrm{DR}}|_{U_Z}$ will be referred to as the *Lagrangian Galois action* on $\mathcal{H}_{\mathrm{DR}}$.

Remark 3.2. Note, in particular, that, unlike the Galois actions considered in [7], Chapter IX, §3; [10], §1 (cf. [12], §1.4.1, for a survey of this theory), the *Lagrangian Galois action depends on the choice of the subgroup* H. In fact, however, the choice of different splittings H does not affect the present theory very much. What is, however, essential here, is the existence of the natural *multiplicative subgroup scheme*:

$$\boldsymbol{\mu}_p|_{U_S} \subseteq E[p]|_{U_S}$$

Indeed, *this is the essential element of the present discussion in a "neighborhood of infinity" that does not exist in the number field case.* We will say more on this later (cf. Remark 3.7 below).

Remark 3.3. Just as was done in [12], §1.4.1, for the usual Galois action, once one has defined the Lagrangian Galois action, one may consider the corresponding *Lagrangian arithmetic Kodaira-Spencer morphism* by looking at the *extent to which the Hodge filtration* $F^r(\mathcal{H}_{\mathrm{DR}})$ *is preserved by the action of* G. We leave it to the reader to make the routine details (entirely similar to [12], §1.4.1) explicit. For more on this Lagrangian arithmetic Kodaira-Spencer morphism, we refer to Corollary 3.6 below.

Next, let us write

$$Z^{\log}; \quad S^{\log}$$

for the *log schemes* obtained by equipping Z (respectively, S) with the log structure defined by the divisor $V(q^{\frac{1}{p}}) \subseteq Z$ (respectively, $V(q) \subseteq S$) and

$$\mathrm{Inf}(Z^{\log} \otimes k/\mathcal{O}_K)$$

for the site of *infinitesimal thickenings* of $Z^{\log} \otimes k$ over $\mathcal{O}_K$. Note that here, we do *not* assume that we are given a divided power structure on the ideal defining a thickening. Since, however, *all the higher* p*-curvatures of* $(\mathcal{V}_H, \nabla_{\mathcal{V}_H})$ *vanish* (cf. Corollary 2.7), it follows that the p-adic completion $(\widehat{\mathcal{V}}_H, \nabla_{\widehat{\mathcal{V}}_H})$ of $(\mathcal{V}_H, \nabla_{\mathcal{V}_H})$ defines a *crystal* on the site $\mathrm{Inf}(Z^{\log} \otimes k/\mathcal{O}_K)$.

In particular, since elements $\sigma \in G$ are all *congruent to the identity morphism on* $Z^{\log}$ modulo $\mathfrak{m}$, it follows from the general theory of crystals on sites of thickenings that σ defines an automorphism

$$\int_\sigma : \widehat{\mathcal{V}}_H \xrightarrow{\sim} \widehat{\mathcal{V}}_H$$

which we denote by $\int_\sigma$, by analogy to *"integration/parallel transport along the (closed) path σ"* in the classical complex case. Now, taking into account the functoriality/naturality of all the definitions involved, we obtain the following *fundamental result* (cf. [10], Theorem 2.4):

Theorem 3.4. (The Crystalline Nature of the Lagrangian Galois Action) *For any $\sigma \in G$, the following diagram commutes:*

$$\begin{array}{ccc} \widehat{\mathcal{V}}_H & \xrightarrow{\Xi_{\widehat{\mathcal{V}}_H}} & \mathcal{L}_\epsilon \otimes_{\mathcal{O}_Z} \mathcal{O}_{(\boldsymbol{\mu}_p)_Z} \\ {\scriptstyle \int_\sigma}\downarrow & & \downarrow{\scriptstyle \sigma} \\ \widehat{\mathcal{V}}_H & \xrightarrow{\Xi_{\widehat{\mathcal{V}}_H}} & \mathcal{L}_\epsilon \otimes_{\mathcal{O}_Z} \mathcal{O}_{(\boldsymbol{\mu}_p)_Z} \end{array}$$

(where the "hats" denote p-adic completion, and the vertical morphisms are the natural morphisms associated to σ).

Put another way, Theorem 3.4 asserts that *the Lagrangian Galois action can be computed via "crystalline methods."* Thus, if one restricts the commutative diagram of Theorem 3.4 to $\mathcal{H}_{\mathrm{DR}} \subseteq \widehat{\mathcal{V}}_H$ and then applies a certain lemma (cf. [10], Lemma 2.6) derived from the theory of the *theta convolution* (cf. [8]; [12], §2) to the effect that $\Xi_{\widehat{\mathcal{V}}_H}$ and $\Xi_{\mathcal{H}_{\mathrm{DR}}}$ have the *same image* in $\mathcal{L}_\epsilon \otimes_{\mathcal{O}_Z} \mathcal{O}_{(\boldsymbol{\mu}_p)_Z}$, one obtains the following result (cf. [10], Corollary 2.5):

Corollary 3.5. (Elimination of Gaussian Poles) *The Lagrangian Galois action on $\mathcal{H}_{\mathrm{DR}}|_{U_Z}$ is, in fact, defined on $\mathcal{H}_{\mathrm{DR}}$ (i.e., without Gaussian poles).*

Finally, we would like to apply Theorem 3.4 to *relate the Lagrangian arithmetic Kodaira-Spencer morphism to the classical Kodaira-Spencer morphism.* To do this, we must consider the Lagrangian Galois action *modulo* $\mathfrak{m}^2$. First, we observe that it follows immediately from Theorems 2.8, 3.4 that for any $\sigma \in G$, the Lagrangian Galois action of σ on $\mathcal{H}_{\mathrm{DR}}$ maps

$$F^1(\mathcal{H}_{\mathrm{DR}}) \to F^3(\mathcal{H}_{\mathrm{DR}}) \otimes_{\mathcal{O}_K} (\mathcal{O}_K/\mathfrak{m}^2)$$

in such a way that modulo $\mathfrak{m}$, $F^1(\mathcal{H}_{\mathrm{DR}})$ maps into $F^1(\mathcal{H}_{\mathrm{DR}})$. In particular, if we compose this map with the projection $F^3 \twoheadrightarrow F^3/F^2$ (cf. the discussion of §2.5), then we obtain an $\mathcal{O}_Z$-linear morphism

$$\begin{aligned} F^1(\mathcal{H}_{\mathrm{DR}}) \otimes_{\mathcal{O}_K} k &\to (F^3/F^2)(\mathcal{H}_{\mathrm{DR}}) \otimes_{\mathcal{O}_K} (\mathfrak{m}/\mathfrak{m}^2) \\ &\xrightarrow{\sim} (\tau_{E_H}^{\otimes 2} \otimes_{\mathcal{O}_Z} F^1(\mathcal{H}_{\mathrm{DR}})) \otimes_{\mathcal{O}_K} (\mathfrak{m}/\mathfrak{m}^2) \end{aligned}$$

— i.e., if we let $\sigma \in G$ *vary*, we obtain a homomorphism

$$\begin{aligned}\kappa_G : G = \mathbb{F}_p(1) &\\ \to Hom_{\mathcal{O}_Z}(F^1(\mathcal{H}_{\mathrm{DR}}) \otimes_{\mathcal{O}_K} k, (\tau_{E_H}^{\otimes 2} \otimes_{\mathcal{O}_Z} F^1(\mathcal{H}_{\mathrm{DR}})) \otimes_{\mathcal{O}_K} (\mathfrak{m}/\mathfrak{m}^2))&\\ = \tau_{E_H}^{\otimes 2} \otimes_{\mathcal{O}_K} (\mathfrak{m}/\mathfrak{m}^2).&\end{aligned}$$

On the other hand, if we *forget about families of elliptic curves* and form the natural *exact sequence of differentials* associated to the triple $Z^{\log} \to S^{\log} \to \mathrm{Spec}(\mathcal{O}_K)$, we obtain an *extension of G-modules*:

$$\begin{aligned}0 \to \Omega_{\mathcal{O}_K/\mathbb{Z}_p} \otimes_{\mathcal{O}_K} \mathcal{O}_Z = (p^{-1} \cdot \mathfrak{m} \cdot \mathcal{O}_Z) \otimes_{\mathcal{O}_K} (\mathcal{O}_K/\mathfrak{m}^{p-2}) \to \Omega_{Z^{\log}/\mathbb{Z}_p}&\\ \to \Omega_{Z^{\log}/S^{\log}} = \Omega_{Z^{\log}/\mathcal{O}_K} \otimes \mathbb{F}_p \to 0&\end{aligned}$$

(where all differentials involving $S^{\log}$, $Z^{\log}$ are understood to be *q-adically continuous*). By considering the extent to which sections of $\Omega_{Z^{\log}/\mathcal{O}_K} \otimes \mathbb{F}_p$ lift to G-invariant sections of $\Omega_{Z^{\log}/\mathbb{Z}_p}$, we thus obtain a homomorphism

$$G \otimes_{\mathbb{F}_p} (\Omega_{Z^{\log}/\mathcal{O}_K} \otimes \mathbb{F}_p) \to (p^{-1} \cdot \mathfrak{m} \cdot \mathcal{O}_Z) \otimes_{\mathcal{O}_K} (\mathcal{O}_K/\mathfrak{m}^{p-2})$$

whose composite with the morphism induced by the projection $\mathfrak{m} \twoheadrightarrow \mathfrak{m}/\mathfrak{m}^2$ is easily verified to be an *isomorphism*

$$\delta : G \otimes_{\mathbb{F}_p} \mathcal{O}_{Z \otimes k} \xrightarrow{\sim} \Theta_{Z^{\log}/\mathcal{O}_K} \otimes_{\mathcal{O}_K} \mathfrak{m}/\mathfrak{m}^2$$

(cf. the theory of [1], I., §4) *relating the Galois group G to the (logarithmic) tangent bundle* $\Theta_{Z^{\log}/\mathcal{O}_K} \overset{\text{def}}{=} Hom_{\mathcal{O}_Z}(\Omega_{Z^{\log}/\mathcal{O}_K}, \mathcal{O}_Z)$ *modulo* $\mathfrak{m}$.

Thus, if we combine δ with κ_G, we obtain a morphism:

$$\kappa_G^{\delta} : \Theta_{Z^{\log}/\mathcal{O}_K} \otimes_{\mathcal{O}_K} k \to \tau_{E_H}^{\otimes 2} \otimes_{\mathcal{O}_K} k$$

On the other hand, if we compute κ_G^{δ} by means of Theorems 2.10, 3.4, it follows that:

Corollary 3.6. (Relation to the Classical Kodaira-Spencer Morphism) *The morphism*

$$\kappa_G^{\delta} : \Theta_{Z^{\log}/\mathcal{O}_K} \otimes_{\mathcal{O}_K} k \to \tau_{E_H}^{\otimes 2} \otimes_{\mathcal{O}_K} k$$

constructed by considering the Lagrangian Galois action modulo $\mathfrak{m}^2$ and applying the natural isomorphism

$$\delta : G \otimes_{\mathbb{F}_p} \mathcal{O}_{Z \otimes k} \xrightarrow{\sim} \Theta_{Z^{\log}/\mathcal{O}_K} \otimes_{\mathcal{O}_K} \mathfrak{m}/\mathfrak{m}^2$$

between the Galois group G and the (logarithmic) tangent bundle modulo $\mathfrak{m}$ is equal to $\frac{1}{2}$ the classical Kodaira-Spencer morphism associated to the family $E_H \to Z$.

Remark 3.7. From the point of view of *applications to diophantine geometry* (cf. [12], §1.5.1), one would like to develop an *analogue of the theory* of the present § over *number fields* — i.e., as opposed to over "formal neighborhoods of infinity" (cf. the base $S \stackrel{\text{def}}{=} \text{Spec}(\mathcal{O}_K[[q]])$), as in the present discussion. Indeed, as discussed in [12], §1.5.1, the main essential "missing link" necessary to apply Hodge-Arakelov theory to diophantine geometry is the *elimination of the Gaussian poles* — i.e., the number field analogue of Corollary 3.5. As discussed above (cf. Remark 3.2), the main ingredient that one needs in order to realize such a theory over number fields is an analogue over number fields of the *multiplicative subgroup scheme*:

$$\boldsymbol{\mu}_p|_{U_S} \subseteq E[p]|_{U_S}$$

— i.e., a *"global multiplicative subspace"* of the Tate module (of an elliptic curve over a number field). A first step towards realizing such a global multiplicative subspace is taken in [10], §3, where a global multiplicative subspace is constructed over the base:

$$(\overline{\mathcal{M}}_{\text{ell}})_{\mathbb{Q}}$$

It is then proposed in *loc. cit.* that perhaps by *restricting* this subspace over $(\overline{\mathcal{M}}_{\text{ell}})_{\mathbb{Q}}$ to a $\mathbb{Q}$-valued point, one may achieve the goal of constructing a global multiplicative subspace for elliptic curves over $\mathbb{Q}$. Unfortunately, this operation of restriction is not so straightforward, since it involves numerous intricacies related to the issue of *keeping track of basepoints* (of the various fundamental groups involved). Moreover, this issue of keeping track of basepoints appears to be related to Grothendieck's *anabelian geometry* (cf. [16]). It is the hope of the author to expose these ideas (cf. [17]) in a future sequel to the present manuscript.

§4. Hodge-Arakelov Theory in Positive Characteristic

In this §, we explain how a certain extension of the theory of §2.2 for ordinary elliptic curves to supersingular elliptic curves in positive characteristic can be used to give an alternate proof of Theorem 1.1. More details on the theory discussed here may be found in [11], §1, 2.

Let

$$E \to S$$

be a *family of elliptic curves* over an $\mathbb{F}_p$-scheme S such that the associated classifying morphism

$$S \to (\mathcal{M}_{\mathrm{ell}})_{\mathbb{F}_p}$$

is *étale*. In the following discussion, we will write $\Phi_S : S \to S$ for the *Frobenius morphism* on S, and denote the result of base-changing objects over S via Φ_S by means of a superscript F. Thus, the morphism

$$[p] : E \to E$$

(multiplication by p) factors as a composite of morphisms:

$$E \xrightarrow{\Phi_E} E^{\mathrm{F}} \xrightarrow{\mathcal{V}} E$$

where Φ_E is the *relative Frobenius morphism* of $E \to S$, and $\mathcal{V}$ is the *Verschiebung morphism*. As is well-known, the kernels of Φ_E and $\mathcal{V}$ are *Cartier dual* to one another:

$$E[\Phi_E] \xrightarrow{\sim} E^{\mathrm{F}}[\mathcal{V}]^*$$

Next, let us observe that the *pull-back*

$$\mathcal{V}^* E^\dagger \to E^{\mathrm{F}}$$

of the *universal extension* $E^\dagger \to E$ via $\mathcal{V}$ admits a *canonical section*:

$$\kappa : E^{\mathrm{F}} \to \mathcal{V}^* E^\dagger$$

Indeed, if we can show that the ω_E-torsor $\mathcal{V}^* E^\dagger \to E^{\mathrm{F}}$ is *trivial locally on* S, then κ may be defined as the unique section that takes the origin $0_{E^{\mathrm{F}}}$ of E^{F} to the origin of $\mathcal{V}^* E^\dagger$ (which is determined by the origin $0_{E^\dagger}$ of $E^\dagger$). Thus, it suffices to show that the ω_E-torsor in question is trivial locally on S. Since $\mathbf{R}^1 f_*(\omega_E)$ (where, by abuse of notation, we denote all structure morphisms to S by f) is a *line bundle* on S, it suffices to prove this local triviality in the case that $E \to S$ is *ordinary*. But then,

$\mathcal{V}^*$ induces an isomorphism $\omega_E \xrightarrow{\sim} \omega_{E^{\mathrm{F}}}$, so it follows that the morphism (again induced by $\mathcal{V}^*$)

$$\mathcal{O}_S \cong \mathbf{R}^1 f_*(\omega_E|_E) \to \mathbf{R}^1 f_*(\omega_{E^{\mathrm{F}}}|_{E^{\mathrm{F}}}) \cong \mathcal{O}_S$$

(where the isomorphisms on the two ends are the "trace maps" of the "residues and duality theory" of E, E^{F}) is given by multiplication by $p = \deg(\mathcal{V})$. Since we are working in characteristic p, this completes the proof that the ω_E-torsor $\mathcal{V}^* E^\dagger \to E^{\mathrm{F}}$ is trivial (locally on S).

Next, let us consider *structure sheaves.* Let us regard $\mathcal{O}_E^\dagger$ as a *quasi-coherent sheaf of $\mathcal{O}_E$-algebras* (via the projection $E^\dagger \to E$). Similarly, $\mathcal{V} : E^{\mathrm{F}} \to E$ allows us to regard $\mathcal{O}_{E^{\mathrm{F}}}$ as a *coherent sheaf of $\mathcal{O}_E$-algebras.* Thus, the morphism $\kappa : E^{\mathrm{F}} \to \mathcal{V}^* E^\dagger$ induces a morphism of quasi-coherent $\mathcal{O}_E$-modules:

$$\kappa^* : \mathcal{O}_E^\dagger \to \mathcal{O}_{E^{\mathrm{F}}}$$

Let us write $\mathcal{O}_{E^\dagger}^{<p} \subseteq \mathcal{O}_E^\dagger$ for the coherent subsheaf consisting of sections whose *torsorial degree* (i.e., relative degree over E) is $< p$. Thus, if we restrict κ^* to $\mathcal{O}_{E^\dagger}^{<p}$, we obtain a morphism

$$\mathcal{O}_{E^\dagger}^{<p} \to \mathcal{O}_{E^{\mathrm{F}}}$$

between *locally free $\mathcal{O}_E$-modules of rank p.* Now one has the following positive characteristic, degree $< p$ version of Theorem 2.2:

Theorem 4.1. (Verschiebung-Theoretic Analogue of the Hodge-Arakelov Comparison Isomorphism) *The morphism*

$$\mathcal{O}_{E^\dagger}^{<p} \to \mathcal{O}_{E^{\mathrm{F}}}$$

is an isomorphism.

Proof. First, we remark that the above discussion extends naturally to *degenerating elliptic curves* and hence is compatible with all natural integral structures in a neighborhood of infinity. For more details on this issue, we refer the reader to [11], §1.

Next, we observe that it essentially follows from Theorem 2.2 that the morphism in question is an *isomorphism over the ordinary locus in S.* Indeed, this is a consequence of the fact that the $\mathbb{F}_p$-vector space of all (set-theoretic) functions on $\mathbb{F}_p$ is spanned by the polynomial function on $\mathbb{F}_p$ of degree $< p$. Thus, in particular, the morphism in question is an isomorphism over a *dense open substack of* $(\overline{\mathcal{M}}_{ell})_{\mathbb{F}_p}$.

Thus, (by working over the *proper* stack $(\overline{\mathcal{M}}_{\mathrm{ell}})_{\mathbb{F}_p}$) it suffices to consider *determinants*. That is to say, it suffices to show (strictly speaking, not just over $(\mathcal{M}_{\mathrm{ell}})_{\mathbb{F}_p}$, but also over $(\overline{\mathcal{M}}_{\mathrm{ell}})_{\mathbb{F}_p}$) that:

$$\det(\mathcal{O}_{E^\dagger}^{<p}) = \det(\mathcal{O}_{E^{\mathrm{F}}}) \in \mathrm{Pic}(E)_{\mathbb{Q}} \stackrel{\mathrm{def}}{=} \mathrm{Pic}(E) \otimes_{\mathbb{Z}} \mathbb{Q}$$

Since $E^\dagger \to E$ is an ω_E-torsor, it follows immediately that

$$\det(\mathcal{O}_{E^\dagger}^{<p}) = \sum_{j=0}^{p-1} j \cdot [\tau_E] = -\frac{1}{2}p(p-1) \cdot [\omega_E]$$

(where we think of $\mathrm{Pic}(E)_{\mathbb{Q}}$ as an *additive* group). On the other hand, since $E^{\mathrm{F}}[\mathcal{V}] \stackrel{\sim}{\to} E[\Phi_E]^*$, we have:

$$\mathcal{O}_{E^{\mathrm{F}}[\mathcal{V}]} \stackrel{\sim}{\to} Hom_{\mathcal{O}_E}(\mathcal{O}_{E[\Phi_E]}, \mathcal{O}_E)$$

(as coherent $\mathcal{O}_E$-modules). Thus, since $E^{\mathrm{F}} \to E$ is a $E^{\mathrm{F}}[\mathcal{V}]$-*torsor*, we conclude that:

$$\begin{aligned}\det(\mathcal{O}_{E^{\mathrm{F}}}) &= \det(\mathcal{O}_{E^{\mathrm{F}}[\mathcal{V}]}) = -\det(\mathcal{O}_{E[\Phi_E]}) \\ &= -\sum_{j=0}^{p-1} j \cdot [\omega_E] = -\frac{1}{2}p(p-1) \cdot [\omega_E]\end{aligned}$$

(since $E[\Phi_E] \subseteq E$ is just an infinitesimal neighborhood of the origin 0_E). This completes the proof. Q.E.D.

Remark 4.2. A version of Theorem 4.1 where "$< p$" is replaced by "$< p^n$" (where $n \geq 1$ is an integer) is given in [11], Theorem 1.1.

Next, we consider *line bundles*. Let

$$\eta \in E(S)$$

be a *torsion point of order prime to p*. Write $\eta^{\mathrm{F}} \stackrel{\mathrm{def}}{=} \Phi_E(\eta) \in E^{\mathrm{F}}(S)$ and set:

$$\mathcal{L}^{\mathrm{F}} \stackrel{\mathrm{def}}{=} \mathcal{O}_{E^{\mathrm{F}}}([\eta^{\mathrm{F}}]); \quad \mathcal{M} \stackrel{\mathrm{def}}{=} \Phi_E^*(\mathcal{L}^{\mathrm{F}})$$

Thus, $\mathcal{L}^{\mathrm{F}}$ (respectively, $\mathcal{M}$) is a *line bundle of degree* 1 (respectively, *degree* p) on E^{F} (respectively, E). Since the order of η is *prime to* p, it follows that η^{F} *does not intersect* $0_{E^{\mathrm{F}}}$, hence that we have an isomorphism

$$f_*(\mathcal{L}^{\mathrm{F}}) \stackrel{\sim}{\to} \mathcal{L}^{\mathrm{F}}|_{0_{E^{\mathrm{F}}}}$$

given by restricting sections over E^{F} to $0_{E^{\mathrm{F}}}$. Now we are ready to apply the method of the above discussion to prove the following *positive characteristic version of Theorem* 1.1:

Theorem 4.3. (Positive Characteristic Version of the Hodge-Arakelov Comparison Isomorphism) *The natural restriction morphisms*

$$\Xi_{E^{\mathrm{F}}} : f_*(\mathcal{L}^{\mathrm{F}}|_{\mathcal{V}^* E^\dagger})^{<p} \to \mathcal{L}^{\mathrm{F}}|_{E^{\mathrm{F}}[\mathcal{V}]}$$

and

$$\Xi_E : f_*(\mathcal{M}|_{[p]^* E^\dagger})^{<p} \to \mathcal{M}|_{E[p]}$$

(*where the superscripted "$< p$'s" denote the subsheaves of sections of torsorial degree $< p$) induced by κ are isomorphisms.*

Proof. First, we observe that by the *theory of theta groups* (applied to descent via the Frobenius morphism $\Phi_E : E \to E^{\mathrm{F}}$) the bijectivity of $\Xi_{E^{\mathrm{F}}}$ is *equivalent* to that of Ξ_E. We refer to [11], §2, for more details.

Next, we observe that this discussion extends to the case of *degenerating elliptic curves* (cf. [11], §2, for more details). Thus, we may work over the proper stack $(\overline{\mathcal{M}}_{\mathrm{ell}})_{\mathbb{F}_p}$, so (just as in the proof of Theorem 4.1) it suffices to show that $\Xi_{E^{\mathrm{F}}}$ is *generically an isomorphism* and that the *determinant bundles* associated to its domain and range define the same element of $\mathrm{Pic}((\overline{\mathcal{M}}_{\mathrm{ell}})_{\mathbb{F}_p})_{\mathbb{Q}}$.

The fact that $\Xi_{E^{\mathrm{F}}}$ (or, equivalently, Ξ_E) is an isomorphism over an open dense substack of $(\overline{\mathcal{M}}_{\mathrm{ell}})_{\mathbb{F}_p}$ follows from the theory of the comparison isomorphism near infinity developed in [7], Chapter V (cf. [11], §2, for more details).

On the other hand, the isomorphism

$$f_*(\mathcal{L}^{\mathrm{F}}) \xrightarrow{\sim} \mathcal{L}^{\mathrm{F}}|_{0_{E^{\mathrm{F}}}}$$

implies that (just as in the proof of Theorem 4.1) the determinants of the domain and range of $\Xi_{E^{\mathrm{F}}}$ are both equal to:

$$-\sum_{j=0}^{p-1} j \cdot [\omega_E] = -\frac{1}{2} p(p-1) \cdot [\omega_E]$$

This completes the proof. Q.E.D.

Remark 4.4. Thus, although in the above proof of Theorem 4.3, we use the (relatively easy) portion of the theory of [7] concerning the comparison isomorphism near infinity, the *intricate degree computations* of [7], Chapter VI, §3, are *not* used in the proof of Theorem 4.3. Also, we observe that although Theorem 4.3 is a positive characteristic result,

reduction modulo p implies Theorem 1.1, at least in the case $d = p$. Moreover, since the truth of the characteristic zero result Theorem 1.1 for arbitrary d is equivalent to the equality of the degrees of two specific line bundles, both of which are *polynomials in* d (cf. the complicated degree computations of [7], Chapter VI), the fact that these two polynomials are equal whenever d is a prime number implies that they are equal for all d. Thus, in summary, *the above technique yields a simple proof of Theorem 1.1 for arbitrary* d. We refer to [11], §2, for more details.

Remark 4.5. One way to interpret the preceding Remark 4.4 is the following:

> *The characteristic p methods (involving the Frobenius and Verschiebung morphisms) of the above discussion yield a new proof of the various combinatorial identities inherent in the computation of degrees in* [7]*, Chapter VI, proof of Theorem* 3.1.

This situation is rather reminiscent of the situation of [14], Chapter V — cf., especially, the second Remark following Corollary 1.3. Namely, in that case, as well, characteristic p methods (involving Frobenius and Verschiebung) give rise to *various nontrivial combinatorial identities.* It would be interesting if this sort of phenomenon could be understood more clearly at a conceptual level.

Remark 4.6. One interesting feature of the above proof is the crucial use of the *Frobenius morphism* $\Phi_E : E \to E^{\mathrm{F}}$. Put another way, this amounts to the use of the subgroup scheme $E[\Phi_E] \subseteq E$ (i.e., the kernel of Φ_E), which, of course, does not exist in characteristic zero. Note that this subgroup scheme is essentially the same as the *"multiplicative subspace"* that played an essential role in the theory surveyed in §3. That is to say, it is interesting to note that just as in the context of §3, the *crucial arithmetic object* that one wants over a number field is a *"global multiplicative subspace,"* in the above proof, the crucial arithmetic object that makes the proof work (in *positive!* characteristic) is the "global multiplicative subspace" $E[\Phi_E] \subseteq E$ (which is defined over all of $(\overline{\mathcal{M}}_{\mathrm{ell}})_{\mathbb{F}_p}$).

Remark 4.7. Another interesting and key point in the above proof is the fact that, unlike the case in characteristic zero (where the structure sheaf of a finite flat group scheme on a proper curve always has degree zero):

> *In positive characteristic, the structure sheaf of a finite flat group scheme on a proper curve can have nonzero degree.*

In fact, it is precisely because of this phenomenon that in order to make the comparison isomorphism hold in characteristic zero over the proper object $(\overline{\mathcal{M}}_{\mathrm{ell}})_{\mathbb{Q}}$, it is necessary to introduce the *Gaussian poles* (cf., e.g., [7], Introduction, §1).

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Refined Cycle Maps

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Abstract.

We explain the theory of refined cycle maps associated to arithmetic mixed sheaves. This includes the case of arithmetic mixed Hodge structures, and is closely related to work of Asakura, Beilinson, Bloch, Green, Griffiths, Müller-Stach, Murre, Voisin and others.

Introduction

One of the most fundamental problems in the theory of algebraic cycles would be Beilinson's conjecture on mixed motives [4], which predicts the bijectivity of the cycle map

$$cl : \mathrm{CH}^p(X)_{\mathbb{Q}} \to \mathrm{Ext}^{2p}_{D^b\mathcal{MM}(X)}(\mathbb{Q}_X, \mathbb{Q}_X(p)) \tag{0.1}$$

for any smooth projective variety X over a field k. Here $\mathrm{CH}^p(X)_{\mathbb{Q}}$ is the Chow group of codimension p algebraic cycles modulo rational equivalence on X with $\mathbb{Q}$-coefficients, and $D^b\mathcal{MM}(X)$ is the bounded derived category of the (conjectural) abelian category of mixed motivic sheaves on X. By the adjoint relation for the structure morphism $a_X : X \to \mathrm{Spec}\, k$, the conjecture would be equivalent to the bijectivity of

$$cl : \mathrm{CH}^p(X)_{\mathbb{Q}} \to \mathrm{Ext}^{2p}_{D^b\mathcal{MM}(\mathrm{Spec}\, k)}(\mathbb{Q}_{\mathrm{Spec}\, k}, (a_X)_*\mathbb{Q}_X(p)), \tag{0.2}$$

because $\mathbb{Q}_X$ should be the pull-back by a_X of the constant object $\mathbb{Q}_{\mathrm{Spec}\, k}$ on $\mathrm{Spec}\, k$. It is known that this conjecture implies many other important conjectures on algebraic cycles, such as those of Murre [36], [37], and Bloch [7].

In the case when k is embeddable into $\mathbb{C}$ (e.g. if k is a number field or $\mathbb{C}$), a natural question would be whether $\mathcal{MM}(\mathrm{Spec}\, k)$ is close to the category of $\mathcal{M}_{\mathrm{SR}}(\mathrm{Spec}\, k)$, the category of *systems of realizations,* which was introduced by Deligne [20] (see also [21], [30]). It is expected that

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the essential image of the natural functor $\mathcal{MM}(\operatorname{Spec} k) \to \mathcal{M}_{\mathrm{SR}}(\operatorname{Spec} k)$ would be quite close to the full subcategory $\mathcal{M}_{\mathrm{SR}}(\operatorname{Spec} k)^{\mathrm{go}}$ consisting of objects of geometric origin (which was introduced in [6] for l-adic sheaves). So the first test of the conjecture would be whether the cycle map

$$(0.3) \qquad cl : \mathrm{CH}^p(X)_{\mathbb{Q}} \to \mathrm{Ext}^{2p}_{D^b\mathcal{M}_{\mathrm{SR}}(\operatorname{Spec} k)^{\mathrm{go}}}(\mathbb{Q}_{\operatorname{Spec} k}, (a_X)_*\mathbb{Q}_X(p))$$

is bijective. We can easily show that the surjectivity of (0.3) is equivalent to the algebraicity of absolute Hodge cycles [20] on any smooth projective k-varieties. See [45], [48]. Thus the surjectivity is essentially reduced to the Hodge conjecture, which is not easy to prove as well-known. (However, during an attempt to solve the conjecture we obtained a germ of a new idea by trying to restrict the Leray spectral sequence to the generic fiber of a morphism [46].)

Since the category of mixed motives should be universal as far as cohomology is concerned, we define the category of systems of realizations as an approximation by endowing the cohomology group with as much structure as possible. However, it has been realized by many people that for a complex algebraic variety (where $k = \mathbb{C}$), its cohomology group has more structure. This was first observed by M. Green, P. Griffiths, and C. Voisin in the study of the image of the Abel-Jacobi map for a generic hypersurface, where we have to use the fact that a complex algebraic variety is actually defined over a finitely generated subring of $\mathbb{C}$. See [25], [26], [27], [55], [56]. (This fact was also essential for the theory of mod p reduction of l-adic sheaves [6].) Then, using the models over the finitely generated subrings of $\mathbb{C}$, it is natural to define the category of *arithmetic mixed Hodge Modules* (or more generally, *arithmetic mixed sheaves*) on a complex algebraic variety X as the inductive limit of the categories of mixed Hodge Modules (or mixed sheaves) on the models of X. In the case the ground field k is a finitely generated field over $\mathbb{Q}$, this was already considered in [46] and [47], 1.9 inspired by the arguments in the Appendix to Lect. 1 of [7], and it is enough to take further the inductive limit over k. In particular, we get for the case $X = \operatorname{Spec}\mathbb{C}$ the category of arithmetic mixed Hodge structures, which is a refinement of mixed Hodge structures (see also [1] where the theory of mixed sheaves [47] is also used in an essential way). It seems that the term "arithmetic mixed Hodge structure" was first used in the work of Green and Griffiths [27], where the $\mathbb{Q}$-structure was not considered because they were mostly interested in infinitesimal variations of Hodge structures.

The main point of the theory of arithmetic mixed sheaves is that the injectivity of the refined cycle map of $\mathrm{CH}^2(X)_{\mathbb{Q}}$ for a smooth projective

complex algebraic variety X can be reduced to the injectivity of the Abel-Jacobi map for codimension two cycles on smooth projective models of X. See (4.2). This shows that an additional hypothesis in [1] is unnecessary. See also [26], [27], [28]. The result can be extended to some case of higher Chow cycles as in (4.5) . However, it should be noted that this category is too big, and the forgetful functor to the category of mixed Hodge structures is not fully faithful (see [51], 2.5). We would have to restrict to the subcategory consisting of objects of "geometric origin" for the study of the surjectivity of the cycle map although it does not seem to cause a big problem for the injectivity. Note that our theory applies also to the category of objects of geometric origin, because it is equivalent to the limit of the category of objects of geometric origin on the models of X. The injectivity of the Abel-Jacobi map for varieties over number fields has been conjectured by Beilinson [3] and Bloch (see also [9]), and is one of the most interesting problems in this area. See also [28].

To show that the obtained new category is really better than the usual one, we can prove that, restricting the refined cycle map to the kernel of the Griffiths' Abel-Jacobi map for codimension two cycles, its image is infinite dimensional if X has a nontrivial global two form. See (4.1). This was inspired by [58], and is a consequence of Bloch's diagonal argument in [7] combined with Murre's result on the Albanese motive [36]. (Some special case is treated in [1] using a different method.) A similar assertion can be proved also for higher Chow groups. See (4.4).

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We review the theory of mixed Hodge Modules and cycle maps in Sections 1 and 2. We define the category of arithmetic mixed sheaves in Sect. 3, and states the main results in Sect. 4. Some examples are given in Sect. 5.

§1. Mixed Hodge Modules

1.1. $\mathcal{D}$-Modules. Let X be a smooth algebraic variety over a filed k of characteristic zero. Then we have the sheaf of algebraic linear differential operators $\mathcal{D}_{X/k}$ which has the increasing filtration F by the order of operators. Since $\mathcal{D}_{X/k} = \mathcal{D}_{X/k'}$ for a finite extension $k' \subset k$, we will write $\mathcal{D}_X$ for $\mathcal{D}_{X/k}$ in the sequel.

Let (M, F) be a coherent filtered $\mathcal{D}_X$-Module. Here we assume always that F is exhaustive and $F_pM = 0$ for $p \ll 0$. Then the coherence of (M, F) means that $\mathrm{Gr}^F M\,(:= \oplus_p \mathrm{Gr}^F_p M)$ is coherent over $\mathrm{Gr}^F \mathcal{D}_X$.

We say that (M,F) is *holonomic* if it is coherent and $\dim \operatorname{supp} \operatorname{Gr}^F M = \dim X$, where $\operatorname{supp} \operatorname{Gr}^F M$ is a subvariety of the cotangent space of X which is naturally isomorphic to $\operatorname{Spec} \operatorname{Gr}^F \mathcal{D}_X$. We say that a $\mathcal{D}_X$-Module is holonomic if it is a coherent $\mathcal{D}_X$-Module and has locally a filtration F such that (M,F) is holonomic. The category of coherent (or holonomic) filtered $\mathcal{D}_X$-Modules will be denoted by $MF_{\mathrm{coh}}(\mathcal{D}_X)$ (or $MF_{\mathrm{hol}}(\mathcal{D}_X)$). Forgetting the filtration, we have $M_{\mathrm{coh}}(\mathcal{D}_X), M_{\mathrm{hol}}(\mathcal{D}_X)$ similarly.

If $k = \mathbb{C}$ and A is a subfield of $\mathbb{C}$, let $\operatorname{Perv}(X,A)$ denote the abelian category of perverse sheaves on X^{an} with A-coefficients and with algebraic stratifications. See [6]. Then we have the de Rham functor

$$\mathrm{DR} : M_{\mathrm{hol}}(\mathcal{D}_X) \to \operatorname{Perv}(X,\mathbb{C}) \tag{1.1.1}$$

defined by $M \to \Omega_X^{\dim X} \otimes^{\mathbf{L}}_{\mathcal{D}_X} M$ (using a natural projective resolution of the right $\mathcal{D}_X$-Module $\Omega_X^{\dim X}$). See e.g. [11]. This functor is exact and faithful.

1.2. Direct images. Let $f : X \to Y$ be a proper morphism of smooth k-varieties. Then the cohomological direct image functor $H^j f_* : MF_{\mathrm{coh}}(\mathcal{D}_X) \to MF_{\mathrm{coh}}(\mathcal{D}_Y)$ is defined by factorizing f by $X \to X \times_k Y \to Y$, where the first morphism is the embedding by the graph of f, and the second is the projection.

If $i : X \to Y$ is a closed embedding of codimension d, let $(y_1, \dots, y_n)$ be a local coordinate system of Y (i.e. it defines an étale map of an open subvariety of Y to $\mathbb{A}^n$) such that $X = \{y_1 = \cdots = y_d = 0\}$. Let $\partial_j = \partial/\partial y_j$ so that $\mathcal{D}_Y$ is locally identified with $\mathcal{O}_Y \otimes_{\mathbb{C}} \mathbb{C}[\partial_1, \dots, \partial_n]$. Then the direct image $i_* M$ is locally isomorphic to $M \otimes_{\mathbb{C}} \mathbb{C}[\partial_1, \dots, \partial_d]$ using the coordinates (see [11]), and the filtration F on $i_* M$ is defined by

$$F_p i_* M = \sum_{\nu \in \mathbb{N}^d} F_{p-|\nu|-d} M \otimes \partial^\nu, \tag{1.2.1}$$

where $\partial^\nu = \prod_{1 \le k \le d} \partial_k^{\nu_k}$ for $\nu = (\nu_1, \dots, \nu_d) \in \mathbb{N}^d$. (This is independent of the choice of the coordinates.)

If $q : X \times_k Y \to Y$ is the projection, we have the relative de Rham complex $\mathrm{DR}_{X \times Y/Y}(M,F)$ such that $F_p \mathrm{DR}_{X \times Y/Y}(M)^j = \Omega^{j+\dim X}_{X \times Y/Y} \otimes_{\mathcal{O}} F_{p+j} M$. Then the cohomological direct image is defined to be the cohomology sheaf of the filtered direct image of $\mathrm{DR}_{X \times Y/Y}(M,F)$ by q. (Note that the filtration on the direct image complex is not necessarily strict, and we take the induced filtration on the cohomology sheaf.)

It is known that holonomic filtered $\mathcal{D}$-Modules are stable by the cohomological direct image under a proper morphism, and the cohomological direct image functors for (filtered) $\mathcal{D}$-Modules and perverse sheaves are compatible with each other via the functor (1.1.1), see e.g. [11], [43].

1.3. Mixed Hodge Modules on complex algebraic varieties. Let X be a smooth complex algebraic variety. We say that $M = ((M_{\mathcal{D}}, F),\ M_{\mathbb{Q}},\ W,\ \alpha)$ is a bifiltered holonomic $\mathcal{D}_X$-Module with rational structure, if $(M_{\mathcal{D}}, F)$ is a holonomic filtered $\mathcal{D}_X$-Module, $M_{\mathbb{Q}}$ is a perverse sheaf with rational coefficients, α is a comparison isomorphism $\mathrm{DR}(M_{\mathcal{D}}) = M_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{C}$ in $\mathrm{Perv}(X, \mathbb{C})$, and W is a pair of finite increasing filtrations on $M_{\mathcal{D}}$ and $M_{\mathbb{Q}}$ compatible with α. Here $\mathrm{Perv}(X, A)$ for $A = \mathbb{Q}$ or $\mathbb{C}$ is the full subcategory of $\mathrm{Perv}(X^{\mathrm{an}}, A)$ (see [6]) consisting of perverse sheaves with algebraic stratifications, and similarly for $D^b_c(X, A)$. A mixed Hodge Module on X is a bifiltered holonomic $\mathcal{D}_X$-Module M with rational structure satisfying the following conditions:

The first condition is that mixed Hodge Modules are defined Zariski-locally. The second is that, restricting a mixed Hodge Module M to an open smooth subvariety Z of $\operatorname{supp} M$ on which K is a (shifted) local system, it is isomorphic to the direct image by the closed immersion $Z \to X \setminus (\operatorname{supp} M \setminus Z)$ of an admissible variation of mixed Hodge structure in the sense of [32], [54] (see also (1.4) below) up to a shift of the weight filtration. Here the converse is also true, and an admissible variation of mixed Hodge structure is a mixed Hodge Module ([44], 3.27). The last condition claims that mixed Hodge Modules are locally obtained inductively by gluing mixed Hodge Modules supported on a divisor and admissible variations of mixed Hodge structures on a smooth closed subvariety in the complement on the divisor:

Let g be a function on X such that the restriction of a bifiltered holonomic $\mathcal{D}_X$-Module with rational structure M to the complement U of $g^{-1}(0)$ is the direct image of an admissible variation of mixed Hodge structure M' on a smooth variety Y by the closed immersion $i : Y \to U$. Let $\psi_{g,1}$ and $\varphi_{g,1}$ be the nearby and vanishing cycle functors with unipotent monodromy [18]. Then the condition is that the nearby and vanishing cycles $\psi_{g,1}M, \varphi_{g,1}M$ are well defined and the obtained $M'' := \varphi_{g,1}M$ is a mixed Hodge Module supported on $g^{-1}(0)$. (The first condition consists of the compatibility of the three filtrations F, W, V and the existence of the relative monodromy filtration, see [44], 2.2.)

Here we have canonical morphisms of mixed Hodge Modules

$$\mathrm{can} : \psi_{g,1} i_* M' \to M'', \quad \mathrm{Var} : M'' \to \psi_{g,1} i_* M'(-1),$$

satisfying the gluing condition

$$\mathrm{Var}\circ\mathrm{can} = N, \tag{1.3.1}$$

where $N = \log T_u$ with $T = T_s T_u$ the Jordan decomposition of the monodromy T. (Here (-1) denotes the Tate twist.) We can show that M is uniquely determined by $(M', M'', \mathrm{can}, \mathrm{Var})$ satisfying the gluing condition (1.3.1). More precisely, we have an equivalence between the category of mixed Hodge Modules on X and the category of $(M', M'', \mathrm{can}, \mathrm{Var})$ satisfying (1.3.1). See [44], 2.28. Furthermore, the corresponding M is uniquely determined by using the above condition on the well-definedness of the nearby and vanishing cycle functors. See [44], 2.8.

We denote by $\mathrm{MHM}(X)$ the category of mixed Hodge Modules on X. It is an abelian category such that any morphism is strictly compatible with the Hodge and weight filtrations F, W in the strong sense. See [43], 5.1.14.

Now let X be a (singular) complex algebraic variety. We consider the category $\mathrm{LE}(X)$ whose objects are closed embeddings $U \to V$ where U is an open subvariety of X and V is smooth. The morphisms are pairs of morphisms between U and between V making a commutative diagram. Here the morphisms of U are assumed to be compatible with the inclusions to X. For $\{U \to V\} \in \mathrm{LE}(X)$, let $\mathrm{MHM}_U(V)$ denote the category of mixed Hodge Modules on V supported on U. Then a mixed Hodge Module on X is a collection of mixed Hodge Modules $M_{U\to V} \in \mathrm{MHM}_U(V)$ for $\{U \to V\} \in \mathrm{LE}(X)$ (which is called the representative of M for $\{U \to V\}$) together with isomorphisms

$$v_* M_{U\to V}|_{V'\setminus(U'\setminus U)} = M_{U'\to V'}|_{V'\setminus(U'\setminus U)}$$

for $(u, v) : \{U \to V\} \to \{U' \to V'\}$ satisfying the usual cocycle condition. (We can define the category of filtered $\mathcal{D}$-Modules on X similarly.)

In the case X is smooth, we can show that this definition is equivalent to the previous one (i.e. we have naturally an equivalence of categories).

Actually, to define a mixed Hodge Module on a singular X, it is not necessary to define $M_{U\to V}$ for all $\{U \to V\}$; it is enough to do so for $\{U \to V\}$ such that the U cover X, but the gluing morphisms are defined by using the closed embeddings $U \cap U' \to V \times V'$.

1.4. Admissible variation of mixed Hodge structure. Let X be a smooth complex algebraic variety, and $\overline{X}$ a smooth compactification of X such that the complement $D := \overline{X} \setminus X$ is a divisor with normal crossings. Let $M = ((M_{\mathcal{O}}; F, W), (M_{\mathbb{Q}}, W))$ be a variation of mixed

Hodge structure on X, where $M_{\mathcal{O}}$ is the underlying $\mathcal{O}_X$-Module with the integrable connection ∇ having regular singularity at infinity, and $M_{\mathbb{Q}}$ is the underlying $\mathbb{Q}$-local system with an isomorphism $\mathrm{Ker}\,\nabla^{\mathrm{an}} = M_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{C}$. By [32], [54], M is admissible if the graded pieces $\mathrm{Gr}_k^W M$ are polarizable variations of Hodge structures with quasi-unipotent local monodromies, and furthermore the following conditions are satisfied:

In the case the local monodromies around D are all unipotent, let $\overline{M}_{\mathcal{O}}$ be Deligne's extension of $M_{\mathcal{O}}$ (see [17]). Then
(i) The filtrations F, W on M are extended to $\overline{M}_{\mathcal{O}}$ so that $\mathrm{Gr}_F^p \mathrm{Gr}_k^W \overline{M}_{\mathcal{O}}$ is a locally free $\mathcal{O}_{\overline{X}}$-Module for any p, k.
(ii) The relative monodromy filtration exists for the local monodromy around each irreducible component of D.

Here we can replace $\overline{M}_{\mathcal{O}}$ with its restriction to $\overline{X} \setminus \mathrm{Sing}\, D$, i.e. it is enough to consider the conditions around the smooth points of D, see [32].

In general, the above conditions should be satisfied for the pull-back of M by a dominant morphism such that the local monodromies of the pull-back are unipotent. More precisely, the condition for admissible variation is analytic local on a compactification of X, and it is enough to take locally a ramified cover of a polydisk as usual. Here we can consider $M_{\mathcal{O}}^{\mathrm{an}}$ instead of $M_{\mathcal{O}}$, because $M_{\mathcal{O}}$ is uniquely determined by $M_{\mathcal{O}}^{\mathrm{an}}$ due to the regularity and GAGA.

1.5. Mixed Hodge Modules on algebraic varieties. Let k be a subfield of $\mathbb{C}$, and X a smooth k-variety. Let $X_{\mathbb{C}} = X \otimes_k \mathbb{C}$. Then a mixed Hodge Module M on X consists of $((M_{\mathcal{D}}, F), W), (M_{\mathbb{Q}}, W)$ and α such that $(((M_{\mathcal{O}} \otimes_k \mathbb{C}, F), W), (M_{\mathbb{Q}}, W), \alpha)$ is a mixed Hodge Module on $X_{\mathbb{C}}$, where $(M_{\mathcal{O}}, F) \in MF_{\mathrm{hol}}(\mathcal{D}_X)$, $M_{\mathbb{Q}} \in \mathrm{Perv}(X_{\mathbb{C}}, \mathbb{Q})$ with a finite increasing filtration W, and α is a comparison isomorphism $\mathrm{DR}(M \otimes_k \mathbb{C}) = K \otimes_{\mathbb{Q}} \mathbb{C}$ in $\mathrm{Perv}(X_{\mathbb{C}}, \mathbb{C})$ which is compatible with W. Here we also assume that polarizations on the graded pieces $\mathrm{Gr}_k^W M$ are defined over k, i.e., they are induced by isomorphisms of filtered $\mathcal{D}_X$-Modules $\mathrm{Gr}_k^W(M_{\mathcal{D}}, F)(k) \simeq \mathbb{D}\mathrm{Gr}_k^W(M_{\mathcal{D}}, F)$ compatible with pairings of perverse sheaves, where $\mathbb{D}$ denotes the dual. This is necessary to assure that the graded pieces are semisimple.

We will denote by $\mathrm{MHM}(X/k)$ the category of mixed Hodge Modules on X/k. This is an abelian category such that every morphism is strictly compatible with F, W in the strong sense. We have naturally the forgetful functors

$$\mathrm{MHM}(X/k) \to \mathrm{MHM}(X_{\mathbb{C}}) \to \mathrm{Perv}(X_{\mathbb{C}}, \mathbb{Q}),$$

which induce

$$D^b\mathrm{MHM}(X/k) \to D^b\mathrm{MHM}(X_{\mathbb{C}}) \to D^b_c(X_{\mathbb{C}},\mathbb{Q}), \tag{1.5.1}$$

using the canonical functor $D^b\mathrm{Perv}(X_{\mathbb{C}},\mathbb{Q}) \to D^b_c(X_{\mathbb{C}},\mathbb{Q})$ in [6].

We can define similarly the notion of admissible variation of mixed Hodge structures on X (i.e., the underlying filtered $\mathcal{O}$-Modules and polarizations are defined over X.) Then we see that mixed Hodge Modules on X are obtained locally by gluing mixed Hodge Modules supported on a divisor and admissible variations of mixed Hodge structure on a smooth closed subvariety in the complement of the divisor as in (1.3).

1.6. Mixed sheaves. In this paper we consider more generally the category of mixed sheaves $\mathcal{M}(X/k)$ in the sense of [47]. However, to simplify the explanation, we assume in this paper that $\mathcal{M}(X/k)$ is either $\mathrm{MHM}(X/k)$ defined above or the category $\mathcal{M}_{\mathrm{SR}}(X/k)$ consisting of *systems of realizations* $((M_{\mathcal{D}};F,W),(M_\sigma,W),(M_l,W))$, where $(M_{\mathcal{D}},F)$ is a holonomic filtered $\mathcal{D}$-Module on X endowed with a finite filtration W, (M_σ,W) for an embedding $\sigma : k \to \mathbb{C}$ is a filtered perverse sheaf on $(X\otimes_{k,\sigma}\mathbb{C})^{\mathrm{an}}$ with $\mathbb{Q}$-coefficients, and (M_l,W) for a prime number l is a filtered perverse l-adic sheaf on $X_{\overline{k}} := X\otimes_k \overline{k}$ with $\mathbb{Q}_l$-coefficients which has a continuous action of the Galois group of $\overline{k}/k$ (i.e. the action is lifted to perverse sheaves with $\mathbb{Z}_l$-coefficients). Furthermore these are endowed with comparison isomorphisms

$$\mathrm{DR}((M_{\mathcal{D}},W)\otimes_{k,\sigma}\mathbb{C}) = (M_\sigma,W)\otimes_{\mathbb{Q}}\mathbb{C}, \quad \epsilon^* i_{\overline{\sigma}}^*(M_l,W) = (M_\sigma,W)\otimes_{\mathbb{Q}}\mathbb{Q}_l$$

for an extension $\overline{\sigma} : \overline{k} \to \mathbb{C}$ of σ (in a compatible way with the action of $\mathrm{Gal}(\overline{k}/k)$, see [30]). Here $A = \mathbb{Q},\overline{k}$ is the algebraic closure of k in $\mathbb{C}$, and $i_{\overline{\sigma}}$: $X_{\mathbb{C}} \to X_{\overline{k}}$ is the canonical morphism. (See [6] for ϵ^*.) In the case $X = \mathrm{Spec}\, k$, $\mathcal{M}_{\mathrm{SR}}(\mathrm{Spec}\, k)$ coincides with the category of systems of realizations introduced by Deligne [20], [21] (and this formulation is due to Jannsen [30]).

For $\mathcal{M}(X/k) = \mathrm{MHM}(X/k)$ or $\mathcal{M}_{\mathrm{SR}}(X/k)$, there exists canonically the base change functor

$$\mathcal{M}(X/k) \to \mathcal{M}(X\otimes_k k'/k') \tag{1.6.1}$$

for a finite extension $k \subset k'$ in a compatible way with the cohomological direct image and pull-back and also with dual and external product, etc. There is also the (canonically defined) forgetful functor

$$\mathcal{M}(X/k) \to \mathrm{MHM}(X/k), \tag{1.6.2}$$

which is compatible with the standard functors as above.

If X/k is smooth and purely d-dimensional, we have the constant object $\mathbb{Q}_{X/k}$ in $D^b\mathcal{M}(X/k)$ which actually belongs to $\mathcal{M}(X/k)[-d]$ by the definition of the perverse sheaf [6]. We have also the Tate twist $\mathbb{Q}_{X/k}(j)$ for $j \in \mathbb{Z}$ (using the cohomology of $\mathbb{P}^{-j}$ for j negative, and taking the dual for j positive).

1.7. Theorem. *For a morphism f of algebraic varieties over k, there exist canonical functors f_*, $f_!$, f^*, $f^!$, $\mathbb{D}$, $\otimes$, $\mathcal{H}om$, etc. between the derived categories $D^b\mathcal{M}(X/k)$ in a compatible way with the corresponding functors between the derived categories $D^b\mathrm{MHM}(X/k)$, $D^b\mathrm{MHM}(X_{\mathbb{C}})$ or $D^b_c(X_{\mathbb{C}},\mathbb{Q})$ via the functors* (1.5.1) *and* (1.6.2).

Proof. This follows from the same argument as in [44]. See also [47]. Indeed, using Beilinson's resolution, the stability by the direct image is reduced to the one by the cohomological direct image for an affine morphism [5]. (If f is quasi-projective, this is especially simple by taking two sets of affine coverings of X associated with general hyperplane sections and using the co-Cech and Cech complexes.) The pull-backs are defined to be the adjoint functors of the direct images. For the existence, we may assume f is either a closed embedding i or a projection p. In the former case, the assertion is reduced to the full faithfulness of the direct image

$$i_* : D^b\mathcal{M}(X/k,\mathbb{Q}) \to D^b\mathcal{M}(Y/k,\mathbb{Q}),$$

which is shown by using the functor ξ_g in [44], 2.22. In the latter case it is enough to show the existence of $a_X^*\mathbb{Q}_{\operatorname{Spec} k/k}$ (using the duality and the external product). But this is represented by any complex $M^\bullet$ having a morphism $\mathbb{Q}_{\operatorname{Spec} k/k} \to (a_{X/k})_*M^\bullet$ in $D^b\mathcal{M}(X/k)$ such that the image of $M^\bullet$ in $D^b_c(X_{\mathbb{C}},\mathbb{Q})$ is isomorphic to $\mathbb{Q}_{X_{\mathbb{C}}}$ and the image of the morphism is identified with the canonical morphism $\mathbb{Q} \to (a_{X_{\mathbb{C}}})_*\mathbb{Q}_{X_{\mathbb{C}}}$. So it exists locally on open subsets which are embeddable into smooth varieties, and we can glue them by using the adjoint morphism for the inclusion of open subvarieties, see [44], 4.4. Q.E.D.

1.8. Definition. For a k-variety X with structure morphism $a_{X/k} : X \to \operatorname{Spec} k$, we define

(1.8.1)

$$\mathbb{Q}_{X/k}(j) = a_{X/k}^*\mathbb{Q}_{\operatorname{Spec} k/k}(j), \quad H^j(X/k,\mathbb{Q}(j)) = H^j(a_{X/k})_*\mathbb{Q}_{X/k}(j).$$

We omit $/k$ in the case $k = \mathbb{C}$.

1.9. Decomposition of the direct images. If X is smooth proper over k, we have a noncanonical isomorphism

$$(1.9.1) \qquad (a_{X/k})_*\mathbb{Q}_{X/k} \simeq \oplus_j H^j(X/k,\mathbb{Q})[-j] \quad \text{in } D^b\mathcal{M}(\operatorname{Spec} k/k).$$

See e.g. [44], 4.5.3.

§2. Cycle Map and Geometric Origin

2.1. Cycle map. Let X be a smooth k-variety. We define an analogue of Deligne cohomology by

$$H^i_{\mathcal{D}}(X/k,\mathbb{Q}(j)) = \mathrm{Ext}^i(\mathbb{Q}_{X/k},\mathbb{Q}_{X/k}(j)) \\ \big(= \mathrm{Ext}^i(\mathbb{Q}_k,(a_{X/k})_*\mathbb{Q}_{X/k}(j))\big), \tag{2.1.1}$$

where the extension groups are taken in the derived category of $\mathcal{M}(X/k)$ or $\mathcal{M}(\mathrm{Spec}\,k/k)$, and the second isomorphism follows from the adjoint relation between the direct image and the pull-back by $a_{X/k}$. (In the case $k=\mathbb{C}$, we will often omit $/k$ to simplify the notation.)

Let $\mathrm{CH}^p(X)_{\mathbb{Q}}$ be the Chow group consisting of codimension p cycles modulo rational equivalence on X with rational coefficients. Then we have naturally the cycle map

$$cl : \mathrm{CH}^p(X)_{\mathbb{Q}} \to H^{2p}_{\mathcal{D}}(X/k,\mathbb{Q}(p)). \tag{2.1.2}$$

If a cycle ζ is represented by an irreducible closed subvariety Z, then $cl(\zeta)$ is defined to be the composition of

$$\mathbb{Q}_{X/k} \to \mathbb{Q}_{Z/k} \to \mathrm{IC}_{Z/k}\mathbb{Q}[-d_{Z/k}]$$

with its dual, by using the dualities

$$\mathbb{D}(\mathbb{Q}_{X/k}) = \mathbb{Q}_{X/k}(d_{X/k})[2d_{X/k}], \quad \mathbb{D}(\mathrm{IC}_{Z/k}\mathbb{Q}) = \mathrm{IC}_{Z/k}\mathbb{Q}(d_{Z/k}),$$

where $\mathrm{IC}_{Z/k}\mathbb{Q}$ is the intersection complex, and $d_{X/k} = \dim X/k$. See [44], 4.5.15. We can show that the cycle map is compatible with the pushdown and the pull-back of cycles. (where a morphism is assumed to be proper in the case of pushdown.) See [45, II].

The composition of (2.1.2) with the natural projection

$$\mathrm{CH}^p(X)_{\mathbb{Q}} \to \mathrm{Hom}\big(\mathbb{Q}_k, H^{2p}(X/k,\mathbb{Q}(p))\big)$$

is the usual cycle map. Let $\mathrm{CH}^p_{\mathrm{hom}}(X)_{\mathbb{Q}}$ be its kernel (which consists of homologically equivalent to zero cycles). Then (2.1.2) induces a generalized Abel-Jacobi map over k:

$$\mathrm{CH}^p_{\mathrm{hom}}(X)_{\mathbb{Q}} \to J^p(X/k)_{\mathbb{Q}} := \mathrm{Ext}^1\big(\mathbb{Q}_k, H^{2p-1}(X/k,\mathbb{Q}(p))\big), \tag{2.1.3}$$

where Ext^1 is taken in $\mathcal{M}(\mathrm{Spec}\,k/k)$.

Let $\mathrm{CH}^p(X,m)_{\mathbb{Q}}$ be Bloch's higher Chow group with rational coefficients of a smooth k-variety X [10]. By [47] we have the cycle map

$$cl : \mathrm{CH}^p(X,m)_{\mathbb{Q}} \to H_{\mathcal{D}}^{2p-m}(X/k,\mathbb{Q}(p)). \tag{2.1.4}$$

If X is smooth proper over k and $m>0$, then this cycle map induces a generalized Abel-Jacobi map over k:

$$\mathrm{CH}^p(X,m)_{\mathbb{Q}} \to \mathrm{Ext}^1\big(\mathbb{Q}_k, H^{2p-m-1}(X/k,\mathbb{Q}(p))\big), \tag{2.1.5}$$

because the Leray spectral sequence for $H_{\mathcal{D}}^{2p-m}(X/k,\mathbb{Q}(p))$ degenerates at E_2 by (1.9.1), and $\mathrm{Hom}\big(\mathbb{Q}_k, H^{2p-m}(X/k,\mathbb{Q}(p))\big)=0$.

2.2. Griffiths' Abel-Jacobi map. If $k=\mathbb{C}$ and $\mathcal{M}(X)=\mathrm{MHM}(X)$, then $H_{\mathcal{D}}^i(X,\mathbb{Q}(j))$ for a smooth projective variety X coincides with Deligne cohomology in the usual sense ([22], [23]), and the cycle map (2.1.2) coincides with Deligne's cycle map. In particular, (2.1.3) coincides with Griffiths' Abel-Jacobi map

$$\mathrm{CH}^p_{\mathrm{hom}}(X) \to J^p(X)\ \big(=\mathrm{Ext}^1_{\mathrm{MHS}}\big(\mathbb{Z}, H^{2p-1}(X,\mathbb{Z}(p))\big)\big) \tag{2.2.1}$$

tensored with $\mathbb{Q}$, where $J^p(X)$ is the Griffiths intermediate Jacobian [29], and the last isomorphism follows from [12].

2.3. Injectivity of the Abel-Jacobi map. It is expected that higher extension groups Ext^i $(i>1)$ should vanish in the (conjectural) category of mixed motives over a number field. Since the category of systems of realizations is an approximation of the category of mixed motives, it is interesting whether the Abel-Jacobi map (2.1.3) is injective in the case k is a number field. Actually Beilinson conjectures the injectivity of the composition of (2.1.3) with the natural morphism:

$$\mathrm{CH}^p_{\mathrm{hom}}(X)_{\mathbb{Q}} \to J^p(X/k)_{\mathbb{Q}} \to J^p(X_{\mathbb{C}})_{\mathbb{Q}} \tag{2.3.1}$$

at least if we restrict it to the subgroup $\mathrm{CH}^p_{\mathrm{alg}}(X)_{\mathbb{Q}}$ consisting of algebraically equivalent to zero cycles [3]. Since the image of $\mathrm{CH}^p_{\mathrm{alg}}(X)_{\mathbb{Q}}$ by (2.1.3) is contained in the algebraic part of the Jacobian, and the restriction of (2.3.1) to $\mathrm{CH}^p_{\mathrm{alg}}(X)_{\mathbb{Q}}$ is defined algebraically (3.10), it would be natural to conjecture the injectivity of (2.3.1) for the algebraically equivalent to zero cycles. However, it may be better to conjecture the injectivity of (2.1.3) in general.

2.4. Geometric origin. We denote by $\mathcal{M}(X/k)^{\mathrm{go}}$ the full subcategory of $\mathcal{M}(X/k)$ consisting of objects of geometric origin (see [6] for the case of perverse sheaves). This is by definition the smallest full subcategory of $\mathcal{M}(X/k)$ which is stable by the standard cohomological functors

$H^jf_*, H^jf_!, H^jf^*, H^jf^!$, etc. and also by subquotients in $\mathcal{M}(X/k)$, and contains the constant object $\mathbb{Q}_{\operatorname{Spec} k/k}$ for $X = \operatorname{Spec} k$. (This satisfies the axiom of mixed sheaves, see [47], 7.1.) Actually, it is enough to assume the stability by the cohomological direct images and pull-backs, because the nearby and vanishing cycle functors are expressed by using the direct images and pull-backs in the same way as in [18] (see e.g. [47], 5.7), and the stability by dual and external product follows from the compatibility with those functors. (More precisely we have (2.5) below.)

We define $H^i_{\mathcal{D}}(X/k, \mathbb{Q}(j))^{\mathrm{go}}$ as in (2.1.1) with $\mathcal{M}(X/k)$ replaced by $\mathcal{M}(X/k)^{\mathrm{go}}$. Note that the natural morphism

$$H^i_{\mathcal{D}}(X/k, \mathbb{Q}(j))^{\mathrm{go}} \to H^i_{\mathcal{D}}(X/k, \mathbb{Q}(j)) \tag{2.4.1}$$

is not injective in general. We have the cycle map

$$cl : \mathrm{CH}^p(X)_{\mathbb{Q}} \to H^{2p}_{\mathcal{D}}(X/k, \mathbb{Q}(p))^{\mathrm{go}} \tag{2.4.2}$$

factorizing (2.1.2). Let

$$J^p(X/k)_{\mathbb{Q}} = \operatorname{Ext}^1_{\mathcal{M}}\big(\mathbb{Q}, H^{2p-1}(X/k, \mathbb{Q}(p))\big),$$
$$J^p(X/k)^{\mathrm{go}}_{\mathbb{Q}} = \operatorname{Ext}^1_{\mathcal{M}^{\mathrm{go}}}\big(\mathbb{Q}, H^{2p-1}(X/k, \mathbb{Q}(p))\big),$$

where $\mathcal{M}$ and $\mathcal{M}^{\mathrm{go}}$ mean $\mathcal{M}(\operatorname{Spec} k/k)$ and $\mathcal{M}(\operatorname{Spec} k/k)^{\mathrm{go}}$. We have a canonical injection

$$J^p(X/k)^{\mathrm{go}}_{\mathbb{Q}} \to J^p(X/k)_{\mathbb{Q}}. \tag{2.4.3}$$

(If $k = \mathbb{C}$ and $\mathcal{M}(X) = \mathrm{MHM}(X)$, we omit $/k$.)

We can show the following:

2.5. Proposition. *For $\mathcal{M} \in \mathcal{M}(X/k)$, it is of geometric origin if and only if for any point of X, there exist an open neighborhood U, a closed embedding $i : U \to Z$, a quasi-projective morphism $\pi : Y \to Z$, and a divisor D on Y such that $i_*\mathcal{M}|_U$ is isomorphic to a subquotient of $H^j\pi_* j_! \mathbb{Q}_{(Y\setminus D)/k}$ in $\mathcal{M}(Z/k)$. Here $j : Y \setminus D \to Y$ denotes the inclusion morphism, and we may assume that D is a divisor with normal crossings on Y.*

(See [47] and also [45, I].)

2.6. Theorem. *Assume $\mathcal{M}(X/k) = \mathcal{M}_{\mathrm{SR}}(X/k)$ in* (1.6). *Then the following assertions are equivalent:*
(i) The cycle map (2.4.2) *is surjective for any smooth projective variety X over k.*

(*ii*) *Absolute Hodge cycles on any smooth projective varieties over* k *are algebraic.*

If $k = \mathbb{C}$ *and* $\mathcal{M}(X) = \mathrm{MHM}(X)$, *then the equivalence holds with absolute Hodge cycles replaced by Hodge cycles, and these are further equivalent to*:

(*iii*) *The images of* (2.2.1) *and* (2.4.3) *coincide for any smooth complex projective varieties.*

(See [45, I] and [48].)

2.7. Remark. By (2.6), the surjectivity of the cycle map (2.4) is reduced to the algebraicity of absolute Hodge cycles, and the latter is easily reduced to the usual Hodge conjecture. To show the last conjecture, it would be natural to consider a morphism of a variety (which is birational to X) to another variety of dimension ≥ 2 (e.g. by taking Lefschetz pencils successively). In [46] we tried to restrict the Leray spectral sequence to each fiber using extension groups. Here the category of usual mixed Hodge structures is not good enough because of the vanishing of higher extension groups. But it is also unclear whether the full subcategory of objects of geometric origin is useful, since their higher extension groups are very difficult to calculate, although they are not expected to vanish.

Another big problem in this attempt is that, even if we could get cycles on fibers, it is not clear whether they come from one cycle on the total space. The difficulty comes from the fact that the higher extension classes do not form a section of some geometric object over the base space as in the case of normal functions.

To solve the last problem it would be natural to consider the generic fiber of the morphism and try to find an argument corresponding to "spreading out" of algebraic cycles (as in [7], p. 1.20). Then we get the idea of taking the inductive limit of the category of mixed sheaves on the pull-back of nonempty open subvarieties of the base space. See [46] and [47], 1.9. Unfortunately, this idea did not work well for the problem mentioned above, because we do not yet have a good category of mixed sheaves (which is strong enough to solve the problem). However, seeing earlier work of M. Green [25] and C. Voisin [55], we notice that the cohomology of a complex algebraic variety has more structure than expressed in the systems of realizations, and find that the converse of the above argument would be possible. A complex algebraic variety X has a model of finite type over a subfield k of $\mathbb{C}$ having a morphism to another integral k-variety whose geometric generic fiber over $\mathrm{Spec}\,\mathbb{C}$ is isomorphic to X. Then it is natural to consider the inductive limit of mixed sheaves on the models of X. See also [1] and especially [56],

p. 194. (Note that a similar idea was essentially used in the theory of mod p reduction for perverse sheaves [6].) Thus we get the notion of arithmetic mixed sheaf which will be explained in the next section.

§3. Arithmetic Mixed Sheaves

3.1. Construction. Let k, K be subfields of $\mathbb{C}$ such that $k \subset K$. Then for a K-variety X, there exists a finitely generated k-subalgebra R of K such that X is defined over R, i.e., there is an R-scheme X_R of finite type such that $X = X_R \otimes_R K$. For a finitely generated smooth k-subalgebra R' of K containing R, let $X_{R'} = X_R \otimes_R R', S' = \operatorname{Spec} R', d_{R'} = \dim_k S'$, and let $k_{R'}$ be the algebraic closure of k in R'. Then $k_{R'}$ coincides with the algebraic closure of k in the function field $k(S')$ of S' (because S' is normal), and $k_{R'}$ is a finite extension of k_R. Furthermore, $S/k_{R'}$ is geometrically irreducible and $S'_{\mathbb{C}} := S' \otimes_{k_{R'}} \mathbb{C}$ is connected. We define

$$\mathcal{M}(X/K)_{\langle k\rangle} = \varinjlim \mathcal{M}(X_{R'}/k_{R'})[-d_{R'}],$$

where the inductive limit is taken over R' as above. Here we denote by $\mathcal{M}(X_{R'}/k_{R'})[-d_{R'}]$ the category of mixed sheaves $\mathcal{M}(X_{R'}/k_{R'})$ shifted by $-d_{R'}$ in the derived category. Note that the shift is necessary due to the normalization of perverse sheaves in [6] which implies that perverse sheaves are stable by the usual pull-back f^* under a smooth morphism f up to the shift of complexes by the relative dimension.

More precisely, we define the order relation $R' < R''$ by the inclusion $R' \subset R''$ together with the smoothness of R'' over R'. Then we have natural functors

$$\begin{aligned}\mathcal{M}(X_{R'}/k_{R'})[-d_{R'}] &\to \mathcal{M}(X_{R'} \otimes_{k_{R'}} k_{R''}/k_{R''})[-d_{R'}] \\ &\to \mathcal{M}(X_{R''}/k_{R''})[-d_{R''}],\end{aligned} \tag{3.1.1}$$

where the first comes from (1.6.1). Note that $R' \otimes_{k_{R'}} k_{R''} \to R''$ is injective because $k_{R'}$ is algebraically closed in the fraction field of R'.

We have the canonical functors

$$\mathcal{M}(X_{R'}/k_{R'})[-d_{R'}] \to \mathrm{MHM}(X_{\mathbb{C}}) \tag{3.1.2}$$

compatible with (3.1.1), because $X_{\mathbb{C}}\,(:= X \otimes_K \mathbb{C})$ is identified with the closed fiber of $X_{R'} \otimes_{k_{R'}} \mathbb{C}$ over the closed point of $S'_{\mathbb{C}}$ defined by the inclusion $R' \subset \mathbb{C}$. So we get the canonical functor

$$\iota : \mathcal{M}(X/K)_{\langle k\rangle} \to \mathrm{MHM}(X_{\mathbb{C}}) \tag{3.1.3}$$

compatible with (3.1.1), (3.1.2). (We will omit $/K$ if $K = \mathbb{C}$.)

We denote by H^i the (shifted) cohomology functor from the derived categories of $\mathcal{M}(X_{R'}/k_{R'})$ and $\mathcal{M}(X/K)_{\langle k\rangle}$ to $\mathcal{M}(X_{R'}/k_{R'})[-d_{R'}]$ and $\mathcal{M}(X/K)_{\langle k\rangle}$ respectively. It is compatible with (3.1.1–3).

In the case $X = \mathbb{C}$, we define

$$\mathcal{M}_{K,\langle k\rangle} = \mathcal{M}(\operatorname{Spec} K/K)_{\langle k\rangle},$$

and it will be denoted by $\mathcal{M}_{\langle k\rangle}$ if $K = \mathbb{C}$. In the case $\mathcal{M}(X) = \mathrm{MHM}(X)$, it is denoted by $\mathrm{MHS}_{K,\langle k\rangle}$, and by $\mathrm{MHS}_{\langle k\rangle}$ when $K = \mathbb{C}$.

We have the canonical functor

$$\iota : \mathcal{M}_{K,\langle k\rangle} \to \mathrm{MHS}, \tag{3.1.4}$$

where the target is the category of graded-polarizable mixed $\mathbb{Q}$-Hodge structures in the usual sense [16].

For $j \in \mathbb{Z}$, we have $\mathbb{Q}_{K,\langle k\rangle}(j) \in \mathcal{M}_{K,\langle k\rangle}$ which is represented by a constant variation of Hodge structure of type $(-j,-j)$ as usual. This is denoted by $\mathbb{Q}_{\langle k\rangle}(j)$ if $K = \mathbb{C}$.

3.2. Remarks. (i) The extension groups in $\mathcal{M}_{K,\langle k\rangle}$ are too big, and the natural functor (3.1.4) is not fully faithful, see [51], 2.5 (ii).

(ii) Let $\mathrm{MHS}_{\nabla/\bar{k}}$ denote the category of mixed Hodge structures whose $\mathbb{C}$-part is endowed with an integrable connection ∇ over $\bar{k}$. Then the functor (3.1.4) factors through $\mathrm{MHS}_{\nabla/\bar{k}}$, and $\mathrm{MHS}_{\langle k\rangle}$ is a full subcategory of $\mathrm{MHS}_{\nabla/\bar{k}}$.

(iii) If K contains $\bar{k}$, we have equivalences of categories

$$\mathrm{MHM}(X/K)_{\langle k\rangle} \to \mathrm{MHM}(X/K)_{\langle \bar{k}\rangle}, \quad \mathrm{MHS}_{K,\langle k\rangle} \to \mathrm{MHS}_{K,\langle \bar{k}\rangle}.$$

induced by $\mathrm{MHM}(X_{R'}/k_{R'}) \to \mathrm{MHM}(X_{R'} \otimes_{k_{R'}} \bar{k}/\bar{k})$. See [51], 2.8.

3.3. Theorem. *The category $\mathcal{M}(X/K)_{\langle k\rangle}$ is an abelian category, and there exist canonical functors f_*, $f_!$, f^*, $f^!$, $\mathbb{D}$, $\otimes$, $\mathcal{H}om$, etc. between the derived categories $D^b\mathcal{M}(X/K)_{\langle k\rangle}$ in a compatible way with the functor ι.*

Proof. This follows from (1.7) by using an analogue of the generic base change theorem in [19]. Q.E.D.

3.4. Refined cycle map. For a K-variety X with structure morphism $a_{X/K} : X \to \operatorname{Spec} K$, we define

$$\mathbb{Q}_{X/K,\langle k\rangle}(j) = a_{X/K}^*\mathbb{Q}_{K,\langle k\rangle}(j) \in D^b\mathcal{M}(X/K)_{\langle k\rangle},$$
$$H^i(X/K, \mathbb{Q}_{\langle k\rangle}(j)) = H^i(a_{X/K})_*\mathbb{Q}_{X/K,\langle k\rangle}(j) \in \mathcal{M}_{K,\langle k\rangle}.$$

The latter is represented by $H^i\pi_*\mathbb{Q}_{X_R/k_R}(j)$ for a model $\pi : X_R \to S = \operatorname{Spec} R$ of X. We define an analogue of Deligne cohomology by

$$H^i_{\mathcal{D}}(X/K, \mathbb{Q}_{\langle k\rangle}(j)) = \operatorname{Ext}^i(\mathbb{Q}_{K,\langle k\rangle}, (a_{X/K})_*\mathbb{Q}_{X/K,\langle k\rangle}(j))$$
$$\big(= \operatorname{Ext}^i(\mathbb{Q}_{X/K,\langle k\rangle}, \mathbb{Q}_{X/K,\langle k\rangle}(j))\big),$$

which is isomorphic to the inductive limit of

$$H^i_{\mathcal{D}}(X_R/k_R, \mathbb{Q}(j)) = \operatorname{Ext}^i(\mathbb{Q}_{X_R/k_R}, \mathbb{Q}_{X_R/k_R}(j)).$$

If $K = \mathbb{C}$, we omit $/K$ or K, to simplify the notation.

If X/K is smooth, we have the refined cycle map

$$cl : \mathrm{CH}^p(X)_{\mathbb{Q}} \to H^{2p}_{\mathcal{D}}(X/K, \mathbb{Q}_{\langle k\rangle}(p)) \tag{3.4.1}$$

by taking the inductive limit of the cycle map in (2.1.2) for models X_R/k_R of X:

$$cl_R : \mathrm{CH}^p(X_R)_{\mathbb{Q}} \to H^{2p}_{\mathcal{D}}(X_R/k_R, \mathbb{Q}_{k_R}(p)). \tag{3.4.2}$$

This means that the cycle map is defined by taking models of cycles.

For smooth K-varieties X, Y, let

$$C^i(X,Y)_{\mathbb{Q}} = \oplus_j \mathrm{CH}^{i+\dim X_j}(X_j \times_K Y)_{\mathbb{Q}},$$

where the X_j are the irreducible components of X. Then the cycle map induces

(3.4.3)

$$C^i(X,Y)_{\mathbb{Q}} \to \operatorname{Ext}^{2i+2\dim X}(\mathbb{Q}_{X\times Y/K,\langle k\rangle}, \mathbb{Q}_{X\times Y/K,\langle k\rangle}(i+\dim X))$$
$$= \operatorname{Hom}((a_{X/K})_*\mathbb{Q}_{X/K,\langle k\rangle}, (a_{Y/K})_*\mathbb{Q}_{Y/K,\langle k\rangle}(i)[2i]),$$

where X is assumed connected. This is compatible with the composition of correspondences, see [45], II, 3.3. In particular, the action of $C^i(X,Y)_{\mathbb{Q}}$ on the Chow groups corresponds by the cycle map to the composition of morphisms with the image of (3.4.3), i.e. for $\Gamma \in C^i(X,Y)_{\mathbb{Q}}$, we have the commutative diagram:

(3.4.4)

$$\begin{array}{ccc} \mathrm{CH}^p(X)_{\mathbb{Q}} & \xrightarrow{cl_*} & \operatorname{Hom}(\mathbb{Q}_{K,\langle k\rangle}, (a_{X/K})_*\mathbb{Q}_{X/K,\langle k\rangle}(p)[2p]) \\ \big\downarrow{\scriptstyle \Gamma_*} & & \big\downarrow{\scriptstyle \Gamma_*} \\ \mathrm{CH}^{p+i}(Y)_{\mathbb{Q}} & \xrightarrow{cl_*} & \operatorname{Hom}(\mathbb{Q}_{K,\langle k\rangle}, (a_{Y/K})_*\mathbb{Q}_{Y/K,\langle k\rangle}(p+i)[2p+2i]) \end{array}$$

We have similarly the refined cycle map for the higher Chow groups

$$cl : \mathrm{CH}^p(X,m)_{\mathbb{Q}} \to H^{2p-m}_{\mathcal{D}}(X/K, \mathbb{Q}_{\langle k\rangle}(p)) \tag{3.4.5}$$

as the limit of (2.1.4).

3.5. Indecomposable higher Chow groups. Let $m = 1$. Then an element of $\mathrm{CH}^p(X,1)$ is represented by $\sum_i (Z_i, g_i)$ where the Z_i are integral closed subvarieties of codimension $p-1$ in X and the g_i are nonzero rational functions on Z_i such that $\sum_i \operatorname{div} g_i = 0$ in X. In particular, we have a well-defined morphism

$$\mathrm{CH}^{p-1}(X)_{\mathbb{Q}} \otimes_{\mathbb{Z}} k^* \to \mathrm{CH}^p(X,1)_{\mathbb{Q}}. \tag{3.5.1}$$

Its image and cokernel are denoted by $\mathrm{CH}^p_{\mathrm{dec}}(X,1)_{\mathbb{Q}}$ and $\mathrm{CH}^p_{\mathrm{ind}}(X,1)_{\mathbb{Q}}$ respectively. Their elements are called decomposable and indecomposable higher cycles respectively.

3.6. Leray Filtration. We have the Leray spectral sequence

$$E_2^{i,j} = \mathrm{Ext}^i\big(\mathbb{Q}_k, H^j(X/K, \mathbb{Q}_{\langle k\rangle}(p))\big) \Rightarrow H_{\mathcal{D}}^{i+j}(X/K, \mathbb{Q}_{\langle k\rangle}(p)),$$

which degenerates at E_2. Indeed, by the decomposition theorem (see e.g. [44], 4.5.3), we have a noncanonical isomorphism

$$(a_{X/K})_* a_{X/K}^* \mathbb{Q}_{K,\langle k\rangle} \simeq \oplus_j H^j(X/K, \mathbb{Q}_{\langle k\rangle})[-j] \quad \text{in } D^b\mathcal{M}_{K,\langle k\rangle}. \tag{3.6.1}$$

We denote by F_L the associated filtration on $H_{\mathcal{D}}^{i+j}(X/K, \mathbb{Q}_{\langle k\rangle}(p))$, and also the filtration on $\mathrm{CH}^p(X)_{\mathbb{Q}}$ induced by the cycle map (3.4.1). This means that $F_L^{r+1}\mathrm{CH}^p(X)_{\mathbb{Q}}$ is the kernel of

$$cl : F_L^r \mathrm{CH}^p(X)_{\mathbb{Q}} \to \mathrm{Ext}^r\big(\mathbb{Q}_{K,\langle k\rangle}, H^{2p-r}(X/K, \mathbb{Q}_{\langle k\rangle}(p)\big),$$

and the cycle map induces injective morphisms

$$\mathrm{Gr}^r_{F_L} cl : \mathrm{Gr}^r_{F_L} \mathrm{CH}^p(X)_{\mathbb{Q}} \to \mathrm{Ext}^r\big(\mathbb{Q}_{K,\langle k\rangle}, H^{2p-r}(X/K, \mathbb{Q}_{\langle k\rangle}(p))\big).$$

3.7. Remark. By definition, $F_L^1\mathrm{CH}^p(X)_{\mathbb{Q}}$ coincides with the subgroup $\mathrm{CH}^p_{\mathrm{hom}}(X)_{\mathbb{Q}}$ consisting of cohomologically equivalent to zero cycles. For $p = 2$, let $\mathrm{CH}^p_{\mathrm{AJ}}(X)_{\mathbb{Q}}$ denote the kernel of the Abel-Jacobi map. Then

$$F_L^2\mathrm{CH}^p(X)_{\mathbb{Q}} \subset \mathrm{CH}^p_{\mathrm{AJ}}(X)_{\mathbb{Q}}. \tag{3.7.1}$$

Indeed, for a model $\pi : X_R \to S = \operatorname{Spec} R$ of X, we have a commutative diagram

$$\begin{array}{ccc} F_L^1\mathrm{CH}^p(X_R)_{\mathbb{Q}} & \longrightarrow & \mathrm{Ext}^1\big(\mathbb{Q}_{S,\langle k\rangle}, H^{2p-1}\pi_*\mathbb{Q}_{X_R/k_R}(p)\big) \\ \downarrow & & \downarrow \\ F_L^1\mathrm{CH}^p(X_{\mathbb{C}})_{\mathbb{Q}} & \longrightarrow & \mathrm{Ext}^1\big(\mathbb{Q}, H^{2p-1}(X_{\mathbb{C}}, \mathbb{Q}(p))\big), \end{array}$$

where the vertical morphisms are induced by $X_{\mathbb{C}} \to X_R$ which is given by the inclusion $R \to \mathbb{C}$. For $p = \dim X$, the equality holds in (3.7.1). This follows from Murre's Chow-Künneth decomposition [36] by using the compatibility of the action of a correspondence (3.4.4), see e.g. [51], 3.6. This can be generalized to algebraically equivalent to zero cycles of any codimension, see (3.9).

We have $\mathrm{Gr}^r_{F_L}\mathrm{CH}^p(X)_{\mathbb{Q}} = 0$ for $r > p$ using (3.4.4), see [45], II. It seems that the filtration F_L coincides with a new filtration of M. Green which was explained in the conference [28]. It is expected that F_L gives a conjectural filtration of Beilinson [3] and Bloch (see also [7], [31]).

3.8. Murre's filtration. Assume that a smooth projective variety X admits the Chow-Künneth decomposition in the sense of Murre [36], [37]. Then $\mathrm{CH}^p(X)_{\mathbb{Q}}$ has Murre's filtration F_M. (See also [31].) For F_L as in (3.6), we can show (see [51], 4.9):

$$F_M \subset F_L \quad \text{and} \quad F_M = F_L \mod \cap_i F_L^i. \tag{3.8.1}$$

In particular, $F_M = F_L$ if the cycle map (3.4.1) for X is injective. The injectivity of (3.4.1) can be used for the construction of the Chow-Künneth decomposition.

In [52], Shuji Saito has constructed a filtration F_{Sh} on $\mathrm{CH}^p(X)_{\mathbb{Q}}$ by induction. If we modify slightly his definition or assume the standard conjectures, we can show that his filtration is contained in F_L and they coincide in the case where the Künneth components of the diagonal are algebraic and the cycle map (3.4.1) is injective for the given X (see [51], 4.9, and also [52, II] where we assume the last two hypotheses for any smooth projective varieties).

3.9. Proposition. *Let* $\mathrm{CH}^p_{\mathrm{alg}}(X)_{\mathbb{Q}}$ *denote the subgroup consisting of algebraically equivalent to zero cycles. Then we have*

$$F_L^2\mathrm{CH}^p_{\mathrm{alg}}(X)_{\mathbb{Q}} = \mathrm{CH}^p_{\mathrm{AJ}}(X)_{\mathbb{Q}} \cap \mathrm{CH}^p_{\mathrm{alg}}(X)_{\mathbb{Q}}. \tag{3.9.1}$$

Proof. Let $J^p(X)_{\mathrm{alg}}$ be the image of $\mathrm{CH}^p_{\mathrm{alg}}(X)$ by the Abel-Jacobi map to $J^p(X)$. Then it has a structure of abelian variety. Furthermore, if X and a cycle ζ are defined over a subfield K of $\mathbb{C}$ which is finitely generated over k, then so are $J^p(X)_{\mathrm{alg}}$ and the image of ζ by the Abel-Jacobi map (enlarging K if necessary), see (3.10) below. Since the functor (3.1.3) is induced by the inclusion $R' \to \mathbb{C}$ which gives a geometric generic point of $\mathrm{Spec}\, R'$, we get the assertion. Q.E.D.

3.10. Algebraic part of the intermediate Jacobian. Let Y be a closed subvariety of X with pure codimension $p-1$, and $\widetilde{Y} \to Y$ a

resolution of singularities. We have the Gysin morphism

$$H^1(\widetilde{Y},\mathbb{Z})(1-p) \to H^{2p-1}(X,\mathbb{Z}). \tag{3.10.1}$$

We assume that Y is sufficiently large so that the image is maximal dimensional. Then $J^p(X)_{\mathrm{alg}}$ coincides with the image of the induced morphism (see (2.2.1)):

$$J^1(\widetilde{Y}) \to J^p(X). \tag{3.10.2}$$

Indeed, for $\zeta \in \mathrm{CH}^p_{\mathrm{alg}}(X)$, there is a one-parameter family $\{\zeta_s\}$ over a connected curve S such that ζ is the difference of ζ_{s_1} and ζ_{s_0} for $s_0, s_1 \in S$, and we may assume that Y contains the supports of ζ_s, because the images of (3.10.1–2) do not change by enlarging Y by the above assumption on Y. More precisely, $\{\zeta_s\}$ comes from a cycle Γ on $S \times X$, and ζ belongs to the image of the composition of a correspondence $\Gamma' \in \mathrm{CH}^1(S \times \widetilde{Y})$ and the pushforward by $\widetilde{Y} \to X$, where Γ' is obtained by pulling back $\Gamma \in \mathrm{CH}^1(S \times Y)$.

In particular, $J^p(X)_{\mathrm{alg}}$ has a structure of abelian variety as a quotient of $\mathrm{Pic}(\widetilde{Y})^0$. (See also [2]). If X and ζ are defined over K, we may assume that so are Y and $\widetilde{Y}$ by enlarging K if necessary. Then the quotient of $\mathrm{Pic}(\widetilde{Y})^0$ is also defined over K. Indeed, the quotient corresponds to a quotient system of realizations of $H^1(\widetilde{Y}_K/K,\mathbb{Z})$, which is defined by using the image of (3.10.1). Here $\widetilde{Y}_K$ is a model of $\widetilde{Y}$ over K, and $H^1(\widetilde{Y}_K/K,\mathbb{Z})$ is a system of realizations with integral coefficients, which is defined in a similar way to (1.6) and (1.8). So the assertion follows because the divisor class is defined in $\mathrm{Pic}(\widetilde{Y}_K)^0$.

3.11. Universal ind-abelian quotient. We can give a more precise description of $\mathrm{CH}^p_{\mathrm{alg}}(X)$ related to [35], [42], when the base field k is an algebraically closed field of characteristic 0. By the above argument, we have

$$\mathrm{CH}^p_{\mathrm{alg}}(X) = \varinjlim \mathrm{CH}^1_{\mathrm{alg}}(Y), \tag{3.11.1}$$

where the inductive limit is taken over closed subvarieties Y of pure codimension $p-1$, and $\mathrm{CH}^1_{\mathrm{alg}}(Y)$ is the image of $\mathrm{CH}^1_{\mathrm{alg}}(\widetilde{Y}) = \mathrm{Pic}(\widetilde{Y})^0$ with $\pi : \widetilde{Y} \to Y$ a resolution of singularities. As is well-known, $\mathrm{Pic}(\widetilde{Y})^0$ is the group of k-valued points of the Picard variety. The latter will be denoted by P_Y, because it is independent of the choice of $\widetilde{Y}$. It is the product of P_{Y_i} for the irreducible components Y_i of Y.

Let Λ be the set of closed subvarieties Y of pure codimension $p-1$ in X. It has a natural ordering by the inclusion relation. For $Y, Y' \in \Lambda$

such that $Y < Y'$, there is a natural morphism of abelian varieties $\lambda_{Y,Y'}$: $P_Y \to P_{Y'}$. So we get an inductive system of abelian varieties $\{P_Y\}$. We say that an inductive subsystem of closed group subschemes $\{P'_Y\}$ of $\{P_Y\}$ is strict if $\lambda^{-1}_{Y,Y'}(P'_{Y'}) = P'_Y$ for any $Y < Y'$. For example, a regular morphism to an abelian variety [42] defines a strict subsystem by taking the kernel. Let $\{P'_Y\}$ be the minimal strict inductive subsystem of closed group subschemes $\{P_Y\}$ such that $P'_Y(k)$ contains the kernel of $P_Y(k) \to \mathrm{CH}^p_{\mathrm{alg}}(X)$ for any Y. Let $\overline{P}_Y = P_Y/P'_Y$. Then $\{\overline{P}_Y\}$ is an inductive system with injective transition morphisms. Let

$$A^p(X)^{\mathrm{ab}} = \varinjlim \overline{P}_Y(k). \tag{3.11.2}$$

This is called the universal ind-abelian quotient of $\mathrm{CH}^p_{\mathrm{alg}}(X)$. Clearly the ind-object $\{\overline{P}_Y\}$ has the universal property for the regular morphisms to abelian varieties (see [42]). By Murre [35], $A^p(X)^{\mathrm{ab}}$ is an abelian variety if $p = 2$. (Indeed, $\dim \overline{P}_Y$ is bounded by using (19) of loc. cit. and taking an abelian subvariety of P_Y whose intersection with P'_Y is finite.) It coincides with the algebraic part of the intermediate Jacobian if furthermore $k = \mathbb{C}$.

§4. Main Results

In this section X is assumed to be a smooth complex projective variety. The first result is the nontriviality of the refined cycle map restricted to the kernel of the Abel-Jacobi map, which is inspired by Voisin's result [58] (see also [1]):

4.1. Theorem. *The image of* $\mathrm{Gr}^2_{F_L} cl$ *in* (3.6) *for* $p = \dim X$ *is an infinite dimensional* $\mathbb{Q}$*-vector space if* $\Gamma(X, \Omega^2_X) \neq 0$. (*See* [51], 4.4.)

This follows from Bloch's diagonal argument combined with Murre's Chow-Künneth decomposition for the Albanese motive. If $\Gamma(X, \Omega^1_X) = 0$, the latter is not necessary, and the argument is rather simple. It is remarked by the referee that a similar assertion holds for the cycle map to the arithmetic de Rham cohomology $H^{2p}_{\mathrm{DR}}(X/k)$ of X which is isomorphic to the inductive limit of the de Rham cohomology $H^{2p}_{\mathrm{DR}}(X_R/k)$ over the models X_R/R of $X/\mathbb{C}$.

We say that a smooth (resp. smooth projective) k-variety Y is a *k-smooth* (resp. *k-smooth projective*) *model* of a complex algebraic variety X, if Y has a morphism to an integral k-variety whose geometric generic fiber over $\operatorname{Spec}\mathbb{C}$ is isomorphic to X. The main point in the theory of arithmetic mixed sheaves is that the injectivity of the refined cycle

map can be reduced to that of the Abel-Jacobi map for smooth projective models over number fields (2.1.3). (This shows that an additional assumption in [1] is unnecessary.)

4.2. Theorem. *Assume k is a number field. Then the cycle map* $\mathrm{CH}^p(X)_{\mathbb{Q}} \to H^{2p}_{\mathcal{D}}(X, \mathbb{Q}_{\langle k\rangle}(p))$ *in* (3.4.1) *is injective if the above Abel-Jacobi map* (2.1.3) *over k_Y is injective for any k-smooth models Y of X, where k_Y is the algebraic closure of k in $\Gamma(Y, \mathcal{O}_Y)$. In the case $p = 2$, the last assumption is reduced to the same injectivity for any k-smooth projective models Y of X.* (*See* [51], 4.6.)

We can describe the image of the usual cycle map to Deligne cohomology assuming the Hodge conjecture, if $H^{2p-1}(X, \mathbb{Q}_{\langle k\rangle})$ is *global section-free* in the sense that the local system on $S_{\mathbb{C}}$ underlying the representative of $H^{2p-1}(X, \mathbb{Q}_{\langle k\rangle})$ on any $S = \operatorname{Spec} R$ has no nontrivial global sections.

4.3. Proposition. *If $H^{2p-1}(X, \mathbb{Q}_{\langle k\rangle})$ is global section-free in the above sense, and the Hodge conjecture for codimension p cycles holds for any k-smooth projective models of X, then the image of the usual cycle map to Deligne cohomology*

$$cl : \mathrm{CH}^p(X)_{\mathbb{Q}} \to H^{2p}_{\mathcal{D}}(X, \mathbb{Q}(p)) \tag{4.3.1}$$

coincides with $\operatorname{Im}\big(H^{2p}_{\mathcal{D}}(X, \mathbb{Q}_{\langle k\rangle}(p)) \to H^{2p}_{\mathcal{D}}(X, \mathbb{Q}(p))\big)$, *and similarly for the Abel-Jacobi map with Deligne cohomology replaced by the intermediate Jacobian.* (*See* [51], 4.1.)

We can show some assertions for higher Chow groups corresponding to (4.1–2). Recall that the coniveau filtration $N^pH^i(X, \mathbb{Q})$ for a smooth proper variety X is defined to be the kernel of $H^i(X, \mathbb{Q}) \to H^i(U, \mathbb{Q})$ for a sufficiently small open subvariety U of X such that $\dim X \setminus U \le p$.

4.4. Theorem. *Assume $N^{p-2}H^{2p-3}(X, \mathbb{Q}) \neq 0$. Then for any positive integer m, there exist $\zeta_i \in \mathrm{CH}^{p-1}_{\mathrm{hom}}(X)_{\mathbb{Q}}$ for $1 \le i \le m$ together with a finitely generated subfield K of $\mathbb{C}$ such that for any complex numbers $\alpha_1, \dots, \alpha_m$ not algebraic over K, the images of $\zeta_i \otimes \alpha_i$ for $1 \le i \le m$ by the composition of* (3.5.1) *and* (3.4.5) *are linearly independent over $\mathbb{Q}$.* *See* [51], 5.2.

4.5. Theorem. *Assume k is a number field. Then the cycle map* (3.4.5) *for $p = 2, m = 1$ is injective if the generalized Abel-Jacobi map* (2.1.5) *for the same p, m over k_Y is injective for any k-smooth projective models Y of X, where k_Y is as in* (4.2). (*See* [51], 5.3.)

Related to the countability of the indecomposable higher Chow group $\mathrm{CH}^2_{\mathrm{ind}}(X, 1)_{\mathbb{Q}}$ in (3.5), we have

4.6. Theorem. *If the cycle map* (3.4.5) *for* X *is injective for* $p = 2, m = 1$, *then it induces an injective morphism* (*see* [51], 5.7) :

(4.6.1)

$$\mathrm{CH}^2_{\mathrm{ind}}(X,1)_{\mathbb{Q}} \to \mathrm{Ext}^1\big(\mathbb{Q}_{\langle k\rangle}, H^2(X,\mathbb{Q}_{\langle k\rangle}(2))/N^1H^2(X,\mathbb{Q}_{\langle k\rangle}(2))\big).$$

For the image of (4.6.1), we can show

4.7. Proposition. *Assume* k *is a number field. Then* (4.6.1) *is surjective if* $H^2(X,\mathbb{Q}_{\langle k\rangle})/\ N^1H^2(X,\mathbb{Q}_{\langle k\rangle})$ *is global section-free as in* (4.3) *and if the Abel-Jacobi map* (2.1.3) *is injective for codimension* 2 *cycles on any* k*-smooth projective models of* X. (*See* [51], 5.11.)

4.8. Remark. By (4.6), Voisin's conjecture on the countability of $\mathrm{CH}^2_{\mathrm{ind}}(X,1)_{\mathbb{Q}}$ (see [57]) can be reduced to the injectivity of (2.1.5) (i.e. to the hypothesis of (4.5)), because we have an analogue of the rigidity argument due to Beilinson [3] and Müller-Stach [34] as follows:

4.9. Proposition. *For a smooth complex projective variety* X, *let* $\widetilde{N}^iH^{2i}(X,\mathbb{Q}_{\langle k\rangle})$ *be the maximal subobject of* $H^{2i}(X,\mathbb{Q}_{\langle k\rangle})$ *which is isomorphic to a direct sum of copies of* $\mathbb{Q}_{\langle k\rangle}(-i)$. *Then the image of the morphism*

(4.9.1)

$$\mathrm{CH}^p(X,1)_{\mathbb{Q}} \to \mathrm{Ext}^1(\mathbb{Q}_{\langle k\rangle}, H^{2p-2}(X,\mathbb{Q}_{\langle k\rangle}(p))/\widetilde{N}^{p-1}H^{2p-2}(X,\mathbb{Q}_{\langle k\rangle}(p)))$$

induced by the cycle map is countable. (*See* [51], 5.9.)

4.10. Remark. We have the reduced higher Abel-Jacobi map

$$\mathrm{CH}^p_{\mathrm{ind}}(X,1)_{\mathbb{Q}} \to \mathrm{Ext}^1\big(\mathbb{Q}, (H^{2p-2}(X,\mathbb{Q})/\mathrm{Hdg}^{p-1}(X))(p)\big) \tag{4.10.1}$$

in the usual sense, where $\mathrm{Hdg}^{p-1}(X)$ denotes the group of Hodge cycles. This is an analogue of (4.6.1). By A. Beilinson [3] and M. Levine [33], it can be described quite explicitly by using currents in a similar way to Griffiths' Abel-Jacobi map (see also [24], [34]). This is generalized to the nonproper smooth case [50]. It is not easy to construct nontrivial indecomposable higher cycles (see [13], [14], [15], [24], [34], [57], etc.) nor nontrivial elements in the image of (4.6.1). An example of a nontrivial higher cycle whose support is the one point compactification of $\mathbb{C}^*$ is given in [50]. This is also an example such that its image by (4.10.1) is not contained in the image of $F^1H^2(X,\mathbb{C})$ (see [15] for another example). It was originally considered in order to find an indecomposable higher cycle on a self-product of an elliptic curve of CM type, where the cycle map to real Deligne cohomology used in [24] does not work.

By [38], [40], the kernel of (4.10.1) is isomorphic to

$$\mathrm{Coker}(K_2(\mathbb{C}(X))_{\mathbb{Q}} \to \varinjlim \mathrm{Hom}_{\mathrm{MHS}}(\mathbb{Q}, H^2(U,\mathbb{Q})(2))), \tag{4.10.2}$$

where the morphism is given by $d\log \wedge d\log$ at the level of integral logarithmic forms, and the inductive limit is taken over nonempty open subvarieties U of X. This isomorphism follows easily from the localization sequence of mixed Hodge structures together with the fact that the residue of $d\log f \wedge d\log g$ coincides with the logarithmic differential of the tame symbol of $\{f, g\}$ up to sign. It is conjectured by Beilinson that (4.10.2) should vanish.

§5. Examples

By (4.2) and (4.5), some major problems are reduced to the injectivity of the generalized Abel-Jacobi maps (2.1.3) and (2.1.5), and this is the crucial point in the theory of arithmetic mixed sheaves. However, it is not easy to verify, for example, the injectivity of (2.1.3) even for a surface X unless $p_g(X) = 0$. The problem seems to be of arithmetic nature, because this injectivity does not hold unless the ground field k is a number field. We try to illustrate the difficulty of the problem in the following example.

5.1. Product of elliptic curves. Let E_i be an elliptic curve with the origin O_i defined over a number field k for $i = 1, 2$. Let $X = E_1 \times E_2$ where the subscript k is omitted to simplify the notation because the base change by $k \to \mathbb{C}$ is not used here. The choice of the origin gives a double cover $E_i \to \mathbb{P}^1$ ramified over four points $\Sigma_i \subset \mathbb{P}^1$, which contain the image of the origin.

Let P_i be k-valued point of E_i, and $\zeta_i = [P_i] - [-P_i]$ for $i = 1, 2$. Put $\zeta = \zeta_1 \times \zeta_2$. Then it belongs to the kernel of the Abel-Jacobi map (i.e. of the Albanese map in this case). So we have to show that a multiple of ζ is rationally equivalent to zero. Consider the involution σ of X defined by $x \to -x$. Then ζ is invariant by this involution, and is identified with a cycle of $X' = X/\sigma$. This X' is a double covering of $S := \mathbb{P}^1 \times \mathbb{P}^1$ ramified over the divisor $D = \Sigma_1 \times \mathbb{P}^1 \cup \mathbb{P}^1 \times \Sigma_2$, and has 16 ordinary double points as well-known. Let τ be the involution associated with the double covering. Then ζ is a τ-anti-invariant cycle, i.e. $\tau^*\zeta = -\zeta$. (Note that any cycle with rational coefficients on X' coincides with a τ-anti-invariant cycle modulo τ-invariant cycles, and τ-invariant cycles are essentially trivial modulo rational equivalence.)

If we consider a curve which is invariant by τ, the description of a rational function on it is complicated. So we try to find a curve C on S together with a rational function g on S such that C can be lifted to a curve C' on X' which is birational to C, and the divisor of the pull-back g' of g to C' coincides with a multiple of ζ in X'. Since S is a self-product of $\mathbb{P}^1$, a curve on S is described explicitly by using an

equation. However, it is not easy to express the condition on birational lifting of the curve. Let x, y be affine coordinates of $\mathbb{A}^2$, which is the complement of two irreducible components of D in $\mathbb{P}^1 \times \mathbb{P}^1$. For $i = 1, 2$, let f_i be the defining equation of $\Sigma_i \cap \mathbb{A}^1$ which is a polynomial of degree 3. Then the restriction of X' over $\mathbb{A}^2$ is given by $z^2 = f_1(x)f_2(y)$, and an example of a liftable curve by $g_1(x,y)^2 = f_1(x)f_2(y)g_2(x,y)^2$, where $g_1(x,y), g_2(x,y)$ are polynomials with no common factors. However, how to choose a rational function on C is still a nontrivial problem. If we take $x - c$ for $c \in k$ such that $\{x = c\}$ is not contained in Σ_1, then we see that $[Q] - [O]$ and hence $[Q] - [\tau Q]$ are rationally equivalent to zero, where Q is the sum of the points (counted with multiplicity) in the intersection of C' with the elliptic curve $\{c\} \times E_2$ (the sum is taken in the elliptic curve E_2), and O is any of the points in the inverse image of D which are rationally equivalent to each other. This Q depends only on the bidegrees of g_1, g_2, because the parameter spaces are rational. In particular, we get only a countable number of Q for each c. It is unclear whether Q is a nontorsion point of E_2.

5.2. Higher cycles on an elliptic curve. Since the higher extension groups should vanish in the category of (conjectural) mixed motives over a number field, it is expected that a higher cycle of the form $\zeta \otimes \alpha$ for $\zeta \in \mathrm{CH}^1_{\mathrm{hom}}(X)$ and $\alpha \in k^*$ (see (3.5)) vanishes in $\mathrm{CH}^2(X,1)_{\mathbb{Q}}$ if X is smooth proper over a number field k (see [8], [39]).

In the case of an elliptic curve E with origin O defined over k, take $P \in E(k)$, and let $P_m = mP \in E(k)$ for $m = 0, 1, 2, 3$. Then we have a rational function f on E such that $\mathrm{div}\, f = 2[P_1] - [P_0] - [P_2]$. Let T_Q denote the translation by $Q \in E(k)$, and $g = T_{-P}^* f$. Considering the tame symbol of $\{cf^3, cg^3\}$ for an appropriate $c \in k^*$, we see that $([P_3] - [P_0]) \otimes \alpha$ vanishes in $\mathrm{CH}^2(X,1)$ for some $\alpha \in k^*$. We can verify that $\alpha \neq 1$ for a general P as follows.

We take the Weierstrass equation $y^2 = x^3 + Ax + B$ so that the origin of E is the point at infinity. Then f is given by the function $T_{-P}^*(x - a)$ where $a = x(P)$ (the value of x at P). Consider the function $(x - a)^2 T_{-P}^* x$. This can be extended to a function on a neighborhood of P. We denote its value at P by $h(P)$. Then $h(P) = h(-P)$ because $x(P) = x(-P)$ and $h(P) = \lim_{Q \to P}(x(Q) - x(P))^2 x(Q - P)$. Furthermore, the above c is given by $h(P)^{-1}$, and α by $x(2P)^9 h(P)^{-3}$. It is easy to see that the last function of P goes to the infinity when P approaches to a point P' such that $2P' = O$. (As remarked by the referee, the above $h(P)$ coincides with $y(P)^2$ by using the Weierstrass $\mathfrak{p}$-function. He also notes that we get a similar result by calculating simply the tame symbol

of $\{f, g\}$ and using the well-definedness of $\mathrm{CH}^1(X) \otimes \mathbb{C} \to \mathrm{CH}^2(X, 1)$, because the tame symbol is bilinear.)

With the notation of (5.1), a cycle ζ of the form $\zeta_1 \times \zeta_2$ with $\zeta_i \in \mathrm{CH}^1(E_i)$ is called decomposable. Assume that the ζ_i are homologically equivalent to zero. Then ζ should be rationally equivalent to zero if the ground field k is a number field. But the situation is rather complicated in general. We explain here a special case where the cycle is detected by the refined cycle map.

5.3. Strictly decomposable cycles. Let X_1 and X_2 be smooth complex projective varieties defined over a subfield k of $\mathbb{C}$, i.e. there exist smooth projective k-varieties $X_{i,k}$ such that $X_i = X_{i,k} \otimes_k \mathbb{C}$. Set $X = X_1 \times X_2, X_k = X_{1,k} \times_k X_{2,k}$. We say that a cycle ζ on X is strictly decomposable if there exist subfields K_i of $\mathbb{C}$ finitely generated over k, together with cycles ζ_i on $X_{i,K_i} := X_{i,k} \otimes_k K_i$ for $i = 1, 2$ such that the algebraic closure k' of k in K_i is independent of i, the canonical morphism $K_1 \otimes_{k'} K_2 \to \mathbb{C}$ is injective, and ζ coincides with the base change of the cycle $\zeta_1 \times_{k'} \zeta_2$ on $X_{1,K_1} \times_{k'} X_{2,K_2} = X_{k'} \otimes_{k'} (K_1 \otimes_{k'} K_2)$ by $K_1 \otimes_{k'} K_2 \to \mathbb{C}$. Here we may assume $k' = k$, replacing k if necessary. We say that a strictly decomposable cycle is of bicodimension (p_1, p_2) if $\operatorname{codim} \zeta_i = p_i$. Put $p = p_1 + p_2$.

Let R_i be a finitely generated smooth k-subalgebra of K_i such that the fraction field is K_i, and ζ_i is defined over R_i. Set $S_i = \operatorname{Spec} R_i \otimes_k \mathbb{C}$. Let $\xi_i^j \in H^{2p_i - j}(X_i, \mathbb{Q}) \otimes H^j(S_i, \mathbb{Q})(p_i)$ be the Künneth components of the cycle class of $\zeta_i \otimes_k \mathbb{C}$ in $H^{2p_i}(X_i \times S_i, \mathbb{Q}(p_i))$. Let $\widetilde{M}_i$ be the pull-back of

$$M_i := H^{2p_i - 1}(X_i, \mathbb{Q})(p_i)$$

by $a_{S_i} : S_i \to \operatorname{Spec} \mathbb{C}$. If $\xi_i^0 = 0$, then $\zeta_i \in F_L^1 \mathrm{CH}^{p_i}(X_{i,k} \otimes_k K_i)_{\mathbb{Q}}$, and $\mathrm{Gr}^1_{F_L} cl_{R_i}(\zeta_i)$ (see (3.4.2)) gives

$$\widetilde{\xi}_i^1 \in \mathrm{Ext}^1(\mathbb{Q}_{S_i}, \widetilde{M}_i).$$

By the Leray spectral sequence, we have an exact sequence

$$0 \to \mathrm{Ext}^1(\mathbb{Q}, M_i) \to \mathrm{Ext}^1(\mathbb{Q}_{S_i}, \widetilde{M}_i) \to \mathrm{Hom}(\mathbb{Q}, H^1(S_i, \mathbb{Q}) \otimes M_i),$$

where the first morphism is given by the pull-back by a_{S_i}. Note that ξ_i^1 coincides with the image of $\widetilde{\xi}_i^1$ in the last term. If $\xi_i^1 = 0$, then $\widetilde{\xi}_i^1$ comes from $\mathrm{Ext}^1(\mathbb{Q}, M_i)$, and we may assume $R_i = k$ as long as $\mathrm{Gr}^1_{F_L} cl_{R_i}(\zeta_i)$ is concerned (e.g. if $p_i = 1$).

We say that a strictly decomposable cycle ζ is *degenerate*, if either $\xi_i^0 = \xi_i^1 = 0$ for both i, or $\xi_i^0 = \widetilde{\xi}_i^1 = 0$ for some i. In the case $p_i = 1$,

it is expected that a degenerate ζ is rationally equivalent to zero by Beilinson's conjecture (2.3) when k is a number field. However, if ζ is nondegenerate, we can detect it by the refined cycle map as follows. (This is by joint work with A. Rosenschon [41].)

5.4. Theorem. *With the above notation, let ζ be a strictly decomposable cycle of codimension p. Then $cl(\zeta) \neq 0$ in the notation of* (3.6). *More precisely, $\zeta \in F_L^r \mathrm{CH}^p(X)_{\mathbb{Q}}$ and $\mathrm{Gr}^r_{F_L} cl(\zeta) \neq 0$ if the number of the i such that $\xi_i^0 = 0$ is r.*

5.5. Remark. The assertion (5.4) was first considered in the case both ξ_1^1 and ξ_2^1 are nonzero, in order to show the nonvanishing of the composition of certain extension classes (see [49]). A. Rosenschon studied Nori's construction of a cycle [53] in the case of a self-product of an elliptic curve without complex multiplication, and obtained a special case of (5.4). Then these two were generalized to (5.4), see [41]. This can be extended to the higher cycle case (loc. cit.) As an application, we can show that $\mathrm{CH}^{p+1}_{\mathrm{ind}}(X_1 \times X_2, 1)_{\mathbb{Q}}$ is uncountable if $\Gamma(X_1, \Omega^1_{X_1}) \neq 0$ and the reduced higher Abel-Jacobi map (see (4.10.1)) for X_2 is not zero. This is a generalization of a result of Gordon and Lewis [24] (see also [14]).

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Independence of ℓ for Intersection Cohomology (after Gabber)

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In his lecture in the conference Algebraic Geometry 2000, O. Gabber explained his proof of the following Theorem 1 that the intersection cohomology of a proper scheme is independent of ℓ.

Theorem 1. *Let X be a proper equidimensional scheme over a finite field $\mathbf{F}_q$ and let $i \in \mathbf{Z}$. For a prime $\ell \nmid q$, let $IH^i(X_{\bar{k}}, \overline{\mathbf{Q}}_\ell)$ be the intersection cohomology of degree i. Then $\det(1 - t\,\mathrm{Fr}, IH^i(X_{\bar{k}}, \overline{\mathbf{Q}}_\ell))$ is with coefficients in $\mathbf{Z}$ and independent of $\ell \nmid q$.*

The aim of this note is to give the proof of Gabber. Some details of the proofs are filled by the author, and he takes the full responsibility for the inaccuracies that may appear in this note. In the talk, Gabber also presented the proofs of other independence of ℓ results which are not contained in this short article.

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§1. Independence for $K(X)$

1.1. Notation

We work over $k = \mathbf{F}_q$. For a prime $\ell \nmid q$, choose an algebraic closure $\overline{\mathbf{Q}}_\ell$. For a scheme X separated of finite type over k, $D^b_c(X, \overline{\mathbf{Q}}_\ell)$ denotes the derived category of $\overline{\mathbf{Q}}_\ell$-sheaves defined in Weil II ([De 4]). This notion of derived category is stable under the six operations $f_!$, f_*, f^*, $f^!$, $\otimes$, $R\,Hom$, and also by the Grothendieck-Verdier dualizing functor $D = D_X$ which we normalize by

$$D_X K = R\,Hom(K,\ f^! \overline{\mathbf{Q}}_\ell) \quad \text{for } f : X \to \operatorname{Spec} k.$$

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By $K(X, \overline{\mathbf{Q}}_\ell)$, we denote the Grothendieck ring of the category of $\overline{\mathbf{Q}}_\ell$-sheaves. For a $\overline{\mathbf{Q}}_\ell$-sheaf $\mathcal{F}$ on X, we denote by $[\mathcal{F}]$ the class in $K(X, \overline{\mathbf{Q}}_\ell)$. For $K \in D_c^b(X)$, we also denote by $[K]$ the class $\sum_{i \in \mathbf{Z}} (-1)^i [H^i(K)]$. $|X|$ denotes the set of the closed points.

For a $\overline{\mathbf{Q}}_\ell$-sheaf $\mathcal{F}$ and $x \in |X|$, we have a geometric Frobenius element $\mathrm{Fr}_x \in \mathrm{Gal}(\overline{k(x)}/k(x))$ acting on the geometric fiber $\mathcal{F}_{\bar{x}}$ localized at x. The local L-function is defined by

$$L_x(\mathcal{F},\ t) = \det(1 - t^{\deg x} \mathrm{Fr}_x,\ \mathcal{F}_{\bar{x}})^{-1}.$$

This definition extends to $K(X, \overline{\mathbf{Q}}_\ell)$ by additivity, and the homomorphism

$$\begin{aligned} K(X, \overline{\mathbf{Q}}_\ell) &\to \prod_{x \in |X|} (1 + t \cdot \overline{\mathbf{Q}}_\ell[[t]])^\times \\ [\mathcal{F}] &\mapsto (L_x(\mathcal{F},\ t))_{x \in |X|} \end{aligned}$$

is injective by the Chebotarev density theorem.

1.2. (E, I)-compatibility

Let E be a field of characteristic 0, and let I be a subset of $\{(\ell, \iota),\ \ell \neq p$ is a prime, $\iota : E \hookrightarrow \overline{\mathbf{Q}}_\ell$ is a field embedding$\}$. For $\alpha = (\ell, \iota) \in I$, we denote the first component by $\ell_\alpha = \ell$, the second by $\iota_\alpha = \iota$.

Definition. *Let X be a separated scheme of finite type over $k = \mathbf{F}_q$. We say that a system $(K_\alpha)_{\alpha \in I} \in \prod_{\alpha \in I} K(X, \overline{\mathbf{Q}}_{\ell_\alpha})$ is (E, I)-compatible if and only if it satisfies the following* a), b):

a) *$L_x(K_\alpha,\ t)$ is E-rational for any $\alpha \in I, x \in |X|$, i.e., belongs to the image of $E[[t]]$ via embedding ι_α,*

b) *For each $x \in |X|$, $L_x(K_\alpha,\ t)$, viewed as an element in $E[[t]]$, coincide for all $\alpha \in I$.*

It is easy to see

$$\begin{aligned} K_\alpha \in K(X, \overline{\mathbf{Q}}_{\ell_\alpha}),\ \alpha \in I \text{ are } (E, I)\text{-compatible} \\ \iff \text{For all } n \geq 1 \text{ and } x \in X(\mathbf{F}_{q^n}), \\ \mathrm{Tr}(\mathrm{Fr}_x,\ K_{\alpha, \bar{x}}) = \iota_\alpha(t_x)\ (\alpha \in I) \text{ for some } t_x \in E. \end{aligned}$$

1.3. Results

The following Theorem 2, Theorem 3 will be proved in §3.

Theorem 2. *Let $(K_\alpha)_{\alpha \in I} \in \prod_{\alpha \in I} K(X, \overline{\mathbf{Q}}_{\ell_\alpha})$ be an (E, I)-compatible system. Then for a morphism $f : X \to Y$ of separated schemes of finite*

type over k, $(f_*K_\alpha)_{\alpha\in I}$, $(f_!K_\alpha)_{\alpha\in I} \in \prod_{\alpha\in I} K(Y, \overline{\mathbf{Q}}_{\ell_\alpha})$ are also (E, I)-compatible. Similar results hold for f^*, $f^!$, $\otimes^{\mathbf{L}}$, $R\,Hom$, D.

For the normalization of perverse sheaves, we follow [BBD].

Theorem 3. *Let $j : U \hookrightarrow X$ be an open immersion and let $(K_\alpha)_{\alpha\in I}$ be an (E, I)-compatible system on U. Assume that each K_α is pure and perverse. Then the system of middle extensions $(j_{!*}K_\alpha)_{\alpha\in I}$ is also (E, I)-compatible.*

§2. Linear reccurence sequences

Later we need an elementary lemma on sequences which can be proved by a high-school student. We recall the following definition:

Definition. Let $X \subset \mathbf{Z}$ be a non-empty subset. Assume X is stable under addition by $\mathbf{N}$, or stable under substraction by $\mathbf{N}$. Let E be a field. An E-valued sequence $(a_n)_{n\in X}$ is called a linear reccurence sequence if it satisfies a difference equation of the form

$$\sum_{i=0}^{r} c_i a_{n+i} = 0 \quad \text{(for all } n \text{ such that } n,\ n+r \in X)$$

for some $r \geq 0$, $c_i \in E$, $c_0 c_r \neq 0$. The set of the E-valued linear reccurence sequences is denoted by $Lr(X,\ E)$.

Then the lemma we need is the following:

Key lemma.

a) *Let E be an algebraically closed field of characteristic 0. Then $Lr(X, E)$ has a basis given by the sequences $e_{\alpha,a} : n \mapsto n^a\alpha^n$ for $\alpha \in E^\times$, $a \in \mathbf{N}$.*

b) *$Lr(X, E)$ is a vector space over E and the canonical map $Lr(\mathbf{Z}, E) \to Lr(X, E)$ is bijective.*

c) *Fix an embedding of fields $E \subset E'$. Assume that $f \in Lr(X,\ E')$ takes values in E. Then f belongs to $Lr(X,\ E)$.*

Proof of Key lemma. a) is well-known, and b) is obvious. We prove c). Assume that X is stable under addition by $\mathbf{N}$. For a field F, let $S : (a_n)_{n\in X} \mapsto (a_{n+1})_{n\in X}$ be the shift operator acting on the set $\mathrm{Map}(X, F)$ of all maps $X \to F$. Then $f = (a_n)_{n\in X} \in \mathrm{Map}(X, F)$ belongs to $Lr(X, F)$ if and only if f satisfies the following conditions (1), (2). Let V be the F-subspace of $\mathrm{Map}(X, F)$ generated by $S^n(f)$ $(n \geq 0)$.

(1) V is finite dimensional over F.

(2) The action of S on V is bijective.

For $f \in \mathrm{Map}(X, E)$, since $\mathrm{Map}(X, E) \otimes_E E' \to \mathrm{Map}(X, E')$ is injective, the conditions (1), (2) with $F = E$ are satisfied if and only if the conditions (1), (2) with $F = E'$ are satisfied. □

§3. Proofs of the theorems

Proof of Theorem 2. For f^* and $\otimes^{\mathbf{L}}$, the claim is obvious. The statement for $f_!$ follows from the Grothendieck trace formula for powers of Frobenius:

$$\mathrm{Tr}(\mathrm{Fr}_y,\ (f_! K_\alpha)_{\bar{y}}) = \sum_{x \in X_y(\mathbf{F}_{q^n})} \mathrm{Tr}(\mathrm{Fr}_x,\ K_{\alpha,\bar{x}}) \quad \text{for all } y \in Y(\mathbf{F}_{q^n}),\ n \geq 1.$$

Since

$$f_* = D_Y f_! D_X, \quad R\,Hom(K, L) = D_X(K \otimes^{\mathbf{L}} D_X(L))$$

by the duality formalism, the claim for f_* and $R\,Hom$ follows from that of D_X. We prove the stability of (E, I)-compatibility under D. It suffices to prove the independence of $\mathrm{Tr}(\mathrm{Fr}_x, D_X K_\alpha)$ at each $x \in X(\mathbf{F}_{q^n})$.

Since our problem is local on X for the Zariski topology, we may assume that X is affine. By taking a closed embedding $i : X \hookrightarrow \mathbf{A}_k^n$, we may also assume that $X = \mathbf{A}_k^n$ is an affine space, since i_* commutes with the duality D. By taking a finite extension of k, we may assume $x \in \mathbf{A}_k^n(k)$, and by translation $x = 0_{\mathbf{A}_k^n}$ is the origin.

Further, we reduce to the case $X = A$, where A is an abelian variety over k, and $x = 0$ is the identity element of $A(k)$: by a definition of smoothness, there is an open set $j : U \hookrightarrow A$ containing 0, with étale morphism $p : U \to \mathbf{A}_k^n$ sending 0 to $0_{\mathbf{A}_k^n}$. Since the problem depends only on henselizations at 0 and $0_{\mathbf{A}_k^n}$, it suffices to prove it for $j_! p^* K$ on A.

Now assume $X = A$, an abelian variety over k. Define functions $f_{\alpha,n},\ g_{\alpha,n} : A(\mathbf{F}_q) \to \overline{\mathbf{Q}}_{\ell_\alpha}$ for $n \geq 1$ by

$$f_{\alpha,n}(a) = \sum_{b \in A(\mathbf{F}_{q^n}),\ T_n(b) = a} \mathrm{Tr}(\mathrm{Fr}_b,\ K_{\alpha,\bar{b}}),$$

$$g_{\alpha,n}(a) = \sum_{b \in A(\mathbf{F}_{q^n}),\ T_n(b) = -a} \mathrm{Tr}(\mathrm{Fr}_b,\ (D_A K_\alpha)_{\bar{b}}),$$

where T_n denotes the trace map $A(\mathbf{F}_{q^n}) \to A(\mathbf{F}_q)$. Note it is sufficient to prove that $g_{\alpha,1}(0)$ belongs to E via ι_α and is independent of α. We

will show that there is a $\overline{\mathbf{Q}}_{\ell_\alpha}$-valued linear reccurence sequence $S_\alpha(a)$ on $\mathbf{Z}$ such that $f_{\alpha,n}(a) = S_\alpha(a)_n$, $g_{\alpha,n}(a) = S_\alpha(a)_{-n}$ for $n \geq 1$. By our assumption, $f_{\alpha,n}(a)$ takes values in E via ι_α for $n \geq 1$ and is independent of a choice of $\alpha \in I$. By Key lemma, the same is true for $g_{\alpha,n}(a)$.

Now we prove the existence of $S_\alpha(a)$.

Let

$$\mathcal{L} : A \to A, \qquad a \mapsto a - \mathrm{Fr}_A(a)$$

denotes the Lang torsor defined by $\mathrm{id}_A - \mathrm{Fr}_A$. The Lang torsor is an $A(\mathbf{F}_q)$-torsor. For a finite dimensional representation

$$\rho : A(\mathbf{F}_q) \to \mathrm{GL}_{\overline{\mathbf{Q}}_{\ell_\alpha}}(V)$$

we have the associated local system $\mathcal{L}_\rho = A \times^{A(\mathbf{F}_q)} \rho$ on A as the contracted product. For $m \geq 1$, it is known that the fiber of $\mathcal{L}_\rho$ at $b \in A(\mathbf{F}_{q^m})$ corresponds to the representation of $\mathrm{Gal}(\overline{\mathbf{F}_{q^m}}/\mathbf{F}_{q^m})$ sending the Frobenius element Fr_b at b to $\rho(T_n(b))$ [De2].

When $\rho : A(\mathbf{F}_q) \to \overline{\mathbf{Q}}_{\ell_\alpha}^\times$ is a character, by the Grothendieck trace formula for $K_\alpha \otimes \mathcal{L}_\rho$ and $D_A(K_\alpha \otimes \mathcal{L}_\rho) = D_A K_\alpha \otimes \mathcal{L}_{\rho^{-1}}$, we have

$$(1) \quad \mathrm{Tr}(\mathrm{Fr}^n,\ R\Gamma(A_{\bar{k}},\ K_\alpha \otimes \mathcal{L}_\rho)) = \sum_{b \in A(\mathbf{F}_{q^n})} \mathrm{Tr}(\mathrm{Fr}_b,\ K_{\alpha,\bar{b}}) \cdot \rho(T_n(b)),$$

$$(1') \quad \mathrm{Tr}(\mathrm{Fr}^n,\ R\Gamma(A_{\bar{k}},\ D_A K_\alpha \otimes \mathcal{L}_{\rho^{-1}})) = \sum_{b \in A(\mathbf{F}_{q^n})} \mathrm{Tr}(\mathrm{Fr}_b,\ (D_A K_\alpha)_{\bar{b}}) \cdot \rho(-T_n(b))$$

for $n \geq 1$.

These two identities are rewritten as follows. Let

$$\mathcal{F}(f)(\rho) = \sum_{a \in A(\mathbf{F}_q)} f(a)\rho(a)$$

be the Fourier transform on $A(\mathbf{F}_q)$. Then

$$\mathrm{Tr}(\mathrm{Fr}^n,\ R\Gamma(A_{\bar{k}},\ K_\alpha \otimes \mathcal{L}_\rho)) = \mathcal{F}(f_{\alpha,n})(\rho),$$
$$\mathrm{Tr}(\mathrm{Fr}^n,\ R\Gamma(A_{\bar{k}},\ D_A K_\alpha \otimes \mathcal{L}_{\rho^{-1}})) = \mathcal{F}(g_{\alpha,n})(\rho)$$

follow from formula (1) and (1').

Let $\alpha_{ij} \in \overline{\mathbf{Q}}_{\ell_\alpha}$, $j \in J_i$ be the Fr-eigenvalues on $H^i_{\mathrm{et}}(A_{\bar{k}},\ K_\alpha \otimes \mathcal{L}_\rho)$. By global duality, $R\Gamma(A_{\bar{k}},\ K_\alpha \otimes \mathcal{L}_\rho)$ and $R\Gamma(A_{\bar{k}},\ D_A K_\alpha \otimes \mathcal{L}_{\rho^{-1}})$ are

dual to each other since A is proper. Hence

$$\mathcal{F}(f_{\alpha,n})(\rho) = \sum_{i\in\mathbf{Z}}(-1)^i \sum_{j\in J_i} \alpha_{ij}^n \quad (n \geq 1),$$
$$\mathcal{F}(g_{\alpha,n})(\rho) = \sum_{i\in\mathbf{Z}}(-1)^i \sum_{j\in J_i} \alpha_{ij}^{-n} \quad (n \geq 1)$$

hold, and it follows that there are $\overline{\mathbf{Q}}_{\ell_\alpha}$-valued linear reccurence sequences $s(\rho)$ on $\mathbf{Z}$ such that $\mathcal{F}(f_{\alpha,n})(\rho) = s(\rho)_n$, $\mathcal{F}(g_{\alpha,n})(\rho) = s(\rho)_{-n}$ for $n \geq 1$. The existence of $S_\alpha(a)$ follows by the inverse Fourier transform.

Remark (by Gabber). In case of curves, Deligne proves the compatibility for Rj_* for any compatible system of ℓ-adic sheaves on open parts ([De1], Théorème 9.8). To get the compatibility at missing points, he uses twists by highly ramified rank one sheaves. The argument here uses twists by unramified rank one sheaves.

Proof of Theorem 3. We may assume $\sharp I = 1$ or 2, and each K_α is pure of weight w. By Gabber's purity theorem [BBD], $j_{!*}K_\alpha$ is also pure of weight w (cf. [De4] for pure complexes).

We put $Y = X \setminus U$, $i : Y \to X$. We prove the claim by a descending induction on $\dim Y$. So it suffices to prove the compatibility near all maximal points of Y. By shrinking Y if necessary, we may assume that Y is smooth of pure dimension d, and the middle extension is calculated as

$$j_{!*}K_\alpha = \tau_{\leq -d-1} j_* K_\alpha$$

for any α. Moreover, we may assume that all cohomology sheaves $\mathcal{H}^q(i^! j_{!*}K_\alpha)$, $\mathcal{H}^q(i^* j_* K_\alpha)$ $(\alpha \in I)$ are smooth on Y. To show the compatibility, it suffices to recover the local L-function $L_y(\tau_{\leq -d-1} j_* K_\alpha, t)$ at any $y \in |Y|$ from $L_y(j_* K_\alpha, t)$, which is already independent of α by Theorem 2.

By purity, for $q \leq -d-1$, $\mathcal{H}^q(j_* K_\alpha) = \mathcal{H}^q(\tau_{\leq -d-1} j_* K_\alpha)$ are punctually mixed of weight $\leq q+w$. By purity again, $i^* D_X j_{!*} K_\alpha = D_Y i^! j_{!*} K_\alpha$ is mixed of weight $\leq -w$, and hence the smooth sheaves $\mathcal{H}^q(i^! j_{!*} K_\alpha)$ on Y are punctually mixed of weight $\geq w + q$. By localization triangle

$$\to i^! j_{!*} K_\alpha \to j_{!*} K_\alpha \to j_* K_\alpha \to i^! j_{!*} K_\alpha[1],$$

we know that the smooth sheaves $\mathcal{H}^q(i^* j_* K_\alpha) = \mathcal{H}^{q+1}(i^! j_{!*} K_\alpha)$ are mixed of weight $\geq w + q + 1$ for $q \geq -d$ on Y. This implies that for $y \in |Y|$, $L_y(\tau_{\leq -d-1} j_* K_\alpha, t)$ is extracted from $L_y(j_* K_\alpha, t)$ as the part of weight $\leq w-d-1$. □

Proof of Theorem 1. By definition,

$$IH^i(X_{\bar{k}}, \overline{\mathbf{Q}}_\ell) = H^i_{\mathrm{et}}(X_{\bar{k}}, IC^{\cdot}_X), \quad IC^{\cdot}_X = (j_{!*}\overline{\mathbf{Q}}_\ell[\dim X])[-\dim X].$$

Here j is the inclusion of a dense smooth subscheme in X. By Theorem 3, $\det(1 - t\,\mathrm{Fr}; R\Gamma(X_{\bar{k}}, IC^{\cdot}_X))$ is with coefficients in $\mathbf{Q}$ and is independent of $\ell \nmid q$. The result for the individual cohomology group $IH^i(X_{\bar{k}}, \overline{\mathbf{Q}}_\ell)$ follows from this by the fact $IH^i(X_{\bar{k}}, \overline{\mathbf{Q}}_\ell)$ is of weight i. For the integrality of the coefficients, we may assume that X is reduced and irreducible. By [dJ], there is an alteration $\pi : Y \to X$ with Y smooth and projective over $\mathbf{F}_q$. By the decomposition theorem, $IC^{\cdot}_X$ is a direct summand of $R\pi_* IC^{\cdot}_Y$, hence the fact that the eigenvalues of Fr on $H^i_{\mathrm{et}}(X_{\bar{k}}, IC^{\cdot}_X)$ are algebraic integers follows from the corresponding statement for Y, which holds by [De3], or [SGA7], XXI 5.5.3. □

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A Proof of the Absolute Purity Conjecture (after Gabber)

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§0. Introduction

This article is an edited version of O. Gabber's talk on his proof of the absolute purity conjecture of A. Grothendieck given at the p-adic conference held in Toulouse in 1994. The details of the proofs given here are supplied by the author following marvelous ideas due to Gabber. The author takes the full responsibility for inaccuracies that may appear in this article.

The absolute purity conjecture is the following.

Conjecture (Grothendieck, [G]). *Let $Y \overset{i}{\hookrightarrow} X$ be a closed immersion of noetherian regular schemes of pure codimension c. Let n be an integer which is invertible on X, and let $\Lambda = \mathbf{Z}/n$. Then*

$$\mathcal{H}^q_Y(\Lambda) \simeq \begin{cases} 0 & \text{for } q \neq 2c, \\ \Lambda_Y(-c) & \text{for } q = 2c. \end{cases}$$

The conjecture has been proved in the following cases:

a) X is smooth over a field k, and Y is also smooth over k ([AGV], exposé XVI, 3.7).
b) X is of equal characteristic ([AGV], exposé XIX for special cases and conditional results, the general case can be deduced from Popescu's general Néron desingularization ([P]) as in §6) .
c) $\dim X \leq 2$ (Gabber (1976), see [Sa], §5, Remark 5.6 for a published proof).

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d) X is of finite type over $\mathbf{Z}$, and any prime divisor l of n satisfies $l \geq \dim X + 2$ ([Thom2], the result in [Thom2] is in fact more general, but in [Thom2] it is necessary to assume that all prime divisors of n are "sufficiently" large).

We explain the outline of the proof by Gabber of the absolute purity conjecture.

The first key step is the construction of the global cycle class in §1

$$\mathrm{cl}(Y) \in H^{2c}_Y(X, \Lambda(c))$$

for X, a noetherian scheme, and $Y \subset X$ a closed local complete intersection subscheme. In the case that Y is an effective Cartier divisor, it is given by the localized first Chern class. In the general case, one uses the blowing up of X along Y, and the cycle class is constructed from the localized Chern classes of the exceptional divisor. The obtained cycle class refines the cycle class defined in [De1], 2.2.

Thanks to the existence of the global cycle class, absolute purity is now reduced to punctual purity (cf. Definition 2.2.1 and Proposition 2.2.4): For a regular strict local ring $\mathcal{O}$ of dimension d with the closed point $i_x : x \to \mathrm{Spec}\,\mathcal{O}$, the cycle class gives an isomorphism

$$\mathrm{cl}(x) : \Lambda_x \simeq i_x^! \Lambda(d)[2d].$$

By using induction on dimension, this is reduced to showing the following vanishing (cf. Proposition 3.1.2)

$$H^p(\mathrm{Spec}\ \mathcal{O}[f^{-1}], \Lambda) = 0 \quad \text{for } p \neq 0, 1,$$

where $f \in m \backslash m^2$ with $m \subset \mathcal{O}$, the maximal ideal. By a reduction step in §6, we may also assume that $\mathcal{O}$ is arithmetic.

The second key step is the affine Lefschetz theorem (§5, Theorem B) on vanishing of cohomology of affine schemes of arithmetic type. It is a generalization of a theorem of M. Artin ([AGV]) in case of algebraic varieties over a field. Thanks to the theorem, we can show the following vanishing (cf. Proposition 5.2.1) by using the induction hypothesis and the duality result in §4

$$H^p(\mathrm{Spec}\ \mathcal{O}[f^{-1}], \Lambda) = 0 \quad \text{for } p \neq 0, 1, d-1, d,$$

where $\mathcal{O}$ and $f \in m$ are the same as before. In order to deal with the remaining vanishing, we invoke in the final key step the Atiyah-Hirzebruch type spectral sequence for the étale K-theory constructed

by Thomason (cf. §7)

$$E_2^{p,q} = \begin{cases} H^p(\text{Spec } \mathcal{O}[f^{-1}], \mathbf{Z}/\ell^\nu(i)) & (q = -2i), \\ 0 & (q \text{ is odd}) \end{cases}$$
$$\Longrightarrow (K/\ell^\nu)_{-p-q}(\mathcal{O}[f^{-1}])[\beta^{-1}].$$

One should note that the idea that the étale K-theory of schemes offers a very strong tool in the proof of the absolute purity conjecture is due to R. Thomason ([Thom2]) who has shown the conjecture with ambiguity of bounded torsion. Thanks to Suslin's computation of K-theory of separably closed field ([Sus]) and Gabber's rigidity theorem for algebraic K-theory ([Ga]), $(K/\ell^\nu)_*(\mathcal{O}[f^{-1}])[\beta^{-1}]$ is computed. This implies the degeneracy of the spectral sequence from which the remaining vanishing of étale cohomology follows.

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Notation. For a scheme X, X_{et} denotes the étale topos of X. For an integer $n \geq 1$ and $\Lambda = \mathbf{Z}/n$, $D^+(X_{\text{et}}, \Lambda)$ denotes the derived category of complexes of Λ-sheaves bounded below. The constant sheaf Λ on X is denoted by Λ_X. All cohomology groups are étale cohomology groups.

§1. Cycle class

1.1. Refined cycle class

Let X be a noetherian scheme, $Y \subset X$ be a closed, local complete intersection subscheme of pure codimension c. For an integer n which is invertible on X, let $\Lambda = \mathbf{Z}/n$. Under this condition, Gabber constructs a cycle class $\text{cl}(Y) \in H^{2c}_Y(X, \Lambda(c))$ without any purity assumption.

In the case of effective Cartier divisors, the class

$$\text{cl}(Y) = c_1(\mathcal{O}(Y)) \in H^2_Y(X, \Lambda(1))$$

is given by the localized first Chern class as in [De1], 2.1. For a morphism $f : X' \to X$ such that $Y' = f^*(Y)$ is also an effective Cartier divisor,

$$f^* \text{cl}(Y) = \text{cl}(Y')$$

holds. In the general case, consider the blowing up

$$\pi : \tilde{X} \to X$$

of X along Y. For the defining ideal $\mathcal{I}_Y$ of Y, $\tilde{X} = \mathrm{Proj} \bigoplus_{m\geq 0} \mathcal{I}_Y^m$. Let E be the exceptional divisor defined by $\mathcal{I}_E = \pi^{-1}(\mathcal{I}_Y)\mathcal{O}_{\tilde{X}}$, and let $\mathcal{N}_{Y|X} = \mathcal{I}_Y/\mathcal{I}_Y^2$ be the conormal sheaf of Y in X.

$$E = \mathrm{Proj} \bigoplus_{m\geq 0} \mathcal{I}_Y^m/\mathcal{I}_Y^{m+1} = \mathbf{P}(\mathcal{N}_{Y|X})$$

is the projective bundle of $\mathcal{N}_{Y|X}$ in the notation of EGA II. $\mathcal{O}(1) = \mathcal{I}_E = \mathcal{O}(-E)$ holds on $\tilde{X}$. Since $\mathcal{O}(1)$ is canonically trivialized on $\tilde{X} \setminus E$, one has the localized first Chern class $c_1(\mathcal{O}(1))$ in $H^2_E(\tilde{X}, \Lambda(1))$.

Lemma 1.1.1. *Assume $c > 0$. There is a canonical isomorphism*

$$\alpha : \bigoplus_{i=1}^{c-1} H^{2(c-i)}(Y,\ \Lambda(c-i)) \oplus H_Y^{2c}(X,\ \Lambda(c)) \xrightarrow{\sim} H_E^{2c}(\tilde{X},\ \Lambda(c)).$$

given by $\alpha((\gamma_i)_{1\leq i\leq c-1}, \gamma) = \sum_{i=1}^{c-1} \gamma_i \cdot c_1(\mathcal{O}(1))^i + \pi^*\gamma$.

Proof. Consider the commutative diagram

$$\begin{array}{ccccc} E & \xrightarrow{\tilde{i}} & \tilde{X} & \xleftarrow{\tilde{j}} & \tilde{X} \setminus E \\ \downarrow{\scriptstyle \pi_Y} & & \downarrow{\scriptstyle \pi} & & \| \\ Y & \xrightarrow{i} & X & \xleftarrow{j} & X \setminus Y. \end{array}$$

We write down the localization triangles on $\tilde{X}$ and X.

$$\begin{aligned} &\to \tilde{i}^!\Lambda_{\tilde{X}} \to \Lambda_E \to \tilde{i}^*\tilde{j}_*\Lambda_{\tilde{X}} \to \\ &\to i^!\Lambda_X \to \Lambda_Y \to i^*j_*\Lambda_X \to . \end{aligned}$$

The first sequence exists on E, and the second on Y. Applying π_{Y*} to the first, we have

$$\mathrm{Cone}(i^!\Lambda_X \to \pi_{Y*}(\tilde{i}^!\Lambda_{\tilde{X}})) = \mathrm{Cone}(\Lambda_Y \to \pi_{Y*}\Lambda_E),$$

since

$$\pi_{Y*}(\tilde{i}^*\tilde{j}_*\Lambda_{\tilde{X}}) = i^*j_*\Lambda_X$$

by the proper base change theorem. Let c' be the image of $c_1(\mathcal{O}(1))$ under $H^2_E(\tilde{X}, \Lambda(1)) \to H^2(\tilde{X}, \Lambda(1)) \to H^2(E, \Lambda(1))$. c' is the first Chern

class of $\mathcal{O}(1)|_E$. Since E is a projective bundle over Y of relative dimension $c-1$, the multiplication by c' induces a canonical decomposition

$$\bigoplus_{i=0}^{c-1} \Lambda_Y(-i)[-2i] \xrightarrow{\sim} \pi_{Y*}\Lambda_E$$

in $D^+(Y_{\text{et}}, \Lambda)$, and hence $\text{Cone}(\Lambda_Y \to \pi_{Y*}\Lambda_E)$ is identified with $\bigoplus_{i=1}^{c-1} \Lambda_Y(-i)[-2i]$. By taking the global section $R\Gamma$, the map α defined in 1.1.1 gives an isomorphism. Q.E.D.

By Lemma 1.1.1, we have an equation

$$c_1(\mathcal{O}(1))^c + \sum_{i=1}^{c} c_i \cdot c_1(\mathcal{O}(1))^{c-i} = 0$$

in $H_E^{2c}(\tilde{X}, \Lambda(c))$. For $0 \leq i \leq c-1$, $c_i \in H^{2i}(Y, \Lambda(i))$ is the i-th Chern class of $\mathcal{N}_{Y|X}^{\vee}$ ([Jou], §3). The constant term $c_c \in H_Y^{2c}(X, \Lambda(c))$ refines the c-th Chern class $c_c(\mathcal{N}_{Y|X}^{\vee})$ in $H^{2c}(Y, \Lambda(c))$.

Definition 1.1.2 (Gabber). $\text{cl}(Y) = c_c \in H_Y^{2c}(X, \Lambda(c))$.

Proposition 1.1.3 (functoriality). *Let X be a noetherian scheme, $Y \subset X$ be a closed, local complete intersection subscheme of pure codimension c. For a morphism $f : X' \to X$ such that $Y' = f^*(Y)$ is also a local complete intersection subscheme of codimension c,*

$$f^* \,\text{cl}(Y) = \text{cl}(Y').$$

Especially, the formation of cl *commutes with any flat pullback.*

Proposition 1.1.4. *Let X be a noetherian scheme. Let D_i $(1 \leq i \leq c)$ be effective Cartier divisors on X crossing normally (i.e., their local defining equations form a regular sequence). If $Y = \bigcap_{1 \leq i \leq c} D_i$, then*

$$\text{cl}(Y) = \bigcup_{i=1}^{c} \text{cl}(D_i).$$

Proof. Let $\tilde{D}_i$ be the strict transform of D_i. $\pi^* D_i = \tilde{D}_i + E$, and hence there is an equality

$$c_1(\mathcal{O}(\pi^* D_i)) + c_1(\mathcal{O}(1)) = c_1(\mathcal{O}(\tilde{D}_i))$$

of localized Chern classes in $H^2_{\pi^* D_i}(\tilde{X}, \Lambda(1))$. By taking the cup product of these classes,

$$\bigcup_{1 \leq i \leq c} (c_1(\mathcal{O}(\pi^* D_i)) + c_1(\mathcal{O}(1))) = \bigcup_{1 \leq i \leq c} c_1(\mathcal{O}(\tilde{D}_i))$$

in $H^{2c}_{\bigcap_{1 \leq i \leq c} \pi^* D_i}(\tilde{X}, \Lambda(c)) = H^{2c}_E(\tilde{X}, \Lambda(c))$. Since $\bigcap_{1 \leq i \leq c} \tilde{D}_i = \emptyset$, $\bigcup_{1 \leq i \leq c} c_1(\mathcal{O}(\tilde{D}_i))$ is zero, and hence we have the following equation

$$\sum_{i=0}^{c} \sigma_i(c_1(\mathcal{O}(\pi^* D_1)), \ldots, c_1(\mathcal{O}(\pi^* D_c))) \cdot c_1(\mathcal{O}(1))^{c-i} = 0$$

in $H^{2c}_E(\tilde{X}, \Lambda(c))$. Here $\sigma_i(\alpha_1, \ldots, \alpha_c)$ is the i-th fundamental symmetric polynomial in $\alpha_1, \ldots, \alpha_c$, and $\sigma_0 = 1$. By the definition of the cycle class,

$$\mathrm{cl}(Y) = \sigma_c(c_1(\mathcal{O}(\pi^* D_1)), \ldots, c_1(\mathcal{O}(\pi^* D_c))) = \bigcup_{i=1}^{c} c_1(\mathcal{O}(\pi^* D_i)).$$

On the other hand, by the functoriality of the localized Chern classes of effective divisors,

$$\bigcup_{i=1}^{c} c_1(\mathcal{O}(\pi^* D_i)) = \pi^* \bigcup_{i=1}^{c} c_1(\mathcal{O}(D_i))$$

via $\pi^* : H^{2c}_Y(X, \Lambda(c)) \to H^{2c}_E(\tilde{X}, \Lambda(c))$. This proves the claim. Q.E.D.

Corollary 1.1.5. *The image of* $\mathrm{cl}(Y)$ *in* $H^0(Y, R^{2c} i^! \Lambda(c))$ *coincides with the class defined in* [De1], 2.2.

1.2. Gysin map

Assume that we are given a closed embedding $Y \subset X$, which is a local complete intersection of pure codimention c. For $K \in D_{tf}(X_{\mathrm{et}}, \Lambda)$ (the index tf means "finite Tor dimension") and $L \in D^+(X_{\mathrm{et}}, \Lambda)$, we have the canonical product

$$i^* K \otimes^{\mathbf{L}} i^! L \to i^!(K \otimes^{\mathbf{L}} L).$$

$\mathrm{cl}(Y)$ defines a morphism $\Lambda \overset{\mathrm{cl}(Y)}{\to} i^! \Lambda(c)[2c]$ in $D^+(Y_{\mathrm{et}}, \Lambda)$, and hence we have the canonical map which we call the *Gysin map* for (i, K)

$$Gys_{(i,K)} : i^* K \to i^* K \otimes^{\mathbf{L}} i^! \Lambda(c)[2c] \to i^! K(c)[2c].$$

Proposition 1.2.1 (Compatibility). *Assume that we are given two embeddings* $Y \overset{i_1}{\hookrightarrow} Z \overset{i_2}{\hookrightarrow} X$ *which are complete intersections of pure codimension* c_1 *and* c_2*, respectively. For the map*

$$i_1^! i_2^* K \overset{i_1^! Gys_{(i_2,K)}}{\to} i_1^! i_2^! K(c_2)[2c_2]$$

induced by the Gysin map

$$i_2^* K \overset{Gys_{(i_2,K)}}{\to} i_2^! K(c_2)[2c_2],$$

the composition of

$$\begin{aligned}(i_2 \cdot i_1)^* K &\overset{Gys_{(i_1,K)}}{\to} i_1^! i_2^* K(c_1)[2c_1] \\ &\overset{i_1^! Gys_{(i_2,K)}(c_1)[2c_1]}{\to} (i_2 \cdot i_1)^! K(c_1+c_2)[2(c_1+c_2)].\end{aligned}$$

is the Gysin map for $(i_2 \cdot i_1, K)$.

Proof. In case that Z and Y are obtained by intersections of Cartier divisors D_i:

$$Y = \bigcap_{i=1}^{c_1+c_2} D_i, \quad Z = \bigcap_{i=1}^{c_2} D_i,$$

1.2.1 follows from Proposition 1.1.4. In what follows in this paper we use Proposition 1.2.1 only in this special case. The general case of Proposition 1.2.1 is shown by using the method in [F], 9.2, but the details are omitted. Q.E.D.

§2. Statement of purity theorem

2.1. Absolute purity theorem

In this section all schemes are over $\operatorname{Spec} \mathbf{Z}[1/n]$, and let $\Lambda = \mathbf{Z}/n$.

Theorem 2.1.1 (Gabber). *Let* $Y \overset{i}{\hookrightarrow} X$ *be a closed immersion of noetherian regular schemes of pure codimension* c*. Then the cycle class defined in* 1.1 *gives an isomorphism*

$$\Lambda_Y \overset{\mathrm{cl}(Y)}{\simeq} i^! \Lambda(c)[2c],$$

in $D^+(Y_{\mathrm{et}}, \Lambda)$*, i.e.,*

$$\mathcal{H}^q_Y(\Lambda) \simeq \begin{cases} 0 & \text{for } q \neq 2c, \\ \Lambda_Y(-c) & \text{for } q = 2c. \end{cases}$$

2.2. Punctual purity

Definition 2.2.1.

a) Let $Y \overset{i}{\hookrightarrow} X$ be a closed immersion of noetherian regular schemes of pure codimension c. We say that the purity holds for (X, Y) if and only if

$$\Lambda_Y \overset{\mathrm{cl}(Y)}{\simeq} i^!\Lambda(c)[2c]$$

is an isomorphism in $D^+(Y_{\mathrm{et}}, \Lambda)$.

b) Let X be a regular scheme, and x be a point of X. We say that the punctual purity holds at x, or X is punctually pure at x if and only if

$$\Lambda_x \overset{\mathrm{cl}(x)}{\simeq} i_x^!\Lambda(d_x)[2d_x]$$

is an isomorphism in $D^+(x_{\mathrm{et}}, \Lambda)$. Here $d_x = \dim \mathcal{O}_{X,x}$ is the local codimension at x, and $i_x^! = i'^*_x \cdot i^!_{\overline{\{x\}}}$ for $\{x\} \overset{i'_x}{\hookrightarrow} \overline{\{x\}} \overset{i_{\overline{\{x\}}}}{\hookrightarrow} X$.

c) Let $\mathcal{O}$ be a regular strict local ring with the maximal ideal m. We say that the punctual purity holds for $(\mathcal{O}, m)$, or $(\mathcal{O}, m)$ is punctually pure, if and only if $\operatorname{Spec}\mathcal{O}$ is punctually pure at the closed point $V(m)$.

Here are very useful propositions to check punctual purity in some special cases.

Proposition 2.2.2. *Let $\mathcal{O}$ be a regular strict local ring with the maximal ideal m, and $\hat{\mathcal{O}}$ be the m-adic completion. Let $X = \operatorname{Spec}\mathcal{O}$, $x = V(m)$, $\hat{X} = \operatorname{Spec}\hat{\mathcal{O}}$, and $\hat{x} = V(m\hat{\mathcal{O}})$. Then X is punctually pure at x if and only if $\hat{X}$ is punctually pure at $\hat{x}$.*

Proof. By the formal base change theorem of Gabber ([Fu], Corollary 6.6.4),

$$H^q(X \setminus \{x\}, \Lambda) \simeq H^q(\hat{X} \setminus \{\hat{x}\}, \Lambda),$$

and hence

$$H^q_x(X, \Lambda) \simeq H^q_{\hat{x}}(\hat{X}, \Lambda)$$

for any $q \in \mathbf{Z}$. Since $\hat{X} \to X$ is faithfully flat, by 1.1.3 $\mathrm{cl}(x)$ is mapped to $\mathrm{cl}(\hat{x})$ under this isomorphism. The claim follows. Q.E.D.

Corollary 2.2.3. *Let X be a noetherian regular scheme of equal characteristic. Then X is punctually pure at every point.*

Proof. It suffices to prove that $(\mathcal{O}, m)$ is punctually pure for any regular strict local ring $\mathcal{O}$ of equal characteristic with the maximal ideal m. By 2.2.2, we may assume that $\mathcal{O}$ is m-adically complete. By the structure theorem of complete regular local rings, there is a local isomorphism

$$\mathcal{O} \simeq \hat{\mathcal{O}}',$$

where $\mathcal{O}'$ is the strict henselization of a polynomial algebra $k[X_1, \ldots, X_N]$ over a separably closed field k at $(X_1, \ldots, X_N)$. By the relative purity theorem ([AGV], exposé XVI, 3.7), $\operatorname{Spec} k[X_1, \ldots, X_N]$ is punctually pure at any k-rational point. Using 2.2.2 again, we finish the proof. Q.E.D.

Proposition 2.2.4. *Let $Y \overset{i}{\hookrightarrow} X$ be a closed immersion of regular schemes of pure codimension c. Any two of the following conditions imply the third.*

a) *The purity holds for (X, Y).*
b) *Y is punctually pure at every point.*
c) *X is punctually pure at every point of Y.*

The proof of "b), c) $\Rightarrow$ a)" will be given in 3.1.3 later.

Proof of a), b) $\Rightarrow$ c). Take any point y of Y. Put $d_y = \dim \mathcal{O}_{X,y}$, $\{y\} \overset{i_y}{\hookrightarrow} Y \overset{i}{\hookrightarrow} X$. From a), we have an isomorphism

$$\Lambda_Y \overset{\mathrm{cl}(Y)}{\simeq} i^! \Lambda(c)[2c],$$

and hence

$$\Lambda \overset{\mathrm{cl}(y)}{\simeq} i_y^! \Lambda_Y(d_y - c)[2d_y - 2c] \overset{i_y^! \mathrm{cl}(Y)(d_y - c)[2(d_y - c)]}{\simeq} (i \cdot i_y)^! \Lambda(d_y)[2d_y].$$

By the compatibility 1.2.1, the composite of the isomorphisms is obtained by the cycle class.

Proof of a), c) $\Rightarrow$ b). It is proved in a similar way as above, so the proof is omitted. Q.E.D.

Proposition 2.2.5. *Let S be a noetherian regular scheme of dimension at most one, and P be a smooth S-scheme. Then the punctual purity holds at every point of P.*

Proof. By 2.2.3, we may assume that S is a trait. Let s be the closed point of S. By 2.2.3 again, it suffices to prove that P is punctually pure

at any point over s. We apply a), b) $\Rightarrow$ c) part of Proposition 2.2.4 (we have already proved this part) with $X = P$, $Y = P_s$. a) is a consequence of the smooth base change theorem ([AGV], XVI, Corollaire 1.2). b) follows from 2.2.3. Q.E.D.

Corollary 2.2.6. *Let V be a strict complete discrete valuation ring, $\mathcal{O} = V[[X_1, \ldots, X_n]]$ be a power series ring over V. Then* $\operatorname{Spec}\mathcal{O}$ *is punctually pure at the closed point.*

This is a consequence of Propositions 2.2.2 and 2.2.5.

2.3. Injectivity

Lemma 2.3.1. *Let $\mathcal{O}$ be a regular strict local ring, and $X =$* $\operatorname{Spec}\mathcal{O}$. *For a regular closed subscheme $Y \subset X$ of codimension c, $\Lambda \overset{\mathrm{cl}(Y)}{\to} H^{2c}_Y(X, \Lambda(c))$ is injective.*

Proof. By considering the composition

$$\Lambda \to H^{2c}_Y(X, \Lambda(c) \to H^{2c}_{\bar{y}}(X_{\bar{y}}, \Lambda(c))$$

where $\bar{y}$ is a geometric generic point of Y, we are reduceed to the case $Y = \{x\}$ where x is the closed point of X and c is the dimension of $\mathcal{O}$. By Proposition 2.2.2, we may assume that $\mathcal{O}$ is a complete local ring. By the structure theorem of complete local rings, there are a coefficient ring C of $\mathcal{O}$ and a surjective local homomorphism $f : C[[X_1, \ldots, X_n]] \twoheadrightarrow \mathcal{O}$. C is a field, or a Cohen ring, namely a complete discrete valuation ring for which a rational prime p is a prime element. Let $P = \operatorname{Spec} C[[X_1, \ldots, X_n]]$, $N = \dim P$. Consider the closed embedding $X \overset{i}{\hookrightarrow} P$ induced by f. P is punctually pure at x by Corollary 2.2.6, and we have the Gysin map

$$H^{2c}_x(X,\ \Lambda(c)) \to H^{2N}_x(P,\ \Lambda(N)) \simeq \Lambda.$$

The last isomorphism is given by the cycle class. By the compatibility of cycle class 1.2.1, the composition of

$$\Lambda \overset{\mathrm{cl}(x)}{\to} H^{2c}_x(X,\ \Lambda(c)) \to \Lambda$$

is the identity, and hence the map is injective. Q.E.D.

§3. Impure cohomology

Definition 3.1.1. Let $\mathcal{O}$ be a regular strict local ring, and $X =$ $\operatorname{Spec}\mathcal{O}$. For a regular closed subscheme $Y \subset X$ of codimension c and

$q \in \mathbf{Z}$, the impure cohomology group $H^q_Y(X, \Lambda)_{\text{impure}}$ is defined as

$$H^q_Y(X,\ \Lambda)_{\text{impure}} = \begin{cases} H^q_Y(X,\ \Lambda) & (q \neq 2c) \\ \text{Coker}(\Lambda(-c) \to H^q_Y(X,\ \Lambda)) & (q = 2c). \end{cases}$$

Here $\Lambda(-c) \overset{\text{cl}(Y)(-c)}{\to} H^{2c}_Y(X, \Lambda)$ is defined by the cycle class. The impure cohomology group $H^q(X \setminus Y,\ \Lambda)_{\text{impure}}$ is defined as $H^{q+1}_Y(X, \Lambda)_{\text{impure}}$ for $q \in \mathbf{Z}$.

By 2.3.1, X is punctually pure at the closed point x if and only if $H^q(X \setminus \{x\}, \Lambda)_{\text{impure}} = \{0\}$ for any $q \in \mathbf{Z}$.

Proposition 3.1.2 (Invariance of impure cohomology groups). *Let $\mathcal{O}$ be a regular strict local ring, $X = \operatorname{Spec} \mathcal{O}$, and x be the closed point of X. Let Y be a non-empty regular closed subscheme different from X. Let n be an integer which is invertible on X, and let $\Lambda = \mathbf{Z}/n$. We assume the following two conditions.*

a) *Y is punctually pure at every point.*
b) *The purity holds for $(X \setminus \{x\}, Y \setminus \{x\})$.*

Then there is a canonical isomorphism

$$H^q(X \setminus \{x\},\ \Lambda)_{\text{impure}} \xrightarrow{\sim} H^q(X \setminus Y,\ \Lambda)_{\text{impure}}$$

for any $q \in \mathbf{Z}$.

Proof. Let $d = \dim \mathcal{O}$, $c = \operatorname{codim}(Y, X)$. $\tilde{X} = X \setminus \{x\}$, $\tilde{Y} = Y \setminus \{x\}$, $\tilde{i} : \tilde{Y} \hookrightarrow \tilde{X}$, $j_x : \tilde{Y} \hookrightarrow Y$, $i_x : \{x\} \hookrightarrow X$. By our assumption b),

$$\Lambda_{\tilde{Y}} \xrightarrow{\sim} \tilde{i}^! \Lambda(c)[2c]$$

by the cycle class. Since the claim is obvious in the case of $c = d$, we may assume that $c < d$. Note that there is a following morphism of localization triangles

$$\begin{array}{ccccc} i_x^! \Lambda_Y(-c)[-2c] & \longrightarrow & i_x^* \Lambda_Y(-c)[-2c] & \longrightarrow & i_x^* j_{x*} \Lambda_{\tilde{Y}}(-c)[-2c] \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} \\ i_x^! \Lambda & \longrightarrow & i_x^* i^! \Lambda & \longrightarrow & i_x^* j_{x*} \tilde{i}^! \Lambda \end{array}$$

which is induced by the cycle class $\Lambda_Y(-c)[-2c] \overset{\text{cl}(Y)(-c)[-2c]}{\to} i^! \Lambda$. First we look at the second line. By assumption a), $H^q(i_x^* j_{x*} \tilde{i}^! \Lambda) = 0$ unless

$q = 2c,\ 2d-1$. The long exact sequence associated to the second line reduces to

$$H^q_x(X,\Lambda) \xrightarrow{\sim} H^q_Y(X,\Lambda) \quad \text{for } q \neq 2c, 2c+1, 2d-1, 2d, \tag{1}$$

$$\begin{aligned} 0 \to H^{2c}_x(X,\Lambda) \to H^{2c}_Y(X,\Lambda) \to & H^{2c}(\tilde{Y}, \tilde{i}^!\Lambda) \\ & \to H^{2c+1}_x(X,\Lambda) \to H^{2c+1}_Y(X,\Lambda), \end{aligned} \tag{2}$$

$$\begin{aligned} H^{2d-1}_x(X,\Lambda) \to & H^{2d-1}_Y(X,\Lambda) \to H^{2d-1}(\tilde{Y}, \tilde{i}^!\Lambda) \\ & \to H^{2d}_x(X,\Lambda) \to H^{2d}_Y(X,\Lambda) \to 0, \end{aligned} \tag{3}$$

where we can add $\to 0$ to the right hand side of (2) and $0 \to$ to the left hand side of (3) in the case $c \neq d-1$. Concerning (2), the composition of

$$\Lambda(-c) \xrightarrow{H^{2c}(\beta)} H^{2c}_Y(X,\Lambda) \to H^{2c}(\tilde{Y}, \tilde{i}^!\Lambda) \xrightarrow{H^{2c}(\gamma)^{-1}}_{\sim} \Lambda(-c)$$

is the identity. So the surjectivity of $H^{2c}_Y(X,\Lambda) \to H^{2c}_{\text{et}}(\tilde{Y}, \tilde{i}^!\Lambda)$ follows, showing that $H^{2c+1}_x(X,\Lambda) \to H^{2c+1}_Y(X,\Lambda)$ is bijective (resp. injective) in the case $c \neq d-1$ (resp. $c = d-1$), and the canonical decomposition

$$H^{2c}_Y(X,\Lambda) \simeq \Lambda(-c) \oplus \operatorname{Ker}(H^{2c}_Y(X,\Lambda) \to \Lambda(-c))$$

induces

$$H^{2c}_x(X,\Lambda) \xrightarrow{\sim} \operatorname{Ker}(H^{2c}_Y(X,\Lambda) \to \Lambda(-c)) \simeq H^{2c}_Y(X,\Lambda)_{\text{impure}}.$$

Concerning (3), we show that

(4) $H^{2d-1}(\tilde{Y}, \tilde{i}^!\Lambda) \to H^{2d}_x(X,\Lambda)$ is injective,

(5) the image of this map is spanned by the cycle class twisted by $\Lambda(-d)$.

Then we finish the proof: it follows that $H^{2d-1}_x(X,\Lambda) \xrightarrow{\sim} H^{2d-1}_Y(X,\Lambda)$ and $H^{2d}_x(X,\Lambda)_{\text{impure}} \xrightarrow{\sim} H^{2d}_Y(X,\Lambda)$.

To prove (4) and (5), consider the following commutative diagram.

$$\begin{array}{ccc} H^{2d-2c-1}(\tilde{Y},\Lambda)(-c) & \xrightarrow{\delta} & H^{2d-2c}_x(Y,\Lambda)(-c) \\ \downarrow{\scriptstyle H^{2d-1}(\gamma)} & & \downarrow{\scriptstyle H^{2d}(\alpha)} \\ H^{2d-1}(\tilde{Y}, \tilde{i}^!\Lambda) & \longrightarrow & H^{2d}_x(X,\Lambda). \end{array}$$

$H^{2d-1}(\gamma)$ is an isomorphism since $\Lambda_{\tilde{Y}} \xrightarrow{\sim} \tilde{i}^!\Lambda(c)[2c]$. δ is an isomorphism by the localization sequence and the fact that the higher cohomology of Y vanishes. $H^{2d-2c}_x(Y,\Lambda)(d-c)$ is spanned by the cycle class ϵ of x in Y, and by the compatibility 1.2.1, $H^{2d}(\alpha)(d)(\epsilon)$ is the cycle class of x in X. (5) is shown. (4) follows from Lemma 2.3.1. This completes the proof of 3.1.2 Q.E.D.

3.1.3. Here we give the proof of the part b), c) $\Rightarrow$ a) of Proposition 2.2.4. Take a point y in Y, and let $X_y = \operatorname{Spec} \mathcal{O}^h_{X,y}$, $Y_y = \operatorname{Spec} \mathcal{O}^h_{Y,y}$. Here $(-)^h$ denotes the henselization. It suffices to prove that the purity holds for (X_y, Y_y) for any $y \in Y$. We proceed by induction on $\delta = \dim \mathcal{O}_{Y,y}$. If δ is zero, the statement is obvious.

Let $\tilde{X}_y = X_y \setminus \{y\}$, $\tilde{Y}_y = Y_y \setminus \{y\}$. By our induction hypothesis,

$$\Lambda_{\tilde{Y}_y} \overset{\mathrm{cl}(\tilde{Y}_y)}{\xrightarrow{\sim}} \tilde{i}^!\Lambda(c)[2c]$$

by the cycle class. By Proposition 3.1.2, the punctual purity of X_y at y implies the purity for (X_y, Y_y). Q.E.D.

Corollary 3.1.4. *In* 3.1.2, *condition* b) *can be replaced by*

b') *X is punctually pure except possibly at the closed point x.*

Proof. By the part b), c) $\Rightarrow$ a) of Proposition 2.2.4 applied to $(X \setminus \{x\}, Y \setminus \{x\})$, the assumption a) of 3.1.2 and b') of 3.1.4 implies condition b) of 3.1.2. Q.E.D.

§4. Duality formalism for arithmetic schemes

In this section, we fix a noetherian regular scheme S of dimension at most one. By an arithmetic S-scheme, we mean a separated scheme X of finite type over S. Let n be an integer which is invertible on S, and let $\Lambda = \mathbf{Z}/n$.

For a morphism of arithmetic S-schemes $f : X \to Y$, we make use of the six operations ([De2])

$$\begin{aligned} f_*, f_! &: D^b_c(X_{\mathrm{et}}, \Lambda) \to D^b_c(Y_{\mathrm{et}}, \Lambda), \\ f^*, f^! &: D^b_c(Y_{\mathrm{et}}, \Lambda) \to D^b_c(X_{\mathrm{et}}, \Lambda), \\ Hom &: D^-_c(X_{\mathrm{et}}, \Lambda) \times D^+_c(X_{\mathrm{et}}, \Lambda) \to D^+_c(X_{\mathrm{et}}, \Lambda), \\ \otimes^{\mathbf{L}} &: D^-_c(X_{\mathrm{et}}, \Lambda) \times D^-_c(X_{\mathrm{et}}, \Lambda) \to D^-_c(X_{\mathrm{et}}, \Lambda). \end{aligned}$$

Here $D^*_c(X_{\mathrm{et}}, \Lambda)$, $D^*_c(Y_{\mathrm{et}}, \Lambda)$ denote the derived categories of the complexes of Λ-sheaves with constructible cohomology sheaves and a suitable boundedness condition.

4.1. Normalization of dualizing complexes

For the structural morphism $f_X : X \to S$ of an arithmetic S-scheme X, let $\delta_{f_X} : X \to \mathbf{Z}$ be the dimension function defined by

$$\delta_{f_X}(x) = \dim \overline{\{f_X(x)\}} + \mathrm{tr.deg}\, k(x)/k(f_X(x)) \quad (x \in X).$$

We put

$$\delta_X = \sup_{x \in X} \delta_{f_X}(x),$$

and call it the *total dimension* of X. For an arithmetic S-scheme X, we say that X is *δ-equidimensional* if and only if δ_Y, where Y is any irreducible component of X, are equal. X is called locally δ-equidimensional if and only if every point of X admits an open neighbourhood which is δ-equidimensional. When X is locally δ-equidimensional, we also denote by δ_X the locally constant function that maps a point x to δ_U of a δ-equidimensional open neighborhood U of x.

For an arithmetic S-scheme X, we put

$$K_X = f_X^! \Lambda(\delta_S)[2\delta_S].$$

Here δ_S is the total dimension of S, which we view as a locally constant function on S. K_X has a finite injective dimension. For any $K \in D_c^b(X, \Lambda)$, define $D_X K$ by

$$D_X K = Hom(K, K_X).$$

By a theorem of Deligne ([De2], 4.3 and 4.7), K_X satisfies the local biduality:

a) $D_X K$ belongs to $D_c^b(X, \Lambda)$.
b) $K \xrightarrow{\sim} D_X D_X K$ for any $K \in D_c^b(X, \Lambda)$.

We call an object K_X of $D_c^b(X, \Lambda)$ having a finite quasi-injective dimension ([G], Définition 1.7) with these two properties a), b), a *dualizing complex* of X. For a connected scheme, dualizing complexes are unique up to a shift, and the twist by a smooth locally free Λ-sheaf of rank one ([G], Théorème 2.1).

We make use of the dualizing complex K_X chosen as above. We will construct the *fundamental class* of X in some special cases.

Let us start with the smooth case. If $f : X \to S$ is smooth of relative dimension d, there is a canonical trace map ([AGV], exposé XVIII, 2.9)

$$Tr_f : R^{2d} f_! f^* K_S(d) \to K_S$$

which induces

$$\Lambda(\delta_X)[2\delta_X] \overset{Tr_f}{\simeq} K_X.$$

Here we regard d and δ_X as a locally constant function on X.

Let X be a δ-equidimensional local complete intersection scheme admitting a closed embedding $X \overset{i}{\hookrightarrow} P$ into a smooth S-scheme P. Then we have the cycle class

$$\Lambda \overset{\mathrm{cl}(X)}{\to} i^!\Lambda(\delta_P - \delta_X)[2(\delta_P - \delta_X)] \overset{i^! Tr_{f_P}(-\delta_X)[-2\delta_X]}{\simeq} i^! f_P^! K_S(-\delta_X)[-2\delta_X],$$

and hence

$$\Lambda(\delta_X)[2\delta_X] \overset{[X]}{\to} K_X$$

as the composition.

Lemma 4.1.1. *Let X be a δ-equidimensional local complete intersection scheme admitting a closed embedding $X \overset{i}{\hookrightarrow} P$ into a smooth S-scheme P. Then $[X]$ is independent of any choice of i. We call $[X]$ the fundamental class of X.*

Proof. The proof is standard. Assume that we have two embeddings $i_s : X \to P_s$ $(s = 1, 2)$. The composition

$$X \overset{(i_1,i_2)}{\to} P_1 \times_S P_2 \overset{\mathrm{pr}_s}{\to} P_s$$

is i_s. By replacing $P_1 \times_S P_2$ by P_1, we may assume that P_1 is a P_2-scheme, $F : P_1 \to P_2$ is a smooth morphism of relative dimension d, and $i_2 = F \cdot i_1$. By the compatibility of the trace map, the composition of

$$\Lambda(\delta_{P_1})[2\delta_{P_1}] \overset{F^* Tr_{f_{P_2}}(d)[2d]}{\to} F^* f_{P_2}^! \Lambda(d)[2d] \overset{Tr_F}{\to} F^! f_{P_2}^! \Lambda$$

is $Tr_{f_{P_1}}$. It suffices to prove

Sublemma 4.1.2. *Let X be a noetherian scheme. Assume that $F : P \to X$ is a smooth morphism of relative dimension d, $Y \overset{i_1}{\hookrightarrow} X$ is a local complete intersection of pure codimension c, and $i_2 : Y \to P$ is a closed immersion such that $F \cdot i_2 = i_1$. Then the composition*

$$\Lambda \overset{\mathrm{cl}(i_2)}{\to} i_2^!\Lambda(d+c)[2(d+c)] \overset{Tr_F(c)[2c]}{\simeq} i_2^! F^!(c)\Lambda[2c]$$

is $\mathrm{cl}(i_1)$. *Here we denote the cycle map $\Lambda \to i^!\Lambda(c)[2c]$ defined for a local complete intersection subscheme $W \overset{i}{\to} Z$ of codimension c as* $\mathrm{cl}(i)$.

Proof. We decompose i_2 as $Y \overset{\tilde{i}_2}{\to} P_Y = P \times_X Y \overset{i_{1P}}{\hookrightarrow} P$.

$$\begin{array}{ccccc} P & \xleftarrow{i_{1P}} & P_Y & \xleftarrow{\tilde{i}_2} & Y \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle F_Y} & & \\ X & \xleftarrow{i_1} & Y & & \end{array}$$

The base change morphism([AGV], expose XVIII (3.1.14.2)) is an isomorphism:

$$i_{1P}^* F^! \Lambda_X \overset{(1)}{\simeq} F_Y^! \Lambda_Y,$$

and the isomorphism (1) is compatible with the trace maps: the identification $Tr_F : \Lambda_P(d)[2d] \simeq F^! \Lambda_X$ induces $Tr_{F_Y} : \Lambda_{P_Y}(d)[2d] \simeq F_Y^! \Lambda_Y$ by the functoriality.

On the other hand, since $F^! \Lambda_X$ is a shift of a Λ-smooth sheaf, we have the canonical isomorphism induced by the canonical product

$$i_{1P}^! F^! \Lambda_X \simeq i_{1P}^! \Lambda_P \otimes^{\mathbf{L}} i_{1P}^* F^! \Lambda_X.$$

So the Gysin map

$$i_{1P}^* F^! \Lambda_X \overset{Gys_{(i_{1P}, F^! \Lambda_X)}}{\to} i_{1P}^! F^! \Lambda_X(c)[2c]$$

is identified with

$$i_{1P}^* F^! \Lambda_X \overset{\mathrm{cl}(i_{1P}) \otimes^{\mathbf{L}} \mathrm{id}_{i_{1P}^* F^! \Lambda_X}}{\to} i_{1P}^! \Lambda_P(c)[2c] \otimes^{\mathbf{L}} i_{1P}^* F^! \Lambda_X,$$

and also with

$$F_Y^! \Lambda_Y \overset{\mathrm{cl}(i_{1P}) \otimes^{\mathbf{L}} \mathrm{id}_{F_Y^! \Lambda_Y}}{\to} i_{1P}^! \Lambda_P(c)[2c] \otimes^{\mathbf{L}} F_Y^! \Lambda_Y$$

by (1). Using this identification and (1),

$$\begin{aligned} (2) \qquad \tilde{i}_2^* i_{1P}^* F^! \Lambda_X(-d)[-2d] &\overset{Gys_{(\tilde{i}_2, i_{1P}^* F^! \Lambda(-d)[-2d])}}{\to} \tilde{i}_2^! i_{1P}^* F^! \Lambda_X \\ &\overset{\tilde{i}_2^! Gys_{(i_{1P}, F^! \Lambda_X)}}{\to} \tilde{i}_2^! i_{1P}^! F^! \Lambda_X(c)[2c] \end{aligned}$$

is identified with

$$\begin{aligned} (3) \qquad \tilde{i}_2^* F_Y^! \Lambda_Y(-d)[-2d] &\overset{Gys_{(\tilde{i}_2, F_Y^! \Lambda_Y(-d)[-2d])}}{\to} \tilde{i}_2^! F_Y^! \Lambda_Y \\ &\overset{\tilde{i}_2^! (\mathrm{cl}(i_{1P}) \otimes^{\mathbf{L}} \mathrm{id}_{F_Y^! \Lambda})}{\to} \tilde{i}_2^! (i_{1P}^! \Lambda_P(c)[2c] \otimes^{\mathbf{L}} F_Y^! \Lambda_Y). \end{aligned}$$

The composition of (2) is

$$Gys_{(i_2,F^!\Lambda_X(-d)[-2d])} = \mathrm{cl}(i_2) \otimes \mathrm{id}_{\tilde{i}_2^* F_Y^! \Lambda(-d)[-2d]}$$

by the compatibility 1.2.1. By the functoriality 1.1.3, $\Lambda_{P_Y} \overset{\mathrm{cl}(i_{1P})}{\to} i_{1P}^! \Lambda_P(c)[2c]$ is canonically isomorphic to $\Lambda_{P_Y} \overset{F_Y^* \mathrm{cl}(i_1)}{\to} F_Y^* i_1^! \Lambda_X(c)[2c]$, and hence the second arrow of (3) is $\tilde{i}_2^! F_Y^! \Lambda_Y \overset{\tilde{i}_2^! F_Y^! \mathrm{cl}(i_1)}{\to} \tilde{i}_2^! F_Y^! i_1^! \Lambda_X(c)[2c]$ which is equal to $\Lambda_Y \overset{\mathrm{cl}(i_1)}{\to} i_1^! \Lambda_X(c)[2c]$.

Since (1) is compatible with the trace map, it suffices to see that the composition of

$$\Lambda_Y \overset{\mathrm{cl}(\tilde{i}_2)}{\to} \tilde{i}_2^! \Lambda_{P_Y}(d)[2d] \overset{\tilde{i}_2^! Tr_{F_Y}}{\simeq} \tilde{i}_2^! F_Y^! \Lambda_Y = \Lambda_Y$$

is the identity of Λ_Y as a constant sheaf on Y.

So we are reduced to the case when $Y = X$, and $i = i_2$ is a section of F. Since the problem is étale local, we may assume that $P = \mathbf{A}_X^N$, and i is the zero section. Using the compatibility 1.2.1 and the compatibility of the trace map with compositions, we may assume that $P = \mathbf{A}_X^1$. It suffices to check it at the maximal points of X, so that we may assume that X is zero dimensional. By considering the geometric closed fibers over X, we reduce to the case when $X = \mathrm{Spec}\, k$, where k is a separably closed field. Since our cycle class coincides with the classical one for divisors, the claim is obvious by the definition of the trace map. Q.E.D.

Proposition 4.1.3. *Let X be a δ-equidimensional regular scheme admitting a closed embedding into a smooth S-scheme. For any $x \in X$, the following two conditions are equivalent:*

a) *The fundamental class $[X]$ gives an isomorphism*

$$i_y^* \Lambda(\delta_X)[2\delta_X] \overset{i_y^*[X]}{\simeq} i_y^* K_X$$

at any generization y of x.

b) *X is punctually pure at any generization y of x.*

Proof. We choose an embedding $X \overset{i}{\hookrightarrow} P$ into a smooth S-scheme P. By 2.2.5, P is punctually pure at every point. Then the claim follows from the implications b), c) $\Rightarrow$ a) and a), c) $\Rightarrow$ b) of Proposition 2.2.4. Q.E.D.

As a corollary, the fundamental class gives an isomorphism if X is regular of equal characteristic by 2.2.3. Since for each point x over $s \in S$, the closure $\overline{\{x\}}$ of x in X_s has a dense affine open regular subscheme of equal characteristic, we have a canonical isomorphism

$$i_x^! K_X \xleftarrow{\sim} \Lambda(\delta_{f_X}(x))[2\delta_{f_X}(x)].$$

Remark. For any δ-equidimensional local complete intersection arithmetic S-scheme X, the fundamental class $[X]$ is defined as follows. Take an affine dense open subscheme U of X. Then it is easy to see that

$$H^{2\delta_X}(X, K_X(-\delta_X)) \xrightarrow{\sim} H^{2\delta_X}(U, K_U(-\delta_X)).$$

Note that $\delta_X = \delta_U$ by the definition. For U, we have $[U] \in H^{2\delta_U}(U, K_U(-\delta_U))$ by taking an embedding into an affine space over S. Then we put $[X] = [U]$. This is independent of any choice of U, and for X which admits an embedding into a smooth S-scheme we have the same fundamental class as before.

For general X, one may take the following formula as the definition of the fundamental class:

$$[X] = \sum_{x \in I_X} \operatorname{length} \mathcal{O}_{X,x} \cdot [U_x] \in H^{2\delta_X}(X, K_X(-\delta_X)).$$

Here I_X is the set of the maximal points of X such that $\delta_{f_X}(x) = \delta_X$, and U_x is an open dense subscheme of the closure $\overline{\{x\}}$ which is a local complete intersection. When X is a δ-equidimensional local complete intersection, the coincidence of the two definitions is shown as follows.

By localization, one can assume that X is irreducible and embedded into an affine space over S. If X dominates an irreducible component of S, one uses the compatibility of cycle classes with proper intersection of cycles ([De1], over a field). In general one may use the formalism of local Chern classes of perfect complexes (as in [Iv]).

4.2. Local duality

We describe the local duality theory for regular local rings of arithmetic type by using another normalization of dualizing complexes.

Definition 4.2.1. We say that a strict local ring $\mathcal{O}$ is of arithmetic type over S if

$$\mathcal{O} \simeq \mathcal{O}_{X,\bar{x}}^{sh}$$

for an arithmetic S-scheme X and a point x of X. Here $\bar{x}$ is a geometric point of X localized at x, and $(-)^{sh}$ denotes the strict henselization.

Let $\mathcal{O}$ be a regular strict local ring of arithmetic type over S. We choose an isomorphism

$$\mathcal{O} \simeq \mathcal{O}^{sh}_{X,\bar{x}}$$

for an affine arithmetic regular S-scheme X and a geometric point $\bar{x}$ above x. Let $X_{\bar{x}} = \operatorname{Spec} \mathcal{O}^{sh}_{X,\bar{x}}$, $\tilde{X}_{\bar{x}} = X_{\bar{x}} \setminus \{x_s\}$, $i_{\bar{x}} : \{x_s\} \hookrightarrow X_{\bar{x}}$, and $j_{\bar{x}} : \tilde{X}_{\bar{x}} \hookrightarrow X_{\bar{x}}$. Here x_s is the closed point of $X_{\bar{x}}$.

Put

$$K^{\mathrm{ren}}_{X_{\bar{x}}} = K_X|_{X_{\bar{x}}}(-\delta_X)[-2\delta_X].$$

Then $K^{\mathrm{ren}}_{X_{\bar{x}}}$ is a dualizing complex of $X_{\bar{x}}$: $K^{\mathrm{ren}}_{X_{\bar{x}}}$ has a finite injective dimension, and the two properties in 4.1 characterizing a dualizing complex are satisfied for this $K_{X_{\bar{x}}}$.

By our renormalization,

$$i^!_{\bar{x}} K^{\mathrm{ren}}_{X_{\bar{x}}} \simeq \Lambda(-d_x)[-2d_x], \quad d_x = \dim \mathcal{O},$$

and there is a canonical map

$$\Lambda \overset{\mathrm{can}}{\to} K^{\mathrm{ren}}_{X_{\bar{x}}}$$

obtained by the fundamental class. From Proposition 4.1.3, we deduce the main result of this section:

Proposition 4.2.2. *For any $y \in X_{\bar{x}}$, the following two conditions are equivalent:*

a) *The canonical map*

$$\Lambda \overset{\mathrm{can}}{\to} K^{\mathrm{ren}}_{X_{\bar{x}}}$$

defined above gives an isomorphism

$$i^*_{y'}\Lambda \overset{i^*_{y'}\,\mathrm{can}}{\simeq} i^*_{y'} K^{\mathrm{ren}}_{X_{\bar{x}}}$$

at any generization y' of y.

b) *$X_{\bar{x}}$ is punctually pure at any generization y' of y.*

4.2.3. Finally we deduce the *local duality theorem* from the biduality of $K^{\mathrm{ren}}_{X_{\bar{x}}}$ ([G], 4.7). For $L \in D^b_c(X_{\bar{x}}, \Lambda)$, we put

$$D_{X_{\bar{x}}} L = Hom(L, K^{\mathrm{ren}}_{X_{\bar{x}}}).$$

For $K \in D^b_c(\tilde{X}_{\bar{x}}, \Lambda)$, apply the biduality to $j_{\bar{x}!}K$, and we get

$$i^!_{\bar{x}} j_{\bar{x}!} K \simeq i^!_{\bar{x}} D_{X_{\bar{x}}} D_{X_{\bar{x}}} j_{\bar{x}!} K \simeq Hom(i^*_{\bar{x}} D_{X_{\bar{x}}} j_{\bar{x}!} K, i^!_{\bar{x}} K^{\mathrm{ren}}_{X_{\bar{x}}})$$

$$\simeq Hom(i_{\bar{x}}^* j_{\bar{x}*} D_{\tilde{X}_{\bar{x}}} K, \Lambda(-d_x)[-2d_x]).$$

Here

$$D_{\tilde{X}_{\bar{x}}} K = Hom(K, K^{\mathrm{ren}}_{\tilde{X}_{\bar{x}}}),\ K^{\mathrm{ren}}_{\tilde{X}_{\bar{x}}} = j_{\bar{x}}^* K^{\mathrm{ren}}_{X_{\bar{x}}}.$$

Using that $i_{\bar{x}}^! j_{\bar{x}!} K \xrightarrow{\sim} i_{\bar{x}}^* j_{\bar{x}*} K[-1]$, we have a perfect pairing of Λ-modules

$$H^q(\tilde{X}_{\bar{x}},\ K) \times H^{2d_x - 1 - q}(\tilde{X}_{\bar{x}},\ D_{\tilde{X}_{\bar{x}}} K) \to \Lambda(-d_x)$$

for any $q \in \mathbf{Z}$.

§5. Vanishing theorems

5.1. Affine Lefschetz theorems

Let S be a noetherian regular scheme of dimension at most one. For an integer n which is invertible on S, let $\Lambda = \mathbf{Z}/n$. Gabber proved the following affine Lefschetz theorems. We will use Theorem C in this paper.

Theorem A. *Let $f : X \to Y$ be an affine morphism of finite type between arithmetic S-schemes. Let $\mathcal{F}$ be a Λ-sheaf on X. Then*

$$\delta_{\mathrm{Supp}\, R^q f_* \mathcal{F}} \leq \delta_{\mathrm{Supp}\, \mathcal{F}} - q.$$

To state the next theorem, we define a dimension function in a general setting. Let Y be the spectrum of a universally catenary local ring, $f : X \to Y$ be a morphism of finite type. For $x \in X$, we put

$$\delta_f(x) = \dim \overline{\{f(x)\}} + \mathrm{tr.deg}\, k(x)/k(f(x)),$$

and $\delta_{X,f} = \sup\{\delta_f(x); x \in X\}$.

Theorem B. *Let Y be the spectrum of a strict local ring of arithmetic type over S, $f : X \to Y$ be an affine morphism of finite type. If $\mathcal{F}$ is a Λ-sheaf on X,*

$$H^q(X,\ \mathcal{F}) = 0$$

for $q > \delta_{\mathrm{Supp}\, \mathcal{F}, f}$.

Especially, Theorem B implies

Theorem C. *Let $\mathcal{O}$ be a strict local ring of arithmetic type over S. For a non-zerodivisor f of $\mathcal{O}$ and a Λ-sheaf $\mathcal{F}$ on $\mathrm{Spec}\, \mathcal{O}[f^{-1}]$,*

$$H^q(\mathrm{Spec}\, \mathcal{O}[f^{-1}], \mathcal{F}) = 0 \quad \text{for } q > \dim \mathcal{O}$$

holds.

Remark. For algebraic varieties over a field, these theorems A–C are due to M. Artin ([AGV], exposé XIV, Théorème 3.1, Corollaire 3.4). Amazingly, the original proof by Artin basically works well in the arithmetic setting, except one important step. The missing step is Theorem C for two dimensional regular local rings of arithmetic type, which is treated by the local duality formalism. See [Il2] for the details.

5.2. Application to purity

The following theorem is an important step towards the proof of the absolute purity conjecture.

Theorem 5.2.1 (Vanishing theorem). *Let $\mathcal{O}$ be a regular strict local ring of arithmetic type, and $X = \operatorname{Spec}\mathcal{O}$. Let Y be a regular divisor of X, and x be the closed point of X. We assume the following two conditions.*

a) *Y is punctually pure at every point.*
b) *X is punctually pure except possibly at x.*

Then

$$H^q(X \setminus Y,\ \Lambda) = 0$$

if $q \neq 0,\ 1,\ d-1,\ d$. Here $d = \dim \mathcal{O}$.

Proof. By Theorem C,

$$H^q(X \setminus Y,\ \Lambda) = 0$$

if $q > d$. By the invariance of impure cohomology groups 3.1.2 (cf. Corollary 3.1.4), we have

$$H^q(X \setminus \{x\},\ \Lambda) = H^q(X \setminus Y,\ \Lambda) = 0$$

for $d < q < 2d-1$. By assumption *b*), the renormalized dualizing complex $K^{\mathrm{ren}}_{X\setminus\{x\}}$ is isomorphic to Λ by Proposition 4.2.2. Hence by the local duality theorem 4.2.3, $H^q(X\setminus\{x\}, \Lambda)$ is the Λ-dual of $H^{2d-1-q}(X\setminus\{x\}, \Lambda)$. Hence we have

$$H^q(X \setminus \{x\},\ \Lambda) = 0$$

for $0 < q < d-1$. By using Proposition 3.1.2 again, we have the desired vanishing. Q.E.D.

§6. Proof of purity: reduction to the arithmetic case

In the following two sections, we prove the absolute purity conjecture. First we make reductions to the arithmetic case over $\mathbf{Z}$ to apply the vanishing theorem and étale K-theories.

6.1. Limit argument

Lemma 6.1.1 (descent lemma). *Let A be a regular local ring with the maximal ideal m and the residue field k, and $I \subset I' \subset A$ be two ideals of A. Assume the following two conditions.*

a) *I is an ideal generated by a part of a regular system of parameters of A of height c, and I'/I is an ideal generated by a part of a regular system of parameters of A/I of height d.*

b) *There is a directed inductive system $\{A_j\}_{j\in J}$ of regular local rings such that each transition map $\varphi_{ij} : A_i \to A_j$ for $i \leq j$ is a local homomorphism, and there is a local isomorphism*

$$A \simeq \varinjlim_{j\in J} A_j.$$

Then there is an index $j_0 \in J$ and two subsystems $\{I_j\}_{j\in J,\ j\geq j_0}$, $\{I'_j\}_{j\in J,\ j\geq j_0}$ of $\{A_j\}_{j\in J,\ j\geq j_0}$ satisfying the following properties:

a') *$I_j \subset I'_j \subset A_j$ are ideals of A_j for $j \geq j_0$. I_j is an ideal generated by a part of a regular system of parameters of A_j of height c, I'_j/I_j is an ideal generated by a part of a regular system of parameters of A_j/I_j of height d for $j \geq j_0$.*

b') *$\varphi_{ij}(I_i)A_j = I_j$, $\varphi_{ij}(I'_i)A_j = I'_j$ for $j \geq i \geq j_0$. $I = I_jA$, $I' = I'_jA$ for any $j \geq j_0$.*

Proof. Let m and m_j $(j \in J)$ be the maximal ideals of A and A_j $(j \in J)$, respectively.

Let f_s $(1 \leq s \leq c)$ be elements of I such that $\{f_s\}_{1\leq s\leq c}$ generate I, and forms a part of a regular system of parameters of A.

Take elements g_t $(1 \leq t \leq d)$ of I' such that $\{g_t \bmod I\}_{1\leq t\leq d}$ generate I'/I, and forms a part of a regular system of parameters of A/I. $\{f_s, g_t\}_{1\leq s\leq c, 1\leq t\leq d}$ forms a part of a regular system of parameters of A.

We identify A with the inductive limit of $\{A_j\}_{j\in J}$. Then $m_jA, j \in J$ generates m, and hence for some $j_0 \in J$ $m_jA = m$ for any $j \geq j_0$.

We take elements $F_s \in m_{j_0}$ $(1 \leq s \leq c)$ and $G_t \in m_{j_0}$ $(1 \leq t \leq d)$ such that F_s and G_t are mapped to f_s and g_t, respectively.

Put $F_s(j) = \varphi_{j_0j}(F_s)$ and $G_t(j) = \varphi_{j_0j}(G_t)$ for $j \geq j_0$. We show that these elements $F_s(j)$ $(1 \leq s \leq c)$ and $G_t(j)$ $(1 \leq t \leq d)$ form a part of a regular system of parameters of A_j. For this, it suffices to see that $c+d$-elements $F_s(j) \bmod m_j^2$ $(1 \leq s \leq c)$ and $G_t(j) \bmod m_j^2$ $(1 \leq t \leq d)$

are linearly independent in m_j/m_j^2. The images of these elements in m/m^2 are $f_s \bmod m^2 (1 \leq s \leq c)$ and $g_t \bmod m^2 (1 \leq t \leq d)$, and span a $c+d$-dimensional subspace of m/m^2. So the linear independence follows.

We put $I_j = (F(j)_s,\ 1 \leq s \leq c)$ and $I'_j = (F_s(j), (1 \leq s \leq c),\ G_t(j), (1 \leq t \leq d)) \subset A_j$ for $j \geq j_0$. I_j, I'_j define regular subschemes of Spec A_j of codimension c and $c+d$, respectively. φ_{ij} induces $\varphi_{ij}(I_i) = I_j$, $\varphi_{ij}(I'_i) = I'_j$ for $j \geq i \geq j_0$. $I_j A = I$, $I'_j A = I'$ for $j \geq j_0$ by the construction. The descent lemma follows. Q.E.D.

Proposition 6.1.2. *Assume that the purity is true for any closed immersion $Y \to X$ of arithmetic regular schemes over a Cohen ring. Then the punctual purity is true for any regular strict local ring.*

Proof. By Proposition 2.2.4, it suffices to show the punctual purity for a regular strict complete local ring $\mathcal{O}$. Let m be the maximal ideal of $\mathcal{O}$, and let $d = \dim \mathcal{O}$. By Corollary 2.2.3, we may assume that $\mathcal{O}$ is of mixed characteristic. Let p be the residue characteristic. By the structure theorem of complete regular local rings, there is a presentation

$$\mathcal{O} \simeq C[[X_1, \ldots, X_n]]/(f), \quad f \in m' \setminus m'^2,\ m' = (p, X_1, \ldots, X_n).$$

Here C is a Cohen ring for k.

Sublemma 6.1.3 *There is a directed inductive system $\{S_j\}_{j \in J}$ of regular local rings satisfying the following properties*:

a) *There is a local C-isomorphism $S_j \simeq C\{\{Y_1, \ldots, Y_{n_j}\}\}$ for some $n_j \in \mathbf{N}$. Here $C\{\{Y_1, \ldots, Y_N\}\}$ is the strict henselization of $C[Y_1, \ldots, Y_N]$ at $(p, Y_1, \ldots, Y_N)$.*

b) *Each transition map $\varphi_{ij} : S_i \to S_j$ for $i \leq j$ is a local C-homomorphism, and*

$$C[[X_1, \ldots, X_n]] \simeq \varinjlim_{j \in J} S_j$$

as a local C-algebra.

Proof. By the theorem of Artin-Rotthaus ([AR]), there is a directed inductive system $\{R_j\}_{j \in J}$ of smooth C-algebras, and $C[[X_1, \ldots, X_n]]$ is isomorphic to the inductive limit: $C[[X_1, \ldots, X_n]] \simeq \varinjlim_{j \in J} R_j$. Let $x_j \in \text{Spec}\, R_j$ be the image of the closed point x of Spec $C[[X_1, \ldots, X_n]]$, and S_j be the henselization of R_j at x_j. $S_j \simeq C\{Y_1, \ldots, Y_{n_j}\}$ for some $n_j \geq 0$ since the residue field $k(x)$ at x is k. The claim follows. Q.E.D.

We return to the proof of 6.1.2. Let $\{S_j\}_{j \in J}$ be a directed inductive system obtained by Sublemma 6.1.3. We apply the descent lemma 6.1.1 with $A = C[[X_1, \ldots, X_n]]$, $A_j = S_j$, $I = fA$ and $I' = m'$. Then

there is an index $j_0 \in J$ and subsystems $\{I_j\}_{j\in J,\ j\geq j_0}$ and $\{I'_j\}_{j\in J,\ j\geq j_0}$ satisfying conditions a') and b') of 6.1.1. Put $\mathcal{O}_j = A_j/I_j$, $\mathcal{J}_j = I'_j/I_j$ for $j \geq j_0$. We have an inductive system $\{\mathcal{O}_j\}_{j\in J, j\geq j_0}$ and $\{\mathcal{J}_j\}_{j\in J, j\geq j_0}$ such that

a) $\mathcal{O}_j$ is a regular local ring.
b) $\mathcal{J}_j$ is an ideal of $\mathcal{O}_j$, and defines a regular subscheme of codimension $\dim \mathcal{O}$.
c) $\mathcal{O} \simeq \varinjlim_{j\in J, j\geq j_0} \mathcal{O}_j$, $\mathcal{J}_j\mathcal{O} = m$.

By condition c),

$$H^q(\operatorname{Spec}\mathcal{O} \setminus V(m), \Lambda) \simeq \varinjlim_{j\in J, j\geq j_0} H^q(\operatorname{Spec}\mathcal{O}_j \setminus V(\mathcal{J}_j), \Lambda),$$

for any $q \in \mathbf{Z}$. $\operatorname{cl}(V(\mathcal{J}_j)) \in H^{2d}_{V(\mathcal{J}_j)}(\operatorname{Spec}\mathcal{O}_j, \Lambda(n)) = H^{2d-1}(\operatorname{Spec}\mathcal{O}_j \setminus V(\mathcal{J}_j), \Lambda(n))$ is mapped to $\operatorname{cl}(V(m))$. So the purity for $(\operatorname{Spec}\mathcal{O}_j,\ V(\mathcal{J}_j))$ implies the punctual purity for $(\mathcal{O}, m)$. Q.E.D.

Proposition 6.1.4. *Assume that the purity is true for any closed immersion $Y \to X$ of arithmetic regular schemes over $\mathbf{Z}$. Then the purity is true for any regular strict local ring of arithmetic type over a Cohen ring.*

Proof. Since any Cohen ring C is absolutely unramified, it is a directed inductive limit of subrings which are regular and essentially of finite type over $\mathbf{Z}$ ([A]). The rest of the limit argument is treated similarly as in the proof of Proposition 6.1.2 using the descent lemma 6.1.1. We omit the details. Q.E.D.

Corollary 6.1.5. *Assume that the puntual purity is true for any pair $(\mathcal{O}, m)$ of a regular strict local ring $\mathcal{O}$ of arithmetic type over $\mathbf{Z}$ and the maximal ideal m. Then the absolute purity conjecture is true.*

§7. Proof of purity: K-theory

We prove the punctual purity for a regular strict local ring of arithmetic type over $\mathbf{Z}$ by the method of local Lefschetz pencils, using induction on the dimension. Our vanishing theorem 5.2.1, which is proved purely by an étale cohomological method using the local duality and a local affine Lefschetz theorem, is not enough to prove the purity. In addition to the information obtained from 5.2.1, we use the relationship between étale cohomology and étale K-theory to get further vanishing.

7.1. Localization in K-theory

For a noetherian scheme X and $q \in \mathbf{Z}$, $K_q(X)$ denotes the K-group of X made from the category of locally free coherent sheaves of $\mathcal{O}_X$-modules on X. We fix a prime ℓ which is invertible on X and an integer $\nu > 0$. We consider the mod ℓ^ν K-theory $(K/\ell^\nu)_q(X)$. In order that it has a good product structure, we assume either $\ell > 3$ or $\ell = 3$ and $\nu > 1$, or $\ell = 2$ and $\nu > 2$. We have the long exact sequence

$$\cdots \to K_q(X) \xrightarrow{\ell^\nu} K_q(X) \to (K/\ell^\nu)_q(X) \to K_{q-1}(X) \xrightarrow{\ell^\nu} K_{q-1}(X) \to \cdots$$

Lemma 7.1.1. *Let $\mathcal{O}$ be a henselian regular local ring, and f be an element in $m \setminus m^2$. For $q \in \mathbf{Z}$, there is a canonical isomorphism*

$$(K/\ell^\nu)_q(\mathcal{O}) \oplus (K/\ell^\nu)_{q-1}(\mathcal{O}) \xrightarrow{\simeq} (K/\ell^\nu)_q(\mathcal{O}[f^{-1}]).$$
$$(x, y) \mapsto \mathrm{res}_q\, x + [f] \cup \mathrm{res}_{q-1}\, y$$

Here $\mathrm{res}_q : (K/\ell^\nu)_q(\mathrm{Spec}\,\mathcal{O}) \to (K/\ell^\nu)_q(\mathrm{Spec}\,\mathcal{O}[f^{-1}])$ *is the restriction map, and* $[f]$ *is the image of* $f \in \mathcal{O}[f^{-1}]^\times$ *under* $K_1(\mathcal{O}[f^{-1}]) \to (K/\ell^\nu)_1(\mathcal{O}[f^{-1}])$.

Proof. By the rigidity theorem of Gabber ([Ga]),

$$(K/\ell^\nu)_q(\mathcal{O}) \overset{\mathrm{res}'_q}{\xrightarrow{\simeq}} (K/\ell^\nu)_q(\mathcal{O}/f\mathcal{O})$$

for any $q \in \mathbf{Z}$ by the restriction map. By the excision sequence in K-theory using the regularity of $\mathcal{O}$ and $\mathcal{O}/f\mathcal{O}$, we have an exact sequence

$$\cdots \to (K/\ell^\nu)_q(\mathcal{O}/f\mathcal{O}) \to (K/\ell^\nu)_q(\mathcal{O})$$
$$\xrightarrow{\mathrm{res}_q} (K/\ell^\nu)_q(\mathcal{O}[f^{-1}]) \xrightarrow{\delta_q} (K/\ell^\nu)_{q-1}(\mathcal{O}/f\mathcal{O}) \to \cdots.$$

Note that $\delta_q([f] \cup \mathrm{res}_{q-1}\, y) = \mathrm{res}'_{q-1}\, y$, and hence the composition of

$$(K/\ell^\nu)_{q-1}(\mathcal{O}) \xrightarrow{[f]\cup} (K/\ell^\nu)_q(\mathcal{O}[f^{-1}]) \xrightarrow{\delta_q} (K/\ell^\nu)_{q-1}(\mathcal{O}/f\mathcal{O})$$

is an isomorphism. Especially, δ_{q+1} is surjective for any q, showing that res_q is injective, and the claim follows. Q.E.D.

Proposition 7.1.2. *Let $\mathcal{O}$ be a regular strict local ring, and f be an element in $m \setminus m^2$. For a prime ℓ which is invertible in $\mathcal{O}$ and $\nu, q \in \mathbf{Z}$, $\nu \geq 0$,*

$$(K/\ell^\nu)_q(\mathcal{O}[f^{-1}])[\beta^{-1}] \xleftarrow{\simeq} \begin{cases} (K/\ell^\nu)_0(\mathcal{O}[f^{-1}])(\frac{q}{2}) & (\textit{if } q \textit{ is even}) \\ (K/\ell^\nu)_1(\mathcal{O}[f^{-1}])(\frac{q-1}{2}) & (\textit{if } q \textit{ is odd}). \end{cases}$$

Here $\beta \in (K/\ell^\nu)_2(\mathcal{O}[f^{-1}])$ is a Bott element.

Proof. Let $k = \mathcal{O}/m$ be the residue field of $\mathcal{O}$. By a theorem of Suslin ([Sus]), $(K/\ell^\nu)_q(k)$ is zero for q odd, and is canonically isomorphic to $\mathbf{Z}/\ell^\nu(i)$ for $q = 2i \geq 0$. Hence the localization by a Bott element induces $(K/\ell^\nu)_q(k) \xrightarrow{\sim} (K/\ell^\nu)_q(k)[\beta^{-1}]$ for $q \geq 0$. By the rigidity theorem of Gabber ([Ga]),

$$(K/\ell^\nu)_q(\mathcal{O}) \xrightarrow{\sim} (K/\ell^\nu)_q(k)$$

by the restriction map for any q, and hence $(K/\ell^\nu)_q(\mathcal{O}) \xrightarrow{\sim} (K/\ell^\nu)_q(\mathcal{O})[\beta^{-1}]$ for $q \geq 0$. Proposition 7.1.2 follows from 7.1.1 since the étale K-theory is mod 2-periodic. Q.E.D.

7.2. Etale K-theory: Conclusion of the proof of purity

Let X be a separated regular scheme having a finite Krull dimension. Assume the following three conditions on X and the prime ℓ.

a) ℓ is invertible on X, $\sqrt{-1}$ is contained in $\Gamma(X, \mathcal{O}_X)$ if $\ell = 2$.
b) Every residue field of X admits a finite Tate-Tsen filtration.
c) There is a uniform bound for the ℓ-cohomological dimensions of the residue fields.

Under these assumptions, R. Thomason constructs the following spectral sequence strongly converging to the étale K-theory of X, which is an analogue of the Atiyah-Hirzebruch spectral sequence for the topological K-theory of topological spaces ([Thom1], a more detailed account is found in [Jar]).

$$E_2^{p,q} = \begin{cases} H^p(X, \mathbf{Z}/\ell^\nu(i)) & (q = -2i), \\ 0 & (q \text{ is odd}) \end{cases} \Longrightarrow (K/\ell^\nu)_{-p-q}(X)[\beta^{-1}].$$

Here β is a Bott element. If a primitive ℓ^ν-th root of unity is contained in $\Gamma(X, \mathcal{O}_X)$, one can choose β from $(K/\ell^\nu)_2(X)$.

Let $\mathcal{O}$ be a regular strict local ring of arithmetic type over $\mathbf{Z}$, and f be an element in $m \setminus m^2$. Then the assumptions a)–c) are satisfied for $\operatorname{Spec} \mathcal{O}[f^{-1}]$, and hence the spectral sequence

$$(*) \qquad \begin{aligned} E_2^{p,q} &= \begin{cases} H^p(\operatorname{Spec}\ \mathcal{O}[f^{-1}], \mathbf{Z}/\ell^\nu(i)) & (q = -2i), \\ 0 & (q \text{ is odd}) \end{cases} \\ &\Longrightarrow (K/\ell^\nu)_{-p-q}(\mathcal{O}[f^{-1}])[\beta^{-1}] \end{aligned}$$

exists by the theory of Thomason.

Theorem 7.2.1. *Let $\mathcal{O}$ be a regular strict local ring of arithmetic type over* $\mathbf{Z}$*, and f be an element in $m \setminus m^2$. Assume that the punctual purity is true for any regular strict local ring of arithmetic type over* $\mathbf{Z}$ *of dimension strictly less than* $\dim \mathcal{O}$*. Then the spectral sequence* $(*)$ *degenerates at E_2.*

Proof. We may assume that $d = \dim \mathcal{O} \geq 1$. By the vanishing theorem 5.2.1, all columns in the E_2-term vanish except for $p = 0, 1, d-1, d$. It follows that any differential $d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1}$ for $r \geq 2$ on the spectral sequence vanishes except possibly for $p = 0, 1$.

Put $U = \operatorname{Spec} \mathcal{O}[f^{-1}]$. For $s \in \mathbf{Z}$, let $\{\mathrm{Fil}^t(K/\ell^\nu)_s(U)[\beta^{-1}]\}_{t \in \mathbf{Z}}$ be the decreasing filtration on $(K/\ell^\nu)_s(U)[\beta^{-1}]$ obtained by the spectral sequence. For $i \in \mathbf{Z}$, we have the edge homomorphism

$$\begin{aligned} e^{0,-2i} : (K/\ell^\nu)_{2i}(U)[\beta^{-1}] &= \mathrm{Fil}^0(K/\ell^\nu)_{2i}(U)[\beta^{-1}] \\ &\to E_2^{0,-2i} = H^0(U, \mathbf{Z}/\ell^\nu(i)). \end{aligned}$$

Since $\mathrm{Gr}^0_{\mathrm{Fil}}(K/\ell^\nu)_{2i-1}(U)[\beta^{-1}] = E_\infty^{0,1-2i} = 0$, we have a homomorphism

$$\begin{aligned} \tilde{e}^{1,-2i} : (K/\ell^\nu)_{2i-1}(U)[\beta^{-1}] &= \mathrm{Fil}^1(K/\ell^\nu)_{2i-1}(U)[\beta^{-1}] \\ &\to E_2^{1,-2i} = H^1(U, \mathbf{Z}/\ell^\nu(i)). \end{aligned}$$

To show the differential $d_r^{p,q}$ on $E_r^{p,q}$ $(r \geq 2)$ is zero for $p = 0, 1$, it suffices to prove the following lemma since it implies that $E_2^{p,q} = E_\infty^{p,q}$ for $p = 0, 1$.

Lemma 7.2.2. *$e^{0,-2i}$ and $\tilde{e}^{1,-2i}$ are isomorphisms for $i \in \mathbf{Z}$.*

Proof. We choose a Bott element β of degree 2. The product with β induces an isomorphism of spectral sequences $E_r^{p,q} \xrightarrow{\sim} E_r^{p,q-2}$ of degree -2, so we may assume that $i = 0$. By Proposition 7.1.2, it suffices to see the two maps

$$\begin{aligned} e^{0,0} &: (K/\ell^\nu)_0(U) \xrightarrow{\sim} (K/\ell^\nu)_0(U)[\beta^{-1}] \to H^0(U, \mathbf{Z}/\ell^\nu) \\ \tilde{e}^{1,0} &: (K/\ell^\nu)_1(U) \xrightarrow{\sim} (K/\ell^\nu)_1(U)[\beta^{-1}] \to H^1(U, \mathbf{Z}/\ell^\nu(1)) \end{aligned}$$

are isomorphisms. For $e^{0,0}$, this is clear.

$$\tilde{e}^{1,0} : \mathcal{O}(U)^\times/(\mathcal{O}(U)^\times)^{\ell^\nu} = (K/\ell^\nu)_1(U) \to H^1(U, \mathbf{Z}/\ell^\nu(1))$$

is identified with the map obtained by the Kummer theory. The claim now follows by $H^1(U, \mathbb{G}_m) = 0$.

Corollary 7.2.3. *$E_\infty^{p,q} = 0$ if $p > 1$.*

Proof. This is clear since $\mathrm{Fil}^t(K/\ell^\nu)_s(U)[\beta^{-1}] = 0$ for $t \geq 1$ if s is even, for $t \geq 2$ if s is odd by Lemma 7.2.2. Q.E.D.

7.2.4. Now we can complete the proof of the absolute purity theorem 2.1.1.

By Corollary 6.1.5, it suffices to prove the punctual purity for any regular strict local ring $\mathcal{O}$ of arithmetic type over $\mathbf{Z}$. We may assume that the coefficient ring Λ is $\mathbf{Z}/\ell^\nu$, where ℓ is a prime which is invertible in $\mathcal{O}$ and $\nu \geq 0$, and that $\nu > 1$ if $\ell = 3$ and $\nu > 2$ if $\ell = 2$. We use induction on $d = \dim \mathcal{O}$. If $d = 0$, the claim is obvious. Assume $d \geq 1$. By our induction hypothesis, the punctual purity is true for any regular strict local ring of arithmetic type over $\mathbf{Z}$ of dimension strictly less than d. We choose an element $f \in m \setminus m^2$. The assumption of theorem 7.2.1 is satisfied, and hence the spectral sequence

$$E_2^{p,q} = H^p(\mathrm{Spec}\,\mathcal{O}[f^{-1}],\ \mathbf{Z}/\ell^\nu(i)) \Longrightarrow (K/\ell^\nu)_{-p-q}(\mathcal{O}[f^{-1}])[\beta^{-1}] \qquad (q = -2i).$$

degenerates at E_2. $E_2^{p,q} = E_\infty^{p,q}$ for any $p, q \in \mathbf{Z}$. For $p \geq 2$, $E_\infty^{p,q} = 0$ by Corollary 7.2.3, and hence

$$H^p(\mathrm{Spec}\,\mathcal{O}[f^{-1}],\ \mathbf{Z}/\ell^\nu) = 0$$

for $p \geq 2$. This implies the purity for $(\mathrm{Spec}\,\mathcal{O},\ V(f))$, and the punctual purity for $(\mathcal{O}, m)$ by Proposition 3.1.2. Q.E.D.

§8. Consequences

Here we list some consequences of the absolute purity theorem.

Consequence. *Let S be a noetherian regular scheme of dimension at most one. Let n be an integer which is invertible on S, and let $\Lambda = \mathbf{Z}/n$. If X is a regular S-scheme of finite type, the dualizing complex K_X normalized as in §4 satisfies*

$$\Lambda(\delta_X)[2\delta_X] \xrightarrow{\sim} K_X.$$

Especially, Λ is a dualizing complex of X.

Consequence (semi-purity). *Let X be a noetherian regular scheme, let n be an integer which is invertible on S, and let $\Lambda = \mathbf{Z}/n$. If $Y \overset{i}{\hookrightarrow} X$ is a closed immersion of codimension $\geq c$,*

$$R^q i^! \Lambda = 0 \quad (q < 2c).$$

Consequence. *Let X be a noetherian regular scheme, let Y be a divisor on X with simple normal crossings, and let $j : X \setminus Y \hookrightarrow X$ be the inclusion. Let n be an integer which is invertible on X, and let $\Lambda = \mathbf{Z}/n$. Then the canonical maps give isomorphisms*

$$R^q j_* \Lambda \simeq \wedge^q R^1 j_* \Lambda \quad \textit{for all } q \in \mathbf{Z},$$

$$R^1 j_* \Lambda \simeq \bigoplus_{i \in I} (\iota_{Y_i})_* \Lambda(-1)$$

where $(Y_i)_{i \in I}$ is the set of irreducible components of Y, and ι_{Y_i} is the inclusion map $Y_i \hookrightarrow X$ for $i \in I$.

Here is a generalization of the above claim in terms of logarithmic schemes (see [Il1] for log étale cohomology).

Consequence (absolute local acyclicity). *Let X be an fs log-scheme such that the underlying scheme X^{cl} is noetherian. Assume that X is log-regular. Let U be the open subset of X defined by*

$$U = \{x \in X; (M_X/\mathcal{O}_X^{\times})_{\bar{x}} = \{1\}\},$$

i.e., U is the maximal open set where the log-structure is trivial. Let n be an integer which is invertible on X, and let $\Lambda = \mathbf{Z}/n$. For a Λ-smooth sheaf $\mathcal{F}$ on $X_{\mathrm{et}}^{\log}$,

$$\mathcal{F} \xrightarrow{\sim} Rj_* j^* \mathcal{F}$$

holds. Here j is the morphism of topoi $U_{\mathrm{et}}^{\mathrm{cl}} = U_{\mathrm{et}}^{\log} \to X_{\mathrm{et}}^{\log}$.

Here is a conditional result which follows from the absolute purity.

Consequence. *Assume that the resolution of singularities of quasi-excellent schemes is true. Let S be a quasi-compact quasi-excellent scheme, and $f : X \to S$ be a finite type morphism. For an integer $n \geq 1$ which is invertible on S, let $\Lambda = \mathbf{Z}/n$. Then for a Λ-constructible sheaf $\mathcal{F}$ on X, $R^q f_* \mathcal{F}$ is Λ-constructible for any $q \in \mathbf{Z}$, and vanishes except for a finite number of q's, i.e., the finiteness theorem is true for quasi-excellent schemes.*

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The Moduli Space of Rational Elliptic Surfaces

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Sur la Formule de Picard-Lefschetz

Luc Illusie

Introduction. La formule de Picard-Lefschetz [SGA 7 XV] exprime la variation locale pour une singularité quadratique ordinaire. En cohomologie étale, sa démonstration, dans le cas de dimension relative impaire, requiert un argument transcendant (loc. cit. 3.3). Nous donnons ici, dans ce même cas, une démonstration purement algébrique. L'idée est de déduire le calcul de la variation de la forme explicite donnée par Rapoport-Zink [RZ] pour le logarithme de la monodromie d'une famille semi-stable, de réduction à deux branches. On se ramène à ce cas par ramification et éclatement. La méthode, inspirée d'un calcul de Steenbrink dans [S], s'applique plus généralement à certaines singularités homogènes.

Dans toute la suite, on fixe un trait strictement local $S = \mathrm{Spec}\ A$. On note $s = \mathrm{Spec}\ k$ le point fermé, p l'exposant caractéristique du corps k, que l'on suppose algébriquement clos, $\eta = \mathrm{Spec}\ K$ le point générique, v la valuation de A, $\mathfrak{m}$ l'idéal maximal de A. On choisit une clôture algébrique $\overline{K}$ de K, on note $\overline{A}$ le normalisé de A dans $\overline{K}$, $\overline{S} = \mathrm{Spec}\ \overline{A}$, $\overline{\eta} = \mathrm{Spec}\ \overline{K}$ le point générique de $\overline{S}$, $\overline{s}$, identifié naturellement à s, le point fermé, $I = \mathrm{Gal}(\overline{K}/K)$ le groupe d'inertie, $P \subset I$ le groupe d'inertie sauvage, pro-p-Sylow de I. On fixe un nombre premier $\ell \neq p$, on note $t_\ell : I \to \mathbb{Z}_\ell(1)(\overline{k})$ le caractère modéré ℓ-adique, donné par $t_\ell(\sigma) = (\sigma(t^{1/\ell^n})/t^{1/\ell^n})_n$ pour $v(t) = 1$. On pose $\Lambda = \mathbb{Z}/\ell^\nu\mathbb{Z}$ $(\nu \geq 1)$.

1. Réduction semi-stable à deux branches

1.1. Soit $f : X \to S$ un morphisme semi-stable (i.e. tel que localement pour la topologie étale, X soit lisse sur $S[x_1, \ldots, x_r]/(x_1 \cdots x_r - t)$, où $v(t) = 1$). Le schéma X est donc régulier, la fibre générique X_η est

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lisse, et la fibre spéciale $D = X_s$ est un diviseur à croisements normaux dans X. On suppose que $D = D_0 + D_1$, où D_0 et D_1 sont lisses sur s et se coupent transversalement suivant $C = D_0 \cap D_1$. On note n la dimension relative de X sur S, qu'on suppose constante ; donc $n = \dim\ D_0 = \dim D_1$, et $\dim C = n - 1$.

On s'intéresse au complexe des cycles proches [SGA 7 XIII 2]

$$\Psi_f := R\Psi_\eta(\Lambda) \in D^+(D \times_s \eta, \Lambda)\ , \tag{1.1.1}$$

défini par $\Psi_f = \bar{i}^* R\bar{j}_* \Lambda$, où $\bar{i} : D \to \overline{X} = X \times_S \overline{S}$ et $\bar{j} : X_{\overline{\eta}} \to \overline{X}$ sont les immersions naturelles, et $D(D \times_s \eta, \Lambda) = D(D, \Lambda[I])$ est la catégorie dérivée des $\Lambda[I]$-modules continus sur D. La structure de Ψ_f est bien connue ([RZ], [I1]). Rappelons les points principaux.

(a) I opère sur Ψ_f à travers t_ℓ, et trivialement sur $H^q(\Psi_f)$ pour tout q ;

(b) On a $H^q\Psi_f = 0$ pour $q > 1$, $H^0\Psi_f = i^*j_*\Lambda = \Lambda_D$ et

$$\begin{aligned} H^1\Psi_f &= i^*(R^1j_*\Lambda)/\Lambda_D(-1) = ((\Lambda_{D_0} \oplus \Lambda_{D_1})/\Lambda_D)(-1) \\ &= ((\Lambda_C \oplus \Lambda_C)/\Lambda_C \text{ diagonal})(-1) \end{aligned}$$

(isomorphe à $\Lambda_C(-1)$ par $(a, b) \mapsto a - b$), où $i : X_s \to X$ et $j : X_\eta \to X$ sont les inclusions, et l'identification de $i^*R^1j_*\Lambda$ à $(\Lambda_{D_0} \oplus \Lambda_{D_1})(-1)$ est donnée par les classes e_0 et e_1 de D_0 et D_1 au sens de [SGA 4 1/2 Cycle 2.1.2] ;

(c) Si $a : D \to s$ désigne la projection et $D(-) = R\underline{Hom}(-, a^!\Lambda_s)$ le foncteur dualisant, l'accouplement canonique

$$\Psi_f \overset{L}{\otimes} \Psi_f(n)[2n] \to a^!\Lambda_s$$

est une dualité parfaite, identifiant Ψ_f à $D(\Psi_f)(-n)[-2n]$ [I1, 4.2].

(d) On dispose, de plus, sur Ψ_f, d'un *opérateur de monodromie* $N : \Psi_f \to \Psi_f(-1)$ et d'une *filtration par le poids* (W_r), définis de la façon suivante ([RZ], [I1]).

(i) Un complexe K de $\Lambda[\mathbb{Z}_\ell(1)]$-modules (continus) sur D, borné et à degrés ≥ 0, représentant Ψ_f étant choisi, on construit, par le procédé de Rapoport-Zink [RZ], rappelé en [I1, 3.6], un bicomplexe $A^{\cdot\cdot}$ de $\Lambda[\mathbb{Z}_\ell(1)]$-modules sur D, borné, concentré dans le premier quadrant, muni d'une augmentation $K \to A^{\cdot\cdot}$ induisant des quasi-isomorphismes sur les colonnes de cohomologie et donc un isomorphisme $K \xrightarrow{\sim} sA^{\cdot\cdot}$ dans $D^+(D, \Lambda[\mathbb{Z}_\ell(1)])$; le bicomplexe $A^{\cdot\cdot}$ est associé (avec différentielles d_1, $(-1)^i d_2$) au bicomplexe naïf dont la q-ième ligne $L^q = (A^{\cdot,q}, d_1)$ est $(\tau_{\geq q+1}(s(K \xrightarrow{T-1} K)(q+1))[q+1]$, où T est un générateur de

$\mathbb{Z}_\ell(1)$; la différentielle verticale d_2 et l'augmentation sont déduites de l'application $x \mapsto xT$ de K dans $K(1)$; la q-ième ligne représente $(\tau_{\geq q+1} i^* R j_* \Lambda(q+1))[q+1]$ (et est donc acyclique pour $q > 1$), et la différentielle d_2 (resp. l'augmentation $K \to L^0$) induit sur les faisceaux de cohomologie l'application $a \mapsto \theta a$, où $\theta \in H^1(\eta, \Lambda)(1)$ est la classe du torseur de Kummer, i.e. l'opposée de la classe de s dans S [SGA 4 1/2 Cycle 2.1.3] (en particulier, $1 \in H^0 K = H^0 \Psi_f = \Lambda$ est envoyé sur $-e_0 - e_1 \in H^0 L^0 = R^1 j_* \Lambda(1)$, et la classe de $e_0 \in H^1 K(1) = H^1 \Psi_f(1)$ sur $(-e_0 - e_1) \wedge e_0 = e_0 \wedge e_1 \in H^1 L^0(1) = R^2 j_* \Lambda(2) = \Lambda^2(R^1 j_* \Lambda(1))$ (avec les notations de (b)).

(ii) On note $W_r A^{\cdot\cdot}$ le sous-complexe de $A^{\cdot\cdot}$ obtenu en appliquant $\tau_{\leq r+q}$ à la q-ième ligne de $A^{\cdot\cdot}$, et $W_r(sA^{\cdot\cdot}) = s(W_r A^{\cdot\cdot})$. Les sous-complexes $W_r sA^{\cdot\cdot}$ de $sA^{\cdot\cdot}$ forment une filtration finie croissante, définissant un objet $(\Psi_f, (W_r))$ de la catégorie dérivée filtrée $D^+F(D, \Lambda[\mathbb{Z}_\ell(1)])$ ayant Ψ_f pour objet non filtré sous-jacent. La filtration (W_r) s'appelle *filtration par le poids*. Le gradué associé est donné (avec des isomorphismes déduits de (i)) par

$$\begin{aligned}
&\mathrm{gr}_i^W \Psi_f = 0 \text{ pour } |i| > -1 \ ; \\
&\mathrm{gr}_{-1}^W \Psi_f = R^2 j_* \Lambda(2)[-1] = \Lambda_C[-1] \ ; \\
&\mathrm{gr}_0^W \Psi_f = R^1 j_* \Lambda(1) = \Lambda_{D_0} \oplus \Lambda_{D_1} \ ; \\
&\mathrm{gr}_1^W \Psi_f = R^2 j_* \Lambda(1)[-1] = \Lambda_C(-1)[-1] \ ,
\end{aligned}$$

où $R^1 j_* \Lambda(1)$ est identifié à $\Lambda_{D_0} \oplus \Lambda_{D_1}$ par $e_0 + e_1$ et $R^2 j_* \Lambda(2)$ à Λ_C par $e_0 \wedge e_1$, avec les notations de (i).

(iii) On note $\nu : A^{\cdot\cdot} \to A^{\cdot - 1, \cdot + 1}(-1)$ l'endomorphisme de bicomplexes donné par $(-1)^{i+j+1}$ fois la projection canonique en bidegré (i, j). (Gabber fait observer que si l'on identifie par la multiplication par $(-1)^{ij}$ en bidegré (i, j) le bicomplexe $A^{\cdot\cdot}$ au bicomplexe déduit de l'ordre (d_2, d_1) (au lieu de (d_1, d_2)) sur les différentielles naïves, le signe dans la définition de ν disparaît.) L'homomorphisme

$$N : \Psi_f \to \Psi_f(-1)$$

de $D(D, \Lambda[\mathbb{Z}_\ell(1)])$ déduit de ν par l'isomorphisme $\Psi_f \xrightarrow{\sim} sA^{\cdot\cdot}$ s'appelle *opérateur de monodromie*. Il est sous-jacent à un homomorphisme de DF envoyant W_r dans W_{r-2} et induisant un isomorphisme

$$N : \mathrm{gr}_1^W \Psi_f \xrightarrow{\sim} \mathrm{gr}_{-1}^W \Psi_f(-1) \ , \tag{1.1.2}$$

qui, par les isomorphismes de (ii), correspond à l'identité (ν est l'identité en bidegré $(1,0)$). On a $N^2 = 0$, et, pour $\sigma \in I$,

$$\sigma - 1 = t_\ell(\sigma)N : \Psi_f \to \Psi_f \ . \tag{1.1.3}$$

(iv) Notons $\mathrm{Per}(D)$ la catégorie des Λ-faisceaux pervers sur D, pour la perversité auto-duale [BBD]. On sait que $\Psi_f[n] \in \mathrm{Per}(D)$ (variante de [I1 4.5] pour $\mathbb{Z}/\ell^\nu\mathbb{Z}$). D'après (1.1.3), l'opérateur de carré nul $N : \Psi_f \to \Psi_f(-1)$ est déterminé par l'action de I (et y commute), et il résulte de (1.1.2) que la filtration par le poids de Ψ_f (comme objet de $DF(D,\Lambda)$) définit la *filtration de monodromie* de $\Psi_f[n]$ (comme objet de la catégorie abélienne $\mathrm{Per}(D)$).

(e) L'isomorphisme de dualité $\Psi_f(n)[n] \xrightarrow{\sim} D(\Psi_f[n])$ de (c) est compatible à l'action de I, donc à celle de N, et par suite induit des isomorphismes $W_i\Psi_f(n)[n] \xrightarrow{\sim} D(\Psi_f[n]/W_{-i-1}\Psi_f[n]) = (W_{-i-1}\Psi_f[n])^\perp$, $\mathrm{gr}_i^W\Psi_f(n)[n] \xrightarrow{\sim} D(\mathrm{gr}_{-i}^W\Psi_f[n])$ dans $\mathrm{Per}(D)$. Pour $i = 1$, ce dernier isomorphisme est compatible aux identifications de (d) (ii) et à la dualité sur C.

1.2. Notons $c : C \to D$, $u_i : D_i \to D$, $w : D - C \to D$ les immersions naturelles. Considérons la suite exacte canonique

$$0 \to w_!w^*\Psi_f \to \Psi_f \to c_*c^*\Psi_f \to 0 \ . \tag{1.2.1}$$

Soit $\sigma \in I$. L'endomorphisme $\sigma - 1$ de Ψ_f induit un endomorphisme du triangle distingué défini par (1.2.1). D'après 1.1 (a) et (b), $w_!w^*\Psi_f = w_!\Lambda_{D-C}$ et $(\sigma - 1)|D - C = 0$. Par suite, $\sigma - 1$ se factorise à travers $c_*c^*\Psi_f$ en un homomorphisme

$$\mathrm{Var}^\sharp(\sigma) : c_*c^*\Psi_f \to \Psi_f \ ; \tag{1.2.2}$$

cette factorisation est unique car
$\mathrm{Hom}^{-1}(w_!w^*\Psi_f, \Psi_f) = \mathrm{Hom}^{-1}(w^*\Psi_f, w^*\Psi_f) = \mathrm{Hom}^{-1}(\Lambda_{D-C}, \Lambda_{D-C}) = 0$. Nous appellerons *variation* l'homomorphisme

$$\mathrm{Var}(\sigma) : c^*\Psi_f \to c^!\Psi_f \tag{1.2.3}$$

déduit de (1.2.2) par adjonction. Considérons d'autre part le triangle distingué canonique

$$c_*c^!\Psi_f \to \Psi_f \to Rw_*w^*\Psi_f \to . \tag{1.2.4}$$

Proposition 1.3. *Le morphisme canonique* $\Psi_f \to \mathrm{gr}_1^W\Psi_f$ *(resp.* $\mathrm{gr}_{-1}^W\Psi_f \to \Psi_f$*) (déduit de* 1.1 (d) (ii)*) se factorise de manière unique (via*

(1.2.1) *et* (1.2.4)) *par un morphisme* $c_*c^*\Psi_f \to \mathrm{gr}_1^W\Psi_f$ (*resp.* $\mathrm{gr}_{-1}^W\Psi_f \to c_*c^!\Psi_f$) *et le carré*

$$\begin{array}{ccc} c_*c^*\Psi_f & \longrightarrow & \mathrm{gr}_1^W\Psi_f \\ {\scriptstyle \mathrm{Var}(\sigma)}\downarrow & & \downarrow{\scriptstyle t_\ell(\sigma)N} \\ c_*c^!\Psi_f & \longleftarrow & \mathrm{gr}_{-1}^W\Psi_f \end{array} \tag{1.3.1}$$

est commutatif.

La première assertion résulte de 1.1 (d) (ii), et la seconde de (1.1.3).

1.4. Pour $\sigma \in I$ et $i = 0, 1$, on définit de manière analogue

$$\mathrm{Var}^\sharp(\sigma) : u_{i*}u_i^*\Psi_f \to \Psi_f \ , \tag{1.4.1}$$

déduit de l'endomorphisme $\sigma - 1$ de Ψ_f et

$$\mathrm{Var}(\sigma) : u_i^*\Psi_f \to u_i^!\Psi_f \tag{1.4.2}$$

déduit de (1.4.1) par dualité. On a des carrés commutatifs

$$\begin{array}{ccccc} \Psi_f & \longrightarrow & u_{i*}u_i^*\Psi_f & \longrightarrow & c_*c^*\Psi_f \\ {\scriptstyle \sigma-1}\downarrow & & {\scriptstyle \mathrm{Var}(\sigma)}\downarrow & & {\scriptstyle \mathrm{Var}(\sigma)}\downarrow \\ \Psi_f & \longleftarrow & u_{i*}u_i^!\Psi_f & \longleftarrow & c_*c^!\Psi_f \ , \end{array} \tag{1.4.3}$$

où les flèches horizontales sont les flèches canoniques.

Proposition 1.5. *Notons* $v_i : D_i - C \to D_i$ *l'inclusion.*

(a) *Le morphisme canonique* $u_i^*\Psi_f \to Rv_{i*}\Lambda$, *défini par la flèche d'adjonction* $u_i^*\Psi_f \to Rv_{i*}v_i^*u_i^*\Psi_f$ *et l'isomorphisme* $v_i^*u_i^*\Psi_f \cong \Lambda_{D_i-C}$, *est un isomorphisme, qui s'insère dans un isomorphisme de triangles distingués*

$$\begin{array}{ccccccc} u_i^*W_0\Psi_f & \longrightarrow & u_i^*\Psi_f & \longrightarrow & \mathrm{gr}_1^W\Psi_f & \longrightarrow & \\ \cong\downarrow & & \cong\downarrow & & \cong\downarrow & & \\ \Lambda_{D_i} & \longrightarrow & Rv_{i*}\Lambda & \longrightarrow & \Lambda_C(-1)[-1] & \longrightarrow & \ , \end{array}$$

où la flèche verticale de droite est $(-1)^{i+1}$ *celle donnée par* 1.1 (d) (ii) *et le triangle inférieur par l'isomorphisme canonique* $R^1v_{i*}\Lambda \cong \Lambda_C(-1)$

(*donné par la classe de cohomologie de C dans D_i* [SGA 4 1/2 Cycle 2.1]).

(b) *Le morphisme canonique $v_{i!}\Lambda \to u_i^!\Psi_f$, défini par la flèche d'adjonction $v_{i!}v_i^*u_i^!\Psi_f \to u_i^!\Psi_f$ et l'isomorphisme $v_i^*u_i^!\Psi_f \cong \Lambda_{D_i-C}$, est un isomorphisme, qui s'insère dans un isomorphisme de triangles distingués*

$$\begin{array}{ccccccc} \mathrm{gr}_{-1}^W\Psi_f & \longrightarrow & u_i^!\Psi_f & \longrightarrow & u_i^!(\Psi_f/W_{-1}\Psi_f) & \longrightarrow & \\ \uparrow\cong & & \uparrow\cong & & \uparrow\cong & & \\ \Lambda_C[-1] & \xrightarrow{-d[-1]} & v_{i!}\Lambda & \longrightarrow & \Lambda_{D_i} & \longrightarrow & , \end{array}$$

où la flèche verticale de gauche est $(-1)^{i+1}$ celle donnée par 1.1 (d) (ii) *et le triangle inférieur est déduit par rotation du triangle défini par la suite exacte canonique $0 \to v_{i!}\Lambda \to \Lambda_{D_i} \to \Lambda_C \to 0$.*

(c) *Le foncteur $R\underline{Hom}(-,\Lambda_{D_i})$ échange, via la (quasi-) autodualité de Ψ_f* (1.1 (c)), *les triangles de* (a) *et* (b).

Le morphisme naturel $Rj_*\Lambda \to \Psi_f$, où $j : X-D \to X$ est l'inclusion, identifie $H^1\Psi_f = \Lambda_C(-1)$ au quotient de $R^1j_*\Lambda = (\Lambda_{D_0} \oplus \Lambda_{D_1})(-1)|C$ par $\Lambda_C(-1)$ diagonal. Par transversalité de D_0 et D_1, on a $Rv_{0*}\Lambda = u_0^*Rj_{1*}\Lambda$, où $j_1 : X - D_1 \to X$ est l'inclusion. Le morphisme naturel $Rv_{0*}\Lambda \to u_0^*Rj_*\Lambda$ qui s'en déduit est l'identité sur $H^0 = \Lambda_{D_0}$ et l'inclusion du facteur $(\Lambda_{D_1}(-1))|C$ (de base $e_1|C$) sur H^1. Le morphisme $Rv_{0*}\Lambda \to u_0^*\Psi_f$ obtenu en composant avec $Rj_*\Lambda \to \Psi_f$ induit l'identité sur H^0 et, sur H^1, est le composé $(\Lambda_{D_1}(-1))|C \hookrightarrow (\Lambda_{D_0}\oplus\Lambda_{D_1})(-1)|C \to ((\Lambda_{D_0} \oplus \Lambda_{D_1})(-1)|C)/(\Lambda_C(-1)$ diagonal), i.e. envoie e_1 sur la classe de $(0,1) = -1 \in \Lambda_C(-1)$ (1.1 (b)), c'est donc un isomorphisme. Par suite, $u_0^*\Psi_f$ est l'image directe de sa restriction à $D_0 - C$, d'où la première assertion de (a). Pour la seconde, le point est la commutativité du carré de droite. Notons e_0 (resp. e_1) la base de $R^1j_{0*}\Lambda(1)$ (resp. $R^1j_{1*}\Lambda(1)$) considérée en 1.1 (b), e la base correspondante de $H^1\Psi_f(1)$, classe de e_0 ou de $-e_1$ modulo Λ diagonal ; dans le diagramme relatif à $i = 0$, l'image de e dans Λ par le composé $H^1\Psi_f(1) \to R^1v_{0*}\Lambda(1) \to \Lambda$ est -1 ; pour calculer l'image de e par l'autre composé, on doit identifier $H^1\Psi_f(1)$ à $\mathrm{gr}_1^W\Psi_f[1](1) = R^2j_*\Lambda(2)$ par 1.1 (d) (ii) : l'image de e est 1. Pour le diagramme relatif à $i = 1$, on trouve le signe opposé. D'où la seconde assertion de (a). La première assertion de 1.5 (b) se déduit de la première de 1.5 (a) par dualité (1.1 (e)). Pour la seconde, le point est la commutativité du carré de gauche. L'argument qui suit est dû à O. Gabber. Il suffit de prouver la commutativité après composition avec la flèche d'adjonction $u_{i*}u_i^!\Psi_f \to \Psi_f$. Dans la construction de

Rapoport-Zink rappelée en 1.1 (d) (i), on peut supposer K concentré en degrés $\in [0,1]$. Le bicomplexe $A^{\cdot\cdot}$ est alors concentré en degrés $(0,0)$, $(0,1)$, $(1,0)$. Le morphisme $v_{i!}\Lambda \to \Psi_f$ se factorise (trivialement) par $H^0\Psi_f$. Il en est de même du morphisme $\mathrm{gr}_{-1}\Psi_f \to \Psi_f$: il est en effet donné par le composé $\tau_{\leq 0}L^1[-1]$ $(=$ $A^{01}[-1])$ $\to sA^{\cdot\cdot} \leftarrow K$, et donc se factorise par $\tau_{\leq 0}L^1[-1]$ $(=$ $R^2j_*\Lambda(2)[-1])$ $\to s(\tau_{\leq 0}L^0 \to \tau_{\leq 0}L^1)$ $(=$ $s(R^1j_*\Lambda(1) \to R^2j_*\Lambda(2)))$ $\leftarrow \tau_{\leq 0}K$ $(=$ $H^0\Psi_f)$ (où les deuxièmes flèches sont les quasi-isomorphismes donnés par l'augmentation $K \to A^{\cdot\cdot}$). Si l'on identifie $H^0\Psi_f$ à Λ_D, $R^1j_*\Lambda(1)$ à $\Lambda_{D_0} \oplus \Lambda_{D_1}$ (au moyen de $e_0 + e_1$) et $R^2j_*\Lambda(2)$ à Λ_C (au moyen de $e_0 \wedge e_1$), ce composé se récrit $\Lambda_C[-1] \to s(\Lambda_{D_0} \oplus \Lambda_{D_1} \xrightarrow{(1,-1)} \Lambda_C) \xleftarrow{(-1,-1)} \Lambda_D$. D'après les conventions standard [SGA 4 1/2, C. D. I §1 2.1, II §1 1.5], c'est donc $d[-1]$, où $d : \Lambda_C \to \Lambda_D[1]$ est la flèche de degré 1 du triangle distingué défini par la suite exacte courte $0 \to \Lambda_D \xrightarrow{(1,1)} \Lambda_{D_0} \oplus \Lambda_{D_1} \xrightarrow{(1,-1)} \Lambda_C \to 0$. Cette suite reçoit la suite exacte canonique $0 \to v_{i!}\Lambda \to \Lambda_{D_i} \to \Lambda_C \to 0$ (définissant $d : \Lambda_C \to v_{i!}\Lambda[1]$) par un morphisme qui est l'identité sur Λ_C et $(-1)^i$ sur les deux autres termes. La commutativité voulue en résulte. Nous omettrons la vérification de (c).

Corollaire 1.6. *Posons* $H^q(-) = H^q(-,\Lambda)$ (resp. $H^q_c(-) = H^q_c(-,\Lambda)$). *Supposons* D_1 propre *sur* s, *et, pour* $\sigma \in I$, *notons* $\mathrm{Var}(\sigma)$: $H^q(D_1 - C) \to H^q_c(D_1 - C)$ *l'homomorphisme défini par* (1.4.2) *via les identifications* $H^q(D_1, u_1^*\Psi_f) \cong H^q(D_1 - C)$ *et* $H^q(D_1, u_1^!\Psi_f) \cong H^q_c(D_1 - C)$ *définies par* 1.5 (a) *et* (b). *On a un carré commutatif*

$$\begin{array}{ccc} H^q(D_1 - C) & \xrightarrow{d} & H^{q-1}(C)(-1) \\ \Big\downarrow{\scriptstyle \mathrm{Var}(\sigma)} & & \Big\downarrow{\scriptstyle t_\ell(\sigma)} \\ H^q_c(D_1 - C) & \xleftarrow[-d]{} & H^{q-1}(C), \end{array} \tag{1.6.1}$$

où les flèches d sont les opérateurs bords des suites exactes longues de cohomologie ($H^{q+1}_c(D_1)$ *étant identifié à* $H^{q-1}(C)(-1)$ *par l'isomorphisme de Gysin*).

Le carré (1.6.1) s'obtient en effet en appliquant $H^q(D_1, -)$ au carré commutatif

$$\begin{array}{ccc} u_1^*\Psi_f & \longrightarrow & \mathrm{gr}_1^W\Psi_f \\ \Big\downarrow{\scriptstyle \mathrm{Var}(\sigma)} & & \Big\downarrow{\scriptstyle Nt_\ell(\sigma)} \\ u_1^!\Psi_f & \longleftarrow & \mathrm{gr}_{-1}^W\Psi_f \end{array} \tag{1.6.2}$$

déduit de (1.3.1) et (1.4.3), via les identifications de 1.5 (a) et (b). Comme on l'a rappelé en (1.1.2), le morphisme induit par $N : \mathrm{gr}_1^W \Psi_f \to \mathrm{gr}_{-1}^W \Psi_f(-1)$ via les identifications $\mathrm{gr}_1^W \Psi_f \cong \Lambda_C(-1)[-1]$, $\mathrm{gr}_{-1}^W \Psi_f \cong \Lambda_C[-1]$ (1.1 (d) (ii)) n'est autre que l'identité, il en est donc de même du morphisme qui s'en déduit de $H^{q-1}(C)(-1)$ dans $H^{q-1}(C)(-1)$.

2. Singularités homogènes et variation

2.1. On considère maintenant un morphisme $f : X \to S$ localement de type fini, plat, de dimension relative n, et lisse le long de X_s hors d'un point $x_0 \in X_s(k)$. On note Ψ_f (resp. Φ_f) le complexe des cycles proches (resp. évanescents) $R\Psi_\eta(\Lambda)$ (resp. $R\Phi(\Lambda)$) $\in D^+(X_s \times_s \eta, \Lambda)$ [SGA 7 XIII 2.1]. Le complexe Φ_f est concentré en $\{x_0\}$. On a de plus le résultat suivant, dont nous ne ferons pas usage :

Proposition 2.2. *Si de plus f est localement d'intersection complète, alors $H^i\Phi_f = 0$ pour $i \neq n$, et $H^n\Phi_f$ est un Λ-module libre de type fini.*

Le cas où S est de caractéristique nulle est traité dans [SGA 7 I 4.6]. Pour le cas général, voir [I2].

2.3. La construction décrite dans ce numéro est inspirée de l'exemple considéré par Steenbrink dans [S, 3.12]. Soient n un entier ≥ 1, e un entier ≥ 2, $F = \Sigma a_\alpha x^\alpha \in A[x_1, \ldots, x_{n+1}]$ un polynôme homogène de degré e, partout non nul, π une uniformisante de A, $u \in A^*$. On fait l'hypothèse suivante :

> (H) le sous-schéma fermé Z de $\mathbb{P}_S^{n+1} = \mathrm{Proj}\, A[x_1, \ldots, x_{n+1}, t]$ d'équation $F - ut^e = 0$ est lisse sur S.

Observons que (H) ne dépend que de F et implique que la section hyperplane $Z \cap (t = 0) \subset \mathbb{P}_S^{n+1}$ (d'équation homogène $F = 0$) est lisse sur S, et que la réciproque est vraie si $(p, e) = 1$. Gabber fait remarquer par ailleurs que, si $(p, e) \neq 1$, la condition (H) ne peut être vérifiée pour $e > 2$ (la relation d'Euler contredirait l'exactitude du complexe de Koszul défini par la suite $(F - ut^e, (\partial F/\partial x_i))$).

Notons X le sous-schéma fermé de $\mathbb{A}_S^{n+1} = \mathrm{Spec}\, A[x_1, \ldots, x_{n+1}]$ défini par l'équation $F - u\pi^e = 0$, $f : X \to S$ la projection. Le morphisme f est plat, de dimension relative n, et lisse hors de l'origine $x_0 = (0, \ldots, 0)$ de $\mathbb{A}_s^{n+1}$. Le complexe Φ_f est donc concentré en x_0 et d'après [SGA 7 XIII 2.4.5], on dispose, pour $\sigma \in I$, du morphisme *variation*

$$\mathrm{Var}(\sigma) : (\Phi_f)_{x_0} \to R\Gamma_{\{x_0\}}(X_s, \Psi_f). \tag{2.3.1}$$

On se propose de décrire ce morphisme à l'aide d'un analogue convenable de la variété des cycles évanescents du cas transcendant.

Soient Y l'éclaté de $\{x_0\}$ dans X, ou, ce qui revient au même, le transformé pur de X dans l'éclaté $(\mathbb{A}_S^{n+1})^\sim$ de $\{x_0\}$ dans $\mathbb{A}_S^{n+1}$. Notons $h : Y \to X$ la projection, et $g = f \circ h : Y \to S$. (Dans le cas considéré par Steenbrink dans (loc. cit.), où $A = \mathbb{C}\{\pi\}$, cette construction équivaut à éclater x_0 dans le schéma (régulier) d'équation $F - \pi = 0$, puis de normaliser le schéma qui s'en déduit par changement de base par $\pi \mapsto u\pi^e$.)

Proposition 2.4. *Le morphisme g a réduction semi-stable. Le diviseur $D = Y_s$ est somme de deux diviseurs D_0 et D_1, lisses sur s, dont l'un, D_1, est irréductible et propre sur s. Le diviseur $D_1 = h^{-1}(x_0)$, diviseur exceptionnel de l'éclatement h, est l'hypersurface de degré e de $\mathbb{P}_s^{n+1} = \operatorname{Proj} k[x_1, \ldots, x_{n+1}, t]$ d'équation $\overline{F} - \overline{u}t^e = 0$, où $(-)^-$ désigne la réduction modulo $\mathfrak{m}$. L'intersection $C = D_0 \cap D_1$, lieu à l'infini du cône d'équation $\overline{F} = 0$ dans $\mathbb{A}_s^{n+1}$, est la section hyperplane $t = 0$ de D_1 dans $\mathbb{P}_s^{n+1}$. Le diviseur D_0, éclaté de $\{x_0\}$ dans la fibre spéciale X_s (ou transformé pur de X_s dans l'éclaté de $\{x_0\}$ dans $\mathbb{A}_s^{n+1}$), est un fibré en droites sur C.*

On recouvre $E = (\mathbb{A}_S^{n+1})^\sim$ par les ouverts standard U_i, $1 \leq i \leq n+2$: $U_i = \operatorname{Spec} A[x_i, y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_{n+1}, t]/(x_i t - \pi)$ $(1 \leq i \leq n+1)$, $U_{n+2} = \operatorname{Spec} A[y_1, \ldots, y_{n+1}]$, la projection canonique $E \to \mathbb{A}_S^{n+1}$ étant donnée par : $x_j \mapsto x_j$ si $i = j$ et $x_i y_j$ pour $j \neq i$ dans U_i $(1 \leq i \leq n+1)$, et par $x_i \mapsto \pi y_i$ $(1 \leq i \leq n+1)$ dans U_{n+2}. Le morphisme $g : Y \to S$ est induit, sur chaque U_i, par la projection canonique. Pour $1 \leq i \leq n+1$, $Y \cap U_i$ est l'intersection, dans $\operatorname{Spec} A[x_i, y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_{n+1}, t]$, des sous-schémas U_i (d'équation $x_i t = \pi$) et V_i d'équation $F(y_1, \ldots, y_{i-1}, 1, y_{i+1}, \ldots, y_{n+1}) - ut^e = 0$. Par l'hypothèse (H), V_i est lisse sur S. Donc, localement, V_i est étale soit sur $A[x_i, (y_j)_{j \neq i, j \neq q}, t]$ (pour $q \in \{1, \ldots, n+1\} - \{i\}$), soit sur $A[x_i, (y_j)_{j \neq i}]$ (avec t inversible). Dans le premier cas, $Y \cap U_i = U_i \cap V_i$ est étale sur $A[x_i, (y_j)_{j \neq i, j \neq q}, t]/(x_i t - \pi)$, et a donc réduction semi-stable sur S. Dans le second, $Y \cap U_i$ est étale sur $A[(y_j)_{j \neq i}]$, donc lisse sur S. Pour $i = n+2$, $Y \cap U_{n+2}$ est le sous-schéma de U_{n+2} d'équation $F(y_1, \ldots, y_{n+1}) - u = 0$, et est donc, par l'hypothèse (H), lisse sur S. Cela prouve la première assertion. Le diviseur D_1 est le lieu à l'infini du cône tangent Γ à X en x_0. Ce cône Γ a pour équation $\overline{F} - \overline{u}t^e = 0$ dans $\operatorname{Spec} k[x_1, \ldots, x_{n+1}, t]$. Son lieu à l'infini est donc lisse, par hypothèse, donc irréductible puisque $n \geq 1$. D'autre part, D_0 est le fibré $\mathbb{V}(\mathcal{O}_C(1))$ sur C, lieu à l'infini du cône $\Gamma \cap (t = 0)$ (d'équation $\overline{F} = 0$ dans $\operatorname{Spec} k[x_1, \ldots, x_{n+1}]$), et C est lisse sur s, soit par suite de la réduction

semi-stable, soit comme conséquence de l'hypothèse (H) comme on l'a observé plus haut.

Définition 2.5. On appelle *variété affine des cycles évanescents* de f en x_0 le s-schéma (affine) $V(f) = V = D_1 - C$.

Le schéma V est lisse sur s, connexe, de dimension n. La terminologie est justifiée par le résultat suivant :

Théorème 2.6. (a) *On a* $H^0(V,\Lambda) = \Lambda$ *et* $H^i(V,\Lambda) = 0$ *pour* $i \neq n, 0$, $H_c^{2n}(V,\Lambda) = \Lambda(-n)$ *et* $H_c^i(V,\Lambda) = 0$ *pour* $i \neq n, 2n$ *; l'accouplement de dualité* $H^n(V,\Lambda) \otimes H_c^n(V,\Lambda) \to \Lambda(-n)$, $(a,b) \mapsto \mathrm{Tr}(ab)$, *est un accouplement parfait entre* Λ*-modules libres de type fini.*

(b) *On a des isomorphismes canoniques* $(\Psi_f)_{x_0} \cong R\Gamma(V,\Lambda)$, $R\Gamma_{\{x_0\}}(X_s,\Psi_f) \cong R\Gamma_c(V,\Lambda)$, *compatibles aux accouplements de dualité naturels* $R\Gamma(V,\Lambda) \overset{L}{\otimes} R\Gamma_c(V,\Lambda)(n)[2n] \to \Lambda$ *et* $(\Psi_f)_{x_0} \overset{L}{\otimes} R\Gamma_{\{x_0\}}(X_s,\Psi_f)(n)[2n] \to R\Gamma_{\{x_0\}}(X_s,a^!\Lambda_s) \xrightarrow{\sim} \Lambda$ [I1, 4.2].

(c) *Pour* $\sigma \in I$, *notons* $\mathrm{Var}(\sigma) : H^n(V,\Lambda) \to H_c^n(V,\Lambda)$ *le morphisme déduit de* $\mathrm{Var}(\sigma) : H^n(\Phi_f)_{x_0} \to H^n_{\{x_0\}}(X_s,\Psi_f)$ *(2.3.1) par les identifications de (b) ; on a un carré commutatif*

$$\begin{array}{ccc} H^n(V,\Lambda) & \xrightarrow{d} & H^{n-1}(C,\Lambda)(-1) \\ {\scriptstyle \mathrm{Var}(\sigma)}\downarrow & & \downarrow{\scriptstyle t_\ell(\sigma)} \\ H_c^n(V,\Lambda) & \xleftarrow{-d} & H^{n-1}(C,\Lambda), \end{array}$$

où la flèche verticale de gauche est définie par (2.3.1) et les flèches d sont les opérateurs bords des suites exactes longues de cohomologie ($H_C^{q+1}(D_1,\Lambda)$ *étant identifié à* $H^{q-1}(C,\Lambda)(-1)$ *par l'isomorphisme de Gysin). De plus, pour* $x \in H^n(V,\Lambda)$ *et* $y \in H^{n-1}(C,\Lambda)$, *on a* $\langle dx, y\rangle_C + (-1)^n\langle x, dy\rangle_V = 0$, $\langle\ ,\ \rangle_C$ *(resp.* $\langle\ ,\ \rangle_V$*) désignant l'accouplement de dualité à valeurs dans* $\Lambda(-n)$ *entre* $H^{n-1}(C,\Lambda)(-1)$ *et* $H^{n-1}(C,\Lambda)$ *(resp.* $H^n(V,\Lambda)$ *et* $H_c^n(V,\Lambda)$*).*

Les assertions de (a) sont bien connues et découlent de la structure de la cohomologie des intersections complètes lisses [SGA 7 XI 1.6] : comme D_1 est une hypersurface lisse de $\mathbb{P}_s^{n+1}$ et que C en est une section hyperplane lisse, l'homomorphisme de Gysin $H^{q-2}(C)(-1) \to H^q(D_1)$ (où $H^q(-) = H^q(-,\Lambda)$) est un isomorphisme pour $0 < q < n$ et un isomorphisme sur un facteur direct pour $q = n$ (envoyant $\xi^{(q-2)/2}$ sur $\xi^{q/2}$ pour q pair, où ξ est la classe d'une section hyperplane), d'où la nullité des $H^q(V)$ pour $0 < q < n$, par la suite longue de cohomologie $\cdots \to H^q(D_1) \to H^q(V) \to H^{q-1}(C)(-1) \to \cdots$; la nullité de

$H^q(V)$ pour $q > n$ vient de ce que V est affine ; enfin, $H^n(V)$, extension de $\mathrm{Ker}(H^{n-1}(C)(-1) \to H^{n+1}(D_1))$ par $\mathrm{Coker}(H^{n-2}(C)(-1) \to H^n(D_1))$, est libre de type fini sur Λ, d'où la dernière assertion de (a). Prouvons (b). Puisque $h : Y \to X$ est propre et est un isomorphisme sur les fibres génériques, on a un isomorphisme canonique [SGA 7 XIII 2.1.7]

$$\Psi_f \cong Rh_*\Psi_g \ . \tag{2.6.1}$$

Par le théorème de changement de base propre, on a donc $(\Psi_f)_{x_0} \cong R\Gamma(D_1, \Psi_g)$. D'autre part, comme g a réduction semi-stable à deux branches (D_0, D_1) (2.4), on peut appliquer 1.5 à g. On a ainsi un isomorphisme canonique $\Psi_g|D_1 \xrightarrow{\sim} Rv_{1*}\Lambda$, où $v_1 : V = D_1 - C \to D_1$ est l'inclusion. Par composition, on obtient le premier des deux isomorphismes canoniques annoncés, $(\Psi_f)_{x_0} \xrightarrow{\sim} R\Gamma(V, \Lambda)$. Considérons d'autre part le carré cartésien

$$\begin{array}{ccc} D_1 & \xrightarrow{u_1} & D\ (= Y_s) \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h} \\ \{x_0\} & \xrightarrow{i_0} & X_s, \end{array}$$

où les flèches horizontales sont les inclusions. D'après [SGA 4 XVIII 3.1.12.3], on a $i_0^! Rh_* \cong Rh_* u_1^!$. On a donc des isomorphismes canoniques

$$\begin{aligned} R\Gamma_{\{x_0\}}(X_s, \Psi_f) &= i_0^! \Psi_f \\ &\cong i_0^! Rh_* \Psi_g \quad (2.6.1) \\ &\cong Rh_* u_1^! \Psi_g \ . \end{aligned}$$

D'autre part, d'après 1.5 appliqué à g, on a $u_1^! \Psi_g \cong v_{1!}\Lambda$, d'où finalement le second isomorphisme canonique annoncé, $R\Gamma_{\{x_0\}}(X_s, \Psi_f) \cong R\Gamma_c(V, \Lambda)$. La compatibilité à la dualité résulte de 1.5 (c) et de la compatibilité de l'autodualité de Ψ aux images directes propres. Il reste à prouver (c). La dernière assertion est classique, et cas particulier de 2.6.6 ci-après. Prouvons la première. Soit $\sigma \in I$. Notons $\mathrm{Var}_g(\sigma) : u_1^* \Psi_g \to u_1^! \Psi_g$ l'homomorphisme défini en (1.4.2). Par ailleurs, soit $j_0 : X_s - \{x_0\} \to X_s$ l'inclusion. L'endomorphisme $\sigma - 1$ de Ψ_f est sous-jacent à un endomorphisme du triangle distingué canonique

$$j_{0!} j_0^* \Psi_f \to \Psi_f \to i_{0*} i_0^* \Psi_f \to \ . \tag{2.6.2}$$

Comme f est lisse hors de x_0, $j_{0!}j_0^*\Psi_f = j_{0!}\Lambda_{X_s-\{x_0\}}$. Donc $\sigma - 1$ s'annule sur $j_{0!}j_0^*\Psi_f$ et se factorise par un homomorphisme

$$\mathrm{Var}_f^\sharp(\sigma) : i_{0*}i_0^*\Psi_f \to \Psi_f \ , \tag{2.6.3}$$

unique car $\mathrm{Hom}^{-1}(j_{0!}j_0^*\Psi_f, \Psi_f) = \mathrm{Hom}^{-1}(\Lambda_{X_s-\{x_0\}}, \Lambda_{X_s-\{x_0\}}) = 0$. Par adjonction, $\mathrm{Var}_f^\sharp(\sigma)$ définit un homomorphisme

$$\mathrm{Var}_f(\sigma) : (\Psi_f)_{x_0} \to i_0^!\Psi_f \ , \tag{2.6.4}$$

qui est le composé de (2.3.1) et du morphisme canonique $(\Psi_f)_{x_0} \to (\Phi_f)_{x_0}$. En particulier, puisque, en vertu de (a) et (b), $H^n(\Psi_f) = H^n(\Phi_f)$, $\mathrm{Var}_f(\sigma)$ induit sur H^n le morphisme variation de [SGA 7 XIII 2.4.5]. Pour achever la preuve de (c), Il suffit donc, compte tenu de 1.6 appliqué à g, d'établir la commutativité du carré

$$\begin{array}{ccc} (\Psi_f)_{x_0} & \xrightarrow{\mathrm{Var}_f(\sigma)} & i_0^!\Psi_f \\ \downarrow & & \downarrow \\ Rh_*u_1^*\Psi_g & \xrightarrow{Rh_*\mathrm{Var}_g(\sigma)} & Rh_*u_1^!\Psi_g \ , \end{array} \tag{2.6.5}$$

où $h : D_1 \to \{x_0\}$ est la projection, et les flèches verticales sont les isomorphismes canoniques (changement de base propre et [SGA 4 XVIII 3.1.12.3]). Par adjonction, il revient au même de vérifier la commutativité du carré

$$\begin{array}{ccc} i_{0*}i_0^*\Psi_f & \xrightarrow{\mathrm{Var}_f^\sharp(\sigma)} & \Psi_f \\ \downarrow & & \downarrow \\ Rh_*u_{1*}u_1^*\Psi_g & \xrightarrow{Rh_*\mathrm{Var}_g^\sharp(\sigma)} & Rh_*\Psi_g \ . \end{array}$$

Comme $\mathrm{Hom}^{-1}(j_{0!}j_0^*\Psi_f, Rh_*\Psi_g) = \mathrm{Hom}^{-1}(\Lambda_{X_s-\{x_0\}}, \Lambda_{X_s-\{x_0\}}) = 0$, il suffit de montrer que le carré déduit par composition avec $\Psi_f \to i_{0*}i_0^*\Psi_f$ est commutatif. Par définition de $\mathrm{Var}_f^\sharp(\sigma)$ et $\mathrm{Var}_g^\sharp(\sigma)$, on est ramené à vérifier la commutativité du carré

$$\begin{array}{ccc} \Psi_f & \xrightarrow{\sigma-1} & \Psi_f \\ \downarrow & & \downarrow \\ Rh_*\Psi_g & \xrightarrow{\sigma-1} & Rh_*\Psi_g \ . \end{array}$$

Mais celle-ci résulte de la fonctorialité de Ψ.

Lemme 2.6.6.[1] *Soient X un schéma, $i : Y \to X$ un sous-schéma fermé, $j : U \to X$ l'ouvert complémentaire, r, s des entiers, $x \in H^r(U, \Lambda)$, $y \in H^s(Y, \Lambda)$. Alors on a, dans $H^{r+s+1}(X, \Lambda)$,*

$$((dx) \cdot y)_X = (-1)^{r+1}(x \cdot dy)_X \ ,$$

avec les notations suivantes : $dx \in H_Y^{r+1}(X, \Lambda)$ (resp. $dy \in H^{s+1}(X, j_!\Lambda)$) est déduit de x (resp. y) par le bord de la suite exacte de cohomologie à supports ; $((dx) \cdot y)_X$ (resp. $(x \cdot dy)_X$) est l'image dans $H^{r+s+1}(X, \Lambda)$, par la flèche canonique, du cup-produit $(dx) \cdot y \in H_Y^{r+s+1}(X, \Lambda)$ (resp. $x \cdot dy \in H^{r+s+1}(X, j_!\Lambda)$).

Rappelons d'abord que, pour A, B, $C \in D^-(X, \Lambda)$ et un accouplement $\pi : A \overset{L}{\otimes} B \to C$, le cup-produit $H^m(X, A) \otimes H^n(X, B) \to H^{m+n}(X, C)$, $a \otimes b \mapsto ab$, défini de la façon usuelle par un produit de cochaînes au moyen d'un relèvement de π à des résolutions convenables de A, B, C, vérifie la formule

(*)

$$(-1)^{mn}ab = \varepsilon \circ (a \overset{L}{\otimes} b) = \varepsilon \circ (\mathrm{Id} \overset{L}{\otimes} b) \circ (a \overset{L}{\otimes} \mathrm{Id}) = \varepsilon \circ (a \overset{L}{\otimes} \mathrm{Id}) \circ (\mathrm{Id} \overset{L}{\otimes} b),$$

où a (resp. b) est considéré comme une flèche $\Lambda_X \to A[m]$ (resp. $\Lambda_X \to B[n]$) de $D(X, \Lambda)$, et

$$\varepsilon : A[m] \overset{L}{\otimes} B[n] \to C[m+n]$$

est composé de l'isomorphisme canonique $A[m] \overset{L}{\otimes} B[n] \xrightarrow{\sim} (A \overset{L}{\otimes} B)[m+n]$ (défini, pour P, $Q \in C(X, \Lambda)$, par l'isomorphisme de Koszul $\Lambda[m] \otimes P \otimes \Lambda[n] \otimes Q \xrightarrow{\sim} \Lambda[m] \otimes \Lambda[n] \otimes P \otimes Q$, via les identifications évidentes $\Lambda[m] \otimes P = P[m]$, $\Lambda[n] \otimes Q = Q[n]$, $\Lambda[m] \otimes \Lambda[n] = \Lambda[m+n]$, qui ne font pas intervenir de signes) et de $\pi[m+n]$.

Considérons le diagramme suivant :

$$\Lambda_X \xrightarrow{y} \Lambda_Y[s] \xrightarrow{\delta[s]} j_!\Lambda_U[s+1] \xrightarrow{x[s+1]} \Lambda_X[r+s+1] \ ,$$

où $\delta : \Lambda_Y \to j_!\Lambda_U[1]$ est le morphisme de degré 1 du triangle distingué $j_!\Lambda_U \to \Lambda_X \to \Lambda_Y \to$ et x (resp. y) est considéré comme une flèche $j_!\Lambda_U \to \Lambda_X[r]$ (resp. $\Lambda_X \to \Lambda_Y[s]$) dans $D(X, \Lambda)$. Posons $f = x[s+1] \circ \delta[s] \circ y$. Par définition, $\delta[s] \circ y = dy \in H^{s+1}(X, j_!\Lambda_U)$. Comme

[1] Je remercie vivement T. Saito et O. Gabber pour leur aide dans la démonstration de ce lemme.

$x \cdot dy = (-1)^{r(s+1)}(dy) \cdot x$ dans $H^{r+s+1}(X, j_!\Lambda_U)$, et donc $(x \cdot dy)_X = (-1)^{r(s+1)}((dy) \cdot x)_X$ dans $H^{r+s+1}(X, \Lambda)$, il résulte de la formule $(*)$ que $f = (x \cdot dy)_X$. On voit de même que $f = ((x[1] \circ \delta) \cdot y)_X$. Pour achever la démonstration, il suffit de vérifier que

$$(**) \qquad x[1] \circ \delta = (-1)^{r+1} dx$$

dans $H_Y^{r+1}(X, \Lambda) = \mathrm{Hom}_{D(X,\Lambda)}(\Lambda_Y, \Lambda_X[r+1])$. Pour cela, choisissons une résolution flasque $C^{\cdot} = (\cdots \to C^i \xrightarrow{d_C} C^{i+1} \to \cdots)$ de Λ_X et un cocycle $\xi \in C^r(U)$ représentant x. Le morphisme $\delta : \Lambda_Y \to j_!\Lambda_U[1]$ est représenté par les morphismes de complexes

$$\Lambda_Y \xleftarrow{q} \text{Cône}(j_!\Lambda_U \longrightarrow \Lambda_X) \xrightarrow{-1} j_!\Lambda_U[1] \ ,$$

où q est le quasi-isomorphisme donné par la projection $\Lambda_X \to \Lambda_Y$. Par suite $x[1] \circ \delta$ est représenté par le composé de q^{-1} et du morphisme de complexes u :

$$\begin{array}{ccccccc}
(0 & \longrightarrow & j_!\Lambda_U & \longrightarrow & \Lambda_X & \longrightarrow & 0) = \text{Cône}(j_!\Lambda_U \longrightarrow \Lambda_X) \\
 & & \downarrow{\scriptstyle -\xi} & & \downarrow{\scriptstyle 0} & & \\
(\cdots & \longrightarrow & C^r & \xrightarrow{(-1)^{r+1}d_C} & C^{r+1} & \longrightarrow & \cdots) = C^{\cdot}[r+1] \ .
\end{array}$$

De même, utilisant la suite exacte $0 \to \mathcal{H}^0_Y(C^{\cdot}) \to C^{\cdot} \to j_*C^{\cdot}_U \to 0$, on voit que $(-1)^{r+1}dx$ est représenté par le composé de q^{-1} et du morphisme de complexes v :

$$\begin{array}{ccccccc}
(0 & \longrightarrow & j_!\Lambda_U & \longrightarrow & \Lambda_X & \longrightarrow & 0) \\
 & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle (-1)^{r+1}d_C\tilde{\xi}} & & \\
(\cdots & \longrightarrow & C^r & \xrightarrow{(-1)^{r+1}d_C} & C^{r+1} & \longrightarrow & \cdots) = C^{\cdot}[r+1] \ ,
\end{array}$$

où $\tilde{\xi} \in C^r(X)$ est un relèvement de ξ. L'identité $(**)$ découle alors de la formule $u - v = dh + hd$, où h est l'homotopie définie par $-\tilde{\xi} : \Lambda_X \to C^r$.

Remarques 2.7. (i) Compte tenu des isomorphismes de 2.6 (b), la nullité des $H^i(V)$ pour $i \neq 0, n$ concorde avec le résultat de 2.2.

(ii) Les résultats de 2.6 impliquent des résultats analogues pour Λ remplacé par $\mathbb{Z}_\ell$. Posons (avec $\Psi_f = \Psi_f(\mathbb{Z}_\ell)$)

$$\begin{aligned}
H^n(\Psi_f)_{\text{prim}} &= \text{Coker } H^n(D_1, \mathbb{Z}_\ell) \to H^n(V, \mathbb{Z}_\ell) \\
&= \text{Im } H^n(V, \mathbb{Z}_\ell) \to H^{n-1}(C, \mathbb{Z}_\ell)(-1),
\end{aligned}$$

$$H^n_{\{x_0\}}(X_s, \Psi_f)^{\text{prim}} = \text{Ker } H^n_c(V, \mathbb{Z}_\ell) \to H^n(D_1, \mathbb{Z}_\ell)$$
$$= \text{Im } H^{n-1}(C, \mathbb{Z}_\ell) \to H^n_c(V, \mathbb{Z}_\ell).$$

On a donc

$$H^n(\Psi_f)_{\text{prim}} = \text{Ker } H^{n-1}(C, \mathbb{Z}_\ell)(-1) \to H^{n+1}(D_1, \mathbb{Z}_\ell)$$
$$= \text{Ker } \xi : H^{n-1}(C, \mathbb{Z}_\ell)(-1) \to H^{n+1}(C, \mathbb{Z}_\ell)$$
$$= H^{n-1}(C, \mathbb{Z}_\ell)^{\text{prim}}(-1),$$

où $H^{n-1}(C, \mathbb{Z}_\ell)^{\text{prim}}$ est la *partie primitive* de $H^{n-1}(C, \mathbb{Z}_\ell)$, égale à $H^{n-1}(C, \mathbb{Z}_\ell)$ si n est pair, et à l'orthogonal de $\xi^{(n-1)/2}$ si n est impair, ξ désignant la classe d'une section hyperplane. De même, $H^n_{\{x_0\}}(X_s, \Psi_f)^{\text{prim}}$ est le *quotient primitif* $H^{n-1}(C, \mathbb{Z}_\ell)_{\text{prim}}$ de $H^{n-1}(C, \mathbb{Z}_\ell)$, un $\mathbb{Z}_\ell$-module libre de type fini, dual de la partie primitive, égal à $H^{n-1}(C, \mathbb{Z}_\ell)$ si n est pair, et à $H^{n-1}(C, \mathbb{Z}_\ell)/\mathbb{Z}_\ell \xi^{(n-1)/2}$ si n est impair. Il résulte du diagramme de 2.6 (c) que $\text{Var}(\sigma)$ induit un morphisme

$$\text{Var}(\sigma)_{\text{prim}} : H^n(\Psi_f)_{\text{prim}} \to H^n_{\{x_0\}}(X_s, \Psi_f)^{\text{prim}} . \tag{2.7.1}$$

Supposons que $t_\ell(\sigma)$ soit un générateur de $\mathbb{Z}_\ell(1)$. Alors, *si n est pair*, $\text{Var}(\sigma)_{\text{prim}}$ *est un isomorphisme*, et *si n est impair*, $\text{Var}(\sigma)_{\text{prim}}$ *est injectif, de conoyau de longueur égale à la valuation ℓ-adique* de e. Le discriminant de la restriction à la partie primitive de la forme d'intersection sur $H^{n-1}(C, \mathbb{Z}_\ell)$ est en effet égal à e (puisque $(\xi^{(n-1)/2} \cdot \xi^{(n-1)/2}) = e$ et que la forme d'intersection sur $H^{n-1}(C, \mathbb{Z}_\ell)$ est unimodulaire, cf. [SGA 7 XIX 5]). J'ignore comment définir directement (2.7.1), sans recours à l'éclatement h.

3. La formule de Picard-Lefschetz en dimension relative impaire

3.1. Soit $n = 2m + 1$ un entier impair, et soit $Q \in A[x_1, \ldots, x_{n+1}]$ une forme quadratique ordinaire [SGA 7 XII 1] : cela signifie que $Q \otimes k$ est non nulle et que la quadrique projective de $\mathbb{P}^n_S$ d'équation $Q = 0$ est lisse, ou encore qu'il existe un changement linéaire de coordonnées $x = Py$, $P \in \text{GL}_{n+1}(A)$, transformant Q en $\sum_{1 \le i \le m+1} y_i y_{m+1+i}$. Soit π une uniformisante de A. Soit X le sous-schéma fermé de $\mathbb{A}^{n+1}_S = \text{Spec } A[x_1, \ldots, x_{n+1}]$ d'équation $Q - \pi = 0$, notons $f : X \to S$ la projection, et, comme en 2.1, Ψ_f (resp. Φ_f) le complexe des cycles proches (resp. évanescents) correspondant. Le morphisme f est lisse hors de l'origine $x_0 = (0, \ldots, 0)$ de $\mathbb{A}^{n+1}_s$, de sorte que Φ_f est concentré

en x_0. Les résultats suivants sont établis par Deligne dans [SGA 7 XV, 2.2, 3.3].

(a) On a $H^i\Psi_f = 0$ pour $i \neq 0,\ n$, $H^i_{\{x_0\}}(X_s, \Psi_f) = 0$ pour $i \neq n,\ 2n$, $H^0\Psi_f = \Lambda_{X_s}$, $H^{2n}_{\{x_0\}}(X_s, \Psi_f(n)) = \Lambda$. Les Λ-modules $H^n(\Psi_f(m+1))_{x_0}$ et $H^n_{\{x_0\}}(X_s, \Psi_f(m))$ sont libres de rang 1, et duaux l'un de l'autre par l'accouplement naturel $\langle\ ,\ \rangle : H^n(\Psi_f(m+1))_{x_0} \otimes H^n_{\{x_0\}}(X_s, \Psi_f(m)) \to H^{2n}_{\{x_0\}}(X_s, \Psi_f(n)) = \Lambda$. Ils ont des bases canoniques $\delta \in H^n_{\{x_0\}}(X_s, \Psi_f(m))$ et $\delta' \in H^n(\Psi_f(m+1)_{x_0})$, définies au signe près, que l'on peut choisir de manière que $\langle \delta', \delta \rangle = 1$. L'application naturelle $\phi : H^n_{\{x_0\}}(X_s, \Psi_f) \to H^n(\Psi_f)_{x_0}$ est nulle.

(b) Pour $\sigma \in I$ et $x \in H^n(\Psi_f(m+1))_{x_0}$, on a (*formule de Picard-Lefschetz*)

$$\mathrm{Var}(\sigma)x = (-1)^{m+1} t_\ell(\sigma)\langle x, \delta\rangle\delta\ . \tag{3.1.1}$$

3.2. Nous allons montrer comment on peut déduire, algébriquement, (a) et (b) de 2.6. En fait, la démonstration de (a) dans [SGA 7 XV] est elle aussi entièrement algébrique, ce n'est que celle de (b) qui fait appel à un argument transcendant.

On peut supposer les coordonnées x_i sur $\mathbb{A}^{n+1}_S$ choisies de manière que $Q = \sum_{1 \leq i \leq m+1} x_i x_{m+1+i}$. Soit $K' \subset \bar{K}$ une extension quadratique séparable de K, totalement ramifiée. Notons $I' = \mathrm{Gal}(\bar{K}/K')$ le sous-groupe (d'indice 2) de I correspondant, A' le normalisé de A dans K', π' une uniformisante de A', $S' = \mathrm{Spec}\ A'$, $f' : X' \to S'$ le morphisme déduit de f par le changement de base $S' \to S$. On a $\pi = u\pi'^2$ dans A', avec $u \in A'^*$, et X' est le sous-schéma fermé de $\mathbb{A}^{n+1}_{S'}$ d'équation $Q - u\pi'^2 = 0$. On a $\Psi_f = \Psi_{f'}$ et l'action de I' sur $\Psi_{f'}$ est induite par celle de I sur Ψ_f. La forme quadratique $Q - ut^2 \in A'[x_1, \ldots, x_{n+1}, t]$ étant ordinaire, l'hypothèse (H) de 2.3 est vérifiée, avec $F = Q$, $e = 2$ et A remplacé par A'. On peut donc appliquer 2.6. La variété des cycles évanescents V de f' est une quadrique affine, complément, dans la quadrique projective lisse D_1 d'équation $Q - \bar{u}t^2 = 0$ dans $\mathbb{P}^{n+1}_s = \mathrm{Proj}\ k[x_1, \ldots, x_{n+1}, t]$, de la section hyperplane C d'équation $t = 0$, quadrique lisse de $\mathbb{P}^n_s = \mathrm{Proj}\ k[x_1, \ldots, x_{n+1}]$ d'équation $Q = 0$. Les assertions de 3.1 (a) découlent donc de 2.6 (a) et (b), compte tenu de la structure de la cohomologie des quadriques [SGA 7 XII]. Comme n est impair, on a, avec les notations de 2.7 (ii) (et en tensorisant sur $\mathbb{Z}_\ell$ par Λ), $H^n(\Psi_{f'})_{x_0} = H^n(V, \Lambda) = H^{n-1}(C, \Lambda)^{\mathrm{prim}}(-1)$, $H^n_{\{x_0\}}(\Psi_{f'}) =$

$H^{n-1}(C,\Lambda)_{\text{prim}}$, et l'application canonique $\phi : H^n_c(V,\Lambda) \to H^n(V,\Lambda)$ est nulle [SGA 7 XII, 3.6]. Choisissons une base δ' de $H^{n-1}(C,\Lambda(m))^{\text{prim}}$ (de la forme $cl(\alpha) - cl(\beta)$, où α et β sont des génératrices de C de types distincts) [SGA 7 XII 3.5], et notons $\delta \in H^{n-1}(C,\Lambda(m))_{\text{prim}} = \text{Hom}(H^{n-1}(C,\Lambda(m))^{\text{prim}},\Lambda)$ la base duale. Comme $\langle\delta',\delta'\rangle = (-1)^m 2$ [SGA 7 XII, 3.3 (iii) (b)], l'image de δ' dans le quotient $H^{n-1}(C,\Lambda(m))_{\text{prim}}$ est $(-1)^m 2\delta$. Notant Var_f (resp. $\text{Var}_{f'}$) la variation relative à f (resp. f') et $t'_\ell : I' \to \mathbb{Z}_\ell(1)$ le caractère canonique relatif à A', il résulte donc de 2.6 (c) que, pour $\tau \in I'$, on a

$$(*) \qquad \text{Var}_{f'}(\tau)\delta' = (-1)^{m+1} t'_\ell(\tau) 2\delta \ .$$

Pour $\sigma \in I$, on a $\sigma^2 \in I'$. Donc, d'après $(*)$,

$$\text{Var}_f(\sigma^2)\delta' = \text{Var}_{f'}(\sigma^2)\delta' = (-1)^{m+1} t'_\ell(\sigma^2) 2\delta \ .$$

Mais $2t'_\ell(\sigma^2) = (\sigma^2((u\pi'^2)^{1/\ell^m})/(u\pi'^2)^{1/\ell^m})_m = (\sigma^2(\pi^{1/\ell^m})/\pi^{1/\ell^m})_m = t_\ell(\sigma^2)$.
On en déduit, pour tout $\sigma \in I$,

$$(**) \qquad \text{Var}_f(\sigma^2)\delta' = (-1)^{m+1} t_\ell(\sigma^2)\delta \ .$$

Mais $t_\ell(\sigma^2) = 2t_\ell(\sigma)$, et comme l'application ϕ est nulle, $\sigma \mapsto \text{Var}_f(\sigma)\delta'$ est un homomorphisme de I dans $H^n_{\{x_0\}}(X_s,\Psi_f(m))$. En particulier, $\text{Var}_f(\sigma^2)\delta' = 2\text{Var}_f(\sigma)\delta'$. On peut donc récrire $(**)$ sous la forme

$$2\text{Var}_f(\sigma)\delta' = (-1)^{m+1} 2t_\ell(\sigma)\delta \ .$$

Passant à la limite sur les $\Lambda = \mathbb{Z}/\ell^\nu\mathbb{Z}$ $(\nu \geq 1)$, on obtient la même formule à coefficients dans $\mathbb{Z}_\ell$. Comme $H^n_{\{x_0\}}(X_s,\Psi_f(\mathbb{Z}_\ell)(m))$ est libre de rang 1 sur $\mathbb{Z}_\ell$, de base δ, on peut diviser par 2, et on en déduit (3.1.1) pour $x = \delta'$, donc pour tout x.

Remarque 3.3. On peut montrer que l'identification de $H^n(\Psi_f)_{x_0}$ à $H^{n-1}(C)^{\text{prim}}(-1)$ utilisée en 3.2 coïncide avec celle décrite dans [SGA 7 XV 2.2.3]. Notons i_1 la première, i_2 la seconde. Rappelons la définition de i_2. Soit $\bar{X}$ la complétion projective de X, d'équation $Q - \pi t^2 = 0$ dans $\mathbb{P}^{n+1}_S = \text{Proj } A[x_1,\ldots,x_{n+1},t]$, $\bar{X}_\infty$ son lieu à l'infini, défini par $t = 0$, quadrique lisse sur S, d'équation $Q = 0$ dans $\mathbb{P}^n_S$, de fibre spéciale $(\bar{X}_\infty)_s = C$. Comme $\bar{X}$ est lisse sur S au voisinage de $\bar{X}_\infty$, la flèche canonique $H^n(X_{\bar{\eta}},\Lambda) \to H^n(X_s,\Psi_f)$ [SGA 7 XIII 2.1.8.3] est un isomorphisme, et, comme expliqué en [SGA 7 XV 2.2.3], la flèche $H^n(X_s,\Psi_f) \to H^n(\Psi_f)_{x_0}$ est un isomorphisme ; on a par ailleurs $H^n(X_{\bar{\eta}},\Lambda) \xrightarrow{\sim} H^{n-1}((\bar{X}_\infty)_{\bar{\eta}},\Lambda)^{\text{prim}}(-1) \xrightarrow{\sim} H^{n-1}(C)^{\text{prim}}(-1)$; l'identification i_2 est composée de ces isomorphismes. On obtiendrait la même

identification i_2 en remplaçant X/S par X'/S', $\bar{X}$ par $\bar{X}'$ (d'équation $Q - u\pi'^2 t^2 = 0$ dans $\mathbb{P}^{n+1}_{S'}$) et appliquant la même suite d'isomorphismes. Pour établir la coïncidence de i_1 et i_2, on peut employer l'argument suivant, dû à T. Saito.

Soit $P \subset \mathbb{P}^{n+1}_{S'} \times \mathbb{P}^{n+1}_{S'}$ l'adhérence schématique du graphe de l'isomorphisme $\phi : \mathbb{P}^{n+1}_{\eta'} \to \mathbb{P}^{n+1}_{\eta'}$, $((x_i), t) \mapsto ((x_i), \pi' t)$ (d'inverse $((x_i), t) \mapsto ((\pi' x_i), t)$). Dans Proj $A'[x_1, \ldots, x_{n+1}, t] \times$ Proj $A'[x'_1, \ldots, x'_{n+1}, t']$, P est défini par les équations $x_i t' = \pi' x'_i t$ $(1 \leq i \leq n+1)$, $x_i x'_j = x'_i x_j$ $(1 \leq i, j \leq n+1)$. On a des projections canoniques $p_i : P \to \mathbb{P}^{n+1}_{S'}$ $(i = 1, 2)$, où p_1 (resp. p_2) identifie P à l'éclaté de x_0 (resp. du sous-schéma fermé d'équations $\pi' = t' = 0$) dans $\mathbb{P}^{n+1}_{S'}$. On en déduit un diagramme

$$\begin{array}{ccccc}
\mathbb{P}^{n+1}_{S'} & \xleftarrow{p_1} & P & \xrightarrow{p_2} & \mathbb{P}^{n+1}_{S'} \\
\cup & & \cup & & \cup \\
\bar{X}' & \longleftarrow & \bar{Z} & \longrightarrow & \bar{W} \\
\cup & & \cup & & \cup \\
\bar{X}'_s & \longleftarrow & \bar{D}_0 \cup D_1 & \longrightarrow & D_1
\end{array} ,$$

où $\bar{W}$ est la quadrique projective (lisse) d'équation $Q(x') - ut'^2 = 0$, et $\bar{Z}$ l'éclaté de $(\bar{W}_\infty)_s := (\pi' = t' = 0)$ dans $\bar{W}$ ou de x_0 dans $\bar{X}'$ (complétion projective de l'éclaté Z de x_0 dans X' considéré en 3.2) ; la ligne inférieure est formée des fibres spéciales ; $\bar{D}_0$ est la complétion projective de D_0, un fibré en droites projectives sur $C = D_0 \cap D_1$, que la projection $\bar{Z} \to \bar{W}$ contracte en $(\bar{W}_\infty)_s$ $(= C)$. Par fonctorialité de $R\Psi$, on obtient un diagramme commutatif d'isomorphismes (pour abréger, Λ est omis, et "0" = "prim")

$$\begin{array}{ccccc}
H^n(X'_{\bar{\eta}}) & \xrightarrow{(2)} & H^n(Z_{\bar{\eta}}) & \longrightarrow & H^n(W_{\bar{\eta}}) \\
\downarrow{\scriptstyle (1)} & & \downarrow{\scriptstyle (3)} & & \downarrow \\
H^n(\Psi_f) & \longrightarrow & H^n(D_1 - C) & \longrightarrow & H^n(W_s) \\
 & & & \longrightarrow & H^n(X'_{\bar{\eta}}) \longrightarrow H^{n-1}((\bar{X}'_\infty)_{\bar{\eta}})^0(-1) \\
 & & & & \downarrow{\scriptstyle (4)} \\
 & & & & H^{n-1}(C)^0(-1),
\end{array}$$

où Z (resp. W) est le complément du diviseur à l'infini $t = 0$. Notons h la flèche composée des isomorphismes horizontaux, et $c : H^n(D_1 - C) \to H^{n-1}(C)^0(-1)$ la flèche composée dans la ligne inférieure, qui est l'isomorphisme canonique. Par définition, $i_2 = (4) \circ h \circ (1)^{-1}$, et $i_1 = c \circ (3) \circ (2) \circ (1)^{-1}$, donc $i_1 = i_2$.

Remarque 3.4. Pour X d'équation $Q - b = 0$, pour $b \in \mathfrak{m} - \{0\}$, on a les mêmes résultats qu'en 3.1 (a) et (b), $t_\ell(\sigma)$ au second membre de (3.1.1) étant remplacé par $\varepsilon_b(\sigma) = v(b)t_\ell(\sigma)$ [SGA 7 XV 3.3.3]. Il est possible qu'une variante des arguments précédents permette de les établir directement.

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Log C^∞-Functions and Degenerations of Hodge Structures

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Introduction

In [27], J.H.M. Steenbrink studied degenerations of Hodge structures. For $f\colon X \longrightarrow \Delta = \{z \in \mathbb{C}\ ;\ |z| < 1\}$ projective and of semi-stable degeneration, he showed that a "limit Hodge structure" appears as the limit of the Hodge structures $H^m(X_t, \mathbb{Z})$ ($m \in \mathbb{Z}$, $t \in \Delta - \{0\}$). In log Hodge theory, as in [23], his theory is interpreted in the form "the higher direct images on Δ of $\mathbb{Z}_X$ carry the natural variations of polarized log Hodge structure."

In this paper, we will generalize the theory of Steenbrink in this form to the theory with coefficients (that is, we will start with general variations of polarized log Hodge structure $\mathcal{H}_\mathbb{Z}$ on X instead of $\mathbb{Z}_X$).

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Our method is different from Steenbrink's. We use "log C^∞-functions" and "log harmonic forms", in the way as we use C^∞-functions and harmonic forms in the case without degeneration in the classical Hodge theory. Our main result is the following. (See Appendix for special terminology of log geometry, if the reader is not familiar with log structures of Fontaine-Illusie. For example, see Appendix 2 for "log smooth fs log analytic space", see Appendix 4 for "log smooth morphism" and for "vertical morphism", and see Appendix 5 for "ket sense". In particular, the word "vertical" in the statement below shows that we assume the degeneration of f and the degeneration of $(\mathcal{H}_{\mathbb{Z}}, \mathcal{M}, (\ ,\ ))$ occur only in the "vertical direction" with respect to f.)

Theorem. *Let X, Y be log smooth fs log analytic spaces, and let $f\colon X \longrightarrow Y$ be a projective log smooth vertical morphism. Let $(\mathcal{H}_{\mathbb{Z}}, \mathcal{M}, (\ ,\ ))$ be a variation of polarized log Hodge structure on X of weight w in the ket sense. Then:*
(1) *The Hodge to de Rham spectral sequence*

$$E_1^{p,q} = R^{p+q} f_* \mathrm{gr}^p(\omega^\bullet_{X/Y}(\mathcal{M})) \Rightarrow E_\infty^m = R^m f_*(\omega^\bullet_{X/Y}(\mathcal{M}))$$

degenerates from E_1 and each $R^m f_* \mathrm{gr}^p(\omega^\bullet_{X/Y}(\mathcal{M}))$ is a locally free $\mathcal{O}_Y$-module on Y_{ket}. Here $\omega^\bullet_{X/Y}(\mathcal{M})$ denotes the de Rham complex with log poles and with coefficients in $\mathcal{M}$.
(2) *For each $m \in \mathbb{Z}$, $\left(R^m f_*^{\log}\mathcal{H}_{\mathbb{Z}},\ R^m f_*\omega^\bullet_{X/Y}(\mathcal{M}),\ (\ ,\ )\right)$ with the Hodge filtration on $R^m f_*\omega^\bullet_{X/Y}(\mathcal{M})$ is a variation of polarized log Hodge structure on Y of weight $w+m$ in the ket sense. Here $(\ ,\ )$ is the intersection form defined by fixing an invertible $\mathcal{O}_X$-module which is relatively very ample with respect to Y.*

In the case where $\mathcal{H}_{\mathbb{Z}} = \mathbb{Z}$ and X is semi-stable over $Y = \Delta$, this gives a new proof of a result of Steenbrink [27] as explained in the above. See 8.13 for the details. The theorem also gives new proofs to results of T. Fujisawa [6], L. Illusie [13] and M. Cailotto [1]. See also Remark 8.12.

In [14], the functoriality of log Riemann-Hilbert correspondences was established, which is a generalization of results of the second author [23], [24], [25], F. Kato [16], and S. Usui [29], [30]. This implies that $R^m f_*(\omega^\bullet_{X/Y}(\mathcal{M}))$ is a locally free $\mathcal{O}_Y$-module in the ket sense, and corresponds to $R^m f_*^{\log}\mathcal{H}_{\mathbb{Z}}$ via the log Riemann-Hilbert correspondence on Y. The above Theorem shows that we can add Hodge filtrations in this functoriality.

See Y. Kawamata-Y. Namikawa [21] for another approach by log method to the degenerations of Hodge structures.

Log C^∞-functions are functions which have, together with their "log derivatives", logarithmic growth at the boundary. After we completed our paper, we learned from Prof. S. Zucker that this notion has been already considered by several authors (for example, [10], [9], [11], [12]) and that contents of the sections 1 and 3 are known.

We are very much thankful to Prof. L. Illusie, Prof. S. Usui, Prof. T. Saito and Prof. S. Zucker for their advice.

§1. Log C^∞-functions

1.1. Let X be an fs log analytic space which is log smooth over $\mathbb{C}$. Let $X_{\mathrm{triv}} = \{x \in X; M_{X,x} = \mathcal{O}_{X,x}^\times\}$, which is an open dense subset of X. (See Appendix.) We define the ring of log C^∞-functions on X as a subring of the ring of C^∞-functions on X_{triv}. When X is a complex manifold M whose log structure is given by a divisor D with normal crossings, the sheaf of log C^∞-functions is the same as $\mathcal{A}^0_{sia}(M, D)$ in [12] (2.2). See also [10], [9], and [11] 3.8.

For a function $f: X_{\mathrm{triv}} \longrightarrow \mathbb{C}$, we say f is of log growth on X if there exists an open covering $(U_\lambda)_\lambda$ of X with an element $t_\lambda \in \Gamma(U_\lambda, M_X^{\mathrm{gp}})$ and an integer $m(\lambda) \geq 0$ for each λ, for which we have

$$|f(x)| \leq |\log |t_\lambda(x)||^{m(\lambda)}$$

for any λ and any $x \in X_{\mathrm{triv}} \cap U_\lambda$.

By a log C^∞-function on X, we mean a C^∞-function $f: X_{\mathrm{triv}} \longrightarrow \mathbb{C}$ having the following property: If U is an open set of X and $(t_j)_{1 \leq j \leq n}$ is a family of elements of $\Gamma(U, M_X^{\mathrm{gp}})$ such that $(d\log(t_j))_{1 \leq j \leq n}$ is an $\mathcal{O}_U$-basis of ω^1_U (= the sheaf of analytic differential forms on U with log poles; see [18] (3.5)), then the following condition (C) is satisfied.
(C) For any $a(j)$, $b(j) \in \mathbb{N}$ $(1 \leq j \leq n)$,

$$\left(\prod_j \left(t_j \cdot \frac{\partial}{\partial t_j}\right)^{a(j)} \left(\overline{t_j} \cdot \frac{\partial}{\partial \overline{t_j}}\right)^{b(j)}\right)(f)$$

is of logarithmic growth on U.

Note that locally on X, a family $(t_j)_{1 \leq j \leq n}$ as above exists and the condition (C) is independent of the choice of such $(t_j)_{1 \leq j \leq n}$.

Example 1.2. (1) When X is a complex manifold whose log structure is given by a divisor with normal crossings, a C^∞-function on X is a log C^∞-function.

(2) A meromorphic function on X is a log C^∞-function on X if and only if it is holomorphic.
(3) For any section t of M_X^{gp}, $\log|t|$ and $(\frac{t}{|t|})^n$ $(n \in \mathbb{Z})$ are log C^∞-functions on X. If t is in M_X, $|t|^c$ $(c \in \mathbb{C}, \mathrm{Re}(c) \geq 0)$ is a log C^∞-function on X.
(4) For any section t of M_X, $|\log|t||^c$ $(c \in \mathbb{C})$ is a log C^∞-function on X outside the points $x \in X$ at which $t_x \in \mathcal{O}_{X,x}^\times$ and $|t(x)| = 1$.

1.3. We show that log C^∞-functions on X form a ring. For this, it is sufficient to show that functions on X_{triv} of log growth on X form a ring. It is sufficient to show that for $x \in X$ and $t_1,\ t_2 \in \Gamma(X, M_X^{\mathrm{gp}})$, there exist an open neighborhood U of x and $t \in \Gamma(U, M_X)$ such that

$$|\log(|t|)| \geq \max(|\log(|t_1|)|, |\log(|t_2|)|) \text{ on } X_{\mathrm{triv}} \cap U.$$

We may assume that $x \notin X_{\mathrm{triv}}$. Take an element s of $M_{X,x}$ whose image in the fs monoid $M_{X,x}/\mathcal{O}_{X,x}^\times$ belongs to the interior of $M_{X,x}/\mathcal{O}_{X,x}^\times$. Then for some $n \geq 1$, $s^n t_1$, $s^n t_1^{-1}$, $s^n t_2$, $s^n t_2^{-1}$ belong to the interior of $M_{X,x}/\mathcal{O}_{X,x}^\times$. Hence $|s^n t_1| < 1$, $|s^n t_1^{-1}| < 1$, $|s^n t_2| < 1$, $|s^n t_2^{-1}| < 1$ on $X_{\mathrm{triv}} \cap U$ for some open neighborhood U of x. This shows

$$|\log(|s^n|)| > \max(|\log(|t_1|)|, |\log(|t_2|)|)$$

on $X_{\mathrm{triv}} \cap U$.

1.4. Let $\mathcal{A}_X$ be the sheaf $U \mapsto \{\log\ C^\infty\text{-functions on } U\}$ of X. Let $V = X_{\mathrm{triv}}$, let C_V^∞ be the sheaf of C^∞-functions on V, and let $j\colon V \longrightarrow X$ be the canonical morphism. Then $\mathcal{A}_X$ is regarded as a subsheaf of $j_* C_V^\infty$. Let $C_V^{\infty,q}$ $(q \in \mathbb{Z})$ be the sheaf of C^∞ q-forms on V. For $p,\ q \in \mathbb{Z}$, define the sheaf $\mathcal{A}_X^{p,q}$ of log C^∞ (p,q)-forms on X to be the image of

$$\mathcal{A}_X \otimes \omega_X^p \otimes \omega_X^q \longrightarrow j_*(C_V^{\infty,p+q}); f \otimes \omega \otimes \eta \mapsto f\omega \wedge \overline{\eta},$$

and for $m \in \mathbb{Z}$, let

$$\mathcal{A}_X^m = \bigoplus_{p+q=m} \mathcal{A}_X^{p,q} \subset j_* C_V^{\infty,m}.$$

Proposition 1.5. *Assume that the underlying analytic space $\overset{\circ}{X}$ of X is Hausdorff. For any $p,\ q \in \mathbb{Z}$, $\mathcal{A}_X^{p,q}$ is a soft sheaf on X.*

Proof. If A is a soft ring, an A-module M is also soft. Hence we can reduce 1.5 to proving that $\mathcal{A}_X$ is soft. Moreover we can assume $X =$

$(\operatorname{Spec}\mathbb{C}[\mathcal{S}])^{\mathrm{an}}$ for some fs monoid $\mathcal{S}$. Then we can find a surjective map $\mathbb{N}^s \longrightarrow \mathcal{S}$ and, hence, a closed immersion $X \overset{i}{\hookrightarrow} Z := (\operatorname{Spec}\mathbb{C}[\mathbb{N}^s])^{\mathrm{an}}$. Then we have a map $i^{-1}\mathcal{A}_Z \longrightarrow \mathcal{A}_X$. Since $C^\infty_{\overset{\circ}{Z}}$ is soft, $C^\infty_{\overset{\circ}{Z}}$-module $\mathcal{A}_Z$ is soft. This implies $\mathcal{A}_X$ is soft. Q.E.D.

Proposition 1.6. *$U \mapsto \mathcal{A}_X(U)$ is a sheaf on X_{ket}. (See [14] or Appendix for the definition of X_{ket}.)*

Proof. It is enough to show that, for a surjective, Kummer log étale morphism $g\colon V \longrightarrow W$ of log smooth fs log analytic spaces, a C^∞-function $f\colon W_{\mathrm{triv}} \longrightarrow \mathbb{C}$ is log C^∞ if and only if $f \circ g$ is log C^∞. This is easily checked with the fact that g is an open map. Q.E.D.

Proposition 1.7. *Let M be a sheaf of $\mathbb{Q}$-vector spaces on X_{ket}. Then $R^q\varepsilon_* M = 0$ for any $q > 0$, where ε is the projection of topoi from X_{ket} to X.*

Proof. See [14]. Q.E.D.

From now, everything is in the ket sense unless the contrary is explicitly stated.

1.8. We define a sheaf $\mathcal{A}_X^{\log}$ on $X^{\log}$ by

$$\mathcal{A}_X^{\log} = \mathcal{O}_X^{\log} \otimes_{\tau^{-1}(\mathcal{O}_X)} \tau^{-1}(\mathcal{A}_X),$$

where τ is the canonical map $X^{\log} \longrightarrow X_{\mathrm{ket}}$. ($\mathcal{O}_X$, $\mathcal{A}_X$, and $\mathcal{O}_X^{\log}$ here are the ket versions.) Note that $\mathcal{A}_X^{\log} \longrightarrow j_*^{\log} C_V^\infty$ is not necessarily injective since $\tau^{-1}\mathcal{A}_X \longrightarrow j_*^{\log} C_V^\infty$ is not. We define

$$\mathcal{A}_X^{p,q,\log} = \mathcal{O}_X^{\log} \otimes_{\tau^{-1}(\mathcal{O}_X)} \tau^{-1}(\mathcal{A}_X^{p,q}) \quad (p,\ q \in \mathbb{Z})$$
$$\mathcal{A}_X^{m,\log} = \mathcal{O}_X^{\log} \otimes_{\tau^{-1}(\mathcal{O}_X)} \tau^{-1}(\mathcal{A}_X^{m}) \quad (m \in \mathbb{Z}).$$

We have a complex conjugate $\mathcal{A}_X^{\log} \longrightarrow \mathcal{A}_X^{\log}$ by extending the complex conjugate of $\mathcal{A}_X$ by $\mathcal{O}_X^{\log} \longrightarrow \mathcal{A}_X^{\log}; \log(f) \mapsto 2 \cdot (\log|f|) - \log(f) \otimes 1$.

Proposition 1.9. *We have*

$$R\tau_*(\mathcal{A}_X^{p,q,\log}) = \mathcal{A}_X^{p,q} \text{ for } p,\ q \in \mathbb{Z}.$$

Proof. It is checked stalkwise that $R\tau_*(\mathcal{O}_X^{\log} \otimes_{\tau^{-1}\mathcal{O}_X} \tau^{-1}M) = M$ for any $\mathcal{O}_X$-module M (cf. [14]). The proposition is a special case of this fact. Q.E.D.

§2. Log C^∞ Hodge decompositions

2.1. In this section, we relate log C^∞-functions to degenerations of polarized Hodge structures. In Theorem 2.6 below, we show that a "variation of polarized log Hodge structure" (VPLH) has a "log C^∞ Hodge decomposition". Here VPLH is a notion which is something like "degenerating variation of polarized Hodge structure" and which matches well the theory of Schmid on nilpotent orbits ([26]). The proof of Theorem 2.6 bases on the theory of Cattani-Kaplan-Schmid on $SL(2)$-orbits ([26], [3]).

In the classical theory, if X is a complex manifold and $\mathcal{H}_\mathbb{Z}$ is a variation of polarized Hodge structure (VPH) on X of weight w, $\mathbb{C} \otimes_\mathbb{Z} \mathcal{H}_{\mathbb{Z},x}$ for each $x \in X$ has Hodge decomposition

$$\mathcal{H}_{\mathbb{C},x} = \bigoplus_{p+q=w} \mathcal{H}^{p,q}_{\mathbb{C},x}$$

where $\mathcal{H}_\mathbb{C} = \mathbb{C} \otimes_\mathbb{Z} \mathcal{H}_\mathbb{Z}$ and $\mathcal{H}^{p,q}_{\mathbb{C},x}$ is the intersection of $\mathrm{Fil}^p(\mathcal{H}_{\mathbb{C},x})$ and the complex conjugate of $\mathrm{Fil}^q(\mathcal{H}_{\mathbb{C},x})$. The $\mathcal{O}_X$-module $\mathcal{O}_X \otimes_\mathbb{Z} \mathcal{H}_\mathbb{Z}$ has a filtration by the definition of VPH, but this $\mathcal{O}_X$-module $\mathcal{O}_X \otimes_\mathbb{Z} \mathcal{H}_\mathbb{Z}$ does not necessarily have a Hodge decomposition (this is because $\mathcal{O}_X$ does not have the complex conjugation). However $C^\infty_X \otimes_\mathbb{Z} \mathcal{H}_\mathbb{Z}$ has a Hodge decomposition

$$C^\infty_X \otimes_\mathbb{Z} \mathcal{H}_\mathbb{Z} = \bigoplus_{p+q=w} (p,q)\text{-part}$$

where (p,q)-part means the intersection of $\mathrm{Fil}^p(C^\infty_X \otimes_\mathbb{Z} \mathcal{H}_\mathbb{Z})$ and the complex conjugate of $\mathrm{Fil}^q(C^\infty_X \otimes_\mathbb{Z} \mathcal{H}_\mathbb{Z})$. Theorem 2.6 states that a similar Hodge decomposition exists also for a VPLH if we replace C^∞-functions by log C^∞-functions.

2.2. Before we discuss VPLH, we review the theory of log Riemann-Hilbert correspondences studied in [18] and [14] (cf. Remark 2.4). Let X be a log smooth fs log analytic space. The log Riemann-Hilbert correspondence relates the following two categories $L_{\mathrm{qunip}}(X)$ and $V_{\mathrm{qnilp}}(X)$. Let $L_{\mathrm{qunip}}(X)$ be the category of locally constant sheaves L of finite dimensional $\mathbb{C}$-vector spaces on $X^{\log}$ such that for any $x \in X$ and $y \in \tau^{-1}(x) \subset X^{\log}$, the action of $\pi_1(\tau^{-1}(x))$ (called the local monodromy at x) on the stalk L_y is quasi-unipotent. On the other hand, let $V_{\mathrm{qnilp}}(X)$ be the category of $\mathcal{O}_X$-modules V on X_{ket} endowed with an integrable connection with log poles

$$\nabla\colon V \longrightarrow \omega^1_X \otimes_{\mathcal{O}_X} V$$

which satisfies the following condition locally on X_{ket} (*cf.* [14]).

There exists a finite family of $\mathcal{O}_X$-submodules $(V_i)_{0\leq i\leq n}$ of V satisfying $\nabla(V_i)\subset \omega_X^1\otimes_{\mathcal{O}_X} V_i$ such that

$$0=V_0\subset V_1\subset\cdots\subset V_n=V$$

and such that for each $1\leq i\leq n$, V_i/V_{i-1} is locally free and the connection induced on V_i/V_{i-1} does not have a pole.

Then we have an equivalence of categories

$$L_{\mathrm{qunip}}(X)\xrightarrow{\sim} V_{\mathrm{qnilp}}(X);\ L\mapsto \tau_*(\mathcal{O}_X^{\log}\otimes_{\mathbb{C}} L)$$

whose converse is given by

$$V\mapsto \mathrm{Ker}\,(\nabla\colon \mathcal{O}_X^{\log}\otimes_{\mathcal{O}_X} V\longrightarrow \omega_X^{1,\log}\otimes_{\mathcal{O}_X} V),$$

where $-\otimes_{\mathcal{O}_X} V=-\otimes_{\tau^{-1}(\mathcal{O}_X)}\tau^{-1}(V)$.

Furthermore, if $L\in L_{\mathrm{qunip}}(X)$ and $V=\tau_*(\mathcal{O}_X^{\log}\otimes_{\mathbb{C}} L)\in V_{\mathrm{qnilp}}(X)$, we have

$$\mathcal{O}_X^{\log}\otimes_{\mathbb{C}} L=\mathcal{O}_X^{\log}\otimes_{\mathcal{O}_X} V.$$

2.3. Now we introduce VPLH. See [19], [20] for generality of log Hodge structures and polarized log Hodge structures (cf. Remark 2.4).

First, we review the definition of VPH. For a complex manifold X and for $w\in\mathbb{Z}$, a VPH on X of weight w is a triple $(\mathcal{H}_{\mathbb{Z}}, F, (\ ,\ ))$ where
∘ $\mathcal{H}_{\mathbb{Z}}$ is a locally constant sheaf of finitely generated $\mathbb{Z}$-modules on X,
∘ F is a descending filtration $(F^p)_{p\in\mathbb{Z}}$ on $\mathcal{O}_X\otimes_{\mathbb{Z}}\mathcal{H}_{\mathbb{Z}}$ by $\mathcal{O}_X$-submodules such that

$$F^p=\mathcal{O}_X\otimes_{\mathbb{Z}}\mathcal{H}_{\mathbb{Z}}\ \text{for}\ p\ll 0,\ \ F^p=0\ \text{for}\ p\gg 0,$$

and each F^p is locally a direct summand of $\mathcal{O}_X\otimes_{\mathbb{Z}}\mathcal{H}_{\mathbb{Z}}$,
∘ $(\ ,\ )$ is a $\mathbb{Q}$-bilinear form $\mathcal{H}_{\mathbb{Q}}\times\mathcal{H}_{\mathbb{Q}}\longrightarrow\mathbb{Q}$,
satisfying the following conditions (1) and (2).
(1) For any $x\in X$, the triple $(\mathcal{H}_{\mathbb{Z},x},\ (\ ,\ )_x, F(x))$ is a polarized Hodge structure of weight w. Here $F(x)$ means the filtration $(\mathbb{C}\otimes_{\mathcal{O}_{X,x}} F_x^p)_{p\in\mathbb{Z}}$ on $\mathbb{C}\otimes_{\mathbb{Z}}\mathcal{H}_{\mathbb{Z},x}$ ($\mathcal{O}_{X,x}\longrightarrow\mathbb{C}$ is given by $f\mapsto f(x)$).
(2) (Griffiths transversality) The connection

$$\nabla=d\otimes 1\colon \mathcal{O}_X\otimes_{\mathbb{Z}}\mathcal{H}_{\mathbb{Z}}\longrightarrow\Omega_X^1\otimes_{\mathbb{Z}}\mathcal{H}_{\mathbb{Z}}$$

sends F^p into $\Omega_X^1\otimes_{\mathcal{O}_X} F^{p-1}$ for any $p\in\mathbb{Z}$.

Now let X be a log smooth fs log analytic space and let $w\in\mathbb{Z}$. A VPLH on X of weight w is a triple $(\mathcal{H}_{\mathbb{Z}},\ \mathcal{M},\ (\ ,\ ))$ where
∘ $\mathcal{H}_{\mathbb{Z}}$ is a locally constant sheaf on $X^{\log}$ of finitely generated $\mathbb{Z}$-modules with quasi-unipotent local monodromies.

$\circ$ $\mathcal{M}$ is the object of $V_{\mathrm{qnilp}}(X)$ corresponding to the object $\mathcal{H}_{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{Z}} \mathcal{H}_{\mathbb{Z}}$ of $L_{\mathrm{qunip}}(X)$, endowed with a descending filtration $(\mathcal{M}^p)_{p \in \mathbb{Z}}$ by $\mathcal{O}_X$-submodules such that

$$\mathcal{M}^p = \mathcal{M} \text{ for } p \ll 0, \ \mathcal{M}^p = 0 \text{ for } p \gg 0,$$

and each $\mathcal{M}^p$ is locally a direct summand of $\mathcal{M}$,
$\circ$ $(\ ,\ )$ is a $\mathbb{Q}$-bilinear form $\mathcal{H}_{\mathbb{Q}} \times \mathcal{H}_{\mathbb{Q}} \longrightarrow \mathbb{Q}$
satisfying the following conditions (1) and (2).
(1) Let $x \in X$, let y be a point of $X^{\log}$ lying over x, and let $\mathrm{sp}(y)$ be the set of all ring homomorphisms $\mathcal{O}_{X,y}^{\log} \longrightarrow \mathbb{C}$ whose restrictions to the subring $\mathcal{O}_{X,x}$ of $\mathcal{O}_{X,y}^{\log}$ coincide with the map $\mathcal{O}_{X,x} \longrightarrow \mathbb{C}$; $f \mapsto f(x)$. Then if $s \in \mathrm{sp}(y)$ and if the map $M_{X,x} \longrightarrow \mathbb{C}^{\times}$; $f \mapsto \exp(s(\log(f)))$ is sufficiently near to the canonical composition

$$M_{X,x} \xrightarrow{\alpha} \mathcal{O}_{X,x} \longrightarrow \mathbb{C}; \ f \mapsto \alpha(f)(x),$$

then $(\mathcal{H}_{\mathbb{Z},y}, \ \mathcal{M}(s), \ (\ ,\ )_y)$ is a polarized Hodge structure of weight w in the classical sense. Here $\log(f)$ is defined in $\mathcal{O}_{X,y}^{\log}/2\pi i\mathbb{Z}$ and $\exp(s(\log(f)))$ is well defined since $\exp(s(2\pi i\mathbb{Z})) = \exp(2\pi i\mathbb{Z}) = 1$, "sufficiently near " is with respect to the topology of simple convergence of the set $\mathrm{Map}(M_{X,x}, \ \mathbb{C})$, and $\mathcal{M}(s) = \mathbb{C} \otimes_{\mathcal{O}_{X,y}} \mathcal{M}_y$ endowed with the induced filtration. ($\mathcal{O}_{X,y}$ (resp. $\mathcal{M}_y$) is the stalk at y of the inverse image of $\mathcal{O}_X$ (resp. $\mathcal{M}$) on $X^{\log}$ by $X^{\log} \longrightarrow X_{\mathrm{ket}}$, $\mathcal{O}_{X,y} \longrightarrow \mathbb{C}$ is $f \mapsto f(y)$, and we identify $\mathcal{M}(s)$ with $\mathbb{C} \otimes_{\mathbb{Z}} \mathcal{H}_{\mathbb{Z},y}$ by

$$\mathcal{M}(s) = \mathbb{C} \otimes_{\mathcal{O}_{X,y}^{\log}} (\mathcal{O}_{X,y}^{\log} \otimes_{\mathcal{O}_{X,y}} \mathcal{M}_y) = \mathbb{C} \otimes_{\mathbb{Z}} \mathcal{H}_{\mathbb{Z},y}$$

where $\mathcal{O}_{X,y}^{\log} \longrightarrow \mathbb{C}$ is s.)
(2) (Griffiths transversality)

$$\nabla(\mathcal{M}^p) \subset \omega_X^1 \otimes_{\mathcal{O}_X} \mathcal{M}^{p-1} \text{ for any } p \in \mathbb{Z}.$$

Sometimes we denote by $(\mathcal{H}_{\mathbb{Z}}, (\mathcal{M}^p)_{p \in \mathbb{Z}}, (\ ,\ ))$ for $(\mathcal{H}_{\mathbb{Z}}, \mathcal{M}, (\ ,\ ))$.

Remark 2.4. In the above 2.2 and 2.3, we work on the ket site. Working on the usual site (of open sets of X) instead, we have the non-ket analogues of 2.2 and 2.3: First, replacing X_{ket} with X (the usual site) in 2.2, we have the definition of the non-ket analogue $V_{\mathrm{nilp}}(X)$ of $V_{\mathrm{qnilp}}(X)$. Then we have the non-ket version of the log Riemann-Hilbert correspondence $L_{\mathrm{unip}}(X) \xrightarrow{\sim} V_{\mathrm{nilp}}(X)$, where $L_{\mathrm{unip}}(X) := \{L \in L_{\mathrm{qunip}}(X) \ ;$ the local monodromies of L are unipotent$\}$. See [18] for the details. Next, replacing X_{ket} with X in 2.3, we have the definition of the non-ket version of VPLH, which is called VPLH in [20].

These non-ket versions relate to ours as follows: First, the functor ι from the category of locally free $\mathcal{O}_X$-modules of finite rank on X ($\mathcal{O}_X$ here is in the non-ket sense) to that of locally free $\mathcal{O}_X$-modules of finite rank on X_{ket} is fully faithful and it induces the categorical equivalence between $V_{\text{nilp}}(X)$ and the full subcategory of $V_{\text{qnilp}}(X)$ consisting of the objects whose "V" belong to the essential image of ι. Further ι induces the equivalence between the category of VPLH in the non-ket sense and the full subcategory of that of VPLH in our sense consisting of the objects whose "$\mathcal{H}_{\mathbb{C}}$" belong to $L_{\text{unip}}(X)$.

The following Proposition 2.5 is a reformulation of the nilpotent orbit theorem of Schmid ([26]).

Proposition 2.5. *Let X be a log smooth fs log analytic space and let $w \in \mathbb{Z}$. Then the restriction to X_{triv} induces an equivalence of categories*

$$\{\textit{VPLH on } X \textit{ of weight } w\} \xrightarrow{\sim} \{\textit{VPH on } X_{\text{triv}} \textit{ of weight } w\}.$$

We show in 2.7–2.9 how Proposition 2.5 is deduced from the nilpotent orbit theorem of Schmid.

See [20] for more details about the relation between nilpotent orbits and polarized log Hodge structures on more general fs log analytic spaces X.

The aim of this section is to prove

Theorem 2.6. *Let X be a log smooth fs log analytic space, let $w \in \mathbb{Z}$, and let $(\mathcal{H}_{\mathbb{Z}}, \mathcal{M}, (\ ,\ ))$ be a VPLH on X of weight w. Then we have*

$$\mathcal{A}_X \otimes_{\mathcal{O}_X} \mathcal{M} = \bigoplus_{p+q=w} \mathcal{M}_{\mathcal{A}}^{p,q}$$

where $\mathcal{M}_{\mathcal{A}}^{p,q}$ is the intersection of $\mathcal{M}_{\mathcal{A}}^{p} = \mathcal{A}_X \otimes_{\mathcal{O}_X} \mathcal{M}^p$ and the complex conjugate of $\mathcal{M}_{\mathcal{A}}^{q}$.

Here the complex conjugation on $\mathcal{A}_X \otimes_{\mathcal{O}_X} \mathcal{M}$ is the one induced by

$$(\text{complex conjugation}) \otimes 1 \text{ on } \mathcal{A}_X^{\log} \otimes_{\mathbb{Z}} \mathcal{H}_{\mathbb{Z}},$$

via the identification

$$\mathcal{A}_X^{\log} \otimes_{\mathbb{Z}} \mathcal{H}_{\mathbb{Z}} = \mathcal{A}_X^{\log} \otimes_{\mathcal{O}_X} \mathcal{M}.$$

2.7. We prove Proposition 2.5 in 2.7–2.9. In there we fix $w \in \mathbb{Z}$ and VPH (resp. VPLH) means VPH (resp. VPLH) of weight w. The fully faithfulness of the restriction functor

$$\{\text{VPLH on } X\} \longrightarrow \{\text{VPH on } X_{\text{triv}}\}$$

is easily seen. Hence it is sufficient to show that a VPH $(\mathcal{H}_{\mathbb{Z}}, F, (\ ,\ ))$ on X_{triv} extends to a VPLH on X. Since $X^{\log}$ is a topological manifold with the boundary $X^{\log} - X_{\mathrm{triv}}$, $\mathcal{H}_{\mathbb{Z}}$ extends uniquely to $X^{\log}$ as a locally constant sheaf, and $(\ ,\ )$ extends also on $X^{\log}$. Denote this extension of $(\mathcal{H}_{\mathbb{Z}}, (\ ,\ ))$ on $X^{\log}$ also by $(\mathcal{H}_{\mathbb{Z}}, (\ ,\ ))$. Since the local monodromy of $\mathcal{H}_{\mathbb{Z}}$ at each point of X is quasi-unipotent by a theorem of Borel [26, 4.5], $\mathcal{H}_{\mathbb{C}}$ is an object of $L_{\mathrm{qunip}}(X)$. Let $\mathcal{M}$ be the object of $V_{\mathrm{qnilp}}(X)$ corresponding to $\mathcal{H}_{\mathbb{C}}$. It remains to show that F extends to a filtration on $\mathcal{M}$ and $(\mathcal{H}_{\mathbb{Z}}, \mathcal{M}, (\ ,\ ))$ satisfies the conditions (1) (2) of VPLH. We prove this in 2.8 in the case where X is a complex manifold and the log structure of X is given by a divisor with normal crossings, and in 2.9 in general.

2.8. Assume that X is a complex manifold whose log structure is given by a divisor with normal crossings. We may assume $X = \Delta^{n+m}$ with the log structure given by the divisor which is the complement of $(\Delta^*)^n \times \Delta^m$. Assume that we are given a VPH $(\mathcal{H}_{\mathbb{Z}}, F, (\ ,\ ))$ on $X_{\mathrm{triv}} = (\Delta^*)^n \times \Delta^m$. We show that it extends to a VPLH on X. As is explained in 2.7, $(\mathcal{H}_{\mathbb{Z}}, (\ ,\ ))$ is extended to $X^{\log}$ and we have an $\mathcal{O}_X$-module $\mathcal{M}$ on X_{ket}. We may assume that the local monodromies of $(\mathcal{H}_{\mathbb{Z}}, F, (\ ,\ ))$ are unipotent. Let

$$U = \text{ the upper half plane } = \{x + yi\ ;\ x,\ y \in \mathbb{R},\ y > 0\},$$
$$\overline{U} = \{x + yi\ ;\ x \in \mathbb{R},\ 0 < y \leq \infty\}.$$

Then we have a commutative diagram

$$\begin{array}{ccccc}
 & & \overline{U}^n \times \Delta^m & \longleftarrow & U^n \times \Delta^m \\
 & & \downarrow & & \downarrow \\
X^{\log} & \cong & (\mathbb{Z}\backslash\overline{U})^n \times \Delta^m & \longleftarrow & (\mathbb{Z}\backslash U)^n \times \Delta^m \\
\tau \downarrow & & (1) \downarrow & & (2) \downarrow \cong \\
X & = & \Delta^{n+m} & \longleftarrow & (\Delta^*)^n \times \Delta^m
\end{array}$$

where $\mathbb{Z}\backslash *$ $(* = U,\ \overline{U})$ means the quotient by the action $z \mapsto z + n$ $(n \in \mathbb{Z})$ of the group $\mathbb{Z}$, (2) is the isomorphism induced by $U \longrightarrow \Delta^*;\ z \mapsto \exp(2\pi i z)$, and (1) is the unique continuous extension of (2). The group $\pi_1(X^{\log}) \cong \pi_1(X_{\mathrm{triv}}) \cong \mathbb{Z}^n$ acts on the stalks of $\mathcal{H}_{\mathbb{Z}}$, and since $\pi_1(X^{\log})$ is commutative, we have a unique action of $\pi_1(X^{\log})$ on $\mathcal{H}_{\mathbb{Z}}$ which induces the original action of $\pi_1(X^{\log})$ on each stalk of $\mathcal{H}_{\mathbb{Z}}$. Let $\gamma_j \in \pi_1((\Delta^*)^n \times \Delta^m)$ $(1 \leq j \leq n)$ be the loop in the j-th Δ^*

around 0 in the clockwise direction, and let N_j be the logarithm of the action of γ_j on $\mathcal{H}_{\mathbb{Z}}$ which is unipotent. It can be shown easily that the inverse image of $\mathcal{M}$ on $\overline{U}^n \times \Delta^m$ is equal to $\mathcal{O}_X \otimes_{\mathbb{Z}} \exp(\sum_{j=1}^n z_j N_j)\mathcal{H}_{\mathbb{Z}}$ in the inverse image of $\mathcal{O}_X^{\log} \otimes_{\mathbb{Z}} \mathcal{H}_{\mathbb{Z}}$ on $\overline{U}^n \times \Delta^m$, where z_j denotes the coordinate function of the j-th U, and regarding z_j as $(2\pi i)^{-1}$ times a logarithm of the coordinate function of the j-th Δ, we regard z_j as a global section of the inverse image of $\mathcal{O}_X^{\log}$ on $\overline{U}^n \times \Delta^m$.

Since $\overline{U}^n \times \Delta^m$ is contractible, the inverse image of $\mathcal{H}_{\mathbb{Z}}$ on $\overline{U}^n \times \Delta^m$ is a constant sheaf. By regarding the inverse image of $\mathcal{H}_{\mathbb{C}}$ on $\overline{U}^n \times \Delta^m$ as a constant $\mathbb{C}$-vector space, let $\overline{D}$ be the set of all descending filtrations $(f^p)_{p\in\mathbb{Z}}$ on this $\mathbb{C}$-vector space and let D be the subset of $\overline{D}$ consisting of $(f^p)_{p\in\mathbb{Z}}$ for which $(\mathcal{H}_{\mathbb{Z}},\ f,\ (\ ,\ ))$ is a PH of weight w (D is a classifying space of polarized Hodge structures of Griffiths). Let

$$\tilde{\phi}\colon U^n \times \Delta^m \longrightarrow D$$

be the map defined by the filtration F. Then by Schmid [26, Section 4], the map

$$U^n \times \Delta^m \longrightarrow \overline{D};\ (z,w) \mapsto \exp(-\sum_{j=1}^n z_j N_j)\tilde{\phi}(z,w)$$

descends to a holomorphic map $\psi\colon (\Delta^*)^n \times \Delta^m \longrightarrow \overline{D}$ and furthermore ψ extends to a holomorphic map $\Delta^{n+m} \longrightarrow \overline{D}$. This implies that the filtration $\exp(-\sum_{j=1}^n z_j N_j)F$ on the inverse image of $\mathcal{O}_{X_{\mathrm{triv}}} \otimes_{\mathbb{Z}} \mathcal{H}_{\mathbb{Z}}$ on $U^n \times \Delta^m$ extends to a filtration F' of $\mathcal{O}_X \otimes_{\mathbb{Z}} \mathcal{H}_{\mathbb{Z}}$ on $\overline{U}^n \times \Delta^m$ by $\mathcal{O}_X$-submodules which are locally direct summands of $\mathcal{O}_X \otimes_{\mathbb{Z}} \mathcal{H}_{\mathbb{Z}}$, and that there is a filtration $(\mathcal{M}^p)_{p\in\mathbb{Z}}$ of $\mathcal{M}$ by $\mathcal{O}_X$-submodules such that the inverse image of $\mathcal{M}^p$ on $\overline{U}^n \times \Delta^m$ is equal to $\exp(\sum_{j=1}^n z_j N_j)F'$. These $\mathcal{M}^p$ are locally direct summands of $\mathcal{M}$. We show that $(\mathcal{H}_{\mathbb{Z}},\ \mathcal{M},\ (\ ,\ ))$ satisfies the condition (1) (2) of VPLH. The Griffiths transversality (2) is checked on X_{triv}. For (1), it is enough to check this at $0 \in \Delta^{n+m}$. Let $a \in X^{\log}$ lie over $0 \in \Delta^{n+m}$, let $s \in \mathrm{sp}(a)$, let b be a lifting of a to $\overline{U}^n \times \Delta^m$, and let $s(z_j) \in \mathbb{C}$ be the image of $z_j \in \mathcal{O}_{X,b}^{\log} = \mathcal{O}_{X,a}^{\log}$ by s. Then the filtration of $\mathcal{M}(s)$ is identified with $\exp(\sum_{j=1}^n s(z_j)N_j)\psi(0)$. Since

$$\mathbb{C}^n \longrightarrow \overline{D};\ (z_j)_{1\le j\le n} \mapsto \exp(\sum_{j=1}^n z_j N_j)\psi(0)$$

is a nilpotent orbit ([26, 4.12], [3, 1.15]), the condition (1) of VPLH is satisfied.

2.9. We prove Proposition 2.5 in general. We may assume that X is an open subspace of the toric variety $(\mathrm{Spec}\,\mathbb{C}[\mathcal{S}])^{\mathrm{an}}$ where $\mathcal{S}$ is a torsion free fs monoid and the log structure of X is given by the divisor which is the complement of $X \cap (\mathrm{Spec}\,\mathbb{C}[\mathcal{S}^{\mathrm{gp}}])^{\mathrm{an}}$. Here $\mathcal{S}^{\mathrm{gp}}$ is the group $\{ts^{-1}\ ;\ t,\ s \in \mathcal{S}\}$ associated to $\mathcal{S}$.

We recall some facts about toric geometry ([22]). Let $\mathbb{Q}_{\geq 0} = \{a \in \mathbb{Q}\ ;\ a \geq 0\}$ regarded as an additive monoid. For a finitely generated $\mathbb{Q}$-cone σ in $\mathrm{Hom}\,(\mathcal{S}, \mathbb{Q}_{\geq 0})$ (i.e., a subset of $\mathrm{Hom}\,(\mathcal{S}, \mathbb{Q}_{\geq 0})$ having the form $\{a_1 h_1 + \cdots + a_r h_r\ ;\ a_j \in \mathbb{Q}_{\geq 0}\}$ for some elements $h_1, \ldots, h_r$ of $\mathrm{Hom}\,(\mathcal{S}, \mathbb{Q}_{\geq 0})$), we have a log smooth fs log analytic space $X_\sigma = X \times_{(\mathrm{Spec}\,\mathbb{C}[\mathcal{S}])^{\mathrm{an}}} (\mathrm{Spec}\,\mathbb{C}[\mathcal{S}_\sigma])^{\mathrm{an}}$ where $\mathcal{S}_\sigma = \{t \in \mathcal{S}^{\mathrm{gp}}\ ;\ h(t) \geq 0 \text{ for all } h \in \sigma\}$. The canonical morphism $f_\sigma \colon X_\sigma \longrightarrow X$ induces an isomorphism $X_\sigma \times_X X_{\mathrm{triv}} \xrightarrow{\sim} X_{\mathrm{triv}}$. If λ is a finite polyhedral cone decomposition of $\mathrm{Hom}\,(\mathcal{S}, \mathbb{Q}_{\geq 0})$, we have a log smooth fs log analytic space $X_\lambda = \cup_{\sigma \in \lambda} X_\sigma$ (open covering) with a proper surjective map $f_\lambda \colon X_\lambda \longrightarrow X$ which induces $X_\lambda \times_X X_{\mathrm{triv}} \xrightarrow{\sim} X_{\mathrm{triv}}$. If λ' is a subdivision of λ, we have a unique morphism $X_{\lambda'} \longrightarrow X_\lambda$ over X.

We endow X_σ and X_λ with the log structures corresponding to the divisors which are the complements of X_{triv}.

Assume that we are given a VPH $(\mathcal{H}_{\mathbb{Z}},\ F,\ (\ ,\ ))$ on X_{triv}.

Claim 2.9.1. *If σ is a simplicial $\mathbb{Q}$-cone (that is, σ is a $\mathbb{Q}$-cone generated by $\dim(\sigma)$ elements), $(\mathcal{H}_{\mathbb{Z}},\ F,\ (\ ,\ ))$ extends to a VPLH on X_σ.*

In fact, there is a finite Galois Kummer log étale covering X'_σ of X_σ such that X'_σ is smooth and such that the reduced part of the complement of the inverse image of X_{triv} in X'_σ is a normal crossing divisor. By 2.8, $(\mathcal{H}_{\mathbb{Z}},\ F,\ (\ ,\ ))$ extends to a VPLH on X'_σ, and by Galois descent, we see that $(\mathcal{H}_{\mathbb{Z}},\ F, (\ ,\ ))$ extends to a VPLH on X_σ.

As in 2.7, we can extend $(\mathcal{H}_{\mathbb{Z}},\ (\ ,\ ))$ to $X^{\log}$ and we have the $\mathcal{O}_X$-module $\mathcal{M}$ on X_{ket}. Let $\overline{D}(\mathcal{M}) \longrightarrow X$ be the space classifying descending filtrations $(\mathcal{F}^p)_{p \in \mathbb{Z}}$ on $\mathcal{M}$ such that all $\mathcal{F}^p$ are locally direct summands of $\mathcal{M}$. ($\overline{D}(\mathcal{M})$ is a (finite disjoint union of) flag manifold bundle(s) over X.) By 2.9.1, for a simplicial $\mathbb{Q}$-cone in $\mathrm{Hom}\,(\mathcal{S}, \mathbb{Q}_{\geq 0})$, the Hodge filtration of the extension of $(\mathcal{H}_{\mathbb{Z}},\ F,\ (\ ,\ ))$ to X_σ defines a morphism $\mu_\sigma \colon X_\sigma \longrightarrow \overline{D}(\mathcal{M})$ over X. Take $\lambda \in \Lambda$ such that for any $\sigma \in \lambda$, $\mathcal{S}_\sigma \cong \mathbb{N}^n \times \mathbb{Z}^m$ for some $m,\ n \geq 0$ (such λ exists by [22] I, Theorem 11). Let $\mu_\lambda \colon X_\lambda \longrightarrow \overline{D}(\mathcal{M})$ be the union of μ_σ $(\sigma \in \lambda)$.

Claim 2.9.2. *μ_λ descends to a section $X \longrightarrow \overline{D}(\mathcal{M})$ of $\overline{D}(\mathcal{M})$.*

If we prove 2.9.2, we have a filtration $(\mathcal{M}^p)_{p \in \mathbb{Z}}$ on $\mathcal{M}$ extending F on X_{triv} such that $\mathcal{M}^p$ are locally direct summands of $\mathcal{M}$. We can then

prove that with this filtration of $\mathcal{M}$, $(\mathcal{H}_{\mathbb{Z}},\ \mathcal{M},\ (\ ,\ ))$ is a VPLH on X. In fact, Griffiths transversality is checked on X_{triv}, and the condition (1) of VPLH follows from the nilpotent orbit theorem of Schmid [26, 4.12] applied to the manifold X_λ.

We prove 2.9.2. It is sufficient to show that $\mu_\lambda(y) = \mu_\lambda(z)$ for any $y,\ z \in X_\lambda$ whose images in X coincide. Fix $x \in X$ and let $X_\lambda(x) = f_\lambda^{-1}(x) \subset X_\lambda$. Let

$$\mathcal{S}_x = \{a \in \mathcal{S}^{\mathrm{gp}}\ ;\ a \in \mathcal{O}_{X,x}\},$$
$$\mathcal{S}_y = \{a \in \mathcal{S}^{\mathrm{gp}}\ ;\ a \in \mathcal{O}_{X_\lambda,y}\} \quad \text{for } y \in X_\lambda(x).$$

Then

$$\mathcal{S} \subset \mathcal{S}_x \subset \mathcal{S}_y \subset \mathcal{S}^{\mathrm{gp}}.$$

For $p = x$ or for $p \in X_\lambda(x)$, let $C(p) = \mathrm{Hom}\,(\mathcal{S}_p, \mathbb{Q}_{\geq 0})$ and regard $C(p)$ as a $\mathbb{Q}$-cone in $\mathrm{Hom}\,(\mathcal{S}^{\mathrm{gp}}, \mathbb{Q})$. Then for $y \in X_\lambda(x)$, $C(y) \subset C(x)$ and the interior $\{h \in C(y)\ ;\ \mathrm{Ker}\,(h\colon \mathcal{S}_y \longrightarrow \mathbb{Q}_{\geq 0}) = (\mathcal{S}_y)^\times\}$ of $C(y)$ is contained in the interior of $C(x)$.

To prove $\mu_\lambda(y_1) = \mu_\lambda(y_2)$ for any $y_1,\ y_2 \in X_\lambda(x)$, it is sufficient to consider the case $C(y_1)$ is a face of $C(y_2)$ (this is because any two points of $X_\lambda(x)$ are connected by a chain of this relation). Let h_1 be an element of the interior of $C(y_1)$. Since h_1 belongs to the topological closure of the interior of $C(y_2)$, by taking a point h_2 of the interior of $C(y_2)$ which is sufficiently near to h_1, we can find a simplicial $\mathbb{Q}$-cone σ in $C(x)$ such that both h_1 and h_2 are contained in the interior of σ and such that $\dim(\sigma) = \dim(C(x))$. Fix such h_1, h_2 and σ.

Take a finite polyhedral cone decomposition λ' of σ such that the corresponding proper birational $X_{\lambda'} \longrightarrow X_\sigma$ has a morphism $X_{\lambda'} \longrightarrow X_\lambda$ over X. The composite maps $X_{\lambda'} \longrightarrow X_\sigma \xrightarrow{\mu_\sigma} \overline{D}(\mathcal{M})$ and $X_{\lambda'} \longrightarrow X_\lambda \xrightarrow{\mu_\lambda} \overline{D}(\mathcal{M})$ coincide because they coincide on X_{triv}. Hence it is sufficient to show that there are elements y_j' of $X_{\lambda'}$ for $j = 1,\ 2$ such that the image of y_j' in X_λ is y_j for $j = 1,\ 2$ and such that the images of y_j' in X_σ coincide.

For $j = 1,\ 2$, let $K_j = \mathrm{Ker}\,(h_j\colon \mathcal{S}^{\mathrm{gp}} \longrightarrow \mathbb{Q})$. Then $K_j \supset (\mathcal{S}_{y_j})^\times$. Extend the homomorphism $(\mathcal{S}_{y_j})^\times \longrightarrow \mathbb{C}^\times;\ f \mapsto f(y_j)$ to a homomorphism $s_j\colon K_j \longrightarrow \mathbb{C}^\times$. For $j = 1,\ 2$, take $\sigma_j \in \lambda'$ such that $h_j \in \sigma_j$ and let y_j' be the point of $X_{\sigma_j} \subset (\mathrm{Spec}\,\mathbb{C}[\mathcal{S}_{\sigma_j}])^{\mathrm{an}}$ characterized by the following property. For $t \in \mathcal{S}_{\sigma_j}$, $t(y_j') = s_j(t)$ if $t \in K_j$ and $t(y_j') = 0$ otherwise. Then the image of y_j' in X_λ coincides with y_j. By $\dim(\sigma) = \dim(C(x))$, we have $(\mathcal{S}_\sigma)^\times = (\mathcal{S}_x)^\times$, and we have $K_j \cap \mathcal{S}_\sigma = (\mathcal{S}_\sigma)^\times$ since h_j is in the interior of σ. Hence for $j = 1,\ 2$ and for $t \in \mathcal{S}_\sigma$, $t(y_j') = t(x)$ if

$t \in (\mathcal{S}_\sigma)^\times$ and $t(y'_j) = 0$ otherwise. Hence the images of y'_1 and y'_2 in X_σ coincide. This completes the proof of Proposition 2.5.

The following proposition is useful in the proof of Theorem 2.6 and also in other places in this paper.

Proposition 2.10 (Cf. [11], 3.8.2). *Let X be a log smooth fs log analytic space, and let $f\colon Z \longrightarrow X$ be a blowing up along log structure. Then*

$$f_*(\mathcal{A}_Z) = \mathcal{A}_X.$$

Proof. It is easy to see that the equality $X_{\mathrm{triv}} = Z_{\mathrm{triv}}$ induces the bijection between the set of functions of log growth on X and that for Z. On the other hand, $\omega^1_Z = \mathcal{O}_Z \otimes_{\mathcal{O}_X} \omega^1_X$ since f is log étale. These imply the desired equality. Q.E.D.

2.11. By 2.10, we can reduce the proof of Theorem 2.6 to the case where X is a manifold and the log structure of X is given by a divisor with normal crossings.

Lemma 2.12. *Let X be a log smooth fs log analytic space and let $f \in \Gamma(X, \mathcal{A}_X)$. Assume that f does not have zero on X_{triv} and that the function $\frac{1}{f}$ on X_{triv} is of log growth on X. Then $\frac{1}{f} \in \Gamma(X, \mathcal{A}_X)$.*

Proof. We may assume that X is an open subspace of $(\operatorname{Spec} \mathbb{C}[\mathcal{S}])^{\mathrm{an}}$ for a torsion free fs monoid $\mathcal{S}$ and the log structure is given by the divisor which is the complement of $X \cap (\operatorname{Spec} \mathbb{C}[\mathcal{S}^{\mathrm{gp}}])^{\mathrm{an}}$. Let $(t_j)_{j \in J}$ be a $\mathbb{Z}$-basis of $\mathcal{S}^{\mathrm{gp}}$ and let

$$\Theta = \{t_j \frac{\partial}{\partial t_j},\ \overline{t_j} \frac{\partial}{\partial \overline{t_j}}\ ;\ j \in J\}.$$

Then 2.12 is reduced to

Claim 2.12.1. *For any $\partial_1, \ldots, \partial_k \in \Theta$, $\partial_1 \cdots \partial_k(\frac{1}{f})$ is contained in the ring generated over $\mathbb{Z}$ by $\{\frac{1}{f},\ \delta_1 \cdots \delta_l(f)\ ;\ l \geq 0,\ \delta_1, \ldots, \delta_l \in \Theta\}$.*

This 2.12.1 is deduced from $\partial(\frac{1}{f}) = f^{-2}\partial(f)$ $(\partial \in \Theta)$ by induction on k.

2.13. We prove Theorem 2.6.

Assume that X is a complex manifold and the log structure of X is given by a divisor with normal crossings. Let $(\mathcal{H}_{\mathbb{Z}},\ \mathcal{M},\ (\ ,\ ))$ be a VPLH on X. We may assume that the local monodromies of $\mathcal{H}_{\mathbb{Q}}$ are unipotent.

It is sufficient to show that the map

$$\mathcal{A}_X \otimes_{\mathcal{O}_X} \mathcal{M}^p \oplus \mathcal{A}_X \otimes_{\mathcal{O}_X} \mathcal{M}^{w+1-p} \longrightarrow \mathcal{A}_X \otimes_{\mathcal{O}_X} \mathcal{M};\ (f,g) \mapsto f + \overline{g}$$

is an isomorphism for any $p \in \mathbb{Z}$. Locally on X, take an $\mathcal{O}_X$-basis $(e_j)_j$ of $\mathcal{M}^p$, an $\mathcal{O}_X$-basis $(e'_k)_k$ of $\mathcal{M}^{w+1-p}$, and an $\mathcal{O}_X$-basis $(e''_l)_l$ of $\mathcal{M}$, and let φ be the matrix which expresses the pair $((e_j)_j, \overline{(e'_k)_k})$ by $(e''_l)_l$. Then $\det(\varphi) \in \mathcal{A}_X$ and $\det(\varphi)$ does not have zero on X_{triv}. It is sufficient to prove $\det(\varphi)^{-1} \in \mathcal{A}_X$. By 2.12, it is enough to show that $\det(\varphi)^{-1}$ is of log growth. Hence it is enough to prove

Claim 2.13.1. *For $p,\ q \in \mathbb{Z}$ such that $p+q=w$, the projector*

$$C^\infty_{X_{\mathrm{triv}}} \otimes_{\mathbb{Z}} \mathcal{H}_{\mathbb{Z}} \longrightarrow (p,q)\text{-part}$$

of the Hodge decomposition on X_{triv} is of log growth on X, that is, in $j_* C^\infty_{X_{\mathrm{triv}}} \otimes_{\mathcal{O}_X} \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{M})$ (j denotes the inclusion $j\colon X_{\mathrm{triv}} \longrightarrow X$), the projector belongs to $A' \otimes_{\mathcal{O}_X} \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{M})$ where A' is the subsheaf of $j_* C^\infty_{X_{\mathrm{triv}}}$ consisting of functions of log growth.

In the following proof of 2.13.1, we use the arguments in section 5 of [3] which were used for the estimate of the Hodge metric of a degenerating VPH ([3, Theorem 5.21], [15]).

We may assume $X = \Delta^{n+m}$ and the log structure of X is given by the divisor which is the complement of $(\Delta^*)^n \times \Delta^m$. As in [3], for a subset I of $\{1,\dots,n\}$ containing n, and for $K > 1$, let

$$(\mathbb{R}^n_{>0})^I_K \subset \mathbb{R}^n_{>0}, \qquad ((\Delta^*)^n)^I_K \subset (\Delta^*)^n$$

($\mathbb{R}_{>0} = \{r \in \mathbb{R}\ ;\ r > 0\}$) be as follows. Write $I = \{i_\alpha\ ;\ 1 \le \alpha \le r\}$, $i_\alpha < i_\beta$ if $\alpha < \beta$. Let

$$\begin{aligned}(\mathbb{R}^n_{>0})^I_K = \{y = (y_j)_j \in \mathbb{R}^n_{>0}\ ;\ & y_{i_\alpha}/y_{i_{\alpha+1}} > K\ (1 \le \alpha \le r, y_{i_{r+1}} \text{ means} \\ & 1),\ K^{-1} \le y_j/y_{i_\alpha} \le K \text{ for any } \alpha\ (1 \le \alpha \le r) \text{ and} \\ & j \text{ such that } i_{\alpha-1} < j < i_\alpha\ (i_0 \text{ means } 0)\},\end{aligned}$$

$$((\Delta^*)^n)^I_K = \{(t_j)_j \in (\Delta^*)^n\ ;\ (-(2\pi)^{-1}\log|t_j|)_{1\le j\le n} \in (\mathbb{R}^n_{>0})^I_K\}.$$

Then, when σ ranges over all permutations on the set $\{1,\dots,n\}$ and I ranges over all subsets of $\{1,\dots,n\}$ containing n, the union $\cup_{\sigma,I}\sigma((\Delta^*)^n)^I_K$ contains a set of the form $V \cap (\Delta^*)^n$ for some neighborhood V of 0 in Δ^n ([3, 5.7]). Hence Claim 2.13.1 is reduced to

Claim 2.13.2. *Fix a subset I of $\{1,\dots,n\}$ containing n. Then if $K > 1$ is sufficiently large, the projectors of the Hodge decomposition*

on X_{triv} *are of log growth at* $0 \in \Delta^{n+m}$ *when they are restricted to* $((\Delta^*)^n)^I_K \times \Delta^m$.

We prove a more precise

Claim 2.13.3. *Fix a subset* I *of* $\{1, \ldots, n\}$ *containing* n. *For* $K > 1$, *let* B^I_K *be the subring of* $(j_* C^\infty_{X_{\mathrm{triv}}})_0$ *consisting of elements which are bounded on* $V \cap (((\Delta^*)^n)^I_K \times \Delta^m)$ *for some open neighborhood* V *of* 0 *in* Δ^{n+m}. *Then, if* $K > 1$ *is sufficiently large, the projectors of the Hodge decomposition on* X_{triv} *are contained in the subring*

$$B^I_K[y_1, \ldots, y_n] \otimes_{\mathcal{O}_{X,0}} \mathrm{End}_{\mathcal{O}_{X,0}}(\mathcal{M}_0)$$

of

$$(j_* C^\infty_{X_{\mathrm{triv}}})_0 \otimes_{\mathcal{O}_{X,0}} \mathrm{End}_{\mathcal{O}_{X,0}}(\mathcal{M}_0)$$

where y_j *are defined by* $z_j = x_j + iy_j$ *with* x_j, y_j *real.*

(If t_j denotes the coordinate function of the j-th Δ, $y_j = -(2\pi)^{-1} \log (|t_j|)$ and it is of log growth.)

Let D, $\overline{D}$, $(N_j)_{1 \le j \le n}$, $\tilde{\phi}\colon U^n \times \Delta^m \longrightarrow D$ and $\psi\colon \Delta^{n+m} \longrightarrow \overline{D}$ be as in 2.8. By regarding the inverse image of $\mathcal{H}_{\mathbb{R}}$ on $U^n \times \Delta^m$ as a constant finite dimensional $\mathbb{R}$-vector space, let $G_{\mathbb{R}}$ be the group of all automorphisms of this $\mathbb{R}$-vector space preserving $(\ ,\ )$. Let $I' := \{1, \ldots, n\} - I$. Let

$$S = \{(u, w)\ ;\ u = (u_j)_{j \in I'},\ u_j \in \mathbb{R}_{>0},\ w \in \Delta^m\}.$$

For $(u, w) \in S$, the pair

$$\left(((\sum_{i_{\alpha-1} < j < i_\alpha} u_j N_j) + N_{i_\alpha})_{1 \le \alpha \le r},\ \psi(0, w) \right)$$

yields a nilpotent orbit, and hence by the theory of $SL(2)$-orbits ([3]), this pair defines a homomorphism

$$\rho_{u,w}\ :\ SL(2, \mathbb{R})^r \longrightarrow G_{\mathbb{R}}$$

of algebraic groups over $\mathbb{R}$. This homomorphism $\rho_{u,w}$ depends real analytically on $(u, w) \in S$. For $a_1, \ldots, a_r \in \mathbb{R}_{>0}$, let

$$t(a_1, \ldots, a_r) = \left(\begin{pmatrix} a_1 & 0 \\ 0 & \frac{1}{a_1} \end{pmatrix}, \ldots, \begin{pmatrix} a_r & 0 \\ 0 & \frac{1}{a_r} \end{pmatrix} \right) \in SL(2, \mathbb{R})^r.$$

By [3, Proposition 5.10], there exists $K_0 > 1$ such that for any $K > K_0$, we can find a compact set C of D and a neighborhood V of 0 in Δ^{n+m} such that

$$\rho_{u(y),w}(t(y_{i_1}^{-1/2}, \ldots, y_{i_r}^{-1/2})) \exp(-\sum_{j=1}^{n} x_j N_j)\tilde{\phi}(z,w) \in C$$

for all $(z,w) \in U^n \times \Delta^m$ such that $(\exp(2\pi i z), w) \in V \cap (((\Delta^*)^n)_K^I \times \Delta^m)$, where $z_j = x_j + iy_j$ with x_j, y_j real and $u(y) = (y_j/y_{i_\alpha})_{j \in I'}$, $i_{\alpha-1} < j < i_\alpha$. This proves 2.13.3.

Example 2.14. We describe an example of the log C^∞ Hodge decomposition, for the VPLH arising from a family $f\colon E \longrightarrow \Delta$ of elliptic curves on Δ^* degenerating at $0 \in \Delta$. Let

$$E = \{(u,v) \in \mathbb{C}^2 \ ; \ |uv| < 1\}/\sim$$

where $\sim$ is the equivalence relation defined as follows: $(u,v) \sim (u',v')$ if and only if either one of the following (1) (2) is satisfied.
(1) $uv = u'v' \neq 0$ and if we denote uv $(= u'v')$ by t, $u' = ut^n$ and $v' = vt^{-n}$ for some $n \in \mathbb{Z}$.
(2) $(u,v) = (c,0)$ and $(u',v') = (0,1/c)$ for some $c \in \mathbb{C}^\times$, or $(u,v) = (0,1/c)$ and $(u',v') = (c,0)$ for some $c \in \mathbb{C}^\times$, or $(u,v) = (u',v')$.

Then E is a complex manifold. Let $f\colon E \longrightarrow \Delta$ be the holomorphic map $(u,v) \mapsto uv$. Then for $t \in \Delta^*$, $f^{-1}(t)$ is identified with the elliptic curve $\mathbb{C}^\times/t^{\mathbb{Z}}$ where we identify the coordinate u on E with the coordinate of $\mathbb{C}^\times$, and $f^{-1}(0)$ is identified with the singular space obtained from $\mathbb{P}^1(\mathbb{C})$ by identifying 0 and ∞. We endow Δ with the log structure corresponding to the divisor $\{0\}$, and E with the log structure corresponding to the divisor $f^{-1}(0)$ with normal crossings.

The family of H^1 of the elliptic curves $\mathbb{C}^\times/t^{\mathbb{Z}}$ forms a VPH on Δ^* and this VPH is extended to a VPLH $(\mathcal{H}_{\mathbb{Z}}, \mathcal{M}, (\ ,\ ))$ on Δ, where

$$\mathcal{H}_{\mathbb{Z}} = R^1 f_*^{\log}\mathbb{Z}, \ \mathcal{M} = R^1 f_*(\omega_{E/\Delta}^\bullet), \ \mathcal{M}^p = R^1 f_*(\omega_{E/\Delta}^{\bullet \geq p})$$

and $(\ ,\ )$ is explained later. The sheaf $\mathcal{H}_{\mathbb{Z}}$ is a locally constant sheaf which is described as follows. Let $\overline{U} \longrightarrow \Delta^{\log} \longrightarrow \Delta$ be as in 2.8. The pull back of the family $E - f^{-1}(0) \longrightarrow \Delta^*$ to U is identified with the family $\{\mathbb{C}/(\mathbb{Z}z + \mathbb{Z})\}_{z \in U}$ of elliptic curves (we identify $\mathbb{C}/(\mathbb{Z}z + \mathbb{Z})$ with $\mathbb{C}^\times/t^{\mathbb{Z}}$, where $t = \exp(2\pi i z)$, by $\exp(2\pi i -)$) and $\mathrm{H}_1(\mathbb{C}/(\mathbb{Z}z + \mathbb{Z}), \mathbb{Z})$ is identified with $\mathbb{Z}z + \mathbb{Z}$. Hence $\mathcal{H}_{\mathbb{Z}} = R^1 f_*^{\log}\mathbb{Z}$ is identified with the local system $\mathcal{H}om_{\mathbb{Z}}(\mathbb{Z}z + \mathbb{Z}, \mathbb{Z})$ where z is regarded as a local section $(2\pi i)^{-1}\log(t)$ of $\mathcal{O}_\Delta^{\log}$ (t denotes here the coordinate function of Δ) and the inverse image

of $\mathcal{H}_{\mathbb{Z}}$ on $\overline{U}$ is identified with the constant sheaf $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}z+\mathbb{Z},\mathbb{Z})$ where z is regarded here as a global section of the inverse image of $\mathcal{O}_{\Delta}^{\log}$ on $\overline{U}$. Let $(e_j)_{j=1,2}$ be the $\mathbb{Z}$-basis of $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}z+\mathbb{Z},\mathbb{Z})$ where e_1 sends z to 1 and 1 to 0, and e_2 sends z to 0 and 1 to 1. Then

$$\Gamma(\Delta,\ R^1f_*\mathbb{Z}) = \Gamma(\Delta^{\log},\ R^1f_*^{\log}\mathbb{Z}) = \mathbb{Z}e_1,\ (R^1f_*\mathbb{Z})_0 = \mathbb{Z}e_1.$$

The $\mathbb{Q}$-bilinear form $(\ ,\ )\colon \mathcal{H}_{\mathbb{Q}}\times\mathcal{H}_{\mathbb{Q}} \longrightarrow \mathbb{Q}$ is the unique anti-symmetric form satisfying $(e_2,e_1)=1$.

Next, $\mathcal{M}$ is a free $\mathcal{O}_{\Delta}$-module of rank 2 with basis (e_1,ω) where

$$\omega = d\log(u) = -d\log(v) \in \Gamma(\Delta,\ f_*\omega^1_{E/\Delta}),$$

and the filtration of $\mathcal{M}$ is described as

$$\mathcal{M}^p = \mathcal{M} \text{ for } p\le 0,\quad \mathcal{M}^p = 0 \text{ for } p\ge 2,$$
$$\mathcal{M}^1 = f_*\omega^1_{E/\Delta} = \mathcal{O}_{\Delta}\cdot\omega.$$

On $\overline{U}$, we have

2.14.1. $\qquad \omega = 2\pi i z e_1 + 2\pi i e_2.$

In fact, the pull back of ω to each elliptic curve $\mathbb{C}/(\mathbb{Z}z+\mathbb{Z})$ for $z\in U$ is $2\pi i ds$ where s is the coordinate of $\mathbb{C}$, and 2.14.1 follows from $\int_0^z 2\pi i ds = 2\pi i z$ and $\int_0^1 2\pi i ds = 2\pi i$.

Now the log C^{∞} Hodge decomposition

$$\mathcal{A}_{\Delta}\otimes_{\mathcal{O}_{\Delta}}\mathcal{M} = \mathcal{M}_{\mathcal{A}}^{1,0}\oplus\mathcal{M}_{\mathcal{A}}^{0,1}$$

is described as follows: $\mathcal{M}_{\mathcal{A}}^{1,0}$ is a free $\mathcal{A}_{\Delta}$-module of rank 1 with basis ω, $\mathcal{M}_{\mathcal{A}}^{0,1}$ is a free $\mathcal{A}_{\Delta}$-module of rank 1 with basis

2.14.2. $\qquad \overline{\omega} = -2\pi i\overline{z}e_1 - 2\pi i e_2.$

The relation with the basis (e_1,ω) of $\mathcal{A}_X\otimes_{\mathcal{O}_X}\mathcal{M}$ is given by

$$\overline{\omega} = -\omega + 2\log(|t|)e_1,$$
$$e_1 = \tfrac{1}{2}\log(|t|)^{-1}\omega + \tfrac{1}{2}\log(|t|)^{-1}\overline{\omega},$$

as is seen from 2.14.1 and 2.14.2. Note that $\log(|t|)$ and $\log(|t|)^{-1}$ are log C^{∞}-functions on Δ, but not C^{∞}-functions on Δ. This tells that C^{∞}-functions are not enough to obtain the Hodge decomposition in the situation of degeneration.

§3. Log $\overline{\partial}$-Poincaré lemma

The purpose of this section is to prove

Theorem 3.1. *Let X be an fs log analytic space which is log smooth over $\mathbb{C}$. Then we have an exact sequence on X_{ket}*

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{A}_X \xrightarrow{\overline{\partial}} \mathcal{A}_X^{0,1} \xrightarrow{\overline{\partial}} \mathcal{A}_X^{0,2} \longrightarrow \cdots,$$

where $\overline{\partial}\colon \mathcal{A}_X^{0,q} \longrightarrow \mathcal{A}_X^{0,q+1}$ is the map induced by $d\colon \mathcal{A}_X^q \longrightarrow \mathcal{A}_X^{q+1}$.

(The non-ket version of this is also true.)

When X is a complex manifold whose log structure is given by a divisor with normal crossings, the non-ket version of this theorem is a case of Proposition (2.2.4) in [12]. The part after 3.3 of this section is essentially included in [12] section 2. See also [10].

3.2. To prove 3.1, first we show that we may assume $X = \Delta^{n+m}$ with the log structure given by the complement of $(\Delta^*)^n \times \Delta^m$. In fact, locally on X, take a blowing up $f\colon Z \longrightarrow X$ along log structure such that Z is a complex manifold and the complement of X_{triv} in Z is a divisor with normal crossings. If Theorem 3.1 is true for Z, then by $Rf_*\mathcal{O}_Z = \mathcal{O}_X$ ([22] I, Corollary 1 c) to Theorem 12, GAGA ([8] XII Théorème 4.2), and 1.7), $Rf_*\mathcal{A}_Z = \mathcal{A}_X$ (1.5, 1.7, and 2.10), and $\mathcal{A}_Z^{p,q} = \mathcal{A}_Z \otimes_{\mathcal{A}_X} \mathcal{A}_X^{p,q}$ (log étaleness of f), Theorem 3.1 is true for X. Hence we may assume that $X = \Delta^{n+m}$ with the log structure as above.

3.3. We fix notation concerning Δ^{n+m}.

For $1 \leq j \leq n+m$, let t_j be the j-th coordinate function of Δ^{n+m}. For $1 \leq j \leq n$, let

$$r_j = |t_j|, \quad u_j = t_j/r_j$$

(u_j is defined on $(\Delta^*)^n \times \Delta^m$). Let

$$\partial_j = t_j \cdot \frac{\partial}{\partial t_j}, \qquad \overline{\partial}_j = \overline{t_j} \cdot \frac{\partial}{\partial \overline{t_j}} \qquad \text{for } 1 \leq j \leq n,$$

$$\partial_j = \frac{\partial}{\partial t_j}, \qquad \overline{\partial}_j = \frac{\partial}{\partial \overline{t_j}} \qquad \text{for } n+1 \leq j \leq n+m.$$

Then for $1 \leq j \leq n$,

3.3.1. $\partial_j = \frac{1}{2}(r_j \frac{\partial}{\partial r_j} + u_j \frac{\partial}{\partial u_j})$, $\overline{\partial}_j = \frac{1}{2}(r_j \frac{\partial}{\partial r_j} - u_j \frac{\partial}{\partial u_j})$.

Let $|\Delta| = \{s \in \mathbb{R} \ ; \ 0 \leq s < 1\}$, $|\Delta^*| = \{s \in \mathbb{R} \ ; \ 0 < s < 1\}$.

We use the theory of Fourier expansions as in [31, Proposition 6.4].

Lemma 3.4. (1) *Via the Fourier expansion*

$$f = \sum_{l \in \mathbb{Z}^n} f_l \cdot \prod_{j=1}^{n} u_j^{l(j)},$$

a C^∞-function f on $(\Delta^*)^n \times \Delta^m$ corresponds bijectively to a family $(f_l)_{l \in \mathbb{Z}^n}$ of C^∞-functions on $|\Delta^*|^n \times \Delta^m$ satisfying the following condition 3.4.1.

3.4.1. *For each $v \in |\Delta^*|^n \times \Delta^m$ and each $a \in \mathbb{N}^n$, there exists $C > 0$ such that*

$$(\prod_{j=1}^{n} |l(j)|^{a(j)}) \cdot |f_l(v)| \leq C$$

for any $l \in \mathbb{Z}^n$.

(2) *Endow Δ^{n+m} with the log structure as in* 3.2. *Then, in the correspondence in* (1), *f is a log C^∞-function on Δ^{n+m} if and only if the family $(f_l)_{l \in \mathbb{Z}^n}$ satisfies the following condition* 3.4.2.

3.4.2. *For each a, $b \in \mathbb{N}^n$, each c, $d \in \mathbb{N}^m$ and each compact subset K of $|\Delta|^n \times \Delta^m$, there exists $C > 0$ and $h \in \mathbb{N}^n$ such that*

$$(\prod_{j=1}^{n} |l(j)|^{a(j)}) \cdot \left| (\prod_{j=1}^{n} (r_j \frac{\partial}{\partial r_j})^{b(j)})(\prod_{j=1}^{m} \partial_{n+j}^{c(j)} \overline{\partial}_{n+j}^{d(j)})(f_l)(r,z) \right|$$
$$\leq C \cdot \prod_{j=1}^{n} |\log(r_j)|^{h(j)}$$

for any $l \in \mathbb{Z}^n$ and any $(r,z) \in K \cap (|\Delta^*|^n \times \Delta^m)$.

Proof. As is well known in the theory of Fourier expansions, a C^∞-function on $(\mathbb{S}^1)^n$ corresponds bijectively to a rapidly decreasing function on $\mathbb{Z}^n$. (1) follows from this. (2) is deduced from the relation 3.3.1 of ∂_j, $\overline{\partial}_j$ and $r_j \frac{\partial}{\partial r_j}$, $u_j \frac{\partial}{\partial u_j}$ $(1 \leq j \leq n)$. Q.E.D.

3.5. We prove that if f is a log C^∞-function on Δ^{n+m} with the log structure as in 3.2 and if $\overline{\partial}(f) = 0$, then f is a holomorphic function on Δ^{n+m}. Let $f = \sum_l f_l \cdot \prod_{j=1}^n u_j^{l(j)}$ be the Fourier expansion of f. Then by $\overline{\partial}_j(f) = 0$ for $1 \leq j \leq n+m$ and by 3.3.1, we have for each $l \in \mathbb{Z}^n$

$$r_j \frac{\partial}{\partial r_j}(f_l) - l(j) f_l = 0 \qquad \text{for } 1 \leq j \leq n,$$
$$\overline{\partial}_j(f_l) = 0 \qquad \text{for } n+1 \leq j \leq n+m.$$

This shows

$$f_l(r,z) = (\prod_{j=1}^{n} r_j^{l(j)}) \cdot h_l(z)$$

where $h_l(z)$ $(z = (t_j)_{n+1 \le j \le n+m})$ is a holomorphic function in $z \in \Delta^m$. The log growth of f shows the log growth of f_l for each $l \in \mathbb{Z}^n$, and this shows $h_l = 0$ unless $l(j) \ge 0$ for all $1 \le j \le n$. Hence

$$f = \sum_{l \in \mathbb{N}^n} (\prod_{j=1}^{n} t_j^{l(j)}) \cdot h_l(z).$$

This and 3.4 (1) show that f is a holomorphic function on Δ^{n+m}.

3.6. By 3.5, for the proof of Theorem 3.1 for $X = \Delta^{n+m}$ with the log structure as in 3.2, it remains to prove $\mathcal{H}^q(\mathcal{A}_X^{0,\bullet}) = 0$ for $q \ge 1$. As in the argument of the proof of the classical $\overline{\partial}$-Poincaré lemma (*cf.* [7, p. 25]), this is reduced to proving the following 3.6.1.

3.6.1. *Let $1 \le k \le n+m$ and let S be a subset of $\{1, \ldots, n+m\}$ which does not contain k. Let f be a log C^∞-function on Δ^{n+m} and assume $\overline{\partial}_j(f) = 0$ for $j \in S$. Then locally on Δ^{n+m}, there exists a log C^∞-function g satisfying*

$$\overline{\partial}_j(g) = 0 \text{ for } j \in S, \quad \text{and } \overline{\partial}_k(g) = f.$$

We prove 3.6.1 in the case $1 \le k \le n$ (resp. $n+1 \le k \le n+m$) in 3.7 (resp. 3.9).

3.7. First assume $1 \le k \le n$. Let $f = \sum_l f_l \cdot \prod_{j=1}^n u_j^{l(j)}$ be the Fourier expansion of f. For each $l \in \mathbb{Z}^n$, define a C^∞-function $g_{f,l}$ on $|\Delta^*|^n \times \Delta^m$ as follows. Fix a positive number $A < 1$. Let $e = l(k)$, and define

$$g_{f,l}(r,z) = 2r_k^e \int_B^{r_k} s^{-e} f_l(r_1, \ldots, r_{k-1}, s, r_{k+1}, \ldots, r_n, z) \frac{ds}{s}$$

where $B = A$ in the case $e \ge 0$ and $B = 0$ in the case $e < 0$. We estimate $g_{f,l}$. Let $a \in \mathbb{N}^n$, let K be a compact subset of $|\Delta|^n \times \Delta^m$, and by putting $b = 0 \in \mathbb{N}^n$ and $c = d = 0 \in \mathbb{N}^m$ in 3.4 (2), let $C > 0$ and $h \in \mathbb{N}^n$ be as in 3.4 (2) for the family $(f_l)_l$. Then by lemma 3.8 below, for any $l \in \mathbb{Z}^n$ and any $(r,z) \in K \cap (|\Delta^*|^n \times \Delta^m)$,

$$(\prod_{j=1}^{n} |l(j)|^{a(j)}) |g_{f,l}(r,z)| \le 2C \cdot \theta(r_k) \cdot \prod_{\substack{j=1 \\ j \ne k}}^{n} |\log(r_j)|^{h(j)}$$

where

$$\theta(r_k) = \begin{cases} h(k)! \cdot \sum_{i=0}^{h(k)} (|\log(r_k)|^i + |\log(A)|^i A^{-e}) & \text{if } e > 0, \\ |\log(r_k)|^{h(k)+1} + |\log(A)|^{h(k)+1} & \text{if } e = 0, \\ h(k)! \cdot \sum_{i=0}^{h(k)} |\log(r_k)|^i & \text{if } e < 0. \end{cases}$$

Hence

$$g_f = \sum_{l \in \mathbb{Z}^n} g_{f,l} \cdot \prod_{j=1}^{n} u_j^{l(j)}$$

is a C^∞-function on $(\Delta^*)^n \times \Delta^m$ and is of log growth on Δ^{n+m}. We can check easily

3.7.1. $\overline{\partial}_k(g_f) = f,$

3.7.2. $D(g_f) = g_{D(f)}$ for $D = \prod_{j=1}^{n+m} \partial_j^{a(j)} \overline{\partial}_j^{b(j)}$

for any $a,\ b \in \mathbb{N}^{n+m}$.

By 3.7.2, we have $\overline{\partial}_j(g_f) = 0$ for $j \in S$. Furthermore, by 3.7.2, what we have proved concerning the log growth of g_f shows that g_f is a log C^∞-function.

Lemma 3.8. *Let $e,\ h \in \mathbb{Z},\ h \geq 0$ and let $B,\ x \in \mathbb{R},\ 0 < x < 1$. Assume $0 < B < 1$ in the case $e \geq 0$, and $B = 0$ in the case $e < 0$. Then*

$$x^e \int_B^x t^{-e} \log(t)^h \cdot \frac{dt}{t}$$

is equal to

$$\begin{cases} -\sum_{i=0}^{h} \frac{h!}{i!} \cdot e^{i-h-1} (\log(x)^i - \log(B)^i (\frac{x}{B})^e) & \text{if } e > 0, \\ \frac{1}{h+1} \cdot (\log(x)^{h+1} - \log(B)^{h+1}) & \text{if } e = 0, \\ -\sum_{i=0}^{h} \frac{h!}{i!} \cdot e^{i-h-1} \log(x)^i & \text{if } e < 0. \end{cases}$$

3.9. We prove the case $n+1 \leq k \leq n+m$ of 3.6.1 by the method in [7]. Fix $v = (v_j)_j \in \Delta^{n+m}$. Take a positive number ϵ such that $|v_j| + \epsilon < 1$

for $1 \leq j \leq n+m$ and $|v_k| + 3\epsilon < 1$. Let

$$U = \{w \in \Delta^{n+m} \ ; \ |w_j - v_j| < \epsilon \quad \text{for } 1 \leq j \leq n+m\},$$
$$K = \{w \in \Delta^{n+m} \ ; \ |w_j - v_j| \leq \epsilon \quad \text{if } 1 \leq j \leq n+m \text{ and } j \neq k,$$
$$|w_k - v_k| \leq 3\epsilon\},$$
$$M = \{z \in \Delta \ ; \ |z - v_k| \leq \epsilon\},$$
$$N = \{(r, \ u) \ ; \ r \in \mathbb{R}, \ u \in \mathbb{C}, \ 0 \leq r \leq 2\epsilon, \ |u| = 1\}.$$

We define a C^∞-function g_f on $U \cap ((\Delta^*)^n \times \Delta^m)$ by

$$g_f(w) = \frac{1}{2\pi i} \int_M (z - w_k)^{-1} f(w_1, \ldots, w_{k-1}, z, w_{k+1}, \ldots, w_{n+m}) dz \wedge d\bar{z}.$$

Since

$$(z - w_k)^{-1} dz \wedge d\bar{z} = -2u^{-2} dr \wedge du,$$

where $r = |z - w_k|$, $u = (z - w_k)/r$, and since

$$M \subset \{w_k + ru \ ; \ (r, \ u) \in N\},$$

we see that the integral defining g_f converges, g_f is a C^∞-function on $U \cap ((\Delta^*)^n \times \Delta^m)$, and

3.9.1.

$$|g_f(w)| \leq \frac{1}{\pi} \int_N |f(w_1, \ldots, w_{k-1}, w_k + ru, w_{k+1}, \ldots, w_{n+m})| \cdot |dr \wedge du|.$$

It is checked easily that $f \mapsto g_f$ satisfies 3.7.1 and 3.7.2. By 3.7.2, $\overline{\partial}_j(g_f) = 0$ for $j \in S$. By 3.7.2, to show that g_f is a log C^∞-function on U, it is sufficient to prove that g_f is of log growth on U. Since K is compact and f is of log growth, there are $C > 0$ and $h \in \mathbb{N}^n$ such that

$$|f(w)| \leq C \cdot \prod_{j=1}^{n} |\log(|w_j|)|^{h(j)}$$

for any $w \in K \cap ((\Delta^*)^n \times \Delta^m)$. By

$$\{(w_1, \ldots, w_{k-1}, w_k + ru, w_{k+1}, \ldots, w_{n+m}) \ ; \ w \in U, \ (r, \ u) \in N\} \subset K,$$

and by 3.9.1,

$$|g_f(w)| \leq 4\epsilon C \cdot \prod_{j=1}^{n} |\log(|w_j|)|^{h(j)}$$

for any $w \in U \cap ((\Delta^*)^n \times \Delta^m)$. This shows that g_f is of log growth on U.

§4. Relative log Poincaré lemma

Here everything is in the ket sense except in the latter part of 4.4.

Let X, Y be fs log analytic spaces which are log smooth over $\mathbb{C}$, and let $f\colon X \longrightarrow Y$ be a log smooth morphism. Let $\mathcal{A}^1_{X/Y}$ be the cokernel of $\mathcal{A}_X \otimes_{\mathcal{A}_Y} \mathcal{A}^1_Y \longrightarrow \mathcal{A}^1_X$, $\mathcal{A}^p_{X/Y} := \bigwedge^p_{\mathcal{A}_X} \mathcal{A}^1_{X/Y}$, and $\mathcal{A}^{p,\log}_{X/Y} := \mathcal{O}^{\log}_X \otimes_{\tau^{-1}(\mathcal{O}_X)} \tau^{-1}(\mathcal{A}^p_{X/Y})$ for each $p \geq 0$.

Theorem 4.1. *Let $f\colon X \longrightarrow Y$ be a log smooth morphism of log smooth fs log analytic spaces. Let x be a point of $X^{\log}$. Assume that f is exact at $\tau(x)$. Then the stalk at x of*

$$0 \longrightarrow (f^{\log})^{-1}(\mathcal{A}^{\log}_Y) \longrightarrow \mathcal{A}^{\log}_X \longrightarrow \mathcal{A}^{1,\log}_{X/Y} \longrightarrow \mathcal{A}^{2,\log}_{X/Y} \longrightarrow \cdots$$

is exact.

For the proof we use;

Proposition 4.2. *Under the same assumption as in Theorem* 4.1, *let $y = f^{\log}(x) \in Y^{\log}$. Assume that the cokernel of $M^{\mathrm{gp}}_{Y,\tau(y)}/\mathcal{O}^{\times}_{Y,\tau(y)} \longrightarrow M^{\mathrm{gp}}_{X,\tau(x)}/\mathcal{O}^{\times}_{X,\tau(x)}$ is torsion free. Then the followings hold.*

(1) *There exists an open neighborhood U_0 of $\tau(y)$ having the following property: For any open neighborhood W of x, there is a continuous map $s\colon U := U_0^{\log} \longrightarrow X^{\log}$ satisfying the following* 4.2.1–4.2.4.

4.2.1. $f^{\log} \circ s = \mathrm{id}_U$.

4.2.2. $s(y) \in W$.

4.2.3. $s(U_{\mathrm{triv}}) \subset X_{\mathrm{triv}}$.

4.2.4. *For any open set V of X, U_1 of U_0 such that $s(U_1^{\log}) \subset V^{\log}$, and for any $g \in \Gamma(V, \mathcal{A}_X)$, $g \circ s$ belongs to $\Gamma(U_1, \mathcal{A}_Y)$.*

(2) *If f is vertical, s can be chosen to satisfy $s(y) = x$.*

4.3. We prove Proposition 4.2. We may assume the following: $X = \mathrm{Spec}\,(\mathbb{C}[\mathcal{T}])^{\mathrm{an}}$, $Y = \mathrm{Spec}\,(\mathbb{C}[\mathcal{S}])^{\mathrm{an}}$ for fs monoids $\mathcal{S}$, $\mathcal{T}$ such that $\mathcal{S} \subset \mathcal{T}$, $\mathcal{S}^{\mathrm{gp}} \cap \mathcal{T} = \mathcal{S}$, $\mathcal{S}^{\times} = \{1\}$, $\mathcal{T}^{\times} = \{1\}$, $\mathcal{T}^{\mathrm{gp}}/\mathcal{S}^{\mathrm{gp}}$ is torsion free, $x \in X^{\log}$ lies over the origin of X, and $f\colon X \longrightarrow Y$ is the natural projection so that $y \in Y^{\log}$ lies over the origin of Y. It is enough to prove the following claim on monoids.

In the rest of this subsection, for $\mathcal{U} = \mathcal{S}$ or $\mathcal{T}$, we denote by $|\mathcal{U}^{\vee}|$ the topological space $\mathrm{Hom}\,(\mathcal{U}, \mathbb{R}^{\mathrm{mult}}_{\geq 0})$. By a log C^{∞}-function on an open U of $|\mathcal{U}^{\vee}|$, we mean a C^{∞}-function $g\colon U \cap \mathrm{Hom}\,(\mathcal{U}, \mathbb{R}_{>0}) \longrightarrow \mathbb{C}$ having

the following property: If $(t_j)_{1\leq j\leq n}$ is a basis of $\mathcal{U}^{\mathrm{gp}}$, for any $a_j \in \mathbb{N}$ $(1 \leq j \leq n)$, $\prod_j \left(t_j \cdot \frac{\partial}{\partial t_j}\right)^{a_j} (g)$ is of log growth on U (cf. 1.1).

Claim. For $\mathcal{S} \subset \mathcal{T}$ as above, let x (resp. y) be the origin of $|\mathcal{T}^\vee|$ (resp. $|\mathcal{S}^\vee|$). Then for any open neighborhood W of x, there is a continuous map $s\colon |\mathcal{S}^\vee| \longrightarrow |\mathcal{T}^\vee|$ satisfying the following 4.2.1′–4.2.4′.

4.2.1′ $f \circ s = \mathrm{id}$, where f is the canonical map $|\mathcal{T}^\vee| \longrightarrow |\mathcal{S}^\vee|$.

4.2.2′ $s(y) \in W$.

4.2.3′ $s(\mathrm{Hom}\,(\mathcal{S}, \mathbb{R}_{>0})) \subset \mathrm{Hom}\,(\mathcal{T}, \mathbb{R}_{>0})$.

4.2.4′ For any open set V of $|\mathcal{T}^\vee|$ and for any log C^∞-function g on V, $g \circ s$ is a log C^∞-function on $s^{-1}(V)$.

Further, if $\mathcal{S} \longrightarrow \mathcal{T}$ is dominating (i.e., any $t \in \mathcal{T}$ divides an element of $\mathcal{S}$), 4.2.2′ can be replaced by $s(y) = x$.

In the rest of this subsection, we prove this claim. By induction on $\mathrm{rank}(\mathcal{T}^{\mathrm{gp}}) - \mathrm{rank}(\mathcal{S}^{\mathrm{gp}})$, we may assume that $\mathrm{rank}(\mathcal{T}^{\mathrm{gp}}) = \mathrm{rank}(\mathcal{S}^{\mathrm{gp}}) + 1$. Fix an embedding $\mathcal{T} \subset \mathcal{S}^{\mathrm{gp}}_{\mathbb{Q}} \oplus \mathbb{Q}$ which sends each $s \in \mathcal{S} \subset \mathcal{T}$ to $(s,\ 0)$. Take a finite family $((a_\lambda,\ e(\lambda)))_{\lambda\in\Lambda}$ $(a_\lambda \in \mathcal{S}^{\mathrm{gp}}_{\mathbb{Q}},\ e(\lambda) \in \{\pm 1\})$ of elements of $\mathcal{T}_{\mathbb{Q}_{\geq 0}} := \mathcal{T} \otimes_{\mathbb{N}} \mathbb{Q}_{\geq 0} \subset \mathcal{S}^{\mathrm{gp}}_{\mathbb{Q}} \oplus \mathbb{Q}$ which together with $\mathcal{S}$ generates $\mathcal{T}_{\mathbb{Q}_{\geq 0}}$. Let

$$\Lambda_+ = \{\lambda \in \Lambda,\ e(\lambda) = 1\}, \quad \Lambda_- = \{\lambda \in \Lambda,\ e(\lambda) = -1\}.$$

Then the exactness $\mathcal{S} = \mathcal{T} \cap \mathcal{S}^{\mathrm{gp}}$ implies the following 4.3.1.

4.3.1. If $(a, 1)$ and $(a', -1)$ belong to $\mathcal{T}_{\mathbb{Q}_{\geq 0}}$, then $aa' \in \mathcal{S}_{\mathbb{Q}_{\geq 0}}$.

The condition "dominating" implies

4.3.2. $\Lambda_+ \neq \emptyset$ and $\Lambda_- \neq \emptyset$.

In the followings, for $\mathcal{U} = \mathcal{S}$ or $\mathcal{T}$, we identify each element of $|\mathcal{U}^\vee|$ with its natural extension in $\mathrm{Hom}\,(\mathcal{U} \otimes_{\mathbb{N}} \mathbb{R}^{\mathrm{add}}_{\geq 0}, \mathbb{R}^{\mathrm{mult}}_{\geq 0})$. We will define $s(h)$ for each $h \in |\mathcal{S}^\vee|$.

In the non-dominating case, define $s(h)$ as follows. We may assume that $\mathcal{S} \neq \{1\}$. We have $\Lambda = \Lambda_+$ or $\Lambda = \Lambda_-$. So assume $\Lambda = \Lambda_+$. Take an element b of $\mathcal{S}$ such that $a_\lambda b$ belongs to the interior of $\mathcal{S}_{\mathbb{Q}_{\geq 0}}$ for any $\lambda \in \Lambda$, and define a homomorphism $\theta\colon \mathcal{T} \longrightarrow \mathcal{S}_{\mathbb{Q}_{\geq 0}}$ by sending $(a_\lambda, 1)$ to $a_\lambda b$. Define $s(h) = h \circ \theta$.

Now we assume that $\mathcal{S} \longrightarrow \mathcal{T}$ is dominating, that is, Λ_+ and Λ_- are non-empty sets. Let $I := \Lambda_+ \times \Lambda_-$. By 4.3.1, we have $s_i := a_\lambda a_\mu \in \mathcal{S}_{\mathbb{Q}_{\geq 0}}$ for any $i = (\lambda, \mu) \in I$.

In the case $h(s_i) = 0$ for any $i \in I$, define $s(h)$ to be the unique homomorphism $\mathcal{T} \longrightarrow \mathbb{R}_{\geq 0}$ which coincides on $\mathcal{S}$ with h and which sends $(a_\lambda,\ e(\lambda))$ to 0 for all $\lambda \in \Lambda$. (It is easy to see that such homomorphism exists.)

Before defining $s(h)$ for $h \in V := \{h \in |\mathcal{S}^\vee| \ ; \ h(s_i) \neq 0 \text{ for some } i \in I\}$, we choose a partition of unity on V, which is subordinate to the covering $(U_i)_{i\in I}$ as follows. Here $U_i := \{h \ ; \ 3h(s_i) > h(s_j) \text{ for any } j \in I\}$. Take a C^∞-function χ on $\mathbb{R}_{\geq 0}$ such that $\chi(t) = 0$ for $t \geq 2$ and $\chi(t) + \chi(t^{-1}) = 1$ for all $t > 0$. For any $i, j \in I$, let $\varphi_{ij} := \chi(s_i^{-1}s_j)$, which is defined on $\{h \ ; \ h(s_i) \neq 0\}$, and $\varphi_i := \prod_{j\neq i} \varphi_{ij}$. Then $\{\varphi_i\}_{i\in I}$ is the desired partition of unity. Note that each φ_i is log C^∞ in the sense explained before the above claim. Now let $c := \prod_{i\in I} s_i^{\varphi_i}$, $a_+ := \prod_{i=(\lambda,\mu)\in I} a_\lambda^{\varphi_i}$, and $a_- := \prod_{i=(\lambda,\mu)\in I} a_\mu^{\varphi_i}$. Then $c(h) = h(a_+a_-) \neq 0$ for any $h \in V$. Let $h \in V$. Define $s(h)$ to be the unique homomorphism $\mathcal{T} \longrightarrow \mathbb{R}_{\geq 0}$ which coincides on $\mathcal{S}$ with h and which sends $(a_\lambda, 1)$ $(\lambda \in \Lambda_+)$ to $h(a_\lambda a_-) \cdot c^{-1/2}$, and $(a_\mu, -1)$ $(\mu \in \Lambda_-)$ to $h(a_+a_\mu) \cdot c^{-1/2}$. (It is easy to see that such homomorphism exists.)

Thus we have defined a map s. It is easy to see that s is continuous and has the desired properties.

4.4. We prove Theorem 4.1. The proof is essentially the same as the proof of the classical Poincaré lemma. We may assume that the following: $X = \mathrm{Spec}\,(\mathbb{C}[\mathcal{T}])^{\mathrm{an}} \times \mathbb{C}^T$, $Y = \mathrm{Spec}\,(\mathbb{C}[\mathcal{S}])^{\mathrm{an}} \times \mathbb{C}^S$ for fs monoids $\mathcal{S}$, $\mathcal{T}$ such that $\mathcal{S} \subset \mathcal{T}$, $\mathcal{S}^{\mathrm{gp}} \cap \mathcal{T} = \mathcal{S}$, $\mathcal{S}^\times = \{1\}$, $\mathcal{T}^\times = \{1\}$ and finite sets $S \subset T$, $x \in X^{\log}$ lies over the origin of X and $f\colon X \longrightarrow Y$ is the natural projection so that $y = f^{\log}(x) \in Y^{\log}$ lies over the origin of Y. We will prove the exactness at $\mathcal{A}^{p,\log}_{X/Y,x}$, $p \geq 0$.

First we reduce to the case where the relative dimension $d := \dim X - \dim Y$ is one by the standard induction argument (cf. [7] p.25) as follows: Supposing that the statement is valid for the case of the relative dimension $< d$, we will prove the case where it is d. We will assume that $\mathcal{S} \neq \mathcal{T}$; the other case is similar. Let $\omega \in \mathcal{A}^{p,\log}_{X/Y,x}$ such that $d\omega = 0$. We will prove that ω comes from $\mathcal{A}^{p-1,\log}_{X/Y,x}$ (resp. $\mathcal{A}^{\log}_{Y,y}$) for $p \geq 1$ (resp. $p = 0$). Take an fs monoid $\mathcal{S}' \subset \mathcal{T}$ such that $\mathcal{S} \subset \mathcal{S}' = \mathcal{S}'^{\mathrm{gp}} \cap \mathcal{T}$ and such that $\mathrm{rank}(\mathcal{S}')^{\mathrm{gp}} = \mathrm{rank}(\mathcal{S})^{\mathrm{gp}} + 1$ and an element $t \in \mathcal{S}'$ such that $t \notin \mathcal{S}^{\mathrm{gp}} \otimes \mathbb{Q}$. Denote $\mathrm{Spec}\,(\mathbb{C}[\mathcal{S}'])^{\mathrm{an}} \times \mathbb{C}^S$ by Y' and the image of x in Y' by y'. Since the image of ω in $\mathcal{A}^{p,\log}_{X/Y',x}$ is closed, the induction hypothesis implies that we may assume that the image is zero if $p \geq 1$; ω lies in $\mathcal{A}^{\log}_{Y',y'}$ if $p = 0$. Hence the case $p = 0$ follows. In the case where $p \geq 1$, we can write ω as $\omega_0 d\log t + \omega_1 d\log \bar{t}$ $(\omega_i \in \mathcal{A}^{p-1,\log}_{X/Y,x}, \ i = 0, 1)$. Similarly we may assume that the image of ω_i in $\mathcal{A}^{p-1,\log}_{X/Y',x}$ is zero $(i = 0, 1)$ if $p \geq 2$; ω_i lies in $\mathcal{A}^{\log}_{Y',y'}$ $(i = 0, 1)$ if $p = 1$. Thus the case $p = 1$ follows. If $p \geq 2$, we can write $\omega = \omega_2 d\log t \wedge d\log \bar{t}$ $(\omega_2 \in \mathcal{A}^{p-2,\log}_{X/Y,x})$.

Then the similar argument shows that ω is exact ($p \geq 3$) or lies in $\mathcal{A}^{2,\log}_{Y'/Y,x}$ ($p = 2$). Thus we may assume that $d = 1$ and $0 \leq p \leq 2$.

Further we may assume that the cokernel of $\mathcal{S}^{\mathrm{gp}} \longrightarrow \mathcal{T}^{\mathrm{gp}}$ is torsion free and it is enough to prove the non-ket version of the statement. In the following we will assume that $\mathcal{S} \neq \mathcal{T}$; the proof for the other case where $S \neq T$ is similar and simpler. Let $\omega \in \mathcal{A}^{p,\log}_{X/Y,x}$ with $d\omega = 0$ and we will prove that ω is exact ($p = 1, 2$) or comes from Y ($p = 0$). In the following, fix an element $t \in \mathcal{T} - \mathcal{S}$ and consider it as a relative coordinate function.

Assume that $p = 0$. Take a base $s_1, \ldots, s_r$ of $\mathcal{S}^{\mathrm{gp}}$. Then ω is regarded as a polynomial in $\mathcal{A}_{X,\tau(x)}[l_1, \ldots, l_r, l]$, where $l_i = \log s_i$ ($1 \leq i \leq r$) and $l = \log t$. Write $\omega = \sum \omega_{i_1 \cdots i_r} l_1^{i_1} \cdots l_r^{i_r}$, $\omega_{i_1 \cdots i_r} \in \mathcal{A}_{X,\tau(x)}[l]$. Then $d\omega = 0$ implies $d\omega_{i_1 \cdots i_r} = 0$ for the highest degree $(i_1, \ldots, i_r)$ in the sense of the lexicographic order. Hence the induction works when $\omega_{i_1 \cdots i_r}$ comes from $\mathcal{A}_{Y,\tau(y)}$. Thus the problem is reduced to show that $d\omega = 0$ for $\omega \in \mathcal{A}_{X,\tau(x)}[l]$ implies $\omega \in \mathcal{A}_{Y,\tau(y)}$. We will show this. By induction of the degree of l with the fact that $df_0 + f_1 d\log t = 0$ implies $f_1 = 0$ for any $f_0 \in \mathcal{A}_{X,\tau(x)}$ and $f_1 \in \mathcal{A}_{Y,\tau(y)}$, we may assume that $\omega \in \mathcal{A}_{X,\tau(x)}$. (We have that, by seeing each fiber near y, the above fact is reduced to another simple fact that $\alpha d\log t$ ($\alpha \in \mathbb{C}$) is not exact on an annulus $\{re^{i\theta}\ ;\ 0 \leq \theta \leq 2\pi,\ R_1 < r < R_2\}$, $R_2 > R_1 \geq 0$ in the complex t-plane unless $\alpha = 0$.) Fix a set of generators $\{t_0 = t, t_1, \ldots, t_s\}$ of $\mathcal{T}$. Then there is a positive real number ε such that ω is defined and $d\omega = 0$ on the neighborhood $X' = \{x \in X\ ;\ |t_i(x)| < \varepsilon \text{ for any } i = 0, \ldots, s\}$ of $\tau(x)$. By Proposition 4.2, we may assume that there exist open neighborhood Y' of $\tau(y)$, a continuous map $s\colon U := Y'^{\log} \longrightarrow X'^{\log} \subset X^{\log}$ satisfying 4.2.1, 4.2.3, and 4.2.4 (U_0 there being replaced with Y'). Then we see that ω comes from $\mathcal{A}_{Y,\tau(y)}$ by 4.2.4.

Next let $p = 1$ or 2. Let $l_1, \ldots, l_r, l$ as above. Write $\omega = \sum \omega_{i_1 \cdots i_r i} l_1^{i_1} \cdots l_r^{i_r} l^i$, $\omega_{i_1 \cdots i_r i} \in \mathcal{A}^p_{X/Y,\tau(x)}$. Then $d\omega = 0$ implies $d\omega_{i_1 \cdots i_r i} = 0$ for the highest degree (in the same sense as above). Hence the induction works and the problem is reduced to show that $\omega \in \mathcal{A}^p_{X/Y,\tau(x)}$ with $d\omega = 0$ comes from $\mathcal{A}^{p-1}_{X/Y,\tau(x)}$. We take the same X', Y' and s as above. In the following, we regard each fiber of $X'_{\mathrm{triv}} \longrightarrow Y'_{\mathrm{triv}}$ as an annulus in the t-plane. Assume that $p = 1$. For $y' \in Y'_{\mathrm{triv}}$, define

$$c(y') = \int_\gamma \omega,$$

where γ is any loop $\{Re^{i\theta}\ ;\ 0 \leq \theta \leq 2\pi\}$, $R > 0$, in the fiber of $X'_{\mathrm{triv}} \longrightarrow Y'_{\mathrm{triv}}$ at y' in the t-plane. Then c is a log C^∞-function on Y'

and $d(c\log(t)) = cd\log(t)$. Replacing ω with $\omega - \frac{c}{2\pi i}d\log(t)$, we may assume that $c = 0$. For $x' \in X'_{\mathrm{triv}}$ which maps into Y', define

$$\phi(x') = \int_\gamma \omega,$$

where γ is any route from $s(f(x'))$ to x' in the fiber $f^{-1}(f(x')) \cap X'_{\mathrm{triv}}$. Then we have $\omega = d\phi$. To show that ϕ is log C^∞, we have to estimate the growth. It is achieved by using special routes; for example, jointed ones of the routes on which either u or v is constant, where $u = \arg(t)$ and $v = \log|t|$.

Assume that $p = 2$ and $\omega = h(u,v)dudv$. Here we take the above u, v as coordinates of the fiber. For $x' \in X'_{\mathrm{triv}}$ which maps into Y', define

$$H(x') = \int_{v(s(f(x')))}^{v(x')} h(u,v)dv.$$

Then H is log C^∞ and $d(-Hdu) = h(u,v)dudv$.

§5. Consequences of the relative log Poincaré lemma

Everything is ket here.

Let $f\colon X \longrightarrow Y$ be as in the beginning of section 4. For an object V of $V_{\mathrm{qnilp}}(X)$, let $\omega^\bullet_{X/Y}(V)$ (resp. $\mathcal{A}^\bullet_{X/Y}(V)$) be the complex $i \mapsto \omega^i_{X/Y} \otimes_{\mathcal{O}_X} V$ (resp. $\mathcal{A}^i_{X/Y} \otimes_{\mathcal{O}_X} V$) $(i \in \mathbb{Z})$ with the differentials induced by those of $\omega^\bullet_{X/Y}$ (resp. $\mathcal{A}^\bullet_{X/Y}$) and the connection $V \longrightarrow \omega^1_X(V) \longrightarrow \omega^1_{X/Y}(V)$.

The aim of this section is to prove the following proposition.

Proposition 5.1. *Let $f\colon X \longrightarrow Y$ be a proper separated log smooth morphism between log smooth fs log analytic spaces. Let V be an object of $V_{\mathrm{qnilp}}(X)$. Then for any $m \in \mathbb{Z}$, the canonical map*

$$\mathcal{A}_Y \otimes_{\mathcal{O}_Y} R^m f_*(\omega^\bullet_{X/Y}(V)) \longrightarrow R^m f_*(\mathcal{A}^\bullet_{X/Y}(V)) = \mathcal{H}^m(f_*(\mathcal{A}^\bullet_{X/Y}(V)))$$

is an isomorphism.

Here the identity of the right hand side is by f_*-acyclicity of $\mathcal{A}^p_{X/Y}(V)$ $(p \in \mathbb{Z})$ which is deduced from Propositions 1.5 and 1.7.

We use the following result on the functoriality of the log Riemann-Hilbert correspondences in [14] (generalization of results of the second author, F. Kato, and S. Usui).

Theorem 5.2. *Let $f: X \longrightarrow Y$, V be as in the hypothesis of Proposition* 5.1. *Let L be the corresponding object to V of $L_{\mathrm{qunip}}(X)$ with respect to the log Riemann-Hilbert correspondence. Then for any $m \in \mathbb{Z}$, we have*:
(1) *$R^m f_*^{\log}(L)$ is an object of $L_{\mathrm{qunip}}(Y)$.*
(2) *$R^m f_*(\omega^\bullet_{X/Y}(V))$, endowed with the Gauss-Manin connection, is an object of $V_{\mathrm{qnilp}}(Y)$.*
(3) *$R^m f_*^{\log}(L)$ and $R^m f_*(\omega^\bullet_{X/Y}(V))$ are in the log Riemann-Hilbert correspondence. In particular,*

$$\mathcal{O}_Y^{\log} \otimes_{\mathbb{C}} R^m f_*^{\log}(L) \cong \mathcal{O}_Y^{\log} \otimes_{\mathcal{O}_Y} R^m f_*(\omega^\bullet_{X/Y}(V)) \quad \text{on } Y^{\log}.$$

(4) $\mathcal{O}_Y^{\log} \otimes_{\mathbb{C}} Rf_*^{\log}(L) \cong \mathcal{O}_Y^{\log} \otimes_{\mathcal{O}_Y} Rf_*(\omega^\bullet_{X/Y}(V)) \quad \text{on } Y^{\log}.$

To prove Proposition 5.1, since the problem is local on Y, we may assume that we have a commutative diagram

$$\begin{array}{ccc} X & \xleftarrow{a} & X' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ Y & \xleftarrow[b]{} & Y', \end{array}$$

where a, b are blowing ups along log structures such that f' is exact ([14]). Further we may assume that $Rf_*(\omega^\bullet_{X/Y}(V))$ is bounded above.

Lemma 5.3. *On $(Y')^{\log}$, we have*

$$\mathcal{A}_{Y'}^{\log} \otimes^{\mathbb{L}}_{\mathcal{O}_Y} Rf_*(\omega^\bullet_{X/Y}(V)) \cong R(f')_*^{\log}(\mathcal{A}^{\bullet,\log}_{X'/Y'}(V)).$$

Proof. This is obtained by the sequence of isomorphisms

$$\begin{aligned} & \mathcal{A}_{Y'}^{\log} \otimes^{\mathbb{L}}_{\mathcal{O}_Y} Rf_*(\omega^\bullet_{X/Y}(V)) \\ \cong\ & \mathcal{A}_{Y'}^{\log} \otimes_{\mathbb{C}} Rf_*^{\log}(L) \\ \cong\ & \mathcal{A}_{Y'}^{\log} \otimes_{\mathbb{C}} R(f')_*^{\log}(L) \\ \cong\ & R(f')_*^{\log}((f'^{\log})^{-1}\mathcal{A}_{Y'}^{\log} \otimes_{\mathbb{C}} L) \\ \cong\ & R(f')_*^{\log}(\mathcal{A}^{\bullet,\log}_{X'/Y'} \otimes_{\mathbb{C}} L) \\ \cong\ & R(f')_*^{\log}(\mathcal{A}^{\bullet,\log}_{X'/Y'}(V)). \end{aligned}$$

Here the first isomorphism is by Theorem 5.2 (4), the second one is by the following Lemma 5.4, the third is by the projection formula, the

fourth is by log Poincaré lemma Theorem 4.1 for $X' \longrightarrow Y'$, and the last isomorphism is the evident one. Q.E.D.

Lemma 5.4. $(b^{\log})^{-1}R^m f_*^{\log}(L) \cong R^m(f')_*^{\log}(L)$ *on* $(Y')^{\log}$. (*The right hand side means* $R^m(f')_*^{\log}((a^{\log})^{-1}(L))$.)

Proof. The both sides are locally constant sheaves (5.2 (1)), and the restrictions of them to Y'_{triv} coincide. Q.E.D.

Lemma 5.5. *On* Y', *we have*

$$\mathcal{A}_{Y'} \otimes^{\mathbb{L}}_{\mathcal{O}_Y} Rf_*(\omega^{\bullet}_{X/Y}(V)) \cong R(f')_*(\mathcal{A}^{\bullet}_{X'/Y'}(V)).$$

Proof. We apply $R\tau_{Y'*}$ to Lemma 5.3. Proposition 1.9 implies that

$$R\tau_{Y'*}(\text{l.h.s. of 5.3}) \cong \mathcal{A}_{Y'} \otimes^{\mathbb{L}}_{\mathcal{O}_Y} Rf_*(\omega^{\bullet}_{X/Y}(V)).$$

On the other hand

$$\begin{aligned} R\tau_{Y'*}(\text{r.h.s. of 5.3}) &\cong R(\tau_{Y'} \circ (f')^{\log})_*(\mathcal{A}^{\bullet,\log}_{X'/Y'}(V)) \\ &= R(f' \circ \tau_{X'})_*(\mathcal{A}^{\bullet,\log}_{X'/Y'}(V)) \\ &\cong R(f')_*(\mathcal{A}^{\bullet}_{X'/Y'}(V)). \end{aligned}$$

Q.E.D.

5.6. Now we prove Proposition 5.1 by applying Rb_* to Lemma 5.5. Propositions 1.5, 1.7, and 2.10 imply that

$$Rb_*(\text{l.h.s. of 5.5}) \cong \mathcal{A}_Y \otimes^{\mathbb{L}}_{\mathcal{O}_Y} Rf_*(\omega^{\bullet}_{X/Y}(V)).$$

On the other hand

$$\begin{aligned} Rb_*(\text{r.h.s. of 5.5}) &\cong R(b \circ f')_*(\mathcal{A}^{\bullet}_{X'/Y'}(V)) \\ &= R(f \circ a)_*(\mathcal{A}^{\bullet}_{X'/Y'}(V)) \\ &\cong Rf_*(\mathcal{A}^{\bullet}_{X/Y}(V)). \end{aligned}$$

Since $R^m f_*(\omega^{\bullet}_{X/Y}(V))$ is locally free (5.2 (2)), we obtain 5.1 by taking $\mathcal{H}^m$.

Remark: The authors do not know whether $\mathcal{A}_Y$ is flat over $\mathcal{O}_Y$ or not.

§6. Log Kähler metrics

Everything is in the ket sense here except in a part of the proof of Proposition 6.4.

6.1. Let X and Y be log smooth fs log analytic spaces and let $X \longrightarrow Y$ be a log smooth morphism.

For p, q, $m \in \mathbb{Z}$, let $\mathcal{A}^{p,q}_{X/Y}$ be the image of $\mathcal{A}^{p,q}_X \longrightarrow \mathcal{A}^{p+q}_{X/Y}$,

$$\mathcal{A}_{X/Y,p,q} = \mathcal{H}om_{\mathcal{A}_X}(\mathcal{A}^{p,q}_{X/Y}, \mathcal{A}_X), \qquad \mathcal{A}_{X/Y,m} = \mathcal{H}om_{\mathcal{A}_X}(\mathcal{A}^m_{X/Y}, \mathcal{A}_X).$$

We call $\mathcal{A}_{X/Y,1}$ the sheaf of log vector fields on X over Y.

6.2. As is easily seen, we have a bijection between the set of Hermitian forms

$$\langle\ ,\ \rangle\colon \mathcal{A}_{X/Y,1,0} \times \mathcal{A}_{X/Y,1,0} \longrightarrow \mathcal{A}_X$$

and the set $\{\omega \in \Gamma(X, \mathcal{A}^{1,1}_{X/Y})\ ;\ \bar{\omega} = -\omega\}$ given by

$$\langle f, g\rangle = (f \wedge \bar{g}, \omega)$$

where $(\ ,\ )$ means the natural pairing between $\mathcal{A}_{X/Y,1,1}$ and $\mathcal{A}^{1,1}_{X/Y}$.

6.3. By a log Hermitian metric on X over Y, we mean a Hermitian form

$$\langle\ ,\ \rangle\colon \mathcal{A}_{X/Y,1,0} \times \mathcal{A}_{X/Y,1,0} \longrightarrow \mathcal{A}_X$$

which is "positive definite" in the following sense: The map

$$\mathcal{A}_{X/Y,1,0} \longrightarrow \mathcal{H}om_{\mathcal{A}_X}(\mathcal{A}_{X/Y,1,0}, \mathcal{A}_X)\ ;\ g \mapsto (f \mapsto \langle f, g\rangle)$$

is an isomorphism and the restriction of $\langle\ ,\ \rangle$ to X_{triv} is positive definite. By a log Kähler metric on X over Y, we mean a log Hermitian metric on X over Y such that the corresponding global section ω of $\mathcal{A}^{1,1}_{X/Y}$ (6.2) satisfies $d\omega = 0$.

Proposition 6.4. *Let $f\colon X \longrightarrow Y$ be a log smooth projective morphism between log smooth fs log analytic spaces, and fix an invertible $\mathcal{O}_X$-module $\mathcal{L}$ that is relatively very ample with respect to Y. Assume that X is a complex manifold and the log structure of X is given by a divisor on X with simple normal crossings having only finite number of irreducible components.*

Then, locally on Y, there exists a log Kähler metric on X over Y such that the class of the corresponding global section of $\mathcal{A}^{1,1}_{X/Y}$ in $\mathcal{H}^2(f_*\mathcal{A}^\bullet_{X/Y})$ coincides with the image of the Chern class of $\mathcal{L}$ under

$$R^2 f_*\mathbb{Z}(1) \longrightarrow R^2 f_*\mathcal{A}^\bullet_{X/Y} \cong \mathcal{H}^2(f_*\mathcal{A}^\bullet_{X/Y}).$$

Proof. The log Kähler metric which we construct below is essentially the same as the metric which appeared in [4] and [31].

Forgetting the log structure of X, take an immersion from X to a projective bundle P over Y such that $\mathcal{L}$ is isomorphic to the pull back of $\mathcal{O}_P(1)$. Let ω_0 be the pull back of the global section of $C^{\infty,1,1}_{P/Y}$ corresponding to the (classical) Kähler metric on P relative to Y. Let D be the divisor on X which gives the log structure of X, and let $(D_j)_j$ be the set of all irreducible components of D. For each j, let f_j be the global section of $M_X/\mathcal{O}_X^\times$ corresponding to D_j, and let $\log(|f_j|)$ be the global section of $\mathcal{A}_X/C^\infty_X$ defined to be the image of f_j under the homomorphism

$$\log(|-|) : M_X^{\mathrm{gp}}/\mathcal{O}_X^\times \longrightarrow \mathcal{A}_X/C^\infty_X.$$

Here $\mathcal{A}_X$ is in the non-ket sense. By the exact sequence

$$0 \longrightarrow C^\infty_X \longrightarrow \mathcal{A}_X \longrightarrow \mathcal{A}_X/C^\infty_X \longrightarrow 0$$

and by $\mathrm{H}^1(X, C^\infty_X) = 0$, there exists a global section s_j of $\mathcal{A}_X$ such that $s_j \equiv -\log(|f_j|) \bmod C^\infty_X$. By replacing s_j by $\frac{1}{2}(s_j + \overline{s}_j) + t_j$ for a C^∞-function t_j on X with sufficiently large positive values, we find s_j such that $s_j > 0$ on X_{triv}. Take a positive real number C, and let

$$\omega = \omega_0 + C \cdot \sum_j \overline{\partial}\partial(\log(s_j))$$

(∂ (resp. $\overline{\partial}$) denotes the part $\mathcal{A}^{p,q}_{X/Y} \longrightarrow \mathcal{A}^{p+1,q}_{X/Y}$ (resp. $\mathcal{A}^{p,q}_{X/Y} \longrightarrow \mathcal{A}^{p,q+1}_{X/Y}$) of d: $\mathcal{A}^{p,q}_{X/Y} \longrightarrow \mathcal{A}^{p+1,q}_{X/Y} \oplus \mathcal{A}^{p,q+1}_{X/Y}$). Then, locally on Y, if C is sufficiently small, ω corresponds to a relative log Kähler metric on X over Y.

Since $\overline{\partial} \circ \partial = d \circ \partial$, we have class (ω) = class (ω_0) in $\mathcal{H}^2(f_*\mathcal{A}^\bullet_{X/Y})$. It is known that class (ω_0) coincides with the image of the Chern class of $\mathcal{L}$. Q.E.D.

§7. Log harmonic forms

Let $f\colon X \longrightarrow Y$ be a projective log smooth vertical morphism between log smooth fs log analytic spaces. Let n be the relative dimension of X over Y (that is, the rank of the locally free sheaf $\omega^1_{X/Y}$ which is a locally constant function on X) and we assume that n is constant.

We assume further that we are given a log Kähler metric on X over Y.

Everything in this section is in the ket sense except in a part of the proof of Proposition 7.6.

Assume that we are given a VPLH $(\mathcal{H}_{\mathbb{Z}}, \mathcal{M}, (\ ,\ ))$ of weight w on X.

7.1. Following the classical theory of Laplacian, we introduce the star operator

$$*\colon \mathcal{A}^m_{X/Y}(\mathcal{M}) \longrightarrow \mathcal{A}^{2n-m}_{X/Y}(\mathcal{M})$$

which is $\mathcal{A}_X$-linear, the δ-operator

$$\delta\colon \mathcal{A}^m_{X/Y}(\mathcal{M}) \longrightarrow \mathcal{A}^{m-1}_{X/Y}(\mathcal{M})\ ;\ \delta = -*d*,$$

where d denotes $\nabla\colon \mathcal{A}^q_{X/Y}(\mathcal{M}) \longrightarrow \mathcal{A}^{q+1}_{X/Y}(\mathcal{M})$, and then the Laplacian

$$\Delta\colon \mathcal{A}^m_{X/Y}(\mathcal{M}) \longrightarrow \mathcal{A}^m_{X/Y}(\mathcal{M})\ ;\ \Delta = d\delta + \delta d.$$

The definition of $*$ is as follows. The Hermitian metric

$$\langle\ ,\ \rangle\colon \mathcal{A}_{X/Y,1,0} \times \mathcal{A}_{X/Y,1,0} \longrightarrow \mathcal{A}_X$$

induces by duality an Hermitian metric

$$\langle\ ,\ \rangle\colon \mathcal{A}^{1,0}_{X/Y} \times \mathcal{A}^{1,0}_{X/Y} \longrightarrow \mathcal{A}_X.$$

Clearly, this Hermitian metric is extended to a unique Hermitian metric

$$\langle\ ,\ \rangle\colon \mathcal{A}^1_{X/Y} \times \mathcal{A}^1_{X/Y} \longrightarrow \mathcal{A}_X$$

having the properties that $\mathcal{A}^{1,0}_{X/Y}$ and $\mathcal{A}^{0,1}_{X/Y}$ are orthogonal under $\langle\ ,\ \rangle$ and $\langle \bar{a}, \bar{b}\rangle$ is the complex conjugate of $\langle a, b\rangle$ for any $a, b \in \mathcal{A}^{1,0}_{X/Y}$. This Hermitian metric on $\mathcal{A}^1_{X/Y}$ is extended naturally to an Hermitian metric

$$\langle\ ,\ \rangle\colon \mathcal{A}^m_{X/Y} \times \mathcal{A}^m_{X/Y} \longrightarrow \mathcal{A}_X$$

for any m.

We have an Hermitian metric

$$\langle\ ,\ \rangle\colon \mathcal{A}^m_{X/Y}(\mathcal{M}) \times \mathcal{A}^m_{X/Y}(\mathcal{M}) \longrightarrow \mathcal{A}_X$$
$$(a \otimes u, b \otimes v) \mapsto \langle a, b\rangle \cdot \langle u, v\rangle\ (a, b \in \mathcal{A}^m_{X/Y}, u, v \in \mathcal{M}_{\mathcal{A}}).$$

Here $\langle u, v\rangle = i^{p-q}(u, \bar{v})$ when $u \in \mathcal{M}^{p,q}_{\mathcal{A}}$ $(p + q = w)$. In this Hermitian metric, the direct summands $\mathcal{A}^{r,s}_{X/Y} \otimes_{\mathcal{A}_X} \mathcal{M}^{p,q}_{\mathcal{A}}$ $(r + s = m,\ p + q = w)$ are orthogonal to each other, and $\langle \bar{u}, \bar{v}\rangle$ coincides with the complex conjugate of $\langle u, v\rangle$ for any $u, v \in \mathcal{A}^m_{X/Y}(\mathcal{M})$.

We define the star operator

$$*\colon \mathcal{A}^m_{X/Y}(\mathcal{M}) \longrightarrow \mathcal{A}^{2n-m}_{X/Y}(\mathcal{M})$$

by the formula

$$u \wedge *(\bar{v}) = \langle u, v \rangle (i\omega)^n \qquad \text{for } u, v \in \mathcal{A}^m_{X/Y}(\mathcal{M})$$

where ω is the global section of $\mathcal{A}^{1,1}_{X/Y}$ corresponding to the log Kähler metric of X over Y, and $\wedge$ denotes the pairing

$$\mathcal{A}^m_{X/Y}(\mathcal{M}) \otimes \mathcal{A}^{2n-m}_{X/Y}(\mathcal{M}) \longrightarrow \mathcal{A}^{2n}_{X/Y}$$

induced by the exterior product $\mathcal{A}^m_{X/Y} \times \mathcal{A}^{2n-m}_{X/Y} \longrightarrow \mathcal{A}^{2n}_{X/Y}$ and the $\mathcal{O}_X$-bilinear form $(\ ,\ ) \colon \mathcal{M} \times \mathcal{M} \longrightarrow \mathcal{O}_X$.

Then we have

7.1.1. $*$ commutes with complex conjugation.

7.1.2. $*(*(u)) = (-1)^p u$ for $u \in \mathcal{A}^p_{X/Y}(\mathcal{M})$.

7.2. We define an $\mathcal{A}_Y$-submodule $\mathrm{har}^m_{X/Y}(\mathcal{M})$ of $f_*\mathcal{A}^m_{X/Y}(\mathcal{M})$, called the sheaf of harmonic m-forms with coefficients in $\mathcal{M}$, by

$$\mathrm{har}^m_{X/Y}(\mathcal{M}) = \mathrm{Ker}\,(\Delta \colon f_*\mathcal{A}^m_{X/Y}(\mathcal{M}) \longrightarrow f_*\mathcal{A}^m_{X/Y}(\mathcal{M})).$$

Then $\mathrm{har}^m_{X/Y}(\mathcal{M})$ coincides with the intersection of the kernels of the two operators

$$d \colon f_*\mathcal{A}^m_{X/Y}(\mathcal{M}) \longrightarrow f_*\mathcal{A}^{m+1}_{X/Y}(\mathcal{M})$$
$$\delta \colon f_*\mathcal{A}^m_{X/Y}(\mathcal{M}) \longrightarrow f_*\mathcal{A}^{m-1}_{X/Y}(\mathcal{M}).$$

In fact, it is clear that $\mathrm{Ker}\,(d) \cap \mathrm{Ker}\,(\delta) \subset \mathrm{Ker}\,(\Delta)$, and the converse inclusion can be checked on Y_{triv}.

The aim of this section is to prove the following log version of the classical direct decomposition theorem.

Theorem 7.3. *For each $m \in \mathbb{Z}$, we have:*
(1) $f_*\mathcal{A}^m_{X/Y}(\mathcal{M}) = \mathrm{har}^m_{X/Y}(\mathcal{M}) \oplus df_*\mathcal{A}^{m-1}_{X/Y}(\mathcal{M}) \oplus \delta f_*\mathcal{A}^{m+1}_{X/Y}(\mathcal{M})$.
(2) $\mathrm{Ker}\,(d \colon f_*\mathcal{A}^m_{X/Y}(\mathcal{M}) \to f_*\mathcal{A}^{m+1}_{X/Y}(\mathcal{M})) = \mathrm{har}^m_{X/Y}(\mathcal{M}) \oplus df_*\mathcal{A}^{m-1}_{X/Y}(\mathcal{M})$.
(3) $\mathrm{Ker}\,(\delta \colon f_*\mathcal{A}^m_{X/Y}(\mathcal{M}) \to f_*\mathcal{A}^{m-1}_{X/Y}(\mathcal{M})) = \mathrm{har}^m_{X/Y}(\mathcal{M}) \oplus \delta f_*\mathcal{A}^{m+1}_{X/Y}(\mathcal{M})$.
(4) $\Delta \colon f_*\mathcal{A}^m_{X/Y}(\mathcal{M}) \longrightarrow f_*\mathcal{A}^m_{X/Y}(\mathcal{M})$ *induces an automorphism of the space* $df_*\mathcal{A}^{m-1}_{X/Y}(\mathcal{M}) \oplus \delta f_*\mathcal{A}^{m+1}_{X/Y}(\mathcal{M})$.

We prove 7.3 after preliminaries on the L^2-metric on $f_*\mathcal{A}^m_{X/Y}(\mathcal{M})$ (7.4) and on Lie derivatives on $f_*\mathcal{A}^m_{X/Y}(\mathcal{M})$ (7.5).

7.4. We define a pairing

$$\langle\langle\ ,\ \rangle\rangle\colon f_*\mathcal{A}^m_{X/Y}(\mathcal{M}) \times f_*\mathcal{A}^m_{X/Y}(\mathcal{M}) \longrightarrow \mathcal{A}_Y$$

(called the L^2-metric on $f_*\mathcal{A}^m_{X/Y}(\mathcal{M})$) by

$$(u, v) \mapsto \left(y \mapsto \int_{X_y} u \wedge *\bar{v}\right) \qquad (y \in Y_{\mathrm{triv}}, X_y = f^{-1}(y)).$$

Here, we have to show that the function $y \mapsto \int_{X_y} u \wedge *\bar{v}$ is log C^∞. This is reduced to the following

Claim: For $u \in \Gamma(X, \mathcal{A}^{2n}_{X/Y})$, the function $y \mapsto \int_{X_y} u$ $(y \in Y_{\mathrm{triv}})$ is a log C^∞-function on Y.

This follows from the fact that the above function coincides with the image of u under

$$f_*\mathcal{A}^{2n}_{X/Y} \longrightarrow \mathcal{H}^{2n}(f_*\mathcal{A}^\bullet_{X/Y}) \cong \mathcal{A}_Y \otimes_{\mathcal{O}_Y} R^{2n}f_*\omega^\bullet_{X/Y} \text{ (by Proposition 5.1)}$$
$$\longrightarrow \mathcal{A}_Y.$$

(The last homomorphism comes from $R^{2n}f_*\omega^\bullet_{X/Y} \longrightarrow \mathcal{O}_Y$ which follows from the fact that the canonical homomorphism $R^{2n}f^{\log}_*(\mathbb{Z}) \longrightarrow \mathbb{Z}$ on Y_{triv} is canonically extended to $Y^{\log}$.)

The pairing $\langle\langle\ ,\ \rangle\rangle$ satisfies

7.4.1. $\langle\langle u, u\rangle\rangle \geq 0$ for any $u \in f_*\mathcal{A}^m_{X/Y}(\mathcal{M})$.

7.5. Let α be a global section of $\mathcal{A}_{X,1} := \mathcal{A}_{X/\mathbb{C},1}$. Then α is identified with a homomorphism of $\mathcal{A}_X$-modules $\mathcal{A}^1_X \longrightarrow \mathcal{A}_X$. We have a homomorphism of $\mathcal{A}_X$-modules

$$i_\alpha\colon \mathcal{A}^q_X \longrightarrow \mathcal{A}^{q-1}_X$$

characterized by

$$i_\alpha(a_1 \wedge \cdots \wedge a_q) =$$
$$\sum_{j=1}^{q}(-1)^{j-1} \cdot \alpha(a_j) \cdot a_1 \wedge \cdots \wedge a_{j-1} \wedge a_{j+1} \cdots \wedge a_q.$$

Define

$$\partial_\alpha = d \circ i_\alpha + i_\alpha \circ d\colon \mathcal{A}^q_X \longrightarrow \mathcal{A}^q_X.$$

Then we have:

7.5.1. $\partial_\alpha(u \wedge v) = \partial_\alpha(u) \wedge v + u \wedge \partial_\alpha(v)$ for any $u \in \mathcal{A}_X^p$, $v \in \mathcal{A}_X^q$ $(p, q \in \mathbb{Z})$.

7.5.2. $d \circ \partial_\alpha = \partial_\alpha \circ d \colon \mathcal{A}_X^p \longrightarrow \mathcal{A}_X^{p+1}$ $(p \in \mathbb{Z})$.

Let

$$\partial_\alpha \colon \mathcal{A}_X^q(\mathcal{M}) \longrightarrow \mathcal{A}_X^q(\mathcal{M})$$

be the additive map characterized by

$$\partial_\alpha(a \otimes u) = \partial_\alpha(a) \otimes u + a \otimes \partial_\alpha(u) \quad (a \in \mathcal{A}_X^q, u \in \mathcal{M})$$

where $\partial_\alpha(u)$ means the image of u under

$$\mathcal{M} \xrightarrow{\nabla} \omega_X^1 \otimes_{\mathcal{O}_X} \mathcal{M} \xrightarrow{\alpha \otimes \mathrm{id}} \mathcal{A}_X \otimes_{\mathcal{O}_X} \mathcal{M}.$$

Then:

7.5.3. $\partial_\alpha(au) = \partial_\alpha(a)u + a\partial_\alpha(u)$ for any $a \in \mathcal{A}_X$ and $u \in \mathcal{A}_X^p(\mathcal{M})$ $(p \in \mathbb{Z})$,

7.5.4. $d \circ \partial_\alpha = \partial_\alpha \circ d \colon \mathcal{A}_X^p(\mathcal{M}) \longrightarrow \mathcal{A}_X^{p+1}(\mathcal{M})$ $(p \in \mathbb{Z})$.

If $\alpha \colon \mathcal{A}_X^1 \longrightarrow \mathcal{A}_X$ sends $\mathcal{A}_Y^1$ into $\mathcal{A}_Y \subset \mathcal{A}_X$, then it is seen from 7.5.1 that $\partial_\alpha \colon \mathcal{A}_X^q(\mathcal{M}) \longrightarrow \mathcal{A}_X^q(\mathcal{M})$ induces $\mathcal{A}_{X/Y}^q(\mathcal{M}) \longrightarrow \mathcal{A}_{X/Y}^q(\mathcal{M})$. We call this induced map $\partial_\alpha \colon \mathcal{A}_{X/Y}^q(\mathcal{M}) \longrightarrow \mathcal{A}_{X/Y}^q(\mathcal{M})$ the Lie derivative defined by α.

Proposition 7.6. *Let $u \in \Gamma(X_{\mathrm{triv}}, \mathcal{A}_{X/Y}^m(\mathcal{M}))$. Then the following (1) and (2) are equivalent.*

(1) $u \in \Gamma(X, \mathcal{A}_{X/Y}^m(\mathcal{M}))$.

(2) *Locally on Y, for any $k \geq 0$ and any sections $\alpha_1, \ldots, \alpha_k$ of $f_*\mathcal{A}_{X,1}$ which send $\mathcal{A}_Y^1$ into $\mathcal{A}_Y$, the section*

$$\langle\langle \partial_{\alpha_1} \circ \cdots \circ \partial_{\alpha_k}(u), \partial_{\alpha_1} \circ \cdots \circ \partial_{\alpha_k}(u) \rangle\rangle$$

of $\mathcal{A}_Y$ on Y_{triv} is of logarithmic growth on Y.

Proof. It is clear that (1) implies (2). We will reduce the converse to the well-known inequality $\sup |f| \leq (2||f||_{L^2} \cdot ||f'||_{L^2})^{\frac{1}{2}}$ for a compactly supported C^∞-function $f \colon \mathbb{R} \longrightarrow \mathbb{R}$, which is a direct consequence of the Schwartz' inequality. We may assume that Y is Hausdorff. We will prove that u is log C^∞ around a point x of X. Let $y = f(x)$. First note that any non-ket germ $\alpha \in \mathcal{A}_{X,1,x}$ which sends $\mathcal{A}_{Y,y}^1$ into $\mathcal{A}_{Y,y}$ is extended to an $\tilde{\alpha} \in (f_*\mathcal{A}_{X,1})_y$ such that $\tilde{\alpha}_{x'}$ sends $\mathcal{A}_{Y,y}^1$ into $\mathcal{A}_{Y,y}$ for any $x' \in f^{-1}(y)$ and $\tilde{\alpha}_x = \alpha$. This is because $\mathcal{A}_Y^1 \otimes_{\mathcal{A}_Y} \mathcal{A}_X$ is a direct summand of $\mathcal{A}_X^1$ and the non-ket version of $\mathcal{A}_{X/Y,1}$ is soft. Then it is enough to show that the section u satisfying (2) is of log growth.

We will prove that u is of log growth around x. Take an open neighborhood U_0 of x, a ket subneighborhood $U \to U_0$, and $t_1, \ldots, t_{n+d} \in \Gamma(U_0, M_X)$ such that $(d\log(t_i))_{1\leq i\leq n}$ is a basis of $\omega^1_{U_0/Y}$; $(d\log(t_i))_{1\leq i\leq n+d}$ is a basis of $\omega^1_{U_0}$; and such that $\mathcal{M}|_U$ is free. Let $u_i = \arg(t_i)$ and $v_i = \log|t_i|$, $1 \leq i \leq n+d$. Let S be the subset $\{\frac{\partial}{\partial u_i}, \frac{\partial}{\partial v_i}\ ;\ 1 \leq i \leq n\}$ of $\{\frac{\partial}{\partial u_i}, \frac{\partial}{\partial v_i}\ ;\ 1 \leq i \leq n+d\}$, the dual basis of $\{du_i, dv_i\}$. Take a compact subneighborhood $K \subset U$ such that for any $\alpha \in S$, there exists an extension $\tilde{\alpha} \in f_*\mathcal{A}_{X,1}$ such that $\tilde{\alpha}$ sends $\mathcal{A}^1_Y$ into $\mathcal{A}_Y$ and such that α and $\tilde{\alpha}$ coincide on a neighborhood of K in U. Fix such an $\tilde{\alpha}$ for each $\alpha \in S$. Further, multiplying u by a log C^∞-function φ on X such that $\varphi \equiv 1$ near x and $\varphi \equiv 0$ outside the image K_0 of K in U_0, we may suppose that $u \equiv 0$ outside K_0.

Consider the metric on $\mathcal{A}^1_{U/Y}$ such that $\{du_1, dv_1, \ldots, du_n, dv_n\}$ is an orthonormal basis with respect to it. Denote by $\langle\ ,\ \rangle'$ the induced metric on $\mathcal{A}^m_{U/Y}(\mathcal{M})$ and by $\langle\langle\ ,\ \rangle\rangle'$ the induced pairing $f_*\mathcal{A}^m_{U_{\mathrm{triv}}/Y_{\mathrm{triv}}}(\mathcal{M}) \times f_*\mathcal{A}^m_{U_{\mathrm{triv}}/Y_{\mathrm{triv}}}(\mathcal{M}) \longrightarrow \mathcal{A}_{Y_{\mathrm{triv}}} : (u,v) \mapsto \left(y \mapsto \int_{K_y} u \wedge *\bar{v}\right)$ $(y \in Y_{\mathrm{triv}}, K_y = f^{-1}(y) \cap K)$.

We prove that u satisfies the following condition $(2)'$.

$(2)'$: For any $\alpha_1, \ldots, \alpha_k \in S$, $\langle\langle \partial_{\alpha_1} \circ \cdots \circ \partial_{\alpha_k}(u), \partial_{\alpha_1} \circ \cdots \circ \partial_{\alpha_k}(u)\rangle\rangle'$ is of log growth.

Taking a function h of log growth on X such that $\langle\langle v,v\rangle\rangle' \leq \sup\limits_{f^{-1}(y)} |h| \cdot \langle\langle v,v\rangle\rangle$ for any $v \in f_*\mathcal{A}^m_{X_{\mathrm{triv}}/Y_{\mathrm{triv}}}(\mathcal{M})$ ($\sup\limits_{f^{-1}(y)}|h|$ denotes the function $y \mapsto \sup\limits_{x\in f^{-1}(y)} |h(x)|$ on Y_{triv}), we see that the function in $(2)'$ is pointwise less than $y \mapsto \sup\limits_{f^{-1}(y)} |h| \cdot \langle\langle \partial_{\tilde{\alpha_1}} \circ \cdots \circ \partial_{\tilde{\alpha_k}}(u), \partial_{\tilde{\alpha_1}} \circ \cdots \circ \partial_{\tilde{\alpha_k}}(u)\rangle\rangle$ $(y \in Y_{\mathrm{triv}})$. Since f is vertical, $\sup|h|$ is of log growth on Y. Thus u satisfies $(2)'$.

The rest is to show that this condition $(2)'$ implies that u is of log growth. Take an orthonormal basis $(e_i)_i$ of $\mathcal{M}_{\mathcal{A}}|_U$. Writing $u = \sum f_i e_i$, we see that $(2)'$ for u implies $(2)'$ for each f_i by induction on k. Hence we may assume that $\mathcal{M} = \mathcal{O}_X$. Then we may assume that $m = 0$. By repeated use of the usual Schwartz' inequality on the real line, we have $\sup\limits_{K_y}|u| \leq c_0 \prod_{I\subset S}\langle\langle \partial_I(u), \partial_I(u)\rangle\rangle'^{2^{-2n}}$ for a positive constant c_0. Here $\partial_I(u) = \partial_{\alpha_1} \circ \cdots \circ \partial_{\alpha_k}(u)$ when we denote by $\alpha_1, \ldots, \alpha_k$ all the distinct elements of I. Hence u is of log growth. Q.E.D.

7.7. We prove Theorem 7.3 (1). Let $j : Y_{\mathrm{triv}} \longrightarrow Y$ be the canonical morphism. Theorem 7.3 (1) is true in the case $Y = Y_{\mathrm{triv}}$ (that implies $X = X_{\mathrm{triv}}$ since f is vertical) by the classical theory (Deligne, [31]).

Hence for $u \in f_*\mathcal{A}^m_{X/Y}(\mathcal{M})$, we have a unique decomposition

$$u = p_h(u) + p_d(u) + p_\delta(u) \qquad \text{in } j_*j^*f_*\mathcal{A}^m_{X/Y}(\mathcal{M})$$

where

$$p_h(u) \in j_*j^*\mathrm{har}^m_{X/Y}(\mathcal{M}),$$
$$p_d(u) \in j_*j^*df_*\mathcal{A}^{m-1}_{X/Y}(\mathcal{M}), \qquad p_\delta(u) \in j_*j^*\delta f_*\mathcal{A}^{m+1}_{X/Y}(\mathcal{M}).$$

By Proposition 5.1 and Theorem 5.2, $\mathcal{H}^m(f_*\mathcal{A}^\bullet_{X/Y}(\mathcal{M})) \cong \mathcal{A}_Y \otimes_{\mathcal{O}_Y} R^m f_*\omega^\bullet_{X/Y}(\mathcal{M})$ is a locally free $\mathcal{A}_Y$-module, and hence the map $\mathcal{H}^m(f_*\mathcal{A}^\bullet_{X/Y}(\mathcal{M})) \longrightarrow j_*j^*\mathcal{H}^m(f_*\mathcal{A}^\bullet_{X/Y}(\mathcal{M}))$ is injective. Hence we have

7.7.1. $j_*j^*df_*\mathcal{A}^{m-1}_{X/Y}(\mathcal{M}) \cap f_*\mathcal{A}^m_{X/Y}(\mathcal{M}) = df_*\mathcal{A}^{m-1}_{X/Y}(\mathcal{M})$,

and (applying the star operator $*$ to this) we also have the similar equality concerning δ. These imply that, for the proof of 7.3 (1), it is sufficient to show that $p_h(u)$, $p_d(u)$, and $p_\delta(u)$ belong to $f_*\mathcal{A}^m_{X/Y}(\mathcal{M})$. We prove that $p_d(u)$ belongs to $f_*\mathcal{A}^m_{X/Y}(\mathcal{M})$. (Then this will show that $p_\delta(u) = (-1)^m * (p_d(*u))$ belongs to $f_*\mathcal{A}^m_{X/Y}(\mathcal{M})$, and hence $p_h(u) = u - p_d(u) - p_\delta(u)$ also belongs to $f_*\mathcal{A}^m_{X/Y}(\mathcal{M})$.)

Let $\alpha_1, \ldots, \alpha_k$ be sections of $f_*\mathcal{A}_{X,1}$ which send $\mathcal{A}^1_Y$ into $\mathcal{A}_Y$. By Proposition 7.6, it is sufficient to show that

$$\langle\langle \partial_{\alpha_1} \circ \cdots \circ \partial_{\alpha_k} \circ p_d(u), \partial_{\alpha_1} \circ \cdots \circ \partial_{\alpha_k} \circ p_d(u) \rangle\rangle \in j_*C^\infty_{Y_{\mathrm{triv}}}$$

is of logarithmic growth. Let

$$l_j = (-1)^m * \circ \partial_{\alpha_j} \circ * - \partial_{\alpha_j} : \mathcal{A}^m_{X/Y}(\mathcal{M}) \longrightarrow \mathcal{A}^m_{X/Y}(\mathcal{M}).$$

Then l_j is a homomorphism of $\mathcal{A}_X$-modules. We have

7.7.2. $\partial_{\alpha_j} \circ p_d = p_d \circ (\partial_{\alpha_j} + l_j - l_j \circ p_d)$ on $j_*j^*f_*\mathcal{A}^m_{X/Y}(\mathcal{M})$.

We prove 7.7.2. Since ∂_{α_j} commutes with d, ∂_{α_j} preserves $j_*j^*df_*\mathcal{A}^{m-1}_{X/Y}(\mathcal{M})$ (resp. $\mathrm{Ker}\,(d\colon\ j_*j^*f_*\mathcal{A}^m_{X/Y}(\mathcal{M}) \longrightarrow j_*j^*f_*\mathcal{A}^{m+1}_{X/Y}(\mathcal{M})) = j_*j^*\mathrm{har}^m_{X/Y}(\mathcal{M}) \oplus j_*j^*df_*\mathcal{A}^{m-1}_{X/Y}(\mathcal{M})$), and this shows the following 7.7.3 (resp. 7.7.4).

7.7.3. $p_d \circ \partial_{\alpha_j} \circ p_d = \partial_{\alpha_j} \circ p_d$ on $j_*j^*f_*\mathcal{A}^m_{X/Y}(\mathcal{M})$.

7.7.4. $(1 - p_\delta) \circ \partial_{\alpha_j} \circ (1 - p_\delta) = \partial_{\alpha_j} \circ (1 - p_\delta)$ on $j_*j^*f_*\mathcal{A}^m_{X/Y}(\mathcal{M})$.

By taking $(-1)^m * \circ (7.7.4) \circ *$, we obtain

$$(1 - p_d) \circ (\partial_{\alpha_j} + l_j) \circ (1 - p_d) = (\partial_{\alpha_j} + l_j) \circ (1 - p_d),$$

which can be rewritten as

7.7.5. $p_d \circ (\partial_{\alpha_j} + l_j) \circ p_d = p_d \circ (\partial_{\alpha_j} + l_j)$.
7.7.2 is obtained by taking (7.7.5) minus (7.7.3).

Now by 7.7.2 and by the fact that $\partial_{\alpha_i} \circ a - a \circ \partial_{\alpha_i} : \mathcal{A}^m_{X/Y}(\mathcal{M}) \longrightarrow \mathcal{A}^m_{X/Y}(\mathcal{M})$ is a homomorphism of $\mathcal{A}_X$-modules for any i and any $\mathcal{A}_X$-homomorphism $a: \mathcal{A}^m_{X/Y}(\mathcal{M}) \longrightarrow \mathcal{A}^m_{X/Y}(\mathcal{M})$, we have:
7.7.6. $\partial_{\alpha_1} \circ \cdots \circ \partial_{\alpha_k} \circ p_d$ is a finite sum of elements of the form

$$s_1 \circ \cdots \circ s_q \circ t_1 \circ \cdots \circ t_r \qquad (q \geq 0, r \geq 0)$$

where each s_i is an operator $j_*j^* f_* \mathcal{A}^m_{X/Y}(\mathcal{M}) \longrightarrow j_*j^* f_* \mathcal{A}^m_{X/Y}(\mathcal{M})$ of the form $p_d \circ a_i$ where a_i is a homomorphism of $\mathcal{A}_X$-modules $\mathcal{A}^m_{X/Y}(\mathcal{M}) \longrightarrow \mathcal{A}^m_{X/Y}(\mathcal{M})$, and each t_i is ∂_{α_j} for some j.

Locally on Y, there exists a log C^∞-function b_i on Y such that
7.7.7. $\langle\langle a_i v, a_i v \rangle\rangle \leq |b_i| \cdot \langle\langle v, v \rangle\rangle$ for any $v \in j_*j^* f_* \mathcal{A}^m_{X/Y}(\mathcal{M})$.
($|b_i|$ denotes the function $y \mapsto |b_i(y)|$ on Y_{triv}.) Since $j_*j^* \text{har}^m_{X/Y}(\mathcal{M})$, $j_*j^* df_* \mathcal{A}^{m-1}_{X/Y}(\mathcal{M})$ and $j_*j^* \delta f_* \mathcal{A}^{m+1}_{X/Y}(\mathcal{M})$ are orthogonal under the pairing

$$\langle\langle\ ,\ \rangle\rangle : j_*j^* f_* \mathcal{A}^m_{X/Y}(\mathcal{M}) \times j_*j^* f_* \mathcal{A}^m_{X/Y}(\mathcal{M}) \longrightarrow j_* C^\infty_{Y_{\text{triv}}}$$

(by the classical theory), we have
7.7.8. $\langle\langle p_d(v), p_d(v) \rangle\rangle \leq \langle\langle v, v \rangle\rangle$ for any $v \in j_*j^* f_* \mathcal{A}^m_{X/Y}(\mathcal{M})$.
By 7.7.7 and 7.7.8, locally on Y, there exists a log C^∞-function b on Y such that

$$\begin{aligned} &\langle\langle s_1 \circ \cdots \circ s_q \circ t_1 \circ \cdots \circ t_r(u), s_1 \circ \cdots \circ s_q \circ t_1 \circ \cdots \circ t_r(u) \rangle\rangle \\ &\leq |b| \cdot \langle\langle t_1 \circ \cdots \circ t_r(u), t_1 \circ \cdots \circ t_r(u) \rangle\rangle. \end{aligned}$$

Since $t_1 \circ \cdots \circ t_r(u)$ is log C^∞, $\langle\langle t_1 \circ \cdots \circ t_r(u), t_1 \circ \cdots \circ t_r(u) \rangle\rangle$ is of log growth. Hence by 7.7.6, $\langle\langle \partial_{\alpha_1} \circ \cdots \circ \partial_{\alpha_k} \circ p_d(u), \partial_{\alpha_1} \circ \cdots \circ \partial_{\alpha_k} \circ p_d(u) \rangle\rangle$ is of log growth.

7.8. We prove Theorem 7.3 (2), (3). Since (3) is obtained by applying the star operator $*$ to (2), it is sufficient to prove (2). Let $u \in f_* \mathcal{A}^m_{X/Y}(\mathcal{M})$, and assume $du = 0$. Then $u - p_h(u)$ is in $j_*j^* df_* \mathcal{A}^{m-1}_{X/Y}(\mathcal{M})$. By 7.7.1, this shows that $u - p_h(u)$ belongs to $df_* \mathcal{A}^{m-1}_{X/Y}(\mathcal{M})$.

7.9. We prove Theorem 7.3 (4). Since $\text{har}^m_{X/Y}(\mathcal{M}) = \text{Ker}\,(\Delta \text{ on } f_* \mathcal{A}^m_{X/Y}(\mathcal{M}))$, the injectivity of Δ on $df_* \mathcal{A}^{m-1}_{X/Y}(\mathcal{M}) \oplus \delta f_* \mathcal{A}^{m+1}_{X/Y}(\mathcal{M})$ is clear. We prove the surjectivity of Δ on $df_* \mathcal{A}^{m-1}_{X/Y}(\mathcal{M}) \oplus \delta f_* \mathcal{A}^{m+1}_{X/Y}(\mathcal{M})$. Let

$u \in df_*\mathcal{A}^{m-1}_{X/Y}(\mathcal{M}) \oplus \delta f_*\mathcal{A}^{m+1}_{X/Y}(\mathcal{M})$. Then of course $u = du_1 + \delta u_2$ for some $u_1 \in f_*\mathcal{A}^{m-1}_{X/Y}(\mathcal{M})$ and $u_2 \in f_*\mathcal{A}^{m+1}_{X/Y}(\mathcal{M})$. Write $p_\delta(u_1) = \delta(v_1)$ and $p_d(u_2) = d(v_2)$ for v_1, $v_2 \in f_*\mathcal{A}^m_{X/Y}(\mathcal{M})$. Then

$$\begin{aligned}\Delta(p_d(v_1) + p_\delta(v_2)) &= (d\delta + \delta d)p_d(v_1) + (d\delta + \delta d)p_\delta(v_2)\\ &= d\delta p_d(v_1) + \delta d p_\delta(v_2)\\ &= d\delta(v_1) + \delta d(v_2)\\ &= du_1 + \delta u_2.\end{aligned}$$

Example 7.10. Take $f\colon E \longrightarrow \Delta$ in 2.14 as $f\colon X \longrightarrow Y$ here. Then X has a log Kähler metric over Y corresponding to the $(1,1)$-form $d\log(u) \wedge d\log(\overline{u}) \in \Gamma(X, \mathcal{A}^{1,1}_{X/Y})$ (u is as in 2.14). For this log Kähler metric, $*\colon \mathcal{A}^m_{X/Y} \longrightarrow \mathcal{A}^{2-m}_{X/Y}$ ($m \geq 0$) are $\mathcal{A}_X$-linear maps which operate on the bases of $\mathcal{A}^m_{X/Y}$ as

$$\begin{aligned}&*(1) = id\log(u) \wedge d\log(\overline{u}),\ *(d\log(u)) = -id\log(u),\\ &*(d\log(\overline{u})) = id\log(\overline{u}),\ *(d\log(u) \wedge d\log(\overline{u})) = -i,\end{aligned}$$

and the Laplacian $\Delta\colon \mathcal{A}^m_{X/Y} \longrightarrow \mathcal{A}^m_{X/Y}$ ($m \geq 0$) are described as

$$\begin{aligned}&\Delta(g) = -2(u\tfrac{\partial}{\partial u})(\overline{u}\tfrac{\partial}{\partial \overline{u}})(g) \text{ for } g \in \mathcal{A}_X,\\ &\Delta(gd\log(u)) = \Delta(g)d\log(u),\ \Delta(gd\log(\overline{u})) = \Delta(g)d\log(\overline{u}),\\ &\Delta(gd\log(u) \wedge d\log(\overline{u})) = \Delta(g)d\log(u) \wedge d\log(\overline{u}) \text{ for } g \in \mathcal{A}_X.\end{aligned}$$

If we take as $(\mathcal{H}_\mathbb{Z}, \mathcal{M}, (\ ,\ ))$ the "unit object" on X ($\mathcal{H}_\mathbb{Z} = \mathbb{Z}$, $\mathcal{M} = \mathcal{O}_X$, and the filtration on $\mathcal{M}$ and $(\ ,\ )$ are the evident ones),

$$\begin{aligned}&\mathrm{har}^0_{X/Y}(\mathcal{M}) = \mathcal{A}_Y,\\ &\mathrm{har}^1_{X/Y}(\mathcal{M}) = \mathcal{A}_Y d\log(u) + \mathcal{A}_Y d\log(\overline{u}),\\ &\mathrm{har}^2_{X/Y}(\mathcal{M}) = \mathcal{A}_Y d\log(u) \wedge d\log(\overline{u}).\end{aligned}$$

Let

$$g = \exp(2\pi i \cdot \log(|u|)/\log(|t|)) \text{ where } t = uv$$

(t is the coordinate function on $Y = \Delta$). Then

$$\Delta(g) = 2\pi^2 \log(|t|)^{-2} g.$$

The inverse of Δ (Green operator) on $\mathrm{Image}(d) + \mathrm{Image}(\delta)$ (Theorem 7.3 (4)) sends g to $(2\pi^2)^{-1}\log(|t|)^2 g$, and we see that Theorem 7.3 (4) is related to the fact that the inverse $(2\pi^2)^{-1}\log(|t|)^2$ of the non-zero eigen value $2\pi^2\log(|t|)^{-2}$ of the Laplacian is of log growth.

§8. Higher direct images of variations of polarized log Hodge structure

Everything here is in the ket sense except 8.11–8.14. In this section we prove:

Theorem 8.1. *Let $f: X \longrightarrow Y$ be a projective vertical log smooth morphism between log smooth fs log analytic spaces. Let $(\mathcal{H}_{\mathbb{Z}}, \mathcal{M}, (\ ,\ ))$ be a VPLH on X of weight w. For $m \in \mathbb{Z}$, let*

$$\mathcal{L}_m = R^m f_*^{\log} \mathcal{H}_{\mathbb{Z}}, \quad \mathcal{V}_m = R^m f_*(\omega^\bullet_{X/Y}(\mathcal{M})),$$

which are in the log Riemann-Hilbert correspondence (Theorem 5.2). For $p \in \mathbb{Z}$, let $\mathrm{Fil}^p(\omega^\bullet_{X/Y}(\mathcal{M}))$ be the subcomplex $(\omega^q_{X/Y} \otimes_{\mathcal{O}_X} \mathcal{M}^{p-q})^q$ of $\omega^\bullet_{X/Y}(\mathcal{M})$, and let $\mathrm{gr}^p(\omega^\bullet_{X/Y}(\mathcal{M})) = \mathrm{Fil}^p(\omega^\bullet_{X/Y}(\mathcal{M}))/\mathrm{Fil}^{p+1}(\omega^\bullet_{X/Y}(\mathcal{M}))$.
(1) *The Hodge to de Rham spectral sequence*

$$E_1^{p,q} = R^{p+q} f_* \mathrm{gr}^p(\omega^\bullet_{X/Y}(\mathcal{M})) \Rightarrow E_\infty^m = \mathcal{V}_m$$

degenerates from E_1, and each $R^m f_* \mathrm{gr}^p(\omega^\bullet_{X/Y}(\mathcal{M}))$ ($m,\ p \in \mathbb{Z}$) is a locally free $\mathcal{O}_Y$-module. Consequently, for any $m,\ p \in \mathbb{Z}$, the canonical map $R^m f_* \mathrm{Fil}^p(\omega^\bullet_{X/Y}(\mathcal{M})) \longrightarrow R^m f_*(\omega^\bullet_{X/Y}(\mathcal{M}))$ is injective and the image is locally an $\mathcal{O}_Y$-direct summand of $R^m f_*(\omega^\bullet_{X/Y}(\mathcal{M}))$.
(2) *Fix an invertible $\mathcal{O}_X$-module $\mathcal{L}$ which is relatively very ample with respect to Y, and for $m \in \mathbb{Z}$, let*

$$(\ ,\ ): \mathcal{L}_{m,\mathbb{Q}} \times \mathcal{L}_{m,\mathbb{Q}} \longrightarrow \mathbb{Q}$$

be the pairing below in 8.2 defined by $\mathcal{L}$. Then, with the Hodge filtration $R^m f_* \mathrm{Fil}^p(\omega^\bullet_{X/Y}(\mathcal{M}))$ on $\mathcal{V}_m$,

$$(\mathcal{L}_m,\ \mathcal{V}_m,\ (\ ,\ ))$$

is a VPLH of weight $w + m$ on Y.

8.2. Let the situation be as in the above theorem.

Define a pairing

$$(\ ,\ ): \mathcal{L}_{m,\mathbb{Q}} \times \mathcal{L}_{m,\mathbb{Q}} \longrightarrow \mathbb{Q}$$

as follows. We will assume that the relative dimension n of X over Y is constant (the beginning of section 7). The definition is obviously extended to the general case. In the case $m \leq n$, define $\mathcal{L}_{m,\mathbb{Q},\mathrm{prim}}$ to be the kernel of

$$c(\mathcal{L})^{n-m+1}: \mathcal{L}_{m,\mathbb{Q}} \longrightarrow \mathcal{L}_{2n-m+2,\mathbb{Q}}.$$

Then for any m,

$$\bigoplus_j \mathcal{L}_{j,\mathbb{Q},\mathrm{prim}} \longrightarrow \mathcal{L}_{m,\mathbb{Q}} \ ; \ (a_j)_j \longmapsto \sum_j c(\mathcal{L})^{(m-j)/2} \cdot a_j$$

is an isomorphism where j ranges over all integers such that $j \leq n$, $j \leq m$, and $j \equiv m \bmod 2$. (This is proved by restricting to Y_{triv}.) Let $(\ ,\ ) \colon \mathcal{L}_{m,\mathbb{Q}} \times \mathcal{L}_{m,\mathbb{Q}} \longrightarrow \mathbb{Q}$ be the unique $\mathbb{Q}$-bilinear form such that the subspaces $c(\mathcal{L})^{(m-j)/2}\mathcal{L}_{j,\mathbb{Q},\mathrm{prim}}$ of $\mathcal{L}_{m,\mathbb{Q}}$ for j as above are orthogonal to each other under $(\ ,\ )$ and

$$(c(\mathcal{L})^{(m-j)/2}u, c(\mathcal{L})^{(m-j)/2}v)$$

for j as above and for $u, v \in \mathcal{L}_{j,\mathbb{Q},\mathrm{prim}}$ is the image of $(-1)^{j(j-1)/2}u \otimes v$ under

$$\begin{aligned} \mathcal{L}_{j,\mathbb{Q}} \otimes \mathcal{L}_{j,\mathbb{Q}} \longrightarrow R^{2j}f_*^{\log}(\mathcal{H}_{\mathbb{Q}} \otimes \mathcal{H}_{\mathbb{Q}}) &\longrightarrow R^{2j}f_*^{\log}\mathbb{Q} \quad (\text{by } (\ ,\ ) \text{ of } \mathcal{H}_{\mathbb{Z}}) \\ &\longrightarrow R^{2n}f_*^{\log}\mathbb{Q} \qquad (\text{by } c(\mathcal{L})^{n-j}) \\ &\longrightarrow \mathbb{Q}. \end{aligned}$$

8.3. Since f is vertical, $X_{\mathrm{triv}} \longrightarrow Y_{\mathrm{triv}}$ is projective. Hence, by Deligne ([5], [31]), the restriction of $(\mathcal{L}_m, (R^m f_* \mathrm{Fil}^p(\omega^\bullet_{X/Y}(\mathcal{M})))_{p\in\mathbb{Z}}, (\ ,\ ))$ to Y_{triv} is a VPH of weight $w+m$. By Proposition 2.5, this VPH extends to a VPLH on Y of weight $w+m$. This extension must have the form $(\mathcal{L}_m, (\mathcal{V}_m^p)_{p\in\mathbb{Z}}, (\ ,\ ))$ for some $\mathcal{O}_Y$-submodules $\mathcal{V}_m^p$ of $\mathcal{V}_m$ such that each $\mathcal{V}_m^p$ is locally free and is locally a direct summand of $\mathcal{V}_m$. We will see below that the canonical map $R^m f_* \mathrm{Fil}^p(\omega^\bullet_{X/Y}(\mathcal{M})) \longrightarrow \mathcal{V}_m^p$ is an isomorphism.

For the proof of 8.1, it is sufficient to prove 8.1 (1). In fact, if we prove 8.1 (1), then we have $R^m f_* \mathrm{Fil}^p(\omega^\bullet_{X/Y}(\mathcal{M})) = \mathcal{V}_m^p$, and hence we obtain 8.1 (2).

Note that 8.1 (1) is a local problem on Y and we may suppose that the relative dimension is constant.

8.4. Locally on Y, take a blowing up $g \colon Z \longrightarrow X$ along the log structure such that Z is a complex manifold and the log structure of Z is given by a divisor with normal crossings whose irreducible components are non-singular and the number of whose irreducible components is finite. Since $Rg_*g^*\mathcal{F} = \mathcal{F}$ for any locally free $\mathcal{O}_X$-module $\mathcal{F}$ of finite rank, the proof of 8.1 (1) is reduced to the case $X = Z$.

In the rest of section 8, we assume that X satisfies the condition on Z in the above and we fix $\mathcal{L}$. Further, shrinking Y, we take and fix a log Kähler metric on X over Y related to $\mathcal{L}$ as in section 7.

8.5. For $p, q \in \mathbb{Z}$ such that $p+q = w+m$, let

$$f_*\mathcal{A}^m_{X/Y}(\mathcal{M})^{p,q} = \bigoplus f_*(\mathcal{A}^{r,s}_{X/Y} \otimes_{\mathcal{A}_X} \mathcal{M}^{j,k}_{\mathcal{A}}) \subset f_*\mathcal{A}^m_{X/Y}(\mathcal{M}),$$

where r, s, j, k range integers satisfying $r+s = m$, $j+k = w$, $r+j = p$, $s+k = q$, and let

$$\mathrm{har}^m_{X/Y}(\mathcal{M})^{p,q} = \mathrm{har}^m_{X/Y}(\mathcal{M}) \cap f_*\mathcal{A}^m_{X/Y}(\mathcal{M})^{p,q} \subset f_*\mathcal{A}^m_{X/Y}(\mathcal{M}).$$

Since $\Delta\colon f_*\mathcal{A}^m_{X/Y}(\mathcal{M}) \longrightarrow f_*\mathcal{A}^m_{X/Y}(\mathcal{M})$ preserves $f_*\mathcal{A}^m_{X/Y}(\mathcal{M})^{p,q}$ for any p, q such that $p+q = w+m$ (this is reduced to the classical situation on Y_{triv} described in [31]), we have

8.5.1. $\mathrm{har}^m_{X/Y}(\mathcal{M}) = \bigoplus_{p+q=w+m} \mathrm{har}^m_{X/Y}(\mathcal{M})^{p,q}$.

8.6. By Griffiths transversality as in [31] pp.420–421, the map $d\colon \mathcal{A}^m_{X/Y}(\mathcal{M}) \longrightarrow \mathcal{A}^{m+1}_{X/Y}(\mathcal{M})$ sends $\mathcal{A}^m_{X/Y}(\mathcal{M})^{p,q}$ $(p+q = w+m)$ into $\mathcal{A}^{m+1}_{X/Y}(\mathcal{M})^{p+1,q} \oplus \mathcal{A}^{m+1}_{X/Y}(\mathcal{M})^{p,q+1}$. Hence d can be written as

$$d = d' + d''$$

in the unique way where d' and d'' are additive maps $\mathcal{A}^m_{X/Y}(\mathcal{M}) \longrightarrow \mathcal{A}^{m+1}_{X/Y}(\mathcal{M})$ such that

$$d'(\mathcal{A}^m_{X/Y}(\mathcal{M})^{p,q}) \subset \mathcal{A}^{m+1}_{X/Y}(\mathcal{M})^{p+1,q}, d''(\mathcal{A}^m_{X/Y}(\mathcal{M})^{p,q}) \subset \mathcal{A}^{m+1}_{X/Y}(\mathcal{M})^{p,q+1}$$

$(p+q = w+m)$. Let

$$\delta'(u) = -*d''*(u), \quad \delta''(u) = -*d'*(u)$$

for $u \in \mathcal{A}^m_{X/Y}(\mathcal{M})$. Then we have:

8.6.1. $d' \circ d' = 0$, $d'' \circ d'' = 0$.

8.6.2. d', d'', δ', δ'' kill $\mathrm{har}^m_{X/Y}(\mathcal{M})$.

8.6.3. $\Delta = 2(d'\delta' + \delta' d') = 2(d''\delta'' + \delta'' d'')$.

8.6.4. $\Delta d' = d'\Delta$, $\Delta d'' = d''\Delta$, $\Delta\delta' = \delta'\Delta$, $\Delta\delta'' = \delta''\Delta$.

These 8.6.1–8.6.4 are proved by restricting to Y_{triv}.

Proposition 8.7. *The canonical map from* $\mathrm{har}^m_{X/Y}(\mathcal{M})$ *to the m-th cohomology sheaf of the complex* $(f_*\mathcal{A}^\bullet_{X/Y}(\mathcal{M}), d')$ *(resp.* $(f_*\mathcal{A}^\bullet_{X/Y}(\mathcal{M}), d'')$*) is an isomorphism.*

Proof. We consider the case of d' (the proof for the case of d'' is similar.) The injectivity can be checked by restricting to Y_{triv} and

reducing to the classical theory (Deligne, [31]). We prove the surjectivity. Let $u \in f_*\mathcal{A}^m_{X/Y}(\mathcal{M})$ and assume $d'u = 0$. By Theorem 7.3 (4), there exists $v \in f_*\mathcal{A}^m_{X/Y}(\mathcal{M})$ such that $u - p_h(u) = \Delta(v)$ (p_h is as in 7.7). It is sufficient to prove that $u - 2d'\delta'v$ belongs to $\mathrm{har}^m_{X/Y}(\mathcal{M})$, that is, $\Delta(u) = 2\Delta d'\delta' v$. We have

$$\begin{aligned}
\Delta d'\delta' v &= d'\delta'\Delta v && (8.6.4)\\
&= d'\delta'(u - p_h(u))\\
&= d'\delta' u && (8.6.2)\\
&= (d'\delta' + \delta' d')u\\
&= \tfrac{1}{2}\Delta u && (8.6.3).
\end{aligned}$$

Q.E.D.

By 8.5.1, Proposition 8.7 shows

Corollary 8.8. *For m, p, $q \in \mathbb{Z}$, such that $p+q = w+m$, the canonical map from $\mathrm{har}^m_{X/Y}(\mathcal{M})^{p,q}$ to the m-th cohomology sheaf of the complex*

$$f_*(\mathcal{A}_X(\mathcal{M})^{w-q,q} \xrightarrow{d'} \mathcal{A}^1_{X/Y}(\mathcal{M})^{w+1-q,q} \xrightarrow{d'} \mathcal{A}^2_{X/Y}(\mathcal{M})^{w+2-q,q} \xrightarrow{d'} \cdots)$$

(resp.

$$f_*(\mathcal{A}_X(\mathcal{M})^{p,w-p} \xrightarrow{d''} \mathcal{A}^1_{X/Y}(\mathcal{M})^{p,w+1-p} \xrightarrow{d''} \mathcal{A}^2_{X/Y}(\mathcal{M})^{p,w+2-p} \xrightarrow{d''} \cdots))$$

is an isomorphism.

8.9. By the Hodge decomposition in 2.6 applied to the VPLH $(\mathcal{L}_m, (\mathcal{V}^p_m)_{p\in\mathbb{Z}}, (\ ,\ ))$ on Y (8.2), we have a Hodge decomposition

8.9.1. $$\mathcal{A}_Y \otimes_{\mathcal{O}_Y} \mathcal{V}_m = \bigoplus_{p+q=w+m} \mathcal{V}^{p,q}_{m,\mathcal{A}}$$

where $\mathcal{V}^{p,q}_{m,\mathcal{A}}$ denotes the intersection of $\mathcal{A}_Y \otimes_{\mathcal{O}_Y} \mathcal{V}^p_m$ and the complex conjugate of $\mathcal{A}_Y \otimes_{\mathcal{O}_Y} \mathcal{V}^q_m$. If we identify $\mathcal{A}_Y \otimes_{\mathcal{O}_Y} \mathcal{V}_m$ with $\mathrm{har}^m_{X/Y}(\mathcal{M})$ via the canonical isomorphism, the decomposition 8.9.1 coincides with the decomposition 8.5.1. (To see this, it is enough to show that the projectors of the direct decompositions coincide, but the coincidence of the projectors can be checked on Y_{triv}, and hence we are reduced to the classical theory on Y_{triv}.) In particular, we have

8.9.2. $$\mathcal{A}_Y \otimes_{\mathcal{O}_Y} \mathcal{V}^p_m/\mathcal{V}^{p+1}_m \cong \mathcal{V}^{p,w+m-p}_{m,\mathcal{A}} \cong \mathrm{har}^m_{X/Y}(\mathcal{M})^{p,w+m-p}.$$

8.10. Now we prove Theorem 8.1 (1).

Fix $p \in \mathbb{Z}$. By the log $\overline{\partial}$-Poincaré lemma on X (3.1), we have an exact sequence of complexes

8.10.1.

$$0 \longrightarrow \mathrm{gr}^p(\omega^\bullet_{X/Y}(\mathcal{M})) \longrightarrow \mathcal{A}_X \otimes_{\mathcal{O}_X} \mathrm{gr}^p(\omega^\bullet_{X/Y}(\mathcal{M}))$$
$$\xrightarrow{\overline{\partial}} \mathcal{A}_X^{0,1} \otimes_{\mathcal{O}_X} \mathrm{gr}^p(\omega^\bullet_{X/Y}(\mathcal{M})) \xrightarrow{\overline{\partial}} \mathcal{A}_X^{0,2} \otimes_{\mathcal{O}_X} \mathrm{gr}^p(\omega^\bullet_{X/Y}(\mathcal{M})) \longrightarrow \cdots .$$

Let $\mathcal{H}^m$ be the m-th cohomology sheaf of the complex

$$f_*(\mathcal{A}_X(\mathcal{M})^{p,w-p} \xrightarrow{d''} \mathcal{A}^1_{X/Y}(\mathcal{M})^{p,w+1-p} \xrightarrow{d''} \mathcal{A}^2_{X/Y}(\mathcal{M})^{p,w+2-p} \to \cdots).$$

From 8.10.1, since $\mathcal{A}_X^{0,q}$ $(q \in \mathbb{Z})$ has a descending filtration whose r-th graded quotient is $\mathcal{A}^{0,q-r}_{X/Y} \otimes_{\mathcal{A}_Y} \mathcal{A}_Y^{0,r}$ for any $r \in \mathbb{Z}$, and since $\mathcal{A}_X \otimes_{\mathcal{O}_X} \omega^m_{X/Y} = \mathcal{A}^{m,0}_{X/Y}$ for $m \in \mathbb{Z}$, we obtain a spectral sequence

8.10.2. $$E_1^{s,t} = \mathcal{A}_Y^{0,s} \otimes_{\mathcal{A}_Y} \mathcal{H}^t \Longrightarrow E_\infty^m = R^m f_* \mathrm{gr}^p(\omega^\bullet_{X/Y}(\mathcal{M}))$$

in which $E_1^{s,t} \longrightarrow E_1^{s+1,t}$ is

$$\overline{\partial}\colon \mathcal{A}_Y^{0,s} \otimes_{\mathcal{A}_Y} \mathcal{H}^t \longrightarrow \mathcal{A}_Y^{0,s+1} \otimes_{\mathcal{A}_Y} \mathcal{H}^t.$$

By Corollary 8.8 and 8.9.2, we have

8.10.3. $$\mathcal{H}^m \cong \mathrm{har}^m_{X/Y}(\mathcal{M})^{p,w+m-p} \cong \mathcal{A}_Y \otimes_{\mathcal{O}_Y} \mathcal{V}^p_m/\mathcal{V}^{p+1}_m.$$

Hence the complex $E_1^{\bullet,m}$ in 8.10.2 is rewritten as

$$\mathcal{A}_Y \otimes_{\mathcal{O}_Y} \mathcal{V}^p_m/\mathcal{V}^{p+1}_m \xrightarrow{\overline{\partial}} \mathcal{A}_Y^{0,1} \otimes_{\mathcal{A}_Y} \mathcal{V}^p_m/\mathcal{V}^{p+1}_m \xrightarrow{\overline{\partial}} \mathcal{A}_Y^{0,2} \otimes_{\mathcal{A}_Y} \mathcal{V}^p_m/\mathcal{V}^{p+1}_m \to \cdots .$$

Hence by the log $\overline{\partial}$-Poincaré lemma on Y (3.1), the spectral sequence 8.10.2 satisfies

$$E_2^{s,m} = \begin{cases} \mathcal{V}^p_m/\mathcal{V}^{p+1}_m & \text{if } s = 0, \\ 0 & \text{if } s \neq 0. \end{cases}$$

Hence the spectral sequence 8.10.2 gives a canonical isomorphism

$$R^m f_* \mathrm{gr}^p(\omega^\bullet_{X/Y}(\mathcal{M})) \cong \mathcal{V}^p_m/\mathcal{V}^{p+1}_m.$$

Hence $R^m f_* \mathrm{gr}^p(\omega^\bullet_{X/Y}(\mathcal{M}))$ is a locally free $\mathcal{O}_Y$-module. The Hodge to de Rham spectral sequence in 8.1 (1) degenerates on Y_{triv} from E_1. Since each E_1-term is a locally free $\mathcal{O}_Y$-module as we have just seen, the degeneration on Y_{triv} implies the degeneration on Y. This completes the proof of Theorem 8.1.

The non-ket version of Theorem 8.1 is deduced directly from it as follows. This is a generalization of results in [23], [25] (but the proof is different).

Theorem 8.11. *Let $f: X \longrightarrow Y$ be as in Theorem* 8.1. *Let $(\mathcal{H}_{\mathbb{Z}}, \mathcal{M}, (\ ,\ ))$ be a VPLH on X in the non-ket sense (cf. Remark* 2.4*). Then the followings hold.*
(1) *The associated Hodge to de Rham spectral sequence (in the classical sense) degenerates from E_1.*
(2) *Assume that the underlying analytic space $\overset{\circ}{Y}$ of Y is smooth. Then each E_1-term of the spectral sequence in* (1) *is locally free.*
(3) *Assume that for any $x \in X$, the cokernel of $M^{\mathrm{gp}}_{Y,f(x)}/\mathcal{O}^{\times}_{Y,f(x)} \longrightarrow M^{\mathrm{gp}}_{X,x}/\mathcal{O}^{\times}_{X,x}$ is torsion free. Then $(\mathcal{L}_m, \mathcal{V}_m := R^m f_*(\omega^{\bullet}_{X/Y}(\mathcal{M})), (\ ,\ ))$, defined similarly as in Theorem* 8.1*, is a VPLH on Y in the non-ket sense, and each E_1-term of the classical spectral sequence in* (1) *is locally free.*

Proof. This is obtained by applying Theorem 8.1 to $(\mathcal{H}_{\mathbb{Z}}, \varepsilon^*\mathcal{M}, (\ ,\ ))$ as follows. Here and hereafter ε denotes the projection from the ket site to the usual site.

(1) By Proposition 1.7, $\varepsilon_* Rf_* \varepsilon^* M = Rf_* M$ for any locally free $\mathcal{O}_X$-module M of finite rank on X, and the Hodge to de Rham spectral sequence associated to $(\mathcal{H}_{\mathbb{Z}}, \mathcal{M}, (\ ,\ ))$ is the direct image by ε of the one associated to $(\mathcal{H}_{\mathbb{Z}}, \varepsilon^*\mathcal{M}, (\ ,\ ))$. Thus the degeneracy follows.

(2) When $\overset{\circ}{Y}$ is smooth, the direct image by ε of a locally free $\mathcal{O}_X$-module of finite rank on X_{ket} is locally free ([14]). This proves (2).

(3) By [14], under the assumption in (3), $R^m f^{\log}_* \mathcal{H}_{\mathbb{C}}$ belongs to $L_{\mathrm{unip}}(Y)$. Hence (3) follows. Note that in this case the spectral sequence in Theorem 8.1 (1) is the pull back to X_{ket} of the classical spectral sequence in the above (1). Q.E.D.

Remark 8.12. In the case where $(\mathcal{H}_{\mathbb{Z}}, \mathcal{M}, (\ ,\ ))$ is the unit ojbect $\mathbb{Z}$, Theorem 8.11 (1)(2) gives alternative proofs of results of
(a) J.H.M. Steenbrink [27], [28] and T. Fujisawa [6] without use of CMHC; and
(b) L. Illusie [13] and M. Cailotto [1] without use of algebraic methods.

8.13. Here we explain a relation between our work and the works of J.H.M. Steenbrink [27] and T. Fujisawa [6] on limit Hodge structures.

Let $Y = \Delta^n$ endowed with the log structure given by $\Delta^n - (\Delta^*)^n$. By the works of Cattani-Kaplan and Schmid, if $(\mathcal{H}_{\mathbb{Z}}, \mathcal{M}, (\ ,\ ))$ is a VPLH on Y of weight w in the non-ket sense, we have a polarized mixed Hodge structure ([3] Definition (2.26)) as follows. Let y be a point of $Y^{\log}$ lying over the origin $0 \in Y$. By identifying $\mathcal{H}_{\mathbb{C},y}$ with $\mathcal{M}(0) = \mathbb{C} \otimes_{\mathcal{O}_{Y,0}} \mathcal{M}_0$, define a descending filtration F on $\mathcal{H}_{\mathbb{C},y}$ by $F^p = \mathcal{M}^p(0)$. Let $W(N)$ be

the weight filtration on $\mathcal{H}_{\mathbb{Q},y}$ associated to the nilpotent operator

$$N = c_1 N_1 + \cdots + c_n N_n \ ; \ \mathcal{H}_{\mathbb{Q}} \longrightarrow \mathcal{H}_{\mathbb{Q}} \quad \text{for } c_1, \ldots, c_n > 0$$

($N_1, \ldots, N_n$ are as in 2.8), and let $W = W(N)[-w]$ be the $-w$ shift of $W(N)$. Then W is independent of the choice of N ([2]), and $(\mathcal{H}_{\mathbb{Z},y}, F, (\ ,\ ), W, N)$ is a polarized mixed Hodge structure for any N as above (cf. [3] 3.4).

Now let $f\colon X \longrightarrow Y = \Delta^n$ be a projective log smooth vertical morphism satisfying the assumption of Theorem 8.11 (3). Assume further that the underlying morphism of f of analytic spaces is flat. Let $(\mathcal{H}_{\mathbb{Z}}, \mathcal{M}, (\ ,\ ))$ be a VPLH on X of weight w in the non-ket sense. By Theorem 8.11 (3), we have a VPLH $(\mathcal{L}_m, \mathcal{V}_m, (\ ,\ ))$ on Y of weight $w+m$ in the non-ket sense. Fix a point $y \in Y^{\log}$ lying over $0 \in Y$ and fix a point $t \in (\Delta^*)^n$. By fixing a path connecting t and y, we identify $\mathrm{H}^m(X_t, \mathcal{H}_{\mathbb{Z}})$ with $\mathcal{L}_{m,y}$ via the isomorphisms

$$\mathrm{H}^m(X_t, \mathcal{H}_{\mathbb{Z}}) \cong (R^m f_*^{\log} \mathcal{H}_{\mathbb{Z}})_t \cong (R^m f_*^{\log} \mathcal{H}_{\mathbb{Z}})_y = \mathcal{L}_{m,y},$$

and identify $\mathrm{H}^m(X_t, \mathcal{H}_{\mathbb{C}})$ with $\mathrm{H}^m(X_0, \mathcal{O}_{X_0} \otimes_{\mathcal{O}_X} \omega^\bullet_{X/Y}(\mathcal{M}))$ via the isomorphisms

$$\mathrm{H}^m(X_t, \mathcal{H}_{\mathbb{C}}) \cong \mathcal{L}_{m,\mathbb{C},y} \cong \mathcal{V}_m(0) \cong \mathrm{H}^m(X_0, \mathcal{O}_{X_0} \otimes_{\mathcal{O}_X} \omega^\bullet_{X/Y}(\mathcal{M})).$$

Applying the above result of Cattani-Kaplan and Schmid to the VPLH $(\mathcal{L}_m, \mathcal{V}_m, (\ ,\ ))$ on Y, we obtain the following result.

Proposition 8.14. *Let $Y = \Delta^n$ and let X and $(\mathcal{H}_{\mathbb{Z}}, \mathcal{M}, (\ ,\ ))$ be as in* 8.13. *Then the map*

$$F^p := \mathrm{H}^m(X_0, \mathrm{Fil}^p(\mathcal{O}_{X_0} \otimes_{\mathcal{O}_X} \omega^\bullet_{X/Y}(\mathcal{M}))) \to \mathrm{H}^m(X_0, \mathcal{O}_{X_0} \otimes_{\mathcal{O}_X} \omega^\bullet_{X/Y}(\mathcal{M}))$$

is injective for any p, and $(\mathrm{H}^m(X_t, \mathcal{H}_{\mathbb{Z}}), F, (\ ,\ ), W, N)$ is a polarized mixed Hodge structure for any N as in 8.13, *where $W = W(N)[-w-m]$ which is independent of such N.*

In the case where $(\mathcal{H}_{\mathbb{Z}}, \mathcal{M}, (\ ,\ ))$ is the unit object $\mathbb{Z}$, this result was obtained by Steenbrink [27] under the assumption that $n = 1$ and X is semistable over $Y = \Delta$. See also Fujisawa's [6] for the case where X is multi-semistable over Y. (For such X, the assumption of 8.11 (3) is satisfied.)

Remark 8.15. The authors hope that Theorem 8.1 would be generalized to the case where the base is not necessarily log smooth over $\mathbb{C}$. When it would be established, it would give a new proof of Lemma 4.1

in [21]. For this, they hope to define the ring of log C^∞-functions $\mathcal{A}_X$ for an fs log analytic space X which need not be log smooth over $\mathbb{C}$ by the following idea: Locally on X, we can take an exact closed immersion $X \longrightarrow Z$ with Z log smooth over $\mathbb{C}$. When we have such an embedding, let I be the ideal of $\mathcal{O}_Z$ which defines X. Then we define $\mathcal{A}_X$ to be the quotient of $\mathcal{A}_Z$ by the ideal generated by I and the complex conjugate of I. The authors do not know that $\mathcal{A}_X$ does not depend on the local choice of $X \longrightarrow Z$. If it is the case, $\mathcal{A}_X$ is defined globally and $\mathcal{A}_X^{\log}$ is also defined by $\mathcal{A}_X^{\log} = \mathcal{O}_X^{\log} \otimes_{\mathcal{O}_X} \mathcal{A}_X$.

Appendix. Terminology in log geometry.

Here we give explanations on special terminologies in log geometry. [18] is a basic reference for what follows.

1. Concerning monoids.

In this paper, a monoid means a commutative monoid with a unit element and a homomorphism of monoids is assumed to respect the unit elements. An fs monoid is a finitely generated monoid $\mathcal{S}$ satisfying the following (i) (ii). (i) $ab = ac$ $(a,\ b,\ c \in \mathcal{S})$ implies $b = c$. (Hence $\mathcal{S}$ is embedded in the associated group $\mathcal{S}^{\mathrm{gp}} := \{ab^{-1}\ ;\ a,\ b \in \mathcal{S}\}$.) (ii) If $a \in \mathcal{S}^{\mathrm{gp}}$ and $a^n \in \mathcal{S}$ for some $n \geq 1$, then $a \in \mathcal{S}$.

2. Concerning log structures.

A log structure on a ringed space $(X,\ \mathcal{O}_X)$ is a sheaf of monoids M endowed with a homomorphism $\alpha\colon M \longrightarrow \mathcal{O}_X$ of sheaves of monoids satisfying $\alpha^{-1}(\mathcal{O}_X^\times) \xrightarrow{\cong} \mathcal{O}_X^\times$ by α. An fs log analytic space is an analytic space over $\mathbb{C}$ endowed with a log structure satisfying a certain "fs condition" (see [18]). In this paper, only "log smooth fs log analytic spaces" appear except in Remark 8.15. A log smooth fs log analytic space is an analytic space with a log structure which is locally isomorphic to an open set of $(\mathrm{Spec}\,\mathbb{C}[\mathcal{S}])^{\mathrm{an}}$ with $\mathcal{S}$ an fs monoid. Here $Y = (\mathrm{Spec}\,\mathbb{C}[\mathcal{S}])^{\mathrm{an}}$ is endowed with the log structure

$$\{f \in \mathcal{O}_Y\ ;\ f \text{ is invertible on } (\mathrm{Spec}\,\mathbb{C}[\mathcal{S}^{\mathrm{gp}}])^{\mathrm{an}}\} = \mathcal{O}_Y^\times \cdot \mathcal{S} \subset \mathcal{O}_Y.$$

For example, if X is a complex manifold and D is a divisor on X with normal crossings, and if X is endowed with the log structure $\{f \in \mathcal{O}_X\ ;\ f$ is invertible outside $D\}$ (called the log structure given by D), then X is a log smooth fs log analytic space.

For a log smooth fs log analytic space X, we denote by M_X its log structure. Let $X_{\mathrm{triv}} := \{x \in X \ ; \ M_{X,x} = \mathcal{O}_{X,x}^{\times}\}$. If X is an open set of $(\mathrm{Spec}\,\mathbb{C}[\mathcal{S}])^{\mathrm{an}}$, then $X_{\mathrm{triv}} = X \cap (\mathrm{Spec}\,\mathbb{C}[\mathcal{S}^{\mathrm{gp}}])^{\mathrm{an}}$.

3. Concerning $(X^{\log}, \mathcal{O}_X^{\log})$.

For an fs log analytic space X, a topological space $X^{\log}$ endowed with a proper map $\tau\colon X^{\log} \longrightarrow X$ is defined (see [18]). If $X = (\mathrm{Spec}\ \mathbb{C}[\mathcal{S}])^{\mathrm{an}} = \mathrm{Hom}\,(\mathcal{S},\ \mathbb{C})$, $X^{\log} = \mathrm{Hom}\,(\mathcal{S},\ \mathbb{R}_{\geq 0}^{\mathrm{mult}}) \times \mathrm{Hom}\,(\mathcal{S},\ \mathbb{S}^1)$ where $\mathbb{S}^1 := \{z \in \mathbb{C}^{\times} \ ; \ |z| = 1\}$ and $\tau\colon X^{\log} \longrightarrow X$ is the map induced by

$$\mathbb{R}_{\geq 0} \times \mathbb{S}^1 \longrightarrow \mathbb{C} \ ; \ (r,\ u) \mapsto ru.$$

We have two important sheaves of rings on $X^{\log}$, the non-ket version of $\mathcal{O}_X^{\log}$ and the ket version of $\mathcal{O}_X^{\log}$ (we use the same notation $\mathcal{O}_X^{\log}$). In this paper, $\mathcal{O}_X^{\log}$ is the ket version unless the contrary is explicitly stated. We explain these two $\mathcal{O}_X^{\log}$ in the case where X is a log smooth fs log analytic space. The inverse image of X_{triv} in $X^{\log}$ is isomorphic to X_{triv} via the canonical projection and hence X_{triv} is identified with an open set of $X^{\log}$. Let $j^{\log}\colon X_{\mathrm{triv}} \longrightarrow X^{\log}$ be the inclusion map. If X is a log smooth fs log analytic space, the non-ket version (resp. ket-version) of $\mathcal{O}_X^{\log}$ is the subring of $j_*^{\log}(\mathcal{O}_{X_{\mathrm{triv}}})$ generated over $\mathcal{O}_X = \tau^{-1}(\mathcal{O}_X)$ locally by $\log(t)$ for $t \in M_X$ (resp. by $\log(t)$, $t^{1/n}$ for $t \in M_X$ and $n \geq 1$). The non-ket version of $\mathcal{O}_X^{\log}$ is used in [18], [19], [20], [23], [24], [25], but the ket version of $\mathcal{O}_X^{\log}$ appear and play an essential role in [14].

4. Concerning morphisms between log smooth fs log analytic spaces.

A morphism between analytic spaces with log structures is defined in the evident way. For log smooth fs log analytic spaces X and Y, a morphism $X \longrightarrow Y$ is the same thing as a morphism of underlying analytic spaces $f\colon X \longrightarrow Y$ satisfying $f(X_{\mathrm{triv}}) \subset Y_{\mathrm{triv}}$. f is said to be vertical if $f^{-1}(Y_{\mathrm{triv}}) = X_{\mathrm{triv}}$. If X and Y are log smooth fs log analytic spaces, a morphism $f\colon X \longrightarrow Y$ is log smooth (resp. log étale) if and only if the following holds locally on X and on Y: There are fs monoids $\mathcal{S}$ and $\mathcal{T}$ and a homomorphism $h\colon \mathcal{S} \longrightarrow \mathcal{T}$ which is injective (resp. which is injective with $\mathcal{T}^{\mathrm{gp}}/h(\mathcal{S}^{\mathrm{gp}})$ finite) such that X is an open set of $(\mathrm{Spec}\,\mathbb{C}[\mathcal{T}])^{\mathrm{an}}$, Y is an open set of $(\mathrm{Spec}\,\mathbb{C}[\mathcal{S}])^{\mathrm{an}}$, and f is induced by h. For a morphism $f\colon X \longrightarrow Y$ of log smooth fs log analytic spaces, we say f is a blowing up along log structure if locally on Y, Y is an open set of $(\mathrm{Spec}\,\mathbb{C}[\mathcal{S}])^{\mathrm{an}}$ for an fs monoid $\mathcal{S}$ and $f\colon X \longrightarrow Y$ is the proper birational morphism associated to a finite polyhedral cone decomposition λ of $\mathrm{Hom}\,(\mathcal{S},\ \mathbb{Q}_{\geq 0})$ such that λ comes from an ideal of $\mathcal{S}$ (cf. 2.9 and [22] I). We say a morphism $f\colon X \longrightarrow Y$ of fs log analytic spaces is exact

at $x \in X$ if the induced homomorphism of fs monoids $M_{Y,y}/\mathcal{O}_{Y,y}^{\times} \longrightarrow M_{X,x}/\mathcal{O}_{X,x}^{\times}$ $(y = f(x))$ is exact (a homomorphism $\mathcal{S} \longrightarrow \mathcal{T}$ of fs monoids is said to be exact if the inverse image of $\mathcal{T}$ under $\mathcal{S}^{\mathrm{gp}} \longrightarrow \mathcal{T}^{\mathrm{gp}}$ coincides with $\mathcal{S}$). f is said to be exact if it is exact at any point of X.

5. Concerning the ket site.

An exact log étale morphism is called also a Kummer log étale morphism. Roughly speaking, "Kummer log étale over X" is something like "nearly étale over X but possibly ramified outside X_{triv}". For a log smooth fs log analytic space X, the Kummer log étale site X_{ket} is the following site: As a category, it is the category of log smooth fs log analytic spaces over X which are Kummer log étale over X. A covering is a surjection. The structure sheaf of X_{ket}; $U \mapsto \mathcal{O}_U(U)$ is denoted by $\mathcal{O}_X$. The canonical morphism of topoi $X^{\log} \longrightarrow X_{\mathrm{ket}}$ is denoted by τ (the same notation as $\tau\colon X^{\log} \longrightarrow X$).

Notations.

$\mathcal{A}_X$	sheaf of log C^∞-functions on X
$\mathcal{A}_X^{\log}$	sheaf of log C^∞-functions on $X^{\log}$
$\mathcal{A}_X^{p,q}$	sheaf of log C^∞ (p,q)-forms
$\mathcal{A}_{X/Y}^{p,\log}$	sheaf of relative log C^∞ p-forms on $X^{\log}$
$\mathcal{A}_{X/Y}^{p,q}$	sheaf of relative log C^∞ (p,q)-forms
$\mathrm{har}_{X/Y}^{m}(\mathcal{M})$	sheaf of harmonic m-forms with coefficients in $\mathcal{M}$
VPH	variation of polarized Hodge structure
VPLH	variation of polarized log Hodge structure
ω_X^p	sheaf of analytic p-forms with log poles
$\omega_{X/Y}^p$	sheaf of relative analytic p-forms with log poles
$\omega_{X/Y}^p(\mathcal{M})$	sheaf of relative analytic p-forms with log poles and with coefficients in $\mathcal{M}$

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Borel-Serre Spaces and Spaces of SL(2)-Orbits

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Introduction

Let $\mathfrak{h}_g$ be the Siegel upper half space of genus g, and Γ an arithmetic subgroup of $\mathrm{Sp}(2g)$. Then the following compactifications of $\Gamma\backslash\mathfrak{h}_g$ were constructed.

(1) Satake-Baily-Borel compactification ([Sa], [BB]).

(2) Borel-Serre compactification ([BS]).

(3) Toroidal compactifications ([AMRT]).

Denote the compactification (2) by $\Gamma\backslash(\mathfrak{h}_g)_{\mathrm{BS}}$ ($\Gamma\backslash\overline{\mathfrak{h}}_g$ in their notation).

Let D be a Griffiths domain, *i.e.*, a classifying space of polarized Hodge structures (cf. §1). In the case of the Hodge numbers $h^{1,0} = h^{0,1} = g$ and $h^{p,q} = 0$ otherwise, D coincides with $\mathfrak{h}_g$. Replacing $\mathfrak{h}_g$ by D, we generalize in this paper the construction of $(\mathfrak{h}_g)_{\mathrm{BS}}$ in two directions. The one is a Borel-Serre space D_{BS} (§2), and the other is a space of

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SL(2)-orbits $D_{\mathrm{SL}(2)}$ (§3). We give remarks about the compactification (1) (3.15, 6.9). In the forthcoming paper [KU2], we will generalize (3) and also (1).

Now let $G_{\mathbf{R}}$, $G_{\mathbf{Z}}$ be the groups in NOTATION below. $G_{\mathbf{R}}$ acts on D transitively. (In the case $D = \mathfrak{h}_g$, $G_{\mathbf{R}} = \mathrm{Sp}(2g, \mathbf{R})$ and $G_{\mathbf{Z}} = \mathrm{Sp}(2g, \mathbf{Z})$.) The construction of the space D_{BS} is similar to that of $(\mathfrak{h}_g)_{\mathrm{BS}}$, and the quotient $\Gamma\backslash D_{\mathrm{BS}}$, by a subgroup Γ of $G_{\mathbf{Z}}$ of finite index, is a compact Hausdorff space (§5). On the other hand, the construction of the space $D_{\mathrm{SL}(2)}$ is based on the theory of SL(2)-orbits in [CKS]. The space $D_{\mathrm{SL}(2)}$ is Hausdorff (§3) but not always locally compact (6.12). The quotient $\Gamma\backslash D_{\mathrm{SL}(2)}$ is Hausdorff (§5), and has a nice property for period maps (3.15) which is an advantage of $\Gamma\backslash D_{\mathrm{SL}(2)}$, that $\Gamma\backslash D_{\mathrm{BS}}$ does not have.

The two spaces D_{BS} and $D_{\mathrm{SL}(2)}$ are not related directly (cf. §6). To remedy this situation, we introduce the spaces $D_{\mathrm{BS,val}}$ (resp. $D_{\mathrm{SL}(2),\mathrm{val}}$), the projective limit of the blowing-ups of D_{BS} (resp. $D_{\mathrm{SL}(2)}$) (2.6, 3.7). We then have the following diagram of topological spaces (3.1 (1)).

$$\begin{array}{ccc} D_{\mathrm{SL}(2),\mathrm{val}} & \hookrightarrow & D_{\mathrm{BS,val}} \\ \downarrow & & \downarrow \\ D_{\mathrm{SL}(2)} & & D_{\mathrm{BS}} \end{array}$$

In the classical situation, that is, when D is a Hermitian symmetric space and its horizontal tangent bundle is the whole tangent bundle (see 6.6; actually, this case is dealt with in [Sa], [BB], [BS], [AMRT]), we have $D_{\mathrm{SL}(2)} = D_{\mathrm{BS}}$ and $D_{\mathrm{SL}(2),\mathrm{val}} = D_{\mathrm{BS,val}}$ except one case (6.7). As a corollary, we derive the canonical surjection from the Borel-Serre compactification $\Gamma\backslash D_{\mathrm{BS}}$ to the Satake compactification $\Gamma\backslash D_S$ (6.9), which was defined by Zucker [Z2] in another way.

In a sequel paper [KU2], we will generalize the theory of toroidal compactifications of $\Gamma\backslash\mathfrak{h}_g$ by Mumford et al. replacing $\mathfrak{h}_g$ by general D, whose summary is in [KU1]. The results of the present paper will be also used there. The whole picture is explained in 3.16 Remark below.

After having written out this paper, the authors were informed by Steven Zucker that a recent work of Borel and Ji [BJ] independently gives generalization of Borel-Serre compactifications.

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NOTATION

Throughout this paper, we use the following notation.

Let H be a $\mathbf{Z}$-module. For $A = \mathbf{Q}, \mathbf{R}, \mathbf{C}$, we denote $H_A := A \otimes_{\mathbf{Z}} H$. We fix a 4-tuple

$$\Phi_0 = (w, (h^{p,q})_{p,q\in\mathbf{Z}}, H_0, \langle\ ,\ \rangle_0)$$

where w is an integer, $(h^{p,q})_{p,q\in\mathbf{Z}}$ is a set of non-negative integers satisfying

$$\begin{cases} h^{p,q} = 0 \text{ for almost all } p,\ q, \\ h^{p,q} = 0 \text{ if } p+q \neq w, \\ h^{p,q} = h^{q,p} \text{ for any } p,\ q, \end{cases}$$

H_0 is a free $\mathbf{Z}$-module of rank $\sum_{p,q} h^{p,q}$, and $\langle\ ,\ \rangle_0$ is a $\mathbf{Q}$-rational non-degenerate $\mathbf{C}$-bilinear form on $H_{0,\mathbf{C}}$ which is symmetric if w is even and anti-symmetric if w is odd. In the case w is even, say $w = 2t$, we assume that the signature (a, b) of $(H_{0,\mathbf{R}}, \langle\ ,\ \rangle_0)$ satisfies

$$a \text{ (resp. } b) = \sum_j h^{t+j,t-j},$$

where j ranges over all even (resp. odd) integers.

Let

$$G_{\mathbf{Z}} := \mathrm{Aut}(H_0, \langle\ ,\ \rangle_0),$$

and for $A = \mathbf{Q}, \mathbf{R}, \mathbf{C}$, let

$$\begin{aligned} G_A &:= \mathrm{Aut}(H_{0,A}, \langle\ ,\ \rangle_0), \\ \mathfrak{g}_A &:= \mathrm{Lie}\, G_A \\ &= \{N \in \mathrm{End}_A(H_{0,A}) \mid \langle Nx, y\rangle_0 + \langle x, Ny\rangle_0 = 0\ (\forall x, \forall y \in H_{0,A})\}. \end{aligned}$$

Following [BS], a *parabolic* subgroup of $G_{\mathbf{R}}$ means a parabolic subgroup of $(G^\circ)_{\mathbf{R}}$, where G° denotes the connected component of G in the Zariski topology containing the unity. (Note that $G^\circ = G$ if w is odd, and $G^\circ = \{g \in G \mid \det(g) = 1\}$ if w is even.)

§1. Classifying spaces of polarized Hodge structures

In this section, we recall polarized Hodge structures, the classifying space D of polarized Hodge structures, horizontal tangent bundles, polarized variations of Hodge structure and the associated period maps (cf. [G]). Let $\Phi_0 = (w, (h^{p,q})_{p,q\in\mathbf{Z}}, H_0, \langle\ ,\ \rangle_0)$ be as in NOTATION.

1.1. A *Hodge structure* of weight w and of Hodge type $(h^{p,q}) = (h^{p,q})_{p,q\in\mathbf{Z}}$ is a pair $(H_{\mathbf{Z}}, F)$ consisting of a free $\mathbf{Z}$-module $H_{\mathbf{Z}}$ of rank

$\sum_{p,q} h^{p,q}$ and of a decreasing filtration F of $H_{\mathbf{C}}$, which satisfy the following two conditions.

(i) $\dim_{\mathbf{C}} F^p = \sum_{r \geq p} h^{r,w-r}$ for all p.

(ii) $H_{\mathbf{C}} = \bigoplus_p H^{p,w-p}$, where $H^{p,w-p} := F^p \cap \overline{F}^{w-p}$.

Here $\bar{}$ is the complex conjugation with respect to $H_{0,\mathbf{R}}$. Note that $\dim H^{p,w-p} = h^{p,w-p}$ for all p.

1.2. A *polarized Hodge structure* of weight w and of Hodge type $(h^{p,q})$ is a triple $(H_{\mathbf{Z}}, \langle\ ,\ \rangle, F)$ consisting of a Hodge structure $(H_{\mathbf{Z}}, F)$ and a $\mathbf{Q}$-rational non-degenerate $\mathbf{C}$-bilinear form $\langle\ ,\ \rangle$ on $H_{\mathbf{C}}$, symmmetric for even w and anti-symmetric for odd w, which satisfy the following two conditions.

(i) $\langle F^p, F^q \rangle = 0$ for $p + q > w$.

(ii) The Hermitian form $H_{0,\mathbf{C}} \times H_{0,\mathbf{C}} \to \mathbf{C}$, $(x, y) \mapsto \langle C_F(x), \overline{y} \rangle$, is positive definite.

Here C_F is the Weil operator which is defined by $C_F(x) := i^{2p-w}x$ for $x \in H^{p,w-p}$. The condition (i) (resp. (ii)) is called the Riemann-Hodge first (resp. second) bilinear relation.

1.3. A *polarized variation of Hodge structure* of weight w and of Hodge type $(h^{p,q})$ on a complex manifold X is a triple $(H_{\mathbf{Z}}, \langle\ ,\ \rangle, F)$ consisting of a local system $H_{\mathbf{Z}}$ on X, of a locally constant $\mathbf{Q}$-rational non-degenerate $\mathbf{C}$-bilinear form $\langle\ ,\ \rangle$ on $\mathbf{C} \otimes H_{\mathbf{Z}}$ and of a decreasing filtration F of $\mathcal{O}_X \otimes H_{\mathbf{Z}}$ by subbundles, which satisfy the following two conditions.

(i) $(H_{\mathbf{Z},x}, \langle\ ,\ \rangle_x, F(x))$ is a polarized Hodge structure of weight w and of Hodge type $(h^{p,q})$ $(\forall x \in X)$.

(ii) Griffiths transversality $\nabla F^p \subset \Omega^1_X \otimes F^{p-1}$ holds $(\forall p)$. ($\nabla := d \otimes \mathrm{id}_{H_{\mathbf{Z}}}$ is the connection of $\mathcal{O}_X \otimes H_{\mathbf{Z}}$.)

Definitions 1.4. The *classifying space D of polarized Hodge structures of type Φ_0* is the set of all decreasing filtrations F on $H_{0,\mathbf{C}}$ such that the triple $(H_0, \langle\ ,\ \rangle_0, F)$ is a polarized Hodge structure of weight w and of Hodge type $(h^{p,q})$.

Note that, by the condition on the signature of $(H_{0,\mathbf{R}}, \langle\ ,\ \rangle_0)$ (see NOTATION), D isnon-empty.

Definitions 1.5. The *compact dual $\check{D}$* of D is the set of all decreasing filtrations F on $H_{0,\mathbf{C}}$ such that the pair (H_0, F) is a Hodge structure

of weight w and of Hodge type $(h^{p,q})$ and that the triple $(H_0, \langle\ ,\ \rangle_0, F)$ satisfies the condition 1.2 (i).

Note that D (resp. $\check{D}$) is homogeneous under $G_{\mathbf{R}}$ (resp. $G_{\mathbf{C}}$) and that D is an open subset of $\check{D}$.

Definition 1.6. Let $F \in D$ and let $T_D(F)$ be the tangent space of D at F. The *horizontal tangent space* $T_D^h(F)$ of D at F is defined as follows:

$$T_D^h(F) = F^{-1}(\mathfrak{g}_{\mathbf{C}})/F^0(\mathfrak{g}_{\mathbf{C}}) \subset T_D(F) = \mathfrak{g}_{\mathbf{C}}/F^0(\mathfrak{g}_{\mathbf{C}}),$$

where $F^r(\mathfrak{g}_{\mathbf{C}}) := \{N \in \mathfrak{g}_{\mathbf{C}} \mid N(F^p) \subset F^{p+r}\ (\forall p \in \mathbf{Z})\}$.

1.7. Let $\Phi_0 = (w, (h^{p,q})_{p,q\in\mathbf{Z}}, H_0, \langle\ ,\ \rangle_0)$ be as in NOTATION. Let X be a connected complex manifold and let $H = (H_{\mathbf{Z}}, \langle\ ,\ \rangle, F)$ be a polarized variation of Hodge structure on X of weight w and of Hodge type $(h^{p,q})$ with $(H_{\mathbf{Z},x}, \langle\ ,\ \rangle_x) \simeq (H_0, \langle\ ,\ \rangle_0)$ for some (hence any) point $x \in X$. Fix such a point x and identify $(H_{\mathbf{Z},x}, \langle\ ,\ \rangle_x) = (H_0, \langle\ ,\ \rangle_0)$. Put $\Gamma := \mathrm{Im}(\pi_1(X) \to G_{\mathbf{Z}})$. Then we have the associated period map

$$\varphi : X \to \Gamma\backslash D. \tag{1}$$

The differential of the period map at $x \in X$ factors through $T_D^h(\tilde{\varphi}(x))$. Here $\tilde{\varphi} : U \to D$ is a local lifting of φ on a neighborhood U of x.

§2. Borel-Serre spaces

2.1. *Summary.* Let $\mathcal{X}$ be the set of all maximal compact subgroups of $G_{\mathbf{R}}$. Then $G_{\mathbf{R}}$ acts transitively on $\mathcal{X}$ by inner automorphisms. Since the normalizer in $G_{\mathbf{R}}$ of each $K \in \mathcal{X}$ is K itself, we have a $G_{\mathbf{R}}$-equivariant isomorphism

$$G_{\mathbf{R}}/K \xrightarrow{\sim} \mathcal{X}, \quad g \mapsto gKg^{-1},$$

for each fixed $K \in \mathcal{X}$. By using this isomorphism, we introduce a topology on $\mathcal{X}$. This topology does not depend on the choice of K. Borel and Serre constructed in [BS] a space $\mathcal{X}_{\mathrm{BS}}$ ($\overline{\mathcal{X}}$ in their notation) which contains $\mathcal{X}$ as an open dense subset. The action of $G_{\mathbf{Z}}$ on $\mathcal{X}$ extends to an action on $\mathcal{X}_{\mathrm{BS}}$. The space $\mathcal{X}_{\mathrm{BS}}$ has the following remarkable properties:

(i) If Γ is a subgroup of $G_{\mathbf{Z}}$ of finite index, then the quotient space $\Gamma\backslash\mathcal{X}_{\mathrm{BS}}$ is compact.

(ii) If Γ is a neat subgroup of $G_{\mathbf{Z}}$, then the projection $\mathcal{X}_{\mathrm{BS}} \to \Gamma\backslash\mathcal{X}_{\mathrm{BS}}$ is a local homeomorphism.

Here Γ being neat means that the subgroup of $\mathbf{C}^\times$ generated by all eigenvalues of all $\gamma \in \Gamma$ is torsion free.

In this section, we enlarge D to get a topological space D_{BS}, which contains D as a dense open subspace, in the same way as $\mathcal{X}$ was enlarged to $\mathcal{X}_{\mathrm{BS}}$. We also construct a topological space $D_{\mathrm{BS,val}}$, as the projective limit of the blowing-ups of D_{BS}, which also contains D as a dense open subspace. These spaces are related by continuous proper surjective maps in the following way:

$$D_{\mathrm{BS,val}} \to D_{\mathrm{BS}} \to \mathcal{X}_{\mathrm{BS}}.$$

2.2. *Borel-Serre action.* Let P be a parabolic subgroup of $G_{\mathbf{R}}$ and P_u be its unipotent radical. Let S_P be the maximal $\mathbf{Q}$-split torus in the center of P/P_u. Let A_P be the connected component of S_P containing the unity. ($S_P \simeq (\mathbf{R}^\times)^r$, $A_P \simeq (\mathbf{R}_{>0})^r$, where $r := \operatorname{rank} S_P$.)

For $K \in \mathcal{X}$, let $\theta_K : G_{\mathbf{R}} \to G_{\mathbf{R}}$ be the Cartan involution associated to the maximal compact subgroup K, *i.e.*, the unique automorphism of $G_{\mathbf{R}}$ characterized by $\theta_K^2 = \mathrm{id}$ and $K = \{g \in G_{\mathbf{R}} \mid \theta_K(g) = g\}$. By [BS], for each $K \in \mathcal{X}$ and $a \in S_P$, there exists a unique element $a_K \in P$ satisfying both of the following.

(i) $(a_K \mod P_u) = a$.

(ii) $\theta_K(a_K) = a_K^{-1}$.

Then the map $S_P \to P$, $a \mapsto a_K$, is a homomorphism of algebraic groups over $\mathbf{R}$. We call a_K ($a \in S_P$) the *Borel-Serre lifting of a at K*.

For $F \in D$, we use the following notation:

$$K_F := \left\{ g \in G_{\mathbf{R}} \;\middle|\; \begin{array}{l} g \text{ preserves the Hermitian} \\ \text{inner product } \langle C_F(\ ), {}^{-}\rangle_0 \end{array} \right\}, \tag{1}$$

$$K'_F := \{g \in G_{\mathbf{R}} \mid gF = F\} \subset K_F,$$

where C_F is the Weil operator in 1.2 (cf. [Sc, §8]). Note that K_F is a maximal compact subgroup of $G_{\mathbf{R}}$ and the Cartan involution θ_{K_F} is given by

$$\theta_{K_F} = \mathrm{Int}(C_F). \tag{2}$$

Note also that we have the canonical $G_{\mathbf{R}}$-equivariant continuous proper map

$$\begin{array}{ccc} D & \longrightarrow & \mathcal{X} \\ \downarrow \wr & & \downarrow \wr \\ G_{\mathbf{R}}/K'_F & \longrightarrow & G_{\mathbf{R}}/K_F, \end{array} \qquad F \longmapsto K_F. \tag{3}$$

For $a \in A_P$ and $K \in \mathcal{X}$ (resp. $F \in D$), we define an action $\circ$ by

$$a \circ K := \mathrm{Int}(a_K)K \quad (\text{resp. } a \circ F := a_{K_F}F). \tag{4}$$

We call this the *Borel-Serre action.*

Lemma 2.3. *For $a \in A_P$ and $p \in P$, we have $a_{\mathrm{Int}(p)K} = \mathrm{Int}(p)a_K$.*

Proof. This follows from the fact

$$\theta_{\mathrm{Int}(p)K} = \mathrm{Int}(p)\theta_K \,\mathrm{Int}(p)^{-1} \tag{1}$$

and the definition of the Borel-Serre liftings in 2.2. Q.E.D.

Lemma 2.4. *For $a \in A_P$, $p \in P$ and $F \in D$, we have $a \circ pF = p(a \circ F)$.*

Proof. By 2.3, we have

$$\begin{aligned} a \circ pF &= a_{K_{pF}}pF = a_{\mathrm{Int}(p)K_F}pF \\ &= (\mathrm{Int}(p)a_{K_F})pF = pa_{K_F}F = p(a \circ F). \end{aligned}$$ Q.E.D.

By 2.4, we see that the Borel-Serre action is indeed an action of A_P on D. In fact, for $a, b \in A_P$ and $F \in D$, we have

$$a \circ (b \circ F) = a \circ (b_{K_F}F) = b_{K_F}(a \circ F) = b_{K_F}a_{K_F}F = (ba)_{K_F}F = (ab) \circ F.$$

It can be verified in a similar way that A_P acts on $\mathcal{X}$ via the Borel-Serre action.

Definition 2.5. The *generalized Borel-Serre space* D_{BS} (resp. *Borel-Serre space* $\mathcal{X}_{\mathrm{BS}}$) is defined by

$$D_{\mathrm{BS}} \ (\text{resp. } \mathcal{X}_{\mathrm{BS}}) := \left\{ (P, Z) \;\middle|\; \begin{array}{l} P \text{ is a } \mathbf{Q}\text{-parabolic subgroup of } G_{\mathbf{R}}, \\ Z \text{ is an } (A_P\circ)\text{-orbit in } D \ (\text{resp. } \mathcal{X}) \end{array} \right\}$$

For an abelian group L and a submonoid V of L, we say V is *valuative* if $V \cup V^{-1} = L$. Also, put $V^\times := V \cap V^{-1}$.

Definition 2.6. We define the space $D_{\mathrm{BS,val}}$ by

$$D_{\mathrm{BS,val}} := \left\{ (T, Z, V) \;\middle|\; \begin{array}{l} T \text{ is an } \mathbf{R}\text{-split torus of } G_{\mathbf{R}}, \\ Z \text{ is a } (T_{>0})\text{-orbit in } D, \\ V \text{ is a valuative submonoid of } X(T), \\ \text{which satisfy the following (i)–(iii)} \end{array} \right\}$$

Here, $T_{>0}$ is the connected component of T containing the unity, and $X(T)$ is the character group of T.

(i) $\theta_{K_F}(t) = t^{-1}$ $(\forall F \in Z, \forall t \in T)$.

(ii) $V^{\times} = \{1\}$.

(iii) Let $H_{0,\mathbf{R}} = \bigoplus_{\chi \in X(T)} H(\chi)$ be the decomposition into eigenspaces $H(\chi) := \{v \in H_{0,\mathbf{R}} \,|\, tv = \chi(t)v \ (\forall t \in T)\}$. Then, for any $\chi \in X(T)$, the $\mathbf{R}$-subspace

$$M_{\chi} := \bigoplus_{\chi' \in \chi V^{-1}} H(\chi')$$

of $H_{0,\mathbf{R}}$ is $\mathbf{Q}$-rational.

2.7. Let M be a finite set of $\mathbf{R}$-subspaces of $H_{0,\mathbf{R}}$ satisfying the following two conditions:

(i) M contains 0 and $H_{0,\mathbf{R}}$ and is totally ordered with respect to the inclusion, *i.e.*,
$M = \{M_j\}_{0 \le j \le m}$, $0 = M_0 \subset M_1 \subset \cdots \subset M_m = H_{0,\mathbf{R}}$.

(ii) $M_j^{\perp} = M_{m-j}$ $(0 \le j \le m)$,
where $M_j^{\perp} := \{v \in H_{0,\mathbf{R}} \mid \langle v, M_j \rangle_0 = 0\}$.

Let $P := \{g \in (G^{\circ})_{\mathbf{R}} \mid gM_j = M_j \ (0 \le j \le m)\}$. Then P is a parabolic subgroup of $G_{\mathbf{R}}$. To see this, it is sufficient to show that $(G^{\circ})_{\mathbf{R}}/P$ is a projective variety. In fact, $(G^{\circ})_{\mathbf{R}}/P$ is identified with the projective variety of finite sets $M' = \{M'_j\}_{0 \le j \le m}$ of $\mathbf{R}$-subspaces of $H_{0,\mathbf{R}}$ satisfying the above conditions (i), (ii) and $\dim M'_j = \dim M_j$ $(0 \le j \le m)$.

If all M_j $(0 \le j \le m)$ are $\mathbf{Q}$-rational, then P is a $\mathbf{Q}$-parabolic subgroup of $G_{\mathbf{R}}$. (It can be shown that any $\mathbf{Q}$-parabolic subgroup of $G_{\mathbf{R}}$ is obtained in this way. See 6.6.)

2.8. For a torus T of $G_{\mathbf{R}}$ and for a valuative submonoid V of $X(T)$, let

$$P_{T,V} := \{g \in (G^{\circ})_{\mathbf{R}} \mid gM_{\chi} = M_{\chi} \ (\forall \chi \in X(T))\},$$

where M_{χ} is defined as in 2.6 (iii).

For $(T, Z, V) \in D_{\mathrm{BS,val}}$, we prove the following two assertions:

(1) $P_{T,V}$ is a $\mathbf{Q}$-parabolic subgroup of $G_{\mathbf{R}}$.

(2) $T \subset P_{T,V}$, the map $T \to P_{T,V}/P_{T,V,u}$ is injective, and the image of this map is contained in S_P.

To prove (1), it is enough to show that the set $\{M_\chi \mid \chi \in X(T)\}$ has the properties (i) and (ii) in 2.7. In fact, for $\chi, \chi' \in X(X)$, we have either $\chi'\chi^{-1} \in V$ or $\chi\chi'^{-1} \in V$, and we have $M_\chi \subset M_{\chi'}$ in the former case and $M_{\chi'} \subset M_\chi$ in the latter case. This proves 2.7 (i). Furthermore, $(M_\chi)^\perp = \sum_{\chi'} M_{\chi'}$, where χ' ranges over all elements of $X(T)$ which are not contained in $\chi^{-1}V$, and hence $(M_\chi)^\perp = M_{\chi'}$ for some χ' by 2.7 (i).

Next, (2) follows from $P/P_u \subset \prod_{\chi \in X(T)} \mathrm{GL}(H(\chi))$ and the fact that T acts on each $H(\chi)$ as scalars.

Now we have maps

$$
\begin{aligned}
&D_{\mathrm{BS,val}} \xrightarrow{\alpha} D_{\mathrm{BS}} \xrightarrow{\beta} \mathcal{X}_{\mathrm{BS}}, \quad \text{where} \\
&\alpha : (T, Z, V) \mapsto (P_{T,V}, A_{P_{T,V}} \circ Z), \\
&\beta : \text{the map induced by 2.2 (3).}
\end{aligned} \tag{3}
$$

2.9. For a $\mathbf{Q}$-parabolic subgroup P of $G_{\mathbf{R}}$, we define

$$
\begin{aligned}
&D_{\mathrm{BS}}(P) := \{(Q, Z) \in D_{\mathrm{BS}} \mid Q \supset P\}, \\
&\mathcal{X}_{\mathrm{BS}}(P) := \{(Q, Z) \in \mathcal{X}_{\mathrm{BS}} \mid Q \supset P\}, \\
&D_{\mathrm{BS,val}}(P) := \{(T, Z, V) \in D_{\mathrm{BS,val}} \mid P_{T,V} \supset P\}.
\end{aligned} \tag{1}
$$

In 2.10–2.14 below, we give preliminaries to define topologies on the spaces D_{BS}, $\mathcal{X}_{\mathrm{BS}}$, $D_{\mathrm{BS,val}}$.

2.10. Let P be a $\mathbf{Q}$-parabolic subgroup of $G_{\mathbf{R}}$. A subset Δ_P of $X(S_P)$ is defined as follows. Let $\tilde{S}_P \subset P$ be any torus which lifts $S_P \subset P/P_u$. Let

$$
\Delta'_P := \left\{ \chi \in X(\tilde{S}_P) \;\middle|\; \begin{array}{l} 0 \neq \exists v \in \mathrm{Lie}(P_u) \text{ such that} \\ \mathrm{Ad}(a)v = \chi(a)^{-1}v \ (\forall a \in \tilde{S}_P) \end{array} \right\}
$$

Identify $X(\tilde{S}_P)$ with $X(S_P)$ via the canonical isomorphism $\tilde{S}_P \xrightarrow{\sim} S_P$. Then Δ'_P is independent of the choice of the liftings $\tilde{S}_P$, and it is a finite subset of $X(S_P)$ which generates $\mathbf{Q} \otimes X(S_P)$ over $\mathbf{Q}$. Then there exists a unique subset Δ_P of Δ'_P satisfying the following two conditions:

(i) $\sharp(\Delta_P) = \mathrm{rank}\, S_P$.

(ii) Δ'_P is contained in the submonoid of $X(S_P)$ generated by Δ_P.

Let P be as above. Let Q be a $\mathbf{Q}$-parabolic subgroup of $G_{\mathbf{R}}$ containing P. Then we have injective maps

$$S_Q \hookrightarrow S_P, \tag{1}$$

$$\Delta_Q \hookrightarrow \Delta_P, \tag{2}$$

obtained as follows, which we regards as inclusion maps. Note $Q \supset P \supset P_u \supset Q_u$. We have that $S_Q \subset P/Q_u$ in Q/Q_u, that the canonical map $S_Q \to P/P_u$ is injective, and that the image of this map is contained in S_P. This gives the injection (1). Let $I := \{\chi \in \Delta_P \mid \chi \text{ annihilates } S_Q\}$, and let $J \subset \Delta_P$ be the complement of I in Δ_P. Then the restriction to S_Q gives a bijection $J \xrightarrow{\sim} \Delta_Q$. The injection (2) is obtained as the composite $\Delta_Q \xleftarrow{\sim} J \hookrightarrow \Delta_P$. It is known that we have a bijection

$$\left\{\begin{matrix}\mathbf{Q}\text{-parabolic subgroup} \\ \text{of } G_{\mathbf{R}} \text{ containing } P\end{matrix}\right\} \xrightarrow{\sim} \{\text{subset of } \Delta_P\}, \quad Q \mapsto \Delta_Q \tag{3}$$

(cf. [BS, §4], [BT, §5], [B, §11]).

2.11. Let $M = \{M_j\}_{0 \le j \le m}$ be as in 2.7, and let P be the associated parabolic subgroup $\{g \in (G^\circ)_{\mathbf{R}} \mid gM_j = M_j \ \ (0 \le j \le m)\}$ of $G_{\mathbf{R}}$. We describe in terms of M the unipotent radical P_u, the quotient P/P_u, and, in the case when all M_j are $\mathbf{Q}$-rational, S_P and Δ_P.

(1) $P_u = \{g \in (G^\circ)_{\mathbf{R}} \mid (g-1)M_j \subset M_{j-1} \ \ (1 \le j \le m)\}$.

(2) $$P/P_u = \left\{ (g_j)_j \in \prod_{1 \le j \le m} \mathrm{GL}(\mathrm{gr}_j^M) \;\middle|\; \begin{matrix} {}^t g_j = g_{m+1-j}^{-1} \ \ (1 \le j \le m), \\ \det(g_{(m+1)/2}) = 1 \text{ if } m \text{ is odd} \end{matrix} \right\}.$$

Here ${}^t(\ )$ is the transpose with respect to the pairing

$$\langle\ ,\ \rangle_0 : \mathrm{gr}_j^M \times \mathrm{gr}_{m+1-j}^M \to \mathbf{R}.$$

Assume now that all M_j $(0 \le j \le m)$ are $\mathbf{Q}$-rational. We say that M is *exceptional* if either one of the following two conditions is satisfied:

(a) w is even, m is even, and $\dim \mathrm{gr}_{(m/2)+1}^M = 1$.

(b) w is even, m is odd, and $\dim \mathrm{gr}_{(m+1)/2}^M = 2$, and there exist elements $e_1, e_2 \in M_{(m+1)/2} \cap H_{0,\mathbf{Q}}$ such that $\langle e_1, e_1\rangle_0 = \langle e_2, e_2\rangle_0 = 0$ and $\langle e_1, e_2\rangle_0 = 1$.

We have

$$(3)\qquad S_P = \left\{ (g_j)_j \in P/P_u \subset \prod_{1\le j\le m} \mathrm{GL}(\mathrm{gr}_j^M) \;\middle|\; \begin{array}{l} g_j \in \mathbf{G_{m,R}} \subset \mathrm{GL}(\mathrm{gr}_j^M) \text{ unless } m \text{ is odd} \\ \text{and } j = (m+1)/2;\ g_{(m+1)/2} = 1 \\ \text{if } m \text{ is odd and } M \text{ is not exceptional} \end{array} \right\}.$$

(4) Δ_P is described in terms of M as follows.
Assume first M is not exceptional. Let $r := m/2$ if m is even, and $r := (m-1)/2$ if m is odd. Then $r = \operatorname{rank} S_P$ and Δ_P consists of

$$(g_j)_{1\le j\le m} \mapsto g_{j+1}g_j^{-1} \quad (1 \le j \le r) \quad (g_j \in \mathbf{G_{m,R}}).$$

Assume next M is exceptional and m is even. Let $r := m/2$. Then $r = \operatorname{rank} S_P$ and Δ_P consitsts of

$$(g_j)_{1\le j\le m} \mapsto g_{j+1}g_j^{-1} \quad (1 \le j \le r-1) \quad \text{and}$$
$$(g_j)_{1\le j\le m} \mapsto g_{r+1}g_{r-1}^{-1}.$$

Assume lastly M is exceptional and m is odd. Let $r := (m+1)/2$. Then $r = \operatorname{rank} S_P$ and Δ_P consitsts of

$$(g_j)_{1\le j\le m} \mapsto g_{j+1}g_j^{-1} \quad (1 \le j \le r-2),$$
$$(g_j)_{1\le j\le m} \mapsto g_{r,1}g_{r-1}^{-1}, \quad \text{and}$$
$$(g_j)_{1\le j\le m} \mapsto g_{r,2}g_{r-1}^{-1},$$

where $g_{r,k} \in \mathbf{G_m}$ $(k = 1, 2)$ are defined by

$$g_r e_k = g_{r,k} e_k \quad \text{in} \quad \mathrm{gr}^M_{(m+1)/2}.$$

2.12. *Identification of* $D_{\mathrm{BS}}(P)$ *with* $D \times^{A_P} \overline{A}_P$. Let P be a $\mathbf{Q}$-parabolic subgroup of $G_{\mathbf{R}}$. Let $\overline{A}_P := \mathrm{Map}(\Delta_P, \mathbf{R}_{\ge 0})$ (= the set of maps $\Delta_P \to \mathbf{R}_{\ge 0}$). Hence $\overline{A}_P \simeq \mathbf{R}^r_{\ge 0}$, $r = \operatorname{rank} S_P$. Via $A_P \xrightarrow{\sim} \mathrm{Map}(\Delta_P, \mathbf{R}_{>0})$, A_P acts on $\overline{A}_P$ in the natural way. Denote $D \times^{A_P} \overline{A}_P := (D \times \overline{A}_P)/A_P$ under the action $a(F, b) = (a \circ F, a^{-1}b)$ $(a \in A_P, (F, b) \in D \times \overline{A}_P)$. Then we have a bijection

$$D_{\mathrm{BS}}(P) \simeq D \times^{A_P} \overline{A}_P, \quad (Q, Z) \longleftrightarrow (F, b),$$

defined as follows. For $(Q, Z) \in D_{\mathrm{BS}}(P)$, F is any element of Z and $b \in \overline{A}_P$ is defined by $b(\chi) = 0$ if $\chi \in \Delta_Q$ and $b(\chi) = 1$ if $\chi \notin \Delta_Q$. For

$(F, b) \in D \times^{A_P} \overline{A}_P$, Q is the $\mathbf{Q}$-parabolic subgroup of $G_{\mathbf{R}}$ containing P such that $\Delta_Q = \{\chi \in \Delta_P \mid b(\chi) = 0\}$, and $Z := \{a \circ F \mid a \in A_P, \chi(a) = b(\chi) \text{ for any } \chi \in \Delta_P - \Delta_Q\}$.

For a $\mathbf{Q}$-parabolic subgroup Q of $G_{\mathbf{R}}$ containing P, we have $A_Q \subset A_P$ and $\overline{A}_Q \subset \overline{A}_P$. The latter follows by extending a map $\Delta_Q \to \mathbf{R}$ to Δ_P by 1. Hence we have a commutative diagram

$$\begin{array}{ccc} D_{\mathrm{BS}}(Q) & \simeq & D \times^{A_Q} \overline{A}_Q \\ \cap & & \cap \\ D_{\mathrm{BS}}(P) & \simeq & D \times^{A_P} \overline{A}_P. \end{array}$$

2.13. *The topological space* $(\mathbf{R}^r_{\geq 0})_{\mathrm{val}}$. We define a topological space $(\mathbf{R}^r_{\geq 0})_{\mathrm{val}}$ $(r \geq 0)$ as follows. Let $r \geq 0$ be an integer, and define a topological space B by

$$B := \varprojlim_{I \in \Psi} \mathrm{Bl}_I((\mathbf{G}_{\mathbf{a},\mathbf{R}})^r), \tag{1}$$

where Ψ denotes the set of all non-zero ideals I of $\mathbf{R}[t_1, \ldots, t_r] = \mathcal{O}((\mathbf{G}_{\mathbf{a},\mathbf{R}})^r)$ generated by some monomials $f_1, \ldots, f_n$, and $\mathrm{Bl}_I((\mathbf{G}_{\mathbf{a},\mathbf{R}})^r)$ means the blowing-up $\mathrm{Proj}(\bigoplus_{s \geq 0} I^s)$ of $(\mathbf{G}_{\mathbf{a},\mathbf{R}})^r$ along I. Define an order in Ψ by: $I \leq J$ if and only if $II' = J$ for some $I' \in \Psi$. For $I \leq II' = J$ with $I = (f_1, \ldots, f_n)$, $I' = (g_1, \ldots, g_m)$, define a morphism

$$\begin{array}{l} \mathrm{Bl}_I((\mathbf{G}_{\mathbf{a},\mathbf{R}})^r) = \bigcup_{1 \leq k \leq n} \mathrm{Spec}\left(\mathbf{R}\left[t_1, \ldots, t_r, \frac{f_1}{f_k}, \ldots, \frac{f_n}{f_k}\right]\right) \\ \qquad \uparrow \\ \mathrm{Bl}_J((\mathbf{G}_{\mathbf{a},\mathbf{R}})^r) = \bigcup_{1 \leq k \leq n, 1 \leq l \leq m} \mathrm{Spec}\left(\mathbf{R}\left[t_1, \ldots, t_r, \frac{f_1 g_1}{f_k g_l}, \ldots, \frac{f_n g_m}{f_k g_l}\right]\right) \end{array}$$

by the inclusions of affine rings

$$\mathbf{R}\left[t_1, \ldots, t_r, \frac{f_1}{f_k}, \ldots, \frac{f_n}{f_k}\right] \hookrightarrow \mathbf{R}\left[t_1, \ldots, t_r, \frac{f_1 g_1}{f_k g_l}, \ldots, \frac{f_n g_m}{f_k g_l}\right]$$

$(1 \leq k \leq n, 1 \leq l \leq m)$. The projective limit in (1) is taken with respect to this ordering.

Since the centers of the blow-ups are outside $(\mathbf{G}_{\mathbf{m},\mathbf{R}})^r$ $(\subset (\mathbf{G}_{\mathbf{a},\mathbf{R}})^r)$, we have a dense open immersion $(\mathbf{G}_{\mathbf{m},\mathbf{R}})^r \hookrightarrow B$. Furthermore, there is a unique action of $(\mathbf{G}_{\mathbf{m},\mathbf{R}})^r$ on B which is compatible with the standard action of $(\mathbf{G}_{\mathbf{m},\mathbf{R}})^r$ on itself. Let $(\mathbf{R}^r_{\geq 0})_{\mathrm{val}}$ be the closure of $\mathbf{R}^r_{>0}$ in B under the composite of open immersions $\mathbf{R}^r_{>0} \subset (\mathbf{G}_{\mathbf{m},\mathbf{R}})^r \hookrightarrow B$. Then the canonical map $(\mathbf{R}^r_{\geq 0})_{\mathrm{val}} \to \mathbf{R}^r_{\geq 0}$ is proper and surjective because so is $B \to (\mathbf{G}_{\mathbf{a},\mathbf{R}})^r$ (cf. [NB, Ch. 1, §10, no. 2, Corollaire 3]). Furthermore the group $\mathbf{R}^r_{>0}$ acts on $(\mathbf{R}^r_{\geq 0})_{\mathrm{val}}$.

Let $\mathbf{N} := \mathbf{Z}_{\geq 0}$. There exists a canonical bijection between $(\mathbf{R}^r_{\geq 0})_{\mathrm{val}}$ and the set of all pairs (V, h) where V is a valuative submonoid of $\mathbf{Z}^r$ containing $\mathbf{N}^r$ and $h : V^\times \to \mathbf{R}_{>0}$ is a group homomorphism. In fact, for a point x of $(\mathbf{R}^r_{\geq 0})_{\mathrm{val}} \subset B$, the corresponding pair (V, h) is defined by

$$(2) \qquad \begin{cases} V := \{m \in \mathbf{Z}^r \mid \prod_j t_j^{m(j)} \text{ is regular at } x\}, \\ h(m) := \big(\prod_j t_j^{m(j)}\big)(x). \end{cases}$$

The inverse map $(V, h) \mapsto x = (x_I)_{I \in \Psi}$, $x_I \in \mathrm{Bl}_I((\mathbf{G}_{\mathbf{a},\mathbf{R}})^r)$, is given as follows. Let $\tilde{h}$ be the extension by zero of h to V. Let $I = (f_1, \ldots, f_n)$, and take $1 \leq k \leq n$ such that the all exponents of $\frac{f_l}{f_k}$ $(1 \leq l \leq n)$, in the expressions as the products of t_j $(1 \leq j \leq r)$, belong to V. Define a point x_I by an $\mathbf{R}$-algebra homomorphism

$$x_I : \mathbf{R}\big[t_1, \ldots, t_r, \tfrac{f_1}{f_k}, \ldots, \tfrac{f_n}{f_k}\big] \to \mathbf{R}, \quad x_I\big(\textstyle\prod_j t_j^{m(j)}\big) := \tilde{h}(m) \quad (m \in V).$$

The action of $a = (a_j)_{1 \leq j \leq r} \in \mathbf{R}^r_{>0}$ sends (V, h) to (V, ah), where ah is defined by

$$(ah)(m) := \big(\textstyle\prod_j a_j^{m(j)}\big) h(m) \quad (m \in V^\times).$$

A directed family of elements (V_λ, h_λ) of $(\mathbf{R}^r_{\geq 0})_{\mathrm{val}}$ converges to $(V, h) \in (\mathbf{R}^r_{\geq 0})_{\mathrm{val}}$ if and only if the following condition is satisfied for each $m \in V$:

(3) $m \in V_\lambda$ for any sufficiently large λ, and $\tilde{h}_\lambda(m)$ converges to $\tilde{h}(m)$.

(For the generality of $(\ )_{\mathrm{val}}$, cf. [K].)

2.14. *Identification of* $D_{\mathrm{BS,val}}(P)$ *with* $D \times^{A_P} (\overline{A}_P)_{\mathrm{val}}$. Let P be a $\mathbf{Q}$-parabolic subgroup of $G_{\mathbf{R}}$. Let $(\overline{A}_P)_{\mathrm{val}}$ be the set of all pairs (V, h) where V is a valuative submonoid of $X(S_P)$ containing Δ_P and $h : V^\times \to \mathbf{R}_{>0}$ is a homomorphism. Then, by writing $\Delta_P = \{\chi_1, \ldots \chi_r\}$ $(r = \mathrm{rank}\, S_P)$, we have a bijection

$$(\overline{A}_P)_{\mathrm{val}} \simeq (\mathbf{R}^r_{\geq 0})_{\mathrm{val}}, \quad (V, h) \mapsto (V', h'),$$

where $V' := \{m \in \mathbf{Z}^r \mid \prod_j \chi_j^{m(j)} \in V\}$ and h' is the map $(V')^\times \to \mathbf{R}_{>0}$, $m \mapsto h(\prod_j \chi_j^{m(j)})$. For the set $D_{\mathrm{BS,val}}(P)$ in 2.9, we have a bijection

$$(1) \qquad D_{\mathrm{BS,val}}(P) \to D \times^{A_P} (\overline{A}_P)_{\mathrm{val}}, \quad (T, Z, V') \mapsto (F, (V, h)),$$

which is given by

$$(2)\quad \begin{cases} F \in Z \ \text{(any element)}, \\ V := (\text{the inverse image of } V' \text{ under } X(S_P) \to X(T)), \\ h : V^\times \to \mathbf{R}_{>0}, \text{ the trivial homomorphism} \\ \qquad\qquad h(m) := 1 \ (\forall m \in V^\times). \end{cases}$$

Here in the definition of V, we regard T as a subtorus of S_P via the composite of the embeddings $T \hookrightarrow S_{P_{T,V'}} \hookrightarrow S_P$ (2.8 and 2.11).

The inverse map $D \times^{A_P} \overline{A}_P \to D_{\mathrm{BS,val}}(P)$, $(F,(V,h)) \mapsto (T,Z,V')$, is given by

$$\begin{cases} T := \begin{pmatrix} \text{the image of the annihilator of } V^\times \text{ in } S_P \\ \text{under the Borel-Serre lifting } S_P \hookrightarrow G_{\mathbf{R}} \text{ at} K_F \end{pmatrix}, \\ Z := \{a \circ F \mid a \in A_P,\ \chi(a) = h(\chi)\ (\forall \chi \in V^\times)\}, \\ V' := (\text{the image of } V \text{ under } X(S_P) \to X(T)) \ \text{(so that } V' \simeq V/V^\times). \end{cases}$$

We show that (T,Z,V') satisfies the conditions (i)–(iii) in 2.6 which define $D_{\mathrm{BS,val}}$. (i) and (ii) are clear. We prove (iii), that is, $M_\chi = \bigoplus_{\chi' \in \chi V'^{-1}} H(\chi')$ is $\mathbf{Q}$-rational for any $\chi \in X(T)$. The image of T in P/P_u is $\mathbf{Q}$-rational, since it is contained in S_P. This shows that uTu^{-1} is $\mathbf{Q}$-rational for some $u \in P_u$. Hence uM_χ is $\mathbf{Q}$-rational for any $\chi \in X(T)$. Hence it is enough to prove $uM_\chi = M_\chi$. For this, it is enough to prove $lM_\chi \subset M_\chi$ for $l := \log u \in \mathrm{Lie}(P_u)$. By decomposing $\mathrm{Lie}(P_u)$ as the direct sum of eigenspaces for the adjoint action of S_P, we may assume that there exists $\chi' \in \Delta'_P$ such that $\mathrm{Ad}(t)l = \chi'(t)^{-1}l$ for any $t \in T$. Since $\Delta_P \subset V$, V' contains the restrictions of elements of Δ'_P to T. Hence $\chi'|_T \in V'$. Hence $lM_\chi \subset M_{\chi(\chi'|_T^{-1})} \subset M_{\chi V'^{-1}} = M_\chi$.

2.15. *Topologies of* D_{BS}, $\mathcal{X}_{\mathrm{BS}}$ *and* $D_{\mathrm{BS,val}}$. Let P be a $\mathbf{Q}$-parabolic subgroup of $G_{\mathbf{R}}$. We have, with the identification of $\overline{A}_P$ and $\mathbf{R}^r_{\geq 0}$,

$$\begin{aligned} &D_{\mathrm{BS}}(P) \simeq D \times^{A_P} \mathbf{R}^r_{\geq 0} \ \text{(see 2.12)}, \\ &\mathcal{X}_{\mathrm{BS}}(P) \simeq \mathcal{X} \times^{A_P} \mathbf{R}^r_{\geq 0} \ \text{(analoguously to the above)}, \\ &D_{\mathrm{BS,val}}(P) \simeq D \times^{A_P} (\mathbf{R}^r_{\geq 0})_{\mathrm{val}} \ \text{(see 2.14)}. \end{aligned}$$

By using these isomorphisms, we introduce a topology on $D_{\mathrm{BS}}(P)$ (resp. $\mathcal{X}_{\mathrm{BS}}(P)$, $D_{\mathrm{BS,val}}(P)$). We introduce the strongest topology on D_{BS} (resp. $\mathcal{X}_{\mathrm{BS}}$, $D_{\mathrm{BS,val}}$) for which the map $D_{\mathrm{BS}}(P) \hookrightarrow D_{\mathrm{BS}}$ (resp. $\mathcal{X}_{\mathrm{BS}}(P) \hookrightarrow \mathcal{X}_{\mathrm{BS}}$, $D_{\mathrm{BS,val}}(P) \hookrightarrow D_{\mathrm{BS,val}}$) is continuous for every $\mathbf{Q}$-parabolic subgroup P of $G_{\mathbf{R}}$. Then, it can be shown as in [BS] that all

these maps $D_{\mathrm{BS}}(P) \hookrightarrow D_{\mathrm{BS}}$, $\mathcal{X}_{\mathrm{BS}}(P) \hookrightarrow \mathcal{X}_{\mathrm{BS}}$, and $D_{\mathrm{BS,val}}(P) \hookrightarrow D_{\mathrm{BS,val}}$ are open embeddings.

2.16. We give here a preparation for the proof of Theorem 2.17. Let P be a $\mathbf{Q}$-parabolic subgroup of $G_{\mathbf{R}}$. As we shall show below, there is a closed subset $\mathcal{X}'$ of $\mathcal{X}$ such that we have homeomorphisms

$$\mathcal{X}' \times A_P \xrightarrow{\sim} \mathcal{X}, \quad (x, a) \mapsto a \circ x, \tag{1}$$

$$D' \times A_P \xrightarrow{\sim} D, \quad (x, a) \mapsto a \circ x, \tag{2}$$

where D' is the inverse image of $\mathcal{X}'$ under the canonical map $D \to \mathcal{X}$ in 2.2 (3). These (1), (2) induce homeomorphisms

$$\begin{aligned} &\mathcal{X}' \times \mathbf{R}^r_{\geq 0} \simeq \mathcal{X}_{\mathrm{BS}}(P), \\ &D' \times \mathbf{R}^r_{\geq 0} \simeq D_{\mathrm{BS}}(P), \\ &D' \times (\mathbf{R}^r_{\geq 0})_{\mathrm{val}} \simeq D_{\mathrm{BS,val}}(P), \end{aligned} \tag{3}$$

where $r = \operatorname{rank} S_P$.

Now we define $\mathcal{X}'$. Let ${}^\circ P$ be the intersection of the kernel of $|\chi| : P \to \mathbf{R}_{>0}$, $a \mapsto |\chi(a)|$, for all homomorphisms χ of algebraic groups $P \to \mathbf{G}_{\mathbf{m},\mathbf{R}}$ defined over $\mathbf{Q}$. Then $P_u \subset {}^\circ P$. By [BS, 1.2], the canonical map $A_P \to P/{}^\circ P$ is an isomorphism of topological groups. Let $|\ | : P \to A_P$ be the composite map $P \to P/{}^\circ P \simeq A_P$. Fix a maximal compact subgroup K of $G_{\mathbf{R}}$. Since $G_{\mathbf{R}} = PK$ (cf. [B, §11]) and since $|\ |$ kills the compact group $P \cap K$, there exists a unique map $G_{\mathbf{R}} \to A_P$ sending pk to $|p|$ $(p \in P, k \in K)$. This map factors through $G_{\mathbf{R}}/K \simeq \mathcal{X}$. Let $\mathcal{X}'$ be the inverse image of $1 \in A_P$ under the induced map $\mathcal{X} \to A_P$. It is seen easily that we have the homeomorphisms (1), (2).

Theorem 2.17. (i) *The maps* $\alpha : D_{\mathrm{BS,val}} \to D_{\mathrm{BS}}$ *and* $\beta : D_{\mathrm{BS}} \to \mathcal{X}_{\mathrm{BS}}$ *in* 2.8 *are proper and surjective.*

(ii) *The spaces* $\mathcal{X}_{\mathrm{BS}}$, D_{BS}, $D_{\mathrm{BS,val}}$ *are Hausdorff and locally compact.*

Proof. (i) follows from the descriptions of $\mathcal{X}_{\mathrm{BS}}(P)$, $D_{\mathrm{BS}}(P)$, $D_{\mathrm{BS,val}}(P)$ in 2.16 (3) together with the fact that $(\mathbf{R}^r_{\geq 0})_{\mathrm{val}} \to \mathbf{R}^r_{\geq 0}$ and $D' \to \mathcal{X}'$ are proper and surjective.

(ii) for $\mathcal{X}_{\mathrm{BS}}$ is proved in [BS]. (ii) for D_{BS}, $D_{\mathrm{BS,val}}$ follows from this and (i). Q.E.D.

§3. Spaces of SL(2)-orbits

3.1. *Summary.* Let $\Delta : (\mathbf{G}_{\mathbf{m},\mathbf{R}})^n \to \mathrm{SL}(2,\mathbf{R})^n$ be the homomorphism defined by

$$(t_1,\dots,t_n) \mapsto \left(\begin{pmatrix} t_1^{-1} & 0 \\ 0 & t_1 \end{pmatrix},\dots,\begin{pmatrix} t_n^{-1} & 0 \\ 0 & t_n \end{pmatrix}\right).$$

For a homomorphism $\rho : \mathrm{SL}(2,\mathbf{C})^n \to G_{\mathbf{C}}$ defined over $\mathbf{R}$, the following three conditions are equivalent:

(1) ρ does not factor through the projection onto any component $\mathrm{SL}(2,\mathbf{C})^m$ of $\mathrm{SL}(2,\mathbf{C})^n$ $(m < n)$.

(2) The induced Lie algebra homomorphism $\rho_* : \mathfrak{sl}(2,\mathbf{C})^{\oplus n} \to \mathfrak{g}_{\mathbf{C}}$ is injective.

(3) The induced Lie algebra homomorphism $\mathrm{Lie}(\mathbf{G}_{\mathbf{m},\mathbf{C}})^n \xrightarrow{\Delta_*} \mathfrak{sl}(2,\mathbf{C})^{\oplus n} \xrightarrow{\rho_*} \mathfrak{g}_{\mathbf{C}}$ is injective.

Let (ρ,φ) be a pair of a homomorphism $\rho : \mathrm{SL}(2,\mathbf{C})^n \to G_{\mathbf{C}}$ defined over $\mathbf{R}$ and a map $\varphi : \mathbf{P}^1(\mathbf{C})^n \to \check{D}$. Throughout this paper, such a pair (ρ,φ) is called an $\mathrm{SL}(2)$-*orbit of rank* n if it satisfies the following three conditions:

(i) ρ satisfies the equivalent conditions (1)–(3).

(ii) $\varphi(gz) = \rho(g)\varphi(z)$ for all $g \in \mathrm{SL}(2,\mathbf{C})^n$ and all $z \in \mathbf{P}^1(\mathbf{C})^n$.

(iii) Let $\mathfrak{h}$ be the upper-half plane and let $\mathbf{i} := (i,\dots,i) \in \mathfrak{h}^n \subset \mathbf{P}^1(\mathbf{C})^n$. Then $\varphi(\mathbf{i}) \in D$, and the associated Lie algebra homomorphism $\mathrm{Lie}(\rho) : \mathfrak{sl}(2,\mathbf{C})^{\oplus n} \to \mathfrak{g}_{\mathbf{C}}$ is a homomorphism of type $(0,0)$ with respect to the Hodge structures induced by the points $\mathbf{i} \in \mathfrak{h}^n$ and $\varphi(\mathbf{i}) \in D$, respectively.

Let (ρ,φ) be an $\mathrm{SL}(2)$-orbit of rank n. We denote by Y_j, $N_j = N_j^-$, N_j^+ the image under the Lie algebra homomorphism $\mathrm{Lie}(\rho)$ of

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

of the j-th factor of $\mathfrak{sl}(2,\mathbf{C})^{\oplus n}$, respectively.

Define a homomorphism

$$\tilde{\rho} : (\mathbf{G}_{\mathbf{m},\mathbf{R}})^n \to G_{\mathbf{R}}, \quad (t_1,\dots,t_n) \mapsto \rho(\Delta(t_1\cdots t_n, t_2\cdots t_n,\dots,t_{n-1}t_n,t_n)). \tag{4}$$

(The map $\tilde{\rho}$ will be related in 3.9 to Borel-Serre liftings.)

In this section, we introduce spaces of $\mathrm{SL}(2)$-orbits $D_{\mathrm{SL}(2)}$ and the projective limit $D_{\mathrm{SL}(2),\mathrm{val}}$ of the blowing-ups of $D_{\mathrm{SL}(2)}$. These spaces,

together with the spaces in the previous section, will form the following diagram:

$$\begin{array}{ccccc} D_{\mathrm{SL}(2),\mathrm{val}} & \hookrightarrow & D_{\mathrm{BS},\mathrm{val}} & & \\ \downarrow & & \downarrow & & \\ D_{\mathrm{SL}(2)} & & D_{\mathrm{BS}} & \to & \mathcal{X}_{\mathrm{BS}}. \end{array} \tag{5}$$

In general, there is no direct relation between $D_{\mathrm{SL}(2)}$ and D_{BS} (see, §6), which is why we introduce $D_{\mathrm{SL}(2),\mathrm{val}}$ and $D_{\mathrm{BS},\mathrm{val}}$.

3.2. *Weight filtrations.* For a nilpotent element $N \in \mathfrak{g}_{\mathbf{R}}$, the *weight filtration associated to* N is the increasing filtration $W = W(N)$ of $H_{0,\mathbf{R}}$ characterized by the following conditions (i), (ii) ([D]).

(i) $NW_k \subset W_{k-2}$ for all $k \in \mathbf{Z}$;

(ii) $N^k : \mathrm{gr}_k^W \xrightarrow{\sim} \mathrm{gr}_{-k}^W$ for all $k \in \mathbf{Z}_{\geq 0}$.

3.3. *Cones.* We fix terminology concerning cones. Let V be an $\mathbf{R}$-vector space. A *cone* in V is a subset σ of V which is closed under addition and under multiplication by elements of $\mathbf{R}_{\geq 0}$ and satisfies $\sigma \cap (-\sigma) = 0$. A subset σ in $\mathfrak{g}_{\mathbf{R}}$ is a *nilpotent cone in* $\mathfrak{g}_{\mathbf{R}}$ if it is a finitely generated cone in $\mathfrak{g}_{\mathbf{R}}$ consisiting of mutually commutative nilpotent elements. Let σ be a nilpotent cone in $\mathfrak{g}_{\mathbf{R}}$. For $A = \mathbf{R}, \mathbf{C}$, we denote by σ_A the A-linear span of σ in $\mathfrak{g}_A$.

Definition 3.4. Let σ be a nilpotent cone in $\mathfrak{g}_{\mathbf{R}}$, and let N_j $(1 \leq j \leq r)$ be its generators over $\mathbf{R}_{\geq 0}$. A subset Z of $\check{D}$ is a *σ-nilpotent orbit* (*resp. σ-nilpotent i-orbit*) if it satisfies the following three conditions for some $F \in Z$.

(i) $Z = \exp(\sigma_{\mathbf{C}})F$ (resp. $Z = \exp(i\sigma_{\mathbf{R}})F$).

(ii) $NF^p \subset F^{p-1}$ $(\forall p, \forall N \in \sigma)$.

(iii) $\exp(\sum_{1 \leq j \leq r} iy_j N_j)F \in D$ $(\forall y_j \gg 0)$.

It is easy to see that, in 3.4, if the conditions (i)–(iii) are satisfied by one $F \in Z$ then they are satisfied by any $F \in Z$. The condition (ii) is called *Griffiths transversality* and the condition (iii) is called *positivity.*

3.5. *Weight filtrations associated to a nilpotent orbit.* We recall here a result of Cattani and Kaplan.

Theorem-Definition ([CK2]). *Let (σ, Z) be a nilpotent i-orbit. Then, for any elements N, N' of the relative interior of σ, the filtrations*

$W(N)$ and $W(N')$ of $H_{0,\mathbf{R}}$ coincide. This common filtration is denoted by $W(\sigma)$.

Note that, as in [CKS, §4], an SL(2)-orbit (ρ, φ) of rank n defines an n-tuple of nilpotent elements $(N_j)_{1\leq j\leq n}$ and a nested family of nilpotent i-orbits $(\sigma_j, Z_j)_{1\leq j\leq n}$ by

$$\begin{aligned} \sigma_j &:= \mathbf{R}_{\geq 0}N_1 + \cdots + \mathbf{R}_{\geq 0}N_j, \\ Z_j &:= \exp(i\sigma_{j,\mathbf{R}})\varphi(\overbrace{0,\ldots,0}^{j}, i, \ldots, i). \end{aligned} \tag{1}$$

We have $W(N_1 + \cdots + N_j) = W(\sigma_j)$ for each $1 \leq j \leq n$.

Definition 3.6. We define $D_{\mathrm{SL}(2),0} := D$ and, for a positive integer n, we define

$$D_{\mathrm{SL}(2),n} := \left\{ (\rho,\varphi) \;\middle|\; \begin{array}{l} (\rho,\varphi) \text{ is an SL(2)-orbit of rank } n, \\ W(\sigma_j) \text{ is } \mathbf{Q}\text{-rational } (1 \leq j \leq n) \end{array} \right\} \Big/ \sim,$$

where $(\rho,\varphi) \sim (\rho',\varphi')$ if and only if there exists $t \in \mathbf{R}^n_{>0}$ such that $\rho' = \mathrm{Int}(\rho(\Delta(t)))(\rho)$ and $\varphi' = \rho(\Delta(t)) \cdot \varphi$. We define

$$D_{\mathrm{SL}(2)} := \bigsqcup_{n\geq 0} D_{\mathrm{SL}(2),n}, \quad D_{\mathrm{SL}(2),\leq r} := \bigsqcup_{0\leq n\leq r} D_{\mathrm{SL}(2),n}.$$

We denote by $[\rho,\varphi]$ the point of $D_{\mathrm{SL}(2)}$ represented by (ρ,φ).

Definition 3.7. For a non-negative integer n, we define

$$D_{\mathrm{SL}(2),\mathrm{val},n} := \left\{ ([\rho,\varphi], Z, V) \;\middle|\; \begin{array}{l} [\rho,\varphi] \in D_{\mathrm{SL}(2),n},\ Z \subset \varphi((\mathbf{R}_{>0}i)^n), \\ V : \text{a valuative submonoid of } X(\mathbf{G}_m^n), \\ \text{which satisfy (i) and (ii) below} \end{array} \right\}.$$

(i) Let $X((\mathbf{G_m})^n)_+$ be the image of $\mathbf{N}^n$ via the canonical isomorphism $\mathbf{Z}^n \simeq X((\mathbf{G_m})^n)$. Then,
$$X((\mathbf{G_m})^n)_+ \subset V \text{ and } X((\mathbf{G_m})^n)_+ \cap V^\times = \{1\}.$$

(ii) Let
$$T := \{t \in (\mathbf{G}_{\mathbf{m},\mathbf{R}})^n \mid \chi(t) = 1\ (\forall \chi \in V^\times)\}.$$
Then Z is a $\tilde{\rho}(T_{>0})$-orbit in D. Here $\tilde{\rho}$ is as in 3.1, and we denote again by $T_{>0}$ the connected component of T containing the unity.

We define
$$D_{\mathrm{SL}(2),\mathrm{val}} := \bigsqcup_{n\geq 0} D_{\mathrm{SL}(2),\mathrm{val},n}.$$

We have the canonical surjection

$$D_{\mathrm{SL}(2),\mathrm{val}} \to D_{\mathrm{SL}(2)}, ([\rho,\varphi], Z, V) \mapsto [\rho,\varphi].$$

After preliminaries in Lemmas 3.8–3.10, we will relate $D_{\mathrm{SL}(2),\mathrm{val}}$ with $D_{\mathrm{BS},\mathrm{val}}$ in 3.11 below.

Lemma 3.8. *Let (ρ,φ) be an* SL(2)*-orbit of rank n and put $\mathbf{r} := \varphi(\mathbf{i})$. Then*

$$\theta_{K_\mathbf{r}}(\rho(\Delta(t))) = \rho(\Delta(t))^{-1} \quad (\forall t \in (\mathbf{G}_{\mathbf{m},\mathbf{R}})^n).$$

Proof. Let Y_j $(1 \le j \le n)$ be as in 3.1. It is enough to show $\theta_{K_\mathbf{r}}(Y_j) = -Y_j$ for all j. We prove this. Here $\theta_{K_\mathbf{r}}$ is regarded as the involution of $\mathfrak{g}_\mathbf{C}$ induced by the Cartan involution $\theta_{K_\mathbf{r}}$ of $G_\mathbf{R}$ at $K_\mathbf{r}$ by abuse of the notation. Let

$$\mathfrak{g}_\mathbf{C} = \bigoplus_s \mathfrak{g}_\mathbf{r}^{s,-s}, \quad \mathfrak{g}_\mathbf{r}^{s,-s} := \{X \in \mathfrak{g}_\mathbf{C} \mid X H_\mathbf{r}^{p,w-p} \subset H_\mathbf{r}^{p+s,w-p-s} \ (\forall p)\}$$

be the Hodge structure on $\mathfrak{g}_\mathbf{C}$ induced by $\mathbf{r}$. Then, by 2.2 (2) and the definition of the Weil operator $C_\mathbf{r}$ in 1.2, $\theta_{K_\mathbf{r}}$ is given by

(1) $\theta_{K_\mathbf{r}}(X) = \sum_s (-1)^s X^{s,-s}$ for $X = \sum_s X^{s,-s} \in \mathfrak{g}_\mathbf{C} = \bigoplus_s \mathfrak{g}_\mathbf{r}^{s,-s}$.

On the other hand, the Hodge decomposition of $H_{\mathfrak{h},\mathbf{C}} = \mathbf{C}^2 = \mathbf{C}e_1 + \mathbf{C}e_2$ corresponding to $i \in \mathfrak{h}$ is

$$H_{\mathfrak{h},\mathbf{C}} = H_\mathfrak{h}^{1,0} \oplus H_\mathfrak{h}^{0,1} = \mathbf{C}(ie_1 + e_2) \oplus \mathbf{C}(-ie_1 + e_2) \ \text{(cf. 6.2 below)},$$

and this induces the Hodge decomposition

$$\begin{aligned}\mathfrak{sl}(2,\mathbf{C}) &= \mathfrak{sl}(2)^{1,-1} \oplus \mathfrak{sl}(2)^{0,0} \oplus \mathfrak{sl}(2)^{-1,1} \\ &= \mathbf{C}\begin{pmatrix} i & 1 \\ 1 & -i \end{pmatrix} \oplus \mathbf{C}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \oplus \mathbf{C}\begin{pmatrix} -i & 1 \\ 1 & i \end{pmatrix}\end{aligned}$$

of $\mathfrak{sl}(2,\mathbf{C})$. Since

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{i}{2}\left(\begin{pmatrix} i & 1 \\ 1 & -i \end{pmatrix} - \begin{pmatrix} -i & 1 \\ 1 & i \end{pmatrix}\right),$$

$Y_j \in \mathfrak{g}_\mathbf{r}^{1,-1} \oplus \mathfrak{g}_\mathbf{r}^{-1,1}$ by 3.1 (ii) for all j. Hence, by (1), $\theta_{K_\mathbf{r}}(Y_j) = -Y_j$ for all j. Q.E.D.

Lemma 3.9. *Let (ρ,φ) and $\mathbf{r}$ be as in* 3.8. *For $1 \le j \le n$, let $W^{(j)} = W(\sigma_j)$ be as in* 3.5 *and let P_j be the $\mathbf{Q}$-parabolic subgroup of $G_\mathbf{R}$ defined by $W^{(j)}$. Then the j-th factor of $\tilde{\rho}$ (cf.* 3.1*),*

$$t_j \mapsto \tilde{\rho}(1,\dots,1,t_j,1,\dots,1),$$

coincides with the Borel-Serre lifting at $K_{\mathbf{r}}$ of the j-th weight map

$$\mathbf{G}_{\mathbf{m},\mathbf{R}} \to P_j/P_{j,u}, \quad t_j \mapsto (t_j^k \text{ on } \mathrm{gr}_k^{W(j)})_k.$$

Proof. This follows from 3.8 and the observation

$$\begin{aligned}\tilde{\rho}(1,\dots,1,t_j,1,\dots,1) &= \rho(\Delta(\overbrace{t_j,\dots,t_j}^{j},1,\dots,1)) \\ &= \exp(\log(t_j)(Y_1+\cdots+Y_j)).\end{aligned}$$ Q.E.D.

Lemma 3.10. *Let (ρ,φ), $\mathbf{r}=\varphi(\mathbf{i})$ be as in 3.8, and W be the family of weight filtrations $(W(\sigma_j))_{1\le j\le n}$ as just before 3.6. Then, an $\mathrm{SL}(2)$-orbit (ρ,φ) of rank n is completely determined by $(W,\mathbf{r})$.*

Proof. By 3.9, $\tilde{\rho}$ is determined by $(W,\mathbf{r})$, and Y_j $(1\le j\le n)$ are determined by $\tilde{\rho}$. Let $T_D(\mathbf{r})$ (resp. $T_{\mathfrak{h}}(i)$) be the tangent space of D at $\mathbf{r}$ (resp. $\mathfrak{h}$ at i). Then we have a commutative diagram

$$\begin{array}{ccc} \mathfrak{g}_{\mathbf{C}} & \xrightarrow{\alpha_{\mathbf{r}}} & T_D(\mathbf{r}) \\ {\scriptstyle \mathrm{Lie}(\rho)}\uparrow & & \uparrow{\scriptstyle d\varphi} \\ \mathfrak{sl}(2,\mathbf{C})^n & \xrightarrow{\alpha_i^n} & T_{\mathfrak{h}}(i)^n, \end{array} \tag{1}$$

where $\alpha_{\mathbf{r}}$ (resp. α_i) is the differential of the morphism $G_{\mathbf{C}} \to \check{D}$, $g \mapsto g\mathbf{r}$, (resp. $\mathrm{SL}(2,\mathbf{C}) \to \mathbf{P}^1(\mathbf{C})$, $g \mapsto gi$,) at 1. Since $-2i\alpha_i \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \alpha_i \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, we have $-2i\alpha_{\mathbf{r}}(N_j) = \alpha_{\mathbf{r}}(Y_j)$ $(1\le j\le n)$. Since the restriction of $\alpha_{\mathbf{r}}$ on $\mathrm{Lie}(P_{j,u})$ is injective, N_j $(1\le j\le n)$ are determined. The N_j^+ are determined by the Y_j and the N_j. Q.E.D.

Theorem 3.11. *There is an injective map*

$$D_{\mathrm{SL}(2),\mathrm{val}} \to D_{\mathrm{BS},\mathrm{val}}, \quad ([\rho,\varphi],Z,V) \mapsto (\tilde{\rho}(T),Z,V'). \tag{1}$$

Here T is the subtorus of $(\mathbf{G}_{\mathbf{m},\mathbf{R}})^n$ in 3.7 (ii) and $V' := \tilde{\rho}_*(V/V^{\times})$, which is regarded as a subset of the character group of $\tilde{\rho}(T)$.

Proof. Let $([\rho,\varphi],Z,V) \in D_{\mathrm{SL}(2),\mathrm{val}}$ and let (T',Z,V') be its image under (1).

We check first the conditions (i)–(iii) in 2.6 hold for (T',Z,V'). (i) follows from 3.8 and 2.3. (ii) is evident. We prove (iii), that is, $M_\chi =$

$\bigoplus_{\chi' \in \chi V'^{-1}} H(\chi')$ is $\mathbf{Q}$-rational for any $\chi \in X(T')$. For $\chi \in X((\mathbf{G_m})^n)$, let

$$\tilde{H}(\chi) := \{v \in H_{0,\mathbf{R}} \mid \tilde{\rho}(t)v = \chi(t)v \ \ (\forall t \in (G_{m,\mathbf{R}})^n)\},$$
$$W_\chi := \bigoplus_{\chi' \in \chi(X((\mathbf{G_m})^n)_+)^{-1}} \tilde{H}(\chi').$$

Let $\chi \in X(T')$ and let $\tilde{\chi} \in X((\mathbf{G_m})^n)$ be an element such that

$$\chi(\tilde{\rho}(t)) = \tilde{\chi}(t) \quad (\forall t \in T).$$

Then

$$M_\chi = \bigoplus_{\chi' \in \tilde{\chi} V^{-1}} \tilde{H}(\chi').$$

Since $X((\mathbf{G_m})^n)_+ \subset V$, we have

$$M_\chi = \sum_{\chi' \in \tilde{\chi} V^{-1}} W_{\chi'}.$$

Hence it is enough to show that W_χ is $\mathbf{Q}$-rational for any $\chi \in X((\mathbf{G_m})^n)$. Write $\chi = (l_1, \ldots, l_n)$ $(l_j \in \mathbf{Z})$ via $X((\mathbf{G_m})^n) = \mathbf{Z}^n$. Then $W_\chi = \bigcap_{1 \le j \le n} W(\sigma_j)_{l_j}$, where $W(\sigma_j)$ is as in 3.5. Since the $W(\sigma_j)_{l_j}$ are $\mathbf{Q}$-rational, so is W_χ. This proves that the map (1) is well-defined.

We prove that (1) is injective. $V \subset X((\mathbf{G_m})^n)$ is the inverse image of V' under the map $X((\mathbf{G_m})^n) \to X(T) \simeq X(T')$. Hence it is sufficient to show that $[\rho, \varphi] \in D_{\mathrm{SL}(2)}$ is determined by (T', Z, V'). Let $\mathbf{r} \in Z$ and take a representative (ρ, φ) with $\varphi(\mathbf{i}) = \mathbf{r}$. To prove that (ρ, φ) is determined by (T', Z, V') and $\mathbf{r}$, it is sufficient to show, by 3.10, that the rank n of (ρ, φ) and the family of weight filtrations $\big(W(N_1 + \cdots + N_j)\big)_{1 \le j \le n}$ associated to (ρ, φ) are determined by (T', Z, V'). Hence it is sufficient to prove that the family $(N_j)_{1 \le j \le n}$ is the unique family of elements of $\mathrm{Lie}(P_{T',V',u})$ having the following properties (i)–(iv).

(i) For $1 \le j \le n$, N_j is a non-zero eigenvector for the adjoint action of T'.

(ii) For $1 \le j \le n$, let $\chi_j : T' \to \mathbf{G}_{\mathbf{m},\mathbf{R}}$ be a character defined by $\mathrm{Ad}(t)N_j = \chi_j(t)N_j$ for $t \in T'$. Then the χ_j are non-trivial and different from each other.

(iii) In the notation (ii) above, $\chi_j \chi_{j+1}^{-1} \in V'$ $(1 \le j \le n-1)$.

(iv) Let $\alpha_{\mathbf{r}} : \mathfrak{g}_{\mathbf{C}} \to T_D(\mathbf{r})$ be the canonical $\mathbf{C}$-linear map 3.10 (1). For $1 \le j \le n$, let $\mathrm{Lie}(\chi_j) : \mathrm{Lie}(T') \to \mathbf{R}$ be the map induced by χ_j. Then

$$-i\alpha_{\mathbf{r}}(A) = \sum_{1 \le j \le n} \mathrm{Lie}(\chi_j)(A)\alpha_{\mathbf{r}}(N_j) \ \text{ for any } A \in \mathrm{Lie}(T').$$

We first show that the family $(N_j)_{1\leq j\leq n}$ associated to (ρ,φ) satisfies these (i)–(iv). We have

$$\mathrm{Ad}(\rho(\Delta(t_1,\dots,t_n)))(N_j)=t_j^{-2}N_j$$

and hence

$$\mathrm{Ad}(\tilde{\rho}(t_1,\dots,t_n))(N_j)=(t_j\cdots t_n)^{-2}N_j. \tag{2}$$

Hence (i) and (iii) are satisfied. Denote by χ_{n+1} the trivial character of T'. Suppose $\chi_j=\chi_k$ for some j, k with $1\leq j\leq k\leq n+1$. Then the character

$$(t_1,\dots,t_n)\mapsto\textstyle\prod_{j\leq l<k}t_l^2$$

of $(\mathbf{G_{m,R}})^n$ is trivial on T and hence belongs to $V^\times$. Since it also belongs to $X((\mathbf{G_m})^n)_+$ and since $X((\mathbf{G_m})^n)_+\cap V^\times=\{1\}$, we have $j=k$. Thus we have proved (ii). Note that

$$\alpha_{\mathbf{r}}(Y_j)=-2i\alpha_{\mathbf{r}}(N_j)\quad(1\leq j\leq n)\quad\text{(cf. Proof of 3.10).} \tag{3}$$

By (2) and (3), we have, for an element $A=\sum_{1\leq j\leq n}b_jY_j$ of $\mathrm{Lie}(T')$,

$$-i\alpha_{\mathbf{r}}(A)=-2\textstyle\sum_{1\leq j\leq n}b_j\alpha_{\mathbf{r}}(N_j)=\sum_{1\leq j\leq n}\mathrm{Lie}(\chi_j)(A)\alpha_{\mathbf{r}}(N_j).$$

Next we prove that a family $(N_j)_{1\leq j\leq n}$ of elements of $\mathrm{Lie}(P_{T',V',u})$ satisfying (i)–(iv) is unique. Since the restriction of $\alpha_{\mathbf{r}}$ on $\mathrm{Lie}(P_{T',V',u})$ is injective, we have:

(4) $\sum_{1\leq j\leq n}\mathrm{Lie}(\chi_j)(A)N_j$ for $A\in\mathrm{Lie}(T')$ is the unique element of $\mathrm{Lie}(P_{T',V',u})$ whose image under $\alpha_{\mathbf{r}}$ coincides with $-i\alpha_{\mathbf{r}}(A)$.

As sets, we have

$$\{\chi_j\}_{1\leq j\leq n}=\left\{\chi\in X(T')\;\middle|\;\begin{array}{l}\text{under the action of }\mathrm{Ad}(T')\\ \text{on }\mathfrak{g}_{\mathbf{C}},\text{the }\chi\text{-component}\\ \text{of }\sum_{1\leq j\leq n}\mathrm{Lie}(\chi_j)(A)N_j\\ (A\in\mathrm{Lie}(T'))\text{ is not zero}\end{array}\right\} \tag{5}$$

Since $(V')^\times=1$, (iii) determines the order, hence the family $(\chi_j)_{1\leq j\leq n}$. For each $1\leq j\leq n$, $\mathrm{Lie}(\chi_j)(A)N_j$ $(A\in\mathrm{Lie}(T'))$ is determined as the χ_j-component of $\sum_{1\leq j\leq n}\mathrm{Lie}(\chi_j)(A)N_j$ $(A\in\mathrm{Lie}(T'))$ under the action of $\mathrm{Ad}(T')$. Since $\mathrm{Lie}(\chi_j)\neq 0$, N_j is determined. Q.E.D.

3.12. *Topologies on $D_{\mathrm{SL}(2)}$, $D_{\mathrm{SL}(2),\mathrm{val}}$.* A family $(W^{(j)})_{1\leq j\leq n}$ of increasing filtrations $W^{(j)}$ of $H_{0,\mathbf{R}}$ is called a *compatible family* if there exists a direct sum decomposition $H_{0,\mathbf{R}}=\bigoplus_{m\in\mathbf{Z}^n}H(m)$ such that $W_k^{(j)}=$

$\bigoplus_{m\in\mathbf{Z}^n,\, m_j\leq k} H(m)$ for any j and k and that $\langle H(m), H(m')\rangle_0 = 0$ if $m+m' \neq 0$. Note that, for $[\rho,\varphi] \in D_{SL(2),n}$, the family of weight filtrations $(W(\sigma_j))_{1\leq j\leq n}$ associated to $[\rho,\varphi]$ in 3.5 is a compatible family.

Let $W = (W^{(j)})_{1\leq j\leq n}$ be a compatible family of $\mathbf{Q}$-rational increasing filtrations $W^{(j)}$ of $H_{0,\mathbf{R}}$. We define the subset $D_{\mathrm{SL}(2)}(W)$ of $D_{\mathrm{SL}(2)}$ by

$$D_{\mathrm{SL}(2)}(W) := \bigcup_{0\leq m\leq n} \left\{ x \in D_{\mathrm{SL}(2),m} \;\middle|\; \begin{array}{l} \exists s_j \in \mathbf{Z}\ (1\leq j\leq m) \text{ such that} \\ 1\leq s_1 < \cdots < s_m \leq n \text{ and} \\ W(\sigma_j) = W^{(s_j)}(\forall j) \end{array} \right\}$$

Here $(W(\sigma_j))_{1\leq j\leq m}$ is the family of weight filtrations associated to $x \in D_{\mathrm{SL}(2),m}$.

We define the subset $D_{\mathrm{SL}(2),\mathrm{val}}(W)$ of $D_{\mathrm{SL}(2),\mathrm{val}}$ by the pull-back of $D_{\mathrm{SL}(2)}(W)$.

Definition 3.13. We define the *topology on* $D_{\mathrm{SL}(2),\mathrm{val}}$ as the weakest one in which the following two families of subsets are open:

(i) The pull-backs on $D_{\mathrm{SL}(2),\mathrm{val}}$ of open subsets of $D_{\mathrm{BS},\mathrm{val}}$.

(ii) The subset $D_{\mathrm{SL}(2),\mathrm{val}}(W)$ for any n and any compatible family of $\mathbf{Q}$-rational increasing filtrations $W = (W^{(j)})_{1\leq j\leq n}$.

Note that the injective map $D_{\mathrm{SL}(2),\mathrm{val}} \to D_{\mathrm{BS},\mathrm{val}}$ is continuous by (i). We induce the quotient topology on $D_{\mathrm{SL}(2)}$ of the above one under the projection $D_{\mathrm{SL}(2),\mathrm{val}} \to D_{\mathrm{SL}(2)}$.

This topology of $D_{\mathrm{SL}(2)}$ has the following property (see 4.19 below). For an SL(2)-orbit of rank n (ρ,φ), $[\rho,\varphi] \in D_{\mathrm{SL}(2)}$ is the limit of

$$\varphi(iy_1,\ldots,iy_n) \in D, \quad \text{as } y_j > 0 \text{ and } \frac{y_j}{y_{j+1}} \to \infty \text{ for } 1\leq \forall j \leq n,$$

where y_{n+1} denotes 1.

Note that the space $D_{\mathrm{SL}(2),\mathrm{val}}$ is Hausdorff by 3.11 and 3.13.

Theorem 3.14. (i) *The canonical map* $D_{\mathrm{SL}(2),\mathrm{val}} \to D_{\mathrm{SL}(2)}$ *is proper and surjective.*

(ii) *The space* $D_{\mathrm{SL}(2)}$ *is Hausdorff.*

The proof of this theorem will be given in §4.

In Remarks 3.15, 3.16 below, we give a rough sketch of the relationship between the present results and the results in [KU1]. Precise descriptions and their proofs will be found in the forthcoming paper [KU2].

3.15 *Remark* (*Relation with period maps*). Let $\overline{X}$ be a connected complex manifold, let Y be a reduced divisor with normal crossings on $\overline{X}$, and let $X := \overline{X} - Y$. Let $H = (H_{\mathbf{Z}}, \langle\ ,\ \rangle, F)$ be a polarized variation of Hodge structure on X of weight w and of Hodge type $(h^{p,q})$. Then, as in 1.7, we have the associated period map

$$\varphi : X \to \Gamma\backslash D. \tag{1}$$

In the forthcoming paper [KU2], by using SL(2)-orbit theorem in [CKS], we will show that φ extends continuously to $\varphi_{\mathrm{SL}(2)}$ and also to $\varphi^{\flat}_{\mathrm{SL}(2)}$ in the following diagram.

$$\begin{array}{ccc} \varprojlim_{I\in\Psi} \mathrm{Bl}_I(\overline{X})^{\log} & \xrightarrow{\varphi_{\mathrm{SL}(2)}} & \Gamma\backslash D_{\mathrm{SL}(2)} \\ \downarrow & & \downarrow \\ \varprojlim_{I\in\Psi} \mathrm{Bl}_I(\overline{X}) & \xrightarrow{\varphi^{\flat}_{\mathrm{SL}(2)}} & \Gamma\backslash D^{\flat}_{\mathrm{SL}(2)}. \end{array} \tag{2}$$

Here, in analogy with 2.13, Ψ denotes the set of all non-zero $\mathcal{O}_{\overline{X}}$-ideals which are locally of forms $I = (f_1, \ldots, f_n)$ generated by some local sections $f_1, \ldots, f_n$ of $\mathcal{O}_{\overline{X}}$ whose zeros are contained in Y, $\mathrm{Bl}_I(\overline{X})$ means the blowing-up of $\overline{X}$ along I, $\mathrm{Bl}_I(\overline{X})^{\log}$ is the topological space defined by the method of [KN], and the projective limit is taken with respect to the ordering of the set Ψ as in 2.13. $D^{\flat}_{\mathrm{SL}(2)}$ is a space of Satake(-Baily-Borel)-Cattani-Kaplan type ([Sa], [BB], [CK]) which is defined as a quotient space of $D_{\mathrm{SL}(2)}$ under the following equivalence relation $\sim$. For $x \in D_{\mathrm{SL}(2),m}$, $y \in D_{\mathrm{SL}(2),n}$,

$$x \sim y \Longleftrightarrow \begin{cases} m = n, \text{ and the associated families of weight,} \\ \text{filtrations coincide, say } W, \text{ and } y \in G_{W,\mathbf{R},u}x. \end{cases}$$

Here $G_{W,\mathbf{R}}$ is the subgroup of $G_{\mathbf{R}}$ preserving all the filtrations in W and $G_{W,\mathbf{R},u}$ is its unipotent radical. (When D is in the classical situation (6.6) except one case (i) in Theorem 6.7, this space $D^{\flat}_{\mathrm{SL}(2)}$ is exactly the same as the 'reductive Borel-Serre space' which was constructed by Zucker in [Z1], [Z3].) Since the centers of the blowing-ups are contained in Y, we have open immersions $X \hookrightarrow \varprojlim_{I\in\Psi} \mathrm{Bl}_I(\overline{X})$, $X \hookrightarrow \varprojlim_{I\in\Psi} \mathrm{Bl}_I(\overline{X})^{\log}$. Note that, when $\overline{X}$ is a unit disc and Y is the origin, we have $\varprojlim_{I\in\Psi} \mathrm{Bl}_I(\overline{X}) = \overline{X}$.

3.16 *Remark* (*Relation with moduli of polarized logarithmic Hodge structures*). In the forthcoming paper [KU2], the diagram 3.1 (5) is

enlarged as

$$
\begin{array}{ccccccc}
 & & & & D_{\mathrm{SL}(2),\mathrm{val}} & \hookrightarrow & D_{\mathrm{BS},\mathrm{val}} \\
 & & & & \downarrow & & \downarrow \\
(1) \qquad D_{\Sigma}^{\sharp} & \leftarrow & D_{\Sigma,\mathrm{val}}^{\sharp} & \rightarrow & D_{\mathrm{SL}(2)} & & D_{\mathrm{BS}} \quad \rightarrow \quad \mathcal{X}_{\mathrm{BS}}. \\
\downarrow & & \downarrow & & \downarrow & & \\
\Gamma\backslash D_{\Sigma} & \leftarrow & D_{\Sigma,\mathrm{val}} & \rightarrow & D_{\mathrm{SL}(2)}^{\flat} & &
\end{array}
$$

We explain roughly the spaces and the morphisms appeared in this diagram (1).

Replacing $\mathfrak{g}_{\mathbf{R}}$ by $\mathfrak{g}_{\mathbf{Q}}$ in 3.3, we can define a *nilpotent cone* σ *in* $\mathfrak{g}_{\mathbf{Q}}$ analogously. In this remark, we always consider nilpotent cones in $\mathfrak{g}_{\mathbf{Q}}$. Let Σ be a *fan* of nilpotent cones in $\mathfrak{g}_{\mathbf{Q}}$, *i.e.*, a set of nilpotent cones in $\mathfrak{g}_{\mathbf{Q}}$ satisfying

$$
\begin{cases}
\sigma \in \Sigma,\ \tau \text{ is a face of } \sigma \Longrightarrow \tau \in \Sigma, \\
\sigma, \sigma' \in \Sigma \Longrightarrow \sigma \cap \sigma' \text{ is a face of } \sigma \text{ and of } \sigma'.
\end{cases}
$$

By using the notions in Definition 3.4 (replacing $\sigma = \sum_j \mathbf{R}_{\geq 0} N_j$ by $\sigma = \sum_j \mathbf{Q}_{\geq 0} N_j$), we define sets

$$
\begin{aligned}
(2) \qquad & D_{\Sigma} \ (\text{resp. } D_{\Sigma}^{\sharp}) \\
& := \{(\sigma, Z) \text{ nilpotent orbit (resp. } i\text{-orbit)} \mid \sigma \in \Sigma,\ Z \subset \check{D}\}.
\end{aligned}
$$

There is a natural map $D_{\Sigma}^{\sharp} \to D_{\Sigma}$, $(\sigma, Z) \mapsto (\sigma, \exp(\sigma_{\mathbf{C}})Z)$. Let Γ be a neat subgroup of $G_{\mathbf{Z}}$. We assume Γ is *strongly compatible* with Σ, *i.e.*, they satisfy

$$
\begin{cases}
\Gamma \in \Gamma, \sigma \in \Sigma \Longrightarrow \Gamma^{-1}\sigma\Gamma \in \Sigma, \\
\Gamma(\sigma) := \Gamma \cap \exp(\sigma) \Longrightarrow \log \Gamma(\sigma) \text{ generates the cone } \sigma.
\end{cases}
$$

Theorem 6.2 in [KU1] says that $\Gamma\backslash D_{\Sigma}$ with a suitable structure of 'generalized fs logarithmic analytic space' is a fine moduli space of 'polarized logarithmic Hodge structures of type (Φ_0, Σ, Γ)', and that $\Gamma\backslash D_{\Sigma}^{\sharp}$, with a suitable structure of ringed space, is isomorphic to 'the ringed space $(\Gamma\backslash D_{\Sigma})^{\log}$ associated to $\Gamma\backslash D_{\Sigma}$'. This space $\Gamma\backslash D_{\Sigma}$ is our generalization of a toroidal compactification (3) in Introduction.

The space $D_{\Sigma,\mathrm{val}}$ and $D_{\Sigma,\mathrm{val}}^{\sharp}$ are certain projective limits of blowing-ups of D_{Σ} and $D_{\Sigma}^{\sharp}$, respectively. The maps $D_{\Sigma,\mathrm{val}} \to D_{\mathrm{SL}(2)}^{\flat}$ and

$D^{\sharp}_{\Sigma,\mathrm{val}} \to D_{\mathrm{SL}(2)}$ are the unique continuous maps extending the identity of D, which are constructed by using SL(2)-orbit theorem in [CKS].

We can derive good properties, such as Hausdorffness, on $\Gamma\backslash D_{\Sigma}$ from good properties 2.1 (i), (ii) on $\mathcal{X}_{\mathrm{BS}}$ along the fundamental diagram (1).

Let $H = (H_{\mathbf{Z}}, \langle\ ,\ \rangle, F)$ be a polarized variation of Hodge structure on X in 3.15. If there is a suitable fan Σ for $\overline{X}$, then the period map 3.15 (1) extends to

$$\begin{aligned} &\overline{X} \to \Gamma\backslash D_{\Sigma}, \\ &\overline{X}^{\log} \to (\Gamma\backslash D_{\Sigma})^{\log} = \Gamma\backslash D^{\sharp}_{\Sigma}, \end{aligned} \tag{3}$$

and to

$$\begin{aligned} &\varprojlim{}_{I\in\Psi}\, \mathrm{Bl}_I(\overline{X}) \to \Gamma\backslash D_{\Sigma,\mathrm{val}} \to \Gamma\backslash D^{\flat}_{\mathrm{SL}(2)}, \\ &\varprojlim{}_{I\in\Psi}\, \mathrm{Bl}_I(\overline{X})^{\log} \to \Gamma\backslash D^{\sharp}_{\Sigma,\mathrm{val}} \to \Gamma\backslash D_{\mathrm{SL}(2)}. \end{aligned} \tag{4}$$

The composite maps (4) are nothing but $\varphi^{\flat}_{\mathrm{SL}(2)}$ and $\varphi_{\mathrm{SL}(2)}$, respectively, in 3.15 (2).

§4. Proof of Theorem 3.14

4.1. *Summary.* This section is devoted to proving Theorem 3.14. To do so, we will introduce a new topology $\mathcal{T}$ on the set $D_{\mathrm{SL}(2)}(W)$ in 3.12 in terms of filters on D associated to points of $D_{\mathrm{SL}(2)}(W)$ by using Cartan decompositions (4.6, 4.8 below). Denote this new topological space by $D_{\mathrm{SL}(2)}(W)_{\mathcal{T}}$, *i.e.*, the underlying set coincides with the one of $D_{\mathrm{SL}(2)}(W)$ but whose topology is $\mathcal{T}$. We will show that the topological space $D_{\mathrm{SL}(2),\mathrm{val}}(W)$ in 3.13 is homeomorphic to $D_{\mathrm{SL}(2)}(W)_{\mathcal{T}} \times_{\mathbf{R}^n_{\geq 0}} (\mathbf{R}^n_{\geq 0})_{\mathrm{val}}$ (4.14 below). From this, we get the homeomorphism $D_{\mathrm{SL}(2)}(W) \xrightarrow{\sim} D_{\mathrm{SL}(2)}(W)_{\mathcal{T}}$ (4.15 below) and the proof of Theorem 3.14 (4.17 below).

4.2. First, we prove two Lemmas 4.3, 4.4 below. Let X_j $(1 \leq j \leq d)$ be indeterminates. A *convergent Lie power series in the* X_j is a power series with respect to the bracket product $[\ ,\ ]$ in the X_j with coefficients in $\mathbf{C}$ which converges if the X_j are elements of a finite-dimensional Lie algebra over $\mathbf{C}$ and sufficiently near 0. The *order* of a convergent Lie power series $f(X_1, \ldots, X_d)$ is the minimum of the degrees of the monomials in $f(X_1, \ldots, X_d)$ whose coefficients are not zero.

Lemma 4.3. *Let X, Y be two indeterminates. Then there exist convergent Lie power serieses $f_-(X,Y)$ and $f_+(X,Y)$ which satisfy the following two conditions.*

(i) $\exp(X+Y) = \exp(f_-(X,Y))\exp(f_+(X,Y))$.

(ii) *Each monomial in* $f_-(X,Y)$ *(resp.* $f_+(X,Y)$*) is of odd (resp. even) degree in* X.

Proof. Let A be the convergent Lie power series of order ≥ 2 defined by

$$\exp(X+Y) = \exp(X+A)\exp(Y).$$

Devide $A = A_- + A_+$ so that each monomial in A_- (resp. A_+) is of odd (resp. even) degree in X. Let A' be the convergent Lie power series of order ≥ 3 defined by

$$\exp(X+Y) = \exp(X+A_-+A')\exp(A_+)\exp(Y).$$

Then $\exp(A_+)\exp(Y) = \exp(B_+)$ for some convergent Lie power series B_+ whose monomials are of even degree in X. Devide $A' = A'_- + A'_+$ so that each monomial in A'_- (resp. A'_+) is of odd (resp. even) degree in X. Continuing this process, we obtain Lie power series $f_-(X,Y)$, $f_+(X,Y)$ which can be checked to be convergent Lie power series. Q.E.D.

Lemma 4.4. *Let* $y \in D_{\mathrm{SL}(2),\mathrm{val},n}$ *and let* $[\rho,\varphi] \in D_{\mathrm{SL}(2),n}$ *be the image of* y. *Let* $P = P_{\tilde{\rho}(T),V'}$ *be the* $\mathbf{Q}$*-parabolic subgroup of* $G_{\mathbf{R}}$ *associated to* y *in* 3.7 *and* 2.8. *Let* $K = K_{\mathbf{r}}$ *be the maximal compact subgroup of* $G_{\mathbf{R}}$ *associated to the point* $\mathbf{r} := \varphi(\mathbf{i})$, *and let* $\theta = \theta_K$ *be the associated Cartan involution. Let* $\mathfrak{g}_{\mathbf{R}} = \bigoplus_l \mathfrak{g}(l)$ *be the decomposition into the eigenspaces under the action of* $(\mathbf{G}_{\mathbf{m},\mathbf{R}})^n$ *through* $\mathrm{Ad}(\tilde{\rho}(\ ))$. *Let* X_l, Y_l *be indeterminates, where the index* l *runs over all* $l \in \mathbf{Z}^n$ *with* $\mathfrak{g}(l) \neq 0$.

Then, there exist convergent Lie power series f_P *and* f_K *in the* X_l *and the* Y_l *which satisfy the following conditions. Let* U *be a sufficiently small neighborhood of* 0 *in* $\mathfrak{g}_{\mathbf{R}}$. *For any* $x \in U$, *writing* $x = \sum_l x_l$, $x_l \in \mathfrak{g}(l)$, *we have*

(i) $f_P\big((x_l)_l, (\theta(x_l))_l\big) \in \mathrm{Lie}\, P$, $f_K\big((x_l)_l, (\theta(x_l))_l\big) \in \mathrm{Lie}\, K$, *and*

(ii) $\exp(x) = \exp\big(f_P\big((x_l)_l, (\theta(x_l))_l\big)\big)\exp\big(f_K\big((x_l)_l, (\theta(x_l))_l\big)\big)$.

Proof. Note that $\mathrm{Lie}(P) = \bigoplus_{-l\in V} \mathfrak{g}(l)$. We write

$$\begin{aligned} x &= \textstyle\sum_l x_l = \sum_{-l\in V} x_l + \sum_{-l\notin V} x_l \\ &= \textstyle\sum_{-l\in V} x_l - \sum_{-l\notin V} \theta(x_l) + \sum_{-l\notin V}(x_l + \theta(x_l)), \\ \exp(x) &= \exp\big(\textstyle\sum_{-l\in V} x_l - \sum_{-l\notin V} \theta(x_l) + A\big) \\ &\qquad \cdot \exp\big(\textstyle\sum_{-l\notin V}(x_l + \theta(x_l))\big), \end{aligned}$$

where A is the convergent Lie power series of order ≥ 2 determined by the above. Then

$$\sum_{-l\in V} x_l - \sum_{-l\notin V} \theta(x_l) \in \mathrm{Lie}(P), \quad \sum_{-l\notin V}(x_l + \theta(x_l)) \in \mathrm{Lie}(K).$$

Do the same for A replacing x, and iterate. In the end, we obtain the desired convergent Lie power series f_P and f_K. Q.E.D.

4.5. *The filter on D associated to a point of $D_{\mathrm{SL}(2)}$.* We use the terminology *filter* in the sense of [NB, §6]. We introduce, in Definition 4.6 below, the filter $\mathcal{F}_x$ on D associated to a point $x \in D_{\mathrm{SL}(2)}$. We shall see later that this $\mathcal{F}_x$ coincides with the filter

$$\{U \cap D \mid U \text{ is a neighborhood of } x \text{ in } D_{\mathrm{SL}(2)}\} \quad (4.15 \text{ below}).$$

Definition 4.6. Let $x \in D_{\mathrm{SL}(2),n}$ and (ρ, φ) be a representative of x. Put $\mathbf{r} = \varphi(\mathbf{i})$. For

$$(1) \qquad \begin{cases} U & : \text{a neighborhood of 1 in } K_{\mathbf{r}}, \\ U' & : \text{a neighborhood of 0 in } \mathbf{R}^n_{\geq 0}, \\ U'' & : \text{a neighborhood of 1 in } G_{\mathbf{R}}, \end{cases}$$

we denote

$$(2) \qquad A(U, U', U'') := \left\{ g\tilde{\rho}(t)k\mathbf{r} \;\middle|\; \begin{array}{l} k \in U,\ t \in \mathbf{R}^n_{>0} \cap U', \\ g \in U'',\ \theta_{\tilde{\rho}(t)k\mathbf{r}}(g) = g^{-1} \end{array} \right\}$$

where $\tilde{\rho}$ is as in 3.1 and $\theta_{\tilde{\rho}(t)k\mathbf{r}}$ is the Cartan involution of $G_{\mathbf{R}}$ associated to the maximal compact subgroup $K_{\tilde{\rho}(t)k\mathbf{r}}$. We define $\mathcal{F}_x$ associated to x as the filter on D whose basis is given by the $A(U, U', U'')$ where U, U' and U'' run over all such neighborhoods as in (1).

As is easily seen, $\mathcal{F}_x$ is independent of the choice of a representative (ρ, φ) of x. Note that, since $\theta_{\tilde{\rho}(t)k\mathbf{r}} = \mathrm{Int}(\tilde{\rho}(t))\theta_{\mathbf{r}}\,\mathrm{Int}(\tilde{\rho}(t))^{-1}$, we have

$$(3) \qquad A(U, U', U'') = \left\{ \tilde{\rho}(t)gk\mathbf{r} \;\middle|\; \begin{array}{l} k \in U,\ t \in \mathbf{R}^n_{>0} \cap U', \\ g \in \mathrm{Int}(\tilde{\rho}(t))^{-1}(U''),\ \theta_{\mathbf{r}}(g) = g^{-1} \end{array} \right\}.$$

Lemma 4.7. *A basis of the filter $\mathcal{F}_x$ is also given by the following family of sets:*

$$(1) \quad B(U, U', U'') := \left\{ g\tilde{\rho}(t)k\mathbf{r} \;\middle|\; \begin{array}{l} k \in U,\ t \in \mathbf{R}^n_{>0} \cap U', \\ \mathrm{Int}(\tilde{\rho}(t))^j(g) \in U'' \ (j = 0, \pm 1) \end{array} \right\}$$

where U, U' and U'' run over all such neighborhoods as 4.6 (1).

Proof. We prove that, for given U, U' and U'', and for sufficiently small V, V' and V'', such as in 4.6 (1), we have

$$A(V,V',V'') \subset B(U,U',U''). \tag{2}$$

$$B(V,V',V'') \subset A(U,U',U''). \tag{3}$$

We prove (2). By the remark just after Definition 4.6, any element of $A(V,V',V'')$ can be written as

$$\tilde{\rho}(t)gk\mathbf{r} \text{ such that } k \in V,\ t \in \mathbf{R}^n_{>0} \cap V',\ \mathrm{Int}(\tilde{\rho}(t))(g) \in V'',\ \theta_{\mathbf{r}}(g) = g^{-1}.$$

When V'' is sufficiently small, there exists $a \in \mathfrak{g}_{\mathbf{R}}$ near 0 such that

$$\mathrm{Int}(\tilde{\rho}(t))(g) = \exp(a), \quad \big(\mathrm{Ad}(\tilde{\rho}(t))\theta_{\mathbf{r}}\,\mathrm{Ad}(\tilde{\rho}(t))^{-1}\big)(a) = -a.$$

Decompose

$$a = \textstyle\sum_{l\in\mathbf{Z}^n} a_l \in \mathfrak{g}_{\mathbf{R}} = \bigoplus_{l\in\mathbf{Z}^n} \mathfrak{g}(l) \tag{4}$$

under the action of $\mathrm{Ad}(\tilde{\rho}(t))$ $(t \in (\mathbf{G}_{\mathbf{m},\mathbf{R}})^n)$. Then we have

$$\mathrm{Int}(\tilde{\rho}(t))(g) = \exp(a), \tag{5}$$

$$g = \mathrm{Int}(\tilde{\rho}(t))^{-1}(\exp(a)) = \exp\big(\textstyle\sum_l t_1^{-l_1}\ldots t_n^{-l_n}a_l\big), \tag{6}$$

$$\mathrm{Int}(\tilde{\rho}(t))^{-1}(g) = \exp\big(\textstyle\sum_l t_1^{-2l_1}\ldots t_n^{-2l_n}a_l\big). \tag{7}$$

We want to see $\mathrm{Int}(\tilde{\rho}(t))^j(g) \in U''$ for $j = 0, \pm 1$. $\mathrm{Int}(\tilde{\rho}(t))(g) \in U''$ is obvious by definition. In order to see $\mathrm{Int}(\tilde{\rho}(t))^{-1}(g) \in U''$, we compute as follows.

$$\begin{aligned} a &= -\big(\mathrm{Ad}(\tilde{\rho}(t))\theta_{\mathbf{r}}\,\mathrm{Ad}(\tilde{\rho}(t))^{-1}\big)(a) = -\textstyle\sum_l \big(\mathrm{Ad}(\tilde{\rho}(t))\theta_{\mathbf{r}}\big)(t_1^{-l_1}\ldots t_n^{-l_n}a_l) \\ &= -\textstyle\sum_l \big(\theta_{\mathbf{r}}\,\mathrm{Ad}(\tilde{\rho}(t))^{-1}\big)(t_1^{-l_1}\ldots t_n^{-l_n}a_l) = -\textstyle\sum_l t_1^{-2l_1}\ldots t_n^{-2l_n}\theta_{\mathbf{r}}(a_l). \end{aligned}$$

Since $\theta_{\mathbf{r}}$ transforms $\mathfrak{g}(l)$ to $\mathfrak{g}(-l)$, we have $a_{-l} = -t_1^{-2l_1}\ldots t_n^{-2l_n}\theta_{\mathbf{r}}(a_l)$, that is,

$$t_1^{-2l_1}\ldots t_n^{-2l_n}a_l = -\theta_{\mathbf{r}}(a_{-l}) \quad (\forall l \in \mathbf{Z}^n). \tag{8}$$

Since a is sufficiently near 0, so is each component a_{-l} and hence so is each $-\theta_{\mathbf{r}}(a_{-l})$. Therefore, by (8) and (7), we have $\mathrm{Int}(\tilde{\rho}(t))^{-1}(g) \in U''$. Finally, $g \in U''$ is proved as follows. Take a basis $\{e_{l,j}\}$ of $\mathfrak{g}_{\mathbf{R}}$ subordinate to the decomposition (4) and write $a_l = \sum_j a_{l,j}e_{l,j}$. Since

$$t_1^{-l_1}\ldots t_n^{-l_n}a_{l,j} = \sqrt{a_{l,j}(t_1^{-2l_1}\ldots t_n^{-2l_n}a_{l,j})} \tag{9}$$

and since $a_{l,j}$ and $t_1^{-2l_1} \dots t_n^{-2l_n} a_{l,j}$ are sufficiently near 0, so is the left-hand-side of (9). Thus we have $g \in U''$ by (6).

We prove (3). By definition, any element of $B(V, V', V'')$ can be written as

$$\tilde{\rho}(t) g k \mathbf{r} \text{ such that } k \in V,\ t \in \mathbf{R}^n_{>0} \cap V',\ \mathrm{Int}(\tilde{\rho}(t))^j(g) \in V'' \text{ for } j = 0, \pm 1.$$

Since V'' is sufficiently small, there exists $b \in \mathfrak{g}_{\mathbf{R}}$ with

$$(10) \qquad g = \exp(b)$$
$$\text{such that } \mathrm{Ad}(\tilde{\rho}(t))^j(b) \text{ are sufficiently near 0 for } j = 0, \pm 1.$$

Let

$$b = b^- + b^+ \in \mathfrak{g}_{\mathbf{R}} = \mathfrak{g}^-_{\mathbf{R}} \oplus \mathfrak{g}^+_{\mathbf{R}}, \quad \mathfrak{g}^{\pm}_{\mathbf{R}} := \{x \in \mathfrak{g}_{\mathbf{R}} \mid \theta_{\mathbf{r}}(x) = \pm x\},$$

be the Cartan decomposition. Then, by 4.3, we have

$$g = \exp(b) = \exp(f_-(b^-, b^+)) \exp(f_+(b^-, b^+)), \quad f_{\pm}(b^-, b^+) \in \mathfrak{g}^{\pm}_{\mathbf{R}}.$$

Since

$$\tilde{\rho}(t) g k \mathbf{r} = \tilde{\rho}(t) \exp(f_-(b^-, b^+)) \exp(f_+(b^-, b^+)) k \mathbf{r},$$

it is enough to show

$$\exp(f_+(b^-, b^+)) k \in U \text{ and } \mathrm{Int}(\tilde{\rho}(t))(\exp(f_-(b^-, b^+))) \in U''.$$

Since b is sufficiently near 0, $\exp(f_+(b^-, b^+)) \in K_{\mathbf{r}}$ is sufficiently near 1. Hence

$$\exp(f_+(b^-, b^+)) k \in U.$$

Since $b^{\pm} = (b \pm \theta_{\mathbf{r}}(b))/2$, we have

$$\mathrm{Ad}(\tilde{\rho}(t))(b^{\pm}) = \frac{\mathrm{Ad}(\tilde{\rho}(t))(b) \pm \theta_{\mathbf{r}}(\mathrm{Ad}(\tilde{\rho}(t))^{-1}(b))}{2}.$$

These are sufficiently near 0 by (10), and hence so is

$$f_-\big(\mathrm{Ad}(\tilde{\rho}(t))(b^-), \mathrm{Ad}(\tilde{\rho}(t))(b^+)\big).$$

Thus

$$\mathrm{Int}(\tilde{\rho}(t))\big(\exp(f_-(b^-, b^+))\big) \in U''. \quad \text{Q.E.D.}$$

4.8. *Topology $\mathcal{T}$ on $D_{\mathrm{SL}(2)}(W)$.* As in 3.12, let $W = (W^{(j)})_{1 \le j \le n}$ be a compatible family of $\mathbf{Q}$-rational increasing filtrations $W^{(j)}$ of $H_{0,\mathbf{R}}$.

For $x \in D_{\mathrm{SL}(2)}(W)$, let $\mathcal{F}_x$ be the filter on D associated to x in 4.6. For an open set U of D, denote

$$\tilde{U} := \{x \in D_{\mathrm{SL}(2)}(W) \mid U \in \mathcal{F}_x\}.$$

We define the *topology* $\mathcal{T}$ *on* $D_{\mathrm{SL}(2)}(W)$ so that its basis of open sets is given by

$$\{\tilde{U} \mid U \text{ is an open set on } D\}.$$

We denote by $D_{\mathrm{SL}(2)}(W)_{\mathcal{T}}$ the topological space whose underlying set coincides with the one of $D_{\mathrm{SL}(2)}(W)$ but whose topology is $\mathcal{T}$. By construction, for $x \in D_{\mathrm{SL}(2)}(W)$, the filter $\mathcal{F}_x$ on D associated to x coincides with the filter

$$\{V \cap D \mid V \text{ is a neighborhood of } x \text{ in } D_{\mathrm{SL}(2)}(W)_{\mathcal{T}}\}.$$

We recall here the definition of 'regular spaces' and a property of a map into a regular space, which will be used in the proofs of 4.12, 4.14 below.

Definition 4.9 ([NB, Ch. 1, §8, no. 4, Definition 2]). A topological space is called *regular* if it is Hausdorff and satisfies the following axiom: Given any closed subset F of X and any point $x \notin F$, there is a neighborhood of x and a neighborhood of F which are disjoint.

We will see that $D_{\mathrm{SL}(2)}(W)$ and $D_{\mathrm{SL}(2),\mathrm{val}}(W)$ are regular spaces. (On the other hand, $D_{\mathrm{SL}(2)}$ and $D_{\mathrm{SL}(2),\mathrm{val}}$ are not necessarily regular.)

Lemma 4.10 ([NB, Ch. 1, §8, no. 5, Theorem 1]). *Let X be a topological space, A a dense subset of X, $f : A \to Y$ a map from A into a regular space Y. A necessary and sufficient condition for f to extend to a continuous map $\overline{f} : X \to Y$ is that, for each $x \in X$, $f(y)$ tends to a limit in Y when y tends to x while remaining in A. The continuous extension $\overline{f}$ of f to X is then unique.*

4.11. Let $W = (W^{(j)})_{1 \le j \le n}$ be a compatible family of $\mathbf{Q}$-rational increasing filtrations $W^{(j)}$ of $H_{0,\mathbf{R}}$ as in 3.12. For valuative submonoid V of $X((\mathbf{G_m})^n) = \mathbf{Z}^n$ containing $X((\mathbf{G_m})^n)_+ = \mathbf{N}^n$, we define a $\mathbf{Q}$-parabolic subgroup P_V of $G_{\mathbf{R}}$ as follows.

Note that the quotient group $G_{W,\mathbf{R}}/G_{W,\mathbf{R},u}$ has the induced effective action on

$$(1) \qquad \bigoplus_{l \in \mathbf{Z}^n} \Big(\bigcap_{1 \le j \le n} W^{(j)}_{l_j} \Big/ \Big(\sum_{1 \le k \le n} \bigcap_{1 \le j \le n} W^{(j)}_{l_j - \delta_{jk}}\Big)\Big),$$

where δ_{jk} is the Kronecker's symbol. Define an action on (1) of $(t_1, \dots, t_n) \in (\mathbf{G}_{\mathbf{m},\mathbf{R}})^n$ by $\bigoplus_{l \in \mathbf{Z}^n} t_1^{l_1} \dots t_n^{l_n}$. This induces a $\mathbf{Q}$-rational group homomorphism

$$\mu : (\mathbf{G}_{\mathbf{m},\mathbf{R}})^n \to G_{W,\mathbf{R}}/G_{W,\mathbf{R},u}. \tag{2}$$

Let $\nu : (\mathbf{G}_{\mathbf{m},\mathbf{R}})^n \to G_{\mathbf{R}}$ be a splitting of W, that is, ν is a homomorphism of algebraic groups over $\mathbf{R}$ such that $W_k^{(j)} = \bigoplus_{l \in \mathbf{Z}^n, l_j \le k} H(l)$ for all j and all k, where $H(l) := \{v \in H_{0,\mathbf{R}} \mid \nu(t)v = \prod_j t_j^{l_j} v \ (\forall t \in (\mathbf{G}_{\mathbf{m},\mathbf{R}})^n)\}$. Then such ν corresponds to a lifting $(\mathbf{G}_{\mathbf{m},\mathbf{R}})^n \to G_{W,\mathbf{R}}$ of μ in (2). Let $T \subset (\mathbf{G}_{\mathbf{m},\mathbf{R}})^n$ be the annihilator of $V^\times$, and let V' be the image of V under $X((\mathbf{G}_{\mathbf{m}})^n) \to X(T) \simeq X(\nu(T))$. (So $V' \simeq V/V^\times$.) Define the parabolic subgroup P_V of $G_{\mathbf{R}}$ by $P_V := P_{\nu(T),V'}$ (see 2.8). Then P_V is independent of the choice of ν as is easily seen. By taking ν defined over $\mathbf{Q}$, we see that P_V is $\mathbf{Q}$-rational.

Let $y = ([\rho, \varphi], Z, V) \in D_{\mathrm{SL}(2),\mathrm{val}}(W)$, let $W' = (W^{(s_j)})_{1 \le j \le m}$ $(0 \le m \le n, 1 \le s_1 < \dots < s_m \le n)$ be the families of weight filtrations associated to $[\rho, \varphi]$, and let $V' \subset X((\mathbf{G}_{\mathbf{m}})^n)$ be the inverse image of V under $X((\mathbf{G}_{\mathbf{m}})^n) \to X((\mathbf{G}_{\mathbf{m}})^m)$, $(a_j)_{1 \le j \le n} \mapsto (a_{s_j})_{1 \le j \le n}$. Then, the $\mathbf{Q}$-parabolic subgroup of $G_{\mathbf{R}}$ associated to the image of y under $D_{\mathrm{SL}(2),\mathrm{val}} \to D_{\mathrm{BS},\mathrm{val}} \to D_{\mathrm{BS}}$ coincides with $P_{V'}$.

Proposition 4.12. *Let $W = (W^{(j)})_{1 \le j \le n}$ be a compatible family of $\mathbf{Q}$-rational increasing filtrations $W^{(j)}$ of $H_{0,\mathbf{R}}$.*

(i) *There exists a continuous map $\beta : D \to \mathbf{R}^n_{>0}$ with the following property. For any splitting $\nu : (\mathbf{G}_{\mathbf{m},\mathbf{R}})^n \to G_{\mathbf{R}}$ of W, we have*

$$\beta(\nu(t)x) = t\beta(x) \quad (\forall x \in D, \forall t \in \mathbf{R}^n_{>0}).$$

(ii) *Let β be as in* (i). *Then the map β extends uniquely to a continuous map $\overline{\beta} : D_{\mathrm{SL}(2)}(W)_{\mathcal{T}} \to \mathbf{R}^n_{\ge 0}$.*

Proof. We prove (i). Take a valuative submonoid V of $X((\mathbf{G}_{\mathbf{m}})^n)$ such that $V \supset X((\mathbf{G}_{\mathbf{m}})^n)_+$ and $V^\times = \{1\}$. (Such V exists. For example, identifying $X((\mathbf{G}_{\mathbf{m}})^n) = \mathbf{Z}^n$ and $X((\mathbf{G}_{\mathbf{m}})^n)_+ = \mathbf{N}^n$, let V be the subset of $X((\mathbf{G}_{\mathbf{m}})^n)$ corresponding to the set of all elements of $\mathbf{Z}^n$ which are ≥ 0 in the lexicographical order of $\mathbf{Z}^n$.) Let $P := P_V$ (see 4.11), let $h : (\mathbf{G}_{\mathbf{m},\mathbf{R}})^n \to S_P \subset P/P_u$ be the canonical homomorphism, and let $\theta : A_P \to \mathbf{R}^n_{>0}$ be a continuous homomorphism such that the composite map $\mathbf{R}^n_{>0} \xrightarrow{h} \to A_P \xrightarrow{\theta} \to \mathbf{R}^n_{>0}$ is the identity map. Fix a maximal compact subgroup K of $G_{\mathbf{R}}$. We define $\beta : D \to \mathbf{R}^n_{>0}$ as the composite map

$$D \to \mathcal{X} \simeq G_{\mathbf{R}}/K \xrightarrow{||} A_P \xrightarrow{\theta} \mathbf{R}^n_{>0},$$

where $|\ |$ is as in 2.16. We show that β has the property stated in (i). Let $x_1 \in D$ be a point lying over $K \in \mathcal{X}$. Let $x \in D$ and write $x = pkx_1$ with $p \in P$ and $k \in K$. Then, for $t \in \mathbf{R}^n_{>0}$, we have

$$\beta(\nu(t)x) = \beta(\nu(t)pkx_1) = \theta(|\nu(t)p|) = \theta(|\nu(t)|)\theta(|p|) = t\theta(|p|) = t\beta(x).$$

We prove (ii). Let $x \in D_{\mathrm{SL}(2)}(W)$, let (ρ, φ) be a representative of x, and let $\mathbf{r} := \varphi(\mathbf{i})$ ($\mathbf{i} = (i, \ldots, i) \in \mathfrak{h}^m$). Let $W' = (W^{(s_j)})_{1 \le j \le m}$ ($1 \le s_1 < \cdots < s_m \le n$) be the family of weight filtrations associated to x, let

$$G_{W',\mathbf{R}} := \{g \in (G^\circ)_{\mathbf{R}} \mid gW^{(s_j)} = W^{(s_j)}\ (1 \le j \le m)\}$$

Put $s := (s_j)_{1 \le j \le m}$. Denote by $\iota_s : (\mathbf{G}_{\mathbf{m},\mathbf{R}})^m \to (\mathbf{G}_{\mathbf{m},\mathbf{R}})^n$ the injection to the s-components, that is, $\iota_s : (a_j)_{1 \le j \le m} \mapsto (b_l)_{1 \le l \le n}$; $b_l = a_j$ if $l = s_j$ ($1 \le j \le m$) and $b_l = 1$ otherwise. Put $\nu_s := \nu \circ \iota_s$. Since both $\tilde{\rho}$ and ν_s split W', there exists a unique

$$u \in G_{W',\mathbf{R},u} \quad \text{such that} \quad \tilde{\rho} = \mathrm{Int}(u)\nu_s. \tag{1}$$

Since the target $\mathbf{R}^n_{\ge 0}$ is a regular space, it is enough to prove, by 4.10, that, for x, (ρ, φ), $\mathbf{r}$ as above, and for directed families $(t_\lambda)_\lambda$, $(g_\lambda)_\lambda$, $(k_\lambda)_\lambda$ such that $t_\lambda \in \mathbf{R}^m_{>0}$, $g_\lambda \in G_{\mathbf{R}}$, $k_\lambda \in K_{\mathbf{r}}$, $\lim_\lambda t_\lambda = 0$, $\lim_\lambda g_\lambda = 1$, $\lim_\lambda k_\lambda = 1$, there exists a limit

$$\lim_\lambda \beta(\tilde{\rho}(t_\lambda)g_\lambda k_\lambda \mathbf{r}) \in \mathbf{R}^n_{\ge 0} \tag{2}$$

Let W' and $u \in G_{W',\mathbf{R},u}$ be as above, and let $u_\lambda := \nu_s(t_\lambda)^{-1} u \nu_s(t_\lambda)$. Then

$$\begin{aligned} \beta(\tilde{\rho}(t_\lambda)g_\lambda k_\lambda \mathbf{r}) &= \beta(u\nu_s(t_\lambda)u^{-1}g_\lambda k_\lambda \mathbf{r}) \\ &= \beta(\nu_s(t_\lambda)u_\lambda u^{-1} g_\lambda k_\lambda \mathbf{r}) = \iota_s(t_\lambda)\beta(u_\lambda u^{-1} g_\lambda k_\lambda \mathbf{r}). \end{aligned}$$

Since ν_s splits W' and $u \in G_{W',\mathbf{R},u}$, we have $\lim_\lambda u_\lambda = 1$. Hence $\lim_\lambda \beta(u_\lambda u^{-1} g_\lambda k_\lambda \mathbf{r}) = \beta(u^{-1}\mathbf{r})$. This proves the existence of the limit (2). Q.E.D.

By the proof of Proposition 4.12 (ii), $\overline{\beta}(x) = \beta(u^{-1}\mathbf{r})$ for $x = [\rho, \varphi] \in D_{\mathrm{SL}(2)}(W)$, where $\mathbf{r} = \rho(\mathbf{i})$ and u is as in (1) above.

Lemma 4.13. *Let $W = (W^{(j)})_{1 \le j \le n}$ be a compatible family of $\mathbf{Q}$-rational increasing filtrations $W^{(j)}$ of $H_{0,\mathbf{R}}$. Then the topology of $D_{\mathrm{SL}(2),\mathrm{val}}(W)$ (as a subspace of $D_{\mathrm{SL}(2),\mathrm{val}}$) coincides with the topology as a subspace of $D_{\mathrm{BS},\mathrm{val}}$.*

Proof. It is sufficient to prove that, for any compatible family W' of $\mathbf{Q}$-rational increasing filtrations of $H_{0,\mathbf{R}}$, there exits an open set U of $D_{\mathrm{BS,val}}$ such that $D_{\mathrm{SL}(2),\mathrm{val}}(W) \cap D_{\mathrm{SL}(2),\mathrm{val}}(W') = D_{\mathrm{SL}(2),\mathrm{val}}(W) \cap U$. As is easily seen, $D_{\mathrm{SL}(2),\mathrm{val}}(W) \cap D_{\mathrm{SL}(2),\mathrm{val}}(W')$ is a finite union of sets of the form $D_{\mathrm{SL}(2),\mathrm{val}}(W'')$ where $W'' = (W^{(s_j)})_{1\leq j\leq m}$ for some m with $0 \leq m \leq n$ and for some $s_1, \ldots, s_m$ with $1 \leq s_1 < \cdots < s_m \leq n$. Hence we may assume that W' itself has the form $W' = (W^{(s_j)})_{1\leq j\leq m}$ $(0 \leq m \leq n,\ 1 \leq s_1 < \cdots < s_m \leq n)$. Assume this. Let $\mathcal{V}$ be the set of all valuative submonoids V of $X((\mathbf{G_m}))^n)$ such that $V \supset X((\mathbf{G_m})^n)_+$ and $V^\times = \{1\}$. Then

$$D_{\mathrm{SL}(2),\mathrm{val}}(W) \subset \bigcup_{V\in\mathcal{V}} D_{\mathrm{BS,val}}(P_V),$$

where P_V is as in 4.11. In fact, for $([\rho,\varphi], Z, V) \in D_{\mathrm{SL}(2),\mathrm{val}}(W)$, let $W' = (W^{(s_j)})_{1\leq j\leq m}$ $(0 \leq m \leq n,\ 1 \leq s_1 < \cdots < s_m \leq n)$ be the family of weight filtrations associated to $[\rho,\varphi]$, and let $V' \in X((\mathbf{G_m})^n)$ be the inverse image of V under $X((\mathbf{G_m})^n) \to X((\mathbf{G_m})^m)$, $(a_j)_{1\leq j\leq n} \mapsto (a_{s_j})_{1\leq j\leq m}$. Take $V'' \in \mathcal{V}$ with $V'' \subset V'$. Then $P_{V'} \supset P_{V''}$ and hence $([\rho,\varphi], Z, V) \in D_{\mathrm{BS,val}}(P_{V'}) \subset D_{\mathrm{BS,val}}(P_{V''})$. We show the existence of V''. Let $l = \mathrm{rank}_{\mathbf{Z}}(V')^\times$ and fix any isomorphism $h : (V')^\times \xrightarrow{\sim} \mathbf{Z}^l$. Define

$$V'' := \{x \in V' \mid x \notin (V')^\times\} \cup \{x \in (V')^\times \mid h(x) \geq 0 \text{ for lexicographical order of } \mathbf{Z}^l\}.$$

Then, $V'' \in \mathcal{V}$ and $V'' \subset V'$.

For $V \in \mathcal{V}$, we define an open subset U_V of $D_{\mathrm{BS,val}}(P_V)$ such that

$$D_{\mathrm{SL}(2),\mathrm{val}}(W') \cap D_{\mathrm{BS,val}}(P_V) = D_{\mathrm{SL}(2),\mathrm{val}}(W) \cap U_V. \tag{1}$$

This will show

$$D_{\mathrm{SL}(2),\mathrm{val}}(W') = D_{\mathrm{SL}(2),\mathrm{val}}(W) \cap \left(\bigcup_{V\in\mathcal{V}} U_V\right).$$

The definition of U_V is as follows. Let $(\mathbf{G_{m,R}})^n \to S_{P_V}$ be the canonical injective homomorphism. Let $(e_j)_{1\leq j\leq n}$ be the standard base of $X((\mathbf{G_m})^n)$. Take $a_j \geq 1$ such that $e_j^{a_j}$ is the restriction of an element $\psi_j \in X(S_{P_V})$ to $(\mathbf{G_{m,R}})^n$ and fix such a_j and ψ_j $(1 \leq j \leq n)$. Let U'_V be the A_{P_V}-stable open set of $(\overline{A}_{P_V})_{\mathrm{val}}$ consisting of all elements (V', h) such that $\psi_{s_j} \in (V')^\times$ for $1 \leq j \leq m$, and let $U_V := D \times^{A_{P_V}} U'_V \subset D \times^{A_{P_V}} (\overline{A}_{P_V})_{\mathrm{val}} = D_{\mathrm{BS,val}}(P_V)$. Then we have (1), as is easily seen. Q.E.D.

Proposition 4.14. *Let $W = (W^{(j)})_{1\leq j\leq n}$ be a compatible family of $\mathbf{Q}$-rational increasing filtrations $W^{(j)}$ of $H_{0,\mathbf{R}}$. Let $\beta : D \to \mathbf{R}^n_{>0}$ and $\overline{\beta} : D_{\mathrm{SL}(2)}(W)_{\mathcal{T}} \to \mathbf{R}^n_{\geq 0}$ be as in* 4.12. *Then there exists a unique homeomorphism*

$$D_{\mathrm{SL}(2),\mathrm{val}}(W) \xrightarrow{\sim} D_{\mathrm{SL}(2)}(W)_{\mathcal{T}} \times_{\mathbf{R}^n_{\geq 0}} (\mathbf{R}^n_{\geq 0})_{\mathrm{val}}$$

which extends the identity map of D.

Proof. Take a splitting $\nu : (\mathbf{G_{m,R}})^n \to G_{\mathbf{R}}$ of W. The homeomorphism in 4.14 is defined by

$$(1) \quad \psi : (x, Z, \overline{V}) \mapsto (x, (V, h)) \quad (x \in D_{\mathrm{SL}(2)}(W), (V, h) \in (\mathbf{R}^n_{\geq 0})_{\mathrm{val}})$$

as follows. Let $W_x = (W^{(s_j)})_{1\leq j\leq m}$ be the family of weight filtrations associated to x. Let (ρ, φ) be a representative of x, and let $\mathbf{r} = \varphi(\mathbf{i})$ $(\mathbf{i} = (i, \ldots, i) \in \mathfrak{h}^m)$. Put $s := (s_j)_{1\leq j\leq m}$. Let

$$\iota_s : (\mathbf{G_{m,R}})^m \hookrightarrow (\mathbf{G_{m,R}})^n, \quad (a_1, \ldots, a_m) \mapsto (b_1, \ldots, b_n)$$

be the map defined by $b_l := a_{s_j}$ if $l = s_j$ $(1 \leq j \leq m)$ and $b_l := 1$ otherwise. Put $\nu_s := \nu \circ \iota_s$. Since both ν_s and $\tilde{\rho}$ split W_x, there exists a unique element u of $G_{W_x,\mathbf{R},u}$ such that $\tilde{\rho} = \mathrm{Int}(u)(\nu_s)$. For $(x, Z, \overline{V}) \in D_{\mathrm{SL}(2),\mathrm{val}}(W)$, the image $(x, (V, h))$ under ψ is defined as follows. First, V is the inverse image of $\overline{V}$ under

$$\mathbf{Z}^n = X((\mathbf{G_m})^n) \to X((\mathbf{G_m})^m), \quad \chi \mapsto \chi \circ \iota_s.$$

Take an element $t \in \mathbf{R}^m_{>0}$ such that $\tilde{\rho}(t\beta(u^{-1}\mathbf{r}))^{-1}\mathbf{r} \in Z$. Define $h : V^{\times} \to \mathbf{R}_{>0}$ by $h(\chi) := \chi(\iota_s(t))$.

The inverse map ψ^{-1} of ψ is given as follows. Let $(x, (V, h)) \in D_{\mathrm{SL}(2)}(W)_{\mathcal{T}} \times_{\mathbf{R}^n_{\geq 0}} (\mathbf{R}^n_{\geq 0})_{\mathrm{val}}$. We define Z and $\overline{V}$ as follows. Let $(e_j)_{1\leq j\leq n}$ be the standard basis of $X((\mathbf{G_m})^n)$. The fact that the image of $(V, h) \in (\mathbf{R}^n_{\geq 0})_{\mathrm{val}}$ in $\mathbf{R}^n_{\geq 0}$ coincides with $\overline{\beta}(x)$ implies the following two assertions.

(2) $e_j \in V - V^{\times}$ if $j \in \{s_1, \ldots, s_m\}$, and $e_j \in V^{\times}$ otherwise.

(3) $\overline{\beta}(x)_j = 0$ if $j \in \{s_1, \ldots, s_m\}$, and
$\overline{\beta}(x)_j = \beta(u^{-1}\mathbf{r})_j = h(e_j) \neq 0$ otherwise.

Here $(\ )_j$ denotes the j-th component of an element of $\mathbf{R}^n_{\geq 0}$. By (2), we see that there exists a unique valuative submonoid $\overline{V}$ of $X((\mathbf{G_m})^m)$ such that $\overline{V} \supset X((\mathbf{G_m})^m)_+$, $\overline{V} \cap X((\mathbf{G_m})^m)_+ = \{1\}$, and such that V is the inverse image of $\overline{V}$ under $X((\mathbf{G_m})^n) \to X((\mathbf{G_m})^m)$, $\chi \mapsto \chi \circ \iota_s$. By (3), we have

(4) $(\beta(u^{-1}\mathbf{r})^{-1}h)(e_j) = 1$ if $1 \leq j \leq n$ and $j \notin \{s_1, \ldots, s_m\}$, and $\beta(u^{-1}\mathbf{r})^{-1}h : V^\times \to \mathbf{R}_{>0}$ factors through $V^\times \to \overline{V}^\times$.

Hence there exists $t \in \mathbf{R}_{>0}^m$ such that

$$\chi(\iota_s(t)) = (\beta(u^{-1}\mathbf{r})^{-1}h)(\chi) \quad (\forall \chi \in V^\times).$$

The Z is defined by

$$Z := \{\tilde{\rho}(t)\mathbf{r} \mid t \in \mathbf{R}_{>0}^m, \chi(\iota_s(t)) = (\beta(u^{-1}\mathbf{r})^{-1}h)(\chi)\ (\forall \chi \in V^\times)\}.$$

It is easy to see that the maps ψ and ψ^{-1} are the inverse to each other and hence ψ is bijective. For the proof of 4.14, it is enough to show that both ψ and ψ^{-1} are continuous.

Assume $\psi(x, Z, \overline{V}) = (x, (V, h))$, let V_0 be a valuative submonoid of $X((\mathbf{G_m})^n)$ such that $V \supset V_0 \supset X((\mathbf{G_m})^n)_+$ and $V_0^\times = \{1\}$, and let $P = P_{V_0}$. Then, since the $\mathbf{Q}$-parabolic subgroup of $G_{\mathbf{R}}$ associated to $(x, Z, \overline{V})$ is P_V (4.11) and P_V contains P, we have $(x, Z, \overline{V}) \in D_{\mathrm{BS,val}}(P)$. The following (5) is checked easily.

(5) The image of $(x, Z, \overline{V})$ under $D_{\mathrm{BS,val}}(P) \xrightarrow{\sim} D \times^{A_P} (\overline{A}_P)_{\mathrm{val}}$ is $(\mathbf{r}, (\hat{V}, \beta(u^{-1}\mathbf{r})^{-1}\hat{h}))$,

where $\hat{V}$ is the inverse image of V under the restriction map $X(S_P) \to X((\mathbf{G_m})^n)$ for the canonical embedding $(\mathbf{G_m})^n \hookrightarrow S_P$, and $\hat{h}$ is the composite $\hat{V}^\times \to V^\times \xrightarrow{h} \mathbf{R}_{>0}$.

Now fix $(x, Z, \overline{V}) \in D_{\mathrm{SL}(2),\mathrm{val}}(W)$ and let

$$(x, (V, h)) \in D_{\mathrm{SL}(2)}(W) \times_{\mathbf{R}_{\geq 0}^n} (\mathbf{R}_{\geq 0}^n)_{\mathrm{val}}$$

be its image under ψ. Fix V_0 as above, and let $P = P_{V_0}$. Fix a representative (ρ, φ) of x, and let $\mathbf{r} = \varphi(\mathbf{i})$.

We prove first that the map ψ^{-1} is continuous. By the fact that $D_{\mathrm{BS,val}}$ is a regular space, and by 4.10 and 4.13, it is enough to prove the following. Let $(y_\lambda)_\lambda$ be a directed family of elements of D which converges to $(x, (V, h))$ in $D_{\mathrm{SL}(2)}(W)_{\mathcal{T}} \times_{\mathbf{R}_{\geq 0}^n} (\mathbf{R}_{\geq 0}^n)_{\mathrm{val}}$. Then $(y_\lambda)_\lambda$ converges to $(\mathbf{r}, (\hat{V}, \beta(u^{-1}\mathbf{r})^{-1}\hat{h}))$ in $D \times^{A_P} (\overline{A}_P)_{\mathrm{val}}$. Since $(y_\lambda)_\lambda$ converges to x in $D_{\mathrm{SL}(2)}(W)_{\mathcal{T}}$,

$$y_\lambda = \tilde{\rho}(t_\lambda) g_\lambda k_\lambda \mathbf{r}, \quad t_\lambda \in \mathbf{R}_{>0}^m,\ g_\lambda \in G_{\mathbf{R}},\ k_\lambda \in K_{\mathbf{r}},$$
$$t_\lambda \to 0,\ \mathrm{Int}(\tilde{\rho}(t_\lambda))^j(g_\lambda) \to 1\ (j = 0, \pm 1),\ k_\lambda \to 1.$$

By 4.4, $g_\lambda = p_\lambda k'_\lambda$, $p_\lambda \in P$, $k'_\lambda \in K_{\mathbf{r}}$, $\mathrm{Int}(\tilde{\rho}(t_\lambda))^j(g_\lambda) \to 1$ $(j = 0, \pm 1)$, $k'_\lambda \to 1$. Put $k''_\lambda := k'_\lambda k_\lambda \in K_{\mathbf{r}}$. We have

$$\begin{aligned} y_\lambda &= \tilde{\rho}(t_\lambda) p_\lambda k''_\lambda \mathbf{r} = \tilde{\rho}(t_\lambda) p_\lambda \tilde{\rho}(t_\lambda)^{-1} \tilde{\rho}(t_\lambda) k''_\lambda \mathbf{r} \\ &= (\tilde{\rho}(t_\lambda) \bmod P_u) \circ \tilde{\rho}(t_\lambda) p_\lambda \tilde{\rho}(t_\lambda)^{-1} k''_\lambda \mathbf{r}, \end{aligned}$$

where $\circ$ is the Bore-Serre action for the $\mathbf{Q}$-parabolic subgroup P. Note that

$$\tilde{\rho}(t_\lambda) p_\lambda \tilde{\rho}(t_\lambda)^{-1} \to 1, \quad k''_\lambda \to 1.$$

It suffices to prove

$$(\tilde{\rho}(t_\lambda) \bmod P_u) \to (\hat{V}, \beta(u^{-1}\mathbf{r})^{-1}\hat{h}) \quad \text{in } (\overline{A}_P)_{\mathrm{val}}.$$

Since $\beta(y_\lambda) \to (V, h)$ in $(\mathbf{R}^n_{\geq 0})_{\mathrm{val}}$ and

$$\begin{aligned} \beta(y_\lambda) &= \beta(\tilde{\rho}(t_\lambda) p_\lambda k''_\lambda \mathbf{r}) = \beta(\nu_s(t_\lambda)\nu_s(t_\lambda)^{-1} u \nu_s(t_\lambda) u^{-1} p_\lambda k''_\lambda \mathbf{r}) \\ &= \iota_s(t_\lambda) \beta(\nu_s(t_\lambda)^{-1} u \nu_s(t_\lambda)) = \iota_s(t_\lambda) \beta(\nu_s(t_\lambda)^{-1} u \nu_s(t_\lambda) u^{-1} p_\lambda k''_\lambda \mathbf{r}), \\ \nu_s(t_\lambda)^{-1} u \nu_s(t_\lambda) &\to 1 \ (\text{by } u \in G_{W_x, \mathbf{R}, u}), \ p_\lambda \to 1, \ k''_\lambda \to 1, \end{aligned}$$

we have

$$\iota_s(t_\lambda) \to (V, \beta(u^{-1}\mathbf{r})^{-1} h) \quad \text{in } (\mathbf{R}^n_{\geq 0})_{\mathrm{val}}.$$

This implies

$$\begin{cases} \chi(\iota_s(t_\lambda)) \to 0 & \text{if } \chi \in V - V^\times, \\ \chi(\iota_s(t_\lambda)) \to (\beta(u^{-1}\mathbf{r})^{-1} h)(\chi) & \text{if } \chi \in V^\times. \end{cases}$$

This shows

$$\begin{cases} \chi(\tilde{\rho}(t_\lambda) \bmod P_u) \to 0 & \text{if } \chi \in \hat{V} - \hat{V}^\times, \\ \chi(\tilde{\rho}(t_\lambda) \bmod P_u) \to (\beta(u^{-1}\mathbf{r})^{-1} \hat{h})(x) & \text{if } \chi \in \hat{V}^\times. \end{cases}$$

Hence $(\tilde{\rho}(t_\lambda) \bmod P_u) \to (\hat{V}, \beta(u^{-1}\mathbf{r})^{-1}\hat{h})$ in $(\overline{A}_P)_{\mathrm{val}}$.

Next we prove that ψ is continuous. Let $(y_\lambda)_\lambda$ be a directed family of elements of $D_{\mathrm{SL}(2),\mathrm{val}}(W)$ converging to $(x, Z, \overline{V})$. Write $y_\lambda = (x_\lambda, Z_\lambda, \overline{V}_\lambda)$, and let $(x_\lambda, (V_\lambda, h_\lambda))$ be the image of y_λ under ψ. We show that x_λ converges to x and (V_λ, h_λ) converges to (V, h). We assume $y_\lambda \in D_{\mathrm{SL}(2),\mathrm{val}}(W_x) \cap D_{\mathrm{BS},\mathrm{val}}(P)$ without loss of generality. Define $(\hat{V}_\lambda, \hat{h}_\lambda) \in (\overline{A}_P)_{\mathrm{val}}$ just as in the definition of $(\hat{V}, \hat{h})$. Since there exist only finitely many possible families of weight filtrations for points

in $D_{\mathrm{SL}(2),\mathrm{val}}(W_x)$, we may assume that all x_λ have a common family of weight filtrations $(W^{(s'_j)})_{1\le j\le m'}$. Put $s' := (s'_j)_{1\le j\le m'}$. Let $\nu_{s'} := \nu\circ\iota_{s'}$ and let $\iota' : (\mathbf{G_m})^{m'} \to (\mathbf{G_m})^m$ be the unique homomorphism with $\iota_{s'} = \iota_s \circ \iota'$. Let $(\rho_\lambda, \varphi_\lambda)$ be a representative of x_λ, and let $\mathbf{r}'_\lambda = \varphi_\lambda(\mathbf{i}')$ $(\mathbf{i}' = (i,\dots,i) \in \mathfrak{h}^{m'})$. Let u_λ be the unique element of $G_{W',\mathbf{R},u}$ with $\tilde{\rho}_\lambda = \mathrm{Int}(u_\lambda)(\nu_{s'})$. The image of y_λ in $D \times^{A_P} (\overline{A}_P)_{\mathrm{val}}$ coincides with $(\mathbf{r}'_\lambda, (\hat{V}_\lambda, \beta(u_\lambda^{-1}\mathbf{r}_\lambda)^{-1}\hat{h}_\lambda))$.

Since $(\mathbf{r}'_\lambda, (\hat{V}_\lambda, \beta(u_\lambda^{-1}\mathbf{r}_\lambda)^{-1}\hat{h}_\lambda))$ converges to $(\mathbf{r}, (\hat{V}, \beta(u^{-1}\mathbf{r})\hat{h}))$ in $D \times^{A_P} (\overline{A}_P)_{\mathrm{val}}$, there exist $a_\lambda \in A_P$ satisfying the following two conditions.

$$(\hat{V}_\lambda, a_\lambda\beta(u_\lambda^{-1}\mathbf{r}_\lambda)^{-1}\hat{h}_\lambda) \to (\hat{V}, \beta(u^{-1}\mathbf{r})^{-1}\hat{h}) \quad \text{in } (\overline{A}_P)_{\mathrm{val}}. \tag{6}$$

$$a_\lambda^{-1} \circ \mathbf{r}_\lambda \to \mathbf{r} \quad \text{in } D. \tag{7}$$

By (6) and (4) (applied also by replacing u, $\mathbf{r}$, h, s by u_λ, $\mathbf{r}_\lambda$, h_λ, s', respectively), the j-th component of the image of a_λ in $\mathbf{R}^n_{>0}$ converges to 1 if $j \notin \{s_1,\dots,s_m\}$. Hence

$$a_\lambda = \iota_s(t_\lambda)b_\lambda,\ t_\lambda \in \mathbf{R}^m_{>0},\ b_\lambda \in A_P,\ b_\lambda \to 1.$$

(We identify here an element of $\mathbf{R}^n_{>0}$ with its canonical image in A_P.) Hence (6) and (7) are rewritten as

$$(\hat{V}_\lambda, \iota_s(t_\lambda)\beta(u_\lambda^{-1}\mathbf{r}_\lambda)^{-1}\hat{h}_\lambda) \to (\hat{V}, \beta(u^{-1}\mathbf{r})^{-1}\hat{h}) \quad \text{in } (\overline{A}_P)_{\mathrm{val}}. \tag{8}$$

$$\iota_s(t_\lambda)^{-1} \circ \mathbf{r}_\lambda \to \mathbf{r} \quad \text{in } D. \tag{9}$$

By (9), we can write

$$\iota_s(t_\lambda)^{-1} \circ \mathbf{r}_\lambda = p_\lambda k_\lambda \mathbf{r},\ \ p_\lambda \in P,\ \ p_\lambda \to 1,\ \ k_\lambda \in K_{\mathbf{r}},\ \ k_\lambda \to 1.$$

Hence

$$\mathbf{r}_\lambda = p_\lambda\tilde{\rho}(t_\lambda)k_\lambda\mathbf{r},\ \ p_\lambda \in P,\ \ p_\lambda \to 1,\ \ k_\lambda \in K_{\mathbf{r}},\ \ k_\lambda \to 1. \tag{10}$$

In (8), we have

$$\begin{aligned}\iota_s(t_\lambda)\beta(u_\lambda^{-1}\mathbf{r}_\lambda)^{-1} &= \beta(\nu_s(t_\lambda)^{-1}u_\lambda^{-1}\mathbf{r}_\lambda)^{-1}\\ &= \beta(\nu_s(t_\lambda)^{-1}u_\lambda^{-1}p_\lambda\tilde{\rho}(t_\lambda)k_\lambda\mathbf{r})^{-1}\\ &= \beta(\nu_s(t_\lambda)^{-1}u_\lambda^{-1}p_\lambda u\nu_s(t_\lambda)u^{-1}k_\lambda\mathbf{r})^{-1}.\end{aligned}$$

We will show

$$\nu_s(t_\lambda)^{-1}u_\lambda^{-1}p_\lambda u\nu_s(t_\lambda) \to 1. \tag{11}$$

This will show

$$\iota_s(t_\lambda)\beta(u_\lambda^{-1}\mathbf{r}_\lambda)^{-1} \to \beta(u^{-1}\mathbf{r})^{-1},$$

and hence, by (8),

$$(\hat{V}_\lambda, \hat{h}_\lambda) \to (\hat{V}, \hat{h}).$$

It follows

$$(V_\lambda, h_\lambda) \to (V, h).$$

We prove (11). Let

$$G_{W',\mathbf{R}} = G_{W',\mathbf{R},u} \rtimes L \tag{12}$$

be the semi-direct decomposition, where

$$L := \{g \in (G^\circ)_\mathbf{R} \mid \nu_{s'}(t)g = g\nu_{s'}(t) \ \ (\forall t \in (\mathbf{G}_{\mathbf{m},\mathbf{R}})^{m'})\}.$$

We claim

$$\begin{aligned} &p_\lambda, p_\lambda u \in G_{W',\mathbf{R}}, \quad u_\lambda^{-1}p_\lambda u \in L, \\ &p_\lambda u = u_\lambda \cdot (u_\lambda^{-1}p_\lambda u) \text{ under the decomposition (12).} \end{aligned} \tag{13}$$

In fact, since the induced homomorphisms $(\mathbf{G}_{\mathbf{m},\mathbf{R}})^{m'} \to P/P_u$ from $\tilde{\rho}_\lambda$ and $\tilde{\rho} \circ \iota'$ coincide and since $\tilde{\rho}_\lambda$ (resp. $\tilde{\rho} \circ \iota'$) is the Borel-Serre lifting at $K_{\mathbf{r}_\lambda}$ (resp. $K_\mathbf{r}$) of this induced homomorphism, we have

$$\tilde{\rho}_\lambda = \mathrm{Int}(p_\lambda)(\tilde{\rho} \circ \iota').$$

Since $\tilde{\rho}_\lambda = \mathrm{Int}(u_\lambda)\nu_{s'}$ and $\tilde{\rho} \circ \iota' = \mathrm{Int}(u)\nu_{s'}$, this proves $u_\lambda^{-1}p_\lambda u \in L$. Since $u_\lambda, u \in G_{W',\mathbf{R}}$, it follows $p_\lambda \in G_{W',\mathbf{R}}$. Hence we have (13).

We prove (11) by using (13). For $\chi \in X((\mathbf{G}_\mathbf{m})^n)$, let

$$H(\chi) := \{v \in H_{0,\mathbf{R}} \mid \nu(t)v = \chi(t)v \ \ (\forall t \in (\mathbf{G}_{\mathbf{m},\mathbf{R}})^n)\}.$$

Let $\chi_1, \chi_2 \in X((\mathbf{G}_\mathbf{m})^n)$, $v \in H_{0,\mathbf{R}}(\chi_1)$. For the proof of (11), it is enough to show

$$(\nu_s(t_\lambda)^{-1}u_\lambda^{-1}p_\lambda u\nu_s(t_\lambda)v)(\chi_2) \to \begin{cases} v & \text{if } \chi_2 = \chi_1, \\ 0 & \text{otherwise,} \end{cases}$$

where (χ_2) means the χ_2-component in the eigenspace decomposition $H_{0,\mathbf{R}} = \bigoplus_{\chi \in X((\mathbf{G}_\mathbf{m})^n)} H(\chi)$. We have

$$(\nu_s(t_\lambda)^{-1}u_\lambda^{-1}p_\lambda u\nu_s(t_\lambda)v)(\chi_2) = \chi_1(\iota_s(t_\lambda))\chi_2(\iota_s(t_\lambda))^{-1}(u_\lambda^{-1}p_\lambda uv)(\chi_2),$$

and the decomposition of $p_\lambda u$ in (13) implies

$$(u_\lambda^{-1} p_\lambda uv)(\chi_2) = \begin{cases} (p_\lambda uv)(\chi_2) & \text{if } (\chi_1\chi_2^{-1}) \circ \iota_{s'} = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Since $p_\lambda u \in P$, we have

$$(p_\lambda uv)(\chi_2) = 0 \quad \text{unless} \quad \chi_1\chi_2^{-1} \in V.$$

Furthermore, since $p_\lambda \to 1$, we have

$$(p_\lambda uv)(\chi_2) \to (uv)(\chi_2).$$

Since $u \in G_{W_x,\mathbf{R},u} \subset P_{V,u}$, we have

$$(uv)(\chi_2) = \begin{cases} v & \text{if } \chi_2 = \chi_1, \\ 0 & \text{if } \chi_2 \neq \chi_1 \text{ and } \chi_1\chi_2^{-1} \in V - V^\times. \end{cases}$$

By these facts, (11) is reduced to the following (14) which we apply to $\chi = \chi_1\chi_2^{-1}$.

(14) Let $\chi \in V$ and assume $\chi \circ \iota_{s'} = 1$. Then $\chi(\iota_s(t_\lambda))$ is bounded. If furthermore $\chi \in V - V^\times$, then $\chi(\iota_s(t_\lambda)) \to 0$.

This (14) follows from (8), and from $(\beta(u_\lambda^{-1}\mathbf{r}_\lambda)^{-1}\hat{h}_\lambda)(\chi) = 1$ by (4) (applied replacing u, $\mathbf{r}$, h, s by u_λ, $\mathbf{r}_\lambda$, h_λ, s', respectively).

It remains to prove that $x_\lambda \to x$ for the $\mathcal{T}$-toplogy. Take $U \in \mathcal{F}_x$. It is enough to show that if λ is sufficiently large then there exist a neighborhood $U_{\lambda,1}$ of 0 in $\mathbf{R}_{\geq 0}^{m'}$, a neighborhood $U_{\lambda,2}$ of 1 in $G_\mathbf{R}$ and a neighborhood $U_{\lambda,3}$ of 1 in $K_{\mathbf{r}_\lambda}$ such that

$$\tilde{\rho}_\lambda(t) g k \mathbf{r}_\lambda \in U$$

for any $t \in \mathbf{R}_{>0}^{m'} \cap U_{\lambda,1}$, any $g \in G_\mathbf{R}$ and any $k \in U_{\lambda,3}$ with $\mathrm{Int}(\rho_\lambda(t))^j(g) \in U_{\lambda,2}$ for $j = 0, \pm 1$. For $t \in \mathbf{R}_{>0}^{m'}$, $g \in G_\mathbf{R}$ and $k \in K_{\mathbf{r}_\lambda}$, we have, by (10) and $\tilde{\rho}_\lambda = \mathrm{Int}(p_\lambda)(\tilde{\rho} \circ \iota')$,

$$\begin{aligned} \tilde{\rho}_\lambda(t) g k \mathbf{r}_\lambda = {} & \tilde{\rho}(t_\lambda \iota'(t)) \, \mathrm{Int}(\tilde{\rho}(t_\lambda \iota'(t)))^{-1}(p_\lambda) \\ & \cdot \mathrm{Int}(p_\lambda \tilde{\rho}(t_\lambda))^{-1}(g) \, \mathrm{Int}(p_\lambda \tilde{\rho}(t_\lambda))^{-1}(k') k_\lambda \mathbf{r}. \end{aligned}$$

Note that $\mathrm{Int}(p_\lambda \tilde{\rho}(t_\lambda))^{-1}(k) \in K_\mathbf{r}$. It is enough to prove that if λ is sufficiently large and if $t \in \mathbf{R}_{>0}^{m'}$ converges to 0 then we have the following two assertions.

$$t_\lambda \iota'(t) \to 0, \tag{15}$$

$$\mathrm{Int}\left(\tilde{\rho}(t_\lambda \iota'(t))\right)^j(p_\lambda) \to 1 \quad \text{for } j = 0, -1, -2. \tag{16}$$

(15) follows from (14), and (16) follows from $p_\lambda \in P \cap G_{W',\mathbf{R}}$ (13). Q.E.D.

Corollary 4.15. *The topology $\mathcal{T}$ on $D_{\mathrm{SL}(2)}(W)$ coincides with the one on $D_{\mathrm{SL}(2)}(W)$ as a quotient space of $D_{\mathrm{SL}(2),\mathrm{val}}(W)$.*

Proof. By 4.14, $D_{\mathrm{SL}(2),\mathrm{val}}(W) \to D_{\mathrm{SL}(2)}(W)_{\mathcal{T}}$ is proper surjective, because so is $(\mathbf{R}^n_{\geq 0})_{\mathrm{val}} \to \mathbf{R}^n_{\geq 0}$. Hence $D_{\mathrm{SL}(2),\mathrm{val}}(W) \to D_{\mathrm{SL}(2)}(W)_{\mathcal{T}}$ is a closed surjective continuous map. This proves 4.15. Q.E.D.

Lemma 4.16. *Let $f : X \to Y$ be a continuous proper surjective map of topological spaces.*

(i) ([NB, Ch. 1, §10, no. 1, Corollaire 2]). *If X is Hausdorff, then so is Y.*

(ii) *If X is regular, then so is Y.*

Proof. We prove (ii). Let F be a closed set of Y and y be a point of Y with $y \notin F$. Then $f^{-1}(y) \cap f^{-1}(F) = \emptyset$. Since X is regular, for each point $x \in f^{-1}(y)$, there exist disjoint open neighborhoods U_x and V_x of x and of the closed set $f^{-1}(F)$, respectively. Since f is proper, $f^{-1}(y)$ is quasi-compact. Hence, there exist finite subsets $\{U_j\}_{1\leq j\leq n}$ of $\{U_x\}_{x\in f^{-1}(y)}$, which cover $f^{-1}(y)$. Let $\{V_j\}_{1\leq j\leq n}$ be the corresponding finite subsets of $\{V_x\}_{x\in f^{-1}(y)}$. Put $U := \bigcup_{1\leq j\leq n} U_j$ and $V := \bigcap_{1\leq j\leq n} V_j$. Then, it is easy to see that $Y - f(X - U)$ and $Y - f(X - V)$ are disjoint open neighborhoods of y and of F, respectively. Hence Y is regular. Q.E.D.

4.17. *Proof of Theorem* 3.14. Since, for each W, $D_{\mathrm{SL}(2),\mathrm{val}}(W) \to D_{\mathrm{SL}(2)}(W)$ is continuous proper surjective, so is $D_{\mathrm{SL}(2),\mathrm{val}} \to D_{\mathrm{SL}(2)}$. It follows that $D_{\mathrm{SL}(2)}$ is Hausdorff by 4.16 (i). Q.E.D.

Proposition 4.18. *$D_{\mathrm{SL}(2),\mathrm{val}}(W)$ and $D_{\mathrm{SL}(2)}(W)$ are regular spaces.*

Proof. Since $D_{\mathrm{BS},\mathrm{val}}$ is regular, so is $D_{\mathrm{SL}(2),\mathrm{val}}(W)$ by 4.13 and hence so is $D_{\mathrm{SL}(2)}(W)$ by 4.16 (ii). Q.E.D.

4.19 *Remark.* We prove that, for an SL(2)-orbit (ρ, φ) of rank n,

$$[\rho, \varphi] = \lim \varphi(iy_1, \ldots, iy_n)$$

as $\frac{y_j}{y_{j+1}} \to \infty$ $(\forall j,\ y_{n+1} = 1)$ in $D_{\mathrm{SL}(2)}$, which is stated after 3.13.

Let W be the family of weight filtration associated to (ρ, φ). Since

$$\varphi(iy_1, \ldots, iy_n) = \tilde{\rho}\Big(\sqrt{\tfrac{y_2}{y_1}}, \ldots, \sqrt{\tfrac{y_{n+1}}{y_n}}\Big)\varphi(\mathbf{i})$$

and $\sqrt{\frac{y_{j+1}}{y_j}} \to 0$, the right-hand-side converges to $[\rho, \varphi]$ in $D_{\mathrm{SL}(2)}(W)_{\mathcal{T}}$ and hence in the topology of $D_{\mathrm{SL}(2)}(W)$ by 4.15.

§5. Actions of $G_{\mathbf{Z}}$

5.1. *Summary.* In this section, we will transport the good properties (i), (ii) in 2.1 of the quotient space $\Gamma\backslash\mathcal{X}_{\mathrm{BS}}$ to other spaces along the diagram 3.1 (1). The main result of this section is the following Theorem 5.2.

Theorem 5.2. (i) *For any subgroup Γ of $G_{\mathbf{Z}}$, all the quotient spaces $\Gamma\backslash D_{\mathrm{BS}}$, $\Gamma\backslash D_{\mathrm{BS,val}}$, $\Gamma\backslash D_{\mathrm{SL}(2),\mathrm{val}}$, $\Gamma\backslash D_{\mathrm{SL}(2)}$ are Hausdorff.*

(ii) *If Γ is a subgroup of $G_{\mathbf{Z}}$ of finite index, then $\Gamma\backslash D_{\mathrm{BS}}$, $\Gamma\backslash D_{\mathrm{BS,val}}$ are compact.*

(iii) *If Γ is a neat subgroup of $G_{\mathbf{Z}}$, then all the projections $D_{\mathrm{BS}} \to \Gamma\backslash D_{\mathrm{BS}}$, $D_{\mathrm{BS,val}} \to \Gamma\backslash D_{\mathrm{BS,val}}$, $D_{\mathrm{SL}(2),\mathrm{val}} \to \Gamma\backslash D_{\mathrm{SL}(2),\mathrm{val}}$, $D_{\mathrm{SL}(2)} \to \Gamma\backslash D_{\mathrm{SL}(2)}$ are local homeomorphisms.*

Before proving this theorem, we recall the notion of 'proper action' and some related results in [NB] which are needed for our present purpose.

Definition 5.3 ([NB, Ch. 3, §4, no. 1, Definition 1]). Let G be a topological group acting continuously on a topological space X. G is said to act properly on X if the map

$$G \times X \to X \times X, \quad (g, x) \mapsto (x, gx),$$

is proper.

Lemma 5.4 (cf. [NB, Ch. 3, §4, no. 2, Proposition 3]). *If a topological group G acts properly on a topological space X, then the quotient space $G\backslash X$ is Hausdorff.*

Lemma 5.5 (cf. [NB, Ch. 3, §4, no. 4, Corollary]). *If a discrete group G acts properly and freely on a Hausdorff space X, then the projection $X \to G\backslash X$ is a local homeomorphism.*

Lemma 5.6 (cf. [NB, Ch. 3, §2, no. 2, Proposition 5]). *Let G be a topological group acting continuously on topological spaces X and X'. Let $\psi : X \to X'$ be an equivariant continuous map.*

(i) *If ψ is surjective and proper, and if G acts properly on X, then G acts properly on X'.*

(ii) *If G acts properly on X' and if X is Hausdorff, then G acts properly on X.*

Now we come back to our situation.

Lemma 5.7. *If Γ is a neat subgroup of $G_{\mathbf{Z}}$, then Γ acts on $D_{\mathrm{SL}(2)}$ freely.*

Proof. Let $x \in D_{\mathrm{SL}(2),n}$, $\gamma \in \Gamma$, and assume $\gamma x = x$. We prove $\gamma = 1$. Let (ρ, φ) be a representative of x. Then $\mathrm{Ad}(\gamma)Y_j = Y_j$ $(1 \leq j \leq n)$. Here the Y_j are the semi-simple elements of $\mathfrak{g}_{\mathbf{R}}$ associated to ρ in 3.1. Put $Y := \sum_{1 \leq j \leq n} Y_j$. Then γ preserves the l-eigenspace $H(l) \subset H_{0,\mathbf{R}}$ of Y for all l. Put $\mathrm{gr}_k := \mathrm{gr}_k^{W^{(n)}}(H_{0,\mathbf{C}})$ and $\mathrm{gr} := \bigoplus_k \mathrm{gr}_k$. Let $F := \varphi(\mathbf{i}) \in D$ and $F(\mathrm{gr})$ be the filtration of gr induced by F. Then, by the assumption, the automorphism $\mathrm{gr}(\gamma)$ of gr induced by γ satisfies $\mathrm{gr}(\gamma)F(\mathrm{gr}) = F(\mathrm{gr})$. Thus we have the following four statements.

(i) $(W^{(n)}[-w], F)$ is an $(N_1 + \cdots + N_n)$-polarized mixed Hodge structure ([Sc]).

(ii) $\gamma W^{(n)} = W^{(n)}$.

(iii) $\mathrm{gr}(\gamma)F(\mathrm{gr}) = F(\mathrm{gr})$.

(iv) If a is an eigenvalue of $\mathrm{gr}(\gamma)$ and if a is a root of 1, then $a = 1$.

We prove $\mathrm{gr}(\gamma) = 1$. Since $F(\mathrm{gr})$ is polarized, the isotropy group of $F(\mathrm{gr})$ is compact, and so $\mathrm{gr}(\gamma)$ is contained in the intersection of a discrete subgroup and a compact subgroup and hence is of finite order. Therefore $\mathrm{gr}(\gamma) = 1$ by (iv).

Now $\gamma = 1$ follows from $\mathrm{gr}(\gamma) = 1$ and the commutativity of γ and Y. Q.E.D.

5.8. *Proof of Theorem* 5.2. We prove (i). $G_{\mathbf{Z}}$ acts on $\mathcal{X}_{\mathrm{BS}}$ properly by [BS]. Since D_{BS}, $D_{\mathrm{BS,val}}$, $D_{\mathrm{SL}(2),\mathrm{val}}$ are Hausdorff by 2.17 (ii), it follows that $G_{\mathbf{Z}}$ acts on these spaces properly by 5.6 (ii). Since $D_{\mathrm{SL}(2),\mathrm{val}} \to D_{\mathrm{SL}(2)}$ is proper and surjective by 3.14 (i), it follows that $G_{\mathbf{Z}}$ acts on $D_{\mathrm{SL}(2)}$ properly by 5.6 (i). Hence, for any subgroup Γ of $G_{\mathbf{Z}}$, all the quotient spaces $\Gamma\backslash D_{\mathrm{BS}}$, $\Gamma\backslash D_{\mathrm{BS,val}}$, $\Gamma\backslash D_{\mathrm{SL}(2),\mathrm{val}}$, $\Gamma\backslash D_{\mathrm{SL}(2)}$ are Hausdorff by 5.4.

We prove (ii). Let Γ be a subgroup of $G_{\mathbf{Z}}$ of finite index. Then $\Gamma\backslash\mathcal{X}_{\mathrm{BS}}$ is compact by [BS]. Since $D_{\mathrm{BS}} \to \mathcal{X}_{\mathrm{BS}}$ and $D_{\mathrm{BS,val}} \to D_{\mathrm{BS}}$ are proper by 2.17 (i), $\Gamma\backslash D_{\mathrm{BS}} \to \Gamma\backslash\mathcal{X}_{\mathrm{BS}}$ and $\Gamma\backslash D_{\mathrm{BS,val}} \to \Gamma\backslash D_{\mathrm{BS}}$ are proper. Hence $\Gamma\backslash D_{\mathrm{BS}}$ and $\Gamma\backslash D_{\mathrm{BS,val}}$ are compact.

We prove (iii). Let Γ be a neat subgroup of $G_{\mathbf{Z}}$. Since Γ acts on $\mathcal{X}_{\mathrm{BS}}$ freely by [BS], so does Γ on D_{BS}, on $D_{\mathrm{BS,val}}$, and on $D_{\mathrm{SL}(2),\mathrm{val}}$

by 3.11. Γ acts on $D_{\mathrm{SL}(2)}$ freely by 5.7. Moreover, all the spaces D_{BS}, $D_{\mathrm{BS,val}}$, $D_{\mathrm{SL}(2),\mathrm{val}}$, $D_{\mathrm{SL}(2)}$ are Hausdorff and acted by Γ properly. Hence all the projections $D_{\mathrm{BS}} \to \Gamma\backslash D_{\mathrm{BS}}$, $D_{\mathrm{BS,val}} \to \Gamma\backslash D_{\mathrm{BS,val}}$, $D_{\mathrm{SL}(2),\mathrm{val}} \to \Gamma\backslash D_{\mathrm{SL}(2),\mathrm{val}}$, $D_{\mathrm{SL}(2)} \to \Gamma\backslash D_{\mathrm{SL}(2)}$ are local homeomorphisms by 5.5. Q.E.D.

§6. Examples and comments

6.1. *Summary.* In this section, we will first give a criterion in Proposition 6.3 for the existence of the canonical map $D_{\mathrm{SL}(2)} \to D_{\mathrm{BS}}$ by using the family of weight filtrations associated to a point of $D_{\mathrm{SL}(2)}$. This criterion explains the reason why we need to introduce the projective limits of blowing-ups $D_{\mathrm{BS,val}}$, $D_{\mathrm{SL}(2),\mathrm{val}}$ of D_{BS}, $D_{\mathrm{SL}(2)}$, respectively, to relate D_{BS} and $D_{\mathrm{SL}(2)}$. Then we will give the list of 'classical situation' in 6.6, and in this situation we will show that $D_{\mathrm{SL}(2)} = D_{\mathrm{BS}}$ and $D_{\mathrm{SL}(2),\mathrm{val}} = D_{\mathrm{BS,val}}$ except one case in Theorem 6.7. As a corollary, we have in 6.9 the canonical surjection from the Borel-Serre space D_{BS} to the Satake space D_S in the 'classical situation'. This map was defined by Zucker [Z2] by another method. Proposition 6.10 gives examples which do not have the canonical map $D_{\mathrm{SL}(2)} \to D_{\mathrm{BS}}$. We will give an example in 6.11 for which $D_{\mathrm{SL}(2)} \subsetneq D_{\mathrm{BS}}$ (hence $\Gamma\backslash D_{\mathrm{SL}(2)}$ is not compact for Γ of finite index in $G_{\mathbf{Z}}$) because the horizontal tangent bundle T_D^h is trivial. Proposition 6.12 gives examples for which $D_{\mathrm{SL}(2)}$ is not locally compact, that is, $D_{\mathrm{SL}(2)}$ has 'slits' influenced by the fact that the isotropy subgroups $K'_{\mathbf{r}}$ are not maximal compact.

6.2. *The case of the upper half-plane* $\mathfrak{h}$. Let $H_{\mathfrak{h}} := \mathbf{Z}^2 = \mathbf{Z}e_1 + \mathbf{Z}e_2$, let $\langle\ ,\ \rangle_{\mathfrak{h}}$ be the anti-symmetric bilinear form on $H_{\mathfrak{h},\mathbf{C}} \times H_{\mathfrak{h},\mathbf{C}}$ characterized by $\langle e_2, e_1\rangle_{\mathfrak{h}} = 1$, and take $(H_{\mathfrak{h}}, \langle\ ,\ \rangle_{\mathfrak{h}})$ as $(H_0, \langle\ ,\ \rangle_0)$. Then

$$\check{D} = \mathbf{P}^1(\mathbf{C});\ F_z \leftrightarrow z = (z_1 : z_2),$$
$$\text{where } F_z^0 = H_{0,\mathbf{C}},\ F_z^1 = \mathbf{C}(z_1e_1 + z_2e_2),\ F_z^2 = 0.$$

Identify $z \in \mathbf{C}$ with $(z : 1) \in \mathbf{P}^1(\mathbf{C})$. Then $D \subset \check{D}$ is identified with the upper-half plane $\mathfrak{h} \subset \mathbf{P}^1(\mathbf{C})$. We have

$$G_{\mathbf{R}} = \mathrm{SL}(2, \mathbf{R}) \supset \mathrm{SO}(2, \mathbf{R}) = K_i = K'_i.$$

The map

$$\mathbf{P}^1(\mathbf{Q}) \to \{P \mid \text{a } \mathbf{Q}\text{-parabolic subgroup of } G_{\mathbf{R}} \text{ with } P \neq G_{\mathbf{R}}\},$$
$$z \mapsto P_z := \{g \in G_{\mathbf{R}} \mid gz = z\},$$

is bijective. We have

$$P_\infty = \left\{ \begin{pmatrix} a^{-1} & b \\ 0 & a \end{pmatrix} \middle| \begin{matrix} a \in \mathbf{R}^\times, \\ b \in \mathbf{R} \end{matrix} \right\}, \quad P_{\infty,u} = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \middle| b \in \mathbf{R} \right\},$$
$$S_{P_\infty} = P_\infty / P_{\infty,u}.$$

The unique element of Δ_{P_∞} sends

$$S_{P_\infty} \ni \left(\begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix} \bmod P_{\infty,u} \right) \longmapsto a^2.$$

If we identiy A_{P_∞} with $\mathbf{R}_{>0}$ by this element of Δ_{P_∞}, we have

$$a \circ (x + iy) = x + ia^{-1}y \quad (a \in \mathbf{R}_{>0}, x \in \mathbf{R}, y \in \mathbf{R}_{>0}).$$

Hence $D_{\mathrm{BS}}(P_\infty)$ is identified with the topological space $\{x + iy \,|\, x \in \mathbf{R}, 0 < y \leq \infty\}$. We have

$$\mathcal{X}_{\mathrm{BS}} = D_{\mathrm{BS}} \xleftarrow{\sim} D_{\mathrm{BS,val}} \xleftarrow{\sim} D_{\mathrm{SL}(2),\mathrm{val}} \xrightarrow{\sim} D_{\mathrm{SL}(2)},$$

and $i\infty \in D_{\mathrm{BS}}(P_\infty)$ is identified with the class of the SL(2)-orbit $(\rho_{\mathfrak{h}}, \varphi_{\mathfrak{h}})$, which we call the *standard* SL(2)*-orbit*, defined by

$$\begin{cases} \rho_{\mathfrak{h}} = \mathrm{id} : \mathrm{SL}(2, \mathbf{C}) \to G_{\mathbf{C}}, \\ \varphi_{\mathfrak{h}}(z) = F_z \quad (z \in \mathbf{P}^1(\mathbf{C})). \end{cases}$$

In fact, it is obvious that $(\mathbf{R}^n_{\geq 0})_{\mathrm{val}} = \mathbf{R}^n_{\geq 0}$ for $n = 1$. Since P is a $\mathbf{Q}$-parabolic subgroup of $G_{\mathbf{R}} = \mathrm{SL}(2, \mathbf{R})$ with $P \neq G_{\mathbf{R}}$, we have $\operatorname{rank} S_P = 1$. Hence, by Proposition 2.15, we have $D_{\mathrm{BS,val}} \xrightarrow{\sim} D_{\mathrm{BS}}$. $D_{\mathrm{SL}(2)} = D_{\mathrm{SL}(2),\leq 1}$, because $\mathfrak{sl}(2, \mathbf{C})^n \to \mathfrak{g}_{\mathbf{C}}$ can not be injective if $n > 1$. Hence, by Proposition 4.14, we have $D_{\mathrm{SL}(2),\mathrm{val}} \xrightarrow{\sim} D_{\mathrm{SL}(2)}$.

Proposition 6.3. *We use the notation in* 2.5, 2.6, 3.6, 3.7. *Let* $x = [\rho, \varphi] \in D_{\mathrm{SL}(2)}$ *and let* $W = (W^{(j)})_{1 \leq j \leq n}$ *be the associated family of weight filtrations, where* $W^{(j)} = W(\sigma_j)$ (3.5).

(i) *The following conditions* (a), (b), (c) *are equivalent.*

(a) *For any* $y, y' \in D_{\mathrm{SL}(2),\mathrm{val}}$ *lying over* x, *the images of* y, y' *in* D_{BS}, *via* $D_{\mathrm{BS,val}}$, *coincide.*

(b) *The subspaces* $W^{(j)}_k$ $(1 \leq j \leq n, k \in \mathbf{Z})$ *are linearly ordered by inclusion.*

(c) *For* $\chi, \chi' \in X((\mathbf{G_m})^n)$ *with* $H(\chi) \neq 0$ *and* $H(\chi') \neq 0$, *either* $\chi\chi'^{-1}$ *or* $\chi'\chi^{-1}$ *is contained in* $X((\mathbf{G_m})^n)_+$.

(ii) *Assume the equivalent conditions in* (i) *are satisfied. Then the* $\mathbf{Q}$*-subgroup* $(G^\circ)_{W,\mathbf{R}} := G_{W,\mathbf{R}} \cap (G^\circ)_{\mathbf{R}}$ *of* $G_{\mathbf{R}}$ *is parabolic and, for any lifting* $y \in D_{\mathrm{SL}(2),\mathrm{val}}$ *of* x*, the image of* y *in* D_{BS}*, via* $D_{\mathrm{BS},\mathrm{val}}$*, coincides with* $((G^\circ)_{W,\mathbf{R}}, A_{(G^\circ)_{W,\mathbf{R}}} \circ \varphi(\mathbf{i}))$.

For the proof of Proposition 6.3, we use the following

Lemma 6.4. *Let* $\mathcal{P}$ *be the set of all* $\mathbf{Q}$*-parabolic subgroups of* $G_{\mathbf{R}}$*, and let* $\mathcal{M}$ *be the set of all finite sets* M *such that* $M = \{M_j \mid 0 \le j \le m\}$*, the* M_j *are* $\mathbf{Q}$*-rational* $\mathbf{R}$*-subspaces of* $H_{0,\mathbf{R}}$*,* $0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_m = H_{0,\mathbf{R}}$ *and* $M_j^\perp = M_{m-j}$ $(0 \le j \le m)$*. Let* $p : \mathcal{M} \to \mathcal{P}$ *be the map defined by* $p(M) := \{g \in (G^\circ)_{\mathbf{R}} \mid gM_j = M_j \ (0 \le j \le m)\}$*. Write*

$$\mathcal{M} = \mathcal{M}_0 \sqcup \mathcal{M}_1 \sqcup \mathcal{M}_2, \quad \textit{where}$$
$$\mathcal{M}_0 := \{M \in \mathcal{M} \mid M \textit{ is not exceptional}\},$$
$$\mathcal{M}_1 := \{M \in \mathcal{M} \mid M \textit{ is exceptional with } m \textit{ even}\},$$
$$\mathcal{M}_2 := \{M \in \mathcal{M} \mid M \textit{ is exceptional with } m \textit{ odd}\}.$$

Here the meaning of 'exceptional' is as in 2.11*. Then*

(i) *Let* $M \in \mathcal{M}_0$ *and* $P = p(M)$*. Then,* M *coincides with the set of all* P*-stable* $\mathbf{R}$*-subspaces of* $H_{0,\mathbf{R}}$*. We have* $p^{-1}(p(M)) = \{M\}$.

(ii) *Let* $P \in p(\mathcal{M}_1 \sqcup \mathcal{M}_2) \subset \mathcal{P}$*. Let* N *be the set of all* P*-stable* $\mathbf{R}$*-subspaces of* $H_{0,\mathbf{R}}$*, and let*

$$M := \{L \in N \mid \forall L' \in N, \textit{ either } L \subset L' \textit{ or } L' \subset L\}.$$

Then, $M \in \mathcal{M}_2$*. Write* $M = \{M_j \mid 0 \le j \le m\}$*,* $0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_m = H_{0,\mathbf{R}}$*, let* e_1*,* e_2 *be as in* 2.11*, and let* $L_k := M_{(m-1)/2} + \mathbf{R}e_k$ *for* $k = 1, 2$*. Then,* $N = M \cup \{L_1, L_2\}$*,* $M \cup \{L_1\}$ *and* $M \cup \{L_2\}$ *belong to* $\mathcal{M}_1$*, and* $p^{-1}(P)$ *consists of the three elements* M*,* $M \cup \{L_1\}$ *and* $M \cup \{L_2\}$.

(iii) $p : \mathcal{M} \to \mathcal{P}$ *is surjective.*

(iv) $p(\mathcal{M}_1) = p(\mathcal{M}_2)$.
p induces a $2 : 1$ *map* $p : \mathcal{M}_1 \to p(\mathcal{M}_1)$*, and a bijection* $p : \mathcal{M}_0 \sqcup \mathcal{M}_2 \xrightarrow{\sim} \mathcal{P}$.

Proof. The proofs of (i) and (ii) are straightforward. It follows that $p(\mathcal{M}_1) = p(\mathcal{M}_2)$, that the map $p : \mathcal{M}_1 \to p(\mathcal{M}_1)$ is $2 : 1$ and that the map $p : \mathcal{M}_0 \sqcup \mathcal{M}_2 \to \mathcal{P}$ is injective. It remains to prove $\mathcal{P} = p(\mathcal{M}_0 \sqcup \mathcal{M}_2)$. We divide our considerations into three cases:

(a) w is odd.

(b) w is even and $\langle\ ,\ \rangle_0 : H_{0,\mathbf{Q}} \times H_{0,\mathbf{Q}} \to \mathbf{Q}$ is not hyperbolic.

(c) w is even, and $\langle\ ,\ \rangle_0 : H_{0,\mathbf{Q}} \times H_{0,\mathbf{Q}} \to \mathbf{Q}$ is hyperbolic (that is, $d := \dim H_{0,\mathbf{Q}}$ is even and there exists a basis $(e_j)_{1\leq j\leq d}$ of $H_{0,\mathbf{Q}}$ such that $\langle e_j, e_k\rangle_0 = 1$ if $j + k = d + 1$ and $\langle e_j, e_k\rangle_0 = 0$ otherwise).

Assume first we are in the case (a). Then there exists $M = \{M_j \mid 0 \leq j \leq m\} \in \mathcal{M}_0$ (m is even, $0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_m = H_{0,\mathbf{R}}$) such that $\dim \mathrm{gr}_j^M = 1$ for $1 \leq j \leq m$. Let $P := p(M)$, $r := m/2$. Then P is a minimal $\mathbf{Q}$-parabolic subgroup of $G_{\mathbf{R}}$. This is because $P/P_u \simeq (\mathbf{G}_{\mathbf{m},\mathbf{R}})^r$ (see 2.11 (2)) and has no $\mathbf{Q}$-parabolic subgroup other than P/P_u itself. We have $r = \mathrm{rank}\, S_P = \sharp(\Delta_P)$. We find 2^r elements M' of $\mathcal{M}_0$ such that $M' \subset M$ (in fact, for each finite subset I of $\{1, \ldots, r\}$, we find M' defined by $M' := \{0, M_j\ (j \in I), M_{m-j}\ (j \in I), H_{0,\mathbf{R}}\}$). Since $p : \mathcal{M}_0 \to \mathcal{P}$ is injective, this shows that there exist at least 2^r elements P' of $p(\mathcal{M}_0)$ such that $P' \supset P$. By 2.10 (3), this shows that any $\mathbf{Q}$-parabolic subgroup P' of $G_{\mathbf{R}}$ with $P' \supset P$ belongs to $p(\mathcal{M}_0)$. Now let P' be any element of $\mathcal{P}$. Take any minimal $\mathbf{Q}$-parabolic subgroup P'' of $G_{\mathbf{R}}$ such that $P' \supset P''$. Then $P = gP''g^{-1}$ for some $g \in G_{\mathbf{Q}}$. Since $gP'g^{-1} \supset P$, we have $gP'g^{-1} \in p(\mathcal{M}_0)$. Hencce $P' \in p(\mathcal{M}_0)$.

Next assume we are in the case (b). Then there exists $M = \{M_j \mid 0 \leq j \leq m\} \in \mathcal{M}_0$ (m is odd, $0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_m = H_{0,\mathbf{R}}$) such that $\dim \mathrm{gr}_j^M = 1$ if $1 \leq j \leq m$ and $j \neq (m+1)/2$ and such that the $\mathbf{R}$-bilinear form $\varphi : \mathrm{gr}^M_{(m+1)/2} \times \mathrm{gr}^M_{(m+1)/2} \to \mathbf{R}$ induced by $\langle\ ,\ \rangle_0$ is anisotropic (that is, $\varphi(x, x) \neq 0$ for any $x \in \mathrm{gr}^M_{(m+1)/2} - \{0\}$). Let $P := p(M)$, $r := (m-1)/2$. Then P is a minimal $\mathbf{Q}$-parabolic subgroup of $G_{\mathbf{R}}$. This is because $P/P_u \simeq (\mathbf{G}_{\mathbf{m},\mathbf{R}})^r \times \mathrm{SO}(\varphi)_{\mathbf{R}}$ (see 2.11 (2)) and has no $\mathbf{Q}$-parabolic subgroup other than P/P_u itself. We have $r = \mathrm{rank}\, S_P = \sharp(\Delta_P)$. Just as in the case (a), we find 2^r elements M' of $\mathcal{M}_0$ such that $M' \subset M$ and then we can deduce $\mathcal{P} = p(\mathcal{M}_0)$.

Assume lastly we are in the case (c). Then there exists $M = \{M_j \mid 0 \leq j \leq m\} \in \mathcal{M}_2$ (m is odd, $0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_m = H_{0,\mathbf{R}}$) such that $\dim \mathrm{gr}_j^M = 1$ if $1 \leq j \leq m$ and $j \neq (m+1)/2$. Let $P := p(M)$, $r := (m+1)/2$. Then P is a minimal $\mathbf{Q}$-parabolic subgroup of $G_{\mathbf{R}}$. This is because $P/P_u \simeq (\mathbf{G}_{\mathbf{m},\mathbf{R}})^r$ (see 2.11 (2)) and has no $\mathbf{Q}$-parabolic subgroup other than P/P_u itself. We have $r = \mathrm{rank}\, S_P = \sharp(\Delta_P)$. Just as in the cases (a), (b), we find 2^r elements M' of $\mathcal{M}_0 \sqcup \mathcal{M}_2$ such that $M' \subset M$ and then we can deduce $\mathcal{P} = p(\mathcal{M}_0 \sqcup \mathcal{M}_2)$. Q.E.D.

6.5. *Proof of Proposition* 6.3. It is easy to see the equivalence of (b) and (c), the implication from (b) and (c) to (a), and the implication from (b) and (c) to the conclusion in (ii).

We prove the implication from (a) to (c). We first prove the following assertion:

(1) Assume (a). Let $\chi, \chi' \in X((\mathbf{G_m})^n)$ which satisfy $H(\chi) \neq 0$, $H(\chi') \neq 0$ and $(\chi\chi'^{-1})^{\pm} \notin X((\mathbf{G_m})^n)_+$. Then, w is even, $\chi' = \chi^{-1}$ and $\dim H(\chi) = \dim H(\chi') = 1$.

Write $\chi = (a_1, \ldots, a_n)$, $\chi' = (a'_1, \ldots, a'_n)$ under the identification $X((\mathbf{G_m})^n) = \mathbf{Z}^n$. Then, there exist j, k such that $a_j > a'_j$, $a_k < a'_k$. Let $s, s' : \{1, \ldots, n\} \to \{1, \ldots, n\}$ be bijections satisfying $s(1) = j$, $s(2) = k$, $s'(1) = k$, $s'(1) = j$, and let V (resp. V') be the valuative submonoid of $X((\mathbf{G_m})^n) = \mathbf{Z}^n$ consisting of all $(b_1, \ldots, b_n)$ $(b_j \in \mathbf{Z})$ such that $(b_{s(1)}, \ldots, b_{s(n)}) \geq 0$ (resp. $(b_{s'(1)}, \ldots, b_{s'(n)}) \geq 0$) for the lexicographical order of $\mathbf{Z}^n$. Then, $V \supset X((\mathbf{G_m})^n)_+$, $V' \supset X((\mathbf{G_m})^n)_+$, $V^\times = V'^\times = \{1\}$. By (a), we have $P_V = P_{V'}$ (see 4.11). Let $\mathcal{M} = \mathcal{M}_0 \sqcup \mathcal{M}_1 \sqcup \mathcal{M}_2$, $\mathcal{P}$, and $p : \mathcal{M} \to \mathcal{P}$ be as in 6.4. Let $L := \sum_{\psi \in \chi V'^{-1}} H(\psi)$, $L' := \sum_{\psi \in \chi' V^{-1}} H(\psi)$. Then, since $\chi' \notin \chi V'^{-1}$ and $\chi \notin \chi' V^{-1}$, we have $H(\chi') \not\subset L$, $H(\chi) \not\subset L'$, $H(\chi) \subset L$, $H(\chi') \subset L'$. Hence $L \not\subset L'$ and $L' \not\subset L$. Since L and L' are stable under $P_V = P_{V'}$, we have, by 6.4, that $P_V = P_{V'} = p(M)$. Write $M = \{M_j \mid 0 \leq j \leq m\}$, $0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_m = H_{0,\mathbf{R}}$. By 6.4, w is even, $L = H(\chi) + M_{(m-1)/2}$, $L' = H(\chi') + M_{(m-1)/2}$, $\dim H(\chi) = \dim H(\chi') = 1$, and $\langle H(\chi), H(\chi')\rangle_0 \neq 0$. Hence we have (1).

To proceed more, we need the following (2), (3).

(2) Let $\theta : \mathbf{Z}^n \to \mathbf{Z}^n$ be the map defined by $\theta(b_1, \ldots, b_n) = (b_1, b_1 + b_2, \ldots, b_1 + \cdots + b_n)$, so that

$$H(\theta(b_1, \ldots, b_n)) = \{v \in H_{0,\mathbf{R}} \mid \rho(\Delta(t))v = t_1^{b_1} \ldots t_n^{b_n} v \ (t \in (\mathbf{G_{m,R}})^n)\}$$

(Δ is as in 3.1). Let $b_1, \ldots, b_n \in \mathbf{Z}$ so that $H(\theta(b_1, \ldots, b_n)) \neq 0$. Then, for any $c_1, \ldots, c_n \in \mathbf{Z}$ with $|c_j| \leq |b_j|$ and $c_j \equiv b_j \mod 2$ $(1 \leq j \leq n)$, we have $H(\theta(c_1, \ldots, c_n)) \neq 0$.

(3) For $b_1, \ldots, b_n \in \mathbf{Z}$, $H(b_1, \ldots, b_n)$ has canonically a Hodge structure of weight $w + b_n$.

(2) is deduced from the following (4) and (5) whose proofs are easy.

(4) Let $b_1, \ldots, b_n \in \mathbf{Z}$. Then,

$$N_j H(\theta(b_1, \ldots, b_n)) \subset H(\theta(b_1, \ldots, b_{j-1}, b_j - 2, b_{j+1}, \ldots b_n)).$$

(5) Let $b_j \in \mathbf{Z}$, $b_j \geq 0$ $(1 \leq j \leq n)$. Then,

$$\prod_{1 \leq j \leq n} N_j^{b_j} : H(\theta(b_1, \ldots, b_n)) \xrightarrow{\sim} H(\theta(-b_1, \ldots, -b_n)).$$

We prove (3). By [Sc], $(W^{(n)}[-w], \varphi(0))$ is a mixed Hodge structure. Hence

$$\varphi(0)(\mathrm{gr}_k^{W^{(n)}}) = \bigoplus_{b \in \mathbf{Z}^{n-1}} H(b, k)$$

has a Hodge structure of weight $w + k$. Since $\tilde{\rho}(t)\varphi(0) = \varphi(0)$ for all $t \in \mathbf{R}_{>0}^n$, each $H(b, k)$ $(b \in \mathbf{Z}^{n-1})$ carries a Hodge structure of weight $w + k$.

Now we complete the proof of $(a) \Rightarrow (c)$. Assume (a). By (1), if (c) is not satisfied, then w is even and there exists $\chi \in X((\mathbf{G_m})^n)$ such that $\dim H(\chi) = \dim H(\chi^{-1}) = 1$ and $\chi \notin X((\mathbf{G_m})^n)_+$, $\chi^{-1} \notin X((\mathbf{G_m})^n)_+$. Write $\chi = (a_1, \ldots, a_n) \in \mathbf{Z}^n$. Then there exist j, k such that $a_j > 0$ and $a_k < 0$. By replacing χ by χ^{-1} if necessary, we may assume $j < k$. We prove

$$a_k = -1 \tag{6}$$

Take l such that $j < l \leq k$ and $a_l < a_{l-1}$. Let $b_j := a_j - a_{j-1}$ $(1 \leq j \leq n$, a_0 means 0). Then $(a_1, \ldots, a_n) = \theta(b_1, \ldots, b_n)$, $b_l < 0$. Define $a_1', \ldots, a_n', b_1', \ldots, b_n' \in \mathbf{Z}$ by $b_j' := b_j$ for $j \neq l$ and $b_l' := b_l + 2$, $a_j' - a_{j-1}' := b_j'$ $(1 \leq j \leq n$, $a_0' = 0)$. Then, $H(a_1', \ldots, a_n') = H(\theta(b_1', \ldots, b_n')) \neq 0$ by (2). If $a_k < -1$, then $a_k' = a_k + 2 < -a_k$, and $a_j' = a_j > -a_j$. Since $H(a_1', \ldots, a_n') \neq 0$ and $H(-a_1, \ldots, -a_n) \neq 0$, (1) shows $a_j' = a_j$ $(1 \leq j \leq n)$. Hence $a_k + 2 = a_k$, a contradiction. Hence we have (6).

We next prove

$$a_l = -1 \quad \text{for} \quad k \leq l \leq n. \tag{7}$$

In fact, if $a_l \neq -1$ for some l with $k \leq l \leq n$, then $(a_1, \ldots, a_n) = \theta(b_1, \ldots, b_n)$ with $b_j \in \mathbf{Z}$ $(1 \leq j \leq n)$, $b_l \neq 0$ for some l with $k \leq l \leq n$. By (5),

$$H(\theta(b_1, \ldots, b_k, -|b_{k+1}|, \ldots, -|b_n|)) \neq 0$$

and hence there exist $a_{k+1}', \ldots, a_n' \in \mathbf{Z}$ such that

$$H(a_1, \ldots, a_k, a_{k+1}', \ldots, a_n') \neq 0$$

and such that $a_l' < a_k$ for some l with $k < l \leq n$. Since $a_j > 0$ and $a_l' \leq -2$, this contradicts (6).

By (7), $a_n = -1$. Hence, by (3), $H(\chi)$ carries a Hodge structure of weight $w-1$, which is odd. This contradicts $\dim H(\chi) = 1$. Hence we have (a) $\Rightarrow$ (c), and Proposition 6.3 is proved. Q.E.D.

6.6. *Classical situation.* Let $F \in D$, and let $T_D(F)$ and $T_D^h(F)$ be the tangent space and horizontal tangent space of D at F, respectively (1.6).

It can be proved that the following (i) and (ii) are equivalent.

(i) For any $F \in D$, $T_D^h(F) = T_D(F)$ and $\dim K'_F = \dim K_F$ (2.2 (1)).

(ii) One of the following (a), (b) is satisfied:

(a) There is $t \in \mathbf{Z}$ such that $w = 2t+1$ and $h^{p,w-p} = 0$ if $p \neq t, t+1$.

(b) There is $t \in \mathbf{Z}$ such that $w = 2t$, $h^{t+1,t-1} \leq \inf\{1, h^{t,t}\}$, and $h^{p,q} = 0$ if $p \geq t+2$.

Note that the condition (i) is independent of the choice of $F \in D$. The equivalence of (i) and (ii) follows by computing dimensions of the subspaces $F^r(\mathfrak{g}_{\mathbf{C}})$ in 1.6 and of the Lie algebras of the following groups.

$$
\begin{aligned}
G_{\mathbf{R}} &\simeq \begin{cases} \mathrm{Sp}(2g, \mathbf{R}) & \text{if } w = 2t+1, \\ O(a, b; \mathbf{R}) & \text{if } w = 2t, \end{cases} \\
K_{\mathbf{r}} &\simeq \begin{cases} U(g) & \text{if } w = 2t+1, \\ O(a, \mathbf{R}) \times O(b, \mathbf{R}) & \text{if } w = 2t, \end{cases} \qquad (1) \\
K'_{\mathbf{r}} &\simeq \begin{cases} \prod_{j \geq 0} U(h^{t+1+j,t-j}) & \text{if } w = 2t+1, \\ \left(\prod_{j>0} U(h^{t+j,t-j})\right) \times O(h^{t,t}, \mathbf{R}) & \text{if } w = 2t, \end{cases}
\end{aligned}
$$

where $g := \operatorname{rank} H_0/2$ if $w = 2t+1$, and a, b are the signature of $(H_{0,\mathbf{R}}, \langle\ ,\ \rangle_0))$ if $w = 2t$. (cf. NOTATION, [U2]).

We say that we are in the *classical situation* if these equivalent conditions (i), (ii) are satisfied. The polarized Hodge structures in (ii) (a) are Tate twists of the first cohomology of polarized abelian varieties, and the primitive part of the second cohomology of a polarized K3 surface belongs to (ii) (b).

Theorem 6.7. *In the classical situation, except in the case* (i) *below, there exist homeomorphisms* $D_{\mathrm{SL}(2)} \xrightarrow{\sim} D_{\mathrm{BS}}$, $D_{\mathrm{SL}(2),\mathrm{val}} \xrightarrow{\sim} D_{\mathrm{BS},\mathrm{val}}$ *extending the identity map of* D.

(i) w *is even,* $\operatorname{rank} H_0 = 4$, *and there exists a* $\mathbf{Q}$*-basis* $(e_j)_{1 \leq j \leq 4}$ *of* $H_{0,\mathbf{Q}}$ *such that* $\langle e_j, e_k \rangle_0 = 1$ *if* $j+k = 5$, *and* $= 0$ *otherwise.*

Proof. In the case $w = 2t$ and $h^{t+s,t-s} = 0$ for any $s \neq t$, D is a one point set and Theorem 6.7 holds trivially. We assume $h^{t+1,t-1} = 1$ in the case $w = 2t$.

We have the following (1).

(1) Let $[\rho, \varphi] \in D_{\mathrm{SL}(2),n}$, and let $W = (W^{(j)})_{1 \leq j \leq n}$ be the associated family of weight filtrations. Let $1 \leq j \leq n$. In the case 6.6 (ii) (a), we have

$$W_{-2}^{(j)} = 0, \ W_1^{(j)} = H_{0,\mathbf{R}}.$$

In the case 6.6 (ii) (b), we have one of the following (b1), (b2).

(b1) $\mathrm{gr}_k^{W^{(j)}} = 0$ unless $k = 0, \pm 1$, and $\dim \mathrm{gr}_k^{W^{(j)}} = 2$ for $k = \pm 1$.

(b2) $\mathrm{gr}_k^{W^{(j)}} = 0$ unless $k = 0, \pm 2$, and $\dim \mathrm{gr}_k^{W^{(j)}} = 1$ for $k = \pm 2$.

This follows from the facts that the filtration $\varphi(\mathbf{i})(\mathrm{gr}_k^{W^{(j)}})$, induced on $\mathrm{gr}_k^{W^{(j)}}$ by $\varphi(\mathbf{i})$, is a Hodge structure of weight $w+k$ for each $k \in \mathbf{Z}$, and that if we denote the Hodge type of this Hodge structure by $(h_k^{p,q})_{p,q \in \mathbf{Z}}$ then $h^{p,w-p} = \sum_k h_k^{p,w+k-p}$.

We next prove the following (2).

(2) Let the notation be as in (1). In the case 6.6 (ii) (a), we have

$$\begin{aligned} 0 \subsetneq W_{-1}^{(1)} \subsetneq W_{-1}^{(2)} \subsetneq \cdots \subsetneq W_{-1}^{(n)} \\ \subset W_0^{(n)} \subsetneq \cdots \subsetneq W_0^{(2)} \subsetneq W_0^{(1)} \subsetneq H_{0,\mathbf{R}}. \end{aligned}$$

In the case 6.6 (ii) (b) with $n \geq 2$, we have $n = 2$, $W^{(1)}$ is of type (b1), $W^{(2)}$ is of type (b2), and

$$\begin{aligned} &0 \subsetneq W_{-2}^{(2)} = W_{-1}^{(2)} \subsetneq W_{-1}^{(1)} \subset W_0^{(1)} \subsetneq W_0^{(2)} = W_1^{(2)} \subsetneq H_{0,\mathbf{R}}, \\ &\dim W_{-2}^{(2)} = \dim W_{-1}^{(1)}/W_{-1}^{(2)} = \dim W_0^{(2)}/W_0^{(1)} \\ &\qquad = \dim H_{0,\mathbf{R}}/W_1^{(2)} = 1. \end{aligned}$$

In fact, in the case 6.6 (ii) (a), since $\mathrm{Ker}(a_1 N_1 + \cdots + a_j N_j) = W_0^{(j)}$ for any $a_1, \ldots, a_j > 0$ (3.5), we have $W_0^{(j')} \supset W_0^{(j)}$ for $1 \leq j' \leq j \leq n$, and hence, by taking $(\ )^{\perp}$, we obtain $W_{-1}^{(j')} \subset W_{-1}^{(j)}$. Since $W^{(j')} \neq W^{(j)}$ for $j' \neq j$, this proves (2) in the case 6.6 (ii) (a).

We consider the case 6.6 (ii) (b). Assume $n \geq 2$. If $1 \leq j \leq n$ and $W^{(j)}$ is of type (b1), $(a_1 N_1 + \cdots + a_j N_j)^2 = 0$ for any $a_1, \ldots, a_j > 0$ and hence $(a_1 N_1 + \cdots + a_{j'} N_{j'})^2 = 0$ for any $j' \leq j$ and any $a_1, \ldots, a_{j'} > 0$.

Hence $W^{(j')}$ for $j' \leq j$ is also of type (b1), and we have $W_0^{(j')} \supsetneq W_0^{(j)}$ for $j' < j$ just as in the case 6.6 (ii) (a). This contradicts the statement about dimensions in (b1) if $j \geq 2$. Hence any $W^{(j)}$ with $j \geq 2$ is of type (b2). If $W^{(j)}$ is of type (b2), $\mathrm{Ker}((a_1N_1 + \cdots + a_jN_j)^2) = W_1^{(j)}$ for any $a_1, \ldots, a_j > 0$. Hence, if $j' < j$ and $W^{(j')}$ and $W^{(j)}$ are of type (b2), we have $W_1^{(j')} \supset W_1^{(j)}$. Since $W^{(j')} \neq W^{(j)}$, we have $W_1^{(j')} \supsetneq W_1^{(j)}$ which contradicts the statement about dimensions in (b2). Hence we have $n = 2$, $W^{(1)}$ is of type (b1), and $W^{(2)}$ is of type (b2). By (1), it remains to prove $W_{-1}^{(1)} \supset W_{-2}^{(2)}$ ($W_0^{(1)} \supset W_1^{(2)}$ follows from this by taking $(\ )^{\perp}$). The condition on the Hodge numbers (ii) (b) shows that the signature of $(H_{0,\mathbf{R}}, \langle\ ,\ \rangle_0)$ is $(d-2, 2)$, where $d = \mathrm{rank}\, H_0$, and hence $H_{0,\mathbf{R}}$ has no 3-dimensional isotropic subspace, *i.e.*, $\mathbf{R}$-subspace on which the restriction of $\langle\ ,\ \rangle_0$ is zero. However, if $W_{-1}^{(1)} \not\supset W_{-1}^{(2)}$, $W_{-1}^{(1)} + W_{-2}^{(2)}$ is a 3-dimensional isotropic subspace as is shown in the following way. From $(aN_1 + bN_2)^3 = 0$ for $a, b \in \mathbf{R}_{>0}$, we have $N_1^iN_2^j = 0$ if $i \geq 0$, $j \geq 0$, $i + j \geq 3$. Since $W_{-1}^{(1)} = \mathrm{Im}\, N_1$ and $W_{-2}^{(2)} = \mathrm{Im}(N_1 + N_2)^2$, by $\langle hx, y\rangle_0 + \langle x, hy\rangle_0 = 0$ for $h \in \mathfrak{g}_{\mathbf{R}}$, we see that $W_{-1}^{(1)} + W_{-2}^{(2)}$ is an isotropic subspace. This proves (2) in the case 6.6 (ii) (b).

By (1) and (2), we have the following.

(3) In the classical situation, the set $\{W_k^{(j)} \mid 1 \leq j \leq n, k \in \mathbf{Z}\}$ is totally ordered, and $(G^{\circ})_{W,\mathbf{R}}$ is a parabolic subgroup of $G_{\mathbf{R}}$.

Hence, by 6.3, we have a continuous map $D_{\mathrm{SL}(2)} \to D_{\mathrm{BS}}$ which extends the identity map of D.

We prove

(4) $D_{\mathrm{SL}(2)} \to D_{\mathrm{BS}}$ is injective.

By 3.10, an SL(2)-orbit (ρ, φ) of rank n is characterized by the associated $(W, \mathbf{r})$. Assume that the points of $D_{\mathrm{SL}(2)}$ determined by $(W, \mathbf{r})$, $(W', \mathbf{r}')$ are sent to the same point in D_{BS}. Then we have

$$(G^{\circ})_{W',\mathbf{R}} = (G^{\circ})_{W,\mathbf{R}}, \quad \mathbf{r}' = a \circ \mathbf{r}\ \ (\exists a \in A_{(G^{\circ})_{W,\mathbf{R}}}).$$

The totally ordered set $\{W_k^{(j)} \mid 1 \leq j \leq n, k \in \mathbf{Z}\}$ is not exceptional (2.11). In fact, if it is exceptional, then $w = 2t$ ($\exists t \in \mathbf{Z}$), $d := \mathrm{rank}\, H_0$ is even, and there exists a $\mathbf{Q}$-basis $(e_j)_{1 \leq j \leq d}$ of $H_{0,\mathbf{Q}}$ such that

$$\langle e_j, e_k\rangle_0 = \begin{cases} 1 & \text{if } j + k = d + 1, \\ 0 & \text{otherwise.} \end{cases}$$

Hence, the signature of $(H_{0,\mathbf{R}}, \langle\ ,\ \rangle_0)$ is $(d/2, d/2)$. On the other hand, it was $(d-2, 2)$, and hence we have $d = 4$ and this would imply that we were in the case (i).

Hence, it follows $W' = W$ from $(G^\circ)_{W',\mathbf{R}} = (G^\circ)_{W,\mathbf{R}}$ and (2). We see also, by (2), that the torus $S_{(G^\circ)_{W,\mathbf{R}}}$ in 2.2 coincides with the torus of the SL(2)-orbit of rank n determined by $(W, \mathbf{r})$. Hence $\mathbf{r}' = a \circ \mathbf{r}$ lies on the torus orbit containing $\mathbf{r}$ of this SL(2)-orbit of rank n. Thus, the points in $D_{\mathrm{SL}(2)}$ determined by $(W, \mathbf{r})$, $(W', \mathbf{r}')$ coincide, as desired.

We prove

(5) $D_{\mathrm{SL}(2)} \to D_{\mathrm{BS}}$ is surjective.

Let $(P, Z) \in D_{\mathrm{BS}}$. Let $M = (0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_m = H_{0,\mathbf{R}}) \in \mathcal{M}$ be a $\mathbf{Q}$-rational increasing filtration of $H_{0,\mathbf{R}}$ such that $P = p(M)$ (see 6.4). Let $n = m/2$ if m is even, and $n = (m-1)/2$ if m is odd.

We first prove (5) in the case 6.6 (ii) (a). Let $e(j) := \dim M_j/M_{j-1}$ for $1 \le j \le n$, and let $e := \sum_{1 \le j \le n} e(j)$. Fix a polarized Hodge structure $(H_1, \langle\ ,\ \rangle_1, F_1)$ of weight 1 whose Hodge type $(h_1^{p,q})_{p,q \in \mathbf{Z}}$ is given by

$$h_1^{p,q} = g - e \ \text{ if } (p,q) = (1,0) \text{ or } (0,1), \quad h_1^{p,q} = 0 \ \text{ otherwise.}$$

Fix an isomorphism

$$\left(H_{\mathfrak{h},\mathbf{Q}}^{\oplus e} \oplus H_{1,\mathbf{Q}}, \langle\ ,\ \rangle_{\mathfrak{h}}^{\oplus e} \oplus \langle\ ,\ \rangle_1\right) \simeq (H_{0,\mathbf{Q}}, \langle\ ,\ \rangle_0),$$

where $(H_{\mathfrak{h},\mathbf{Q}}, \langle\ ,\ \rangle_{\mathfrak{h}})$ is as in 6.2, and take this isomorphism as an identification. Let (ρ, φ) be the SL(2)-orbit of rank n defined by

$$\rho(g_1, \ldots, g_n) := \left(\bigoplus_{1 \le j \le n} \rho_{\mathfrak{h}}(g_j)^{\oplus e(j)}\right) \oplus \mathrm{id},$$
$$\varphi(z_1, \ldots, z_n) := \left(\bigoplus_{1 \le j \le n} \varphi_{\mathfrak{h}}(z_j)^{\oplus e(j)}\right)(-t) \oplus F_1(-t),$$

where $(\rho_{\mathfrak{h}}, \varphi_{\mathfrak{h}})$ is the standard SL(2)-orbit in 6.2, and $(-t)$ means the Tate twist. Then the family W of weight filtrations of (ρ, φ) satisfies $M = \{W_k^{(j)} \mid 1 \le j \le n, k \in \mathbf{Z}\}$ and hence $(G^\circ)_{W,\mathbf{R}} = P$. Let $\mathbf{r} = \varphi(\mathbf{i})$. Since $K_{\mathbf{r}} = K'_{\mathbf{r}}$, we have $D = P \cdot \mathbf{r}$ by $G_{\mathbf{R}} = PK_{\mathbf{r}}$, and hence tere is $p \in P$ such that $p\mathbf{r} \in Z$. The group $P = (G^\circ)_{W,\mathbf{R}}$ acts on $D_{\mathrm{SL}(2)}(W)$, and the image of $p[\rho, \varphi] \in D_{\mathrm{SL}(2)}(W)$ in D_{BS} is (P, Z).

Next we prove (5) in the case 6.6 (ii) (b). Since $G_{\mathbf{R}} \simeq O(h^{t,t}, 2; \mathbf{R})$, if $P \neq G_{\mathbf{R}}$ then we have one of the following (c), (d), (e).

(c) $n = 1$, $\dim M_1 = 2$.

(d) $n = 1$, $\dim M_1 = 1$.

(e) $n = 2$, $\dim M_1 = \dim M_2/M_1 = 1$.

In the case (d), since $G_{\mathbf{R}} \simeq O(h^{t,t}, 2; \mathbf{R})$ and $\dim M_1 < 2$, there is an element $l \in M_{m-1} \cap H_{0,\mathbf{Q}}$ such that $\langle l, l\rangle_0 < 0$. Fix such l. Take a $\mathbf{Q}$-subspace L of $H_{0,\mathbf{Q}}$ such that

$$\begin{aligned} &M_{m-1} = M_1 \oplus L_{\mathbf{R}} && \text{in the case (c),}\\ &M_{m-1} \cap (l^{\perp}) = M_1 \oplus L_{\mathbf{R}} && \text{in the case (d),}\\ &M_{m-2} = M_2 \oplus L_{\mathbf{R}} && \text{in the case (e).} \end{aligned}$$

Then, $\dim_{\mathbf{Q}} L = d - 4$ in the cases (c), (e), $\dim_{\mathbf{Q}} L = d - 3$ in the case (d) ($d = \operatorname{rank} H_0$), and the restriction of $\langle\ ,\ \rangle_0$ to L is non-degenerate in any case. Fix a polarized Hodge structure $(H_1, \langle\ ,\ \rangle_1, F_1)$ of weight w satisfying the following conditions.

$$\begin{aligned} &(H_{1,\mathbf{Q}}, \langle\ ,\ \rangle_1) \simeq (L, \text{the restriction of } \langle\ ,\ \rangle_0),\\ &h_1^{p,q} = 0 \ \text{ for } (p,q) \neq (t,t). \end{aligned}$$

In the case (c), fix also a polarized Hodge structure $(H_2, \langle\ ,\ \rangle_2, F_2)$ of weight 1 whose Hodge type $(h_2^{p,q})_{p,q\in\mathbf{Z}}$ is given by

$$h_2^{p,q} = 1 \ \text{ if } (p,q) = (1,0) \text{ or } (0,1), \quad h_2^{p,q} = 0 \ \text{ otherwise.}$$

Then $(H_{0,\mathbf{Q}}, \langle\ ,\ \rangle_0)$ is isomorphic to

$$\begin{aligned} &\left(H_{\mathfrak{h},\mathbf{Q}} \otimes_{\mathbf{Q}} H_{2,\mathbf{Q}} \oplus H_{1,\mathbf{Q}},\ \langle\ ,\ \rangle_{\mathfrak{h}} \otimes \langle\ ,\ \rangle_2 \oplus \langle\ ,\ \rangle_1\right) && \text{in the case (c),}\\ &\left(\operatorname{Sym}^2_{\mathbf{Q}}(H_{\mathfrak{h},\mathbf{Q}}) \oplus H_{1,\mathbf{Q}},\ -\langle l,l\rangle_0 \operatorname{Sym}^2(\langle\ ,\ \rangle_{\mathfrak{h}}) \oplus \langle\ ,\ \rangle_1\right) && \text{in the case (d),}\\ &\left(H_{\mathfrak{h},\mathbf{Q}}^{\otimes 2} \oplus H_{1,\mathbf{Q}},\ \langle\ ,\ \rangle_{\mathfrak{h}}^{\otimes 2} \oplus \langle\ ,\ \rangle_1\right) && \text{in the case (e).} \end{aligned}$$

Here $\operatorname{Sym}^k(\langle\ ,\ \rangle_{\mathfrak{h}})$ is defined by

$$\left(\textstyle\prod_{1\le j\le k} x_j, \prod_{1\le j\le k} y_j\right) \mapsto \sum_{\sigma\in\mathfrak{S}_k} \prod_{1\le j\le k} \langle x_j, \sigma y_j\rangle_{\mathfrak{h}},$$

and $-\langle l,l\rangle_0 \operatorname{Sym}^2(\langle\ ,\ \rangle_{\mathfrak{h}})$ means $-\langle l,l\rangle_0$ times $\operatorname{Sym}^2(\langle\ ,\ \rangle_{\mathfrak{h}})$. We fix this isomorphism and take it as an identification. Let (ρ, φ) be the SL(2)-orbit of rank 1 in the cases (c), (d), and the SL(2)-orbit of rank 2 in the case (e), defined respectively by

$$\begin{cases} \rho(g) := \rho_{\mathfrak{h}}(g) \otimes 1_{H_2} \oplus 1_{H_1},\\ \varphi(z) := (\varphi_{\mathfrak{h}}(z) \otimes F_2)(1-t) \oplus F_1 \end{cases} \quad \text{in the case (c),}$$

$$\begin{cases} \rho(g) := \operatorname{Sym}^2(\rho_{\mathfrak{h}}(g)) \oplus 1_{H_1},\\ \varphi(z) := (\operatorname{Sym}^2(\varphi_{\mathfrak{h}}(z)))(1-t) \oplus F_1 \end{cases} \quad \text{in the case (d),}$$

$$\begin{cases} \rho(g_1, g_2) := \rho_{\mathfrak{h}}(g_1) \otimes \rho_{\mathfrak{h}}(g_2) \oplus 1_{H_1},\\ \varphi(z_1, z_2) := (\varphi_{\mathfrak{h}}(z_1) \otimes \varphi_{\mathfrak{h}}(z_2))(1-t) \oplus F_1, \end{cases} \quad \text{in the case (e).}$$

Then the family W of weight filtrations of (ρ, φ) satisfies $M = \{W_k^{(j)} \mid 1 \leq j \leq n, k \in \mathbf{Z}\}$ and hence $(G^\circ)_{W,\mathbf{R}} = P$. It can be checked that $G_{\mathbf{R}} = G_{W,\mathbf{R}} K'_{\mathbf{r}}$, where $\mathbf{r} = \varphi(\mathbf{i})$. This implies $D = G_{W,\mathbf{R}} \cdot \mathbf{r}$, and hence there is $p \in G_{W,\mathbf{R}}$ such that $p \cdot \mathbf{r} \in Z$. The group $G_{W,\mathbf{R}}$ acts on $D_{\mathrm{SL}(2)}(W)$, and the image of $p[\rho, \varphi] \in D_{\mathrm{SL}(2)}(W)$ in D_{BS} is (P, Z).

Finally we prove

(6) $D_{\mathrm{SL}(2)} \to D_{\mathrm{BS}}$ and $D_{\mathrm{SL}(2),\mathrm{val}} \to D_{\mathrm{BS},\mathrm{val}}$ are homeomorphisms.

From the coincidence of the tori in the proof of (4), we see that, for $x \in D_{\mathrm{BS}}$, the map from the inverse image of x in $D_{\mathrm{SL}(2),\mathrm{val}}$ to the the inverse image of x in $D_{\mathrm{BS},\mathrm{val}}$ is bijective. Hence $D_{\mathrm{SL}(2),\mathrm{val}} \to D_{\mathrm{BS},\mathrm{val}}$ is bijective. By (3), this map is a homeomorphism. This shows that the bijection $D_{\mathrm{SL}(2)} \to D_{\mathrm{BS}}$ is also a homeomorphism. Q.E.D.

6.8. *Remark.* In the case 6.7 (i), we have $D_{\mathrm{SL}(2),\mathrm{val}} = D_{\mathrm{BS},\mathrm{val}}$ as topological spaces, and we have a continuous surjection $D_{\mathrm{SL}(2)} \to D_{\mathrm{BS}}$ extending the identity map of D. This map $D_{\mathrm{SL}(2)} \to D_{\mathrm{BS}}$ is not injective. In fact,

$$(H_{0,\mathbf{Q}}, \langle\ ,\ \rangle_0) \simeq (H_{\mathfrak{h},\mathbf{Q}} \otimes_{\mathbf{Q}} H_{\mathfrak{h},\mathbf{Q}}, \langle\ ,\ \rangle_{\mathfrak{h}} \otimes \langle\ ,\ \rangle_{\mathfrak{h}}),$$

and if we take this isomorphism as an identification, we have two $\mathrm{SL}(2)^2$-orbits (ρ, φ), (ρ', φ') defined by

$$\rho(g_1, g_2) := \rho_{\mathfrak{h}}(g_1) \otimes \rho_{\mathfrak{h}}(g_2), \quad \varphi(z_1, z_2) := (\varphi(z_1) \otimes \varphi(z_2))(1 - t),$$
$$\rho'(g_1, g_2) := \rho_{\mathfrak{h}}(g_2) \otimes \rho_{\mathfrak{h}}(g_1), \quad \varphi'(z_1, z_2) := (\varphi(z_2) \otimes \varphi(z_1))(1 - t),$$

whose images in D_{BS} coincide but $[\rho, \varphi] \neq [\rho', \varphi']$.

6.9. *Relation with Satake compactifications.* In the classical situation, we have a compactification $\Gamma\backslash D_S$ of $\Gamma\backslash D$ defined by Satake for a subgroup Γ of $G_{\mathbf{Z}}$ of finite index ([Sa]). The space D_S is the set of all pairs (W, F), where W is a $\mathbf{Q}$-rational increasing filtration of $H_{0,\mathbf{R}}$ and $F = (F_{(j)})_{j \in \mathbf{Z}}$ is a family of decreasing filtrations $F_{(j)}$ of the $\mathbf{C}$-vector spaces $\mathbf{C} \otimes_{\mathbf{R}} \mathrm{gr}_j^W$ $(j \in \mathbf{Z})$, satisfying the following condition (i).

(i) There exist an integer $n \geq 0$ and an element $[\rho, \varphi]$ of $D_{\mathrm{SL}(2),n}$ such that the n-th weight filtration $W(N_1 + \cdots + N_n)$ of $[\rho, \varphi]$ coincides with W, and such that, for some $\tilde{F} \in \varphi(\mathbf{C}^n) \subset \check{D}$, the filtration of $\mathbf{C} \otimes_{\mathbf{R}} \mathrm{gr}_j^W$ induced by $\tilde{F}$ (which is independent of the choice of $\tilde{F}$) coincides with $F_{(j)}$ for any $j \in \mathbf{Z}$.

Except the case 6.7 (i), by composing the evident surjection $D_{\mathrm{SL}(2)} \to D_S$ with the isomorphism $D_{\mathrm{SL}(2)} \simeq D_{\mathrm{BS}}$ in 6.7, we obtain a canonical surjection $D_{\mathrm{BS}} \to D_S$. (In the case 6.7 (i), by using (1) and (2) in

the proof of Theorem 6.7, we can see that the map $D_{\mathrm{BS}} \to D_S$ factors through the surjection $D_{\mathrm{SL}(2)} \to D_{\mathrm{BS}}$.) This map $D_{\mathrm{BS}} \to D_S$ was defined by Zucker [Z2] by another method.

Propositon 6.10. *Assume one of the following* (i), (ii) *is satisfied for some* $t \in \mathbf{Z}$.

(i) $w = 2t+1$, $h^{t+1,t} \geq 2$, $h^{t+2,t-1} \neq 0$.

(ii) $w = 2t$, $h^{t,t} \geq 3$, $h^{t+1,t-1} \geq 2$, *and there is a* $\mathbf{Q}$*-vector subspace of* $H_{0,\mathbf{Q}}$ *of dimension* 3 *on which the restriction of* $\langle\ ,\ \rangle_0$ *is zero.*

Then there is no continuous map $D_{\mathrm{SL}(2)} \to D_{\mathrm{BS}}$ *which extends the identity map of* D.

Proof. First we consider the case (i). Fix a polarized Hodge structure $(H_1, \langle\ ,\ \rangle_1, F_1)$ of weight w whose Hodge type $(h_1^{p,q})_{p,q\in\mathbf{Z}}$ is given by

$$h_1^{p,q} = h^{p,q} - \begin{cases} 2 & \text{if } (p,q) = (t+1,t) \text{ or } (t,t+1), \\ 1 & \text{if } (p,q) = (t+2,t-1) \text{ or } (t-1,t+2), \\ 0 & \text{otherwise.} \end{cases}$$

Fix an isomorphism

$$\begin{aligned} &\left(H_{\mathfrak{h},\mathbf{Q}} \otimes_{\mathbf{Q}} \mathrm{Sym}^2_{\mathbf{Q}}(H_{\mathfrak{h},\mathbf{Q}}) \oplus H_{1,\mathbf{Q}},\ \langle\ ,\ \rangle_{\mathfrak{h}} \otimes \mathrm{Sym}^2(\langle\ ,\ \rangle_{\mathfrak{h}}) \oplus \langle\ ,\ \rangle_1\right) \\ &\simeq (H_{0,\mathbf{Q}}, \langle\ ,\ \rangle_0), \end{aligned}$$

where $(H_{\mathfrak{h},\mathbf{Q}}, \langle\ ,\ \rangle_{\mathfrak{h}})$ is as in 6.2, and take this as an identification. Let (ρ, φ) be the SL(2)-orbit of rank 2 defined by

$$\begin{aligned} \rho(g_1, g_2) &:= \rho_{\mathfrak{h}}(g_1) \otimes \mathrm{Sym}^2(\rho_{\mathfrak{h}}(g_2)) \oplus 1_{H_1}, \\ \varphi(z_1, z_2) &:= \varphi_{\mathfrak{h}}(z_1) \otimes \mathrm{Sym}^2(\varphi_{\mathfrak{h}}(z_2))(1-t) \oplus F_1. \end{aligned}$$

Then this SL(2)-orbit of rank 2 does not satisfy the condition 6.3 (i) (c). In fact, $\rho(\Delta(t_1,t_2))$ acts on $(e_1 \otimes e_2^2, 0)$ (resp. $(e_2 \otimes e_1^2, 0)$) by $t_1^{-1}t_2^2$ (resp. $t_1 t_2^{-2}$), hence $\tilde{\rho}(t_1,t_2)$ acts on $(e_1 \otimes e_2^2, 0)$ (resp. $(e_2 \otimes e_1^2, 0)$) by $t_1^{-1}t_2$ (resp. $t_1 t_2^{-1}$). It follows $(e_1 \otimes e_2^2, 0) \in H(-1,1)$ and $(e_2 \otimes e_1^2, 0) \in H(1,-1)$.

Next we consider the case (ii). Let L be a $\mathbf{Q}$-vector subspace of $H_{0,\mathbf{Q}}$ of dimension 3 on which the restriction of $\langle\ ,\ \rangle_0$ is zero. Since $h^{t+1,t-1} + h^{t-1,t+1} > 3$, there is an element $l \in L^{\perp} \subset H_{0,\mathbf{Q}}$ such that $\langle l, l\rangle_0 < 0$. Let L' be a $\mathbf{Q}$-subspace of $(L + \mathbf{Q}l)^{\perp} \subset H_{0,\mathbf{Q}}$ such that $L \oplus L' = (L + \mathbf{Q}l)^{\perp}$. Then we have $\dim_{\mathbf{Q}} L' = \dim_{\mathbf{Q}} H_{0,\mathbf{Q}} - 7$, and the restriction of $\langle\ ,\ \rangle_0$ to $L'_{\mathbf{C}}$ is non-degenerate. Fix polarized Hodge structures $(H_1, \langle\ ,\ \rangle_1, F_1)$ of

weight w and $(H_2, \langle\ ,\ \rangle_2, F_2)$ of weight 1 having the following properties: The Hodge types $(h_j^{p,q})_{p,q\in\mathbf{Z}}$ of $(H_j, \langle\ ,\ \rangle_j, F_j)$ for $j = 1, 2$ are given by

$$h_1^{p,q} = h^{p,q} - \begin{cases} 2 & \text{if } (p,q) = (t,t), \\ 1 & \text{if } (p,q) = (t+1, t-1) \text{ or } (t-1, t+1), \\ 0 & \text{otherwise.} \end{cases}$$

$$h_2^{p,q} = \begin{cases} 1 & \text{if } (p,q) = (1,0) \text{ or } (0,1), \\ 0 & \text{otherwise,} \end{cases}$$

and

$$(H_{1,\mathbf{Q}}, \langle\ ,\ \rangle_1) \simeq (L', \text{the restriction of } \langle\ ,\ \rangle_0).$$

Then there is an isomorphism

$$\begin{aligned} &\big(H_{\mathfrak{h},\mathbf{Q}} \otimes_{\mathbf{Q}} H_{2,\mathbf{Q}} \oplus \mathrm{Sym}^2_{\mathbf{Q}}(H_{\mathfrak{h},\mathbf{Q}}) \oplus H_{1,\mathbf{Q}}, \\ &\qquad \langle\ ,\ \rangle_{\mathfrak{h}} \otimes \langle\ ,\ \rangle_2 \oplus (-\langle l, l\rangle_0 \,\mathrm{Sym}^2(\langle\ ,\ \rangle_{\mathfrak{h}})) \oplus \langle\ ,\ \rangle_1\big) \\ &\simeq (H_{0,\mathbf{Q}}, \langle\ ,\ \rangle_0). \end{aligned}$$

We take this as an identification. Let (ρ, φ) be the SL(2)-orbit of rank 2 defined by

$$\begin{aligned} &\rho(g_1, g_2) := \rho_{\mathfrak{h}}(g_1) \otimes 1_{H_2} \oplus \mathrm{Sym}^2(\rho_{\mathfrak{h}}(g_2)) \oplus 1_{H_1}, \\ &\varphi(z_1, z_2) := (\varphi_{\mathfrak{h}}(z_1) \otimes F_2)(1-t) \oplus \mathrm{Sym}^2(\varphi_{\mathfrak{h}}(z_2))(1-t) \oplus F_1. \end{aligned}$$

Then this SL(2)-orbit of rank 2 does not satisfy the condition in 6.3 (i) (c). In fact, for any element $x \in H_2$, $(e_2 \otimes x, 0, 0) \in H(1,1)$ and $(0, e_2^2, 0) \in H(0,2)$. Q.E.D.

Example 6.11 and Proposition 6.12 below show that, for a subgroup Γ of $G_{\mathbf{Z}}$ of finite index, $\Gamma\backslash D_{\mathrm{SL}(2)}$ is not necessarily compact in general, and furthermore not necessarily locally compact in general.

6.11. *Example.* Consider the case $h^{5,0} = h^{0,5} = 1$ and $h^{p,q} = 0$ otherwise. This is satisfied by the polarized Hodge structure associated to a modular form of weight 6. In this case, D is identified with the upper half plane $\mathfrak{h}$, which is the Griffiths domain of the case $h^{1,0} = h^{0,1} = 1$ and $h^{p,q} = 0$ otherwise. We have $D_{\mathrm{BS}} = \mathfrak{h}_{\mathrm{BS}}$, but $D_{\mathrm{SL}(2)} = \mathfrak{h}$, as follows from the condition 3.1 (ii).

Propositon 6.12. *Assume one of the following* (i), (ii) *is satisfied for some* $t \in \mathbf{Z}$.

(i) $w = 2t+1$, $h^{t+1,t} \neq 0$, *and* $h^{s,w-s} \neq 0$ *for some* $s > t+1$.

(ii) $w = 2t$, $h^{t,t} \geq 2$, $h^{t+1,t-1} \geq 1$, $\sum_{j\geq 1} h^{t+j,t-j} \geq 2$, *and there is a* $\mathbf{Q}$*-vector subspace of* $H_{0,\mathbf{Q}}$ *of dimension* 2 *on which the restriction of* $\langle\ ,\ \rangle_0$ *is zero.*

Then $D_{\mathrm{SL}(2)}$ *is not locally compact. More precisely, there are an open set* U *of* D_{BS} *and* V *of* $D_{\mathrm{SL}(2)}$ *such that the inverse image* U' *of* U *in* $D_{\mathrm{BS,val}}$ *and the inverse image* V' *of* V *in* $D_{\mathrm{SL}(2),\mathrm{val}}$ *satisfy*

$$U' \xrightarrow{\sim} U,\quad V' \xrightarrow{\sim} V,\quad V' = U' \cap D_{\mathrm{SL}(2),\mathrm{val}},$$

and such that there are integers $m > l \geq 0$ *and a commutative diagram*

$$\begin{array}{ccc} U' & \simeq & \mathbf{R}^m \times \mathbf{R}_{\geq 0} \\ \cup & & \cup \\ V' & \simeq & (\mathbf{R}^m \times \mathbf{R}_{>0}) \cup (\mathbf{R}^l \times 0) \\ \cup & & \cup \\ U' \cap D & \simeq & \mathbf{R}^m \times \mathbf{R}_{>0}. \end{array}$$

Note that the subspace $(\mathbf{R}^m \times \mathbf{R}_{>0}) \cup (\mathbf{R}^l \times 0)$ *of* $\mathbf{R}^m \times \mathbf{R}_{\geq 0}$ *is not locally compact.*

Proof. Fix a $\mathbf{Q}$-rational $\mathbf{R}$-subspace L of $H_{0,\mathbf{R}}$ satisfying the following condition. In the case (i), $\dim L = 1$. In the case (ii), $\dim L = 2$ and $\langle\ ,\ \rangle_0$ is zero on L. Let P be the $\mathbf{Q}$-parabolic subgroup $\{g \in (G^\circ)_{\mathbf{R}} \mid gL = L\}$ of $G_{\mathbf{R}}$, and let W be the $\mathbf{Q}$-rational filtration of $H_{0,\mathbf{R}}$ defined by

$$W_{-2} := 0 \subset W_{-1} := L \subset W_0 := L^\perp \subset W_1 := H_{0,\mathbf{R}}.$$

Then we have

$$(G^\circ)_{W,\mathbf{R}} = P,\quad D_{\mathrm{BS,val}}(P) \cap D_{\mathrm{SL}(2),\mathrm{val}} = D_{\mathrm{SL}(2),\mathrm{val}}(W),$$
$$D_{\mathrm{BS,val}}(P) \xrightarrow{\sim} D_{\mathrm{BS}}(P),\quad D_{\mathrm{SL}(2),\mathrm{val}}(W) \xrightarrow{\sim} D_{\mathrm{SL}(2)}(W).$$

On the other hand, in the case (i), fix a polarized Hodge structure $(H_1, \langle\ ,\ \rangle_1, F_1)$ of weight w whose Hodge type $(h_1^{p,q})_{p,q\in\mathbf{Z}}$ is given by

$$h_1^{p,q} = h^{p,q} - \begin{cases} 1 & \text{if } (p,q) = (t+1,t) \text{ or } (t,t+1), \\ 0 & \text{otherwise.} \end{cases}$$

In the case (ii), fix polarized Hodge structures $(H_j, \langle\ ,\ \rangle_j, F_j)$ $(j = 1, 2)$ of weight w for $j = 1$ and of weight 1 for $j = 2$, respectively, having the following properties: The Hodge types $(h_j^{p,q})_{p,q\in\mathbf{Z}}$ of $(H_j, \langle\ ,\ \rangle_j, F_j)$ for

$j = 1, 2$ are given by

$$h_1^{p,q} = h^{p,q} - \begin{cases} 2 & \text{if } (p,q) = (t,t), \\ 1 & \text{if } (p,q) = (t+1, t-1) \text{ or } (t-1, t+1), \\ 0 & \text{otherwise.} \end{cases}$$

$$h_2^{p,q} = \begin{cases} 1 & \text{if } (p,q) = (1,0) \text{ or } (0,1), \\ 0 & \text{otherwise,} \end{cases}$$

and

$$(H_{1,\mathbf{Q}}, \langle\ ,\ \rangle_1) \simeq (L', \text{the restriction of } \langle\ ,\ \rangle_0)$$

for some $\mathbf{Q}$-subspace L' of $H_{0,\mathbf{Q}}$ such that $L \oplus L'_{\mathbf{R}} = L^{\perp}$. Then we have an isomorphism

$$(H_{0,\mathbf{Q}}, \langle\ ,\ \rangle_0)$$
$$\simeq \begin{cases} (H_{\mathfrak{h},\mathbf{Q}} \oplus H_{1,\mathbf{Q}}, \langle\ ,\ \rangle_{\mathfrak{h}} \oplus \langle\ ,\ \rangle_1) & \text{in the case (i),} \\ (H_{\mathfrak{h},\mathbf{Q}} \otimes_{\mathbf{Q}} H_{2,\mathbf{Q}} \oplus H_{1,\mathbf{Q}}, \langle\ ,\ \rangle_{\mathfrak{h}} \otimes \langle\ ,\ \rangle_2 \oplus \langle\ ,\ \rangle_1) & \text{in the case (ii),} \end{cases}$$

which sends L onto $\mathbf{Q}e_1 \oplus \{0\}$ in the case (i) (resp. $e_1 \otimes H_{2,\mathbf{Q}} \oplus \{0\}$ in the case (ii)) and L' onto $H_{1,\mathbf{Q}}$. Fix this isomorphism and take it as an identification. Let (ρ, φ) be the SL(2)-orbit defined by

$$\begin{cases} \rho(g) := \rho_{\mathfrak{h}}(g) \oplus 1_{H_1} \\ \varphi(z) := \varphi_{\mathfrak{h}}(z)(1-t) \oplus F_1 \end{cases} \quad \text{in the case (i),}$$

$$\begin{cases} \rho(g) := \rho_{\mathfrak{h}}(g) \otimes 1_{H_2} \oplus 1_{H_1}, \\ \varphi(z) := (\varphi_{\mathfrak{h}}(z) \otimes F_2)(1-t) \oplus F_1, \end{cases} \quad \text{in the case (ii).}$$

We claim

$$D_{\mathrm{SL}(2),\mathrm{val}}(W) = D \cup P[\rho, \varphi] \quad \text{in } D_{\mathrm{BS},\mathrm{val}}(P). \tag{1}$$

In fact, let (ρ', φ') be an SL(2)-orbit of rank 1 whose weight filtration is W. Since $\tilde{\rho}$ and $\tilde{\rho}'$ split W, there is an element $p \in P_u$ such that $\tilde{\rho}' = \mathrm{Int}(p)\tilde{\rho}$. The Hodge types $\varphi(i)(\mathrm{gr}_j^W)$ and of $\varphi'(i)(\mathrm{gr}_j^W)$ coincide for each j. (In the case (i) (resp. (ii)), it is $(t+1, t+1)$ (resp. $(t+1, t) + (t, t+1)$) for $j = 1$, (t, t) (resp. $(t, t-1) + (t-1, t)$) for $j = -1$, and $(h_1^{p,q})$ for $j = 0$.) Hence by [U1, 3.16 (iii)], there is an element $q \in (G^{\circ})_{\mathbf{R}}$ which commutes with $\tilde{\rho}' = \mathrm{Int}(p)\tilde{\rho}$ and satisfies $\varphi'(\mathbf{i}) = qp\varphi(\mathbf{i})$. By 3.10, we have $\rho' = \mathrm{Int}(qp)\rho$, $\varphi' = qp\varphi$. Since $qp \in P$, this proves (1).

Let D' be the subspace of D defined in 2.16 with respect to the maximal compact subgroup $K_{\mathbf{r}}$ of $G_{\mathbf{R}}$. Then, (1) shows that we have a commutative diagram of topological spaces

$$\begin{array}{ccccc} D_{\mathrm{BS,val}}(P) & = & D_{\mathrm{BS}}(P) & \simeq & D' \times \mathbf{R}_{\geq 0} \\ \cup & & & & \cup \\ D_{\mathrm{SL}(2),\mathrm{val}}(W) & = & D_{\mathrm{SL}(2)}(W) & \simeq & (D' \times \mathbf{R}_{>0}) \cup ({}^{\circ}P \cdot \mathbf{r} \times 0), \end{array}$$

where ${}^{\circ}P$ is as in 2.16 and $[\rho, \varphi] \in D_{\mathrm{SL}(2)}(W)$ corresponds to $\mathbf{r} \times 0$. Let

$$\begin{aligned} m &:= \dim D' = \dim D - 1, \\ l &:= \dim({}^{\circ}P \cdot \mathbf{r}) = \dim(P \cdot \mathbf{r}) - 1. \end{aligned}$$

By [U2, 3.12 (iii)], we have $\dim(P \cdot \mathbf{r}) < \dim D$ under the assumption of Proposition 6.12. This proves that the statement of 6.12 holds for some neighborhood U of the image of $[\rho, \varphi]$ in D_{BS}. Q.E.D.

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The Moduli Space of Curves of Genus 4 and Deligne-Mostow's Complex Reflection Groups

Shigeyuki Kondō

§1. Introduction

In this note we shall show that the moduli space of curves of genus 4 is birational to an arithmetic quotient of the 9-dimensional complex ball and that the arithmetic subgroup is commensurable with one of Deligne-Mostow's complex reflection groups related to hypergeometric functions. Let C be a non-hyperelliptic curve of genus 4. Then its canonical model is the intersection of a quadric Q and a cubic S in $\mathbf{P}^3$. Let X be the minimal resolution of the triple cover of Q branced along C which is a $K3$ surface with an automorphism σ of order 3. The period domain of the pairs (X, σ) is a 9-dimensional complex ball $\mathcal{B}$. This gives an isomorphism between the moduli space of non-hyperelliptic curves of genus 4 and an arithmetic quotient $(\mathcal{B} \setminus \mathcal{H})/\Gamma$ where $\mathcal{H}$ is the union of hyperplanes of $\mathcal{B}$ and Γ is an arithmetic subgroup of $Aut(\mathcal{B})$ (§2, Theorem 1). We remark that $\mathcal{H}$ consists of two components $\mathcal{H}_n$ and $\mathcal{H}_h$ so that a generic point of $\mathcal{H}_n$ (resp. $\mathcal{H}_h$) corresponds to a curve of genus 4 with a node (resp. a hyperelliptic curve of genus 4 plus a point on the quotient of the hyperelliptic curve by the hyperelliptic involution) (§3, Theorem 2). The method works in some other cases, for example, the moduli space of universal curves of genus 2, 3 or del Pezzo surfaces of degree 1–4 (see Remarks 1–6).

The above $K3$ surface X has the structure of an elliptic fibration $\pi : X \longrightarrow \mathbf{P}^1$ which is induced from a ruling on Q. The automorphism σ acts on each fiber of π as an automorphism of order 3, and hence the functional invariant of π is constant. Moreover, for a generic X, this fibration has twelve singular fibers of type II in the sense of Kodaira [Ko],

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and hence this fibration gives twelve points on $\mathbf{P}^1$. This suggests a relation between Γ and Deligne-Mostow's complex reflection groups [DM], [M1], [M2]. In fact, in §4, Theorem 3, we shall show that our group Γ is commensurable with the largest Γ_μ in Deligne-Mostow's list where

$$\mu = (\mu_i) = (\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6})$$

(No.1 in Deligne-Mostow's list [M2] and No.10 in Thurston's list [T]).

In [K2], we showed that the moduli space of curves of genus three is also birational to an arithmetic quotient of the 6-dimensional complex ball. In this case we take the 4-cyclic cover of $\mathbf{P}^2$ branched along a smooth plane quartic curve. Then we have a $K3$ surface with an automorphism of order 4. However this arithmetic subgroup does not appear in Deligne-Mostow's list (the corresponding $K3$ surface has no elliptic fibration invariant under the automorphism of order 4). We remark that the moduli space of curves of hyperelliptic curves of genus 3 or plane quartic curves with a node is birational to an arithmetic quotient of the 5-dimensional complex ball ([K2], §4, 5). Both of these arithmetic subgroups are commensurable with the group Γ_μ where

$$\mu = (\mu_i) = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$$

(No.8 in Deligne-Mostow's list [M2] and No.3 in Thurston's list [T]).

For recent works related to this paper, we refer the reader to Allcock, Carlson, Toledo [ACT1], [ACT2], van Geemen, Izadi [vG], [vGI], Heckman, Looijenga [HL], Vakil [V].

In this paper we shall use the following notation: A *lattice* L is a free $\mathbf{Z}$-module of finite rank endowed with an integral non-degenerate symmetric bilinear form $\langle , \rangle$. A lattice L is *even* if $\langle x, x \rangle$ is even for each $x \in L$, and *unimodular* if its discriminant is ± 1. For a lattice L, we denote by L^* the dual of L, and by A_L the quotient group L^*/L. Let L be an even lattice. We extend the bilinear form on L to the one on L^* and define

$$q_L : A_L \longrightarrow \mathbf{Q}/2\mathbf{Z}, \quad q_L(x + L) = \langle x, x \rangle + 2\mathbf{Z}, \quad (x \in L^*)$$

which is called the *discriminant quadratic form*. We denote by A_m, D_n, E_k $(m \geq 1,\ n \geq 4,\ k = 6, 7, 8)$ the negative definite lattice which is defined by the Cartan matrix of type A_m, D_n, E_k respectively. We also denote by U the lattice of signature $(1, 1)$ defined by the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

For a lattice L and an integer m, $L(m)$ is the lattice whose bilinear form is the one on L multiplied by m.

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§2. A ball quotient structure

In this section, we shall show that the moduli space of curves of genus 4 is birational to an arithmetic quotient of a 9-dimensional complex ball by using the periods of $K3$ surfaces with an automorphism of order 3.

Let C be a smooth non-hyperelliptic curve of genus 4. First we assume that C has no vanishing theta constants. Then its canonical model is the complete intersection of a smooth quadric Q and a cubic S in $\mathbf{P}^3$. Let X be the 3-cyclic cover of Q branched along C which is a $K3$ surface with an automorphism σ of order 3. We denote by L the second cohomology group $H^2(X, \mathbf{Z})$. Together with the cup product, L has the structure of a lattice which is even, unimodular and of signature $(3, 19)$. Let E (resp. F) be the inverse image of a general fiber of one of the rulings of Q (resp. another ruling of Q). Then E, F are smooth elliptic curves with $\langle E, F \rangle = 3$. Since σ has a fixed curve, σ acts on $H^0(X, \Omega_X^2)$ as a multiplication by a cube root ω of unity (Nikulin [N2], §5). We remark that $U(3) = H^2(X, \mathbf{Z})^{\langle\sigma\rangle}$. We have a 9-dimensional family of $K3$ surfaces with an automorphism of order 3. The transcendental lattice of a generic member of this family has rank 20 (see the definition of the period domain $\mathcal{B}$ in this section). Hence if C is generic, then the Picard lattice S_X of X is generated by E and F, and isometric to $U(3)$.

Next we consider the case that C has a vanishing theta constant. Then its canonical model is the complete intersection of a quadric cone Q' and a cubic S. By taking the minimal resolution of the triple covering of Q' branched along C, we have a $K3$ surface X' with an automorphism σ of order 3. Let R_1, R_2, R_3 be three smooth rational curves obtained by resolution of three rational double points of type A_1 of the triple cover of Q'. Let F be the pull back of a fiber of the ruling of Q'. Then F and $R_1 + R_2 + R_3$ generate the invariant part $H^2(X', \mathbf{Z})^{\langle\sigma\rangle}$ which is isomorphic to $U(3)$. For generic C with a vanishing theta constant, the Picard lattice of X' is generated by F, R_1, R_2, R_3 and isomorphic to $U \oplus A_2(2)$. By a result of Brieskorn [B], (X', σ) is a deformation of (X, σ). Thus the action of σ on the cohomology group does not depend on the condition whether C has a vanishing theta constant or not. It is

known that σ^* fixes no non-zero vectors in $T \otimes \mathbf{Q}$ (Nikulin [N2], Theorem 3.1).

Let

$$T = U(3) \oplus U \oplus E_8 \oplus E_8$$

which is isometric to the orthogonal complement of $H^2(X, \mathbf{Z})^{\langle\sigma\rangle}$ ($\simeq U(3)$) in L. Let e, f (resp. e', f') be a basis of $U(3)$ (resp. U) with $e^2 = f^2 = 0, \langle e, f\rangle = 3$ (resp. $(e')^2 = (f')^2 = 0, \langle e', f'\rangle = 1$). Let ρ_1 be an isometry of $U(3) \oplus U$ defined by

$$\rho_1(e) = -2e + 3e', \quad \rho_1(f) = f + 3f',$$

$$\rho_1(e') = -e + e', \quad \rho_1(f') = -f - 2f'.$$

Note that ρ_1 is of order 3, has no non-zero fixed vectors in $U(3) \oplus U$ and acts on the discriminat of $U(3) \oplus U$ trivially. On the other hand, it is known that the root lattice E_8 can be regarded as a complex lattice defined over the Eisenstein integers (Allcock [A], §5). In other words, there exists an isometry ρ_2 on E_8 of order 3 which has no non-zero fixed vectors in E_8. We denote by ρ the isometry of $T = U(3) \oplus U \oplus E_8 \oplus E_8$ defined by

$$\rho = (\rho_1, \rho_2, \rho_2).$$

Note that ρ has order 3 and has no non-zero fixed vectors. Since ρ acts on T^*/T trivially, it can be extended to an isometry of $L = U \oplus U \oplus U \oplus E_8 \oplus E_8$ which acts on the orthogonal complement $U(3)$ of T trivially. For simplicity, we also denote by the same ρ this isometry of L.

Now we shall define the period domain for the above $K3$ surfaces. Let X be a $K3$ surface with an automorphism σ of order 3 of above type. A *marking* of the pair (X, σ) is an isometry

$$\alpha_X : H^2(X, \mathbf{Z}) \longrightarrow L$$

with $\alpha_X \circ \sigma^* \circ \alpha_X^{-1} = \rho$ or ρ^{-1}. In the proof of Theorem 1, we shall show the existence of a marking for each pair. (Note that to prove the existence of a marking, it is enough to show this for some pair (X, σ) because our 9-dimensional family of $K3$ surfaces is irreducible). Let

$$T \otimes \mathbf{C} = T_\omega \oplus T_{\bar{\omega}}$$

be the decomposition into eigenspaces $T_\omega, T_{\bar{\omega}}$ of ρ with eigenvalues $\omega, \bar{\omega}$ respectively. Put

$$\mathcal{B} = \{z \in \mathbf{P}(T_\omega) : \langle z, \bar{z}\rangle > 0\}, \quad \Gamma = \{\phi \in O(T) : \phi \circ \rho = \rho \circ \phi\}.$$

Note that $\mathcal{B}$ is a bounded symmetric domian of type $I_{1,9}$, that is, a 9-dimensional complex ball. Also note that if $z \in \mathcal{B}$, then $\langle z, z\rangle = 0$. Hence $\mathcal{B}$ is contained in a 18-dimensional bounded symmetric domain $\mathcal{D}$ of type IV:

$$\mathcal{D} = \{z \in \mathbf{P}(T \otimes \mathbf{C}) : \langle z, z\rangle = 0, \langle z, \bar{z}\rangle > 0\}.$$

We call a vector r in T with $r^2 = -2$ a *root*. For a root r, we define

$$H_r = \{\omega \in \mathcal{B} : \langle r, \omega\rangle = 0\}, \quad \mathcal{H} = \bigcup_r H_r$$

where r varies over the set of all roots. It is known that for each X with an automorphism of order 3, $(H^2(X, \mathbf{Z})^{<\sigma^*>})^{\perp} \cap S_X$ contains no (-2)-vectors (Namikawa [Na], Theorem 3.10). Hence a marked $K3$ surface (X, σ, α_X) of above type determines the point $\alpha_X(\omega_X)$ in $\mathcal{B} \setminus \mathcal{H}$ where ω_X is a nowhere vanishing holomorphic 2-form on X. Thus $\mathcal{B} \setminus \mathcal{H}$ is the period domian of marked $K3$ surfaces of above type.

We remark that the natural map from $O(U(3))$ to $O(q_{U(3)})$ is surjective. In fact $O(q_{U(3)})$ is isomorphic to $(\mathbf{Z}/2\mathbf{Z})^2$. The following isometries of $U(3)$ induce generators of $O(q_{U(3)})$: let e, f be a basis of $U(3)$ with $e^2 = f^2 = 0, \langle e, f\rangle = 3$. Then, with respect to this basis, the involutions

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

generate $O(q_{U(3)})$. This implies that the restriction map from $O(L)$ to $O(T)$ is surjective (Nikulin [N1] Proposition 1.6.1). Since $\rho \mid U(3) = 1$, we can easily see that the natural map from

$$\tilde{\Gamma} = \{\gamma \in O(L) : \gamma \circ \rho = \rho \circ \gamma\}$$

to Γ is surjective.

Theorem 1. *$(\mathcal{B} \setminus \mathcal{H})/\Gamma$ is isomorphic to the moduli space of non-hyperelliptic curves of genus 4.*

Proof. Let $z \in \mathcal{B}$ $(\subset \mathcal{D})$. It follows from the surjectivity of the period map (Kulikov [Ku], Persson, Pinkham [PP]) that there exist a $K3$ surface X and an isometry

$$\alpha_X : H^2(X, \mathbf{Z}) \longrightarrow L$$

with $\alpha_X(\omega_X) = z$ where ω_X is a nowhere vanishing holomorphic 2-form on X.

In the following, we shall show that if $z \in \mathcal{B} \setminus \mathcal{H}$, then X has an automorphism σ with $\alpha_X \circ \sigma^* \circ \alpha_X^{-1} = \rho$, and X is obtained as a 3-cyclic cover of Q or Q' branched along a smooth curve C of genus 4 where Q is a smooth quadric and Q' is a quadric cone. We may identify $H^2(X, \mathbf{Z})$ with L by α_X.

First we remark that by the assumption $z \notin \mathcal{H}$, $(L^{\langle\rho\rangle})^\perp \cap z^\perp$ contains no roots. Hence ρ is induced from an automorphism σ of X of order 3 (Namikawa [Na], Theorem 3.10). Since σ acts on $H^0(X, \Omega_X^2)$ as a multiplication by ω, it acts on the tangent space of a fixed point as

$$\begin{pmatrix} 1 & 0 \\ 0 & \omega \end{pmatrix}, \quad \begin{pmatrix} \omega^2 & 0 \\ 0 & \omega^2 \end{pmatrix}.$$

Hence the fixed locus of σ consists of the disjoint union of smooth curves and isolated fixed points. Note that the trace of ρ on L is -8. It follows from the topological Lefschetz fixed point formula that σ fixes a smooth curve C of genus $g(C) > 1$. Then by the Hodge index theorem, C is a unique fixed curve with $g(C) > 0$. On the other hand, $L^{\langle\rho\rangle} = U(3)$ contains no (-2)-vectors, σ has only one fixed curve C. Let k be the number of isolated fixed points of σ. Then by the topological Lefschetz fixed point formula, we have

$$k + 2 - 2g(C) = -6.$$

This implies $g(C) \geq 4$. Let $\{e, f\}$ be a basis of $L^{\langle\rho\rangle} = U(3)$ with $e^2 = f^2 = 0, \langle e, f\rangle = 3$. By the Riemann-Roch theorem, we may assume that both e and f are effective.

Claim. Either e or f is nef.

Proof of Claim. Assume that e is not nef. Then there exists a smooth rational curve R with $\langle R, e\rangle < 0$. Since $L^{\langle\rho\rangle}$ contains no (-2)-vectors, $R = r' + r''$ where $r' \in U(3)^*$ and $r'' \in (U(3)^\perp)^*$, $r'' \neq 0$. Since $(L^{\langle\rho\rangle})^\perp \cap z^\perp$ is negative definite, $(r'')^2 < 0$. Put $r' = (me + nf)/3$ $(m, n \in \mathbf{Z})$. Since $\langle R, e\rangle < 0$, $n < 0$. If $m \leq 0$, then r' is not effective because we assume that e and f are effective. This contradicts the fact that $3r' = R + \sigma(R) + \sigma^2(R)$ is effective. Hence $m > 0$. By the equation $-2 = 2mn/3 + (r'')^2$ and $(r'')^2 < 0$, we have $(m, n) = (1, -1), (1, -2), (2, -1)$. In the last two cases, we have $(3r')^2 = (R + \sigma(R) + \sigma^2(R))^2 = -12$. On the other hand, $R^2 = -2$ and $R \neq \sigma(R)$, and hence $(R + \sigma(R) + \sigma^2(R))^2 \geq -6$. This is a contradiction. Thus $(m, n) = (1, -1)$ and $e - f = 3r'$ is effective. On the other hand, if f is not nef, the same argument shows that $f - e$ is effective. This is a contradiction. Hence we have proved the assertion. Q.E.D. for Claim.

Thus we may assume that e is nef , in other words, it gives an elliptic fibration

$$\pi : X \longrightarrow \mathbf{P}^1.$$

Let E be a general fiber of π. Let $C = ae + bf$. Then $6ab = C^2 = 2g(C) - 2 \geq 6$. If $b > 1$, then $C \cdot E \geq 6$. Since C is the fixed curve of σ, σ acts on the base of π trivially. Hence σ acts on E as an automorphism. Now by applying the Hurwitz formula to the pair (E, σ), we have a contradiction. Hence $b = 1$. Thus we have the following two cases:

Case 1. f is nef.

Case 2. f is not nef.

In Case 1, $a = b = 1$, $g(C) = 4$ and $k = 0$. By taking the quotient of X by σ, we have a smooth quadric surface $X/\langle\sigma\rangle$.

In Case 2, there exists a smooth rational curve R so that $\langle R, f\rangle < 0$. Since $U(3)$ contains no (-2)-vectors, R is not contained in $U(3)$. Let $R_1 = \sigma(R), R_2 = \sigma^2(R)$. Then $R + R_1 + R_2 \in U(3)$. By the same argument as in the proof of the claim, $R + R_1 + R_2 = -e + f$. Hence $\langle R, R_1\rangle = 0$. Since C is the fixed locus of σ, $\langle C, R\rangle = 0$. Hence $a = b = 1$, $g(C) = 4$ and $k = 0$. Now by taking the quotient by σ and contracting the (-2)-curve which is the image of R, R_1, R_2, we have a quadric cone.

Finally we shall show that the isomorphism class of C is uniquely determined by its image in $(\mathcal{B} \setminus \mathcal{H})/\Gamma$. Let C and C' be non-hyperelliptic curves of genus 4. Let (X, σ), (X', σ') be the corresponding $K3$ surfaces with automorphisms σ, σ' of order 3. Assume that their periods are the same in $(\mathcal{B} \setminus \mathcal{H})/\Gamma$. Since the natural map $\tilde{\Gamma} \longrightarrow \Gamma$ is surjective, there exists a Hodge isometry

$$\varphi : H^2(X, \mathbf{Z}) \longrightarrow H^2(X', \mathbf{Z})$$

with $\varphi \circ \sigma^* = (\sigma')^* \circ \varphi$. Since $(H^2(X, \mathbf{Z})^{<\sigma^*>})^\perp \cap S_X$ contains no (-2)-vectors, C is an ample class. Obviously $\varphi(C) = C'$. It now follows from the Torelli theorem for $K3$ surfaces (Piatetskii-Shapiro, Shafarevich [PS]) that there exists an isomorphism $f : X' \longrightarrow X$ with $f^* = \varphi$ and $f \circ \sigma' = \sigma \circ f$. By taking the quotient of X, X' by σ, σ' respectively, f induces an isomorphism between the canonical models of C and C'. Q.E.D.

Remark 1. The following was suggested by I. Dolgachev. In the same way as above, we can see that the moduli spaces $\mathcal{M}_{2,1}$, $\mathcal{M}_{3,1}$ of pointed curves of genus 2 and 3 have a ball quotient structure. Let (C, q) be a pair of a smooth curve of genus 2 (resp. genus 3) and $q \in C$. Then the linear system $\mid K_C + 2q \mid$ gives a plane quartic curve $\bar{C}$ with a cusp (resp. a curve $\bar{C}$ of bidegree (3,3) with a cusp in a smooth quadric Q).

By taking the 4-cyclic cover of $\mathbf{P}^2$ (resp. the triple cover of Q) branched along $\bar{C}$ and then by taking the minimal resolution of rational double point, we have a $K3$ surface with an automorphism of order 4 (resp. order 3). This correspondence gives a birational map from $\mathcal{M}_{2,1}$ (resp. $\mathcal{M}_{3,1}$) to an arithmetic quotient of 4-dimensional complex ball (resp. 7-dimensional complex ball). In case $\mathcal{M}_{2,1}$, the Picard lattice of a generic member X is isomorphic to $U \oplus D_4 \oplus D_4 \oplus A_1^2$. The pencil of lines on $\mathbf{P}^2$ through q induces an elliptic pencil of X with one singular fibers of type I_0^* and six singular fibers of type III. In case $\mathcal{M}_{3,1}$, the Picard lattice of a generic member Y is isomorphic to $U(3) \oplus D_4$, and a ruling of Q induces an elliptic pencil on Y which has one singular fiber of type I_0^* and 9 singular fibers of type II.

Remark 2. Let C be a plane quartic curve and let q be a flex. Then by considering the map $\mid K_C + 2q \mid$ as above, we can see that the moduli space $\mathcal{M}_{3,flex}$ of plane quartic curves with a flex is birational to an arithmetic quotient of the 6-dimensional complex ball. Let X be a generic $K3$ surface appearing in this family. Then its transcendental lattice, together with an automorphism of order 3, has the structure of a complex lattice defined over the Eisenstein integers $\mathbf{Z}[\omega]$. On the other hand, in [K2], we showed that the moduli space $\mathcal{M}_3$ of curves of genus 3 is birational to an arithmetic quotient of the 6-dimensional complex ball by taking the 4-cyclic cover of $\mathbf{P}^2$ branched along a plane quartic curve. In this case, the transcendental lattice, together with an automorphism of order 4, has the structure of a complex lattice defined over the Gaussian integers $\mathbf{Z}[\sqrt{-1}]$. By forgetting a flex we have a map

$$\mathcal{M}_{3,flex} \longrightarrow \mathcal{M}_3$$

of degree 24. The author does not know the relation between two complex ball quotient structures.

Remark 3. Let C be a general smooth curve of genus 6. It is known that the canonical model of C ($\subset \mathbf{P}^5$) lies on a unique del Pezzo surface R of degree 5 (Kollár, Schreyer [KS], Shepherd-Barron [SB]). By taking the double covering of R branched along C, we have a $K3$ surface X with the covering transformation σ. Let $p : R \longrightarrow \mathbf{P}^2$ be a blow up at 4 points. We can easily see that the Picard lattice S of a generic X is isomorphic to $\langle 2 \rangle \oplus A_1^4$ where $\langle 2 \rangle$ is generated by the pullback of the class of a line in $\mathbf{P}^2$ and A_1^4 correspond to 4 exceptional curves on R. It is known that $O(q_S) \simeq O^-(4, \mathbf{F}_2) \simeq S_5$ (Morrison-Saito [MS], Corollary 2.4, Lemma 2.5). Let T be the transcendental lattice of X. Let $\mathcal{D}$ be a bounded symmetric domain of type IV associated to T, and

let $\Gamma = O(T)$. Then we can see that the moduli space of curves of genus 6 is birational to the arithmetic quotient $\mathcal{D}/\Gamma$. The author does not know whether the moduli of curves of genus 5 has a similar description as an arithmetic quotient or not.

§3. Discriminant locus

In this section we shall discuss the discriminant locus $\mathcal{H}$. Let $r \in T$ with $r^2 = -2$. By using the equation

$$\rho^2 + \rho + 1_T = 0,$$

we have $\langle r, \rho(r) \rangle = 1$. Let Λ_r be the lattice generated by r and $\rho(r)$. Obviously $\Lambda_r \simeq A_2$. Let $\Lambda_r^{\perp}$ be the orthogonal complement of Λ_r in T and let M be the orthogonal complement of $\Lambda_r^{\perp}$ in L. We remark here that ρ acts on T^*/T because $\rho \mid H^2(X, \mathbf{Z})^{\langle \rho \rangle} = 1$ (Nikulin [N1], Corollary 1.5.2). Also ρ acts on Λ_r^*/Λ_r trivially. This follows from the fact that $(r + 2\rho(r))/3$ is a generator of Λ_r^*/Λ_r and $\rho(r + 2\rho(r)) \equiv r + 2\rho(r)$ *mod* $3\Lambda_r$.

Lemma 1. *There are two possibilities*:

Case (i). $M \simeq U(3) \oplus A_2$ *and* $\Lambda_r^{\perp} \simeq U(3) \oplus U \oplus E_8 \oplus E_6$

Case (ii). $M \simeq U \oplus A_2$ *and* $\Lambda_r^{\perp} \simeq U \oplus U \oplus E_8 \oplus E_6$.

Proof. First note that M contains $S \oplus \Lambda_r$ as a sublattice of finite index where $S = L^{\langle \rho \rangle} \simeq U(3)$. M is determined by the isotropic subgroup $I = M/(S \oplus \Lambda_r)$ of $A_S \oplus A_{\Lambda_r}$ with respect to the discriminant quadratic form $q_S \oplus q_{\Lambda_r}$ (Nikulin [N1], Proposition 1.4.1). Since $A_S \oplus A_{\Lambda_r} \simeq (\mathbf{Z}/3\mathbf{Z})^3$, $I = \{0\}$ or $I = \mathbf{Z}/3\mathbf{Z}$. In the case $I = \mathbf{Z}/3\mathbf{Z}$, by using Nikulin [N1], Corollary 1.5.2, we can see that $q_M(a) = -2/3$ where a is a generator of $I = \mathbf{Z}/3\mathbf{Z}$. Now the assertion follows from Nikulin [N1], Theorem 1.14.2. Q.E.D.

It follows that $\mathcal{H}$ decomposes into two pieces $\mathcal{H}^n$ and $\mathcal{H}^h$ so that the first case $\mathcal{H}^n$ corresponds to the case (i) in Lemma 1. In the following, we shall study $K3$ surfaces whose periods are in $\mathcal{H}$.

Case (i): We shall show that the Case (i) in Lemma 1 corresponds to a curve in Q of bidegree $(3,3)$ with a node.

Example 1. Let C be a curve in a smooth quadric Q of bidegree $(3,3)$ with a node p. Let L_1, L_2 be the two lines through p. First blow up at p and denote by E the exceptional curve. Next blow up the two points in which E and the proper transform of C meet. Then take the 3-cyclic

cover X' branched along the proper transforms of C and E. Then X' contains an exceptional curve of the first kind which is the pullback of the proper transform of E. By contracting this exceptional curve to a point q, we have a $K3$ surface X. On X, there are 4 smooth rational curves F_1, F_2, F_3, F_4 which are the inverse images of L_1, L_2 and the exceptional curves which appeared in the second blow up. They meet together at one point q. Each triple of F_j defines an elliptic pencil with singular fiber of type IV and a 3-section. For generic C as above, these 4 curves F_j generate the Picard lattice of X isometric to $U(3) \oplus A_2$, where $U(3)$ is generated by $F_1 + F_2 + F_3$, $F_1 + F_2 + F_4$ and A_2 is generated by F_1, F_2. It is known that X has a finite group of automorphisms and the F_i (i =1,2,3,4) are all the smooth rational curves on X (e.g., Nikulin [N3], §4, p.661).

Next we shall show that a generic point of $\mathcal{H}^n$ corresponds to a $K3$ surface mentioned in Example 1. Let z be a generic point in $\mathcal{H}^n$ which is orthogonal to a root $r \in T$ with $\Lambda_r^{\perp} \simeq U(3) \oplus U \oplus E_8 \oplus E_6$. Let Y be the $K3$ surface whose period is z. Then the Picard lattice of Y is isomorphic to $M \simeq U(3) \oplus A_2$. Since the dual graph of all smooth rational curves on Y depends only on the Picard lattice, Y contains exactly 4 smooth rational curves F_j' ($1 \leq j \leq 4$) which form the same dual graph as that of F_j on X: $F_j' \cdot F_k' = 1, (j \neq k)$. Let ρ' be the isometry of L given by

$$\rho' = (1_M, \rho \mid \Lambda_r^{\perp}).$$

Obviously $(L^{\langle \rho' \rangle})^{\perp} \cap z^{\perp} = 0$, and hence it is induced from an automorphism σ' of order 3 (Namikawa [Na], Theorem 3.10). On the other hand, by the topological Lefschetz fixed point formula, σ' fixes a smooth curve C' of genus $g(C') > 1$. Now take any triple, for example, F_1', F_2', F_3' and consider the linear system $\mid F_1' + F_2' + F_3' \mid$ which defines an elliptic pencil $\pi : Y \longrightarrow \mathbf{P}^1$. By the Hodge index theorem, each fiber meets C', and hence σ' acts on the base of π trivially. Thus σ' acts on a general fiber as an automorphism of order 3, and hence the functional invariant of π is equal to 0. Hence the singular fiber $F_1' + F_2' + F_3'$ is of type IV. This implies that the four F_i' ($i = 1, 2, 3, 4$) meet each other at one point q. Since σ' acts on M trivially, σ' preserves each F_i'. Since σ' acts on a fiber as an automorphism of order 3, it fixes 3 points on it. Hence C' meets a fiber at three points. Now we can easily conclude that C' meets each F_i' at one point, the fixed point set of σ' consists of $\{q\}$ and C', and $g(C') = 3$. Thus Y is obtained by the same way as X in Example 1.

Case (ii): We shall show that the Case (ii) in Lemma 1 corresponds to a smooth hyperelliptic curve of genus 4 plus a point.

Example 2. Let C be a hyperelliptic curve of genus 4. Then C is given by the equation

$$y^2 = \prod_{i=1}^{10}(x_0 - \lambda_i x_1)$$

which is unique up to automorphisms of $\mathbf{P}^1$. Let $(x_0 : x_1, y_0 : y_1)$ be a bi-homogeneous coordinate of $\mathbf{P}^1 \times \mathbf{P}^1$. Then C can be embeded in $\mathbf{P}^1 \times \mathbf{P}^1$ as follows:

$$y_0^2 \cdot \prod_{i=1}^{5}(x_0 - \lambda_i x_1) + y_1^2 \cdot \prod_{i=6}^{10}(x_0 - \lambda_i x_1) = 0.$$

Let E be the divisor defined by $y_0 = 0$. Let L be a general fiber of the ruling given by

$$p : (x_0 : x_1, y_0 : y_1) \longrightarrow (x_0 : x_1).$$

Note that the fiber given by $x_0 = \lambda_i x_1$ is tangent to C. By taking elementary transformations at the intersection of C and E, we have the Hirzebruch surface $\mathbf{F}_5$. Let C', E' be the proper transform of C, E respectively. Let R be a rational surface obtained by blowing up $\mathbf{F}_5$ at three points which are the intersection of C', E' and L. Let C'', E'', L'' be the proper transform of C', E', L respectively. Let Y' be the 3-cyclic cover of R branched along the divisor $C'' + E'' + L''$. The inverse image of L'' is a (-1)- curve. By contracting this we have a $K3$ surface Y. We can see that the ruling p induces a structure of an elliptic pencil

$$\pi : Y \longrightarrow \mathbf{P}^1$$

which has one singular fiber of type IV and 10 singular fibers of type II, and a section. Here the singular fiber of type IV corresponds to L, ten singular fibers of type II corresponds to fibers $x_0 + \lambda_i x_1$ and E corresponds to a section of π. For a generic C, the Picard lattice of Y is generated by three components of the singular fiber of type IV and a section, which is isomorphic to $U \oplus A_2$. In this case, $Aut(Y)$ is finite and Y contains exactly 4 smooth rational curves (e.g., Nikulin [N3], §4, p.661).

Next we shall show that a generic point of $\mathcal{H}^h$ corresponds to a $K3$ surface mentioned in Example 2. Let z be a generic point in $\mathcal{H}^h$ which

is orthogonal to a root $r \in T$ with $\Lambda_r^{\perp} \simeq U \oplus U \oplus E_8 \oplus E_6$. Let Y' be the $K3$ surface whose period is z. Then the Picard lattice of Y' is isomorphic to $M \simeq U \oplus A_2$. Since the dual graph of all non-singular rational curves on Y' depends only on the Picard lattice, Y' contains exactly 4 smooth rational curves F_j $(0 \leq j \leq 3)$ which form the same dual graph as that of Example 2. Here we assume that F_1, F_2, F_3 form the dual graph of a singular fiber of type I_3 or of type IV, and F_0 meets F_1. Then the linear system $\mid F_1 + F_2 + F_3 \mid$ defines an elliptic pencil $\pi : Y \longrightarrow \mathbf{P}^1$ which has one singular fiber of type I_3 or of type IV, and a section F_0. Let ρ' be the isometry of L given by

$$\rho' = (1_M, \rho \mid \Lambda_r^{\perp}).$$

Obviously $(L^{\langle \rho' \rangle})^{\perp} \cap z^{\perp} = 0$, and hence ρ' is induced from an automorphism σ' of order 3 (Namikawa [Na], Theorem 3.10). On the other hand, by the topological Lefschetz fixed point formular, σ' fixes a smooth curve C' of genus $g(C') > 1$. By the Hodge index theorem, each fiber meets C', and hence σ' acts on the base of π trivially. Hence F_0 is a fixed curve of σ'. Also σ' acts on a general fiber as an automorphism of order 3, and hence the functional invariant is a constant. Hence the singular fiber $F_1 + F_2 + F_3$ is of type IV and all irreducible singular fibers are of type II. Thus π has one singular fiber of type IV and 10 singular fibers of type II. Since σ' acts on a general fiber as an automorphism of order 3, it fixes 3 points on a general fiber, i.e., C' meets a fiber at two points. Let q be the singular point of the fiber $F_1 + F_2 + F_3$. Then we can easily conclude that C' meets each F_2, F_3 at one point $(\neq q)$ and the fixed point set of σ' consists of $\{q\}$, F_0 and C'. It follows from the topological Lefschetz fixed point formula that the genus of C' is 4. Therefore Y is obtained by the same way as X in Example 2. Thus we have

Theorem 2. *A generic point in $\mathcal{H}^n$ (resp. in $\mathcal{H}^h$) corresponds to a curve in Q of bidegree $(3,3)$ with a node (resp. a pair of a hyperelliptic curve of genus 4 and a point on the quotient of the hyperelliptic curve by the hyperelliptic involution).*

Remark 4. (i) The above theorem 2 tells us that $\mathcal{B}/\Gamma$ looks like a blow up of the moduli space of curves of genus 4 along the hyperelliptic locus.

(ii) There is a family of codimension 1 in the moduli space of curves of genus 4 which consists of smooth curves with a vanishing theta null. In this case, the corresponding generic $K3$ surface contains smooth rational curves, but the covering transformation does not fix them (see the proof of Theorem 1, *Case* 2) and the periods of these (generic) $K3$ are

contained in $\mathcal{D} \setminus \mathcal{H}$. The Picard lattice of a generic $K3$ surface in this family is isometric to

$$U \oplus A_2(2).$$

This lattice contains $U(3) \oplus A_2(2)$ as a sublattice of finite index and the factor $U(3)$ corresponds to the Picard lattice of a generic member of 9-dimensional family.

Remark 5. Recall that the anti-bicanonical map of a del Pezzo surface of degree 1 is the double cover of a quadric cone in $\mathbf{P}^3$ branched along the canonical curve C of genus 4 (Demazure [D]). Then C has a vanishing theta null. This gives a birational map from the moduli space of del Pezzo surfaces of degree 1 to the moduli space of curves of genus 4 with vanishing theta constant. Thus the moduli space of del Pezzo surfaces of degree 1 can be written as a ball quotient, too. Heckman, Looijenga [HL] and Vakil [V] studied this case from a different point of view. In case of del Pezzo surfaces of degree 2 or 3, van Geemen (unpublished), Kondo [K2], Allcock, Carlson, Toledo [ACT1],[ACT2] gave a ball quotient structure for this moduli space. In the case of del Pezzo surfaces of degree 4, the moduli space is also a ball quotient (e.g. Heckman, Looijenga [HL]): A del Pezzo surface of degree 4 is the complete intersection of two quadrics Q_1, Q_2 in $\mathbf{P}^4$. The discriminant locus of the pencil of quadrics defined by Q_1 and Q_2 gives five points on $\mathbf{P}^1$. This gives a correspondence between the moduli space of del Pezzo surfaces of degree 4 and a compact arithmetic quotient of 2-dimensional complex ball which appeared in Shimura [S], Terada [Te], Deligne-Mostow [DM].

Remark 6. Del Pezzo surfaces of degree 4 are also related to $K3$ surfaces with an automorphism of order 5. Let C be the plane quintic curve defined by

$$y^5 = \prod_{i=1}^{5}(x - \xi_i z)$$

which corresponds to five points $\{(\xi_i : 1)\}$ on $\mathbf{P}^1$ (see Remark 5). Let σ be an automorphism of $\mathbf{P}^2$ given by

$$\sigma(x, y, z) = (x, \zeta y, z)$$

where ζ is a primitive 5-th root of unity. Let L be the line defined by $y = 0$ which is fixed under the action of σ. Let X be the minimal resolution of the double cover branched along the sextic curve $C + L$. Then X is a $K3$ surface and σ can be lifted to an automorphism of X of order 5. We can see that the period domain of these $K3$ surfaces is a 2-dimensional complex ball. The pencil of lines through $(0 : 1 : 0)$ lifts

to a pencil of curves of genus 2 on the $K3$ surface which has two base points. A general member of this pencil is a smooth curve of genus 2 with an automorphism of order 5 and this pencil contains five singular members corresponding to five lines

$$x - \xi_i z = 0, (1 \leq i \leq 5).$$

§4. Deligne-Mostow's complex reflection groups

In this section we shall show that Γ is commensurable with an arithmetic subgroup of $\mathbf{P}U(1,9)$ which appeared in Deligne-Mostow's list. The idea of the proof of Theorem 3 is due to T. Terasoma.

Let $\{\infty, 0, 1, x_2, ..., x_{d+1}\}$ be a set of $d+3$-distinct points in $\mathbf{P}^1$. For each integer i, with $0 \leq i \leq d+1$, and $i = \infty$, let μ_i be a real number such that the following equality holds:

$$\sum_i \mu_i = 2.$$

In [DM], for each $\mu = \{\mu_i\}$, Deligne and Mostow defined a subgroup Γ_μ of the automorphism group of a d-dimensional complex ball, which is the monodromy group of a hypergeometric equation.

Let $S_1 \subset S = \{\infty, 0, 1, ..., d+1\}$ and assume that $\mu_s = \mu_t$ for all $s, t \in S_1$. We assume that $\mu_s > 0$ for all $s \in S$ and $\{\mu_s\}$ satisfies the condition

(ΣINT): *For all $s \neq t$ such that $\mu_s + \mu_t < 1$, $(1 - \mu_s - \mu_t)^{-1}$ is an integer if s or t is not in S_1, and a half-integer if $s, t \in S_1$.*

Deligne and Mostow [DM],[M1] showed that this condition is a sufficient condition for which Γ_μ is a *lattice* in $\mathbf{P}U(1,d)$, i.e., Γ_μ is discrete and has cofinite volume. Conversely if Γ_μ is discrete and $d > 3$, then μ satisfies the condition (ΣINT) (Mostow [M2]). In [DM], [M1], Deligne and Mostow determined all such μ and listed them in case $d \geq 5$. Note that Thurston gave a correction of their list ([T]).

Now we shall show that our group Γ is commensurable with Γ_μ, where $\mu = \{\mu_i\}, \mu_i = 1/6$ for all $i = \infty, 0, 1, ..., 10$ (No.1 in Deligne-Mostow's list [M2] and No.10 in Thurston's list [T]).

Let C be a curve defined by

$$y^6 = \prod_{i=1}^{12} (x - \xi_i).$$

Let σ be the covering transformation of C over $\mathbf{P}^1$. Consider the action of σ on $H^1(C, \mathbf{C})$. Let $H^1(C, \mathbf{C})_{-\omega}$ be the eigenspace of σ with eigenvector $-\omega$ where ω is a primitive cube root of unity. Let $(,)$ be the symplectic form on $H^1(C, \mathbf{Z})$. Then $\psi(x, y) = \sqrt{-3}(x, \bar{y})$, $x, y \in H^1(C, \mathbf{C})_{-\omega}$, is a hermitian form on $H^1(C, \mathbf{C})_{-\omega}$. This space is defined over the Eisenstein integers $\mathbf{Z}[\omega]$. Deligne and Mostow showed that the signature of ψ is $(1, 9)$ and that Γ_μ is an arithmetic subgroup of $(\mathbf{P}U(H^1(C, \mathbf{C})_{-\omega}), \psi)$.

In the following we shall use the same notation as in §2. Recall that the $K3$ surface X has an elliptic fibration π induced from a ruling of the smooth quadric Q. For a generic X, π has twelve singular fibers of type II in the sense of Kodaira [Ko]. Note that π has no sections. Let $S = U(3)$ be the Picard lattice for a generic X and let $T = U(3) \oplus U \oplus E_8 \oplus E_8$ be the transcendental lattice. Since $\rho \mid S = 1$, ρ acts trivially on the discriminant group $S^*/S \simeq T^*/T$. Let $\alpha \in T^*/T$ be the non-zero isotropic vector corresponding to the class of a fiber of π under the canonical isomorphism $S^*/S \simeq T^*/T$. By adding vectors in T^* representing α to T, we have an even lattice T' which contains T as a sublattice of index 3. Hence T' is unimodular and isometric to $U \oplus U \oplus E_8 \oplus E_8$. Since ρ fixes α, ρ induces an isometry ρ'' of T' of order 3. We denote by ρ' the isometry $-\rho''$ of order 6. By the surjectivity of the period map for $K3$ surfaces, there exists a $K3$ surface Y whose transcendental lattice is isometric to T' and whose period is the same as that of X. Since T' is unimodular, ρ' can be extended to an isometry of $H^2(Y, \mathbf{Z})$ acting on $(T')^{\perp} (\simeq U)$ trivially. Hence it follows from the Torelli theorem that ρ' is represented by an automorphism of order 6. Since the Picard lattice of Y is isomorphic to U, Y has an elliptic fibration π' with a unique section (Kondo [K1], Lemma 2.1). If ρ' acts on the base of π' non trivially, then the set of fixed points of ρ' is contained in two fibers. On the other hand the Lefschetz number of ρ' is -6 (see the proof of Theorem 1). With this observation it follows from the topological Lefschetz fixed point formula that ρ' acts on the base trivially. Thus every smooth fiber has an automorphism of order 6 which fixes the intersection point with the section. Hence the functional invariant of this elliptic fibration is a constant. Since $Pic(Y) \simeq U$, every singular fiber is irreducible, and hence it is of type II. We remark that $\pi' : Y \longrightarrow \mathbf{P}^1$ is nothing but the Jacobian fibration of π.

By the theory of elliptic surfaces with a section (Kodaira [Ko]), we can easily see that there exists a Galois cover of Y with the Galois group $G \simeq \mathbf{Z}/6\mathbf{Z}$, which is birational to $C \times E$ where C is a $\mathbf{Z}/6\mathbf{Z}$-cover of $\mathbf{P}^1$ ramified at 12 points and E is an elliptic curve with a complex multiplication ω (a cube root of unity). We denote by σ (resp. τ) an automorphism of C (resp. E) of order 6. We may assume that (σ, τ) is

a generator of G. Denote by f the rational map from $C \times E$ to Y. Then

$$f^*(T' \otimes \mathbf{Q}) \simeq (H^1(C, \mathbf{Q}) \otimes H^1(E, \mathbf{Q}))^G$$

and

$$(H^1(C, \mathbf{C}) \otimes H^1(E, \mathbf{C}))^G \simeq H^1(C)_{-\omega} \otimes H^1(E)_{-\bar{\omega}} \oplus H^1(C)_{-\bar{\omega}} \otimes H^1(E)_{-\omega}.$$

Moreover the action of ρ' on $T' \otimes \mathbf{Q}$ is compatible with that of τ on $f^*(T' \otimes \mathbf{Q})$. The above isomorphisms give an isomorphism between two hermitian spaces $T'_{-\omega} = \{x \in T' \otimes \mathbf{C} : \rho'(x) = -\omega x\}$ and $H^1(C, \mathbf{C})_{-\omega}$, which is defined over $\mathbf{Q}(\omega)$. Since both Γ_μ and

$$\Gamma' = \{g \in O(T') : g \circ \rho' = \rho' \circ g\}$$

are arithmetic, Γ_μ is commensurable with Γ'. On the other hand, by definition of T', Γ is commensurable with Γ', and hence with Γ_μ. Thus we have

Theorem 3. *Γ is commensurable with Γ_μ where*

$$\mu = (\mu_i) = (\tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}).$$

Remark 7. In [A], Allcock proved that Γ_μ in Theorem 3 is isomorphic to the hyperbolic reflection group of the complex lattice over the Eisenstein integers $\mathbf{Z}[\omega]$ whose real form is $U \oplus U \oplus E_8 \oplus E_8$. We also remark that van Geemen gave a similar correspondence between the curve C as above and some $K3$ surface (see [vG], Example 3.11).

Remark 8. In [K2], we showed that the moduli space of curves of genus 3 is birational to a ball quotient by taking the 4-cyclic cover of $\mathbf{P}^2$ branched along a plane quartic curve. The corresponding discrete group does not appear in Deligne-Mostow's list (the corresponding $K3$ surface has no elliptic fibration invariant under the action of the automorphism of order 4). However, for example, in case of hyperelliptic curves of genus 3 or plane quartic curves with a node, the corresponding generic $K3$ surface has an elliptic fibration with 8 singular fibers of type III (see [K2], §5). The same argument as above shows that the corresponding arithmetic subgroup is commensurable with Γ_μ where

$$\mu = (\mu_i) = (\tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4})$$

(No. 8 in Deligne-Mostow's list [M2] and No. 3 in Thurston's list [T]). Allcock informed the author that he showed the commensurability in this

case by a different way. Shiga [Sh] suggested a relation between Deligne-Mostow's complex reflection groups and elliptic $K3$ surfaces with a section in some special cases. In the case of del Pezzo surfaces of degree 4 (see Remark 6), the pencil of curves of genus 2 with an order 5 automorphism on the $K3$ surface works like the elliptic pencil in the above cases.

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Infinitesimal Logarithmic Torelli Problem for Degenerating Hypersurfaces in $\mathbb{P}^n$

Shuji Saito

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§0. Introduction

Recently Kato and Usui ([KU]) have constructed partial compactifications of period domains by using logarithmic Hodge structures. It motivates us to study extended period maps and their Torelli problems. The purpose of this article is to give some results on the infinitesimal logarithmic Torelli problem of extended period maps for degenerating hypersurfaces.

We first set up the following problem in the logarithmic algebraic geometry. Let k be a field of characteristic zero and let $\underline{S}_0 = (\mathrm{Spec}(k), N_0)$ be the standard log point where the log structure N_0 is defined by $\mathbb{N} \to k; 1 \to 0$. Let $f_0 : (Z, M_Z) \to \underline{S}_0$ be a log smooth morphism of semistable type whose underlying morphism is proper and flat of relative dimension m. Let ω_{Z/S_0} be the sheaf of logarithmic differentials

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of $(Z, M_Z)/\underline{S}_0$ introduced by [K]. It is a locally free $\mathcal{O}_Z$-module and we put

$$\omega^q_{Z/S_0} = \overset{q}{\wedge}\omega_{Z/S_0} \quad \text{and} \quad \theta_{Z/S_0} = \mathcal{H}om_{\mathcal{O}_Z}(\omega_{Z/S_0}, \mathcal{O}_Z).$$

The main object of study is the map

$$d\rho_Z \ : \ H^1(Z, \theta_{Z/S_0}) \to \bigoplus_{0 \le p \le m-1} \mathrm{Hom}\big(H^p(Z, \omega^{m-p}_{Z/S_0}), H^{p+1}(Z, \omega^{m-1-p}_{Z/S_0})\big)$$

that is given rise to by the pairing $H^1(Z, \theta_{Z/S_0}) \otimes H^p(Z, \omega^{m-p}_{Z/S_0}) \to H^{p+1}(Z, \omega^{m-1-p}_{Z/S_0})$ induced by the contraction $\theta_{Z/S_0} \otimes \omega^q_{Z/S_0} \to \omega^{q-1}_{Z/S_0}$ and cup product. We say that the infinitesimal logarithmic Torelli holds for $(Z, M_Z)/\underline{S}_0$ when $d\rho_Z$ is injective.

Now we consider the case $k = \mathbb{C}$. In [KU] $d\rho_Z$ is interpreted as the logarithmic differential of an extended period map and the injectivity implies that the period map is an embedding on the universal deformation space of f_0. In order to explain the implication more precisely, we consider the following concrete situation. Let $B = \{t \in \mathbb{C} \mid |t| < 1\}$ be the unit disc and let $\mathbb{P}^{m+1}$ be the complex projective space with homogeneous coordinate $X_0, \ldots, X_{m+1}$. We will construct a certain moduli space over B that parametrizes families of hypersurfaces $X \subset \mathbb{P}^{m+1} \times B$ over B such that $X \to B$ is smooth outside the point $\{t = X_1 = \cdots X_{m+1} = 0\}$ with some prescribed type of singularity at the point. Every member X/B of the moduli space has a standard desingularization $\widetilde{X} \to X$ such that $\widetilde{X}$ has strictly semistable reduction over B. It gives rise to a log smooth morphism $(Z, M_Z) \to \underline{S}_0$ where Z is the central fiber of $\widetilde{X} \to B$ and M_Z is associated to the embedding $Z \hookrightarrow \widetilde{X}$. Our main result Theorem(2-1) asserts under suitable assumptions that the infinitesimal logarithmic Torelli holds for $(Z, M_Z)/\underline{S}_0$. Letting $\overline{B}^*$ be the universal covering of $B - \{0\}$, it implies that the limiting Hodge structure on $H^m(\widetilde{X} \times_B \overline{B}^*, \mathbb{Q})$ defined by Steenbrink [St] determines (Z, M_Z) locally on the moduli space up to isomorphisms of log schemes over $\underline{S}_0$ (see Theorem(2-2) for the more precise statement).

The proof of the main theorem follows the line of thoughts developed by Griffiths [Grif]. The point is to express $H^1(Z, \theta_{Z/S_0})$ and $H^p(Z, \omega^q_{Z/S_0})$ by Jacobian rings and observe that $d\rho_Z$ is induced by multiplication of polynomials. In order to develop the generalized Jacobian rings in our logarithmic context, we apply the Green's technique of Koszul complexes ([G, Lecture 4]) to a certain toric variety.

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§1. A moduli space of degenerating hypersurfaces in $\mathbb{P}^n$

In the whole paper we fix the base field k of characteristic zero, a variable t over k and a discrete valuation ring Λ over $k[t]$ such that t is a prime element of Λ and that $\Lambda/(t) \simeq k$. Let $\mathbb{P}_\Lambda = \mathrm{Proj}(\Lambda[X_0, \ldots, X_n])$ be the projective space of dimension $n \geq 3$ over $\mathrm{Spec}(\Lambda)$ and let $\mathbb{A}_\Lambda = \mathrm{Spec}(\Lambda[x_1, \ldots, x_n])$ with $x_i = X_i/X_0$ be the affine subspace. We are going to study hypersurfaces in $\mathbb{P}_\Lambda$ which are smooth over $\mathrm{Spec}(\Lambda)[1/t]$ and whose fibers over $\mathrm{Spec}(\Lambda/(t))$ has isolated singularity of some prescribed type at the origin of $\mathbb{A}_\Lambda$. We need introduce some notations.

Definition(1-1). We fix an integer $s \geq 1$.

(1) Let $q \geq 0$ be an integer. Let $P_\Lambda^q \subset \Lambda[X_0, \ldots, X_n]$ be the Λ-module of homogeneous polynomials of degree q. Let $A_\Lambda = \Lambda[x_1, \ldots, x_n]$ and $A_\Lambda^{\leq q}$ be the subspace of polynomials of degree $\leq q$. We constantly use the identification $P_\Lambda^q \xrightarrow{\sim} A_\Lambda^{\leq q}$; $G(X_0, X_1, \ldots, X_n) \to G(1, x_1, \ldots, x_n)$.

(2) For an integer $\nu \geq 0$ let $\mathfrak{m}_\Lambda(\nu) \subset A_\Lambda$ be the ideal generated by the elements

$$t^\beta x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} \quad \text{with } \beta + \alpha_n + s \sum_{1 \leq i \leq n-1} \alpha_i \geq \nu$$

and define

$$P_\Lambda^q(\nu) = \mathrm{Ker}(P_\Lambda^q \to A_\Lambda/\mathfrak{m}_\Lambda(\nu)) = A_\Lambda^{\leq q} \cap \mathfrak{m}_\Lambda(\nu).$$

Now we fix an integer $d > 0$ for the degree of our hypersurfaces and $\delta > 0$ for the multiplicity of designated singularity. Our object to study is hypersurfaces defined by an equation $F \in P_\Lambda^d(s\delta)$ (s is already fixed in Definition(1-1)). By definition $F \in P_\Lambda^d$ lies in $P_\Lambda^d(s\delta)$ if $f = F(1, x_1, \ldots, x_n) \in A_\Lambda^{\leq d}$ is written in the form

$$f = \sum_{\alpha_1 + \cdots + \alpha_n \leq d} a_{\alpha_1, \ldots, \alpha_n} x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} \quad (a_{\alpha_1, \ldots, \alpha_n} \in \Lambda),$$

$$|a_{\alpha_1, \ldots, \alpha_n}| + \sum_{1 \leq i \leq n-1} s\alpha_i + \alpha_n \geq s\delta$$

where $|\lambda|$ denotes the normalized additive valuation of $\lambda \in \Lambda$.

Remark(1-1). A typical example of such f is

$$f = x_1^{\delta} + \cdots + x_{n-1}^{\delta} + x_n^{s\delta} + t^{s\delta} + h \quad \text{with } h \in A_{\Lambda}^{\leq d} \cap \mathfrak{m}_{\Lambda}(s\delta + 1).$$

where we assume $s\delta < d$. The reason why I choose such special type of degenerating hypersurfaces for our stydy of the logarithmic Torelli problem is not so much more than the convenience of computation. By using toric geometry we will construct a birational map $\widetilde{\mathbb{P}}_{\Lambda} = \widetilde{\mathbb{P}}_{\Lambda}(\Delta) \to \mathbb{P}_{\Lambda}$ associated to a certaion cone decomposition Δ in such a way that the inverse images of the generic members of degenerating hypersurfaces defined by equations in $P_{\Lambda}^{d}(s\delta)$ have semistable reduction over $\mathrm{Spec}(\Lambda)$. A change of Δ will bring about that of the type of equations, and vice versa. I believe that with more effort, one should be able to carry out the similar computation for degenerating hypersurfaces defined by equations of more general type. For example one may fix appropriate rational numbers $\delta_0, \delta_1, \ldots, \delta_n$ to consider equations of the form

$$f = \sum_{\alpha_1 + \cdots + \alpha_n \leq d} a_{\alpha_1, \ldots, \alpha_n} x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} \quad (a_{\alpha_1, \ldots, \alpha_n} \in \Lambda),$$

$$\delta_0 |a_{\alpha_1, \ldots, \alpha_n}| + \sum_{1 \leq i \leq n} \delta_i \alpha_i \geq 1.$$

Now we construct a moduli space over $\mathrm{Spec}(\Lambda)$ parametrizing hypersurfaces in $\mathbb{P}_{\Lambda}$ of the above type which are smooth in the logarithmic sense (cf. Theorem(1-1)). We start with the following.

Definition(1-2). We fix integers d, δ, r satisfying $r = s\delta$ and $\delta \geq 2$ and $d \geq r$. Let $M = \mathbb{P}(P_{\Lambda}^{d}(r)^{*})$ be the projective space bundle over $\mathrm{Spec}(\Lambda)$ associated to the Λ-dual of $P_{\Lambda}^{d}(r)$. For a morphism $p : \mathrm{Spec}(\Lambda) \to M$ we let $X_p \subset \mathbb{P}_{\Lambda}$ be the corresponding hypersurface. Note that X_p passes through $0 = (t, x_1, \ldots, x_n) \in \mathbb{A}_{\Lambda}$ where X_p is singular.

The next task is to construct an appropriate simultaneous desingularization of X_p. For this we use the terminology of toric geometry that we refer to [F]. Let N be the lattice of rank $n+1$ with the standard basis

$$e_0 = (1, 0, \ldots, 0),\ e_1 = (0, 1, \ldots, 0), \cdots, e_n = (0, 0, \ldots, 1)$$

where the vectors are $n+1$-dimensional. Let $M = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ with dual pairing $\langle\ ,\ \rangle : M \times N \to \mathbb{Z}$. For a cone σ in $N_{\mathbb{R}}$ write

$$U_{\sigma} = \mathrm{Spec}(k[M_{\sigma}]) \quad \text{with } M_{\sigma} = \{u \in M \mid \langle u, v \rangle \geq 0\ (\forall v \in \sigma)\}.$$

For a fan Δ in N let

$$X(\Delta) = \bigcup_{\sigma \in \Delta} U_\sigma$$

be the corresponding toric variety. In particular we have

$$\mathbb{A} := X(\Delta_0) = \mathrm{Spec}(k[t, x_1, \ldots, x_n])$$

for $\Delta_0 = \{\text{faces of } \sigma_0\}$ with $\sigma_0 = \mathbb{R}_{\geq 0}e_0 + \cdots + \mathbb{R}_{\geq 0}e_n$ where $t, x_1, \ldots, x_n$ are the elements in $k[M_{\sigma_0}]$ corresponding to the dual basis of $\{e_0, \ldots, e_n\}$. For a fan Δ which is a subdivision of Δ_0 we have the natural morphism $\pi : X(\Delta) \to \mathbb{A}$ arising from the natural map $(N, \Delta) \to (N, \Delta_0)$.

Definition(1-3). Let $s \geq 1$ be as in Definition(1-1).

(1) Let Δ_s be the fan consisting of the following cones and their faces

$$\sigma_s^+,\ \sigma_s^-,\ \sigma_{0,\hat{i}}, \sigma_{k,\hat{i}}^+,\ \sigma_{k,\hat{i}}^- \ (1 \leq k \leq s-1, 1 \leq i \leq n-1),$$

where, denoting $v_k = (1, k, \ldots, k, 1)$,

$$\begin{aligned}
\sigma_{k,\hat{i}}^+ =& \mathbb{R}_{\geq 0}e_0 + \mathbb{R}_{\geq 0}v_k + \mathbb{R}_{\geq 0}v_{k+1} + \mathbb{R}_{\geq 0}e_1 \\
&+ \cdots + \widehat{\mathbb{R}_{\geq 0}e_i} + \cdots + \mathbb{R}_{\geq 0}e_{n-1}, \\
\sigma_{k,\hat{i}}^- =& \mathbb{R}_{\geq 0}e_n + \mathbb{R}_{\geq 0}v_k + \mathbb{R}_{\geq 0}v_{k+1} + \mathbb{R}_{\geq 0}e_1 \\
&+ \cdots + \widehat{\mathbb{R}_{\geq 0}e_i} + \cdots + \mathbb{R}_{\geq 0}e_{n-1}, \\
\sigma_{0,\hat{i}} =& \mathbb{R}_{\geq 0}e_0 + \mathbb{R}_{\geq 0}e_n + \mathbb{R}_{\geq 0}v_1 + \mathbb{R}_{\geq 0}e_1 \\
&+ \cdots + \widehat{\mathbb{R}_{\geq 0}e_i} + \cdots + \mathbb{R}_{\geq 0}e_{n-1}, \\
\sigma_s^+ =& \mathbb{R}_{\geq 0}e_0 + \mathbb{R}_{\geq 0}v_s + \mathbb{R}_{\geq 0}e_1 + \cdots + \mathbb{R}_{\geq 0}e_{n-1}, \\
\sigma_s^- =& \mathbb{R}_{\geq 0}e_n + \mathbb{R}_{\geq 0}v_s + \mathbb{R}_{\geq 0}e_1 + \cdots + \mathbb{R}_{\geq 0}e_{n-1}.
\end{aligned}$$

Let $\widetilde{\mathbb{A}} := X(\Delta_s)$ with $\pi : \widetilde{\mathbb{A}} \to \mathbb{A}$ be the corresponding toric variety. Note that if $s = 1$, π is the blow up of $\mathbb{A}$ with center at $0 = (t, x_1, \ldots, x_n)$.

(2) Let $\mathbb{P} = \mathrm{Proj}(k[t][X_0, \ldots, X_n])$ be the projective n-space over $\mathrm{Spec}(k[t])$ and fix the embedding $\mathbb{A} \hookrightarrow \mathbb{P}$ over $\mathrm{Spec}(k[t])$ via $x_i = X_i/X_0$. Putting

$$\widetilde{\mathbb{P}} = \widetilde{\mathbb{A}} \cup \bigcup_{1 \leq i \leq n} \mathbb{A}_i \quad \text{with } \mathbb{A}_i = \{X_i \neq 0\} \subset \mathbb{P}$$

patched together in an evident manner, we get $\pi : \widetilde{\mathbb{P}} \to \mathbb{P}$ that fits into the following cartesian diagram

$$\begin{array}{ccc} \widetilde{\mathbb{A}} & \hookrightarrow & \widetilde{\mathbb{P}} \\ \downarrow \pi & & \downarrow \pi \\ \mathbb{A} & \hookrightarrow & \mathbb{P}. \end{array}$$

Proposition(1-1). *$\widetilde{\mathbb{A}}$ is smooth over k, π is proper and $\pi : \widetilde{\mathbb{A}} - \pi^{-1}(0) \xrightarrow{\sim} \mathbb{A} - \{0\}$ where $0 = (t, x_1, \ldots, x_n) \in \mathbb{A}$. The reduced part $\mathbb{E}$ of $\pi^{-1}(0)$ is a simple normal crossing divisor on $\widetilde{\mathbb{A}}$. We can write $\mathbb{E} = \bigcup_{1 \leq k \leq s} \mathbb{E}_k$ as a divisor on $\widetilde{\mathbb{A}}$, where $\mathbb{E}_k \subset \widetilde{\mathbb{A}}$ is an irreducible smooth divisor characterized by the following property:*

$$\mathbb{E} \cap U_\sigma \subset \begin{cases} \mathbb{E}_k \cup \mathbb{E}_{k+1} & \text{if } \sigma = \sigma^{\pm}_{k,\hat{i}} \text{ with } 1 \leq k \leq s-1 \\ \mathbb{E}_1 & \text{if } \sigma = \sigma_{0,\hat{i}} \\ \mathbb{E}_s & \text{if } \sigma = \sigma^{\pm}_s. \end{cases}$$

Proof. Easy and left to the readers. Q.E.D.

Definition(1-4). Let

$$\begin{array}{ccc} \widetilde{\mathbb{A}}_\Lambda & \hookrightarrow & \widetilde{\mathbb{P}}_\Lambda \\ \downarrow \pi & & \downarrow \pi \\ \mathbb{A}_\Lambda & \hookrightarrow & \mathbb{P}_\Lambda \end{array}$$

be the base change of the diagram in Definition(1-3)(2) via $\mathrm{Spec}(\Lambda) \to \mathrm{Spec}(k[t])$. For $X_p \subset \mathbb{P}_\Lambda$ as in Definition(1-2), we denote by $\widetilde{X}_p \subset \widetilde{\mathbb{P}}_\Lambda$ the proper transform of X_p by $\pi : \widetilde{\mathbb{P}}_\Lambda \to \mathbb{P}_\Lambda$.

Theorem(1-1). *There exists a Zariski open subset $M_{ls} \subset M$ (the locus of "log-smooth points") characterized by the property that a morphism $p : \mathrm{Spec}(\Lambda) \to M$ factors through M_{ls} if and only if the following condition $(*)$ holds:*
$(*)$ $\widetilde{X}_p$ is regular and $X_p \to \mathrm{Spec}(\Lambda)$ is smooth outside 0 and $\widetilde{X}_p$ is a strict semi-stable reduction over $\mathrm{Spec}(\Lambda)$.

Moreover $M_{ls} \subset M$ is strictly dense, where a dense open immersion $U \hookrightarrow V$ of Λ-schemes is strictly dense if $U \otimes_\Lambda k \hookrightarrow V \otimes_\Lambda k$ is dense.

Fix a morphism $p : \mathrm{Spec}(\Lambda) \to M$ and let $F \in P^d_\Lambda(r)$ be a corresponding non-zero polynomial. Let $f = F(1, x_1, \ldots, x_n) \in A_\Lambda$. By

Definition(1-1) we can write

$$(1\text{-}1)\quad f = \sum_{\alpha_1+\cdots+\alpha_n\le d} a_{\alpha_1,\ldots,\alpha_n} t^{\beta} x_1^{\alpha_1} x_2^{\alpha_2}\cdots x_n^{\alpha_n} \quad (a_{\alpha_1,\ldots,\alpha_n}\in\Lambda),$$
$$\beta = \beta(\alpha_1,\ldots,\alpha_n) = \max\{0, r - s(\alpha_1+\cdots+\alpha_{n-1}) - \alpha_n\}.$$

Note $M = \mathrm{Proj}(\Lambda[a_{\alpha_1,\ldots,\alpha_n}])$ by taking the coefficients of f as a homogeneous coordinate. Put

$$(1\text{-}2)\quad \begin{cases} f^{hom} = \displaystyle\sum_{s(\alpha_1+\cdots+\alpha_{n-1})+\alpha_0+\alpha_n=r} a_{\alpha_1,\ldots,\alpha_n} x_0^{\alpha_0} x_1^{\alpha_1} x_2^{\alpha_2}\cdots x_n^{\alpha_n} \\ \qquad\qquad \in \Lambda[x_0,x_1,\ldots,x_n], \\ f_0^{hom} = (f^{hom} \mod t) \in k[x_0,x_1,\ldots,x_n]. \end{cases}$$

Grading $k[x_0,\ldots,x_n]$ by $\deg x_i = s$ for $1\le i\le n-1$ and $\deg x_0 = \deg x_n = 1$, f_0^{hom} is homogeneous of degree r. Let $\mathbb{P}(Q) := \mathrm{Proj}(k[x_0,x_1,\ldots,x_n])$ be the weighted projective space of type $Q = (1,s,\ldots,s,1)$. Now Theorem(1-1) is an easy consequence of Theorem(1-2) below.

Theorem(1-2). *X_p satisfies the condition $(*)$ of Theorem(1-1) if and only if the following conditions hold.*

(1) *$\partial F/\partial X_0,\ldots,\partial F/\partial X_n$ have no common zero in $\mathbb{P}_\Lambda - \{0\}$.*
(2) *$x_0\partial f_0^{hom}/\partial x_0, \partial f_0^{hom}/\partial x_1,\ldots,\partial f_0^{hom}/\partial x_n$ have no common zero in $\mathbb{P}(Q)$.*

Remark(1-2). For f as Rem.(1-1), we have $f_0^{hom} = x_0^r + x_1^\delta + \cdots + x_{n-1}^\delta + x_n^r$ and we easily see that it satisfies Theorem(1-2)(2). Hence it corresponds to a morphism $\mathrm{Spec}(\Lambda)\to M_{ls}$ if we take h sufficiently general so that f satisfies the condition of Theorem(1-2)(1).

Proposition(1-3). *Let $V_1\subset\mathbb{P}(Q)$ be the hypersurface defined by $f_0^{hom}=0$ and let $V_2 = V_1\cap\{x_0=0\}$ and $V_3 = V_1\cap\{x_0=x_n=0\}$. The condition of Theorem(1-2)(2) is equivalent to the following condition: If $s=1$, V_1 and V_2 are regular. If $s\ge 2$, V_3 and $V_1\cap(D^+(x_0)\cup D^+(x_n))$ and $V_2\cap D^+(x_n)$ are regular where $D^+(x_i) = \mathbb{P}(Q) - \{x_i=0\}$.*

Proof. The proof is standard and left to the readers. Q.E.D.

Now we prove Theorem(1-2). Let $H_t\subset\mathbb{P}_\Lambda$ be defined by $\{t=0\}$ and let $\widetilde{H}_t\subset\widetilde{\mathbb{P}}_\Lambda$ be its proper transform. In view of Proposition(1-1)

the conditions of Theorem(1-1) are equivalent to the conditions that $X_p \to \mathrm{Spec}(\Lambda)$ is smooth outside 0 and that:

(*): $\widetilde{X}_p \cap \mathbb{E}_k$, $\widetilde{X}_p \cap \mathbb{E}_k \cap \mathbb{E}_{k+1}$ $(1 \leq k \leq s-1)$, $\widetilde{X}_p \cap \widetilde{H}_t$, $\widetilde{X}_p \cap \widetilde{H}_t \cap \mathbb{E}_k$ and $\widetilde{X}_p \cap \widetilde{H}_t \cap \mathbb{E}_k \cap \mathbb{E}_{k+1}$ intersect transversally.

It is standard that the first condition is equivalent to Theorem(1-2)(1). We show the equivalence of the condition (*) and the condition of Proposition(1-3). We assume $s \geq 2$ and leave the (easier) case $s = 1$ to the readers. Recall that we have

$$\widetilde{\mathbb{A}}_\Lambda = \bigcup_{\sigma \in \Delta_s} U_\sigma \to \mathbb{A}_\Lambda = \mathrm{Spec}(\Lambda[x_1, x_2, \ldots, x_n]),$$

where we write U_σ for $U_\sigma \times_{\mathrm{Spec}(k[t])} \mathrm{Spec}(\Lambda)$. We describe the condition (*) on each open subset U_σ for $\sigma = \sigma_{0,\hat{i}}, \sigma^{\pm}_{k,\hat{i}}, \sigma^{\pm}_{s}$. For simplicity we assume $n = 3$ while the argument in general case is the same.

Case $\sigma = \sigma_{0,\hat{i}}$ *or* $\sigma^{\pm}_{k,\hat{i}}$. We take for example $\sigma = \sigma^{+}_{k,\hat{2}} = \mathbb{R}_{\geq 0} v_k + \mathbb{R}_{\geq 0} v_{k+1} + \mathbb{R}_{\geq 0} e_0 + \mathbb{R}_{\geq 0} e_1$ with $1 \leq k \leq s-1$. We have $U_\sigma = \mathrm{Spec}(R_\sigma)$ where

$$R_\sigma = \Lambda[y_1, y_2, y_3, y_4]/(t - y_1 y_2 y_3)) \quad \text{with} \quad \begin{cases} x_1 = y_1^k y_2^{k+1} y_4 \\ x_2 = y_1^k y_2^{k+1} \\ x_n = y_1 y_2. \end{cases}$$

Note that $\mathbb{E} \cap U_\sigma = (\mathbb{E}_k \cup \mathbb{E}_{k+1}) \cap U_\sigma$ and $\mathbb{E}_k$, $\mathbb{E}_{k+1}$, $\widetilde{H}_t$ are defined by $y_1 = 0$, $y_2 = 0$, $y_3 = 0$ respectively on U_σ. First assume $k \leq s-2$. An easy calculation shows that $f = (y_1^k y_2^{k+1})^\delta (g + y_1 y_2 h)$ with $h \in R_\sigma$ and

$$g = g(y_4) = \sum_{\alpha_1 + \alpha_2 = \delta, \alpha_n = 0} a_{\alpha_1, \alpha_2, 0} \cdot y_4^{\alpha_1}.$$

We note that

$$(*1) \qquad x_2^\delta g\Big(\frac{x_1}{x_2}\Big) = f^{hom}(0, x_1, x_2, 0)$$

Now the condition (*) on U_σ is equivalent to the regularity of $R_\sigma/(y_1, y_2, y_3, g)$. By (*1) it is equivalent to the regularity of $V_3 \cap D^+(x_2)$ where V_3 is as in Proposition(1-3). The similar computations in cases $\sigma = \sigma_{0,\hat{i}}$ or $\sigma^{\pm}_{k,\hat{i}}$ show that the condition (*) on U_σ for these σ is equivalent to the regularity of V_3. In case $k = s-1$ we see that $f = (y_1^{s-1} y_2^s)^\delta (g +$

y_1y_2h) with $h \in R_\sigma$ and

$$\begin{aligned} g &= g(y_1, y_3, y_4) \\ &= \sum_{s(\alpha_1+\alpha_2)+\alpha_n \le r} a_{\alpha_1,\alpha_2,\alpha_n} \cdot y_1^{\delta-(\alpha_1+\alpha_2)} y_3^{r-s(\alpha_1+\alpha_2)-\alpha_n} y_4^{\alpha_1}. \end{aligned}$$

We note that

$$(*2) \qquad \begin{aligned} x_2^\delta g\Big(\frac{x_n^s}{x_2}, \frac{x_0}{x_n}, \frac{x_1}{x_2}\Big) &= f^{hom}(x_0, x_1, x_2, x_n), \\ g(0, y_3, y_4) &= \sum_{\alpha_1+\alpha_2=\delta, \alpha_n=0} a_{\alpha_1,\alpha_2,0} \cdot y_4^{\alpha_1}. \end{aligned}$$

Now the condition $(*)$ on U_σ is equivalent to the regularity of $R_\sigma/(y_i, g)$ and $R_\sigma/(y_3, y_i, g)$ with $i = 1, 2$ and that of $R_\sigma/(y_1, y_2, g)$ and $R_\sigma/(y_1, y_2, y_3, g)$. By $(*1)$ and $(*2)$ it is equivalent to the regularity of $V_3 \cap D^+(x_2)$ and $V_1 \cap (D^+(x_2) \cap D^+(x_n))$ and $V_2 \cap (D^+(x_2) \cap D^+(x_n))$. The similar computations in cases $\sigma = \sigma^\pm_{k,\hat{i}}$ with $k = s-1$ show that the condition $(*)$ on U_σ for these σ is equivalent to the regularity of V_3 and $V_1 \cap (D^+(x_0) \cup D^+(x_n)) \cap (D^+(x_1) \cup D^+(x_2))$ and $V_2 \cap D^+(x_n) \cap (D^+(x_1) \cup D^+(x_2))$.

Case $\sigma = \sigma_s^\pm$. We take for example $\sigma = \sigma_s^+ = \mathbb{R}_{\ge 0}v_s + \mathbb{R}_{\ge 0}e_1 + \mathbb{R}_{\ge 0}e_2 + \mathbb{R}_{\ge 0}e_0$. We have $U_\sigma = \mathrm{Spec}(R_\sigma)$ where

$$R_\sigma = \Lambda[u_0, u_1, u_2, u_3]/(t - u_0u_3) \quad \text{with} \quad \begin{cases} x_1 = u_3^s u_1 \\ x_2 = u_3^s u_2 \\ x_n = u_3. \end{cases}$$

Note that $\mathbb{E} \cap U_\sigma = \mathbb{E}_s \cap U_\sigma$ and $\mathbb{E}_s$ and $\widetilde{H}_t$ are defined by $u_3 = 0$ and $u_0 = 0$ on U_σ respectively. An easy calculation shows that $f = u_3^r(g + u_3h)$ with $h \in R_\sigma$ and

$$g = g(u_0, u_1, u_2) = \sum_{s(\alpha_1+\alpha_2)+\alpha_n \le r} a_{\alpha_1,\alpha_2,\alpha_n} u_0^{r-s(\alpha_1+\alpha_2)-\alpha_n} u_1^{\alpha_1} u_2^{\alpha_2}.$$

We note

$$(*3) \qquad x_n^r g\Big(\frac{x_0}{x_n}, \frac{x_1}{x_n^s}, \frac{x_2}{x_n^s}\Big) = f^{hom}(x_0, x_1, x_2, x_n).$$

Now the condition $(*)$ on U_σ is equivalent to the regularity of $R_\sigma/(u_3, g)$ and $R_\sigma/(u_0, u_3, g)$. By $(*3)$ it is equivalent to the regularity of $V_1 \cap D^+(x_n)$ and $V_2 \cap D^+(x_n)$. The similar computation in case $\sigma = \sigma_s^-$

shows that the condition $(*)$ on U_σ for these σ is equivalent to the regularity of $V_1 \cap (D^+(x_0) \cup D^+(x_n))$ and $V_2 \cap D^+(x_n)$. This completes the proof of Theorem(1-2). Q.E.D.

As an immediate consequence of the above proof we get the following.

Proposition(1-4). *For* $p : \mathrm{Spec}(\Lambda) \to M_{ls}$, $\pi^{-1}(X_p) = \widetilde{X}_p + \delta \cdot \sum_{1 \leq k \leq s} k\mathbb{E}_k$ *as a divisor on* $\widetilde{\mathbb{P}}_\Lambda$.

§2. Injectivity of infinitesimal period map

Fix a morphism $p : \mathrm{Spec}(\Lambda) \to M_{ls}$ and let $X \subset \mathbb{P}_\Lambda$ and $\widetilde{X} \subset \widetilde{\mathbb{P}}_\Lambda$ be the corresponding hypersurface and its proper transform with the projection $\pi : \widetilde{X} \to X$. Let $Z \subset \widetilde{X}$ be the closed fiber of $\widetilde{X} \to \mathrm{Spec}(\Lambda)$.

Definition(2-1). Let $\omega_{\widetilde{X}/S} = \Omega_{\widetilde{X}/S}(\log Z)$ be the sheaf of logarithmic differentials of the semistable family $\widetilde{X}$ over $S := \mathrm{Spec}(\Lambda)$ in the sense of [St]. By [St] $\omega_{\widetilde{X}/S}$ is a locally free $\mathcal{O}_{\widetilde{X}}$-module. We put

$$\omega^q_{\widetilde{X}/S} = \overset{q}{\wedge}\omega_{\widetilde{X}/S} \quad \text{and} \quad \theta_{\widetilde{X}/S} = \mathcal{H}om_{\mathcal{O}_{\widetilde{X}}}(\omega_{\widetilde{X}/S}, \mathcal{O}_{\widetilde{X}}).$$

We also define a locally free $\mathcal{O}_Z$-modules

$$\omega_{Z/S_0} = \omega_{\widetilde{X}/S} \otimes_\Lambda k, \quad \omega^q_{Z/S_0} = \overset{q}{\wedge}\omega_{Z/S_0}, \quad \theta_{Z/S_0} = \mathcal{H}om_{\mathcal{O}_Z}(\omega_{Z/S_0}, \mathcal{O}_Z).$$

Remark(2-1). In the language of log geometry (cf. [K]), $\omega_{\widetilde{X}/S}$ (resp. ω_{Z/S_0}) is the sheaf of logarithmic differentials of the log smooth morphisms $(\widetilde{X}, M_{\widetilde{X}}) \to (\mathrm{Spec}(\Lambda), N_\Lambda)$ (resp. $(Z, M_Z) \to (\mathrm{Spec}(k), N_0)$). Here N_0 and N_Λ are the log structure defined by $\mathbb{N} \to \Lambda \to k$; $1 \to t \to 0$ and $M_{\widetilde{X}}$ is associated to the embedding $Z \subset \widetilde{X}$ and M_Z is its inverse image on Z.

Let $m = n - 1$ be the relative dimension of $\widetilde{X}/S$ and let

$$H^1(Z, \theta_{Z/S_0}) \otimes H^p(Z, \omega^{m-p}_{Z/S_0}) \to H^{p+1}(Z, \omega^{m-1-p}_{Z/S_0})$$

be the map induced by the contraction $\theta_{Z/S_0} \otimes \omega^q_{Z/S_0} \to \omega^{q-1}_{Z/S_0}$ and the cup product. It induces

$$H^1(Z, \theta_{Z/S_0}) \overset{d\rho_Z}{\to} \bigoplus_{0 \leq p \leq m-1} \mathrm{Hom}(H^p(Z, \omega^{m-p}_{Z/S_0}), H^{p+1}(Z, \omega^{m-1-p}_{Z/S_0})).$$

Theorem(2-1). *Let the assumption be as in Definition*(1-1) *and Definition*(1-2). *Assume the following conditions*:

(i) $d \geq r+1$ *and either* $n \geq 4$ *or* $n = 3$ *and* $d \neq 4$.
(ii) $\delta < n$.
(iii) $H^0(Z, \theta_{Z/S_0}) = 0$.

The map $d\rho_Z$ *is injective if* n *is even or* $\delta \geq 3$. *If* n *is odd and* $\delta = 2$ *then* $\dim_k(\mathrm{Ker}(d\rho_Z)) = r-1$.

Remark(2-2). We will see (cf. Remark(4-1)) that the singularity $(X, 0)$ is canonical if and only if $\delta < n$ or $\delta = n$, $s = 1$. Thus Theorem(2-1) suggests that the canonicality of the singularity plays an important role in the infinitesimal logarithmic Torelli problem.

Concerning the second assumption of Theorem (2-1), we have the following.

Proposition(2-1). *The condition* $H^0(Z, \theta_{Z/S_0}) = 0$ *defines strictly dense open subset* M_{ls}^{st} *of* M_{ls}.

Proposition(2-1) follows from Proposition(2-2)(2) below that will be proven in §4. Indeed, by the semicontinuity, the vanishing of $H^0(Z, \theta_{Z/S_0})$ is an open condition on the moduli space. Thus it suffices to show that there exists a morphism $p : \mathrm{Spec}(\Lambda) \to M_{ls}$ satisfying the condition. Indeed we may take for example

$$F = X_0^{d-\delta}(X_1^\delta + \cdots X_{n-1}^\delta) + X_0^{d-r}X_n^r + \Phi_d + t^r G \in P_\Lambda^d(r)$$

where Φ_d (resp. G) is a sufficiently general homogeneous polynomials of degree d in $X_1, \ldots, X_n$ (resp. in $X_0, X_1, \ldots, X_n$) (cf. Theorem(1-2)).

Proposition(2-2). *Assume* $d \geq r$ *and* $\delta < n$.

(1) $H^j(Z, \theta_{Z/S_0}) = 0$ *for* $j \geq 2$ *if either* $j \leq n-3$ *or* $j = n-2$ *and* $d \neq n+1$ *or* $j = n-1$ *and* $d \leq n+1$.
(2) *Let* $F \in P_\Lambda^d(r)$ *be a polynomial defining* $X \subset \mathbb{P}_\Lambda$ *and put* $F_0 = (F \mod t) \in k[X_0, \ldots, X_n]$. *Then* $H^0(Z, \theta_{Z/S_0}) = 0$ *if either* $s \geq 2$ *and*

$$X_j \partial F_0/\partial X_i \ (1 \leq i, j \leq n-1), \ X_j \partial F_0/\partial X_i \ (1 \leq j \leq n, \ i = 0, n)$$

are linearly independent over k, *or if* $s = 1$ *and*

$$X_j \partial F_0/\partial X_i \ (1 \leq i, j \leq n), \ X_j \partial F_0/\partial X_0 \ (0 \leq j \leq n)$$

are linearly independent over k.

In the rest of this section we give a geometric implication of Theorem(2-1). We suppose that the readers are familiar with basic notions of log geometry.

Let M_{ls}^{st} be as Proposition(2-1) and put $M_{ls,0}^{st} = M_{ls}^{st} \otimes_{\Lambda} k$. Note that M_{ls}^{st} is smooth over $\mathrm{Spec}(\Lambda)$. The construction of §1 gives us the following cartesian diagram of log schemes

$$\begin{array}{ccc} (Z^{univ}, M_{Z^{univ}}) & \hookrightarrow & (\widetilde{X}^{univ}, M_{\widetilde{X}^{univ}}) \\ \downarrow f_0^{univ} & & \downarrow f^{univ} \\ (M_{ls,0}^{st}, N_0) & \hookrightarrow & (M_{ls}^{st}, N_{\Lambda}) \end{array} \tag{2-1}$$

with the following properties: Let

$$\underline{S}_0 = (\mathrm{Spec}(k), N_0) \hookrightarrow \underline{S} = (\mathrm{Spec}(\Lambda), N_{\Lambda})$$

denote the exact closed immersion of log schemes, where N_0 and N_{Λ} are defined in Remark(2-1).

(a): N_{Λ} and N_0 in (2-1) are defined in the same way as Remark(2-1).

(b): $Z^{univ} \subset \widetilde{X}^{univ}$ is a simple normal crossing divisor defined by $t = 0$.

(c): $M_{\widetilde{X}^{univ}}$ is associated to the embedding $Z^{univ} \subset \widetilde{X}^{univ}$ and $M_{Z^{univ}}$ is its inverse image.

(d): f^{univ} and f_0^{univ} are log smooth of semistable type.

(e): Let $p : \underline{S} \to (M_{ls}^{st}, N_{\Lambda})$ be an exact closed immersion and let $p_0 : \underline{S}_0 \to (M_{ls,0}^{st}, N_0)$ be the induced exact closed immersion. By pulling back the above diagram via p and p_0, we get the following cartesian diagram of log smooth morphisms

$$\begin{array}{ccc} (Z, M_Z) & \hookrightarrow & (\widetilde{X}, M_{\widetilde{X}}) \\ \downarrow & & \downarrow \\ \underline{S}_0 & \hookrightarrow & \underline{S} \end{array}$$

where the underlying morphisms are those associated to p as in §1 and $M_{\widetilde{X}}$ and M_Z are defined as Remark(2-1).

By (a) $(M_{ls}^{st}, N_{\Lambda})$ is a log scheme over $\underline{S}$. A morphism $p_0 : \mathrm{Spec}(k) \to M_{ls,0}^{st}$ extends in the unique way to an exact closed immersion $p_0 : \underline{S}_0 \to (M_{ls,0}^{st}, N_0)$ over $\underline{S}_0$. By pulling back f_0^{univ} in (2-1) via p_0 we get the log smooth morphism

$$f_{p_0} : (Z_{p_0}, M_{Z_{p_0}}) \to \underline{S}_0.$$

Now assume $k = \mathbb{C}$ and let $\Lambda \subset \mathbb{C}[[t]]$ be the ring of convergent formal power series. By [Mat] and [FK3] f_{p_0} gives rise to a log Hodge

structure $H(p_0)$ over $\underline{S}_0$ that is underlain by

$$(R^m f_{p_0*}^{log}\mathbb{Q}, R^m f_{p_0*}\omega^{\cdot}_{Z_{p_0}/S_0}),$$

a pair of a local system on $\underline{S}_0^{log}(\simeq \{t \in \mathbb{C} \mid |t| = 1\})$ and a $\mathbb{C}$-vector space with the descending filtration given by subspaces $R^m f_{p_0*}\omega^{\geq q}_{Z_{p_0}/S_0} \subset R^m f_{p_0*}\omega^{\cdot}_{Z_{p_0}/S_0}$. We note that $H(p_0)$ is determined by the limiting Hodge structure defined by Steenbrink [St] on the space

$$H^m(\widetilde{X}^{an} \times_{B_\epsilon} \overline{B}^*_\epsilon, \mathbb{Q}).$$

Here $\widetilde{X}^{an} \to B_\epsilon := \{t \in \mathbb{C} \mid |t| < \epsilon\}$ with sufficiently small $\epsilon > 0$ is the morphism of complex analytic space that arises from $\widetilde{X} \to \mathrm{Spec}(\Lambda)$ corresponding to a lift $p : \mathrm{Spec}(\Lambda) \to M^{st}_{ls}$ of p_0 over $\mathrm{Spec}(\Lambda)$ and $\overline{B}^*_\epsilon$ is the universal covering of $B_\epsilon - \{0\}$.

Theorem(2-2). *Assume the following conditions*:

(i) $d \geq r+1$ *and either* n *is even or* $\delta \geq 3$.
(ii) $\delta < n$
(iii) $n \geq 4$.

Locally on $M^{st}_{ls,0}$, $H(p_0)$ *determines* $(Z_{p_0}, M_{Z_{p_0}})$ *up to isomorphisms of log schemes over* $\underline{S}_0$.

Proof Fix an exact closed immersion $p_0 : \underline{S}_0 \to (M^{st}_{ls,0}, N_0)$ and denote simply by $f_0 : (Z_0, M_{Z_0}) \to \underline{S}_0$ the log smooth morphism obtained by pulling back f_0^{univ} via p_0. We recall the logarithmic deformation theory of f_0 (cf. [KN], [FK1] and [FK2]). Let $\mathcal{C}_k$ (resp. $\mathcal{C}_k^\wedge$) be the category of artinian (resp. Noetherian complete) local k-algebra with residue field k. Let $L\mathcal{C}_k$ (resp. $L\mathcal{C}_k^\wedge$) be the category of pairs $(\underline{T}, 0_{\underline{T}})$ where $\underline{T} = (\mathrm{Spec}(A), M)$ with $A \in \mathcal{C}_k$ is a log scheme (resp. $\underline{T} = (\mathrm{Spf}(A), M)$ with $A \in \mathcal{C}_k^\wedge$ is a formal log scheme) whose log structure M is isomorphic to the inverse image of the log structure on $\underline{S}_0$ via $\mathrm{Spec}(A) \to \mathrm{Spec}(k)$ and $0_{\underline{T}} : \underline{S}_0 \hookrightarrow \underline{T}$ is an exact closed immersion whose underlying morphisms come from $\mathcal{C}_k$ (resp. $\mathcal{C}_k^\wedge$). One defines the functor $D_{Z_0/S_0} : L\mathcal{C}_k \to Sets$ by setting $D_{Z_0/S_0}(\underline{T})$ to be the set of isomorphism classes of log smooth liftings of f_0 to $\underline{T}$ (cf. [FK2, Definition 4.1]). We remark that by definition the log structure $\alpha : M \to A$ for an object $\underline{T} = (\mathrm{Spec}(A), M) \in L\mathcal{C}_k$ is isomorphic to $\mathbb{N} \oplus A^* \to A$; $(n, u) \to 0^n u$. Hence D_{Z_0/S_0} deals only with locally trivial deformations of f_0. By [FK1, §8], [FK2, §4] and [FK4] we have the following facts:

(a): D_{Z_0/S_0} is pro-represented by an object

$$\underline{\mathcal{T}} = (\mathcal{T} = \mathrm{Spf}(R), N_{\mathcal{T}})$$

in $LC_k^\wedge$ with the universal family $\varphi : (\mathcal{X}, M_\mathcal{X}) \to (\mathcal{T}, N_\mathcal{T})$. Moreover R is formally smooth over k. By definition the fiber of φ over $0_{\underline{T}} : \underline{S}_0 \to \underline{T}$ is f_0.

(b): Let $k[\epsilon]$ be the ring of dual numbers and let N_ϵ be the inverse image of the log structure N_0 on S_0 via the map $\mathrm{Spec}(k[\epsilon]) \to \mathrm{Spec}(k)$ induced by the canonical map $k \to k[\epsilon]$. By definition $\underline{S}_\epsilon := (\mathrm{Spec}(k[\epsilon]), N_\epsilon)$ with the canonical exact closed immersion $\underline{S}_0 \hookrightarrow \underline{S}_\epsilon$ induced by the residue map $k[\epsilon] \to k$ is an object in LC_k. For an object $\underline{T}$ in LC_k or $LC_k^\wedge$ we call $T_0(\underline{T}) = \mathrm{Hom}_{LC_k^\wedge}(\underline{S}_\epsilon, \underline{T})$ the logarithmic tangent space of $\underline{T}$ at $0_{\underline{T}}$, where the space on the right hand side denotes the set of morphisms $\underline{S}_\epsilon \to \underline{T}$ in LC_k. The logarithmic Kodaira-Spencer map induces the isomorphism

$$H^1(Z_0, \theta_{Z_0/S_0}) \xrightarrow{\sim} T_0(\underline{T}) = D_{Z_0/S_0}(\underline{S}_\epsilon).$$

(c): For a morphism

$$\phi : \underline{T} = (\mathrm{Spf}(R), M) \to \underline{T}' = (\mathrm{Spf}(R'), M')$$

in $LC_k^\wedge$, we define its logarithmic differential of ϕ to be the map

$$d\phi : T_0(\underline{T}) \to T_0(\underline{T}').$$

If R and R' are formally smooth over k and if $d\phi$ is injective, then the underlying morphism $\mathrm{Spf}(R) \to \mathrm{Spf}(R')$ is an embedding.

The pro-representability follows from the assumption $H^0(Z_0, \theta_{Z_0/S_0}) = 0$ and the rigidity of D_{Z_0/S_0} (cf. [FK2, §3 and §4]). The formally smoothness of R over k is a consequence of the fact that D_{Z_0/S_0} has no obstruction that follows from Proposition(2-2)(1) (with $j = 2$) and the assumption (ii) (In cases $n = 4$, $d = 5$, $H^2(Z_0, \theta_{Z_0/S_0}) \neq 0$ so that one needs an extra argument that we omit).

Now we assume $k = \mathbb{C}$ and let $\varphi^{an} : (\mathcal{X}^{an}, M_{\mathcal{X}^{an}}) \to \underline{\mathcal{T}}^{an} = (\mathcal{T}^{an}, N_{\mathcal{T}^{an}})$ be the corresponding morphism of log analytic spaces over $\mathbb{C}$. By the universality there is an open neighborhood $V \subset M_{ls,0}^{st}$ of p_0 and a strict morphism of log analytic spaces $g : (V, N_{0|V}) \to \underline{\mathcal{T}}^{an}$ mapping p_0 to 0 such that the restriction to V of f_0^{univ} in the diagram (2-1) is isomorphic to the pullback of φ^{an} via g. By the theory of logarithmic Hodge structures and their moduli space (cf. [KU], [Mat], [FK3] and [Us]) φ^{an} gives rise to the extended period map

$$\rho : \underline{\mathcal{T}}^{an} \to \Gamma \backslash D_\Sigma$$

where the space on the right hand side is the classifying space of logarithmic Hodge structures of suitable type equipped with its canonical

log structure. We have $\rho(g(p'_0)) = [H(p'_0)]$ for $p'_0 : \mathrm{Spec}(\mathbb{C}) \to V \subset M^{st}_{ls}$. Theorem(2-1) implies that under the assumption of Theorem(2-2) the logarithmic differential $d\rho$ at $0_{\underline{T}}$ of ρ is injective, which implies ρ is an embedding. This proves the assertion of Theorem(2-2). Q.E.D.

§3. Jacobian rings of degenerating hypersurfaces

Let the assumption be as in §2. In this section we express the cup product

$$(3\text{-}1) \qquad H^1(\widetilde{X}, \theta_{\widetilde{X}/S}) \otimes_\Lambda H^p(\widetilde{X}, \omega^{m-p}_{\widetilde{X}/S}) \to H^{p+1}(\widetilde{X}, \omega^{m-p-1}_{\widetilde{X}/S})$$

$$(3\text{-}2) \quad H^p(\widetilde{X}, \omega^{m-p}_{\widetilde{X}/S}) \otimes_\Lambda H^{m-p-1}(\widetilde{X}, \omega^{p+1}_{\widetilde{X}/S}) \to H^{m-1}(\widetilde{X}, \omega^1_{\widetilde{X}/S} \otimes \omega^m_{\widetilde{X}/S})$$

in terms of Jacobian rings and we prove Theorem(2-1).

Definition(3-1). Let $F \in P^d_\Lambda(r)$ be an equation defining $X \subset \mathbb{P}_\Lambda$.

(1) For an integer $q \geq 0$ we write $B^q_F = P^q_\Lambda / J^q_F$ where

$$J^q_F = \Big\{ \sum_{0 \leq i \leq n} H_i \partial F/\partial X_i \;\Big|\; H_i \in P^{q-d+1}_\Lambda \Big\} \subset P^q_\Lambda.$$

(2) For integers $q, \nu \geq 0$ we write $B^q_F(\nu) = P^q_\Lambda(\nu)/J^q_F(\nu)$ where

$$J^q_F(\nu) = \Big\{ \sum_{0 \leq i \leq n} H_i \partial F/\partial X_i \;\Big|\; H_i \in P^{q-d+1}_\Lambda(\nu - r + \mu_i) \Big\} \subset J^q_F,$$

where $\mu_i = s$ if $1 \leq i \leq n-1$ and $\mu_n = 1$ and $\mu_0 = 0$. Note that $J^q_F(\nu) \subset P^q_\Lambda(\nu)$ since $\partial F/\partial X_i \in P^{d-1}_\Lambda(r - \mu_i)$.

(3) Write $f = F(1, x_1, x_2, \ldots, x_n) \in A_\Lambda$. For integers $q, \nu \geq 0$ we define

$$R^q_F(\nu) = \mathrm{Ker}(B^q_F \to A_\Lambda/(I_f + \mathfrak{m}_\Lambda(\nu))),$$

where $I_f = \big\{ g_0 f + \sum_{1 \leq i \leq n} g_i \partial f/\partial x_i \mid g_i \in A_\Lambda \ (0 \leq i \leq n) \big\}$.

Lemma(3-1). *The natural map $\iota : B^q_F(\nu) \to R^q_F(\nu)$ is injective if $B^q_F(\nu)$ is torsion free as a Λ-module. It is surjective if $q - \nu \geq d - r + s - 2$.*

Proof. It is easy to see $\mathrm{Ker}(\iota)$ is torsion, which implies the first assertion immediately. To show the second assertion it suffices to show

the surjectivity of $P_\Lambda^q(\nu) \to R_F^q(\nu)$. We have the following commutative diagram of exact sequences

$$\begin{array}{ccccccc} & & & & 0 & & 0 \\ & & & & \downarrow & & \downarrow \\ & & & & J_F^q & \overset{\alpha}{\to} & I_f + \mathfrak{m}_\Lambda(\nu)/\mathfrak{m}_\Lambda(\nu) \\ & & & & \downarrow & & \downarrow \\ 0 \to & P_\Lambda^q(\nu) & \to & P_\Lambda^q & \to & & A_\Lambda/\mathfrak{m}_\Lambda(\nu) \\ & \downarrow & & \downarrow & & & \downarrow \\ 0 \to & R_F^q(\nu) & \to & B_F^q & \to & & A_\Lambda/I_f + \mathfrak{m}_\Lambda(\nu) \\ & & & \downarrow & & & \downarrow \\ & & & 0 & & & 0 \end{array}$$

By the diagram the desired assertion follows from the surjectivity of α. Note that under $P_\Lambda^q \overset{\sim}{\to} A_\Lambda^{\leq q}$

$$J_F^q \overset{\sim}{\to} \Big\{ h_0(d \cdot f - \sum_{1 \leq i \leq n} x_i \partial f/\partial x_i) + \sum_{1 \leq i \leq n} h_i \partial f/\partial x_i \;\Big|\; h_i \in A_\Lambda^{\leq q-(d-1)} \Big\}.$$

Take $\phi = g_0 f + \sum_{1 \leq i \leq n} g_i \partial f/\partial x_i \in I_f$ with $g_i \in A_\Lambda$. We have

$$\phi = \frac{1}{d} g_0 \Big(d \cdot f - \sum_{1 \leq i \leq n} x_i \partial f/\partial x_i \Big) + \sum_{1 \leq i \leq n} \Big(g_i + \frac{1}{d} g_0 x_i \Big) \partial f/\partial x_i.$$

We may write

$$\frac{1}{d} g_0 = h_0 + g_0', \;\; g_i + \frac{1}{d} g_0 x_i = h_i + g_i' \;\; (1 \leq i \leq n)$$

with $h_i \in A_\Lambda^{\leq q-(d-1)}$ and $g_i' \in (x_1, \ldots, x_n)^{q-d+2} \subset \mathfrak{m}_\Lambda(q-d+2)$ for $0 \leq i \leq n$. The assumption of Lemma(3-1) implies $\nu \leq q-d+2+r-\mu_i$ (cf. Definition(3-1)(2)) and we have

$$g_i' \partial f/\partial x_i \in \mathfrak{m}_\Lambda(q-d+2+r-\mu_i) \subset \mathfrak{m}_\Lambda(\nu).$$

Since $h_0\big(d \cdot f - \sum_{1 \leq i \leq n} x_i \partial f/\partial x_i\big)$, $h_i \partial f/\partial x_i \in J_F^q$, this completes the proof. Q.E.D.

Theorem(3-1). *Assume $n \geq 3$ and $d \geq r+1$.*

(1) *For $0 \leq p \leq m := n-1$ there is the natural isomorphism of free Λ-modules*

$$\phi_F^p \;:\; B_F^{\kappa(p)}(\nu(p)) \overset{\sim}{\to} R_F^{\kappa(p)}(\nu(p)) \overset{\sim}{\to} H^p(\widetilde{X}, \omega_{\widetilde{X}/S}^{m-p})_{prim}$$

where $\kappa(p) = d(p+1) - n - 1$ *and* $\nu(p) = s(\delta(p+1) - n + 1) - 1$. *Here the primitive part* $H^p(\widetilde{X}, \omega^q_{\widetilde{X}/S})_{prim}$ *is defined to be*

$$\mathrm{Ker}\big(H^p(\widetilde{X}, \omega^q_{\widetilde{X}/S}) \to H^p(X_\eta, \Omega^q_{X_\eta/\eta})/H^p(X_\eta, \Omega^q_{X_\eta/\eta})_{prim}\big),$$

where X_η/η *is the generic fiber of* X/S.

(2) *Assume* $\delta < n$ *and either* $n \geq 4$ *or* $n = 3$ *and* $d \neq 4$. *There is the natural isomorphism of free* Λ*-modules*

$$\phi_F^{tan*} \ : \ B_F^{d^*}(r^*) \xrightarrow{\sim} R_F^{d^*}(r^*) \xrightarrow{\sim} H^{m-1}(\widetilde{X}, \omega^1_{\widetilde{X}/S} \otimes \omega^m_{\widetilde{X}/S}),$$

where $d^* = dn - 2(n+1)$ *and* $r^* = rn - s(2n-2) - 2 = s(\delta n - 2n + 2) - 2$.

(3) *Assume* $H^0(Z, \theta_{Z/S_0}) = 0$ *and* $\delta < n$. *There is the natural injective homomorphism of free* Λ*-modules*

$$\phi_F^{tan} \ : \ R_F^d(r) \to H^1(\widetilde{X}, \theta_{\widetilde{X}/S}).$$

It is an isomorphism if either $n \geq 4$ *or* $n = 3$ *and* $d \neq 4$. *There is an exact sequence*

$$0 \to B_F^d(r) \to R_F^d(r) \to \bigoplus_{1 \leq i \leq n-1} \mathrm{Coker}(A_\Lambda^{\leq 1} \to A_\Lambda/\mathfrak{m}_\Lambda(s)) \to 0.$$

In particular $B_F^d(r) \xrightarrow{\sim} R_F^d(r)$ *if and only if* $s \leq 2$.

Moreover the cup products (3-1) *and* (3-2) *are compatible with the multiplication of the Jacobian rings.*

The proof of Theorem(3-1) is given in §4.

Remark(3-1). *Under the assumption* $d \geq r + 1$ *we can verify* $\kappa(p) - \nu(p)$, $d^* - r^* \geq d - r + s - 2$. *Hence Lemma*(3-1) *implies the first isomorphisms of Theorem*(3-1)(1) *and* (2) *if we already know that the groups on the left hand side are torsion free.*

We have the following auxiliary result that will be proven in §4.

Proposition(3-1). *Assume* $d \geq r$ *and* $j \geq 2$ *and* $\delta < n$. *Then* $H^j(\widetilde{X}, \theta_{\widetilde{X}/S})$ *is a free* Λ*-module and it vanishes if either* $j \leq n - 3$ *or* $j = n - 2$ *and* $d \neq n + 1$ *or* $j = n - 1$ *and* $d \leq n + 1$.

To deduce Theorem(2-1) from Theorem(3-1) we need the following.

Lemma(3-2). *Assume* $d \geq r + 1$ *and* $n \geq 3$. *For an integer* $0 \leq p \leq n - 2$ *let*

$$\psi_p \ : \ P_\Lambda^{\kappa(p)}(\nu(p)) \otimes P_\Lambda^{\kappa(n-2-p)}(\nu(n-2-p)) \to P_\Lambda^{d^*}(r^*)$$

be the multiplication. Recall that $P_\Lambda^{d^*}(r^*)$ *is generated over* k *by those polynomials of the form* $\phi_\gamma x_n^\alpha t^\beta$ *where* $s\gamma+\alpha+\beta \geq r^*$ *and* $\alpha+\gamma \leq d^*$ *and* ϕ_γ *is a homogeneous polynomial of degree* γ *in* $x_1, \ldots, x_{n-1}$. *Assume that* $\nu(p) \geq 0$ *and* $\nu(n-2-p) \geq 0$.

(1) *If* $s = 1$, ψ_p *is surjective. If* $s \geq 2$, $\mathrm{Im}(\psi_p)$ *contains all the above polynomials except for* $\phi_{\mu-1} x_n^\alpha t^\beta$ *with* $\alpha+\beta = s-2$ *where* $\mu = (\delta-2)n+2$.

(2) *The composite of* ψ_p *with the projection* $P_\Lambda^{d^*}(r^*) \to B_F^{d^*}(r^*)$ *is surjective.*

Proof. First we deduce Lemma(3-2)(2) from (1). We may assume $s \geq 2$ and fix $\phi_{\mu-1} x_n^\alpha t^\beta$ with $\alpha+\beta = s-2$ as in (1). By Definition(3-1)(2) $J_F^{d^*}(r^*)$ contains the polynomials

$$x_n^\alpha t^\beta \sum_{1 \leq i \leq n-1} h_i \partial f / \partial x_i$$

where h_i is a homogeneous polynomial of degree $\mu-\delta$ in $x_1, \ldots, x_{n-1}$ and f is as in Definition(3-1)(3). In fact we need note that the assumptions $d \geq r+1$ and $\delta \geq 2$ imply $\mu-\delta = (\delta-2)(n-1) \geq 0$ and $d^* - ((s-2)+(\mu-\delta)+(d-1)) = (n-1)(d-\delta) - s - 1 \geq 0$. We can write

$$f = \sum_{a,b,c \geq 0} \Phi_{a,b,c} x_n^a t^b$$

where $\Phi_{a,b,c}$ is a homogeneous polynomial of degree c in $x_1, \ldots, x_{n-1}$ and $a, b, c \geq 0$ are integers satisfying $sc+a+b \geq r$ and $c+a \leq d$. By Theorem(1-2) and Proposition(1-3) $\Phi_{0,0,\delta}$ is non-degenerate. By Macaulay's theorem ([D, §2]) any homogeneous polynomial of degree $> (\delta-2)(n-1) = \mu-\delta$ in $x_1, \ldots, x_{n-1}$ is in the homogeneous ideal generated by $\partial \Phi_{0,0,\delta}/\partial x_i$ with $1 \leq i \leq n-1$. Noting $\mu-1 > \mu-\delta \geq 0$ if $\delta \geq 2$, we can thus find h_i homogeneous of degree $\mu-\delta$ in $x_1, \ldots, x_{n-1}$ such that

$$\phi_{\mu-1} x_n^\alpha t^\beta \equiv \sum_{1 \leq i \leq n-1} \sum_{(a,b,c) \neq (0,0,\delta)} x_n^{\alpha+a} t^{\beta+b} h_i \partial \Phi_{a,b,c}/\partial x_i \mod J_F^{d^*}(r^*).$$

Thus the desired assertion follows from Lemma(3-2)(1).

Next we prove Lemma(3-2)(1). We prove only the statement in case $s \geq 2$ and leave the (easier) case $s = 1$ to the readers. Put

$$\begin{cases} \kappa_1 = \kappa(p), \ \kappa_2 = \kappa(n-2-p), \ \nu_1 = \nu(p), \ \nu_2 = \nu(n-2-p) \\ \lambda_1 = \delta(p+1) - (n-1), \ \lambda_2 = \delta(n-p-1) - (n-1) \end{cases}$$

We note

$$\begin{cases} d^* = \kappa_1 + \kappa_2,\ \mu = \lambda_1 + \lambda_2,\ \lambda_1, \lambda_2 \geq 1 \\ r^* = s\mu - 2,\ \nu_1 = s\lambda_1 - 1,\ \nu_2 = s\lambda_2 - 1. \end{cases}$$

We have

$$P_\Lambda^{d^*}(r^*) = \sum_{\epsilon \leq \mu} P_\Lambda^{d^*}(r^*)_\epsilon \quad \text{and} \quad P_\Lambda^{\kappa_i}(\nu_i) = \sum_{\tau_i \leq \lambda_i} P_\Lambda^{\kappa_i}(\nu_i)_{\tau_i}\ (i = 1, 2).$$

Here $P_\Lambda^{d^*}(r^*)_\epsilon \subset P_\Lambda^{d^*}(r^*)$ and $P_\Lambda^{\kappa_i}(\nu_i)_{\tau_i} \subset P_\Lambda^{\kappa_i}(\nu_i)$ are the submodule generated by those polynomials of the form

$$(*) \qquad \phi_{\mu-\epsilon} x_n^\alpha t^\beta \quad \text{with } \alpha + \beta \geq s\epsilon - 2 \text{ and } \alpha \leq d^* - \mu + \epsilon,$$

$$\phi_{\lambda_i - \tau_i} x_n^{\alpha_i} t^{\beta_i} \quad \text{with } \alpha_i + \beta_i \geq s\tau_i - 1 \text{ and } \alpha_i \leq \kappa_i - \lambda_i + \tau_i,$$

respectively where ϕ_γ is as in Lemma(3-2). When $\tau_1 + \tau_2 = \epsilon$, the multiplication induces

$$\psi_{\epsilon, \tau_1, \tau_2} : P_\Lambda^{\kappa_1}(\nu_1)_{\tau_1} \otimes_\Lambda P_\Lambda^{\kappa_1}(\nu_1)_{\tau_1} \to P_\Lambda^{d^*}(r^*)_\epsilon.$$

We note that the assumption $d \geq r + 1$, $n \geq 3$ and $s \geq 2$ implies that $d^* > r^*$ and $\kappa_i > \nu_i$ $(i = 1, 2)$, so that $d^* - \mu + \epsilon > s\epsilon - 2$ and $\kappa_i - \lambda_i + \tau_i > s\tau_i - 1$. From this we see that $\psi_{\epsilon, \tau_1, \tau_2}$ is surjective if either $s\epsilon - 2 \geq 0$, $s\tau_i - 1 \geq 0$ $(i = 1, 2)$ or $s\epsilon - 2 \leq 0$, $s\tau_i - 1 \leq 0$ $(i = 1, 2)$. Unless $\epsilon = 1$, for a given $\epsilon \leq \mu$ we can find $\tau_i \leq \lambda_i$ $(i = 1, 2)$ such that $\tau_1 + \tau_2 = \epsilon$ and that the above condition is satisfied. Thus we get $P_\Lambda^{d^*}(r^*)_\epsilon \subset \mathrm{Im}(\psi_p)$ if $\epsilon \neq 1$. When $\epsilon = 1$, it is easy to see that $\mathrm{Im}(\psi_{1,1,0})$ contains all the polynomials of the form $(*)$ except $\alpha + \beta = s - 2$. This completes the proof. Q.E.D.

Now we deduce Theorem(2-1) from Theorem(3-1) and Lemma(3-2). We use the following duality theorem.

Theorem(3-2). *Let $f : \widetilde{X} \to S = \mathrm{Spec}(\Lambda)$ be the natural morphism. For a locally free $\mathcal{O}_{\widetilde{X}}$-module $\mathcal{F}$, we have the isomorphism in the derived category of bounded complexes of $\mathcal{O}_S$-modules.*

$$\mathbb{R}f_* \mathbb{R}\mathcal{H}om_{\mathcal{O}_{\widetilde{X}}}(\mathcal{F}, \omega_{\widetilde{X}/S}^m[m]) \xrightarrow{\sim} \mathbb{R}\mathcal{H}om_{\mathcal{O}_S}(\mathbb{R}f_*\mathcal{F}, \mathcal{O}_S).$$

Writing $\mathcal{F}^\vee = \mathcal{H}om_{\mathcal{O}_{\widetilde{X}}}(\mathcal{F}, \mathcal{O}_{\widetilde{X}})$, it gives rise to the exact sequence

$$\begin{aligned} 0 \to \mathrm{Ext}_\Lambda^1(H^{m+1-p}(\widetilde{X}, \mathcal{F}), \Lambda) \to & H^p(\widetilde{X}, \mathcal{F}^\vee \otimes \omega_{\widetilde{X}/S}^m) \\ & \to \mathrm{Hom}(H^{m-p}(\widetilde{X}, \mathcal{F}), \Lambda) \to 0. \end{aligned}$$

Proof. This is a consequence of [H]. The key fact is $f^!\mathcal{O}_S = \omega^m_{\widetilde{X}/S}[m]$. Q.E.D.

By [St] $H^p(\widetilde{X}, \omega^q_{\widetilde{X}/S})$ is a free Λ-module. By Proposition(3-1) the first two assumptions of Theorem(2-1) imply that $H^2(\widetilde{X}, \theta_{\widetilde{X}/S})$ is a free Λ-module. By the long exact sequence induced by

$$0 \to \theta_{\widetilde{X}/S} \xrightarrow{t} \theta_{\widetilde{X}/S} \to \theta_{Z/S_0} \to 0,$$

the last assumption of Theorem(2-1) and the freeness of $H^2(\widetilde{X}, \theta_{\widetilde{X}/S})$ imply that $H^1(\widetilde{X}, \theta_{\widetilde{X}/S})$ is a free Λ-module and

$$H^1(\widetilde{X}, \theta_{\widetilde{X}/S}) \otimes_\Lambda k \xrightarrow{\sim} H^1(Z, \theta_{Z/S_0}).$$

Hence Theorem(3-1) and Theorem(3-2) imply that the dual of $d\rho_Z$ of Theorem(2-1) is equal to the multiplication

$$\Psi : \bigoplus_{0 \le p \le n-2} B_F^{\kappa(p)}(\nu(p)) \otimes_\Lambda B_F^{\kappa(n-2-p)}(\nu(n-2-p)) \otimes_\Lambda k \to B_F^{d^*}(r^*) \otimes_\Lambda k.$$

We easily see that $\nu(p) \ge 0$ and $\nu(n-2-p) \ge 0$ if and only if $\frac{n}{\delta} - 1 \le p \le n - 1 - \frac{n}{\delta}$. If n is even or $\delta \ge 3$, there exists $0 \le p \le n-2$ satisfying the condition. Hence Theorem(2-1) in this case follows from Lemma(3-2). Now assume that $\delta = 2$ and n is odd. We note $r^* = r - 2 = 2s - 2$. For $0 \le \forall p \le n-2$, we have either $\nu(p) \ge 2s-1$ or $\nu(n-2-p) \ge 2s-1$ so that the image of multiplication

$$\psi_p : P_\Lambda^{\kappa(p)}(\nu(p)) \otimes_\Lambda P_\Lambda^{\kappa(n-2-p)}(\nu(n-2-p)) \to P_\Lambda^{d^*}(r^*) = P_\Lambda^{d^*}(2s-2)$$

is contained in $P_\Lambda^{d^*}(2s-1)$. Taking $q = \frac{n-3}{2}$, we have $0 \le q \le n-2$, $\nu(q) = -1$ and $\nu(n-2-q) = 2s-1$. By the same argument as the proof of Lemma(3-2) we can prove $\mathrm{Im}(\psi_q) = P_\Lambda^{d^*}(2s-1)$. This shows that

$$(*) \qquad \mathrm{Coker}(\Psi) \xrightarrow{\sim} P_\Lambda^{d^*}(2s-2)/P_\Lambda^{d^*}(2s-1) + J_F^{d^*}(2s-2).$$

If $s = 1$, we easily see that the right hand side is of dimension 1 over k. Assume $s \ge 2$. A direct computation shows that $P_\Lambda^{d^*}(2s-2)/P_\Lambda^{d^*}(2s-1)$ is a k-linear space with a basis

$t^\mu x_n^\nu$ with $\mu + \nu = 2s-2$, and $x_i t^a x_n^b$ with $1 \le i \le n-1$, $a+b = s-2$.

By Definition(3-1)(2) the image of $J_F^{d^*}(2s-2)$ in $P_\Lambda^{d^*}(2s-2)/P_\Lambda^{d^*}(2s-1)$ is generated by the classes of $t^a x_n^b \partial f/\partial x_j$ with $1 \leq j \leq n-1$ and $a+b=s-2$. We can write

$$f = \sum_{\substack{0\leq\gamma\leq 2 \\ \nu+\mu=s(2-\gamma)}} \phi_{\gamma,\nu,\mu} t^\nu x_n^\mu + g \quad \text{with } g \in P_\Lambda^d(r+1) = P_\Lambda^d(2s+1)$$

where $\phi_{\gamma,\nu,\mu}$ is homogeneous of degree γ in $x_1, \ldots, x_{n-1}$. By Theorem(1-2) and Proposition(1-3), $\phi_2 = \phi_{2,0,0}$ is non-degenerate and hence x_i for $1 \leq \forall i \leq n-1$ is a linear combination of $\partial\phi_2/\partial x_1, \ldots, \partial\phi_2/\partial x_{n-1}$. Noting $t^a x_n^b \partial g/\partial x_j \in P_\Lambda^{d^*}(2s-1)$ for $1 \leq j \leq n-1$ and $a+b=s-2$, this shows that the right hand side of $(*)$ is generated by $t^\mu x_n^\nu$ with $\mu+\nu = 2s-2$ and they are linearly independent over k. This completes the proof of Theorem(2-1). Q.E.D.

§4. Proof of main results

In this section we prove Theorem(3-1), Proposition(2-2) and Proposition(3-1). The key is Proposition(4-1) and Proposition(4-2) below. We maintain the assumption in §2.

Definition(4-1). Let the notation be as in Definition(1-3). Let $H_i \subset \mathbb{P}$ for $1 \leq i \leq n$ be the hyperplane $X_i = 0$ and let $H_t \subset \mathbb{P}$ be defined by $t=0$. As a divisor on $\widetilde{\mathbb{P}}$ write $\pi^{-1}H_i = \widetilde{H}_i + \mathbb{E}(i)$ and $\pi^{-1}H_t = \widetilde{H}_t + \mathbb{E}(t)$ where $\widetilde{H}_i$ and $\widetilde{H}_t$ are the proper transforms of H_i and H_t respectively. From the computation in the proof of Theorem(1-2) we see (cf. Proposition(1-1))

$$\mathbb{E}(i) = \sum_{1\leq k\leq s} k\mathbb{E}_k \text{ if } 1 \leq i \leq n-1 \quad \text{and} \quad \mathbb{E}(n) = \mathbb{E}(t) = \sum_{1\leq k\leq s} \mathbb{E}_k.$$

The first key result concerns the cohomology of the sheaf

$$\mathcal{F} := \pi^*\mathcal{O}_{\mathbb{P}_\Lambda}(\ell) \otimes \mathcal{O}_{\widetilde{\mathbb{P}}_\Lambda}(-a\mathbb{E}(t) - b\mathbb{E}(1)) \quad (\ell, a, b \in \mathbb{Z}).$$

Recall the notation in Definition(1-1). Let $\mathfrak{m}(a,b) \subset k[t, x_1, \ldots, x_n]$ be the ideal generated by

$$t^\beta x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$$
$$\text{with } \beta + \alpha_n + k\Big(\sum_{1\leq i\leq n-1} \alpha_i\Big) \geq a + kb \quad \text{for } 1 \leq \forall k \leq s.$$

Let $\mathfrak{m}_\Lambda(a,b) = \mathfrak{m}(a,b) \otimes_{k[t]} \Lambda \subset A_\Lambda$ and write $P_\Lambda^\ell(a,b) = \mathrm{Ker}(P_\Lambda^\ell \to A_\Lambda/\mathfrak{m}_\Lambda(a,b))$. We note $P_\Lambda^\ell(a,b) = P_\Lambda^\ell$ if $a, b \leq 0$ and that $\mathfrak{m}_\Lambda(a,b) = \mathfrak{m}_\Lambda(a+sb)$ if $a \leq 0$.

Proposition(4-1). *Let $\mathbb{E} \subset \widetilde{\mathbb{P}}_\Lambda$ be the exceptional divisor of $\pi : \widetilde{\mathbb{P}}_\Lambda \to \mathbb{P}_\Lambda$.*

(1) *Letting $i : 0 \to \mathbb{P}_\Lambda$ be the inclusion, we have the exact sequences*

$$0 \to \pi_* \mathcal{F} \to \mathcal{O}_{\mathbb{P}_\Lambda}(\ell) \to i_*(A_\Lambda/\mathfrak{m}_\Lambda(a,b)) \to 0.$$

(2) $H^0(\widetilde{\mathbb{P}}_\Lambda, \mathcal{F}) = P^\ell_\Lambda(a,b)$.

(3) *Assume $a \geq -2$, $b \geq 1-n$ and $a+b \geq -n$. Then we have $R^\nu \pi_* \mathcal{F} = 0$ if $\nu \geq 1$.*

(4) *Under the same assumption as (3) we have:*

$$H^\nu(\widetilde{\mathbb{P}}_\Lambda, \mathcal{F}) = 0 \text{ if } 2 \leq \nu \neq n \text{ or } \nu = n,\ \ell \geq -n,$$
$$H^1(\widetilde{\mathbb{P}}_\Lambda, \mathcal{F}) = \mathrm{Coker}(P^\ell_\Lambda \to A_\Lambda/\mathfrak{m}_\Lambda(a,b)),$$
$$H^1(\widetilde{\mathbb{P}}_\Lambda, \mathcal{F}) = 0 \text{ if } \ell \geq \max\{a+sb, a+b\} - 1.$$

(5) *Under the same assumption as (3) we have*

$$H^i_{\mathbb{E}}(\widetilde{\mathbb{P}}_\Lambda, \mathcal{F}) \xrightarrow{\sim} \begin{cases} 0 & \text{if } 0 \leq i \neq 1, n+1 \\ A_\Lambda/\mathfrak{m}_\Lambda(a,b) & \text{if } i = 1. \end{cases}$$

The proof of Proposition(4-1) is given in §5.

Definition(4-2). Let $\omega_{\widetilde{\mathbb{P}}_\Lambda/S}$ be the sheaf of logarithmic differentials of the semistable family $\widetilde{\mathbb{P}}_\Lambda$ over $\mathrm{Spec}(\Lambda)$ in the sense of [St]. Let $\omega_{\widetilde{\mathbb{P}}_\Lambda/S}(\log \widetilde{X})$ be the sheaf of logarithmic differentials with additional logarithmic pole along $\widetilde{X}$. By [St] these are locally free $\mathcal{O}_{\widetilde{\mathbb{P}}_\Lambda}$-modules. Let $\theta_{\widetilde{\mathbb{P}}_\Lambda/S}$ (resp. $\theta_{\widetilde{\mathbb{P}}_\Lambda/S}(-\log \widetilde{X})$) be the $\mathcal{O}_{\widetilde{\mathbb{P}}_\Lambda}$-dual of $\omega_{\widetilde{\mathbb{P}}_\Lambda/S}$ (resp. $\omega_{\widetilde{\mathbb{P}}_\Lambda/S}(\log \widetilde{X})$). We write

$$\omega^p_{\widetilde{\mathbb{P}}_\Lambda/S} = \overset{p}{\wedge} \omega_{\widetilde{\mathbb{P}}_\Lambda/S} \quad \text{and} \quad \omega^p_{\widetilde{\mathbb{P}}_\Lambda/S}(\log \widetilde{X}) = \overset{p}{\wedge} \omega_{\widetilde{\mathbb{P}}_\Lambda/S}(\log \widetilde{X}).$$

Recall the Euler exact sequence

$$0 \to \Omega^1_{\mathbb{P}_\Lambda/S} \to \bigoplus_{0 \leq i \leq n} \mathcal{O}_{\mathbb{P}_\Lambda}(-1) \cdot dX_i \to \mathcal{O}_{\mathbb{P}_\Lambda} \to 0 \tag{4-1}$$

and its $\mathcal{O}_{\mathbb{P}_\Lambda}$-dual

$$0 \to \mathcal{O}_{\widetilde{\mathbb{P}}_\Lambda} \to \Sigma \to T_{\mathbb{P}_\Lambda/S} \to 0,$$

where $\Sigma = \bigoplus_{0 \leq i \leq n} \mathcal{O}_{\mathbb{P}_\Lambda}(1)$. The second key result gives us the similar exact sequences for $\omega_{\widetilde{\mathbb{P}}_\Lambda/S}$. Write $\mathcal{L} = \mathcal{O}_{\widetilde{\mathbb{P}}_\Lambda}(\widetilde{X})$. By Proposition(1-4) we have the canonical isomorphism

$$\mathcal{L} \simeq \pi^* \mathcal{O}_{\mathbb{P}_\Lambda}(d) \otimes \mathcal{O}_{\widetilde{\mathbb{P}}_\Lambda}(-\delta \cdot \mathbb{E}(1)). \tag{4-2}$$

Proposition(4-2). *We have the exact sequences*

$$0 \to \omega_{\widetilde{\mathbb{P}}_\Lambda/S} \to \pi^*\mathcal{O}_{\mathbb{P}_\Lambda}(-1)\cdot dX_0 \oplus \bigoplus_{1\le i\le n} \mathcal{O}_{\widetilde{\mathbb{P}}_\Lambda}(\mathbb{E}(i)) \otimes \pi^*\mathcal{O}_{\mathbb{P}_\Lambda}(-1)\cdot dX_i \to \mathcal{O}_{\widetilde{\mathbb{P}}_\Lambda} \to 0,$$

and its $\mathcal{O}_{\widetilde{\mathbb{P}}_\Lambda}$*-dual*

$$0 \to \mathcal{O}_{\widetilde{\mathbb{P}}_\Lambda} \xrightarrow{\iota} \pi^*\mathcal{O}_{\mathbb{P}_\Lambda}(1)\cdot\partial_0 \oplus \bigoplus_{1\le i\le n} \pi^*\mathcal{O}_{\mathbb{P}_\Lambda}(1)\otimes\mathcal{O}_{\widetilde{\mathbb{P}}_\Lambda}(-\mathbb{E}(i))\cdot\partial_i \to \theta_{\widetilde{\mathbb{P}}_\Lambda/S} \to 0$$

where $\iota(1) = \sum_{0\le i\le n} X_i\partial_i$. *Denoting by* $\widetilde{\Sigma}$ *the sheaf at the middle of the second exact sequence, we have the following exact sequence*

$$0 \to \theta_{\widetilde{\mathbb{P}}_\Lambda/S}(-\log\widetilde{X}) \to \widetilde{\Sigma} \xrightarrow{j_F} \mathcal{L} \to 0,$$

where $j_F(\partial_i) = \partial F/\partial X_i \in P_\Lambda^{d-1}(r-\mu_i)$ *for* $0 \le i \le n$ (*cf. Definition*(3-1)(2)) *that induces by* (4-2) *and Proposition*(4-1)(2) *the map*

$$\pi^*\mathcal{O}_{\mathbb{P}_\Lambda}(1) \to \mathcal{L} \ (i=0)$$

$$\textit{and} \quad \pi^*\mathcal{O}_{\mathbb{P}_\Lambda}(1)\otimes\mathcal{O}_{\widetilde{\mathbb{P}}_\Lambda}(-\mathbb{E}(i)) \to \mathcal{L} \ (1\le i\le n).$$

The proof will be given in the next section. We also need the following auxiliary results.

Proposition(4-3). *We have the exact sequences*

$$0 \to \omega^p_{\widetilde{\mathbb{P}}_\Lambda/S} \to \omega^p_{\widetilde{\mathbb{P}}_\Lambda/S}(\log\widetilde{X}) \to \omega^{p-1}_{\widetilde{X}/S} \to 0,$$

$$0 \to \theta_{\widetilde{\mathbb{P}}_\Lambda/S}\otimes\mathcal{L}^{-1} \to \theta_{\widetilde{\mathbb{P}}_\Lambda/S}(-\log\widetilde{X}) \to \theta_{\widetilde{X}/S} \to 0.$$

Proof. The proof is standard and left to the readers. Q.E.D.

Proposition(4-4). *We have the natural isomorphism*

$$\omega^m_{\widetilde{X}/S} \xrightarrow{\sim} \pi^*(\mathcal{O}_{\mathbb{P}_\Lambda}(d-n-1)) \otimes \mathcal{O}_{\widetilde{\mathbb{P}}_\Lambda}(\mathbb{E}(t) + (n-1-\delta)\mathbb{E}(1)) \otimes \mathcal{O}_{\widetilde{X}}.$$

Proof. By the first sequence of Proposition(4-2)

$$\omega^n_{\widetilde{\mathbb{P}}_\Lambda/S}(\log\widetilde{X}) = \mathcal{L}\otimes\mathcal{E} \tag{4-3}$$

$$\text{with } \mathcal{E} = \pi^*(\mathcal{O}_{\mathbb{P}_\Lambda}(-n-1)) \otimes \mathcal{O}_{\widetilde{\mathbb{P}}_\Lambda}(\mathbb{E}(t) + (n-1)\mathbb{E}(1)).$$

Thus Proposition(4-4) follows from Proposition(4-3). Q.E.D.

Remark(4-1). Proposition(4-4) immediately implies the following fact: The singularity $(X,0)$ is canonical if and only if either $\delta \leq n-1$ or $\delta = n,\ s=1$. If $\delta \geq n+2$, $(X,0)$ is not log-canonical.

Now we prove Theorem(3-1) and Proposition(3-1). First we show Theorem(3-1)(1). By [St] we know that $H^p(\widetilde{X}, \omega^q_{\widetilde{X}/S})$ is a free Λ-module. By Remark(3-1) it suffices to show $B_F^{\kappa(p)}(\nu(p)) \xrightarrow{\sim} H^p(\widetilde{X}, \omega^q_{\widetilde{X}/S})_{prim}$. We note

$$H^q(\mathbb{P}_\Lambda, \Omega^p_{\mathbb{P}_\Lambda/S}) \xrightarrow{\sim} H^q(\widetilde{\mathbb{P}}_\Lambda, \omega^p_{\widetilde{\mathbb{P}}_\Lambda/S}) \quad \text{for } \forall p,q \geq 0. \tag{4-4}$$

To see this it suffices to show $\pi_*(\omega^p_{\widetilde{\mathbb{P}}_\Lambda/S}) = \Omega^p_{\mathbb{P}_\Lambda/S}$ and $R^\nu\pi_*(\omega^p_{\widetilde{\mathbb{P}}_\Lambda/S}) = 0$ for $\forall \nu \geq 1$. The first sequence of Proposition(4-2) induces the following exact sequence

$$0 \to \omega^p_{\widetilde{\mathbb{P}}_\Lambda/S} \to \overset{p}{\wedge}\widetilde{\Sigma}^* \to \overset{p-1}{\wedge}\widetilde{\Sigma}^* \to \cdots \to \widetilde{\Sigma}^* \to \mathcal{O}_{\widetilde{\mathbb{P}}_\Lambda} \to 0,$$

where $\widetilde{\Sigma}^*$ is the $\mathcal{O}_{\widetilde{\mathbb{P}}_\Lambda}$-dual of $\widetilde{\Sigma}$. By Proposition(4-1)(3) and (1), $R^\nu\pi_*(\overset{\mu}{\wedge}\widetilde{\Sigma}^*) = 0$ for $\forall \nu \geq 1$, $\forall \mu \geq 0$ and $\pi_*(\overset{\mu}{\wedge}\widetilde{\Sigma}^*) = \overset{\mu}{\wedge}\Sigma^*$ with $\Sigma^* = \bigoplus_{0\leq i\leq n} \mathcal{O}_{\mathbb{P}_\Lambda}(-1)$. This shows the second assertion and that we have the exact sequence

$$0 \to \pi_*\omega^p_{\widetilde{\mathbb{P}}_\Lambda/S} \to \overset{p}{\wedge}\Sigma^* \to \overset{p-1}{\wedge}\Sigma^* \to \cdots \Sigma^* \to \mathcal{O}_{\mathbb{P}_\Lambda} \to 0.$$

Compared with the similar exact sequence induced by (4-1), this proves $\pi_*\omega^p_{\widetilde{\mathbb{P}}_\Lambda/S} = \Omega^p_{\mathbb{P}_\Lambda/S}$. By (4-4) and Proposition(4-3) and the Bott vanishing of $H^q(\mathbb{P}_\Lambda, \Omega^q_{\mathbb{P}_\Lambda/S})$, Theorem(3-1)(1) follows from the isomorphism

$$B_F^{\kappa(p)}(\nu(p)) \xrightarrow{\sim} H^p(\widetilde{\mathbb{P}}_\Lambda, \omega^{n-p}_{\widetilde{\mathbb{P}}_\Lambda/S}(\log\widetilde{X})).$$

By (4-3) we have $\overset{p}{\wedge}\theta_{\widetilde{\mathbb{P}}_\Lambda/S}(-\log\widetilde{X}) \otimes \mathcal{L} \otimes \mathcal{E} = \omega^{n-p}_{\widetilde{\mathbb{P}}_\Lambda/S}(\log\widetilde{X})$. Therefore the last sequence of Proposition(4-2) gives rise to the following exact sequence

$$0 \to \omega^{n-p}_{\widetilde{\mathbb{P}}_\Lambda/S}(\log\widetilde{X}) \to \mathcal{C}_p^0 \to \mathcal{C}_p^1 \to \cdots \to \mathcal{C}_p^p \to 0,$$

where $\mathcal{C}_p^a = \overset{p-a}{\wedge}\widetilde{\Sigma} \otimes \mathcal{L}^{a+1} \otimes \mathcal{E}$ for $0 \leq a \leq p$. Thus we obtain the spectral sequence

$$E_1^{a,b} = H^b(\widetilde{\mathbb{P}}_\Lambda, \mathcal{C}_p^a) \Rightarrow H^{a+b}(\widetilde{\mathbb{P}}_\Lambda, \omega^{n-p}_{\widetilde{\mathbb{P}}_\Lambda/S}(\log\widetilde{X})).$$

By definition $\mathcal{C}_p^a$ is the direct sum of $\pi^*\mathcal{O}_{\mathbb{P}_\Lambda}(\ell) \otimes \mathcal{O}_{\widetilde{\mathbb{P}}_\Lambda}(-\alpha\mathbb{E}(t) - \beta\mathbb{E}(1))$ with

$$\begin{cases} \ell = d(a+1) - n - 1 + p - a, \\ \beta = \delta(a+1) + p - a - (n-1) - \epsilon \text{ with } \epsilon = 0 \text{ or } 1 \text{ or } 2, \\ \alpha = -1 \text{ or } 0 \text{ and } (\alpha, \epsilon) \neq (0,0), \end{cases}$$

We have $\ell \geq -n$ and $\beta \geq 1 - n$ by the assumption that $d \geq 3$, $\delta \geq 2$ and $p \geq a \geq 0$. By Proposition(4-1)(4), $E_1^{a,b} = 0$ for $\forall b \geq 1$ if

$$d(a+1) - n - 1 + p - a \geq c(\delta(a+1) + p - a - (n-1)) - 1 \quad \text{for } c = 1, s.$$

It is easy to see that this holds under the assumption $d \geq r+1$, $s \geq 1$ and $p \leq n-1$. Thus the spectral sequence degenerates at E_2 and we get the isomorphism

$$H^p(\widetilde{\mathbb{P}}_\Lambda, \omega^{n-p}_{\widetilde{\mathbb{P}}_\Lambda/S}(\log \widetilde{X})) \xrightarrow{\sim} \mathrm{Coker}(H^0(\widetilde{\mathbb{P}}_\Lambda, \widetilde{\Sigma} \otimes \mathcal{L}^p \otimes \mathcal{E}) \xrightarrow{j_F} H^0(\widetilde{\mathbb{P}}_\Lambda, \mathcal{L}^{p+1} \otimes \mathcal{E})).$$

This shows the desired isomorphism by Proposition(4-1)(2) and Definition(3-1)(2). Q.E.D.

Next we show Proposition(3-1). First we show

$$H^\nu(\widetilde{\mathbb{P}}_\Lambda, \theta_{\widetilde{\mathbb{P}}_\Lambda/S}(-\log \widetilde{X})) = 0 \quad \text{for } \nu \geq 2 \tag{4-5}$$

Indeed this follows from the second sequence of Proposition(4-2) and the fact $H^\nu(\widetilde{\mathbb{P}}_\Lambda, \mathcal{L}) = H^{\nu+1}(\widetilde{\mathbb{P}}_\Lambda, \widetilde{\Sigma}) = 0$ for $\forall \nu \geq 1$ by Proposition(4-1)(4). We note that the assumption $d \geq r$ is used for the vanishing of $H^1(\widetilde{\mathbb{P}}_\Lambda, \mathcal{L})$. By (4-5) Proposition(4-3) implies

$$H^j(\widetilde{X}, \theta_{\widetilde{X}/S}) \xrightarrow{\sim} H^{j+1}(\widetilde{\mathbb{P}}_\Lambda, \theta_{\widetilde{\mathbb{P}}_\Lambda/S} \otimes \mathcal{L}^{-1}) \quad \text{for } j \geq 2.$$

By using the exact sequence

$$0 \to \mathcal{L}^{-1} \to \widetilde{\Sigma} \otimes \mathcal{L}^{-1} \to \theta_{\widetilde{\mathbb{P}}_\Lambda/S} \otimes \mathcal{L}^{-1} \to 0$$

coming from Proposition(4-2), the same argument as the proof of (4-4) shows

$$\begin{aligned} H^i(\widetilde{\mathbb{P}}_\Lambda, \theta_{\widetilde{\mathbb{P}}_\Lambda/S} \otimes \mathcal{L}^{-1}) &\xrightarrow{\sim} H^i(\mathbb{P}_\Lambda, T_{\mathbb{P}_\Lambda/S}(-d)) \\ &\xrightarrow{\sim} H^i(\mathbb{P}_\Lambda, \Omega^{n-1}_{\mathbb{P}_\Lambda/S}(n+1-d)) \quad \text{for } \forall i \geq 0. \end{aligned} \tag{4-6}$$

We note that the assumption $\delta < n$ is used to get

$$R^\nu \pi_* \mathcal{L}^{-1} = R^\nu \pi_*(\widetilde{\Sigma} \otimes \mathcal{L}^{-1}) = 0 \quad \text{for } \forall \nu \geq 1$$

by applying Proposition(4-1)(3). Noting that the last group in (4-6) is torsion free and that it vanishes if either $i \leq n-2$ or $i = n-1$ and $d \neq n+1$ or $i = n$ and $d \leq n+1$, it implies Proposition(3-1). Q.E.D.

Next we show Theorem(3-1)(2). Proposition(3-1) implies that $H^2(\widetilde{X}, \theta_{\widetilde{X}/S})$ is free, which implies by Theorem(3-2) that $H^{m-1}(\widetilde{X}, \omega^1_{\widetilde{X}/S} \otimes \omega^m_{\widetilde{X}/S})$ is free. By Proposition(4-3) and Proposition(4-4) we have the exact sequence

$$0 \to \omega^2_{\widetilde{\mathbb{P}}_\Lambda/S} \otimes \mathcal{L} \otimes \mathcal{E} \to \omega^2_{\widetilde{\mathbb{P}}_\Lambda/S}(\log \widetilde{X}) \otimes \mathcal{L} \otimes \mathcal{E} \to \omega^1_{\widetilde{X}/S} \otimes \omega^m_{\widetilde{X}/S} \to 0.$$

By the same argument as the proof of Theorem(3-1)(1) we can show the isomorphsim

$$B_F^{d^*}(r^*) \xrightarrow{\sim} H^{n-2}(\widetilde{\mathbb{P}}_\Lambda, \omega^2_{\widetilde{\mathbb{P}}_\Lambda/S}(\log \widetilde{X}) \otimes \mathcal{L} \otimes \mathcal{E}).$$

By the same argument as the proof of (4-4) we can show

$$H^q(\widetilde{\mathbb{P}}_\Lambda, \omega^2_{\widetilde{\mathbb{P}}_\Lambda/S} \otimes \mathcal{L} \otimes \mathcal{E}) \xrightarrow{\sim} H^q(\mathbb{P}_\Lambda, \Omega^2_{\mathbb{P}_\Lambda/S}(d-n-1))$$

and it vanishes for $1 \leq \forall q \neq n$ unless $q = 2$ and $d = n+1$. It implies

$$H^{n-2}(\widetilde{\mathbb{P}}_\Lambda, \omega^2_{\widetilde{\mathbb{P}}_\Lambda/S}(\log \widetilde{X}) \otimes \mathcal{L} \otimes \mathcal{E}) \xrightarrow{\sim} H^{m-1}(\widetilde{X}, \omega^1_{\widetilde{X}/S} \otimes \omega^m_{\widetilde{X}/S})$$

if $n \geq 4$ or $n = 3, d \neq 4$. Hence Theorem(3-1)(2) is proven. Q.E.D.

Next we prove Theorem(3-1)(3). Let

$$H^1(\widetilde{\mathbb{P}}_\Lambda, \theta_{\widetilde{\mathbb{P}}_\Lambda/S}(-\log \widetilde{X})) \to H^1(\widetilde{X}, \theta_{\widetilde{X}/S})$$

be the map induced by the last sequence of Proposition(4-3). By (4-6) it is injective if $n \geq 3$ and an isomorphism if either $n \geq 4$ or $n = 3$ and $d \neq 4$. Therefore it suffices to show the following.

Proposition(4-5). *Assume $d \geq r$. There are the exact sequences*

$$0 \to B_F^d(r) \to H^1(\widetilde{\mathbb{P}}_\Lambda, \theta_{\widetilde{\mathbb{P}}_\Lambda/S}(-\log \widetilde{X})) \to \bigoplus_{1 \leq i \leq n-1} A_\Lambda/\mathfrak{m}_\Lambda(s) + A_\Lambda^{\leq 1} \to 0,$$

$$H^1(\widetilde{\mathbb{P}}_\Lambda, \theta_{\widetilde{\mathbb{P}}_\Lambda/S}(-\log \widetilde{X})) \xrightarrow{\iota} B_F^d \to A_\Lambda/(\mathfrak{m}_\Lambda(r) + I_f)$$

and $\mathrm{Ker}(\iota)$ *is torsion.*

Proof. The first sequence follows immediately from the last sequence of Proposition(4-2) and Proposition(4-1). To show the second we consider the localization sequence

$$H^1_{\mathbb{E}}(\widetilde{\mathbb{P}}_\Lambda, \Theta) \to H^1(\widetilde{\mathbb{P}}_\Lambda, \Theta) \to H^1(\mathbb{U}, \Theta_{|\mathbb{U}}) \to H^2_{\mathbb{E}}(\widetilde{\mathbb{P}}_\Lambda, \Theta),$$

where $\Theta = \theta_{\widetilde{\mathbb{P}}_\Lambda/S}(-\log\widetilde{X})$ and $\mathbb{U} = \widetilde{\mathbb{P}}_\Lambda - \mathbb{E} = \mathbb{P}_\Lambda - \{0\}$. We know that $H^i_{\mathbb{E}}(\widetilde{\mathbb{P}}_\Lambda, \Theta)$ is a torsion Λ-module if $i \geq 1$. Thus it suffices to show

$$H^1(\mathbb{U}, \theta_{\widetilde{\mathbb{P}}_\Lambda/S}(-\log\widetilde{X})_{|\mathbb{U}}) \xrightarrow{\sim} B_F^d$$

and

$$H^2_{\mathbb{E}}(\widetilde{\mathbb{P}}_\Lambda, \theta_{\widetilde{\mathbb{P}}_\Lambda/S}(-\log\widetilde{X})) \xrightarrow{\sim} A_\Lambda/\mathfrak{m}_\Lambda(r) + I_f.$$

The first isomorphism follows easily from the exact sequence

$$0 \to \theta_{\widetilde{\mathbb{P}}_\Lambda/S}(-\log\widetilde{X})_{|\mathbb{U}} \to \bigoplus_{0\leq i\leq n} \mathcal{O}_{\mathbb{P}_\Lambda}(1)_{|\mathbb{U}} \to \mathcal{O}_{\mathbb{P}_\Lambda}(d)_{|\mathbb{U}} \to 0$$

induced by the last sequence of Proposition(4-2). It also induces the exact sequence

$$H^1_{\mathbb{E}}(\widetilde{\mathbb{P}}_\Lambda, \widetilde{\Sigma}) \to H^1_{\mathbb{E}}(\widetilde{\mathbb{P}}_\Lambda, \mathcal{L}) \to H^2_{\mathbb{E}}(\widetilde{\mathbb{P}}_\Lambda, \theta_{\widetilde{\mathbb{P}}_\Lambda/S}(-\log\widetilde{X})) \to H^2_{\mathbb{E}}(\widetilde{\mathbb{P}}_\Lambda, \widetilde{\Sigma}).$$

Hence the second isomorphism follows from Proposition(4-1)(5). Q.E.D.

Finally we show Proposition(2-2). Proposition(2-2)(1) is a direct consequence of Proposition(3-1). To show Proposition(2-2)(2) we start with the following exact sequence induced by Proposition(4-3):

$$0 \to \theta_{\widetilde{\mathbb{P}}_\Lambda/S} \otimes \mathcal{L}^{-1} \otimes_\Lambda k \to \theta_{\widetilde{\mathbb{P}}_\Lambda/S}(-\log\widetilde{X}) \otimes_\Lambda k \to \theta_{Z/S_0} \to 0.$$

By (4-6) it induces $H^0(\widetilde{\mathbb{P}}_{\Lambda,0}, \theta_{\widetilde{\mathbb{P}}_\Lambda/S}(-\log\widetilde{X}) \otimes_\Lambda k) \xrightarrow{\sim} H^0(Z, \theta_{Z/S_0})$ where $\widetilde{\mathbb{P}}_{\Lambda,0}$ is the special fiber of $\widetilde{\mathbb{P}}_\Lambda$. To compute the left hand side we use the exact sequence induced by the last sequence of Proposition(4-2):

$$(*) \qquad 0 \to \theta_{\widetilde{\mathbb{P}}_\Lambda/S}(-\log\widetilde{X}) \otimes_\Lambda k \to \widetilde{\Sigma} \otimes_\Lambda k \to \mathcal{L} \otimes_\Lambda k \to 0.$$

Writing $\mathcal{F} = \pi^*\mathcal{O}_{\mathbb{P}_\Lambda}(1) \otimes \mathcal{O}_{\widetilde{\mathbb{P}}_\Lambda}(-\mathbb{E}(i)))$, we have the following exact sequence by Proposition(4-1)

$$0 \to \pi_*\mathcal{F} \to \mathcal{O}_{\mathbb{P}_\Lambda}(1) \to i_* A_\Lambda/\mathfrak{m}_\Lambda(\mu_i) \to 0$$

where μ_i is as in Definition(3-1)(2). It induces the exact sequence

$$\begin{aligned} 0 \to A_\Lambda/\mathfrak{m}_\Lambda(\mu_i)[t] \to & H^0(\mathbb{P}_{\Lambda,0}, (\pi_*\mathcal{F}) \otimes_\Lambda k) \\ & \to H^0(\mathbb{P}_{\Lambda,0}, \mathcal{O}_{\mathbb{P}_\Lambda}(1) \otimes_\Lambda k) \to (A_\Lambda/\mathfrak{m}_\Lambda(\mu_i)) \otimes_\Lambda k \end{aligned}$$

where $\mathbb{P}_{\Lambda,0}$ is the special fiber of $\mathbb{P}_\Lambda$ and $M[t] = \mathrm{Ker}(M \xrightarrow{t} M)$ for a Λ-module M. Hence we get the exact sequences

$$\begin{aligned} 0 \to & A_\Lambda/\mathfrak{m}_\Lambda(\mu_i)[t] \\ & \to H^0(\widetilde{\mathbb{P}}_{\Lambda,0}, (\pi^*\mathcal{O}_{\mathbb{P}_\Lambda}(1) \otimes \mathcal{O}_{\widetilde{\mathbb{P}}_\Lambda}(-\mathbb{E}(i))) \otimes_\Lambda k) \to P_k^1(\mu_i) \to 0, \end{aligned}$$

by noting that $(\pi_*\mathcal{F})\otimes_\Lambda k = \pi_*(\mathcal{F}\otimes_\Lambda k)$ since $R^1\pi_*\mathcal{F} = 0$ by Proposition(4-1)(3). Here, for integers $q, \nu \geq 0$ we write $P_k^q(\nu) = \mathrm{Ker}(P_k^q \to A_k/\mathfrak{m}_k(\nu))$ with $P_k^q = P_\Lambda^q \otimes_\Lambda k$, $A_k = A_\Lambda \otimes_\Lambda k$ and $\mathfrak{m}_k(\nu) \subset A_k$ is the image of $\mathfrak{m}_\Lambda(\nu)$. By the same argument we get the exact sequence

$$0 \to A_\Lambda/\mathfrak{m}_\Lambda(r)[t] \to H^0(\widetilde{\mathbb{P}}_{\Lambda,0}, \mathcal{L} \otimes_\Lambda k) \to P_k^d(r) \to 0.$$

Combining $(*)$ with the last two exact sequences, we get the exact sequence

$$0 \to T_0 \to H^0(\widetilde{\mathbb{P}}_{\Lambda,0}, \theta_{\widetilde{\mathbb{P}}_\Lambda/S}(-\log \widetilde{X}) \otimes_\Lambda k) \to T_1$$

where T_0 is the kernel of the map

$$\begin{aligned} \bigoplus_{1\leq i\leq n} A_\Lambda/\mathfrak{m}_\Lambda(\mu_i)[t] &\to A_\Lambda/\mathfrak{m}_\Lambda(r); \\ (h_i)_{1\leq i\leq n} &\mapsto \sum_{1\leq i\leq n} h_i \cdot \partial f/\partial x_i \mod \mathfrak{m}_\Lambda(r) \end{aligned}$$

and T_1 is the kernel of the map

$$\bigoplus_{0\leq i\leq n} P_k^1(\mu_i) \to P_k^d(r);\ (H_i)_{0\leq i\leq n} \to \sum_{0\leq i\leq n} H_i \cdot \partial F_0/\partial X_i$$

and hence it vanishes under the assumption of Proposition(2-2)(2). Thus it suffices to show $T_0 = 0$. In what follows we assume $s \geq 2$ and leave the case $s = 1$ to the readers. We see

$$T_0 = \left\{ \partial = \sum_{1\leq i\leq n-1} \sum_{\substack{\mu+\nu=s \\ \mu\geq 1,\nu\geq 0}} a_{i,\mu,\nu} t^{\mu-1} x_n^\nu \partial/\partial x_i + b\partial/\partial x_n \;\middle|\; a_{i,\mu,\nu},\ b \in k,\ \partial f \in \mathfrak{m}_\Lambda(r) \right\}$$

It is easy to see that $T_0 = 0$ if and only if there is no non-trivial relation such as

$$(**) \qquad \sum_{1\leq i\leq n-1} \sum_{\substack{\mu+\nu=s \\ \mu\geq 1,\nu\geq 0}} a_{i,\mu,\nu} x_0^{\mu-1} x_n^\nu \partial f_0^{hom}/\partial x_i + b\partial f_0^{hom}/\partial x_n = 0$$

in $k[x_0, x_1, \ldots, x_n]$ where f_0^{hom} is as (1-2) in §1. First assume $b \neq 0$. Let $I \subset k[x_0, \ldots, x_n]$ be the ideal generated by

$$x_0 \partial f_0^{hom}/\partial x_0, \partial f_0^{hom}/\partial x_1, \ldots \partial f_0^{hom}/\partial x_n.$$

The relation implies that I is generated only by n elements so that $\mathbb{P}(Q) \supset \mathrm{Sup}(k[x_0, \ldots, x_n]/I) \neq \emptyset$. This contradicts Theorem(1-2). Next assume $b = 0$. We can write $f_0^{hom} = \sum_{s\alpha+\nu+\mu=r} \Phi_{\alpha,\nu,\mu} x_0^\nu x_n^\mu$ where $\Phi_{\alpha,\nu,\mu}$ is homogeneous of degree α in $x_1, \ldots, x_{n-1}$. Then, putting $\Phi_\delta = \Phi_{\delta,0,0}$, $(**)$ implies

$$0 = \sum_{\substack{\mu+\nu=s \\ \mu\geq 1, \nu \geq 0}} x_0^{\mu-1} x_n^\nu \Big(\sum_{1\leq i\leq n-1} a_{i,\mu,\nu} \partial\Phi_\delta/\partial x_i \Big)$$

$$+ (\text{terms of degree} \leq \delta - 2 \text{ in } x_1, \ldots, x_{n-1}).$$

Since $\Phi_\delta = f_0^{hom}(0, x_1, \ldots, x_{n-1}, 0)$ is non-degenerated by Theorem(1-2) and Proposition(1-3), it implies $a_{i,\mu,\nu} = 0$ for $\forall i, \nu, \mu$ and the proof is completed. Q.E.D.

§5. Proof of key propositions

In this section we prove Proposition(4-1) and Proposition(4-2). First we show Proposition(4-1). By the projection formula

$$R^\nu \pi_* \mathcal{F} = \mathcal{O}_{\mathbb{P}_\Lambda}(\ell) \otimes R^\nu \pi_* \mathcal{O}_{\widetilde{\mathbb{P}}_\Lambda}(-a\mathbb{E}(t) - b\mathbb{E}(1)) \quad \text{for } \forall \nu \geq 0.$$

Hence Proposition(4-1)(1) and (3) follow from Proposition(5-1) below. Proposition(4-1)(2) is a direct consequence of (1). Next we show Proposition(4-1)(4). Proposition(4-1)(3) implies $H^\nu(\widetilde{\mathbb{P}}_\Lambda, \mathcal{F}) \xrightarrow{\sim} H^\nu(\mathbb{P}_\Lambda, \pi_* \mathcal{F})$ for $\forall \nu \geq 0$. Therefore Proposition(4-1)(1) implies the following isomorphism and the exact sequence

$$H^\nu(\mathbb{P}_\Lambda, \pi_* \mathcal{F}) \xrightarrow{\sim} H^\nu(\mathbb{P}_\Lambda, \mathcal{O}_{\mathbb{P}_\Lambda}(\ell)) \quad \text{for } \nu \geq 2,$$

$$0 \to H^0(\mathbb{P}_\Lambda, \pi_* \mathcal{F}) \to H^0(\mathbb{P}_\Lambda, \mathcal{O}_{\mathbb{P}_\Lambda}(\ell))$$
$$\to A_\Lambda/\mathfrak{m}_\Lambda(a, b) \to H^1(\mathbb{P}_\Lambda, \pi_* \mathcal{F}) \to 0.$$

Thus Proposition(4-1)(4) follow from the standard vanishing of $H^\nu(\mathbb{P}_\Lambda, \mathcal{O}_{\mathbb{P}_\Lambda}(\ell))$ except that the vanishing of $H^1(\widetilde{\mathbb{P}}_\Lambda, \mathcal{F})$ requires Lemma(5-1) below. Finally, to show Proposition(4-1)(5), we use the localization sequence

$$\cdots \to H^i_{\mathbb{E}}(\widetilde{\mathbb{P}}_\Lambda, \mathcal{F}) \to H^i(\widetilde{\mathbb{A}}_\Lambda, \mathcal{F}) \to H^i(\widetilde{\mathbb{A}}_\Lambda - \mathbb{E}, \mathcal{F}) \to \cdots .$$

By [H, §9 Theorem9.1] we have

$$H^i(\widetilde{\mathbb{A}}_\Lambda - \mathbb{E}, \mathcal{F}) = H^i(\mathbb{A}_\Lambda - \{0\}, \mathcal{O}_{\mathbb{A}_\Lambda}) = \begin{cases} 0 & \text{if } i \neq 0, n, \\ A_\Lambda & \text{if } i = 0. \end{cases}$$

Thus Proposition(4-1)(5) follows from Proposition(5-1). Q.E.D.

Lemma(5-1). *The map* $P_\Lambda^\ell \to A_\Lambda/\mathfrak{m}_\Lambda(a,b)$ *is surjective if* $\ell \geq \max\{a+sb, a+b\} - 1$.

Proof. This follows from the fact that $(x_1, \ldots, x_n)^\nu \subset \mathfrak{m}_\Lambda(a,b)$ with $\nu = \max\{a+sb, a+b\}$. Q.E.D.

Proposition(5-1). *Let the notation be as in Definition*(1-3). *Let* a, b *be integers.*

(1) $H^0(\widetilde{\mathbb{A}}, \mathcal{O}_{\widetilde{\mathbb{A}}}(-a\mathbb{E}(t) - b\mathbb{E}(1))) = \mathfrak{m}(a,b) \subset k[t, x_1, \ldots, x_n]$.

(2) $H^i(\widetilde{\mathbb{A}}, \mathcal{O}_{\widetilde{\mathbb{A}}}(-a\mathbb{E}(t) - b\mathbb{E}(1))) = 0$ *for* $\forall i \geq 1$ *if* $a \geq -2$, $b \geq 1-n$ *and* $a + b \geq -n$.

Proof. Proposition(5-1)(1) follows from the standard description of the space of global sections of line bundles on toric varieties (cf. [F,§3]). For the proof of Proposition(5-1)(2) we need a preparation. Let $\mathbb{A}^\ell = \mathrm{Spec}(k[x_1, \ldots, x_\ell])$ be an affine space over k. A refinement of $\mathbb{A}^\ell$ is the proper morphism of toric varieties $X(\Delta) \to X(\Delta_0) = \mathbb{A}^\ell$ where $\Delta_0 = \{\sigma_0\}$ is as in §1 and Δ is a refinement of Δ_0. We use the following standard fact from toric geometry (cf. [F, §3.5, Proposition]).

Proposition(5-2). *Let* $X(\Delta) \to \mathbb{A}^\ell$ *be a refinement and let* $\mathcal{F} = \mathcal{O}_{X(\Delta)}(D)$ *with a Cartier divisor* D *such that* $\mathcal{F}$ *is generated by global sections. Then* $H^i(X(\Delta), \mathcal{F}) = 0$ *for* $\forall i \geq 1$.

Now we show Proposition(5-1)(2). First assume $a \geq 0$. If $b \geq 0$, $\mathcal{O}_{\widetilde{\mathbb{A}}}(-a\mathbb{E}(t) - b\mathbb{E}(1))$ is generated by global sections. Hence the vanishing follows from Proposition(5-2). If $1 - n \leq b \leq -1$ we use the exact sequence

$$0 \to \mathcal{O}_{\widetilde{\mathbb{A}}}(-a\mathbb{E}(t) - b\mathbb{E}(1)) \to \mathcal{O}_{\widetilde{\mathbb{A}}}(-a\mathbb{E}(t) - (b+1)\mathbb{E}(1)) \to \mathcal{O}_{\widetilde{H}_{n-1}}(-a\mathbb{E}_{\widetilde{H}_{n-1}}(t) - (b+1)\mathbb{E}_{\widetilde{H}_{n-1}}(1)) \to 0$$

that follows from $\mathcal{O}_{\widetilde{\mathbb{A}}}(\mathbb{E}(1)) \overset{x_{n-1}}{\tilde{\to}} \mathcal{O}_{\widetilde{\mathbb{A}}}(-\widetilde{H}_{n-1})$, where $\widetilde{H}_i$ is as in Definition(4-1) and $\mathbb{E}_{\widetilde{H}_i}(*)$ is the pull back of $\mathbb{E}(*)$ to $\widetilde{H}_i$. Note that

$\widetilde{H}_{n-1}$ is a refinement of the affine space H_{n-1} and that the map

$$H^0(\mathcal{O}_{\widetilde{\mathbb{A}}}(-a\mathbb{E}(t)-(b+1)\mathbb{E}(1))) \\ \to H^0(\mathcal{O}_{\widetilde{H}_{n-1}}(-a\mathbb{E}_{\widetilde{H}_{n-1}}(t)-(b+1)\mathbb{E}_{\widetilde{H}_{n-1}}(1)))$$

is surjective by Proposition(5-1)(2) and its variant for the toric variety $\widetilde{H}_{n-1}$. The desired vanishing is now reduced to the case $b = 0$ by induction.

In case $a = -1$ we use the exact sequence

$$0 \to \mathcal{O}_{\widetilde{\mathbb{A}}}(\mathbb{E}(t) - b\mathbb{E}(1)) \to \mathcal{O}_{\widetilde{\mathbb{A}}}(-b\mathbb{E}(1)) \to \mathcal{O}_{\widetilde{H}_t}(-b\mathbb{E}_{\widetilde{H}_t}(1)) \to 0$$

that follows from $\mathcal{O}_{\widetilde{\mathbb{A}}}(\mathbb{E}(t)) \overset{t}{\tilde{\to}} \mathcal{O}_{\widetilde{\mathbb{A}}}(-\widetilde{H}_t)$. Thus we are reduced to the case $a = 0$ by the same argument as before.

In case $a = -2$ we use the exact sequence

$$0 \to \mathcal{O}_{\widetilde{\mathbb{A}}}(2\mathbb{E}(t) - b\mathbb{E}(1)) \to \mathcal{O}_{\widetilde{\mathbb{A}}}(\mathbb{E}(t) - b\mathbb{E}(1)) \\ \to \mathcal{O}_{\widetilde{H}_n}(\mathbb{E}_{\widetilde{H}_n}(t) - b\mathbb{E}_{\widetilde{H}_n}(1)) \to 0$$

that follows from $\mathcal{O}_{\widetilde{\mathbb{A}}}(\mathbb{E}(t)) \overset{x_n}{\tilde{\to}} \mathcal{O}_{\widetilde{\mathbb{A}}}(-\widetilde{H}_n)$. Thus we are reduced to the case $a = -1$ by the same argument as before. This completes the proof of Proposition(5-1)(2). Q.E.D.

Next we show Proposition(4-2). Recall the notation of Definition(1-3). We start with the exact sequence

$$0 \to \Omega_{\mathbb{P}/k}\Big(\log \sum_{1\le i\le n} H_i + H_t\Big) \\ \to \mathcal{O}_{\mathbb{P}}(-1)\cdot dX_0 \oplus \bigoplus_{1\le i\le n} \mathcal{O}_{\mathbb{P}}\frac{dX_i}{X_i} \oplus \mathcal{O}_{\mathbb{P}}\frac{dt}{t} \to \mathcal{O}_{\widetilde{\mathbb{P}}} \to 0.$$

It is easy to see

$$\pi^*\Omega_{\mathbb{P}/k}\Big(\log \sum_{0\le i\le n} H_i + H_t\Big) \tilde{\to} \Omega_{\widetilde{\mathbb{P}}/k}\Big(\log \sum_{1\le i\le n} \widetilde{H}_i + \widetilde{H}_t + \mathbb{E}\Big).$$

Hence we get the exact sequence

$$0 \to \Omega_{\widetilde{\mathbb{P}}/k}\Big(\log \sum_{1\le i\le n} \widetilde{H}_i + \widetilde{H}_t + \mathbb{E}\Big) \\ \to \pi^*\mathcal{O}_{\mathbb{P}}(-1)\cdot dX_0 \oplus \bigoplus_{1\le i\le n} \mathcal{O}_{\widetilde{\mathbb{P}}}\frac{dX_i}{X_i} \oplus \mathcal{O}_{\widetilde{\mathbb{P}}}\frac{dt}{t} \to \mathcal{O}_{\widetilde{\mathbb{P}}} \to 0.$$

Noting $\mathcal{O}_{\widetilde{\mathbb{P}}}(\widetilde{H}_i) \otimes \mathcal{O}_{\widetilde{\mathbb{P}}}(\mathbb{E}(i)) = \pi^*\mathcal{O}_{\mathbb{P}}(H_i) \overset{x_i}{\tilde{\to}} \pi^*\mathcal{O}_{\mathbb{P}}(1)$, it induces the exact sequence

$$\begin{aligned} 0 \to & \Omega_{\widetilde{\mathbb{P}}/k}(\log \widetilde{H}_t + \mathbb{E}) \\ & \to \pi^*\mathcal{O}_{\mathbb{P}}(-1) \cdot dX_0 \oplus \bigoplus_{1 \le i \le n} \mathcal{O}_{\widetilde{\mathbb{P}}}(\mathbb{E}(i)) \otimes \pi^*\mathcal{O}_{\mathbb{P}}(-1) \cdot dX_i \oplus \mathcal{O}_{\widetilde{\mathbb{P}}} \frac{dt}{t} \\ & \qquad \to \mathcal{O}_{\widetilde{\mathbb{P}}} \to 0. \end{aligned}$$

The first exact sequence of Proposition(4-2) is an immediate consequence of this. To deduce the last exact sequence from the first one, we use the commutative diagram

$$\begin{array}{ccccccc} 0 \to & \mathcal{O}_{\widetilde{\mathbb{P}}_\Lambda} & \to & \widetilde{\Sigma} & \to & \theta_{\widetilde{\mathbb{P}}_\Lambda/S} & \to 0 \\ & \| & & \downarrow j_F & & \downarrow i_F & \\ 0 \to & \mathcal{O}_{\widetilde{\mathbb{P}}_\Lambda} & \to & \mathcal{L} & \to & \mathcal{L} \otimes \mathcal{O}_{\widetilde{X}} & \to 0 \end{array}$$

where i_F is induced by j_F. It suffices to show that i_F is surjective and that its kernel is $\theta_{\widetilde{\mathbb{P}}_\Lambda/S}(-\log \widetilde{X})$. Restricted on $\widetilde{\mathbb{P}}_\Lambda - \mathbb{E} = \mathbb{P}_\Lambda - \{0\}$, the assertion follows from the condition Theorem(1-2)(1). Thus it suffices to verify it on each U_σ with $\sigma \in \Delta_s$. Restricted on $\widetilde{\mathbb{A}}_\Lambda$, $\mathcal{L}_{|\widetilde{\mathbb{A}}_\Lambda} = \mathcal{O}_{\widetilde{\mathbb{A}}_\Lambda}(-\delta\mathbb{E}(1))$ and we have

$$i_F(\partial) = \text{the class of } \partial f \in \mathcal{O}_{\widetilde{\mathbb{A}}_\Lambda}(-\delta\mathbb{E}(1)) \quad \text{for } \partial \in \theta_{\widetilde{\mathbb{P}}_\Lambda/S}.$$

Taking $\sigma = \sigma^+_{k,\hat{i}}$, we have (cf. the proof of Theorem(1-2))

$$U_\sigma = \mathrm{Spec}(R_\sigma) \quad \text{with } R_\sigma = \Lambda[y_1, \ldots, y_{n+1}]/(t - y_1y_2y_3).$$

The exceptional divisors on U_σ are defined by $y_1 = 0$ and $y_2 = 0$. We can write $f = (y_1^k y_2^{k+1})^\delta \cdot \tilde{f}$ in R_σ where $\tilde{f}$ is an equation of $\widetilde{X} \cap U_\sigma \subset U_\sigma$ and $\mathcal{O}_{U_\sigma}(-\delta\mathbb{E}(1)) = (y_1^k y_2^{k+1})^\delta \mathcal{O}_{U_\sigma}$. We have locally on U_σ

$$\begin{cases} \theta_{\widetilde{\mathbb{P}}_\Lambda/S} = \Big\{ \displaystyle\sum_{1 \le i \le 3} a_i \cdot y_i \partial/\partial y_i + \sum_{4 \le j \le n+1} a_j \cdot \partial/\partial y_j \;\Big|\; \\ \qquad\qquad a_i,\ a_j \in \mathcal{O}_{U_\sigma},\ a_1 + a_2 + a_3 = 0 \Big\}, \\ \theta_{\widetilde{\mathbb{P}}_\Lambda/S}(-\log \widetilde{X}) = \{\partial \in \theta_{\widetilde{\mathbb{P}}_\Lambda/S} \mid \partial \tilde{f} \in \tilde{f}\mathcal{O}_{U_\sigma}\}. \end{cases}$$

We compute

$$\begin{cases} (y_1^k y_2^{k+1})^{-\delta} \cdot y_i \partial f/\partial y_i \equiv y_i \partial \tilde{f}/\partial y_i \mod (\tilde{f}) \quad (i = 1, 2), \\ (y_1^k y_2^{k+1})^{-\delta} \cdot \partial f/\partial y_j = \partial \tilde{f}/\partial y_j \quad (3 \le j \le n+1), \end{cases}$$

The condition of Theorem(1-1) on U_σ implies the regularity of

$$R_\sigma/(\tilde{f}),\ R_\sigma/(\tilde{f}, y_3),\ R_\sigma/(\tilde{f}, y_i),\ R_\sigma/(\tilde{f}, y_3, y_i)\ (i = 1, 2),$$
$$R_\sigma/(\tilde{f}, y_1, y_2),\ R_\sigma/(\tilde{f}, y_1, y_2, y_3).$$

By a standard argument this implies the desired assertion restricted on U_σ. The same computation shows the assertion on U_σ for other σ and the proof is complete. Q.E.D.

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Appendix

Atsushi Ikeda

This is an appendix for the paper *"Infinitesimal logarithmic Torelli problem for degenerating hypersurfaces in* $\mathbb{P}^n$*"* by S. Saito. In Theorem (2-1), the injectivity of $d\rho_Z$ was proved for degenerating hypersurfaces. But $d\rho_Z$ is not injective in case n is odd and $\delta = 2$. We want to know the meaning of the exceptional cases. It is explained here, by using the extended period map, which is defined by K. Kato and S. Usui.

When we fix integers $\delta \geq 2$, $s \geq 1$ and $d \geq s\delta$, and general coefficients $a_\alpha \in \mathbb{C}$, a strict semistable degeneration $\tilde{X} \to B$ of hypersurfaces in $\mathbb{P}^{m+1}$ over the unit disk is constructed in Section 1. We denote the central fiber by $Z = Z_0 \cup Z_1 \cup \cdots \cup Z_s$, where $Z_0 = \tilde{X} \cap \tilde{H}_t$ and $Z_k = \tilde{X} \cap \mathbb{E}_k$.

Proposition 1. *Assume* $d \geq s\delta + 1$. *The mixed Hodge structure on* $H^m(Z, \mathbb{Q})$ *satisfies*

- $\mathrm{Gr}_l^W H^m(Z, \mathbb{Q}) = 0$ *if* $l \leq m - 2$,
- $\mathrm{Gr}_{m-1}^W H^m(Z, \mathbb{Q}) \simeq H^{m-1}_{\mathrm{prim}}(Z_0 \cap Z_s, \mathbb{Q})$.

$Z_0 \cap Z_s$ *is a nonsingular hypersurface of degree* δ *in* $\tilde{H}_t \cap \mathbb{E}_s \cong \mathbb{P}^m$.

Proof. The spectral sequence

$$E_1^{p,q} = H^q(Z^{[p]}, \mathbb{Q}) \Rightarrow H^{p+q}(Z, \mathbb{Q})$$

defines the weight filtration on $H^i(Z, \mathbb{Q})$, where $Z^{[p]} = \coprod_{0 \leq i_0 < \cdots < i_p \leq s} Z_{i_0} \cap \cdots \cap Z_{i_p}$. Let $\tilde{\mathbb{P}}_o = \tilde{H}_t \cup \mathbb{E}_1 \cup \cdots \cup \mathbb{E}_s$ be the central fiber of $\tilde{\mathbb{P}}_B \to B$. We know that $\tilde{\mathbb{P}}_o^{[p]} = Z^{[p]} = \emptyset$ for $p \geq 3$, so $\mathrm{Gr}_l^W H^m(Z, \mathbb{Q}) = 0$ for $l \leq m - 3$.

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We recall that

$$\begin{cases} \tilde{\mathbb{P}}_o^{[2]} = \coprod_{1\leq k\leq s-1} \tilde{H}_t \cap \mathbb{E}_k \cap \mathbb{E}_{k+1}, \\ \tilde{\mathbb{P}}_o^{[1]} = \coprod_{1\leq k\leq s} \tilde{H}_t \cap \mathbb{E}_k \amalg \coprod_{1\leq k\leq s-1} \mathbb{E}_k \cap \mathbb{E}_{k+1}, \end{cases}$$

and

$$\begin{cases} Z^{[2]} = \coprod_{1\leq k\leq s-1} Z_0 \cap Z_k \cap Z_{k+1}, \\ Z^{[1]} = \coprod_{1\leq k\leq s} Z_0 \cap Z_k \amalg \coprod_{1\leq k\leq s-1} Z_k \cap Z_{k+1}. \end{cases}$$

Here we remark that $Z_0 \cap Z_k \cap Z_{k+1}$, $Z_0 \cap Z_k$ and $Z_k \cap Z_{k+1}$ are not connected if $m = 2$ and $1 \leq k \leq s-1$.

The semistable degeneration is constructed by

$$\begin{array}{ccccccccc} B \times \mathbb{P}^{m+1} = & \mathbb{P}_B & \xleftarrow{\pi_1} & \mathbb{P}_B^{(1)} & \xleftarrow{\pi_2} & \mathbb{P}_B^{(2)} & \xleftarrow{\pi_3} \cdots \xleftarrow{\pi_s} & \mathbb{P}_B^{(s)} & = \tilde{\mathbb{P}}_B \\ & \cup & & \cup & & \cup & & \cup & \\ & X & \longleftarrow & X^{(1)} & \longleftarrow & X^{(2)} & \longleftarrow \cdots \longleftarrow & X^{(s)} & = \tilde{X}, \end{array}$$

where π_1 is the blowing-up of $\mathbb{P}_B$ along the singular point $p \in X$, π_k is the blowing-up of $\mathbb{P}_B^{(k-1)}$ along the singular locus $L_{k-1} \cong \mathbb{P}^1$ of $X^{(k-1)}$, and $X^{(k)}$ is the proper transform of X. $\mathbb{E}_k$ is the proper transform in $\tilde{\mathbb{P}}_B$ of the exceptional set of π_k. $\tilde{H}_t \subset \tilde{\mathbb{P}}_B$ is the proper transform of $H_t = \{t = 0\} \subset \mathbb{P}_B$.

$Z_k \cap Z_{k+1}$ is isomorphic to $\mathbb{P}^1 \times (Z_0 \cap Z_k \cap Z_{k+1})$, and contains $Z_0 \cap Z_k \cap Z_{k+1}$ as a fiber of the projection $\mathbb{P}^1 \times (Z_0 \cap Z_k \cap Z_{k+1}) \to \mathbb{P}^1$. So the restriction $H^{m-2}(Z_k \cap Z_{k+1}) \to H^{m-2}(Z_0 \cap Z_k \cap Z_{k+1})$ is surjective. Since $Z_k \cap Z_{k+1}$ meet only Z_0, this shows $\mathrm{Gr}_{m-2}^W H^m(Z, \mathbb{Q}) = 0$.

There is a commutative diagram

$$\begin{array}{ccccccc} H^{m-1}(\tilde{\mathbb{P}}_o^{[0]}) & \to & H^{m-1}(\tilde{\mathbb{P}}_o^{[1]}) & \to & H^{m-1}(\tilde{\mathbb{P}}_o^{[2]}) & \to & H^{m-1}(\tilde{\mathbb{P}}_o^{[3]}) = 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H^{m-1}(Z^{[0]}) & \to & H^{m-1}(Z^{[1]}) & \to & H^{m-1}(Z^{[2]}) & \to & H^{m-1}(Z^{[3]}) = 0, \end{array}$$

where the horizontal sequences are complex, and $\mathrm{Gr}_{m-1}^W H^{m-1+i}(Z, \mathbb{Q})$ is the i-th cohomology of the second sequence. So $\mathrm{Gr}_{m-1}^W H^m(Z, \mathbb{Q}) \simeq H^{m-1}_{\mathrm{prim}}(Z_0 \cap Z_s, \mathbb{Q})$ is proved by the following:

1. $H^{m-1}(\tilde{\mathbb{P}}_o^{[2]}) \simeq H^{m-1}(Z^{[2]})$.

2. $\mathrm{Coker}(H^{m-1}(\tilde{\mathbb{P}}_o^{[1]}) \to H^{m-1}(Z^{[1]})) \simeq H^{m-1}_{\mathrm{prim}}(Z_0 \cap Z_s)$.
3. The composition $H^{m-1}(Z^{[0]}) \to H^{m-1}(Z^{[1]}) \to H^{m-1}_{\mathrm{prim}}(Z_0 \cap Z_s)$ is zero.
4. $\mathrm{Gr}^W_{m-1}H^m(\tilde{\mathbb{P}}_o) = 0$ and $\mathrm{Gr}^W_{m-1}H^{m+1}(\tilde{\mathbb{P}}_o) = 0$. (This means that the first sequence in the diagram is exact.)

Let X be defined in $B \times \mathbb{P}^{m+1}$ by the equation

$$\sum_{\alpha_0+\cdots+\alpha_{m+1}=d} a_\alpha t^{\max\{0,s\delta-s(\alpha_1+\cdots+\alpha_m)-\alpha_{m+1}\}} X_0^{\alpha_0}\cdots X_{m+1}^{\alpha_{m+1}} = 0.$$

We define a hypersurface $Y \subset \mathbb{P}^m$ by

$$\sum_{\alpha_1+\cdots+\alpha_m=\delta,\alpha_{m+1}=0} a_\alpha X_1^{\alpha_1}\cdots X_m^{\alpha_m} = 0,$$

which is singular at $[1:0:\cdots:0]$. Let $\tilde{Y} \subset \tilde{\mathbb{P}^m}$ be the desingularization by the blowing-up at the point, and Y_0 be the hyperplane section of Y by $\{X_0 = 0\}$. Y is the projective cone over Y_0, and $\tilde{Y}$ is a $\mathbb{P}^1$-bundle over Y_0.

For $1 \le k \le s-1$, there are isomorphisms

$$\begin{array}{ccc} Z_0 \cap Z_k \cap Z_{k+1} & \subset & \tilde{H}_t \cap \mathbb{E}_k \cap \mathbb{E}_{k+1} \\ \downarrow\cong & & \downarrow\cong \\ Y_0 & \subset & \mathbb{P}^{m-1}, \end{array}$$

$$\begin{array}{ccc} Z_k \cap Z_{k+1} & \subset & \mathbb{E}_k \cap \mathbb{E}_{k+1} \\ \downarrow\cong & & \downarrow\cong \\ \mathbb{P}^1 \times Y_0 & \subset & \mathbb{P}^1 \times \mathbb{P}^{m-1} \end{array} \quad \textit{and} \quad \begin{array}{ccc} Z_0 \cap Z_k & \subset & \tilde{H}_t \cap \mathbb{E}_k \\ \downarrow\cong & & \downarrow\cong \\ \tilde{Y} & \subset & \tilde{\mathbb{P}^m}. \end{array}$$

So we have $H^{m-1}(\tilde{H}_t \cap \mathbb{E}_k \cap \mathbb{E}_{k+1}) \simeq H^{m-1}(Z_0 \cap Z_k \cap Z_{k+1})$, $H^{m-1}(\mathbb{E}_k \cap \mathbb{E}_{k+1}) \simeq H^{m-1}(Z_k \cap Z_{k+1})$ and $H^{m-1}(\tilde{H}_t \cap \mathbb{E}_k) \simeq H^{m-1}(Z_0 \cap Z_k)$.

$Z_0 \cap Z_s$ is isomorphic to the hypersurface in $\tilde{H}_t \cap \mathbb{E}_s \cong \mathbb{P}^m$ defined by

$$\sum_{s\delta=s(\alpha_1+\cdots+\alpha_m)+\alpha_{m+1}} a_\alpha X_0^{\delta-(\alpha_1+\cdots+\alpha_m)} X_1^{\alpha_1}\cdots X_m^{\alpha_m} = 0.$$

The map $H^{m-1}(\tilde{H}_t \cap \mathbb{E}_s) \to H^{m-1}(Z_0 \cap Z_s)$ has the cokernel $H^{m-1}_{\mathrm{prim}}(Z_0 \cap Z_s, \mathbb{Q})$. We have proved (1) and (2).

$\mathbb{E}_s$ is isomorphic to the hypersurface in $\mathbb{P}^1 \times \mathbb{P}^{m+1}$ defined by $u_0^s X_0 = u_1^s X_{m+1}$, where u_i is the parameter of $\mathbb{P}^1$. In this identification, Z_s is defined by

$$\begin{cases} u_0^s X_0 = u_1^s X_{m+1}, \\ \sum_{s\delta \geq s(\alpha_1+\cdots+\alpha_m)+\alpha_{m+1}} a_\alpha u_0^{s\delta-\alpha_{m+1}} u_1^{\alpha_{m+1}} X_1^{\alpha_1} \cdots X_m^{\alpha_m} X_{m+1}^{\delta-(\alpha_1+\cdots+\alpha_m)} = 0. \end{cases}$$

This shows that Z_s is a very ample divisor in $\mathbb{E}_s$. So the left vertical map in the diagram

$$\begin{array}{ccc} H^{m-1}(\mathbb{E}_s) & \to & H^{m-1}(\tilde{H}_t \cap \mathbb{E}_s) \\ \downarrow \simeq & & \downarrow \\ H^{m-1}(Z_s) & \to & H^{m-1}(Z_0 \cap Z_s) \end{array}$$

is an isomorphism. Hence $H^{m-1}(Z_s) \to H^{m-1}_{\mathrm{prim}}(Z_0 \cap Z_s)$ is the zero map.

By Lemma 2, the left vertical map in the diagram

$$\begin{array}{ccc} H^{m-1}(\tilde{H}_t) & \to & H^{m-1}(\tilde{H}_t \cap \mathbb{E}_s) \\ \downarrow \simeq & & \downarrow \\ H^{m-1}(Z_0) & \to & H^{m-1}(Z_0 \cap Z_s) \end{array}$$

is an isomorphism. Hence $H^{m-1}(Z_0) \to H^{m-1}_{\mathrm{prim}}(Z_0 \cap Z_s)$ is the zero map, and (3) is proved.

Because the monodromy of $\tilde{\mathbb{P}}_B \to B$ is trivial, (4) is proved by same argument in the proof of Corollary 3, using Clemens-Schmid exact sequence. Q.E.D.

Lemma 2. *If $d \geq s\delta + 1$, then $H^{m-1}(\tilde{H}_t) \simeq H^{m-1}(Z_0)$.*

Proof. $\tilde{H}_t$ is obtained by

$$\mathbb{P}^{m+1} \cong H_t = H_t^{(0)} \overset{\pi_1}{\leftarrow} H_t^{(1)} \overset{\pi_2}{\leftarrow} \cdots \overset{\pi_s}{\leftarrow} H_t^{(s)} = \tilde{H}_t,$$

where π_1 is the blowing-up along the point p, and π_k is the blowing-up along the point $L_{k-1} \cap H_t^{(k-1)}$ for $2 \leq k \leq s$, and we set $\pi = \pi_1 \circ \cdots \circ \pi_s$. We denote by $E'_k \subset H_t^{(k)}$ the exceptional divisor of π_k, and by $E_k \subset \tilde{H}_t$ its proper transform in $\tilde{H}_t$. If H is a hyperplane in H_t, then $a\pi_1^*(H) + b(\pi_1^*(H) - E'_1)$ is a very ample divisor in $H_t^{(1)}$ for $a, b \in \mathbb{Z}_{>0}$. In case $s = 1$, $Z_0 \sim d\pi^*(H) - \delta E_1 = (d-\delta)\pi^*(H) + \delta(\pi^*(H) - E_1)$ is very ample in $\tilde{H}_t$, hence Lemma is proved.

We assume $s \geq 2$. In this case, Z_0 is not ample in $\tilde{H}_t$. By the exact sequence

$$H_c^{m-1}(\tilde{H}_t \setminus Z_0) \to H^{m-1}(\tilde{H}_t) \to H^{m-1}(Z_0) \to H_c^{m-1}(\tilde{H}_t \setminus Z_0),$$

we want to show $H_c^{m-1}(\tilde{H}_t \setminus Z_0) = 0$ and $H_c^{m-1}(\tilde{H}_t \setminus Z_0) = 0$. Let $Z_0^{(k)}$ be the proper transform of $Z_0^{(k-1)}$ by π_k, where $Z_0^{(0)} = H_t \cap X$. Since $H_t \cong \mathbb{P}^{m+1}$, $E'_k \cong \mathbb{P}^m$ and

$$H_t^{(k)} \setminus (Z_0^{(k)} \cup E'_k) \cong H_t^{(k-1)} \setminus Z_0^{(k-1)},$$

$H_c^{m-1}(\tilde{H}_t \setminus Z_0) = 0$ is proved by the exact sequence

$$\begin{aligned} H_c^{m-2}(E'_k \setminus E'_k \cap Z_0^{(k)}) &\to H_c^{m-1}(H_t^{(k)} \setminus (Z_0^{(k)} \cup E'_k)) \\ &\to H_c^{m-1}(H_t^{(k)} \setminus Z_0^{(k)}) \to H_c^{m-1}(E'_k \setminus E'_k \cap Z_0^{(k)}), \end{aligned}$$

inductively.

By $E_k = \tilde{H}_t \cap \mathbb{E}_k \cong \tilde{\mathbb{P}}^m$ and $E_k \cap Z_0 \cong \tilde{Y}$, we have

$$H_c^m(E_k \setminus (E_k \cap Z_0)) = 0,$$

for $1 \leq k \leq s-1$, and these contain $E_{k-1} \cap E_k \cong \mathbb{P}^{m-1}$ and $E_{k-1} \cap E_k \cap Z_0 \cong Y_0$ as a section of the $\mathbb{P}^1$-bundle structure, so we can see

$$\begin{aligned} H_c^{m-1}(E_k \setminus (E_k \cap Z_0)) &\simeq H_c^{m-1}((E_{k-1} \cap E_k) \setminus (E_{k-1} \cap E_k \cap Z_0)) \\ &\simeq H_{\mathrm{prim}}^{m-2}(Y_0), \end{aligned}$$

$$H_c^{m-1}(E_k \setminus ((E_k \cap Z_0) \cup (E_{k-1} \cap E_k))) = 0,$$

$$H_c^m(E_k \setminus ((E_k \cap Z_0) \cup (E_{k-1} \cap E_k))) = 0,$$

for $2 \leq k \leq s-1$. By the exact sequence

$$\begin{aligned} &H_c^{m-1}(E_k \setminus ((E_k \cap Z_0) \cup (E_{k-1} \cap E_k))) \\ &\to H_c^m(\tilde{H}_t \setminus (Z_0 \cup E_1 \cup \cdots \cup E_k)) \to H_c^m(\tilde{H}_t \setminus (Z_0 \cup E_1 \cup \cdots \cup E_{k-1})) \\ &\qquad \to H_c^m(E_k \setminus ((E_k \cap Z_0) \cup (E_{k-1} \cap E_k))), \end{aligned}$$

we have

$$H_c^m(\tilde{H}_t \setminus (Z_0 \cup E_1 \cup \cdots \cup E_{s-1})) \simeq H_c^m(\tilde{H}_t \setminus (Z_0 \cup E_1)),$$

inductively, and by the exact sequence

$$0 = H_c^{m-1}(\tilde{H}_t \setminus Z_0) \to H_c^{m-1}(E_1 \setminus (E_1 \cap Z_0)) \to H_c^m(\tilde{H}_t \setminus (Z_0 \cup E_1)) \\ \to H_c^m(\tilde{H}_t \setminus Z_0) \to H_c^m(E_1 \setminus (E_1 \cap Z_0)) = 0,$$

we have the exact sequence

$$0 \to H_{\mathrm{prim}}^{m-2}(Y_0) \to H_c^m(\tilde{H}_t \setminus (Z_0 \cup E_1 \cup \cdots \cup E_{s-1})) \\ \to H_c^m(\tilde{H}_t \setminus Z_0) \to 0.$$

To see $H_c^m(\tilde{H}_t \setminus Z_0) = 0$,

$$H_{\mathrm{prim}}^{m-2}(Y_0) \simeq H_c^m(\tilde{H}_t \setminus (Z_0 \cup E_1 \cup \cdots \cup E_{s-1}))$$

is proved in the following. We consider the rational map

$$\begin{array}{ccc} H_t \cong \mathbb{P}^{m+1} & \cdots \to & \mathbb{P}^{2m} \\ [X_0 : \cdots : X_{m+1}] & \mapsto & [X_1^s : \cdots : X_{m+1}^s : X_0^{s-1}X_1 : \cdots : X_0^{s-1}X_m] \\ & & = [y_1 : \cdots : y_{m+1} : z_1 : \cdots : z_m], \end{array}$$

which has the elimination of indeterminacy $\phi : \tilde{H}_t \to \mathbb{P}^{2m}$. Let H_t' be the image of $\pi \times \phi : \tilde{H}_t \to H_t \times \mathbb{P}^{2m}$, and $W \subset H_t \times \mathbb{P}^{2m}$ be the subvariety defined by $X_1 = \cdots = X_{m+1} = y_1 = \cdots = y_{m+1} = 0$, which is contained in H_t'. The birational morphism $\pi \times \phi : \tilde{H}_t \to H_t'$ has the exceptional set $E_1 \cup \cdots \cup E_{s-1}$, and $(\pi \times \phi)(E_1 \cup \cdots \cup E_{s-1}) = W$. We can see

$$\begin{cases} \pi^* \mathcal{O}_{H_t}(1) \simeq \mathcal{O}_{\tilde{H}_t}(\pi^* H), \\ \phi^* \mathcal{O}_{\mathbb{P}^{2m}}(1) \simeq \mathcal{O}_{\tilde{H}_t}(s(\pi^* H) - E_1 - 2E_2 - \cdots - sE_s). \end{cases}$$

Because Z_0 is linearly equivalent to $d\pi^*(H) - \delta E_1 - 2\delta E_2 - \cdots - s\delta E_s$ in $\tilde{H}_t$, $\mathcal{O}_{H_t'}(Z_0') \simeq (\mathcal{O}_{H_t}(d - s\delta) \boxtimes \mathcal{O}_{\mathbb{P}^{2m}}(\delta)) \otimes \mathcal{O}_{H_t'}$, where $Z_0' = (\pi \times \phi)(Z_0)$. By the assumption $d \geq s\delta + 1$, Z_0' is a very ample divisor in H_t', so we have

$$H_c^{m-1}(W \setminus (W \cap Z_0')) \simeq H_c^m(H_t' \setminus (Z_0' \cup W)).$$

Since $W \cap Z_0'$ is isomorphic to Y_0 in $\mathbb{P}^{m-1} \cong W$, we have $H_c^{m-1}(W \setminus (W \cap Z_0')) \simeq H_{\mathrm{prim}}^{m-2}(Y_0)$. $\pi \times \phi$ induces the isomorphism

$$\tilde{H}_t \setminus (Z_0 \cup E_1 \cup \cdots \cup E_{s-1}) \cong H_t' \setminus (Z_0' \cup W),$$

so $H_c^m(\tilde{H}_t \setminus (Z_0 \cup E_1 \cup \cdots \cup E_{s-1})) \simeq H_{\mathrm{prim}}^{m-2}(Y_0)$ is proved. Q.E.D.

The degeneration $\tilde{X} \to B$ defines the limit Hodge structure $H^i_{\lim}$.

Corollary 3. *Assume $d \geq s\delta + 1$. The mixed Hodge structure on $H^m_{\lim}$ satisfies*

- $\mathrm{Gr}^W_l H^m_{\lim} = 0$ *if* $l \leq m-2$.
- $\mathrm{Gr}^W_{m-1} H^m_{\lim} \simeq H^{m-1}_{\mathrm{prim}}(Z_0 \cap Z_s, \mathbb{Q})$.

If we denote the logarithm of the monodromy by $N : H^m_{\lim} \to H^m_{\lim}$, then $N^2 = 0$, and we have $N = 0$ if and only if m is even and $\delta = 2$.

Proof. We use the Clemens-Schmid exact sequence

$$\mathrm{Gr}^W_{l-2m-2} H_{m+2}(Z, \mathbb{Q}) \to \mathrm{Gr}^W_l H^m(Z, \mathbb{Q}) \to \mathrm{Gr}^W_l H^m_{\lim} \xrightarrow{[N]} \mathrm{Gr}^W_{l-2} H^m_{\lim}.$$

Since $W_{-m-3} H_{m+2}(Z, \mathbb{Q}) = 0$, $\mathrm{Gr}^W_l H^m(Z, \mathbb{Q}) \simeq \mathrm{Gr}^W_l H^m_{\lim}$ is proved by the sequence and Proposition 1, inductively for $l \leq m-1$. By the property of the weight filtration on $H^m_{\lim}$,

$$\mathrm{Gr}^W_{m+i} H^m_{\lim} \xrightarrow{[N^i]}{\sim} \mathrm{Gr}^W_{m-i} H^m_{\lim},$$

the condition $N^i = 0$ is equivalent to $W_{m-i} H^m_{\lim} = 0$, hence the monodromy statement is proved. Since $Z_0 \cap Z_s$ is a hypersurface of degree δ in $\mathbb{P}^m$, $H^{m-1}_{\mathrm{prim}}(Z_0 \cap Z_s, \mathbb{Q}) = 0$ if and only if m is even and $\delta = 2$. Q.E.D.

Corollary 4. *If m is even and $\delta = 2$, then $d\rho_Z$ is not injective.*

Proof. We have the extended period map $\phi : B \to \Gamma \backslash D_\sigma$, where D is the period domain, σ is the nilpotent cone $\mathbb{Q}_{\geq 0} \cdot N$, and Γ is the subgroup of $\mathrm{Aut}(H^m_{\lim}, <, >)$ generated by the monodromy. The log differential of the extended period map satisfies the commutative diagram

$$\begin{array}{ccc} T^{\log}_B(o) = \mathbb{C} \cdot t\frac{\partial}{\partial t} & \xrightarrow{d\phi} & T^h_{\Gamma \backslash D_\sigma}(\phi(o)) \\ \downarrow & & \downarrow \\ H^1(Z, \theta_{Z/S_o}) & \xrightarrow{d\rho_Z} & \bigoplus_{1 \leq p \leq m} \mathrm{Hom}(H^{m-p}(Z, \omega^p_{Z/S_o}), H^{m-p+1}(Z, \omega^{p-1}_{Z/S_o})). \end{array}$$

If $N = 0$, then the extended period domain $\Gamma \backslash D_\sigma$ is equal to D. Because the log structure of D is trivial, $d\phi(t\frac{\partial}{\partial t})$ must be zero. On the other hand, the image of $t\frac{\partial}{\partial t}$ by the log Kodaira-Spencer map is not zero in $H^1(Z, \theta_{Z/S_o})$. So $d\rho_Z$ is not injective. Q.E.D.

Remark 5. If $m \geq 3$ or $d \neq 3$, then the period map $\phi : B \to \Gamma\backslash D_\sigma$ is injective, by the local Torelli for smooth hypersurfaces. But the log differential $d\phi$ is zero if m is even and $\delta = 2$. The injectivity of ϕ does not necessary imply the injectivity of $d\phi$.

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