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Levi-Flat Minimal Hypersurfaces in Two-dimensional Complex Space Forms

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Abstract.

The purpose of this article is to classify the real hypersurfaces in complex space forms of dimension 2 that are both Levi-flat and minimal. The main results are as follows:

When the curvature of the complex space form is nonzero, there is a 1-parameter family of such hypersurfaces. Specifically, for each one-parameter subgroup of the isometry group of the complex space form, there is an essentially unique example that is invariant under this one-parameter subgroup.

On the other hand, when the curvature of the space form is zero, i.e., when the space form is $\mathbb{C}^2$ with its standard metric, there is an additional 'exceptional' example that has no continuous symmetries but is invariant under a lattice of translations. Up to isometry and homothety, this is the unique example with no continuous symmetries.

Introduction

A real hypersurface $\Sigma^3 \subset \mathbb{C}^2$ is *Levi-flat* [CM] if it is foliated by complex curves. (If such a foliation exists, it is necessarily unique.) Thus, a Levi-flat hypersurface in $\mathbb{C}^2$ is essentially a 1-parameter family of complex curves in $\mathbb{C}^2$. If one imposes the further condition that the hypersurface be *minimal*, there is, in addition to the obvious example of a real hyperplane, the deleted cone $C^* \subset \mathbb{C}^2 \setminus \{(0,0)\}$ defined by

$$|z_1|^2 - |z_2|^2 = 0, \quad |z_1|^2 + |z_2|^2 > 0.$$

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This article was begun during a June 1999 visit at the IHES. I would like to thank the IHES for its hospitality. The idea for this article came to me during a conversation with Mikhail Gromov and Gennadi Henkin, who asked (perhaps idly) whether any nontrivial examples of the kind mentioned in the title exist. I thank them for their stimulating conversation.

This cone is foliated by the (punctured) lines $z_1 = \lambda z_2$ with $|\lambda| = 1$ and hence is Levi-flat. Since C^* is the cone on the Clifford torus, it is also minimal as a submanifold of $\mathbb{C}^2$.

It is not obvious that there are any examples of minimal, Levi-flat hypersurfaces in $\mathbb{C}^2$ that are distinct from these up to rigid motion. The condition of being either minimal or Levi-flat constitutes a single non-linear second order PDE for the hypersurface Σ. A short calculation shows that the combined conditions form a second order system that is not involutive in Cartan's sense. In fact, by Cartan's classification [Ca] of the involutive second order systems for one function of three variables, there is no second order equation that is in involution with the minimal hypersurface equation for a hypersurface in a Riemannian 4-manifold. Thus, describing the solutions of such a system requires analysis that goes beyond an application of the Cartan-Kähler theorem.

In this article, I carry out this analysis, classify the solutions of this overdetermined system, both locally and globally, and show that there are many other examples. Since it is no harder to do the analysis for the general two-dimensional complex space form, I do the computations in this more general setting. While the calculations were guided by certain concepts from exterior differential systems, this article has been written so that no knowledge of this subject is required of the reader beyond the (elementary) Frobenius theorem on integrable plane fields. Nevertheless, the reader who wonders how some of the calculations in §2 could be motivated might want to consult [BCG, Chapter VI]. General references on calculations via the moving frame could also be helpful, in which case I recommend [Sp] or [Gr].

The results can be described as follows: Each local solution extends to a unique maximal solution and the space of maximal solutions is finite dimensional, breaking up into two or three different families.

The members of the first family are those hypersurfaces Σ whose complex leaves are totally geodesic in the ambient space form. In flat space, there are only two such examples up to isometry: the hyperplane and the Clifford cone constructed above. When the space form has positive sectional curvature and hence is $\mathbb{P}^2$ with its standard Fubini-Study metric up to a constant scale factor, there is only one example up to rigid motion. Its closure in $\mathbb{P}^2$ has one singular point, near which it resembles the Clifford cone in flat space. When the space form has negative sectional curvature and hence is the complex hyperbolic 2-ball $\mathbb{B}^2$ (i.e., the noncompact dual of $\mathbb{P}^2$) up to a scale factor, there are three distinct examples up to isometry. The closure of one of these examples has a singular point, near which it resembles the Clifford cone. The other two examples are nonsingular, complete, embedded hypersurfaces. For details, see §3.1.

The remaining two families are somewhat more difficult to describe explicitly. The structure equations for the second family show that each such example Σ^3 is invariant under a one-parameter group of isometries of the ambient space and that this one-parameter group acts on the hypersurface Σ preserving each of its complex leaves. Conversely, each one-parameter group of isometries of the ambient space preserves a family of holomorphic curves that foliates the ambient space in the complement of the fixed point set. Up to ambient isometry, there is a unique one-parameter family of these curves whose union is a minimal hypersurface. The minimal Levi-flat hypersurfaces constructed in this way that do not belong to the first family constitute the members of the second family. In §3.2, I construct these hypersurfaces explicitly for each conjugacy class of one-parameter subgroup of the isometry group of the ambient space form. The examples in this second family often have some sort of singular locus and can be either real algebraic or transcendental, see §3.2.

The third family is the most difficult to describe explicitly. It only exists when the ambient curvature is zero, i.e., in the case of $\mathbb{C}^2$ itself. Up to holomorphic isometry and homothety, there is only one such example and it is periodic with respect to a lattice $\Lambda \subset \mathbb{C}^2$ of type F_4. The quotient hypersurface $\Sigma^3 \subset \mathbb{C}^2/\Lambda$ has quite interesting properties. Its complex leaves are compact Riemann surfaces of genus 3 and the 1-parameter family of genus 3 surfaces that makes up this hypersurface is a nontrivial variation in moduli. The formula that defines the embedding of Σ into the abelian variety $\mathbb{C}^2/\Lambda$ is essentially a quotient of the Abel-Jacobi mapping on each complex leaf. There is reason to believe that this hypersurface is an open dense subset of a 'real algebraic' hypersuface in the algebraic variety $\mathbb{C}^2/\Lambda$, but I have not verified this in detail. I would like to thank Dave Morrison for a helpful conversation about the algebraic geometry of this example.

§1. Two-Dimensional Complex Space Forms

This section introduces the structure equations for complex space forms of dimension 2 and establishes the notation that will be used for the remainder of the article. For further discussion of these models, the reader might consult [He] or [KN].

1.1. The group G_R

Let R be a real number and let $G_R \subset \mathrm{SL}(3,\mathbb{C})$ be the connected subgroup whose Lie algebra $\mathfrak{g}_R$ consists of the matrices of the form

$$\begin{pmatrix} ir_1 & -R\bar{x} & -R\bar{y} \\ x & ir_2 & -\bar{z} \\ y & z & -i(r_1+r_2) \end{pmatrix}$$

where r_1 and r_2 are real and x, y, and z are complex.

When $R \neq 0$, this is the identity component of the set of unimodular matrices $\mathbf{g}$ that satisfy ${}^t\bar{\mathbf{g}} H_R \mathbf{g} = H_R$, where

$$H_R = {}^t\overline{H_R} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & R & 0 \\ 0 & 0 & R \end{pmatrix}.$$

In this case, H_R defines a nondegenerate Hermitian inner product $\langle,\rangle_R$ on $\mathbb{C}^3$. Even the matrix H_0 defines a (very degenerate) Hermitian inner product $\langle,\rangle_0$ on $\mathbb{C}^3$ and G_0 preserves it.

1.2. The complex space form $\mathbb{P}^2_R$

The set $\mathbb{P}^2_R \subset \mathbb{P}^2$ consisting of the lines through $0 \in \mathbb{C}^3$ on which $\langle,\rangle_R$ is positive is a homogeneous space of G_R. Write the general element of G_R as

$$\mathbf{g} = \begin{pmatrix} \mathbf{e}_0 & \mathbf{e}_1 & \mathbf{e}_2 \end{pmatrix},$$

where the columns $\mathbf{e}_i$ of $\mathbf{g}$ are to be regarded as $\mathbb{C}^3$-valued functions on G_R. The map $\pi\colon G_R \to \mathbb{P}^2_R$ defined by $\pi(\mathbf{g}) = \mathbb{C}\cdot\mathbf{e}_0$ is a submersion. The fibers of π are the left cosets of the connected subgroup $K \subset G_R$ whose Lie algebra consists of matrices of the form

$$\begin{pmatrix} ir_1 & 0 & 0 \\ 0 & ir_2 & -\bar{z} \\ 0 & z & -i(r_1+r_2) \end{pmatrix}.$$

The group K is compact and isomorphic to the nontrivial double cover of $\mathrm{U}(2)$. In particular, $\mathbb{P}^2_R \simeq G_R/K$ as a homogeneous space.

1.3. The structure equations

Write the left invariant Maurer-Cartan form on G_R in the form

$$\gamma = \mathbf{g}^{-1}d\mathbf{g} = \begin{pmatrix} i\tau & -R\bar{\eta} & -R\bar{\omega} \\ \eta & i(\phi+\tau) & -\bar{\sigma} \\ \omega & \sigma & -i(\phi+2\tau) \end{pmatrix},$$

so that the *first structure equation* becomes

$$\begin{pmatrix} d\mathbf{e}_0 & d\mathbf{e}_1 & d\mathbf{e}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{e}_0 & \mathbf{e}_1 & \mathbf{e}_2 \end{pmatrix} \begin{pmatrix} i\tau & -R\bar{\eta} & -R\bar{\omega} \\ \eta & i(\phi+\tau) & -\bar{\sigma} \\ \omega & \sigma & -i(\phi+2\tau) \end{pmatrix}.$$

There exist on $\mathbb{P}^2_R$ a unique metric ds^2 and a ds^2-orthogonal complex structure J with corresponding Kähler form Ω for which

$$\pi^*(ds^2) = \eta \circ \bar{\eta} + \omega \circ \bar{\omega} \quad \text{and} \quad \pi^*(\Omega) = \tfrac{i}{2}(\eta \wedge \bar{\eta} + \omega \wedge \bar{\omega}).$$

The *second structure equation* $d\gamma = -\gamma \wedge \gamma$ shows that this Kähler structure has constant holomorphic sectional curvature $4R$. (I.e., the Gauss curvature of any totally geodesic complex curve in $\mathbb{P}^2_R$ is $4R$.)

From now on, the fibration $\pi\colon G_R \to \mathbb{P}^2_R$ will be taken as the standard unitary bundle structure for the Kähler geometry of $\mathbb{P}^2_R$. (Strictly speaking, of course, this is not quite correct since one should first divide out by the center of G_R, a cyclic subgroup of order 3, but for simplicity, I will not do this. It should not cause any confusion.)

§2. Real Hypersurfaces

Let Σ^3 be a connected, smooth, embedded[1] real hypersurface in $\mathbb{P}^2_R$. The preimage $B_0 = \pi^{-1}(\Sigma)$ is a principal K-bundle over Σ. From now on, all the forms on G_R are to be understood as pulled back to B_0.

2.1. First invariants

Since Σ is a hypersurface, there will be one linear relation among the real and imaginary parts of the two 1-forms η and ω. Let $B_1 \subset B_0$ be the subset where this relation is $\eta = \bar{\eta}$. Then B_1 is a union of left K_1-cosets where $K_1 \simeq S^1$ is the group of matrices of the form

$$E_\theta = \begin{pmatrix} e^{i\theta} & 0 & 0 \\ 0 & e^{i\theta} & \\ 0 & 0 & e^{-2i\theta} \end{pmatrix}.$$

From now on, all the forms on B_0 are to be understood as pulled back to B_1. In addition to the relation $\eta = \bar{\eta}$, there will be relations of the form

$$\begin{aligned} \phi &= H\eta - ia\omega + i\bar{a}\bar{\omega} \\ \sigma &= \ c\eta + iL\omega - 2s\bar{\omega} \end{aligned}$$

[1]All these calculations will be local, so embeddedness is not a serious restriction.

for some functions a, c, $H = \bar{H}$, L, and s on B_1. (The choice of numerical coefficients is cosmetic.) By the structure equations,

$$d\eta = -i\phi \wedge \eta + \bar{\sigma} \wedge \omega = -(a\omega - \bar{a}\bar{\omega}) \wedge \eta + \overline{(c\eta + iL\omega - 2s\bar{\omega})} \wedge \omega.$$

Since η is real, the imaginary part of the right hand expression must vanish. I.e.,

$$L = \bar{L} \quad \text{and} \quad c = -2\bar{a}.$$

Let $R_\theta\colon B_1 \to B_1$ denote right action by the matrix E_θ. Then η, H, and L are invariant under R_θ while

$$R_\theta^*\omega = e^{3i\theta}\omega, \quad R_\theta^* a = e^{-3i\theta}a, \quad R_\theta^* s = e^{6i\theta}s.$$

Note that quantities such as η, $a\omega$, $\bar{s}\omega^2$, L, H, $|a|^2$, and $|s|^2$ are π-semibasic and invariant under R_θ and so can be considered to be well defined as functions or forms on Σ.

2.1.1. Levi-flatness. The equation $\eta = 0$ defines the preimage in B_1 of the bundle of complex tangent spaces to Σ. Consequently, Σ will be Levi-flat if and only if $\eta \wedge d\eta = 0$. However, by the structure equations and the relations just derived,

$$\eta \wedge d\eta = iL\eta \wedge \omega \wedge \bar{\omega}.$$

Thus, Levi-flatness is equivalent to the condition $L = 0$. From now on, I will assume that Σ is Levi-flat.

2.1.2. Minimality. The induced metric on Σ pulls back to B_1 to be the quadratic form $\eta^2 + \omega \circ \bar{\omega}$, while the second fundamental form II satisfies

$$\pi^*(\mathrm{II}) = c_1 H\eta^2 + \mathrm{Re}(c_2 a\omega) \circ \eta + \mathrm{Re}(c_3 \bar{s}\omega^2)$$

for some nonzero constants c_1, c_2, and c_3 (the explicit values will not be important for what follows). In particular, H is the the mean curvature function of Σ (up to some universal constant multiple), i.e., Σ is minimal if and only if H vanishes identically on B_1. From now on, I will assume that Σ is minimal (as well as Levi-flat).

2.2. Differential consequences of the structure equations

At this point, the forms on B_1 satisfy the reality condition $\eta = \bar{\eta}$, the nondegeneracy condition $\eta \wedge \omega \wedge \bar{\omega} \neq 0$, and the relations

$$\begin{aligned} \phi &= -ia\omega + i\bar{a}\bar{\omega}, \\ \sigma &= -2\bar{a}\eta - 2s\bar{\omega}. \end{aligned}$$

Thus, γ pulled back to B_1 has the form

$$\gamma = \begin{pmatrix} i\tau & -R\eta & -R\bar\omega \\ \eta & i\tau + a\omega - \bar a\bar\omega & 2a\eta + 2\bar s\omega \\ \omega & -2\bar a\eta - 2s\bar\omega & -2i\tau - a\omega + \bar a\bar\omega \end{pmatrix}.$$

The structure equation $d\gamma = -\gamma\wedge\gamma$ expands to the relations

$$\begin{aligned} d\tau &= iR\omega\wedge\bar\omega, \\ d\eta &= (a\omega + \bar a\bar\omega)\wedge\eta, \\ d\omega &= (3i\tau - \bar a\bar\omega)\wedge\omega + 2s\bar\omega\wedge\eta, \end{aligned}$$

and implies the existence of complex-valued functions x and y on B_1 so that

$$\begin{aligned} da &= -3ia\tau - 6\bar a\bar s\eta + (x - 3a^2)\omega - \left(\tfrac12 R - |a|^2 + 2|s|^2\right)\bar\omega, \\ ds &= \quad 6is\tau + \bar x\eta + 3sa\omega + y\bar\omega. \end{aligned}$$

Remark 1. These equations imply strong conditions about the vanishing locus of s on each complex leaf $L\subset\Sigma$. In a small neighborhood U of any point $p\in L$, one can choose a complex coordinate z so that, on $B_L = \pi^{-1}(L)$, there is a nonzero function h so that $\omega = h\pi^*(dz)$ holds on B_L. Correspondingly, there will be a function f on U so that $\bar a\bar\omega = \pi^*(f_{\bar z}\,d\bar z)$ and a function g on U so that $\bar s\omega^2 = \pi^*(g\,dz^2)$. The above structure equations then imply that the product $e^{-3f}g$ is holomorphic in z. Consequently, the quadratic form $\bar s\omega^2$ is a nonvanishing multiple of a holomorphic quadratic form on L and so either vanishes identically or else only vanishes at discrete points of L and then only to finite order. Note that $|s|^2$ vanishes identically on a complex leaf if and only if that leaf is totally geodesic in $\mathbb{P}^2_R$.

Remark 2. It will be useful to understand the metric $\omega\circ\bar\omega$ induced on the complex leaves, in particular, the Gauss curvature of this induced metric. Now, the equation for $d\omega$ can be written in the form

$$d\omega = -i\rho\wedge\omega + 2s\bar\omega\wedge\eta$$

where $\rho = -3\tau + i(a\omega - \bar a\bar\omega)$. The equation

$$d\rho \equiv -\tfrac{i}{2}(4R - 8|s|^2)\omega\wedge\bar\omega \mod \eta$$

then shows that the function $K = 4(R - 2|s|^2)$ restricts to each complex leaf to be its Gauss curvature.

2.2.1. First case. Using the structure equations to expand the identity $d(da) = 0$ and then reducing the result modulo ω yields

$$sx - \bar{s}\bar{x} = 0.$$

There are now two cases to consider. First, suppose that s vanishes identically. Then so do x and y, and the remaining structure equation for a is

$$da = -3ia\tau - 3a^2\omega - (\tfrac{1}{2}R - |a|^2)\bar{\omega}.$$

Differentiating this equation just yields an identity. Thus, the system

$$\begin{aligned} d\tau &= iR\omega \wedge \bar{\omega} \\ d\eta &= (a\omega + \bar{a}\bar{\omega}) \wedge \eta \\ d\omega &= (3i\tau - \bar{a}\bar{\omega}) \wedge \omega \\ da &= -3ia\tau - 3a^2\omega - (\tfrac{1}{2}R - |a|^2)\bar{\omega} \end{aligned} \tag{1}$$

is differentially closed[2] and describes the class of solutions Σ for which the complex leaves are totally geodesic. This class will be analyzed in the next section, after all of the integrability conditions have been found for the remaining cases.

2.2.2. Second and third cases. Suppose now that s does not vanish identically. Since Σ is real analytic and connected and since $|s|^2$ is well-defined on Σ, there is a dense open set $\Sigma^* \subset \Sigma$ on which $|s|^2 > 0$. On the bundle $B_1^* = \pi^{-1}(\Sigma^*) \cap B_1$, which is a dense open subset of B_1, write $x = \bar{s}p$, where p is real. The structure equations are now

$$\begin{aligned} da &= -3ia\tau - 6\bar{a}\bar{s}\eta + (\bar{s}p - 3a^2)\omega - \big(\tfrac{1}{2}R - |a|^2 + 2|s|^2\big)\bar{\omega} \\ ds &= \quad 6is\tau + sp\eta + 3sa\omega + sy\bar{\omega} \end{aligned}$$

(where, to simplify equations to follow, I have replaced the former y by sy, which is permissible since s is nonzero).

Now, a cannot vanish identically. If it were to do so, then the above equations would imply $p = 0$ and $R = -4|s|^2 < 0$ (since s is nonzero). The equation for ds would then simplify to $ds = 6is\tau + sy\bar{\omega}$. Differentiating the relation $R + 4|s|^2 = 0$ then shows that $y = 0$, in turn implying

[2] I.e., the exterior derivatives of these equations are identities. Of course, it then follows from Cartan's generalization of Lie's Third Fundamental Theorem that there are solutions to these equations, but the explicit computations in the next section will make recourse to Cartan's theorem unnecessary. This same comment applies to the other two cases that will turn up in the next subsubsection.

that $ds = 6is\tau$, which then implies that $d\tau = 0$, contradicting the structure equation for τ since $R \neq 0$. By the real analyticity and connectedness of Σ, it follows that $|a|^2$ is nonzero on a dense open set $\Sigma^{**} \subset \Sigma^*$ and I can restrict attention to the corresponding subbundle B_1^{**}, which I will do from now on. Thus, a is nonzero on B_1^{**}.

Now, the structure equations plus the reality of p yield

$$0 = \frac{d(da) \wedge \bar{\omega}}{\bar{s}} + \frac{d(d\bar{a}) \wedge \omega}{s} = 6(\bar{a}\bar{y} - ay)\eta \wedge \omega \wedge \bar{\omega}.$$

Thus ay is real, implying that there exists a function $q = \bar{q}$ for which $y = \bar{a}(q+3)$. (Writing $q+3$ instead of q here simplifies the following formulae.) Expanding the identity $d(da) = 0$ and using the reality of p implies that p satisfies the equation

$$\begin{aligned} dp = (2R - 64|a|^2 + 8|s|^2 - 6|a|^2 q - p^2)\eta \\ - (ap + 24\bar{a}\bar{s} + 2\bar{a}\bar{s}q)\omega - (\bar{a}p + 24as + 2asq)\bar{\omega}. \end{aligned}$$

By this structure equation and the reality of q,

$$0 = \frac{d(ds) \wedge a\omega}{s} + \frac{d(d\bar{s}) \wedge \bar{a}\bar{\omega}}{\bar{s}} = 4q(a^2 s - \bar{a}^2 \bar{s})\eta \wedge \omega \wedge \bar{\omega}.$$

Thus, either q or the imaginary part of $a^2 s$ vanishes identically. These two cases will be considered separately.

First, suppose that $a^2 s$ is real and introduce a real-valued function $t = \bar{t}$ so that $s = \bar{a}^2 t$. Using the reality of t and expanding the identities $0 = d(da) = d(ds) = d(dp)$ yields

$$q = R + 4|a|^2 + 2|a|^2 pt + |a|^4 t^2$$

plus a formula for dt. The result is structure equations of the form

$$\begin{aligned}
d\tau &= iR\omega \wedge \bar{\omega} \\
d\eta &= (a\omega + \bar{a}\bar{\omega}) \wedge \eta \\
d\omega &= (3i\tau - \bar{a}\bar{\omega}) \wedge \omega + 2\bar{a}^2 t\bar{\omega} \wedge \eta \\
da &= -3ia\tau - 6|a|^2 at\eta + a^2(tp-3)\omega - \left(\tfrac{1}{2}R - |a|^2 + 2|a|^4 t^2\right)\bar{\omega} \qquad (2) \\
dp &= -(4R + 16|a|^2 + 16|a|^4 t^2 + 12|a|^2 pt + p^2)\eta \\
&\qquad - \big(p + 8|a|^2 t + 2t(R + 4|a|^4 t^2 + 2|a|^2 pt)\big)(a\omega + \bar{a}\bar{\omega}) \\
dt &= t(p + 12|a|^2 t)\eta + t(1 + 4|a|^2 t^2 + R/|a|^2)(a\omega + \bar{a}\bar{\omega}).
\end{aligned}$$

Differentiating these equations yields only identities, so this represents a set of solutions. These will be analyzed below. This system is compatible with the relation $t = 0$, in which case the structure equations specialize to (1), the first solution found. Thus, the solutions (1) can be regarded as special cases of (2).

On the other hand, if $q \equiv 0$, then the structure equations yield $d\big(d(s)\big) = 6sR\omega\wedge\bar\omega$, so this case can only occur when $R = 0$. Assuming this, the structure equations found so far are

$$
\begin{aligned}
d\tau &= 0\\
d\eta &= (a\omega + \bar a\bar\omega)\wedge\eta\\
d\omega &= (3i\tau - \bar a\bar\omega)\wedge\omega + 2s\bar\omega\wedge\eta\\
da &= -3ia\tau - 6\bar a\bar s\eta + (\bar s p - 3a^2)\omega + \big(|a|^2 - 2|s|^2\big)\bar\omega\\
ds &= s(6i\tau + p\eta + 3a\omega + 3\bar a\bar\omega)\\
dp &= \big(8|s|^2 - 64|a|^2 - p^2\big)\eta - (ap + 24\bar a\bar s)\omega - (\bar a p + 24as)\bar\omega.
\end{aligned}
\tag{3}
$$

Differentiating these equations yield only identities, so this represents a class of solutions that exist only in the case $R = 0$. These will be analyzed below. Since $a^2 s$ is not, in general, real for these solutions, they are not special cases of (2), although when $s = 0$, these solutions do specialize to the $t = 0$ solutions of (2) in the case $R = 0$. These special solutions are the only overlap between the two.

§3. Existence of Solutions

In this section, I will prove general existence results that assure that there are solutions to the equations (1), (2) and (3). In each case, this will be followed by an analysis of the equations that allows a complete description of the corresponding solutions.

3.1. Solutions of type 1

3.1.1. Existence via the Frobenius theorem. Let $M^{10} = G_R\times\mathbb{C}$ and let $\mathbf{g}\colon M\to G_R$ and $\mathbf{a}\colon M\to\mathbb{C}$ be the projections onto the factors. I will regard forms on G_R or $\mathbb{C}$ as forms on M via the pullbacks under these two maps and will not notate the pullback explicitly. Let $\mathcal{I}_1$ be the exterior ideal on M generated by the linearly independent real-valued 1-forms $\theta_1,\dots,\theta_6$ where

$$
\begin{aligned}
\theta_1 &= i(\bar\eta - \eta)\\
\theta_2 &= \phi + i\mathbf{a}\omega - i\bar{\mathbf{a}}\bar\omega\\
\theta_3 + i\theta_4 &= \sigma + 2\bar{\mathbf{a}}\eta\\
\theta_5 + i\theta_6 &= d\mathbf{a} + 3i\mathbf{a}\tau + 3\mathbf{a}^2\omega + \big(\tfrac12 R - |\mathbf{a}|^2\big)\bar\omega.
\end{aligned}
$$

The structure equations $d\gamma = -\gamma\wedge\gamma$ imply that $\mathcal{I}_1$ is differentially closed. Thus, the Frobenius theorem implies that M is foliated by 4-dimensional integral manifolds of $\mathcal{I}_1$. Each leaf $L \subset M$ is the image of a bundle $B_1 \subset G_R$ of a minimal Levi-flat hypersurface Σ satisfying equations (1) under the embedding $\mathrm{id} \times a\colon B_1 \to G_R \times \mathbb{C}$. This gives an abstract description of the solutions of type (1).

Since G_R acts by left translation on $G_R \times \mathbb{C}$ preserving the ideal $\mathcal{I}_1$, this left action permutes the integral manifolds, and two integral manifolds are equivalent under this action if and only if they correspond to congruent hypersurfaces in $\mathbb{P}^2_R$. In particular, two leaves L_1 and L_2 represent equivalent solutions if and only if they satisfy $\mathbf{a}(L_1) = \mathbf{a}(L_2)$. Note that this happens if and only if the two images $\mathbf{a}(L_1)$ and $\mathbf{a}(L_2)$ have nonempty intersection.

3.1.2. Explicit description of the solutions. On any connected solution to (1), the structure equations imply

$$\begin{aligned} 4da \wedge d\bar{a} &= \big(R + 4|a|^2\big)\big((R - 8|a|^2)\omega \wedge \bar{\omega} + 6i\tau \wedge (a\omega + \bar{a}\bar{\omega})\big) \\ d\big(R + 4|a|^2\big) &= -2\big(R + 4|a|^2\big)(a\omega + \bar{a}\bar{\omega}). \end{aligned}$$

It follows that for any leaf L of $\mathcal{I}_1$, either the function $R+4|\mathbf{a}|^2$ vanishes identically or else $\mathbf{a}\colon L \to \mathbb{C}$ is an immersion.

Now, when $R > 0$, the only possibility is that $\mathbf{a}\colon L \to \mathbb{C}$ is an immersion everywhere. Moreover, using the left action of G_R plus the existence of a leaf through any point of $G_R \times \mathbb{C}$, it follows that $\mathbf{a}\colon L \to \mathbb{C}$ is a surjective submersion for every leaf. In particular, all of the leaves are equivalent under the action of G_R. Since $|s|^2$ vanishes identically on L, it follows that the complex leaves of Σ are totally geodesic in $\mathbb{P}^2_R$, which is, up to a constant scale factor, isometric to $\mathbb{CP}^2$ endowed with the Fubini-Study metric. Thus, Σ must be a 1-parameter family of complex lines in $\mathbb{CP}^2$. In fact, Σ must be congruent to the smooth locus C_1^* of the 'cone'

$$C_1 = \left\{ \begin{bmatrix} z \\ w \\ e^{ir}w \end{bmatrix} \in \mathbb{CP}^2 \;\middle|\; r \in \mathbb{R},\ [z,w] \in \mathbb{CP}^1 \right\}.$$

It is evident that C_1^* is both Levi-flat and minimal. Note that C_1 has only one singular point (the intersection of the complex lines that foliate it) and is otherwise smooth.

When $R = 0$, so that $\mathbb{P}^2_0$ is isometric to $\mathbb{C}^2$ with the standard flat metric, there are two possibilities. The first possibility is that $|a|^2$ vanishes identically, in which case the corresponding Σ is congruent to a real hyperplane:

$$H_0 = \left\{ \begin{bmatrix} 1 \\ z \\ r \end{bmatrix} \in \mathbb{P}^2_0 \;\middle|\; r \in \mathbb{R},\ z \in \mathbb{C} \right\}.$$

The second possibility is that $|a|^2$ never vanishes. By the same sort of argument made for the case of positive holomorphic sectional curvature, one sees that all of these cases are equivalent to the smooth part of the cone

$$C_0 = \left\{ \begin{bmatrix} 1 \\ z \\ e^{ir} z \end{bmatrix} \in \mathbb{P}^2_0 \;\middle|\; r \in \mathbb{R},\ z \in \mathbb{C} \right\}.$$

When $R < 0$, there is no loss of generality in setting $R = -1$, so I will do so for this discussion. Then

$$\mathbb{P}^2_{-1} = \left\{ \begin{bmatrix} 1 \\ z^1 \\ z^2 \end{bmatrix} \in \mathbb{CP}^2 \;\middle|\; \left|z^1\right|^2 + \left|z^2\right|^2 < 1 \right\}$$

is the hyperbolic complex 2-ball and there are three possibilities, depending on the sign of $R + 4|a|^2 = 4|a|^2 - 1$.

The solutions with $4|a|^2 - 1 > 0$ are all congruent to the smooth part of the hyperbolic version of the cone:

$$C_{-1} = \left\{ \begin{bmatrix} 1 \\ z \\ e^{ir} z \end{bmatrix} \in \mathbb{P}^2_{-1} \;\middle|\; r \in \mathbb{R},\ \sqrt{2}\,|z| < 1 \right\}.$$

This cone has one singular point. The leaves of $dr = 0$ are the complex leaves, each one biholomorphic to a punctured disk.

All solutions with $4|a|^2 - 1 = 0$ are congruent to the 'horosphere' solution

$$S_{-1} = \left\{ \begin{bmatrix} 1 \\ z \\ ir(1-z) \end{bmatrix} \in \mathbb{P}^2_{-1} \;\middle|\; r \in \mathbb{R},\ |z|^2 + r^2|1-z|^2 < 1 \right\}.$$

The complex leaves in S_{-1} are the leaves of $dr = 0$ in the chosen parametrization. All of these complex leaves intersect at one point on the boundary of the ball. This solution can be interpreted as a limit of the cone C_{-1} as one moves the singular point of the cone out to the boundary of $\mathbb{P}^2_R$ in $\mathbb{P}^2$.

All solutions with $4|a|^2 - 1 < 0$ are congruent to the hyperbolic version of the hyperplane solution, namely

$$H_{-1} = \left\{ \begin{bmatrix} 1 \\ z \\ r \end{bmatrix} \in \mathbb{P}^2_0 \;\middle|\; r \in \mathbb{R},\ r^2 + |z|^2 < 1 \right\}.$$

This completes the list of solutions of the system (1).

3.2. Solutions of type 2

Consider the solutions of the system (2). To avoid repetition, I am going to consider only solutions for which t is non-zero, since the solutions with t vanishing identically have already been accounted for as solutions of type (1).

3.2.1. Existence via the Frobenius theorem. Let $M^{12} = G_R \times \mathbb{C}^* \times \mathbb{R} \times \mathbb{R}$, and let $\mathbf{g}\colon M \to G_R$, $\mathbf{a}\colon M \to \mathbb{C}^*$, $\mathbf{p}\colon M \to \mathbb{R}$, and $\mathbf{t}\colon M \to \mathbb{R}$ be the projections onto the first through fourth factors, respectively. Let $\mathcal{I}_2$ be the exterior ideal on M generated by the linearly independent real-valued 1-forms $\theta_1, \ldots, \theta_8$ where

$$\begin{aligned}
\theta_1 &= i(\bar\eta - \eta) \\
\theta_2 &= \phi + i\mathbf{a}\omega - i\bar{\mathbf{a}}\bar\omega \\
\theta_3 + i\theta_4 &= \sigma + 2\bar{\mathbf{a}}\eta + 2\bar{\mathbf{a}}^2\mathbf{t}\bar\omega \\
\theta_5 + i\theta_6 &= d\mathbf{a} + 3i\mathbf{a}\tau + 6|\mathbf{a}|^2\mathbf{a}\mathbf{t}\eta \\
&\qquad - \mathbf{a}^2(\mathbf{t}\mathbf{p} - 3)\omega + \big(\tfrac{1}{2}R - |\mathbf{a}|^2 + 2|\mathbf{a}|^4\mathbf{t}^2\big)\bar\omega \\
\theta_7 &= d\mathbf{p} + 4(R + 4|\mathbf{a}|^2 + 4|\mathbf{a}|^4\mathbf{t}^2 + 3|\mathbf{a}|^2\mathbf{p}\mathbf{t} + \mathbf{p}^2)\eta \\
&\qquad + \big(\mathbf{p} + 2\mathbf{t}(R + 4|\mathbf{a}|^2 + 4|\mathbf{a}|^4\mathbf{t}^2 + 2|\mathbf{a}|^2\mathbf{p}\mathbf{t})\big)(\mathbf{a}\omega + \bar{\mathbf{a}}\bar\omega) \\
\theta_8 &= d\mathbf{t} - \mathbf{t}\big((\mathbf{p} + 12|\mathbf{a}|^2\mathbf{t})\eta + (1 + 4|\mathbf{a}|^2\mathbf{t}^2 + R/|\mathbf{a}|^2)(\mathbf{a}\omega + \bar{\mathbf{a}}\bar\omega)\big).
\end{aligned}$$

(The reason for the restriction $\mathbf{a} \neq 0$ is the division by $|\mathbf{a}|^2$ in the last formula.) The structure equations show that the ideal $\mathcal{I}_2$ is closed under exterior differentiation, so M is foliated by 4-dimensional integral manifolds of $\mathcal{I}_2$.

By construction, each leaf $L \subset M$ is the image of the bundle $B_1^{**} \subset G_R$ over the nondegenerate part Σ^{**} of a minimal Levi-flat hypersurface Σ satisfying equations (2) under the embedding

$$\mathrm{id} \times a \times p \times t\colon B_1 \longrightarrow G_R \times \mathbb{C}^* \times \mathbb{R} \times \mathbb{R} = M.$$

Since G_R acts by left translation on $G_R \times \mathbb{C}^* \times \mathbb{R} \times \mathbb{R}$ preserving the ideal $\mathcal{I}_2$, this left action permutes its integral manifolds, and two integral manifolds are equivalent under this action if and only if they correspond to congruent hypersurfaces in $\mathbb{P}^2_R$. In particular, two leaves L_1 and L_2 represent equivalent solutions if and only if they satisfy $(\mathbf{a},\mathbf{p},\mathbf{t})(L_1) = (\mathbf{a},\mathbf{p},\mathbf{t})(L_2)$.

In fact, in order for two leaves L_1 and L_2 to be equivalent under G_R, it suffices that the two image sets $(\mathbf{a},\mathbf{p},\mathbf{t})(L_1)$ and $(\mathbf{a},\mathbf{p},\mathbf{t})(L_2)$ in $\mathbb{C}^* \times \mathbb{R} \times \mathbb{R}$ have a nonempty intersection. To see why this is so, note that if L_i contains (g_i, a, p, t), then the submanifold L described by

$$L = \{(g_2 {g_1}^{-1} g, b, q, u) \mid (g, b, q, u) \in L_1\}$$

contains $(g_2, a, p, t) \in L_2$, is evidently a maximal integral manifold of $\mathcal{I}_2$, and so must equal L_2. In particular, in order to classify the solutions up to rigid motion, it would suffice to determine the partition of $\mathbb{C}^* \times \mathbb{R} \times \mathbb{R}$ into the images $(\mathbf{a},\mathbf{p},\mathbf{t})(L)$ as L ranges over the leaves of $\mathcal{I}_2$. Moreover, this argument shows that the fibers of the map $(\mathbf{a},\mathbf{p},\mathbf{t})\colon L \to \mathbb{C}^* \times \mathbb{R} \times \mathbb{R}$ are the orbits of the action on L of the ambient symmetry group of the corresponding solution Σ^{**}.

The structure equations imply that the function $\mathbf{t}$ cannot vanish anywhere on a leaf L unless it vanishes identically on L. As mentioned at the begining of this subsection, the leaves on which $\mathbf{t}$ vanishes identically are of type (1) and so can be set aside in this discussion. For the rest of this subsection, the assumption that $\mathbf{t}$ is nonvanishing on L will be in force.

3.2.2. Symmetries of the solutions. One might expect the images $(\mathbf{a},\mathbf{p},\mathbf{t})(L)$ to have dimension 4, at least at 'generic' points, since each leaf L has dimension 4. However, equations (2) imply that $da \wedge d\bar{a} \wedge dp \wedge dt$ vanishes identically. Consequently, the rank of $(\mathbf{a},\mathbf{p},\mathbf{t})\colon L \to \mathbb{C}^* \times \mathbb{R} \times \mathbb{R}$ is strictly less than 4 at all points, implying that the fibers of this map (and hence the symmetry group of L) must have positive dimension.

It is not hard to make these fibers explicit. By the structure equations (2), the (real) nowhere vanishing vector field Y on L that satisfies

$$\begin{aligned}
\tau(Y) &= R + 2|a|^2(pt - 4) + 4|a|^4 t^2 \\
\eta(Y) &= 0 \\
\omega(Y) &= 6i\bar{a}
\end{aligned}$$

also satisfies $da(Y) = dp(Y) = dt(Y) = 0$. The structure equations also show that, for the generic value $(a_0, p_0, t_0) \in \mathbb{C}^* \times \mathbb{R} \times \mathbb{R}$, the leaf L whose $(\mathbf{a},\mathbf{p},\mathbf{t})$-image contains (a_0, p_0, t_0) has the property that $(d\mathbf{a}, d\mathbf{p}, d\mathbf{t})$ has rank 3 along the preimage of (a_0, p_0, t_0). In particular, Y spans the tangent to the fiber at such points.

Given this, it would not be surprising to find that Y can be scaled so as to become a symmetry vector field. In fact, one finds that the flow of $X = e^f Y$ preserves the coframing (τ, ρ, ω) on L if and only if f satisfies the equation

$$df = 8|a|^2 t\eta - (pt-2)(a\omega + \bar{a}\bar{\omega}).$$

Now, by the structure equations, the right hand side of this equation is a closed 1-form on L. This shows that, at least locally (or, more precisely, on some covering space of L), a scaling factor e^f exists making $X = e^f Y$ a symmetry vector field. Moreover, this f is unique up to the addition of a constant.

This implies that any solution hypersurface $\Sigma \subset \mathbb{P}^2_R$ whose structure equations are of the form (2) must actually be invariant under a one-parameter group of isometries of $\mathbb{P}^2_R$, i.e., a one-parameter subgroup of G_R. Moreover, because $\eta(Y) = 0$, this one-parameter subgroup can be chosen (if it is not actually unique) so that it preserves each complex leaf in Σ.

3.2.3. Explicit solutions invariant under a given 1-parameter subgroup. A one-parameter subgroup of isometries of $\mathbb{P}^2_R$ isz of the form $\{e^{tz} \mid t \in \mathbb{R}\}$ for some $z \neq 0$ in $\mathfrak{g}_R$. There is a unique holomorphic vector field Z on $\mathbb{P}^2_R$ whose real part is the infinitesimal generator of the action of the subgroup $\{e^{tz} \mid t \in \mathbb{R}\}$. If $U_z \subset \mathbb{P}^2_R$ denotes the open set that is the complement of the fixed locus of the flow e^{tz}, then U_z is foliated by complex curves that are the 'integral curves' of the holomorphic flow generated by Z. By the above discussion, the nondegenerate part Σ^{**} of any solution Σ of type (2) will be swept out by a (real) one-parameter family of integral curves of Z for some isometric flow e^{tz}. Since, by construction, the complex leaves of a solution of type (2) are not totally geodesic, this shows that $z \in \mathfrak{g}_R$ must be chosen so that the Z-integral curves in U_z are not totally geodesic. I will refer to a $z \in \mathfrak{g}_R$ with this property as *nondegenerate.*

Conversely, starting with any one-parameter subgroup e^{tz} of isometries of $\mathbb{P}^2_R$ and considering the corresponding holomorphic foliation of $U_z \subset \mathbb{P}^2_R$ by complex curves, one can construct e^{tz}-invariant Levi-flat hypersurfaces in U_z by taking the union of any (real) one-parameter family of complex leaves of this foliation. It now suffices to show that one can choose this one-parameter family in such a way that the resulting hypersurface will be minimal. I am going to show that this can always be done, essentially in only one way up to isometry, and that, when z is nondegenerate in the sense of the previous paragraph, this always yields a solution Σ of type (2). Thus, the solutions of type (2) correspond to the conjugacy classes of nondegenerate one-parameter subgroups of isometries of $\mathbb{P}^2_R$.

First, consider the case where $R > 0$. Without essential loss of generality, I can assume that $R = 1$, so that $G_R = G_1 = \mathrm{SU}(3)$. Every one-parameter subgroup of SU(3) is semi-simple and hence conjugate to a diagonal subgroup generated by a nonzero element

$$z = \begin{pmatrix} i\lambda_0 & 0 & 0 \\ 0 & i\lambda_1 & 0 \\ 0 & 0 & i\lambda_2 \end{pmatrix} \quad \text{where} \quad \lambda_0 + \lambda_1 + \lambda_2 = 0.$$

The corresponding vector field on $\mathbb{C}^3$ (which is also well-defined on $\mathbb{P}^2_R \simeq \mathbb{P}^2$) can be written in terms of unitary holomorphic coordinates $z = (z^a)$ as the real part of the holomorphic vector field

$$Z = i\lambda_0 z^0 \frac{\partial}{\partial z^0} + i\lambda_1 z^1 \frac{\partial}{\partial z^1} + i\lambda_2 z^2 \frac{\partial}{\partial z^2}.$$

The holomorphic integral curve of Z through $c = [c^a] \in \mathbb{P}^2_R$ is of the form

$$\left\{ \begin{bmatrix} c^0 e^{i\lambda_0 w} \\ c^1 e^{i\lambda_1 w} \\ c^2 e^{i\lambda_2 w} \end{bmatrix} \;\middle|\; w \in \mathbb{C} \right\}.$$

This will be a point or a line for all such c if and only if two of the λ_i are equal. In such a case, the integral curves of Z are open subsets of lines through a fixed point in $\mathbb{P}^2$. Thus any minimal Levi-flat hypersurface whose complex leaves are integral curves of Z will be of type (1). Set this case aside and, from now on, assume that z is nondegenerate, i.e., that the λ_i are mutually distinct.

Since radial dilation has no effect on the projective space, the flow the vector field Z induces on $\mathbb{P}^2$ is the same as that of the vector field

$$Z' = i(\lambda_1 - \lambda_0) z^1 \frac{\partial}{\partial z^1} + i(\lambda_2 - \lambda_0) z^2 \frac{\partial}{\partial z^2}$$

and this, in turn, will have the same holomorphic integral curves in $\mathbb{P}^2$ as

$$Z'' = z^1 \frac{\partial}{\partial z^1} + \lambda z^2 \frac{\partial}{\partial z^2} \quad \text{where} \quad \lambda = \frac{(\lambda_2 - \lambda_0)}{(\lambda_1 - \lambda_0)} \neq 0,\ 1.$$

The nonlinear integral curves of this vector field are of the form

$$\left\{ \begin{bmatrix} 1 \\ c e^{w} \\ e^{\lambda w} \end{bmatrix} \;\middle|\; w \in \mathbb{C} \right\},$$

where c is any nonzero complex constant. Thus, a Levi-flat hypersurface whose complex leaves are integral curves of this vector field can be locally parametrized in the form

$$\Sigma = \left\{ \begin{bmatrix} 1 \\ e^{w+x(r)+iy(r)} \\ e^{\lambda w} \end{bmatrix} \;\middle|\; w \in \mathbb{C},\ r \in I \right\},$$

where $x + iy\colon I \to \mathbb{C}$ is some smooth immersion of an interval $I \subset \mathbb{R}$. Brute force calculation then yields that such a hypersurface is minimal if and only if y is a constant function. Thus, up to a holomorphic isometry, such a minimal Levi-flat hypersurface is an open subset of the hypersurface

$$\Sigma_\lambda = \left\{ \begin{bmatrix} 1 \\ e^{w+r} \\ e^{\lambda w} \end{bmatrix} \;\middle|\; w \in \mathbb{C},\ r \in \mathbb{R} \right\}.$$

Note that Σ_λ is congruent to $\Sigma_{1/\lambda}$ but that, otherwise, the Σ_λ are mutually noncongruent. When λ is irrational, this hypersurface is dense in $\mathbb{P}^2$, but when $\lambda = p/q$ where p ($\neq 0$, q) and $q > 0$ are integers without common factors, this hypersurface is dense in an algebraically defined hypersurface that is singular at the point $z^1 = z^2 = 0$ but can also be singular along the entire lines $z^1 = 0$ and $z^2 = 0$, depending on the values of p and q. A typical such hypersurface is defined by an equation of the form $\mathrm{Im}\big((\bar{z}^0)^{p+q}(z^1)^p(z^2)^q\big) = 0$.

Next, consider the case where $R = 0$, i.e., when $\mathbb{P}^2_R = \mathbb{P}^2_0$ is isometric to $\mathbb{C}^2$ with its standard flat metric. Let (z^1, z^2) be unitary holomorphic linear coordinates on $\mathbb{C}^2$. A nonzero vector field whose flow is a holomorphic isometry on $\mathbb{C}^2$ is then conjugate via an action of G_0 to a constant multiple of the real part of either

$$Z = iz^1\frac{\partial}{\partial z^1} + i\lambda z^2\frac{\partial}{\partial z^2} \quad \text{or} \quad Z = i\frac{\partial}{\partial z^1} + i\lambda z^2\frac{\partial}{\partial z^2}$$

for some real number λ. In the first case, the holomorphic integral curves of Z will be lines in $\mathbb{C}^2$ if and only if $\lambda = 0$ or 1 while, in the second case, the holomorphic integral curves of Z will be lines in $\mathbb{C}^2$ if and only if $\lambda = 0$. These are the degenerate values that will be set aside, as these degenerate cases lead to the hyperplane or Clifford cone solutions that have already been discussed in the previous subsection.

Consider the first type of vector field with $\lambda \neq 0$ or 1. Any holomorphic integral curve of Z that is not contained in a line in $\mathbb{C}^2$ is of the form

$$\left\{ \begin{pmatrix} ce^{w} \\ e^{\lambda w} \end{pmatrix} \;\middle|\; w \in \mathbb{C} \right\},$$

where $c \in \mathbb{C}$ is a nonzero constant. A smooth Levi-flat hypersurface $\Sigma^3 \subset \mathbb{C}^2$ whose complex leaves consist of such integral curves can be locally parametrized in the form

$$\Sigma^3 = \left\{ \begin{pmatrix} e^{w+x(r)+iy(r)} \\ e^{\lambda w} \end{pmatrix} \,\middle|\, w \in \mathbb{C},\ r \in I \right\},$$

where $x + iy\colon I \to \mathbb{C}$ is some smooth immersion of an interval $I \subset \mathbb{R}$. Brute force calculation then yields that such a hypersurface is minimal if and only if y is a constant function. Consequently, it follows that, up to a holomorphic isometry, the connected solutions of this kind are all equivalent to open subsets of the immersed hypersurface

$$\Sigma_\lambda = \left\{ \begin{pmatrix} e^{w+r} \\ e^{\lambda w} \end{pmatrix} \,\middle|\, w \in \mathbb{C},\ r \in \mathbb{R} \right\}.$$

If λ is irrational, then Σ_λ is dense in $\mathbb{C}^2$ and the (implicitly described) immersion given above is an embedding. On the other hand, if $\lambda = p/q$ where $p \neq 0$ and $q > 0$ are distinct integers without common factors, then this immersion is not an embedding. Moreover, $\Sigma_{p/q}$ is dense in an algebraic real hypersurface, namely

$$\begin{aligned} (z^1)^p(\bar{z}^2)^q - (\bar{z}^1)^p(z^2)^q &= 0 \quad \text{when } p > 0, \\ (\bar{z}^1)^{-p}(\bar{z}^2)^q - (z^1)^{-p}(z^2)^q &= 0 \quad \text{when } p < 0. \end{aligned}$$

Note that these hypersurfaces are cones that are singular at the origin and along the axes except when p or q equals 1.

Consider the second type of vector field with $\lambda \neq 0$. Any holomorphic integral curve of Z that is not contained in a line in $\mathbb{C}^2$ is of the form

$$\left\{ \begin{pmatrix} w + c \\ e^{\lambda w} \end{pmatrix} \,\middle|\, w \in \mathbb{C} \right\},$$

where $c \in \mathbb{C}$ is a nonzero constant. A smooth Levi-flat hypersurface $\Sigma^3 \subset \mathbb{C}^2$ whose complex leaves consist of such integral curves can be locally parametrized in the form

$$\Sigma^3 = \left\{ \begin{pmatrix} w + x(r) + iy(r) \\ e^{\lambda w} \end{pmatrix} \,\middle|\, w \in \mathbb{C},\ r \in I \right\},$$

where $x + iy\colon I \to \mathbb{C}$ is some smooth immersion of an interval $I \subset \mathbb{R}$. Brute force calculation then yields that such a hypersurface is minimal if and only if y is a constant function. Consequently, up to a holomorphic isometry followed by a homothety, the connected solutions of this kind are open subsets of the closed embedded hypersurface

$$\Sigma = \left\{ \begin{pmatrix} w \\ re^{w} \end{pmatrix} \,\middle|\, w \in \mathbb{C},\ r \in \mathbb{R} \right\}.$$

(In this parametrization, the complex leaf given by $r = 0$ does not belong to Σ^*, as defined in §2.2.2.) This hypersurface can be defined implicitly by the equation

$$\mathrm{Im}(z^2 e^{-z^1}) = 0$$

and is evidently transcendental.

Finally, consider the case $R < 0$, where, without essential loss of generality, it suffices to consider only the case $R = -1$. The equivalence classes of one-dimensional subspaces of $\mathfrak{su}(2,1) = \mathfrak{g}_{-1}$ under the adjoint action are more complicated in this case. The elements z that have an eigenvector that is $\langle,\rangle_{-1}$-positive (and whose associated flow, therefore, has a fixed point in $\mathbb{P}^2_{-1}$) can be diagonalized in the form

$$z = \begin{pmatrix} i\lambda_0 & 0 & 0 \\ 0 & i\lambda_1 & 0 \\ 0 & 0 & i\lambda_2 \end{pmatrix} \quad \text{where} \quad \lambda_0 + \lambda_1 + \lambda_2 = 0.$$

If z has no $\langle,\rangle_{-1}$-positive eigenvector, then it must have a null eigenvector. In this case, the most generic possibility is for z to have three distinct eigenvalues, in which case two of the eigenvalues cannot be purely imaginary and their corresponding eigenvectors must be $\langle,\rangle_{-1}$-null. Consequently, one can normalize these eigenvectors and show that, up to a (real) multiple, z is conjugate to an element of the form

$$z = \begin{pmatrix} i\lambda & 1 & 0 \\ 1 & i\lambda & 0 \\ 0 & 0 & -2i\lambda \end{pmatrix} \quad \text{where } \lambda \in \mathbb{R}.$$

If z has a double eigenvalue with a unique $\langle,\rangle_{-1}$-null corresponding eigenvector and a $\langle,\rangle_{-1}$-negative eigenvector, then, up to a (real) multiple, z is conjugate to an element of the form

$$z = \begin{pmatrix} i(\lambda+1) & -i & 0 \\ i & i(\lambda-1) & 0 \\ 0 & 0 & -2i\lambda \end{pmatrix} \quad \text{where } 0 \neq \lambda \in \mathbb{R}.$$

If z has a triple eigenvalue, i.e., is nilpotent, then either $z^2 \neq 0$, in which case it is conjugate to an element of the form

$$z = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix},$$

or else $z^2 = 0$ (the most degenerate case), in which case it is conjugate to an element of the form

$$z = \begin{pmatrix} i & -i & 0 \\ i & -i & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Among these five cases, the holomorphic flow on $\mathbb{P}^2_{-1}$ corresponding to e^{tz} will have all integral curves be totally geodesic in two cases. In the case where z is diagonalizable, this happens when $\{\lambda_0, \lambda_1, \lambda_2\}$ are not distinct. Among the nondiagonalizable cases, this happens only for the last case, i.e., when $z^2 = 0$. These cases will be set aside, as they have already been treated in the discussion of type (1) solutions.

Now, in the diagonalizable case, the analysis proceeds exactly along the lines of the elliptic case and there is no need to give details. The end result is that a connected Levi-flat minimal hypersurface whose complex leaves are invariant under a nondegenerate diagonalizable flow is congruent to an open subset of the hypersurface

$$\Sigma_\lambda = \left\{ \begin{bmatrix} 1 \\ e^{w+r} \\ e^{\lambda w} \end{bmatrix} \;\middle|\; w \in \mathbb{C},\ r \in \mathbb{R},\ \left|e^{w+r}\right|^2 + \left|e^{\lambda w}\right|^2 < 1 \right\} \subset \mathbb{P}^2_{-1},$$

where λ is a real constant not equal to 0 or 1. This hypersurface has an algebraic defining equation if and only if λ is rational.

The next case, where z has two distinct $\langle,\rangle_{-1}$-null eigenvectors, can be analyzed in a similar manner and one finds that a connected Levi-flat minimal hypersurface whose complex leaves are invariant under the associated holomorphic flow is congruent to an open subset of the hypersurface

$$\Sigma'_\lambda = \left\{ \begin{bmatrix} e^{(1+i\lambda)(w+r)} + e^{(-1+i\lambda)w} \\ e^{(1+i\lambda)(w+r)} - e^{(-1+i\lambda)w} \\ 1 \end{bmatrix} \;\middle|\; w \in \mathbb{C},\ r \in \mathbb{R} \right\} \subset \mathbb{P}^2_{-1},$$

where λ is a real constant. When $\lambda = 0$, this is a real curve in the pencil of conics that pass through two points on the boundary of $\mathbb{P}^2_{-1}$ and have given tangents there. When λ is nonzero, the curves $r = r_0$ are not algebraic. (Of course, w and r must satisfy an inequality in order that the formula given in this description represent at point in $\mathbb{P}^2_{-1}$, but it is not useful to make this inequality explicit for the purposes at hand.)

In the case where z has a double eigenvalue (and not a triple one), a similar analysis shows that a connected Levi-flat minimal hypersurface whose complex leaves are invariant under the associated holomorphic flow is congruent to an open subset of the hypersurface

$$\Sigma''_\mu = \left\{ \begin{bmatrix} w + r + 1 \\ w + r - 1 \\ e^{\mu w} \end{bmatrix} \middle| \, w \in \mathbb{C}, \ r \in \mathbb{R}, \ 4r > e^{\mu(w+\bar{w})} - 2(w + \bar{w}) \right\} \subset \mathbb{P}^2_{-1},$$

where μ is a nonzero real constant.

In the final nondegenerate case, where the symmetry generator $z \in \mathfrak{g}_{-1}$ satisfies $z^2 \neq 0$ but $z^3 = 0$, the nonlinear integral curves of the associated holomorphic flow are conics (i.e., rational curves of degree 2) in $\mathbb{P}^2$, all tangent at a point on the boundary of $\mathbb{P}^2_{-1} \subset \mathbb{P}^2$. Brute force calculation shows that any Levi-flat minimal hypersurface whose complex leaves are invariant under such a flow is congruent to the hypersurface

$$\Sigma = \left\{ \begin{bmatrix} w^2 + r + 1 \\ w^2 + r - 1 \\ 2w \end{bmatrix} \middle| \, w \in \mathbb{C}, \ r \in \mathbb{R}, \ 4(\operatorname{Im} w)^2 < r \right\} \subset \mathbb{P}^2_{-1}.$$

Details will be left to the reader.

3.3. Solutions of type 3

Finally, consider the solutions of the system (3). To avoid repetition, I will set aside the cases where the solution reduces to one of type (1). This means that the solution has $s \neq 0$, which, by the structure equations (3), implies that s is nowhere vanishing.

3.3.1. Existence via the Frobenius theorem. Let $M^{13} = G_0 \times \mathbb{C} \times \mathbb{C} \times \mathbb{R}$, and let $\mathbf{g}\colon M \to G_0$, $\mathbf{a}\colon M \to \mathbb{C}$, $\mathbf{s}\colon M \to \mathbb{C}$, and $\mathbf{p}\colon M \to \mathbb{R}$ be the projections onto the first through fourth factors, respectively. Let $\mathcal{I}_3$ be the exterior ideal on M generated by the linearly independent real-valued 1-forms $\theta_1, \ldots, \theta_9$ where

$$\begin{aligned}
\theta_1 &= i(\bar{\eta} - \eta) \\
\theta_2 &= \phi + i\mathbf{a}\omega - i\bar{\mathbf{a}}\bar{\omega} \\
\theta_3 + i\theta_4 &= \sigma + 2\bar{\mathbf{a}}\eta + 2\mathbf{s}\bar{\omega} \\
\theta_5 + i\theta_6 &= d\mathbf{a} + 3i\mathbf{a}\tau + 6\bar{\mathbf{a}}\bar{\mathbf{s}}\eta - (\bar{\mathbf{s}}\mathbf{p} - 3\mathbf{a}^2)\omega - (|\mathbf{a}|^2 - 2|\mathbf{s}|^2)\bar{\omega} \\
\theta_7 + i\theta_8 &= d\mathbf{s} - \mathbf{s}\big(6i\tau + \mathbf{p}\eta + \mathbf{a}\omega + \bar{\mathbf{a}}\bar{\omega}\big)
\end{aligned}$$

$$\theta_9 = d\mathbf{p} - \big(8|\mathbf{s}|^2 - 64|\mathbf{a}|^2 - \mathbf{p}^2\big)\eta \\ + (\mathbf{ap} + 24\bar{\mathbf{a}}\bar{\mathbf{s}})\omega + (\bar{\mathbf{a}}\mathbf{p} + 24\mathbf{as})\bar{\omega}.$$

By the structure equations, the ideal $\mathcal{I}_3$ is closed under exterior differentiation, so M is foliated by 4-dimensional integral manifolds of $\mathcal{I}_3$.

By construction, each leaf $L \subset M$ is the image of the bundle $B_1^{**} \subset G_0$ over the nondegenerate part Σ^{**} of a minimal Levi-flat hypersurface Σ satisfying equations (3) under the embedding

$$\mathrm{id} \times a \times s \times p\colon B_1^{**} \longrightarrow G_0 \times \mathbb{C} \times \mathbb{C} \times \mathbb{R} = M.$$

Since G_0 acts by left translation on $G_0 \times \mathbb{C}^* \times \mathbb{R} \times \mathbb{R}$ preserving the ideal $\mathcal{I}_3$, this left action permutes its integral manifolds, and two integral manifolds are equivalent under this action if and only if they correspond to congruent hypersurfaces in $\mathbb{P}^2_0 \simeq \mathbb{C}^2$. In particular, two leaves L_1 and L_2 represent equivalent solutions if and only if they satisfy $(\mathbf{a},\mathbf{s},\mathbf{p})(L_1) = (\mathbf{a},\mathbf{s},\mathbf{p})(L_2)$.

In fact, in order for two leaves L_1 and L_2 to be equivalent under G_0, it suffices that the two image sets $(\mathbf{a},\mathbf{s},\mathbf{p})(L_1)$ and $(\mathbf{a},\mathbf{s},\mathbf{p})(L_2)$ in $\mathbb{C} \times \mathbb{C} \times \mathbb{R}$ have a nonempty intersection. To see why this is so, note that if L_i contains (g_i, a, s, p), then the submanifold L described by

$$L = \big\{ (g_2 {g_1}^{-1} g, b, q, u) \bigm| (g, b, q, u) \in L_1 \big\}$$

contains $(g_2, a, s, p) \in L_2$, is evidently a maximal integral manifold of $\mathcal{I}_3$, and so must equal L_2. In particular, in order to classify the solutions up to rigid motion, it would suffice to determine the partition of $\mathbb{C} \times \mathbb{C} \times \mathbb{R}$ into the images $(\mathbf{a},\mathbf{s},\mathbf{p})(L)$ as L ranges over the leaves of $\mathcal{I}_3$.

3.3.2. First integrals and the symmetry of solutions. Now, it would be reasonable to expect the images $(\mathbf{a},\mathbf{s},\mathbf{p})(L)$ to have dimension 4, at least at 'generic' points, since each leaf L has dimension 4. In fact, by the argument in the previous paragraph, it is evident that the fibers of the map $(\mathbf{a},\mathbf{s},\mathbf{p})\colon L \to \mathbb{C} \times \mathbb{C} \times \mathbb{R}$ are the orbits of the action on L of the ambient symmetry group of the corresponding solution Σ^{**}.

Consider the quantities[3]

$$\mathbf{A} = \tfrac{1}{9}|\mathbf{s}|^{2/3}\big(48|\mathbf{a}|^2 + 12|\mathbf{s}|^2 + \mathbf{p}^2\big), \\ \mathbf{B} = \tfrac{1}{27}|\mathbf{s}|\big(216\mathbf{a}^2\mathbf{s} + 216\bar{\mathbf{a}}^2\bar{\mathbf{s}} + 72|\mathbf{a}|^2\mathbf{p} - 36|\mathbf{s}|^2\mathbf{p} + \mathbf{p}^3\big).$$

The structure equations show that the 1-form $d(\mathbf{A}^3 - \mathbf{B}^2)$ lies in $\mathcal{I}_3$, which implies that the image $(\mathbf{a},\mathbf{s},\mathbf{p})(L)$ of any $\mathcal{I}_3$-leaf L lies in a level

[3]The significance of these quantities will become clear in the analysis to be carried out below.

set of $\mathbf{F} = \mathbf{A}^3 - \mathbf{B}^2$, a homogeneous polynomial of degree 8 in the variables $\mathbf{a}$, $\bar{\mathbf{a}}$, $\mathbf{s}$, $\bar{\mathbf{s}}$, and $\mathbf{p}$.

Calculation shows that $\mathbf{F} \geq 0$, with equality exactly along the 3-dimensional cone $C_0 \subset \mathbb{C} \times \mathbb{C} \times \mathbb{R}$ defined by the equations

$$0 = \mathbf{a}^2\mathbf{s} - \bar{\mathbf{a}}^2\bar{\mathbf{s}} = 8|\mathbf{a}|^2|\mathbf{s}|^2 - 4|\mathbf{s}|^4 - 2\mathbf{a}^2\mathbf{s}\mathbf{p} - 2\bar{\mathbf{a}}^2\bar{\mathbf{s}}\mathbf{p} + \mathbf{p}^2|\mathbf{s}|^2.$$

In particular, the $\mathcal{I}_3$-leaves that lie in $G_0 \times C_0$ represent either solutions of type (1) or of type (2), and have already been analysed in the previous subsections. Moreover, 0 is the only critical value of $\mathbf{F}$ on $\mathbb{C}\times\mathbb{C}\times\mathbb{R} \simeq \mathbb{R}^5$. The remaining level sets of $\mathbf{F}$ are smooth, connected hypersurfaces. In fact, because $\mathbf{F}$ is a homogeneous polynomial of degree 8, it follows that all of the positive level sets are diffeomorphic by homothety.

A rather laborious calculation using the structure equations above shows that for any $\mathcal{I}_3$-leaf L on which $\mathbf{F} = c^2 > 0$, the rank of the map $(\mathbf{a}, \mathbf{s}, \mathbf{p})\colon L \to \mathbb{C} \times \mathbb{C} \times \mathbb{R}$ is 4, i.e., that $(\mathbf{a}, \mathbf{s}, \mathbf{p})\colon L \to \mathbf{F}^{-1}(c^2)$ is a local diffeomorphism. The existence theorem proved above via the Frobenius theorem coupled with the G_0-invariance of $\mathcal{I}_3$ shows that this map must actually be a (surjective) covering map. Thus, there is a 1-parameter family of noncongruent solutions of type (3), one for each positive level set of $\mathbf{F}$.

3.3.3. The effect of homothety. While the members of this 1-parameter family are mutually incongruent by isometries, it turns out that they are congruent via homothety. To see this, note that if X is a vector field on $\mathbb{C}^2 \simeq \mathbb{P}^2_0$ that generates dilation about a fixed point, then X lifts to a vector field Y on G_0 that satisfies

$$\mathfrak{L}_Y\tau = \mathfrak{L}_Y\phi = \mathfrak{L}_Y\sigma = 0, \quad \mathfrak{L}_Y\eta = c\eta, \quad \mathfrak{L}_Y\omega = c\omega$$

for some nonzero (real) constant c. The vector field Y can then be lifted to a vector field Z on $G_0 \times \mathbb{C} \times \mathbb{C} \times \mathbb{R}$ so that it satisfies the same equations above as Y does but also satisfies

$$\mathfrak{L}_Z\mathbf{a} = -c\mathbf{a}, \quad \mathfrak{L}_Z\mathbf{s} = -c\mathbf{s}, \quad \mathfrak{L}_Z\mathbf{p} = -c\mathbf{p}.$$

It then follows from the formulae for the generators of $\mathcal{I}_3$ that the flow of Z leaves $\mathcal{I}_3$ invariant and therefore permutes the leaves of $\mathcal{I}_3$. Since $\mathbf{F}$ is homogeneous of degree 8 in $(\mathbf{a}, \mathbf{s}, \mathbf{p})$, it follows that $\mathfrak{L}_Z\mathbf{F} = -8c\mathbf{F}$. In particular, the flow of Z acts as homothety on the level sets of $\mathbf{F}$. Thus, any two solutions on which $\mathbf{F}$ is positive are congruent via homothety in $\mathbb{C}^2 \simeq \mathbb{P}^2_0$.

In this situation, it is therefore reasonable to restrict attention to the leaves that lie in the level set $\mathbf{F} = 1$. It is the solution corresponding to such a leaf that I am now going to describe. Note that the isometry group preserving such a solution is necessarily discrete.

3.3.4. Local integration of the equations. Suppose that one has a minimal Levi-flat hypersurface $\Sigma \subset \mathbb{C}^2$ for which the bundle B_1^{**} satisfies equations (3). Since s is nonzero on B_1^{**}, the structure equations show that there is a submanifold $B_2 \subset B_1^{**}$ defined as the set on which s is real and positive and that B_2 is a 6-fold cover of Σ^{**}. From now on, all functions and forms are to be regarded as pulled back to B_2.

The reality of s and the structure equation

$$s^{-1}ds = 6i\tau + p\eta + 3a\omega + 3\bar{a}\bar{\omega}$$

imply that $\tau = 0$. Combining this with the structure equation $d\eta = (a\omega + \bar{a}\bar{\omega}) \wedge \eta$ yields

$$d\big(s^{-1/3}\eta\big) = 0.$$

The structure equations also imply that the quantities

$$A = \tfrac{1}{9}s^{2/3}\big(48|a|^2 + 12s^2 + p^2\big),$$
$$B = \tfrac{1}{27}s\big(216a^2s + 216\bar{a}^2s + 72|a|^2p - 36s^2p + p^3\big)$$

introduced earlier satisfy equations of the form

$$dA = -4B\ s^{-1/3}\eta$$
$$dB = -6A^2s^{-1/3}\eta.$$

The assumption that the hypersurface Σ correspond to an $\mathcal{I}_3$-leaf on which $\mathbf{F}$ is identically equal to 1 is equivalent to the equation $A^3 - B^2 = 1$, so there is a unique function θ on B_2 with values in the open interval $(-\pi/2, \pi/2)$ for which

$$A = \sec^{2/3}\theta > 0 \quad \text{and} \quad B = -\tan\theta.$$

The above differential equations for A and B now imply that

$$s^{-1/3}\eta = \tfrac{1}{6}\sec^{2/3}\theta\, d\theta.$$

In particular, $d\theta$ never vanishes on B_2 and is a nonzero multiple of η.

The structure equations now imply that

$$d(s^{1/3}\omega) = \big(2s^{5/3}\bar{\omega} - \tfrac{1}{3}ps^{2/3}\omega\big) \wedge \tfrac{1}{6}\sec^{2/3}\theta\, d\theta.$$

It follows that any point $q \in B_2$ has a neighborhood U_0 on which there exists a complex valued function z, uniquely defined up to the addition of a (complex) function of t, and a complex function L, uniquely defined once z is chosen, so that

$$s^{1/3}\omega = dz + \tfrac{1}{6}L\sec^{2/3}\theta\, d\theta.$$

(Introducing such a coefficient in the $L\,d\theta$ term simplifies later calculations.) Because $dz \wedge d\bar{z} \wedge d\theta = s^{2/3}\omega \wedge \bar{\omega} \wedge d\theta \neq 0$, it follows that $(z, \theta)\colon U_0 \to \mathbb{C} \times \mathbb{R}$ is a local diffeomorphism. By restricting to an appropriate neighborhood $U_1 \subset U_0$ of q, I can assume that $(z, \theta)\colon U_1 \to \mathbb{R} \times \mathbb{C}$ defines a rectangular coordinate system (not necessarily centered on q). Write $z = x + iy$ where x and y are real-valued. Given the ambiguities in the choice of the coordinate system, partial differentiation with respect to z (or x or y) is coordinate independent although partial differentiation with respect to θ is not.

In these coordinates, the above structure equation for $d\big(s^{1/3}\omega\big)$ now becomes

$$dL \wedge \sec^{2/3}\theta\,d\theta = \big(2s^{4/3}d\bar{z} - \tfrac{1}{3}ps^{1/3}dz\big) \wedge \sec^{2/3}\theta\,d\theta,$$

so

$$L_z = -\tfrac{1}{3}ps^{1/3} \quad \text{and} \quad L_{\bar{z}} = 2s^{4/3}.$$

Set $u = s^{1/3}$. Then

$$u^{-1}du = \tfrac{1}{3}s^{-1}ds \equiv a\omega + \bar{a}\bar{\omega} \equiv u^{-1}\big(a\,dz + \bar{a}\,d\bar{z}\big) \bmod d\theta,$$

so $a = u_z$. The structure equation for da now gives

$$da \equiv (u^3p - 3a^2)u^{-1}dz + (|a|^2 - 2u^6)u^{-1}d\bar{z} \bmod d\theta,$$

so it follows that $u_{zz} = a_z = (u^3p - 3{u_z}^2)u^{-1}$, which can be written in the form

$$(u^4)_{zz} = 4u^5p.$$

Since p is real and since $v_{zz} = \frac{1}{4}(v_{xx} - v_{yy}) + \frac{i}{2}v_{xy}$ for any function v on U_1, it follows that $(u^4)_{xy} = 0$. Consequently, there exist functions f and g defined on the rectangles $(x, \theta)(U_1)$ and $(y, \theta)(U_1)$ in $\mathbb{R}^2$ so that

$$u^4 = f(x, \theta) - g(y, \theta) > 0.$$

These functions are unique up to the addition of a function of θ, i.e., one could replace $\big(f(x,\theta), g(y,\theta)\big)$ by $\big(f(x,\theta)+h(\theta), g(y,\theta)+h(\theta)\big)$ for some h defined on the interval $\theta(U_1)$, but this is the only ambiguity in the choice of these two functions.

Now, the equation for da also implies the equation $u_{z\bar{z}} = a_{\bar{z}} = (|u_z|^2 - 2u^6)u^{-1}$, which can be written in the form

$$(u^4)_{z\bar{z}} = u^{-4}\big|(u^4)_z\big|^2 - 8(u^4)^2.$$

Using the expression already found for u^4 plus the formulae $v_{z\bar z} = \frac{1}{4}(v_{xx}+v_{yy})$ and $|v_z|^2 = \frac{1}{4}\big({v_x}^2+{v_y}^2\big)$, this equation can be written in the form

$$f_{xx}(x,\theta)-g_{yy}(y,\theta)=\frac{f_x(x,\theta)^2+g_y(y,\theta)^2}{f(x,\theta)-g(y,\theta)}-32\big(f(x,\theta)-g(y,\theta)\big)^2.$$

Now, setting $v=f-g$, this can be written in the form

$$v_{xx}+v_{yy}=\frac{{v_x}^2+{v_y}^2}{v}-32v^2$$

and rearranged to give

$$\left(\frac{v_x}{v}\right)_x=\frac{{v_y}^2}{v^2}+\frac{v_{yy}}{v}-32v.$$

Since $v_{xy}=0$, both v_y and v_{yy} are constant in x. Thus, multiplying this equation by $2v_x/v$ and integrating with respect to x yields

$$\left(\frac{v_x}{v}\right)^2=C(y,\theta)-\frac{{v_y}^2}{v^2}-\frac{2v_{yy}}{v}-64v.$$

for some function C on $(y,\theta)(U_1)$. Now, multiplying by v^2 and substituting $v=f-g$, this can be written in the form

$$f_x(x,\theta)^2=2a_0(y,\theta)+12a_1(y,\theta)f(x,\theta)+48a_2(y,\theta)f(x,\theta)^2-64f(x,\theta)^3$$

for some functions a_0, a_1, and a_2 on $(y,\theta)(U_1)$. (The choice of numerical coefficients is cosmetic.)

Now, if the functions a_i really did depend on y, differentiating this equation with respect to y would then force $f(x,\theta)$ to be constant in x, making f_x vanish identically. This would, in turn, imply that $a=u_z=-\frac{1}{2}iu_y$ is purely imaginary, so that the quantity a^2s would be real. However, going back to the analysis in §2.2.2, this can only happen for solutions of type (2). Since the goal of this section is analyse the solutions of type (3) that have not already been accounted for by those of type (1) or (2), this case can therefore be set aside.

Thus, f satisfies an equation of the form

$$f_x(x,\theta)^2=2a_0(\theta)+12a_1(\theta)f(x,\theta)+48a_2(\theta)f(x,\theta)^2-64f(x,\theta)^3$$

for some functions a_0, a_1, and a_2 on $\theta(U_1)$. A similar analysis shows that there are functions b_0, b_1, and b_2 on $\theta(U_1)$ for which

$$g_y(y,\theta)^2=-2b_0(\theta)-12b_1(\theta)g(y,\theta)-48b_2(\theta)g(y,\theta)^2+64g(y,\theta)^3.$$

Moreover, substituting these relations and their derivatives back into the original equation for v, it follows that $b_0 = a_0$, $b_1 = a_1$, and $b_2 = a_2$. Thus,

$$\begin{aligned} f_x(x,\theta)^2 &= 2a_0(\theta) + 12a_1(\theta)f(x,\theta) + 48a_2(\theta)f(x,\theta)^2 - 64f(x,\theta)^3, \\ g_y(y,\theta)^2 &= -2a_0(\theta) - 12a_1(\theta)g(y,\theta) - 48a_2(\theta)g(y,\theta)^2 + 64g(y,\theta)^3. \end{aligned}$$

By replacing $\big(f(x,\theta), g(y,\theta)\big)$ with $\big(f(x,\theta) - \frac{1}{4}a_2(\theta), g(y,\theta) - \frac{1}{4}a_2(\theta)\big)$, it can be arranged that $a_2 \equiv 0$. This removes the ambiguity in the choice of f and g.

At this point, f and g satisfy the equations

$$\begin{aligned} f_x(x,\theta)^2 &= 2a_0(\theta) + 12a_1(\theta)f(x,\theta) - 64f(x,\theta)^3 \\ g_y(y,\theta)^2 &= -2a_0(\theta) - 12a_1(\theta)g(y,\theta) + 64g(y,\theta)^3 \end{aligned}$$

as well as equations

$$\begin{aligned} f_{xx}(x,\theta) &= 6a_1(\theta) - 96f(x,\theta)^2, \\ g_{yy}(y,\theta) &= -6a_1(\theta) + 96g(y,\theta)^2. \end{aligned}$$

This information can now be substituted back into the previous formulae, yielding

$$\begin{aligned} s &= \big(f(x,\theta) - g(y,\theta)\big)^{3/4} \\ a &= \tfrac{1}{8}\big(f(x,\theta) - g(y,\theta)\big)^{-3/4}\big(f_x(x,\theta) + i g_y(y,\theta)\big) \\ p &= -6\big(f(x,\theta) - g(y,\theta)\big)^{-1/4}\big(f(x,\theta) + g(y,\theta)\big). \end{aligned}$$

Using these formulae, the definitions of A and B, and the equations satisfied by f and g, it now follows that

$$a_1(\theta) = A = \sec^{2/3}\theta \quad \text{and} \quad a_0(\theta) = B = -\tan\theta.$$

The previous formula for dL now simplifies to

$$dL \equiv 4f(x,\theta)dx + 4g(y,\theta)dy \bmod d\theta,$$

so that $L = F(x,\theta) + iG(y,\theta)$ for functions F and G satisfying $F_x = 4f$ and $G_y = 4g$. All this information combines to yield the formulae

$$\begin{aligned} \tau &= 0 \\ \eta &= \tfrac{1}{6}\big(f(x,\theta) - g(y,\theta)\big)^{1/4}\sec^{2/3}\theta\, d\theta \\ \omega &= \big(f(x,\theta) - g(y,\theta)\big)^{-1/4}\big(dz + \tfrac{1}{6}\sec^{2/3}\theta\big(F(x,\theta) + iG(y,\theta)\big)d\theta\big). \end{aligned}$$

Now, the cubic polynomial

$$p(\lambda,\theta) = -2\tan\theta + 12\sec^{2/3}\theta\,\lambda - 64\lambda^3$$

has three real, distinct roots in λ. In fact, defining

$$\begin{aligned} r_1(\theta) &= \tfrac{1}{2}\sin\left(\tfrac{1}{3}\theta - \tfrac{2}{3}\pi\right)\sec^{1/3}\theta,\\ r_2(\theta) &= \tfrac{1}{2}\sin\left(\tfrac{1}{3}\theta \qquad\right)\sec^{1/3}\theta,\\ r_3(\theta) &= \tfrac{1}{2}\sin\left(\tfrac{1}{3}\theta + \tfrac{2}{3}\pi\right)\sec^{1/3}\theta, \end{aligned}$$

one has $r_1(\theta) < r_2(\theta) < r_3(\theta)$ when $-\pi/2 < \theta < \pi/2$ and

$$p(\lambda,\theta) = -64\big(\lambda - r_1(\theta)\big)\big(\lambda - r_2(\theta)\big)\big(\lambda - r_3(\theta)\big).$$

Now, the differential equations on $f(x,\theta)$ and $g(y,\theta)$ coupled with the inequality $g(x,\theta) < f(y,\theta)$ imply the inequalities

$$r_1(\theta) < g(y,\theta) < r_2(\theta) < f(x,\theta) < r_3(\theta).$$

Moreover the differential equation for f (resp. g) can now be used to extend its range of definition from $(x,\theta)(U_1)$ (resp. $(y,\theta)(U_1)$) to all of $\mathbb{R}\times(-\pi/2,\pi/2)$. The extended functions satisfy

$$r_1(\theta) \le g(y,\theta) \le r_2(\theta) \le f(x,\theta) \le r_3(\theta)$$

and the periodicity relations

$$\begin{aligned} f\big(x + 2\rho_+(\theta),\theta\big) &= f(x,\theta)\\ g\big(y + 2\rho_-(\theta),\theta\big) &= g(y,\theta) \end{aligned}$$

where the functions $\rho_\pm$ are defined by the elliptic integrals

$$\begin{aligned} \rho_+(\theta) &= \frac{1}{8}\int_{r_2(\theta)}^{r_3(\theta)} \frac{da}{\sqrt{\big(r_3(\theta)-a\big)\big(a-r_2(\theta)\big)\big(a-r_1(\theta)\big)}},\\ \rho_-(\theta) &= \frac{1}{8}\int_{r_1(\theta)}^{r_2(\theta)} \frac{da}{\sqrt{\big(r_3(\theta)-a\big)\big(r_2(\theta)-a\big)\big(a-r_1(\theta)\big)}}. \end{aligned}$$

Note, by the way, that $\rho_+(-\theta) = \rho_-(\theta) > 0$ for θ in $(-\pi/2,\pi/2)$.

Using these extended functions, I can now modify x and y by adding functions of θ so as to arrange that

$$g(0,\theta) = r_2(\theta) = f(0,\theta).$$

This makes the coordinates (x, y, θ) unique up to replacement by coordinates of the form

$$(x^*, y^*, \theta^*) = \big(x + 2m\rho_+(\theta), y + 2n\rho_-(\theta), \theta\big)$$

for some integers m and n. These formulae will be important in the discussion of discrete symmetries that will be undertaken below.

The functions f and g are now uniquely defined on the entire strip $\mathbb{R} \times (-\pi/2, \pi/2)$ by the requirement that they satisfy the second order equations with initial conditions

$$\begin{aligned} f_{xx}(x,\theta) &= 6\sec^{2/3}\theta - 96f(x,\theta)^2, \quad f(0,\theta) = r_2(\theta), \quad f_x(0,\theta) = 0, \\ g_{yy}(y,\theta) &= -6\sec^{2/3}\theta + 96g(y,\theta)^2, \quad g(0,\theta) = r_2(\theta), \quad g_y(0,\theta) = 0. \end{aligned}$$

Then $u(x,y,\theta)^4 = f(x,\theta) - g(y,\theta) \geq 0$ is doubly periodic on $\mathbb{R} \times \mathbb{R} \times (-\pi/2, \pi/2)$ in the obvious sense and is strictly positive except along the curves $C_{m,n}$ of the form $(x,y,\theta) = \big(2m\rho_+(\theta), 2n\rho_-(\theta), \theta\big)$ for any integers m and n. The vanishing near these lines is very simple: Along $C_{0,0}$, i.e., the line $(x,y,\theta) = (0,0,\theta)$, there are convergent Taylor expansions

$$f(x,\theta) = r_2(\theta) + \sum_{k=1}^{\infty} c_k(\theta)x^{2k}, \quad g(y,\theta) = r_2(\theta) + \sum_{k=1}^{\infty}(-1)^k c_k(\theta)y^{2k},$$

implying that there is a smooth function $\tilde{u}$ on $\mathbb{R} \times \mathbb{R} \times (-\pi/2, \pi/2)$ satisfying $\tilde{u}(0,0,\theta) = c_1(\theta) = 3\sec^{2/3}\theta\big(1 - 4\sin^2(\tfrac{1}{3}\theta)\big) > 0$ for which

$$u(x,y,\theta)^4 = (x^2 + y^2)\tilde{u}(x,y,\theta).$$

By the periodicity relations, the description of the vanishing of u near the other curves $C_{m,n}$ follows from this one.

Now, examining the coefficient of $d\theta$ in the formula for ds yields the relation

$$g_y G - 6\cos^{2/3}\theta\, g_\theta - 8g^2 = f_x F - 6\cos^{2/3}\theta\, f_\theta - 8f^2.$$

The left hand side of this relation is independent of x while the right hand side is indepdendent of y, so that each side is a function of θ only. Evaluating either side at $x = y = 0$ then yields

$$\begin{aligned} g_y G - 6\cos^{2/3}\theta\, g_\theta - 8g^2 &= f_x F - 6\cos^{2/3}\theta\, f_\theta - 8f^2 \\ &= -6\cos^{2/3}\theta\, r_2'(\theta) - 8r_2(\theta)^2. \end{aligned}$$

Of course, this allows one to solve for F and G away from the places where f_x and g_y vanish, yielding formulae of the form

$$\begin{aligned} F &= \big[6\cos^{2/3}\theta\,(f_\theta - r_2'(\theta)) + 8(f^2 - r_2(\theta)^2)\big]/f_x \\ G &= \big[6\cos^{2/3}\theta\,(g_\theta - r_2'(\theta)) + 8(g^2 - r_2(\theta)^2)\big]/g_y. \end{aligned}$$

Since $f_x(x,\theta) = 0$ if and only if x is an integer multiple of $\rho_+(\theta)$ and $g_y(y,\theta) = 0$ if and only if y is an integer multiple of $\rho_-(\theta)$, this gives integration-free formulae for F and G that are valid over a dense open set. Moreover, differentiating the relations above with respect to x or y and using the identities $F_x = 4f$ and $G_y = 4g$ yields

$$g_{yy}G - 6\cos^{2/3}\theta\, g_{y\theta} - 12gg_y = f_{xx}F - 6\cos^{2/3}\theta\, f_{x\theta} - 12ff_y = 0.$$

Since f_x and f_{xx} do not vanish simultaneously, and since g_y and g_{yy} do not vanish simultaneously, these relations together with the relations above yield explicit smooth formulae for F and G over all of $\mathbb{R}\times(-\pi/2,\pi/2)$. In particular, these formulae imply that $F(0,\theta) = G(0,\theta) = 0$, so that F and G can also be described by

$$F(x,\theta) = 4\int_0^x f(\xi,\theta)d\xi, \quad G(y,\theta) = 4\int_0^y f(\xi,\theta)d\xi.$$

The integration-free formulae yield pseudo-periodicity relations for F and G: Differentiating

$$f\big(x + 2\rho_+(\theta),\theta\big) = f(x,\theta)$$

with respect to θ shows that f_θ satisfies the pseudo-periodicity relation

$$f_\theta\big(x + 2\rho_+(\theta),\theta\big) - f_\theta(x,\theta) = -2f_x(x,\theta)\rho_+'(\theta).$$

Consequently, F satisfies the pseudo-periodicity relation

$$F\big(x + 2\rho_+(\theta),\theta\big) - F(x,\theta) = -12\rho_+'(\theta)\cos^{2/3}\theta.$$

Similarly,

$$G\big(y + 2\rho_-(\theta),\theta\big) - G(y,\theta) = -12\rho_-'(\theta)\cos^{2/3}\theta.$$

At this point, all the structure equations in (3) are identities.

3.3.5. Global structure of the solution. The local information derived in the previous subsubsection can now be used to give a global description of the corresponding minimal Levi-flat hypersurface in $\mathbb{C}^2$. To begin, define r_i for $i = 1, 2$, and 3 and $\rho_\pm$ as functions on $(-\pi/2, \pi/2)$ by the already listed formulae. Then, define functions f and g on $\mathbb{R} \times (-\pi/2, \pi/2)$ by the differential equations with initial conditions:

$$\begin{aligned} f_{xx}(x,\theta) &= 6\sec^{2/3}\theta - 96f(x,\theta)^2, \quad f(0,\theta) = r_2(\theta), \quad f_x(0,\theta) = 0, \\ g_{yy}(y,\theta) &= -6\sec^{2/3}\theta + 96g(y,\theta)^2, \quad g(0,\theta) = r_2(\theta), \quad g_y(0,\theta) = 0. \end{aligned}$$

Note that f is even and periodic of period $2\rho_+(\theta)$ in its first argument while g is even and periodic of period $2\rho_-(\theta)$ in its first argument. Moreover, these functions automatically satisfy the first order equations

$$\begin{aligned} f_x(x,\theta)^2 &= 2\tan\theta + 12\sec^{2/3}\theta\, f(x,\theta) - 64f(x,\theta)^3, \\ g_y(y,\theta)^2 &= -2\tan\theta - 12\sec^{2/3}\theta\, g(y,\theta) + 64g(y,\theta)^3. \end{aligned}$$

Define F and G on the same domain by

$$F(x,\theta) = \int_0^x 4f(\xi,\theta)d\xi, \quad G(y,\theta) = \int_0^y 4g(\xi,\theta)d\xi.$$

Let $D = \mathbb{R} \times \mathbb{R} \times (-\pi/2, \pi/2)$ and let $D^* \subset D$ be the complement of the curves

$$C_{m,n} = \big\{\big(2m\rho_+(\theta), 2n\rho_-(\theta), \theta\big) \;\big|\; \theta \in (-\tfrac{\pi}{2}, \tfrac{\pi}{2})\big\}.$$

Finally, define functions and 1-forms on D^* by the formulae

$$\begin{aligned} s &= \big(f(x,\theta) - g(y,\theta)\big)^{3/4}, \\ a &= \tfrac{1}{8}\big(f(x,\theta) - g(y,\theta)\big)^{-3/4}\big(f_x(x,\theta) + ig_y(y,\theta)\big), \\ p &= -6\big(f(x,\theta) - g(y,\theta)\big)^{-1/4}\big(f(x,\theta) + g(y,\theta)\big), \\ \eta &= \tfrac{1}{6}\big(f(x,\theta) - g(y,\theta)\big)^{1/4}\sec^{2/3}\theta\, d\theta, \\ \omega &= \big(f(x,\theta) - g(y,\theta)\big)^{-1/4}\big(dz + \tfrac{1}{6}\sec^{2/3}\theta\,\big(F(x,\theta) + iG(y,\theta)\big)d\theta\big), \\ \tau &= 0. \end{aligned}$$

Then the structure equations (3) are satisfied on D^*. In particular, setting $\sigma = -2\bar{a}\eta - 2s\bar{\omega}$ and $\phi = -ia\omega + i\bar{a}\bar{\omega}$, the $\mathfrak{g}_0$-valued 1-form

$$\gamma = \begin{pmatrix} 0 & 0 & 0 \\ \eta & i\phi & -\bar{\sigma} \\ \omega & \sigma & -i\phi \end{pmatrix}$$

satisfies $d\gamma = -\gamma \wedge \gamma$.

By the usual moving frame argument [Gr], it follows that, if $U \subset D^*$ is any simply connected domain in D^*, then there is a map $\mathbf{g}\colon U \to G_0$, unique up to left translation by a constant, so that $\mathbf{g}^{-1}d\mathbf{g} = \gamma_{|U}$. The projection $\mathbf{g}K\colon U \to G_0/K = \mathbb{C}^2$ is then an immersion of U into $\mathbb{C}^2$ as a minimal Levi-flat hypersurface of type (3). However, this argument does not provide a description of the topology or global properties of the solution. It is to this description that I now turn.

The group $\mathbb{Z}^2$ acts on D preserving D^* via the maps

$$\Phi_{m,n}(x, y, \theta) = \big(x + 2m\rho_+(\theta), y + 2n\rho_-(\theta), \theta\big).$$

Denote the $\mathbb{Z}^2$-orbit of (x, y, θ) by $[x, y, \theta] \in N$. The periodicity relations on f and g combined with the pseudo-periodicity relations on F and G imply $\Phi_{m,n}^*\gamma = \gamma$. (In fact, all the quantitites s, a, p, η, ω, and τ $(= 0)$ are invariant under this $\mathbb{Z}^2$-action.) Thus γ is well-defined on the quotient space $N^* = D^*/\mathbb{Z}^2$, which is diffeomorphic to a punctured torus cross an open interval.

On $N^* \times G_0$, thought of as a trivialized principal left G_0-bundle over N^*, consider the $\mathfrak{g}_0$-valued connection 1-form

$$\psi = dg\, g^{-1} - g\gamma g^{-1} = g\big(g^{-1}dg - \gamma\big)g^{-1}.$$

Since $d\gamma = -\gamma \wedge \gamma$, it follows that $d\psi = \psi \wedge \psi$, i.e., that ψ is flat. Consequently, $N^* \times G_0$ is foliated by ψ-leaves, each of which is a smooth submanifold $L \subset N^* \times G_0$ such that projection onto the first factor is a covering map and such that any two leaves differ by left action in the G_0-factor by a constant element of G_0. For any such leaf L, we can regard the functions s, a, p and 1-forms η and ω as being well defined on L via pullback from the projection $L \to N^*$.

The map $(g, a, s, p)\colon L \to M^{13} = G_0 \times \mathbb{C} \times \mathbb{C} \times \mathbb{R}$ then immerses L as an $\mathcal{I}_3$-leaf lying in the locus $\mathbf{F} = 1$. By the construction of γ and the development that led up to it, the image of L is a complete $\mathcal{I}_3$-leaf. Thus, the topology of the leaves will be known once the covering map $L \to D^*$ and the projection $g\colon L \to G_0$ are understood.

The projection $g\colon L \to G_0$ is simply a diffeomorphism. This follows because, on L, the $\mathfrak{g}_0$-valued 1-form $g^{-1}dg$ is simply γ, which determines the forms ω and η and the functions s, a, and p. The construction of the coordinate system (x, y, θ) from (η, ω, s, a, p) shows that this suffices to recover the map $(x, y, \theta)\colon L \to D^*$ up to the action of $\mathbb{Z}^2$, which is the same as recovering $[x, y, \theta]\colon L \to N^*$ and hence the full embedding of L into $N^* \times G_0$. In particular, this implies that (g, a, s, p) is an embedding.

Now, a leaf L is just the holonomy bundle of ψ through each of its points. For the sake of concreteness, choose $n_0 = [\rho_0, \rho_0, 0] \in N^*$ as basepoint, where $\rho_0 = \rho_+(0) = \rho_-(0)$ and let $L \subset N^* \times G_0$ be the leaf of ψ that passes through (n_0, I_3). The intersection $L \cap (\{n_0\} \times G_0)$ is then of the form $\{n_0\} \times \Gamma$ where $\Gamma \subset G_0$ is the holonomy subgroup of ψ and this is what must be computed. The calculations below will actually determine the ψ-monodromy homomorphism $\pi_1(N^*, n_0) \to G_0$, whose image is Γ.

Since N^* is an interval cross a punctured torus, $\pi_1(N^*, n_0)$ is generated by the loops $X\colon [0, 2\rho_0] \to N^*$ and $Y\colon [0, 2\rho_0] \to N^*$ defined by

$$X(x) = [x + \rho_0, \rho_0, 0], \quad Y(y) = [\rho_0, y + \rho_0, 0].$$

To compute the ψ-monodromy around these two loops, information about the behavior of the functions f and g when $\theta = 0$ will be used. To begin, note that $r_1(0) = -\sqrt{3}/4$, $r_2(0) = 0$, and $r_3(0) = \sqrt{3}/4$ and observe that, by the symmetry properties of f and g, there is a $2\rho_0$-periodic function v on $\mathbb{R}$ that satisfies

$$v(t) = f(t + \rho_0, 0) + \sqrt{3}/4 = \sqrt{3}/4 - g(t + \rho_0, 0)$$

for all t. In fact, v is defined by the conditions that it satisfy both the initial condition $v(0) = \sqrt{3}/2$ and the Weierstraß-type differential equation

$$\big(v'(t)\big)^2 = 64 v(t)\big(\sqrt{3}/2 - v(t)\big)\big(v(t) - \sqrt{3}/4\big).$$

Note that v is positive, satisfying $\sqrt{3}/4 \le v(t) \le \sqrt{3}/2$, and that v is an even function on $\mathbb{R}$. In particular, satisfies $v(2\rho_0 - t) = v(t)$, a fact that will be used below.

Now, from the definition of X it follows that

$$X^*(\gamma) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2\big(v(x)\big)^{1/2} dx \\ \big(v(x)\big)^{-1/4} dx & -2\big(v(x)\big)^{1/2} dx & 0 \end{pmatrix}.$$

Consider the $g_X\colon [0, 2\rho_0] \to G_0$ that satisfies ${g_X}^{-1} dg_X = X^*\gamma$ and $g_X(0) = \mathrm{I}_3$. Because $X^*\gamma$ takes values in $\mathfrak{g}_0 \cap \mathfrak{sl}(3, \mathbb{R})$, the map g_X has values in $G_0 \cap \mathrm{SL}(3, \mathbb{R})$ and so can be written in the form

$$g_X(x) = \begin{pmatrix} 1 & 0 & 0 \\ u_1(x) & \cos\varphi(x) & \sin\varphi(x) \\ u_2(x) & -\sin\varphi(x) & \cos\varphi(x) \end{pmatrix},$$

where the functions u_1, u_2, and φ on $[0, 2\rho_0]$ are defined by the ODE system

$$\begin{aligned} u_1'(x) &= \sin\varphi(x)\,\bigl(v(x)\bigr)^{-1/4}, & u_1(0) &= 0,\\ u_2'(x) &= \cos\varphi(x)\,\bigl(v(x)\bigr)^{-1/4}, & u_2(0) &= 0,\\ \varphi'(x) &= \qquad 2\,\bigl(v(x)\bigr)^{1/2}, & \varphi(0) &= 0. \end{aligned}$$

The ODE that v satisfies suggests a change of variables eliminating the explicit x-dependence, yielding

$$\begin{aligned} \varphi(2\rho_0) &= \int_0^{2\rho_0} 2\bigl(v(x)\bigr)^{1/2}dx\\ &= 4\cdot\frac{1}{8}\int_{\sqrt{3}/4}^{\sqrt{3}/2} \frac{v^{1/2}dv}{\sqrt{v\bigl(\sqrt{3}/2-v\bigr)\bigl(v-\sqrt{3}/4\bigr)}}\\ &= \frac{\pi}{2}. \end{aligned}$$

Thus φ defines a diffeomorphism $\varphi\colon [0, 2\rho_0] \to [0, \pi/2]$ that, because of the symmetries of v, has the symmetry $\varphi(2\rho_0 - x) = \pi/2 - \varphi(x)$. In turn, this implies that $u_i'(x) > 0$ for all $x \in (0, 2\rho_0)$ and, by a straightforward change of variables, that $u_1(2\rho_0) = u_2(2\rho_0) = r$ for some[4] $r > 0$.

This implies that $g_X(2\rho_0) = h_X$ where

$$h_X = \begin{pmatrix} 1 & 0 & 0\\ r & 0 & 1\\ r & -1 & 0 \end{pmatrix}.$$

This h_X represents the holonomy of ψ around the loop X. (Note that it is possible to compute the map g_X and hence the holonomy h_X by quadratures in this manner because $X^*\gamma$ takes values in a solvable subalgebra of $\mathfrak{g}_0$.)

A similar argument for Y gives

$$Y^*(\gamma) = \begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & -2i\bigl(v(y)\bigr)^{1/2}dy\\ \bigl(v(y)\bigr)^{-1/4}dy & -2i\bigl(v(y)\bigr)^{1/2}dy & 0 \end{pmatrix}.$$

Carrying out the same sort of analysis as was applied to X leads to the conclusion that if $g_Y\colon [0, 2\rho_0] \to G_0$ is the map that satisfies

[4] For the curious: Numerical calculation yields $\rho_0 \approx 0.498083225$ and $r \approx .565201447$.

${g_Y}^{-1}dg_Y = Y^*\gamma$ and $g_Y(0) = \mathrm{I}_3$, then $g_Y(2\rho_0) = h_Y$ where

$$h_Y = \begin{pmatrix} 1 & 0 & 0 \\ -ir & 0 & -i \\ r & -i & 0 \end{pmatrix}.$$

Thus h_Y represents the holonomy of ψ around the loop Y.

Now, setting

$$\mathbf{v} = \begin{pmatrix} 0 \\ r \end{pmatrix} \neq 0,$$

and

$$\mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{i} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \mathbf{j} = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \quad \mathbf{k} = \mathbf{ij} = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix},$$

it follows that

$$h_X = \begin{pmatrix} 1 & 0 \\ (\mathbf{1}+\mathbf{i})\mathbf{v} & \mathbf{i} \end{pmatrix}, \quad h_Y = \begin{pmatrix} 1 & 0 \\ (\mathbf{1}+\mathbf{j})\mathbf{v} & \mathbf{j} \end{pmatrix}.$$

Noting that $\mathbf{i}^2 = \mathbf{j}^2 = -\mathbf{1}$ while $\mathbf{k} = \mathbf{ij} = -\mathbf{ji}$, it is evident that ${h_X}^4 = {h_Y}^4 = \mathrm{I}_3$ and that any iterated product of the matrices h_X and h_Y is of the form

$$h = \begin{pmatrix} 1 & 0 \\ (a_0\mathbf{1} + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k})\mathbf{v} & \mathbf{q} \end{pmatrix}$$

where $\mathbf{q}$ lies in $\{\pm\mathbf{1}, \pm\mathbf{i}, \pm\mathbf{j}, \pm\mathbf{k}\}$ and the a_i are integers whose sum is even. In particular, the subgroup $\Gamma \subset G_0$ generated by h_X and h_Y is discrete. Moreover, the homomorphism $\Gamma \to \{\pm\mathbf{1}, \pm\mathbf{i}, \pm\mathbf{j}, \pm\mathbf{k}\}$ defined by $h \mapsto \mathbf{q}$ in the above notation is surjective. It is not difficult to establish that the kernel $\hat{\Lambda}$ of this homomorphism consists exactly of the matrices of the form

$$\begin{pmatrix} 1 & 0 \\ 2(a_0\mathbf{1} + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k})\mathbf{v} & \mathbf{1} \end{pmatrix}$$

where the a_i are integers whose sum is even. Since $\mathbf{v} \neq 0$, the set $\Lambda \subset \mathbb{C}^2$ consisting of the vectors $2(a_0\mathbf{1}+a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k})\mathbf{v}$ where the a_i are integers whose sum is even is a lattice in $\mathbb{C}^2$, i.e., a discrete abelian subgroup of rank 4. Up to rotation and scaling, Λ is a lattice of type F_4. In what follows, it will be useful to identify Λ with $\hat{\Lambda} \subset G_0$ via the identification

$$2(a_0\mathbf{1} + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k})\mathbf{v} \longmapsto \begin{pmatrix} 1 & 0 \\ 2(a_0\mathbf{1} + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k})\mathbf{v} & \mathbf{1} \end{pmatrix},$$

so I will do this henceforth without explicit comment.

Let $\hat{K} \subset \pi_1(N^*, n_0)$ denote the normal subgroup of index 8 consisting of those homotopy classes of loops whose ψ-holonomy lies in $\hat{\Lambda}$ and let $\hat{N}^* \to N^*$ denote the 8-fold covering space corresponding to $\hat{K}$. I am going to show that there is a way of 'completing' $\hat{N}^*$ in a natural way so that each of the complex leaves of $\hat{N}^*$ (i.e., the leaves of $\eta = 0$) is realized as a compact Riemann surface of genus 3 punctured at four points. I will then examine to what extent the functions and forms s, a, p, η, and ω extend smoothly across these punctures.

Ultimately, the goal is to show that $\mathbb{C}^2/\Lambda$ contains a minimal Levi-flat hypersurface whose complex leaves are (compact) Riemann surfaces of genus 3.

Let $\tilde{N}$ be the quotient of D by the action of $(2\mathbb{Z})^2$, i.e., the index 4 subgroup of $\mathbb{Z}^2$ generated by the transformations $\Phi_{2m,2n}$, and let $\tilde{N}^* \subset \tilde{N}$ be the image of $D^* \subset D$ under this quotient action. Let $\langle x, y, \theta\rangle \in \tilde{N}$ denote the equivalence class of $(x, y, \theta) \in D$ under the action of $(2\mathbb{Z})^2$. Any product of a finite sequence drawn from $\{h_X, h_Y\}$ that contains an odd number of copies of either h_X or h_Y will be an $h \in \Gamma$ whose corresponding $\mathbf{q}$ lies in $\{\pm\mathbf{i}, \pm\mathbf{j}, \pm\mathbf{k}\}$. Consequently, the quotient map $\tilde{N}^* \to N^*$ defines a 4-fold cover of N^* that is, itself, a 2-fold quotient of $\hat{N}^*$. I.e., there is a sequence of coverings

$$\hat{N}^* \xrightarrow{2-1} \tilde{N}^* \xrightarrow{4-1} N^*$$

corresponding to the inclusion of subgroups $\{\mathbf{1}\} \subset \{\pm\mathbf{1}\} \subset \{\pm\mathbf{1}, \pm\mathbf{i}, \pm\mathbf{j}, \pm\mathbf{k}\}$. The commutator loop $Y^{-1} * X^{-1} * Y * X$ is closed in $\tilde{N}^*$ and this is a loop over which the cover $\hat{N}^* \to \tilde{N}^*$ is non-trivial since this loop does not lie in $\hat{K}$.

It will be useful to construct a embedding of $\tilde{N}^*$ into $\mathbb{CP}^2 \times (-\pi/2, \pi/2)$. Consider the meromorphic solution $\mathfrak{p}$ on $\mathbb{C} \times (-\pi/2, \pi/2)$ to the second order holomorphic differential equation with initial conditions

$$\mathfrak{p}_{zz}(z,\theta) = 6\sec^{2/3}\theta - 96\mathfrak{p}(z,\theta)^2, \quad \mathfrak{p}(0,\theta) = r_2(\theta), \quad \mathfrak{p}_z(0,\theta) = 0.$$

Of course, $\mathfrak{p}$ is a version of the Weierstrass $\mathfrak{p}$-function. It satisfies the first order differential equation

$$\mathfrak{p}_z(z,\theta)^2 = -2\tan\theta + 12\sec^{2/3}\theta\,\mathfrak{p}(z,\theta) - 64\mathfrak{p}(z,\theta)^3.$$

Moreover $\mathfrak{p}(x,\theta) = f(x,\theta)$ when x is real and $\mathfrak{p}(iy,\theta) = g(y,\theta)$ when y is real, as follows easily from the Chain Rule. Now, $\mathfrak{p}$ is doubly periodic and even:

$$\mathfrak{p}\big(z + 2\rho_+(\theta)\big) = \mathfrak{p}\big(z + 2i\rho_-(\theta)\big) = \mathfrak{p}(-z) = \mathfrak{p}(z).$$

and also assumes the special values

$$\begin{aligned} \mathfrak{p}\big(i\rho_-(\theta)\big) &= r_1(\theta), & \mathfrak{p}(0) &= r_2(\theta), \\ \mathfrak{p}\big(\rho_+(\theta)\big) &= r_3(\theta), & \mathfrak{p}\big(\rho_+(\theta) + i\rho_-(\theta)\big) &= \infty. \end{aligned}$$

In fact, $\mathfrak{p}$ has a double pole at $\rho_+(\theta) + i\rho_-(\theta)$ and no other singularities in the fundamental rectangle. Moreover, $\mathfrak{p}_z$ has simple zeros at 0, $\rho_+(\theta)$, $i\rho_-(\theta)$ and a triple pole at $\rho_+(\theta) + i\rho_-(\theta)$.

Now consider, for each fixed θ in the interval $(-\pi/2, \pi/2)$, the quadratic form

$$ds_\theta^2 = \big(f(x,\theta) - g(y,\theta)\big)(dx^2 + dy^2).$$

By the earlier analysis of the vanishing locus of $u\colon D \to \mathbb{R}$, this quadratic form defines a conformal pseudo-metric on $\mathbb{C}$ that branches to order 1 at the points of the lattice

$$\Lambda_\theta = \big\{2m\rho_+(\theta) + i\,2n\rho_-(\theta) \mid m,\, n \in \mathbb{Z}\big\} \subset \mathbb{C}$$

and is periodic with respect to this lattice. Since ds_θ^2 is invariant under reflection in the x-axis and the y-axis, the lines $x = m\rho_+(\theta)$ and $y = n\rho_-(\theta)$ for integer m and n are geodesics in this metric.

The structure equations show that ds_θ^2 has constant Gauss curvature $K = 16$, and so must be induced by pullback from the standard metric on the Riemann sphere with this curvature. In particular, there is a meromorphic function w on the z-plane so that

$$f(x,\theta) - g(y,\theta) = \frac{|w'(z)|^2}{4\big(1 + |w(z)|^2\big)^2}.$$

The function w must ramify to order 1 at each of the points of Λ_θ and must carry the geodesics $x = 2m\rho_+(\theta)$ and $y = 2n\rho_-(\theta)$ onto a single geodesic on the Riemann sphere. (Since they intersect at right angles in the z-plane and the intersection point is a branch point of w of order 2, the image geodesics must meet at an angle of π and hence must lie along the same geodesic on the sphere.) This information is not enough to make the function w unique; it only determines w up to composition with an isometric rotation of the Riemann sphere. However, adding the requirements that $w(0) = 0$ and that $w''(0)$ be real and positive do make w unique, so this will be assumed from now on.

Because the geodesic segment $t\rho_+(\theta)$ for $0 \le t \le 2$ is congruent to the geodesic segment $t\rho_+(\theta) + 2i\rho_-(\theta)$ for $0 \le t \le 2$, and because the geodesic segment $it\rho_-(\theta)$ for $0 \le t \le 2$ is congruent to the geodesic segment $2\rho_+(\theta) + it\rho_-(\theta)$ for $0 \le t \le 2$, and because there are no ramification points of w in the interior of the fundamental rectangle, it follows that the normalized w must satisfy

$$w\big(2\rho_+(\theta)\big)w\big(2i\rho_-(\theta)\big) = -1$$

with $w\big(2\rho_+(\theta)\big)$ real and positive and $w\big(2\rho_+(\theta) + 2i\rho_-(\theta)\big) = \infty$. Pursuing this analysis, it follows without much difficulty that w must be doubly periodic with periods $4\rho_+(\theta)$ and $4i\rho_-(\theta)$ and have one double pole at $2\rho_+(\theta) + 2i\rho_-(\theta)$ in the fundamental rectangle of $2\Lambda_\theta$.

By the usual properties of doubly periodic meromorphic functions on the plane, only one function w with all these properties exists. It can be written in terms of $\mathfrak{p}$ as

$$w(z,\theta) = \frac{\mathfrak{p}(\frac{1}{2}z,\theta) - r_2(\theta)}{\sqrt{\big(r_3(\theta) - r_2(\theta)\big)\big(r_2(\theta) - r_1(\theta)\big)}}.$$

The Weierstraß-type equation for $\mathfrak{p}$ shows that w itself satisfies the Weierstraß-type equation

$$(w_z)^2 = 16b(\theta)w - 48r_2(\theta)w^2 - 16b(\theta)w^3$$

where

$$b(\theta) = \sqrt{\big(r_3(\theta) - r_2(\theta)\big)\big(r_2(\theta) - r_1(\theta)\big)} > 0.$$

By symmetry considerations, w must map the boundary of the rectangle $\mathcal{R}$ with vertices 0, $2\rho_+(\theta)$, $2\rho_+(\theta) + 2i\rho_-(\theta)$, and $2i\rho_-(\theta)$ to the real line plus ∞ on the Riemann sphere and do so in a one-to-one and onto manner. Consequently w establishes a biholomorphism between the interior of $\mathcal{R}$ and the upper half plane. Because of the symmetry of the boundary values, particularly the identity $w\big(2\rho_+(\theta)\big)w\big(2i\rho_-(\theta)\big) = -1$, it follows that w must map $\rho_+(\theta) + i\rho_-(\theta)$, the center of $\mathcal{R}$, to the center of the upper half plane (endowed with its usual metric of constant *positive* curvature), i.e., that

$$w\big(\rho_+(\theta) + i\rho_-(\theta), \theta\big) = i.$$

Using this information, it is not difficult to deduce that

$$w_z\big(\rho_+(\theta) + i\rho_-(\theta), \theta\big) = 4\sqrt{3r_2(\theta) + 2b(\theta)i}.$$

(In view of the Weierstraß equation, the only problem is to fix the ambiguity of the sign of this square root, but this is not difficult. I mean the one with positive imaginary part.)

It follows that there is a well-defined map $\Psi\colon \tilde{N} \to \mathbb{CP}^2 \times (-\pi/2, \pi/2)$ satisfying

$$\Psi\big(\langle x, y, \theta\rangle\big) = \big([1, w(x+iy,\theta), w_z(x+iy,\theta)], \theta\big).$$

Note, in particular, that $\Psi\big(\langle \rho_+(\theta), \rho_-(\theta), \theta\rangle\big) = \big(\big[1, i, 4\sqrt{3r_2(\theta)+2b(\theta)i}\,\big], \theta\big)$. The image of Ψ is the locus $\tilde{E} \subset \mathbb{CP}^2 \times (-\pi/2, \pi/2)$ consisting of points $\big([Z_0, Z_1, Z_2], \theta\big)$ that satisfy the equation

$$Z_0 {Z_2}^2 = 16b(\theta){Z_0}^2 Z_1 - 48r_2(\theta) Z_0 {Z_1}^2 - 16b(\theta){Z_1}^3.$$

Let $\tilde{E}_\theta \subset \mathbb{CP}^2$ be the smooth plane cubic curve so that $\tilde{E}_\theta \times \{\theta\} = \tilde{E} \cap \big(\mathbb{CP}^2 \times \{\theta\}\big)$. This is an elliptic curve and will be referred to as the θ-slice of $\tilde{E}$. By the discussion already given plus elementary properties of elliptic curves, Ψ is a diffeomorphism from $\tilde{N}$ to $\tilde{E}$. Moreover, $\Psi(\tilde{N}^*) = \tilde{E}^*$, which is defined as the complement in $\tilde{E}$ of the three points on each $\tilde{E}_\theta$ that lie on the line $Z_2 = 0$ together with the point at infinity (i.e., the flex tangent on the line $Z_0 = 0$) on each $\tilde{E}_\theta$.

Now, the double cover $\hat{N}^* \to \tilde{N}^* \simeq \tilde{E}^*$ is nontrivial around each of these missing points in each θ-slice. Consider the smooth plane quartic family $\hat{E} \subset \mathbb{CP}^2 \times (-\pi/2, \pi/2)$ consisting of points $\big([W_0, W_1, W_2], \theta\big)$ that satisfy the equation

$${W_2}^4 = 16b(\theta){W_0}^3 W_1 - 48r_2(\theta){W_0}^2{W_1}^2 - 16b(\theta) W_0 {W_1}^3.$$

The map that takes $\big([W_0, W_1, W_2], \theta\big) \in \hat{E}$ to $\big([(W_0)^2, W_0 W_1, (W_2)^2], \theta\big) \in \tilde{E}$ is a branched double cover over each $\tilde{E}_\theta$. The branch locus over each $\tilde{E}_\theta$ consists of the four points on $\tilde{E}_\theta$ that do not belong to $\tilde{E}^*$. Let $\hat{E}^* \subset \hat{E}$ be the inverse image of $\tilde{E}^*$ under this smooth mapping.

Now the double cover $\hat{E}^* \to \tilde{E}^* \simeq \tilde{N}^*$ is nontrivial exactly along the same curves as the double cover $\hat{N}^* \to \tilde{N}^*$. Thus, there is a diffeomorphism $\hat{\Psi}\colon \hat{N}^* \to \hat{E}^*$ that identifies the two double covers and this $\hat{\Psi}$ is unique up to composition with the deck transformation $\big([W_0, W_1, W_2], \theta\big) \to \big([W_0, W_1, -W_2], \theta\big)$ of the covering $\hat{E}^* \to \tilde{E}^*$. From now on, I will fix a choice of $\hat{\Psi}$ and use it to identify $\hat{N}^*$ with $\hat{E}^*$.

Each of the θ-slices $\hat{E}_\theta \subset \hat{E}$ is a nonsingular plane quartic and hence is a nonhyperelliptic Riemann surface of genus 3 [GH, Chapter 2]. In fact, the functions

$$w = \frac{W_1}{W_0}, \quad v = \frac{W_2}{W_0}$$

are smooth and well-defined on $\hat{E}^*$, restricting to each $\hat{E}_\theta$ to become meromorphic functions with poles located at the point at 'infinity' given by the intersection of $\hat{E}_\theta$ with the line $W_0 = 0$. The 1-forms

$$\alpha_1 = \frac{w\,dw}{v^3}, \quad \alpha_2 = \frac{dw}{v^3}, \quad \alpha_3 = \frac{v\,dw}{v^3} = \frac{dw}{v^2}$$

restrict to each $\hat{E}_\theta$ to be a basis for the holomorphic 1-forms on $\hat{E}_\theta$. Note that α_3 is actually invariant under the deck transformation $(w, v, \theta) \mapsto (w, -v, \theta)$ of the covering $\hat{E}^* \to \tilde{E}^*$ and hence is well-defined as a 1-form on $\tilde{E}^*$. This 1-form restricts to each $\tilde{E}_\theta$ to become the nontrivial holomorphic differential on that elliptic curve. Note that α_1 and α_2 have no common zeroes: In fact, α_2 has only one zero, which is of order 4, and this occurs at the common pole of w and v. Since w has a pole of order exactly 4 at this point, it follows that α_1 does not vanish there.

Let $\hat{n}(\theta) = \big([1, i, 2\sqrt[4]{3r_2(\theta) + 2b(\theta)i}\,], \theta\big)$ and consider the multivalued 'function' on $\hat{E}$ 'defined' by the abelian integral

$$\begin{aligned}\vartheta([1,w,v],\theta) = \begin{pmatrix}\vartheta_1([1,w,v],\theta)\\ \vartheta_2([1,w,v],\theta)\end{pmatrix} &= \int_{\hat{n}(\theta)}^{([1,w,v],\theta)} \sqrt{2}\begin{pmatrix} w\\ 1\end{pmatrix}\frac{dw}{v^3}\\ &= \int_{\hat{n}(\theta)}^{([1,w,v],\theta)} \begin{pmatrix}\sqrt{2}\,\alpha_1\\ \sqrt{2}\,\alpha_2\end{pmatrix},\end{aligned}$$

where the integral is to be computed along a path joining $\hat{n}(\theta)$ to $([1,w,v],\theta) \in \hat{E}$ that lies entirely in $\hat{E}_\theta$. Of course, the value of this integral depends on the homology class of the path joining the two endpoints, so this is not well-defined as a function on $\hat{E}$. The ambiguity in the definition of ϑ will be determined below. For the time being, consider ϑ as being defined on a suitable cover $\check{E} \to \hat{E}$. Since α_1 and α_2 do not have any common zeroes, this map is an immersion on each $\hat{E}_\theta$.

Now consider the functions

$$A = \frac{\bar{v}}{v\sqrt{1+|w|^2}}, \quad B = \frac{-\bar{v}w}{v\sqrt{1+|w|^2}}$$

defined on $\hat{E}^*$. They satisfy $|A|^2 + |B|^2 = 1$, so the function

$$h = \begin{pmatrix}1 & 0 & 0\\ 0 & \bar{A}\big(\hat{n}(\theta)\big) & -B\big(\hat{n}(\theta)\big)\\ 0 & \bar{B}\big(\hat{n}(\theta)\big) & A\big(\hat{n}(\theta)\big)\end{pmatrix}^{-1}\begin{pmatrix}1 & 0 & 0\\ \vartheta_1 & \bar{A} & -B\\ \vartheta_2 & \bar{B} & A\end{pmatrix}$$

takes values in G_0 and is well-defined on the open set $\check{E}^* \subset \check{E}$ that is the inverse image of $\hat{E}^*$ under the cover $\check{E} \to \hat{E}$. (The purpose of the first matrix is to arrange that $h\big(\hat{n}(\theta)\big) = \mathrm{I}_3$ for all θ, which will be needed below.)

Since the first factor in h depends only on θ, computation yields

$$h^{-1}dh \equiv \begin{pmatrix} 0 & 0 & 0 \\ A\,d\vartheta_1 + B\,d\vartheta_2 & A\,d\bar{A} + B\,d\bar{B} & B\,dA - A\,dB \\ \bar{A}\,d\vartheta_2 - \bar{B}\,d\vartheta_1 & \bar{A}\,d\bar{B} - \bar{A}\,d\bar{A} & \bar{A}\,dA + \bar{B}\,dB \end{pmatrix} \bmod d\theta.$$

Since $d\vartheta_i \equiv \sqrt{2}\,\alpha_i \bmod d\theta$ for $i = 1, 2$, it follows that

$$\left.\begin{aligned} A\,d\vartheta_1 + B\,d\vartheta_2 &\equiv 0 \\ \bar{A}\,d\vartheta_2 - \bar{B}\,d\vartheta_1 &\equiv \omega \end{aligned}\right\} \quad \bmod d\theta.$$

(This last follows from the identities

$$\omega \equiv (f-g)^{-1/4}dz \equiv \frac{\sqrt{2(1+|w|^2)}}{|w'(z)|^{1/2}}\frac{dw}{w'(z)} \equiv \frac{\sqrt{2(1+|w|^2)}}{|v|}\frac{dw}{v^2} \quad \bmod d\theta,$$

together with the definitions of A and B. The reader can now probably see why the factor of $\sqrt{2}$ was introduced into the definition of ϑ.) Moreover,

$$\begin{aligned} \bar{A}\,d\bar{B} - \bar{B}\,d\bar{A} &= -\bar{A}^2\,d(\bar{B}/\bar{A}) = \frac{-v^2}{|v|^2(1+|w|^2)}\,d\bar{w} \\ &= \frac{-|v|^2}{(1+|w|^2)}\frac{d\bar{w}}{\bar{v}^2} \equiv -2s\omega \equiv \sigma \quad \bmod d\theta. \end{aligned}$$

By these results, there exists a real-valued 1-form ϕ^* so that

$$h^{-1}dh \equiv \begin{pmatrix} 0 & 0 & 0 \\ 0 & i\phi^* & -\bar{\sigma} \\ \omega & \sigma & -i\phi^* \end{pmatrix} \bmod d\theta.$$

The matrix on the right is almost γ. In fact, I claim that it is congruent to γ modulo $d\theta$. Since η is a multiple of $d\theta$ by definition, the only thing to check is whether $\phi^* \equiv \phi$ modulo $d\theta$. However, this follows immediately from the structure equations, which show that $d\sigma \equiv 2i\phi \wedge \sigma \bmod d\theta$ while the very fact that ϕ^* appears where it does in $h^{-1}dh$ shows that $d\sigma \equiv 2i\phi^* \wedge \sigma \bmod d\theta$. Comparing these two relations and using the fact that ϕ and ϕ^* are real then yields $\phi^* \equiv \phi \bmod d\theta$, as desired. (Alternatively, one can simply carry out the computations and compare the results.)

It has now been shown that $h^{-1}dh \equiv \gamma \bmod d\theta$. Now, γ is well-defined on $\hat{E}^*$, not just on $\check{E}^*$, so it follows that $h^{-1}dh$ is well defined on each $\hat{E}_\theta$ and has the same holonomy as γ on each $\hat{E}_\theta$. Now, it has already been shown that the holonomy of γ on $\hat{E}^*$ lies in the discrete subgroup $\hat{\Lambda} \subset G_0$ and the inclusion $\hat{E}^*_\theta \hookrightarrow \hat{E}^*$ induces and isomorphism on fundamental groups. Consequently, there is a well-defined mapping

$$\hat{\Lambda}h \colon \hat{E} \to \hat{\Lambda}\backslash G_0.$$

Note that the quotient is via the left action and not the right action. In particular, the canonical left-invariant form on G_0 is well-defined on $\hat{\Lambda}\backslash G_0$.

Now, consider the $\mathfrak{g}_0$-valued 1-form κ that is well-defined on $\check{E}^*$ by the formula $\kappa = h\gamma h^{-1} - dh\, h^{-1}$. Since

$$\kappa = h(\gamma - h^{-1}dh)h^{-1} \equiv 0 \bmod d\theta$$

since $d\kappa = -\kappa\wedge\kappa = 0$, and since κ vanishes when restricted to each $\check{E}^*_\theta$, it must be a 1-form in θ alone. In fact, a computation using the properties of f and g shows that

$$\hat{n}^*(\kappa) = \begin{pmatrix} 0 & 0 & 0 \\ \frac{1}{6}\bigl(r_3(\theta) - r_1(\theta)\bigr)^{1/4}\sec^{2/3}\theta\, d\theta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

In particular, $\kappa = k^{-1}dk$ where

$$k = \begin{pmatrix} 1 & 0 & 0 \\ m(\theta) & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and where m satisfies $m(0) = 0$ and $m'(\theta) = \frac{1}{6}\bigl(r_3(\theta) - r_1(\theta)\bigr)^{1/4}\sec^{2/3}\theta$.

Since the elements of the form $k(\theta)$ commute with all of the elements of $\hat{\Lambda}$, it now follows that $\gamma = g^{-1}dg$ where $[g] = \hat{\Lambda}kh$ is well defined on $\hat{E}^*$ as a map into $\hat{\Lambda}\backslash G_0$. Since $\hat{\Lambda}\backslash G_0/K \simeq \mathbb{C}^2/\Lambda$, the map

$$\begin{aligned}&\Phi([1,w,v],\theta)\\ &\equiv \begin{pmatrix} A\bigl(\hat{n}(\theta)\bigr) & B\bigl(\hat{n}(\theta)\bigr) \\ -\bar{B}\bigl(\hat{n}(\theta)\bigr) & \bar{A}\bigl(\hat{n}(\theta)\bigr) \end{pmatrix} \begin{pmatrix} \vartheta_1([1,w,v],\theta) \\ \vartheta_2([1,w,v],\theta) \end{pmatrix} + \begin{pmatrix} m(\theta) \\ 0 \end{pmatrix} \quad \bmod \Lambda\end{aligned}$$

is well-defined as a map $\Phi\colon \hat{E} \to \mathbb{C}^2/\Lambda$.

From the formulae that went into its definition, Φ is an immerison on $\hat{E}^*$ whose image is a Levi-flat minimal hypersurface in $\mathbb{C}^2/\Lambda$ of type (3). Moreover $\Phi(\hat{E}_\theta) \subset \mathbb{C}^2/\Lambda$ is a complex leaf in this hypersurface and is immersed as a compact Riemann surface of genus 3.

Now, Φ is not an immersion near the four curves $v = 0$ in $\hat{E}$. (These are the curves that intersect each $\hat{E}_\theta$ in the four branch points.) In fact, it collapses each of these curves to a point, as can be seen by doing a local computation. Let these points be labeled $P_i \in \mathbb{C}^2/\Lambda$ for $i = 1$ to 4.

A possible 'algebraic' structure. Now $\mathbb{C}^2/\Lambda$ is a complex torus that has nontrivial divisors, for example, the genus 3 Riemann surfaces $\Phi(\hat{E}_\theta)$. It follows that $\mathbb{C}^2/\Lambda$ is an Abelian variety (actually, this also follows from the explicit description of Λ as a lattice of type F_4 that has already been given). In particular, $\mathbb{C}^2/\Lambda$ is an algebraic surface. By a standard Riemann-Roch calculation [GH, Chapter 4], one can show that the curves in the connected family of $C_\theta = \Phi(\hat{E}_\theta)$ that pass through the points P_i form a pencil, i.e., the moduli M of such curves is a $\mathbb{CP}^1$. In fact, regarding θ as a complex parameter in the formula for Φ gives a local real parameter on M near $\theta = 0$. Evidently, the curve M admits an antiholomorphic involution for which the curves C_θ are fixed points. Of course, any antiholomorphic involution of $\mathbb{CP}^1$ that has fixed points is conjugate via an automorphism of $\mathbb{CP}^1$ to the standard conjugation fixing an $\mathbb{RP}^1 \subset \mathbb{CP}^1$. Thus, it would appear that the image $\Sigma = \Phi(\hat{E}) \subset \mathbb{C}^2/\Lambda$ is a dense open set in an 'algebraic' real hypersurface $\bar{\Sigma} \subset \mathbb{C}^2/\Lambda$ that is the union of the curves in M that are fixed under the antiholomorphic involution. Presumably, the singular curves in the pencil M are unions of elliptic curves embedded in $\mathbb{C}^2/\Lambda$ linearly and are therefore the totally geodesic complex leaves in $\bar{\Sigma}$. It would be interesting to know whether or not the only singularities of $\bar{\Sigma}$ are the four points P_i and whether or not these singular points really do resemble cones on the Clifford torus, as they appear to.

References

[BCG] R. Bryant, et al., *Exterior Differential Systems*, Springer-Verlag, New York, 1991.

[Ca] É. Cartan, *Sur les systèmes en involution d'équations aux dérivées partielles du second ordre a une fonction inconnue de trois variables indépendantes*, Bull. Soc. Math. France, **39** (1911), 352–443. (Reprinted in Cartan's Collected Works, Part II.)

[CM] S.-S. Chern and J. Moser, *Real hypersurfaces in complex manifolds*, Acta Math., **133** (1974), 219–271.

[Gr] P. Griffiths, *On Cartan's method of Lie groups and moving frames as applied to existence and uniqueness questions in differential geometry*, Duke Math J., **41** (1974), 775–814.

[GH] P. Griffiths and J. Harris, *Principles of Algebraic Geometry*, John Wiley & Sons, New York, 1978.

[He] S. Helgason, Differential Geometry, Lie Groups, and Symmetric Spaces, Academic Press, Princeton 1978.

[KN] S. Kobayashi and K. Nomizu, Foundations of Differential Geometry, vol. II, John Wiley & Sons, New York 1963.

[Sp] M. Spivak, A Comprehensive Introduction to Differential Geometry, Publish or Perish, Inc., Wilmington, Del. 1979. (For a discussion of the moving frame, especially see Volume III, Chapters 1 and 2.)

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Partially Integrable Almost CR Manifolds of CR Dimension and Codimension Two

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Abstract.

We extend the results of [11] on embedded CR manifolds of CR dimension and codimension two to abstract partially integrable almost CR manifolds. We prove that points on such manifolds fall into three different classes, two of which (the hyperbolic and the elliptic points) always make up open sets. We prove that manifolds consisting entirely of hyperbolic (respectively elliptic) points admit canonical Cartan connections. More precisely, these structures are shown to be exactly the normal parabolic geometries of types $(PSU(2,1) \times PSU(2,1), B \times B)$, respectively $(PSL(3,\mathbb{C}), B)$, where B indicates a Borel subgroup. We then show how general tools for parabolic geometries can be used to obtain geometric interpretations of the torsion part of the harmonic components of the curvature of the Cartan connection in the elliptic case.

§1. Introduction

For non-degenerate real hypersurfaces in $\mathbb{C}^{n+1}$ (or more generally in complex manifolds) there is a nice geometric setup based on Cartan connections. The Levi form in any point of such a hypersurface is a non-degenerate Hermitian form, so up to isomorphism (in an obvious sense) it is determined by its signature. In particular, this signature is constant on connected components of M, so without loss of generality one may assume that it is constant on M. This can be interpreted as follows: The tangent spaces of M come with a canonical filtration provided by the maximal complex subspaces, and the constancy of the Levi form provides a reduction of the structure group of the associated graded bundle of the tangent bundle to a reductive subgroup of the group $PSU(p+1, q+1)$ of CR automorphisms of the homogeneous

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model. This observation is the basis for the construction of a canonical Cartan connection for non-degenerate CR manifolds of hypersurface type in [5] and [12], although in these two papers the associated graded vector bundle to the tangent bundle and the reduction of its structure group do not show up explicitly. In the paper [3], reductions of the above type are used as the basis of a general construction of normalized Cartan connections, which immediately shows that in the CR case, canonical Cartan connections do not only exist for (integrable) CR manifolds but more generally for partially integrable almost CR structures (still non-degenerate and of hypersurface type), which was also observed in [12]. Also, in the approach of [3] embeddability plays no role at all, it is only the filtration of the tangent bundle and the reduction of structure group of the associated graded vector bundle that is used.

For CR structures of higher codimension (i.e. real submanifolds of higher codimension in complex manifolds) the situation is much more complicated. The main problem is that the Levi form at a point now has values in a real vector space of dimension bigger than one, and the classification of such forms up to isomorphism is much more difficult. More drastically, one may have continuously varying isomorphism classes in general, so even the associated graded vector spaces to the tangent spaces are not locally isomorphic (under an isomorphism preserving the Levi form). In such cases, there seems to be no hope for geometric structures similar to Cartan connections.

There are, however, a few cases in which the set of isomorphism classes is still discrete, and which thus are more manageable. The simplest of these cases is the case of CR dimension two and codimension two, which is the one that will be treated in this paper. The basic examples of such structures are provided by non-degenerate codimension two submanifolds in $\mathbb{C}^4$, whose tangent spaces are not complex subspaces. In this case, there are three possible isomorphism classes of Levi-forms, referred to as hyperbolic, elliptic, and exceptional (to avoid confusion with parabolic geometries, we do not use the classical name "parabolic" for this class). Moreover, being of hyperbolic or elliptic type are open conditions, so one may study local properties around such points by restricting to manifolds all of whose points are of fixed type. In the paper [11], the authors used a simple normal form argument to show that embedded elliptic and hyperbolic CR manifolds of CR dimension and codimension two are examples of parabolic geometries, as discussed for example in [13], [3]. The general theory developed in the latter two papers as well as in [9] and [14] then shows that these manifolds carry a canonical Cartan connection. It is known in general that the harmonic part of the curvature of the Cartan connection is a complete obstruction against local flatness. This harmonic part splits into several irreducible components, and [11] contains geometric interpretations of those components which are of torsion type. It is also shown the latter paper, that some of these torsions vanish automatically in the embedded case, so [11] partly deals with the more general abstrac parabolic geometries of appropriate type, but without relating them to (almost) CR structures.

In this paper, we extend the approach of [11] to abstract CR structures. It turns out that, as in the hypersurface case, the integrability condition needed to obtain a parabolic geometry is exactly partial integrability, which is a quite simple and natural condition in the real picture. It is interesting that partial integrability is also exactly the condition under which the paper [8] develops analogs of Webster-Tanaka connections for higher codimension almost CR manifolds. We give an independent proof that for non-degenerate partially integrable almost CR manifolds of CR dimension two and codimension two (i.e. six dimensional manifolds endowed with a rank two complex subbundle in the tangent bundle which satisfying partial integrability and a non-degeneracy condition) the possible Levi-brackets (i.e. the bundle maps induced by the Lie bracket) fall into three different classes, called hyperbolic, elliptic and exceptional. Next, we show that the oriented manifolds all of whose points are hyperbolic (elliptic) are exactly the normal parabolic geometries of type $(PSU(2,1) \times PSU(2,1), B \times B)$ (respectively of type $(PSL(3,\mathbb{C}), B)$), where B indicates a Borel subgroup. Thus we prove that reducing to the connected component of the structure group used in [11] exactly corresponds to picking an orientation. While the passage from the CR structure to the parabolic geometry is rather straightforward in the hyperbolic case, there is a more tricky point in the elliptic case, which involves flipping of the almost complex structure on the subbundle. Next, we give a geometric interpretation of the harmonic curvature components in the elliptic case, which were discussed only briefly in [11]. We derive some improved interpretations of torsion components as well as complete proofs for results whose proofs where only sketched in [11]. For example we prove, that torsion-free elliptic manifolds are automatically real analytic and therefore embeddable. Throughout the presentation we stress the fact that these interpretations can be obtained by applying standard tools available for any parabolic geometry. We do not include a discussion of the hyperbolic case, since this was treated in detail in [11].

§2. Partially integrable almost CR manifolds of CR dimension and codimension two

2.1. almost CR manifolds

We start by considering an (abstract) almost CR manifold of CR dimension k and codimension l, i.e. a smooth manifold M of real dimension $2k+l$, together with a rank k complex subbundle $HM \subset TM$, the *CR subbundle* of M. Since we shall meet a different almost complex structure on HM which is more important in the sequel, we will denote the corresponding almost complex structure on H by $\tilde{J}$ or by $\tilde{J}^H$. A smooth map between two almost CR manifolds is called a *CR map*, if and only if its derivative in each point maps the CR subspace to the CR subspace and the restriction to the CR subspace is complex linear.

Let $QM := TM/HM$ be the quotient bundle, which is a real vector bundle of rank l over M and let $q\colon TM \to QM$ be the canonical quotient map. For two smooth sections ξ, η of HM the value of $q([\xi,\eta])$ in a point $x \in M$ depends only on the values of ξ and η in x, so this induces a bilinear, skew symmetric bundle map $\mathcal{L}\colon HM \times HM \to QM$, the *Levi-bracket* of M. Any CR map $f\colon M_1 \to M_2$ induces a homomorphism $QM_1 \to QM_2$ of vector bundles, which we also denote by Tf. A (local) *CR diffeomorphism* between manifolds of the same CR dimension and codimension is a (locally defined) CR map, which is a diffeomorphism (onto its image). If $f\colon M_1 \to M_2$ is a local CR diffeomorphism, then $f^*\mathcal{L}^{M_2} = \mathcal{L}^{M_1}$, i.e. $\mathcal{L}^{M_2}(T_xf\cdot\xi, T_xf\cdot\eta) = T_xf\cdot\mathcal{L}^{M_1}(\xi,\eta)$ for all elements ξ, $\eta \in HM_1$.

Definition. An almost CR manifold M is called *partially integrable* if the Levi bracket is totally real, i.e. $\mathcal{L}(\tilde{J}\xi, \tilde{J}\eta) = \mathcal{L}(\xi,\eta)$.

This partial integrability condition shows up as condition (A1) in [12, p. 170] (included into the definition of an almost CR manifold) and it plays an important role in [8]. To see why it is called partial integrability, one has to pass to the complexified tangent bundle $T_{\mathbb{C}}M = TM \otimes \mathbb{C}$ of M. Since $HM \subset TM$ is a complex subbundle, its complexification $H_{\mathbb{C}}M = HM \otimes \mathbb{C}$ splits into a holomorphic and an antiholomorphic part, $H_{\mathbb{C}}M = H_{1,0}M \oplus H_{0,1}M$. A typical smooth section of $H_{1,0}M$ is of the form $\xi - i\tilde{J}\xi$ for a smooth section ξ of HM. Taking two such sections and computing their bracket, we obtain

$$[\xi,\eta] - [\tilde{J}\xi, \tilde{J}\eta] - i([\tilde{J}\xi,\eta] + [\xi,\tilde{J}\eta]).$$

Now the condition of partial integrability as stated above is obviously equivalent to the fact that both the real and the imaginary part of this are again sections of HM, so it is equivalent to the bracket of two sections of $H_{1,0}M$ being a section of $H_{\mathbb{C}}M$. Recall that an almost CR manifold is called *integrable* or a *CR manifold* if and only if the bundle $H_{1,0}M$ is integrable, i.e. the bracket of two sections of $H_{1,0}M$ is again a section of $H_{1,0}M$. Assuming partial integrability, this condition can be nicely expressed without using complexifications as follows: By partial integrability, for two sections ξ, η of HM, both $[\xi,\eta]-[\tilde{J}\xi,\tilde{J}\eta]$ and $[\xi,\tilde{J}\eta]+[\tilde{J}\xi,\eta]$ are again sections of HM. Thus, we may define the *Nijenhuis-tensor* $\tilde{N}$ of M by

$$\tilde{N}(\xi,\eta)=[\xi,\eta]-[\tilde{J}\xi,\tilde{J}\eta]+\tilde{J}([\tilde{J}\xi,\eta]+[\xi,\tilde{J}\eta]).$$

The usual proof shows that this is bilinear over smooth functions, so it defines a smooth section of the bundle $H^*M\otimes H^*M\otimes HM$, where H^*M is the bundle dual to HM. Moreover, from the definition it follows immediately that $\tilde{N}$ is skew symmetric and conjugate linear in both variables. Clearly, integrability of $H_{1,0}M$ is equivalent to vanishing of the Nijenhuis tensor. Finally note that since $H_{0,1}M=\overline{H_{1,0}M}$, integrability of $H_{1,0}M$ and of $H_{0,1}M$ are equivalent.

2.2. Embedded almost CR manifolds

The typical examples of almost CR manifolds are certain submanifolds in smooth manifolds endowed with an almost complex structure. Suppose that $\mathcal{M}$ is a smooth manifold of real dimension $2k+2l$ endowed with an almost complex structure $J^{\mathcal{M}}$, and let $M\subset\mathcal{M}$ be a smooth submanifold of codimension l. For a point $x\in M$, consider the subspace $H_xM=T_xM\cap J^{\mathcal{M}}(T_xM)$. By construction, this is a complex subspace of T_xM (with the complex structure $\tilde{J}$ given by the restriction of $J^{\mathcal{M}}$). If the complex dimension of this spaces is equal to k for all $x\in M$, then the union of the H_xM defines a smooth subbundle $HM\subset TM$, which makes M into an almost CR manifold of CR dimension k and codimension l.

In fact, any almost CR manifold arises in that way. To see this, consider the quotient bundle $\pi\colon QM\to M$, which is a vector bundle of real rank l, into which M canonically embeds as the zero section. Choosing a linear connection on this vector bundle, we get an associated isomorphism $TQM\cong VQM\oplus\pi^*TM$, where VQM is the vertical tangent bundle and π^*TM is the pullback of the tangent bundle on M. Since the connection is linear, the zero section is covariantly constant, which implies that along the zero section, the horizontal complement to the vertical tangent space coincides with the image of the tangent map of the zero section. Let us further choose a splitting of the projection $q\colon TM\to QM$, which gives us an isomorphism $TM\cong HM\oplus QM$ of vector bundles. For a vector bundle, the vertical bundle is canonically isomorphic to the pullback of the original bundle, so putting all our choices and observations together, we see that we get an isomorphism from TQM to $\pi^*QM \oplus \pi^*HM \oplus \pi^*QM$, such that the first Q-factor corresponds to the vertical bundle. Now define an endomorphism J of TQM by $J(\xi,\eta,\zeta) = (-\zeta, \tilde{J}\eta, \xi)$ in the above decomposition, where $\tilde{J}$ comes from the almost complex structure on HM. Obviously, $J^2 = -\,\mathrm{id}$, so this defines an almost complex structure on the manifold QM. From the above observation on the horizontal subspaces in the points of the zero section we conclude that viewing M as a submanifold of QM, then in the above splitting the tangent space to M consists of all elements of the form $(0,\eta,\zeta)$ and hence the maximal complex subspace consists of all elements of the form $(0,\eta,0)$, so we exactly obtain the given subbundle HM.

If $\mathcal{M}$ is actually a complex manifold, then M is automatically a CR manifold. However, the converse question, whether any CR manifold can be embedded into a complex manifold is much more subtle, and it is well known that the answer is negative in general. There is however one case in which one gets a fairly simple general result. Although this is certainly well known to specialists, it seems that it is not easy to find a general proof in the literature, so we include the argument, which is based on the proof for the hypersurface case in [6].

Proposition. *Let $(M, HM, \tilde{J})$ be a real analytic CR manifold, i.e. M is a real analytic manifold, $HM \subset TM$ is a real analytic subbundle and the bundle map $\tilde{J}\colon HM \to HM$ is real analytic. Then M is locally embeddable, i.e. any point $x \in M$ has an open neighborhood U in M which embeds into $\mathbb{C}^{k+l}$ in such a way that HM becomes the subbundle of maximal complex subspaces.*

Proof. Choosing local coordinates on M in such a way that in each point the subspace generated by the last l coordinate vector fields is transversal to the subbundle HM, we may reduce to the case where M is an open subset $U \subset \mathbb{R}^{2k} \times \mathbb{R}^l$ such that for each $x \in U$ we have $H_xU \cap T_x\mathbb{R}^l = \{0\}$. We view $\mathbb{R}^{2k} \times \mathbb{R}^l$ as a subspace of $\mathbb{R}^{2k} \times \mathbb{C}^l$, and denote the coordinates on the latter space by $x_1, \dots, x_{2k}$, $x_{2k+1} + iy_{2k+1}, \dots, x_{2k+l} + iy_{2k+l}$. In particular, we may view the coordinate vector fields $\frac{\partial}{\partial x_j}$ as a $\mathbb{C}$-Basis of $TU \otimes \mathbb{C}$.

Since HM is a real analytic subbundle of TU and $\tilde{J}$ is real analytic, too, the subbundle $H_{0,1}U \subset TU \otimes \mathbb{C}$ is real analytic, and possibly shrinking U, we find a real analytic frame $\xi_1, \dots, \xi_k$ for this bundle, i.e. $\xi_j = \sum_{m=1}^{2k+l} a_{jm} \frac{\partial}{\partial x_m}$ for real analytic functions $a_{jm}\colon U \to \mathbb{C}$. Now there exists a neighborhood $\tilde{U}$ of U in $\mathbb{R}^{2k} \times \mathbb{C}^l$, such that all the functions a_{jm} extend to $\tilde{U}$ as functions holomorphic in the $\mathbb{C}^l$-factor. Next, we define sections $\tilde{\xi}_1, \dots, \tilde{\xi}_k$ of $T\tilde{U} \otimes \mathbb{C}$ by $\tilde{\xi}_j = \sum_{m=1}^{2k+l} \tilde{a}_{jm} \frac{\partial}{\partial x_m}$, and for $j = 1, \dots, l$ we put $\tilde{\xi}_{k+j} := \frac{\partial}{\partial x_{2k+j}} + i \frac{\partial}{\partial y_{2k+j}}$.

Since the values of $\xi_1, \dots, \xi_k$ in each point of U are linearly independent, we conclude that (possibly shrinking $\tilde{U}$) also the values of $\tilde{\xi}_1, \dots, \tilde{\xi}_k$ in each point of $\tilde{U}$ are linearly independent, which easily implies that $\tilde{\xi}_1, \dots, \tilde{\xi}_{k+l}$ span a rank $k + l$ complex subbundle V of $T\tilde{U} \otimes \mathbb{C}$. Next, we claim that $V \cap \bar{V} = \{0\}$: Suppose that $\lambda_1, \dots, \lambda_{k+l}, \mu_1, \dots, \mu_{k+l} \in \mathbb{C}$ are such that in a point of U we have

$$\lambda_1 \tilde{\xi}_1 + \dots + \lambda_{k+l} \tilde{\xi}_{k+l} = \mu_1 \bar{\tilde{\xi}}_1 + \dots + \mu_{k+l} \bar{\tilde{\xi}}_{k+l}.$$

Looking at the coefficients of the fields $\frac{\partial}{\partial y_j}$ one immediately concludes that $\mu_j = -\lambda_j$ for $j = k+1, \dots, k+l$. But then the equation reduces to

$$\mu_1 \bar{\tilde{\xi}}_1 + \dots + \mu_k \bar{\tilde{\xi}}_k - \lambda_1 \tilde{\xi}_1 - \dots - \lambda_k \tilde{\xi}_k = 2\lambda_{k+1} \frac{\partial}{\partial x_{k+1}} + \dots + 2\lambda_{k+l} \frac{\partial}{\partial x_{k+l}}.$$

The left hand side of this equation lies in $HU \otimes \mathbb{C}$, while the right hand side lies in the complex span of the fields $\frac{\partial}{\partial x_j}$ for $j = 2k + 1, \dots, 2k + l$. But by our assumptions, these two spaces are transversal, so the two sides have to vanish individually. Since $H_{1,0}M \cap \bar{H}_{1,0}M = \{0\}$, we thus conclude that $V \cap \bar{V} = \{0\}$ along U, so possibly shrinking $\tilde{U}$ again, we may assume that this holds on the whole of $\tilde{U}$.

Hence V defines an almost complex structure on $\tilde{U}$, and since U is just the subset on which $y_{2k+1} = \dots = y_{2k+l} = 0$, one immediately concludes that for $x \in U$ the subspace $(T_x U \otimes \mathbb{C}) \cap V_x$ coincides with the fiber of $H_{0,1}U$ at x. To finish the proof, we only have to show that the distribution V is involutive, since then $\tilde{U}$ is a complex manifold and thus locally isomorphic to $\mathbb{C}^{k+l}$. But by assumption $H_{0,1}U$ is involutive, so any bracket $[\xi_i, \xi_j]$ is a linear combination (with real analytic coefficients) of ξ_m's. Since the $\tilde{\xi}$ are defined as partial holomorphic extensions of the ξ, we conclude that for $i, j \leq k$, any bracket $[\tilde{\xi}_i, \tilde{\xi}_j]$ can be written as a linear combination of $\tilde{\xi}_m$ with $m = 1, \dots, k$. On the other hand, we obviously have $[\tilde{\xi}_i, \tilde{\xi}_j] = 0$ if $i, j > k$, while for $i \leq k$ and $j > k$ we get the same result since the coefficients of $\tilde{\xi}_i$ are holomorphic in the last l factors, and the coefficients of $\tilde{\xi}_j$ are constant. Q.E.D.

2.3. Non-degeneracy

Let $(M, HM, \tilde{J})$ be a partially integrable almost CR manifold with Levi-bracket $\mathcal{L}$. Consider the dual Q^*M of the quotient bundle QM.

If $x \in M$ is a point and $\psi \in Q^*_xM$ is any element, we consider the totally real, skew symmetric bilinear map $\mathcal{L}^\psi : H_xM \times H_xM \to \mathbb{R}$ defined by $\mathcal{L}^\psi(\xi, \eta) = \psi(\mathcal{L}(\xi, \eta))$. A point $x \in M$ is called *non-degenerate* if $\mathcal{L}(\xi, \eta) = 0$ for all $\eta \in T_xM$ implies $\xi = 0$ and for each nonzero element $\psi \in Q^*_xM$ the map $\mathcal{L}^\psi$ is non-zero. The first condition just means that the Levi bracket at x is non-degenerate as a bilinear map, while the second condition is a coordinate-free version of linear independence of the components of $\mathcal{L}$. Obviously, non-degeneracy is an open condition, so if x is non-degenerate then there is an open neighborhood of x in M in which all points are non-degenerate. Since any CR diffeomorphism preserves the Levi brackets, it also preserves non-degeneracy, i.e. maps non-degenerate points to non-degenerate points. From now on, we will only consider non-degenerate partially integrable almost CR manifolds, i.e. manifolds all of whose points are non-degenerate.

Let us again compare this to the complexified picture. Here one usually considers the quotient bundle $Q_\mathbb{C}M = T_\mathbb{C}M/H_\mathbb{C}M$, which is a rank l complex bundle, and the *Levi-form* $\mathcal{H}$: $H_{1,0}M \times H_{1,0}M \to Q_\mathbb{C}M$ induced by $(\xi, \eta) \mapsto q_\mathbb{C}(\frac{i}{2}[\xi, \bar{\eta}])$. This turns out to be a hermitian form, and the usual non-degeneracy condition is just that $\mathcal{H}$ is non-degenerate and its components are linearly independent. A simple computation then shows that assuming partial integrability, the Levi bracket $\mathcal{L}$ corresponds exactly to the imaginary part of $\mathcal{H}$ under the identification $HM \to H_{1,0}M$ given by mapping ξ to $\xi - iJ\xi$. Now obviously non-degeneracy of $\mathcal{H}$ is equivalent to non-degeneracy of its imaginary part, while (complex) linear independence of the components of $\mathcal{H}$ is equivalent to (real) linear independence of the components of its imaginary part, so we recover the usual conditions.

2.4. The case of CR dimension and codimension two

For partially integrable almost CR manifolds of general CR dimension k and codimension l, the classification of possible Levi brackets up to the obvious notion of isomorphism is fairly complicated. In the special case $k = l = 2$ however, we shall see below that there are only three possible cases. This is quite well known, see e.g. [10] but since we will need detailed information about the classification in the further constructions we will reproduce it here in a form convenient for our purposes.

The key to this classification is to consider the bilinear maps $\mathcal{L}^\psi_x$: $H_xM \times H_xM \to \mathbb{R}$ for nonzero elements $\psi \in Q^*_xM$. By our non-degeneracy assumption, these maps are all nonzero, skew symmetric, and totally real, so since $k = 2$ they are either non-degenerate, or there is a complex subspace H^ψ_xM of complex dimension one in H_xM

such that $\mathcal{L}^\psi(\xi,\eta)=0$ for all $\eta\in H_xM$ if and only if $\xi\in H_x^\psi M$. Note that clearly both the question whether $\mathcal{L}_x^\psi$ is degenerate and the space $H_x^\psi M$ in the degenerate case depend only on the class $[\psi]$ of ψ in the projectivization $P(Q_x^*M)\cong\mathbb{R}P^1$ of Q_x^*M. We write $\mathcal{L}_x^{[\psi]}$ for the class of forms corresponding to $[\psi]$ and in the case when $\mathcal{L}_x^{[\psi]}$ is degenerate, we write $H_x^{[\psi]}M$ for the corresponding subspace of H_xM.

Proposition. *Let M be a (non-degenerate) partially integrable almost CR manifold of CR dimension and codimension two and let $x\in M$ be a point. Then there are exactly three possibilities:*

1. *There are two points $[\psi_1]\neq[\psi_2]\in P(Q_x^*M)$ such that $\mathcal{L}^\psi: H_xM\times H_xM\to\mathbb{R}$ is degenerate if and only if $\psi\in[\psi_1]$ or $\psi\in[\psi_2]$. In this case, we have $H_xM=H_x^{[\psi_1]}M\oplus H_x^{[\psi_2]}M$ and the point x is called hyperbolic.*
2. *There is one point $[\psi_0]\in P(Q_x^*M)$ such that $\mathcal{L}^\psi$ is degenerate if and only if $\psi\in[\psi_0]$. In this case, the point x is called exceptional.*
3. *$\mathcal{L}^\psi$ is non-degenerate for all nonzero elements $\psi\in Q_x^*M$. In this case, the point x is called elliptic.*

Proof. Let us first assume that $[\psi_1]\neq[\psi_2]\in P(Q_x^*M)$ are such that $\mathcal{L}^{\psi_1}$ and $\mathcal{L}^{\psi_2}$ are degenerate. If $\xi\in H_x^{[\psi_1]}M\cap H_x^{[\psi_2]}M$ then $\mathcal{L}^{\psi_1}(\xi,\eta)=\mathcal{L}^{\psi_2}(\xi,\eta)=0$ for all $\eta\in H_xM$. But since $[\psi_1]\neq[\psi_2]$, the elements ψ_1 and ψ_2 form a basis of Q_x^*M, so $\mathcal{L}^\psi(\xi,\eta)=0$ for all $\psi\in Q_x^*M$. Thus, $\xi=0$ by non-degeneracy of $\mathcal{L}$, and consequently $H_xM=H_x^{[\psi_1]}M\oplus H_x^{[\psi_2]}M$.

If we assume $\psi\in Q_x^*M$ is another nonzero element such that $\mathcal{L}^\psi$ is degenerate, and such that $[\psi]$ is different from both $[\psi_1]$ and $[\psi_2]$, then on one hand, we may write $\mathcal{L}_x^\psi=a\mathcal{L}_x^{\psi_1}+b\mathcal{L}_x^{\psi_2}$ for nonzero real numbers a and b. On the other hand, $H_x^{[\psi]}M$ is complementary both to $H_x^{[\psi_1]}M$ and to $H_x^{[\psi_2]}M$ and thus for each element $\xi\in H_x^{[\psi_1]}M$, there is a unique element $\varphi(\xi)\in H_x^{[\psi_2]}M$, such that $\xi+\varphi(\xi)\in H_x^{[\psi]}M$, and φ is a linear isomorphism. Since $\mathcal{L}^\psi=a\mathcal{L}_x^{\psi_1}+b\mathcal{L}_x^{\psi_2}$, $\xi\in H_x^{[\psi_1]}M$ and $\varphi(\xi)\in H_x^{[\psi_2]}M$, we see that $\xi+\varphi(\xi)\in H_x^{[\psi]}M$ simply means that $0=a\mathcal{L}_x^{\psi_1}(\varphi(\xi),\eta)+b\mathcal{L}_x^{\psi_2}(\xi,\eta)$, for all η in H_xM. But inserting η from $H_x^{[\psi_1]}M$ we get $\mathcal{L}_x^{\psi_2}(\xi,\eta)=0$, for all ξ, $\eta\in H_x^{[\psi_1]}M$. Thus $\mathcal{L}_x^{\psi_2}$ would be identically zero, a contradiction. Q.E.D.

Remarks. (1) Clearly, the separation into these three classes is invariant under CR diffeomorphisms, so any CR diffeomorphism maps a point from one of the three classes to a point of the same class. Moreover, by definition, the properties of being hyperbolic or elliptic are stable, so any hyperbolic (elliptic) point has an open neighborhood in M which

consists entirely of hyperbolic (elliptic) points. Consequently, for local questions one may restrict to manifolds, all of whose points are either hyperbolic or elliptic.
(2) Traditionally, exceptional points are called parabolic points, but in view of the fact that parabolic geometries can be used to describe the hyperbolic and elliptic cases, we prefer to avoid this name. With these exceptional points, the situation is more complicated. On one hand, there are manifolds consisting entirely of exceptional points (for example an appropriate quadric), but it may also happen that any neighborhood of an exceptional point contains elliptic and/or hyperbolic points. Also, even the case in which all points are exceptional cannot be studied using parabolic geometries. It should also be mentioned that the exceptional points are more degenerate than the hyperbolic and elliptic ones. Namely, in our combination of dimensions, one can also consider the following (even more natural) non-degeneracy condition: The Levi bracket $\mathcal{L}$ may be considered as a smooth section of the bundle $\bigwedge^2_{\mathbb{R}} H^*M \otimes QM$. Hence, $\mathcal{L} \wedge \mathcal{L}$ is a smooth section of $\bigwedge^4_{\mathbb{R}} H^*M \otimes S^2QM$ and thus can be viewed as a quadratic polynomial on Q^*M defined up to scale. If one requires these polynomials to be non-degenerate, then only hyperbolic and elliptic points survive, see [2]. In the latter paper it is also shown that parts of the basic theory we develop here, carry over to the case where $HM \subset TM$ is only a real rank four vector bundle (i.e. without an almost complex structure).

2.5. The hyperbolic case

We next want to identify hyperbolic partially integrable almost CR manifolds of CR dimension and codimension two with a parabolic geometry. To avoid having to deal with non-connected groups, we restrict to the case when M is oriented, which definitely makes no problems locally. So let us assume that M is oriented and that all points of M are hyperbolic.

By proposition 2.4, for each point $x \in M$ we find the two distinguished classes $[\psi_1]$, $[\psi_2] \in P(Q^*_xM)$. Moreover, in local coordinates these are the solutions of a smooth equation, which implies that the classes $[\psi] \in P(Q^*M)$ which have the property that $\mathcal{L}^{[\psi]}$ is degenerate form a smooth submanifold of the bundle $P(Q^*M)$. Since M is hyperbolic, for any point $x \in M$ there are exactly two points in this submanifold lying over x, and since the projection $P(Q^*M) \to M$ is a surjective submersion, it restricts to a local diffeomorphism from the submanifold to M, so we get a two-sheeted covering of M. Hence, we can choose local smooth sections ψ_1 and ψ_2 of Q^*M which represent the distinguished classes in each point of their domain of definition. But then the null spaces of $\mathcal{L}^{\psi_1}$ and $\mathcal{L}^{\psi_2}$ form smooth subbundles H^1M and H^2M of HM, which are independent of the choice of the sections ψ_1 and ψ_2 (up to their numbering). So locally, we get a splitting $HM = H^1M \oplus H^2M$ of HM into a sum of complex line bundles. Now for $i = 1, 2$ let ξ_i be a local non vanishing section of H^iM and consider the local vector fields $\{\xi_1, J\xi_1, [\xi_1, J\xi_1], \xi_2, J\xi_2, [\xi_2, J\xi_2]\}$ on M. We claim that these form a local frame for TM. By construction, $\{\xi_i, J\xi_i\}$ is a local frame for H^iM, and since $HM = H^1M \oplus H^2M$ these four vector fields together form a local frame for HM. Applying $q\colon TM \to QM$ on the other hand, kills ξ_i and $J\xi_i$ for $i = 1, 2$ and maps the remaining vector fields to $\mathcal{L}(\xi_1, J\xi_1)$ and $\mathcal{L}(\xi_2, J\xi_2)$, respectively. Moreover, by definition $\psi_1(\mathcal{L}(\xi_1, J\xi_1)) = 0$. On the other hand, $\psi_1(\mathcal{L}(\xi_2, J\xi_2))$ must be nonzero, since $\psi_1 \circ \mathcal{L} = \mathcal{L}^{\psi_1}$ is nonzero, but has H^1M as its null space. Similarly, ψ_2 vanishes on $\mathcal{L}(\xi_2, J\xi_2)$ but is nonzero on $\mathcal{L}(\xi_1, J\xi_1)$, which implies our claim.

The question whether the above local frame is positively or negatively oriented is independent of the choice of ξ_1 and ξ_2. Indeed, one could even choose an arbitrary linearly independent section η_1 instead of $J\xi_1$, since exchanging the two sections changes the sign of the Lie bracket. Thus, we get a preferred order of the two subbundles, which we indicate by calling them H^+M and H^-M. Furthermore, choosing corresponding sections $\psi_\pm$ of Q^*M, they fit together globally, which shows that the two-fold covering constructed above is trivial. In particular, this means that we globally have $HM = H^+M \oplus H^-M$ and we find global smooth nonzero section $\psi_\pm$ of Q^*M representing the distinguished classes of functionals. Moreover, putting $Q^+M := \ker(\psi_-)$ and $Q^-M := \ker(\psi_+)$ we get a decomposition $QM = Q^+M \oplus Q^-M$ as a sum of real line bundles, which is independent of the choice of $\psi_\pm$, so it is intrinsic to M. In this language, we can rephrase the definition of the subbundles $H^\pm M$ as $\xi \in H^\pm M$ if and only if $\mathcal{L}(\xi, \eta) \in Q^\pm M$ for all $\eta \in HM$. Hence, $\mathcal{L}$ vanishes on $H^+M \times H^-M$, so it splits as $\mathcal{L} = \mathcal{L}^+ \oplus \mathcal{L}^-$, where $\mathcal{L}^\pm\colon H^\pm M \times H^\pm M \to Q^\pm M$.

2.6. Hyperbolic almost CR manifolds as parabolic geometries

Passing from the data $(M, H^+M, H^-M, QM = Q^+M \oplus Q^-M)$ to a parabolic geometry is now rather straightforward. Consider $\mathbb{C}^3$ with the hermitian inner product $\langle z, w\rangle = z_1\bar{w}_3 + z_3\bar{w}_1 + z_2\bar{w}_2$. This is clearly non-degenerate and has signature $(2, 1)$. Let $G := PSU(2, 1)$ be the quotient of the group of all complex automorphisms of $\mathbb{C}^3$ respecting this inner product by its center, and let $\mathfrak{su}(2,1)$ be the Lie algebra of this group. Then this is exactly the space of all complex 3×3-matrices, which are skew hermitian with respect to the above inner product. One easily computes directly that $\mathfrak{su}(2,1)$ is exactly the space of all matrices of the form $\begin{pmatrix} A & Z & iz \\ X & -2i\,\mathrm{im}(A) & -\bar{Z} \\ ix & -\bar{X} & -\bar{A} \end{pmatrix}$, where X, A, $Z \in \mathbb{C}$ and x, $z \in \mathbb{R}$. This Lie algebra gets a grading of the form $\mathfrak{g} = \mathfrak{g}_{-2} \oplus \cdots \oplus \mathfrak{g}_2$ by $\begin{pmatrix} \mathfrak{g}_0 & \mathfrak{g}_1 & \mathfrak{g}_2 \\ \mathfrak{g}_{-1} & \mathfrak{g}_0 & \mathfrak{g}_1 \\ \mathfrak{g}_{-2} & \mathfrak{g}_{-1} & \mathfrak{g}_0 \end{pmatrix}$. In particular, $\mathfrak{g}_{\pm 1} \cong \mathbb{C}$ and $\mathfrak{g}_{\pm 2} \cong \mathbb{R}$. Next, we define $G_0 \subset B \subset G$ as the subgroups of all elements whose adjoint action preserves the grading respectively the corresponding filtration of $\mathfrak{g}$. This means that g lies in G_0 (respectively B) if and only if $\mathrm{Ad}(g)(\mathfrak{g}_i) \subset \mathfrak{g}_i$ (respectively $\subset \mathfrak{g}_i \oplus \cdots \oplus \mathfrak{g}_2$) for all $i = -2, \dots, 2$. Note that B is actually a Borel subgroup of G.

Now one verifies directly, that any element of G_0 must be the class of a diagonal matrix, and a diagonal matrix lies in $SU(2,1)$ if and only if its entries on the diagonal are $(a, \bar{a}/a, 1/\bar{a})$ for $a \in \mathbb{C} \setminus \{0\}$. The Lie bracket $\mathfrak{g}_{-1} \times \mathfrak{g}_{-1} \to \mathfrak{g}_{-2}$ is just given by $(X,Y) \mapsto \bar{Y}X - \bar{X}Y$. A simple computation shows that the adjoint action of G induces an isomorphism from G_0 to the group of complex linear isomorphisms of $\mathfrak{g}_{-1}$ (which is isomorphic to $\mathbb{C} \setminus \{0\}$) and the action on $\mathfrak{g}_{-2}$ is chosen in such a way that it is compatible with the Lie bracket.

Now consider a product $G^+ \times G^-$ of two copies of G, with the corresponding Lie algebra $\mathfrak{g}^+ \oplus \mathfrak{g}^-$. Then the adjoint action is componentwise, we get a grading of $\mathfrak{g}^+ \oplus \mathfrak{g}^-$ and the subgroup of elements whose adjoint action preserves this grading (respectively the corresponding filtration) is exactly $G_0^+ \times G_0^-$ (respectively $B^+ \times B^-$). Now suppose that M is an oriented hyperbolic partially integrable almost CR manifold of CR dimension and codimension two. Let $\mathcal{G}_0$ be the complex frame bundle of the complex vector bundle $H^+M \oplus H^-M$ viewed as being modeled on $\mathfrak{g}^+_{-1} \oplus \mathfrak{g}^-_{-1}$, i.e. the fiber of $\mathcal{G}_0$ over $x \in M$ is exactly the set of all pairs (u^+, u^-), where $u^\pm \colon \mathfrak{g}^\pm_{-1} \to H^\pm_x M$ is a complex linear isomorphism. For any $b \in G_0$, $\mathrm{Ad}(b) \colon \mathfrak{g}_{-1} \to \mathfrak{g}_{-1}$ is a complex linear isomorphism which respects the splitting, so $(u^+, u^-) \cdot b := (u^+, u^-) \circ \mathrm{Ad}(b)$ defines right action of G_0 on $\mathcal{G}_0$, which obviously is free and transitive on each fiber, thus making $p_0 \colon \mathcal{G}_0 \to M$ into a G_0-principal bundle.

Next, we define a filtration $V\mathcal{G}_0 \subset T^{-1}\mathcal{G}_0 \subset T\mathcal{G}_0$, where $V\mathcal{G}_0$ denotes the vertical bundle, by putting $T^{-1}\mathcal{G}_0 := (Tp_0)^{-1}(HM)$. We get a canonical one-form $\theta_{-2} \in \Omega^1(\mathcal{G}_0, \mathfrak{g}_{-2})$ as follows: Take a point

$u = (u^+, u^-) \in \mathcal{G}_0$. Then $u^+ : \mathfrak{g}^+_{-1} \to H^+_x M$ is a complex linear isomorphism, so there exist unique linear isomorphism $\tilde{u}^+ : \mathfrak{g}^+_{-2} \to Q^+_x M$ such that $\mathcal{L}_x(u^+(X), u^+(Y)) = \tilde{u}^+([X, Y])$, and similarly we get linear isomorphism $\tilde{u}^- : \mathfrak{g}^-_{-2} \to Q^-_x M$. For a tangent vector $\xi \in T_u\mathcal{G}_0$, consider $Tp_0 \cdot \xi \in T_x M$, and define

$$\theta_{-2}(\xi) := \big((\tilde{u}^+)^{-1}(q_+(Tp_0 \cdot \xi)), (\tilde{u}^-)^{-1}(q_-(Tp_0 \cdot \xi))\big),$$

where $q_\pm : TM \to Q^\pm M$ are the canonical projections. It is easy to see that this is a smooth one-form, and by construction its kernel in a point u is exactly the space $T^{-1}_u \mathcal{G}_0$. Moreover, since the action of G_0 on $\mathfrak{g}$ is compatible with the Lie bracket, it follows that for $(v^+, v^-) = (u^+, u^-) \cdot b$ we get $(\tilde{v}^+, \tilde{v}^-) = (\tilde{u}^+, \tilde{u}^-) \circ \mathrm{Ad}(b)$. Since $Tp_0 \circ Tr^b = Tp_0$, this implies that $(r^b)^* \theta_{-2} = \mathrm{Ad}(b^{-1}) \circ \theta_{-2}$, so θ_{-2} is equivariant.

Similarly, we get a canonical section $\theta_{-1} \in \Gamma(L(T^{-1}\mathcal{G}_0, \mathfrak{g}_{-1}))$. Namely, if $\xi \in T^{-1}_u \mathcal{G}_0$, then $Tp_0 \cdot \xi \in H_{p_0(u)} M$, so we uniquely decompose this as $\xi^+ + \xi^-$ with $\xi^\pm \in H^\pm_{p_0(u)} M$, and we define

$$\theta_{-1}(\xi) := \big((u^+)^{-1}(\xi^+), (u^-)^{-1}(\xi^-)\big).$$

Again, this is visibly smooth and its kernel in a point u is exactly the vertical tangent space $V_u \mathcal{G}_0$. Again since $Tp_0 \circ Tr^b = Tp_0$, the definition of the action of G_0 immediately implies that $(r^b)^* \theta_{-1} = \mathrm{Ad}(b^{-1}) \circ \theta_{-1}$. Consequently, $\theta = (\theta_{-2}, \theta_{-1})$ is a frame form of length one in the sense of [3, 3.2]. Moreover, by [3, proposition 4.2] this frame form satisfies the structure equation, so $(\mathcal{G}_0, \theta)$ is a P-frame bundle of degree one in the sense of [3, 3.6].

Theorem. *If M is an oriented hyperbolic partially integrable almost CR manifold of CR dimension and codimension two, then there is a canonical principal bundle $p : \mathcal{G} \to M$ with structure group $B \times B$, where B is the Borel subgroup in $PSU(2,1)$ endowed with a normal Cartan connection $\omega \in \Omega^1\big(\mathcal{G}, \mathfrak{su}(2,1) \times \mathfrak{su}(2,1)\big)$.*

Conversely, such a principal bundle and Cartan connection over a six dimensional smooth manifold M canonically make M into an oriented hyperbolic partially integrable almost CR manifolds of CR dimension and codimension two. These two constructions actually describe an equivalence of categories.

Proof. The first part is a special case of the main result of [3] or of the prolongation procedure of [9]. Moreover, the uniqueness part of this result implies that a local CR diffeomorphism $M_1 \to M_2$ lifts to a homomorphism $\Phi : \mathcal{G}_1 \to \mathcal{G}_2$ of principal bundles such that $\Phi^* \omega_2 = \omega_1$.

Conversely, let us assume that $p\colon \mathcal{G} \to M$ is a B-principal bundle over a smooth six dimensional manifold M endowed with a normal Cartan connection ω. For a point $x \in M$ choose a point $u \in \mathcal{G}$ with $p(u) = x$. Consider the component ω_{-2} of the Cartan connection which has values in $\mathfrak{g}_{-2}$. Then $\omega_{-2}(u)\colon T_u\mathcal{G} \to \mathfrak{g}^+_{-2} \oplus \mathfrak{g}^-_{-2}$ is a surjective linear map which vanishes on the vertical subbundle, so it induces a linear map $T_xM \to \mathfrak{g}^+_{-2} \oplus \mathfrak{g}^-_{-2}$. Let us denote by H_xM the kernel of this map and by Q_xM the quotient space T_xM/H_xM. Then we get a decomposition $Q_xM = Q^+_xM \oplus Q^-_xM$ by putting $Q^\pm_xM$ the subspaces mapped to $\mathfrak{g}^\pm_{-2}$. Further, the restriction of $\omega_{-1}(u)$ to the kernel of $\omega_{-2}(u)$ has as kernel exactly the vertical tangent space, so it descends to a linear isomorphism $H_xM \cong \mathfrak{g}^+_{-1} \oplus \mathfrak{g}^-_{-1}$, which gives us a decomposition $H_xM = H^+_xM \oplus H^-_xM$ into two one-dimensional complex spaces. If we choose a different point $\tilde{u}$ instead of u, then there is a unique element $b \in P$ such that $\tilde{u} = u{\cdot}b$ and equivariancy of the Cartan connection reads as $\omega(\tilde{u})(Tr^b{\cdot}\xi) = \mathrm{Ad}(b^{-1})(\omega(u)(\xi))$. Since $p \circ r^b = p$, this implies that the maps induced on tangent spaces of M change only by composition with $\mathrm{Ad}(b^{-1})$. Now B consists of elements respecting the filtration of $\mathfrak{g}$ so the adjoint action respects the set of elements with trivial $\mathfrak{g}_{-2}$-component, and thus the subspace H_xM remains unchanged. Moreover, on this subspace, the action of B factors through G_0 (see [3, 2.12]), which implies that also the complex structure and the decomposition of H_xM remain unchanged. Since the adjoint action of any element of G respects the decomposition $\mathfrak{g} = \mathfrak{g}^+ \oplus \mathfrak{g}^-$, also the decomposition of Q_xM remains unchanged.

Since the structures are independent of the choice of the point u, we can now use a local smooth section of $p\colon \mathcal{G} \to M$ to construct the data locally around a point and by uniqueness, they fit together globally, showing that we actually get a smooth subbundle HM, with a decomposition $H^+M \oplus H^-M$ into a sum of complex line bundles, as well as a decomposition $QM = Q^+M \oplus Q^-M$ of the quotient bundle $QM = TM/HM$. In particular, (M, HM) is an almost CR manifold of CR dimension and codimension two. The fact that the Cartan connection is normal implies that the underlying frame form of length one satisfies the structure equations, which in turn by [3, propositon 4.2] implies that the Levi bracket on M coincides with the algebraic bracket coming from the G_0-structure underlying the principal bundle $\mathcal{G}$. In particular, this implies that the Levi bracket is totally real, so (M, HM) is partially integrable. Moreover, we get that the bilinear map $\mathcal{L}$ is non-degenerate, it satisfies $\mathcal{L}(H^\pm M, HM) \subset Q^\pm M$ and for any point $x \in M$ the image of $\mathcal{L}$ contains Q^+_xM and Q^-_xM, and thus all of QM. But this immediately implies that for each nonzero $\psi \in Q^*_xM$ the map $\mathcal{L}^\psi$ is nonzero, so

(M, HM) is non-degenerate. Denoting by $\pi_\pm\colon QM \to Q^\pm M$ the canonical projections and choosing any local trivializations $\varphi_\pm$ of $Q_\pm$ we see that $\psi_\pm = \varphi_\pm \circ \pi_\pm$ are two linearly independent functionals such that $\mathcal{L}^{\psi_\pm}$ is degenerate (with null space $H^\pm$), so we see that (M, HM) is hyperbolic.

Finally, assume that $\mathcal{G}_1 \to M_1$ and $\mathcal{G}_2 \to M_2$ are two such principal bundles endowed with normal Cartan connections, and $\Phi\colon \mathcal{G}_1 \to \mathcal{G}_2$ is a homomorphism of principal bundles which covers a local diffeomorphism $f\colon M_1 \to M_2$. From the above construction of the CR structures on the M_i it is then immediate that $\Phi^*\omega_2 = \omega_1$ implies that $f\colon M_1 \to M_2$ is a CR map, and thus a local CR diffeomorphism, which establishes the equivalence of categories. Q.E.D.

2.7. The elliptic case

Next, we will consider the case of oriented elliptic partially integrable almost CR manifolds of CR dimension and codimension two. So we have $(M, HM, \tilde{J})$ such that for any point $x \in M$ and any nonzero element $\psi \in Q^*_x M$ the form $\mathcal{L}^\psi_x\colon H_xM \times H_xM \to \mathbb{R}$ introduced in 2.4 is non-degenerate. Note that since the complex subbundle $HM \subset TM$ is canonically oriented, choosing an orientation of M is equivalent to choosing an orientation of the quotient bundle QM.

Proposition. *Let $(M, HM, \tilde{J})$ be an oriented elliptic partially integrable almost CR manifold of CR dimension and codimension two with quotient bundle QM.*
(1) *There is a unique almost complex structure J^Q on the bundle QM which is compatible with the orientation of M and has the property that for each point $x \in M$ there is a nonzero element $\eta \in H_xM$ such that $\mathcal{L}_x(\tilde{J}\xi, \eta) = J^Q\mathcal{L}_x(\xi, \eta)$ for all $\xi \in H_xM$.*
(2) *If we define $H^\pm_x M$ to be the subspaces consisting of all $\eta \in H_xM$ such that $\mathcal{L}_x(\tilde{J}\xi, \eta) = \pm J^Q\mathcal{L}_x(\xi, \eta)$ for all $\xi \in H_xM$, then these subspaces fit together to form smooth subbundles $H^\pm M \subset HM$, which both are complex line bundles and have the property that $HM = H^+M \oplus H^-M$. Also, the Levi-bracket $\mathcal{L}$ vanishes on $H^+M \times H^+M$ and on $H^-M \times H^-M$.*
(3) *If we define a new almost complex structure J on HM by $J|_{H^+M} = -\tilde{J}$ and $J|_{H^-M} = \tilde{J}$, then with respect to the almost complex structures J and J^Q the Levi bracket $\mathcal{L}\colon HM \times HM \to QM$ is complex bilinear.*

Proof. Consider the complexification $Q_{\mathbb{C}}M = QM \otimes \mathbb{C}$ and the map $\mathcal{H}\colon HM \times HM \to Q_{\mathbb{C}}M$ defined by $\mathcal{H}(\xi, \eta) = \mathcal{L}(\tilde{J}\xi, \eta) + i\mathcal{L}(\xi, \eta)$. Using the fact that $\mathcal{L}$ is totally real, one immediately verifies that this is a $Q_{\mathbb{C}}M$-valued hermitian form on HM.

Next, for a point $x \in M$ consider the (complex) dual $(Q_{\mathbb{C}})^*_x M$ of the fiber of $Q_{\mathbb{C}}M$ at x. By definition, an element ψ of this space is a $\mathbb{C}$-linear map $(Q_{\mathbb{C}})_x M \to \mathbb{C}$. Similarly to 2.4 above, we can now consider $\mathcal{H}^\psi_x = \psi \circ \mathcal{H}_x \colon H_xM \times H_xM \to \mathbb{C}$. This is not a hermitian form any more, but it still is complex linear in the first and conjugate linear in the second variable, and we may still ask whether it is degenerate or non-degenerate. Moreover, as before the question whether $\mathcal{H}^\psi_x$ is degenerate or not, as well as the null space in the case where it is degenerate, depends only on the class of ψ in the (complex) projectivization $\mathcal{P}((Q_{\mathbb{C}})^*_x M) \cong \mathbb{C}P^1$. As before, we use square brackets to indicate the class in a projectivization. Choosing a real basis of Q_xM (and considering the corresponding basis of the complexification) we can split $\mathcal{H}$ into two components $\mathcal{H}_1$, $\mathcal{H}_2$, and choosing further a complex basis ξ_1, ξ_2 of H_xM, we can consider the corresponding Hermitian matrices H_1, H_2 in this basis. Moreover, from the basis of $(Q_{\mathbb{C}})_xM$, we get homogeneous coordinates on $\mathcal{P}((Q_{\mathbb{C}})^*_x M)$ and in this picture the condition that $\mathcal{H}^\psi$ is degenerate exactly means that we have a solution $(\lambda : \mu)$ of the homogeneous polynomial of degree two given by $\det(\lambda H_1 + \mu H_2) = 0$. Since the matrices H_i are Hermitian, for any solution (λ, μ) of this equation, also $(\bar\lambda, \bar\mu)$ is a solution. By definition, $\mathcal{H}^\psi_x(\xi, \eta) = 0$ if and only if $\psi(\mathcal{L}(\tilde J\xi, \eta)) = i\psi(\mathcal{L}(\xi, \eta))$. Now consider the image of $Q_xM \subset Q_xM \otimes \mathbb{C}$ under ψ, which is a real linear subspace of $\mathbb{C}$. If this is a proper subspace, then the above equation can only hold if the two sides both vanish. Since M is elliptic, we conclude that $\mathcal{H}^\psi_x$ is non-degenerate if $\psi(Q_xM) \neq \mathbb{C}$.

In particular, $\det(\lambda H_1 + \mu H_2)$ is not identically zero, so there are exactly two points $[\psi]$, $[\bar\psi] \in \mathcal{P}((Q_{\mathbb{C}})^*_x M)$ representing the solutions. From above, we know that the restrictions of ψ and $\bar\psi$ to Q_xM are injective, so both maps define linear isomorphisms $Q_xM \to \mathbb{C}$. Clearly, exactly one of the maps ψ and $\bar\psi$ is orientation preserving as a real linear map from Q_xM to $\mathbb{C}$ (with the canonical orientation), so we assume that ψ has this property. Thus, we get a complex structure J^Q_x on the vector space Q_xM, which clearly depends on the class $[\psi]$ only.

Using the formula from above, we see that the condition that $\mathcal{H}^\psi$ is degenerate exactly means that there is a nonzero element $\eta \in H_xM$ such that $\psi(\mathcal{L}(\tilde J\xi, \eta)) = i\psi(\mathcal{L}(\xi, \eta))$ for all $\xi \in H_xM$. By definition of J^Q, the last equation just reads as $\mathcal{L}(\tilde J\xi, \eta) = J^Q\mathcal{L}(\xi, \eta)$. Similarly, for an element η to lie in the null space of $\mathcal{H}^{\bar\psi}$ is equivalent to $\mathcal{L}_x(\tilde J\xi, \eta) = -J^Q\mathcal{L}_x(\xi, \eta)$, so the two null spaces are exactly $H^\pm_x M$ as defined in the theorem. By construction, they are both complex subspaces and nonzero, and their intersection is zero since $\mathcal{L}$ is non-degenerate, so the only possibility is that they both are of complex dimension one and $H_xM = H_x^+M \oplus H_x^-M$, so (1) follows.

To see that we get smooth subbundles $H^\pm M \subset HM$, one just has to note that the above constructions depend smoothly on the point x. Indeed, $\mathcal{H}$ is a smooth form $HM \times HM \to Q_\mathbb{C}M$. Choosing smooth local frames for HM and QM and the corresponding homogeneous coordinates on $\mathcal{P}((Q_\mathbb{C})^*M)$, we see that $(x, \lambda, \mu) \mapsto \det(\lambda H_1(x) + \mu H_2(x))$ is a smooth function, which is regular since the polynomial has different roots in each point, so the solutions form a smooth submanifold in $\mathcal{P}((Q_\mathbb{C})^*M)$, which by construction is a two-fold covering of M. The condition on the orientation distinguishes the two sheets of the covering, so we get a smooth section $M \to \mathcal{P}((Q_\mathbb{C})^*M)$ whose value in each point is exactly the class leading to the almost complex structure J_x^Q. Thus $J^Q : QM \to QM$ is smooth and hence an almost complex structure. Finally, choosing a local section ψ of $Q_\mathbb{C}^*M$ whose class in each point x is the distinguished element of $\mathcal{P}((Q_\mathbb{C})_x^*M)$, we get a smooth vector bundle map $HM \to L(HM, \mathbb{C})$ defined by $\eta_x \mapsto \mathcal{H}_x^\psi(\_, \eta)$, and H^+M is exactly the kernel of this bundle map, which we already know is of constant rank, so it is a smooth subbundle. Similarly, one deals with H^-M.

By construction, if one considers the restriction of the Levi-bracket $\mathcal{L}$ to $H^+M \times H^+M$, then this is complex bilinear and skew symmetric, so it vanishes since H^+M has complex rank one. Similarly, $\mathcal{L}$ vanishes on $H^-M \times H^-M$, so (2) is proved.

Thus, the only part of the Levi-bracket that has to be considered is its restriction to $H^+M \times H^-M$. By construction, this is conjugate linear in the first and complex linear in the second variable, so switching the almost complex structure on H^+M, it becomes complex bilinear. Q.E.D.

2.8. An equivalence of categories

The structure (M, H^+M, H^-M, J^Q) obtained in proposition 2.7 is preserved under orientation preserving local CR diffeomorphisms. Suppose that $(M_k, HM_k, \tilde{J}_k)$ are oriented elliptic partially integrable almost CR manifolds and $f : M_1 \to M_2$ is an orientation preserving (local) CR diffeomorphism. For $x \in M_1$ consider $f(x) \in M_2$ and the induced linear isomorphism $T_xf : Q_xM_1 \to Q_xM_2$. Pulling back the complex structure on Q_xM_2 obtained from proposition 2.7(1) to Q_xM_1, we obtain a complex structure which is compatible with the orientation since f was assumed to be orientation preserving and the restriction of T_xf to the CR tangent spaces is complex linear and thus orientation preserving. Choosing $\eta \in H_xM_1$ such that $T_xf \cdot \eta \in H_{f(x)}^+M_2$, we conclude from $f^*\mathcal{L}^{M_2} = \mathcal{L}^{M_1}$ that $\mathcal{L}_x(\_,\eta)$ is complex linear for the pulled back structure, so by the uniqueness in proposition 2.7(1), we see that $T_xf\colon Q_xM_1 \to Q_xM_2$ is complex linear for the structures obtained from proposition 2.7(1). Moreover, the above argument also shows that $T_xf(H^\pm_xM_1) \subset H^\pm_{f(x)}M_2$. But then the fact that $T_xf\colon HM_1 \to HM_2$ is complex linear for the structures $\tilde{J}$ implies that it is also complex linear for the structures J obtained by proposition 2.7(3).

Conversely, assume that M is a smooth manifold of dimension 6 equipped with two complementary complex line bundles $H^\pm M \subset TM$, an almost complex structure J^Q on the quotient $QM = TM/HM$, where $HM = H^+M \oplus H^-M$, which is such that the bundle map $\mathcal{L}\colon HM \times HM \to QM$ induced by the Lie bracket is complex bilinear and non-degenerate. Then on HM consider the almost complex structure $\tilde{J}$ defined by flipping the complex structure on H^+M and keeping the complex structure on H^-M. The Levi bracket $\mathcal{L}\colon HM \times HM \to QM$ is by assumption non-degenerate. For ξ, $\eta \in HM$, we can split $\xi = \xi_+ + \xi_-$ and similarly for η and we compute

$$\begin{aligned}
\mathcal{L}(\tilde{J}\xi, \tilde{J}\eta) &= \mathcal{L}(-J\xi_+ + J\xi_-, -J\eta_+ + J\eta_-)\\
&= -\mathcal{L}(J\xi_-, J\eta_+) - \mathcal{L}(J\xi_+, J\eta_-)\\
&= \mathcal{L}(\xi_-, \eta_+) + \mathcal{L}(\xi_+, \eta_-) = \mathcal{L}(\xi, \eta),
\end{aligned}$$

where we have used that $\mathcal{L}$ is complex bilinear for J and $H^\pm M$ are isotropic for $\mathcal{L}$ since they have complex dimension one. Thus $(M, HM, \tilde{J})$ is a partially integrable almost CR manifold of CR dimension and codimension two. Finally, consider nonzero elements $\psi \in Q^*_xM$ and $\xi \in H_xM$. As before, we may split $\xi = \xi_+ + \xi_-$, and let us assume without loss of generality that $\xi_+ \neq 0$. Then $\mathcal{L}(\xi_+, \_)$ restricts to a nonzero complex linear (and thus surjective) map $H^-_xM \to Q_xM$, so we can find $\eta \in H^-_xM$ such that $\psi(\mathcal{L}(\xi_+, \eta)) \neq 0$. But since $\eta \in H^-_xM$, we have $\mathcal{L}(\xi, \eta) = \mathcal{L}(\xi_+, \eta)$, which implies that $\mathcal{L}^\psi$ is non-degenerate, and hence $(M, HM, \tilde{J})$ is elliptic. Finally, if $(M_j, H^+M_j, H^-M_j, J^Q_j)$ are such manifolds for $j = 1, 2$ and $f\colon M_1 \to M_2$ is a (local) diffeomorphism such that Tf restricts to complex linear isomorphism $H^\pm M_1 \to H^\pm M_2$ then f defines a (local) CR diffeomorphism from $(M_1, HM_1, \tilde{J}_1)$ to $(M_2, HM_2, \tilde{J}_2)$. Note that the condition on Tf preserving the subbundles $H^\pm$ implies that we get an induced map $Tf\colon QM_1 \to QM_2$, which is automatically complex linear since $Tf\cdot\mathcal{L}(\xi, \eta) = \mathcal{L}(Tf\cdot\xi, Tf\cdot\eta)$ because $\mathcal{L}$ is induced by the Lie bracket. Thus we get

Theorem. *The category of elliptic partially integrable almost CR manifolds of CR dimension and codimension two and local CR diffeomorphisms is equivalent to the category whose objects are six dimensional manifolds M endowed with two complementary complex line bundles $H^{\pm}M \subset TM$ and an almost complex structure on $QM = TM/HM$ such that the Levi bracket $\mathcal{L}\colon HM \times HM \to QM$ is non-degenerate and complex bilinear, and whose morphisms are the local diffeomorphisms whose derivative in each point restricts to complex linear isomorphisms between the subbundles $H^{\pm}M$.*

2.9. Real parabolic geometries of type $(PSL(3,\mathbb{C}), B)$

To show that the category from theorem 2.8 above is equivalent to a category of normal parabolic geometries is now strictly parallel to the hyperbolic case, so we are more brief on that. Consider first the Lie algebra $\mathfrak{g} = \mathfrak{sl}(3,\mathbb{C})$ (as a real Lie algebra), let $\mathfrak{g}_0$ be its Cartan subalgebra, i.e. the subalgebra of all trace free diagonal matrices and let $\mathfrak{b}$ be its Borel subalgebra, i.e. the subalgebra of all trace free upper triangular matrices. As a module over $\mathfrak{g}_0$, $\mathfrak{g}$ decomposes as a direct sum of $\mathfrak{g}_0$ and the root spaces, and we write this decomposition as $\mathfrak{g} = \mathfrak{g}_{-2} \oplus (\mathfrak{g}_{-1}^{-} \oplus \mathfrak{g}_{-1}^{+}) \oplus \mathfrak{g}_0 \oplus (\mathfrak{g}_1^{-} \oplus \mathfrak{g}_1^{+}) \oplus \mathfrak{g}_2$ defined by $\begin{pmatrix} \mathfrak{g}_0 & \mathfrak{g}_1^{+} & \mathfrak{g}_2 \\ \mathfrak{g}_{-1}^{+} & \mathfrak{g}_0 & \mathfrak{g}_1^{-} \\ \mathfrak{g}_{-2} & \mathfrak{g}_{-1}^{-} & \mathfrak{g}_0 \end{pmatrix}$.
Clearly, this makes $\mathfrak{g}$ into a graded Lie algebra, i.e. $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$. The subalgebra $\mathfrak{b}$ is exactly the non-negative part in this grading, so the adjoint action of $\mathfrak{b}$ never moves down in the grading, which implies that the corresponding filtration is $\mathfrak{b}$-invariant. By the grading property, we have in particular the bracket $\mathfrak{g}_{-1} \times \mathfrak{g}_{-1} \to \mathfrak{g}_{-2}$. Like all brackets in $\mathfrak{g}$ this is complex bilinear, and hence $\mathfrak{g}_{-1}^{+}$ and $\mathfrak{g}_{-1}^{-}$ are isotropic, while the restriction of the bracket to $\mathfrak{g}_{-1}^{+} \times \mathfrak{g}_{-1}^{-}$ is non-degenerate.

Next, consider the adjoint group $G = PSL(3,\mathbb{C})$ of $\mathfrak{g}$. We define subgroups $G_0 \subset B \subset G$ as the groups of those elements whose adjoint actions preserve the grading respectively the filtration on $\mathfrak{g}$. According to the general theory (see [3, proposition 2.9]), G_0 has Lie algebra $\mathfrak{g}_0$ and B has Lie algebra $\mathfrak{b}$, so it is a Borel subgroup in G. The group G is the quotient of $SL(3,\mathbb{C})$ by its center, which is just the third roots of unity times the identity matrix, so we will usually work in $SL(3,\mathbb{C})$ keeping in mind that we work modulo the center. Now it is easy to verify that for $g \in G_0$, any representative in $SL(3,\mathbb{C})$ must be diagonal. If a, b, c are the diagonal entries (and $abc = 1$), then one immediately verifies that the adjoint action on $\mathfrak{g}_{-1}^{+}$, $\mathfrak{g}_{-1}^{-}$ and $\mathfrak{g}_{-2}$ is given by multiplication by $a^{-1}b$, $b^{-1}c$ and $a^{-1}c$, respectively. Taking $0 \neq \lambda$, $\mu \in \mathbb{C}$ we see that putting $a = (\lambda^{-2}\mu^{-1})^{1/3}$, $b = (\lambda\mu^{-1})^{1/3}$ and $c = (\lambda\mu^2)^{1/3}$ we obtain an element that acts on $\mathfrak{g}_{-1}^{+}$ by multiplication with λ and on $\mathfrak{g}_{-1}^{-}$ by μ, while the action on $\mathfrak{g}_{-2}$ is fixed by compatibility with the Lie bracket. Moreover, one easily sees that by this condition the diagonal matrix with entries a, b, c is uniquely determined up to multiplication with a third root of unity times the identity matrix. Thus we see that the adjoint action identifies G_0 with the group of pairs φ_+, φ_-, where $\varphi_\pm$ is a complex linear isomorphism of $\mathfrak{g}_{-1}^{\pm}$, and the action on $\mathfrak{g}_{-2}$ is fixed by compatibility with the Lie bracket.

Now let M be a smooth manifold of dimension 6 equipped with two complementary complex line bundles $H^{\pm}M \subset TM$, an almost complex structure on $QM = TM/HM$ such that the Levi bracket $\mathcal{L}\colon HM \times HM \to QM$ is non-degenerate and complex bilinear. As in the hyperbolic case we consider the complex frame bundle $\mathcal{G}_0$ of $H^+M \oplus H^-M$ as modeled on $\mathfrak{g}_{-1}$, and via the adjoint action we can view this as a principal G_0-bundle. Denoting elements of $\mathcal{G}_0$ as (u^+, u^-), where $u^{\pm}\colon \mathfrak{g}_{-1}^{\pm} \to H_x^{\pm}M$ is a complex linear isomorphism, we now get a unique complex linear isomorphism $\tilde{u}\colon \mathfrak{g}_{-2} \to Q_xM$ such that $\mathcal{L}(u^+(X), u^-(Y)) = \tilde{u}([X,Y])$, which allows us to define a form $\theta_{-2} \in \Omega^1(\mathcal{G}_0, \mathfrak{g}_{-2})$ as in the hyperbolic case. As there one shows that the form is equivariant and its kernel is exactly $T^{-1}\mathcal{G}_0 = (Tp_0)^{-1}(HM)$. On the other hand, the definition of $\theta_{-1} \in \Gamma(L(T^{-1}\mathcal{G}_0, \mathfrak{g}_{-1}))$ is completely the same as in the hyperbolic case, and also the properties are verified in the same way. Hence we again get a frame form $(\theta_{-2}, \theta_{-1})$ of length one on $\mathcal{G}_0$, which satisfies the structure equations, since $\mathcal{L}$ is induced by the Lie bracket of vector fields, and we get:

Theorem. *For any elliptic partially integrable almost CR manifold M there exists a canonical principal bundle $p\colon \mathcal{G} \to M$ with group B equipped with a normal Cartan connection $\omega \in \Omega^1(\mathcal{G}, \mathfrak{sl}(3,\mathbb{C}))$. Conversely, a principal B-bundle $\mathcal{G}$ over a smooth 6-dimensional manifold M endowed with a normal Cartan connection makes M canonically into an oriented elliptic partially integrable almost CR manifold. These constructions actually give rise to an equivalence of categories.*

Proof. Existence of $\mathcal{G}$ and ω for an elliptic partially integrable almost CR manifold follows from the main result of [3] or the procedure of [9] or (with a reinterpretation of the underlying structure along the lines of [3, 4.4]) from the original procedure of Tanaka, see [13]. In view of theorem 2.8, the converse is completely analogous to the hyperbolic case. Explicitly, $HM = Tp(\ker(\omega_{-2}))$, the almost complex structure on QM is induced by ω_{-2} and the splitting of HM, as well as the almost complex structure J on HM are induced by ω_{-1}. (To get back to a

CR picture, one has to flip the almost complex structure on H^+M as in 2.8.)

Also, establishing the equivalence of categories is done exactly like in the hyperbolic case. Q.E.D.

§3. Interpretations of torsions in the elliptic case

One of the main advantages of having a canonical Cartan connection is that this offers a conceptual approach to obstructions against local flatness. It is well known in general (see e.g. [3, proposition 4.12] for a proof in the setting of parabolic geometries) that the curvature of a Cartan connection is a complete obstruction against local flatness. In our setting this means that a point $x \in M$ has a neighborhood which is CR diffeomorphic to an open subset of the flat model (which is a quadric in our case) if and only if the curvature of the Cartan connection vanishes identically locally around x. In the case of normal Cartan connection there is a further refinement of that. In this section we will show how general tools for parabolic geometries can be used to give geometric interpretations of these obstructions. Since this has been done in the hyperbolic case in some detail in [11], we restrict to the elliptic case here, in which we get several new results and improvements compared to the latter paper.

3.1. The curvature of the normal Cartan connection

As a start, we have to describe the normalization condition on our Cartan connections in a little more detail. For the Cartan connection $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$, we define the *curvature function* $\kappa\colon \mathcal{G} \to L(\bigwedge^2 \mathfrak{g}_-, \mathfrak{g})$, where $\mathfrak{g}_- = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$ and the exterior product is over $\mathbb{R}$, by $\kappa(u)(X,Y) = d\omega(\omega_u^{-1}(X), \omega_u^{-1}(Y)) + [X,Y]$. Recall that by definition, $\omega_u\colon T_u\mathcal{G} \to \mathfrak{g}$ is a linear isomorphism, so $\omega_u^{-1}(X)$ makes sense. This curvature function captures all the information about the curvature of the Cartan connection. Note that κ splits into homogeneous degrees with respect to the grading as $\kappa = \kappa^{(0)} + \cdots + \kappa^{(6)}$, where $\kappa^{(0)}$ maps $\bigwedge^2 \mathfrak{g}_{-1}$ to $\mathfrak{g}_{-2}$, $\kappa^{(1)}$ maps $\bigwedge^2 \mathfrak{g}_{-1}$ to $\mathfrak{g}_{-1}$ and $\mathfrak{g}_{-2} \otimes \mathfrak{g}_{-1}$ to $\mathfrak{g}_{-2}$, and so on. The first normalization condition on ω is that $\kappa^{(0)}$ is identically zero.

To formulate the second normalization condition, observe that $L(\bigwedge^2 \mathfrak{g}_-, \mathfrak{g})$ is just the second chain group in the standard complex computing the (real!) Lie algebra cohomology $H^*_{\mathbb{R}}(\mathfrak{g}_-, \mathfrak{g})$. The spaces in this complex are just the spaces $L(\bigwedge^k_{\mathbb{R}} \mathfrak{g}_-, \mathfrak{g})$ of k-linear, alternating maps $\mathfrak{g}_-^k \to \mathfrak{g}$, and the differential $\partial\colon L(\bigwedge^k \mathfrak{g}_-, \mathfrak{g}) \to L(\bigwedge^{k+1} \mathfrak{g}_-, \mathfrak{g})$ is defined by the usual formula

$$\partial\varphi(X_0,\dots,X_k)=\sum_{i=0}^{k}(-1)^i[X_i,\varphi(X_0,\dots,\hat{X}_i,\dots,X_k)]$$
$$+\sum_{i<j}(-1)^{i+j}\varphi([X_i,X_j],X_0,\dots,\hat{X}_i,\dots,\hat{X}_j,\dots,X_k).$$

For our purpose the essential fact is that extending a construction of Kostant (see [7]) one can show that there is a natural adjoint $\partial^*\colon L(\bigwedge^k\mathfrak{g}_-,\mathfrak{g})\to L(\bigwedge^{k-1}\mathfrak{g}_-,\mathfrak{g})$ to the Lie algebra differential, see [3, 2.5, 2.6]. It turns out that both ∂ and ∂^* are differentials, so $\partial^2=(\partial^*)^2=0$, defining the Laplacian $\Box=\partial\partial^*+\partial^*\partial$ one gets a Hodge decomposition $L(\bigwedge^k\mathfrak{g}_-,\mathfrak{g})=\mathrm{Im}(\partial)\oplus\mathrm{Ker}(\Box)\oplus\mathrm{Im}(\partial^*)$ and $\mathrm{Ker}(\Box)$ is naturally isomorphic to the k-th cohomology group $H^k_{\mathbb{R}}(\mathfrak{g}_-,\mathfrak{g})$. A formula for ∂^* can be easily obtained from the fact that it is the dual map to another Lie algebra differential. For this, one has to note that the Killing form on $\mathfrak{g}$ induces a duality of $\mathfrak{g}_0$-modules between $\mathfrak{g}_-$ and $\mathfrak{g}_+$. We shall only need the formula for $\partial^*\colon L(\bigwedge^2\mathfrak{g}_-,\mathfrak{g})\to L(\mathfrak{g}_-,\mathfrak{g})$, which is given by

$$\partial^*\varphi(X)=\sum_\alpha\big([Z_\alpha,\varphi(X_\alpha,X)]+\tfrac{1}{2}\varphi([Z_\alpha,X]_-,X_\alpha)\big),$$

where $\{X_\alpha\}$ is a basis of $\mathfrak{g}_-$, $\{Z_\alpha\}$ is the dual basis of $\mathfrak{g}_+$ and $[\ ,\ ]_-$ denotes the $\mathfrak{g}_-$-component of the bracket, see [3, 2.5]. The second normalization condition on the Cartan connection is that the curvature function has values in $\mathrm{Ker}(\partial^*)=\mathrm{Ker}(\Box)\oplus\mathrm{Im}(\partial^*)$. Note that from the formulae it is obvious that both ∂ and ∂^* preserve homogeneous degrees, so this conditions can be applied to various homogeneous degrees of κ separately. Also, this implies that the cohomology groups split as direct sums $H^k(\mathfrak{g}_-,\mathfrak{g})=\sum_l H^k_{(l)}(\mathfrak{g}_-,\mathfrak{g})$ according to homogeneous degree.

For later use, we note one further refinement: Since $\mathfrak{g}$ is a complex representation of $\mathfrak{g}_-$, we can also view $L(\bigwedge^k_{\mathbb{R}}\mathfrak{g}_-,\mathfrak{g})\cong\bigwedge^k_{\mathbb{R}}\mathfrak{g}_-^*\otimes_{\mathbb{R}}\mathfrak{g}$ as $(\bigwedge^k_{\mathbb{R}}\mathfrak{g}_-^*\otimes_{\mathbb{R}}\mathbb{C})\otimes_{\mathbb{C}}\mathfrak{g}$. Now $\bigwedge^k_{\mathbb{R}}\mathfrak{g}_-^*\otimes_{\mathbb{R}}\mathbb{C}$ can be identified with $\bigwedge^k_{\mathbb{C}}(\mathfrak{g}_-^*\otimes_{\mathbb{R}}\mathbb{C})$ and the splitting of the complexification of $\mathfrak{g}_-^*$ into a holomorphic and an antiholomorphic part induces a splitting $L(\bigwedge^k_{\mathbb{R}}\mathfrak{g}_-,\mathbb{C})=\oplus_{p+q=k}L^{p,q}(\mathfrak{g}_-,\mathbb{C})$, which in turn induces a similar splitting for $L(\bigwedge^k_{\mathbb{R}}\mathfrak{g}_-,\mathfrak{g})$. Explicitly, $L^{1,0}(\mathfrak{g}_-,\mathfrak{g})$ and $L^{0,1}(\mathfrak{g}_-,\mathfrak{g})$ are the spaces of complex linear respectively conjugate linear maps. More generally, $L^{k,0}(\mathfrak{g}_-,\mathfrak{g})$ and $L^{0,k}(\mathfrak{g}_-,\mathfrak{g})$ are the spaces of those k-linear alternating maps which are complex linear respectively complex anti-linear in each variable. The last bit of information that we will need is that $L^{1,1}(\mathfrak{g}_-, \mathfrak{g})$ is exactly the set of those bilinear, alternating maps $\varphi\colon \mathfrak{g}_- \times \mathfrak{g}_- \to \mathfrak{g}$ which have the property that $\varphi(iX, iY) = \varphi(X,Y)$ for all X, $Y \in \mathfrak{g}_-$.

Thus, the final splitting of the spaces in the standard complex looks as $L(\bigwedge^k \mathfrak{g}_-, \mathfrak{g}) = \bigoplus_{p+q=k;l} L^{p,q}_{(l)}(\mathfrak{g}_-, \mathfrak{g})$, where l refers to the homogeneous degree. The fact that the brackets in $\mathfrak{g}$ all are complex bilinear together with the fact that ∂ preserves homogeneous degrees now implies that $\partial(L^{p,q}_{(l)}(\mathfrak{g}_-, \mathfrak{g})) \subset L^{p+1,q}_{(l)}(\mathfrak{g}_-, \mathfrak{g})$. For all fixed q and l we have a subcomplex $L^{*,q}_{(l)}(\mathfrak{g}_-, \mathfrak{g})$, and the cohomology groups finally split as $H^k(\mathfrak{g}_-, \mathfrak{g}) = \bigoplus_{p+q=k;l} H^{p,q}_{(l)}(\mathfrak{g}_-, \mathfrak{g})$. From the construction of the adjoint ∂^* one can verify that also ∂^* is compatible with bidegrees, so $\partial^*(L^{p,q}_{(l)}(\mathfrak{g}_-, \mathfrak{g})) \subset L^{p-1,q}_{(l)}(\mathfrak{g}_-, \mathfrak{g})$. In particular, ∂^* is identically zero on $L^{0,k}_{(l)}(\mathfrak{g}_-, \mathfrak{g})$ for all k, l.

The final point we have to observe is the Bianchi identity for the curvature of a Cartan connection, which can be used to compute $\partial \circ \kappa$. By [3, proposition 4.9] this reads as

$$(\partial \circ \kappa)(X,Y,Z) = \sum_{\text{cycl}} \big(\kappa(\kappa_-(X,Y),Z) + (\omega^{-1}(X)\cdot\kappa)(Y,Z)\big).$$

Here the sum is over all cyclic permutations of the arguments, κ_- is the component of κ in $\mathfrak{g}_-$, and in the last term we use the vector field $\omega^{-1}(X)$ to differentiate the function κ. The importance of this identity is that if we consider a fixed homogeneous degree on the left hand side, then only lower homogeneous degrees can enter on the right hand side. In particular, the lowest nonzero homogeneous degree of κ must have values in $\operatorname{Ker}(\partial) \cap \operatorname{Ker}(\partial^*) = \operatorname{Ker}(\Box)$ which is isomorphic to an appropriate cohomology group. This can be used to show (see [3, proposition 4.10]) that κ vanishes if and only if its harmonic part (i.e. its component in $\operatorname{Ker}(\Box)$) vanishes. Moreover, from the formula one can obviously get informations on possible bidegrees of $\partial \circ \kappa$ if the possible bidegrees of lower homogeneous components are known.

3.2. Harmonic curvature components

The relevant cohomology group $H^2(\mathfrak{g}_-, \mathfrak{g})$ for our case has been computed in [11]. Expressed in terms of bidegrees the results read as follows:

1. The only nonzero components in $H^2(\mathfrak{g}_-, \mathfrak{g})$ are $H^{0,2}_{(1)}$, $H^{1,1}_{(1)}$, and $H^{2,0}_{(4)}$.
2. $H^{0,2}_{(1)}$ splits as a B_0-module into two one-dimensional components, which are represented by maps $\mathfrak{g}_{-2} \times \mathfrak{g}^{\pm}_{-1} \to \mathfrak{g}_{-2}$.

3. $H^{1,1}_{(1)}$ splits into four one-dimensional components which are represented by totally real maps $\bigwedge^2 \mathfrak{g}^+_{-1} \to \mathfrak{g}^-_{-1}$, and $\bigwedge^2 \mathfrak{g}^-_{-1} \to \mathfrak{g}^+_{-1}$ respectively by maps $\mathfrak{g}^+_{-1} \times \mathfrak{g}^-_{-1} \to \mathfrak{g}^-_{-1}$ and $\mathfrak{g}^-_{-1} \times \mathfrak{g}^+_{-1} \to \mathfrak{g}^+_{-1}$, which are complex linear in the first and conjugate linear in the second variable (this last point needs a slightly closer look on the description of cohomologies in [11]).
4. $H^{2,0}_{(4)}$ splits into two one dimensional components represented by complex bilinear maps $\mathfrak{g}_{-2} \times \mathfrak{g}^+_{-1} \to \mathfrak{g}^+_1$ respectively $\mathfrak{g}_{-2} \times \mathfrak{g}^-_{-1} \to \mathfrak{g}^-_1$.

In particular, we have six irreducible torsion-type components (the components homogeneous of degree one) and two curvature-type components. Note moreover, that the homogeneous component $\kappa^{(1)}$ of the curvature function is harmonic by the Bianchi identity, so only the components indicated in 2 and 3 of the above list may actually be nonzero.

Our next task is to give geometric interpretations of the harmonic components of the curvature which are of torsion-type. Most of this has already been done in [11], but we partly have different interpretations and partly simpler and more complete proofs. The main tool for deducing these interpretations is the following general result on parabolic geometries which is a variant of [11, lemma 2.10]:

Lemma. *Let $(p\colon \mathcal{G} \to M, \omega)$ be a parabolic geometry, i.e. a principal P-bundle endowed with a $\mathfrak{g}$-valued Cartan connection, where $P \subset G$ is a parabolic subgroup, and let κ be the curvature function of ω. Suppose that $x \in M$ is a point and $u \in \mathcal{G}$ is such that $p(u) = x$. Then there is a neighborhood U of $x \in M$ and an extension operator $\xi_x \mapsto \xi$ from the tangent space T_xM to the set of local vector fields defined on U, which is compatible with all structures on TM carried over from $\mathfrak{g}_-$ using ω, and which has the following property: For ξ_x, $\eta_x \in T_xM$ let X, $Y \in \mathfrak{g}_-$ be the unique elements such that $\xi_x = T_up\cdot\omega_u^{-1}(X)$ and $\eta_x = T_up\cdot\omega_u^{-1}(Y)$. Then*

$$[\xi,\eta](x) = T_up\cdot\omega_u^{-1}\big([X,Y] - \kappa(u)(X,Y)\big).$$

Proof. (see [11, lemma 2.10]) Note first that X and Y are well defined since the difference of two lifts of a tangent vector on M lies in the vertical bundle, which is mapped by ω to $\mathfrak{b}$. Denoting by Fl^ξ the flow of a vector field ξ, we consider the map $\varphi(X) := \mathrm{Fl}_1^{\omega^{-1}(X)}(u)$, which is defined on a neighborhood V of 0 in $\mathfrak{g}_-$. The tangent map in zero of $p \circ \varphi\colon V \to M$ is a linear isomorphism, so possibly shrinking V we may assume that φ and $p \circ \varphi$ are both diffeomorphisms onto their images. Denoting $U := p(\varphi(V))$, we get a smooth local section $\sigma\colon U \to \varphi(V)$, since p has to restrict to a diffeomorphism $\varphi(V) \to U$.

Now given $\xi_x \in T_xM$ we define the extension $\xi \in \mathfrak{X}(U)$ by $\xi(y) := Tp\cdot\big(\omega(\sigma(y))^{-1}(X)\big)$. Clearly, this extension operator is compatible with all structures on TM carried over from $\mathfrak{g}_-$ via ω. To verify the condition on the brackets, let us start with two tangent vectors ξ_x and η_x and the corresponding elements X, $Y \in \mathfrak{g}_-$. Any point in $p^{-1}(U)$ can be uniquely written as $\sigma(y)b$ for $y \in U$ and $b \in P$. Defining a local vector field $\tilde{\xi}$ on $p^{-1}(U)$ by $\tilde{\xi}(\sigma(y)\cdot b) := Tr^b\cdot\big(\omega(\sigma(y))^{-1}(X)\big)$, we obviously get a projectable vector field, which projects onto ξ and similarly we get a field $\tilde{\eta}$ projecting onto η. Hence, $[\xi, \eta](x) = T_up\cdot[\tilde{\xi}, \tilde{\eta}](u)$. Moreover, $u = \sigma(p)$, so the flow lines of $\tilde{\xi}$ and $\tilde{\eta}$ through u stay in the image of σ, and thus $\omega(\tilde{\xi})$ is constant along the flow line of $\tilde{\eta}$ through u and vice versa. But then the definition of the exterior derivative implies that $\omega([\tilde{\xi}, \tilde{\eta}])(u) = -d\omega(u)(\tilde{\xi}(u), \tilde{\eta}(u))$, and since $\tilde{\xi}(u) = \omega(u)^{-1}(X)$ and similarly for η, the result follows from the definition of κ. Q.E.D.

The main upshot of this result is that whenever one has a tensorial quantity which can be expressed as a combination of Lie brackets of vector fields (i.e. whenever one has an expression in Lie brackets which becomes linear over smooth functions in all arguments), then one can compute the value of the corresponding tensor on tangent vectors using the extensions described in the proposition, and thus immediately get an interpretation of the tensor in terms of the curvature function κ.

3.3. The Nijenhuis tensor

Let $(M, HM, \tilde{J})$ be an elliptic partially integrable almost CR manifold. Then there is an obvious first candidate for a torsion type object, namely the Nijenhuis tensor $\tilde{N}\colon HM \times HM \to HM$ introduced in 2.1. As we have seen, $\tilde{N}$ is skew symmetric and conjugate linear (with respect to $\tilde{J}$) in both arguments, so its restriction to $H^+M \times H^+M$ and to $H^-M \times H^-M$ vanishes since both these subbundles are of complex rank one, so what remains is $\tilde{N}\colon H^+M \times H^-M \to HM$, which we can split as $\tilde{N}^+ + \tilde{N}^-$ according to the splitting of the values. Then both components are conjugate linear with respect to $\tilde{J}$ in both arguments, so with respect to J they both become sesquilinear, which fits well to two of the harmonic curvature components described above. In fact, vanishing of the parts of the Nijenhuis tensor is equivalent to vanishing of the corresponding irreducible components of the curvature function:

Proposition. *Let κ be the curvature function of the canonical Cartan connection. Then the restriction of $\kappa^{(1)}$ to $\mathfrak{g}^+_{-1} \times \mathfrak{g}^-_{-1}$ has values in $\mathfrak{g}^+_{-1}$ (respectively $\mathfrak{g}^-_{-1}$) if and only if $\tilde{N}^- = 0$ (respectively $\tilde{N}^+ = 0$). In particular, M is integrable and thus a CR manifold if and only if $\kappa^{(1)}$ vanishes on $\mathfrak{g}^+_{-1} \times \mathfrak{g}^-_{-1}$.*

Proof. For tangent vectors $\xi_x \in H_x^+M$ and $\eta_x \in H_x^-M$, the Nijenhuis tensor can be computed as

$$\tilde{N}(\xi_x,\eta_x) = [\xi,\eta](x) - [\tilde{J}\xi,\tilde{J}\eta](x) + \tilde{J}\big([\tilde{J}\xi,\eta](x) + [\xi,\tilde{J}\eta](x)\big),$$

for any extensions ξ, η to local smooth vector fields. Splitting the result into $\tilde{N}^+ + \tilde{N}^-$, we may replace the $\tilde{J}'s$ by plus or minus J. Denoting by $\pi_\pm \colon H_xM \to H_x^\pm M$ the canonical projections, we obtain the following expressions for $\tilde{N}^+(\xi_x,\eta_x)$ and $\tilde{N}^-(\xi_x,\eta_x)$

$$\pi_+\big([\xi,\eta](x) + [J\xi,J\eta](x)\big) - J\pi_+\big(-[J\xi,\eta](x) + [\xi,J\eta](x)\big)$$
$$\pi_-\big([\xi,\eta](x) + [J\xi,J\eta](x)\big) + J\pi_-\big(-[J\xi,\eta](x) + [\xi,\tilde{J}\eta](x)\big).$$

Let us start by discussing $\tilde{N}^+$. Choose a point $u \in \mathcal{G}$ with $p(u) = x$, and let X, $Y \in \mathfrak{g}_-$ be the elements corresponding to ξ_x and η_x as in lemma 3.2. Then $X \in \mathfrak{g}_{-1}^+$ and $Y \in \mathfrak{g}_{-1}^-$. By lemma 3.2 and complex bilinearity of the bracket on $\mathfrak{g}$, the element $[\xi,\eta](x) + [J\xi,J\eta](x)$ is simply given by $-T_up\cdot\big(\omega_u^{-1}(\kappa(X,Y) + \kappa(iX,iY))\big)$. Moreover, since $\omega_u^{-1}(\mathfrak{b})$ is killed by T_up, we may in this equation as well replace κ by $\kappa^{(1)}$, and since we know from 3.2 that the restriction of $\kappa^{(1)}$ to $\mathfrak{g}_{-1}^+ \times \mathfrak{g}_{-1}^-$ is sesquilinear, we may replace $\kappa^{(1)}(X,Y) + \kappa^{(1)}(iX,iY)$ by $2\kappa^{(1)}(X,Y)$. Thus, $\pi_+\big([\xi,\eta](x) + [J\xi,J\eta](x)\big)$ is obtained by projecting down ω_u^{-1} of the $\mathfrak{g}_{-1}^+$-component of $-2\kappa^{(1)}(X,Y)$. Similarly, one sees that $-J\pi_+\big(-[J\xi,\eta](x)+[\xi,J\eta](x)\big)$ is obtained by projecting down ω_u^{-1} of the $\mathfrak{g}_{-1}^+$-component of $-2i\kappa^{(1)}(iX,Y)$. But from 3.2 we know that this component is conjugate linear in the first variable, so we see that $\frac{1}{4}\tilde{N}^+(\xi_x,\eta_x)$ is obtained by projecting down the $\mathfrak{g}_{-1}^+$-component of $-\kappa^{(1)}(X,Y)$, so we see that vanishing of $\tilde{N}^+$ is equivalent to the fact that $\kappa^{(1)}(X,Y) \in \mathfrak{g}_{-1}^-$ for all $X \in \mathfrak{g}_{-1}^+$ and $Y \in \mathfrak{g}_{-1}^-$. Similarly, one deals with $\tilde{N}^-$. Q.E.D.

Remark. Note that embedded partially integrable almost CR structures (i.e. manifolds for which the almost CR structure comes from an embedding into a *complex* manifold of appropriate dimension) are automatically CR, so the restriction of $\kappa^{(1)}$ to $\mathfrak{g}_{-1}^+ \times \mathfrak{g}_{-1}^-$ vanishes automatically in the embedded case.

3.4. The other torsions of type $(1,1)$

The other two $(1,1)$-components in the homogeneous part of degree one of the curvature are even easier to interpret. Recall that by construction both H^+M and H^-M are isotropic with respect to the (complex bilinear) Levi-bracket. Consequently, for two sections ξ, $\eta \in \Gamma(H^+M)$, the Lie bracket $[\xi,\eta]$ is a section of HM, so we can project it to H^-M. Obviously, the result is bilinear over smooth functions, so there is a well defined tensorial map $T^+\colon \bigwedge^2_{\mathbb{R}} H^+M \to H^-M$ defined by $T^+(\xi,\eta) = \pi_-([\xi,\eta])$ for smooth sections ξ, η as above. Note that since H^+M is of complex rank one and T^+ is skew symmetric, it must be automatically totally real. Clearly T^+ vanishes identically if and only if the Lie bracket of two sections of H^+M is again a section of H^+M, i.e. if and only if H^+M is integrable. Similarly, we obtain a bundle map $T^-\colon \bigwedge^2 H^-M \to H^+M$ whose vanishing is equivalent to integrability of the bundle H^-M.

Proposition. *Let κ be the curvature function of the canonical Cartan connection. Then the restriction of $\kappa^{(1)}$ to $\bigwedge^2 \mathfrak{g}^+_{-1}$ (respectively to $\bigwedge^2 \mathfrak{g}^-_{-1}$) vanishes if and only if the subbundle $H^+M \subset TM$ (respectively $H^-M \subset TM$) is integrable.*

Proof. Consider tangent vectors ξ_x, $\eta_x \in H^+_xM$ and a point $u \in \mathcal{G}$ with $p(u) = x$. Then by definition the corresponding elements X, $Y \in \mathfrak{g}_-$ from lemma 3.2 lie in $\mathfrak{g}^+_{-1}$. Moreover, the extensions ξ, η provided by lemma 3.2 are sections of H^+M, and we have $[\xi,\eta](x) = -T_up\cdot\omega_u^{-1}(\kappa(X,Y))$, since $[X,Y] = 0$. Again, since vertical elements are killed by Tp, we may replace κ by $\kappa^{(1)}$ in this expression. Moreover, from 3.2 we know that $\kappa^{(1)}(X,Y) \in \mathfrak{g}^-_{-1}$ for X, $Y \in \mathfrak{g}^+_{-1}$, so the projection coincides with $\pi_-([\xi,\eta](x)) = T^+(\xi_x,\eta_x)$. Consequently, vanishing of the restriction of $\kappa^{(1)}$ to $\bigwedge^2 \mathfrak{g}^+_{-1}$ is equivalent to vanishing of T^+ and thus to integrability of H^+M. The other component is treated similarly. Q.E.D.

3.5. Torsions of type $(0,2)$

To interpret the remaining two components of $\kappa^{(1)}$ it is convenient (although not formally necessary) to construct first an almost complex structure J on M, which combines the almost complex structures J on HM and J^Q on QM. Using the canonical Cartan connection ω it is clear how to get such an extension. In fact, for each point $u \in \mathcal{G}$, we get an isomorphism $\mathfrak{g}_- \to T_{p(u)}M$ defined by $X \mapsto T_up\cdot\omega_u^{-1}(X)$ and thus a complex structure on T_xM, where $x = p(u)$. Moreover, changing the point u to $u\cdot g$ for $g \in B$, equivariancy of ω implies that the new isomorphism is given by composing the old one with $\mathrm{Ad}(g)$, which is complex linear, so the complex structure on T_xM is canonical. Clearly, this defines a smooth almost complex structure J on M. Moreover, this has the property that the bundle maps $H^\pm M \to TM$ and $TM \to QM$ are complex linear, since the structures on the other spaces are also induced by ω.

Now we can easily characterize this almost complex structure: To do this, consider an arbitrary almost complex structure $\hat{J}$ on M such that $H^{\pm}M \to TM$ and $q\colon TM \to QM$ are complex linear. For a vector field ξ on M and a section η of HM consider the expression $q([\hat{J}\xi,\eta]) - J^Q q([\xi,\eta])$. Since $q(\eta) = 0$ this is linear over smooth functions in η and since $q(\hat{J}\xi) = J^Q q(\xi)$ is also linear over smooth functions in ξ, so it defines a tensor $TM \times HM \to QM$. Moreover, for $\xi \in HM$, the tensor is given by $\mathcal{L}(J\xi,\eta) - J^Q(\mathcal{L}(\xi,\eta)) = 0$ by complex bilinearity of the Levi bracket. Hence we can factor to QM in the first variable and splitting $HM = H^+M \oplus H^-M$ we obtain two tensors $S^{\pm}\colon QM \times H^{\pm}M \to QM$. Note that these tensors by construction are conjugate linear in the first variable.

Proposition. *The almost complex structure J on M induced by the canonical Cartan connection ω is the unique almost complex structure on M such that the bundle maps $H^{\pm}M \to TM$ and $TM \to QM$ are complex linear and such that the tensors $S^{\pm}\colon QM \times H^{\pm}M \to QM$ induced by $(\xi,\eta) \mapsto q([J\xi,\eta]) - J^Q([\xi,\eta])$ are both conjugate linear in the second variable. Moreover, the restriction of $\kappa^{(1)}$ to $\mathfrak{g}_{-2} \times \mathfrak{g}^+_{-1}$ (respectively $\mathfrak{g}_{-2} \times \mathfrak{g}^-_{-1}$) vanishes if and only if S^+ (respectively S^-) is identically zero.*

Finally, vanishing of both S^+ and S^- is equivalent to integrability of the almost complex structure J on M.

Proof. Let us first verify that the almost complex structure induced by ω has the stated property. Take $\xi_x \in T_xM$ and $\eta_x \in H_xM$ and a point $u \in \mathcal{G}$ with $p(u) = X$. Let $X \in \mathfrak{g}_-$ and $Y \in \mathfrak{g}_{-1}$ be the corresponding elements from lemma 3.2, and ξ, η the extensions provided by lemma 3.2. Then $J\xi$ is the extension of $J_x\xi_x$ provided by lemma 3.2, so $[J\xi,\eta](x) = -T_up\cdot\omega_u^{-1}\big(\kappa(iX,Y)+[iX,Y]\big)$ and $[\xi,\eta](x) = -T_up\cdot\omega_u^{-1}\big(\kappa(X,Y) + [X,Y]\big)$. For the classes provided by q, only the $\mathfrak{g}_{-2}$-component of the result is significant, so we can again replace κ by $\kappa^{(1)}$, and we see that our tensor is given by taking the class of the image under T_up of

$$-\omega_u^{-1}\big(\kappa^{(1)}(iX_{-2},Y) + [iX,Y]_{-2} - i\kappa^{(1)}(X_{-2},Y) - i[X,Y]_{-2}\big),$$

where the subscripts -2 indicate the component in $\mathfrak{g}_{-2}$. Since the bracket in $\mathfrak{g}$ is complex bilinear, the bracket terms cancel. Moreover, from 3.2 we know that the restriction of $\kappa^{(1)}$ to $\mathfrak{g}_{-2} \times \mathfrak{g}_{-1}$ is conjugate linear in both variables. Now conjugate linearity in the first variable implies that the tensor is actually given by taking the class of the projection of $\omega_u^{-1}(2\kappa^{(1)}(X_{-2},Y))$, and conjugate linearity in the second variable then implies our claim. Moreover, the equivalent conditions for vanishing of components of $\kappa^{(1)}$ are obvious from this computation.

To prove the uniqueness of J, assume that $\hat{J}$ is another almost complex structure on M such that $H^{\pm}M \to TM$ and $TM \to QM$ are complex linear. For a vector field ξ on M we have by assumption $q(J\xi) = q(\hat{J}\xi)$, so there is a smooth section ξ' of HM such that $\hat{J}\xi = J\xi + \xi'$. But then $q([\hat{J}\xi, \eta]) = q([J\xi, \eta]) + \mathcal{L}(\xi', \eta)$ for all η, and the second term is complex linear in η. Since $\xi' \mapsto \mathcal{L}(\xi', \_)$ induces an isomorphism between HM and the bundle of complex linear maps $HM \to QM$ by non-degeneracy and complex bilinearity of the Levi bracket, the uniqueness follows.

The final statement is a little more subtle to prove. If both S^+ and S^- vanish, then from above we know that the restriction of $\kappa^{(1)}$ to $\mathfrak{g}_{-2} \times \mathfrak{g}_{-1}$ vanishes, so from 3.2 we conclude that $\kappa^{(1)}$ is totally real and has values in $\mathfrak{g}_{-1}$ only. Now take two elements ξ_x, $\eta_x \in T_xM$ and a point $u \in \mathcal{G}$ with $p(u) = x$, and let X, $Y \in \mathfrak{g}_-$ be the corresponding elements from lemma 3.2. Applying lemma 3.2 one sees that the value $N(\xi_x, \eta_x)$ of the Nijenhuis tensor of J on ξ_x and η_x is given by

$$-T_u p \cdot \omega_u^{-1}\big(\kappa(X,Y) - \kappa(iX,iY) + i\big(\kappa(iX,Y) + \kappa(X,iY)\big)\big),$$

which obviously equals $-4T_u p \cdot \omega_u^{-1}(\kappa_{0,2}(X,Y))$, where $\kappa_{0,2}$ denotes the component of κ which is conjugate linear in both arguments, and the terms involving brackets in $\mathfrak{g}$ do not show up since this bracket is complex bilinear. From above, we know that in this expression only components of κ of homogeneity ≥ 2 may enter. Moreover, since vertical elements are killed by Tp, we only have to consider components of $\kappa_{0,2}(X,Y)$ which lie in $\mathfrak{g}_-$. Now the homogeneous component of degree l of κ maps $\bigwedge^2_{\mathbb{R}} \mathfrak{g}_{-1}$ to $\mathfrak{g}_{l-2}$, $\mathfrak{g}_{-2} \otimes \mathfrak{g}_{-1}$ to $\mathfrak{g}_{l-3}$ and $\bigwedge^2_{\mathbb{R}} \mathfrak{g}_{-2}$ to $\mathfrak{g}_{l-4}$. Moreover, maps which are conjugate linear in both arguments automatically vanish on $\bigwedge^2 \mathfrak{g}_{-2}$ since $\mathfrak{g}_{-2}$ is of complex dimension one. Thus we see that the only relevant contribution to the above expression could come from the part of $\kappa^{(2)}_{0,2}$ which maps $\mathfrak{g}_{-2} \otimes \mathfrak{g}_{-1}$ to $\mathfrak{g}_{-1}$. The only other possible component of $\kappa^{(2)}_{0,2}$ maps $\bigwedge^2 \mathfrak{g}_{-1}$ to $\mathfrak{g}_0$.

Specialized to homogeneity two, the Bianchi identity from 3.1 tells us that $(\partial \circ \kappa^{(2)})(X,Y,Z)$ can be computed as

$$\sum_{\text{cycl}} \big(\kappa^{(1)}(\kappa^{(1)}(X,Y),Z) + (\omega^{-1}(Z) \cdot \kappa^{(1)})(X,Y)\big).$$

Since $\kappa^{(1)}$ has values in $\mathfrak{g}_{-1}$, the same must hold for $\partial \circ \kappa^{(2)}$ by this formula. Consequently, $\partial \circ \kappa^{(2)}$ has to vanish on $\mathfrak{g}_{-2} \otimes \bigwedge^2 \mathfrak{g}_{-1}$ since

∂ preserves homogeneities. Moreover, since we have observed in 3.1 that $\partial(L^{p,q}(\mathfrak{g}_-,\mathfrak{g})) \subset L^{p+1,q}(\mathfrak{g}_-,\mathfrak{g})$, we conclude that also $\partial \circ \kappa^{(2)}_{0,2}$ must vanish on $\mathfrak{g}_{-2} \otimes \bigwedge^2 \mathfrak{g}_{-1}$. For $X \in \mathfrak{g}_{-2}$ and Y, $Z \in \mathfrak{g}_{-1}$ we have $[X,Y] = [X,Z] = 0$ and $\kappa^{(2)}_{0,2}([Y,Z],X) = 0$ since $\kappa^{(2)}_{0,2}$ must vanish on $\bigwedge^2 \mathfrak{g}_{-2}$, and inserting the definition of ∂, we see that $(\partial \circ \kappa^{(2)}_{0,2})(X,Y,Z) = 0$ is equivalent to

$$[Y,\kappa^{(2)}_{0,2}(X,Z)] = [X,\kappa^{(2)}_{0,2}(Y,Z)] + [Z,\kappa^{(2)}_{0,2}(X,Y)].$$

Replacing X by iX the same equation must hold. On the other hand, doing that multiplies the left hand side by $-i$, the first term in the right hand side by i and the second term in the right hand side by $-i$, so we conclude that $[X,\kappa^{(2)}_{0,2}(Y,Z)] = 0$ and thus $[Y,\kappa^{(2)}_{0,2}(X,Z)] = [Z,\kappa^{(2)}_{0,2}(X,Y)]$ for all $X \in \mathfrak{g}_{-2}$ and Y, $Z \in \mathfrak{g}_{-1}$. But replacing Y by iY in this equation, the left hand side gets multiplied by i and the right hand side by $-i$, so we must have $[Y,\kappa^{(2)}_{0,2}(X,Z)] = 0$ for all X, Y, Z, and thus the restriction of $\kappa^{(2)}_{0,2}$ to $\mathfrak{g}_{-2} \otimes \mathfrak{g}_{-1}$ vanishes by non-degeneracy of the bracket and the result follows. Q.E.D.

3.6. Torsion free elliptic CR manifolds

We conclude the discussion of geometric interpretations of the torsion-type components of the curvature of the canonical Cartan connection by discussing the case where all torsion type components vanish simultaneously. In this case, since by 3.1 the lowest nonzero homogeneous component of κ must be harmonic, we immediately see that there are no nonzero components of homogeneity less than four, and moreover $\kappa^{(4)}$ is complex bilinear and its only nonzero components are $\mathfrak{g}_{-2} \otimes \mathfrak{g}^+_{-1} \to \mathfrak{g}^+_1$ and $\mathfrak{g}_{-2} \otimes \mathfrak{g}^-_{-1} \to \mathfrak{g}^-_1$. But indeed, much more can be said in this case:

Theorem. *Suppose that M is an elliptic partially integrable almost CR manifold of CR dimension and codimension two such that $\tilde{N}^\pm = T^\pm = S^\pm = 0$. Then the almost complex structure on $\mathcal{G}$ induced by ω is integrable, and the projection $p\colon \mathcal{G} \to M$ is holomorphic, so $\mathcal{G}$ is a holomorphic principal B-bundle over M. Moreover, the Cartan connection $\omega \in \Omega^1(\mathcal{G},\mathfrak{g})$ is a holomorphic $(1,0)$-form, so $(p\colon \mathcal{G} \to M,\omega)$ is a complex parabolic geometry of type $(PSL(3,\mathbb{C}),B)$. Conversely, any complex parabolic geometry of that type is torsion free when viewed as a real parabolic geometry.*

Proof. The Cartan connection ω defines a trivialization $T\mathcal{G} \cong \mathcal{G} \times \mathfrak{g}$ of the tangent bundle of $\mathcal{G}$, so since $\mathfrak{g}$ is a complex vector space, it induces an almost complex structure $J^{\mathcal{G}}$ on $\mathcal{G}$. The almost complex structure J on M was defined via ω, so it follows immediately that $p\colon \mathcal{G} \to M$ has complex linear derivative. Moreover, since $S^{\pm} = 0$, the almost complex structure J is integrable by proposition 3.4. To prove the first statement, we only have to show that torsion freeness implies integrability of $J^{\mathcal{G}}$. For $X \in \mathfrak{g}$, let us denote by $\tilde{X} \in \mathfrak{X}(\mathcal{G})$ the vector field $\omega^{-1}(X)$. The definition of the curvature function together with the fact that the curvature of ω is horizontal easily implies that $\omega([\tilde{X},\tilde{Y}]) = -\kappa(X_-,Y_-) - [X,Y]$, where the subscript $-$ denotes the $\mathfrak{g}_-$-component. Using this and the fact that the bracket in $\mathfrak{g}$ is complex bilinear, one now concludes that the Nijenhuis-tensor on $\mathcal{G}$ evaluated on $\tilde{X}$ and $\tilde{Y}$ is just $-\frac{1}{4}\kappa_{0,2}(X_-,Y_-)$, where $\kappa_{0,2}$ denotes the component of the curvature which is conjugate linear in both arguments.

Once we have shown integrability of $J^{\mathcal{G}}$, we know that ω is a smooth $(1,0)$-form, and holomorphicity of this form is equivalent to $\bar{\partial}\omega = 0$, i.e. to $d\omega$ being a $(2,0)$-form. (Mistakenly, it was claimed in [11] that holomorphicity of ω is trivially satisfied.) But since the bracket on $\mathfrak{g}$ is complex bilinear, the fact that $d\omega$ is of type $(2,0)$ is equivalent to $\kappa(u)$ being complex bilinear for any $u \in G$. Hence proving complex bilinearity of κ suffices to prove the theorem. One possibility to prove this is to eliminate first the possibilities of having a nontrivial component of type $(0,2)$ in κ and then eliminating possible $(1,1)$-components using a pretty involved analysis of the Bianchi identity, similar to the proof of proposition 3.5. Following an idea of [1] on strengthening the Bianchi identity, there is a neat way around all that using fairly heavy tools:

Since the curvature function κ is an equivariant map $\mathcal{G} \to \bigwedge^2 \mathfrak{g}_- \otimes \mathfrak{g}$, it can be viewed as an element of $\Omega^2(M,\mathcal{A})$, where $\mathcal{A} = \mathcal{G} \times_B \mathfrak{g}$ is the adjoint tractor bundle. In [4], the twisted exterior derivative $d_{\mathfrak{g}} : \Omega^i(M,\mathcal{A}) \to \Omega^{i+1}(\mathcal{A})$ is constructed. On the other hand, the Cartan connection induces a principal connection $\tilde{\omega}$ on $\mathcal{G} \times_B G$ and since $\mathcal{A}$ can be viewed as associated to that bundle, we get an induced covariant exterior derivative $d^{\tilde{\omega}}$ between the same spaces. In [4, section 2] it is shown that in the torsion free case these two operators coincide. Moreover, the curvature of $\tilde{\omega}$ is also given by κ, so the Bianchi identity for principal connections implies $d^{\tilde{\omega}}(\kappa) = 0$. Now since ω is normal, the curvature κ has ∂^*-closed values and the harmonic part κ_0 may simply be viewed as the image of κ under the bundle map π_H corresponding to the projection $\ker(\partial^*) \to \ker(\partial^*)/\operatorname{im}(\partial^*)$, see [4, section 2]. This projection is splitted by an invariant differential operator L, whose construction is one of the main achievements of the paper [4]. In [4, Lemma 2.7] it is shown that this operator is characterized by $\pi_H(L(s)) = s$ and $\partial^* \circ d_{\mathfrak{g}} \circ L = 0$, which implies that $\kappa = L(\kappa_0)$ in the torsion free case.

But then [4, Theorem 2.5] shows that κ has values in the B-submodule generated by the values of κ_0, and since κ_0 has only complex bilinear values, the generated B-module consists of complex bilinear maps only. Q.E.D.

Corollary. *Let $(M, HM, \tilde{J})$ be a torsion free elliptic CR manifold of CR dimension and codimension two. Then M is a real analytic manifold and the subbundle $HM \subset TM$ and the endomorphism $\tilde{J}\colon HM \to HM$ are real analytic. In particular, M is automatically locally embeddable.*

Proof. By proposition 3.5, M is a complex manifold, and thus in particular real analytic. Moreover, by the last theorem, the Cartan bundle $p\colon \mathcal{G} \to M$ is a holomorphic principal bundle and the Cartan connection $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$ is holomorphic. Hence, $\omega^{-1}(\mathfrak{g}_{-1} \oplus \cdots \oplus \mathfrak{g}_2)$ is a real analytic subbundle of $T\mathcal{G}$. By construction, this projects onto the subbundle $HM \subset TM$, which thus also is real analytic. Similarly, the subbundles $H^{\pm}M$ and the almost complex structure J is analytic, since they all are induced from ω. Since $\tilde{J}$ is obtained by keeping J on H^-M and flipping it on H^+M, it is real analytic, too. Embeddability then follows from proposition 2.2. Q.E.D.

References

[1] D. M. J. Calderbank and T. Diemer, *Differential invariants and curved Bernstein-Gelfand-Gelfand sequences*, J. Reine Angew. Math., **537** (2001), 67–103.

[2] A. Čap and M. G. Eastwood, Some special geometry in dimension six, ESI preprint 851, electronically available at http://www.esi.ac.at.

[3] A. Čap and H. Schichl, *Parabolic Geometries and Canonical Cartan Connections*, Hokkaido Math. J., **29**, No. 3 (2000), 453–505.

[4] A. Čap, J. Slovák and V. Souček, *Bernstein-Gelfand-Gelfand sequences*, Ann. of Math., **154**, No. 1 (2001), 97–113.

[5] S. S. Chern and J. Moser, *Real hypersurfaces in complex manifolds*, Acta Math., **133** (1974), 219–271.

[6] H. Jacobowitz, *An Introduction to CR Structures*, AMS Mathematical Surveys and Monographs, 32, 1990.

[7] B. Kostant, *Lie algebra cohomology and the generalized Borel-Weil theorem*, Ann. Math., **74**, No. 2 (1961), 329–387.

[8] R. I. Mizner, *Almost CR structures, f-structures, almost product structures and associated connections*, Rocky Mt. J. Math., **23**, No. 4 (1993), 1337–1359.

[9] T. Morimoto, *Geometric structures on filtered manifolds*, Hokkaido Math. J., **22** (1993), 263–347.

[10] G. Schmalz, *Remarks on CR-manifolds of codimension* 2 *in* $\mathbb{C}^4$, Proceedings Winter School Geometry and Physics, Srní, 1998, Supp. Rend. Circ. Matem. Palermo, **59** (1999), 171–180.

[11] G. Schmalz and J. Slovák, *The geometry of hyperbolic and elliptic CR-manifolds of codimension two*, Asian J. Math., **4**, No. 3 (2000), 565–598.

[12] N. Tanaka, *On non-degenerate real hypersurfaces, graded Lie algebras and Cartan connections*, Japan J. Math., **2** (1976), 131–190.

[13] N. Tanaka, *On the equivalence problem associated with simple graded Lie algebras*, Hokkaido Math. J., **8** (1979), 23–84.

[14] K. Yamaguchi, *Differential Systems Associated with Simple Graded Lie Algebras*, Advanced Stud. in Pure Math., **22** (1993), 413–494.

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Some Remarks on the Infinitesimal Rigidity of the Complex Quadric

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Introduction

Let (X, g) be a compact Riemannian symmetric space. We say that a symmetric 2-form h on X satisfies the zero-energy condition if for all closed geodesics γ of X the integral

$$\int_\gamma h = \int_0^L h(\dot\gamma(s), \dot\gamma(s))ds$$

of h over γ vanishes, where $\dot\gamma(s)$ is the tangent vector to the geodesic γ parametrized by its arc-length and L is the length of γ. A Lie derivative of the metric g always satisfies the zero-energy condition. The space (X, g) said to be infinitesimally rigid if the only symmetric 2-forms on X satisfying the zero-energy condition are the Lie derivatives of the metric g.

Michel introduced the notion of infinitesimal rigidity in the context of the Blaschke conjecture, and proved that the real projective spaces $\mathbb{RP}^n$, with $n \geq 2$, and the flat tori of dimension ≥ 2 are infinitesimally rigid (see [17], [18] and [2]). Michel and Tsukamoto demonstrated the infinitesimal rigidity of the complex projective space $\mathbb{CP}^n$ of dimension $n \geq 2$ (see [17], [21], [6] and [7]); in fact, they proved that all the projective spaces which are not isometric to a sphere are infinitesimally rigid.

In [7] and [9], we showed that the complex quadric Q_n of dimension n is infinitesimally rigid when $n \geq 4$. In the monograph [12], we shall give a complete proof of the infinitesimal rigidity of the complex quadric Q_3 of dimension 3, which relies on the Guillemin rigidity of the Grassmannian of 2-planes in $\mathbb{R}^{n+2}$ proved in [10] and on results of Tela Nlenvo [20].

In this note, we present outlines of some new proofs of the infinitesimal rigidity of the complex quadric Q_n of dimension $n \geq 4$; the

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complete proofs shall appear in [12]. In particular, we show that the infinitesimal rigidity of the quadric Q_3 implies that all the quadrics Q_n, with $n \geq 4$, are infinitesimally rigid. The new proof of the infinitesimal rigidity of the complex quadric Q_n of dimension $n \geq 5$ presented here is quite different from the one found in [7] and follows some of the lines of the proof for the infinitesimal rigidity of the complex quadric Q_4 given in [9].

§1. Symmetric spaces

Let (X, g) be a Riemannian manifold. We denote by T and T^* its tangent and cotangent bundles. By $\bigotimes^k T^*$, $S^l T^*$, $\bigwedge^j T^*$, we shall mean the k-th tensor product, the l-th symmetric product and the j-th exterior product of the vector bundle T^*. If α, $\beta \in T^*$, we identify the symmetric product $\alpha \cdot \beta$ with the element $\alpha \otimes \beta + \beta \otimes \alpha$ of $\bigotimes^2 T^*$. If E is a vector bundle over X, we denote by $E_{\mathbb{C}}$ its complexification, by $\mathcal{E}$ the sheaf of sections of E over X and by $C^\infty(E)$ the space of global sections of E over X. If ξ is a vector field on X and β is a section of $\bigotimes^k T^*$ over X, we denote by $\mathcal{L}_\xi \beta$ the Lie derivative of β along ξ. Let $g^\sharp \colon T^* \to T$ be the isomorphism determined by the metric g.

Let $B = B_X$ be the sub-bundle of $\bigwedge^2 T^* \otimes \bigwedge^2 T^*$ consisting of those tensors $u \in \bigwedge^2 T^* \otimes \bigwedge^2 T^*$ satisfying the first Bianchi identity

$$u(\xi_1, \xi_2, \xi_3, \xi_4) + u(\xi_2, \xi_3, \xi_1, \xi_4) + u(\xi_3, \xi_1, \xi_2, \xi_4) = 0,$$

for all ξ_1, ξ_2, ξ_3, $\xi_4 \in T$. Let H denote the sub-bundle of $T^* \otimes B$ consisting of those tensors $v \in T^* \otimes B$ which satisfy the relation

$$v(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5) + v(\xi_2, \xi_3, \xi_1, \xi_4, \xi_5) + v(\xi_3, \xi_1, \xi_2, \xi_4, \xi_5) = 0,$$

for all ξ_1, ξ_2, ξ_3, ξ_4, $\xi_5 \in T$.

Let

$$\mathrm{Tr} \colon S^2 T^* \to \mathbb{R}, \qquad \mathrm{Tr} \colon \bigwedge^2 T^* \otimes \bigwedge^2 T^* \to \bigotimes^2 T^*$$

be the trace mappings defined by

$$\mathrm{Tr}\, h = \sum_{j=1}^n h(t_j, t_j), \qquad (\mathrm{Tr}\, u)(\xi, \eta) = \sum_{j=1}^n u(t_j, \xi, t_j, \eta),$$

for $h \in S^2 T^*_x$, $u \in \bigwedge^2 T^* \otimes \bigwedge^2 T^*_x$ and ξ, $\eta \in T_x$, where $x \in X$ and $\{t_1, \ldots, t_n\}$ is an orthonormal basis of T_x. It is easily seen that

$$\mathrm{Tr}\, B \subset S^2 T^*.$$

We denote by $S_0^2T^*$ the sub-bundle of S^2T^* equal to the kernel of the trace mapping $\mathrm{Tr}\colon S^2T^* \to \mathbb{R}$.

We now introduce various differential operators associated to the Riemannian manifold (X, g). First, let ∇ be the Levi-Civita connection of (X, g). The Killing operator

$$D_0\colon \mathcal{T} \to S^2\mathcal{T}^*$$

of (X, g) sends $\xi \in \mathcal{T}$ into $\mathcal{L}_\xi g$. The Killing vector fields of (X, g) are the solutions $\xi \in C^\infty(T)$ of the equation $D_0\xi = 0$. Consider the first-order differential operator

$$\mathrm{div}\colon S^2\mathcal{T}^* \to \mathcal{T}^*$$

and the Laplacian

$$\overline{\Delta}\colon S^2\mathcal{T}^* \to S^2\mathcal{T}^*$$

defined by

$$(\mathrm{div}\, h)(\xi) = -\sum_{j=1}^{n}(\nabla h)(t_j, t_j, \xi),$$

$$(\overline{\Delta}h)(\xi, \eta) = -\sum_{j=1}^{n}(\nabla^2 h)(t_j, t_j, \xi, \eta),$$

for $h \in C^\infty(S^2T^*)$, ξ, $\eta \in T_x$, where $x \in X$ and $\{t_1, \dots, t_n\}$ is an orthonormal basis of T_x. The formal adjoint of D_0 is equal to $2g^\sharp \cdot \mathrm{div}\colon S^2\mathcal{T}^* \to \mathcal{T}$. Since D_0 is elliptic, if X is compact, we therefore have the orthogonal decomposition

$$C^\infty(S^2T^*) = D_0C^\infty(T) \oplus \{h \in C^\infty(S^2T^*) \mid \mathrm{div}\, h = 0\} \tag{1.1}$$

(see [1]).

Let $\mathcal{R}(h)$ be the Riemann curvature tensor, as defined in [5, §4], and $\mathrm{Ric}(h)$ be the Ricci tensor of a metric h on X, which is are sections of B and S^2T^*, respectively. We set $R = \mathcal{R}(g)$ and $\mathrm{Ric} = \mathrm{Ric}(g)$; we have $\mathrm{Ric} = -\mathrm{Tr}\, R$. We also consider the curvature tensor $\widetilde{R}$ which is the section of $\bigwedge^2 T^* \otimes T^* \otimes T$ related to R by

$$g(\widetilde{R}(\xi_1, \xi_2, \xi_3), \xi_4) = R(\xi_1, \xi_2, \xi_3, \xi_4),$$

for ξ_1, ξ_2, ξ_3, $\xi_4 \in T$. Let

$$\mathcal{R}'_g\colon S^2\mathcal{T}^* \to \mathcal{B}$$

be the linear differential operator of order 2 which is the linearization along g of the non-linear operator $h \mapsto \mathcal{R}(h)$, where h is a Riemannian metric on X. The invariance of the operator $h \mapsto \mathcal{R}(h)$ leads us to the formula

$$\mathcal{R}'_g(\mathcal{L}_\xi g) = \mathcal{L}_\xi R, \tag{1.2}$$

for all $\xi \in \mathcal{T}$.

We now suppose that (X, g) is an Einstein manifold and we write $\mathrm{Ric} = \lambda g$, with $\lambda \in \mathbb{R}$. We consider the morphism of vector bundles $L\colon S^2T^* \to S^2T^*$ determined by

$$L(\alpha \cdot \beta)(\xi, \eta) = 2\big(R(\xi, g^\sharp\alpha, \eta, g^\sharp\beta) + R(\xi, g^\sharp\beta, \eta, g^\sharp\alpha)\big),$$

for α, $\beta \in T^*$ and ξ, $\eta \in T$, and the Lichnerowicz Laplacian

$$\Delta\colon S^2\mathcal{T}^* \to S^2\mathcal{T}^*$$

of [16] defined by

$$\Delta h = \overline{\Delta} h + 2\lambda h + Lh,$$

for $h \in S^2\mathcal{T}^*$. If X is compact, in [1] Berger-Ebin define the space $E(X)$ of infinitesimal Einstein deformations of the metric g by

$$E(X) = \{h \in C^\infty(S^2T^*) \mid \operatorname{div} h = 0,\ \operatorname{Tr} h = 0,\ \Delta h = 2\lambda h\}$$

(see also Koiso [14]); by definition, the space $E(X)$ is contained in an eigenspace of the Lichnerowicz Laplacian Δ, which is a determined elliptic operator, and is therefore finite-dimensional.

For the remainder of this section, we shall suppose that (X, g) is a connected locally symmetric space. We consider the sub-bundle $\tilde{B} = \tilde{B}_X$ of B, which is the infinitesimal orbit of the curvature and whose fiber at $x \in X$ is

$$\tilde{B}_x = \{(\mathcal{L}_\xi R)(x) \mid \xi \in \mathcal{T}_x \text{ with } (\mathcal{L}_\xi g)(x) = 0\}.$$

We denote by $\alpha\colon B \to B/\tilde{B}$ the canonical projection and we consider the second-order differential operator

$$D_1\colon S^2\mathcal{T}^* \to \mathcal{B}/\tilde{\mathcal{B}}$$

introduced in [5] and determined by

$$(D_1 h)(x) = \alpha(\mathcal{R}'_g(h - \mathcal{L}_\xi g))(x),$$

for $x \in X$ and $h \in S^2\mathcal{T}^*_x$, where ξ is an element of $\mathcal{T}_x$ satisfying $h(x) = (\mathcal{L}_\xi g)(x)$. Using (1.2), it is easily seen that this operator is well-defined and that

$$D_1 \cdot D_0 = 0.$$

Thus we may consider the complex

$$C^\infty(T) \xrightarrow{D_0} C^\infty(S^2T^*) \xrightarrow{D_1} C^\infty(B/\tilde{B}). \tag{1.3}$$

In [5] and [12], we prove the following result:

Theorem 1.1. *Suppose that (X, g) is a symmetric space of compact type. If the equality*

$$H \cap (T^* \otimes \tilde{B}) = \{0\} \tag{1.4}$$

holds, the sequence (1.3) *is exact.*

If (X, g) has constant curvature, according to [5] we have

$$\tilde{B} = \{0\}; \tag{1.5}$$

in this case, the operator D_1 is equal to the one introduced by Calabi [3].

Let Y be a connected totally geodesic submanifold of X; we denote by i the natural imbedding of Y into X. Let $g_Y = i^*g$ be the Riemannian metric on Y induced by g. Then (Y, g_Y) is a connected locally symmetric space. For $x \in Y$, we consider the mapping $i^*\colon B_x \to B_{Y,x}$; in [7] and [12], we show that

$$i^*\tilde{B}_x \subset \tilde{B}_{Y,x}.$$

If Y has constant curvature, by (1.5) we know that $\tilde{B}_Y = \{0\}$, and so we infer that

$$i^*\tilde{B} = \{0\}. \tag{1.6}$$

The following lemma is proved in [12] (see also Lemma 1.2 of [7]).

Lemma 1.1. *Assume that (X, g) is a connected locally symmetric space. Let Y, Z be totally geodesic submanifolds of X; suppose that Z is a submanifold of Y of constant curvature. Let h be a section of S^2T^* over X. Let $x \in Z$ and u be an element of B_x such that $(D_1h)(x) = \alpha u$. If the restriction of h to the submanifold Y is a Lie derivative of the metric on Y induced by g, then the restriction of u to the submanifold Z vanishes.*

§2. Criteria for infinitesimal rigidity

Let (X, g) be a compact locally symmetric space. As we remarked in the introduction, if ξ is a vector field on X, the symmetric 2-form $\mathcal{L}_\xi g$ on X satisfies the zero-energy condition. From this fact and the decomposition (1.1), we obtain:

Proposition 2.1. *Let X be a compact locally symmetric space. Assume that any symmetric 2-form h, which satisfies the zero-energy condition and the relation* $\operatorname{div} h = 0$, *vanishes. Then the space X is infinitesimally rigid.*

We now assume that (X, g) is a symmetric space of compact type. Then there is a Riemannian symmetric pair (G, K) of compact type, where G is a compact, connected semi-simple Lie group and K is a closed subgroup of G such that the space X is isometric to the homogeneous space G/K endowed with a G-invariant metric. We identify X with G/K.

Let $\mathcal{F}$ be a family of closed connected totally geodesic surfaces of X which is invariant under the group G. Then the set $N_{\mathcal{F}}$ consisting of those elements of B, which vanish when restricted to the submanifolds belonging to $\mathcal{F}$, is a sub-bundle of B. According to formula (1.6), we see that

$$\tilde{B} \subset N_{\mathcal{F}};$$

we shall identify $N_{\mathcal{F}}/\tilde{B}$ with a sub-bundle of $B/\tilde{B}$. If $\beta\colon B/\tilde{B} \to B/N_{\mathcal{F}}$ is the canonical projection, we consider the differential operator

$$D_{1,\mathcal{F}} = \beta D_1 : S^2\mathcal{T}^* \to \mathcal{B}/\mathcal{N}_{\mathcal{F}}.$$

Let $\mathcal{F}'$ be a family of closed connected totally geodesic submanifolds of X. We denote by $\mathcal{L}(\mathcal{F}')$ the subspace of $C^\infty(S^2T^*)$ consisting of all symmetric 2-forms h satisfying the following condition: for all submanifolds $Z \in \mathcal{F}'$, the restriction of h to Z is a Lie derivative of the metric of Z induced by g. If every submanifold of X belonging to $\mathcal{F}'$ is infinitesimally rigid, then a symmetric 2-form h on X satisfying the zero-energy condition belongs to $\mathcal{L}(\mathcal{F}')$; indeed, the restriction of h to a submanifold $Z \in \mathcal{F}'$ also satisfies the zero-energy condition.

From Lemma 1.1, we obtain:

Proposition 2.2. *Let (X, g) be a symmetric space of compact type. Let $\mathcal{F}$ be a family of closed connected totally geodesic surfaces of X which is invariant under the group G, and let $\mathcal{F}'$ be a family of closed connected totally geodesic submanifolds of X. Assume that each surface of X belonging to $\mathcal{F}$ is contained in a submanifold of X belonging to $\mathcal{F}'$. A symmetric 2-form h on X belonging to $\mathcal{L}(\mathcal{F}')$ satisfies the relation $D_{1,\mathcal{F}}h = 0$.*

Theorem 2.1. *Let (X, g) be a symmetric space of compact type. Let $\mathcal{F}$ be a family of closed connected totally geodesic surfaces of X which is invariant under the group G, and let $\mathcal{F}'$ be a family of closed connected totally geodesic submanifolds of X. Assume that every submanifold of*

X belonging to $\mathcal{F}'$ is infinitesimally rigid; assume that each surface of X belonging to $\mathcal{F}$ is contained in a submanifold of X belonging to $\mathcal{F}'$. Suppose that the relation (1.4) *and the equality*

$$N_{\mathcal{F}} = \tilde{B} \tag{2.1}$$

hold. Then the symmetric space X is infinitesimally rigid.

Proof. Let h be a symmetric 2-form h on X satisfying the zero-energy condition. According to our hypothesis on the family $\mathcal{F}'$, we know that h belongs to $\mathcal{L}(\mathcal{F}')$. From Proposition 2.1, we obtain the relation $D_{1,\mathcal{F}}h = 0$. According to the equality (2.1), we therefore see that $D_1 h = 0$. By the relation (1.4) and Theorem 1.1, the sequence (1.3) is exact, and so h is a Lie derivative of the metric g.

We now assume that (X, g) is an irreducible symmetric space of compact type; then X is an Einstein manifold and we have $\mathrm{Ric} = \lambda g$, where λ is a positive real number. The following result appears in [12].

Theorem 2.2. *Let (X, g) be an irreducible symmetric space of compact type. Let $\mathcal{F}$ be a family of closed connected totally geodesic surfaces of X which is invariant under the group G, and let $\mathcal{F}'$ be a family of closed connected totally geodesic submanifolds of X. Let E be a G-invariant sub-bundle of $S^2_0T^*$. Assume that each surface of X belonging to $\mathcal{F}$ is contained in a submanifold of X belonging to $\mathcal{F}'$, and suppose that the relation*

$$\mathrm{Tr}\, N_{\mathcal{F}} \subset E \tag{2.2}$$

holds. Let h be a symmetric 2-form on X satisfying the relations

$$\mathrm{div}\, h = 0, \qquad D_{1,\mathcal{F}}h = 0.$$

Then we may write

$$h = h_1 + h_2,$$

where h_1 is an element of $E(X)$ and h_2 is a section of E; moreover, if h also satisfies the zero-energy condition, we may require that h_1 and h_2 satisfy the zero-energy condition.

Proof. Since $\mathrm{Tr}\, E = \{0\}$ and since the relation (2.2) holds, by Lemma 2.1 of [11], with $N = N_{\mathcal{F}}$, we see that $\mathrm{Tr}\, h = 0$ and that

$$\Delta h - 2\lambda h \in C^\infty(E).$$

A variant of Proposition 4.2 of [11], with $\mu = 2\lambda$, gives us the desired result.

§3. The complex quadric

We suppose that X is the complex quadric Q_n, with $n \geq 2$, which is the complex hypersurface of complex projective space $\mathbb{CP}^{n+1}$ defined by the homogeneous equation

$$\zeta_0^2 + \zeta_1^2 + \cdots + \zeta_{n+1}^2 = 0,$$

where $\zeta = (\zeta_0, \zeta_1, \ldots, \zeta_{n+1})$ is the standard complex coordinate system of $\mathbb{C}^{n+2}$. Let g be the Kähler metric on X induced by the Fubini-Study metric $\tilde{g}$ on $\mathbb{CP}^{n+1}$ of constant holomorphic curvature 4. We denote by J the complex structure of X or of $\mathbb{CP}^{n+1}$.

The group $SU(n+2)$ acts on $\mathbb{C}^{n+2}$ and $\mathbb{CP}^{n+1}$ by holomorphic isometries. Its subgroup $G = SO(n+2)$ leaves the submanifold X of $\mathbb{CP}^{n+1}$ invariant; in fact, the group G acts transitively and effectively on the Riemannian manifold (X, g) by holomorphic isometries. It is easily verified that X is isometric to the homogeneous space

$$SO(n+2)/SO(2) \times SO(n)$$

of the group $SO(n+2)$, which is a Hermitian symmetric space of compact type; when $n \geq 3$, this space is irreducible. We also know that (X, g) is an Einstein manifold; its Ricci tensor is given by

$$\mathrm{Ric} = 2ng. \tag{3.1}$$

We now recall some results of Smyth [19]. The second fundamental form C of the complex hypersurface X of $\mathbb{CP}^{n+1}$ is a symmetric 2-form with values in the normal bundle of X in $\mathbb{CP}^{n+1}$. We denote by S the bundle of unit vectors of this normal bundle.

Let x be a point of X and ν be an element of S_x. We consider the element h_ν of $S^2T_x^*$ defined by

$$h_\nu(\xi, \eta) = \tilde{g}(C(\xi, \eta), \nu),$$

for all ξ, $\eta \in T_x$. Since $\{\nu, J\nu\}$ is an orthonormal basis for the fiber of the normal bundle of X in $\mathbb{CP}^{n+1}$ at the point x, we see that

$$C(\xi, \eta) = h_\nu(\xi, \eta)\nu + h_{J\nu}(\xi, \eta)J\nu,$$

for all ξ, $\eta \in T_x$. If μ is another element of S_x, we have

$$\mu = \cos\theta \cdot \nu + \sin\theta \cdot J\nu, \tag{3.2}$$

with $\theta \in \mathbb{R}$. We consider the symmetric endomorphism K_ν of T_x determined by

$$h_\nu(\xi, \eta) = g(K_\nu \xi, \eta),$$

for all ξ, $\eta \in T_x$. Since our manifolds are Kähler, we have

$$C(\xi, J\eta) = JC(\xi, \eta),$$

for all ξ, $\eta \in T_x$; from this relation, we deduce the equalities

$$K_{J\nu} = JK_\nu = -K_\nu J. \tag{3.3}$$

It follows that h_ν and $h_{J\nu}$ are linearly independent. By (3.3), we see that h_ν belongs to $(S^2T^*)^-$. If μ is the element (3.2) of S_x, it is easily verified that

$$K_\mu = \cos\theta \cdot K_\nu + \sin\theta \cdot JK_\nu. \tag{3.4}$$

From the Gauss equation, the expression for the Riemann curvature tensor of $\mathbb{CP}^{n+1}$ (endowed with the metric $\tilde{g}$) and the relation (3.3), we obtain the equality

$$\begin{aligned} \tilde{R}(\xi,\eta)\zeta = g(\eta,\zeta)\xi - g(\xi,\zeta)\eta + g(J\eta,\zeta)J\xi - g(J\xi,\zeta)J\eta \\ - 2g(J\xi,\eta)J\zeta + g(K_\nu\eta,\zeta)K_\nu\xi - g(K_\nu\xi,\zeta)K_\nu\eta \\ + g(JK_\nu\eta,\zeta)JK_\nu\xi - g(JK_\nu\xi,\zeta)JK_\nu\eta, \end{aligned} \tag{3.5}$$

for all ξ, η, $\zeta \in T_x$. From (3.3), we infer that the trace of the endomorphism K_ν of T_x vanishes. According to this last remark and formulas (3.3) and (3.5), we see that

$$\mathrm{Ric}(\xi, \eta) = -2g(K_\nu^2\xi, \eta) + 2(n+1)g(\xi, \eta),$$

for all ξ, $\eta \in T_x$. From (3.1), it follows that K_ν is an involution. We call K_ν the *real structure* of the quadric associated to the unit normal ν.

We denote by T_ν^+ and T_ν^- the eigenspaces of K_ν corresponding to the eigenvalues $+1$ and -1, respectively. Then by (3.3), we infer that J induces isomorphisms of T_ν^+ onto T_ν^- and of T_ν^- onto T_ν^+, and that

$$T_x = T_\nu^+ \oplus T_\nu^- \tag{3.6}$$

is an orthogonal decomposition. If ϕ is an element of the group G, we have

$$C(\phi_*\xi, \phi_*\eta) = \phi_*C(\xi, \eta),$$

for all ξ, $\eta \in T$. Thus, if μ is the tangent vector $\phi_*\nu$ belonging to $S_{\phi(x)}$, we see that

$$h_\mu(\phi_*\xi, \phi_*\eta) = h_\nu(\xi, \eta),$$

for all ξ, $\eta \in T_x$, and hence that

$$K_\mu\phi_* = \phi_*K_\nu \tag{3.7}$$

on T_x. Therefore ϕ induces isomorphisms

$$\phi_*\colon T^+_\nu \to T^+_\mu, \qquad \phi_*\colon T^-_\nu \to T^-_\mu.$$

We now decompose the homogeneous bundle S^2T^* of symmetric 2-forms on X into G-invariant sub-bundles following [8]. The complex structure of X induces a decomposition

$$S^2T^* = (S^2T^*)^+ \oplus (S^2T^*)^-$$

of the bundle S^2T^*, where $(S^2T^*)^+$ is the sub-bundle of Hermitian forms and $(S^2T^*)^-$ is the sub-bundle of skew-Hermitian forms. We consider the sub-bundle L of $(S^2T^*)^-$ introduced in [8], whose fiber at $x \in X$ is equal to

$$L_x = \{h_\mu \mid \mu \in S_x\};$$

according to (3.4), this fiber L_x is generated by the elements h_ν and $h_{J\nu}$ and so the sub-bundle L of $(S^2T^*)^-$ is of rank 2. We denote by $(S^2T^*)^{-\perp}$ the orthogonal complement of L in $(S^2T^*)^-$.

For $h \in (S^2T^*)^+_x$, we define an element $K_\nu(h)$ of $S^2T^*_x$ by

$$K_\nu(h)(\xi,\eta) = h(K_\nu\xi, K_\nu\eta),$$

for all ξ, $\eta \in T_x$. Using (3.3) and (3.5), we see that $K_\nu(h)$ belongs to $(S^2T^*)^+$ and does not depend on the choice of the unit normal ν. We thus obtain a canonical involution of $(S^2T^*)^+$ over all of X, which gives us the orthogonal decomposition

$$(S^2T^*)^+ = (S^2T^*)^{++} \oplus (S^2T^*)^{+-}$$

into the direct sum of the eigenbundles $(S^2T^*)^{++}$ and $(S^2T^*)^{+-}$ corresponding to the eigenvalues $+1$ and -1, respectively, of this involution. We easily see that

$$(S^2T^*)^{++}_x = \{h \in (S^2T^*)^+_x \mid h(\xi, J\eta) = 0, \text{ for all } \xi,\eta \in T^+_\nu\},$$
$$(S^2T^*)^{+-}_x = \{h \in (S^2T^*)^+_x \mid h(\xi, \eta) = 0, \text{ for all } \xi,\eta \in T^+_\nu\}.$$

The metric g is a section of $(S^2T^*)^{++}$ and generates a line bundle $\{g\}$, whose orthogonal complement in $(S^2T^*)^{++}$ is the sub-bundle $(S^2T^*)^{++}_0$ consisting of the traceless symmetric tensors of $(S^2T^*)^{++}$. We thus obtain the G-invariant orthogonal decomposition

$$S^2T^* = L \oplus (S^2T^*)^{-\perp} \oplus \{g\} \oplus (S^2T^*)^{++}_0 \oplus (S^2T^*)^{+-}; \tag{3.8}$$

using the relation (3.7), we easily see that this decomposition is G-invariant.

Let x_0 be a fixed point of X and let K be the subgroup of G equal to the isotropy group of the point x_0. Let $\mathfrak{g}$ denote the complexification of the Lie algebra $\mathfrak{so}(n+2)$ of G. The fibers at x_0 of the sub-bundles of S^2T^* appearing in the decomposition (3.8) and their complexifications are K-modules.

We write

$$E_1 = (S^2T^*)^{++}_{0,\mathbb{C}}, \qquad E_2 = L_{\mathbb{C}}, \qquad E_3 = (S^2T^*)^{-\perp}_{\mathbb{C}}.$$

In [12], we prove the following result:

Lemma 3.1. *Let X be the complex quadric Q_n, with $n \geq 3$.*

(i) *We have*

$$\mathrm{Hom}_K(\mathfrak{g}, E_{j,x_0}) = \{0\},$$

for $j = 1, 2, 3$.

(ii) *If $n \neq 4$, we have*

$$\dim \mathrm{Hom}_K(\mathfrak{g}, (S^2T^*)^{+-}_{\mathbb{C},x_0}) = 1.$$

(iii) *If $n = 4$, we have*

$$\dim \mathrm{Hom}_K(\mathfrak{g}, (S^2T^*)^{+-}_{\mathbb{C},x_0}) = 2.$$

From Lemma 3.1 and the decomposition (3.8), we deduce that

$$\dim \mathrm{Hom}_K(\mathfrak{g}, S^2_0T^*_{\mathbb{C},x_0}) = 1 \tag{3.9}$$

when $n \neq 4$, and that

$$\dim \mathrm{Hom}_K(\mathfrak{g}, S^2_0T^*_{\mathbb{C},x_0}) = 2 \tag{3.10}$$

when $n = 4$.

In [12], it is shown that the following proposition is a consequence of Lemma 3.1 and the equalities (3.9) and (3.10).

Proposition 3.1. *Let X be the complex quadric Q_n, with $n \geq 3$. If $n \neq 4$, we have*

$$E(X) = \{0\}.$$

If $n = 4$, we have

$$E(X) \subset C^\infty((S^2T^*)^{+-}).$$

When $n \neq 4$, the vanishing of the space $E(X)$ was first proved by Koiso (see [14] and [15]).

§4. Totally geodesic submanifolds of the quadric

In this section, we suppose that X is the complex quadric Q_n, with $n \geq 3$. We first introduce various families of closed connected totally geodesic submanifolds of X. Let x be a point of X and ν be an element of S_x.

If $\{\xi, \eta\}$ is an orthonormal set of vectors of T_ν^+, according to formula (2.5) we see that the set $\mathrm{Exp}_x F$ is a closed connected totally geodesic surface of X, whenever F is the subspace of T_x generated by one of following families of vectors:

(A_1) $\{\xi, J\eta\}$;
(A_2) $\{\xi + J\eta, J\xi - \eta\}$;
(A_3) $\{\xi, J\xi\}$;
(A_4) $\{\xi, \eta\}$.

Let $\{\xi, \eta\}$ be an orthonormal set of vectors of T_ν^+. According to [4], if F is generated by the family (A_2) (resp. the family (A_3)) of vectors, the surface $\mathrm{Exp}_x F$ is isometric to the complex projective line $\mathbb{CP}^1$ with its metric of constant holomorphic curvature 4 (resp. curvature 2). Moreover, if F is generated by the family (A_1), the surface $\mathrm{Exp}_x F$ is isometric to a flat torus. In [12], we verify that, if F is generated by the family (A_4), the surface $\mathrm{Exp}_x F$ is isometric to a sphere of constant curvature 2.

For $1 \leq j \leq 4$, we denote by $\tilde{\mathcal{F}}^{j,\nu}$ the set of all closed totally geodesic surfaces of X which can be written in the form $\mathrm{Exp}_x F$, where F is a subspace of T_x generated by a family of vectors of type (A_j).

If ε is a number equal to ± 1 and if ξ, η, ζ are unit vectors of T_ν^+ satisfying

$$g(\xi, \eta) = g(\xi, \zeta) = 3g(\eta, \zeta) = \varepsilon \frac{3}{5},$$

and if F is the subspace of T_x generated by the vectors

$$\{\xi + J\zeta, \eta + \varepsilon J(\xi - \eta) - J\zeta\},$$

according to (2.5) we also see that the set $\mathrm{Exp}_x F$ is a closed connected totally geodesic surface of X. Moreover, according to [4] this surface is isometric to a sphere of constant curvature $2/5$. We denote by $\tilde{\mathcal{F}}^{5,\nu}$ the set of all such closed totally geodesic surfaces of X.

If $\{\xi_1, \xi_2, \xi_3, \xi_4\}$ is an orthonormal set of vectors of T_ν^+ and if F is the subspace of T_x generated by the vectors

$$\{\xi_1 + J\xi_2, \xi_3 + J\xi_4\},$$

according to (2.5) we see that the set $\mathrm{Exp}_x F$ is a closed connected totally geodesic surface of X. Moreover, according to [4] this surface is isometric to the real projective plane $\mathbb{RP}^2$ of constant curvature 1. Clearly such submanifolds of X only occur when $n \geq 4$. We denote by $\tilde{\mathcal{F}}^{6,\nu}$ the set of all such closed totally geodesic surfaces of X.

If $\{\xi_1, \xi_2, \xi_3, \xi_4\}$ is an orthonormal set of vectors of T^+_ν and if F is the subspace of T_x generated by the vectors

$$\{\xi_1 + J\xi_2, J\xi_1 - \xi_2, \xi_3 + J\xi_4, J\xi_3 - \xi_4\},$$

according to (2.5) we see that the set $\mathrm{Exp}_x F$ is a closed connected totally geodesic submanifold of X. Moreover, this submanifold is isometric to the complex projective plane $\mathbb{CP}^2$ of constant holomorphic curvature 4. Clearly such submanifolds of X only occur when $n \geq 4$. We denote by $\tilde{\mathcal{F}}^{7,\nu}$ the set of all such closed totally geodesic submanifolds of X.

When $n \geq 4$, clearly a surface belonging to the family $\tilde{\mathcal{F}}^{2,\nu}$ or to the family $\tilde{\mathcal{F}}^{6,\nu}$ is contained in a closed totally geodesic submanifold of X belonging to the family $\tilde{\mathcal{F}}^{7,\nu}$. In fact, the surfaces of the family $\tilde{\mathcal{F}}^{2,\nu}$ (resp. the family $\tilde{\mathcal{F}}^{6,\nu}$) correspond to complex lines (resp. to linearly imbedded real projective planes) of the submanifolds of X belonging to the family $\tilde{\mathcal{F}}^{7,\nu}$ viewed as complex projective planes.

Let W be a subspace of T^+_ν of dimension $k \geq 2$; by (3.6), we may consider the subspace $F = W \oplus JW$ of T_x of dimension $2k$, which is stable under J. The set $\mathrm{Exp}_x F$ is a closed connected totally geodesic complex submanifold of X; in [12], we show that it isometric to the quadric Q_k of dimension k. Let $\mathcal{F}'$ be the G-invariant family of all closed connected totally geodesic submanifolds of X which are isometric to the quadric Q_3 of dimension 3.

Let Z be a surface belonging to the family $\tilde{\mathcal{F}}^{j,\nu}$, with $1 \leq j \leq 5$. We may write $Z = \mathrm{Exp}_x F$, where F is an appropriate subspace of T_x. Clearly, this space F is contained in a subspace of T_x which can be written in the form $W \oplus JW$, where W is a subspace of T^+_ν of dimension 3. Therefore Z is contained in a submanifold of X belonging to $\mathcal{F}'$.

For $1 \leq j \leq 7$, we consider the G-invariant families

$$\tilde{\mathcal{F}}^j = \bigcup_{\substack{\nu \in S_x \\ x \in X}} \mathcal{F}^{j,\nu}$$

of closed connected totally geodesic submanifolds of X. When $n \geq 4$, we know that a surface belonging to the family $\tilde{\mathcal{F}}^2$ is contained in a closed totally geodesic submanifold of X belonging to the family $\tilde{\mathcal{F}}^7$. We write

$$\mathcal{F}_1 = \tilde{\mathcal{F}}^1 \cup \tilde{\mathcal{F}}^3 \cup \tilde{\mathcal{F}}^4, \qquad \mathcal{F}_2 = \tilde{\mathcal{F}}^1 \cup \tilde{\mathcal{F}}^2 \cup \tilde{\mathcal{F}}^6,$$
$$\mathcal{F}_3 = \tilde{\mathcal{F}}^1 \cup \tilde{\mathcal{F}}^2 \cup \tilde{\mathcal{F}}^4 \cup \tilde{\mathcal{F}}^5.$$

We have seen that a surface belonging to the family $\tilde{\mathcal{F}}^j$, with $1 \leq j \leq 5$, is contained in a closed totally geodesic submanifold of X belonging to the family $\mathcal{F}'$.

In [4], Dieng classifies all closed connected totally geodesic surfaces of X and proves the following:

Proposition 4.1. *If $n \geq 3$, then the family of all closed connected totally geodesic surfaces of X is equal to $\mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3$.*

In fact, the family $\tilde{\mathcal{F}}^1$ is equal to the set of all maximal flat totally geodesic tori of X.

We now describe some of the relationships between the families of closed totally geodesic surfaces of X introduced above, the G-invariant sub-bundles of S^2T^* and the infinitesimal orbit of the curvature $\tilde{B}$. If $\mathcal{F}$ is a G-invariant family of closed connected totally geodesic surfaces of X, we denote by $N_{\mathcal{F}}$ the sub-bundle of B consisting of those elements of B which vanish when restricted to the submanifolds of $\mathcal{F}$.

For $j = 1, 2, 3$, we set

$$N_j = N_{\mathcal{F}_j}.$$

According to formula (1.6), we see that

$$\tilde{B} \subset N_j,$$

for $j = 1, 2, 3$.

The following lemma, proved in [12], will not be required here.

Lemma 4.1. *For $n \geq 3$, we have*

$$\operatorname{Tr} N_1 \subset (S^2T^*)^{+-}.$$

In [12], we prove Proposition 4.2; on the other hand, Proposition 4.3 is given by Proposition 5.1 of [8].

Proposition 4.2. *For $n \geq 5$, we have*

$$\operatorname{Tr} N_2 = L.$$

Proposition 4.3. *For $n = 4$, we have*

$$\operatorname{Tr} N_2 \subset L \oplus (S^2T^*)^{+-}.$$

In [6], Dieng shows that an element of N_3 vanishes when restricted to a surface of X belonging to the family $\tilde{\mathcal{F}}^3$ and proves the following result:

Proposition 4.4. *For $n \geq 3$, we have*

$$N_3 = \tilde{B}.$$

When $n \geq 3$, Dieng [4] shows that

$$H \cap (T^* \otimes N_3) = \{0\},$$

and then deduces the relation (1.4) for the complex quadric X from Proposition 4.4; thus, we have the following result:

Proposition 4.5. *For $n \geq 3$, we have*

$$H \cap (T^* \otimes \tilde{B}) = \{0\}.$$

From Proposition 4.5 and Theorem 1.1, we deduce the exactness of the sequence (1.3) for the complex quadric $X = Q_n$, with $n \geq 3$.

§5. Infinitesimal rigidity of the quadric

The sub-bundle $L_{\mathbb{C}}$ of $S^2T^*_{\mathbb{C}}$ is a homogeneous bundle over X; thus $C^\infty(L_{\mathbb{C}})$ is a G-module. Let γ be an element of the set $\hat{G}$ of equivalence classes of irreducible G-modules over $\mathbb{C}$, and let V_γ be an irreducible G-module which is a representative of γ. In [12], we show that the isotypic component $C^\infty_\gamma(L_{\mathbb{C}})$ of the G-module $C^\infty(L_{\mathbb{C}})$ corresponding to γ is a G-submodule of $C^\infty(L_{\mathbb{C}})$ isomorphic to k copies of V_γ, where k is equal either to 0 or 2. When $k = 2$, we also describe an explicit basis for the subspace W_γ of dimension 2 generated by the highest weight vectors of the G-module $C^\infty_\gamma(L_{\mathbb{C}})$; we then consider the action of the differential operator $\operatorname{div}\colon S^2T^*_{\mathbb{C}} \to T^*_{\mathbb{C}}$ on the elements of W_γ and prove that the induced mapping $\operatorname{div}\colon W_\gamma \to C^\infty(T^*_{\mathbb{C}})$ is injective. Since the restriction $\operatorname{div}\colon L_{\mathbb{C}} \to T^*_{\mathbb{C}}$ is a homogeneous differential operator, from these facts we deduce the following result:

Proposition 5.1. *Let X be the complex quadric Q_n, with $n \geq 3$. A section h of L over X, which satisfies the relation* $\operatorname{div} h = 0$, *vanishes identically.*

The essential aspects of the proof of following proposition were first given by Dieng in [4].

Proposition 5.2. *The infinitesimal rigidity of the quadric Q_3 implies that all the quadrics Q_n, with $n \geq 3$, are infinitesimally rigid.*

Proof. We consider the G-invariant family $\mathcal{F}_3$ of closed connected totally geodesic surfaces of X and the family $\mathcal{F}'$ of closed connected totally geodesic submanifolds of X isometric to the quadric Q_3 of §4. We have seen that each surface belonging to the family $\mathcal{F}_3$ is contained in a totally geodesic submanifold of X belonging to the family $\mathcal{F}'$. Assume that we know that the quadric Q_3 is infinitesimally rigid; then every submanifold of X belonging to $\mathcal{F}'$ is infinitesimally rigid; moreover, by Propositions 4.4 and 4.5, the families $\mathcal{F} = \mathcal{F}_3$ and $\mathcal{F}'$ satisfy the hypotheses of Theorem 2.1. From this last theorem, we deduce the infinitesimal rigidity of X.

We consider the families $\tilde{\mathcal{F}}^1$, $\tilde{\mathcal{F}}^2$, $\tilde{\mathcal{F}}^6$ and $\tilde{\mathcal{F}}^7$ of closed connected totally geodesic submanifolds of X. We set

$$\mathcal{F}'' = \tilde{\mathcal{F}}^1 \cup \tilde{\mathcal{F}}^6 \cup \tilde{\mathcal{F}}^7.$$

We consider the G-invariant family

$$\mathcal{F} = \mathcal{F}_2 = \tilde{\mathcal{F}}^1 \cup \tilde{\mathcal{F}}^2 \cup \tilde{\mathcal{F}}^6$$

of totally geodesic surfaces of X and the sub-bundle $N_2 = N_{\mathcal{F}_2}$ of B, introduced in §4, and the corresponding differential operator

$$D_{1,\mathcal{F}} \colon S^2\mathcal{T}^* \to \mathcal{B}/\mathcal{N}_2.$$

We recall that a submanifold of X belonging to $\tilde{\mathcal{F}}^1$ (resp. to $\tilde{\mathcal{F}}^6$) is a surface isometric to the flat 2-torus (resp. to the real projective plane $\mathbb{RP}^2$), while a submanifold of X belonging to $\tilde{\mathcal{F}}^7$ is isometric to the complex projective space $\mathbb{CP}^2$. Each surface belonging to $\tilde{\mathcal{F}}^2$ is contained in a submanifold of X belonging to the family $\tilde{\mathcal{F}}^7$; therefore each surface of X belonging to $\mathcal{F}$ is contained in a submanifold of X belonging to the family $\mathcal{F}''$. In the introduction, we mentioned that a flat 2-tori, the real projective plane $\mathbb{RP}^2$ and the complex projective space $\mathbb{CP}^2$ are infinitesimally rigid symmetric spaces. Thus every submanifold of X belonging to $\mathcal{F}''$ is infinitesimally rigid. Hence a symmetric 2-form h on X satisfying the zero-energy condition belongs to $\mathcal{L}(\mathcal{F}'')$; by Proposition 2.2, the 2-form h verifies the relation

$$D_{1,\mathcal{F}}h = 0.$$

Proposition 5.3. *Let h be a symmetric 2-form on quadric $X = Q_n$, with $n \geq 4$, satisfying the zero-energy condition and the relation $\operatorname{div} h = 0$. Then when $n \geq 5$, the symmetric form h is a section of the vector bundle L; when $n = 4$, it is a section of the vector bundle $L \oplus (S^2T^*)^{+-}$.*

Proof. We know that h belongs to $\mathcal{L}(\mathcal{F}'')$. We suppose that $n \geq 5$ (resp. that $n = 4$). According to Proposition 4.2 (resp. to Proposition 4.3), we see that the hypotheses of Theorem 2.2 hold, with $E = L$ (resp. with $E = L \oplus (S^2T^*)^{+-}$). By Proposition 3.1, we know that $E(X) = \{0\}$ (resp. that $E(X) \subset C^\infty((S^2T^*)^{+-})$). Then Theorem 2.2 tells us that h is a section of L (resp. of $L \oplus (S^2T^*)^{+-}$).

The following result is proved in [9] (see also [12]):

Proposition 5.4. *Let X be the quadric Q_4. A section h of the vector bundle $L \oplus (S^2T^*)^{+-}$ satisfying the relations*

$$\operatorname{div} h = 0, \qquad D_{1,\mathcal{F}} h = 0$$

vanishes identically.

We now prove the infinitesimal rigidity of the quadric $X = Q_n$, with $n \geq 4$, using Propositions 5.1, 5.3 and 5.4. In the case $n = 4$, this proof appears in [9]. Let h be a symmetric 2-form on the quadric $X = Q_n$, with $n \geq 4$, satisfying the zero-energy condition and the relation $\operatorname{div} h = 0$. When $n \geq 5$, Proposition 5.3 tells us that h is a section of L; by Proposition 5.1, we see that h vanishes identically. When $n = 4$, Proposition 5.3 tells us that h is a section of $L \oplus (S^2T^*)^{+-}$, and, as we saw above, Proposition 2.2 gives us the relation $D_{1,\mathcal{F}} h = 0$; by Proposition 5.4, we see that h vanishes. Then Proposition 2.1 gives us the infinitesimal rigidity of X.

References

[1] M. Berger and D. Ebin, *Some decompositions of the space of symmetric tensors on a Riemannian manifold*, J. Differential Geom., **3** (1969), 379–392.

[2] A. Besse, Manifolds all of whose geodesics are closed, Ergeb. Math. Grenzgeb., Bd. **93**, Springer-Verlag, Berlin, Heidelberg, New York, 1978.

[3] E. Calabi, *On compact, Riemannian manifolds with constant curvature.* I, Proc. Sympos. Pure Math., **3** (1961), Amer. Math. Soc. Providence, RI, 155–180.

[4] Y. Dieng, *Quelques résultats de rigidité infinitésimale pour les quadriques complexes*, C. R. Acad. Sci. Paris Sér. I Math., **304** (1987), 393–396.

[5] J. Gasqui and H. Goldschmidt, *Déformations infinitésimales des espaces riemanniens localement symétriques.* I, Adv. in Math., **48** (1983), 205–285.

[6] J. Gasqui and H. Goldschmidt, *Déformations infinitésimales des espaces riemanniens localement symétriques.* II. *La conjecture infinitésimale*

de Blaschke pour les espaces projectifs complexes, Ann. Inst. Fourier (Grenoble), **34** 2 (1984), 191–226.

[7] J. Gasqui and H. Goldschmidt, *Rigidité infinitésimale des espaces projectifs et des quadriques complexes*, J. Reine Angew. Math., **396** (1989), 87–121.

[8] J. Gasqui and H. Goldschmidt, *On the geometry of the complex quadric*, Hokkaido Math. J., **20** (1991), 279–312.

[9] J. Gasqui and H. Goldschmidt, *The infinitesimal rigidity of the complex quadric of dimension four*, Amer. Math. J., **116** (1994), 501–539.

[10] J. Gasqui and H. Goldschmidt, *Radon transforms and spectral rigidity on the complex quadrics and the real Grassmannians of rank two*, J. Reine Angew. Math., **480** (1996), 1–69.

[11] J. Gasqui and H. Goldschmidt, *The Radon transform and spectral rigidity of the Grassmannians*, Contemp. Math., **251** (2000), 205–221.

[12] J. Gasqui and H. Goldschmidt, Radon transforms and the rigidity of the Grassmannians (to appear).

[13] H. Goldschmidt, *On the infinitesimal rigidity of the complex quadric*, Contemp. Math., **140** (1992), 49–63.

[14] N. Koiso, *Rigidity and stability of Einstein metrics — The case of compact symmetric spaces*, Osaka J. Math., **17** (1980), 51–73.

[15] N. Koiso, *Rigidity and infinitesimal deformability of Einstein metrics*, Osaka J. Math., **19** (1982), 643–668.

[16] A. Lichnerowicz, *Propagateurs et commutateurs en relativité générale*, Inst. Hautes Études Sci. Publ. Math., **10** (1961).

[17] R. Michel, *Problèmes d'analyse géométrique liés à la conjecture de Blaschke*, Bull. Soc. Math. France, **101** (1973), 17–69.

[18] R. Michel, (a) *Un problème d'exactitude concernant les tenseurs symétriques et les géodésiques*; (b) *Tenseurs symétriques et géodésiques*, C. R. Acad. Sci. Sér. A, **284** (1977), 183–186; 1065–1068.

[19] B. Smyth, *Differential geometry of complex hypersurfaces*, Ann. of Math., **85** (1967), 246–266.

[20] L. Tela Nlenvo, *Formes impaires et un problème de rigidité infinitésimale pour la quadrique complexe de dimension* 3, Thèse de Doctorat, Université de Grenoble I, 1997.

[21] C. Tsukamoto, *Infinitesimal Blaschke conjectures on projective spaces*, Ann. Sci. École Norm. Sup. (4), **14** (1981), 339–356.

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On One-Parametric Families of Bäcklund Transformations

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Abstract.

In the context of the cohomological deformation theory, infinitesimal description of one-parametric families of Bäcklund transformations of special type including classical examples is given. It is shown that any family of such a kind evolves in the direction of a nonlocal symmetry shadow in the sense of [9].

Introduction

The role of Bäcklund transformations in constructing exact solutions of nonlinear partial differential equations is well known, see [2] and [13] and relevant references therein, for example. A general scheme is illustrated by classical works by Bäcklund and Bianchi. Namely, for the sine-Gordon equation

$$u_{xy} = \sin u \tag{1}$$

Bäcklund constructed a system of differential relations $\mathcal{B}(u, v; \lambda) = 0$ depending on a real parameter $\lambda \in \mathbb{R}$ and satisfying the following property: if $u = u(x, y)$ is a solution of (1), then v is a solution of the same equation and vice versa. Using this result, Bianchi showed that if a known solution u_0 is given and solutions u_1, u_2 satisfy the relations $\mathcal{B}(u_0, u_i; \lambda_i) = 0$, $i = 1, 2$, then there exists a solution u_{12} which satisfies $\mathcal{B}(u_1, u_{12}; \lambda_2) = 0$, $\mathcal{B}(u_2, u_{12}; \lambda_1) = 0$ and is expressed in terms of u_0, u_1, u_2 in terms of relatively simple equalities. This is the so-called *Bianchi permutability theorem*, or *nonlinear superposition principle*. This scheme was successfully applied to many other "integrable" equations.

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Quite naturally, a general problem arises: given an arbitrary PDE $\mathcal{E}$, when are we able to implement a similar construction? This question is closely related to another problem of a great importance in the theory of integrable systems, the problem of insertion of a nontrivial "spectral parameter" to the initial equation. In this paper, we mainly deal with the first problem referring the reader to the yet unpublished work by M. Marvan [10], where the second problem is analyzed.

Our approach to solution lies in the framework of the geometrical theory of nonlinear PDE, and the first section of the paper contains a brief introduction to this theory, including its nonlocal aspects (the theory of coverings), see [1], [7], [8] and [9]. The second section deals with cohomological invariants of nonlinear PDE naturally associated to the equation structure. Our main concern here is the relation between this cohomology theory and deformations of the structure [4], [5] and [8]. In the third section, we give a geometrical definition of Bäcklund transformations and using cohomological techniques prove the main result of the paper describing infinitesimal part of one-parameter families of Bäcklund transformations.

§1. Equations and coverings

Let us recall basic facts from the geometry of nonlinear PDE, [1], [7] and [8].

Consider a smooth manifold M, $\dim M = n$, and a locally trivial smooth vector bundle $\pi\colon E \to M$. Denote by $\pi_k\colon J^k(\pi) \to M$, $k = 0, 1, \ldots, \infty$, the corresponding bundles of jets. A *differential equation* of order k, $k < \infty$, in the bundle π is a smooth submanifold $\mathcal{E} \subset J^k(\pi)$. To any equation $\mathcal{E}$ there corresponds a series of its *prolongations* $\mathcal{E}^s \subset J^{k+s}(\pi)$ and the *infinite prolongation* $\mathcal{E}^\infty \subset J^\infty(\pi)$. We consider below *formally integrable* equations, which means that all $\mathcal{E}^s$ are smooth manifolds and the natural projection $\pi_\mathcal{E} = \pi_\infty|_{\mathcal{E}^\infty}\colon \mathcal{E}^\infty \to M$ is a smooth bundle. For any $s > 0$ there also exist natural bundles

$$\mathcal{E}^\infty \xrightarrow{\pi_{\mathcal{E},s}} \mathcal{E}^s \xrightarrow{\pi_{\mathcal{E},s,s-1}} \mathcal{E}^{s-1} \xrightarrow{\pi_{\mathcal{E},s-1}} M \tag{2}$$

whose composition equals $\pi_\mathcal{E}$. The space $J^\infty(\pi)$ is endowed with an integrable distribution[1] denoted by $\mathcal{C}\mathrm{D}(\pi)$. Namely, any point $\theta \in J^\infty(\pi)$ is, by definition, represented in the form $[f]_x^\infty$, $x = \pi_\infty(\theta) \in M$, where f is a (local) section of π such that the graph M_f^∞ of its infinite jet

[1]Integrability in this context means that $\mathcal{C}\mathrm{D}(\pi)$ satisfies the Frobenius condition: $[\mathcal{C}\mathrm{D}(\pi), \mathcal{C}\mathrm{D}(\pi)] \subset \mathcal{C}\mathrm{D}(\pi)$.

passes through θ while $[f]_x^\infty$ is the class of (local) sections f' such that the graph of f' is tangent to the graph of f at $f(x) \in E$ with infinite order.

Then the tangent plane $T_\theta M_f^\infty$ is independent of the choice of f and we set $\mathcal{C}\mathrm{D}(\pi)_\theta = T_\theta M_f^\infty$. The distribution $\mathcal{C}\mathrm{D}(\pi)$ is n-dimensional and is called the *Cartan distribution* on $J^\infty(\pi)$. Since, by construction, all planes of the Cartan distribution are horizontal (with respect to π_∞) and n-dimensional, a connection $\mathcal{C}\colon \mathrm{D}(M) \to \mathrm{D}(\pi)$ is determined, where $\mathrm{D}(M)$ and $\mathrm{D}(\pi)$ denote the modules of vector fields on M and $J^\infty(\pi)$ respectively. This connection is flat and is called the *Cartan connection.*

Remark 1. In fact, the bundle π_∞ possesses a stronger structure than just a flat connection. Namely, for any vector bundles ξ and η over M and a linear differential operator Δ acting from ξ to η, a linear differential operator $\mathcal{C}\Delta$ acting from the pullback $\pi_\infty^*(\xi)$ to $\pi_\infty^*(\eta)$ is defined in a natural way. The correspondence $\Delta \mapsto \mathcal{C}\Delta$ is linear, preserves composition, and the Cartan connection is its particular case.

Both the Cartan distribution and the Cartan connection are restricted to the spaces $\mathcal{E}^\infty$ and bundles $\pi_\mathcal{E}$ respectively. The corresponding objects are denoted by $\mathcal{C}\mathrm{D}(\mathcal{E}^\infty)$ and $\mathcal{C} = \mathcal{C}_\mathcal{E}\colon \mathrm{D}(M) \to \mathrm{D}(\mathcal{E}^\infty)$, where $\mathrm{D}(\mathcal{E}^\infty)$ is the module of vector fields on $\mathcal{E}^\infty$. The characteristic property of the Cartan distribution $\mathcal{C}\mathrm{D}(\mathcal{E}^\infty)$ on $\mathcal{E}^\infty$ is that its maximal integral manifolds are solutions of the equation $\mathcal{E}$ and vice versa. The connection form $U_\mathcal{E} \in \mathrm{D}(\Lambda^1(\mathcal{E}^\infty))$ of the connection $\mathcal{C}_\mathcal{E}$ is called the *structural element* of the equation $\mathcal{E}$. Here $\mathrm{D}(\Lambda^1(\mathcal{E}^\infty))$ denotes the module of derivations $C^\infty(\mathcal{E}^\infty) \to \Lambda^1(\mathcal{E}^\infty)$ with the values in the module of one-forms on $\mathcal{E}^\infty$.

Denote by $\mathrm{D}_\mathcal{C}(\mathcal{E}^\infty)$ the module

$$\mathrm{D}_\mathcal{C}(\mathcal{E}^\infty) = \{X \in \mathrm{D}(\mathcal{E}^\infty) \mid [X, \mathcal{C}\mathrm{D}(\mathcal{E}^\infty)] \subset \mathcal{C}\mathrm{D}(\mathcal{E}^\infty)\}.$$

Then $\mathrm{D}_\mathcal{C}(\mathcal{E}^\infty)$ is a Lie algebra with respect to commutator of vector fields and due to integrability of the Cartan distribution $\mathcal{C}\mathrm{D}(\mathcal{E}^\infty)$ is its ideal. The quotient Lie algebra $\operatorname{sym}\mathcal{E} = \mathrm{D}_\mathcal{C}(\mathcal{E}^\infty)/\mathcal{C}\mathrm{D}(\mathcal{E}^\infty)$ is called the *algebra of (higher) symmetries* of the equation $\mathcal{E}$. Denote by $\mathrm{D}^v(\mathcal{E}^\infty)$ the module of $\pi_\mathcal{E}$-vertical vector fields on $\mathcal{E}^\infty$. Then in any coset $X \bmod \mathcal{C}\mathrm{D}(\mathcal{E}^\infty) \in \operatorname{sym}\mathcal{E}$ there exists a unique vertical element and this element is called a *(higher) symmetry* of $\mathcal{E}$.

Remark 2. It may so happen that a coset $X \bmod \mathcal{C}\mathrm{D}(\mathcal{E}^\infty)$ contains a representative X' which is projectable to a vector field X'_s on $\mathcal{E}^s$ by $\pi_{\mathcal{E},s}$ for some $s < \infty$. In this case, X' is called a *classical (infinitesimal) symmetry* of $\mathcal{E}$ and possesses trajectories in $\mathcal{E}^\infty$. The corresponding diffeomorphisms preserve solutions of $\mathcal{E}$ and are called *finite symmetries.*

We now pass to a generalization of the above described geometrical theory, the theory of coverings [1, 8, 9]. Let $\tau\colon W \to \mathcal{E}^\infty$ be a smooth fiber bundle, the manifold W being equipped with an integrable distribution $\mathcal{C}_\tau \mathrm{D}(W) = \mathcal{C}\mathrm{D}(W) \subset \mathrm{D}(W)$ of dimension $n = \dim M$. Then τ is called a *covering* over $\mathcal{E}$ (or over $\mathcal{E}^\infty$), if for any point $\theta \in W$ one has $\tau_*(\mathcal{C}\mathrm{D}(W)_\theta) = \mathcal{C}\mathrm{D}(\mathcal{E}^\infty)_{\tau(\theta)}$. Equivalently, a covering structure in the bundle τ is determined by a flat connection $\mathcal{C}_\tau\colon \mathrm{D}(M) \to \mathrm{D}(W)$ satisfying $\tau_* \circ \mathcal{C}_\tau = \mathcal{C}_\mathcal{E}$. Let $U_\tau \in \mathrm{D}(\Lambda^1(W))$ be the corresponding connection form. We call it the *structural element* of the covering τ.

Example 1 (see [12]). Let $\mathcal{E} \subset J^k(\pi)$ be an equation. Consider the tangent bundle $T\mathcal{E}^\infty \to \mathcal{E}^\infty$ and the subbundle $\pi_\mathcal{E}^v\colon T^v\pi_\mathcal{E} \to \mathcal{E}^\infty$, where $T^v\pi_\mathcal{E}$ consists of $\pi_\mathcal{E}$-vertical vectors. Hence, the module of sections for $\pi_\mathcal{E}^v$ consists of $\pi_\mathcal{E}$-vertical vector fields on $\mathcal{E}^\infty$.

Then $\pi_\mathcal{E}^v$ carries a natural covering structure. Namely, for any vector field $X \in \mathrm{D}(M)$ and a vertical vector field Y we set $X(Y) = [\mathcal{C}_\mathcal{E}(X), Y]$. Thus, X acts on sections of $\pi_\mathcal{E}^v$ and defines a vector field $\mathcal{C}_{\pi_\mathcal{E}^v}(X)$ on $T^v\pi_\mathcal{E}$ projected by $(\pi_\mathcal{E}^v)_*$ to $\mathcal{C}_\mathcal{E}(X)$. The connection $\mathcal{C}_{\pi_\mathcal{E}^v}$ is well defined in this way and is projected to the connection $\mathcal{C}_\mathcal{E}$.

Given two coverings $\tau_i\colon W_i \to \mathcal{E}^\infty$, $i = 1, 2$, we say that a smooth mapping $F\colon W_1 \to W_2$ is a *morphism* of τ_1 to τ_2, if

(i) F is a morphism of fiber bundles,
(ii) F_* takes the distribution $\mathcal{C}_{\tau_1}\mathrm{D}(W_1)$ to $\mathcal{C}_{\tau_2}\mathrm{D}(W_2)$ (equivalently, $F_* \circ \mathcal{C}_{\tau_1} = \mathcal{C}_{\tau_2}$).

A morphism F is said to be an *equivalence*, if it is a diffeomorphism.

Similar to the case of infinitely prolonged equations, we can define the Lie algebra $\mathrm{D}_{\mathcal{C}_\tau}(W)$ such that $\mathcal{C}_\tau\mathrm{D}(W)$ is its ideal and introduce the algebra of *nonlocal τ-symmetries* as the quotient $\operatorname{sym}_\tau \mathcal{E} = \mathrm{D}_{\mathcal{C}_\tau}(W)/\mathcal{C}_\tau\mathrm{D}(W)$. Again, in any coset $X \bmod \mathcal{C}_\tau\mathrm{D}(W) \in \operatorname{sym}_\tau \mathcal{E}$ there exists a unique $(\pi_\mathcal{E} \circ \tau)$-vertical representative and it is called a *nonlocal τ-symmetry* of the equation $\mathcal{E}$.

Obviously, one can introduce the notion of a covering over covering, etc. In particular, the subbundle $(\pi_\mathcal{E} \circ \tau)^v\colon T^v(\pi_\mathcal{E} \circ \tau) \to W$ of $(\pi_\mathcal{E} \circ \tau)$-vertical vectors (cf. Example 1) is a covering over W. It is also easily seen that the corresponding integrable distribution is tangent to the submanifold $T^v\tau \subset T^v(\pi_\mathcal{E} \circ \tau)$ of τ-vertical vectors and therefore $\tau^v\colon T^v\tau \to W$ is a covering over W as well. Note that the correspondence $\tau \Rightarrow \tau^v$ determines a covariant functor in the category of coverings.

We shall now reinterpret the concepts of a symmetry and nonlocal symmetry using the results of [12]. Namely, one has

Proposition 1. *Let $\mathcal{E}$ be an equation and $\tau\colon W \to \mathcal{E}^\infty$ be a covering over it. Then*:

(1) *There is a one-to-one correspondence between symmetries of $\mathcal{E}$ and sections $\varphi\colon \mathcal{E}^\infty \to T^v\pi_{\mathcal{E}}$ of the bundle $\pi_{\mathcal{E}}^v\colon T^v\pi_{\mathcal{E}} \to \mathcal{E}^\infty$ such that φ_* takes the Cartan distribution on $\mathcal{E}^\infty$ to that on $T^v\pi_{\mathcal{E}}$.*

(2) *There is a one-to-one correspondence between nonlocal τ-symmetries of $\mathcal{E}$ and sections ψ of the bundle $(\pi_{\mathcal{E}}\circ\tau)^v\colon T^v(\pi_{\mathcal{E}}\circ\tau) \to W$ such that ψ_* takes the Cartan distribution on W to that on $T^v(\pi_{\mathcal{E}}\circ\tau)$.*

Let us say that a mapping $s\colon W \to T^v\pi_{\mathcal{E}}$ is a τ-*shadow* of a nonlocal symmetry (cf. [1], [8] and [9]), if $\pi_{\mathcal{E}}^v \circ s = \tau$ and s_* preserves the Cartan distribution.

Example 2. Every symmetry φ considered as a section $\varphi\colon \mathcal{E}^\infty \to T^v\pi_{\mathcal{E}}$ determines a shadow $\varphi\circ\tau$.

Proposition 2 (The shadow reconstruction theorem). *For an arbitrary covering $\tau\colon W \to \mathcal{E}^\infty$ and a τ-shadow $s\colon W \to T^v\pi_{\mathcal{E}}$ there exists a covering $\tau'\colon W' \to W$ and a nonlocal $\tau\circ\tau'$-symmetry $s'\colon W' \to T^v(\pi_{\mathcal{E}}\circ\tau\circ\tau')$ such that the diagram*

$$
\begin{tikzcd}
 & T^v\pi_{\mathcal{E}} \arrow[d, "\pi_{\mathcal{E}}^v"'] & & T^v(\pi_{\mathcal{E}}\circ\tau\circ\tau') \arrow[ll, "(\tau\circ\tau')_*"'] \arrow[d, shift right, "(\pi_{\mathcal{E}}\circ\tau\circ\tau')^v"'] \\
M & \mathcal{E}^\infty \arrow[l, "\pi_{\mathcal{E}}"'] & W \arrow[l, "\tau"'] \arrow[ul, "s"'] & W' \arrow[l, "\tau'"'] \arrow[u, shift right, "s'"']
\end{tikzcd}
\tag{3}
$$

is commutative. In other words, any shadow can be reconstructed up to a nonlocal symmetry in some new covering.

Proof. Consider the following commutative diagram:

$$
\begin{tikzcd}
 & T^v\pi_{\mathcal{E}} \arrow[d, "\pi_{\mathcal{E}}^v"'] & T^v(\pi_{\mathcal{E}}\circ\tau) \arrow[l] \arrow[d, "(\pi_{\mathcal{E}}\circ\tau)^v"] & T^v(\pi_{\mathcal{E}}\circ\tau\circ\tau^v) \arrow[l] \arrow[d, "(\pi_{\mathcal{E}}\circ\tau\circ\tau^v)^v"] & \cdots \arrow[l] \\
M & \mathcal{E}^\infty \arrow[l, "\pi_{\mathcal{E}}"] & W \arrow[l, "\tau"] \arrow[ul, "s"] & T^v\tau \arrow[l, "\tau^v"] \arrow[ul, "s_*"] & T^v\tau^v \arrow[l, "(\tau^v)^v"] \arrow[ul, "(s_*)_*"'] & \cdots \arrow[l]
\end{tikzcd}
$$

and let us set $\tau_0 = \tau$, $\tau_{i+1} = \tau_i^v$, $W_0 = W$, $W_i = T^v\tau_i$, $s_0 = s$, $s_{i+1} = (s_i)_*$, where $s_* = ds$. Then the above diagram is infinitely continued to the right, while by setting $\bar\tau_i = \tau_1\circ\cdots\circ\tau_i$ and passing to the inverse limit, we obtain Diagram 3 with $\tau' = \bar\tau_\infty$, $s' = s_\infty$, and $W' = W_\infty$. $\square$

§2. $\mathcal{C}$-complex and deformations

We now pass to describe a cohomological theory naturally related to covering structures and supplying their important invariants, cf. [5] and [8].

Let W be a smooth manifold and $\mathrm{D}(\Lambda^i(W))$ denote the $C^\infty(W)$-module of $\Lambda^i(W)$-valued derivations $C^\infty(W) \to \Lambda^i(W)$. For any element $\Omega \in \mathrm{D}(\Lambda^i(W))$ one can define the *inner product* operation

$$\mathrm{i}_\Omega \colon \Lambda^j(W) \to \Lambda^{i+j-1}(W),$$

also denoted by $\Omega \lrcorner \rho$, $\rho \in \Lambda^*(W)$, and the *Lie derivative* along Ω:

$$\mathrm{L}_\Omega = [\mathrm{i}_\Omega, d] \colon \Lambda^j(W) \to \Lambda^{i+j}(W),$$

where $[\mathrm{i}_\Omega, d]$ denotes the *graded* commutator.

Then for any two elements Ω, $\Theta \in \mathrm{D}(\Lambda^*(W))$ we can introduce their *Frölicher-Nijenhuis bracket* by setting

$$[\![\Omega, \Theta]\!](f) = \mathrm{L}_\Omega(\Theta(f)) - (-1)^{ij}\mathrm{L}_\Theta(\Omega(f)),$$

where $f \in C^\infty(W)$ and i, j are degrees of Ω and Θ respectively[2]. Note also that two other operations are defined: we can multiply elements of $\mathrm{D}(\Lambda^*(W))$ by forms $\rho \in \Lambda^*(W)$ and insert elements of $\mathrm{D}(\Lambda^*(W))$ into each other. Namely, set

$$(\rho \wedge \Theta)(f) = \rho \wedge (\Theta(f)), \quad (\mathrm{i}_\Omega\Theta)(f) = \mathrm{i}_\Omega(\Theta(f)).$$

The basic properties of the above introduced operations are formulated in

Proposition 3 (see [4]). *Let* $\Omega \in \mathrm{D}(\Lambda^i(W))$, $\Theta \in \mathrm{D}(\Lambda^j(W))$, $\rho \in \Lambda^k(W)$, *and* $\eta \in \Lambda^l(W)$. *Then*:

(i) $\mathrm{i}_\Omega(\rho \wedge \eta) = \mathrm{i}_\Omega(\rho) \wedge \eta + (-1)^{(i-1)k}\rho \wedge \mathrm{i}_\Omega(\eta)$;

(ii) $\mathrm{i}_\Omega(\rho \wedge \Theta) = \mathrm{i}_\Omega(\rho) \wedge \Theta + (-1)^{(i-1)k}\rho \wedge \mathrm{i}_\Omega(\Theta)$;

(iii) $[\mathrm{i}_\Omega, \mathrm{i}_\Theta] = \mathrm{i}_{[\![\Omega,\Theta]\!]^{\mathrm{rn}}}$, *where*

$$[\![\Omega, \Theta]\!]^{rn} = \mathrm{i}_\Omega\Theta - (-1)^{(i-1)(j-1)}\mathrm{i}_\Theta\Omega$$

is the Richardson-Nijenhuis bracket *of* Ω *and* Θ;

(iv) $\mathrm{L}_\Omega(\rho \wedge \eta) = \mathrm{L}_\Omega(\rho) \wedge \eta + (-1)^{ik}\omega \wedge \mathrm{L}_\Omega(\eta)$;

(v) $\mathrm{L}_{\rho\wedge\Omega} = \rho \wedge \mathrm{L}_\Omega + (-1)^{i+k}d\omega \wedge \mathrm{i}_\Omega$;

(vi) $[\mathrm{L}_\Omega, d] = 0$;

---

[2]We say that i is the degree of Ω, if $\Omega \in \mathrm{D}(\Lambda^i(W))$.

(vii) $[\mathrm{L}_\Omega, \mathrm{L}_\Theta] = \mathrm{L}_{[\![\Omega,\Theta]\!]}$;
(viii) $[\![\Omega,\Theta]\!] + (-1)^{ij}[\![\Theta,\Omega]\!] = 0$;
(ix) $[\![\Omega,[\![\Theta,\Xi]\!]]\!] = [\![[\![\Omega,\Theta]\!],\Xi]\!] + (-1)^{ij}[\![\Omega,[\![\Theta,\Xi]\!]]\!]$, *where* $\Xi \in \mathrm{D}(\Lambda^m(W))$;
(x) $[\mathrm{L}_\Omega, \mathrm{i}_\Theta] = \mathrm{i}_{[\![\Omega,\Theta]\!]} - (-1)^{i(j+1)}\mathrm{L}_{\Theta \lrcorner \Omega}$;
(xi) $\Xi \lrcorner [\![\Omega,\Theta]\!] = [\![\Xi \lrcorner \Omega,\Theta]\!] + (-1)^{i(m+1)}[\![\Omega,\Xi \lrcorner \Theta]\!] + (-1)^i[\![\Xi,\Omega]\!] \lrcorner \Theta - (-1)^{(i+1)j}[\![\Xi,\Theta]\!] \lrcorner \Omega$;
(xii) $[\![\Omega,\rho\wedge\Theta]\!] = (\mathrm{L}_\Omega\rho)\wedge\Theta - (-1)^{(i+1)(j+k)}d\rho\wedge\mathrm{i}_\Theta\Omega + (-1)^{ik}\rho\wedge[\![\Omega,\Theta]\!]$.

In particular, from Proposition 3 (ix) it follows that for $\Omega \in \mathrm{D}(\Lambda^1(W))$ satisfying the *integrability property*

$$[\![\Omega,\Omega]\!] = 0 \tag{4}$$

the mapping

$$\partial_\Omega = [\![\Omega,\cdot\,]\!] \colon \mathrm{D}(\Lambda^i(W)) \to \mathrm{D}(\Lambda^{i+1}(W))$$

is a differential, i.e., $\partial_\Omega \circ \partial_\Omega = 0$, and thus we obtain the complex

$$0 \to \mathrm{D}(W) \to \cdots \to \mathrm{D}(\Lambda^i(W)) \xrightarrow{\partial_\Omega} \mathrm{D}(\Lambda^{i+1}(W)) \to \cdots \tag{5}$$

Assume now that the manifold W is fibered by $\xi\colon W \to M$ and a connection ∇ is given in the bundle ξ. Then the following fact is valid:

Proposition 4 (cf. [3]).

$$[\![U_\nabla, U_\nabla]\!] = 2R_\nabla,$$

where U_∇ is the connection form and R_∇ is the curvature.

Consequently, if ∇ is a flat connection, i.e., $R_\nabla = 0$, then $\Omega = U_\nabla$ enjoys the integrability property (4) and to any flat connection a complex of the form (5) corresponds. In this case, we shall use the notation $\partial_\Omega = \partial_\nabla$.

Now, we pass to the case of our main interest: let ξ be the composition $W \xrightarrow{\tau} \mathcal{E}^\infty \xrightarrow{\pi_\mathcal{E}} M$, τ being a covering over $\mathcal{E}$, and ∇ be the Cartan connection $\mathcal{C}_\tau$ associated to the covering structure. We include in consideration the case $W = \mathcal{E}^\infty$, $\tau = \mathrm{id}$, and $\mathcal{C}_\tau = \mathcal{C}_\mathcal{E}$. Let us restrict complex (5) to *vertical* derivations, i.e., to derivations

$$\mathrm{D}^v(\Lambda^i(W)) = \{\Omega \in \mathrm{D}(\Lambda^i(W)) \mid \Omega(f) = 0,\ \forall f \in C^\infty(M)\}.$$

By construction, U_τ (or $U_\mathcal{E}$) lies in $\mathrm{D}^v(\Lambda^1(W))$ (resp., in $\mathrm{D}^v(\Lambda^1(\mathcal{E}^\infty))$), while from the definition of the Frölicher-Nijenhuis bracket it follows that the differential in (5) preserves vertical derivations. The vertical part of (5) will be denoted by

$$0 \to \mathrm{D}^v(W) \to \cdots \to \mathrm{D}^v(\Lambda^i(W)) \xrightarrow{\partial_\tau} \mathrm{D}^v(\Lambda^{i+1}(W)) \to \cdots \tag{6}$$

or

$$0 \to \mathrm{D}^v(\mathcal{E}^\infty) \to \cdots \to \mathrm{D}^v(\Lambda^i(\mathcal{E}^\infty)) \xrightarrow{\partial_\mathcal{E}} \mathrm{D}^v(\Lambda^{i+1}(\mathcal{E}^\infty)) \to \cdots, \tag{7}$$

when the equation is considered as is. The cohomology of (6) (resp., of (7)) is denoted by $H_\mathcal{C}(\mathcal{E};\tau)$ (resp., by $H_\mathcal{C}(\mathcal{E})$) and is called the *$\mathcal{C}$-cohomology* of the covering τ (resp., of the equation $\mathcal{E}$). The following fundamental result is valid:

Theorem 1 (cf. [5]). *Let $\mathcal{E} \subset J^k(\pi)$ be a formally integrable equation and $\tau\colon W \to \mathcal{E}^\infty$ be a covering over $\mathcal{E}$. Then:*

(1) *The module $H^0_\mathcal{C}(\mathcal{E};\tau)$ is isomorphic to the Lie algebra $\mathrm{sym}_\tau\,\mathcal{E}$ of nonlocal τ-symmetries (resp., $H^0_\mathcal{C}(\mathcal{E})$ is isomorphic to $\mathrm{sym}\,\mathcal{E}$).*
(2) *The module $H^1_\mathcal{C}(\mathcal{E};\tau)$ is identified with equivalence classes of nontrivial infinitesimal deformations of the covering structure U_τ (resp., of the equation structure $U_\mathcal{E}$).*
(3) *The module $H^2_\mathcal{C}(\mathcal{E};\tau)$ consists of obstructions to prolongation of infinitesimal deformations up to formal ones.*

Remark 3. Of course, if U_λ is a deformation of the equation structure, the condition that $dU_\lambda/d\lambda|_{\lambda=0}$ lies in $\ker\partial_\mathcal{E}$ is not sufficient for this deformation to be trivial. Nevertheless, the following fact is obviously valid:

Proposition 5. *Let U_λ be a smooth deformation of the equation structure $U = U_\mathcal{E}$ satisfying the condition*

$$\frac{dU_\lambda}{d\lambda} = [\![X_\lambda, U_\lambda]\!], \tag{8}$$

where X_λ is a smooth vector field on $\mathcal{E}^\infty$ for any λ with smooth dependence on λ. Then U_λ is uniquely defined by (8) and is of the form

$$U_\lambda = \exp\Big(\int_0^\lambda X_\mu\,d\mu\Big)U,$$

where the left and right hand sides are understood as formal series. In this sense, U_λ is formally trivial.

Let us now consider the mapping $\mathrm{L}_{U_\tau}\colon \Lambda^i(W) \to \Lambda^{i+1}(W)$ and denote it by $d_\mathcal{C}$. Since the element U_τ is integrable, one has the identity

$d_\mathcal{C} \circ d_\mathcal{C} = 0$. We call $d_\mathcal{C}$ the *vertical*, or *Cartan differential* associated to the covering structure. Due to Proposition 3 (vi), $[d, d_\mathcal{C}] = 0$ and consequently the mapping $d_h = d - d_\mathcal{C}$ is also a differential and $[d_h, d_\mathcal{C}] = 0$. The differential d_h is called the *horizontal differential*, while the pair $(d_h, d_\mathcal{C})$ forms a bicomplex with the total differential d. The corresponding spectral sequence coincides with the Vinogradov $\mathcal{C}$-spectral sequence for the covering τ, [1], [8] and [14].

Denote by $\Lambda^1_h(W)$ the submodule in $\Lambda^1(W)$ spanned by $\operatorname{im} d_h$ and by $\mathcal{C}^1\Lambda(W)$ the submodule generated by $\operatorname{im} d_\mathcal{C}$. Then the direct sum decomposition $\Lambda^1(W) = \Lambda^1_h(W) \oplus \mathcal{C}\Lambda^1(W)$ takes place and generates the decomposition

$$\Lambda^i(W) = \bigoplus_{p+q=i} \mathcal{C}^p\Lambda(W) \otimes \Lambda^q_h(W) = \bigoplus_{p+q=i} \Lambda^{p,q}(W),$$

where

$$\mathcal{C}^p\Lambda(W) = \underbrace{\mathcal{C}^1\Lambda(W) \wedge \cdots \wedge \mathcal{C}^1\Lambda(W)}_{p \text{ times}}, \quad \Lambda^q_h(W) = \underbrace{\Lambda^1_h(W) \wedge \cdots \wedge \Lambda^1_h(W)}_{q \text{ times}}.$$

Then $d_\mathcal{C}\colon \Lambda^{p,q}(W) \to \Lambda^{p+1,q}(W)$, $d_h\colon \Lambda^{p,q}(W) \to \Lambda^{p,q+1}(W)$ and, moreover, as it follows from Proposition 3 (xi), $\partial_\tau\colon \mathrm{D}^v(\Lambda^{p,q}(W)) \to \mathrm{D}^v(\Lambda^{p,q+1}(W))$.

Remark 4. The complex $(\Lambda^q_h(W), d_h)$ is called the *horizontal complex* of the covering τ, while its cohomology is the *horizontal cohomology* of τ. It is worth to note that d_h in this case is obtained from the de Rham differential on the manifold M by applying the operation $\mathcal{C} = \mathcal{C}_\tau$ (see Remark 1). From Proposition 3 (xii) it follows that the $\mathcal{C}$-cohomology of τ is a graded module over the graded algebra of horizontal cohomology.

§3. Bäcklund transformations and the main result

Following [9], let us give a geometric definition of Bäcklund transformations. Let $\mathcal{E}_i \subset J^{k_i}(\pi_i)$, $i = 1, 2$, be two differential equations and $\tau_i\colon W \to \mathcal{E}_i^\infty$ be coverings with the same total space W endowed with an integrable distribution $\mathcal{C}\mathrm{D}(W)$. Then the diagram

$$\begin{array}{ccc} & W & \\ {}^{\tau_1}\swarrow & & \searrow^{\tau_2} \\ \mathcal{E}_1^\infty & & \mathcal{E}_2^\infty \end{array}$$

is called a *Bäcklund transformation* between the equations $\mathcal{E}_1^\infty$ and $\mathcal{E}_2^\infty$. A point $w \in W$ is called *generic*, if the intersection of the tangent planes to the fibres of τ_1 and τ_2 passing through w is trivial. Now, if $s \subset \mathcal{E}_1^\infty$ is a solution of $\mathcal{E}_1$ and $\tau_1^{-1}(s)$ contains a generic point, then there exists a neighborhood $\mathcal{U}$ of this point such that $\tau_2(\mathcal{U} \cap \tau_1^{-1}(s))$ is fibered by solutions of $\mathcal{E}_2$. Thus, Bäcklund transformations really determine a correspondence between solutions.

The following construction is equivalent to the definition of Bäcklund transformations. Let $\tau_i\colon W_i \to \mathcal{E}_i^\infty$, $i = 1, 2$, be two coverings and $F\colon W_1 \to W_2$ be a diffeomorphism taking the distribution $\mathcal{C}_{\tau_1}\mathrm{D}(W)$ to $\mathcal{C}_{\tau_2}\mathrm{D}(W)$. Then $\mathcal{B} = (W, \tau_1, \tau_2 \circ F, \mathcal{E}_1, \mathcal{E}_2)$ is a Bäcklund transformation and any Bäcklund transformations is obtained in this way.

Remark 5. It is important to stress here that if F is an isomorphism of coverings, then the Bäcklund transformation obtained in such a way is *trivial* in the sense of its action on solutions. Thus, we are interested in mappings F which are isomorphisms of manifolds with distributions, but not morphisms of coverings.

Assume now that a smooth family $F_\lambda\colon W_1 \to W_2$ of diffeomorphisms is given, then it generates a family $\mathcal{B}_\lambda$ of Bäcklund transformations. Our aim is to describe such families in sufficiently efficient terms. One way to construct these objects is given by the following

Example 3 (see [9]). Consider an equation $\mathcal{E}$, a covering $\tau\colon W \to \mathcal{E}^\infty$ over it, and a finite symmetry $A\colon \mathcal{E}^\infty \to \mathcal{E}^\infty$. Let $\bar{A}\colon W \to W$ be a diffeomorphic lifting of A to W such that

$$\tau \circ \bar{A} = A \circ \tau. \tag{9}$$

Denote by $\bar{A}_*\mathcal{C}_\tau\mathrm{D}(W)$ the image of the distribution $\mathcal{C}_\tau\mathrm{D}(W)$ under $\bar{A}$. Then, by obvious reasons, $\bar{A}_*\mathcal{C}_\tau\mathrm{D}(W)$ determines a covering structure $U_\tau^{\bar{A}}$ in W and if $\tilde{A}$ is another lifting of A, then the structures $U_\tau^{\bar{A}}$ and $U_\tau^{\tilde{A}}$ are equivalent. Thus, $\mathcal{B}_A = (W, \tau, A \circ \tau, \mathcal{E})$ is a Bäcklund transformation for $\mathcal{E}$.

Let X be a classical infinitesimal symmetry of $\mathcal{E}$ and $A_\lambda = \exp(\lambda X)\colon \mathcal{E}^\infty \to \mathcal{E}^\infty$ the corresponding one-parameter group of transformations lifted to $\mathcal{E}^\infty$. Then, using the above construction, we obtain a one-parametric family of Bäcklund transformations $\mathcal{B}_\lambda = \mathcal{B}_{A_\lambda}$.

In fact, the families of Bäcklund transformations obtained in the previous example are in a sense "counterfeit", since, due to (9), their action on solutions reduces to the action of symmetries A_λ. To get a "real" Bäcklund transformation, one needs to add into considerations an additional diffeomorphism $F\colon W \to W$ such that $\tau \circ F$ is a covering, i.e., τ_* projects $F_*(\mathcal{C}_\tau \mathrm{D}(W))$ onto $\mathcal{C}\mathrm{D}(\mathcal{E}^\infty)$, but F does not preserve the fibres of τ.

Example 4. Consider the infinite prolongation $\mathcal{E}^\infty$ of the sine-Gordon equation

$$u_{xy} = \sin u$$

and the trivial bundle

$$\tau\colon W = \mathcal{E}^\infty \otimes \mathbb{R} \to \mathcal{E}^\infty$$

with a coordinate v along fibres. Then the distribution $\mathcal{C}_\lambda \mathrm{D}(W)$ spanned by the vector fields $D_x + X$ and $D_t + T$, where $D_x = \mathcal{C}(\partial/\partial x)$, $D_t = \mathcal{C}(\partial/\partial t)$ are total derivatives and

$$X = \Big(-u_x + 2\lambda \sin\frac{u-v}{2}\Big)\frac{\partial}{\partial v},$$
$$T = \Big(u_t + \frac{2}{\lambda}\sin\frac{u+v}{2}\Big)\frac{\partial}{\partial v},$$

$\lambda \neq 0$, determines a one-dimensional covering structure on the bundle τ. The manifold W with this distribution is isomorphic to the infinite prolongation of the system

$$\begin{aligned} v_x &= -u_x + 2\lambda \sin\frac{u-v}{2}, \\ v_t &= u_t + \frac{2}{\lambda}\sin\frac{u+v}{2}. \end{aligned} \tag{10}$$

This system has a finite symmetry $F_\lambda\colon W \to W$ acting on the coordinates as follows $F_\lambda^*(u) = v$, $F_\lambda^*(v) = u$, $F_\lambda^*(x) = -x$, $F_\lambda^*(t) = -t$. Consider also the group $A_\lambda\colon x \mapsto \lambda x$, $t \mapsto \lambda^{-1}t$ of scale symmetries of the sine-Gordon equation and denote by $\bar{A}_\lambda\colon W \to W$ the diffeomorphic lifting of A_λ acting trivially on the coordinate v. Then

$$\mathcal{C}_\lambda \mathrm{D}(W) = \bar{A}_{\lambda,*}(\mathcal{C}_1 \mathrm{D}(W)), \quad F_\lambda = \bar{A}_\lambda \circ F_1 \circ \bar{A}_{\lambda^{-1}},$$

and $(W, \tau, \tau \circ \bar{A}_{-1} \circ F_\lambda, \mathcal{E}^\infty)$ is the family of the classical Bäcklund transformations: if $u(x,t)$ is a solution of $\mathcal{E}$ then every solution $v(x,t)$ of (10) satisfies the sine-Gordon equation as well.

Example 5. Consider now the potential KdV equation $\mathcal{E}$

$$u_t = -u_{xxx} - 3u_x^2$$

and the bundle

$$\tau\colon W = \mathcal{E}^\infty \otimes \mathbb{R} \to \mathcal{E}^\infty$$

with a coordinate v along fibres. The distribution $\mathcal{C}_\lambda \mathrm{D}(W)$ generated by $D_x + X$ and $D_t + T$, where

$$X = -\Big(u_x + \frac{1}{2}(v-u)^2 + 2\lambda\Big)\frac{\partial}{\partial v},$$
$$T = \big(u_{xxx} + u_x^2 - 4\lambda u_x - 8\lambda^2 + 2u_{xx}(u-v) + (u_x - 2\lambda)(u-v)^2\big)\frac{\partial}{\partial v},$$

$\lambda \in \mathbb{R}$, defines a one-dimensional covering structure on τ. There is a finite symmetry $F_\lambda \colon W \to W$, $F_\lambda^*(u) = v$, $F_\lambda^*(v) = u$, preserving $\mathcal{C}_\lambda \mathrm{D}(W)$. Then $(W, \tau, \tau \circ F_\lambda, \mathcal{E}^\infty)$ is the one-parameter family of Bäcklund transformations

$$v_x = -u_x - \frac{1}{2}(v-u)^2 - 2\lambda,$$
$$v_t = u_{xxx} + u_x^2 - 4\lambda u_x - 8\lambda^2 + 2u_{xx}(u-v) + (u_x - 2\lambda)(u-v)^2$$

constructed by Wahlquist and Estabrook [15]. Consider the group

$$A_\lambda \colon u(x,t) \mapsto u(x - 6\lambda t, t) + \lambda x - 3\lambda^2 t \tag{11}$$

of symmetries of the potential KdV equation and denote by $\bar{A}_\lambda \colon W \to W$ the diffeomorphic lifting of A_λ acting on v as follows

$$\bar{A}_\lambda^*(v) = v - (\lambda x - 3\lambda^2 t).$$

Then we similarly have

$$\mathcal{C}_\lambda \mathrm{D}(W) = \bar{A}_{\lambda,*}(\mathcal{C}_0 \mathrm{D}(W)) \text{ and } F_\lambda = \bar{A}_\lambda \circ F_0 \circ \bar{A}_{-\lambda}.$$

Let us denote by

$$\mathrm{D}^g(\Lambda^i(W)) = \{\Omega \in \mathrm{D}^v(\Lambda^i(W)) \mid \Omega(f) = 0,\ \forall f \in C^\infty(\mathcal{E}^\infty)\}$$

the module of τ-vertical derivations.

Lemma 1. *The modules* $\mathrm{D}^g(\Lambda^i(W))$ *are invariant with respect to the differential* ∂_τ:

$$\partial_\tau\big(\mathrm{D}^g(\Lambda^i(W))\big) \subset \mathrm{D}^g(\Lambda^{i+1}(W)).$$

Proof. Let $\Omega \in \mathrm{D}^g(\Lambda^i(W))$ and $f \in C^\infty(\mathcal{E}^\infty)$. Then due to the definition of the Frölicher-Nijenhuis bracket one has

$$(\partial_\tau(\Omega))(f) = [\![U_\tau, \Omega]\!](f) = \mathrm{L}_{U_\tau}(\Omega(f)) - (-1)^\Omega \mathrm{L}_\Omega(U_\tau(f)).$$

The first summand vanishes, since $\Omega \in \mathrm{D}^g(\Lambda^i(W))$. On the other hand, $U_\tau(f) = U_{\mathcal{E}}(f)$ and consequently is a one-form on $\mathcal{E}^\infty$. Hence, the second summand vanishes as well. □

Denote by $\partial_g\colon \mathrm{D}^g(\Lambda^i(W)) \to \mathrm{D}^g(\Lambda^{i+1}(W))$ the restriction of ∂_τ to $\mathrm{D}^g(\Lambda^i(W))$ and by

$$\partial_s\colon \mathrm{D}^s(\Lambda^i(W)) \to \mathrm{D}^s(\Lambda^{i+1}(W))$$

the corresponding quotient complex, where, by definition,

$$\mathrm{D}^s(\Lambda^i(W)) = \mathrm{D}^v(\Lambda^i(W))/\mathrm{D}^g(\Lambda^i(W)).$$

Then the short exact sequence of complexes

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & \mathrm{D}^g(W) & \xrightarrow{\partial_g} & \mathrm{D}^g(\Lambda^1(W)) \to \cdots \to & & \mathrm{D}^g(\Lambda^i(W)) & \xrightarrow{\partial_g} & \mathrm{D}^g(\Lambda^{i+1}(W)) \to \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & \mathrm{D}^v(W) & \xrightarrow{\partial_\tau} & \mathrm{D}^v(\Lambda^1(W)) \to \cdots \to & & \mathrm{D}^v(\Lambda^i(W)) & \xrightarrow{\partial_\tau} & \mathrm{D}^v(\Lambda^{i+1}(W)) \to \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & \mathrm{D}^s(W) & \xrightarrow{\partial_s} & \mathrm{D}^s(\Lambda^1(W)) \to \cdots \to & & \mathrm{D}^s(\Lambda^i(W)) & \xrightarrow{\partial_s} & \mathrm{D}^s(\Lambda^{i+1}(W)) \to \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0 & & 0
\end{array}$$

is defined.

Denote by $H^i_g(\mathcal{E};\tau)$ and $H^i_s(\mathcal{E};\tau)$ the cohomology of the top and bottom lines respectively. Then one has the long exact cohomology sequence

$$\begin{aligned}
&0 \to H^0_g(\mathcal{E};\tau) \to H^0_{\mathcal{C}}(\mathcal{E};\tau) \to H^0_s(\mathcal{E};\tau)\\
&\xrightarrow{\phi} H^1_g(\mathcal{E};\tau) \to H^1_{\mathcal{C}}(\mathcal{E};\tau) \to H^1_s(\mathcal{E};\tau) \to \cdots\\
&\to H^i_g(\mathcal{E};\tau) \to H^i_{\mathcal{C}}(\mathcal{E};\tau) \to H^i_s(\mathcal{E};\tau) \to \cdots,
\end{aligned} \tag{12}$$

where ϕ is the connecting homomorphism.

Similar to Theorem 1, we have the following result:

Proposition 6. *In the situation above one has*:

(1) *The module $H^0_g(\mathcal{E};\tau)$ consists of "gauge" symmetries in the covering τ, i.e., of nonlocal τ-symmetries vertical with respect to the projection τ.*
(2) *The module $H^0_s(\mathcal{E};\tau)$ coincides with the set of τ-shadows in the covering τ.*
(3) *The module $H^1_g(\mathcal{E};\tau)$ consists of equivalence classes of deformations of the covering structure U_τ acting trivially on the equation structure $U_{\mathcal{E}}$.*

Now, combining the last result with exact sequence (12), we obtain the following fundamental theorem:

Theorem 2. *Let $\tau\colon W \to \mathcal{E}$ be a covering and $A_\lambda\colon W \to W$ be a smooth family of diffeomorphisms such that $A_0 = \mathrm{id}$ and $\tau_\lambda = \tau \circ A_\lambda\colon W \to \mathcal{E}$ is a covering for any $\lambda \in \mathbb{R}$. Then U_{τ_λ} is of the form*

$$U_{\tau_\lambda} = U_\tau + \lambda[\![U_\tau, X]\!] + O(\lambda^2), \tag{13}$$

where X is a τ-shadow, i.e., all smooth families corresponding to the covering τ are infinitesimally identified with $\operatorname{im}\partial_s$.

Proof. The family of coverings τ_λ is a deformation of τ. Since we work with deformations which leave the equation structure unchanged, then, by Proposition 6, their infinitesimal parts are elements of $H^1_g(\mathcal{E};\tau)$. Let Ω be such an element.

Now, by Remark 5, the deformation we are dealing with is to be trivial as a deformation of W endowed with the structure U_τ. On the infinitesimal level, this means that the image of Ω in $H^1(\mathcal{E};\tau)$ should vanish. But by exactness of (12) we see that $\Omega = \phi(X)$ for some $X \in H^0_s(\mathcal{E};\tau)$. It now suffices to note that by construction of the connecting homomorphism, $\phi(X) = [\![U_\tau, X]\!]$.

The family A_λ allows us to find the shadow explicitly. Namely, we obviously have

$$\begin{aligned}\left.\frac{d}{d\lambda}\right|_{\lambda=0} U_{\tau_\lambda} &= \left.\frac{d}{d\lambda}\right|_{\lambda=0} A_{\lambda,*}(\mathrm{L}_{U_\tau})\\ &= \left.\frac{d}{d\lambda}\right|_{\lambda=0} A^*_\lambda \circ \mathrm{L}_{U_\tau} \circ (A^*_\lambda)^{-1} = [\mathrm{L}_Y, \mathrm{L}_{U_\tau}] = \mathrm{L}_{[\![Y,U_\tau]\!]},\end{aligned}$$

where

$$Y = \left.\frac{dA_\lambda}{d\lambda}\right|_{\lambda=0} \in \mathrm{D}(W).$$

Hence, infinitesimal action is given by the Frölicher-Nijenhuis bracket. In the coset $Y \bmod \mathcal{C}_\tau\mathrm{D}(W)$ there exists a unique $(\pi_{\mathcal{E}} \circ \tau)$-vertical representative X, and the corresponding element $[X] \in H^0_s(\mathcal{E};\tau)$ is the required shadow. □

Remark 6. Consider the one-parameter families of coverings τ_λ and τ'_λ from Examples 4 and 5 respectively. The classical infinitesimal symmetries corresponding to the one-parameter groups A_λ of finite symmetries are

$$x\frac{\partial}{\partial x} - t\frac{\partial}{\partial t} \qquad \text{for the sine-Gordon equation,}$$

$$x\frac{\partial}{\partial u} - 6t\frac{\partial}{\partial x} \qquad \text{for the potential KdV equation.}$$

The corresponding higher symmetries are shadows in τ_1 and τ_0' respectively (see Example 2). These shadows determine the infinitesimal parts of the families U_{τ_λ} and $U_{\tau'_\lambda}$ according to Theorem 2.

Remark 7. Denote by $\mathcal{C}ov(\tau)$ the "manifold" of all coverings obtained from the covering τ by the above described way. Then from exactness of (12) it follows that the "tangent plane" to $\mathcal{C}ov(\tau)$ at τ is identified with the space $\operatorname{shad}_\tau \mathcal{E} / \overline{\operatorname{sym}}_\tau \mathcal{E}$, where $\operatorname{shad}_\tau \mathcal{E} = H^0_s(\mathcal{E};\tau)$ is the space of all τ-shadows. Finally, the space $\overline{\operatorname{sym}}_\tau \mathcal{E} = \operatorname{sym}_\tau \mathcal{E} / \operatorname{sym}^g_\tau \mathcal{E}$ is the quotient of all τ-symmetries over gauge ones.

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References

[1] A. V. Bocharov, V. N. Chetverikov, S. V. Duzhin, N. G. Khor′kova, I. S. Krasil′shchik, A. V. Samokhin, Yu. N. Torkhov, A. M. Verbovetsky and A. M. Vinogradov, Symmetries and Conservation Laws for Differential Equations of Mathematical Physics, American Mathematical Society, Providence, RI, 1999. Edited and with a preface by Krasil′shchik and Vinogradov.

[2] R. K. Dodd, J. C. Eilbeck, J. D. Gibbons and H. C. Morris, Solitons and Nonlinear Wave Equations, Academic Press, 1982.

[3] I. S. Krasil′shchik, *Algebras with flat connections and symmetries of differential equations*, In "Lie groups and Lie algebras", pp. 407–424. Kluwer Acad. Publ., Dordrecht, 1998.

[4] I. S. Krasil′shchik, *Cohomology background in geometry of PDE*, In "Secondary calculus and cohomological physics (Moscow, 1997)", Contemporary Math., **219**, 121–139, Amer. Math. Soc., Providence, RI, 1998.

[5] I. S. Krasil′shchik, *Some new cohomological invariants for nonlinear differential equations*, Differential Geom. Appl., **2** (1992) no. 4, 307–350.

[6] I. S. Krasil′shchik, *Notes on coverings and Bäcklund transformations*, Preprint of the Erwin Schroedinger International Inst. for Math. Phys. 1995, no. 260, Wien, 15 pp. `http://www.esi.ac.at/ESI-Preprints.htm`.

[7] I. S. Krasil′shchik, V. V. Lychagin and A. M. Vinogradov, Geometry of Jet Spaces and Nonlinear Partial Differential Equations, Gordon and Breach, New York, 1986.

[8] I. S. Krasil′shchik and A. M. Verbovetsky, Homological methods in equations of mathematical physics, Open Education and Sciences, Opava, 1998. E-print `math.DG/9808130`.

[9] I. S. Krasil′shchik and A. M. Vinogradov, *Nonlocal trends in the geometry of differential equations: symmetries, conservation laws, and Bäcklund transformations*, Acta Appl. Math., **15** (1989) no. 1-2, 161–209.

[10] M. Marvan, *On the horizontal gauge cohomology and non-removability of the spectral parameter*, Acta Appl. Math. (to appear).

[11] M. Marvan, *Some local properties of Bäcklund transformations* Acta Appl. Math., **54** (1998), 1–25.

[12] M. Marvan, *Another look on recursion operators*, In "Differential geometry and applications (Brno, 1995)", pp. 393–402. Masaryk Univ., Brno, 1996. E-print `www.emis.de/proceedings/6ICDGA/`.

[13] C. Rogers and W. F. Shadwick, Bäcklund transformations and their applications, Academic Press, New York, 1982.

[14] A. M. Vinogradov, *The C-spectral sequence, Lagrangian formalism, and conservation laws. I. The linear theory. II. The nonlinear theory*, J. Math. Anal. Appl., **100** (1984), 1–129.

[15] H. D. Wahlquist and F. B. Estabrook, *Bäcklund transformation for solitons of the Korteweg-de Vries equation*, Phys. Rev. Lett., **31** (1973), 1386–1390.

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Submanifolds with Degenerate Gauss Mappings in Spheres

Goo Ishikawa, Makoto Kimura and Reiko Miyaoka

§1. Introduction

Let M be a connected l-dimensional C^∞ manifold. An immersion $f\colon M \to S^n$ to the sphere (resp. $f\colon M \to \mathbb{RP}^n$ to the projective space) is called *tangentially degenerate* (or, *developable*, or, *strongly parabolic*) if its Gauss mapping $\gamma\colon M \to G_{l+1}(\mathbb{R}^{n+1})$ has rank $< l$. Here $G_{l+1}(\mathbb{R}^{n+1})$ denotes the Grassmannian of $(l+1)$-dimensional linear subspaces in $\mathbb{R}^{n+1}$. A submanifold of S^n or $\mathbb{RP}^n$ is called *tangentially degenerate* (or, *developable*, or, *strongly parabolic*) if so is the inclusion.

In the present paper we construct new examples of tangentially degenerate compact submanifolds satisfying the equality for the inequality proved by Ferus [19]. Remark that, if we have a tangentially degenerate immersed submanifold in S^n then, via the canonical double covering $\pi\colon S^n \to \mathbb{RP}^n$, we have a tangentially degenerate immersed submanifold in $\mathbb{RP}^n$.

Remark also that the notion of tangential degeneracy is invariant under the projective transformations. Recall that $\mathbb{RP}^n = G_1(\mathbb{R}^{n+1})$ and $S^n = \widetilde{G}_1(\mathbb{R}^{n+1})$ (oriented Grassmannian) have natural projective structures, respectively. In fact, M. A. Akivis clearly stated in [3] and [4] that the study of tangentially degenerate submanifolds belongs to projective geometry. Then our standpoint is as follows: We do not need the metric structures on them for the formulation of the results, while, for the proofs of the results, we use freely the metric structures.

Let M^l be compact and connected, and $f\colon M \to S^n$ a tangentially degenerate immersion. Denote by r the maximal rank of the Gauss

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mapping γ. Then, by [19], we know that there exists a number $F(l)$ (Ferus number), depending only on the dimension l of M, such that, if $r < F(l)$ then $r = 0$, therefore $M = S^l$ and $f(M)$ is a great l-sphere in S^n. The Ferus number $F(l)$ is defined by

$$F(l) := \min\{k \mid A(k) + k \geq l\},$$

where $A(k)$ is the Adams number: the maximal number of linearly independent vector fields over the sphere S^{k-1}.

In other words, if $f\colon M^l \to S^n$ is a tangentially degenerate immersion of rank r, and $f(M)$ is not a great l-sphere, then $F(l) \leq r$ (Ferus inequality).

Then the problem we are concerning with is the following:

Problem. Is the inequality $r < F(l)$ best possible for the implication $r = 0$? Do there exist tangentially degenerate immersions $M^l \to S^n$ with $r = F(l)$? Moreover can we classify tangentially degenerate immersions $M^l \to S^n$ with $r = F(l)$?

In contrast to the tangentially degenerate submanifolds in $\mathbb{R}^n$ such as cylinders, cones or tangent developable of space curves, all of which have singularities when considered in $\mathbb{RP}^n$, we construct many *compact* tangentially degenerate submanifolds in the sphere, some of which even satisfy the Ferus equality. In §2 we recall, in more detail, the Ferus inequality on tangentially degenerate immersions. In §3, degeneracy of the Gauss mapping of isoparametric hypersurfaces and their focal submanifolds in the sphere is discussed as a model case. These provide us with an infinitely many tangentially degenerate submanifolds in the sphere, which satisfy the Ferus equality for $(l,r) = (2^p + 1, 2^p)$ or $(2^q + 3, 2^q)$ for $p \geq 2$, $q \geq 3$. This answers the first and the second problem affirmatively. In §9, we construct further homogeneous examples with $r = F(l)$ for $(l,r) = (3,2)$, $(5,4)$, $(6,4)$, $(9,8)$, $(10,8)$, $(12,8)$, $(17,16)$, $(18,16)$, $(20,16)$, $(25,14)$, $(26,24)$, $(28,24)$, and possibly more. In order to describe the idea of the construction, in §4, the Stiefel manifold $V_2(\mathbb{R}^{n+1})$ of orthonormal 2-vectors is introduced as a circle bundle over a complex quadric Q^{n-1}. Then in §5, we study complex submanifolds $\varphi\colon \Sigma \to Q^{n-1}$, and using this, in §6, we construct a natural map $\Phi\colon M = \varphi^* V_2(\mathbb{R}^{n+1}) \to S^n$ from the pullback bundle over Σ to the sphere, such that the image of Φ is the union of l-parameter family of great circles in S^n. Then we show that, on the open set of maximal rank of $d\Phi$ in M, each fiber of the circle bundle $\varphi^* V_2(\mathbb{R}^{n+1}) \to \Sigma$ lies in the kernel of the differential of the Gauss mapping of Φ, so Φ is tangentially degenerate. Examples in this context are given in §7.

In §8, we recall *austere* submanifolds defined by Harvey and Lawson [23]. They proved that from austere submanifolds in spheres, one can construct *special Lagrangian* and volume minimizing varieties in complex Euclidean spaces. Then we show that if a complex submanifold $\varphi\colon \Sigma \to Q^{n-1}$ is *first order isotropic*, then the corresponding map $\Phi\colon \varphi^* V_2(\mathbb{R}^{n+1}) \to S^n$ is always an immersion and is austere. In §9, we give further examples of homogeneous *austere* submanifolds M^l, which can be considered as a generalization of *Cartan's isoparametric hypersurfaces*, see §2. More precisely, M is the total space of $\mathbb{KP}^1$-bundle over 2-plane Grassmannian $G_2(\mathbb{K}^{n+1})$ where $\mathbb{K} = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ and each fiber of the bundle lies in the kernel of the Gauss mapping of M.

In §10, we give a classification of tangentially degenerate hypersurfaces in S^4 with $r = 2$. Moreover, using Bryant's result [9], we construct an example of tangentially degenerate immersions $M^3 \to S^4$ such that the rank of Gauss mapping is not constant.

§2. Ferus inequality for submanifolds with degenerate Gauss mapping

The proof of the Ferus inequality is achieved by considering the Levi-Civita connection of the ordinary metric on S^n, the co-nullity operator, and a matrix Riccati-type equation [18] and [19]. Here we review the outline of the proof: Let $f\colon M^l \to S^n$ be a tangentially degenerate immersion. Assume that the induced metric on M from f is complete. It is the case if M is compact. Now, assume $0 < r < l$, for the maximal rank r of the Gauss mapping of f. Let $D \subset TM$ denote the *Monge-Ampère distribution*, namely the kernel of the differential of Gauss mapping, along the open subset U of M consisting of points where the Gauss mapping has the maximal rank r. Remark that $f(U)$ is a union of totally geodesic spheres of dimension $l - r$. Take x from U. Then we get a linear mapping $D_x^\perp - \{0\} \to (D_x^\perp)^{l-r+1}$ defined by $Y \mapsto (Y, C_{X_1}Y, \ldots, C_{X_{l-r}}Y)$, where $X_1, \ldots, X_{l-r}$ are basis of D_x and $C_X Y := -\operatorname{pr}(\nabla_Y X)$ denotes the co-nullity operator, $\operatorname{pr}\colon T_xM \to D_x^\perp$ being the projection. Then, by the assumption $r \neq 0$ and by an argument on Riccati equation, we conclude that Y, $C_{X_1}Y, \ldots, C_{X_{l-r}}Y$ are linearly independent. Thus we have $l - r \leq A(r)$, Adams number. Therefore $l \leq A(r) + r$.

It is well known ([6] and [19]) that the Adams number is determined by

$$A((2k+1)2^{c+4d}) = 2^c + 8d - 1, \quad \text{where } 0 \leq c \leq 3,\ 0 \leq d.$$

Therefore the Ferus number $F(l)$ is given for $l \leq 24$, by

(1) $$F(l) = \{\text{the highest power of 2 not larger than } l\}.$$

Remark that $F(l) = l - \mu(l)$ and $A(l) = \rho(l) - 1$, using the original notations in [6] and [19].

Regard S^n as the unit hypersphere $S^n(1)$ in $\mathbb{R}^{n+1}$ with the ordinary metric. For a submanifold M in $S^n(1)$, the *index of relative nullity* $\nu(x)$ at $x \in M$, introduced by Chern and Kuiper [17] and [30], is defined as the dimension of

$$\{X \in T_x(M) \mid \sigma(X, Y) = 0 \text{ for any } Y \in T_x(M)\},$$

σ being the second fundamental form. Notice that the rank $r(x)$ of differential mapping $d\gamma_x \colon T_x M \to T_{\gamma(x)} G_{l+1}(\mathbb{R}^{n+1})$ at $x \in M$ is related to $\nu(x)$ by $r(x) = l - \nu(x)$, because

(2) $$\ker d\gamma_x = \{X \in T_x(M) \mid \sigma(X, Y) = 0 \text{ for any } Y \in T_x(M)\}.$$

Therefore the minimum ν of $\nu(x)$ over $x \in M$ is equal to $l - r$.

Theorem 2.1 ([6] , [12] and [19]). *Let M be a complete submanifold of n-dimensional unit sphere with* $\dim_{\mathbb{R}} M = l$. *If $r < F(l)$, then $r = 0$ and M is totally geodesic.*

For instance, in view of (1), we see that (cf. [6])

(i) if l is a power of 2, $r < l$ implies $r = 0$,
(ii) if $l = 3$, $r < 2$ implies $r = 0$,
(iii) if $5 \leq l \leq 7$, $r < 4$ implies $r = 0$,
(iv) if $9 \leq l \leq 15$, $r < 8$ implies $r = 0$,
(v) if $17 \leq l \leq 24$, $r < 16$ implies $r = 0$,
(vi) if $25 \leq l \leq 31$, $r < 24$ implies $r = 0$.

We have examples of compact connected tangentially degenerate embedded hypersurfaces $M^3 \subset S^4$, $M^6 \subset S^7$, $M^{12} \subset S^{13}$, $M^{24} \subset S^{25}$ with $r = 2, 4, 8, 16$ respectively; *Cartan hypersurfaces* [12] and [27]. Remark that $F(3) = 2$, $F(6) = 4$, $F(12) = 8$ and $F(24) = 16$. Each of them is defined by a real cubic polynomial, and it is a closed orbit of projective actions of SO(3), SU(3), Sp(3), F_4 on $S^n = \widetilde{G}_1(\mathbb{R}^{n+1})$, $n = 4, 7, 13, 25$, respectively. Their projective dual $M^\vee = \gamma(M) \in G_1(\mathbb{R}^{(n+1)*}) = \mathbb{RP}^{n*}$ are the images of Veronese embeddings of projective planes $\mathbb{KP}^2$, for $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ (the Cayley's octonians) respectively. The latter are real parts with respect to properly chosen real forms of *Severi varieties* in complex projective spaces, [45]. The reason of this coincidence has not been explained fully, as far as the authors know.

In [26], the first author showed that a homogeneous compact hypersurface in $\mathbb{RP}^n$ which is tangentially degenerate is projectively equivalent to a hyperplane or a Cartan hypersurface. This result is the motivation of our study. Cartan hypersurfaces are well-known as isoparametric hypersurfaces, and we give some observation concerning this. Moreover, motivated by their sphere bundle structure, we construct many examples of tangentially degenerate submanifolds (§7, §9). Simple examples are obtained as follows:

Lemma 2.2. *Let $\Sigma^k \subset \mathbb{CP}^n$ be a complex submanifold of complex dimension k. Consider the Hopf fibration $\pi\colon S^{2n+1}(\subset \mathbb{C}^{n+1}) \to \mathbb{CP}^n$, and set $M^{2k+1} := \pi^{-1}(\Sigma) \subset S^{2n+1}$. Then M is a submanifold with degenerate Gauss mapping of S^{2n+1}. If Σ is compact and not a complex projective subspace, then the rank of Gauss mapping is equal to $2k$.*

Remark that the codimension of M^{2k+1} equals to $2n+1-(2k+1) = 2(n-k) \geq 2$. In some cases this construction provides examples of compact tangentially degenerate submanifolds satisfying the equality $r = F(l)$: $(l, r) = (3,2)$, $(5,4)$, $(9,8)$, $(17,16)$, $(25,24)$ (see §7, §9).

Proof of Lemma 2.2. The Gauss mapping $\gamma\colon M \to G_{2k+2}(\mathbb{R}^{2n+2})$ is decomposed into the projection $\pi|_M\colon M \to \Sigma$, the complex Gauss mapping $\gamma_{\mathbb{C}}\colon \Sigma \to G_{k+1}(\mathbb{C}^{n+1})$ and the natural inclusion $G_{k+1}(\mathbb{C}^{n+1}) \hookrightarrow G_{2k+2}(\mathbb{R}^{2n+2})$. Therefore M is tangentially degenerate; γ is degenerate along the Hopf fibers. We refer to the following:

Theorem 2.3 ([2] and [21]). *Let Σ^k be a k-dimensional compact complex submanifold in $\mathbb{CP}^n$ and let $\gamma_{\mathbb{C}}\colon \Sigma \to G_{k+1}(\mathbb{C}^{n+1})$ be the complex Gauss mapping of Σ in $\mathbb{CP}^n$. If the rank of $\gamma_{\mathbb{C}}$ is less than* $\dim_{\mathbb{C}} \Sigma$, *then Σ is necessarily a complex projective subspace $\mathbb{CP}^k$ in $\mathbb{CP}^n$.*

§3. Examples related to isoparametric hypersurfaces

Hypersurfaces in the sphere are tangentially degenerate if they have zero principal curvature. In the simplest case where the principal curvatures are constant, i.e. in the case of isoparametric hypersurfaces (see [43] for general facts), the principal curvatures are given by

$$\lambda_i = \cot\left(\theta_0 + \frac{\pi(i-1)}{g}\right), \quad 0 < \theta_0 < \frac{\pi}{g}, \quad i = 1, \ldots, g$$

where $g = 1$, 2, 3, 4, 6. Then the tangentially degenerate isoparametric hypersurfaces are

(i) $g = 1$ and M is a great hypersphere

(ii) $g = 3$ and M is the Cartan hypersurfaces ([12]).

Each isoparametric hypersurface has two focal submanifolds $M_\pm$, and some of them are tangentially degenerate. It is well known that all the shape operators S_N of $M_\pm$ have constant eigenvalues given by

(i) 0 for $g = 2$
(ii) $\pm 1/\sqrt{3}$ for $g = 3$
(iii) ± 1, 0 for $g = 4$
(iv) $\pm\sqrt{3}$, $\pm 1/\sqrt{3}$, 0 for $g = 6$

When $g = 2$, $M_\pm$ are totally geodesic subspheres hence tangentially degenerate. Other possibilities are when $g = 4$ or 6. If the kernel of the shape operators have a common non-trivial vector, they are tangentially degenerate. When $g = 6$ and M is homogeneous, both focal submanifolds are tangentially degenerate [35]. Note that they are given by singular orbits of the linear isotropy representation of the rank two symmetric spaces $G_2/SO(4)$ and $G_2 \times G_2/G_2$. Moreover, these satisfy the Ferus equality for $(l, r) = (5, 4)$, $(10, 8)$.

Examples of isoparametric hypersurfaces with $g = 4$ are given in [39] and [20]. Take the principal curvatures λ_1, λ_2, λ_3, λ_4 so that m_1 and m_2 are the multiplicities of λ_{odd} and λ_{even} where $m_1 \leq m_2$. Examples in [39] are classified in [20] as

(i) Homogeneous ones of Clifford type:

$$(m_1, m_2) = (1, k),\ (2, 2k-1),\ (4, 4k-1),\ (9, 6)$$

(ii) Homogeneous ones with $(m_1, m_2) = (2, 2)$ and $(4, 5)$
(iii) Inhomogeneous ones of Clifford type: $(m_1, m_2) = (3, 4k)$, $(7, 8k)$

Following [20], we assume $\dim M_- = 2m_1 + m_2$ and $\dim M_+ = m_1 + 2m_2$. Now we summarize results and remarks on tangential degeneracy of these examples.

Proposition 3.1. *Let M be a homogeneous isoparametric hypersurface with $g = 4$ and $(m_1, m_2) = (1, k-2)$, $k \geq 3$, $(2, 2k-3)$, $(4, 4k-5)$, $k \geq 2$. When $(m_1, m_2) = (1, k-2)$, the focal submanifolds M_+ is tangentially degenerate with $(l, r) = (2k-3, 2k-4)$; while M_- is not. When $(m_1, m_2) = (2, 2k-3)$, $(4, 4k-5)$, M_- is tangentially degenerate with $(l, r) = (2k+1, 2k)$, $(4k+3, 4k)$, respectively; while M_+ is not. In particular, there exist infinitely many tangentially degenerate homogeneous submanifolds in the sphere, which satisfy the Ferus equality.*

On the last assertion, we can easily show that for $p \geq 1$ and $q \geq 2$, $F(2^p + 1) = 2^p$ and $F(2^q + 3) = 2^q$ hold, hence examples are given by M_- of isoparametric hypersurfaces with $(m_1, m_2) = (2, 2^p - 3)$, $p \geq 2$, and $(4, 2^q - 5)$, $q \geq 3$.

Proposition 3.2. *Let M be a homogeneous isoparametric hypersurface with $g = 4$. When $(m_1, m_2) = (2, 2)$, the focal submanifolds M_+ is tangentially degenerate with $(l, r) = (6, 4)$, satisfying the Ferus equality; while M_- is not. When $(m_1, m_2) = (4, 5)$, M_- is tangentially degenerate with $(l, r) = (13, 12)$; while M_+ is not.*

Corollary 3.3. *The focal submanifold M_+ of homogeneous isoparametric hypersurface M with $g = 4$ and $(m_1, m_2) = (1, k-2)$, $(2, 2)$, and M_- with $(2, 2k-3)$, $(4, 4k-5)$, $(4, 5)$ is foliated by totally geodesic subspheres of dimension* 1, 2 *and* 1, 3, 1, *respectively. Along these subspheres, the tangent space is parallel.*

Remark 3.4. As for the case (iii), M_+ is not tangentially degenerate, since by Theorem 5.8 in [20], there exists a point $x \in M_+$ at which $d(x) = \dim \bigcap_n \operatorname{Ker} S_N = 0$. In the case $(m_1, m_2) = (3, 4k)$, M_- is homogeneous (6.4 in [20]). We will discuss these and the remaining homogeneous case with $(m_1, m_2) = (9, 6)$ in another occasion, as well as all other inhomogeneous examples of Clifford type. The tangential degeneracy of M_+ for $(m_1, m_2) = (1, k-2)$ and of M_- for $(2, 2k-3)$, $(4, 5)$ follows from Lemma 2.2, since the odd dimensional focal submanifolds given by singular orbits of the linear isotropy representation of a Hermitian symmetric space of rank two is tangentially degenerate. Then use the classification in [24]. In the Appendix, we give a systematic proof of the propositions, which is applicable to all the homogeneous cases.

§4. Stiefel manifolds and complex quadrics

Let W be a real vector space with Euclidean inner product $\langle\ ,\ \rangle$. By a 2-*frame* in W we mean an ordered set of 2 orthonormal vectors in W. Let $V_2(W)$ be the space of 2-frames in W, i.e.,

$$V_2(W) = \{(\mathbf{f}_1, \mathbf{f}_2) \in W \times W \mid \langle \mathbf{f}_\alpha, \mathbf{f}_\beta \rangle = \delta_{\alpha\beta}\ (\alpha,\ \beta = 1,\ 2)\}. \tag{3}$$

Then $V_2(W)$ is a Stiefel manifold with $\dim_{\mathbb{R}} V_2(W) = 2\dim_{\mathbb{R}} W - 3$. The tangent space $T_{(\mathbf{f}_1, \mathbf{f}_2)} V_2(W)$ is

$$\mathbb{R}(-\mathbf{f}_2, \mathbf{f}_1) \oplus \{(\mathbf{x}_1, \mathbf{x}_2) \in W \times W \mid \mathbf{x}_1,\ \mathbf{x}_2 \perp \operatorname{span}\{\mathbf{f}_1, \mathbf{f}_2\}\}.$$

The inner product on $W \times W$ defined by

$$\langle (\mathbf{x}_1, \mathbf{x}_2), (\mathbf{y}_1, \mathbf{y}_2) \rangle = \langle \mathbf{x}_1, \mathbf{y}_1 \rangle + \langle \mathbf{x}_2, \mathbf{y}_2 \rangle$$
$$\text{for} \quad (\mathbf{x}_1, \mathbf{x}_2),\ (\mathbf{y}_1, \mathbf{y}_2) \in W \times W$$

induces a Riemannian metric $\widetilde{g}$ on $V_2(W)$.

Let $\widetilde{G}_2(W)$ be the space of oriented 2-planes in W. Then $V_2(W)$ is a principal fiber bundle over $\widetilde{G}_2(W)$ with structure group S^1 and projection map $\pi\colon V_2(W) \to \widetilde{G}_2(W)$ defined by

$$\pi((\mathbf{f}_1, \mathbf{f}_2)) = \operatorname{span}\{\mathbf{f}_1, \mathbf{f}_2\}.$$

For each $(\mathbf{f}_1, \mathbf{f}_2) \in V_2(W)$, the fiber $\pi^{-1}(\pi(\mathbf{f}_1, \mathbf{f}_2))$ is

$$\{(\cos\theta\, \mathbf{f}_1 - \sin\theta\, \mathbf{f}_2, \sin\theta\, \mathbf{f}_1 + \cos\theta\, \mathbf{f}_2) \in V_2(W) \mid \theta \in S^1\}.$$

The Riemannian metric $\widetilde{g}$ on $V_2(W)$ is invariant by the structure group. Thus we may define a Riemannian metric g on $\widetilde{G}_2(W)$ such that π is a Riemannian submersion.

The distribution given by

$$\begin{aligned} &T'_{(\mathbf{f}_1,\mathbf{f}_2)}(V_2(W)) \\ &= \{(\mathbf{x}_1, \mathbf{x}_2) \in T_{(\mathbf{f}_1,\mathbf{f}_2)}(V_2(W)) \mid \mathbf{x}_1,\ \mathbf{x}_2 \perp \operatorname{span}\{\mathbf{f}_1, \mathbf{f}_2\}\} \end{aligned} \tag{4}$$

defines a connection in the principal fiber bundle $V_2(W)(\widetilde{G}_2(W), S^1)$, because $T'_{(\mathbf{f}_1,\mathbf{f}_2)}$ is complementary to the subspace $\mathbb{R}(-\mathbf{f}_2, \mathbf{f}_1)$ tangent to the fiber through $(\mathbf{f}_1, \mathbf{f}_2)$, and invariant under the S^1-action. The natural projection π of $V_2(W)$ onto $\widetilde{G}_2(W)$ induces a linear isomorphism of $T'_{(\mathbf{f}_1,\mathbf{f}_2)}(V_2(W))$ onto $T_p(\widetilde{G}_2(W))$, where $\pi((\mathbf{f}_1, \mathbf{f}_2)) = p$. The complex structure $\widetilde{J}$ on $T'_{(\mathbf{f}_1,\mathbf{f}_2)}(V_2(W))$ defined by

$$(\mathbf{x}_1, \mathbf{x}_2) \mapsto (-\mathbf{x}_2, \mathbf{x}_1) \tag{5}$$

induces a canonical complex structure J on $\widetilde{G}_2(W)$ through $d\pi$.

Let $\widetilde{Q}(\mathbb{C}^{m+1})$ be a submanifold of $S^{2m+1}(\sqrt{2}\,)$ defined by

$$\widetilde{Q}(\mathbb{C}^{m+1}) = \{\mathbf{z} \in S^{2m+1}(\sqrt{2}\,) \mid {}^t\mathbf{z}\mathbf{z} = 0\}. \tag{6}$$

There is an identification between $\widetilde{Q}(\mathbb{C}^{m+1})$ and $V_2(\mathbb{R}^{m+1})$ as:

$$\widetilde{Q}(\mathbb{C}^{m+1}) \ni \mathbf{z} \mapsto (\operatorname{Re}\mathbf{z}, \operatorname{Im}\mathbf{z}) \in V_2(\mathbb{R}^{m+1}).$$

Then $\widetilde{G}_2(\mathbb{R}^{m+1})$ is identified with the complex quadric

$$Q^{m-1} = \{\pi(\mathbf{z}) \in \mathbb{CP}^m \mid \mathbf{z} \in \widetilde{Q}(\mathbb{C}^{m+1})\}, \tag{7}$$

such that the following diagram is commutative:

$$\begin{array}{ccc} \widetilde{Q}(\mathbb{C}^{m+1}) & \xrightarrow{\ \sim\ } & V_2(\mathbb{R}^{m+1}) \\ {\scriptstyle\pi}\big\downarrow & & \big\downarrow{\scriptstyle\pi} \\ Q^{m-1} & \xrightarrow{\ \sim\ } & \widetilde{G}_2(\mathbb{R}^{m+1}). \end{array}$$

§5. Complex submanifolds in complex quadrics

Consider an isometric immersion $\varphi\colon \Sigma \to Q^{n-1} \cong \widetilde{G}_2(\mathbb{R}^{n+1})$ of a Riemannian manifold Σ with $\dim_{\mathbb{R}} \Sigma = l$ to the complex quadric. Let $\eta\colon U' \to V_2(\mathbb{R}^{n+1})$ be a local cross section of the circle bundle $\pi\colon V_2(\mathbb{R}^{n+1}) \to \widetilde{G}_2(\mathbb{R}^{n+1})$ on an open set $U' \subset \widetilde{G}_2(\mathbb{R}^{n+1})$ and let U be an open neighborhood of a point p in Σ such that $\varphi(U) \subset U'$. Denote

$$(\eta \circ \varphi)(q) = (\mathbf{f}_1(q), \mathbf{f}_2(q)) \quad \text{for} \quad q \in U \subset \Sigma. \tag{8}$$

Here $\mathbf{f}_\alpha$ is an $\mathbb{R}^{n+1}$-valued function on U with $\langle \mathbf{f}_\alpha, \mathbf{f}_\beta \rangle = \delta_{\alpha\beta}$ (α, $\beta = 1, 2$). Write differential maps of $\mathbf{f}_\alpha\colon U \to \mathbb{R}^{n+1}$ ($\alpha = 1, 2$) as

$$d\mathbf{f}_1(X) = \lambda(X)\mathbf{f}_2 + \mathbf{p}(X), \quad d\mathbf{f}_2(X) = -\lambda(X)\mathbf{f}_1 + \mathbf{q}(X) \qquad \text{for} \quad X \in T_q(\Sigma), \tag{9}$$

where λ is a 1-form on U, and $\mathbf{p}$, $\mathbf{q}$ are $\mathbb{R}^{n+1}$-valued 1-forms on U such that $\mathbf{p}(X)$, $\mathbf{q}(X) \perp \operatorname{span}\{\mathbf{f}_1, \mathbf{f}_2\}$. Hence differentials of $\eta \circ \varphi$ and φ are given by

$$\begin{aligned} d(\eta \circ \varphi)(X) &= (d\mathbf{f}_1(X), d\mathbf{f}_2(X)) \\ &= \big(\lambda(X)\mathbf{f}_2 + \mathbf{p}(X), -\lambda(X)\mathbf{f}_1 + \mathbf{q}(X)\big), \\ d\varphi(X) &= (d\pi \circ d(\eta \circ \varphi))(X) \\ &= d\pi(\mathbf{p}(X), \mathbf{q}(X)). \end{aligned} \tag{10}$$

Suppose that (Σ^m, J) is a Kähler manifold with $\dim_{\mathbb{C}} \Sigma = m$ and $\varphi\colon \Sigma^m \to Q^{n-1} \cong \widetilde{G}_2(\mathbb{R}^{n+1})$ ($m < n-1$) is a holomorphic isometric immersion. Then (5), (10) and $d\varphi \circ J = J \circ d\varphi$ imply

$$\begin{aligned} d\pi(\mathbf{p}(JX), \mathbf{q}(JX)) = d\varphi(JX) = J d\varphi(X) &= J d\pi(\mathbf{p}(X), \mathbf{q}(X)) \\ &= d\pi(-\mathbf{q}(X), \mathbf{p}(X)), \end{aligned}$$

and $\mathbf{q}(X) = -\mathbf{p}(JX)$. Hence

$$d\varphi(T_q(\Sigma)) = \{d\pi(\mathbf{p}(X), -\mathbf{p}(JX)) \mid X \in T_q(\Sigma)\}.$$

With respect to the metrics on Σ and Q^{n-1}, we have

$$\begin{aligned} \langle d\varphi(X), d\varphi(Y) \rangle &= \langle \mathbf{p}(X), \mathbf{p}(Y) \rangle + \langle \mathbf{p}(JX), \mathbf{p}(JY) \rangle \\ &= \langle X, Y \rangle. \end{aligned} \tag{11}$$

Let ∇^G and ∇^V be the Levi-Civita connections on $\widetilde{G}_2(\mathbb{R}^{n+1})$ and $V_2(\mathbb{R}^{n+1})$, respectively. If X and Y are vector fields on Σ, then we have

$\nabla^G_{d\varphi(X)} d\varphi(Y) = d\pi(\nabla^V_{d\varphi(X)'} d\varphi(Y)')$ where $d\varphi(X)'$ and $d\varphi(Y)'$ are the basic vector fields corresponding to $d\varphi(X)$ and $d\varphi(Y)$, respectively. Let σ^φ denote the second fundamental form of φ. Then we can see that

$$\begin{aligned}\sigma^\varphi(X,Y) = \text{component of } & d\pi((\nabla_X \mathbf{p})Y, -(\nabla_X \mathbf{p})JY) \\ & \text{orthogonal to } d\varphi(T_q(\Sigma)) \text{ in } T_{\varphi(q)}(\widetilde{G}_2(\mathbb{R}^{n+1})),\end{aligned}$$

where $\nabla\mathbf{p}$ denotes the covariant derivative of $\mathbf{p}\colon T_q(\Sigma) \to \mathbb{R}^{n+1}$. Let $\mathbf{s}\colon T_q(\Sigma) \times T_q(\Sigma) \to \mathbb{R}^{n+1}$ be a bilinear mapping defined by

$$\begin{aligned}\mathbf{s}(X,Y) = \text{the component of } & (\nabla_X \mathbf{p})Y \text{ orthogonal to} \\ & \{\mathbf{p}(X) \mid X \in T_q(\Sigma)\} \text{ in } \mathbb{R}^{n+1}.\end{aligned} \tag{12}$$

Then $\sigma^\varphi(X,Y) = \sigma^\varphi(Y,X)$ and $\sigma^\varphi(X,Y) + \sigma^\varphi(JX,JY) = 0$ imply that

$$\mathbf{s}(X,Y) = \mathbf{s}(Y,X), \quad \mathbf{s}(X,Y) + \mathbf{s}(JX,JY) = 0. \tag{13}$$

§6. Submanifolds with degenerate Gauss mapping in spheres

Let $\varphi\colon \Sigma \to Q^{n-1} \cong \widetilde{G}_2(\mathbb{R}^{n+1})$ be a mapping from a differentiable manifold Σ with $\dim_{\mathbb{R}} \Sigma = l$ to the complex quadric, and let $\pi_\varphi\colon \varphi^* V_2(\mathbb{R}^{n+1}) \to \Sigma$ be the pullback bundle of the circle bundle $\pi\colon V_2(\mathbb{R}^{n+1}) \to \widetilde{G}_2(\mathbb{R}^{n+1})$ with respect to φ:

$$\begin{array}{ccc} \varphi^* V_2(\mathbb{R}^{n+1}) & \xrightarrow{\psi} & V_2(\mathbb{R}^{n+1}) \\ {\scriptstyle\pi_\varphi}\downarrow & & \downarrow{\scriptstyle\pi} \\ \Sigma & \xrightarrow{\varphi} & \widetilde{G}_2(\mathbb{R}^{n+1}). \end{array} \tag{14}$$

Let $\Phi\colon \varphi^* V_2(\mathbb{R}^{n+1}) \to S^n(1)$ be the mapping defined by

$$\Phi = \mathrm{pr}_1 \circ \psi, \tag{15}$$

where $\psi\colon \varphi^* V_2(\mathbb{R}^{n+1}) \to V_2(\mathbb{R}^{n+1})$ is the bundle mapping in (14) and $\mathrm{pr}_1\colon V_2(\mathbb{R}^{n+1}) \to S^n(1)$ is the projection given by

$$\mathrm{pr}_1(\mathbf{f}_1, \mathbf{f}_2) = \mathbf{f}_1.$$

Then we have

$$\Phi(\varphi^* V_2(\mathbb{R}^{n+1})) = \bigcup_{p\in\Sigma} \{\cos\theta\,\mathbf{f}_1 + \sin\theta\,\mathbf{f}_2 \mid \pi(\mathbf{f}_1,\mathbf{f}_2) = \varphi(p),\ \theta \in S^1\}.$$

Hence $\Phi(\varphi^* V_2(\mathbb{R}^{n+1}))$ is a union of (real) l-parameter family of great circles in $S^n(1)$.

By the local triviality of the circle bundle $\pi_\varphi\colon \varphi^* V_2(\mathbb{R}^{n+1}) \to \Sigma$, every point p of Σ has an open neighborhood U such that there is a diffeomorphism $\phi_U : \pi_\varphi^{-1}(U) \to S^1 \times U$ defined by $\phi_U(u) = (\theta(u), \pi_\varphi(u))$ where θ is a mapping of $\pi_\varphi^{-1}(U)$ into S^1 satisfying $\theta(ua) = \theta(u)a$ for all $u \in \pi_\varphi^{-1}(U)$ and $a \in S^1$. Let η' be a local cross section of $\pi_\varphi\colon \varphi^* V_2(\mathbb{R}^{n+1}) \to \Sigma$ on U, and denote $(\psi \circ \eta')(q) = (\mathbf{f}_1(q), \mathbf{f}_2(q))$ for $q \in U$. Using (7), we see that oriented 2-plane spanned by $(\mathbf{f}_1(q), \mathbf{f}_2(q)) \in V_2(\mathbb{R}^{m+1})$ is identified with $\varphi(q) \in Q^{n-1}$. We define the mapping $\Phi_U = \Phi \circ \phi_U^{-1}\colon S^1 \times U \to S^n$, where Φ is defined by (15). Then Φ_U is written as

$$\Phi_U(\theta, q) = \cos\theta\, \mathbf{f}_1(q) + \sin\theta\, \mathbf{f}_2(q), \quad (\theta, q) \in S^1 \times U. \tag{16}$$

Let X be a tangent vector of Σ at q in $U \subset \Sigma$. Then (9) and (16) yield that the differential of Φ_U is

$$d\Phi_U(\partial/\partial\theta, 0) = -\sin\theta\, \mathbf{f}_1 + \cos\theta\, \mathbf{f}_2, \tag{17}$$

$$\begin{aligned} d\Phi_U(0, X) &= \cos\theta\,(\lambda(X)\mathbf{f}_2 + \mathbf{p}(X)) + \sin\theta\,(-\lambda(X)\mathbf{f}_1 + \mathbf{q}(X)) \\ &= \lambda(X)\, d\Phi_U(\partial/\partial\theta, 0) + \cos\theta\, \mathbf{p}(X) + \sin\theta\, \mathbf{q}(X). \end{aligned} \tag{18}$$

Let $\mathbf{e}_1, \dots, \mathbf{e}_l$ be an orthonormal basis of the tangent space $T_q(\Sigma)$ at $q \in U \subset \Sigma$. Then the mapping Φ_U is non-singular at $(\theta, q) \in S^1 \times U$ if and only if

$$\Phi_U \wedge d\Phi_U(\partial/\partial\theta, 0) \wedge d\Phi_U(0, \mathbf{e}_1) \wedge \cdots \wedge d\Phi_U(0, \mathbf{e}_l) \neq 0. \tag{19}$$

By (16) and (17) we have

$$\Phi_U \wedge d\Phi_U(0, \partial/\partial\theta) = \mathbf{f}_1 \wedge \mathbf{f}_2. \tag{20}$$

With respect to $\mathbb{R}^{n+1}$-valued 1-forms $\mathbf{p}$, $\mathbf{q}$ on $U \subset \Sigma$ defined by (10), denote $\mathbf{p}(\mathbf{e}_j) = \mathbf{p}_j$ and $\mathbf{q}(\mathbf{e}_j) = \mathbf{q}_j$ for $j = 1, \dots, l$, and put

$$\Psi_j = \cos\theta\, \mathbf{p}_j + \sin\theta\, \mathbf{q}_j \in T_{\Phi_U(\theta,q)}(S^n) \quad \text{for} \quad j = 1, \dots, l. \tag{21}$$

Then by (18) and (20), (19) is equivalent to

$$\Psi_1 \wedge \cdots \wedge \Psi_l \neq 0. \tag{22}$$

Consequently we obtain

Proposition 6.1. *The mapping* $\Phi_U\colon S^1 \times U \to S^n(1)$ *defined by* (16) *is non-singular at* (q,θ) *if and only if at* $q \in U \subset \Sigma$, *the mapping* $\varphi\colon \Sigma \to Q^{n-1}$ *satisfies*

$$\begin{gathered}\cos^l\theta\,\mathbf{p}_1\wedge\cdots\wedge\mathbf{p}_l\\ +\cos^{l-1}\theta\sin\theta\,(\mathbf{q}_1\wedge\mathbf{p}_2\wedge\cdots\wedge\mathbf{p}_l+\cdots+\mathbf{p}_1\wedge\cdots\wedge\mathbf{p}_{l-1}\wedge\mathbf{q}_l)\\ \cdots\\ +\cos\theta\sin^{l-1}\theta\,(\mathbf{p}_1\wedge\mathbf{q}_2\wedge\cdots\wedge\mathbf{q}_l+\cdots+\mathbf{q}_1\wedge\cdots\wedge\mathbf{q}_{l-1}\wedge\mathbf{p}_l)\\ +\sin^l\theta\,\mathbf{q}_1\wedge\cdots\wedge\mathbf{q}_l\neq 0.\end{gathered}$$

Suppose that (Σ^m, J) is a Kähler manifold with $\dim_{\mathbb{C}}\Sigma = m$ and $\varphi\colon \Sigma^m \to Q^{n-1}$ $(m < n-1)$ is a holomorphic isometric immersion. Let $\{\mathbf{e}_{2k-1}, \mathbf{e}_{2k} = J\mathbf{e}_{2k-1} \mid k = 1,\ldots,m\}$ be an orthonormal basis of the tangent space $T_q(\Sigma^m)$ at $q \in U \subset \Sigma^m$. Then we obtain

$$\begin{aligned}\Psi_{2k-1}\wedge\Psi_{2k} &= (\cos\theta\,\mathbf{p}_{2k-1}+\sin\theta\,\mathbf{q}_{2k-1})\wedge(\cos\theta\,\mathbf{p}_{2k}+\sin\theta\,\mathbf{q}_{2k})\\ &= (\cos\theta\,\mathbf{p}_{2k-1}-\sin\theta\,\mathbf{p}_{2k})\wedge(\cos\theta\,\mathbf{p}_{2k}+\sin\theta\,\mathbf{p}_{2k-1})\\ &= \mathbf{p}_{2k-1}\wedge\mathbf{p}_{2k}, \quad (k=1,\ldots,m)\end{aligned}$$

and $\Psi_1\wedge\cdots\wedge\Psi_{2m} = \mathbf{p}_1\wedge\cdots\wedge\mathbf{p}_{2m}$. By (22) we get

Proposition 6.2. *Let* (Σ^m, J) *be a Kähler manifold of* $\dim_{\mathbb{C}}\Sigma = m$ *and let* $\varphi\colon \Sigma^m \to Q^{n-1}$ $(m < n-1)$ *be a holomorphic immersion. Then the mapping* $\Phi\colon \varphi^*V_2(\mathbb{R}^{n+1}) \to S^n(1)$ *defined by* (15) *is non-singular at each point in* $\pi_\varphi^{-1}(q)$ *if and only if at* $q \in \Sigma^m$, φ *satisfies*

$$\mathbf{p}_1\wedge\cdots\wedge\mathbf{p}_{2m}\neq 0. \tag{23}$$

Remark 6.3. Jensen-Rigoli-Yang [28] studied holomorphic curves $(m = 1)$ in complex quadrics. A point p in the holomorphic curve $\varphi\colon \Sigma^1 \to Q^{n-1}$ is called a *real point* if $\mathbf{p}_1\wedge\mathbf{p}_2 = 0$ at p. They showed that if any $p \in \Sigma^1$ is a real point, then $\varphi(\Sigma^1)$ is contained in a totally geodesic Q^1 in Q^{n-1}.

Suppose that a holomorphic immersion $\varphi\colon \Sigma^m \to Q^{n-1}$ satisfies (23) at each point of Σ^m. Let $V = d\phi_U^{-1}(\partial/\partial\theta, 0)$ be a tangent vector of the fiber $\pi_\varphi^{-1}(q)$ of the submersion $\pi_\varphi\colon \varphi^*V_2(\mathbb{R}^{n+1}) \to \Sigma$ at $q \in \Sigma$. Then $\mathbb{R}\,d\Phi(V) + \mathrm{span}\{\Psi_j \mid j = 1,\ldots,2m\} = \mathbb{R}\,d\Phi_U(\partial/\partial\theta, 0) + \{d\Phi_U(0,X) \mid X \in T_q(\Sigma)\}$. Denote σ^Φ the second fundamental form of the immersion $\Phi\colon \varphi^*V_2(\mathbb{R}^{n+1}) \to S^n(1)$. Since each fiber $\pi_\varphi^{-1}(q)$ is a great circle of $S^n(1)$, we have

$$\sigma^\Phi(V,V) = 0. \tag{24}$$

On the other hand, if we denote D the Euclidean connection of $\mathbb{R}^{n+1}$, then using (21) we get

$$D_{\partial/\partial\theta}\Psi_{2k-1} = -\sin\theta\,\mathbf{p}_{2k-1} - \cos\theta\,\mathbf{p}_{2k},$$
$$D_{\partial/\partial\theta}\Psi_{2k} = -\sin\theta\,\mathbf{p}_{2k} + \cos\theta\,\mathbf{p}_{2k-1},$$

$(k = 1, \dots, m)$ and both of these terms are contained in the tangent space of $\Phi(\varphi^* V_2(\mathbb{R}^{n+1}))$. Hence we obtain

$$\sigma^\Phi(V, d\phi_U^{-1}(0, X)) = 0 \quad \text{for} \quad X \in T_q(\Sigma). \tag{25}$$

Combining with (2), (24) and (25), we obtain a generalization of Lemma 2.2:

Theorem 6.4. *Let $\varphi\colon \Sigma^m \to Q^{n-1}$ $(m < n-1)$ be a holomorphic immersion from a Kähler manifold Σ^m to the complex quadric for which* (23) *holds. Then with respect to the immersion $\Phi\colon \varphi^* V_2(\mathbb{R}^{n+1}) \to S^n(1)$ given by* (15), *any tangent line of the fiber $\pi_\varphi^{-1}(p)$ at each $u \in \pi_\varphi^{-1}(p)$, $p \in \Sigma^m$ lies in the kernel of differential of the Gauss mapping of φ. Hence Φ is tangentially degenerate.*

Remark 6.5. Let $\pi\colon S^{2m+1}(1) \to \mathbb{CP}^m$ be the Hopf fibration, and let $f\colon M \to \mathbb{CP}^m$ be an isometric immersion from a *real* l-dimensional Riemannian manifold M. Then there is a natural immersion $F\colon \pi^{-1}(M) \to S^{2m+1}(1)$ such that the following diagram is commutative:

$$\begin{tikzcd} \pi^{-1}(M) \arrow[r, "F"] \arrow[d, "\pi"'] & S^{2m+1}(1) \arrow[d, "\pi"] \\ M \arrow[r, "f"] & \mathbb{CP}^m. \end{tikzcd} \tag{26}$$

It can be seen that any tangent line of each fiber $\pi^{-1}(p)$ $(p \in M)$ of the S^1-bundle $\pi\colon \pi^{-1}(M) \to M$ lies in

$$\{X \in T_x(\pi^{-1}M) \mid \sigma^F(X, Y) = 0 \text{ for any } Y \in T_x(M)\}, \quad (\pi(x) = p)$$

if and only if l is even and M is a complex submanifold of $\mathbb{CP}^m$.

Let $\varphi\colon \Sigma^k \to Q^{2m}$ $(k < 2m)$ be a holomorphic immersion from a Kähler manifold Σ^k with $\dim_{\mathbb{C}} \Sigma = k$ to the complex quadric Q^{2m} satisfying (23). Then the immersion $\Phi\colon \varphi^* V_2(\mathbb{R}^{n+1}) \to S^{2m+1}(1)$ defined by (15) is congruent to the inverse image of a complex submanifold M^k in $\mathbb{CP}^m$ if and only if $\varphi(\Sigma^k)$ is contained in a totally geodesic $\mathbb{CP}^m$ in Q^{2m} [16] and [29]. In fact the set of fibers of the Hopf fibration π is identified with totally geodesic $\mathbb{CP}^m$ in Q^{2m}.

§7. Examples

Now we give examples of homogeneous complex submanifolds of complex quadrics, which satisfy the assumption of Theorem 6.4.

Let $M(m+1,\mathbb{C})$ be the space of all complex $(m+1)\times(m+1)$ matrices, and let

$$\mathrm{Sym}^{\mathbb{C}}(m+1) = \{A \in M(m+1,\mathbb{C}) \mid {}^tA = A\},$$
$$\mathrm{Sym}_0^{\mathbb{C}}(m+1) = \{A \in \mathrm{Sym}^{\mathbb{C}}(m+1) \mid \operatorname{trace} A = 0\}.$$

Then $\mathrm{Sym}^{\mathbb{C}}(m+1)$ (resp. $\mathrm{Sym}_0^{\mathbb{C}}(m+1)$) can be considered as a real $(m+1)(m+2)$ (resp. $m(m+3)$)-dimensional vector space with the inner product given by

$$\langle A_1, A_2\rangle = \frac{1}{2}\,\mathrm{Re}(\operatorname{trace}(A_1A_2^*)). \tag{27}$$

Similarly, let

$$\mathrm{Sym}^{\mathbb{R}}(m+1) = \{A \in M(m+1,\mathbb{R}) \mid {}^tA = A\},$$
$$\mathrm{Sym}_0^{\mathbb{R}}(m+1) = \{A \in \mathrm{Sym}^{\mathbb{R}}(m+1) \mid \operatorname{trace} A = 0\}.$$

Then $\mathrm{Sym}_0^{\mathbb{R}}(m+1)$ can be considered as a real $m(m+3)/2$-dimensional vector space with the inner product given by

$$\langle B_1, B_2\rangle = \frac{1}{2}\,\mathrm{Re}(\operatorname{trace}(B_1B_2)). \tag{28}$$

Let $S_r(\mathrm{Sym}^{\mathbb{C}}(m+1))$ (resp. $S_r(\mathrm{Sym}_0^{\mathbb{C}}(m+1))$) be the hypersphere of radius $r>0$ in $\mathrm{Sym}^{\mathbb{C}}(m+1)$ (resp. $\mathrm{Sym}_0^{\mathbb{C}}(m+1)$), and let $\mathbb{P}(\mathrm{Sym}^{\mathbb{C}}(m+1))$ (resp. $\mathbb{P}(\mathrm{Sym}_0^{\mathbb{C}}(m+1))$) be the complex projective space which is considered as the quotient space of $S_r(\mathrm{Sym}^{\mathbb{C}}(m+1))$ (resp. $S_r(\mathrm{Sym}_0^{\mathbb{C}}(m+1))$) obtained by identifying v with λv, where $\lambda \in \mathbb{C}$ with $|\lambda| = 1$. Let $\pi\colon S_r(\mathrm{Sym}^{\mathbb{C}}(m+1)) \to \mathbb{P}(\mathrm{Sym}^{\mathbb{C}}(m+1))$ (resp. $\pi\colon S_r(\mathrm{Sym}_0^{\mathbb{C}}(m+1)) \to \mathbb{P}(\mathrm{Sym}_0^{\mathbb{C}}(m+1))$) be the *Hopf fibration.*

Define a mapping $\widetilde{\varphi}_m\colon S^{2m+1}(\sqrt{2}) \to \mathrm{Sym}^{\mathbb{C}}(m+1)$ as

$$\widetilde{\varphi}_m(\mathbf{z}) = \frac{1}{\sqrt{2}}\mathbf{z}\,{}^t\mathbf{z} = \frac{1}{\sqrt{2}}\begin{pmatrix} z_0^2 & z_0z_1 & \cdots & z_0z_m \\ z_1z_0 & z_1^2 & \cdots & z_1z_m \\ \vdots & \vdots & \ddots & \vdots \\ z_mz_0 & z_mz_1 & \cdots & z_m^2 \end{pmatrix} \tag{29}$$

for $\mathbf{z} = (z_j) \in S^{2m+1}(\sqrt{2})$. Then it can be verified that $\langle \widetilde{\varphi}_m(\mathbf{z}), \widetilde{\varphi}_m(\mathbf{z})\rangle = 1$ with respect to (27) for $\mathbf{z} \in S^{2m+1}(\sqrt{2})$ and $\widetilde{\varphi}_m$ induces a mapping

φ_m of $\mathbb{CP}^m$ into $\mathbb{P}(\mathrm{Sym}^{\mathbb{C}}(m+1)) \cong \mathbb{CP}^{m(m+3)/2}$ such that the following diagram is commutative:

$$\begin{array}{ccc} S^{2m+1}(\sqrt{2}) & \xrightarrow{\widetilde{\varphi}_m} & S_1(\mathrm{Sym}^{\mathbb{C}}(m+1)) \\ \pi\downarrow & & \downarrow\pi \\ \mathbb{CP}^m & \xrightarrow{\varphi_m} & \mathbb{P}(\mathrm{Sym}^{\mathbb{C}}(m+1)). \end{array}$$

φ_m is nothing but the *complex Veronese embedding.*

If we restrict $\widetilde{\varphi}_m$ to the submanifold $\widetilde{Q}(\mathbb{C}^{m+1})$ which is given by (6), then the image $\widetilde{\varphi}_m(\widetilde{Q}(\mathbb{C}^{m+1}))$ is contained in the submanifold

$$\widetilde{Q}(\mathrm{Sym}_0^{\mathbb{C}}(m+1)) = \{A \in S_1(\mathrm{Sym}_0^{\mathbb{C}}(m+1)) \mid \operatorname{trace} A^2 = 0\}. \tag{30}$$

Putting $W = \mathrm{Sym}_0^{\mathbb{C}}(m+1)$ in the argument in §4, we have a commutative diagram:

$$\begin{array}{ccc} \widetilde{Q}(\mathrm{Sym}_0^{\mathbb{C}}(m+1)) & \xrightarrow{\sim} & V_2(\mathrm{Sym}_0^{\mathbb{R}}(m+1)) \\ \pi\downarrow & & \downarrow\pi \\ Q(\mathrm{Sym}_0^{\mathbb{C}}(m+1)) & \xrightarrow{\sim} & \widetilde{G}_2(\mathrm{Sym}_0^{\mathbb{R}}(m+1)), \end{array}$$

where we can write

$$Q(\mathrm{Sym}_0^{\mathbb{C}}(m+1)) = \{\pi(A) \in \mathbb{P}(\mathrm{Sym}_0^{\mathbb{C}}(m+1)) \mid \operatorname{trace} A^2 = 0\}.$$

Consider a Riemannian metric on $\widetilde{Q}(\mathrm{Sym}_0^{\mathbb{C}}(m+1))$ which is induced by the inclusion into $\mathrm{Sym}_0^{\mathbb{C}}(m+1)$ of (27). Then we have the Riemannian metric on $Q(\mathrm{Sym}_0^{\mathbb{C}}(m+1))$ such that the fibering $\pi\colon \widetilde{Q}(\mathrm{Sym}_0^{\mathbb{C}}(m+1)) \to Q(\mathrm{Sym}_0^{\mathbb{C}}(m+1))$ is a Riemannian submersion. Hence $\widetilde{\varphi}_m$ induces a mapping φ_m of Q^{m-1} into $Q(\mathrm{Sym}_0^{\mathbb{C}}(m+1)) \cong Q^{(m-1)(m+4)/2}$ such that the following diagram is commutative:

$$\begin{array}{ccc} \widetilde{Q}(\mathbb{C}^{m+1}) & \xrightarrow{\widetilde{\varphi}_m} & \widetilde{Q}(\mathrm{Sym}_0^{\mathbb{C}}(m+1)) \\ \pi\downarrow & & \downarrow\pi \\ Q^{m-1} & \xrightarrow{\varphi_m} & Q(\mathrm{Sym}_0^{\mathbb{C}}(m+1)). \end{array} \tag{31}$$

We can see that the image of $\widetilde{Q}(\mathbb{C}^{m+1})$ under $\widetilde{\varphi}_m$ is given by

$$\widetilde{\varphi}_m(\widetilde{Q}(\mathbb{C}^{m+1})) = \{A \in \widetilde{Q}(\mathrm{Sym}_0^{\mathbb{C}}(m+1)) \mid A^2 = 0,\ AA^*A = 4A\}.$$

The special orthogonal group $\mathrm{SO}(m+1)$ acts on Q^{m-1} isometrically as

$$\pi(\mathbf{z}) \mapsto \pi(T\mathbf{z}), \quad (T \in \mathrm{SO}(m+1),\ \mathbf{z} \in \widetilde{Q}(\mathbb{C}^{m+1})). \tag{32}$$

This action is transitive [30, p. 279] and the isotropy subgroup at the point $o = \pi({}^t(1, i, 0, \ldots, 0)) \in Q^{m-1}$ is $\mathrm{SO}(2) \times \mathrm{SO}(m-1)$. $\mathrm{SO}(m+1)$ acts on $Q(\mathrm{Sym}_0^{\mathbb{C}}(m+1))$ isometrically as

$$\pi(A) \mapsto \pi(TAT^{-1}), \quad (T \in \mathrm{SO}(m+1),\ A \in \widetilde{Q}(\mathrm{Sym}_0^{\mathbb{C}}(m+1))). \tag{33}$$

By (29) and (31), we have

$$\varphi_m(\pi(T\mathbf{z})) = \pi(T\varphi_m(\mathbf{z})T^{-1}).$$

Hence *the embedding* $\varphi_m\colon Q^{m-1} \to Q(\mathrm{Sym}_0^{\mathbb{C}}(m+1))$ *given by* (29) *and* (31) *is equivariant with respect to the actions* (32) *and* (33) *of* $\mathrm{SO}(m+1)$.

We calculate differential of $\varphi_m\colon Q^{m-1} \to Q(\mathrm{Sym}_0^{\mathbb{C}}(m+1))$ at $\pi(\mathbf{f}_1 + \mathbf{i}\mathbf{f}_2) \in Q^{m-1}$. Here we identify (31) with

$$\begin{array}{ccc} V_2(\mathbb{R}^{m+1}) & \xrightarrow{\widetilde{\varphi}_m} & V_2(\mathrm{Sym}_0^{\mathbb{R}}(m+1)) \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ \widetilde{G}_2(\mathbb{R}^{m+1}) & \xrightarrow{\varphi_m} & \widetilde{G}_2(\mathrm{Sym}_0^{\mathbb{R}}(m+1)). \end{array}$$

Let $t \mapsto (\mathbf{f}_1(t), \mathbf{f}_2(t))$ be a curve in $V_2(\mathbb{R}^{m+1})$ such that $(\mathbf{f}_1(0), \mathbf{f}_2(0)) = (\mathbf{f}_1, \mathbf{f}_2)$ and $(\mathbf{f}_1'(0), \mathbf{f}_2'(0)) = (\mathbf{x}_1, \mathbf{x}_2) \in T'_{(\mathbf{f}_1, \mathbf{f}_2)}(V_2(\mathbb{R}^{m+1}))$ (cf. (4)). Then we have

$$\widetilde{\varphi}_m(\mathbf{f}_1(t), \mathbf{f}_2(t)) = \frac{1}{\sqrt{2}}\big(\mathbf{f}_1(t){}^t\mathbf{f}_1(t) - \mathbf{f}_2(t){}^t\mathbf{f}_2(t), \mathbf{f}_1(t){}^t\mathbf{f}_2(t) + \mathbf{f}_2(t){}^t\mathbf{f}_1(t)\big)$$

and

$$\begin{aligned} d\widetilde{\varphi}_m(\mathbf{x}_1, \mathbf{x}_2) &= \frac{d}{dt}\bigg|_{t=0} \widetilde{\varphi}_m(\mathbf{f}_1(t), \mathbf{f}_2(t)) \\ &= \frac{1}{\sqrt{2}}\big(\mathbf{x}_1{}^t\mathbf{f}_1 + \mathbf{f}_1{}^t\mathbf{x}_1 - \mathbf{x}_2{}^t\mathbf{f}_2 - \mathbf{f}_2{}^t\mathbf{x}_2, \mathbf{x}_1{}^t\mathbf{f}_2 + \mathbf{f}_1{}^t\mathbf{x}_2 + \mathbf{x}_2{}^t\mathbf{f}_1 + \mathbf{f}_2{}^t\mathbf{x}_1\big) \\ &\in T'_{\widetilde{\varphi}_m(\mathbf{f}_1, \mathbf{f}_2)}\big(V_2(\mathrm{Sym}_0^{\mathbb{R}}(m+1))\big). \end{aligned}$$

Hence

$$\begin{aligned} &d\varphi_m(d\pi(\mathbf{x}_1, \mathbf{x}_2)) \\ &= \frac{1}{\sqrt{2}} d\pi\big(\mathbf{x}_1{}^t\mathbf{f}_1 + \mathbf{f}_1{}^t\mathbf{x}_1 - \mathbf{x}_2{}^t\mathbf{f}_2 - \mathbf{f}_2{}^t\mathbf{x}_2, \mathbf{x}_1{}^t\mathbf{f}_2 + \mathbf{f}_1{}^t\mathbf{x}_2 + \mathbf{x}_2{}^t\mathbf{f}_1 + \mathbf{f}_2{}^t\mathbf{x}_1\big) \end{aligned}$$

implies that φ_m is holomorphic, i.e., $d\varphi_m \circ J = J \circ d\varphi_m$. With respect to the notation (9), we obtain

$$\mathbf{p}(d\pi(\mathbf{x}_1, \mathbf{x}_2)) = \frac{1}{\sqrt{2}}(\mathbf{x}_1{}^t\mathbf{f}_1 + \mathbf{f}_1{}^t\mathbf{x}_1 - \mathbf{x}_2{}^t\mathbf{f}_2 - \mathbf{f}_2{}^t\mathbf{x}_2) \in \mathrm{Sym}_0^{\mathbb{R}}(m+1),$$

and

$$\begin{aligned}\langle \mathbf{p}(d\pi(\mathbf{x}_1, \mathbf{x}_2)), \mathbf{p}(d\pi(\mathbf{y}_1, \mathbf{y}_2))\rangle &= \frac{1}{2}({}^t\mathbf{x}_1\mathbf{y}_1 + {}^t\mathbf{x}_2\mathbf{y}_2) \\ &= \frac{1}{2}\langle d\pi(\mathbf{x}_1, \mathbf{x}_2), d\pi(\mathbf{y}_1, \mathbf{y}_2)\rangle. \end{aligned} \tag{34}$$

In particular, the holomorphic embedding $\varphi_m \colon Q^{m-1} \to Q(\mathrm{Sym}_0^{\mathbb{C}}(m+1))$ satisfies (23). Therefore, applying Theorem 6.4, we obtain homogeneous examples of tangentially degenerate submanifolds:

$$\Phi \colon \varphi_m^* V_2(\mathbb{R}^{m+1}) \to S^{(m^2+3m-2)/2}(1)$$

.

§8. Austere submanifolds in spheres

A submanifold M in a Riemannian manifold is called *austere* [23] if for each normal vector ξ, the set of eigenvalues of the shape operator A_ξ is invariant under multiplication of -1. Clearly austere submanifolds are minimal, and they are closely related to *special Lagrangian submanifolds* (see also [10]). In fact, Harvey and Lawson showed (Theorem 3.17 in [23]) that from any compact austere submanifold of S^n, one can construct an $n+1$-dimensional special Lagrangian cone of least mass in $\mathbb{R}^{2n+2}$. In this section we will show that if Σ is a complex submanifold of *first order isotropic* in Q^{n-1}, then the corresponding submanifold M in S^n with 2-parameter family of great circles is austere, as well as tangentially degenerate.

We will use notations of §5 and §6. Let $(\Sigma^m, \langle\ ,\ \rangle, J)$ be a Kähler manifold of $\dim_{\mathbb{C}} \Sigma = m$ and let $\varphi\colon \Sigma^m \to Q^{n-1} \cong G_2(\mathbb{R}^{n+1})$ $(m < n-1)$ be a holomorphic isometric immersion, i.e., (11) holds. We say that holomorphic immersion $\varphi\colon \Sigma \to Q^{n-1}$ is *first order isotropic* (cf. (9), (18) and (34)) if

$$\langle \mathbf{p}(X), \mathbf{p}(Y)\rangle = \frac{1}{2}\langle X, Y\rangle. \tag{35}$$

This condition is independent of the choice of local cross section $\eta'\colon U \to \varphi^* V_2(\mathbb{R}^{n+1})$ for an open subset $U \subset \Sigma$. Moreover, using (11) and (35), we see that

Proposition 8.1. *Let $\varphi\colon \Sigma \to Q^{n-1}$ be a holomorphic isometric immersion of a Kähler manifold and let $\iota\colon Q^{n-1} \to \mathbb{CP}^n$ be the inclusion. Then φ is first order isotropic if and only if $d(\iota \circ \varphi)(X)$ is an isotropic vector for each $X \in T\Sigma$.*

For holomorphic curves (i.e., 1-dimensional complex submanifolds) in the complex quadric, the definition of *first order isotropic* is given by Jensen-Rigoli-Yang [28]. In this case, (23) holds for φ, and by Proposition 6.1, the mapping $\Phi\colon \varphi^* V_2(\mathbb{R}^{n+1}) \to S^n(1)$ defined by (15) is an immersion. Let $\Phi_U = \Phi \circ \phi_U^{-1}\colon S^1 \times U \to S^n$ be the local expression of Φ defined by (16). For each $\theta \in S^1$, we define a linear mapping $\Psi_\theta\colon T_q(\Sigma) \to T_{\Phi_U(\theta,q)}(S^n)$ as

$$\Psi_\theta(X) = \cos\theta\,\mathbf{p}(X) + \sin\theta\,\mathbf{q}(X) = \cos\theta\,\mathbf{p}(X) - \sin\theta\,\mathbf{p}(JX).$$

By (18), we have $\Psi_\theta(X) = d\Phi_{U(\theta,q)}(0,X) - \lambda(X)d\Phi_{U(\theta,q)}(\partial/\partial\theta,0)$. Then (17), (18), (23) and $\mathbf{q}(X) = -\mathbf{p}(JX)$, imply

$$\begin{aligned} d\Phi_U(T_{(\theta,q)}(S^1 \times U)) &= \mathbb{R}\,d\Phi_U(\partial/\partial\theta) \oplus \{\mathbf{p}(X) \mid X \in T_q(\Sigma)\} \\ &= \mathbb{R}\,d\Phi_U(\partial/\partial\theta) \oplus \{\Psi_\theta(X) \mid X \in T_q(\Sigma)\}. \end{aligned}$$

With respect to the submersion $\pi_\varphi\colon \varphi^* V_2(\mathbb{R}^{n+1}) \to \Sigma$, $d\phi_U^{-1}(\partial/\partial\theta)$ is a unit vertical vector and $\{\Psi_\theta(X) \mid X \in T_q(\Sigma)\}$ is the image of horizontal subspace in $T_{\phi_U^{-1}(\theta,q)}(\varphi^* V_2(\mathbb{R}^{n+1}))$ under $d\Phi$. Using (35) we see that for each θ, $\langle \Psi_\theta(X), \Psi_\theta(Y)\rangle = (1/2)\langle X, Y\rangle$ and in particular Ψ_θ is injective. Hence we have

Proposition 8.2. *Let $\varphi\colon \Sigma \to Q^{n-1}$ be a first order isotropic holomorphic isometric immersion from a Kähler manifold to the complex quadric, and let $\Phi\colon \varphi^* V_2(\mathbb{R}^{n+1}) \to S^n$ be the corresponding immersion defined by* (15). *Then restriction of the differential of the projection $\pi_\varphi\colon \varphi^* V_2(\mathbb{R}^{n+1}) \to \Sigma$ to the horizontal subspaces is a homothety with respect to the metric on $\varphi^* V_2(\mathbb{R}^{n+1})$ induced by Φ.*

Let D denote the Euclidean covariant derivative on $\mathbb{R}^{n+1}$. Then we obtain

$$\begin{aligned} D_{\Psi_\theta(X)}\Psi_\theta(Y) = \cos\theta\,&\{(\nabla_X \mathbf{p})Y + \mathbf{p}(\nabla_X Y) + \lambda(X)\mathbf{p}(JY)\} \\ &- \sin\theta\,\{(\nabla_X \mathbf{p})JY + \mathbf{p}(\nabla_X(JY)) - \lambda(X)\mathbf{p}(Y)\}. \end{aligned}$$

Then (25) yields

$$\sigma^\Phi\big(d\phi_U^{-1}(0,X), d\phi_U^{-1}(0,Y)\big) = \cos\theta\,\mathbf{s}(X,Y) - \sin\theta\,\mathbf{s}(X,JY)$$

where X, $Y \in T_q(\Sigma)$. Hence by virtue of (13), we obtain
(36)
$$\sigma^{\Phi}\left(d\phi_U^{-1}(0,X), d\phi_U^{-1}(0,Y)\right) + \sigma^{\Phi}\left(d\phi_U^{-1}(0,JX), d\phi_U^{-1}(0,JY)\right) = 0.$$

As a generalization of Theorem 1 in [29], we get

Theorem 8.3. *Let $\varphi\colon \Sigma \to Q^{n-1}$ be a first order isotropic holomorphic isometric immersion from a Kähler manifold to the complex quadric. Then the corresponding immersion $\Phi\colon \varphi^* V_2(\mathbb{R}^{n+1}) \to S^n$ defined by* (15) *is austere.*

Proof. Let U be an open subset in Σ and let $\Phi_U = \Phi \circ \phi_U^{-1}\colon S^1 \times U \to S^n$ be the local expression of Φ as (16). By Proposition 8.2, at each point $u = \phi_U^{-1}(\theta, q) \in \pi_{\varphi}^{-1}(U)$, we can choose an orthonormal basis $\{V, \mathbf{e}_1, \ldots, \mathbf{e}_{2m}\}$ of the tangent space $T_u(\varphi^* V_2(\mathbb{R}^{n+1}))$ such that $V = d\phi_U^{-1}(\partial/\partial\theta, 0)$ is a unit vertical vector and $\mathbf{e}_j = \mathcal{H}\, d\phi_U^{-1}(0, \mathbf{v}_j)$, $(j = 1, \ldots, 2m)$ are horizontal vector with respect to the submersion $\pi_{\varphi}\colon \varphi^* V_2(\mathbb{R}^{n+1}) \to \Sigma$, where $\mathcal{H}$ denotes the horizontal component of the tangent vector and $\mathbf{v}_1, \ldots, \mathbf{v}_{2m}$ is a basis of orthogonal vectors of $T_q(\Sigma)$ with $J\mathbf{v}_{2k-1} = \mathbf{v}_{2k}$ for $k = 1, \ldots, m$. Then by Theorem 6.4 and (36), we can see that Φ is an austere immersion. $\square$

Remark 8.4. It is well-known (cf. [11] and [28]) that there is a one-to-one correspondence between *totally isotropic holomorphic curves* in Q^{2m-1} and *pseudoholomorphic surfaces* (*superminimal surfaces, or isotropic minimal surfaces*) in S^{2m}. Hence from full minimal 2-spheres in S^{2m}, we can construct 3-dimensional full austere submanifolds in S^{2m} [29]. As for tangential degeneracy, see §10.

§9. Examples satisfying Ferus equality in spheres

In this section, we give further examples of homogeneous submanifolds with degenerate Gauss mapping, and list up those which satisfy Ferus equality.

Let $\mathbb{K}$ be the field $\mathbb{R}$ of real numbers, the field $\mathbb{C}$ of complex numbers or the field $\mathbb{H}$ of quaternions. In the natural way, $\mathbb{R} \subset \mathbb{C} \subset \mathbb{H}$. The conjugate of each element $q \in \mathbb{H}$ is defined as follows:

$$\bar{q} = q_0 - q_1\mathbf{i} - q_2\mathbf{j} - q_3\mathbf{k} \quad \text{for} \quad q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$$

where $\{\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ is usual basis for $\mathbb{H}$. Define a number d by

$$d = \begin{cases} 1 & \text{if } \mathbb{K} = \mathbb{R}, \\ 2 & \text{if } \mathbb{K} = \mathbb{C}, \\ 4 & \text{if } \mathbb{K} = \mathbb{H}. \end{cases}$$

Let $\mathbf{x} \in \mathbb{K}^{n+1}$ be a column vector. The usual inner product on $\mathbb{K}^{n+1} = \mathbb{R}^{(n+1)d}$ is given by

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathrm{Re}(\mathbf{x}^* \mathbf{y}) \quad \text{for} \quad \mathbf{x},\, \mathbf{y} \in \mathbb{K}^{n+1}$$

where $\mathrm{Re}(\mathbf{x}^*\mathbf{y})$ denotes the real part of $\mathbf{x}^*\mathbf{y}$.

The projective space $\mathbb{KP}^n$ over $\mathbb{K}$ is considered as the quotient space of the unit $((n+1)d-1)$-dimensional sphere $S^{(n+1)d-1} = \{\mathbf{x} \in \mathbb{K}^{n+1} \mid \mathbf{x}^*\mathbf{x} = 1\}$ obtained by identifying $\mathbf{x}$ with $\mathbf{x}\lambda$ where $\lambda \in S^{d-1} = \{t \in \mathbb{K} \mid |t| = 1\}$. Let $\pi\colon S^{(n+1)d-1} \to \mathbb{KP}^n$ be the Hopf fibration and denote $\pi(\mathbf{x}) = [\mathbf{x}] \in \mathbb{KP}^n$ for $\mathbf{x} \in S^{(n+1)d-1}$. Then the canonical metric in $\mathbb{KP}^n$ is the invariant metric such that π is a Riemannian submersion.

Let

$$V_2(\mathbb{K}^{n+1}) = \{(\mathbf{u}_1, \mathbf{u}_2) \in S^{(n+1)d-1} \times S^{(n+1)d-1} \mid \mathbf{u}_1^*\mathbf{u}_2 = 0\}$$

be the Stiefel manifold over $\mathbb{K}$. Then $T_{(\mathbf{u}_1,\mathbf{u}_2)}(V_2(\mathbb{K}^{n+1}))$, the tangent space at $(\mathbf{u}_1, \mathbf{u}_2) \in V_2(\mathbb{K}^{n+1})$ is

$$\begin{aligned} T_{(\mathbf{u}_1,\mathbf{u}_2)}(V_2(\mathbb{K}^{n+1})) = \{&(\mathbf{x}_1, \mathbf{x}_2) \in \mathbb{K}^{n+1} \times \mathbb{K}^{n+1} \mid \\ &\langle \mathbf{x}_1, \mathbf{u}_1 \rangle = \langle \mathbf{x}_2, \mathbf{u}_2 \rangle = 0,\ \mathbf{x}_1^*\mathbf{u}_2 + \mathbf{u}_1^*\mathbf{x}_2 = 0\}. \end{aligned}$$

The subspaces $T_0(\mathbf{u}_1, \mathbf{u}_2)$, $T_\lambda(\mathbf{u}_1, \mathbf{u}_2)$, $T_\mu(\mathbf{u}_1, \mathbf{u}_2)$ of $T_{(\mathbf{u}_1,\mathbf{u}_2)}(V_2(\mathbb{K}^{n+1}))$ are defined as

$$\begin{aligned} T_0(\mathbf{u}_1, \mathbf{u}_2) &= \{(\mathbf{x}_1, \mathbf{x}_2) \in \mathbb{K}^{n+1} \times \mathbb{K}^{n+1} \mid \mathbf{x}_\alpha^*\mathbf{u}_\beta = 0 \text{ for } \alpha,\, \beta = 1,\, 2\} \\ T_\lambda(\mathbf{u}_1, \mathbf{u}_2) &= \{(-\mathbf{u}_2\bar{\lambda}, \mathbf{u}_1\lambda) \in \mathbb{K}^{n+1} \times \mathbb{K}^{n+1} \mid \lambda \in \mathbb{K}\} \\ T_\mu(\mathbf{u}_1, \mathbf{u}_2) &= \{(\mathbf{u}_1\mu_1, \mathbf{u}_2\mu_2) \in \mathbb{K}^{n+1} \times \mathbb{K}^{n+1} \mid \mu_1,\, \mu_2 \in \mathrm{Im}\,\mathbb{K}\}. \end{aligned}$$

Then we have $T_{(\mathbf{u}_1,\mathbf{u}_2)}(V_2(\mathbb{K}^{n+1})) = T_0(\mathbf{u}_1, \mathbf{u}_2) \oplus T_\lambda(\mathbf{u}_1, \mathbf{u}_2) \oplus T_\mu(\mathbf{u}_1, \mathbf{u}_2)$.

Put

$$F_2(\mathbb{K}^{n+1}) = \{([\mathbf{u}_1],\, [\mathbf{u}_2]) \in \mathbb{KP}^n \times \mathbb{KP}^n \mid \mathbf{u}_1^*\mathbf{u}_2 = 0\}.$$

Then $V_2(\mathbb{K}^{n+1})$ is a principal fiber bundle over $F_2(\mathbb{K}^{n+1})$ with structure group $S^{d-1} \times S^{d-1}$ and projection map $\pi\colon V_2(\mathbb{K}^{n+1}) \to F_2(\mathbb{K}^{n+1})$ defined by

$$\pi(\mathbf{u}_1, \mathbf{u}_2) = ([\mathbf{u}_1], [\mathbf{u}_2]).$$

The distribution $T_0(\mathbf{u}_1, \mathbf{u}_2) \oplus T_\lambda(\mathbf{u}_1, \mathbf{u}_2)$ defines a connection in the principal fiber bundle $V_2(\mathbb{K}^{n+1})(F_2(\mathbb{K}^{n+1}), S^{d-1} \times S^{d-1})$, because this is complementary to the subspace $T_\mu(\mathbf{u}_1, \mathbf{u}_2)$ tangent to the fiber through

$(\mathbf{u}_1, \mathbf{u}_2)$, and invariant under the $S^{d-1} \times S^{d-1}$-action. The projection π of $V_2(\mathbb{K}^{n+1})$ onto $F_2(\mathbb{K}^{n+1})$ induces a linear isomorphism of $T_0(\mathbf{u}_1, \mathbf{u}_2) \oplus T_\lambda(\mathbf{u}_1, \mathbf{u}_2)$ onto $T_p(F_2(\mathbb{K}^{n+1}))$, where $\pi((\mathbf{u}_1, \mathbf{u}_2)) = p$. We mention that $F_2(\mathbb{K}^{n+1})$ is considered as the total space of "tautological $\mathbb{KP}^1$-bundle" over 2-plane Grassmann manifold $\widetilde{G}_2(\mathbb{K}^{n+1})$ with projection $\pi(\mathbf{u}_1, \mathbf{u}_2) \mapsto \mathrm{span}_{\mathbb{K}}\{\mathbf{u}_1, \mathbf{u}_2\}$.

Let $\mathfrak{M}^{\mathbb{K}}(n+1)$ be the space of all $(n+1) \times (n+1)$ matrices over $\mathbb{K}$. The inner product on $\mathfrak{M}^{\mathbb{K}}(n+1) = \mathbb{R}^{(n+1)^2 d}$ is defined as

$$\langle A, B\rangle = \frac{1}{2} \operatorname{Re} \operatorname{trace}(AB^*) \quad \text{for} \quad A,\ B \in \mathfrak{M}^{\mathbb{K}}(n+1).$$

Let

$$\begin{aligned}
\mathrm{Herm}^{\mathbb{K}}(n+1) &= \{A \in \mathfrak{M}^{\mathbb{K}}(n+1) \mid A^* = A\} \\
\mathrm{Herm}_0^{\mathbb{K}}(n+1) &= \{A \in \mathrm{Herm}^{\mathbb{K}}(n+1) \mid \operatorname{trace} A = 0\} \\
S(\mathrm{Herm}_0^{\mathbb{K}}(n+1)) &= \{A \in \mathrm{Herm}_0^{\mathbb{K}}(n+1) \mid \langle A, A\rangle = 1\} \\
U^{\mathbb{K}}(n+1) &= \{P \in \mathfrak{M}^{\mathbb{K}}(n+1) \mid P^*P = E\}
\end{aligned}$$

where $A^* = {}^t\bar{A}$ and E is the identity matrix. $U^{\mathbb{K}}(n+1)$ acts on $V_2(\mathbb{K}^{n+1})$, $F_2(\mathbb{K}^{n+1})$ and $\widetilde{G}_2(\mathbb{K}^{n+1})$ transitively as

$$\begin{aligned}
P \cdot (\mathbf{u}_1, \mathbf{u}_2) &= (P\mathbf{u}_1, P\mathbf{u}_2) \text{ for } (\mathbf{u}_1, \mathbf{u}_2) \in V_2(\mathbb{K}^{n+1}) \\
P \cdot \pi(\mathbf{u}_1, \mathbf{u}_2) &= \pi(P\mathbf{u}_1, P\mathbf{u}_2) \text{ for } \pi(\mathbf{u}_1, \mathbf{u}_2) \in F_2(\mathbb{K}^{n+1}) \\
P \cdot \mathrm{span}_{\mathbb{K}}\{\mathbf{u}_1, \mathbf{u}_2\} &= \mathrm{span}_{\mathbb{K}}\{P\mathbf{u}_1, P\mathbf{u}_2\} \text{ for } \mathrm{span}_{\mathbb{K}}\{\mathbf{u}_1, \mathbf{u}_2\} \in \widetilde{G}_2(\mathbb{K}^{n+1})
\end{aligned}$$

and $P \in U^{\mathbb{K}}(n+1)$. Hence as homogeneous spaces, we have

$$\begin{aligned}
V_2(\mathbb{K}^{n+1}) &= U^{\mathbb{K}}(n+1)/U^{\mathbb{K}}(n-1) \\
F_2(\mathbb{K}^{n+1}) &= U^{\mathbb{K}}(n+1)/U^{\mathbb{K}}(1) \times U^{\mathbb{K}}(1) \times U^{\mathbb{K}}(n-1) \\
\widetilde{G}_2(\mathbb{K}^{n+1}) &= U^{\mathbb{K}}(n+1)/U^{\mathbb{K}}(2) \times U^{\mathbb{K}}(n-1).
\end{aligned}$$

Define a map $\widetilde{\varphi}_n^{\mathbb{K}} \colon V_2(\mathbb{K}^{n+1}) \to S(\mathrm{Herm}_0^{\mathbb{K}}(n+1))$ as follows:

$$\widetilde{\varphi}_n^{\mathbb{K}}(\mathbf{u}_1, \mathbf{u}_2) = \mathbf{u}_1\mathbf{u}_1^* - \mathbf{u}_2\mathbf{u}_2^* \quad \text{for} \quad (\mathbf{u}_1, \mathbf{u}_2) \in V_2(\mathbb{K}^{n+1}). \tag{37}$$

It is easy to see that $\widetilde{\varphi}_n^{\mathbb{K}}$ gives a map $\varphi_n^{\mathbb{K}} \colon F_2(\mathbb{K}^{n+1}) \to S(\mathrm{Herm}_0^{\mathbb{K}}(n+1))$ such that $\widetilde{\varphi}_n^{\mathbb{K}} = \varphi_n^{\mathbb{K}} \circ \pi$. For simplicity, we denote $\varphi = \varphi_n^{\mathbb{K}}$.

$U^{\mathbb{K}}(n+1)$ acts on $\mathrm{Herm}_0^{\mathbb{K}}(n+1)$ orthogonally as

$$P(A) = PAP^* \quad \text{for} \quad P \in U^{\mathbb{K}}(n+1),\ A \in \mathrm{Herm}_0^{\mathbb{K}}(n+1).$$

It is well-known that for each $A \in \mathrm{Herm}_0^{\mathbb{K}}(n+1)$, there exists $P \in U^{\mathbb{K}}(n+1)$ that PAP^* is a diagonal matrix whose elements are real numbers. The map φ is equivariant, since

$$\begin{aligned}\varphi(P\cdot\pi(\mathbf{u}_1,\mathbf{u}_2)) &= \varphi(\pi(P\mathbf{u}_1,P\mathbf{u}_2)) = \widetilde{\varphi}_n^{\mathbb{K}}(P\mathbf{u}_1,P\mathbf{u}_2)\\ &= (P\mathbf{u}_1)(P\mathbf{u}_1)^* - (P\mathbf{u}_2)(P\mathbf{u}_2)^* = P\varphi(\pi(\mathbf{u}_1,\mathbf{u}_2))P^*\\ &\quad \text{for any } \pi(\mathbf{u}_1,\mathbf{u}_2)\in F_2(\mathbb{K}^{n+1}).\end{aligned}$$

Take an element $(\mathbf{x}_1,\mathbf{x}_2)\in T_0(\mathbf{u}_1,\mathbf{u}_2)$ with $\|\mathbf{x}_1\|=\|\mathbf{x}_2\|=1$. Then the curve $t\mapsto \pi(\mathbf{u}_1\cos t+\mathbf{x}_1\sin t, \mathbf{u}_2\cos t+\mathbf{x}_2\sin t)$ is tangent to $d\pi(\mathbf{x}_1,\mathbf{x}_2)$ at $\pi(\mathbf{u}_1,\mathbf{u}_2)$ in $F_2(\mathbb{K}^{n+1})$. We have

$$\begin{aligned}&\varphi\big(\pi(\mathbf{u}_1\cos t+\mathbf{x}_1\sin t, \mathbf{u}_2\cos t+\mathbf{x}_2\sin t)\big)\\ &= (\mathbf{u}_1\mathbf{u}_1^*-\mathbf{u}_2\mathbf{u}_2^*)\cos^2 t + (\mathbf{x}_1\mathbf{x}_1^*-\mathbf{x}_2\mathbf{x}_2^*)\sin^2 t\\ &\quad + (\mathbf{u}_1\mathbf{x}_1^*+\mathbf{x}_1\mathbf{u}_1^*-\mathbf{u}_2\mathbf{x}_2^*-\mathbf{x}_2\mathbf{u}_2^*)\cos t\sin t.\end{aligned} \tag{38}$$

Thus we obtain

$$\begin{aligned}d\varphi(d\pi(\mathbf{x}_1,\mathbf{x}_2)) &= \mathbf{u}_1\mathbf{x}_1^*+\mathbf{x}_1\mathbf{u}_1^*-\mathbf{u}_2\mathbf{x}_2^*-\mathbf{x}_2\mathbf{u}_2^*\\ &\quad \text{for} \quad (\mathbf{x}_1,\mathbf{x}_2)\in T_0(\mathbf{u}_1,\mathbf{u}_2)\end{aligned} \tag{39}$$

and

$$\|d\varphi(d\pi(\mathbf{x}_1,\mathbf{x}_2))\| = \sqrt{2}. \tag{40}$$

Take an element $(-\mathbf{u}_2\bar{\lambda},\mathbf{u}_1\lambda)\in T_\lambda(\mathbf{u}_1,\mathbf{u}_2)$ with $|\lambda|=1$. Then the curve $t\mapsto\pi(\mathbf{u}_1\cos t-\mathbf{u}_2\bar{\lambda}\sin t, \mathbf{u}_2\cos t+\mathbf{u}_1\lambda\sin t)$ is tangent to $d\pi(-\mathbf{u}_2\bar{\lambda},\mathbf{u}_1\lambda)$ at $\pi(\mathbf{u}_1,\mathbf{u}_2)$ in $F_2(\mathbb{K}^{n+1})$. We have

$$\begin{aligned}&\varphi\big(\pi(\mathbf{u}_1\cos t-\mathbf{u}_2\bar{\lambda}\sin t, \mathbf{u}_2\cos t+\mathbf{u}_1\lambda\sin t)\big)\\ &= (\mathbf{u}_1\mathbf{u}_1^*-\mathbf{u}_2\mathbf{u}_2^*)\cos 2t - (\mathbf{u}_1\lambda\mathbf{u}_2^*+\mathbf{u}_2\bar{\lambda}\mathbf{u}_1^*)\sin 2t.\end{aligned} \tag{41}$$

Hence we get

$$\begin{aligned}d\varphi(d\pi(-\mathbf{u}_2\bar{\lambda},\mathbf{u}_1\lambda)) &= -2(\mathbf{u}_1\lambda\mathbf{u}_2^*+\mathbf{u}_2\bar{\lambda}\mathbf{u}_1^*)\\ &\quad \text{for} \quad (-\mathbf{u}_2\bar{\lambda},\mathbf{u}_1\lambda)\in T_\lambda(\mathbf{u}_1,\mathbf{u}_2)\end{aligned} \tag{42}$$

and

$$\|d\varphi(d\pi(-\mathbf{u}_2\bar{\lambda},\mathbf{u}_1\lambda))\| = 2. \tag{43}$$

By (40) and (43), the map φ is an immersion. If $\widetilde{\varphi}_n^{\mathbb{K}}(\mathbf{u}_1,\mathbf{u}_2)=\widetilde{\varphi}_n^{\mathbb{K}}(\mathbf{v}_1,\mathbf{v}_2)$ holds for $(\mathbf{u}_1,\mathbf{u}_2)$, $(\mathbf{v}_1,\mathbf{v}_2)\in V_2(\mathbb{K}^{n+1})$, then $\mathbf{u}_1\mathbf{u}_1^*-\mathbf{u}_2\mathbf{u}_2^*=\mathbf{v}_1\mathbf{v}_1^*-\mathbf{v}_2\mathbf{v}_2^*$

and $(\mathbf{u}_1\mathbf{u}_1^* - \mathbf{u}_2\mathbf{u}_2^*)^2 = (\mathbf{v}_1\mathbf{v}_1^* - \mathbf{v}_2\mathbf{v}_2^*)^2$ imply $\mathbf{u}_1\mathbf{u}_1^* = \mathbf{v}_1\mathbf{v}_1^*$ and $\mathbf{u}_2\mathbf{u}_2^* = \mathbf{v}_2\mathbf{v}_2^*$. By comparing the components, we see that $\mathbf{v}_1 = \mathbf{u}_1\mu_1$ and $\mathbf{v}_2 = \mathbf{u}_2\mu_2$ for some μ_1, $\mu_2 \in \mathbb{K}$ with $|\mu_1| = |\mu_2| = 1$ and hence φ is an embedding.

Let σ be the second fundamental form of φ. Take a tangent vector $(-\mathbf{u}_2\bar{\lambda}, \mathbf{u}_1\lambda) \in T_\lambda(\mathbf{u}_1, \mathbf{u}_2)$ ($|\lambda| = 1$) at $(\mathbf{u}_1, \mathbf{u}_2) \in V_2(\mathbb{K}^{n+1})$. (41) implies

$$\begin{aligned} &\left.\frac{d^2}{dt^2}\right|_{t=0} \varphi\big(\pi(\mathbf{u}_1 \cos t - \mathbf{u}_2\bar{\lambda}\sin t, \mathbf{u}_2\cos t + \mathbf{u}_1\lambda\sin t)\big) \\ = &-4(\mathbf{u}_1\mathbf{u}_1^* - \mathbf{u}_2\mathbf{u}_2^*), \end{aligned}$$

which is proportional to $\varphi(\pi(\mathbf{u}_1, \mathbf{u}_2))$. Hence we get $\sigma(d\pi(X), d\pi(X)) = 0$ for any $X \in T_\lambda(\mathbf{u}_1, \mathbf{u}_2)$ and

$$\sigma(d\pi(X), d\pi(Y)) = 0 \quad \text{for any} \quad X,\, Y \in T_\lambda(\mathbf{u}_1, \mathbf{u}_2), \tag{44}$$

by polarization. Put $(\mathbf{x}_1, \mathbf{x}_2) \in T_0(\mathbf{u}_1, \mathbf{u}_2)$ with $\|\mathbf{x}_1\| = \|\mathbf{x}_2\| = 1$. Using (38), we obtain

$$\begin{aligned} &\left.\frac{d^2}{dt^2}\right|_{t=0} \varphi\big(\pi(\mathbf{u}_1 \cos t + \mathbf{x}_1\sin t, \mathbf{u}_2\cos t + \mathbf{x}_2\sin t)\big) \\ &= 2(-\mathbf{u}_1\mathbf{u}_1^* + \mathbf{u}_2\mathbf{u}_2^* + \mathbf{x}_1\mathbf{x}_1^* - \mathbf{x}_2\mathbf{x}_2^*). \end{aligned}$$

By taking the component which is orthogonal to $\varphi(\pi(\mathbf{u}_1, \mathbf{u}_2))$, and using (39) and (42), we have

$$\sigma\big(d\pi(\mathbf{x}_1, \mathbf{x}_2), d\pi(\mathbf{x}_1, \mathbf{x}_2)\big) = 2(\mathbf{x}_1\mathbf{x}_1^* - \mathbf{x}_2\mathbf{x}_2^*)$$

for any $(\mathbf{x}_1, \mathbf{x}_2) \in T_0(\mathbf{u}_1, \mathbf{u}_2)$. Hence

$$\begin{aligned} \sigma\big(d\pi(\mathbf{x}_1, \mathbf{x}_2), d\pi(\mathbf{y}_1, \mathbf{y}_2)\big) &= \mathbf{x}_1\mathbf{y}_1^* + \mathbf{y}_1\mathbf{x}_1^* - \mathbf{x}_2\mathbf{y}_2^* - \mathbf{y}_2\mathbf{x}_2^* \\ &\text{for any } (\mathbf{x}_1, \mathbf{x}_2),\, (\mathbf{y}_1, \mathbf{y}_2) \in T_0(\mathbf{u}_1, \mathbf{u}_2) \end{aligned} \tag{45}$$

holds by polarization. Finally we consider the curve in $V_2(\mathbb{K}^{n+1})$ defined as:

$$\begin{aligned} t \mapsto \big(\mathbf{u}_1 \cos t + ((\mathbf{x}_1 - \mathbf{u}_2\bar{\lambda})/\sqrt{2}\,)\sin t, \mathbf{u}_2\cos t + ((\mathbf{x}_2 + \mathbf{u}_1\lambda)/\sqrt{2}\,)\sin t\big) \\ ((\mathbf{x}_1, \mathbf{x}_2) \in T_0(\mathbf{u}_1, \mathbf{u}_2),\ \|\mathbf{x}_1\| = \|\mathbf{x}_2\| = 1 \text{ and } |\lambda| = 1), \end{aligned}$$

which is tangent to $\big((\mathbf{x}_1, \mathbf{x}_2) + (-\mathbf{u}_2\bar{\lambda}, \mathbf{u}_1\lambda)\big)/\sqrt{2}$ at $(\mathbf{u}_1, \mathbf{u}_2)$. Then we have

$$\begin{aligned} &\left.\frac{d^2}{dt^2}\right|_{t=0} \varphi\Big(\pi\Big(\mathbf{u}_1\cos t + \Big(\frac{\mathbf{x}_1 - \mathbf{u}_2\bar{\lambda}}{\sqrt{2}}\Big)\sin t, \mathbf{u}_2\cos t + \Big(\frac{\mathbf{x}_2 + \mathbf{u}_1\lambda}{\sqrt{2}}\Big)\sin t\Big)\Big) \\ &= -3(\mathbf{u}_1\mathbf{u}_1^* - \mathbf{u}_2\mathbf{u}_2^*) + (\mathbf{x}_1\mathbf{x}_1^* - \mathbf{x}_2\mathbf{x}_2^*) \\ &\quad - \big((\mathbf{x}_1\lambda)\mathbf{u}_2^* + \mathbf{u}_2(\mathbf{x}_1\lambda)^* + (\mathbf{x}_2\bar{\lambda})\mathbf{u}_1^* + \mathbf{u}_1(\mathbf{x}_2\bar{\lambda})^*\big). \end{aligned}$$

By (39), we get

$$\begin{aligned}&\frac{1}{2}\sigma\big(d\pi(\mathbf{x}_1,\mathbf{x}_2)+d\pi(-\mathbf{u}_2\bar{\lambda},\mathbf{u}_1\lambda),d\pi(\mathbf{x}_1,\mathbf{x}_2)+d\pi(-\mathbf{u}_2\bar{\lambda},\mathbf{u}_1\lambda)\big)\\&=\mathbf{x}_1\mathbf{x}_1^*-\mathbf{x}_2\mathbf{x}_2^*.\end{aligned}$$

Then (44) and (45) yield

$$\sigma\big(d\pi(\mathbf{x}_1,\mathbf{x}_2),d\pi(X)\big)=0 \quad \text{for any } (\mathbf{x}_1,\mathbf{x}_2)\in T_0(\mathbf{u}_1,\mathbf{u}_2) \text{ and } X\in T_\lambda(\mathbf{u}_1,\mathbf{u}_2). \tag{46}$$

Hence the index of relative nullity [30] of the embedding $\varphi\colon F_2(\mathbb{K}^{n+1})\to S(\mathrm{Herm}_0^{\mathbb{K}}(n+1))$ is equal to $\dim_{\mathbb{R}} T_\lambda(\mathbf{u}_1,\mathbf{u}_2)=d$ at any $x=\pi(\mathbf{u}_1,\mathbf{u}_2)\in F_2(\mathbb{K}^{n+1})$.

Our examples $\varphi\colon F_2(\mathbb{K}^{n+1})\to S(\mathrm{Herm}_0^{\mathbb{K}}(n+1))=S^m$ satisfying Ferus equality $r=F(l)$ (or equivalently $\nu=\mu(l)$) for $l\le 32$ are as follows:

Embedding	l	r	ν	m
$\varphi_2^{\mathbb{R}}\colon F_2(\mathbb{R}^3)\to S(\mathrm{Herm}_0^{\mathbb{R}}(3))$	3	2	1	4
$\varphi_3^{\mathbb{R}}\colon F_2(\mathbb{R}^4)\to S(\mathrm{Herm}_0^{\mathbb{R}}(4))$	5	4	1	8
$\varphi_2^{\mathbb{C}}\colon F_2(\mathbb{C}^3)\to S(\mathrm{Herm}_0^{\mathbb{C}}(3))$	6	4	2	7
$\varphi_5^{\mathbb{R}}\colon F_2(\mathbb{R}^6)\to S(\mathrm{Herm}_0^{\mathbb{R}}(6))$	9	8	1	19
$\varphi_3^{\mathbb{C}}\colon F_2(\mathbb{C}^4)\to S(\mathrm{Herm}_0^{\mathbb{C}}(4))$	10	8	2	14
$\varphi_2^{\mathbb{H}}\colon F_2(\mathbb{H}^3)\to S(\mathrm{Herm}_0^{\mathbb{H}}(3))$	12	8	4	13
$\varphi_9^{\mathbb{R}}\colon F_2(\mathbb{R}^{10})\to S(\mathrm{Herm}_0^{\mathbb{R}}(10))$	17	16	1	53
$\varphi_5^{\mathbb{C}}\colon F_2(\mathbb{C}^6)\to S(\mathrm{Herm}_0^{\mathbb{C}}(6))$	18	16	2	33
$\varphi_3^{\mathbb{H}}\colon F_2(\mathbb{H}^4)\to S(\mathrm{Herm}_0^{\mathbb{H}}(4))$	20	16	4	26
$\varphi_{13}^{\mathbb{R}}\colon F_2(\mathbb{R}^{14})\to S(\mathrm{Herm}_0^{\mathbb{R}}(14))$	25	24	1	103
$\varphi_7^{\mathbb{C}}\colon F_2(\mathbb{C}^8)\to S(\mathrm{Herm}_0^{\mathbb{C}}(8))$	26	24	2	62
$\varphi_4^{\mathbb{H}}\colon F_2(\mathbb{H}^5)\to S(\mathrm{Herm}_0^{\mathbb{H}}(5))$	28	24	4	43

Note that $\varphi_2^{\mathbb{K}}\colon F_2(\mathbb{K}^3)\to S(\mathrm{Herm}_0^{\mathbb{K}}(3))=S^{3d+1}$ is nothing but the Cartan's isoparametric (minimal) hypersurfaces.

With respect to $\varphi\colon F_2(\mathbb{K}^{n+1})\to S(\mathrm{Herm}_0^{\mathbb{K}}(n+1))$, (45) implies

$$\begin{aligned}&\sigma\big(d\pi(\mathbf{x},0),d\pi(\mathbf{y},0)\big)+\sigma\big(d\pi(0,\mathbf{x}),d\pi(0,\mathbf{y})\big)\\&=\sigma\big(d\pi(\mathbf{x},0),d\pi(0,\mathbf{y})\big)=0,\end{aligned}\tag{47}$$

where $\mathbf{x}^*\mathbf{u}_1 = \mathbf{x}^*\mathbf{u}_2 = \mathbf{y}^*\mathbf{u}_1 = \mathbf{y}^*\mathbf{u}_2 = 0$ at any $\pi(\mathbf{u}_1, \mathbf{u}_2) \in F_2(\mathbb{K}^{n+1})$. Hence the embedding φ is austere (§8) and minimal.

Remark 9.1. When $\mathbb{K} = \mathbb{R}$, the submanifold $F_2(\mathbb{R}^{n+1})$ is given in §7. Also when $\mathbb{K} = \mathbb{C}$ or $\mathbb{K} = \mathbb{H}$, the submanifold $F_2(\mathbb{K}^{n+1})$ is obtained as a pull-back bundle over 2-plane Grassmannian $G_2(\mathbb{K}^{n+1})$ of canonical d-dimensional sphere bundle over real Grassmannian $G^{\mathbb{R}}_{d+1}(\mathrm{Herm}_0^{\mathbb{K}}(n+1))$ with respect to the embedding:

Case $\mathbb{K} = \mathbb{C}$: $\quad G_2(\mathbb{C}^{n+1}) \to G^{\mathbb{R}}_3(\mathrm{Herm}_0^{\mathbb{C}}(n+1))$,

$$\mathrm{span}_{\mathbb{C}}\{\mathbf{u}, \mathbf{v}\} \mapsto \mathrm{span}_{\mathbb{R}}\{\mathbf{u}\mathbf{u}^* - \mathbf{v}\mathbf{v}^*, \mathbf{u}\mathbf{v}^* + \mathbf{v}\mathbf{u}^*, \mathbf{i}(\mathbf{u}\mathbf{v}^* - \mathbf{v}\mathbf{u}^*)\}$$

Case $\mathbb{K} = \mathbb{H}$: $\quad G_2(\mathbb{H}^{n+1}) \to G^{\mathbb{R}}_5(\mathrm{Herm}_0^{\mathbb{H}}(n+1))$,

$$\begin{aligned}\mathrm{span}_{\mathbb{H}}\{\mathbf{u}, \mathbf{v}\} \mapsto \mathrm{span}_{\mathbb{R}}\{&\mathbf{u}\mathbf{u}^* - \mathbf{v}\mathbf{v}^*, \mathbf{u}\mathbf{v}^* + \mathbf{v}\mathbf{u}^*,\\ &\mathbf{u}\mathbf{i}\mathbf{v}^* - \mathbf{v}\mathbf{i}\mathbf{u}^*, \mathbf{u}\mathbf{j}\mathbf{v}^* - \mathbf{v}\mathbf{j}\mathbf{u}^*, \mathbf{u}\mathbf{k}\mathbf{v}^* - \mathbf{v}\mathbf{k}\mathbf{u}^*\},\end{aligned}$$

where $\mathbf{u}^*\mathbf{u} = \mathbf{v}^*\mathbf{v} = 1$ and $\mathbf{u}^*\mathbf{v} = 0$. For $\mathbb{K} = \mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$, $\varphi\colon F_2(\mathbb{K}^{n+1}) \to S(\mathrm{Herm}_0^{\mathbb{K}}(n+1))$ satisfies B. Y. Chen's equality (cf. Theorem 4.1 in [15])

$$\delta\big((n-1)d, (n-1)d\big) = (n-1)(nd+1)d$$
$$(n = (2n-1)d,\ n_1 = n_2 = (n-1)d,\ H = 0,\ \epsilon = 1).$$

§10. Hypersurfaces with degenerate Gauss mappings in the four dimensional sphere

In this section, we study the simplest case $n = 4$, $l = 3$, $r = F(3) = 2$.

Recall that the Cartan hypersurface $M^3 \subset S^4$ is a homogeneous space of SO(3) and written as $M = \mathrm{O}(3)/\big(\mathrm{O}(1) \times \mathrm{O}(1) \times \mathrm{O}(1)\big)$. The Gauss mapping $\gamma\colon M^3 \to G_4(\mathbb{R}^5) \cong G_1(\mathbb{R}^{5*}) = \mathbb{RP}^{4*}$ into the dual projective space, has the constant rank 2. Moreover its image $\gamma(M)$, that is the projective dual in this case, is a linear projection $\mathbb{RP}^2 \subset \mathbb{RP}^{4*}$ of the Veronese surface $\mathbb{RP}^2 \subset \mathbb{RP}^{5*}$ in the sense of algebraic geometry [26]: The Veronese surface has the crucial property that its secant variety is of positive codimension in $\mathbb{RP}^{5*}$ (cf. [45]). Notice that it lifts to the Veronese surface $i\colon \mathbb{RP}^2 \hookrightarrow \widetilde{G}_4(\mathbb{R}^5) \cong \widetilde{G}_1(\mathbb{R}^{5*}) = S^{4*}$, in the sense of differential geometry. The liftability means just that M is orientable. Consider the double covering $\pi\colon S^2 \to \mathbb{RP}^2$ and take the fiber product $\widetilde{M}$ of π and $\gamma\colon M \to \mathbb{RP}^2$:

$$\begin{array}{ccccc}\widetilde{M} & \longrightarrow & S^2 & & \\ {\scriptstyle \Pi}\downarrow & & \downarrow{\scriptstyle \pi} & & \\ M & \xrightarrow{\ \gamma\ } & \mathbb{RP}^2 & \xrightarrow{\ i\ } & S^{4*}.\end{array}$$

We call $\widetilde{M}$ the *doubled Cartan hypersurface.* Then we have the tangentially degenerate immersion $\widetilde{M} \to S^4$, that is the composition of the double covering $\Pi\colon \widetilde{M} \to M$ and the inclusion $M \subset S^4$. $\widetilde{M}$ is connected and realized by $\mathrm{O}(3)/(\mathrm{SO}(1) \times \mathrm{O}(1) \times \mathrm{O}(1))$ as a homogeneous space. Also remark, by the spherical-projective duality, that $\widetilde{M}$ is the total space of the associated $\widetilde{G}_1(\mathbb{R}^2) = S^1$ bundle over S^2 to the normal bundle of the immersion $i \circ \pi\colon S^2 \to S^{4*}$.

Then we give the following characterization of the diffeomorphism type of compact connected tangentially degenerate hypersurfaces in S^4, using the result of Asperti [5].

Theorem 10.1. *Let M' be a compact connected 3-dimensional manifold, and $f'\colon M'^3 \to S^4$ a tangentially degenerate immersion. Assume that the rank of the Gauss mapping of f' is everywhere 2. Then, there exists a tangentially degenerate immersion $f\colon \widetilde{M} \to S^4$ from the doubled Cartan hypersurface $\widetilde{M}$ with $f(\widetilde{M}) = f'(M')$.*

Proof. We may assume M' is orientable. If not, we take its connected double covering. Then the Gauss mapping $\gamma(f')\colon M' \to \mathbb{RP}^{4*}$ of f' lifts to $\widetilde{\gamma}(f')\colon M' \to S^{4*}$. Then $\operatorname{rank}\widetilde{\gamma}(f')$ is identically 2. Therefore $\widetilde{\gamma}(f')$ is decomposed into π and j, where $\pi\colon M' \to N'$ is an S^1-fibration and $j\colon N' \to S^{4*}$ is an immersion. Since M' is connected, N' is connected. Moreover we may assume N' is orientable without loss of generality. (If not, we may take connected double covers of M' and N'). Since $\widetilde{\gamma}(f')$ is an immersion, we see that the second fundamental form of j has no singular quadratic form [27] and [26]. Then, j has non-vanishing normal curvature, relatively to the ordinary metric. Now we recall that any oriented immersed surfaces in S^4 with non-vanishing normal curvature is parameterized by an immersion $S^2 \to S^4$ with the normal Euler number ± 4 [5]. Thus we can assume that $N' = S^2$ and the normal Euler number of j is equal to 4. If it is equal to -4, we take another lifting $\widetilde{\gamma}(f')$, changing the orientation of M'. Now consider the associated $\widetilde{G}_1(\mathbb{R}^2) = S^1$ bundle M'' over S^2 to the normal bundle of the immersion $j\colon S^2 \to S^{4*}$. Then we see that the Euler number of the S^1-bundle $M'' \to S^2$ is equal to 4. Since the diffeomorphism type of such bundles is uniquely determined, and since also the immersion $\widetilde{M} \to S^4$ induces the S^1 fibration $\widetilde{M} \to S^2$ with Euler number 4 as well, we see that there exists a diffeomorphism $\rho\colon \widetilde{M} \to M''$. On the other hand, by the projective duality, if we set $f = f'' \circ \rho\colon \widetilde{M} \to S^4$, then we see that f is a tangentially degenerate immersion and $f(\widetilde{M}) = f'(M')$. $\square$

Lastly we proceed to construct an example of tangentially degenerate immersions from a compact submanifold M of dimension 3 to S^4, the rank of whose Gauss mapping is not constant 2.

Recall that for a Riemannian surface Σ, a holomorphic immersion $\varphi\colon \Sigma \to Q^{n-1}$ is called *first-order isotropic* if the complex derivative $\varphi'\colon \Sigma \to \mathbb{CP}^n$ lies in Q^{n-1} again. This condition is equivalent to that the *tangent developable*, the union of tangent lines, to φ is contained in Q^{n-1} (cf. Proposition 8.1). A holomorphic immersion $\varphi\colon \Sigma \to Q^{n-1}$ *has no real point* if $\varphi'(\Sigma) \cap \mathbb{RP}^n = \emptyset$, which is the case for first-order isotropic immersion. Using the notation in §8, let $M = \varphi^* V_2(\mathbb{R}^{n+1})$ be the pull-back bundle over the Riemannian surface Σ, and let Φ be given by (15).

Theorem 10.2 ([29]). *If $\varphi\colon \Sigma \to Q^{n-1}$ has no real point, then $f\colon M^3 \to S^n$ is a tangentially degenerate immersion. If $\varphi\colon \Sigma \to Q^{n-1}$ is a first-order isotropic immersion, then $f\colon M^3 \to S^n$ is a minimal tangentially degenerate immersion, with respect to the ordinary metric on S^n.*

Now, in the case $n = 4$, there exist first-order isotropic holomorphic immersions (unramified) $\varphi\colon S^2(= \mathbb{CP}^1) \to Q^3 \subset \mathbb{CP}^4$, [9]. Thus we have

Proposition 10.3. *There exist a minimal tangentially degenerate immersion $f\colon M^3 \to S^4$ such that M is a circle bundle over S^2, and that the oriented Gauss mapping $\widetilde{\gamma}\colon M \to \widetilde{G}_4(\mathbb{R}^5) = S^4$ splits into a fibration $M \to S^2$ and a ramified minimal immersion $X\colon S^2 \to S^4$. The rank of $\widetilde{\gamma}$ is not constant 2.*

Proof. Take $\gamma_3\colon S^2 \to Q^3$ of [9], page 237. The corresponding complex contact curve $\lambda_3\colon S^2 \to \mathbb{CP}^3$ has the ramification degree 2. Therefore the induced minimal immersion $X = \pi \circ \lambda_3\colon S^2 \to S^4$ is ramified as well, and X is a parameterization of the image of $\widetilde{\gamma}$. □

Remark that, in [8], it is proved that there exist minimal immersions $\Sigma \to S^4$ from any *compact* Riemann surface Σ of arbitrary genus. Then, by taking their directrix, we have first-order isotropic homomorphic mappings $\varphi\colon \Sigma \to Q^3$, however, in general, ramified.

§11. Appendix

Here we give proofs of Propositions in §3. Let D^1, D^2, D^3, D^4 be the curvature distributions with respect to the principal curvatures λ_1, λ_2, λ_3, λ_4. Using an orthonormal basis of each D^i, take an orthonormal frame $e_1, \dots, e_{2(m_1+m_2)}$ of M. Then the shape operator B_ζ of M_- is given by [33] and [34]

$$B_\zeta = c\begin{pmatrix} 0 & B_{12} & B_{13} \\ {}^tB_{12} & 0 & B_{23} \\ {}^tB_{13} & {}^tB_{23} & 0 \end{pmatrix}, \quad c = (\sin(\operatorname{arccot}\lambda_4))^{-1}$$

where $B_{ij} = \left(\Lambda^l_{k4}/(\lambda_i - \lambda_4)\right)$, $1 \leq i,\ j \leq 3$ is the $m_i \times m_j$ matrix with

$$\Lambda^l_{k4} = \langle \nabla_{e_k} e_4, e_l \rangle, \quad e_k \in D^i,\ e_l \in D^j$$

and $\zeta \in D^4$ is a unit normal vector for M_-.

Similarly, the shape operator C_ζ of M_+ is given by

$$C_\zeta = c'\begin{pmatrix} 0 & C_{23} & C_{24} \\ {}^tC_{23} & 0 & C_{34} \\ {}^tC_{24} & {}^tC_{34} & 0 \end{pmatrix}, \quad c' = (\sin(\operatorname{arccot}\lambda_1))^{-1}$$

where $C_{ij} = \left(\Lambda^l_{k1}/(\lambda_i - \lambda_1)\right)$, $2 \leq i,\ j \leq 4$ is the $m_i \times m_j$ matrix with

$$\Lambda^l_{k1} = \langle \nabla_{e_k} e_1, e_l \rangle, \quad e_k \in D^i,\ e_l \in D^j$$

and $\zeta \in D^1$ is a unit normal vector for M_+.

M_- or M_+ is tangentially degenerate if there exists a frame with respect to which a certain row (then column) of B_ζ (C_ζ) in the middle block vanishes for all $\zeta = e_4$ (e_1). This is because, if we take $\eta = \cos\theta\, p + \sin\theta\, \xi_p$ where ξ_p is the unit normal of M at $p \in M$ and $\theta = \operatorname{arccot}\lambda_4$, ($\operatorname{arccot}\lambda_1$, respectively), B_η is given by $\operatorname{diag}(1_{m_2}, 0_{m_1}, -1_{m_2})$ ($C_\eta = \operatorname{diag}(1_{m_1}, 0_{m_2}, -1_{m_1})$, respectively), (see [33]) and we should have $\bigcap_{n \in T^\perp M_-} \operatorname{Ker} B_n \neq \{0\}$, ($\bigcap_{T^\perp M_+} \operatorname{Ker} C_n \neq \{0\}$, respectively). Note that any $n \in T^\perp M_-$ ($n \in T^\perp M_+$) can be written as a combination of η and vectors in D^4 (D^1).

By the argument of 3.1 in [33], the Codazzi equation implies

$$\Lambda^k_{ij}(\lambda_j - \lambda_k) = \Lambda^i_{jk}(\lambda_k - \lambda_i) = \Lambda^j_{ki}(\lambda_i - \lambda_j)$$

hence, vanishing of Λ^k_{ij} where λ_i, λ_j, λ_k are distinct, depends only on the set $\{i, j, k\}$. We calculate Λ^k_{ij} for necessary indices. For this we need root vectors of the symmetric Lie algebras given in section 4 of [39].

For $(m_1, m_2) = (1, k-2)$, $(2, 2k-3)$, $(4, 4k-5)$, the Lie algebra is

$$\mathfrak{g} = \{A \in \mathfrak{gl}(k+2, \mathbb{K}) \mid T(A) = 0,\ {}^t\bar{A}\Phi + \Phi A = 0\}$$

where $\mathbb{K} = \mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, respectively, and

$$\Phi = \begin{pmatrix} 1_2 & 0 \\ 0 & -1_k \end{pmatrix}.$$

The Cartan involution is given by $\sigma(A) = -{}^t\bar{A}$ hence the ± 1 eigenspaces $\mathfrak{k}$, $\mathfrak{p}$ are, respectively

$$\mathfrak{k} = \left\{ \begin{pmatrix} Y & 0 \\ 0 & Z \end{pmatrix} \,\middle|\, {}^t\bar{Y} + Y = 0,\ {}^t\bar{Z} + Z = 0 \right\}$$
$$\mathfrak{p} = \left\{ \hat{X} = \begin{pmatrix} 0 & {}^t\bar{X} \\ X & 0 \end{pmatrix} \,\middle|\, X = (x_{ij}) \in M_{k,2}(\mathbb{K}) \right\}.$$

We use a metric

$$\langle \hat{X}, \hat{Y} \rangle = \Re\,\mathrm{Tr}({}^t\bar{X}Y).$$

A maximal abelian subalgebra $\mathfrak{a}$ of $\mathfrak{p}$ is given by

$$\mathfrak{a} = \{H(\xi_1, \xi_2) = \xi_1(E_{31} + E_{13}) + \xi_2(E_{42} + E_{24}) \mid \xi_1,\, \xi_2 \in \mathbb{R}\},$$

where we use the standard basis E_{ij}. Following the notation in [39], Σ_*^+ consists of

$$\gamma_1 = \xi_1 - \xi_2,\ \gamma_2 = \xi_1,\ \gamma_3 = \xi_1 + \xi_2,\ \gamma_4 = \xi_2.$$

The corresponding root vectors in $M_{r+2}(\mathbb{K})$ are, respectively, given by (we omit $\hat{}$)

$$\begin{aligned}
X_1 &= x_{21}(E_{41} + E_{23}) + \bar{x}_{21}(E_{14} + E_{32}), \quad (\bar{x}_{21} = x_{12}) \\
X_2^i &= x_{i-2\,1}E_{i1} + \bar{x}_{i-2\,1}E_{1i}, \quad 5 \le i \le k+2 \\
X_2' &= x_{11}(-E_{13} + E_{31}), \quad (x_{11} + \bar{x}_{11} = 0) \\
X_3 &= y_{21}(E_{41} - E_{23}) + \bar{y}_{21}(E_{14} - E_{32}) \quad (\bar{y}_{21} = -y_{12}) \\
X_4^i &= x_{i-2\,2}E_{i-2\,2} + \bar{x}_{i2}E_{2i}, \quad 5 \le i \le k+2 \\
X_4' &= x_{22}(-E_{24} + E_{42}), \quad (x_{22} + \bar{x}_{22} = 0).
\end{aligned}$$

Here x_{ij}, $y_{ij} \in \mathbb{K}$ and noting that X_2' and X_4' are trivial for $\mathbb{K} = \mathbb{R}$, we have

$$(m_1, m_2) = \begin{cases} (1, k-2) & \mathbb{K} = \mathbb{R} \\ (2, 2k-3) & \mathbb{K} = \mathbb{C} \\ (4, 4k-5) & \mathbb{K} = \mathbb{H}. \end{cases}$$

Putting $T_i = [H, X_i] \in \mathfrak{k}$ up to constant, we get

$$\begin{aligned}
T_1 &= \bar{x}_{21}(E_{12} + E_{34}) - x_{21}(E_{21} + E_{43}) \\
T_2^i &= \bar{x}_{i-2\,1}E_{3i} - x_{i-2\,1}E_{i3}, \quad 5 \le i \le k+2 \\
T_2' &= x_{11}(E_{11} - E_{33}), \quad (x_{11} + \bar{x}_{11} = 0) \\
T_3 &= \bar{y}_{21}(-E_{12} + E_{34}) + y_{21}(E_{21} - E_{43}) \\
T_4^i &= \bar{x}_{i-2\,2}E_{4i} - x_{i-2\,2}E_{i4}, \quad 5 \le i \le k+2 \\
T_4' &= x_{22}(E_{22} - E_{44}), \quad (x_{22} + \bar{z}_{22} = 0).
\end{aligned}$$

By the argument in [42] and in [33] and [34], we know that

$$\nabla_{X_i^j} X_k^l \sim [T_i^j, X_k^l],$$

where $\sim$ means be equal up to constant.

Noting that $[E_{ij}, E_{kl}] = \delta_{jk}E_{il} - \delta_{li}E_{kj}$, we obtain

$$\begin{aligned}
\nabla_{X_3} X_1 \sim [T_3, X_1] = \big[&\bar{y}_{21}(-E_{12} + E_{34}) + y_{21}(E_{21} - E_{43}), \\
&x_{21}(E_{41} + E_{23}) + \bar{x}_{21}(E_{14} + E_{32})\big]
\end{aligned}$$

which vanishes only when $\mathbb{K} = \mathbb{R}$, and then we have

$$C_\zeta = c' \begin{pmatrix} 0 & 0 & C_{24} \\ 0 & 0 & 0 \\ {}^tC_{24} & 0 & 0 \end{pmatrix},$$

hence M_+ is tangentially degenerate. When $\mathbb{K} = \mathbb{C}$ and $\mathbb{H}$, noting that $\langle E_{ij}, E_{kl} \rangle = \delta_{ik}\delta_{jl}$, we know that $\nabla_{X_3} X_1$ has X_2' and X_4' components, hence M_+ is not tangentially degenerate.

Next, we have

$$\nabla_{X_2^i} X_4^j \sim [T_2^i, X_4^j] = \big[\bar{x}_{i-2\,1}E_{3i} - x_{i-2\,1}E_{i3}, x_{j-2\,2}E_{j2} + \bar{x}_{j-2\,2}E_{2j}\big]$$

which has X_1 and X_3 components if $i = j$, hence M_- is not tangentially degenerate if $\mathbb{K} = \mathbb{R}$. On the other hand, when $\mathbb{K} = \mathbb{C}$ and $\mathbb{H}$,

$$\begin{aligned}
\nabla_{X_2'} X_4^j &\sim [T_2', X_4^j] = \big[x_{21}(E_{11} - E_{33}), x_{j-2\,2}E_{j2} + \bar{x}_{j-2\,2}E_{2j}\big] = 0 \\
\nabla_{X_2'} X_4' &\sim [T_2', X_4'] = \big[x_{21}(E_{11} - E_{33}), x_{22}(-E_{24} + E_{42})\big] = 0
\end{aligned}$$

implies that the row in B_ζ corresponding to the vector X_2' vanishes identically, hence M_- is tangentially degenerate with $(l, r) = (2m_1 + m_2, m_1 + m_2 + 1)$. Thus we obtain Proposition 3.1.

Next we consider the case $(m_1, m_2) = (2, 2)$, $(4, 5)$. By [39], we have

$$\mathfrak{g} = \{A \in \mathfrak{gl}(5, \mathbb{H}) \mid {}^t\bar{A}\Psi + \Psi A = 0,\ \Psi = \sqrt{-1}\,1_5\} = \mathfrak{k} + \mathfrak{p},$$

where

$$\begin{cases} \mathfrak{k} = \hat{\mathfrak{u}}(5) \\ \mathfrak{p} = \{jZ \mid Z \in M_5(\mathbb{C}),\ {}^t\bar{Z} = -Z\}, \end{cases}$$

$$\begin{cases} \mathfrak{k} = \mathfrak{o}(5) \\ \mathfrak{p} = \{\sqrt{-1}\,Z \mid Z \in M_5(\mathbb{R}),\ {}^tZ = -Z\}, \end{cases}$$

and the former corresponds to $(m_1, m_2) = (4, 5)$. We identify $jZ \mapsto Z$, and $\sqrt{-1}\,Z \mapsto Z$, respectively. We use

$$\langle Z, W\rangle = -\frac{1}{2}\Re\,\mathrm{Tr}(Z\bar{W}).$$

A maximal abelian subalgebra of $\mathfrak{p}$ is given by

$$\mathfrak{a} = \{H(\xi_1, \xi_2) = \xi_1(E_{21} - E_{12}) + \xi_2(E_{43} - E_{34}) \mid \xi_1,\ \xi_2 \in \mathbb{R}\}.$$

Then Σ_*^+ is as before and the corresponding root vectors are given by

$$\begin{aligned}
&X_1^1 = z_{13}(E_{13} + E_{24}) - \bar{z}_{13}(E_{31} + E_{42}),\\
&X_1^2 = z_{14}(E_{14} - E_{23}) - \bar{z}_{14}(E_{41} - E_{32}),\\
&X_2^1 = z_{15}E_{15} - \bar{z}_{15}E_{51}, \quad X_2^2 = z_{25}E_{25} - \bar{z}_{25}E_{52},\\
&\quad (X_2^3 = \sqrt{-1}\,s(-E_{11} + E_{22})),\\
&X_3^1 = w_{13}(E_{13} - E_{24}) - \bar{w}_{13}(E_{31} - E_{42}),\\
&X_3^2 = w_{14}(E_{14} + E_{23}) - \bar{w}_{14}(E_{41} + E_{32}),\\
&X_4^1 = z_{35}E_{35} - \bar{z}_{35}E_{53}, \quad X_4^2 = z_{45}E_{45} - \bar{z}_{45}E_{54},\\
&\quad (X_4^3 = \sqrt{-1}\,t(E_{33} - E_{44})),
\end{aligned}$$

where $x_{ij},\ y_{ij} \in \mathbb{R}$ or $\mathbb{C}$ and $s,\ t \in \mathbb{R}$, and in the real case, $X_2^3 = X_4^3 = 0$. Now putting $T_i^j = [H, X_i^j]$ up to constant, we obtain

$$\begin{aligned}
&T_1^1 = z_{13}(-E_{14} + E_{23}) - \bar{z}_{13}(-E_{41} + E_{32}),\\
&T_1^2 = z_{14}(E_{13} + E_{24}) - \bar{z}_{14}(E_{31} + E_{42}),\\
&T_2^1 = z_{15}E_{25} - \bar{z}_{15}E_{52}, \quad T_2^2 = -z_{25}E_{15} + \bar{z}_{25}E_{51},\\
&\quad (T_2^3 = \sqrt{-1}\,s(E_{12} + E_{21})),\\
&T_3^1 = w_{13}(E_{14} + E_{23}) - \bar{w}_{13}(E_{41} + E_{32}),\\
&T_3^2 = w_{14}(-E_{13} + E_{24}) - \bar{w}_{14}(-E_{31} + E_{42}),\\
&T_4^1 = z_{35}E_{45} - \bar{z}_{35}E_{54}, \quad T_4^2 = z_{45}E_{35} - \bar{z}_{45}E_{53},\\
&\quad (T_4^3 = \sqrt{-1}\,t(E_{34} + E_{43})),
\end{aligned}$$

From these, we have

$$
\begin{aligned}
[T_3^1, X_1^1] &= \bigl[w_{13}(E_{14}+E_{23}) - \bar{w}_{13}(E_{41}+E_{32}),\\
&\qquad z_{13}(E_{13}+E_{24}) - \bar{z}_{13}(E_{31}+E_{42})\bigr] = 0\\
[T_3^2, X_1^2] &= \bigl[w_{14}(-E_{13}+E_{24}) - \bar{w}_{14}(-E_{31}+E_{42}),\\
&\qquad z_{14}(E_{14}-E_{23}) - \bar{z}_{14}(E_{41}-E_{32})\bigr] = 0\\
[T_3^1, X_1^2] &= \bigl[w_{13}(E_{14}+E_{23}) - \bar{w}_{13}(E_{41}+E_{32}),\\
&\qquad z_{14}(E_{14}-E_{23}) - \bar{z}_{14}(E_{41}-E_{32})\bigr]\\
&= -w_{13}\bar{z}_{14}(E_{11}-E_{22}) - \bar{w}_{13}z_{14}(E_{44}-E_{33})\\
[T_3^2, X_1^1] &= \bigl[w_{14}(-E_{13}+E_{24}) - \bar{w}_{14}(-E_{31}+E_{42}),\\
&\qquad z_{13}(E_{13}+E_{24}) - \bar{z}_{13}(E_{31}+E_{42})\bigr]\\
&= -w_{14}\bar{z}_{13}(-E_{11}+E_{22}) - \bar{w}_{14}z_{13}(-E_{33}+E_{44}).
\end{aligned}
$$

Note that the latter two vanishes when $\mathbb{K}=\mathbb{R}$, hence M_+ is tangentially degenerate. When $\mathbb{K}=\mathbb{C}$, each vector has distinct one of X_2^3 and X_4^3 components, thus M_+ is not tangentially degenerate.

On the other hand, we obtain

$$
\begin{aligned}
&[T_2^1, X_4^1] = z_{15}\bar{z}_{35}E_{23} + \bar{z}_{15}z_{35}E_{32},\\
&[T_2^1, X_4^2] = -z_{15}\bar{z}_{45}E_{24} + \bar{z}_{15}z_{45}E_{42}, \quad [T_2^1, X_4^3] = 0,\\
&[T_2^2, X_4^1] = z_{25}\bar{z}_{35}E_{13} - \bar{z}_{25}z_{35}E_{31},\\
&[T_2^2, X_4^2] = z_{25}\bar{z}_{45}E_{14} - \bar{z}_{25}z_{45}E_{41}, \quad [T_2^2, X_4^3] = 0,\\
&[T_2^3, X_4^1] = 0, \quad [T_2^3, X_4^2] = 0, \quad [T_2^3, X_4^3] = 0.
\end{aligned}
$$

When $\mathbb{K}=\mathbb{R}$, the last three are trivial, and the four non-vanishing vectors have distinct one of X_1^1, X_1^2 and X_3^1, X_3^2 components, thus M_- is not tangentially degenerate. When $\mathbb{K}=\mathbb{C}$, from the last three, the row corresponding to X_2^3 in B_ζ vanishes, hence M_- is tangentially degenerate. This completes the proof of Proposition 3.2.

References

[1] K. Abe, *A complex analogue of Hartman-Nirenberg cylinder theorem*, J. Diff. Geom., **7** (1972), 453–460.

[2] K. Abe, *Applications of a Riccati type differential equation to Riemannian manifolds with totally geodesic distributions*, Tôhoku Math. Journ., **25** (1973), 425–444.

[3] M. A. Akivis, *On multidimensional strongly parabolic surfaces*, Izv. Vyssh. Uchebn. Zaved. Mat., **5** (1987), 3–10 = Soviet Math., **31-5** (1987), 1–11.

[4] M. A. Akivis and V. V. Goldberg, Projective differential geometry of submanifold, North-Holland Math. Lib., **49**, North-Holland, Amsterdam, 1993.

[5] A. C. Asperti, *Immersions of surfaces into 4-dimensional spaces with nonzero normal curvature*, Ann. Mat. Pura Appl., **125** (1980), 313–328.

[6] A. A. Borisenko, *Extrinsic geometry of strongly parabolic multidimensional submanifolds*, Uspekhi Mat. Nauk, **52** (1997), no.6 (318); translation in Russian Math. Surveys, **52** (1997), no. 6, 1141–1190.

[7] A. A. Borisenko, *Extrinsic geometry of parabolic and saddle multidimensional submanifolds*, Russ. Math. Surveys, **53-6** (1998), 1111–1158.

[8] R. L. Bryant, *Conformal and minimal immersions of compact surfaces into the 4-sphere*, J. Diff. Geom., **17** (1982), 455–473.

[9] R. L. Bryant, *Surfaces in conformal geometry*, Proc. Symp. Pure Math., **48** (1988), 227–240.

[10] R. L. Bryant, *Some remarks on the geometry of austere submanifolds*, Boletim da Sociedade Brasileira de Matematica (Nova Sie), **21** (1991), 133–157.

[11] E. Calabi, *Minimal immersions of surfaces in euclidean spheres*, J. Diff. Geom., **1** (1967), 111–125.

[12] É. Cartan, *Sur des familles remarquables d'hypersurfaces isoparamétriques dans les espaces sphériques*, Math. Z., **45** (1939), 335–367.

[13] A. Cayley, *Mémoire sur les coubes à double courbure et les surfaces développables*, Journal de Mathématique Pure et Appliquées (Liouville), **10** (1845), 245–250 = The Collected Mathematical Papers vol. I, pp. 207–211.

[14] T. Cecil and P. Ryan, Tight and Taut Immersions of Manifolds, Research Notes in Math., **107**, Pitman, London, 1985.

[15] B. Y. Chen, *Strings of Riemannian invariants, inequalities, ideal immersions and their applications*, Proc. Third Pacific Rim Geom. Conf., International Press, Cambridge, (1998), 7–60.

[16] B. Y. Chen and T. Nagano, *Totally geodesic submanifolds of symmetric spaces I*, Duke Math. J., **44** (1977), 745–755.

[17] S. S. Chern and N. Kuiper, *Some theorems on the isometric imbedding of compact Riemann manifolds in Euclidean space*, Ann. Math., **56** (1952), 422–430.

[18] D. Ferus, *On the type number of hypersurfaces in spaces of constant curvature*, Math. Ann., **187** (1970), 310–316.

[19] D. Ferus, *Totally geodesic foliations*, Math. Ann., **188** (1970), 313–316.

[20] D. Ferus, H. Karcher and H. F. Münzner, *Cliffordalgebren und neue isoparametrische Hyperflächen*, Math. Z., **177** (1981), 479–502.

[21] P. Griffith and J. Harris, *Algebraic geometry and local differentiable geometry*, Ann. Sci. Éc. Norm. Sup. (4), **12** (1979), 355–432.

[22] M. Gromov, Partial Differential Relations, Ergebnisse der Mathematik und ihrer Grenzgebiete, 3.Folge, Bd. **9**, Springer-Verlag, 1986.

[23] R. Harvey and H. B. Lawson, *Calibrated Geometries*, Acta. Math., **148** (1982), 47–157.

[24] S. Helgason, Differential Geometry, Lie Groups and Symmetric spaces, New York, Academic Press, 1978.

[25] G. Ishikawa, *Singularities of Developable Surfaces*, London Math. Soc. Lecture Note Series, Cambridge Univ. Press, vol. 263, pp. 403–418, 1999.

[26] G. Ishikawa, *Developable hypersurfaces and homogeneous spaces in a real projective space*, Lobachevskii J. Math., **3** (1999), 113–125.

[27] G. Ishikawa and T. Morimoto, *Solution surfaces of Monge-Ampère equations*, Differential Geometry and its Applications., **14** (2001), 113–124.

[28] G. R. Jensen, M. Rigoli and K. Yang, *Holomorphic curves in the complex quadric*, Bull. Austral. Math. Soc., **35** (1987), 125–148.

[29] M. Kimura, *Minimal immersions of some circle bundles over holomorphic curves in complex quadric to sphere*, Osaka J. Math., **37** (2000), 883–903.

[30] S. Kobayashi and K. Nomizu, Foundations in Differential Geometry II, Wiley-Interscience, New York, 1969.

[31] J. M. Landsberg, *On second fundamental forms of projective varieties*, Invent. Math., **117** (1994), 303–315.

[32] J. M. Landsberg, *On degenerate secant and tangential varieties and local differential geometry*, Duke Math. Journal, **85-3** (1996), 605–634.

[33] R. Miyaoka, *The linear isotropy group of $G_2/SO(4)$, the Hopf fibering and isoparametric hypersurfaces*, Osaka J. Math., **30** (1993), 179–202.

[34] R. Miyaoka, *A new proof of the homogeneity of isoparametric hypersurfaces with $(g, m) = (6, 1)$*, Geometry and Topology of Submanifolds X (eds. C. H. Chen, A. M. Li, U. Simon, L. Verstraelen, C. P. Wang, M. Wiehe) 178–199 (2000), World Scientific.

[35] R. Miyaoka, *Homogeneity of isoparametric hypersurfaces with six principal curvatures*, preprint.

[36] T. Morimoto, *La géométrie des équations de Monge-Ampère*, C. R. Acad. Sci. Paris, **289** (1979), A-25–A-28.

[37] T. Morimoto, *Monge-Ampère equations viewed from contact geometry*, Banach Center Publications, **39**, "Symplectic Singularities and Geometry of Gauge Fields" 1998, pp. 105–120.

[38] V. Yu. Ovsienko, *Projective structures and contact forms*, Funct. Anal. Appl., **32-3** (1994), 187–197.

[39] H. Ozeki and M. Takeuchi, *On some types of isoparametric hypersurfaces in spheres II*, Tôhoku Math. J., **28** (1976), 7–55.

[40] V. Yu. Rovenskii, *Characterization of totally geodesic submanifolds of S^n and $\mathbb{CP}^n$*, Ukrain. Geom. Sb., **28** (1985), 106–116 = J. Soviet Math., **48** (1990), 87–94.

[41] K. Sakamoto, *Planar geodesic immersions*, Tôhoku. Math. J., **29** (1977), 25–56.

[42] R. Takagi and T. Takahashi, *On the principal curvatures of homogeneous hypersurfaces in a sphere*, Diff. Geom. in honor of K. Yano, Kinokuniya, Tokyo, 1972, pp. 469–481.

[43] G. Thorbergsson, *A survey on isoparametric hypersurfaces and their generalizations*, Handbook of Differential Geometry (eds. F. Dillen and P. Verstraelen), Elsevier Science, 2000, pp. 963–995.

[44] H. Wu, *Complete developable submanifolds in real and complex Euclidean spaces*, International J. Math., **6-3** (1995), 461–489.

[45] F. L. Zak, Tangents and Secants of Algebraic Varieties, Translations of Math. Monographs **127**, Amer. Math. Soc., Providence, 1993.

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Complete Integrability of the Coupled KdV-mKdV System

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Abstract.

The coupled KdV-mKdV system arises as the classical part of one of superextensions of the KdV equation. For this system, we prove its complete integrability, i.e., existence of a recursion operator and of infinite series of symmetries.

Introduction

There are several supersymmetric extensions of the classical Korteweg-de Vries equation (KdV) [5], [8] and [9]. One of them is of the form (the so-called $N = 2$, $A = 1$ extension [3])

$$\begin{aligned}
u_t = & -u_3 + 6uu_1 - 3\varphi\varphi_2 - 3\psi\psi_2 - 3ww_3 - 3w_1w_2 + 3u_1w^2 + 6uww_1 \\
& + 6\psi\varphi_1 w - 6\varphi\psi_1 w - 6\varphi\psi w_1, \\
\varphi_t = & -\varphi_3 + 3\varphi u_1 + 3\varphi_1 u - 3\psi_2 w - 3\psi_1 w_1 + 3\varphi_1 w^2 + 6\varphi w w_1, \\
\psi_t = & -\psi_3 + 3\psi u_1 + 3\psi_1 u + 3\varphi_2 w + 3\varphi_1 w_1 + 3\psi_1 w^2 + 6\psi w w_1, \\
w_t = & -w_3 + 3w^2 w_1 + 3uw_1 + 3u_1 w,
\end{aligned}$$

where u and w are classical (even) independent variables while φ and ψ are odd ones (here and below the numerical subscript at an unknown variable denotes it derivative over x of the corresponding order). Being completely integrable itself, this system gives rise to an interesting system of even equations

$$\begin{aligned}
u_t &= -u_3 + 6uu_1 - 3ww_3 - 3w_1w_2 + 3u_1w^2 + 6uww_1, \\
w_t &= -w_3 + 3w^2w_1 + 3uw_1 + 3u_1w,
\end{aligned} \tag{1}$$

which can be considered as a sort coupling between the KdV (with respect to u) and the modified KdV (with respect to w) equations. In

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fact, setting $w = 0$, we obtain

$$u_t = -u_3 + 6uu_1,$$

while for $u = 0$ we have

$$w_t = -w_3 + 3w^2 w_1.$$

In what follows, we prove complete integrability, cf. [2], of system (1) by establishing existence of infinite series of symmetries and/or conservation laws. Toward this end we construct a recursion operator using the techniques of deformation theory introduced in [4] and extensively described and exemplified in [5].

In the first section of the paper the theoretical background is introduced. The second section deals with particular computations and description of basic results.

§1. Geometrical and algebraic background

Here we briefly describe the geometrical theory of partial differential equations [1], [6] and algebraic foundations of computational approach to recursion operators [4], [5].

Let $\pi\colon E \to M$ be a locally trivial vector bundle and $\pi_k\colon J^k(\pi) \to M$, $k = 0, 1, \dots, \infty$, be the bundles of its k-jets. A (nonlinear) partial differential equation (PDE) of order k is a submanifold $\mathcal{E} \subset J^k(\pi)$, $k < \infty$. Its lth prolongation is a subset $\mathcal{E}^l \subset J^{k+l}(\pi)$. There exist natural mappings $\pi_{k+l+1,k+l}\colon \mathcal{E}^{l+1} \to \mathcal{E}^l$, and $\mathcal{E}$ is said to be formally integrable, if all $\mathcal{E}^l$ are smooth manifolds while $\pi_{k+l+1,k+l}$ are smooth fiber bundles. Below, only formally integrable equations are considered.

The inverse limit $\mathcal{E}^\infty \subset J^\infty(\pi)$ of the system $\{\mathcal{E}^l, \pi_{k+l,k+l-1}\}$ is called the infinite prolongation of $\mathcal{E}$ and we consider the bundle $\pi_{\mathcal{E}}\colon \mathcal{E}^\infty \to M$. This bundle enjoys the following characteristic property:

Proposition 1 (see [6]). *Let $\pi_i\colon E_i \to M$, $i = 1, 2$, be two locally trivial vector bundles and $\Delta\colon \Gamma(\pi_1) \to \Gamma(\pi_2)$ be a linear differential operator acting from sections of π_1 to sections of π_2. Then there exists a unique differential operator $\mathcal{C}\Delta\colon \Gamma(\pi_{\mathcal{E}}^*(\pi_1)) \to \Gamma(\pi_{\mathcal{E}}^*(\pi_2))$ such that*

$$j_\infty(s)^* \circ \mathcal{C}\Delta = \Delta \circ j_\infty(s)^* \tag{2}$$

for any formal solution s of the equation $\mathcal{E}$. The correspondence $\Delta \mapsto \mathcal{C}\Delta$ is $C^\infty(\mathcal{E}^\infty)$-linear and complies with the composition of differential operators:

$$\mathcal{C}(\Delta_2 \circ \Delta_1) = \mathcal{C}\Delta_2 \circ \mathcal{C}\Delta_1, \tag{3}$$

where $\Delta_1\colon \Gamma(\pi_1) \to \Gamma(\pi_2)$, $\Delta_2\colon \Gamma(\pi_2) \to \Gamma(\pi_3)$ *are linear differential operators,* $\pi_3\colon E_3 \to M$ *being a third vector bundle.*

As a corollary of Proposition 1, we get

Proposition 2. *The bundle* $\pi_{\mathcal{E}}$ *possesses a natural flat connection.*

Proof. It suffices to take the trivial bundle $\mathbf{1}_M\colon M \times \mathbb{R} \to M$ for the bundles π_1 and π_2 and an arbitrary vector field X for the operator Δ. Flatness is a consequence of (3). □

Let us denote by $\mathrm{D}(N)$ the $C^\infty(N)$-module of vector fields on a manifold N.

Definition 1. The connection $\mathcal{C}\colon \mathrm{D}(M) \to \mathrm{D}(\mathcal{E}^\infty)$ is called the *Cartan connection* on $\mathcal{E}^\infty$.

Denote by $\mathcal{C}\mathrm{D}(\mathcal{E}^\infty) \subset \mathrm{D}(\mathcal{E}^\infty)$ the horizontal distribution on $\mathcal{E}^\infty$ with respect to the Cartan connection (the *Cartan distribution*) and by $\mathrm{D}_{\mathcal{C}}(\mathcal{E}^\infty)$ the normalizer of $\mathcal{C}\mathrm{D}(\mathcal{E}^\infty)$ in $\mathrm{D}(\mathcal{E}^\infty)$. Then, since $\mathcal{C}$ is flat, $\mathcal{C}\mathrm{D}(\mathcal{E}^\infty)$ is integrable in a formal Frobenius sense and thus $\mathcal{C}\mathrm{D}(\mathcal{E}^\infty)$ is an ideal in $\mathrm{D}_{\mathcal{C}}(\mathcal{E}^\infty)$.

Definition 2. The quotient Lie algebra $\operatorname{sym}\mathcal{E} = \mathrm{D}_{\mathcal{C}}(\mathcal{E}^\infty)/\mathcal{C}\mathrm{D}(\mathcal{E}^\infty)$ is called the *algebra of higher symmetries* of the equation $\mathcal{E}$.

The Cartan connection in $\pi_{\mathcal{E}}$ determines the splitting

$$\mathrm{D}(\mathcal{E}^\infty) = \mathrm{D}^v(\mathcal{E}^\infty) \oplus \mathcal{C}\mathrm{D}(\mathcal{E}^\infty), \tag{4}$$

where $\mathrm{D}^v(\mathcal{E}^\infty)$ denotes the module of $\pi_{\mathcal{E}}$-vertical vector fields, and for any coset $S \in \operatorname{sym}\mathcal{E}$ there exists a unique vertical representative. In its turn, such a representative is uniquely determined by a section $\varphi \in \Gamma(\pi_{\mathcal{E}}^*(\pi))$ (generating section) satisfying the defining equation

$$\ell_{\mathcal{E}}\varphi = 0 \tag{5}$$

and vice versa. Here $\ell_{\mathcal{E}}$ is the universal linearization operator for $\mathcal{E}$ restricted to $\mathcal{E}^\infty$. Due to this fact, we shall identify the solutions of (5) with higher symmetries of $\mathcal{E}$.

The connection form $U_{\mathcal{E}}$ (the *structural element* of $\mathcal{E}$) of the Cartan connection $\mathcal{C}$ is an element of the module $\mathrm{D}^v(\Lambda^1(\mathcal{E}^\infty))$ of $\Lambda^1(\mathcal{E}^\infty)$-valued vertical derivations. Thus, we can introduce an operator $\partial_{\mathcal{E}}\colon \mathrm{D}^v(\Lambda^i(\mathcal{E}^\infty)) \to \mathrm{D}^v(\Lambda^{i+1}(\mathcal{E}^\infty))$ defined by

$$\partial_{\mathcal{E}}\Omega = [\![U_{\mathcal{E}}, \Omega]\!], \quad \Omega \in \mathrm{D}^v\left(\Lambda^i(\mathcal{E}^\infty)\right),$$

where $[\![\cdot,\cdot]\!]$ is the Frölicher-Nijenhuis bracket. Since the Cartan connection is flat, one has $[\![U_\mathcal{E},U_\mathcal{E}]\!]=0$, from where it follows that $\partial_\mathcal{E}\circ\partial_\mathcal{E}=0$. Thus we obtain a complex $(\mathrm{D}^v(\Lambda^i(\mathcal{E}^\infty)),\partial_\mathcal{E})$, whose cohomology is called the $\mathcal{C}$*-cohomology* of $\mathcal{E}$ and is denoted by $H^\bullet_\mathcal{C}(\mathcal{E})$. It is easy to see that $H^0_\mathcal{C}(\mathcal{E})=\operatorname{sym}\mathcal{E}$, while $H^1_\mathcal{C}(\mathcal{E})$, by standard reasons, is identified with classes of nontrivial infinitesimal deformations of $U_\mathcal{E}$ (or, which is the same, of the equation structure).

If Ω and Θ are elements of $\mathrm{D}^v(\Lambda^i(\mathcal{E}^\infty))$ and $\mathrm{D}^v(\Lambda^j(\mathcal{E}^\infty))$ respectively, their *contraction* $\Omega\lrcorner\Theta$ is defined as an element of $(\mathrm{D}^v(\Lambda^{i+j-1}(\mathcal{E}^\infty))$ This operation is inherited by the $\mathcal{C}$-cohomology groups. In particular, if $\varphi\in\operatorname{sym}\mathcal{E}$ and $\mathcal{R}\in H^1_\mathcal{C}(\mathcal{E})$, then $\varphi\lrcorner\mathcal{R}=\mathcal{R}\varphi$ is a symmetry again. In other words, the module $H^1_\mathcal{C}(\mathcal{E})$ acts on the Lie algebra of higher symmetries.

Let $\mathcal{C}\Lambda^1(\mathcal{E}^\infty)\subset\Lambda^1(\mathcal{E}^\infty)$ be the submodule of one-forms on $\mathcal{E}^\infty$ vanishing on the Cartan distribution. Then one has the direct sum decomposition

$$\Lambda^1(\mathcal{E}^\infty)=\mathcal{C}\Lambda^1(\mathcal{E}^\infty)\oplus\Lambda^1_h(\mathcal{E}^\infty)$$

dual to (4), where $\Lambda^1_h(\mathcal{E}^\infty)$ is the submodule of *horizontal* forms. This splitting is also inherited by $H^1_\mathcal{C}(\mathcal{E}^\infty)$ and an element $\mathcal{R}\in H^1_\mathcal{C}(\mathcal{E}^\infty)$ acts nontrivially on $\operatorname{sym}\mathcal{E}$ if only it corresponds to a derivation from $\mathrm{D}^v(\mathcal{C}\Lambda^1(\mathcal{E}^\infty))$. Moreover, it can be shown that $\operatorname{im}\partial_\mathcal{E}\cap\mathrm{D}^v(\mathcal{C}\Lambda^1(\mathcal{E}^\infty))=0$ and consequently nontrivial actions can be found by solving the equation

$$\partial_\mathcal{E}(\mathcal{R})=0. \tag{6}$$

Solutions of (6) are called *recursion operators* for symmetries. Any recursion operator $\mathcal{R}$ is uniquely determined by an element $\omega_\mathcal{R}\in\mathcal{C}\Lambda^1(\mathcal{E}^\infty)\otimes\Gamma(\pi_\mathcal{E}(\pi))$ satisfying the defining equation

$$\ell^{[1]}_\mathcal{E}(\omega_\mathcal{R})=0, \tag{7}$$

where $\ell^{[1]}_\mathcal{E}$ is the extension of the operator $\ell_\mathcal{E}$ to the module $\mathcal{C}\Lambda^1(\mathcal{E}^\infty)\otimes\Gamma(\pi_\mathcal{E}(\pi))$. If $\varphi\in\Gamma(\pi^*_\mathcal{E}(\pi))$ is a symmetry and $S_\varphi\in\mathrm{D}^v(\mathcal{E}^\infty)$ is the corresponding vertical vector field, then the action of $\mathcal{R}$ on φ is given by

$$\mathcal{R}\varphi=S_\varphi\lrcorner\omega_\mathcal{R}. \tag{8}$$

Local coordinates

Let $\mathcal{U}\subset M$ be a coordinate neighborhood in M such that the bundle π trivializes over $\mathcal{U}$, $x_1,\dots,x_n$ be local coordinates in $\mathcal{U}$ and $u^1,\dots,u^m$ be coordinates along the fiber in a given trivialization. Then the *adapted coordinates* in $\pi_\infty^{-1}(\mathcal{U}) \subset J^\infty(\pi)$ are given by the functions u^j_σ, $j = 1, \dots, m$, $\sigma = i_1 \dots i_k$, $1 \le i_\alpha \le n$, uniquely defined by

$$j_\infty(f)^*(u^j_\sigma) = \frac{\partial^{|\sigma|} f^j}{\partial x_{i_1} \cdots \partial x_{i_k}}$$

for any local section $f = (f^1, \dots, f^m) \in \Gamma(\pi|_{\mathcal{U}})$. Then the Cartan connection in $J^\infty(\pi)$ is given by

$$\mathcal{C}\Big(\frac{\partial}{\partial x_i}\Big) = D_i = \frac{\partial}{\partial x_i} + \sum_{j,\sigma} u^j_{\sigma i} \frac{\partial}{\partial u^j_\sigma}, \tag{9}$$

where D_i are the so-called *total derivatives.* The structural element in this case is given by the formula

$$U_\pi = \sum_{j,\sigma} \omega^j_\sigma \otimes \frac{\partial}{\partial u^j_\sigma}, \tag{10}$$

where $\omega^j_\sigma = du^j_\sigma - \sum_i u^j_{\sigma i}\, dx_i$ are the *Cartan forms* constituting a basis in the module $\mathcal{C}\Lambda^1(J^\infty(\pi))$. The forms ω^j_σ may be rewritten as $\omega^j_\sigma = d_{\mathcal{C}} u^j_\sigma$, where $d_{\mathcal{C}}$ is the *Cartan differential* acting on $f \in C^\infty(J^\infty(\pi))$ by

$$d_{\mathcal{C}} f = \sum \frac{\partial f}{\partial u^j_\sigma} \omega^j_\sigma.$$

This differential restricts to any submanifold $\mathcal{E}^\infty \subset J^\infty(\pi)$ and for infinite prolongations (10) transforms to

$$U_{\mathcal{E}} = \sum_I d_{\mathcal{C}} u_I \otimes \frac{\partial}{\partial u_I}, \tag{11}$$

where $\{u_I\}$ spans the set of *internal coordinates* in $\mathcal{E}^\infty$.

If an equation $\mathcal{E} \subset J^k(\pi)$ is given by the system of equalities

$$\begin{cases} F^1(x_1, \dots, x_n, \dots, u^j_\sigma, \dots) = 0, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ F^r(x_1, \dots, x_n, \dots, u^j_\sigma, \dots) = 0, \end{cases}$$

then its infinite prolongation is defined by

$$D_\sigma F^\alpha = 0, \quad \alpha = 1, \dots,\ r,\ 0 \le |\sigma| < \infty,$$

where $D_\sigma = D_{i_1} \circ \dots \circ D_{i_k}$ for $\sigma = i_1 \dots i_k$. Then the universal linearization operator corresponding to this system is of the form

$$\ell_F = \left\| \sum_\sigma \frac{\partial F^\alpha}{\partial u^j_\sigma} D_\sigma \right\|. \tag{12}$$

The operator $\ell_{\mathcal{E}}$ is obtained from (12) by rewriting it in internal coordinates.

Example (Evolutionary $1+1$ equations). Consider a system $\mathcal{E}$ of evolutionary equations

$$\begin{cases} u_t^1 = F^1(x,t,u^1,\dots,u_k^m), \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ u_t^m = F^r(x,t,u^1,\dots,u_k^m), \end{cases} \tag{13}$$

where $x = x_1$, $t = x_2$ and $u_l^j = \partial^l u^j/\partial x^l$. Then the functions x, $t,\dots,$ $u_l^j,\dots$ can be taken for internal coordinates on $\mathcal{E}^\infty$. The total derivatives are written down as

$$D_x = \frac{\partial}{\partial x} + \sum_{l,j} u_{l+1}^j \frac{\partial}{\partial u_l^j}, \quad D_t = \frac{\partial}{\partial t} + \sum_{l,j} D_x^l(F^j)\frac{\partial}{\partial u_l^j}$$

in these coordinates. The Cartan forms on $\mathcal{E}^\infty$ are

$$\omega_l^j = du_l^j - u_{l+1}^j\,dx - D_x^l(F^j)\,dt,$$

while the structural element for $\mathcal{E}$ is given by

$$U_{\mathcal{E}} = \sum_{l,j} \omega_l^j \otimes \frac{\partial}{\partial u_l^j}. \tag{14}$$

The restriction $\ell_{\mathcal{E}}$ of the universal linearization operator to $\mathcal{E}^\infty$ is of the form

$$\ell_{\mathcal{E}} = \left\| \sum_{l,j} \frac{\partial F^\alpha}{\partial u_l^j} D_x^l - \mathbf{E}D_t \right\|,$$

where $\mathbf{E}$ is the identity matrix. So, a vector function $\varphi = (\varphi^1,\dots,\varphi^m)$, $\varphi^j = \varphi^j(x,t,\dots,u_l^j,\dots)$, is a symmetry of $\mathcal{E}$ if and only if

$$\sum_{l,j} \frac{\partial F^\alpha}{\partial u_l^j} D_x^l \varphi^j = D_t\varphi^\alpha \tag{15}$$

for all $\alpha = 1,\dots,m$. The corresponding vector field is given by the formula

$$\partial_\varphi = \sum_{l,j} D_x^l(\varphi^j)\frac{\partial}{\partial u_l^j}.$$

In a similar way, recursion operators are determined by vector-valued forms $\omega_{\mathcal{R}} = (\omega_{\mathcal{R}}^1, \dots, \omega_{\mathcal{R}}^m)$, where $\omega_{\mathcal{R}}^j = \sum_{l,\alpha} \psi_\alpha^{jl} \omega_l^\alpha$, $\psi_\alpha^{jl} \in C^\infty(\mathcal{E}^\infty)$, satisfying the equations

$$\sum_{l,j} \frac{\partial F^\alpha}{\partial u_l^j} D_x^l \omega_{\mathcal{R}}^j = D_t \omega_{\mathcal{R}}^\alpha. \tag{16}$$

To compute the left and right sides of (16), it suffices to note that

$$D_x \omega_l^j = \omega_{l+1}^j, \quad D_t \omega_l^j = D_x^l d_{\mathcal{C}} F^j = D_x^l \sum_{\alpha, be} \frac{\partial F^j}{\partial u_\beta^\alpha} \omega_\beta^\alpha.$$

Remark 1. Let $\omega = (\omega^1, \dots, \omega^m)$, $\omega^j = \sum_{l,\alpha} \psi_\alpha^{jl} \omega_l^\alpha$, be a vector-valued Cartan form on $\mathcal{E}^\infty$. Then for any vector-valued function $\varphi = (\varphi^1, \dots, \varphi^m)$ on $\mathcal{E}^\infty$ the action $\mathcal{R}_\omega \colon \varphi \mapsto \mathcal{R}_\omega \varphi$ is defined by $\mathcal{R}_\omega \varphi = \partial_\varphi \lrcorner\, \omega$. In local coordinates, this action is expressed by the formula

$$(\mathcal{R}_\omega \varphi)^j = \sum_{l,\alpha} \psi_\alpha^{jl} D_x^l(\varphi^\alpha). \tag{17}$$

Operators of this type (i.e., expressed in terms of total derivatives) are called $\mathcal{C}$-*differential* (or *total differential*) operators.

Remark 2. As it was mentioned above, operators of the form (17), provided ω satisfies (16), take symmetries of the equation at hand to symmetries of the same equation. In other words, one has $\mathcal{R}_\omega \colon \ker \ell_{\mathcal{E}} \to \ker \ell_{\mathcal{E}}$. Under not very restrictive conditions on the equation $\mathcal{E}$, this is equivalent to the operator equality

$$\ell_{\mathcal{E}} \circ \mathcal{R}_\omega = \mathcal{A} \circ \ell_{\mathcal{E}},$$

where $\mathcal{A}$ is a $\mathcal{C}$-differential operator. Taking formally adjoint, one obtains

$$\mathcal{R}_\omega^* \circ \ell_{\mathcal{E}}^* = \ell_{\mathcal{E}}^* \circ \mathcal{A}^*,$$

which means, that $\mathcal{A}^*$ is a recursion operator for generating functions of conservation laws [10].

Nonlocal setting

In practice, when solving (7), one usually finds no nontrivial solutions, though the equation $\mathcal{E}$ may possess recursion operators.

Example (The Burgers equation). Consider the equation

$$u_t = u_{xx} + u u_x.$$

It is known to possess a recursion operator of the form

$$\mathcal{R} = D_x + \frac{1}{2}u_0 + \frac{1}{2}u_1 D_x^{-1}. \tag{18}$$

Nevertheless, the only solution of the equation

$$\ell_{\mathcal{E}}^{[1]}\omega \equiv (D_x^2 + u_0 D_x + u_1 - D_t)\omega = 0 \tag{19}$$

for $\omega = \psi_0\omega_0 + \dots + \psi_k\omega_k$ is $\alpha\omega_0$, $\alpha \in \mathbb{R}$, which provides the trivial action $\mathcal{R}_\omega \colon \varphi \mapsto \alpha\varphi$.

To resolve this apparent contradiction, let us extend the algebra $C^\infty(\mathcal{E}^\infty)$ with an additional element u_{-1} and set

$$\begin{aligned}
&D_x u_{-1} = u_0,\\
&D_t u_{-1} = u_1 + \frac{1}{2}u_0^2,\\
&d_{\mathcal{C}} u_{-1} \equiv \omega_{-1} = du_{-1} - u_0\,dx - \Big(u_1 + \frac{1}{2}u_0^2\Big)dt.
\end{aligned} \tag{20}$$

Then, solving (19) for $\omega = \psi_{-1}\omega_{-1} + \psi_0\omega_0 + \dots + \psi_k\omega_k$, we obtain a two-parametric solution

$$\omega = \alpha\omega_0 + \beta\Omega, \quad \Omega = \omega_1 + \frac{1}{2}u_0\omega_0 + \frac{1}{2}u_1\omega_{-1}.$$

Then the action $\varphi \mapsto \mathcal{R}_\Omega\varphi = \partial_\varphi \lrcorner\, \Omega$ coincides exactly with (18).

This example reflects a general scheme of computations which arises in a lot of applications [5] and is used in Section 2. Namely, in search of recursion operators for symmetries we extend the algebra $C^\infty(\mathcal{E}^\infty)$ with a new set of variables $w^1, \dots, w^r, \dots$ (so-called *nonlocal variables*) and respectively extend the total derivatives to vector fields

$$\bar{D}_i = D_i + \sum_\alpha X_i^\alpha \frac{\partial}{\partial w^\alpha}, \quad i = 1, \dots, n, \tag{21}$$

in such a way that

$$[\bar{D}_i, \bar{D}_j] \equiv [D_i, X_j] + [X_i, D_j] + [X_i, X_j] = 0, \tag{22}$$

where $X_l = \sum_\alpha X_l^\alpha \partial/\partial w^\alpha$.

Having a solution $X_1, \dots, X_n$ of (21), we obtain an integrable distribution on the space $\mathcal{E}^\infty \times \mathbb{R}^N$, where N is the number of nonlocal variables (the case $N = \infty$ is included). The projection $\tau \colon \overline{\mathcal{E}^\infty} = \mathcal{E}^\infty \times \mathbb{R}^N \to \mathcal{E}^\infty$ is called a *covering* over $\mathcal{E}$ and N is called its *dimension* (for an invariant geometrical definition see [7]). Similar to the local case, we define the Lie algebra $\mathrm{sym}_\tau \mathcal{E} = \mathrm{D}_\mathcal{C}(\overline{\mathcal{E}^\infty})/\mathcal{C}\mathrm{D}(\overline{\mathcal{E}^\infty})$ of *nonlocal* τ-symmetries.

We introduce *Cartan forms*

$$\theta^j = dw^j - \sum_{i=1}^{n} X_i^j \, dx_i, \quad j = 1, \dots, N,$$

corresponding to nonlocal variables on $\overline{\mathcal{E}^\infty}$. The module of all Cartan forms on $\overline{\mathcal{E}^\infty}$ is denoted by $\mathcal{C}\Lambda^1(\overline{\mathcal{E}^\infty})$. We also extend the universal linearization operator $\ell_\mathcal{E}$ to $\overline{\mathcal{E}^\infty}$ just by changing the total derivatives D_i to $\bar{D}_i$. Let us now consider two equations, associated to this extension:

$$\bar{\ell}_\mathcal{E}\varphi = 0 \quad \text{and} \quad \bar{\ell}_\mathcal{E}^{[1]}\Omega = 0,$$

where $\varphi \in \Gamma((\pi_\mathcal{E} \circ \tau)^*\pi)$ and $\Omega \in \Gamma((\pi_\mathcal{E} \circ \tau)^*\pi) \otimes \mathcal{C}\Lambda^1(\overline{\mathcal{E}^\infty})$. Solutions of the first equation are called τ-*shadows* of nonlocal symmetries, while solutions of the second one are said to be τ-*shadows* of recursion operators in the covering τ. The following result establishes relations of shadows to symmetries and recursion operators:

Theorem 1 (see [5] and [7]). *Let $\tau\colon \overline{\mathcal{E}^\infty} \to \mathcal{E}^\infty$ be a covering. Then:*

1. *If φ is a τ-shadow, then there exists a covering*

$$\bar{\tau}\colon \overline{\overline{\mathcal{E}^\infty}} \to \overline{\mathcal{E}^\infty} \xrightarrow{\tau} \mathcal{E}^\infty$$

such that φ reconstructs up to a nonlocal $\bar{\tau}$-symmetry.

2. *If φ is a nonlocal τ-symmetry and $\mathcal{R}$ is a recursion operator shadow, then $\mathcal{R}\varphi$ is a symmetry shadow.*

Remark 3. Among all one-dimensional coverings over a given equation there exists a special class consisting of those ones, for which the fields $X_1, \dots, X_n$ in (22) are independent of nonlocal variables (so-called *abelian coverings*). To any such a covering, one can put into correspondence a differential form on $\mathcal{E}^\infty$:

$$\omega_\tau = \sum_{l=1}^{n} X_l \, dx_l.$$

This form is closed with respect to the so-called *horizontal differential* $d_h = \mathcal{C}d$ (cf. Proposition 1). Vice versa, to any such a form there corresponds a covering of the above mentioned type. Moreover, two closed forms determine the same class in the cohomology group $H^1(d_h)$

if and only if the corresponding coverings are equivalent. In particular, if $n = 2$, the group $H^1(d_h)$ coincides with the group of conservation laws of the equation $\mathcal{E}$ [10]. Thus, to construct a covering under consideration is the same as to find a conservation law. This fact is used in the computations below.

§2. Basic computations and results

In this section we shall discuss the complete integrability of the KdV-mKdV system given in (1), i.e.,

$$(23)\qquad \begin{aligned} u_t &= -u_3 + 6uu_1 - 3ww_3 - 3w_1w_2 + 3u_1w^2 + 6uww_1, \\ w_t &= -w_3 + 3w^2w_1 + 3uw_1 + 3u_1w. \end{aligned}$$

In order to demonstrate the complete integrability of this system, we shall construct the recursion operator for symmetries of this coupled system, leading to infinite hierarchies of symmetries and, most probably, of conservation laws. Due to the very special form of the final results, it seems that integrability of this system, which looks quite ordinary, has not been discussed before or elsewhere. In order to do this, we shall discuss conservation laws in Subsection 2.1 leading to the necessary nonlocal variables.

In Subsection 2.2 we shall discuss local and nonlocal symmetries of the system, while in Subsection 2.3 we construct the recursion operator or deformation of the equation structure (14).

2.1. Conservation laws and nonlocal variables

Here we shall construct conservation laws for (23) in order to arrive at an abelian covering of the coupled KdV-mKdV system as was shown for Burgers equation (20). So we construct $X = X(x, t, u, \dots, w \dots)$, $T = T(x, t, u, \dots, w \dots)$ such that

$$(24)\qquad D_x(T) = D_t(X)$$

and in a similar way we construct nonlocal conservation laws by the requirement

$$(25)\qquad \bar{D}_x(\bar{T}) = \bar{D}_t(\bar{X}),$$

where $\bar{D}_*$ is defined by (9); moreover $\bar{X}$, $\bar{T}$ are dependent on local variables x, t, $u, \dots, w, \dots$ as well as the already determined nonlocal variables, denoted here by p_* or $p_{*,*}$, which are associated to the conservation laws (X, T) by the formal definition

$$D_x(p_*) = (p_*)_x = X,$$
$$D_t(p_*) = (p_*)_t = T.$$

Proceeding in this way, we obtained the following set of nonlocal variables

$$p_{0,1},\ p_{0,2},\ p_1,\ p_{1,1},\ p_{1,2},\ p_{2,1},\ p_3,\ p_{3,1},\ p_{3,2},\ p_{4,1},\ p_5, \tag{26}$$

where their defining equations are given by

$$\begin{aligned}
(p_1)_x &= u,\\
(p_1)_t &= 3u^2 + 3uw^2 - u_2 - 3ww_2,\\
(p_{0,1})_x &= w,\\
(p_{0,1})_t &= 3uw + w^3 - w_2,\\
(p_{0,2})_x &= p_1,\\
(p_{0,2})_t &= -6p_3 - u_1,\\
(p_{1,1})_x &= \cos(2p_{0,1})p_1 w + \sin(2p_{0,1})w^2,\\
(p_{1,1})_t &= \cos(2p_{0,1})(3p_1 uw + p_1 w^3 - p_1 w_2 + uw_1 - u_1 w - w^2 w_1)\\
&\quad + \sin(2p_{0,1})(4uw^2 + w^4 - 2ww_2 + w_1^2),\\
(p_{1,2})_x &= \cos(2p_{0,1})w^2 - \sin(2p_{0,1})p_1 w,\\
(p_{1,2})_t &= \cos(2p_{0,1})(4uw^2 + w^4 - 2ww_2 + w_1^2)\\
&\quad + \sin(2p_{0,1})(-3p_1 uw - p_1 w^3 + p_1 w_2 - uw_1 + u_1 w + w^2 w_1),\\
(p_{2,1})_x &= \frac{1}{2}\big(4\cos(2p_{0,1})p_{1,1}w^2 - 4\sin(2p_{0,1})p_1 p_{1,1}w + w(p_1^2 - 2u + w^2)\big),\\
(p_{2,1})_t &= \frac{1}{2}\big(4\cos(2p_{0,1})p_{1,1}(4uw^2 + w^4 - 2ww_2 + w_1^2)\\
&\quad + 4\sin(2p_{0,1})\\
&\qquad \times p_{1,1}(-3p_1 uw - p_1 w^3 + p_1 w_2 - uw_1 + u_1 w + w^2 w_1)\\
&\quad + 3p_1^2 uw + p_1^2 w^3 - p_1^2 w_2 + 2p_1 uw_1 - 2p_1 u_1 w - 2p_1 w^2 w_1\\
&\quad - 8u^2 w - uw^3 + 2uw_2 - 2u_1 w_1 + 2u_2 w + w^5 + 3w^2 w_2),\\
(p_3)_x &= \frac{1}{2}(-u^2 - uw^2 + ww_2),\\
(p_3)_t &= \frac{1}{2}(-4u^3 - 9u^2 w^2 + 2uu_2 - 3uw^4 + 11uww_2 - uw_1^2 - u_1^2\\
&\quad + u_1 ww_1 + 4u_2 w^2 + 6w^3 w_2 + 3w^2 w_1^2 - ww_4 + w_1 w_3 - w_2^2),
\end{aligned}$$

$$
\begin{aligned}
(p_{3,1})_x = \frac{1}{12}\bigl(&\cos(2p_{0,1})\\
&\quad\times w(p_1^3 - 6p_1u + 39p_1w^2 - 24p_{1,1}p_{1,2}w + 12p_3 + 6u_1)\\
&+ 2\sin(2p_{0,1})w(12p_1p_{1,1}p_{1,2} + 18p_1w_1 + 2w^3 + 3w_2)\\
&+ 6p_{1,2}w(-p_1^2 + 2u - w^2)\bigr),\\
(p_{3,2})_x = \frac{1}{12}\bigl(&2\cos(2p_{0,1})w(12p_1p_{1,1}p_{1,2} - 18p_1w_1 - 2w^3 - 3w_2)\\
&+ \sin(2p_{0,1})\\
&\quad\times w(p_1^3 - 6p_1u + 39p_1w^2 + 24p_{1,1}p_{1,2}w + 12p_3 + 6u_1)\\
&+ 6p_{1,1}w(-p_1^2 + 2u - w^2)\bigr),\\
(p_{4,1})_x = \frac{1}{48}\bigl(&8\cos(2p_{0,1})w(p_1^3p_{1,2} + 12p_1p_{1,1}^2p_{1,2} - 6p_1p_{1,2}u\\
&+ 3p_1p_{1,2}w^2 - 12p_{1,1}p_{1,2}^2w + 18p_{1,1}uw - 4p_{1,1}w^3 - 6p_{1,1}w_2\\
&+ 12p_{1,2}p_3 + 6p_{1,2}u_1)\\
&+ 8\sin(2p_{0,1})w(p_1^3p_{1,1} + 12p_1p_{1,1}p_{1,2}^2 - 6p_1p_{1,1}u + 3p_1p_{1,1}w^2\\
&+ 12p_{1,1}^2p_{1,2}w + 12p_{1,1}p_3 + 6p_{1,1}u_1 - 18p_{1,2}uw + 4p_{1,2}w^3\\
&+ 6p_{1,2}w_2)\\
&+ w(-p_1^4 - 24p_1^2p_{1,1}^2 - 24p_1^2p_{1,2}^2 + 12p_1^2u - 6p_1^2w^2 - 48p_1p_3\\
&- 24p_1u_1 + 48p_{1,1}^2u - 24p_{1,1}^2w^2 + 48p_{1,2}^2u - 24p_{1,2}^2w^2\\
&- 60u^2 + 44uw^2 + 24u_2 - 13w^4 + 6ww_2)\bigr),\\
(p_5)_x = \frac{1}{6}(&12u^3 + 24u^2w^2 - 6uu_2 + 6uw^4 - 30uww_2 - 3u_2w^2\\
&- 8w^3w_2 + 6ww_4).
\end{aligned}
$$

In the previous equations, we skipped explicit formulas for $(p_{3,1})_t$, $(p_{3,2})_t$, $(p_{4,1})_t$, and $(p_5)_t$, because they are too massive, though quite important for the setting to be well defined and in order to avoid ambiguities. The reader is referred to the Appendix for them.

It is quite a striking result that functions $\cos(2p_{0,1})$, $\sin(2p_{0,1})$ appear in the presentation of the conservation laws and their associated nonlocal variables.

We should note that p_1, $p_{0,1}$, p_3, p_5 arise from **local conservation laws** and we shall call p_1, $p_{0,1}$, p_3, p_5 *nonlocalities of first order.*

In a similar way we see that $p_{0,2}$, $p_{1,1}$, $p_{1,2}$ arise from **nonlocal conservation laws**, where their x- and t-derivatives are dependent on the first order nonlocalities. For this reason $p_{0,2}$, $p_{1,1}$, $p_{1,2}$ are called *nonlocalities of second order.*

Proceeding in this way $p_{2,1}$, $p_{3,1}$, $p_{3,2}$, $p_{4,1}$ constitute *nonlocalities of third order.*

2.2. Local and nonlocal symmetries

In this section we shall present results for the construction of local and nonlocal symmetries of system (23). In order to construct these symmetries, we consider the system of partial differential equations obtained by the infinite prolongation of (23) together with the covering by the nonlocal variables

$$p_{0,1},\ p_{0,2},\ p_1,\ p_{1,1},\ p_{1,2},\ p_{2,1},\ p_3,\ p_{3,1}, p_{3,2},\ p_{4,1},\ p_5.$$

So, in the augmented setting governed by (23), their total derivatives and the equations given in Subsection 2.1 we construct symmetries $Y = (Y^u, Y^w)$ which have to satisfy the symmetry condition

$$\bar{\ell}_{\mathcal{E}} Y = 0.$$

From this condition we obtained the following symmetries

$$Y_{0,1},\ Y_{1,1},\ Y_{1,2},\ Y_{1,3},\ Y_{2,1},\ Y_{3,1},\ Y_{3,2},\ Y_{3,3},$$

where generating functions $Y^u_{*,*}$, $Y^w_{*,*}$ are given as

$$\begin{aligned}
Y^u_{0,1} &= 3t(6uu_1 + 6uww_1 + 3u_1w^2 - u_3 - 3ww_3 - 3w_1w_2) + xu_1 + 2u,\\
Y^w_{0,1} &= 3t(3uw_1 + 3u_1w + 3w^2w_1 - w_3) + xw_1 + w,\\
Y^u_{1,1} &= u_1,\\
Y^w_{1,1} &= w_1,\\
Y^u_{1,2} &= \cos(2p_{0,1})(2uw - w_2) + \sin(2p_{0,1})(u_1 + 2ww_1),\\
Y^w_{1,2} &= -\cos(2p_{0,1})u - \sin(2p_{0,1})w_1,\\
Y^u_{1,3} &= \cos(2p_{0,1})(u_1 + 2ww_1) + \sin(2p_{0,1})(-2uw + w_2),\\
Y^w_{1,3} &= -\cos(2p_{0,1})w_1 + \sin(2p_{0,1})u,\\
Y^u_{2,1} &= \frac{1}{2}\big(2\cos(2p_{0,1})(p_{1,1}u_1 + 2p_{1,1}ww_1 - 2p_{1,2}uw + p_{1,2}w_2)\\
&\quad + 2\sin(2p_{0,1})(-2p_{1,1}uw + p_{1,1}w_2 - p_{1,2}u_1 - 2p_{1,2}ww_1)\\
&\quad + 2p_1uw - p_1w_2 + 2uw_1 + 3u_1w + 2w^2w_1 - w_3\big),\\
Y^w_{2,1} &= \frac{1}{2}\big(2\cos(2p_{0,1})(-p_{1,1}w_1 + p_{1,2}u) + 2\sin(2p_{0,1})(p_{1,1}u + p_{1,2}w_1)\\
&\quad - p_1u + u_1 + ww_1\big),\\
Y^u_{3,1} &= \frac{1}{3}\big(6uu_1 + 6uww_1 + 3u_1w^2 - u_3 - 3ww_3 - 3w_1w_2\big),
\end{aligned}$$

$$
\begin{aligned}
Y^w_{3,1} &= \frac{1}{3}\big(3uw_1 + 3u_1w + 3w^2w_1 - w_3\big),\\
Y^u_{3,2} &= \frac{1}{8}\big(\cos(2p_{0,1})(-2p_1^2uw + p_1^2w_2 - 4p_1uw_1 - 6p_1u_1w\\
&\quad - 4p_1w^2w_1 + 2p_1w_3 + 8p_{1,1}p_{1,2}u_1 + 16p_{1,1}p_{1,2}ww_1 - 8p_{1,2}^2uw\\
&\quad + 4p_{1,2}^2w_2 - 4p_{2,1}u_1 - 8p_{2,1}ww_1 + 10u^2w + 6uw^3 - 8uw_2\\
&\quad - 14u_1w_1 - 8u_2w - 11w^2w_2 - 14ww_1^2 + 2w_4)\\
&\quad + 2\sin(2p_{0,1})(-8p_{1,1}p_{1,2}uw + 4p_{1,1}p_{1,2}w_2 - 2p_{1,2}^2u_1 - 4p_{1,2}^2ww_1\\
&\quad + 4p_{2,1}uw - 2p_{2,1}w_2 + 6uu_1 + 10uww_1 + 3u_1w^2 - u_3 + 2w^3w_1\\
&\quad - 3ww_3 - 5w_1w_2)\\
&\quad + 4p_{1,2}(2p_1uw - p_1w_2 + 2uw_1 + 3u_1w + 2w^2w_1 - w_3)\big),\\
Y^w_{3,2} &= \frac{1}{8}\big(\cos(2p_{0,1})(p_1^2u - 2p_1u_1 - 2p_1ww_1 - 8p_{1,1}p_{1,2}w_1 + 4p_{1,2}^2u\\
&\quad + 4p_{2,1}w_1 - 4u^2 - 3uw^2 + 2u_2 + 4ww_2 + 2w_1^2)\\
&\quad + 2\sin(2p_{0,1})\\
&\qquad \times (4p_{1,1}p_{1,2}u + 2p_{1,2}^2w_1 - 2p_{2,1}u - 3uw_1 - 3u_1w - 3w^2w_1 + w_3)\\
&\quad + 4p_{1,2}(-p_1u + u_1 + ww_1)\big),\\
Y^u_{3,3} &= \frac{1}{8}\big(2\cos(2p_{0,1})(2p_{1,1}^2u_1 + 4p_{1,1}^2ww_1 - 4p_{2,1}uw + 2p_{2,1}w_2 - 6uu_1\\
&\quad - 10uww_1 - 3u_1w^2 + u_3 - 2w^3w_1 + 3ww_3 + 5w_1w_2)\\
&\quad + \sin(2p_{0,1})(-2p_1^2uw + p_1^2w_2 - 4p_1uw_1 - 6p_1u_1w - 4p_1w^2w_1\\
&\quad + 2p_1w_3 - 8p_{1,1}^2uw + 4p_{1,1}^2w_2 - 4p_{2,1}u_1 - 8p_{2,1}ww_1 + 10u^2w\\
&\quad + 6uw^3 - 8uw_2 - 14u_1w_1 - 8u_2w - 11w^2w_2 - 14ww_1^2 + 2w_4)\\
&\quad + 4p_{1,1}(2p_1uw - p_1w_2 + 2uw_1 + 3u_1w + 2w^2w_1 - w_3)\big),\\
Y^w_{3,3} &= \frac{1}{8}\big(2\cos(2p_{0,1})(-2p_{1,1}^2w_1 + 2p_{2,1}u + 3uw_1 + 3u_1w + 3w^2w_1 - w_3)\\
&\quad + \sin(2p_{0,1})(p_1^2u - 2p_1u_1 - 2p_1ww_1 + 4p_{1,1}^2u + 4p_{2,1}w_1 - 4u^2\\
&\quad - 3uw^2 + 2u_2 + 4ww_2 + 2w_1^2)\\
&\quad + 4p_{1,1}(-p_1u + u_1 + ww_1)\big).
\end{aligned}
$$

2.3. Recursion operator

Here we present the recursion operator $\mathcal{R}$ for symmetries for this case obtained as a higher symmetry in the Cartan covering of system of equations (1) augmented by equations governing the nonlocal variables (26). As explained in the previous section, the recursion operator is in effect the a deformation of the equation structure (11).

As demonstrated there, this deformation is a form-valued vector field (or a vectorfield-valued one-form) and has to satisfy

$$\bar{\ell}_{\mathcal{E}}^{[1]}\mathcal{R} = 0. \tag{27}$$

In order to arrive at a nontrivial result as was explained for Burgers' equation too (cf. Example 1), we have to introduce nonlocal variables

$$p_{0,1},\ p_{0,2},\ p_1,\ p_{1,1},\ p_{1,2},\ p_{2,1},\ p_3,\ p_{3,1}, p_{3,2},\ p_{4,1},\ p_5$$

and their associated Cartan contact forms

$$\omega_{p_{0,1}},\ \omega_{p_{0,2}},\ \omega_{p_1},\ \omega_{p_{1,1}},\ \omega_{p_{1,2}}, \omega_{p_{2,1}},\ \omega_{p_3},\ \omega_{p_{3,1}},\ \omega_{p_{3,2}},\ \omega_{p_{4,1}}, \omega_{p_5}.$$

The final result, which is dependent on the nonlocal Cartan forms

$$\omega_{p_{0,1}},\ \omega_{p_1},\ \omega_{p_{1,1}},\ \omega_{p_{1,2}},$$

is given by

$$\mathcal{R} = R^u\frac{\partial}{\partial u} + R^w\frac{\partial}{\partial w} + \cdots, \tag{28}$$

where the components R^u, R^w are given by

$$\begin{aligned}
R_u &= \omega_{u_2}(-1) + \omega_u(4u + w^2)\\
&\quad + \omega_{w_2}(-2w) + \omega_{w_1}(-w_1) + \omega_w(3uw - 2w_2)\\
&\quad + \omega_{p_{1,2}}(-\cos(2p_{0,1})(u_1 + 2ww_1) + \sin(2p_{0,1})(2uw - w_2))\\
&\quad + \omega_{p_{1,1}}(\cos(2p_{0,1})(-2uw + w_2) - \sin(2p_{0,1})(u_1 + 2ww_1))\\
&\quad + \omega_{p_1}(2u_1 + ww_1)\\
&\quad + \omega_{p_{0,1}}(2p_1uw - p_1w_2 + 2uw_1 + 3u_1w + 2w^2w_1 - w_3),\\
R_w &= \omega_{w_2}(-1) + \omega_w(2u + w^2) + \omega_u(2w)\\
&\quad + \omega_{p_{1,2}}(\cos(2p_{0,1})w_1 - \sin(2p_{0,1})u)\\
&\quad + \omega_{p_{1,1}}(\cos(2p_{0,1})u + \sin(2p_{0,1})w_1)\\
&\quad + \omega_{p_1}(w_1) + \omega_{p_{0,1}}(-p_1u + u_1 + ww_1).
\end{aligned} \tag{29}$$

We shall now present this result in a more conventional form which appeals to expressions using operators of the form D_x and D_x^{-1}. In order to do this, we first split (29) into the so-called local part and nonlocal parts, consisting of terms associated to ω_{u_2}, ω_u, ω_{w_2}, ω_{w_1}, ω_w and those associated to $\omega_{p_{1,2}}$, $\omega_{p_{1,1}}$, ω_{p_1}, $\omega_{p_{0,1}}$ respectively. The first part will account for D_x presentation, while the second one accounts for the D_x^{-1} part.

Due to the action of contraction $\partial_\varphi \lrcorner \mathcal{R}$, the local part is given by the following matrix operator:

$$\begin{bmatrix} -D_x^2 + 4u + w^2 & -2wD_x^2 - w_1 D_x + 3uw - 2w_2 \\ 2w & -D_x^2 + 2u + w^2 \end{bmatrix}.$$

The nonlocal part will be split into parts associated to ω_{p_1}, $\omega_{p_{0,1}}$ and $\omega_{p_{1,2}}$, $\omega_{p_{1,1}}$, respectively. The first one is given as

$$\begin{bmatrix} (2u_1 + ww_1)D_x^{-1} & (2p_1uw - p_1w_2 + 2uw_1 + 3u_1w + 2w^2w_1 - w_3)D_x^{-1} \\ w_1 D_x^{-1} & (-p_1u + u_1 + ww_1)D_x^{-1} \end{bmatrix}.$$

To deal with the last part, let us introduce the notation:

$$\begin{aligned} A_1 &= \cos(2p_{0,1})(-2uw + w_2) - \sin(2p_{0,1})(u_1 + 2ww_1), \\ A_2 &= \cos(2p_{0,1})u + \sin(2p_{0,1})w_1, \\ B_1 &= -\cos(2p_{0,1})(u_1 + 2ww_1) + \sin(2p_{0,1})(2uw - w_2), \\ B_2 &= \cos(2p_{0,1})w_1 - \sin(2p_{0,1})u, \end{aligned}$$

being the coefficients at $\omega_{p_{1,1}}$ and $\omega_{p_{1,2}}$ in (29).

According to the presentations of $(p_{1,1})_x$ and $(p_{1,2})_x$, i.e.,

$$\begin{aligned} (p_{1,1})_x &= \cos(2p_{0,1})p_1w + \sin(2p_{0,1})w^2 \\ (p_{1,2})_x &= \cos(2p_{0,1})w^2 - \sin(2p_{0,1})p_1w, \end{aligned}$$

we introduce their partial derivatives with respect to $p_{0,1}$, p_1, and w as

$$\begin{aligned} \alpha_1 &= -2p_1w\sin(2p_{0,1}) + 2w^2\cos(2p_{0,1}), \\ \alpha_2 &= w\cos(2p_{0,1}), \\ \alpha_3 &= p_1\cos(2p_{0,1}) + 2w\sin(2p_{0,1}), \\ \beta_1 &= -2w^2\sin(2p_{0,1}) - 2p_1w\cos(2p_{0,1}), \\ \beta_2 &= -w\sin(2p_{0,1}), \\ \beta_3 &= 2w\cos(2p_{0,1}) - p_1\sin(2p_{0,1}). \end{aligned}$$

From this we arrive in a straightforward way at the last nonlocal part of the recursion operator, i.e.,

$$\begin{bmatrix} A_1D_x^{-1}\alpha_2D_x^{-1} & A_1D_x^{-1}(\alpha_1D_x^{-1} + \alpha_3) \\ A_2D_x^{-1}\alpha_2D_x^{-1} & A_2D_x^{-1}(\alpha_1D_x^{-1} + \alpha_3) \end{bmatrix} + \begin{bmatrix} B_1D_x^{-1}\beta_2D_x^{-1} & B_1D_x^{-1}(\beta_1D_x^{-1} + \beta_3) \\ B_2D_x^{-1}\beta_2D_x^{-1} & B_2D_x^{-1}(\beta_1D_x^{-1} + \beta_3) \end{bmatrix}$$

So, in the final form we obtain the recursion operator as

$$
\begin{aligned}
\mathcal{R} = &\begin{bmatrix} -D_x^2 + 4u + w^2 & -2wD_x^2 - w_1 D_x + 3uw - 2w_2 \\ 2w & -D_x^2 + 2u + w^2 \end{bmatrix} \\
&+ \begin{bmatrix} (2u_1 + ww_1)D_x^{-1} & KD_x^{-1} \\ w_1 D_x^{-1} & (-p_1 u + u_1 + ww_1)D_x^{-1} \end{bmatrix} \\
&+ \begin{bmatrix} A_1 D_x^{-1}\alpha_2 D_x^{-1} & A_1 D_x^{-1}(\alpha_1 D_x^{-1} + \alpha_3) \\ A_2 D_x^{-1}\alpha_2 D_x^{-1} & A_2 D_x^{-1}(\alpha_1 D_x^{-1} + \alpha_3) \end{bmatrix} \\
&+ \begin{bmatrix} B_1 D_x^{-1}\beta_2 D_x^{-1} & B_1 D_x^{-1}(\beta_1 D_x^{-1} + \beta_3) \\ B_2 D_x^{-1}\beta_2 D_x^{-1} & B_2 D_x^{-1}(\beta_1 D_x^{-1} + \beta_3) \end{bmatrix},
\end{aligned}
$$

where $K = 2p_1 uw - p_1 w_2 + 2uw_1 + 3u_1 w + 2w^2 w_1 - w_3$.

§3. Conclusion

We gave an outline of the theory of deformations of the equation structure of differential equations, leading to the construction of recursion operators for symmetries of such equations. The extension of this theory to the nonlocal setting of differential equations is essential for getting nontrivial results. The theory has been applied to the construction of the recursion operator for symmetries for a coupled KdV-mKdV system, leading to a highly nonlocal result for this system. Moreover the appearance of nonpolynomial nonlocal terms in all results, e.g., conservation laws, symmetries and recursion operator is striking and reveals some unknown and intriguing underlying structure of the equations.

Appendix

Here we present explicit formulas for $(p_{3,1})_t$, $(p_{3,2})_t$, $(p_{4,1})_t$, and $(p_5)_t$:

$$
\begin{aligned}
(p_{3,1})_t = \frac{1}{12}\big(&\cos(2p_{0,1})(3p_1^3 uw + p_1^3 w^3 - p_1^3 w_2 + 3p_1^2 uw_1 - 3p_1^2 u_1 w \\
&- 3p_1^2 w^2 w_1 - 24p_1 u^2 w + 105 p_1 uw^3 + 6p_1 uw_2 - 6p_1 u_1 w_1 \\
&+ 6p_1 u_2 w + 39 p_1 w^5 - 27 p_1 w^2 w_2 - 96 p_{1,1} p_{1,2} uw^2 \\
&- 24 p_{1,1} p_{1,2} w^4 + 48 p_{1,1} p_{1,2} ww_2 - 24 p_{1,1} p_{1,2} w_1^2 + 36 p_3 uw \\
&+ 12 p_3 w^3 - 12 p_3 w_2 - 12u^2 w_1 + 48 uu_1 w + 39 uw^2 w_1 + 3u_1 w^3 \\
&- 6u_1 w_2 + 6u_2 w_1 - 6u_3 w - 9w^4 w_1 - 12 w^2 w_3 \\
&- 18 ww_1 w_2 + 6w_1^3) \\
&+ 2\sin(2p_{0,1})(36 p_1 p_{1,1} p_{1,2} uw + 12 p_1 p_{1,1} p_{1,2} w^3
\end{aligned}
$$

$$
\begin{aligned}
&- 12p_1p_{1,1}p_{1,2}w_2 + 54p_1uww_1 + 54p_1u_1w^2 + 54p_1w^3w_1\\
&- 18p_1ww_3 + 12p_{1,1}p_{1,2}uw_1 - 12p_{1,1}p_{1,2}u_1w - 12p_{1,1}p_{1,2}w^2w_1\\
&- 9u^2w^2 + 18uw^4 + 27uww_2 - 9uw_1^2 + 9u_1ww_1 + 3u_2w^2\\
&+ 2w^6 - 11w^3w_2 + 12w^2w_1^2 - 3ww_4 + 3w_1w_3 - 3w_2^2)\\
&+ 6p_{1,2}(-3p_1^2uw - p_1^2w^3 + p_1^2w_2 - 2p_1uw_1 + 2p_1u_1w + 2p_1w^2w_1\\
&+ 8u^2w + uw^3 - 2uw_2 + 2u_1w_1 - 2u_2w - w^5 - 3w^2w_2)\big),
\end{aligned}
$$

$$
\begin{aligned}
(p_{3,2})_t = \frac{1}{12}\big(&2\cos(2p_{0,1})(36p_1p_{1,1}p_{1,2}uw + 12p_1p_{1,1}p_{1,2}w^3\\
&- 12p_1p_{1,1}p_{1,2}w_2 - 54p_1uww_1 - 54p_1u_1w^2 - 54p_1w^3w_1\\
&+ 18p_1ww_3 + 12p_{1,1}p_{1,2}uw_1 - 12p_{1,1}p_{1,2}u_1w - 12p_{1,1}p_{1,2}w^2w_1\\
&+ 9u^2w^2 - 18uw^4 - 27uww_2 + 9uw_1^2 - 9u_1ww_1 - 3u_2w^2\\
&- 2w^6 + 11w^3w_2 - 12w^2w_1^2 + 3ww_4 - 3w_1w_3 + 3w_2^2)\\
&+ \sin(2p_{0,1})(3p_1^3uw + p_1^3w^3 - p_1^3w_2 + 3p_1^2uw_1 - 3p_1^2u_1w\\
&- 3p_1^2w^2w_1 - 24p_1u^2w + 105p_1uw^3 + 6p_1uw_2 - 6p_1u_1w_1\\
&+ 6p_1u_2w + 39p_1w^5 - 27p_1w^2w_2 + 96p_{1,1}p_{1,2}uw^2\\
&+ 24p_{1,1}p_{1,2}w^4 - 48p_{1,1}p_{1,2}ww_2 + 24p_{1,1}p_{1,2}w_1^2 + 36p_3uw\\
&+ 12p_3w^3 - 12p_3w_2 - 12u^2w_1 + 48uu_1w + 39uw^2w_1 + 3u_1w^3\\
&- 6u_1w_2 + 6u_2w_1 - 6u_3w - 9w^4w_1 - 12w^2w_3\\
&- 18ww_1w_2 + 6w_1^3)\\
&+ 6p_{1,1}(-3p_1^2uw - p_1^2w^3 + p_1^2w_2 - 2p_1uw_1 + 2p_1u_1w + 2p_1w^2w_1\\
&+ 8u^2w + uw^3 - 2uw_2 + 2u_1w_1 - 2u_2w - w^5 - 3w^2w_2)\big),
\end{aligned}
$$

$$
\begin{aligned}
(p_{4,1})_t = \frac{1}{48}\big(&8\cos(2p_{0,1})(3p_1^3p_{1,2}uw + p_1^3p_{1,2}w^3 - p_1^3p_{1,2}w_2 + 3p_1^2p_{1,2}uw_1\\
&- 3p_1^2p_{1,2}u_1w - 3p_1^2p_{1,2}w^2w_1 + 36p_1p_{1,1}^2p_{1,2}uw + 12p_1p_{1,1}^2p_{1,2}w^3\\
&- 12p_1p_{1,1}^2p_{1,2}w_2 - 24p_1p_{1,2}u^2w - 3p_1p_{1,2}uw^3 + 6p_1p_{1,2}uw_2\\
&- 6p_1p_{1,2}u_1w_1 + 6p_1p_{1,2}u_2w + 3p_1p_{1,2}w^5 + 9p_1p_{1,2}w^2w_2\\
&+ 12p_{1,1}^2p_{1,2}uw_1 - 12p_{1,1}^2p_{1,2}u_1w - 12p_{1,1}^2p_{1,2}w^2w_1\\
&- 48p_{1,1}p_{1,2}^2uw^2 - 12p_{1,1}p_{1,2}^2w^4 + 24p_{1,1}p_{1,2}^2ww_2 - 12p_{1,1}p_{1,2}^2w_1^2\\
&+ 72p_{1,1}u^2w^2 + 18p_{1,1}uw^4 - 54p_{1,1}uww_2 + 18p_{1,1}uw_1^2\\
&- 18p_{1,1}u_1ww_1 - 24p_{1,1}u_2w^2 - 4p_{1,1}w^6 - 32p_{1,1}w^3w_2\\
&- 24p_{1,1}w^2w_1^2 + 6p_{1,1}ww_4 - 6p_{1,1}w_1w_3 + 6p_{1,1}w_2^2
\end{aligned}
$$

$$
\begin{aligned}
&+36p_{1,2}p_3uw+12p_{1,2}p_3w^3-12p_{1,2}p_3w_2-12p_{1,2}u^2w_1\\
&+48p_{1,2}uu_1w+39p_{1,2}uw^2w_1+3p_{1,2}u_1w^3\\
&-6p_{1,2}u_1w_2+6p_{1,2}u_2w_1-6p_{1,2}u_3w-9p_{1,2}w^4w_1\\
&-12p_{1,2}w^2w_3-18p_{1,2}ww_1w_2+6p_{1,2}w_1^3)\\
&+8\sin(2p_{0,1})(3p_1^3p_{1,1}uw+p_1^3p_{1,1}w^3-p_1^3p_{1,1}w_2+3p_1^2p_{1,1}uw_1\\
&-3p_1^2p_{1,1}u_1w-3p_1^2p_{1,1}w^2w_1+36p_1p_{1,1}p_{1,2}^2uw\\
&+12p_1p_{1,1}p_{1,2}^2w^3-12p_1p_{1,1}p_{1,2}^2w_2-24p_1p_{1,1}u^2w-3p_1p_{1,1}uw^3\\
&+6p_1p_{1,1}uw_2-6p_1p_{1,1}u_1w_1+6p_1p_{1,1}u_2w+3p_1p_{1,1}w^5\\
&+9p_1p_{1,1}w^2w_2+48p_{1,1}^2p_{1,2}uw^2+12p_{1,1}^2p_{1,2}w^4-24p_{1,1}^2p_{1,2}ww_2\\
&+12p_{1,1}^2p_{1,2}w_1^2+12p_{1,1}p_{1,2}^2uw_1-12p_{1,1}p_{1,2}^2u_1w\\
&-12p_{1,1}p_{1,2}^2w^2w_1+36p_{1,1}p_3uw+12p_{1,1}p_3w^3-12p_{1,1}p_3w_2\\
&-12p_{1,1}u^2w_1+48p_{1,1}uu_1w+39p_{1,1}uw^2w_1+3p_{1,1}u_1w^3\\
&-6p_{1,1}u_1w_2+6p_{1,1}u_2w_1-6p_{1,1}u_3w-9p_{1,1}w^4w_1-12p_{1,1}w^2w_3\\
&-18p_{1,1}ww_1w_2+6p_{1,1}w_1^3-72p_{1,2}u^2w^2-18p_{1,2}uw^4\\
&+54p_{1,2}uww_2-18p_{1,2}uw_1^2+18p_{1,2}u_1ww_1+24p_{1,2}u_2w^2\\
&+4p_{1,2}w^6+32p_{1,2}w^3w_2+24p_{1,2}w^2w_1^2-6p_{1,2}ww_4\\
&+6p_{1,2}w_1w_3-6p_{1,2}w_2^2)-3p_1^4uw-p_1^4w^3+p_1^4w_2-4p_1^3uw_1\\
&+4p_1^3u_1w+4p_1^3w^2w_1-72p_1^2p_{1,1}^2uw-24p_1^2p_{1,1}^2w^3+24p_1^2p_{1,1}^2w_2\\
&-72p_1^2p_{1,2}^2uw-24p_1^2p_{1,2}^2w^3+24p_1^2p_{1,2}^2w_2+48p_1^2u^2w+6p_1^2uw^3\\
&-12p_1^2uw_2+12p_1^2u_1w_1-12p_1^2u_2w-6p_1^2w^5-18p_1^2w^2w_2\\
&-48p_1p_{1,1}^2uw_1+48p_1p_{1,1}^2u_1w+48p_1p_{1,1}^2w^2w_1-48p_1p_{1,2}^2uw_1\\
&+48p_1p_{1,2}^2u_1w+48p_1p_{1,2}^2w^2w_1-144p_1p_3uw-48p_1p_3w^3\\
&+48p_1p_3w_2+48p_1u^2w_1-192p_1uu_1w-156p_1uw^2w_1\\
&-12p_1u_1w^3+24p_1u_1w_2-24p_1u_2w_1+24p_1u_3w+36p_1w^4w_1\\
&+48p_1w^2w_3+72p_1ww_1w_2-24p_1w_1^3+192p_{1,1}^2u^2w\\
&+24p_{1,1}^2uw^3-48p_{1,1}^2uw_2+48p_{1,1}^2u_1w_1-48p_{1,1}^2u_2w-24p_{1,1}^2w^5\\
&-72p_{1,1}^2w^2w_2+192p_{1,2}^2u^2w+24p_{1,2}^2uw^3-48p_{1,2}^2uw_2\\
&+48p_{1,2}^2u_1w_1-48p_{1,2}^2u_2w-24p_{1,2}^2w^5\\
&-72p_{1,2}^2w^2w_2-48p_3uw_1+48p_3u_1w+48p_3w^2w_1-252u^3w\\
&-36u^2w^3+60u^2w_2-144uu_1w_1+240uu_2w-7uw^5
\end{aligned}
$$

$$
\begin{aligned}
&+ 342uw^2w_2 + 24uww_1^2 + 144u_1^2w + 228u_1w^2w_1 + 94u_2w^3\\
&- 24u_2w_2 + 24u_3w_1 - 24u_4w - 13w^7 + 47w^4w_2 - 4w^3w_1^2\\
&- 78w^2w_4 - 60ww_1w_3 - 120ww_2^2 + 60w_1^2w_2\big),\\
(p_5)_t = \frac{1}{6}\big(&54u^4 + 180u^3w^2 - 72u^2u_2 + 126u^2w^4 - 282u^2ww_2\\
&- 12u^2w_1^2 - 84uu_1ww_1 - 174uu_2w^2 + 6uu_4 + 18uw^6\\
&- 300uw^3w_2 - 90uw^2w_1^2 + 66uww_4 + 6uw_1w_3 + 48uw_2^2\\
&+ 42u_1^2w^2 - 6u_1u_3 + 12u_1w^3w_1 + 42u_1ww_3 - 48u_1w_1w_2\\
&+ 6u_2^2 - 39u_2w^4 + 162u_2ww_2 - 48u_2w_1^2 + 48u_3ww_1 + 21u_4w^2\\
&- 42w^5w_2 - 12w^4w_1^2 + 35w^3w_4 + 120w^2w_1w_3\\
&+ 195w^2w_2^2 + 120ww_1^2w_2 - 6ww_6 - 30w_1^4 + 6w_1w_5 - 6w_2w_4\big).
\end{aligned}
$$

References

[1] A. V. Bocharov, V. N. Chetverikov, S. V. Duzhin, N. G. Khor′kova, I. S. Krasil′shchik, A. V. Samokhin, Yu. N. Torkhov, A. M. Verbovetsky and A. M. Vinogradov, Symmetries and Conservation Laws for Differential Equations of Mathematical Physics, American Mathematical Society, Providence, RI, 1999. Edited and with a preface by Krasil′shchik and Vinogradov.

[2] R. K. Dodd, J. C. Eilbeck, J. D. Gibbons and H. C. Morris, Solitons and Nonlinear Wave Equations, Academic Press, 1982.

[3] P. H. M. Kersten, *Supersymmetries and recursion operators for $N = 2$ supersymmetric KdV-equation*, RIMS Kokyuroku, **1150**, Kyoto Univ. Kyoto, Japan, 2000, pp. 153–161.

[4] I. S. Krasil′shchik, *Some new cohomological invariants for nonlinear differential equations*, Differential Geom. Appl., **2** (1992), no. 4, 307–350.

[5] I. S. Krasil'shchik and P. H. M. Kersten, Symmetries and Recursion Operators for Classical and Supersymmetric Differential Equations, Kluwer Acad. Publ. Dordrecht/Boston/London, 2000.

[6] I. S. Krasil′shchik, V. V. Lychagin and A. M. Vinogradov, Geometry of Jet Spaces and Nonlinear Partial Differential Equations, Gordon and Breach, New York, 1986.

[7] I. S. Krasil′shchik and A. M. Vinogradov, *Nonlocal trends in the geometry of differential equations: symmetries, conservation laws, and Bäcklund transformations*, Acta Appl. Math., **15** (1989), no. 1-2, 161–209.

[8] S. Krivonos and A. Sorin, *Extended $N = 2$ supersymmetric matrix $(1, s)$-KdV hierarchies*, Phys. Lett., **A251** (1999), 109.

[9] P. Mathieu, *Open problems for the super KdV equation*, AARMS-CRM Workshop on Baecklund and Darboux transformations. The Geometry of Soliton Theory. June 4–9, 1999. Halifax, Nova Scotia.

[10] A. M. Vinogradov, *Local symmetries and conservation laws*, Acta Appl. Math., **3** (1984), 21–78.

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An Approach to the Cartan Geometry I: Conformal Riemann Manifolds

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Introduction

As is well known F. Klein extracted the essence of the classical geometry by saying that the geometry is the study of properties invariant under the transformations of Lie groups on homogenous spaces. This includes for instance the euclidean geometry and the conformal euclidean geometry. However, this geometry is too rigid to treat geometric objects we meet in reality. B. Riemann was thus led to introduce his geometry generalizing the euclidean geometry.

It is a natural question to ask how to generalize the Riemann's work to the case of an arbitrary classical geometry which is a homogenous space $X = G/H$, where G is a Lie group and H is its closed subgroup. We call any such generalization a structure modeled after the classical geometry G/H.

E. Cartan [1] gave an answer by introducing "a generalized space". Namely, instead of the space X together with the action of G on X, he considers the projection $\rho_G \colon G \to X$. There is on G the invariant 1-form, say ω_G, valued in the Lie algebra $\mathbf{g}$ of G. He associate to the classical geometry G/H the pair (G, ρ_G, ω_G), which is in todays language a Cartan connection ω_G on a principal H-bundle G over X. We recover the homogenous space structure of X because the graphs of the transformations of G are the integral submanifolds of the differential system $\pi_1^*\omega_G - \pi_2^*\omega_G$ on $G \times G$, where π_1 (resp. π_2) is the projection to the first (resp. second) component of $G \times G$. By the structure equation of the Lie algebra we have

$$(1) \qquad d\omega_G + \frac{1}{2}[\omega_G, \omega_G] = 0.$$

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Cartan's generalized space structures are deformations of the above structure. Namely, in todays language, a generalized space structure on a manifold M is a pair (E, ρ, ω) of a principal H-bundle E over M with the projection ρ and a Cartan connection ω on E. We call it a Cartan structure modeled after the homogenous space G/H. E will be called the frame bundle. It has the curvature form

$$K = d\omega + \frac{1}{2}[\omega, \omega]. \tag{2}$$

While developing the modern terminology for Cartan's work, C. Ehresmann [3] made an interesting comment on the problem by saying that a structure modeled after G/H is a space where a homogenous space G/H is attached to each point. We interpret this as saying that, on such a space, neighborhoods of each point are identified infinitesimally (up to certain order) with a neighborhood of a reference point in G/H.

Since Cartan's work many answers to our question are introduced, including the use of Cartan's theory of equivalence and infinite Lie groups. In this note we mainly view the development evolved around these two view points of Cartan and Ehresmann.

We note that the parameter space of structures modeled after G/H on a manifold is obviously infinite dimensional. Therefore we can think of two approaches to the problem. One is to develop a way to write down all such structures and the other is to find a good way to pick one nice such structure.

A variation of the first order infinitesimal version of Ehresmann's view was started by S. S. Chern [2] under the name G-structure. This G refers to a linear Lie group, not to our G, but more related to our H. Actually a slightly limited case of the G-structure was already considered by H. Weyl as a generalization of the general relativity. The theory of G-structures seems to mainly concerned with the first approach. For surveys see for example S. Kobayashi [5] and T. Ochiai [12].

There are also a lot of works with respect to the second approach. The Levi-Chivita's Riemann geometry may be viewed, in retrospect, as the first satisfactory fusion of Cartan's and Eheresman's viewpoints in the second approach. This is the Cartan connection, with vanishing torsion, on the orthonormal frame bundle. The case for the conformal geometry was worked out by H. Weyl [18], who extracted the conformally invariant components of the curvature tensor called Weyl tensor. CR geometry created by E. Cartan [2], N. Tanaka [16], and S. S. Chern-J. Moser [5] can be considered as the case where the model structure is the unit complex ball with the holomorphic automorphism group.

Generalizing his pioneering work on CR structures, N. Tanaka [14]–[16] introduced structures closely related to Cartan's. His work was further developed by T. Morimoto [10], [11] and K. Yamaguchi [17]. The works of A. Cap and J. Slovak for the higher codimensional CR structures are in this volume. There is also a work of R. Miyaoka [9] on the Lie's sphere geometry.

In the cases of the conformal geometry and CR geometry on a manifold M, we may follow the analogy with the Riemann geometry and construct the bundle, say E_1, using the first order Ehresmann approach. However, it is a principal H/H_1-bundle for a normal subgroup H_1 of H. We have to enlarge E_1 to a principal H-bundle, say E.

Our attempt to develop a general method to include the cases of Riemann, conformal, and CR as special cases was first outlined in [6] and completed in [7]. It is further developed by Y. Liu [8]. We constructed the above E by applying the Cartan's method of prolongation to E_1. However, the traditional approach is to use 2-jets as was done, for example, in Kobayashi [5] and Ogiue [13]. Namely, E_1 may be naturally regarded as embedded in J^1, the space of the 1-jets of maps of G/H to M at a reference point $e \in G/H$. We also have the space of 2-jets J^2, and the projection ρ: $J^2 \to J^1$. We construct a section $E_1 \to J^2$. Then E is defined as the subspace of J^2 consisting of the orbits of H-action passing points of the image of E^1 in J^2.

In this paper, we use the Ehresmann approach of the second order and construct E as a quotient space $\rho^{-1}(E_1) \to E$ with a commutative diagram:

$$
\begin{array}{ccc}
J^1 & \leftarrow & J^2 \\
\uparrow & & \uparrow \\
E_1 & \leftarrow & \rho^{-1}E_1 \\
\| & & \downarrow \\
E_1 & \leftarrow & E
\end{array}
\tag{3}
$$

When we construct a principal H-bundle, say E_2, so that $E_1 \leftarrow E_2 \subset \rho^{-1}E_1 \subset J^2$. then the vertical downarrow in (3) will induce an isomorphism $E_2 \to E$. Therefore our frame bundle is isomorphic to the traditional one.

Once the frame bundle is constructed, we work locally and find a Cartan connection by imposing conditions on the curvature form. In Kobayashi [5] this was done using the canonical forms of J^2. We can adopt this method in our frame work. However, we used here a direct method using the definition of the Cartan connections.

The curvature is valued in the Lie algebra $\mathbf{g}$ of G, which has the grading: $\mathbf{g} = \mathbf{g}_{(-1)} + \mathbf{g}_{(0)} + \mathbf{g}_{(1)}$. We designate a suitable subspace

$\mathbf{g_n} \subset \mathbf{g}_{(0)} + \mathbf{g}_{(1)}$. A Cartan connections on E is called normal when the cuvature takes value in $\mathbf{g_n}$.

In our case the set of normal connections is a family of isomorphic Cartan connections depending on one arbitrary function. It turns out that the Weyl tensor is independent of the connections in the family. Therefore we obtain a unique Weyl curvature form. However, to construct a Cartan connection globally we need to choose locally one connection from the above family in such a way they match up. We do this is this paper.

In §1 we review the case of the homogenous conformal Riemann geometry. We write down several formulas which will be used later. In §2 we construct the frame bundle and the normal Cartan connections along the line mentioned above in the case of conformal geometry. We also show that the $\mathbf{g}_{(1)}$-part of the normal Cartan connections are obtained using the conformal covariant derivative of Weyl tensor. In the end we construct a global normal conformal Cartan connection.

The literature for the conformal connection is too numerous and very difficult to give a complete reference. As a result we listed only a few which we quoted in this paper. We beg perdon for the omission.

The author is greately benefited by the discussions with Professor Keizo Yamaguchi.

§1. The Homogeous Conformal Space

We fix a nondegenerate $m \times m$ matrix

$$(1) \qquad (\underline{h}_{ij}), \quad i, j = 1, \ldots, m.$$

We consider the conformal euclidean geometry based on the metric on $\mathbf{R}^m$ given by

$$(2) \qquad \langle dx, dx \rangle = \underline{h}_{ij}\, dx^i\, dx^j.$$

A) Let $\mathbf{R}^{m+2}$ be the euclidean space with the standard chart:

$$(3) \qquad (\xi^0, \ldots, \xi^{m+1}) = (\xi^0, \xi', \xi^{m+1}), \quad \xi' = (\xi^1, \ldots, \xi^m),$$

from which we remove the origin obtaining the punctured euclidean space $\dot{\mathbf{R}}^{m+2}$. Dividing by the non-zero scalar mutiplication operation, we obtain the projective space

$$(4) \qquad \rho\colon \dot{\mathbf{R}}^{m+2} \to \dot{\mathbf{R}}^{m+2}/\mathbf{C}^* = \mathbf{RP}^{m+1}.$$

Denote by $[\xi] = [\xi^0, \ldots, \xi^{m+1}]$ the homogenous coordinate of $\mathbf{RP}^{m+1}$.

Consider the hypersurface Φ^m in $\mathbf{RP}^{m+1}$ given by

$$\Phi^m\colon \phi(\xi) = \langle \xi', \xi' \rangle - 2\xi^0 \xi^{m+1} = 0, \tag{5.1}$$

where

$$\langle \xi', \xi' \rangle = \underline{h}_{ij} \xi^i \xi^j. \tag{5.2}$$

We embed $\mathbf{R}^m$ in Φ^m by

$$\mathbf{R}^m \ni x \to \left[1, x, \frac{1}{2}\langle x, x\rangle\right] \in \Phi^m. \tag{6}$$

B) $\mathbf{R}^m$ itself is not the homogeous space. Its closure Φ^m is the homogenous conformal space, given as follows:

Denote by $\tilde{G}$ the subgroup of $GL(\mathbf{R}, m+2)$ consisting of all matrix g satisfying:

$$\det g = 1, \quad \phi(g(\xi)) = \phi(\xi). \tag{7}$$

Let G be the subgroup of the projective transformation group induced by $\tilde{G}$. In view of (5.1) we find that G preseeves Φ^m and acts as a transformation group of Φ^m.

We find that $\tilde{G}$ decomposes to the product of the translation group and the isotropy group. Namely,

$$\tilde{G} = L \cdot H, \tag{8}$$

$$L = \left\{ l(y) = \begin{pmatrix} 1 & 0 & 0 \\ y & I & 0 \\ w & y^* & I \end{pmatrix} : y = (y^1, \ldots, y^m)^{\mathrm{tr}},\ w = \frac{1}{2}\langle y, y \rangle \right\} \tag{9}$$

where $(y^*)_j = \underline{h}_{jk} y^k$, and H consists of matrixes of the form

$$h = h(a, t, \beta) = \begin{pmatrix} a & \gamma & b \\ 0 & t & \beta \\ 0 & 0 & a^{-1} \end{pmatrix}, \quad \text{where} \tag{10}$$

$$\begin{gathered} \det t = 1, \quad tt^* = I, \quad \beta = (\beta^1, \ldots, \beta^m)^{\mathrm{tr}}, \\ \gamma_l = a(\beta^* t)_l, \quad \frac{b}{a} = \frac{1}{2}\langle \beta, \beta \rangle, \end{gathered} \tag{11}$$

where $(t^*)^i_j = \underline{h}^{ik} \underline{h}_{jl} t^l_k$ and I is the identity $m \times m$-matrix. It is convenient to consider a smaller group where $a > 0$. The Lie algebra $\mathbf{g}$ of G has the grading:

$$\mathbf{g} = \mathbf{g}_{(-1)} + \mathbf{g}_{(0)} + \mathbf{g}_{(1)}, \quad \text{where,} \tag{12}$$

$$\mathbf{g}_{(-1)} = \left\{ \{\dot{y}\}_{(-1)} = \left(\frac{d(l(s\dot{y}))}{ds}\right)_{s=0} : \dot{y} \in \mathbf{R}^m \right\}, \tag{13.1}$$

$$\mathbf{g}_{(0)} = \mathbf{R}\pi + \{o(m)\}, \quad \text{where for } \dot{t} \in o(m) \tag{13.2}$$

$$\{\dot{t}\} = \left(\frac{dh(1, I + s\dot{t}, 0)}{ds}\right)_{s=0}, \quad \pi = \left(\frac{dh(e^s, I, 0)}{ds}\right)_{s=0}, \tag{13.3}$$

$$\mathbf{g}_{(1)} = \left\{ \{\dot{\beta}\}_{(1)} = \left(\frac{dh(1, I, s\dot{\beta})}{ds}\right)_{s=0} : \dot{\beta} \in \mathbf{R}^m \right\}. \tag{13.4}$$

$$\mathbf{h} = \mathbf{g}_{(0)} + \mathbf{g}_{(1)} \text{ is the Lie algebra of } H. \tag{14}$$

We find by calculation

$$\begin{aligned} \mathrm{Ad}(h^{-1})\{\dot{y}\}_{(-1)} &= \{at^*\dot{y}\}_{(-1)} - a\langle \dot{y}, \beta\rangle \pi \\ &\quad + \{at^*\dot{y} \otimes \beta^* t - at^*\beta \otimes \dot{y}^* t\} + \{t^*(b\dot{y} - a\langle \beta, \dot{y}\rangle\beta)\}_{(1)}, \\ \mathrm{Ad}(h^{-1})\pi &= \pi + \{t^*\beta\}_{(1)} \\ \mathrm{Ad}(h^{-1})\{\dot{t}\} &= \{t^*\dot{t}t\} + \{t^*\dot{t}\beta\}_{(1)} \\ \mathrm{Ad}(h^{-1})\{\dot{\beta}\}_{(1)} &= \{a^{-1}t^*\dot{\beta}\}_{(1)}. \end{aligned} \tag{15}$$

In terms of the decomposition (8) the action of $g \in G$ on $x \in \mathbf{R}^m \subset \Phi^m$ is given by

$$T_g x = y + \frac{1}{\lambda}\left(tx + \frac{1}{2}\langle x, x\rangle\beta\right), \quad \text{where} \tag{16.1}$$

$$\lambda = a\left(1 + \langle tx, \beta\rangle + \frac{1}{4}\langle \beta, \beta\rangle\langle x, x\rangle\right). \tag{16.2}$$

We see now that G acts transitively on Φ^m.

We regard Φ^m as a conformal Riemann manifold as follows: We consider a metric on

$$\tilde{\Phi}^{m+1} = \rho^{-1}(\Phi^m) \tag{17}$$

given by

$$ds^2 = \langle d\xi', d\xi'\rangle - 2d\xi^0\, d\xi^{n+1}. \tag{18}$$

For simplicity we set

$$F = \mathbf{R}^m. \tag{19}$$

We have a chart $\xi = (c, x)$ on $\rho^{-1}F$ given by

$$(\xi) = \left(c, cx, \frac{1}{2}c\langle x, x\rangle\right). \tag{20}$$

Then we find that the metric on $\rho^{-1}F$ induced by ds^2 is

$$c^2(ds)_F, \quad \text{where} \quad (ds)_F = \langle dx, dx\rangle. \tag{21}$$

In view of (7) the metric ds^2 on $\tilde{\Phi}^{m+1}$ is invariant under the action of the matrix g. We find by (19), (16), and (10) that

$$(h\xi) = \Bigl(c_h, c_h T_h x, \frac{1}{2}c_h\langle T_h x, T_h x\rangle\Bigr), \quad c_h = c\lambda. \tag{22}$$

Hence $h\xi$ has the coordinate $(c_h, T_h x)$. Then it follows by (19)–(20) that $\lambda^2\langle dx, dx\rangle = \langle T_h\, dx, T_h\, dx\rangle$. Since $T_{l(y)}$ is a translation, it follows that

$$(T_g)^*(ds)_F = \lambda^{-2}(ds)_F. \tag{23}$$

We conclude that the action of G on Φ^m is conformal.

C) Denote by $J_0^p(F)$ the space of 2-jets at the reference point 0 of maps of neighborhoods of $0 \in F$ into F. $J_0^p(F)$ has the standard chart $(y, \dots, p^k_{j_1\dots j_q}, \dots)$, where $q \le p$. If $J \in J_0^p(\mathbf{R}^m)$ is represented by a map $f = \bigl(f^1(x), \dots, f^m(x)\bigr)$

$$y = f(0), \quad p_{j_1\dots j_q}(J) = \frac{\partial^q f}{\partial x^{j_1}\cdots\partial x^{j_q}}(0). \tag{24}$$

We find by calculation

$$\begin{aligned} p^k_j(T_h) &= \frac{1}{a}t^k_j, \\ p^k_{jl}(T_h) &= \frac{1}{a}\underline{h}_{jl}\beta^k - \frac{1}{a}\underline{h}_{iq}(t^k_j t^i_l + t^k_l t^i_j)\beta^q. \end{aligned} \tag{25}$$

We note that $J_0^1(T_h)$ gives informations on a, t in (10), and we need J_0^2 to get β. We note also that we reach β more quickly by using the conformal factor λ^{-2} of T_h (cf. (23) and (16)), i.e.

$$\frac{\partial\lambda}{\partial x^j}(0) = a\underline{h}_{jk}(t^{-1})^k_l\beta^l. \tag{26}$$

E) In view of (9)–(10) the Maurer-Cartan form:

$$\Omega_G = \begin{pmatrix} (\Omega_G)^0_0 & \dots (\Omega_G)^0_k & \dots (\Omega_G)^0_{m+1} \\ (\Omega_G)^j_0 & \dots (\Omega_G)^j_k & \dots (\Omega_G)^j_{m+1} \\ (\Omega_G)^{m+1}_0 & \dots (\Omega_G)^{m+1}_k & \dots (\Omega_G)^{m+1}_{m+1} \end{pmatrix} \tag{27}$$

has the relations

$$\begin{gathered} (\Omega_G)^0_0 + (\Omega_G)^{m+1}_{m+1} = 0, \quad (\Omega_G)^{m+1}_k = \underline{h}_{kl}(\Omega_G)^l_0, \\ (\Omega_G)^0_k = \underline{h}_{kl}(\Omega_G)^l_{m+1}, \quad \underline{h}_{jl}(\Omega_G)^l_k + \underline{h}_{kl}(\Omega_G)^l_j = 0, \\ (\Omega_G)^0_{m+1} = (\Omega_G)^{m+1}_0 = 0. \end{gathered} \tag{28}$$

Since $\Omega_G = g^{-1}\,dg$, we also find by (8)–(10) that

$$(29)\quad \begin{aligned} (\Omega_G)^0_0 &= d\log a - a\langle\beta, dy\rangle,\\ (\Omega_G)^j_0 &= a(t^*dy)^j = a\underline{h}^{jk}\underline{h}_{li}t^i_k\,dy^l,\\ (\Omega_G)^j_k &= (t^*dt)^j_k + a(t^*dy)^j(\beta^*t)_k - a(t^*\beta)^j(dy^*t)_k\\ &= \underline{h}^{jl}\underline{h}_{iq}t^q_l(dt^i_k + a\underline{h}_{pr}t^p_k(\beta^r\,dy^i - \beta^i\,dy^r)),\\ (\Omega_G)^j_{m+1} &= (t^*d\beta)^j + (t^*\beta)^j\,d\log a + b(t^*dy)^j - a(t^*\beta)^j\langle\beta, dy\rangle\\ &= \underline{h}^{jl}\underline{h}_{ki}t^i_l\big(d\beta^k + \beta^k(d\log a - a\langle\beta, dy\rangle) + b\,dy^k\big). \end{aligned}$$

We use ω_H ro denote the Maurer-Cartan form of H. We also set $\omega_H = h^{-1}\,dh$. Since ω_H is obtained by setting $dy = 0$ in the above

$$(30)\quad \begin{aligned} &(\omega_H)^0_0 = d\log a, \quad (\omega_H)^j_0 = 0, \quad (\omega_H)^j_k = (t^*dt)^j_k = \underline{h}^{jl}\underline{h}_{iq}t^q_l\,dt^i_k,\\ &(\omega_H)^j_{m+1} = (t^*d\beta)^j + (t^*\beta)^j\,d\log a = \underline{h}^{jl}\underline{h}_{ki}t^i_l(d\beta^k + \beta^k\,d\log a). \end{aligned}$$

Note that $(\Omega_G)^0_0, \ldots, (\Omega_G)^j_0, \ldots, (\Omega_G)^j_k\ (j > k), \ldots, (\Omega_G)^j_{m+1}$ form a base.

The structure equations:

$$(31)\qquad d(\Omega_G)^r_s + (\Omega_G)^r_t \wedge (\Omega_G)^t_s = 0$$

$(r,\ s,\ t = 0,\ 1, \ldots, m+1)$ is rewritten, due to the reltion (28), as

$$(32)\quad \begin{aligned} &d(\Omega_G)^0_0 + \underline{h}_{jk}(\Omega_G)^j_{m+1} \wedge (\Omega_G)^k_0 = 0,\\ &d(\Omega_G)^j_0 + \{(\Omega_G)^j_k - \delta^j_k(\Omega_G)^0_0\} \wedge (\Omega_G)^k_0 = 0,\\ &d(\Omega_G)^j_k + (\Omega_G)^j_i \wedge (\Omega_G)^i_k\\ &\qquad + \underline{h}_{kl}\{(\Omega_G)^j_0 \wedge (\Omega_G)^l_{m+1} - (\Omega_G)^l_0 \wedge (\Omega_G)^j_{m+1}\} = 0,\\ &d(\Omega_G)^j_{m+1} + \{(\Omega_G)^j_i + \delta^j_i(\Omega_G)^0_0\} \wedge (\Omega_G)^i_{m+1} = 0. \end{aligned}$$

When we regsard Ω_G as a 1-form valued in the Lie algebra $\mathbf{g}$ of G, the adjoint action of H transforms the components of Ω_G. In fact by (15)

$$(33)\quad \begin{aligned} (\mathrm{Ad}(h^{-1})\Omega_G)^i_0 &= a(t^*)^i_j(\Omega_G)^j_0,\\ (\mathrm{Ad}(h^{-1})\Omega_G)^0_0 &= (\Omega_G)^0_0 - a\underline{h}_{jk}\beta^j(\Omega_G)^k_0,\\ (\mathrm{Ad}(h^{-1})\Omega_G)^i_j &= (t^*)^i_k t^l_j(\Omega_G)^k_l\\ &\qquad + \big(a(t^*)^i_k(\beta^*t)_j - a(t^*\beta)^i t^l_j\underline{h}_{lk}\big)(\Omega_G)^k_0,\\ (\mathrm{Ad}(h^{-1})\Omega_G)^i_{m+1} &= (t^*)^i_j\Big\{a^{-1}(\Omega_G)^j_{m+1} + \beta^l(\Omega_G)^j_l + \beta^j(\Omega_G)^0_0\\ &\qquad - a\beta^j\underline{h}_{lk}\beta^l(\Omega_G)^k_0 + \frac{a}{2}\langle\beta, \beta\rangle(\Omega_G)^j_0\Big\}. \end{aligned}$$

§2. Conformal Riemann Geometry

We consider the conformal Riemann geometry on a manifold M based on a Riemann metric $(ds^2)_M$. We study the local aspect of the metric near a reference point, say P_0. Fix a chart $x = (x^1, \ldots, x^m)$ on a neighborhood of P_0, $x(P_0) = 0$. We write

$$(ds^2)_M = g_{ij}(x)dx^i\, dx^j. \tag{1}$$

We assume that the matrix $(g_{ij}(x))$ is conjugate to $\underline{h}_{ij}$ given in §1 (1). As in §1 we denote by F the model conformal structure. We also use F to denote a neighborhood of 0 of the model structure. $(ds)_F = \langle dy, dy \rangle = \underline{h}_{ij}\, dy^i\, dy^j$, where y is the standard chart of F. $(\underline{h}^{ij})$ is the inverse matrix of $(\underline{h}_{ij})$.

A) Let $q = q_{ij}(y)dy^i\, dy^j$ be a quadratic form. We set

$$\mathbf{tr}\, q = \underline{h}^{ij} q_{ij}(y). \tag{2}$$

Let f be a map of F into M.

Definition 1. We say that f is an attaching map of M at $f(0)$ when there is a function $c > 0$ on F such that

$$f^*(ds^2)_M - c(ds^2)_F = O(1), \quad \mathbf{tr}\big(f^*(ds^2)_M - c(ds^2)_F\big) = O(2), \tag{3}$$

where $O(l)$ denotes terms in the ideal generated by $y^{i_1} \cdots y^{i_l}$. c will be called the conformal factor of the attaching map. When f satisfies the first equation in (3), we say that f is an attaching map of order 1.

We claim that, for any attching map g of order 1 and for any linear form $c = c_l y^l$ in y, there is an attaching map f with the conformal factor $c + c_0$ such that the 1-jets of f and g agree. The constant c_0 in the above is determined by g. Namely, for an unknown f we set

$$f^i(y) = x_0^i + p_j^i y^j + \frac{1}{2} p_{jk}^i y^j y^k + \cdots. \tag{4}$$

Then

$$f^*(ds^2)_M = g_{st}(f(y))(p_i^s + p_{ik}^s y^k)(p_j^t + p_{jl}^t y^l)dy^i\, dy^j + O(2). \tag{5}$$

Hence the equation for f to be an attaching map as above is given by

$$\begin{gathered} g_{kl}(x_0)p_i^k p_j^l = c_0 \underline{h}_{ij}, \quad \text{and with} \quad G_k^i = \underline{h}^{ij} g_{kl}(x_0) p_j^l, \\ 2G_k^i p_{it}^k + \underline{h}^{ij} \frac{\partial g_{kl}}{\partial y^s}(x_0) p_i^k p_j^l p_t^s = mc_t. \end{gathered} \tag{6}$$

Since the matrix (G^i_j) is non-singular, our assertion follows easily.

We say that attaching maps f_1, f_2 at x_0 with the conformal factors c_1, c_2 are equivalent when

$$j^1_0 f_1 = j^1_0 f_2, \quad j^1_0 c_1 = j^1_0 c_2. \tag{7}$$

The equivalent classes of attaching maps will be called the frames of M. Denote by E the set of the frames of M. Let E_1 be the space of 1-jets at 0 of attaching maps. Clearly we have the projection $E \to E_1$:

$$E \ni \text{the class of } f \to j^1_0 f \in E_1, \quad \text{and} \tag{8}$$

$$E_1 = \mathbf{R}^+(\text{the frame bundle of the metric } (ds^2)_M). \tag{9}$$

Note by (6) that c_0 is determined by the 1-jets information. Hence E is a manifold with a standard chart:

$$(E_1, c_1, \ldots, c_m). \tag{10}$$

B) If f is an attaching map at x_0, $f \circ T_h$ is also an attaching map at x_0 because T_h is a conformal map of F. We denote by (the class of f)$\circ R_h$ the above frame. Let the class of f has the standard chart $(x_0, p^i_j, c_1, \ldots c_m)$. Since the conformal factor of $f \circ T_h$ is $\lambda^{-2} c \circ T_h$, we see by (23)–(26) §1, the class of $f \circ T_h$ has the standard chart:

$$\begin{gathered}\Big(x_0, p^i_k \frac{1}{a} t^k_j, c'_1, \ldots, c'_m\Big), \quad \text{where} \\ c'_j = a^{-3} c_i t^i_j - 2a^{-2} c_0 \underline{h}_{kl} t^k_j \beta^l, \quad m c_0 = \underline{h}^{ij} g_{kl}(x_0) p^k_i p^l_j.\end{gathered} \tag{11}$$

We thus have the operation of H on E. In particular, E is a principal H-bundle, where the R_h action of H in the standard chart is given by the above formula.

C) We next discuss local trivializations of E. Let $\mathbf{f}(x)$ be a local section of E. Then the induced local trivialization of E is given by

$$F \times H \ni (x, h) \to \mathbf{f}(x) \circ R_h. \tag{12}$$

Denote by

$$\big(x, p^i_j(x), c_1(x), \ldots, c_m(x)\big) \tag{13}$$

the standard chart of $\mathbf{f}(x)$. Then we see by (11) that in the standard chart the above local trivialization has the expression:

$$\begin{gathered}\Big(x, p^i_k(x) \frac{1}{a} t^k_j, c_1, \ldots, c_m\Big), \quad \text{where} \\ c_j = a^{-3} c_i(x) t^i_j - 2a^{-2} c_0(x) \underline{h}_{kl} t^k_j \beta^l, \\ m c_0(x) = \underline{h}^{ij} g_{kl}(x) p^k_i(x) p^l_j(x).\end{gathered} \tag{14}$$

Let us change the local section $\mathbf{f}(x)$ to $\mathbf{f}^\sharp(x) = \mathbf{f}(x) \circ R_{h(x)}$, inducing a new local chart $(x, h^\sharp)$. We see by (12) that

$$h = h(x)h^\sharp, \tag{15}$$

$$a = a(x)a^\sharp, \quad t = t(x)t^\sharp, \quad \beta = t(x)\beta^\sharp + \frac{1}{a^\sharp}\beta(x). \tag{16}$$

D) It is known how to construct a unique Cartan connection locally on E. Nevertheless, we want to go over the construction, because we wish to take up the problem of constructing such Cartan connection globally.

We fix a local trivialization of E induced by a local section $\mathbf{f}(x)$ of E. We work on the domain of the above section and call it M. We use the induced chart (x, h) of E.

We first follow the analogy with the Riemann manifold and construct 1-form Ω_F on E valued in $\mathbf{R}^m = F$. These are the first order coframes of the conformal structure. Namely, we note that E_1 is the space of 1-jets of the first order attaching maps. Hence they are linear maps $T_0F \to TM$. Their dual may be regarded as F-valued 1-forms Ω_F on E_1. Composing with the projection $E \to E_1$, we thus have a well defined 1-form: Ω_F on E.

In terms of our chart (cf. (12) §1)

$$\Omega_F = \mathrm{Ad}(h^{-1})w_F, \quad w_F = (\ldots, w_F^j, \ldots), \quad w_F^j = p^{-1}(x)^j_k\, dx^k. \tag{17.1}$$

When we set $\Omega_F = (\Omega_F^1, \ldots, \Omega_F^m)$,

$$\Omega_F^j = a(t^{-1})^j_k p^{-1}(x)^k_l\, dx^l. \tag{17.2}$$

Note by (6) that

$$\begin{aligned} &(\ldots, w_F^j, \ldots) \text{ is a section of the 1-st order coframe bundle} \\ &\text{of the metric } \frac{1}{mc_0(x)}(ds^2)_M. \end{aligned} \tag{17.3}$$

E) A Cartan connection on M has the expression

$$\Omega = \mathrm{Ad}(h^{-1})w + h^{-1}\, dh, \quad w \text{ is a } \mathbf{g}\text{-valued 1-form on } W. \tag{18}$$

Note that we have the projection $\rho_F \colon \mathbf{g} \to \mathbf{g}/\mathbf{h} = F$. By a Cartan connection of the conformal structure we mean a Cartan connection Ω such that

$$\rho\Omega = \Omega_F. \tag{19}$$

Hence the Cartan connections of F are of the form

$$\Omega = \mathrm{Ad}(h^{-1})(w_F + w_H) + h^{-1}\,dh, \tag{20}$$
$$\text{where } w_H \text{ is an 1-form valued in } \mathbf{h}.$$

To determine Ω we have to determine w_H. We do this by using the curvature of Ω.

F) The curvature form K of Ω is given by

$$K = d\Omega + \frac{1}{2}[\Omega, \Omega] = \mathrm{Ad}(h^{-1})k, \quad \text{where } k = dw + \frac{1}{2}[w, w]. \tag{21}$$

We set (cf. (13) §1)

$$w_H = w_\pi \pi + \{w_0\} + \{w_{\mathbf{h}}\}_{(1)}, \quad \text{where} \tag{22.1}$$

$$\begin{gathered} w_\pi \text{ is } \mathbf{R}\text{-valued}, \quad w_0 = ((w_0)^i_j) \text{ is } o(m)\text{-valued}, \\ w_{\mathbf{h}} = (w^1_{\mathbf{h}}, \dots, w^m_{\mathbf{h}}) \text{ is } \mathbf{R}^m\text{-valued}. \end{gathered} \tag{22.2}$$

In the above, $o(m)$ is with respect to the quadratic form (1) §1. In view of (32) §1 we then find that

$$k = \{k_F\}_{(-1)} + k_\pi \pi + \{k_0\} + \{k_{\mathbf{h}}\}_{(1)}, \quad \text{where} \tag{23.1}$$

$$\begin{aligned} k^j_F &= dw^j_F + ((w_0)^j_k - w_\pi \delta^j_k) \wedge w^k_F, \\ k_\pi &= dw_\pi + \underline{h}_{jk} w^j_{\mathbf{h}} \wedge w^k_F, \\ (k_0)^j_k &= d(w_0)^j_k + (w_0)^j_l \wedge (w_0)^l_k + \underline{h}_{kl}(w^j_F \wedge w^l_{\mathbf{h}} + w^j_{\mathbf{h}} \wedge w^l_F), \\ k^j_{\mathbf{h}} &= dw^j_{\mathbf{h}} + ((w_0)^j_k + w_\pi \delta^j_k) \wedge w^k_{\mathbf{h}}, \end{aligned} \tag{23.2}$$

We note that, since w_0 is $o(m)$-valued, k_0 defined by the above formula is also $o(m)$-valued.

G) We first examine the case when

$$K^j_F = 0, \quad K_\pi = 0, \tag{24.1}$$

which is (by (15) §1 and (21)) equivalent to the conditions:

$$k^j_F = 0, \quad k_\pi = 0. \tag{24.2}$$

We set

$$w_\pi = w_{\pi l} w^l_F, \quad (w_0)^j_k = (w_0)^j_{kl} w^l_F, \quad w^j_{\mathbf{h}} = (w_{\mathbf{h}})^j_l w^l_F. \tag{25}$$

We first write down the condition on w_0, w_π which is equivalent to the condition: $k_F = 0$. In view of the formula for w^j_F in (17.1) we find by calculation that

$$(26)\qquad \begin{aligned} &dw^j_F + q^j_k \wedge w^k_F = 0, \quad \text{where} \\ &q^j_k = q^j_{kl} w^l_F, \quad q^j_{kl} = (p^{-1})^j_{j_1} \frac{\partial p^{j_1}_k}{\partial x^i} p^i_l. \end{aligned}$$

Therefore $k^j_F = 0$ if and only if we can find $A^j_{kl}(y)$ such that

$$(27)\qquad (w_0)^j_k - w_\pi \delta^j_k = q^j_k + A^j_{kl} w^l_F, \quad A^j_{kl} = A^j_{lk}.$$

Since $\underline{h}_{jk}(w_0)^j_i + \underline{h}_{ji}(w_0)^j_k = 0$, we can eliminate w_0 in the above. We thus find that the condition (26) implies that

$$(28)\qquad \begin{aligned} &\underline{h}_{jk} A^j_{kl} + \underline{h}_{jl} A^j_{ki} = r_{kli}, \quad \text{where} \\ &r_{kli} = -(\underline{h}_{jk} q^j_{li} + \underline{h}_{jl} q^j_{ki} + 2\underline{h}_{kl} w_{\pi i}). \end{aligned}$$

As in the case of Riemann geometry, this equation has the unique solution. Namely,

$$(29)\qquad \begin{aligned} A^j_{kl} = &-\frac{1}{2}(q^j_{kl} + q^j_{lk}) + \frac{1}{2}\underline{h}^{jj_1}\underline{h}_{kk_1}(q^{k_1}_{lj_1} - q^{k_1}_{j_1 l}) \\ &+ \frac{1}{2}\underline{h}^{jj_1}\underline{h}_{ll_1}(q^{l_1}_{kj_1} - q^{l_1}_{j_1 k}) - \delta^j_k w_{\pi l} - \delta^j_l w_{\pi k} + \underline{h}^{ji}\underline{h}_{kl} w_{\pi i}. \end{aligned}$$

Therefore, it follows by (26) that

$$(30.1)\qquad \begin{aligned} (w_0)^j_{kl} = &\frac{1}{2}(q^j_{kl} - q^j_{lk}) + \frac{1}{2}\underline{h}^{jj_1}\underline{h}_{kk_1}(q^{k_1}_{lj_1} - q^{k_1}_{j_1 l}) \\ &+ \frac{1}{2}\underline{h}^{jj_1}\underline{h}_{ll_1}(q^{l_1}_{kj_1} - q^{l_1}_{j_1 k}) - \delta^j_l w_{\pi k} + \underline{h}^{ji}\underline{h}_{kl} w_{\pi i}. \end{aligned}$$

We check by calculation that the above w_0 is $o(m)$-valued. We thus find that for an arbitrary choice of w_π there is an unique w_0 for which $k_F = 0$. Recalling the construction of the Levi-Civita connections, in view of (17.3) we may rewrite (30.1) as

$$(30.2)\qquad (w_0)^j_k = (w^\sharp_0)^j_k + H^{ji}_{lk} w_{\pi i} w^l_F, \quad \text{where} H^{ji}_{lk} = \underline{h}^{ji}\underline{h}_{kl} - \delta^j_l \delta^i_k.$$

where $(w^\sharp_0)^j_k$ is the $o(m)$-part of the Levi-Civita Cartan connection of the metric $(1/c(x))(ds^2)_M$.

We see by (23.2) that $k_\pi = 0$ if and only if

$$(31)\qquad (dw_\pi)_{jl} = \frac{1}{2}\big(\underline{h}_{jk}(w^k_{\mathbf{h}})_l - \underline{h}_{lk}(w^k_{\mathbf{h}})_j\big), \quad dw_\pi = (dw_\pi)_{jl} w^j_F \wedge w^l_F.$$

H) It remains to determine $\underline{h}_{jk}(w^k_{\mathbf{h}})_l + \underline{h}_{lk}(w^k_{\mathbf{h}})_j$. The formula for k_0 in (23.2) suggests that we may be able to obtain the above term using k_0 and w_0. In fact, when we set

$$\text{(32)} \qquad d(w_0)^j_k + (w_0)^j_l \wedge (w_0)^l_k = W^j_{kli} w^l_F \wedge w^i_F, \quad W^j_{kli} + W^j_{kil} = 0,$$

we find by calculation that

$$\text{(33)} \quad ((k_0)^j_k)_{il} = W^j_{kil} + \frac{1}{2}(\delta^j_i \underline{h}_{kk_1} w^{k_1}_{\mathbf{h}l} - \delta^j_l \underline{h}_{kk_1} w^{k_1}_{\mathbf{h}i} + \underline{h}_{kl} w^j_{\mathbf{h}i} - \underline{h}_{ki} w^j_{\mathbf{h}l}).$$

Therefore

$$\text{(34)} \qquad ((k_0)^j_k)_{ij} = W^j_{kij} + \frac{1}{2}(2-m)\underline{h}_{kl} w^l_{\mathbf{h}i} - \underline{h}_{ki} w^j_{\mathbf{h}j}.$$

In order to eliminate $w^j_{\mathbf{h}j}$ in the above, we multiply $\underline{h}^{ki}$ and add in k, i. We find

$$\text{(35)} \qquad \underline{h}^{ki}((k_0)^j_k)_{ij} = \underline{h}^{ki} W^j_{kij} + (1-m) w^j_{\mathbf{h}j}.$$

It then follows by calculation that

$$\text{(36)} \qquad \begin{aligned} A^{k_1 i_1}_{ki}\{((k_0)^j_{k_1})_{i_1 j} - W^j_{k_1 i_1 j}\} &= \frac{1}{2}(1-m)\underline{h}_{kl} w^l_{\mathbf{h}i}, \\ A^{jl}_{ki} &= \delta^j_k \delta^l_i + \frac{1}{1-m}\underline{h}_{ki}\mathbf{h}^{jl}. \end{aligned}$$

Therefore we see that $\underline{h}_{kl} w^l_{\underline{h}i} + \underline{h}_{il} w^l_{\underline{h}k}$ is determined by k_0 and W. The condition for k_0 becomes simpler when we note as in the case of Riemann geometry that

$$\text{(37)} \qquad ((k_0)^j_k)_{lj} = ((k_0)^j_l)_{kj}.$$

The above follows by taking the exterior derivative of $0 = k^j_F$ in (23.2) and using the formulas in (23.2). In the end the terms containing $w_{\mathbf{h}}$ cancel out.

We impose the condition:

$$\text{(38)} \qquad ((k_0)^i_j)_{ki} = 0.$$

Since the above condition is equivalent to the condition:

$$\text{(39)} \qquad ((K_0)^i_j)_{ki} = 0, \quad \text{where} (K_0)^i_j = ((K_0)^i_j)_{ki} \Omega^k_F \wedge \Omega^i_F,$$

this is a well defined curvature condition. We find by (36)

$$\text{(40)} \qquad \underline{h}_{kl} w^l_{\mathbf{h}i} + \underline{h}_{il} w^l_{\mathbf{h}k} = \frac{2}{m-1} A^{k_1 i_1}_{ki}\{W^j_{k_1 i_1 j} + W^j_{i_1 k_1 j}\}.$$

Therefore it follows by (31)

$$\underline{h}_{kl} w^l_{\mathbf{h}i} = \frac{1}{2}(dw_\pi)_{ki} + \frac{1}{m-1} A^{k_1 i_1}_{ki} \{W^j_{k_1 i_1 j} + W^j_{i_1 k_1 j}\}. \tag{41}$$

We conclude

(42) Proposition. *For an arbitrary 1-form w_π in* (22.1) *there is an unique conformal Cartan connection* (20), (22) *satisfying the conditions*:

$$K_F = 0, \quad K_\pi = 0, \quad ((K_0)^i_j)_{ki} = 0.$$

w_0 *and* $w_{\mathbf{h}}$ *of the connection is given by* (30.2) *and* (41).

The above connections will be called normal conformal Cartan connections.

I) We next find an expression of the curvatures. For simplicity we set

$$\underline{h}_{kj} w^j_{\mathbf{h}l} = w_{[kl]} + w_{\langle kl \rangle}, \quad w_{[kl]} = w_{[lk]}, \ w_{\langle kl \rangle} = -w_{\langle lk \rangle}. \tag{43}$$

By (31) and (40)

$$w_{\langle kl \rangle} = (dw_\pi)_{kl}, \quad w_{[kl]} = \frac{1}{m-1} A^{k_1 i_1}_{kl} \{W^j_{k_1 i_1 j} + W^j_{i_1 k_1 j}\}. \tag{44}$$

We then find by (33)

$$((k_0)^j_k)_{il} = W^j_{kil} + \frac{1}{2}(H^{js}_{lk}\delta^t_i - H^{js}_{ik}\delta^t_l)(w_{[st]} + w_{\langle st \rangle}). \tag{45}$$

To calculate W^j_{kil}, we see by (30.2)

$$dw^j_{0k} = dw^{\sharp j}_{0k} + H^{ji}_{lk}\, d(w_{\pi i}) \wedge w^l_F + w_{\pi i} H^{ji}_{lk}\, dw^l_F. \tag{46}$$

Set

$$\begin{gathered} d(w_{\pi k}) = (w_{\pi [kl]} + w_{\pi \langle kl \rangle}) w^l_F, \\ w_{\pi [kl]} = w_{\pi [lk]}, \quad w_{\pi \langle kl \rangle} = -w_{\pi \langle lk \rangle}. \end{gathered} \tag{47}$$

Since $dw_\pi = d(w_{\pi l}) \wedge w^l_F + w_{\pi i}\, dw^i_F$, we note by (44) that

$$w_{\langle kl \rangle} = w_{\pi \langle lk \rangle} - w_{\pi i}\frac{1}{2}\big((w^i_{0l})_k - (w^i_{0k})_l - \delta^i_l w_{\pi k} + \delta^i_k w_{\pi l}\big). \tag{48}$$

Therefore

$$\begin{aligned} d(w^j_{0k}) &= d(w^{\sharp j}_{0k}) + H^{ji}_{qk}(w_{\pi [ip]} + w_{\langle pi \rangle}) w^p_F \wedge w^q_F \\ &\quad + w_{\pi i}\Big\{ H^{jl}_{qk}\frac{1}{2}\big((w^i_{0l})_p - w_{\pi p}\delta^i_l - (w^i_{0p})_l + w_{\pi l}\delta^i_p\big) \\ &\qquad + H^{ji}_{lk}\big(w_{\pi p}\delta^l_q - (w^l_{0q})_p\big)\Big\} w^p_F \wedge w^q_F. \end{aligned} \tag{49}$$

We also have by (30.2)

$$w^j_{0l} \wedge w^l_{0k} = w^{\sharp j}_{0l} \wedge w^{\sharp l}_{0k} + w_{\pi i}\{H^{li}_{qk}(w^{\sharp j}_{0l})_p + H^{ji}_{pl}(w^{\sharp l}_{0k})_q + H^{ji}_{pl}H^{lm}_{qk}w_{\pi m}\}w^p_F \wedge w^q_F. \tag{50}$$

Hence by (32) we find that

$$W^j_{kpq} = R^j_{kpq} + \frac{1}{2}\{H^{ji}_{qk}(w_{\pi[ip]} + w_{\langle pi\rangle}) - H^{ji}_{pk}(w_{\pi[iq]} + w_{\langle qi\rangle})\} + w_{\pi i}P^{ij}_{kpq}, \tag{51}$$

where $R^j_{lpq}w^p_f \wedge w^q_F$ is the curvature form of the metric $(1/c(x))(ds^2)_M$, and

$$\begin{aligned} P^{ij}_{kpq} = &\frac{1}{4}H^{jl}_{qk}\big((w^i_{0l})_p - w_{\pi p}\delta^i_l - (w^i_{0p})_l + w_{\pi l}\delta^i_p\big) \\ &- \frac{1}{4}H^{jl}_{pk}\big((w^i_{0l})_q - w_{\pi q}\delta^i_l - (w^i_{0q})_l + w_{\pi l}\delta^i_q\big) \\ &+ \frac{1}{2}H^{ji}_{lk}\big((w^l_{0p})_q - w_{\pi q}\delta^l_p - (w^l_{0q})_p + w_{\pi p}\delta^l_q\big) \\ &+ \frac{1}{2}\big(H^{li}_{qk}(w^{\sharp j}_{0l})_p - H^{li}_{pk}(w^{\sharp j}_{0l})_q + H^{ji}_{pl}(w^{\sharp l}_{0k})_q - H^{ji}_{ql}(w^{\sharp l}_{0k})_p\big) \\ &+ \frac{1}{2}(H^{ji}_{pl}H^{lm}_{qk} - H^{ji}_{ql}H^{lm}_{pk})w_{\pi m}. \end{aligned} \tag{52}$$

Therefore we find by (45) that

$$((k_0)^j_k)_{pq} = R^j_{kpq} + \frac{1}{2}(H^{js}_{qk}\delta^t_p - H^{js}_{pk}\delta^t_q)(w_{[st]} + w_{\pi[st]}) + w_{\pi i}P^{ij}_{kpq}. \tag{53}$$

Summing in $j = q$ in the above, we find by (38)

$$(H^{js}_{jk}\delta^t_p - H^{js}_{pk}\delta^t_j)(w_{[st]} + w_{\pi[st]}) = -2R^j_{kpj} - 2w_{\pi i}P^{ij}_{kpj}. \tag{54}$$

It turns out by (32) that for an indeterminant X_{st} symmetric in s, t

$$Y_{kp} = (H^{js}_{jk}\delta^t_p - H^{js}_{pk}\delta^t_j)X_{st} = (2-m)X_{kp} - \underline{h}_{kp}\underline{h}^{st}X_{st}. \tag{55}$$ Hence

$$X_{st} = K^{kp}_{st}Y_{kp}, \quad \text{where } K^{kp}_{st} = \frac{1}{2-m}\delta^k_s\delta^p_t + \frac{1}{2}\frac{1}{(m-1)(m-2)}\underline{h}_{st}\underline{h}^{kp}. \tag{56}$$

Therefore we find that

$$((k_0)^j_k)_{pq} = R^j_{kpq} - (H^{js}_{qk}\delta^t_p - H^{js}_{pk}\delta^t_q)K^{kl}_{st}R^i_{kli} + w_{\pi i}\tilde{P}^{ij}_{kpq}, \tag{57}$$ where

$$\tilde{P}^{ij}_{kpq} = P^{ij}_{kpq} - (H^{js}_{qk}\delta^t_p - H^{js}_{pk}\delta^t_q)K^{kl}_{st}R^{ir}_{klr}. \tag{58}$$

We find by calculation that

$$w_{\pi i}\tilde{P}^{ij}_{kpq} = 0. \quad \text{Hence} \tag{59}$$

$$((k_0)^j_k)_{pq} = R^j_{kpq} - (H^{js}_{qk}\delta^t_p - H^{js}_{pk}\delta^t_q)K^{kp}_{st}R^r_{kpr}. \tag{60}$$

Note that

$$\begin{aligned}(H^{js}_{qk}\delta^t_p - H^{js}_{pk}\delta^t_q)K^{il}_{st}X_{il} &= \frac{1}{m-2}(H^{jl}_{pk}X_{lq} - H^{jl}_{qk}X_{lp}) \\ &+ \frac{1}{(m-1)(m-2)}(\underline{h}_{qk}\delta^j_p - \underline{h}_{pk}\delta^j_q)\underline{h}^{il}X_{il}.\end{aligned} \tag{61}$$

Then the formula (60) is rewritten as the classical formula for the Weyl tensor:

$$\begin{aligned}((k_0)^j_k)_{pq} = R^j_{kpq} &+ \frac{1}{m-2}(\delta^j_p R_{kq} - \delta^j_q R_{kp} + \underline{h}_{kq}\underline{h}^{jl}R_{lp} \\ &- \underline{h}_{kp}\underline{h}^{jl}R_{lq}) + \frac{R}{(m-1)(m-2)}(\delta^j_p\underline{h}_{kq} - \delta^j_p\underline{h}^{kp}), \quad \text{where}\end{aligned} \tag{62}$$

$$R_{kl} = R^j_{klj}, \quad R = \underline{h}^{kl}R_{kl}. \tag{63}$$

(cf. formula (28.12) Chapter 2, Eisenhart [4], where the chart coframe dx^j is used. We used the orthonormal coframe w^j_F.)

J) There is an a priori ground why the cancelation (59) takes place. This is a reflection of the fact that for normal conformal connections we have $k_\pi = 0$. In fact, for arbitrary $\mathbf{R}^m$-valued function $\beta(x)$ let us consider a Cartan connection $\underline{\Omega}$ given by

$$\underline{w}(x) = \mathrm{Ad}\big(h(1, I, \beta(x))^{-1}\big)w(x). \tag{64}$$

in (18). We see by (15) §1

$$\underline{w}_\pi(x) = w_\pi(x) - \underline{h}_{ij}\beta^j(x)w^i_F(x). \tag{65}$$

By (21) the new curvature is given by

$$\underline{k}(x) = \mathrm{Ad}\big(h(1, I, \beta(x))^{-1}\big)k(x). \tag{66}$$

We find by (15) §1 that this is a conformal Cartan connection and

$$\underline{k}_\pi = k_\pi = 0, \quad \underline{k}_0(x) = k_0(x), \quad \underline{k}_{\mathbf{h}} = k_{\mathbf{h}} + k_0\beta(x). \tag{67}$$

Therefore $\underline{\Omega}$ is a normal connection. When $w_\pi = 0$, we see by (53) that the formula for k_0 is given by (60). We see by the above, when

$w_\pi = \underline{h}_{ij}\beta^j(x)w_F^i(x)$, the formula for k_0 is still given by (60). This means that (59) must be true.

The above formula also prove the followings: Let $\underline{k}_{\mathbf{h}}(x)$ be the $\mathbf{g}_{(1)}$-part of the curvature form when $\underline{w}_\pi = 0$. Then

$$k_{\mathbf{h}}^j = \underline{k}_{\mathbf{h}}^j + \underline{h}^{ki}w_{\pi i}(k_0)_k^j. \tag{68}$$

(69) Proposition. *Any two normal conformal Cartan connections are isomorphic.*

K) We next write down the expression of $k_{\mathbf{h}}$. In view of (68) it is enough to consider the case $w_\pi = 0$. We then find by (41) and (36) that with $w_{\mathbf{h}}^j = w_{\mathbf{h}k}^j w_F^k$

$$\begin{aligned} w_{\mathbf{h}k}^j &= \frac{1}{m-1}\underline{h}^{ji}A_{ik}^{pq}(W_{pqr}^r + W_{qpr}^r) \\ &= \frac{1}{m-1}\underline{h}^{jp}(W_{pkr}^r + W_{kpr}^r) - \frac{2}{(m-1)^2}\delta_k^j\underline{h}^{pq}W_{pqr}^r. \end{aligned} \tag{70}$$

Since $w_{\pi[ip]} = 0$ by (47) and $w_{\langle ip\rangle} = 0$ by (44) when $w_\pi = 0$ we see by (51)

$$W_{kpr}^r = R_{kp}. \tag{71}$$

Therefore we find

$$w_{\mathbf{h}}^j = \frac{2}{m-1}\Big(\underline{h}^{jl}R_{lk} - \frac{1}{m-1}\delta_k^j R\Big)w_F^k. \tag{72}$$

It then follows by (23.2) and (68)

$$\begin{aligned} k_{\mathbf{h}}^j = \frac{2}{m-1}\Big\{&\underline{h}^{jl}(dR_{lk} - R_{li}(w_0)_k^i) + \underline{h}^{rl}R_{lk}(w_0)_r^j \\ &- \frac{1}{m-1}\delta_k^j dR\Big\} \wedge w_F^k + \underline{h}^{ki}w_{\pi i}(k_0)_k^j. \end{aligned} \tag{73}$$

We can also express $k_{\mathbf{h}}^j$ by k_0 and its derivatives, provided $m > 3$. By (21) (or by calculation) we find by (23.2) that

$$d(k_0)_k^j = (k_0)_l^j \wedge (w_0)_k^l - (w_0)_l^j \wedge (k_0)_k^l + \underline{h}_{kl}(k_{\mathbf{h}}^j \wedge w_F^l - w_F^j \wedge k_{\mathbf{h}}^l). \tag{74}$$

Noting that for $\alpha = \alpha_{jk}\gamma^j \wedge \gamma^k$ with $\alpha_{jk} = -\alpha_{kj}$ and $\beta = \beta_l\gamma^l$

$$\alpha \wedge \beta = \frac{1}{3}(\alpha_{jk}\beta_l + \alpha_{kl}\beta_j + \alpha_{lj}\beta_k)\gamma^j \wedge \gamma^k \wedge \gamma^l, \tag{75}$$

we find by (38) that

$$
\begin{aligned}
3((dk_0)^j_k)_{pqj} &= ((k_0)^j_l)_{pq}((w_0)^l_k)_j - ((k_0)^l_k)_{pq}((w_0)^j_l)_j \\
&\quad - ((k_0)^l_k)_{qj}((w_0)^j_l)_p - ((k_0)^l_k)_{jp}((w_0)^j_l)_q \\
&\quad + \underline{h}_{kl}\{(k^j_{\mathbf{h}})_{qj}\delta^l_p + (k^j_{\mathbf{h}})_{jp}\delta^l_q - (m-3)(k^l_{\mathbf{h}})_{pq}.
\end{aligned}
\tag{76}
$$

Therefore

$$
(3-m)k^i_{\mathbf{h}} - 2(k^j_{\mathbf{h}})_{pj} w^p_F \wedge w^i_F = \tilde{d}k^i_0, \quad \text{where} \tag{77}
$$

$$
\begin{aligned}
(\tilde{d}k^i_0)_{pq} w^p_F \wedge w^q_F &= \underline{h}^{il}\{3((dk_0)^j_l)_{pqj} w^p_F \wedge w^q_F - ((w_0)^r_l)_j (k_0)^j_r \\
&\quad + ((w_0)^j_r)_j (k_0)^r_l + 2(w_0)^j_r \wedge ((k_0)^r_l)_{pj} w^p_F\}.
\end{aligned}
\tag{78}
$$

We then conclude that

$$
k^i_{\mathbf{h}} = \frac{1}{3-m}\Big\{\tilde{d}k^i_0 + \frac{1}{2-m}(\tilde{d}k^j_0)_{pj} w^p_F \wedge w^i_F\Big\}. \tag{79}
$$

For future use we rewrite the formula for $\tilde{d}k^i_0$ in (78). We set, with the proper symmetry, $k_0 = (k_0)_{ij} w^i_F \wedge w^j_F$, $dk_0 = (dk_0)_{ijl} w^i_F \wedge w^j_F \wedge w^l_F$, $df = (df)_i w^i_F$ for a function f. Then

$$
\begin{aligned}
3(dk_0)_{ijl} &= (d(k_0)_{jl})_i + (d(k_0)_{li})_j + (d(k_0)_{ij})_l + (k_0)_{ir}(w^r_{0l})_j \\
&\quad - (k_0)_{ir}(w^r_{0j})_l + (k_0)_{jr}(w^r_{0i})_l - (k_0)_{jr}(w^r_{0l})_i \\
&\quad + (k_0)_{lr}(w^r_{0j})_i - (k_0)_{lr}(w^r_{0i})_j \\
&\quad + 2\{(k_0)_{ij}(w_\pi)_l + (k_0)_{jl}(w_\pi)_i + (k_0)_{li}(w_\pi)_j\}.
\end{aligned}
\tag{80}
$$

Therefore by (38)

$$
\begin{aligned}
3(dk^l_{0i})_{pql} &= (d(k^l_{0i})_{pq})_l + (k^l_{0i})_{pr}(w^r_{0l})_q - (k^l_{0i})_{pr}(w^r_{0q})_l \\
&\quad + (k^l_{0i})_{qr}(w^r_{0p})_l - (k^l_{0i})_{qr}(w^r_{0l})_p + 2(k^l_{0i})_{pq}(w_\pi)_l.
\end{aligned}
\tag{81}
$$

We then find by (78)

$$
\begin{aligned}
\underline{h}_{il}\,\tilde{d}k^l_0 &= (d(k^l_{0i})_{pq})_l w^p_F \wedge w^q_F - (w^r_{0i})_l (k_0)^l_r + (w^j_{0l})_j (k_0)^l_i \\
&\quad + 2(w^r_{0p})_l w^p_F \wedge (k^l_{0i})_{qr} w^q_F + 2(w_\pi)_l k^l_{0i}.
\end{aligned}
\tag{82}
$$

J) We will show that $k_{\mathbf{h}}$ is also obtained by the conformal covariant derivatives of K_0. We first recall the definitions. This is valid for any principal H-bundle E with a Cartan connection Ω (18) given in terms of a local trivialization of E. We are considering any homogenous space G/H.

Consider a curve $\mathbf{f}_t = (x_t, h_t)$ in E. We denote its tangent vectors $\dot{\mathbf{f}}_t$ by

$$(83) \qquad (\dot{x}_t, \dot{h}_t) \quad \text{where} \quad \dot{x}_t = dx/dt,\ h_{t+\epsilon} \equiv h_t(I + \epsilon \dot{h}_t)$$

(mod. ϵ^2). $\dot{h}_t$ is $\mathbf{h}$-valued. Let $\Omega(\dot{\mathbf{f}}_t)$ be the evaluation of Ω at $\dot{\mathbf{f}}_t$. Then

$$(84) \qquad \Omega(\dot{\mathbf{f}}_t) = \mathrm{Ad}((h_t)^{-1}) w(x_t, dx(\dot{x}_t)) + \dot{h}_t.$$

$\mathbf{f}_t$ is called the parallel displacement of $\mathbf{f}_0$ over the curve $x(t)$ in M when

$$(85) \qquad \rho_{\mathbf{h}} \Omega(\dot{\mathbf{f}}_t) = 0.$$

Clearly, given $x(t)$ and $\mathbf{f}_0$ there is an unique parallel displacement. Namely, h_t is obtained by solving the ordinary differential equation:

$$(86) \qquad \dot{h}_t = -\mathrm{Ad}((h_t)^{-1}) w_H(x_t, dx(\dot{x}_t)).$$

Let $\mathbf{f}_t$ be a parallel displacement and h_1 be a fixed element in H. Then $R_{h_1}\mathbf{f}$ is also a parallel displacement. Hence it is enough to consider the case: $\mathbf{f}_0 = (x_0, I)$.

Let $\mathcal{X}_t$ be a vector field along the curve $\mathbf{f}_t$. When we express $\mathcal{X}_t = (X_t, \dot{\phi}_t)$ with $\dot{\phi}_t \in \mathbf{h}$ as in (83), $\Omega(\mathcal{X}_t) = \mathrm{Ad}((h_t)^{-1}) w(x_t, dx(X_t)) + \dot{\phi}_t$. We say that $\mathcal{X}_t$ is the parallel displacement of $\mathcal{X}_0$ along $\mathbf{f}_t$ when for all t

$$(87) \qquad \Omega(\mathcal{X}_t) = \Omega(\mathcal{X}_0).$$

This means that X_t is determined by the equation:

$$(88.1) \qquad \begin{aligned} &\rho_{\mathbf{g}/\mathbf{h}} \mathrm{Ad}((h_t)^{-1}) w_F(x_t, dx(X_t)) \\ &= \rho_{\mathbf{g}/\mathbf{h}} \mathrm{Ad}((h_0)^{-1}) w_F(x_0, dx(X_0)), \end{aligned}$$

and $\dot{\phi}_t$ is determined by the equation:
(88.2)

$$\begin{aligned} &\mathrm{Ad}((h_t)^{-1}) w_H(x_t, dx(X_t)) + \rho_{\mathbf{h}} \mathrm{Ad}((h_t)^{-1}) w_F(x_t, dx(X_t)) + \dot{\phi}_t \\ = \ &\mathrm{Ad}((h_0)^{-1}) w_H(x_0, dx(X_0)) + \rho_{\mathbf{h}} \mathrm{Ad}((h_0)^{-1}) w_F(x_0, dx(X_0)) + \dot{\phi}_0. \end{aligned}$$

Let $\mathcal{V} = (V, \dot{\psi})$ be a vector field on E, where $V = V^j(x,h)\partial/\partial x^j$, $\dot{\psi} = \dot{\psi}(x,h) \in \mathbf{h}$. Pick a tangent vector $\dot{x}_0$ of M at x_0 and $\mathbf{f}_0 \in E$ over x_0. By a conformal covariant derivative at $\mathbf{f}_0$ of $\mathcal{V}$ to the direction $\dot{x}_0$ is defined as follows: Take a curve x_t in M such that the tangent vector at $t = 0$ is $\dot{x}_0$. Let $\mathbf{f}_t$ be the parallel displacement of $\mathbf{f}_0$ and $\mathcal{V}_t$ be the parallel displacement of $\mathcal{V}_{\mathbf{f}_0}$ along $\mathbf{f}_t$. Then

$$(89) \qquad \nabla_{\dot{x}_0} \mathcal{V}_{\mathbf{f}_0} = \lim_{t \to 0} \frac{1}{t}(\mathcal{V}_{\mathbf{f}_t} - \mathcal{V}_t).$$

Let Θ be a differential g-form on E. For a vector field $\mathcal{X}$ on E we define the covariant derivative $\nabla_{\mathcal{X}}\Theta$ of T by $\mathcal{X}$ as follows: For any vector fields $\mathcal{V}_1, \dots, \mathcal{V}_g$

$$\begin{aligned} \nabla_{\mathcal{X}}\Theta(\mathcal{V}_1, \dots, \mathcal{V}_g) = \mathcal{X}(\Theta(\mathcal{V}_1, \dots, \mathcal{V}_g)) - \Theta(\nabla_{\mathcal{X}}\mathcal{V}_1, \dots, \mathcal{V}_g) \\ - \dots - \Theta(\mathcal{V}_1, \dots, \nabla_{\mathcal{X}}\mathcal{V}_g). \end{aligned} \tag{90}$$

Let e^γ be a base of $\mathbf{g}$. Set $\Omega = \Omega_\gamma e^\gamma$. Denote by $\dots, \mathcal{W}^\gamma, \dots$ the base of vector fields dual to $\dots, \Omega_\gamma, \dots$. For any parallel displacement $\mathbf{f}_t$ of a frame we see clearly $(\mathcal{W}^\gamma)_{\mathbf{f}_t}$ is the parallel displacement of $(\mathcal{W}^\gamma)_{\mathbf{f}_0}$ along $\mathbf{f}_t$. Therefore for any vector field $\mathcal{X}$

$$\nabla_{\mathcal{X}}\Theta(\mathcal{W}_1, \dots, \mathcal{W}_g) = \mathcal{X}(\Theta(\mathcal{W}_1, \dots, \mathcal{W}_g)). \tag{91}$$

We now consider the case of a normal conformal Cartan connection. We set $K_0 = \rho_{(0)} \operatorname{Ad}(h^{-1})k_0$, where $\rho_{(0)}$ is the projection to the degree 0 part of the grading (12) §1, and calculate $\nabla_{\mathcal{W}_l} K_0$. We have by (15) §1

$$w_F^j = (p^{-1})_k^j(x)dx^k, \quad w_F^l(x, dx(\mathcal{W}_j)) = a^{-1}t_j^l, \tag{92}$$

$$\mathcal{W}_l = a^{-1}p(x)_r^i t_l^r \frac{\partial}{\partial x^i} + W_{l\pi}\pi + \{W_{l0}\} + \{W_{l\mathbf{h}}\}_{(1)}, \tag{93.1}$$

where

$$W_{l\pi}(x, h) = \underline{h}_{kr} t_l^k \beta^r - w_{\pi i}(x) a^{-1} t_l^i, \tag{93.2}$$

$$\begin{aligned} &W_{l0}(x, h) = t^*\big(\beta \otimes (t_l)^* - t_l \otimes \beta^* - a^{-1}t_l^i(w_0)_i(x)\big)t, \\ &\text{with } t_l = (t_l^1, \dots, t_l^m). \end{aligned} \tag{93.3}$$

The above means that as differential operators

$$W_{l\pi}(x, I) = -w_{\pi l}(x)\Big(\frac{\partial}{\partial a}\Big)_{a=1}, \tag{93.4}$$

$$W_{l0}(x, I) = -(w_{0k}^j)_l(x)\Big(\frac{\partial}{\partial t_k^j}\Big)_{t=I}, \tag{93.5}$$

We then find that

$$K_0(\mathcal{W}_p, \mathcal{W}_q) = \frac{1}{2}a^{-2}(t_p^j t_q^r - t_q^j t_p^r)t^*(k_0)_{jr}t. \tag{94}$$

By calculation we now find by (82) and (91) that

$$\underline{h}_{il}\, \tilde{d}k_0^l = ((\nabla_{\mathcal{W}_l} K_0)_i^l)_{(x,I)}. \tag{95}$$

K) The normal conformal Cartan connections, defined locally, depend on arbitrary functions w_π. They determine a unique class up to local isomorphism. In order to define globally a normal conformal Cartan connection, we have to choose w_π for each local trivialization of the conformal frame bundle in such a way that they match up on the intersections on the domains of trivializations.

Let $(x, \underline{h})$ be a local trivialization. Then for a $\mathbf{h}$-valued function $h(x)$ we have

$$h = h(x)\underline{h}. \tag{96}$$

For a normal conformal Cartan connection Ω we have two expressions:

$$\Omega = \mathrm{Ad}(h^{-1})w + h^{-1}\,dh = \mathrm{Ad}(\underline{h}^{-1})\underline{w} + \underline{h}^{-1}\,d\underline{h}. \tag{97}$$

Therefore we find by (15) §1 and (30) §1

$$\underline{w}^i_F = a(x)t^*(x)^i_j w^j_F(x), \tag{98.1}$$

$$\underline{w}_\pi(x) = w_\pi(x) - a(x)\underline{h}_{jk}\beta(x)^k w^j(x) + d\log a(x), \tag{98.2}$$

where as in (10) §1 we set $h(x) = h\big(a(x), t(x), \beta(x)\big)$.

To find such w_π as above we recall that our chart (x, h) is induced by a section $\mathbf{f}(x) = \big(x, p^i_j(x), c_1(x), \ldots, c_m(x)\big)$ (cf. (12)–(13)) of the frame bundle. We also have $c_0(x) = \underline{h}^{ij} g_{kl}(x) p^k_i(x) p^l_j(x)$. The chart $(x, \underline{h})$ is induced by $\underline{\mathbf{f}}(x) = R_{h(x)}\mathbf{f}(x) = \big(x, \underline{p}^i_j(x), \underline{c}_1(x), \ldots, \underline{c}_m(x)\big)$. Hence by (11)

$$\begin{aligned} \underline{c}_j(x) &= \frac{1}{a(x)^3} t^l_j(x) c_l(x) - \frac{2}{a(x)^2}\underline{h}_{il} t^i_j(x)\beta^l(x)c_0(x), \\ \underline{c}_0 &= \frac{1}{a(x)^2} c_0(x). \end{aligned} \tag{99}$$

We then find by (98.1) that

$$w_\pi = -\frac{1}{2} d\log c_0 + \frac{1}{2}\frac{c_l}{c_0} w^l_F \tag{100}$$

obeys the transformation law (98). When the above w_π is chosen we call it the global normal conformal Cartan connection.

References

[1] E. Cartan, *Notices sur les Travaux Scientifiques*, Oeuvres, t. I.1. 17–90.

[2] S. S. Chern, *The geometry of G-structures*, Bull. Amer. Math. Soc., **72** (1966), 162–219.

[3] C. Ehresmann, *Les connexions infinitesimales dans un espace fibre differentiable*, in Colloque de Topologie. [Bruxelles,1950], 29–55.

[4] L. S. Eisenhart, Riemannian Geometry, Princeton University Press, 1950.

[5] S. Kobayashi, Tansformation groups in differential geometry, Springer, 1972.

[6] M. Kuranishi, *Cartan connections and CR structures with non-degenerate Levi-forms*, Ast. Soc. Math. de France (1985), 273–288.

[7] M. Kuranishi, *CR geometry and Cartan geometry*, Forum Math., **7** (1995), 147–206.

[8] Y. Liu, *Cartan geoemtry, CR geometry and complex geometry*, Thesis, Columbia University, 1998.

[9] R. Miyaoka, *Lie contact structures and conformal structures*, Kodai Math. J., **14** (1991), 42–71.

[10] T. Morimoto, *Sur la probléme d'équivalence des structures géométriques*, Japan. J. Math., **9-2** (1983), 293–372.

[11] T. Morimoto, *Geometric structures on filtered manifolds*, Hokkaido Math. J., **22** (1993), 263–347.

[12] T. Ochiai, *Geometry associated with semi-simple flat homogenous spaces*, Trans. A.M.S., **152** (1970), 1–33.

[13] K. Ogiue, *Theory of conformal connections*, Kodai Sem. Rep., **19** (1967), 193–224.

[14] N. Tanaka, *On differential systems, graded Lie algebras and pseudo-groups*, J. Math. Kyoto Univ., **10** (1970), 1–82.

[15] N. Tanaka, *On nondegenerate real hypersurfaces, graded Lie algebras and Cartan connections*, Japan. J. Math., **2** (1976), 131–190.

[16] N. Tanaka, *On the equivalence problems associated with simple graded Lie algebras*, Hokkaido Math. J., **8** (1979), 23–84.

[17] K. Yamaguchi, *Differential systems associated with simple graded Lie algebras*, Advanced Studies in Pure Math., **22** (1993), 413–494.

[18] H. Weyl, Zur Infinitesimalgeometrie: Einordung der projectiven und konformen Ausfassung, Göttinger Nachrichten (1912), pp. 99–112.

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Differential Algebra and Differential Geometry

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§1. Introduction

There are now two theories devoted to partial differential equations in the algebraic or analytic domain:

On one side, the theory of involutive differential systems, based on the Cartan-Kähler theorem, and developed namely by Matsushima, Kuranishi, Guillemin-Singer-Sternberg, Quillen, Goldschmidt. This theory is of constant use in differential geometry, f.i. in the study of the "equivalence problems" in the sense of E. Cartan.

On the other side, the "differential algebra" of Ritt, Kolchin, and others, which studies the differential ideals and their properties of finiteness, dimension, etc. cf. [Ri 1], [Ko]. There is a nice application by Buium [Bu] to some problems of algebraic geometry on functions fields. A part from this application, this theory seems to have had practically no contact with geometry, especially with differential geometry; compare f.i. the bibliographies of [Ko] and [B-C-G 3]: their intersection is empty; see however [Po].

It seems to me that a mutual interaction should be useful for both theories. For instance, with the help of the ideas of Ritt, one can prove rather easily the "generic involutiveness" of analytic systems of p.d.e.'s; see a precise statement in §3. Hopefully, the result could be useful in several contexts, namely in the theory of Lie groupoids and in differential Galois theory; I will develop this point elsewhere.

On the opposite side, I mention only the following fact: differential algebraists use classically Riquier-Janet theory of "passive orthonormic systems" rather than Cartan involutiveness. But it seems that they are now becoming aware of this last theory; see f.i. [Se].

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§2. D-varieties

(2.1). — For a general theory of analytic p.d.e.'s, one needs a context generalizing both analytic spaces and D-modules (which correspond to the linear case). Here, I will describe the formalism adopted in [Ma 1]; later, I discuss briefly some other possibilities.

A few words of informal explanations: we are interested to systems of equations of the form $f_k(x_i, \partial^\alpha y_j) = 0$, $1 \leq i \leq n$, $1 \leq j \leq p$, $\alpha = (\alpha_1, \ldots, \alpha_n)$, $|\alpha| \leq l$ with, as usual $\partial_i = \frac{\partial}{\partial x_i}$, $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$; the f_k are supposed analytic in all the variables, which we denote x_i, y_j^α.

In the course of the study, one has to differentiate the equations, with the usual derivations: $D_i f = \frac{\partial f}{\partial x_i} + \sum \frac{\partial f}{\partial y_j^\alpha} y_j^{\alpha+\varepsilon_i}$, $\varepsilon_i = (0, \ldots, 1, \ldots, 0)$. We note the following fact: suppose f of order $\leq k$, i.e. the y_j^α occuring in f verify $|\alpha| \leq l$. Then, when we differentiate f as many times as we want, the y_j^α, $|\alpha| \geq l+1$ occur only *in polynomial form.* Now, by the usual trick of adding some derivatives as new functions, we can suppose that all our equations are polynomial in the y_j^α, $|\alpha| \geq 1$.

This is the point of view adopted by Ritt himself in [Ri 2]. However, he considers only local situations, and we want global objects; this explains the definitions below.

In this pages, as in [Ma 1], I will call (analytic) *variety* what is called in the literature $\mathbb{C}$-analytic space in the sense of Grothendieck [Gr]; a priori, I will not suppose a variety smooth, and not even reduced (= without nilpotent elements). However, in the applications to differential equations, only the reduced case will be really interesting.

Let Y be a variety; I note $|Y|$ the underlying topological space, and $\mathcal{O}_Y$ the structural sheaf. By definition, an *affine variety Z over Y* is defined by the ringed space $(|Y|, \mathcal{A})$, with $\mathcal{A}$ and $\mathcal{O}_Y$-algebra of locally finite presentation, i.e. verifying the following property: over a small open set $U \subset |Y|$, one has

$$\mathcal{A} = \mathcal{O}_Y[t_1, \ldots, t_n]/(f_1, \ldots, f_m), \quad \text{with} \quad f_i \in \Gamma(U, \mathcal{O}_Y[t_1, \ldots, t_n]).$$

If we have two such varieties over Y; $Z = (|Y|, \mathcal{A})$ and $Z' = (|Y|, \mathcal{A}')$, a morphism $Z \to Z'$ is, of course, a morphism of $\mathcal{O}_Y$-algebras: $\mathcal{A}' \to \mathcal{A}$; if Z' is affine over Y', one defines in the same way a morphism $Z \to Z'$ over a morphism on $Y \to Y'$. I will write often $\mathcal{O}_Z$ instead of $\mathcal{A}$, although this is a little bit confusing ($\mathcal{O}_Z$ is a sheaf on $|Y|$); on the other hand, I denote Z^{an} the analytic space specan Z [Ho]. If π denotes the projection $|Z^{\text{an}}| \to |Y|$, one has a natural map $\pi^{-1}\mathcal{O}_Z \to \mathcal{O}_{Z^{\text{an}}}$. If we have a morphism $Z \to Z'$ of affine varieties over Y, we say that "Z is an affine variety over Z'". As usual in algebraic geometry, we say that "$Z \to Z'$

is dominant" if the corresponding morphism of (sheaves of) rings is *injective*. Note that this does not imply that $Z^{\mathrm{an}} \to Z'^{\mathrm{an}}$ is surjective: this is only true generically, in an obvious sense.

(2.2). — Now, let Y_0 be a variety: a "projective system of affine varieties over Y_0" is a collection $Y_i \xrightarrow{\psi_i} Y_0$ of affine varieties over Y_0 $(i = 1,\ 2, \ldots)$, with a family of morphisms $Y_i \xrightarrow{\pi_i} Y_{i-1}$; one has $\psi_{i-1}\pi_i = \psi_i$ $(i \geq 1)$, and $\psi_1 = \pi_1$; we say that $\{Y_i\}$ is an *affine provariety over* Y_0 if the morphisms φ_i are dominant. If we have two affine provarieties $\{Y_i\}$, $\{Z_i\}$, a strict morphism is defined by an analytic morphism $u_0 \colon Y_0 \to Z_0$ and morphisms $u_i \colon Y_i \to Z_i$ over u_0, with the condition of commutativity of the obvious diagram

$$\begin{array}{ccc} Y_i & \longrightarrow & Y_{i-1} \\ \downarrow & & \downarrow \\ Z_i & \longrightarrow & Z_{i-1} \end{array}$$

(I use the word "strict", since there is a weaker notion of morphisms; cf. loc. cit., or below).

(2.3). — Now, I can define a D-variety, as the object naturally associated to a system of analytic p.d.e's (including all its prolongations); it is defined by the following datas

i) A variety X, which is supposed non singular (and, in particular, reduced)
ii) A variety Y_0, may be singular, provided with a morphism $p \colon Y_0 \to X$
iii) An affine provariety $Y = (Y_i, \pi_i)$ over Y_0; we note $\mathcal{O}_Y = \varinjlim \mathcal{O}_{Y_i}$
iv) A derivation (or "connection") $D \colon \mathcal{O}_Y \to p^{-1}\Omega^1_X \otimes_{p^{-1}\mathcal{O}_X} \mathcal{O}_Y$, where Ω^1_X denotes, the differential 1-forms over X.

These datas are submitted to conditions which will be described below. Before to do it, we need a definition: if we have two such datas (X, Y, D) and (X, Z, D), with same X, a strict D-morphism is defined by a morphism $u_0 \colon Y_0 \to Z_0$ of analytic varieties, commuting with the projections over X, and a strict morphism $u \colon Y \to Z$ over u_0; these datas should commute with the derivation D.

Suppose now (X, Y, D) given; let X' (resp. Y_0') an open subvariety of X (resp. Y_0), with $p|Y_0'| \subset |X'|$; one defines in an obvious way the restriction (X', Y', D) of (X, Y, D) to (X', Y_0').

With these definitions, a D-variety is a system (X, Y, D) with the following property: for every $y \in |Y_0|$, one can find a pair (X', Y_0'), with $y \in |Y_0'|$ such that the restriction of (X, Y, D) à (X', Y_0') is strictly isomorphic to a model which we will now describe.

To do that, we give

i) An open set $U \subset \mathbb{C}^n$, of coordinates $(x_1, \dots, x_n)$

ii) An open set $V \subset \mathbb{C}^n$, of coordinates $(y_1, \dots, y_p)$.

We put $\mathcal{A}_0 = \mathcal{O}_{U \times V}$, the sheaf of holomorphic functions on $U \times V$. For $l \geq 1$, we put $\mathcal{A}_l = \mathcal{O}_{U \times V}[y_j^\alpha]$, $\alpha = (\alpha_1, \dots, \alpha_n)$, $1 \leq |\alpha| \leq l$ and $\mathcal{A} = \varinjlim \mathcal{A}_l$.

On $\mathcal{A}$, one has a natural derivation $Df = \sum dx_i \otimes D_i f$, D_i as in (2.1).

Now, let $\mathcal{J}$ be a sheaf of ideals of $\mathcal{A}$, which is *differential*, e.g. stable by the D_i's and *pseudocoherent*, e.g. the $\mathcal{J}_l = \mathcal{J} \cap \mathcal{A}_l$ are coherent.

Now the model is as follows: one takes $X = U$, $Y_0 =$ the closed analytic subspace of $U \times V$ defined by $\mathcal{J}_0$; p is induced by the projection $U \times V \to U$; for $i \geq 1$, Y_i is the affine variety over Y_0 defined by $\mathcal{A}_l / \mathcal{J}_l$. Finally, D is defined in the obvious way by the "D" given on $\mathcal{A}$.

A D-variety is reduced if all the Y_i are reduced $=$ the corresponding sheaves have no nilpotent element. If we have a D-variety (X, Y, D), $Y = \{Y_i\}$, one defines naturally its "reduction" (X, Y^{red}, D), with $Y^{\text{red}} = \{Y_i^{\text{red}}\}$. Using the local model, the reader will find the following interpretation of a reduced D-variety, in terms of differential equations: the points of $|Y_l^{\text{an}}|$ are the jets of order l of solutions, and the points of $|Y^{\text{an}}| = \varprojlim |Y_l^{\text{an}}|$ are the formal solutions. Note that the maps $|Y_l^{\text{an}}| \to |Y_{l-1}^{\text{an}}|$ are only generically surjective; this explain the interest of the results of the next section.

§3. Formal integrability and generic involutiveness

To express these properties, it is simpler to work in a local model: so, let U, V, $\mathcal{A}$ and $\mathcal{J}$ as before, and suppose that $\mathcal{J}$ is reduced, i.e. the $\mathcal{J}_l$ are equal to their radical. One has the following theorems

Theorem 3.1. *Let $U' \subset U$ and $V' \subset V$ be polycylinders relatively compacts in U and V; then there exists $l \geq 1$ and $f \in \Gamma(U' \times V', \mathcal{A}_l)$ with the following properties*

i) *On $U' \times V'$, f is injective on $\mathcal{A}/\mathcal{J}$.*

ii) *Outside of $f = 0$, $\mathcal{J}_l$ is involutive, and the $\mathcal{J}_k$, $(k \geq l+1)$ are the prolongation of $\mathcal{J}_l$.*

For the notion of involutiveness (smoothness+formal integrability+ acyclicity), I refer f.i. to Goldschmidt [Go]; I leave it to the reader to translate these notions in the present context; this translation can be made in terms of $Y_k^{\mathrm{an}} - \{f = 0\}$, or more precisely in terms of $\mathcal{A}[f^{-1}]$ and $\mathcal{J}[f^{-1}]$.

Theorem 3.2. *An increasing sequence of differential pseudocoherent and reduced ideals of $\mathcal{A}$ is stationary on every relatively compact $U' \times V' \subset U \times V$.*

The first theorem express the "generic involutiveness" of reduced differential ideals; the second theorem is the version in our context of the finiteness theorem of Ritt-Raudenbush. We note that (3.2) has been already proved by Ritt [Ri 2] in the case where we take germs at a point $a \in U \times V$.

Theorems 3.1 and 3.2 are proved simultaneously; the main lines of the proof can be found in [Ma 2]; complete proofs will be given later. Roughly speaking, the idea is the following: we take for U' and V' *closed* polydiscs, instead of open ones; according to Cartan-Oka theorems, $\mathcal{J}$ is determined on $U' \times V'$ by its global sections; and, according to Frisch theorem, the $\Gamma(U' \times V', \mathcal{A}_l)$ are *noetherian rings*; so, we have only to study reduced differential ideals $\mathfrak{p}$ of $\Gamma(U' \times V', \mathcal{A})$; one proves successively the following results, which imply easily (3.1) and (3.2):

i) Theorem 3.1 is true when $\mathfrak{p}$ is prime.
ii) Any increasing sequence of reduced differential ideals $\mathfrak{p}$ is stationary.
iii) Any such $\mathfrak{p}$ is a finite intersection of primes.

The main point is i). Then ii) follows by an argument of differential algebra to be found, f.i. in [Ka]. Finally, ii) $\Rightarrow$ iii) is standard.

§4. General morphisms

In many problems, one has to consider two kinds of transformations which cannot be represented by the "strict morphisms" considered in §2.

A) Transformations of the type $z_k = f_k(x, y_j^\alpha)$ [and, of course $z_k^\alpha = D^\alpha f_k(x, y_j^\beta)$, $D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}$]. These transformations are called classically "Lie Bäcklund transformations".

B) Change of independent variables, f.i. Legendre transformation where y' is taken as the new independent variable; this is more generally the case when the system is given as an exterior differential system "with independence condition" in the sense of [B-C-G 3].

Concerning **A)**, let me first mention that these transformations are very simple to express in a more special context, the "affine D-varieties": they are given by families (X, Y, D), with X $\mathbb{C}$-analytic smooth, and the system $X \leftarrow Y_0 \leftarrow \cdots \leftarrow Y_l \leftarrow \cdots$ an affine provariety over X; the local models are given by pseudocoherent differential ideals $\mathcal{J}$ of $\mathcal{O}_X[y_j^\alpha]$, $|\alpha| \geq 0$. [In other words, these "varieties" represent differential systems which are polynomial in all the $\partial^\alpha y_j$. This context is sufficient for many applications; but, f.i. it would not contain the equation $y' = e^y$.]

If we have another affine D-variety (X, Z, D), with the same basis X, a Lie-Bäcklund transformation or "morphism" $(X, Y, D) \to (X, Z, D)$ is simply given by a morphism of $\mathcal{O}_X$-algebras $u\colon \mathcal{O}_Z \to \mathcal{O}_Y$, commuting with D. In the interesting cases there will be an $l \geq 0$ such that $u(\mathcal{O}_{Z_0}) \subset \mathcal{O}_{Y_l}$, and therefore $u(\mathcal{O}_{Z_k}) \subset \mathcal{O}_{Y_{l+k}}$, $k \geq 0$ (use commutation with D); we will say that "u is of order $\leq l$". One can express this in another way; call $Y(l)$ the affine system over X defined by $Y(l)_k = Y_{k+l}$; then a morphism of order $\leq l$, $(X, Y, D) \to (X, Z, D)$ is simply defined by a strict morphism $(X, Y(l), D) \to (X, Z, D)$; if $m \geq l$, a morphism of order on $(X, Y(m), D) \to (X, Z, D)$ is identified with the preceding one if it is obtained by composition with the obvious morphism $(X, Y(m), D) \to X, Y(l), D)$ given by the identity on the structure sheaf (note that both spaces have the same structure sheaf).

In the context of D-varieties, I will copy the last procedure: I define $Y(l)$ by $Y(l)_0 = Y_l^{\mathrm{an}}$, the "analytic spectrum" of Y_l, and $Y(l)_k = Y_l^{\mathrm{an}} \times_{Y_l} Y_{l+k}$, the "analytisation up to order l" (see [Ma 1] for more details). Then the morphisms are defined as the previous morphisms of finite order. This analytisation procedure is a little bit unpleasant; but, due to the good properties of analytisation, things behave in a reasonable good way. For instance, one can prove that the characteristic variety is invariant, outside of the zero section, by general isomorphisms cf. [Ma 1]; this generalizes the well-known result of "independence of the filtration" of the characteristic variety in the linear case (= in the theory of D-modules).

Concerning **B)**, the point of view of affine D-varieties is obviously irrelevant, since any change of the independent variables would destroy the affine structure. On the other hand, to include such changes, the category of D-varieties is "suitable" in the following sense: it has to be enlarged, but no new local model is needed; cf. [Ma 1], §4; this is the main reason for which I have adopted this point of view.

Another point of view which is also adapted to **A)** and **B)** consists in a separation of "analytic" and "algebraic" variables in Y_0. More precisely, one takes Y_0' analytic over X, and Y_0 affine over Y_0', then $Y_l \to \cdots \to Y_1 \to Y_0 \to Y_0'$ an affine provariety; the local models are made with such Y's, like in the preceding cases (I omit the details); here, $Y(l)$ is defined in the following way: $Y(l)_0' = Y_0'$; $Y(l)_k = Y_{l+k}$, $k \geq 0$; no analytisation is required.

One could improve also this model by taking Y_0 algebraic over Y_0', e.g. a relative schema in the sense of [Ha] (this is made by gluing affine models, as schemes are defined by gluing affine varieties or schemes; but here the "gluing" is more sophisticated, and require 2-categories; therefore the simplicity of morphism is "compensated" by a greater difficulty of definition). Of course, as in local models, $Y_0 \to Y_0'$ is affine, and there is here nothing new locally; f.i. the results of §3 are still true.

Generally speaking, it seems to me that the "good" definition of D-varieties one should take depends on the problem to be studied. I will give an example in a forth coming paper about "Lie groupoids", i.e. p.d.e's in the space of *invertible* jets $X \to X$, whose solutions form a groupoid (the invertibility of jets forces to change slightly the previous definitions). Here, the generic involutiveness shows that, at the general points, they coincide with the "infinite groups" of Lie and Cartan. But the consideration of singular points seems to me very important, and essentially overlooked in the literature.

Literature

[B-C-G 3] R. Bryant, S. S. Chern, R. Gardner, H. Goldschmidt and P. Griffiths, Exterior differential systems, MSRI Publ., **18**, Springer, 1992.

[Bu] A. Buium, Differential algebra and diophantine geometry, Hermann, 1994.

[Go] H. Goldschmidt, *Integrability criteria for systems of non-linear partial differential equations*, J. Differential Geom., **1** (1967), 269–307.

[Gr] A. Grothendieck, *Techniques de construction en géométrie analytique*, Séminaire H. Cartan, **13** (1960/61), exposés 7–17, Benjamin, 1967.

[Ha] M. Hakim, Topos annelés et schémas relatifs, Springer, 1972.

[Ho] C. Houzel, *Géométrie analytique locale*, Séminaire H. Cartan, **13** (1960/61), exposés 18–21, Benjamin, 1967.

[Ka] I. Kaplanski, An introduction to differential algebra, Hermann, Paris, 1957.

[Ko] E. R. Kolchin, Differential algebra and algebraic groups, Acad. Press New-York, 1973.

[Ma 1] B. Malgrange, *La variété caractéristique d'un système différentiel analytique*, Ann. Inst. Fourier (Grenoble), **50-2** (2000), 491–518.

[Ma 2] B. Malgrange, *L'involutivité générique des systèmes différentiels analytiques*, C. R. Acad. Sci. Paris Sér. I Math., **326-1** (1998), 863–866.

[Po] J. P. Pommaret, Systems of partial differential equations and Lie pseudo groups, Gordon and Breach, London, 1978.

[Ri 1] J. Ritt, Differential algebra, Coll. publications, **33**, Amer. Math. Soc. Dover, 1966.

[Ri 2] J. Ritt, *Systems of differential equations*, I-theory of ideals, Amer. J. of Math, **60**, (1938), 535–548.

[Se] W. M. Seiler, Analysis and application of the formal theory of partial differential equations, Thesis, Lancaster University, 1994.

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Lie Algebras, Geometric Structures and Differential Equations on Filtered Manifolds

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Introduction

Since Lie, Klein and Cartan, there has been a great deal of progress in understanding deep relations between groups, geometry and differential equations.

In this paper we give a survey on some recent development made by systematic studies from the view point of nilpotent geometry on transformation groups (or rather Lie algebras), geometric structures and differential equations, placing ourselves on filtered manifolds.

A filtered manifold is a differential manifold M endowed with a filtration $\{\mathfrak{f}^p\}_{p\in\mathbb{Z}}$ consisting of subbundles $\mathfrak{f}^p$ of the tangent bundle TM such that

i) $\mathfrak{f}^p \supset \mathfrak{f}^{p+1}$,
ii) $\mathfrak{f}^0 = 0, \quad \bigcup_{p\in\mathbb{Z}} \mathfrak{f}^p = TM$,
iii) $[\underline{\mathfrak{f}}^p, \underline{\mathfrak{f}}^q] \subset \underline{\mathfrak{f}}^{p+q}$ for all $p, q \in \mathbb{Z}$,

where $\underline{\mathfrak{f}}^p$ denotes the sheaf of the germs of sections of $\mathfrak{f}^p$.

This notion of a filtered manifold has arisen from a fundamental paper of Tanaka [Tan70] on differential systems that he elaborated, inspired by the deep work of Cartan, especially by [Car10].

Of fundamental importance is the fact that, to a filtered manifold $(M, \mathfrak{f})$, there is associated at each point x of M the nilpotent graded Lie algebra $gr\mathfrak{f}_x$, as the first order approximation at x to the filtered manifold, where $gr\mathfrak{f}_x = \bigoplus \mathfrak{f}^p_x/\mathfrak{f}^{p+1}_x$. It should be remarked that if the filtration is trivial (i.e., $\mathfrak{f}^{-1} = TM$) then $gr\mathfrak{f}_x$ is nothing but the tangent space T_xM regarded as an abelian Lie algebra.

To study various objects on filtered manifolds by letting the tangent nilpotent Lie algebras play the usual rôle of the tangent spaces may be

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called nilpotent geometry or nilpotent analysis. The generalization from the abelian to the nilpotent allows one to develop more refined theories than the anterior from much wider perspectives.

We now describe the contents of this paper.

In Section 0 we give basic definitions and notation about filtered manifolds.

In Section 1 we study transformation groups on filtered manifolds, confining ourselves only to the algebraic parts of transitive infinitesimal transformation groups.

We know well about the structures of transitive filtered Lie algebras (algebraic abstractions of transitive transformation groups) through the work of Guillemin-Sternberg [GS64], Singer-Sternberg [SS65], Kobayashi-Nagano [KN66] and others.

If a transitive transformation group on a filtered manifold preserves the tangential filtration, then the transitive Lie algebra corresponding to the transformation group admits a natural filtration compatible with the tangential filtration and more refined than the usual filtration, the former deriving from the "weighted" Taylor expansion and the latter from the usual Taylor expansion. This leads to the notion of a transitive filtered Lie algebra of depth $\mu(\geq 1)$ introduced in [Mor88]. If $\mu = 1$ it reduces to the usual one.

The fundamental problem is to understand how the structure of a transitive filtered Lie algebra $(L, \{L^p\})$ can be determined from the truncated structure L/L^k which consists of finite dimensional data, or certain information up to finite orders. This is a prototype of the problems that we encounter in the study of geometric structures and differential equations.

By extending the theory of Guillemin-Sternberg to the transitive filtered Lie algebras of depth greater than one, we have a fairly complete answer to the above algebraic problem.

We shall introduce the notion of weighted involutivity by using generalized Spencer cohomology groups. This notion plays a fundamental rôle not only in the study of filtered Lie algebras, but also in the study of geometric structures and differential equations.

In Section 2 we study geometric structures on filtered manifold and explain our general method to treat equivalence problems.

The key concept is what we introduced as C-fibre in [Mor83] and as tower in [Mor93] (the latter is a refinement of the former to apply to filtered manifolds).

Roughly speaking, it is a principal fibre bundle P over a manifold M with structure group G endowed with a 1-form θ taking values in a vector space E which defines an absolute parallelism on P and satisfies some natural conditions. In particular, we assume E is a G-module containing the Lie algebra $\mathfrak{g}$ of G as a G-submodule and θ is an equivariant map.

It should be noticed that P is not necessarily finite dimensional and that E is not necessarily a Lie algebra. The reader who is familiar with Cartan connections will notice that if E is a Lie algebra containing $\mathfrak{g}$ the tower introduced above is just a principal fibre bundle with a Cartan connection. Therefore the notion of a tower is a generalization of that of a Cartan connection. However, we might rather say that the former precedes (and is more basic than) the latter; the notion of a tower seems to represent some general heuristic ideas of Cartan which appeared in his papers of infinite groups [Car04], [Car05], [Car08], [Car09] much earlier than his notion of éspace généralisé.

The category of the towers has an advantage that it is well adapted to deal with all possible (virtual) symmetries of geometric structures. Moreover, the notion of differentiation is geometrically well represented by the associated filtration on the bundle P (and on the group G). If a tower (P, M, G) on a filtered manifold $(M, \mathfrak{f})$ is compatible with the tangential filtration $\mathfrak{f}$, it admits another natural filtration on P and on G associated with $\mathfrak{f}$, which is a filtration deriving from "weighted order", and it is this filtration that plays an important rôle when we treat towers on filtered manifolds.

In the framework of tower we shall construct a unified scheme to treat geometric structures on filtered manifolds. Any geometric structure on a filtered manifold may be regarded as a tower or as a truncated tower on a filtered manifold. Given a geometric structure on a filtered manifold, we shall show the general procedure to find the invariants of the structure. When we study geometric structures, it is important to distinguish the difference between the intransitive and the transitive, and that between infinite type and finite type. It should be remarked that a structure of infinite type in the usual sense can be of finite type with respect to the weighted filtration associated with the filtered manifold.

To treat a transitive geometric structure on a filtered manifold of infinite type, we introduce a notion of weighted involutivity and clarify the procedure to find all the invarints of the structure, which is exactly a geometrical version of the procedure to determine a transitive filtered Lie algebra from a truncated Lie algebra.

Though we can also treat the intransitive infinite cases, we will not enter into the discussion rather complicated.

For a geometric structure of finite type, it is important in applications to construct a Cartan connection associated to it. But the construction has been usually difficult and technical. Our method also answers this question. We shall give a criterion (probably best possible) for the existence of a Cartan connection and a unified algorithm to construct the Cartan connection.

In Section 3 we study general systems of non-linear partial differential equations on filtered manifolds.

Now we just recall the development of general theories on analytic systems of non-linear partial differential equations. As seen easily, any system of differential equations can be brought to an exterior differential system. Cartan established for the first time a general existence theorem in the framework of exterior differential systems [Car04]. He introduced the notion of a Pfaff system in involution and obtained the solutions by successive use of the Caucy-Kowalevski theorem, which was generalized to any involutive exterior differential system by Kähler [Käh34] as well known as the Cartan-Kähler theorem.

The modern theory of these systems was initiated by Kuranishi to establish the so-called Cartan-Kuranishi prolongation theorem [Kur57]. Using Ehresman's theory of jet, Spencer introduced fundamental tools to treat systems in jet formulation (cf. [Spe69]). In this framework Goldschmidt [Gol67] and Quillen [Qui64] established a formal theory of systems of partial differential equations clarifying in modern language the notion of involutivity; the sufficient condition for the existence of formal solutions. Malgrange [Mal72] gave an elegant proof for the existence of analytic solutions to an involutive analytic system by using the "privileged neighbourhood theorem" of Grauert.

Now in studying differential equations on a filtered manifold, it is the notion of weighted orders for differential operators associated with the filtered manifold that will play the principal rôle.

We shall first introduce the notion of a weighted jet bundle for a vector bundle on a filtered manifold, and establish a formal theory in terms of weighted jet bundles analogously to the formal theory of Goldschmidt. We introduce a notion of weighted involutivity for the system, which gives sufficient condition in order that the formal solutions can be constructed in some regular way by "weighted Taylor expansion".

Next we consider the problem of convergence. Since a weightedly involutive system is in general not involutive in the ordinary sense, we cannot expect in general the existence of analytic solutions.

Without loss of generality we will work on a standard filtered manifold, namely a nilpotent Lie group N whose Lie algebra $\mathfrak{n}$ is graded: $\mathfrak{n} = \bigoplus_{p=1}^{\mu} \mathfrak{n}_p$.

We first establish the following theorem:

For a weightedly involutive analytic system, there exists a formal solution satisfying a certain type of estimate, called Gevrey estimate.

The Gevrey estimate is expressed in terms of the filtration of the filtered manifold and is a little weaker than the analyticity estimate. We can then define the class of formal Gevrey functions on a filtered manifold.

For the proof of the above theorem we employ Malgrange's method after generalizing the privileged neighbourhood theorem to the universal enveloping algebra of a nilpotent Lie algebra $\mathfrak{n}$.

We then study geometric properties of formal Gevrey functions on a graded nilpotent Lie group N, which leads to the following remarkable theorem:

If the Lie algebra $\mathfrak{n}$ is generated by $\mathfrak{n}_1$ (Hörmander condition), then the formal Gevrey function on N are analytic.

Combining the above theorems, we finally establish the following existence theorem (a generalization of the Cartan-Kähler theorem):

Consider an analytic system of non-linear partial differential equations of weighted order k on a graded nilpotent Lie group N with a Lie algebra $\mathfrak{n} = \bigoplus_{p=1}^{\mu} \mathfrak{n}_p$. Suppose that the Lie algebra $\mathfrak{n}$ is generated by $\mathfrak{n}_1$, and that the system is weightedly involutive. Then there exists an analytic solution for any prescribed weighted k-jet solution.

It should be noted that the class of the weightedly involutive systems is much larger than that of the ordinary involutive systems and contains a wide class of differential equations with singularities.

Our primary purpose of this paper is to try to make clear intrinsic relations underlying three objects; Lie algebras, geometric structures, and differential equations on filtered manifolds. We, therefore, will not enter into the details of each subjects, and not intend to give a complete proof of each statement, referring for them to our papers ([Mor83], [Mor88], [Mor90], [Mor93], [Mor95], [Mor0x]), on which our discussions are mainly based.

§0. Filtered manifolds

0.1. Definitions.

A *tangential filtration* $\mathfrak{f}$ on a differentiable manifold M is a sequence $\{\mathfrak{f}^p\}_{p\in\mathbb{Z}}$ of subbundles of the tangent bundle TM of M such that the following conditions are satisfied:

i) $\mathfrak{f}^p \supset \mathfrak{f}^{p+1}$,
ii) $\mathfrak{f}^0 = 0, \quad \bigcup_{p\in\mathbb{Z}} \mathfrak{f}^p = TM$,
iii) $[\underline{\mathfrak{f}}^p, \underline{\mathfrak{f}}^q] \subset \underline{\mathfrak{f}}^{p+q}, \quad$ for all $\quad p,\ q \in \mathbb{Z}$,

where $\underline{\mathfrak{f}}^p$ denotes the sheaf of the germs of sections of $\mathfrak{f}^p$.

A *filtered manifold* is a differentiable manifold M equipped with a tangential filtration $\mathfrak{f}$. We shall denote the filtered manifold by $(M, \mathfrak{f})$ or often by the bold letter $\mathbb{M}$ and its tangential filtration by $\{\mathfrak{f}^p\}$, $\{\mathfrak{f}^p TM\}$ or $\{T^p\mathbb{M}\}$.

An isomorphism of a filtered manifold $\mathbb{M}$ onto a filtered manifold $\mathbb{M}'$ is a diffeomorphism $\varphi \colon M \to M'$ such that $\varphi_* T^p\mathbb{M} = T^p\mathbb{M}'$ for all $p \in \mathbb{Z}$, where φ_* denotes the differential of φ.

If $\mathbb{M}$ is a filtered manifold, by definition there is an integer $\mu \geq 0$ such that $T^{-\mu}\mathbb{M} = TM$. The minimum of such integers is called the depth of $\mathbb{M}$.

Let $\mathbb{M}$ be a filtered manifold. The tangential filtration $\{T^p\mathbb{M}\}$ defines on each tangent space T_xM, $x \in M$, the induced filtration $\{T^p_x\mathbb{M}\}$. We denote by $T_x\mathbb{M}$ this filtered vector space $(T_xM, \{T^p_x\mathbb{M}\})$. Now by setting

$$gr_p T_x\mathbb{M} = T^p_x\mathbb{M}/T^{p+1}_x\mathbb{M},$$

we form a graded vector space:

$$grT_x\mathbb{M} = \bigoplus_{p\in\mathbb{Z}} gr_p T_x\mathbb{M}.$$

This vector space carries a natural bracket operation induced from the Lie bracket of vector fields: For $\xi \in gr_pT_x\mathbb{M}$, $\eta \in gr_qT_x\mathbb{M}$, take local cross-sections X, Y of $T^p\mathbb{M}$, $T^q\mathbb{M}$ resp. such that $\xi \equiv X_x (\operatorname{mod} T^{p+1}_x\mathbb{M})$, $\eta \equiv Y_x (\operatorname{mod} T^{q+1}_x\mathbb{M})$, and define

$$[\xi, \eta] \equiv [X, Y]_x \qquad \operatorname{mod} T^{p+q+1}_x\mathbb{M}.$$

It is then easy to see that this bracket operation is well defined and makes $grT_x\mathbb{M}$ a Lie algebra. Clearly we have:

i) $[gr_pT_x\mathbb{M}, gr_qT_x\mathbb{M}] \subset gr_{p+q}T_x\mathbb{M}$,

ii) $gr_pT_x\mathbb{M} = 0 \qquad$ for $p \geq 0$.

This graded Lie algebra $grT_x\mathbb{M}$ is called the *symbol algebra* of $\mathbb{M}$ at x ([Tan70]), and may be consider as the tangent space (algebra) at x of the filtered manifold $\mathbb{M}$.

We say that a filtered manifold $\mathbb{M}$ is *regular* of type $\mathfrak{m}$ if the symbol algebras $grT_x\mathbb{M}$ are all isomorphic to a graded Lie algebra $\mathfrak{m}$.

0.2. Some examples.

1) Trivial filtration. A differentiable manifold M itself may be regarded as a filtered manifold equipped with the trivial filtration defined by $\mathfrak{f}^p_{tr}TM = TM$ for $p < 0$ and $\mathfrak{f}^q_{tr}TM = 0$ for $q \geq 0$. The symbol algebras $grT_x\mathbb{M}$ of this trivial filtered manifold is nothing but the tangent space T_xM regarded as an abelian Lie algebra with trivial gradation.

2) Standard filtered manifold. Let $\mathfrak{n}$ be a finite-dimensional nilpotent Lie algebra endowed with a gradation $\mathfrak{n} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{n}_p$ such that

i) $[\mathfrak{n}_p, \mathfrak{n}_q] \subset \mathfrak{n}_{p+q}$,
ii) $\mathfrak{n}_p = 0 \qquad p \geq 0$.

Let N be a Lie group whose Lie algebra is $\mathfrak{n}$. Set $\mathfrak{n}^p = \bigoplus_{i\geq p} \mathfrak{n}_i$ and identify $N \times \mathfrak{n}^p$ with a left invariant subbundle of TN, then $\{N \times \mathfrak{n}^p\}_{p\in\mathbb{Z}}$ is a tangential filtration on N. The filtered manifold $\mathbb{N} = (N, \{N \times \mathfrak{n}^p\})$ is called a *standard filtered manifold* of type $\mathfrak{n}$.

3) Tangential filtration derived from a regular differential system [Tan70]. Let D be a differential system on a differentiable manifold M, that is, a subbundle of the tangent bundle of M. Then there is associated a sequence of subsheaves $\{\mathcal{D}^p\}_{p<0}$ of $\underline{TM}$, called the derived systems of D, which is defined inductively by:

$$\begin{cases} \mathcal{D}^{-1} = \underline{D}, \\ \mathcal{D}^{p-1} = \mathcal{D}^p + [\mathcal{D}^p, \mathcal{D}^{-1}] \qquad (p < 0). \end{cases}$$

It then holds that:

$$[\mathcal{D}^p, \mathcal{D}^q] \subset \mathcal{D}^{p+q} \qquad \text{for} \quad p,\ q < 0.$$

Now suppose that the derived systems $\mathcal{D}^p$ are all vector bundles, that is, there are subbundles $D^p \subset TM$ such that $\underline{D^p} = \mathcal{D}^p$ for all $p < 0$ (in this case the differential system D is called regular [Tan70]). Then there exists a minimum integer $\mu \geq 1$ such that $D^p = D^{-\mu}$ for all $p \leq -\mu$. Setting

$$\mathfrak{f}^p TM = \begin{cases} 0 & (p \geq 0) \\ D^p & (-1 \geq p \geq -\mu) \\ TM & (p \leq -\mu - 1), \end{cases}$$

we have a filtered manifold $(M, \mathfrak{f})$ derived from the regular differential system D. If $D^{-\mu} = TM$, we say that the tangential filtration $\mathfrak{f}$ is generated by the differential system D. If $D^{-\mu} \subsetneqq TM$, then $D^{-\mu}$ is completely integrable and defines a foliation on M. In particular, if D is completely integrable the filtered manifold $\mathbb{M}$ is nothing but a foliated manifold.

If a filtered manifold $\mathbb{M}$ (or $\mathbb{M}'$) is derived from a differential system D on M (resp. D' on M'), then $\mathbb{M}$ and $\mathbb{M}'$ are isomorphic if and only if (M, D) and (M', D') are isomorphic, that is, there is a diffeomorphism $\varphi\colon M \to M'$ such that $\varphi_* D = D'$.

4) Higher order contact manifold (cf. [Yam82]). Let $\pi\colon M \to N$ be a fibred manifold. Let $J^k(M,N)$ be the bundle of k-jets of cross-sections of π. On this jet bundle we have a sequence of canonical differential systems $\{D^p\}$ called the higher order contact structure. In local coordinates it is expressed as follows: Let $(x^1,\dots,x^n)$, $(x^1,\dots,x^n,y^1,\dots,y^m)$ be local coordinates of N and M respectively. Then $(x^1,\dots,x^n,\dots,p^i_\alpha,\dots)$, where $p^i_\alpha = \frac{\partial^{|\alpha|}y^i}{\partial x^\alpha}$ with $\alpha = (\alpha_1,\dots,\alpha_n)$, $|\alpha| \le k$, gives a local coordinate system of $J^k(M,N)$ called a canonical coordinates system. Put

$$\omega^i_\alpha = dp^i_\alpha - \sum_{j=1}^{n} p^i_{\alpha+1_j}\, dx^j$$

for $|\alpha| \le k-1$, with $\alpha + 1_j = (\alpha_1,\dots,\alpha_j+1,\dots,\alpha_n)$, and define $D^p (p \le -1)$ by the following Pfaff equations:

$$D^p\colon \quad \omega^i_\alpha = 0 \qquad (i = 1,\dots,n \quad |\alpha| \le k+p).$$

It is easy to see that D^p are well-defined subbundles of $TJ^k(M,N)$ and satisfy:

i) $\underline{D}^{p-1} = \underline{D}^p + [\underline{D}^p, \underline{D}^{-1}]$,
ii) $D^p = TJ^k(M,N) \qquad \text{for} \quad p \le -k-1$.

We thus obtain a canonical tangential filtration $\{D^p\}$ on $J^k(M,N)$ of depth $k+1$ generated by D^{-1}. It should be noted that if $\dim M = n+1$, $\dim N = n$ and $k = 1$ then $J^1(M,N)$ is a contact manifold having D^{-1} as its contact structure.

0.3. We shall often use the following notation and terminologies throughout this paper without explicit mention in each place.

For filtered objects (vector spaces V, W, Lie groups G etc.) we denote by $\{\mathfrak{f}^p\}$ not only their filtrations but also the induced filtrations defined naturally on various associated spaces, for instance:

$$\mathfrak{f}^p(V \oplus W) = \mathfrak{f}^p V \oplus \mathfrak{f}^p W$$
$$\mathfrak{f}^p(V \otimes W) = \sum_{r+s=p} \mathfrak{f}^r V \otimes \mathfrak{f}^s W$$
$$\mathfrak{f}^p(G/\mathfrak{f}^h G) = \mathfrak{f}^p G/\mathfrak{f}^p G \cap \mathfrak{f}^h G$$
$$\mathfrak{f}^p \operatorname{Hom}(V,W) = \{\alpha \in \operatorname{Hom}(V,W) \mid \alpha(\mathfrak{f}^i V) \subset \mathfrak{f}^{i+p} W \quad \forall i\}$$
$$\mathfrak{f}^p GL(V) = \{\alpha \in GL(V) \mid \alpha - 1_V \in \mathfrak{f}^p \operatorname{Hom}(V,V)\}.$$

When we take a quotient space, for instance $\mathfrak{f}^0 G/\mathfrak{f}^p G$, we often write it simply as $\mathfrak{f}^0 G/\mathfrak{f}^p$.

If $\mathfrak{v} = \bigoplus \mathfrak{v}_p$, $\mathfrak{w} = \bigoplus \mathfrak{w}_p$ are graded vector spaces, we set

$$\mathrm{Hom}(\mathfrak{v}, \mathfrak{w})_p = \{\alpha \in \mathrm{Hom}(\mathfrak{v}, \mathfrak{w}) \mid \alpha(\mathfrak{v}_i) \subset \mathfrak{w}_{i+p} \quad \forall i\},$$

that is, the set of all linear maps of degree p.

§1. Transitive Lie algebras on filtered manifolds

1.1. Let us begin with the following definition:

Definition 1.1. A transitive filtered Lie algebra (TFLA) of depth $\mu(\geq 1)$ is a Lie algebra L equipped with a filtration $\{L^p\}_{p\in\mathbb{Z}}$ satisfying:

i) $L = L^{-\mu}$
ii) $L^p \supset L^{p+1}$
iii) $[L^p, L^q] \subset L^{p+q}$
iv) $\dim L^p/L^{p+1} < \infty$
v) $\bigcap_{p\in\mathbb{Z}} L^p = 0$
vi) $L^{p+1} = \{x \in L^p; [x, L^a] \subset L^{p+a+1}$ for all $a < 0\}$ for $p \geq 0$.

To justify the above definition, some remarks are in order.

Remark 1. The so-called continuous groups that Lie studied are in modern language the pseudo-groups of transformations on manifolds which are defined by systems of partial differential equations. In other words, a continuous group of Lie is a pseudo-group of transformations that leave invariant certain geometric structure on a manifold. To study such pseudo-groups, in particular, infinite dimensional ones to which there are no good global representatives such as finite dimensional Lie groups, it is usually more convenient to study infinitesimal objects, namely Lie algebra subsheaf $\mathcal{L}$ of the Lie algebra sheaf $\underline{TM}$ of the germs of local vector fields on a manifold M. The Lie algebra sheaf $\mathcal{L}$ is said to be transitive if the evaluation map $\mathcal{L}_x \to T_xM$ is surjective for all $x \in M$, where $\mathcal{L}_x$ denotes the stalk of $\mathcal{L}$ at x.

Let $\mathcal{L}$ be a Lie algebra sheaf on M. We can associate to each x the formal algebra L_x defined as follows [SS65]: Let, for $k \geq 0$, $\mathfrak{f}^k\mathcal{L}_x$ be the subalgebra of $\mathcal{L}_x$ consisting of all germs $[X]_x$ at x of sections X of $\mathcal{L}$ such that X vanishes at x to order k. Then put $L_x = \operatorname{proj\,lim}_{k\to\infty} \mathcal{L}_x/\mathfrak{f}^k\mathcal{L}_x$. The formal algebra L_x has a natural filtration $\{L_x^p\}_{p\in\mathbb{Z}}$, where we put $L_x^p = \operatorname{proj\,lim}_{k\to\infty} \mathfrak{f}^p\mathcal{L}_x/\mathfrak{f}^{p+k}\mathcal{L}_x$, and $L_x^q = L_x$ for $q < 0$. It is then easy to see that $\{L_x^p\}_{p\in\mathbb{Z}}$ satisfies all the conditions of Definition 1.1 with $\mu = 1$ except (vi), which is also satisfied if $\mathcal{L}$ is transitive. Thus the formal algebra L_x of a transitive Lie algebra sheaf $\mathcal{L}$ is a transitive filtered Lie algebra of depth 1. It is well-known that under the category of analyticity and under certain regularity condition $\mathcal{L}$ is locally uniquely determined by its formal algebra L_x at a point x. In these contexts, and in particular, in connection with the classification of simple infinite Lie algebras, the transitive filtered Lie algebras of depth 1 were well studied ([GS64], [SS65], [KN66], [Hay70], etc.).

Remark 2. Suppose now we are given a transitive Lie algebra sheaf $\mathcal{L}$ on a filtered manifold $(M, \mathfrak{f})$ and suppose that $\mathcal{L}$ leaves invariant the filtration $\mathfrak{f}$. Then we can introduce on its formal algebra L_x another filtration more refined than the original one and well adapted to the underlying filtration $\mathfrak{f}$: We define for $q \leq 0$ $\hat{\mathcal{L}}_x^q$ to be the subspace of $\mathcal{L}_x$ consisting of all germs $[X]_x$ at x of sections X of $\mathcal{L}$ such that $X_x \in \mathfrak{f}_x^q$ and set $\hat{L}_x^q$ to be the its image on L_x. For $p > 0$ we define $\hat{L}_x^p$ by the condition (vi) of Definition 1.1, replacing L^k by $\hat{L}_x^k$. In this way we have a transitive filtered Lie algebra $\{\hat{L}_x^p\}$ of depth μ. (If we use the notion of weighted jet bundle introduced in Section 3, we can better understand the meaning of this new filtration.)

The remarks above will motivate to study the transitive filtered Lie algebras of depth greater than 1, which not only leads us to a natural generalization of Guillemin-Sternberg but also becomes a good guide to studying geometric structures and differential equations on filtered manifolds.

In this section we study the transitive filtered Lie algebras of depth greater than 1, and we shall see how a filtered Lie algebra can be constructed from finite dimensional data i.e., its truncated Lie algebra.

Let L be a transitive filtered Lie algebra of depth μ. Let $grL = \bigoplus_{p\in\mathbb{Z}} gr_pL$ be its associated graded Lie algebra, where $gr_pL = L^p/L^{p+1}$. Then grL is a transitive graded Lie algebra of depth μ in the following sense:

Definition 1.2. A graded Lie algebra $g = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ is called transitive graded Lie algebra (TGLA) of depth μ if it satisfies the following conditions:

i) $\mathfrak{g}_p = 0$ for $p < -\mu$
ii) $\dim \mathfrak{g}_p < \infty$
iii) For $i \geq 0$, $x_i \in \mathfrak{g}_i$, if $[x_i, \mathfrak{g}_-] = 0$ then $x_i = 0$,

where we set $\mathfrak{g}_- = \bigoplus_{p<0} \mathfrak{g}_p$.

Let us recall the notion of prolongation concerning a TGLA ([GS64], [Tan70]). For this we first give the following:

Definition 1.3. Let k be an integer or ∞. A truncated graded Lie algebra of order k is a graded vector space $\mathfrak{g}(k) = \bigoplus_{p\leq k} \mathfrak{g}_p$ equipped with a bracket operation (skew-symmetric bilinear map)

$$[\ ,\ ]\colon \mathfrak{g}_p \times \mathfrak{g}_q \to \mathfrak{g}_{p+q}$$

defined partially for p, q, $p+q \leq k$, satisfying the partial Jacobi identity:

$$\mathfrak{S}[[x_p, y_q], z_r] = 0$$

for $x_p \in \mathfrak{g}_p$, $y_q \in \mathfrak{g}_q$, $z_r \in \mathfrak{g}_r$, whenever p, q, $p+q$, $q+r$, $r+p$, $p+q+r \leq k$, where $\mathfrak{S}$ denotes the cyclic sum in x_p, y_q, z_r. If moreover the conditions (1) (2) (3) of Definition 1.2 are satisfied, $\mathfrak{g}(k)$ is called truncated transitive graded Lie algebra (truncated TGLA) of order k of depth μ.

Note that a truncated TGLA of order ∞ is just a TGLA. If $\mathfrak{g}(k) = \bigoplus_{p\leq k} \mathfrak{g}_p$ is a truncated TGLA of order k, then for each integer $l \leq k$, $\bigoplus_{p\leq l} \mathfrak{g}_p$ becomes a truncated TGLA of order l with respect to the induced bracket operation, which we will denote by $\operatorname{Trun}_l \mathfrak{g}(k)$. Morphisms of truncated TFLA's can be defined in the natural manner. In particular, a homomorphism $\varphi\colon \mathfrak{h}(k) \to \mathfrak{g}(k)$ will be called an embedding if φ induces an isomorphism of $\mathfrak{h}_- = \bigoplus_{p<0} \mathfrak{h}_p$ onto $\mathfrak{g}_- = \bigoplus_{p<0} \mathfrak{g}_p$. Note that an embedding is necessarily injective.

Now let us define the prolongation of a truncated TGLA $\mathfrak{g}(k) = \bigoplus_{p\leq k} \mathfrak{g}_p$ of order $k \geq -1$. Put $\mathfrak{g}_- = \bigoplus_{p<0} \mathfrak{g}_p$ and define $\operatorname{Der}_{k+1} \mathfrak{g}(k)$ to be the vector space consisting of all $\alpha \in \operatorname{Hom}(\mathfrak{g}_-, \mathfrak{g}(k))$ such that

$$\begin{cases} \alpha(\mathfrak{g}_p) \subset \mathfrak{g}_{p+k+1} \quad (p < 0) \\ \alpha([x, y]) = [\alpha(x), y] + [x, \alpha(y)], \quad \text{for } x,\ y \in \mathfrak{g}_- \end{cases}$$

and we set

$$p\mathfrak{g}(k) = \mathfrak{g}(k) \oplus \operatorname{Der}_{k+1} \mathfrak{g}(k).$$

It is then easy to see that there exists a unique bracket operation on $p\mathfrak{g}(k)$ which makes $p\mathfrak{g}(k)$ into a truncated TGLA of order $k+1$ such that $\operatorname{Trun}_k(p\mathfrak{g}(k)) = \mathfrak{g}(k)$ and $[\alpha, x] = \alpha(x)$ for $\alpha \in \operatorname{Der}_{k+1} \mathfrak{g}(k)$, $x \in \mathfrak{g}_-$.

Iterating this construction, we obtain a truncated TGLA $p^i\mathfrak{g}(k)$ $(= p(p^{i-1}\mathfrak{g}(k))$ of order $k+i$, and a TGLA $p^\infty\mathfrak{g}(k)$ $(= \operatorname{inj\,lim} p^i\mathfrak{g}(k))$. Thus we have:

Proposition 1.1 (Tanaka). *For a truncated TGLA $\mathfrak{g}(k)$ of order k, there exists, uniquely up to isomorphism, a truncated TGLA $p^i\mathfrak{g}(k)$ of order $k + i (0 \leq i \leq \infty)$ which satisfies the following conditions:*

i) $\operatorname{Trun}_k(p^i\mathfrak{g}(k)) = \mathfrak{g}(k)$

ii) *If* $\mathfrak{h}(k+i)$ *is a truncated TGLA of order* $k+i$ *and if there is an embedding* $\psi_k\colon \mathrm{Trun}_k\,\mathfrak{h}(k+i) \to \mathfrak{g}(k)$, *then there exists a unique embedding* $\psi_{k+i}\colon \mathfrak{h}(k+i) \to p^i\mathfrak{g}(k)$ *such that* $\psi_{k+i}\,|_{\mathrm{Trun}_k\,\mathfrak{h}(k+i)} = \psi_k$.

The truncated TGLA $p^i\mathfrak{g}(k)$ is called the prolongation of $\mathfrak{g}(k)$. We will often denote $p^\infty\mathfrak{g}(k)$ by $\mathrm{Prol}\,\mathfrak{g}(k)$. We say also that a TGLA $\mathfrak{g}$ is the prolongation of $\mathrm{Trun}_k\,\mathfrak{g}$ if $\mathfrak{g} = \mathrm{Prol}\,\mathrm{Trun}_k\,\mathfrak{g}$. Note that, by the proposition above, $\mathfrak{g}$ can be always identified with a graded subalgebra of $\mathrm{Prol}\,\mathrm{Trun}_k\,\mathfrak{g}$.

1.2. Generalized Spencer cohomology groups. Now we define a cohomology group associated with a TGLA $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}}\mathfrak{g}_p$. We set

$$\mathfrak{m} = \mathfrak{g}_- = \bigoplus_{p<0}\mathfrak{g}_p,$$

which is a nilpotent subalgebra of $\mathfrak{g}$, and consider the cohomology group associated with the adjoint representation of $\mathfrak{m}$ on $\mathfrak{g}$, namely the cohomology group $H(\mathfrak{m},\mathfrak{g}) = \bigoplus H^p(\mathfrak{m},\mathfrak{g})$ of the cochain complex $(C(\mathfrak{m},\mathfrak{g}) = \oplus C^p(\mathfrak{m},\mathfrak{g}), \partial)$, where

$$C^p(\mathfrak{m},\mathfrak{g}) = \mathrm{Hom}\Big(\bigwedge^p \mathfrak{m},\mathfrak{g}\Big)$$

and the coboudary operator $\partial\colon \mathrm{Hom}(\bigwedge^p\mathfrak{m},\mathfrak{g}) \to \mathrm{Hom}(\bigwedge^{p+1}\mathfrak{m},\mathfrak{g})$ is defined by

$$\begin{aligned}(\partial\omega)(X_1,\dots,X_{p+1}) &= \sum_{i=1}^{n+1}(-1)^{i-1}[X_i,\omega(X_1,\dots,\hat{X}_i,\dots,X_{p+1})]\\ &+ \sum_{1\le i<j\le p+1}(-1)^{i+j}\omega([X_i,X_j],X_1,\dots,\hat{X}_i,\dots,\hat{X}_j,\dots,X_{p+1})\end{aligned}$$

for $\omega \in \mathrm{Hom}(\bigwedge^p\mathfrak{m},\mathfrak{g})$, $X_1,\dots,X_{p+1}\in\mathfrak{m}$. Since both $\mathfrak{m}$ and $\mathfrak{g}$ are graded, we can define a bigradation $\bigoplus H^p_r(\mathfrak{m},\mathfrak{g})$ of $H(\mathfrak{m},\mathfrak{g})$ as follows: Denote by $\mathrm{Hom}(\bigwedge^p\mathfrak{m},\mathfrak{g})_r$ the set of all homogeneous p-cochain ω of degree r (i.e., $\omega(\mathfrak{g}_{a_1}\wedge\cdots\wedge\mathfrak{g}_{a_p}) \subset \mathfrak{g}_{a_1+\cdots+a_p+r}$ for any $a_1,\dots,a_p<0$), and set $C_r(\mathfrak{m},\mathfrak{g}) = \mathrm{Hom}(\bigwedge\mathfrak{m},\mathfrak{g})_r = \bigoplus_p \mathrm{Hom}(\bigwedge^p\mathfrak{m},\mathfrak{g})_r$. Note that ∂ preserves the degree. Hence $C_r(\mathfrak{m},\mathfrak{g})$ is a subcomplex and the direct sum decomposition

$$C(\mathfrak{m},\mathfrak{g}) = \bigoplus C_r(\mathfrak{m},\mathfrak{g})$$

yields the decomposition of the cohomology group:

$$H(\mathfrak{m},\mathfrak{g}) = \bigoplus H_r(\mathfrak{m},\mathfrak{g}) = \bigoplus H^p_r(\mathfrak{m},\mathfrak{g}).$$

This cohomology group $H(\mathfrak{m},\mathfrak{g})$ was introduced by Tanaka [Tan79] with another gradation:

$$H^{s,p}(\mathfrak{m},\mathfrak{g}) = H^p_{s+p-1}(\mathfrak{m},\mathfrak{g}).$$

It should be remarked that if the depth $\mu = 1$, then $\mathfrak{m}$ is abelian and $H^{s,p}(\mathfrak{m},\mathfrak{g})$ is known as the Spencer cohomology group. The following theorem generalizes the well-known result in the case $\mu = 1$ to the case of arbitrary μ [Mor88].

Theorem 1.1. *Let $\mathfrak{g}$ be a TGLA of depth μ. Then there exists an integer r_0 such that $H_r(\mathfrak{m},\mathfrak{g}) = 0$ for all $r \geq r_0$.*

The proof is based on the fact that the universal enveloping algebra of a finite dimensional Lie algebra is Noetherian.

For a concrete criterion in terms of quasi-regular bases for the vanishing of the cohomology group, see [Mor91].

1.3. Truncated transitive filtered Lie algebras. Let A be a vector space. For α, $\beta \in \mathrm{Hom}(\bigwedge^2 A, A)$ define $\alpha \circ \beta \in \mathrm{Hom}(\bigwedge^3 A, A)$ by

$$(\alpha\circ\beta)(x,y,z) = \mathfrak{S}\alpha(\beta(x,y),z),$$

where $\mathfrak{S}$ denotes the cyclic sum in x, y, $z \in A$. Define then a quadratic map

$$J\colon \mathrm{Hom}\Big(\bigwedge^2 A, A\Big) \to \mathrm{Hom}\Big(\bigwedge^3 A, A\Big)$$

by $J(\gamma) = \gamma\circ\gamma$ for $\gamma \in \mathrm{Hom}(\bigwedge^2 A, A)$. Note that to define a Lie algebra structure on A is equivalent to taking a $\gamma \in \mathrm{Hom}(\bigwedge^2 A, A)$ satisfying $J(\gamma) = 0$.

Now if A is endowed with a descending filtration $\{A^p\}_{p\in\mathbb{Z}}$, then $\mathrm{Hom}(\bigwedge^r A, A)$ has the natural filtration $\{\mathrm{Hom}(\bigwedge^r A, A)^k\}$, where $\mathrm{Hom}(\bigwedge^r A, A)^k$ consists of all $\alpha \in \mathrm{Hom}(\bigwedge^r A, A)$ satisfying $\alpha(A^{p_1}\wedge\cdots\wedge A^{p_r}) \subset A^{p_1+\cdots+p_r+k}$ for any $(p_1,\dots,p_r) \in \mathbb{Z}^r$. Let us introduce on $\mathrm{Hom}(\bigwedge^r A, A)^0$ another filtration $\{I^k\,\mathrm{Hom}(\bigwedge^r A, A)^0\}_{k\in\mathbb{Z}}$ by defining $I^k\,\mathrm{Hom}(\bigwedge^r A, A)^0$ to be the subspace of $\mathrm{Hom}(\bigwedge^r A, A^k)^0$ which consists of all $\alpha \in \mathrm{Hom}(\bigwedge^r A, A)^0$ such that

$$\alpha(A^{p_1}\wedge\cdots\wedge A^{p_r}) \subset A^{p_1^*+\cdots+p_r^*+k}$$

for any $(p_1,\dots,p_r) \in \mathbb{Z}^r$, where we set $p^* = \mathrm{Min}\{p,0\}$.

It is easy to check that if $\alpha-\beta \in I^k\,\mathrm{Hom}(\bigwedge^2 A, A)^0$ for $\alpha,\beta \in \mathrm{Hom}(\bigwedge^2 A, A)^0$ then $J(\alpha)-J(\beta) \in I^k\,\mathrm{Hom}(\bigwedge^3 A, A)^0$. Therefore if we put

$$\Big[\mathrm{Hom}\Big(\bigwedge^r A, A\Big)^0\Big]^{[k]} = \mathrm{Hom}\Big(\bigwedge^r A, A\Big)^0 \Big/ I^{k+1}\,\mathrm{Hom}\Big(\bigwedge^r A, A\Big)^0,$$

we have the induced map

$$J\colon \left[\mathrm{Hom}\left(\bigwedge^2 A, A\right)^0\right]^{[k]} \to \left[\mathrm{Hom}\left(\bigwedge^3 A, A\right)^0\right]^{[k]}$$

defined by $J\alpha^{[k]} = (J(\alpha))^{[k]}$ for $\alpha \in \mathrm{Hom}(\bigwedge^2 A, A)^0$, where $\beta^{[k]}$ denotes the equivalence class of $\beta \in \mathrm{Hom}(\bigwedge^r A, A)^0$ modulo $I^{k+1}\,\mathrm{Hom}(\bigwedge^r A, A)^0$.

Definition 1.4. A truncated filtered Lie algebra of order k is a vector space A endowed with a descending filtration $\{A^p\}_{p\in\mathbb{Z}}$ and a truncated bracket $\gamma^{[k]} \in [\mathrm{Hom}(\bigwedge^2 A, A)^0]^{[k]}$ satisfying the following conditions:

i) $A^{k+1} = 0$
ii) $J(\gamma^{[k]}) = 0$ (truncated Jacobi identity)

Note that if $A(k) = (A, \{A^p\}, \gamma^{[k]})$ is a truncated filtered Lie algebra then grA has the induced structure of truncated graded Lie algebra, which will be denoted by $grA(k)$.

Definition 1.5. A truncated filtered Lie algebra $A(k)$ is called a truncated transitive filtered Lie algebra (truncated TFLA) if $grA(k)$ is transitive.

Note that a truncated TFLA of order ∞ is just a TFLA.

If $A(k)$ is a truncated TFLA of order $k(\leq \infty)$ then for each $l \leq k$, we have a truncated TFLA of order l denoted by $\mathrm{Trun}_l\, A(k)$ by passage to the quotient $A(k)/A^{l+1}$.

Homomorphisms of truncated TFLA's are defined in the natural manner. Note that a homomorphism $\varphi\colon A(k) \to B(k)$ of truncated TFLA's gives rise to a homomorphism $gr\varphi\colon grA(k) \to grB(k)$ of truncated GLA's. We say φ is an embedding if so is $gr\varphi$.

For a truncated TFLA $A(k)$, the cohomology group $H^p_r(grA(k))$ will be defined to be $H^p_r((\mathrm{Prol}\, grA(k))_-, \mathrm{Prol}\, grA(k))$.

1.4. Now we are in a position to state main structure theorems:

Theorem 1.2. *Let $A(k)$ be a truncated TFLA of order $k \geq 0$. Assume that*

$$H^2_r(grA(k)) = 0 \quad \textit{and} \quad H^3_s(grA(k)) = 0$$

for $r \geq k+1$ and $s \geq \mathrm{Max}\{k+1, 2\}$. Then there exists, uniquely up to isomorphism, a complete TFLA L such that

$$\mathrm{Trun}_k\, L = A(k) \quad \textit{and} \quad grL = \mathrm{Prol}\, grA(k).$$

Theorem 1.3. *Let L be a complete TFLA and k a non-negative integer such that*

$$H^1_r(grL) = H^2_r(grL) = 0 \quad \textit{for} \quad r \geq k+1.$$

If there is an embedding $\psi_k\colon \mathrm{Trun}_k K \to \mathrm{Trun}_k L$ for a TFLA K, then there exists an embedding $\varphi\colon K \to L$ such that $\mathrm{Trun}_k \varphi = \psi_k$ Moreover, two such embeddings differ by an inner automorphism of L which fixes $\mathrm{Trun}_k L$.

Theorem 1.4. *Let L_1 and L_2 be complete TFLA's and k a non-negative integer such that*

$$H^1_r(grL_i) = H^2_r(grL_i) = 0 \quad \textit{for} \quad i = 1,\ 2 \quad \textit{and} \quad r \geq k+1.$$

Then L_1 and L_2 are isomorphic if and only if so are $\mathrm{Trun}_k L_1$ and $\mathrm{Trun}_k L_2$.

Corollary 1.5. *If L is a complete TFLA satisfying*

$$H^1_r(grL) = H^2_r(grL) = 0 \quad \textit{for} \quad r \geq 1,$$

then L is graded, that is, isomorphic to the completion of grL.

For the proofs see [Mor88]. The above theorems as well as their proofs clarify how a TFLA is constructed step by step from a truncated TFLA of lower order.

§2. Geometric structures on filtered manifolds

2.0. The main problem that we discuss in this section is the equivalence problem: It is, given two geometric structures, to obtain criteria to decide whether they are (locally) equivalent. For this problem, geometrically, the main task is to determine the complete invariants of a given structure.

The general equivalence problem was first posed by Lie to find the differential invariants under the action of a finite or infinite dimensional Lie group and was investigated during the last quarter of the 19th century by Lie himself, Halphen, Tresse, Wilczynski etc., mainly for various classes of differential equations.

At the beginning of the 20th century Cartan invented a powerful method for the equivalence problem by combining Lie group theory and the method of moving frames, and applied it to his extensive work, especially to his brilliant work in 1900s on the theory of infinite groups and on geometric studies of differential equations. Later in 1920s he also introduced the notion of éspace généralisé, what is nowadays called Cartan connection. He wrote:

In the wake of the movement of ideas which followed the general theory of relativity, I was led to introduce the notion of new geometries, more general than Riemannian geometry, and playing with respect to the different Klein geometries the same role as the Riemannian geometries play with respect to Euclidean space. The vast synthesis that I realized in this way depends of course on the ideas of Klein formulated in his celebrated Erlangen programme while at the same time going far beyond it since it includes Riemannian geometry, which had formed a completely isolated branch of geometry, within the compass of a very general scheme in which the notion of group still plays a fundamental role. (E. Cartan [Car31] p. 58. The translation is borrowed from [Sha97].)

However, it took a rather long time until after the second world war that Cartan's fundamental ideas came to be rigorously formulated and developed into modern theory by the work of Ehresmann, Chern, Kuranishi, Spencer and others. In particular, Chern [Che54], Guillemin, Singer and Sternberg ([GS64], [SS65]) formulated the equivalence problem as that of G-structures and clarified many of Cartan's ideas.

Meanwhile, inspired by the deep work of Cartan, Tanaka elaborated skilled methods to construct Cartan connections through his studies on conformal and projective connections, on CR geometry and on geometry of differential equations. In particular, he developed fundamental work on differential systems. ([Tan62], [Tan70], [Tan76] and [Tan79].)

Pursuing a more complete treatment of equivalence problem, Morimoto introduced the notion of C-fibre [Mor83] and then extended this as that of tower on filtered manifolds [Mor93], which gives us wider perspectives to develop a unified theory on the equivalence problem, in particular, to lead us to a basic notion of weighted involutivity and to a general criterion for the existence of Cartan connection.

In this section we will introduce the notion of a tower and explain the method to treat the equivalence problem, laying emphasis on the conceptual aspects and referring to [Mor93] for technical details.

2.1. Let us begin with some reflection on differentiation. Let f be a function of $x_1, \ldots, x_n$. The partial derivatives $\frac{\partial f}{\partial x_i}$ are nothing but the coefficients of df:

$$df = f_1\, dx_1 + \cdots + f_n\, dx_n.$$

If we have in mind a certain geometric structure and if we have no reason to choose a special system of coordinates but a certain family of coframes $\omega_1, \ldots, \omega_n$ invariantly associated to the structure, it will be better to differentiate with respect to the coframes:

$$df = f_1\omega_1 + \cdots + f_n\omega_n.$$

But then f_i will be a function not only of $(x_1, \ldots, x_n)$ but also of new parameters $\lambda_1, \ldots, \lambda_l$ if coframes $\omega_1, \ldots, \omega_n$ depend on l-parameters $\lambda_1, \ldots, \lambda_l$.

Next if we have a distinguished family of coframes of the space $x_1, \ldots, x_n$, $\lambda_1, \ldots, \lambda_l$, say, $\omega_1, \ldots, \omega_n$, $\pi_1, \ldots, \pi_l$, where π_p are 1-forms expressed in terms of $x_1, \ldots, x_n$, $\lambda_1, \ldots, \lambda_l$ and new parameters $\mu_1, \ldots$, μ_m, we get second order derivatives with respect to these coframes:

$$df_i = \sum f_{ij}\omega_j + \sum f_{i;p}\pi_p,$$

where f_{ij}, $f_{i;p}$ are now functions of x, λ, μ.

If each family of coframes is taken in an invariant way, then the parameter spaces $\{\lambda_1, \ldots, \lambda_l\}$, $\{\lambda_1, \ldots, \lambda_l, \mu_1, \ldots, \mu_m\}$ will all form Lie groups.

Iterating this procedure to higher orders, we may arrive to a space to which we need no longer to add new parameters, or we have to continue the procedure infinitely.

This leads us to consider as a model of the space in which we finally arrive after the above procedure the following objects: $(P, M, G, E, \rho, \theta)$, where P is a principal fibre bundle over a manifold M with the structure group G equipped with an absolute parallelism, that is, a 1-form θ taking values in a vector space E such that $\theta_z\colon T_zP \to E$ is an isomorphism for all z.

It will be natural to assume that there is a representation ρ of G on E and satisfies the following conditions:

(T1) $$R_a^*\theta = \rho(a)^{-1}\theta \quad \text{for } a \in G,$$

where R_a denotes the right translation by a.
(T2) There is an exact sequence of G-modules:

$$0 \to \mathfrak{g} \to E,$$

where the Lie algebra $\mathfrak{g}$ of G is regarded as a G-module by the adjoint action of G on $\mathfrak{g}$.

(T3) $$\theta(\widetilde{A}) = A \quad \text{for } A \in \mathfrak{g},$$

where $\widetilde{A}$ denotes the vector field on P induced by the right translations $\{R_{\exp tA}\}$.

The structure $(P, M, G, E, \rho, \theta)$ as above, with some additional assumptions mentioned later on, will be called a *tower* on M with (algebraic) *skeleton* (G, E, ρ).

If E happens to be a finite dimensional Lie algebra $\mathfrak{l}$ containing $\mathfrak{g}$ as a Lie subalgebra, then the tower is just a principal fibre bundle with a Cartan connection. If moreover the structure function γ defined by the structure equation;

$$d\theta + \frac{1}{2}\gamma(\theta \wedge \theta) = 0$$

is constant, then the tower represents locally a homogeneous space: $L \to L/G$, where L is a Lie group with Lie algebra $\mathfrak{l}$ and θ is the Maurer-Cartan form of L.

It should be noted that towers P (and hence G, E) may be infinite dimensional.

Example. The infinite order frame bundle $\mathcal{F}^\infty(M)$ of a differentiable manifold M is defined to be the set of all infinite order jet $j_0^\infty f$, where $f\colon \mathbb{R}^n \to M$ is a local diffeomorphism from a neighbourhood of the origin $0 \in \mathbb{R}^n$ into M. This is a principal fibre bundle (of infinite dimension) over M of which the structure group $G^\infty(\mathbb{R}^n)$ is the group consisting of all $j_0^\infty g$, where $g\colon (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ is a local diffeomorphism with $g(0) = 0$. Let $L = J_0^\infty T\mathbb{R}^n$ denote the set of all ∞-jet at 0 of local vector fields on a neighbourhood of 0 in $\mathbb{R}^n$, i.e., the Lie algebra of all formal vector fields at 0. Since the tangent space of $\mathcal{F}^\infty(M)$ at $j_0^\infty f$ may be identified with $J_{f(0)}^\infty TM$, the map

$$j(f_*^{-1})\colon J_{f(0)}^\infty TM \to J_0^\infty T\mathbb{R}^n$$

defines a L-valued 1-form θ on $\mathcal{F}^\infty(M)$.

What is important is that every tower has a natural filtration which represents implicitly the notion of "differentiation". Let (P, M, G, θ, E) be a tower. Then there is a canonical filtration $\mathfrak{f}_{tr}$ of G and $\mathfrak{g}$ (and then on the tangent bundle of P) defined inductively by the following exact sequences:

$$1 \longrightarrow \mathfrak{f}_{tr}^{p+1}G \longrightarrow G \longrightarrow GL(E/\mathfrak{f}_{tr}^p\mathfrak{g}) \tag{2.1}$$

with $\mathfrak{f}_{tr}^0 G = G$ and $\mathfrak{f}_{tr}^p\mathfrak{g}$ denoting the Lie algebra of $\mathfrak{f}_{tr}^p G$. This is, so to speak, the filtration according to the Taylor expansion.

It is natural to assume the action of G is *formally effective* in the following sense:

$$\bigcap_{p\in\mathbb{Z}} \mathfrak{f}_{tr}^p G = \{e\}. \tag{T4}$$

Now let us turn our attention to filtered manifolds. A tower (P, M, G, θ) with skeleton (E, G, ρ) is called a tower on a filtered manifold $\mathbb{M}$ if there is a filtration $\{\mathfrak{f}^p E/\mathfrak{g}\}$ of $E/\mathfrak{g}$ invariant under the action of G and if θ preserves the filtrations, that is, for all $z \in P$ the map $T_{\pi(z)}M \to E/\mathfrak{g}$ induced by θ_z preserves the filtrations.

In this case the skeleton (E, G, ρ) leaves invariant the filtration $\{\mathfrak{f}^p E/\mathfrak{g}\}$, and we can introduce another filtration $\{\mathfrak{f}^p G\}$ of G by the following exact sequence:

$$(2.2) \qquad 1 \longrightarrow \mathfrak{f}^{p+1}G \longrightarrow G \longrightarrow GL(E/\mathfrak{f}^p\mathfrak{g})/\mathfrak{f}^{p+1}GL(E/\mathfrak{f}^p\mathfrak{g}),$$

with $\mathfrak{f}^0 G = G$, where the filtration $\{\mathfrak{f}^k\}$ of $E/\mathfrak{f}^p\mathfrak{g}$ is the induced one from that of $E/\mathfrak{g}$ for $k \leq 0$ and from $\mathfrak{f}^k\mathfrak{g}$ for $k \geq 0$, and the filtration $\mathfrak{f}^j GL(E/\mathfrak{f}^p\mathfrak{g})$ is the natural one induced from that of $E/\mathfrak{f}^p\mathfrak{g}$.

This filtration derives from the notion of weighted order of the filtered manifold and will be used exclusively in studying the towers on the filtered manifold.

So far we have not specified the category in which possibly infinite dimensional objects P, G, E are considered. By virtue of the filtrations introduced above, we see the proper category is that of projective limits of finite dimensional objects, where we can freely speak of principal bundles, Lie groups, their Lie algebras, differential forms etc. as in the finite dimensional case.

2.2. To define the morphisms of towers on filtered manifolds, we make the following convention:

We choose once for all one filtered vector space $\mathbb{V} = (V, \{V^p\})$ for each isomorphic class of filtered vector spaces, for instance, $V = \mathbb{R}^{\dim V}$, $V^p = \mathbb{R}^{\dim V^p}$ with fixed standard inclusions, and we assign to each filtered manifold $\mathbb{M}$ one such filtered vector space $\mathbb{V}$ called the type (filtered vector space) of $\mathbb{M}$ such that $T_x\mathbb{M}$ is isomorphic to $\mathbb{V}$ for all $x \in M$. Thus two filtered manifolds are of the same type if they are isomorphic.

If P is a tower on a filtered manifold $\mathbb{M}$ with skeleton (E, G), then $E/\mathfrak{g}$ is isomorphic to the type space $\mathbb{V}$ of $\mathbb{M}$. We shall always fix one splitting $\mathbb{V} \to E$ for the skeleton (E, G) so that we have:

$$(\mathrm{T5}) \qquad E = V \oplus \mathfrak{g}$$

Now let $\mathbb{M}$ and $\mathbb{M}'$ be filtered manifolds of the same type $\mathbb{V}$. Let $P(\mathbb{M}, G, \theta)$ and $P(\mathbb{M}', G', \theta')$ be towers on $\mathbb{M}$ and $\mathbb{M}'$ respectively. A morphism of towers P to P' is a homomorphism of principal fibre bundles $\varphi\colon P \to P'$ with the induced diffeomorphism $\varphi^{(-1)}\colon M \to M'$ and the induced Lie homomorphism $\iota\colon G \to G'$ such that

$$\varphi^*\theta' = \iota_* \circ \theta,$$

where ι_* denotes the induced map:

$$\iota_*(=\mathrm{Id}_V + \iota_*)\colon E(= V \oplus \mathfrak{g}) \to E'(= V \oplus \mathfrak{g}').$$

A morphism φ will be referred to as an isomorphism if φ is a diffeomorphism, and as an embedding if $M = M'$ and $\varphi^{(-1)} = \mathrm{Id}_M$.

The category of towers has the following remarkable properties:

Proposition 2.1. *For a filtered manifold* $\mathbb{M}$ *there exists a universal tower on* $\mathbb{M}$ *such that any tower on* $\mathbb{M}$ *is uniquely embedded in the universal tower.*

We denote the universal tower of the filtered manifold $\mathbb{M}$ of type $\mathbb{V}$ by $(\mathcal{R}(\mathbb{M}), M, G(\mathbb{V}), \theta_{\mathcal{R}})$ and its skeleton by $(E(\mathbb{V}), G(\mathbb{V}))$.

Using the filtration introduced by (2.2), we set:

$$\mathcal{R}^{(k)}(\mathbb{M}) = \mathcal{R}(\mathbb{M})/\mathfrak{f}^{k+1},$$

the quotient bundle by the action of $\mathfrak{f}^{k+1}G(\mathbb{V})$. It is a principal bundle over M with structure group $G^{(k)}(\mathbb{V}) = G(\mathbb{V})/\mathfrak{f}^{k+1}$ and is referred to as the *non-commutative frame bundle* of $\mathbb{M}$ of (weighted) order $k+1$.

We say that a principal subbundle $P^{(k)}$ of $\mathcal{R}^{(k)}(\mathbb{M})$ is *adapted* if there exists a tower P on $\mathbb{M}$ such that $P^{(k)} = P/\mathfrak{f}^{k+1}$. We then have:

Proposition 2.2. *For an adapted subbundle* $P^{(k)}$ *of* $\mathcal{R}^{(k)}(\mathbb{M})$ *there exists a unique universal tower* $\mathcal{R}P^{(k)}$ *such that* $\mathcal{R}P^{(k)}/\mathfrak{f}^{k+1} = P^{(k)}$ *and any tower* Q *on* $\mathbb{M}$ *is embedded in* $\mathcal{R}P^{(k)}$ *if* $Q/\mathfrak{f}^{k+1} \subset P^{(k)}$.

We have also:

Proposition 2.3. *If* $f\colon \mathbb{M} \to \mathbb{M}'$ *is an isomorphism of filtered manifolds, it canonically induces isomorphisms* $\mathcal{R}f\colon \mathcal{R}(\mathbb{M}) \to \mathcal{R}(\mathbb{M}')$ *and* $\mathcal{R}^{(k)}f\colon \mathcal{R}^{(k)}(\mathbb{M}) \to \mathcal{R}^{(k)}(\mathbb{M}')$. *Moreover if* $P^{(k)}$ *is an adapted subbundle of* $\mathcal{R}^{(k)}(\mathbb{M})$, *then* $(\mathcal{R}^{(k)}f)P^{(k)}$ *is also adapted and* $(\mathcal{R}f)(\mathcal{R}P^{(k)}) = \mathcal{R}((\mathcal{R}^{(k)}f)P^{(k)})$.

The tower $\mathcal{R}P^{(k)}$ is called the universal tower prolonging $P^{(k)}$ or the universal prolongation of $P^{(k)}$. We set

$$\#P^{(k)} = \mathcal{R}P^{(k)}/\mathfrak{f}^{k+2}$$

and call it also the prolongation of $P^{(k)}$.

The above universal properties completely characterize the inductive construction of $\mathcal{R}^{(p)}(\mathbb{M})$ and $\mathcal{R}P^{(k)}/\mathfrak{f}^{p+1}$ to obtain $\mathcal{R}(\mathbb{M})$ and $\mathcal{R}P^{(k)}$ as their projective limits.

The first order frame bundle $\mathcal{R}^{(0)}(\mathbb{M})$ of a filtered manifold $\mathbb{M}$ of type $\mathbb{V}$ is given as follows: Let $\hat{\mathcal{R}}^{(0)}(\mathbb{M})$ be the set of all linear frames

$$\hat{z}\colon \mathbb{V} \to T_x\mathbb{M}$$

preserving the filtrations. It is a principal fibre bundle over $\mathbb{M}$ whose structure group is $\mathfrak{f}^0GL(\mathbb{V})$, the group of all filtration preserving linear isomorphisms. Then we see

$$\mathcal{R}^{(0)}(\mathbb{M}) = \hat{\mathcal{R}}^{(0)}(\mathbb{M})/\mathfrak{f}^1,$$

namely, the quotient bundle by the action of $\mathfrak{f}^1GL(\mathbb{V})$, which is a principal fibre bundle over $\mathbb{M}$ with structure group $G^{(0)}(\mathbb{V}) = \mathfrak{f}^0GL(\mathbb{V})/\mathfrak{f}^1$.

In other word, $\mathcal{R}^{(0)}(\mathbb{M})$ is a the set of all isomorphisms of graded vector spaces

$$z\colon gr\mathbb{V} \to grT_x\mathbb{M}.$$

Then the inductive construction can be carried out by the following properties:

(1) Every subbundle of $\mathcal{R}^{(0)}(\mathbb{M})$ is adapted.

(2) For $k > 0$, a subbundle $P^{(k)}$ of $\mathcal{R}^{(k)}(\mathbb{M})$ is adapted if and only if so is $P^{(k-1)} = P^{(k)}/\mathfrak{f}^k$ and $P^{(k)}$ is a subbundle of $\#P^{(k-1)}$.

(3) To every adapted subbundle $(P^{(k)}, G^{(k)})$ of $\mathcal{R}^{(k)}(\mathbb{M})$, there is associated the principal fibre bundle $(\hat{\#}P^{(k)}, \hat{\#}G^{(k)})$ over $\mathbb{M}$ consisting of filtration preserving linear isomorphisms

$$\hat{z}^{k+1}\colon V \oplus \mathfrak{g}^{(k)} \to T_{z^k}P^{(k)} \quad (\text{with } z^k \in P^{(k)})$$

such that

i) $\hat{z}^{k+1}(A) = \widetilde{A}_{z^k} \quad$ for $A \in \mathfrak{g}^{(k)}$,

ii) $[\hat{z}^k] = z^k$, where $[\hat{z}^k]$ denotes the equivalence class modulo $\mathfrak{f}^{k+1}$ and $\hat{z}^k$ is defined by the commutative diagramme:

$$\begin{array}{ccc} V \oplus \mathfrak{g}^{(k)} & \xrightarrow{\hat{z}^{k+1}} & T_{z^k}P^{(k)} \\ \downarrow & & \downarrow{\scriptstyle \pi_*} \\ V \oplus \mathfrak{g}^{(k-1)} & \xrightarrow{\hat{z}^{k}} & T_{z^{k-1}}P^{(k-1)} \end{array}$$

(4) For an adapted subbundle $P^{(k)}$ of $\mathcal{R}^k(\mathbb{M})$,

$$\#P^{(k)} = \hat{\#}P^{(k)}/\mathfrak{f}^{k+2}.$$

Then $\mathcal{R}(\mathbb{M})$ and $\mathcal{R}P^{(k)}$ are obtained by

$$\mathcal{R}(\mathbb{M}) = \operatorname*{proj\,lim}_{i\to\infty} \#^i \mathcal{R}^{(0)}(\mathbb{M}), \quad \mathcal{R}P^{(k)} = \operatorname*{proj\,lim}_{i\to\infty} \#^i P^{(k)}.$$

We remark that if M is a trivial filtered manifold then $\mathcal{R}(M)$ has a system of local coordinates $(x^i_{j_1,\dots,j_m})$ with $1 \le i,\ j_1, \dots, j_m \le \dim M$, $m = 0,\ 1,\ 2, \dots$, (the introduction of new variables which stand for the higher order derivatives, but without any commutation relations), while the usual infinite order frame bundle $\mathcal{F}^\infty(M)$ is embedded in $\mathcal{R}(M)$ by the equation $x^i_{j_{\sigma(1)},\dots,j_{\sigma(m)}} = x^i_{j_1,\dots,j_m}$ for all permutations σ. This is the reason why $\mathcal{R}(M)$ is called the non-commutative frame bundle of M and has a great advantage for studying curved spaces.

For instance, if the filtration is not trivial, since the filtered manifold $\mathbb{M}$ itself may not be even locally trivial, we have a priori no counter-part to $\mathcal{F}(M)$ associated to $\mathbb{M}$, but we always have the non-commutative frame bundle $\mathcal{R}(\mathbb{M})$ which is large enough to contain all curved structures.

2.3. Structure functions.

We now introduce the structure function of a tower. Let $(P, \mathbb{M}, G, \theta)$ be a tower on a filtered manifold $\mathbb{M}$ with skeleton (E, G). Since θ defines an absolute parallelism on P, there exists a unique $\operatorname{Hom}(\bigwedge^2 E, E)$-valued function γ on P which satisfies the following structure equation:

$$d\theta + \frac{1}{2}\gamma(\theta, \theta) = 0. \tag{2.3}$$

This function γ, referred to as the structure function of the tower P, has the following properties:

Proposition 2.4. *Let γ be the structure function of a tower $(P, \mathbb{M}, G, \theta)$. Then*

i) $\gamma(z)(A, X) = A \cdot X$ *for $z \in P$, $A \in \mathfrak{g}$, $X \in E$.*
ii) $\gamma(za)(X, Y) = a^{-1}\gamma(z)(aX, aY)$ *for $z \in P$, $a \in G$, X, $Y \in E$.*
iii) *If $\varphi \colon P \longrightarrow P'$ is a morphism of towers then $\varphi^*\gamma' = \gamma$, where γ' denotes the structure function of P'.*

If we denote also ρ the natural representation of G on $\operatorname{Hom}(\bigwedge^2 E, E)$, then the above formula (2) is written as

$$R_a^*\gamma = \rho(a)^{-1}\gamma.$$

Since $E = V \oplus \mathfrak{g}$, we have the direct sum decomposition:

$$\operatorname{Hom}\Big(\bigwedge^2 E, E\Big) = \operatorname{Hom}\Big(\bigwedge^2 V, E\Big) \oplus \operatorname{Hom}(\mathfrak{g} \otimes E, E).$$

Note that $\mathrm{Hom}(\bigwedge^2 V, E)$ is a G-invariant subspace of $\mathrm{Hom}(\bigwedge^2 E, E)$, while $\mathrm{Hom}(\mathfrak{g} \otimes E, E)$ is not invariant. Let β denote the element of $\mathrm{Hom}(\mathfrak{g} \otimes E, E)$ given by the action of $\mathfrak{g}$ on E:

$$\beta(A, X) = A \cdot X \quad \text{for } A \in \mathfrak{g},\ X \in E.$$

Then the representation ρ induces the affine representation of G on the affine subspace $\beta + \mathrm{Hom}(\bigwedge^2 V, E)$, and the structure function γ is a G-equivariant map from P to the affine space $\beta + \mathrm{Hom}(\bigwedge^2 V, E)$. The $\mathrm{Hom}(\bigwedge^2 V, E)$-valued function c given by

$$\gamma = \beta + c \tag{2.4}$$

is therefore the crucial part of γ and also called the structure function of P.

We next introduce the structure function of a truncated tower. Let $(P^{(k)}, \mathbb{M}, G^{(k)})$ be a truncated tower, that is, an adapted subbundle of $\mathcal{R}^{(k)}(\mathbb{M})$. Let $(P, \mathbb{M}, G, \theta)$ be any tower prolonging $P^{(k)}$, i.e., $P/\mathfrak{f}^{k+1} = P^{(k)}$ and γ its structure function. Let $\{\mathfrak{f}^p \mathrm{Hom}(\bigwedge^2 E, E)\}$ be the natural induced filtration. First of all we see that the structure function of a tower takes values in $\mathfrak{f}^0 \mathrm{Hom}(\bigwedge^2 E, E)$.

To define the structure function of $P^{(k)}$ we put

$$\mathrm{Hom}\Big(\bigwedge^2 E, E\Big)^{[k]} = \mathfrak{f}^0 \mathrm{Hom}\Big(\bigwedge^2 E, E\Big)/I^{k+1}. \tag{2.5}$$

Here $\{I^k\}$ is the filtration of $\mathrm{Hom}(\bigwedge^2 E,\ E)$, the same filtration as we used for truncated Lie algebras (cf. §1, 1.3):

$$\alpha \in I^k \mathrm{Hom}\Big(\bigwedge^2 E, E\Big) \iff \alpha(\mathfrak{f}^p E \wedge \mathfrak{f}^q E) \subset \mathfrak{f}^{p^* + q^* + k} E \qquad \forall p,\ q \in \mathbb{Z},$$

where $p^* = p$ for $p < 0$ and $p^* = 0$ for $p \geq 0$. Note that

$$\mathrm{Hom}\Big(\bigwedge^2 E, E\Big)^{[k]} \cong \mathfrak{f}^0 \mathrm{Hom}\Big(\bigwedge^2 E^{(k)}, E^{(k)}\Big)/I^{k+1}.$$

Note also that G acts on $\mathrm{Hom}(\bigwedge^2 E, E)^{[k]}$ with $\mathfrak{f}^{k+1}G$ acting trivially. Hence the structure function γ of P induces a map $\gamma^{[k]}$ which makes the following diagramme commutative:

$$\begin{array}{ccc} P & \xrightarrow{\gamma} & \mathrm{Hom}(\bigwedge^2 E, E) \\ \downarrow & & \downarrow \\ P^{(k)} & \xrightarrow{\gamma^{[k]}} & \mathrm{Hom}(\bigwedge^2 E, E)^{[k]} \end{array}$$

Moreover, with respect to the induced representation $\rho^{[k]}$ of $G^{(k)}$ on $\mathrm{Hom}(\bigwedge^2 E, E)^{[k]}$, we have

$$\gamma^{[k]}(za) = \rho^{[k]}(a^{-1})\gamma^{[k]}(z) \qquad \text{for } z \in P^{(k)},\ a \in G^{(k)}.$$

It should be noted that $\gamma^{[k]}$ does not depend of the choice of the tower P but only on $P^{(k)}$. We can also decompose $\gamma^{[k]}$ as

$$\gamma^{[k]} = \beta^{[k]} + c^{(k)}, \tag{2.6}$$

where $c^{(k)}$ is the $\mathfrak{f}^0 \mathrm{Hom}(\bigwedge^2 V, E^{(k-1)})/\mathfrak{f}^{k+1}$-component and $\beta^{[k]}$ the projection of β.

The function $\gamma^{[k]}$ as well as $c^{(k)}$ will be referred to as the structure function of $P^{(k)}$. Summarizing the above discussion, we have:

Proposition 2.5. *The structure function $\gamma^{[k]}$ of a truncated tower $(P^{(k)}, \mathbb{M}, G^{(k)})$ is a $G^{(k)}$-equivariant map*

$$\gamma^{[k]} \colon P^{(k)} \longrightarrow \mathrm{Hom}\Big(\bigwedge^2 E, E\Big)^{[k]} \cong \mathfrak{f}^0 \,\mathrm{Hom}\Big(\bigwedge^2 E^{(k)}, E^{(k)}\Big)/I^{k+1},$$

and if $\varphi^{(k)} \colon P^{(k)} \longrightarrow P'^{(k)}$ is an adapted homomorphism then

$$(\varphi^{(k)})^* \gamma'^{[k]} = \gamma^{[k]}.$$

Let $(P, M, \pi; \theta)$ be a tower with skeleton (E, G). Let us see what the tower P looks like when the structure function γ is constant.

Assume that γ is constant. Applying the exterior differentiation to the structure equation (2.3), we have

$$\gamma(\gamma(\theta, \theta), \theta) = 0,$$

which implies $\gamma(\in \mathrm{Hom}(\bigwedge^2 E, E))$ satisfies the Jacobi identify:

$$\mathfrak{S}\gamma(\gamma(x, y), z) = 0, \quad x,\ y,\ z \in E.$$

Hence the filtered vector space E, endowed with the bracket operation given by γ, becomes a Lie algebra. Moreover, as easily seen, it is a transitive filtered Lie algebra. Thus,

Proposition 2.6. *If the structure function γ of a tower P with skeleton (E, G) is constant, then (E, γ) is a transitive filtered Lie algebra.*

Thus a tower $(P, M, G; \theta)$ with constant structure function γ is an analogue of a homogeneous space $\tilde{G}/G$ with $\tilde{G}$ a Lie group possibly infinite dimensional and G its closed Lie subgroup.

We have also:

Proposition 2.7. *If the structure function $\gamma^{[k]}$ of a truncated tower $(P^{(k)}, M, G^{(k)})$ is constant, then $(E^{(k)}, \gamma^{[k]})$ is a truncated transitive filtered Lie algebra.*

It should be noted that the constancy of the structure function γ or $\gamma^{[k]}$ has a strong effect to reduce the "size" of G or $G^{(k)}$ (hence P and $P^{(k)}$) just as the passage from a tensor algebra to an enveloping algebra.

2.4. Equivalence problems.

Without much loss of generality, we may define a geometric structure of weighted order k on a filtered manifold $\mathbb{M}$ to be an adapted subbundle $P^{(k)}$ of $\mathcal{R}(\mathbb{M})^{(k)}$, which will be alternatively called a truncated tower of order k on $\mathbb{M}$.

Two geometric structures $(P^{(k)}, \mathbb{M}, G^{(k)})$ and $(P'^{(k)}, \mathbb{M}', G'^{(k)})$ are said to be isomorphic (or equivalent) if there exists an isomorphism $\varphi^{(k)}\colon P^{(k)} \longrightarrow P'^{(k)}$ of adapted subbundles. This is equivalent to saying that there exists an isomorphism $f\colon \mathbb{M} \longrightarrow \mathbb{M}'$ of filtered manifolds such that the lift $\mathcal{R}^{(k)}f\colon \mathcal{R}^{(k)}(\mathbb{M}) \longrightarrow \mathcal{R}^{(k)}(\mathbb{M}')$ sends $P^{(k)}$ onto $P'^{(k)}$.

We say that $P^{(k)}$ and $P'^{(k)}$ are locally isomorphic (or locally equivalent) at $(x, x') \in M \times M'$ if there exist neighbourhoods U, U' of x, x' respectively and an isomorphism of filtered manifolds $f\colon U \longrightarrow U'$ such that $f(x) = x'$ and that

$$\mathcal{R}^{(k)}f(P^{(k)}|_U) = P'^{(k)}|_{U'}.$$

Given a geometric structure of order $k+1$ on a filtered manifold $\mathbb{M}$, that is, a truncated tower $(P^{(k)}, \mathbb{M}, G^{(k)})$. The general procedure to find the invariants of $P^{(k)}$ proceeds as follows:

Since the structure function $\gamma^{[k]}\colon P^{(k)} \longrightarrow \mathrm{Hom}(\bigwedge^2 E, E)^{[k]}$ is a $G^{(k)}$-equivariant map, the image $\gamma^{[k]}$ decomposes into $G^{(k)}$-orbits. Suppose that it consists of a single $G^{(k)}$-orbit. Then choose a $\mathring{\gamma} \in \gamma^{[k]}(P^{(k)})$, and reduce $P^{(k)}$ to obtain $Q^{(k)} = (\gamma^{[k]})^{-1}(\mathring{\gamma})$. Note that a different choice of $\mathring{\gamma}$ yields a conjugate subbundle.

Note also that $Q^{(k)}$ may not be adapted. If it is not adapted we take an $l(< k)$ such that $Q^{(l)} = Q^{(k)}/\mathfrak{f}^{l+1}$ is adapted. (If $Q^{(k)} \to P^{(l-1)}$ is surjective then $Q^{(l)}$ is adapted. $Q^{(0)}$ is always adapted.)

Next we prolong $Q^{(l)}$ to get $\#Q^{(l)}$ and iterate this procedure.

In the course of the procedure, if the image of a structure function happens to contain more than one orbits, the geometric structure is intransitive. To treat the intransitive cases, we have to generalize our formulation to treat principal bundles whose structure groups $G^{(k)}$

may vary with parameters. For detailed discussion we refer to [Kis79], [Mor83] and [Mor93].

If the structure is transitive, then the finiteness theorem (Theorem 1.1) of generalized Spencer cohomology group assures that after a finite number of prolongation and reduction we will arrive at what we call a (weightedly) involutive truncated tower:

Definition 2.1. An adapted subbundle $(P^{(k)}, M, G^{(k)})$ of $\mathcal{R}^{(k)}(\mathbb{M})$ (namely, a truncated tower) is called weightedly involutive if the following conditions are satisfied:

i) The structure function $\gamma^{[k]}$ is constant.

ii) $H^2(grE^{(k)})_r = 0 \quad \text{for } r \geq k+1.$

Note that, in the definition above, since $\gamma^{[k]}$ is constant, $grE^{(k)}$ becomes a transitive truncated graded Lie algebra, so that it makes sense to speak of the cohomology group $H(\operatorname{Prol} grE^{(k)})$, which is denoted simply by $H(grE^{(k)})$.

We shall often use the adjective "involutive" in the extended sense of "weightedly involutive" if it is clear from the context.

Then we have:

Theorem 2.1. *For an involutive truncated tower $P^{(k)}$, we can construct, in a natural manner, a tower P with constant structure function such that $P/\mathfrak{f}^{k+1} = P^{(k)}$.*

In fact, by the vanishing of the cohomology group, it can be shown that the image of the structure function $\gamma^{[k+1]}$ of $\#P^{(k)}$ consists of a single orbit, moreover that the reduction $P^{(k+1)} = (\gamma^{[k+1]})^{-1}(\mathring{\gamma})$ is adapted and involutive for any $\mathring{\gamma} \in \operatorname{Im} \gamma^{[k+1]}$. Iterating this, we obtain a tower P with constant structure function: $P = \operatorname{proj\,lim}_l P^{(k+l)}$

Thus, after we reach an involutive tower the prolongation and reduction procedure proceeds automatically and there appear no essentially new invariants.

It should be remarked that the way of constructing $P^{(k+l)}$ from $P^{(k)}$ just correspond to the way in which truncated transitive Lie algebra $(E^{(k+l)}, \gamma^{(k+l)})$ is algebraically constructed from $(E^k, \gamma^{(k)})$.

Now, assuming the analyticity, we solve the local equivalence problem of involutive truncated towers.

Theorem 2.2. *Let $\mathbb{M}$ and $\mathbb{M}'$ be filtered manifolds of type $\mathbb{V}$, and let $(P^{(k)}, \mathbb{M}, G^{(k)})$ and $({P'}^{(k)}, \mathbb{M}', {G'}^{(k)})$ be involutive subbundle of $\mathcal{R}^{(k)}(\mathbb{M})$ and $\mathcal{R}^{(k)}(\mathbb{M}')$ with structure functions $\gamma^{[k]}$ and ${\gamma'}^{[k]}$ respectively. Then under the assumption of analyticity the following two conditions are equivalent:*

1) *$G^{(k)} = G'^{(k)}$ and $\gamma^{[k]} = \gamma'^{[k]}$*
2) *For any $(p, p') \in P^{(k)} \times P'^{(k)}$, there exist open neighbourhoods U and U' of $\pi(p)$ and $\pi'(p')$ respectively (π and π' denote the projections $P^{(k)} \to M$ and $P'^{(k)} \to M'$), and a filtration preserving analytic homomorphism $\varphi\colon U \to U'$ such that $(\mathcal{R}^{(k)}\varphi)(P^{(k)}|_U) = P'^{(k)}|_{U'}$ and that $(\mathcal{R}^{(k)}\varphi)(p) = p'$.*

To prove the theorem, it might seem rather natural to use the theory of differential equations on filtered manifolds that we shall discuss in the next section. However, the usual Cartan Kähler theorem suffices to prove it, since the structure is actually transitive and has no singularities in this case.

2.5. Cartan connections.

2.5.0. What we nowadays call Cartan connection was first introduced by E. Cartan as "éspace généralisé". It is a curved space modeled after a homogeneous space. Let us recall the definition.

Let $\mathfrak{l}$ be a Lie algebra and $\mathfrak{k}$ a Lie subalgebra of $\mathfrak{l}$. Let K be a Lie group with Lie algebra $\mathfrak{k}$ equipped with a representation $\rho\colon K \to GL(\mathfrak{l})$ such that the differential $\rho_*\colon \mathfrak{k} \to \mathfrak{gl}(\mathfrak{g})$ coincides with the adjoint representation of $\mathfrak{k}$ on $\mathfrak{l}$. By abuse of notation this representation ρ will be denoted by Ad.

Let $P(M, K)$ be a principal fibre bundle over a manifold M with structure group K. A Cartan connection in P of type $(\mathfrak{l}, K)$ is a 1-form θ on P with values in $\mathfrak{l}$ satisfying the following conditions:

i) $\theta\colon T_zP \to \mathfrak{l}$ is an isomorphism for all $z \in P$.
ii) $R_a^*\theta = Ad(a)^{-1}\theta$ for $a \in K$.
iii) $\theta(\widetilde{A}) = A$ for $A \in \mathfrak{k}$.

We know various examples of Cartan connections hitherto obtained: Riemannian, conformal, projective (cf. [Kob72]), or strongly pseudoconvex CR-structures [Tan62], and more generally certain geometric structures associated with simple graded Lie algebras [Tan79].

It then naturally arises the following question: Given a geometric structure Γ on a manifold M, is it possible to construct a principal bundle over M and a Cartan connection θ in P in such a way that (P, θ) is canonically associated with Γ?

First of all it should be remarked that our frame bundle $\mathcal{R}(M)$ has the universal property also for the Cartan connections: Assume that the pair $(\mathfrak{l}, K)$ is formally effective (see (T4) in §2, 2.1). By choosing a complementary subspace V of $\mathfrak{l}$ to $\mathfrak{k}$ we can view $(\mathfrak{l}, K, Ad)$ as a skeleton over V. Then it is clear that a Cartan connection (P, M, K, θ) of type

$(\mathfrak{l}, K)$ is a tower over M. Hence, by Proposition 2.1, there exists a unique embedding $\iota\colon P \to \mathcal{R}(M)$ such that $\iota^*\theta_{\mathcal{R}} = \theta$.

Thus the problem of finding a Cartan connection is reduced to the problem of constructing, for a given tower Q (or truncated tower $Q^{(k)}$ of order k) on a filtered manifold (M, F), a sub-tower $P(M, \theta, E)$ in a canonical way so that E becomes a Lie algebra.

In the next subsections we will give a general criterion and a unified method to construct Cartan connections.

2.5.1. First we need to introduce reduced frame bundles. Let $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{m}_p$ be a graded Lie algebra. We say that a filtered manifold $\mathbb{M}$ is regular (of type $\mathfrak{m}$) if the symbol algebra $grT_x\mathbb{M}$ $(x \in \mathbb{M})$ are all isomorphic (to $\mathfrak{m}$) as graded Lie algebras.

Let $(\mathcal{R}^{(0)}(\mathbb{M}), \mathbb{M}, G^{(0)}(\mathbb{V}))$ be the first order frame bundle of $\mathbb{M}$. Then, we have immediately:

Proposition 2.8. *A filtered manifold is regular if and only if the structure function of $\mathcal{R}^{(0)}(\mathbb{M})$ takes its values in a single $G^{(0)}(\mathbb{V})$-orbit.*

Given a filtered manifold $\mathbb{M}$ regular of type $\mathfrak{m}$, we shall identify $\mathfrak{m}$ with the type filtered vector space $\mathbb{V}$ and also with $gr\mathbb{V}$ (as filtered vector space or as graded vector space). Let $\gamma^{[0]} = \beta^{[0]} + c^{(0)}$ be the structure function of $\mathcal{R}^{(0)}(\mathbb{M})$. Then $c^{(0)}$ may be considered as taking values in $\mathrm{Hom}(\bigwedge^2 \mathfrak{m}, \mathfrak{m})_0$. Let $c^{(0)}_{\mathfrak{m}}$ be the bilinear map which defines the bracket operation of $\mathfrak{m}$. We set

$$\mathcal{R}^{(0)}(\mathbb{M}, \mathfrak{m}) = \{z \in \mathcal{R}^{(0)}(m) \mid c^{(0)}(z) = c^{(0)}_{\mathfrak{m}}\}.$$

Then $\mathcal{R}^{(0)}(\mathbb{M}, \mathfrak{m})$ is a principal subbundle of $\mathcal{R}^{(0)}(\mathbb{M})$. Its structure group, denoted by $G^{(0)}(\mathfrak{m})$, consists of all automorphisms of the graded Lie algebra $\mathfrak{m}$. In other words, $\mathcal{R}^{(0)}(\mathbb{M}, \mathfrak{m})$ is nothing but the set of all isomorphisms $z\colon \mathfrak{m} \to grT_x\mathbb{M}$ of graded Lie algebras.

We shall denote by $\mathcal{R}(\mathbb{M}, \mathfrak{m})$ the universal tower $\mathcal{R}\mathcal{R}^{(0)}(\mathbb{M}, \mathfrak{m})$ prolonging $\mathcal{R}^{(0)}(\mathbb{M}, \mathfrak{m})$, and by $(E(\mathfrak{m}), G(\mathfrak{m}))$ its skeleton. Hence $E(\mathfrak{m}) = \mathfrak{m} \oplus \mathfrak{g}(\mathfrak{m})$, where $\mathfrak{g}(\mathfrak{m})$ is the Lie algebra of $G(\mathfrak{m})$. We set $\mathcal{R}^{(k)}(\mathbb{M}, \mathfrak{m}) = \mathcal{R}(\mathbb{M}, \mathfrak{m})/\mathfrak{f}^{k+1}$ and call it the reduced frame bundle of $\mathbb{M}$ of order $k+1$.

Now let us examine the structure function of $\mathcal{R}(\mathbb{M}, \mathfrak{m})$. We define a bilinear map

$$\beta_{\mathfrak{m}} \quad = [\ ,\ ]\colon E(\mathfrak{m}) \times E(\mathfrak{m}) \to E(\mathfrak{m})$$

by

$$\begin{cases} [u,v] = [u,v]_{\mathfrak{m}} & \text{(the bracket of } \mathfrak{m}) \\ [A,B] = [A,B]_{\mathfrak{g}(\mathfrak{m})} & \text{(the bracket of } \mathfrak{g}(\mathfrak{m})) \\ [A,x] = Ax & \text{(the action of } \mathfrak{g}(\mathfrak{m}) \text{ on } E(\mathfrak{m})) \end{cases}$$

for u, $v \in \mathfrak{m}$, $x \in E(\mathfrak{m})$, and A, $B \in \mathfrak{g}(\mathfrak{m})$. Note that this bracket does not satisfy the Jacobi identity. Recall that $G(\mathfrak{m})$ has the natural representation on $\mathrm{Hom}(\bigwedge^2 \mathfrak{m}, E(\mathfrak{m})$, and note that the subspace $\mathfrak{f}^1 \mathrm{Hom}(\bigwedge^2 \mathfrak{m}, E(\mathfrak{m}))$ is $G(\mathfrak{m})$-invariant. Moreover it is easy to see that the equivalence class of $\beta_{\mathfrak{m}} \mod \mathfrak{f}^1 \mathrm{Hom}(\bigwedge^2 \mathfrak{m}, E(\mathfrak{m}))$ is fixed by the induced action of $G(\mathfrak{m})$ on the quotient space. Hence $G(\mathfrak{m})$ has the induced affine representation on the affine space $\beta_{\mathfrak{m}} + \mathfrak{f}^1 \mathrm{Hom}(\bigwedge^2 \mathfrak{m}, E(\mathfrak{m}))$. Hence:

Proposition 2.9. *The structure function $\gamma_{\mathcal{R}(\mathbb{M},\mathfrak{m})}$ of $\mathcal{R}(\mathbb{M},\mathfrak{m})$ is a $G^{(k)}$-equivariant map from $\mathcal{R}(\mathbb{M},\mathfrak{m})$ to the affine space*

$$\beta_{\mathfrak{m}} + \mathfrak{f}^1 \mathrm{Hom}\Big(\bigwedge^2 \mathfrak{m}, E(\mathfrak{m})\Big).$$

We therefore write

$$\gamma_{\mathcal{R}(\mathbb{M},\mathfrak{m})} = \beta_{\mathfrak{m}} + \hat{c} \tag{2.7}$$

with $\hat{c}$ an $\mathfrak{f}^1 \mathrm{Hom}(\bigwedge^2 \mathfrak{m}, E(\mathfrak{m}))$-valued function on $\mathcal{R}(\mathbb{M},\mathfrak{m})$.

In applications, most of the first order geometric structures are defined as subbundles $P^{(0)}$ of the reduced frame bundle $\mathcal{R}^{(0)}(\mathbb{M},\mathfrak{m})$. Thus the prolongation $\mathcal{R}P^{(0)}$ is contained in $\mathcal{R}(\mathbb{M},\mathfrak{m})$ as an adapted subbundle. Clearly the structure function of an adapted subbundle of $\mathcal{R}(\mathbb{M},\mathfrak{m})$ satisfies the same properties as in Proposition 3.5.3.

2.5.2. Criterion for the existence of Cartan connections.

Now we can formulate our problem for geometric structures of first order as follows:

Let $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{m}_p$ be a graded Lie algebra and $G^{(0)}$ a Lie subgroup of $G^{(0)}(\mathfrak{m})$. Let $\mathbb{M}$ be a filtered manifold regular of type $\mathfrak{m}$. Given a principal subbundle $(P^{(0)}, M, G^{(0)})$ of $\mathcal{R}^{(0)}(\mathbb{M},\mathfrak{m})$ with structure group $G^{(0)}$. We ask whether there exists a Cartan connection (P,θ) naturally associated with $P^{(0)}$.

A little more generally, let us pose the problem for higher order geometric structures. Consider a transitive graded Lie algebra $\mathfrak{l} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{l}_p$ and write:

$$\mathfrak{l} = \mathfrak{m} \oplus \mathfrak{k},$$

where we put:

$$\mathfrak{m} = \bigoplus_{p<0} \mathfrak{l}_p, \quad \mathfrak{k} = \bigoplus_{p\geq 0} \mathfrak{l}_p.$$

Let $G(\mathfrak{m})$ be the structure group of $\mathcal{R}(\mathbb{M}, \mathfrak{m})$ and $\mathfrak{g}(\mathfrak{m})$ its Lie algebra. Then from the universal property of $G(\mathfrak{m})$ it turns out that $\mathfrak{k}$ is a Lie subalgebra of $\mathfrak{g}(\mathfrak{m})$.

Let ν be an integer (≥ 0) such that $\mathfrak{l}$ is a prolongation of $\bigoplus_{p\leq\nu} \mathfrak{l}$, that is, $H^1_r(\mathfrak{m}, \mathfrak{l}) = 0$ for $r \geq \nu + 1$.

Let $G(\mathfrak{m})^{(\nu)}$ be the structure group of $\mathcal{R}^{(k)}(\mathbb{M}, \mathfrak{m})$, i.e. $G(\mathfrak{m})^{(\nu)} = G(\mathfrak{m})/\mathfrak{f}^{\nu+1}$ and $\mathfrak{g}(\mathfrak{m})^{(\nu)}$ its Lie algebra. Then $\mathfrak{k}^{(\nu)} = \mathfrak{k}/\mathfrak{f}^{\nu+1} = \bigoplus_{p=0}^{\nu} \mathfrak{l}_p$ is a subalgebra of $\mathfrak{g}(\mathfrak{m})^{(\nu)}$.

Now given a Lie subgroup $K^{(\nu)}$ of $G^{(\nu)}(\mathfrak{m})$ with Lie algebra $\mathfrak{k}^{(\nu)}$. Let K denotes the maximal subgroup of $G(\mathfrak{m})$ such that its Lie algebra is $\mathfrak{k}$ and that $K/\mathfrak{f}^{\nu+1} = K^{(\nu)}$, which we call the prolongation of $K^{(\nu)}$. Note that K acts on $\mathfrak{l}$ naturally.

Now we consider the following complex:

$$\cdots \to \operatorname{Hom}(\mathfrak{m}, \mathfrak{l}) \xrightarrow{\partial} \operatorname{Hom}\Big(\bigwedge^2 \mathfrak{m}, \mathfrak{l}\Big) \xrightarrow{\partial} \operatorname{Hom}\Big(\bigwedge^3 \mathfrak{m}, \mathfrak{l}\Big) \xrightarrow{\partial} \cdots$$

Since $\mathfrak{m}$ is a Lie subalgebra of $\mathfrak{l}$, the coboundary operator ∂ is defined as usual. Note that the group K acts on $\operatorname{Hom}(\bigwedge^{\bullet} \mathfrak{m}, \mathfrak{l})$ and preserves the filtration $\{\mathfrak{f}^p \operatorname{Hom}(\bigwedge^{\bullet} \mathfrak{m}, \mathfrak{l})\}$.

It being prepared,

Definition 2.2. We say that a Lie subgroup $K^{(\nu)} \subset G^{(\nu)}(\mathfrak{m})$ satisfies the condition (C) if there exists a subspace

$$W \subset \mathfrak{f}^1 \operatorname{Hom}\Big(\bigwedge^2 \mathfrak{m}, \mathfrak{l}\Big)$$

such that

i) $\mathfrak{f}^1 \operatorname{Hom}(\bigwedge^2 \mathfrak{m}, \mathfrak{l}) = W \oplus \partial \mathfrak{f}^1 \operatorname{Hom}(\bigwedge^1 \mathfrak{m}, \mathfrak{l})$,

ii) W is stable under the actions of K on $\mathfrak{f}^1 \operatorname{Hom}(\bigwedge^2 \mathfrak{m}, \mathfrak{l})$.

If $P^{(\nu)}(\mathbb{M}, K^{(\nu)})$ is an adapted subbundle of $\mathcal{R}^{(\nu)}(\mathbb{M}, \mathfrak{m})$, then the structure function $\gamma^{[\nu]}$ is, according to the decomposition (2.7), written as:

$$\gamma^{[\nu]} = [\beta_{\mathfrak{m}}]^{[\nu]} + \hat{c}^{(\nu)},$$

where $\hat{c}^{(\nu)}$ takes values in

$$\mathfrak{f}^1 \operatorname{Hom}\Big(\bigwedge^2 \mathfrak{m}, \mathfrak{m} + \mathfrak{k}^{(\nu)}\Big) \Big/ I^{\nu+1} \Big(= \mathfrak{f}^1 \operatorname{Hom}\Big(\bigwedge^2 \mathfrak{m}, \mathfrak{m} + \mathfrak{k}^{(\nu)}\Big) \Big/ \mathfrak{f}^{\nu+1}\Big).$$

Now we can state:

Theorem 2.3. *The notation being as above, let $\mathfrak{l}\,(=\mathfrak{m}+\mathfrak{k})$ be a transitive graded Lie algebra with $H^1_r(\mathfrak{m},\mathfrak{l}) = 0$ for $r \geq \nu+1$. Let $K^{(\nu)}$ be a Lie subgroup of $G^{(\nu)}(\mathfrak{m})$ and K its prolongation. Suppose that $K^{(\nu)}$ satisfies the condition* (C). *Then for any filtered manifold $\mathbb{M}$ regular of type $\mathfrak{m}$ and for each adapted subbundle $P^{(\nu)}$ of $\mathcal{R}^{(\nu)}(\mathbb{M},\mathfrak{m})$ with structure group $K^{(\nu)}$ such that its structure function $\hat{c}^{(\nu)}$ takes value in $W^{(\nu)}(=W/I^{\nu+1})$, we can construct a tower $P \subset \mathcal{R}(\mathbb{M},\mathfrak{m})$ in such a way that*

i) *P is a tower on $\mathbb{M}$ with skeleton $(\mathfrak{l}, K)$,*
ii) *The structure function $\hat{c}$ of P takes values in W.*
iii) *The assignment $P^{(\nu)} \rightarrowtail P$ is compatible with equivalences.*

Thus (P,θ) is a Cartan connection of type $(\mathfrak{l},K)$ associated with $P^{(\nu)}$, where θ is the canonical form of P.

Proof. The construction of P proceeds by induction. Assume that we have constructed for an $l(\geq \nu)$ an adapted subbundle $P^{(l)}(\mathbb{M},K^{(l)})$ of $\mathcal{R}^{(l)}(\mathbb{M},\mathfrak{m})$ with structure group $K^{(l)} = K/\mathfrak{f}^{l+1}$ so that $P^{(l)}/\mathfrak{f}^{\nu+1} = P^{(\nu)}$ and that the structure function $\hat{c}^{(l)}$ takes values in $W^{(\nu)}$. We put

$$P^{(l+1)} = \{z \in \#P^{(l)} \mid \hat{c}^{(l+1)}(z) \in W^{(l+1)}\},$$

where $\#P^{(l)}$ is the prolongation of $P^{(l)}$ and $\hat{c}^{(l+1)}$ its structure function. Passing to the limit, we obtain: $P = \operatorname{proj\,lim} P^{(l)}$. □

The following proposition is useful in application.

Proposition 2.10. *The notation being as above, let $\mathfrak{l}(=\mathfrak{m}+\mathfrak{k})$ be a transitive graded Lie algebra with $H^1_r(\mathfrak{m},\mathfrak{l}) = 0$ for $r \geq \nu+1$. Let $K^{(\nu)}$ be a Lie subgroup of $G^{(\nu)}(\mathfrak{m})$ and K its prolongation and let $K^{(0)} = K/\mathfrak{f}^1$. Assume that $\mathfrak{l}$ is finite dimensional and that there exists a positive definite symmetric bilinear form*

$$(\ ,\ )\colon \mathfrak{l}\times\mathfrak{l} \to \mathbb{R}$$

satisfying:

i) $(\mathfrak{l}_p,\mathfrak{l}_q) = 0$ *if $p \neq q$.*
ii) *There exists $\tau\colon \mathfrak{k} \to \mathfrak{l}$ such that*

$$\begin{cases} \tau(\mathfrak{l}_p) \subset \mathfrak{l}_{-p} & \text{for } p \geq 0 \\ ([A,x],y) = (x,[\tau(A),y]) & \text{for } x,\ y \in \mathfrak{l},\ A \in \mathfrak{k}. \end{cases}$$

iii) *There exists $\tau_0\colon K^{(0)} \to K^{(0)}$ such that*

$$(ax,y) = (x,\tau_0(a)y) \qquad \text{for } x,\ y \in \mathfrak{l},\ a \in K^{(0)}.$$

Then $K^{(\nu)}$ *satisfies the condition* (C).

The proof is same as that given in [Mor93], though we give here a statement in a little more general form so as to apply to higher order structures(cf. [CS00]).

If the Lie algebra $\mathfrak{l}$ is semi-simple the conditions of Proposition 2.10 are satisfied. Thus Theorem 2.3 together with Proposition 2.10 covers all the existence theorems for Cartan connection hitherto known.

Proposition 2.10 also applies to the case where $\mathfrak{l}$ is given as a semi-direct product of a simple Lie algebra and its irreducible representation, which appears in the geometry of holonomic systems of differential equations ([Tan82], [Tan89], [DKM99]). For further detailed geometric studies based on Cartan connections, see [SY98], [Yam93], [Yam99], [Yat92].

§3. Differential equations on filtered manifolds

3.0. In this section we develop a general study of differential equations on filtered manifolds. Let $\mathbb{M}$ be a filtered manifold. Recalling that the filtration $\mathfrak{f}$ satisfies

$$TM = \mathfrak{f}^{-\mu} \supset \mathfrak{f}^{-\mu+1} \supset \cdots \supset \mathfrak{f}^{-1} \supset \mathfrak{f}^{0} = 0,$$

we say that a local vector field X on $\mathbb{M}$ is of *weighted order* $\leq k$ if X is a section of $\mathfrak{f}^{-k}$. The minimum of such k is called the weighted order of X and denoted by w-ord$X$. A differential operator P on $\mathbb{M}$ is said to be of *weighted order* $\leq k$ if $P = \sum X_1 \cdots X_r$ (locally) for local vector fields $X_1, \ldots, X_r$ and if $\sum \text{w-ord}\, X_i \leq k$. The minimum of such k is called the weighted order of P and denoted by w-ord P. Since it is only under the inequality that we actually use the notion of order, we shall often say, by abuse of terminology, that w-ord $P = k$ if w-ord $P \leq k$. This notion of weighted order, which well accords with the filtration of a filtered manifold, was rather implicit and disguised into algebraic or geometric appearances when we studied transformation groups and geometric structures on filtered manifolds in the preceding sections, but will become explicit and play a fundamental rôle in this section.

3.1. Formal theory. We shall explain rapidly how to treat a system of differential equations on a filtered manifold by using weighted orders and introduce the notion of weighted involutivity, a sufficient condition for the system to be formally integrable. A detailed account together with some geometric applications will be treated in [Mor0x].

Let us first introduce the notion of a *weighted jet bundle*. Consider a filtered vector bundle $(E, \mathfrak{f})$ over a filtered manifold $(M, \mathfrak{f})$, that is, a vector bundle E over M of finite rank equipped with a filtration consisting of subbundles $\mathfrak{f} = \{E^p\}_{p \in \mathbb{Z}}$ and satisfying:

i) $E^p \supset E^{p+1}$

ii) There exist integers ν_I, ν_T such that $E^{\nu_I} = E$, $E^{\nu_T+1} = 0$.

Let $\underline{E}$ denote the sheaf of local sections of E and $\underline{E}_a$ the stalk over $a \in M$. First we define a filtration $\{\mathfrak{f}^k \underline{E}_a\}$ of $\underline{E}_a$ by setting $\mathfrak{f}^k \underline{E}_a$ to be the subspace of $\underline{E}_a$ consisting of $s \in \underline{E}_a$ such that

$$(P\langle \alpha^i, s\rangle)(a) = 0$$

for any differential operator P and any section α^i of the annihilating bundle $(E^{i+1})^\perp$ of E^{i+1} whenever

$$\text{w-ord}\, P + i < k.$$

We then define:

$$\mathfrak{J}^k E = \bigcup_{a \in M} \mathfrak{J}^k_a E, \quad \mathfrak{J}^k_a E = \underline{E}_a / \mathfrak{f}^{k+1} \underline{E}_a.$$

We denote by $\mathfrak{j}^k$ and $\mathfrak{j}^k_a$ the natural projections $\underline{E} \to \mathfrak{J}^k E$ and $\underline{E}_a \to \mathfrak{J}^k_a E$ respectively. It is easy to see that $\mathfrak{J}^k E$ is a vector bundle over M.

There is a natural filtration of $\mathfrak{J}^k E$ defined by $\mathfrak{f}^l \mathfrak{J}^k E = 0$ for $l \geq k+1$ and by the following exact sequences for $l \leq k$:

$$0 \longrightarrow \mathfrak{f}^{l+1} \mathfrak{J}^k E \longrightarrow \mathfrak{J}^k E \xrightarrow{\pi_{kl}} \mathfrak{J}^l E \longrightarrow 0,$$

where π_{kl} are the natural projections.

The vector bundle $\mathfrak{J}^k E$ equipped with this filtration will be called the weighted jet bundle of order k of $(E, \mathfrak{f})$ over $(M, \mathfrak{f})$.

Note that if $E^{\nu_I} = E$ then

$$\mathfrak{J}^{\nu_I - 1} E = 0, \quad \mathfrak{J}^{\nu_I} E = E^{\nu_I}/E^{\nu_I+1}, \quad \mathfrak{f}^{\nu_I} \mathfrak{J}^k E = \mathfrak{J}^k E.$$

Note also that if the filtrations of M and E are trivial, that is $T^{-1}M = TM$, $E = E^0 \supset E^1 = 0$, then the weighted jet bundle $\mathfrak{J}^k E$ reduces to the usual jet bundle $J^k E$. But it should be noted that $\mathfrak{J}^l \mathfrak{J}^k E$ and $J^l J^k E$ are different since the former respects the filtration of $\mathfrak{J}^k E$ but the latter does not.

The subbundle $\mathfrak{f}^k \mathfrak{J}^k E$ is called the *symbol* of $\mathfrak{J}^k E$. Let us describe it more explicitly. For $x \in M$, let $grT_x\mathbb{M}$, grE_x be the associated graded Lie algebra, graded vector space of $(T_x M, \mathfrak{f})$ and of $(E_x, \mathfrak{f})$ respectively, and $U(grT_x\mathbb{M})$ the universal enveloping algebra of $grT_x\mathbb{M}$. Let U_l denote the set of all homogeneous elements of degree l ($\deg\xi = \sum p_i$ if $\xi = A_1\cdots A_m$ with $A_i \in gr_{p_i}T_x\mathbb{M}$.) We denote $\mathrm{Hom}(U(grT_x\mathbb{M}), grE_x)_k$ the set of all linear map $f\colon U \to grE_x$ of degree k, namely $f(U_l) \subset gr_{l+k}E_x$. Then we have the following fundamental exact sequence of bundle maps:

$$0 \longrightarrow \mathrm{Hom}(U(grT\mathbb{M}), grE)_k \longrightarrow \mathfrak{J}^k E \longrightarrow \mathfrak{J}^{k-1}E \longrightarrow 0. \tag{3.1}$$

If the filtrations of M and E are trivial the above exact sequence reduces to the well-known one:

$$0 \longrightarrow E \otimes S^kT^*M \longrightarrow J^kE \longrightarrow J^{k-1}E \longrightarrow 0.$$

Now some elementary properties are in order:

(1) As easily seen, the map $\mathfrak{j}^k_x\colon \underline{E}_x \to \underline{\mathfrak{J}^k E}_x$ preserves the filtration, that is

$$\mathfrak{j}^k_x(\mathfrak{f}^{l+1}\underline{E}_x) \subset \mathfrak{f}^{l+1}\underline{\mathfrak{J}^k E}_x$$

for $l \in \mathbb{Z}$. Hence we have the bundle map:

$$\iota\colon \mathfrak{J}^l E \to \mathfrak{J}^l\mathfrak{J}^k E. \tag{3.2}$$

(2) If $\varphi\colon \mathbb{E} \to \mathbb{F}$ is a bundle map of degree r, that is, $\varphi(E^p) \subset F^{p+r}$ for all p, then we have the bundle map:

$$\mathfrak{j}^k\varphi\colon \mathfrak{J}^k E \to \mathfrak{J}^{k+r}F.$$

Now we are going to study differential equations on a filtered manifold, confining our discussion to the linear case for the sake of simplicity. It is not difficult to extend the following formulation to the non-linear case.

Let $\mathbb{E}$, $\mathbb{F}$ be filtered vector bundles over a filtered manifold $\mathbb{M}$. A bundle map (of degree r)

$$\varphi\colon \mathfrak{J}^k E \to F$$

is a linear differential operator of weighted order k and the kernel of φ, denoted by R, is a system of linear differential equations. A section s of E is a solution of R if $\varphi(\mathfrak{j}^k s) = 0$.

Without loss of generality we may assume that φ is of degree 0 and $E^{k+1} = F^{k+1} = 0$.

If $\varphi\colon \mathfrak{J}^k E \to F$ is a bundle map of degree 0, it induces bundle maps for $i \leq k$:

$$\begin{tikzcd}
0 \arrow[r] & R \arrow[r] \arrow[d] & \mathfrak{J}^k E \arrow[r, "\varphi"] \arrow[d] & F \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & R^i \arrow[r] & \mathfrak{J}^i E \arrow[r, "\varphi^i"] & F^{(i)} \arrow[r] & 0,
\end{tikzcd}$$

where we set $F^{(i)} = F/F^{i+1}$. We call φ^i (or R^i) differential operator (or equation) associated with φ (or R resp.).

By the following commutative diagram we define $\sigma(R^i)$ and call it the *symbol* of degree i of R:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & \sigma(R^i) \arrow[r] \arrow[d] & \mathrm{Hom}(U, grE)_i \arrow[r] \arrow[d] & gr_i F \arrow[d] \\
0 \arrow[r] & R^i \arrow[r] \arrow[d] & \mathfrak{J}^i E \arrow[r, "\varphi^i"] \arrow[d] & F^{(i)} \arrow[d] \\
0 \arrow[r] & R^{i-1} \arrow[r] & \mathfrak{J}^{i-1} E \arrow[r, "\varphi^{i-1}"] \arrow[d] & F^{(i-1)} \arrow[d] \\
 & & 0 & 0.
\end{tikzcd}$$

A bundle map $\varphi\colon \mathfrak{J}^k E \to F$ of degree 0 gives rise to bundle maps,

$$p^l\varphi\colon \mathfrak{J}^l E \to \mathfrak{J}^l F \tag{3.3}$$

for all $l \in \mathbb{Z}$, called the prolongation of φ, defined by the following commutative diagram:

$$\begin{tikzcd}
\mathfrak{J}^l \mathfrak{J}^k E \arrow[r, "\mathfrak{j}^l\varphi"] & \mathfrak{J}^l F \\
\mathfrak{J}^l E \arrow[r, "p^l\varphi"] \arrow[u, "\iota"] & \mathfrak{J}^l F. \arrow[u, "\mathrm{Id}"']
\end{tikzcd}$$

We put $\bar{\varphi}^l = p^l\varphi$ and $\bar{R}^l = \mathrm{Ker}\,\bar{\varphi}^l$, and call $\bar{\varphi} = \{\bar{\varphi}^l\}_{l\in\mathbb{Z}}$ or $\bar{R} = \{\bar{R}^l\}l \in \mathbb{Z}$ the prolongation of φ or R. We have:

1) For $i \leq l$, $\bar{R}^i$ is the associated equations of order i to $\bar{R}^l$.
2) If $l \geq k$, the prolongation of $\bar{R}^l$ coincides with $\bar{R}$.
3) For $i \leq k$, $R^i \supset \bar{R}^i$.

Before entering to the study of symbols of $\bar{R}$, we just have a look at the following graded spaces:

$$\mathrm{Hom}(U, grE) = \bigoplus_p \mathrm{Hom}(U, grE)_p,$$

whose dual space is identified with:

$$U \otimes (grE)^* = \bigoplus_q (U \otimes (grE)^*)_q,$$

where

$$(U \otimes (grE)^*)_q = \bigoplus_{a+b=q} U_a \otimes (grE)^*_b,$$

and

$$(grE)^*_b = (gr_{-b}E)^*.$$

In particular,

$$\mathrm{Hom}(U, grE)^*_p \cong (U \otimes (grE)^*)_{-p}.$$

$\mathrm{Hom}(U, grE)$ is a right U-graded module and $U \otimes (grE)^*$ a left U-graded module by means of the formula:

$$\langle t, \xi(\eta \otimes \alpha) \rangle = \langle t\xi, \eta \otimes \alpha \rangle$$

for $t \in \mathrm{Hom}(U, grE)$, $\xi \in U$, $\eta \otimes \alpha \in U \otimes (grE)^*$, and we have

$$\mathrm{Hom}(U, grE)_p U_q \subset \mathrm{Hom}(U, grE)_{p+q}.$$

Now we set

$$\mathfrak{s}_l = \begin{cases} \sigma(R^l) & \text{for } l \leq k \\ \mathrm{Hom}(U, grE)_k & \text{for } l > k, \end{cases} \qquad \bar{\mathfrak{s}}_l = \sigma(\bar{R}^l)$$

and let $\mathfrak{s}_l^\perp$ and $\bar{\mathfrak{s}}_l^\perp$ be the null spaces of $\mathfrak{s}_l$ and $\bar{\mathfrak{s}}_l$ in $(U \otimes (grE)^*)_{-l}$ respectively, and

$$\mathfrak{s} = \bigoplus \mathfrak{s}_l, \quad \bar{\mathfrak{s}} = \bigoplus \bar{\mathfrak{s}}_l$$
$$\mathfrak{s}^\perp = \bigoplus \mathfrak{s}_l^\perp, \quad \bar{\mathfrak{s}}^\perp = \bigoplus \bar{\mathfrak{s}}_l^\perp.$$

Then we have:

Proposition 3.1.

1) $\bar{\mathfrak{s}}^{\perp}$ *is a left* U*-module generated by* $\mathfrak{s}^{\perp}$; $\bar{\mathfrak{s}}^{\perp} = U\mathfrak{s}^{\perp}$.
2) $\bar{\mathfrak{s}}$ *is a right* U*-module. Furthermore,* $t \in \bar{\mathfrak{s}}_l$ *for* $t \in \mathrm{Hom}(U, grE)_l$ *if and only if* $t \in \mathfrak{s}_l$ *and if* $t\xi \in \bar{\mathfrak{s}}_{l+\deg\xi}$ *for any* $\xi \in U$.

Since $\bar{\mathfrak{s}} = \bigoplus \bar{\mathfrak{s}}_l$ is a right U-module, we can consider the following differential chain complex:

$$\mathrm{Hom}\Big(\bigwedge^p grT\mathbb{M}, \bar{\mathfrak{s}}\Big)_r \xrightarrow{\partial} \mathrm{Hom}\Big(\bigwedge^{p+1} grT\mathbb{M}, \bar{\mathfrak{s}}\Big)_r$$

defined by:

$$\begin{aligned}
&(\partial\omega)(X_1, \ldots, X_{p+1}) \\
&\quad = \sum (-1)^i \omega(X_1, \ldots, \hat{X}_i, \ldots, X_{p+1}) X_i \\
&\qquad + \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_{p+1})
\end{aligned}$$

for $\omega \in \mathrm{Hom}(\bigwedge^p grT\mathbb{M}, \bar{\mathfrak{s}})_r$ and $X_1, \ldots, X_{p+1} \in grT\mathbb{M}$.

The cohomology group $H^p_r(grT\mathbb{M}, \bar{\mathfrak{s}})$ is the generalized Spencer cohomology group. We can deduce from Theorem 1.1 the following:

Theorem 3.1. *For any* $x_0 \in M$, *there exists a neighbourhood* $\mathcal{U}$ *of* x_0 *and an integer* r_0 *such that*

$$H^p_r(gr_xT\mathbb{M}, \bar{\mathfrak{s}}_x) = 0$$

for $r \geq r_0$ *and all* p, *and for all* $x \in \mathcal{U}$.

Now we have:

Theorem 3.2. *Given a differential operator of weighted order* k

$$0 \to R^k \to \mathfrak{J}^k E \xrightarrow{\varphi} F.$$

Let $\bar{R}^i$ *be the ith prolongation. Assume that there exists an integer* $l_0 (\geq k)$ *which satisfies the following conditions*:

i) $\bar{R}^i$ *are vector bundles (i.e., rank constant) for* $i \leq l_0$.
ii) $\bar{R}^{l_0} \to \bar{R}^{l_0-1} \to \cdots \to \bar{R}^{l_0-\mu}$ *are all surjective.*
iii) $H^2_r(grT\mathbb{M}, \bar{\mathfrak{s}}) = 0$ *for* $r \geq l_0 + 1$.

Then for any $l > l_0$ *it holds*:

i) $\bar{R}^l$ *is a vector bundle.*

ii) $\bar{R}^l \to \bar{R}^{l-1}$ *is surjective.*

In particular, $\bar{R}^\infty(= \operatorname{proj\,lim} \bar{R}^l) \to \bar{R}^{l_0}$ *is surjective. Therefore the equation* R^k *has a formal solution for any prescribed* l_0*-jet in* $\bar{R}^{l_0}$.

This theorem gives a criterion for the existence of formal solutions. We say that the equations $\bar{R}$ is *weightedly involutive* if the conditions of Theorem 3.2 hold.

This theorem can be extended to the non-linear systems of differential equations.

For a single equation the criterion of weighted involutivity is easy: It is weightedly involutive if one of the highest order terms does not vanish.

3.2. By the preceding discussion, we have shown that any weightedly involutive system of differential equations on a filtered manifold has formal solutions. Now we turn to the problem of convergence under the category of analyticity. First we shall show that a weightedly involutive system has not always an analytic solution, but does have a formal solution satisfying a certain estimate, namely a formal Gevrey solution. Next, studying geometric properties of formal Gevrey functions, we shall show that if the filtered manifold satisfies the Hörmander condition then the formal Gevrey functions proves to be analytic functions, which, in turn, establish a general existence theorem of analytic solutions for a weightedly involutive analytic system of differential equations on a filtered manifold.

For our purpose, we may assume without loss of generality that our filtered manifold is a standard one, that is, a nilpotent Lie group N whose Lie algebra $\mathfrak{n}$ is graded: $\mathfrak{n} = \bigoplus_{p=1}^{\mu} \mathfrak{n}_p$ (for convenience's sake we reverse the gradation) and that the filtration of E is trivial. Choose a basis $\{X_1, \dots, X_n\}$ of $\mathfrak{n}$ such that $\{X_{d(p-1)+1}, \dots, X_{d(p)}\}$ is a basis of $\mathfrak{n}_p$, where $d(p) = \sum_{i=1}^{p} \dim \mathfrak{n}_i$. We define a weight function

$$w\colon \{1, \dots, n\} \to \{1, \dots, \mu\}$$

by the condition: $X_i \in \mathfrak{n}_{w(i)}$ for all i. For $I = (i_1, \dots, i_l) \in \{1, \dots, n\}^l$, we set

$$X_I = X_{i_1} \cdots X_{i_l}, \ w(X_I) = w(I) = \sum_{a=1}^{l} w(i_a).$$

We will regard X_I as a left invariant differential operator on N of weighted order $w(I)$. For a function F in a neighbourhood of a point $o \in N$, the values $\{(X_I F)(o) : w(I) \leq k\}$ determine the weighted k-jet $\mathrm{j}_o^k F$ of F at o.

Now a general system of non-linear differential equations of weighted order k on N can be written as:

$$\text{(E)} \qquad \Phi(x, (X_I F)(x)) = 0, \quad w(I) \le k, \ x \in N,$$

where Φ and F are vector valued functions taking values in some vector spaces, say, W and V respectively.

We say a weighted l-jet F^l at $o \in N$ $(l \ge k)$ is an l-jet solution of (E) if

$$X_J \Phi(x, (X_I F)(x)) \mid_{x=o} = 0$$

for all J such that $w(J) \le l - k$. We say also that an l-jet solution is *strongly prolongeable* (with respect to the weighted order) if for all m-jet solution $F^m (m \ge l)$ such that $\mathrm{j}_o^l F^m = F^l$ there exists $(m+1)$-jet solution F^{m+1} satisfying $\mathrm{j}_o^m F^{m+1} = F^m$.

We remark that if the system (E) is weightedly involutive then any k-jet solution is strongly prolongeable.

Example 3.1. Let $N = \mathbb{R}^2$ be an abelian Lie algebra with non-trivial gradation $\mathfrak{n} = \mathfrak{n}_1 \oplus \mathfrak{n}_2$, where $\mathfrak{n}_1 = \langle \frac{\partial}{\partial x} \rangle, \mathfrak{n}_2 = \langle \frac{\partial}{\partial t} \rangle$ in terms of coordinates (x, t) of $\mathbb{R}^2$.

Regarding N as a standard filtered manifold, we consider on N the following equation:

$$(3.4) \qquad \left(\frac{\partial}{\partial t} - a(x,t) \frac{\partial^2}{\partial x^2} \right) F = b \left(x, t, \frac{\partial F}{\partial x} \right).$$

The left-hand side of this equation is homogeneous of weighted order 2 with respect to this filtration and weightedly involutive since the coefficient of $\frac{\partial}{\partial t}$ does not vanish $(= 1)$. Hence any 2-jet solution is strongly prolongeable in the weighted sense. But if the function $a(x, t)$ vanishes at a point then the equation is not Kowalevskayan and is not involutive in the usual sense around the point. Therefore jet solutions are not always strongly prolongeable in the ordinary sense.

Example 3.2. Let (x, y, z) be coordinates of $\mathbb{R}^3$ and consider the vector fields:

$$X = \frac{\partial}{\partial x} - \frac{1}{2} y \frac{\partial}{\partial z}, \quad Y = \frac{\partial}{\partial y} + \frac{1}{2} x \frac{\partial}{\partial z}, \quad Z = \frac{\partial}{\partial z}.$$

Since $[X, Y] = Z$ and the other brackets are trivial, $\{X, Y, Z\}$ span a graded Lie algebra $\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2$, where $\mathfrak{h}_1$ is spanned by X, Y and $\mathfrak{h}_2$ by Z. We identify $\mathbb{R}^3$ with the Heisenberg group H with Lie algebra $\mathfrak{h}$. Consider a differential equation:

$$(3.5) \qquad (Z + aX^2 + bXY + cY^2)F = h(x, y, z, XF, YF),$$

where a, b, c, are functions of x, y, z. Since the left-hand side of the equation is homogeneous of weighted order two, the equation is weightedly involutive. But it is not involutive in the usual sense at the points where a, b, c simultaneously vanish.

Now a fundamental problem is, given an l_0-jet solution F^{l_0} with $l_0 \geq k$ of (E), to find an analytic solution F of (E) such that $\mathrm{j}_o^{l_0} F = F^{l_0}$.

However, there are not always such analytic solutions as seen for the equation (3.4) in Example 3.1. In fact, it is easy to choose function $a(x,t)$ and b satisfying $a(0,0) = 0$ in such a way that there exists an l-jet solution at $(0,0)$ which cannot be prolonged to any analytic solution.

Now we introduce the formal Gevrey functions on a graded nilpotent Lie group N. A formal function F at $o \in N$ is called *formal Gevrey* if there exist positive constants C, ρ such that

$$|(X_I F)(o)| \leq C w(I)!\, \rho^{w(I)} \quad \text{for all multi-index } I. \tag{3.6}$$

The first fundamental theorem we obtain is the following:

Theorem 3.3. *Given an equation* (E). *Assume that $\Phi(x, y_I)$ is formal Gevrey with respect to x and analytic with respect to y_I at (o, y_I^0). If $F^k = (y_I^0) \in \mathfrak{J}^k(N \times V)$ is a k-jet solution of* (E) *and strongly prolongeable in the weighted sense, there exists a formal Gevrey solution F of* (E) *such that $\mathrm{j}_o^k F = F^k$.*

In the case of Example 3.1 above, the theorem asserts that for any prescribed 2-jet solution there exists a formal solution of (3.4) satisfying the estimate:

$$\left| \left(\left(\frac{\partial}{\partial x}\right)^i \left(\frac{\partial}{\partial t}\right)^j F \right)(o) \right| \leq C(i+2j)!\, \rho^{i+2j},$$

which means that F is analytic in x but Gevrey of order 2 in t. In this case it is rather easy to find such a solution by direct calculation. However, already in the case of Example 3.2, it is hard to find a formal Gevrey solution for a prescribed jet solution by direct calculation. In general, difficulties occur in finding a good algorithm to determine inductively an $(l+1)$-jet solution from a given l-jet solution since F^{l+1} is not uniquely determined from F^l.

To prove the theorem, following the method of Malgrange [Mal72], we employ the privileged neighbourhood theorem [Mal77]. In order that we generalize the privileged neighbourhood theorem to the universal enveloping algebra $U(\mathfrak{n})$ of $\mathfrak{n}$ as follows:

Any element of $U(\mathfrak{n})$ being a linear combinations of $X_I = X_{i_1} \cdots X_{i_l}$, we set

$$U_a(\mathfrak{n}) = \left\{ P = \sum c_I X_I \in U(\mathfrak{n}) \mid w(X_I) = a \right\}$$

and $\hat{U}(\mathfrak{n}) = \widehat{\bigoplus} U_a(\mathfrak{n})$ be the completion of $U(\mathfrak{n})$.

For $\rho = (\rho_1, \ldots, \rho_n)$ with $\rho_i > 0$, define a pseudo-norm $|\cdot|_\rho$ of $\hat{U}(\mathfrak{n})$: If $P_a \in U_a(\mathfrak{n})$ we set

$$|P_a|_\rho = \inf \sum |c_I| \rho^{w(I)},$$

where the infimum is taken over all expressions $P_a = \sum_{w(I)=a} c_I X_I$ of P_a. For $P = \sum P_a \in \hat{U}(\mathfrak{n})$ we put $|P|_\rho = \sum |P_a|_\rho$ and set

$$\begin{aligned} \mathcal{U}(\rho) &= \{ P \in \hat{U}(\mathfrak{n}) \mid |P|_\rho < \infty \} \\ \mathcal{U} &= \bigcup_{\rho > 0} \mathcal{U}(\rho). \end{aligned}$$

We see that $\mathcal{U}$ is noetherian and that

$$|PQ|_\rho \leq |P|_\rho |Q|_\rho \quad \text{for } P,\ Q \in \mathcal{U}(\rho).$$

Now let

$$u \colon \mathcal{U}^{m'} = (\mathcal{U} \times \cdots \times \mathcal{U}) \longrightarrow \mathcal{U}^m$$

be a left $\mathcal{U}$-linear map. We say after Malgrange [Mal77] that an $\mathbb{R}$-linear map $\lambda \colon \mathcal{U}^m \to \mathcal{U}^{m'}$ is a scission of u if $u\lambda u = u$ and that λ is adapted to a polydisk $\mathbb{P}(\rho) = \{(x_1, \ldots, x_n); |x_i| < \rho_i\}$ if there exists $C > 0$ such that

$$|\lambda P|_\rho \leq |P|_\rho$$

for all $P \in \mathcal{U}^m$. Then we have:

Theorem 3.4 (A non-commutative version of privileged neighbourhood theorem)**.** *Let $u \colon \mathcal{U}^{m'} \to \mathcal{U}^m$ be a left $\mathcal{U}$-linear map. Then one can find a scission λ of u having the following property: The set of all $\mathbb{P}(\rho)$ to which λ is adapted forms a system of fundamental neighbourhoods of zero in $\mathbb{R}^n$.*

This theorem enables us to find a formal solution satisfying a Gevrey estimate.

It then naturally arises the question what the formal Gevrey functions are.

Let us first examine it in the simplest case of three-dimensional Heisenberg group (Example 3.2). A formal Gevrey function on H at 0 is a formal function F at 0 satisfying:

$$|(X^i Y^j Z^k F)(0)| \leq C i!\, j!\, (2k)!\, \rho^{i+j+2k}.$$

Let D be the contact distribution generated by X and Y, that is,

$$D_p = \{v \in T_p\mathbb{R}^3 \mid \langle \omega, v \rangle = 0\},$$

where

$$\omega = dz - \frac{1}{2} x\, dy + \frac{1}{2} y\, dx.$$

Roughly speaking, a formal Gevrey function is analytic along the contact distribution, or a little more precisely, if $\gamma(t)$ is an analytic integral curve of D with $\gamma(0) = 0$ and if F is formal Gevrey at 0 then $F \circ \gamma$ is analytic since every higher order derivatives can be expressed in terms of those of γ and $(X^i Y^j F)(0)$ which satisfy analytic estimates. On the other hand, Chow's theorem [Cho40] implies that, since X, Y and their bracket generate the tangent space at all point, any point p can be joined to 0 by an integral curve of D. This suggests that a formal Gevrey function is already something which looks like a "real" function; we might attempt to define the value $F(p)$ to be $F(\gamma(t_1))$ by taking an integral curve γ with $\gamma(0) = 0$, $\gamma(t_1) = p$. However, the value might depend on the curve chosen.

To choose "nice" integral curves we will take curves of minimal length by making use of sub-riemannian geometry.

We define an inner product g on the subbundle D by $g = (dx)^2 + (dy)^2$. Then the length L of an integral curve $\gamma(t) (a \leq t \leq b)$ is given by:

$$\int_a^b g(\dot\gamma, \dot\gamma)^{1/2}\, dt.$$

If $(x(t), y(t), z(t))$, $a \leq t \leq b$, is an integral curve joining 0 and (x_0, y_0, z_0) then

$$z_0 = \int_a^b dz = \frac{1}{2} \int_a^b x\, dy - y\, dx$$

$$L(\gamma) = \int_a^b (\dot x^2 + \dot y^2)^{1/2}\, dt.$$

Now in this case we can easily solve the variation problem: If a curve $\gamma(t) = (x(t), y(t), z(t))$, $a \leq t \leq b$, is of minimal length among the integral curves with the fixed endpoints then the projection $\bar\gamma(t) = (x(t), y(t))$ must be an arc of a circle.

This in mind, we define the exponential mapping:

$$\psi\colon \mathbb{R}^3 \ni (\lambda, \theta, t) \to (x, y, z) \in \mathbb{R}^3$$

by

$$(x, y) = \frac{1}{\lambda} e^{i(\theta+\pi/2)}(1 - e^{i\lambda t}),$$
$$z = \frac{1}{2}\int x\,dy - y\,dx.$$

Then for each fixed λ, θ, the curve $\psi(\lambda, \theta, t)$ gives a geodesic of this subriemannian metric.

Now let F be a formal Gevrey function at 0. Then, as noted in the above, $F \circ \psi(\lambda, \theta, t)$ is analytic in t in a neighbourhood of 0. But thanks to the good parametrization of integral curves, we see moreover that $F \circ \psi$ is analytic in λ, θ, t at $t = 0$.

Here we recall Gabrièlov's theorem ([Gab73], [Izu89], [Tou90]) which just applies to our situation:

Theorem 3.5 (Gabrièlov). *If $\Psi\colon X \to Y$ is an analytic map of generically maximal rank of analytic spaces. Then a formal function F at $y \in Y$ is convergent if the pull-back $F \circ \Psi$ is convergent at a point $x \in \Psi^{-1}(y)$*

Hence we see that a formal Gevrey function on the Heisenberg Lie group turns out to be an analytic function.

The consideration above generalizes to:

Theorem 3.6. *Let N be a Lie group with Lie algebra $\mathfrak{n} = \bigoplus_{i=1}^{\mu} \mathfrak{n}_i$. If $\mathfrak{n}$ is generated by $\mathfrak{n}_1$, that is, $\mathfrak{n}_{i+1} = [\mathfrak{n}_1, \mathfrak{n}_i]$ for $i > 0$ (Hörmander condition), then the formal Gevrey functions on N are analytic.*

The outline of a proof is as follows. Let θ be the Maurer-Cartan form of N taking values in $\mathfrak{n}$, which decomposes as: $\theta = \sum \theta_i$ with θ_i taking values in $\mathfrak{n}_i$. We identify the cotangent bundle T^*N with $N \times \mathfrak{n}^*$ by assigning $(x, \lambda) \in N \times \mathfrak{n}^*$ to $\lambda \circ \theta_x$. Then the Liouville form is given by $\Theta = \lambda \circ \theta = \sum \lambda_i \circ \theta^i$, where $\lambda = \sum \lambda_i$ with $\lambda_i \in \mathfrak{n}^*$. The symplectic form is given by:

$$\Omega = d\Theta = \sum d\lambda_i \wedge \theta^i + \sum \lambda_i \wedge d\theta^i.$$

Choosing an inner product $(\ ,\ )$ on $\mathfrak{n}_1^*$, we set

$$H = \frac{1}{2}(\lambda_1, \lambda_1)$$

and let X_H be the hamiltonian vector field associated with the energy function H, i.e., the vector field determined by $X_H \rfloor d\Theta = -dH$. Let φ_t be the flow generated by X_H and

$$\Phi \colon N \times \mathfrak{n}^* \times \mathbb{R} \to N \times \mathfrak{n}^*$$

the map defined by $\Phi(x, \lambda, t) = \varphi_t(x, \lambda)$ on a neighbourhood U of $N \times \mathfrak{n}^* \times \{0\}$ Let $\iota \colon \mathfrak{n}^* \times \mathbb{R} \to \{o\} \times \mathfrak{n}^* \times \mathbb{R} \subset N \times \mathfrak{n}^* \times \mathbb{R}$ be the canonical injection and $\pi \colon N \times \mathfrak{n}^* \to N$ the canonical projection and put $\Psi = \pi \circ \Phi \circ \iota$. This map $\Psi \colon \mathfrak{n}^* \times \mathbb{R} \to N$ is the polar form of the exponential map associated to the energy H and $\Psi(\lambda, t)$ gives an extremal for each fixed λ. Then we have

Proposition 3.2. *If F is a formal Gevrey function at o of N the pull-back $\Psi^* F$ is convergent at every point $(\lambda, 0) \in \mathfrak{n}^* \times \{0\}$*

We will also have:

Proposition 3.3. *If $\mathfrak{n}$ is generated by $\mathfrak{n}_1$ then Ψ is generically of maximal rank.*

Then Theorem 3.6 follows from above two propositions and Gabrièlov's theorem.

We notice however that our original proof of Proposition 3.3 is not complete since it uses a delicate theorem of Strichard ([Str86] and [Str89]). Nevertheless, it is quite plausible that Proposition 3.3 can be verified concretely from the structure of a nilpotent graded Lie group.

While B. Jakubczyk has communicated to the author a simpler proof of Theorem 3.6 [Jak00]. His idea is as follows: Let $X_1, \ldots, X_{n_1}$ be a basis of $\mathfrak{n}_1$. In view of Chow's theorem and Sard's theorem we see that there exist $i_1, \ldots, i_s \in \{1, \ldots, n_1\}$ such that the map $\varphi \colon \mathbb{R}^s \to N$ given by

$$\varphi(t_1, \ldots, t_s) = (\exp t_1 X_{i_1} \cdots \exp t_s X_{i_s})(o)$$

is generically of maximal rank and it is easy to see that the properties of Proposition 3.2 and 3.3 are satisfied for this map φ. Hence the theorem follows from Gabrièlov's theorem.

Finally we have established the following:

Theorem 3.7. *Let* (E) *be an analytic system of non-linear partial differential equations of weighted order k on a graded nilpotent Lie group N with a Lie algebra $\mathfrak{n} = \bigoplus_{p=1}^{\mu} \mathfrak{n}_p$. Assume that $\mathfrak{n}$ is generated by $\mathfrak{n}_1$. If $F^k \in \mathfrak{J}^k_o(N \times V)$ is a weighted k-jet solution of* (E) *at $o \in N$ and strongly prolongeable, then there exists an analytic solution F of* (E) *defined in a neighbourhood of o such that $\mathfrak{j}^k_o F = F^k$.*

Theorem 3.8. *Let* (E) *be an analytic system of non-linear partial differential equations of weighted order* k *on a graded nilpotent Lie group* N *with a Lie algebra* $\mathfrak{n} = \bigoplus_{p=1}^{\mu} \mathfrak{n}_p$. *Assume that* $\mathfrak{n}$ *is generated by* $\mathfrak{n}_1$. *If* (E) *is weightedly involutive, then there exists an analytic solution for any prescribed weighted* k*-jet solution.*

It should be remarked that the above theorems apply to a wide class of systems of non-linear partial differential equations with singularities.

Thus we are led to a non-trivial generalization of the Cartan-Kähler theorem by the nilpotent analysis.

References

[CS00] A. Čap and H. Schichl, *Parabolic geometries and canonical Cartan connections*, Hokkaido Math. J., **29** (2000), 453–505.

[Car04] É. Cartan, *Sur la structure des groupes infinis de transformations*, Ann. École Norm. Sup., **21** (1904), 153–206.

[Car05] É. Cartan, *Sur la structure des groupes infinis de transformations*, Ann. École Norm. Sup., **22** (1905), 219–308.

[Car08] É. Cartan, *Les sous-groupes des groupes continus de transformations*, Ann. École Norm. Sup., **25** (1908), 57–194.

[Car09] É. Cartan, *Les groupes de transformations continus, infinis, simples*, Ann. École Norm. Sup., **26** (1909), 93–161.

[Car10] É. Cartan, *Les systèmes de Pfaff à cinq variables et les équations aux*, dérivées partielles du second ordre Ann. École Norm. Sup., **27** (1910), 109–192.

[Car31] É. Cartan, *Notice sur les travaux scientifiques*, Élie Cartan Oeuvres complètes, Partie 1, Volume 1, 1952.

[Che54] S. S. Chern, *Pseudo-groupes continus infinis*, Colloque de Geom. Diff., Strasbourg, (1954), 119–136.

[Cho40] W. L. Chow, *Über Systeme Von Linearen Partial Differentialgleichungen*, Erster Ordnung Math. Ann., **117** (1940–1941), 98–105.

[DKM99] B. Doubrov, B. Komrakov and T. Morimoto, *Equivalence of holonomic differential equations*, Lobachevskii Journal of Mathematics, **3** (1999), 39–71.

[Ehr53] C. Ehresmann, *Introduction à la théorie des structures infinitésimal et des pseudo-groupes de Lie*, Colloque International de CNRS, Strasbourg, 1953.

[Gab73] A. M. Gabrièlov, *Formal relations between analytic functions*, Ivz. Akad. Nauk SSSR, **37** (1973), 1056–1088.

[Gol67a] H. Goldschmidt, *Existence theorems for analytic linear partial differential equations*, Ann. of Math., **86** (1967), 246–270.

[Gol67b] H. Goldschmidt, *Integrability criteria for systems of non-linear partial differential equations*, J. Differential Geom., **1** (1967), 269–307.

[GS64] V. W. Guillemin and S. Sternberg, *An algebraic model of transitive differential geometry*, Bull. Amer. Math. Soc., **70** (1964), 16–47.

[Hay70] I. Hayashi, *Embedding and existence theorems of infinite Lie algebras*, J. Math. Soc. Japan, **22** (1970), 1–14.

[Kis79] K. Kiso, *Local properties of intransitive infinite Lie algebra sheaves*, Japan. J. Math., **5** (1979), 101–155.

[Izu89] S. Izumi, *The rank condition and convergence of formal functions*, Duke Math. J., **59** (1989), 241–264.

[Jak00] B. Jakubczyk, *Convergence of formal power series along vector fields and their commutators*, Ann. Polonici Math., **74** (2000), 117–132.

[Käh34] E. Kähler, Einfürhung in die Theorie der Systeme von Differentialgleichungen, Teubner, Leipzig, 1934.

[Kob72] S. Kobayashi, Transformation groups in differential geometry, Springer-Verlag, 1972.

[KN66] S. Kobayashi and T. Nagano, *On filtered Lie algebras and geometric structures,* IV, J. of Math. and Mech., **15** (1966), 163–175.

[Kur57] M. Kuranishi, *On E. Cartan's prolongtion theorem of exterior differential systems*, Amer. J. Math., **79** (1957), 1–47.

[Kur62] M. Kuranishi, Lectures on exterior differential systems, Tata Institute of Fundamental Research, Bombay, 1962.

[Mal72] B. Malgrange, *Equations de Lie* II, J. of Differential Geometry, **7** (1972), 117–141.

[Mal77] B. Malgrange, *Frobenius avec singularités* (2. *Le cas général*), Inventiones Math., **39** (1977), 67–89.

[Mor83] T. Morimoto, *Sur le problème d'équivalence des structures géométriques*, Japan. J. Math., **9-2** (1983), 293–372.

[Mor88] T. Morimoto, *Transitive Lie algebras admitting differential systems*, Hokkaido Math. J., **17** (1988), 45–81.

[Mor90] T. Morimoto, *Théorème de Cartan-Kähler dans une classe de fonctions formelles Gevrey*, C. R. Acad. Sci. Paris, **311** (1990), 433–436.

[Mor91] T. Morimoto, *Generalized Spencer cohomology groups and quasi-regular bases*, Tokyo J. Math., **14** (1991), 165–179.

[Mor93] T. Morimoto, *Geometric structures on filtered manifolds*, Hokkaido Math. J., **22** (1993), 263–347.

[Mor95] T. Morimoto, *Théorème d'existence de solutions analytiques pour des systèmes d'équations aux dérivées partielles non-linéaires avec singularités*, C. R. Acad. Sci. Paris, **321** (1995), 1491–1496.

[Mor97] T. Morimoto, *Monge-Ampère equations viewed from contact geometry*, Banach Center Publications "Symplectic singularities and geometry of gauge fields", **39** (1997), 105–120.

[Mor0x] T. Morimoto, *Differential equations on filtered manifolds*, in preparation.

[Qui64] D. Quillen, *Formal properties of over-determined systems of linear partial differential equations*, Ph. D. thesis, Harvard University, 1964.

[Sha97] R. W. Sharpe, Differential geometry, Cartan's generalization of Klein's Erlangen Program, Springer, 1997.

[SS65] I. M. Singer and S. Sternberg, *On the infinite groups of Lie and Cartan,* I, J. Analyse Math., **15** (1965), 1–114.

[Spe69] D. C. Spencer, *Overdetermined systems of linear partial differential equations*, Bull. Amer. Math. Soc., **75** (1969), 179–239.

[Ste64] S. Sternberg, Lectures on differential geometry, Prentice-Hall, 1964.

[Str86] R. S. Strichartz, *Sub-riemannian geometry*, J. Differential Geometry, **24** (1986), 221–263.

[Str89] R. S. Strichartz, *Corrections to "Sub-riemannian geometry"*, J. Differential Geometry, **30** (1989), 595–596.

[SY98] H. Sato and A. Y. Yoshikawa, *Third order ordinary differential equations and Legendre connections*, J. Math. Soc. Japan, **50** (1998), 993–1013.

[Tan62] N. Tanaka, *On the pseudo-conformal geometry of hypersurfaces of the space of n complex variables*, J. Math. Soc. Japan, **14** (1962), 397–429.

[Tan70] N. Tanaka, *On differential systems, graded Lie algebras and pseudo-groups*, J. Math. Kyoto Univ., **10** (1970), 1–82.

[Tan76] N. Tanaka, *On non-degenerate real hypersurfaces, graded Lie algebras and Cartan connections*, Japanese J. Math., **2** (1976), 131–190.

[Tan79] N. Tanaka, *On the equivalence problems associated with simple graded Lie algebras*, Hokkaido Math. J., **8** (1979), 23–84.

[Tan82] N. Tanaka, *On geometry and integration of systems of second order ordinary differential equations*, (in Japanese), Proc. Symposium on Differential Geometry, (1982), 194–205.

[Tan89] N. Tanaka, *Geometric theory of ordinary differential equations*, Report of Grant-in-Aid for Science Research MESC Japan (1989).

[Tou90] J. C. Tougeron, *Sur les racines d'une polynome à coéfficients séries formelles*, in Real analytic and algebraic geometry (ed. by M. Galbiati and A. Tognoli), Lecture Note in Math. (Springer), **1420** (1990), 325–363.

[Yam82] K. Yamaguchi, *Contact geometry of higher order*, Japan. J. Math., **8** (1982), 109–176.

[Yam93] K. Yamaguchi, *Differential systems associated with simple graded Lie algebras*, Adv. Studies in Pure Math., **22** (1993), 413–494.

[Yam99] K. Yamaguchi, *G_2-geometry of overdetermined systems of second order*, Trends in Math. (Analysis and geometry in several complex variables) (1999), 289–314.

[Yat92] T. Yatsui, *On completely reducible transitive graded*, Lie algebras of finite depth, Japan. J. Math., **18** (1992), 291–329.

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Cayley Transforms and Symmetry Conditions for Homogeneous Siegel Domains

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Abstract.

In this article we first present a family of Cayley transforms of a homogeneous Siegel domain. We then give characterizations of symmetric Siegel domains among homogeneous Siegel domains in terms of norm equalities involving the Cayley transforms. Applications of these characterizations to analysis on Siegel domains (the Berezin transforms and the Poisson kernel) are also exhibited.

Introduction

Homogeneous Siegel domains are very interesting objects for researchers working in geometry or in analysis (or in both areas). This class of domains contains Hermitian symmetric spaces, and as one sees from the study for symmetric spaces as developed in [12], [28] or else, the presence of symmetry makes the algebraic and geometric structure rich and the analysis fertile. Occasionally some of well-known facts for Hermitian symmetric spaces drastically fail to hold upon loss of symmetry. Several of these phenomena are provided in the paper [3] as striking contrasts with symmetric Siegel domains. The results announced in this article lie in the same direction, and give analytic-geometric grounds to some properties of the Laplace-Beltrami operator which the present author came across during the study of Berezin transforms. Specifically, we exhibit

(1) a norm equality which leads us to the equivalence between the commutativity of the Berezin transform with the Laplace-Beltrami operator and the symmetry of the domain,

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(2) a norm equality accounting for the equivalence between the vanishing of the Poisson kernel under the Laplace-Beltrami operator and the symmetry of the domain.

Each of the two norm equalities involves a Cayley transform which is a visible generalization of the linear fractional one in $\mathbb{C}$ mapping the right half plane onto the unit disk. Our two Cayley transforms as well as Penney's [24] differ slightly from each other in general, but coincide up to positive scalar multiples if the domain is symmetric. The Cayley transforms are presented in Section 2 in a unified manner. In the notation used there, the Cayley transform needed for the norm equality in (1) above is $\mathcal{C}_{2\mathbf{d}+\mathbf{b}}$ (see Section 3 for $\mathbf{d}$ and $\mathbf{b}$), and the one for (2) is $\mathcal{C}_{\mathbf{d}+\mathbf{b}}$. Penney's in [24] is expressed as $\mathcal{C}_{\mathbf{d}}$. An explicit formula for the inverse Cayley transform is also given in this article. The references are [19] and [22].

The norm equalities are presented in Section 3. We outline the proof of them briefly in this article. The details are quite technical, and we refer the reader to the papers [20] and [23].

Applications to the Berezin transforms and to the Poisson kernel are exhibited in Sections 4 and 5, respectively. The proofs of these results are found in [21] and [23].

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§1. Preliminaries

Let V be a finite-dimensional real vector space and Ω an open convex cone in V containing no entire line. We put $W := V_{\mathbb{C}}$, the complexification of V. The conjugation in W with respect to the real form V is written as $w \mapsto w^*$. Let U be another complex vector space of finite dimension. Let $Q\colon U \times U \to W$ be an Ω-positive Hermitian sesquilinear ($\mathbb{C}$-linear in the first variable and antilinear in the second) map. We have

$$\begin{cases} Q(u', u) = Q(u, u')^* & (u,\ u' \in U), \\ Q(u, u) \in \overline{\Omega} \setminus \{0\} & \text{for all } u \in U \setminus \{0\}. \end{cases}$$

The *Siegel domain* D corresponding to these data is defined to be

$$D := \{(u, w) \in U \times W;\ w + w^* - Q(u, u) \in \Omega\}.$$

We always assume that D is homogeneous, that is, the *Lie* group $\mathrm{Hol}(D)$ of holomorphic automorphisms of D acts transitively on D.

By [25], we can find a split solvable Lie group G which acts simply transitively on D. In G we also have a subgroup H acting linearly and simply transitively on the cone Ω. Let $\mathfrak{g} := \mathrm{Lie}(G)$ and $\mathfrak{h} := \mathrm{Lie}(H)$, the Lie algebras of G and H respectively. Since G is diffeomorphic to the complex manifold D, we have an integrable almost complex structure J on $\mathfrak{g}$. Moreover there is a linear form ω on $\mathfrak{g}$ such that $\langle x \,|\, y \rangle_\omega := \langle [Jx, y], \omega \rangle$ defines a J-invariant positive definite inner product on $\mathfrak{g}$. Such linear forms ω are said to be *admissible*. Structure theory of $\mathfrak{g}$ in [25] or [26] tells us that the orthogonal complement $\mathfrak{a}$ of the derived algebra $\mathfrak{n} := [\mathfrak{g}, \mathfrak{g}]$ is an abelian subalgebra such that $\mathfrak{a}$ acts semisimply on $\mathfrak{g}$ by adjoint representation. This gives us a root space decomposition $\mathfrak{g} = \mathfrak{a} + \sum_{\alpha \in \Delta} \mathfrak{n}_\alpha$, where Δ is a finite subset of $\mathfrak{a}^*$ explained shortly and

$$\mathfrak{n}_\alpha := \{x \in \mathfrak{n};\ [h, x] = \langle h, \alpha \rangle x \quad \text{for all } h \in \mathfrak{a}\}.$$

The dimension $r := \dim \mathfrak{a}$ is called the *rank* of $\mathfrak{g}$. The Lie algebra $\mathfrak{g}$ always possesses a direct product of r copies of $(ax+b)$-algebra, that is, a basis $H_1, \ldots, H_r$ of $\mathfrak{a}$ such that if we put $E_k := -JH_k$, then $[H_j, E_k] = \delta_{jk} E_k$. Let $\alpha_1, \ldots, \alpha_r$ be the basis of $\mathfrak{a}^*$ dual to $H_1, \ldots, H_r$. Then the elements of Δ, which we call the *roots* of $\mathfrak{g}$, are of the following form (not all possibilities need occur):

$$\begin{array}{llll} \frac{1}{2}(\alpha_m + \alpha_k) & (1 \leqq k < m \leqq r), & \frac{1}{2}(\alpha_m - \alpha_k) & (1 \leqq k < m \leqq r), \\ \frac{1}{2}\alpha_k & (1 \leqq k \leqq r), & \alpha_k & (1 \leqq k \leqq r). \end{array}$$

Moreover we have $\mathfrak{n}_{\alpha_k} = \mathbb{R} E_k$. With $\mathfrak{g}(1/2) := \sum_{i=1}^r \mathfrak{n}_{\alpha_i/2}$ and

$$\mathfrak{g}(0) := \mathfrak{a} \oplus \sum_{m>k} \mathfrak{n}_{(\alpha_m - \alpha_k)/2}, \quad \mathfrak{g}(1) := \sum_{i=1}^r \mathfrak{n}_{\alpha_i} \oplus \sum_{m>k} \mathfrak{n}_{(\alpha_m + \alpha_k)/2},$$

we have the eigenspace decomposition $\mathfrak{g} = \mathfrak{g}(0) + \mathfrak{g}(1/2) + \mathfrak{g}(1)$ of $\mathrm{ad}(H_1 + \cdots + H_r)$, which gives a gradation $[\,\mathfrak{g}(i), \mathfrak{g}(j)\,] \subset \mathfrak{g}(i+j)$, where we understand $\mathfrak{g}(i) = 0$ for $i > 1$. Then, we can take $\mathfrak{g}(1)$ as V, $(\mathfrak{g}(1/2), -J)$ as U (the subspace $\mathfrak{g}(1/2)$ being J-invariant), and $G(0) := \exp \mathfrak{g}(0)$ as the group H. Set $E := E_1 + \cdots + E_r$. The cone Ω can be taken as the H-orbit $H(E)$, and we have a diffeomorphism from H onto Ω by the orbit map $h \mapsto hE$. The sesquilinear map Q is then written as

$$(1.1) \qquad Q(u, u') = \frac{1}{2}\left([Ju, u'] - i[u, u']\right) \quad (u,\ u' \in U).$$

Finally we take $\mathsf{e} := (0, E)$ as a base point of D, so that we have a diffeomorphism $g \mapsto g \cdot \mathsf{e}$ from G onto D.

§2. Family of Cayley transforms

Let us put $A := \exp \mathfrak{a}$ and set for $t = (t_1, \ldots, t_r) \in \mathbb{R}^r$

$$a(t) := \exp(t_1 H_1 + \cdots + t_r H_r). \tag{2.1}$$

For every $\mathbf{s} = (s_1, \ldots, s_r) \in \mathbb{R}^r$ let $\chi_{\mathbf{s}}$ be the one-dimensional representation of A defined by $\chi_{\mathbf{s}}(a(t)) = \exp(\sum_k s_k t_k)$. Let $N := \exp \mathfrak{n}$ be the subgroup corresponding to $\mathfrak{n}$. Then $G = N \rtimes A$, and $\chi_{\mathbf{s}}$ extends to a positive character of G by defining $\chi_{\mathbf{s}}(n) = 1$ for $n \in N$. Let $\Delta_{\mathbf{s}}$ be the function on Ω obtained by the transfer of $\chi_{\mathbf{s}}|_H$, that is, $\Delta_{\mathbf{s}}(hE) := \chi_{\mathbf{s}}(h)$ $(h \in H)$. Evidently we have

$$\Delta_{\mathbf{s}}(hx) = \chi_{\mathbf{s}}(h)\Delta_{\mathbf{s}}(x) \qquad (h \in H,\ x \in \Omega).$$

We know that $\Delta_{\mathbf{s}}$ extends to a holomorphic function on the tube domain $\Omega + iV$ (cf. for instance [14, Corollary 2.5]).

For $\mathbf{c} \in \mathbb{R}^r$, we write $\mathbf{c} > 0$ if $c_j > 0$ for all $j = 1, \ldots, r$. Let $\mathbf{c} \in \mathbb{R}^r$ with $\mathbf{c} > 0$. Denote by D_v $(v \in V)$ the directional differentiation in the direction $v \in V$: $D_v f(x) := \frac{d}{dt} f(x + tv)\big|_{t=0}$. Define a map $\mathcal{I}_{\mathbf{c}} \colon \Omega \to V^*$ through

$$\langle v, \mathcal{I}_{\mathbf{c}}(x) \rangle := -D_v \log \Delta_{-\mathbf{c}}(x) \quad (v \in V,\ x \in \Omega).$$

By [22] we know that $\mathcal{I}_{\mathbf{c}}$ is a bijection of Ω onto the dual cone Ω^*, where

$$\Omega^* := \big\{\xi \in V^*;\ \langle x, \xi \rangle > 0 \quad \text{for all } x \in \overline{\Omega} \setminus \{0\}\big\}.$$

Moreover, $\mathcal{I}_{\mathbf{c}}$ extends analytically to a rational map $W \to W^*$ (cf. [7, Satz 1.3.3] or [22]). To get a map inverse to $\mathcal{I}_{\mathbf{c}}$ we set for every $\mathbf{s} \in \mathbb{R}^r$

$$E^*_{\mathbf{s}} := s_1 E^*_1 + \cdots + s_r E^*_r,$$

where $\langle E_i, E^*_j \rangle = \delta_{ij}$ and the E^*_j's are considered as elements of V^* by putting 0 on the orthogonal complement of $\mathbb{R}E_1 + \cdots + \mathbb{R}E_r$ in V relative to the inner product $\langle \cdot \,|\, \cdot \rangle_\omega$. The group H acts also on Ω^* simply transitively by the contragredient action $h \cdot \xi := \xi \circ h^{-1}$ $(h \in H,\, \xi \in V^*)$. We have $\mathcal{I}_{\mathbf{c}}(E) = E^*_{\mathbf{c}}$, which we choose as a base point of Ω^*. Define a function $\Delta^*_{\mathbf{c}}$ on Ω^* by $\Delta^*_{\mathbf{c}}(h \cdot E^*_{\mathbf{c}}) := \chi_{\mathbf{c}}(h)$ $(h \in H)$. Let $\mathcal{I}^*_{\mathbf{c}}$ be a map $\Omega^* \to V$ obtained by

$$\langle \mathcal{I}^*_{\mathbf{c}}(\xi), f \rangle = -D_f \log \Delta^*_{\mathbf{c}}(\xi) \quad (\xi \in \Omega^*,\ f \in V^*).$$

Then, $\mathcal{I}^*_{\mathbf{c}}$ turns out to be a bijection of Ω^* onto Ω, and extends analytically to a rational map $W^* \to W$.

Proposition 2.1 ([22]).

(1) *One has* $\mathcal{I}_{\mathbf{c}}^* = \mathcal{I}_{\mathbf{c}}^{-1}$, *so that* $\mathcal{I}_{\mathbf{c}}$ *is a birational map.*

(2) $\mathcal{I}_{\mathbf{c}}$ *is holomorphic on the tube domain* $\Omega + iV$, *and* $\mathcal{I}_{\mathbf{c}}^*$ *on* $\Omega^* + iV^*$.

(3) $\mathcal{I}_{\mathbf{c}}(\Omega + iV)$ *is contained in the holomorphic domain of* $\mathcal{I}_{\mathbf{c}}^*$, *and* $\mathcal{I}_{\mathbf{c}}^*(\Omega^* + iV^*)$ *in the holomorphic domain of* $\mathcal{I}_{\mathbf{c}}$.

Remark 2.2. In general, we cannot have $\mathcal{I}_{\mathbf{c}}(\Omega + iV) \subset \Omega^* + iV^*$. See for example [19, §5].

Now we define our Cayley transform. Regarding $E_{\mathbf{c}}^*$ canonically as a complex linear form on W, we first set for $w \in W$

$$C_{\mathbf{c}}(w) := E_{\mathbf{c}}^* - 2\mathcal{I}_{\mathbf{c}}(w + E). \tag{2.2}$$

This is for the tube domain $\Omega + iV$, and the image $C_{\mathbf{c}}(\Omega + iV)$ is in W^*. Our Cayley transform $\mathcal{C}_{\mathbf{c}}$ for the type II domain D is defined to be

$$\mathcal{C}_{\mathbf{c}}(u, w) := \big(2\langle Q(u, \cdot), \mathcal{I}_{\mathbf{c}}(w + E)\rangle, C_{\mathbf{c}}(w)\big), \tag{2.3}$$

where $u \in U$ and $w \in W$. The image $\mathcal{C}_{\mathbf{c}}(D)$ lies in $U^\dagger \oplus W^*$, where $U^\dagger$ denotes the space of antilinear forms on U.

Let us describe the inverse map of $\mathcal{C}_{\mathbf{c}}$. We first introduce a real inner product on V by $\langle x \,|\, y \rangle_{\mathbf{c}} := \langle [Jx, y], E_{\mathbf{c}}^* \rangle$. We note here that $\langle x \,|\, y \rangle_{\mathbf{c}} = D_x D_y \log \Delta_{-\mathbf{c}}(E)$. We extend it to a complex bilinear form on $W \times W$, which we denote by the same symbol. For each $f \in W^*$, an element $\iota_{\mathbf{c}}(f) \in W$ is defined by requiring that $\langle w, f \rangle = \langle w \,|\, \iota_{\mathbf{c}}(f) \rangle_{\mathbf{c}}$ for any $w \in W$. On the other hand,

$$(u_1 \,|\, u_2)_{\mathbf{c}} = 2\langle Q(u_1, u_2) \,|\, E \rangle_{\mathbf{c}}$$

defines a Hermitian inner product on U. Then, $\iota_{\mathbf{c}}(F)$ ($F \in U^\dagger$) is the element in U such that $\langle u, F \rangle = (\iota_{\mathbf{c}}(F) \,|\, u)_{\mathbf{c}}$ for any $u \in U$. Now for each $w \in W$, we obtain a complex linear operator $\varphi_{\mathbf{c}}(w)$ ($w \in W$) on U through the formula

$$(\varphi_{\mathbf{c}}(w) u_1 \,|\, u_2)_{\mathbf{c}} = 2\langle Q(u_1, u_2) \,|\, w \rangle_{\mathbf{c}}. \tag{2.4}$$

Theorem 2.3 ([22]).

(1) *The image* $\mathcal{C}_{\mathbf{c}}(D)$ *is bounded.*

(2) $\mathcal{C}_{\mathbf{c}}$ *maps* D *onto* $\mathcal{C}_{\mathbf{c}}(D)$ *birationally and biholomorphically, and one has for* $f \in W^*$ *and* $F \in U^\dagger$

$$C_{\mathbf{c}}^{-1}(f) = 2\mathcal{I}_{\mathbf{c}}^*(E_{\mathbf{c}}^* - f) - E,$$

$$\mathcal{C}_{\mathbf{c}}^{-1}(F, f) = \big(2\varphi_{\mathbf{c}}(E - \iota_{\mathbf{c}}(f))^{-1}(\iota_{\mathbf{c}}(F)), C_{\mathbf{c}}^{-1}(f)\big).$$

§3. Norm equalities

We suppose from now on that our Siegel domain D is irreducible. The Bergman metric of D induces a Hermitian inner product $(\cdot\,|\,\cdot)$ on the tangent space $T_{\mathsf{e}}(D) = U + W$. Recall that by (2.3) the image $\mathcal{C}_{\mathbf{c}}(D)$ is contained in the space $U^{\dagger} + W^{*}$, on which we import a Hermitian inner product $(\cdot\,|\,\cdot)$ from $U + W$ canonically. Let $\|\cdot\|$ be the corresponding norm.

We put for $j = 1, \dots, r$

$$d_j := \operatorname{tr} \operatorname{ad}_{\mathfrak{g}(1)}(H_j), \quad b_j := \operatorname{tr} \operatorname{ad}_{\mathfrak{g}(1/2)}(H_j). \tag{3.1}$$

Setting $\mathbf{d} := (d_1, \dots, d_r)$ and $\mathbf{b} := (b_1, \dots, b_r)$, we consider the Cayley transform $\mathcal{C}_{2\mathbf{d}+\mathbf{b}}$. Recall that the domain D is said to be symmetric if for every $z \in D$, there exists an involutive holomorphic automorphism σ_z of D such that z is an isolated fixed point of σ_z.

Theorem 3.1 ([20]). *The norm equality*

$$\|\mathcal{C}_{2\mathbf{d}+\mathbf{b}}(g \cdot \mathsf{e})\| = \|\mathcal{C}_{2\mathbf{d}+\mathbf{b}}(g^{-1} \cdot \mathsf{e})\|$$

holds for any $g \in G$ if and only if D is symmetric.

Since $\mathcal{C}_{\mathbf{c}}(\mathsf{e}) = 0$ for any $\mathbf{c} > 0$, Theorem 3.1 can be rephrased as

Theorem 3.2. *The norm equality*

$$\|h \cdot 0\| = \|h^{-1} \cdot 0\|$$

holds for any $h \in \mathcal{C}_{2\mathbf{d}+\mathbf{b}} \circ G \circ \mathcal{C}_{2\mathbf{d}+\mathbf{b}}^{-1}$ if and only if $\mathcal{C}_{2\mathbf{d}+\mathbf{b}}(D)$ is symmetric.

We first indicate the proof of the "if part" of Theorem 3.2 by granting that $\mathcal{D} := \mathcal{C}_{2\mathbf{d}+\mathbf{b}}(D)$ is the *Harish-Chandra realization* of a Hermitian symmetric space if D is symmetric. In this case $\operatorname{Hol}(\mathcal{D})$ is a semisimple Lie group, and we denote by G its connected component of the identity. Let K be the stabilizer of G at the origin. Then K is a maximal compact subgroup of G. Put $\mathsf{A} := \mathcal{C}_{2\mathbf{d}+\mathbf{b}} \circ A \circ \mathcal{C}_{2\mathbf{d}+\mathbf{b}}^{-1}$. We have a Cartan decomposition $\mathsf{G} = \mathsf{KAK}$. Every element $h \in \mathsf{G}$ is written as $h = k_1 \mathsf{a}(t) k_2$, where $k_1, k_2 \in \mathsf{K}$ and $\mathsf{a}(t) := \mathcal{C}_{2\mathbf{d}+\mathbf{b}} \circ a(t) \circ \mathcal{C}_{2\mathbf{d}+\mathbf{b}}^{-1} \in \mathsf{A}$ with $a(t)$ as in (2.1). The only thing to be noted is that K is a closed subgroup of the unitary group. Therefore $\|h \cdot 0\| = \|h^{-1} \cdot 0\|$ if and only if $\|\mathsf{a}(t) \cdot 0\|$ is invariant under $t \mapsto -t$. But this is clear from the fact that

$$\mathsf{a}(t) \cdot 0 = \sum_{j=1}^{r} (2d_j + b_j) \Big(\tanh \frac{t_j}{2} \Big) E_j^{*},$$

where we note that, D being irreducible and symmetric, both d_j and b_j are independent of j.

The proof of the "only if part" of Theorem 3.1 requires not only deep results due to Satake and Dorfmeister about characterizations of symmetric Siegel domains but hard computations. We also need a criterion for an irreducible Siegel domain to be quasisymmetric published in [4]. By saying that D is *quasisymmetric*, we mean that the cone Ω is selfdual with respect to the inner product $\langle v_1 \,|\, v_2 \rangle_{2\mathbf{d}+\mathbf{b}} = D_{v_1} D_{v_2} \log \Delta_{-2\mathbf{d}-\mathbf{b}}(E)$ $(v_1, v_2 \in V)$, see the formula (4.1) below for the Bergman kernel of D and the paper [5]. On the other hand, we define a non-associative product $v_1 v_2$ in V by

$$\langle v_1 v_2 \,|\, v_3 \rangle_{2\mathbf{d}+\mathbf{b}} = -\frac{1}{2} D_{v_1} D_{v_2} D_{v_3} \log \Delta_{-2\mathbf{d}-\mathbf{b}}(E).$$

Then, D is quasisymmetric if and only if this is a Jordan algebra product by [5, Theorem 2.1] or by the proof of [4, Proposition 3].

Proposition 3.3 (D'Atri and Dotti Miatello [4]). *D is quasisymmetric if and only if the following two conditions are satisfied:*

(1) $\dim \mathfrak{n}_{(\alpha_m+\alpha_k)/2}$ $(m > k)$ *is independent of* k, m,

(2) $\dim \mathfrak{n}_{\alpha_j/2}$ *is independent of* j.

The validity of the norm equality in Theorem 3.1 for elements g in $\exp \mathfrak{g}'$, where $\mathfrak{g}'$ varies over rank two or rank three subalgebras of $\mathfrak{g}$, together with Proposition 3.3, reduces D to a quasisymmetric domain after a lot of computations. Then we have a Jordan algebra structure in V. Furthermore, due to Dorfmeister, the linear map $\varphi := \varphi_{2\mathbf{d}+\mathbf{b}} \colon W \to \mathrm{End}_{\mathbb{C}}\, U$ defined by (2.4) for $\mathbf{c} = 2\mathbf{d} + \mathbf{b}$ turns out to be a Jordan $*$-representation, see [5] and [19]. The final reduction to a symmetric domain is to show that the Jordan structure in V and the Jordan representation φ come naturally from a Hermitian Jordan triple system (see Satake [28] and Dorfmeister [5]). Actually we use the following criterion described in [2, Corollary 1, p. 332]:

Proposition 3.4 (Dorfmeister). *Suppose that D is quasisymmetric. Then, D is symmetric if and only if there is a complete set of primitive idempotents $f_1, \dots, f_r$ in the Jordan algebra V such that with $U_k := \varphi(f_k)U$ we have $\varphi(Q(u_1, u_2))u_1 = 0$ for all $u_1 \in U_1$ and $u_2 \in U_2$.*

To verify this criterion, we consider $\mathfrak{n}_D := \mathfrak{g}(1) + \mathfrak{g}(1/2)$. It is at most 2-step nilpotent in view of our gradation of $\mathfrak{g}$. Let $N_D = \exp \mathfrak{n}_D$ be the corresponding connected and simply connected nilpotent Lie group contained in G. Writing the elements of N_D as $n(a, b)$ $(a \in \mathfrak{g}(1)$,

$b \in \mathfrak{g}(1/2)$), we see by the Campbell-Hausdorff formula that the group operation is described as

$$n(a,b)n(a',b') = n(a + a' - \operatorname{Im} Q(b,b'), b + b').$$

The group N_D acts on $U + W$ through affine transformations:

$$n(a,b) \cdot (u,w) = \Big(u + b, w + ia + \frac{1}{2}Q(b,b) + Q(u,b)\Big),$$

where $(u,w) \in U \times W$. Now verification of the criterion in Proposition 3.4 is done by inspecting the validity of the norm equality for the elements $g = n(0,u_k)n(0,u_j)$, where $u_k \in \mathfrak{n}_{\alpha_k/2}$ and $u_j \in \mathfrak{n}_{\alpha_j/2}$.

Let us proceed to the second norm equality. We know that the Silov boundary Σ of D is described as

$$\Sigma = \{(u,w) \in U \times W;\ 2\operatorname{Re} w = Q(u,u)\}. \tag{3.2}$$

Clearly Σ is the N_D-orbit $N_D \cdot 0$. Indeed the orbit map $n \mapsto n \cdot 0$ gives a diffeomorphism of N_D onto Σ. On the other hand, we denote by β the *Koszul form* on $\mathfrak{g}$ given by

$$\langle x, \beta \rangle = \operatorname{tr}(\operatorname{ad}(Jx) - J\operatorname{ad}(x)) \quad (x \in \mathfrak{g}). \tag{3.3}$$

It is known by [18, Théorème 1] that $\langle\, x \mid y \,\rangle_\beta := \langle [Jx,y], \beta \rangle$ is the real part of the Hermitian inner product of $\mathfrak{g}$ induced by the Bergman metric of D up to a positive number multiple. In particular, β is admissible.

Let $\Psi \in \mathfrak{g}$ be the element such that $\langle\, x \mid \Psi \,\rangle_\beta = \operatorname{tr}\operatorname{ad}(x)$ holds for any $x \in \mathfrak{g}$. We know that $\Psi \in \mathfrak{a}$. For any $\mathbf{s} \in \mathbb{R}^r$, let $\alpha_{\mathbf{s}} \in \mathfrak{a}^*$ be the element determined by $\chi_{\mathbf{s}}(\exp T) = \exp\langle T, \alpha_{\mathbf{s}} \rangle$ for any $T \in \mathfrak{a}$. We now consider the Cayley transform $\mathcal{C}_{\mathbf{d}+\mathbf{b}}$.

Theorem 3.5 ([23]). *The norm equality*

$$\|\mathcal{C}_{\mathbf{d}+\mathbf{b}}(\zeta)\|^2 = \langle \Psi, \alpha_{\mathbf{d}+\mathbf{b}} \rangle$$

holds for any $\zeta \in \Sigma$ if and only if the domain D is symmetric.

Here also we first outline the proof of the "if part" of Theorem 3.5. Suppose that D is symmetric. Then, since D is irreducible, one knows that both d_j and b_j in (3.1) are independent of j. Thus we can consider $\mathcal{D} := \mathcal{C}_{\mathbf{d}+\mathbf{b}}(D)$ as the Harish-Chandra realization of a bounded symmetric domain. Let G denote, as before, the connected semisimple Lie group $\operatorname{Hol}(\mathcal{D})^\circ$. Note that $\mathcal{C}_{\mathbf{d}+\mathbf{b}}(0) = -E^*_{\mathbf{d}+\mathbf{b}}$ in view of (2.3). Let K be the stabilizer of G at the origin. K is a maximal compact subgroup of G as well as a subgroup of the unitary group. By Theorem 3 in [16, p. 179],

$\mathcal{C}_{\mathbf{d}+\mathbf{b}}(\Sigma)$ is contained in the Silov boundary of $\mathcal{D}$, which equals the K-orbit $\mathsf{K}\cdot(-E^*_{\mathbf{d}+\mathbf{b}})$ (cf. Corollary in [16, p. 155]). Therefore $\|\mathcal{C}_{\mathbf{d}+\mathbf{b}}(\zeta)\|$ is independent of $\zeta \in \Sigma$, and the equality $\|E^*_{\mathbf{d}+\mathbf{b}}\|^2 = \langle \Psi, \alpha_{\mathbf{d}+\mathbf{b}} \rangle$ is readily verified because d_j and b_j are independent of j.

To outline the proof of the "only if part", we need the complexification $G_{\mathbb{C}}$ of G. Let $G(0)_{\mathbb{C}}$ be the subgroup of $G_{\mathbb{C}}$ corresponding to the subalgebra $\mathfrak{g}(0)_{\mathbb{C}}$ of $\mathfrak{g}_{\mathbb{C}}$. We rely on the following proposition for the analysis of the norm equality in Theorem 3.5.

Proposition 3.6 ([19]). *There exists a real analytic map $\eta\colon V \to G(0)_{\mathbb{C}}$ such that $\eta(y)E = E + iy$ and $\eta(0) = e$, the identity element of $G(0)_{\mathbb{C}}$.*

Note that if $y \in V$, then we have $(0, iy) \in \Sigma$. For $j < k$, let us write V_{kj} instead of $\mathfrak{n}_{(\alpha_k+\alpha_j)/2}$ for simplicity. Since $\mathcal{I}_{\mathbf{d}+\mathbf{b}}(E+iy) = \eta(y)\cdot E^*_{\mathbf{d}+\mathbf{b}}$, we can compute $\|\mathcal{C}_{\mathbf{d}+\mathbf{b}}(0, iy)\|^2 = \|C_{\mathbf{d}+\mathbf{b}}(iy)\|^2$ for y in V_{kj}, in $V_{lj} + V_{lk}$, or in $V_{kj} + V_{lj}$ for $j < k < l$. Then, the validity of the norm equality for these $\zeta = (0, iy)$, together with Proposition 3.3, yields that D is quasisymmetric, though the computations are by no means trivial. Once we reduce D to a quasisymmetric domain, we have a Jordan algebra structure in V and a Jordan $*$-representation $\varphi := \varphi_{\mathbf{d}+\mathbf{b}}$ of W just as in the previous discussion. The final reduction of D to a symmetric domain is done by using Proposition 3.4 and by analyzing the norm equality for

$$\zeta = \Big(u_j + u_k, \frac{1}{2}Q(u_j + u_k, u_j + u_k) + i\,\mathrm{Im}\,Q(u_j, u_k)\Big) \qquad (j < k),$$

where $u_j \in \mathfrak{n}_{\alpha_j/2}$ and $u_k \in \mathfrak{n}_{\alpha_k/2}$.

§4. Berezin transforms

Let us first consider the Laplace-Beltrami operator on D determined by the Bergman metric of D. Since D is diffeomorphic to our split solvable Lie group G, we have the corresponding Laplace-Beltrami operator on G, which is, up to a positive number multiple, the operator $\mathcal{L}_\beta$ defined by the left invariant Riemannian metric on G induced by the real inner product $\langle x \,|\, y \rangle_\beta$, where β is the Koszul form (3.3). To express $\mathcal{L}_\beta$ in terms of elements of the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$, we need to fix our notation. If we regard an element $X \in U(\mathfrak{g})$ as a left invariant differential operator on G, we write $\widetilde{X}$, whereas we add nothing to X when we regard X as a right invariant differential operator on G. Thus if $X \in \mathfrak{g}$, we have for smooth functions f on G

$$\widetilde{X}f(x) = \frac{d}{dt}f\big(x\exp(tX)\big)\Big|_{t=0}, \quad Xf(x) = \frac{d}{dt}f\big(\exp(-tX)x\big)\Big|_{t=0}.$$

Let $\Psi \in \mathfrak{g}$ be as in Theorem 3.5. Though Proposition 4.1 below holds for any connected Lie group, we write it down here in our situation. Let $2N := \dim \mathfrak{g}$.

Proposition 4.1 (Urakawa [29]). *One has* $\mathcal{L}_\beta = -\widetilde{\Lambda} + \widetilde{\Psi}$, *where*

$$\Lambda := X_1^2 + \cdots + X_{2N}^2 \in U(\mathfrak{g})$$

with an orthonormal basis $\{X_j\}_{j=1}^{2N}$ *of* $\mathfrak{g}$ *relative to* $\langle \cdot \,|\, \cdot \rangle_\beta$.

Let κ be the Bergman kernel of D. We have, up to a positive number multiple,

$$\kappa(z_1, z_2) = \Delta_{-2\mathbf{d}-\mathbf{b}}(w_1 + w_2^* - Q(u_1, u_2)), \tag{4.1}$$

where $z_j = (u_j, w_j) \in D$. The *Berezin kernel* A_λ ($\lambda \in \mathbb{R}$) on D is given by

$$A_\lambda(z_1, z_2) := \left(\frac{|\kappa(z_1, z_2)|^2}{\kappa(z_1, z_1)\, \kappa(z_2, z_2)} \right)^\lambda \quad (z_1,\ z_2 \in D).$$

It is G-invariant:

$$A_\lambda(g \cdot z_1, g \cdot z_2) = A_\lambda(z_1, z_2) \quad (g \in G).$$

We put $a_\lambda(g) := A_\lambda(g \cdot \mathsf{e}, \mathsf{e})$ ($g \in G$). We see easily that $a_\lambda \in L^1(G)$ with respect to the left Haar measure provided that λ is greater than some positive number $\lambda_0 < 1$ (λ_0 can be given explicitly). We have $a_\lambda(g) = a_\lambda(g^{-1})$. Consider the space $L^2(G)$ on G for the left Haar measure. The *Berezin transform* B_λ ($\lambda > \lambda_0$), when transferred to $L^2(G)$, is given by the convolution operator

$$B_\lambda f(x) := \int_G f(y) a_\lambda(y^{-1}x)\, dy = f * a_\lambda(x) \qquad (f \in L^2(G)).$$

The integral is absolutely convergent by a standard argument.

Theorem 4.2 ([21]). *Let* $\lambda > \lambda_0$ *be fixed. Then,* B_λ *commutes with* $\mathcal{L}_\beta$ *if and only if* D *is symmetric.*

We indicate here how Theorem 4.2 is derived from Theorem 3.1.

(1) B_λ commutes with $\mathcal{L}_\beta \iff (-\widetilde{\Lambda} + \widetilde{\Psi}) a_\lambda = (-\Lambda + \Psi) a_\lambda$.

(2) Since $a_\lambda(g) = a_\lambda(g^{-1})$, we have $\widetilde{X} a_\lambda(g) = X a_\lambda(g^{-1})$ for all $X \in U(\mathfrak{g})$ and $g \in G$.

Therefore we have

$$B_\lambda \text{ commutes with } \mathcal{L}_\beta \iff (\Lambda - \Psi) a_\lambda(g) = (\Lambda - \Psi) a_\lambda(g^{-1}) \ (\forall g \in G).$$

On the other hand, after a somewhat lengthy calculation we get

$$(\Lambda - \Psi)a_\lambda(g) = \lambda\, a_\lambda(g)\big(\lambda\|\mathcal{C}_{2\mathbf{d}+\mathbf{b}}(g\cdot \mathsf{e})\|^2 - \langle \Psi, \alpha_{2\mathbf{d}+\mathbf{b}}\rangle\big).$$

Thus Theorem 4.2 follows from Theorem 3.1.

§5. Poisson kernel

Let $S(z, z')$ (z, $z' \in D$) be the Szegö kernel of the Siegel domain D. It is the reproducing kernel of the Hardy space over D (cf. [11], [17]). We have, up to a positive number multiple,

$$S(z_1, z_2) = \Delta_{-\mathbf{d}-\mathbf{b}}(w_1 + w_2^* - Q(u_1, u_2)), \tag{5.1}$$

where $z_j = (u_j, w_j) \in D$. Let Σ be the Silov boundary (3.2) of D. The boundary Σ is stable under the (affine) action of $G = N_D \rtimes H$. We note that the value $S(z, \zeta)$ for $z \in D$ and $\zeta \in \Sigma$ is obtained by a simple substitution in (5.1). The *Poisson kernel* $P(z, \zeta)$ ($z \in D$, $\zeta \in \Sigma$) is defined to be

$$P(z, \zeta) := \frac{|S(z,\zeta)|^2}{S(z,z)}.$$

In what follows we set $P_\zeta^G(g) := P(g\cdot \mathsf{e}, \zeta)$ ($g \in G$). Let $\mathcal{L}_\beta$ be the Laplace-Beltrami operator on G introduced in the previous section. The following theorem is known.

Theorem 5.1 (Hua-Look-Korányi-Xu). *$\mathcal{L}_\beta P_\zeta^G = 0$ for any $\zeta \in \Sigma$ if and only if the domain D is symmetric.*

Hua and Look [13] gave a proof of the "if part" for the classical domains by direct and case-by-case calculations, and Korányi [15] for the case of general symmetric Siegel domains. Korányi's proof is via the mean-value property and actually shows a stronger property that the Poisson kernel is annihilated by *any* invariant differential operator without constant term. The "only if part" is due to [30]. However, Xu's proof is hardly traceable at least for the present author. The formula in Theorem 5.2 below clarifies the computation of Xu, and indeed gives it a geometric meaning. The formula together with Theorem 3.5 also yields a direct proof of Theorem 5.1. We remark here that since

$$P(g\cdot z, \zeta) = \chi_{-\mathbf{d}-\mathbf{b}}(g) P(z, g^{-1}\cdot \zeta) \qquad (g \in G),$$

it holds that $\mathcal{L}_\beta P_\zeta^G = 0$ for any $\zeta \in \Sigma$ if and only if $\mathcal{L}_\beta P_\zeta^G(e) = 0$ for any $\zeta \in \Sigma$.

Theorem 5.2 ([23]). *One has*

$$\mathcal{L}_\beta P_\zeta^G(e) = \big(-\|\mathcal{C}_{\mathbf{d}+\mathbf{b}}(\zeta)\|^2 + \langle \Psi, \alpha_{\mathbf{d}+\mathbf{b}}\rangle\big) P_\zeta^G(e).$$

References

[1] E. Damek, A. Hulanicki and R. Penney, *Admissible convergence for the Poisson-Szegö integrals*, J. Geom. Anal., **5** (1995), 49–76.

[2] J. E. D'Atri and J. Dorfmeister, *Flat totally geodesic submanifolds of quasisymmetric Siegel domains*, Geom. Dedicata, **28** (1988), 321–336.

[3] J. E. D'Atri, J. Dorfmeister and Zhao Yan Da, *The isotropy representation for homogeneous Siegel domains*, Pacific J. Math., **120** (1985), 295–326.

[4] J. E. D'Atri and I. Dotti Miatello, *A characterization of bounded symmetric domains*, Trans. Amer. Math. Soc., **276** (1983), 531–540.

[5] J. Dorfmeister, *Quasisymmetric Siegel domains and the automorphisms of homogeneous Siegel domains*, Amer. J. Math., **102** (1980), 537–563.

[6] J. Dorfmeister, *Homogeneous Siegel domains*, Nagoya Math. J., **86** (1982), 39–83.

[7] J. Dorfmeister and M. Koecher, *Relative Invarianten und nicht-assoziative Algebren*, Math. Ann., **228** (1977), 147–186.

[8] J. Dorfmeister and M. Koecher, *Reguläre Kegel*, Jber. Deutsch. Math.-Verein., **81** (1979), 109–151.

[9] M. Engliš, *Berezin transform and the Laplace-Beltrami operator*, St. Petersburg Math. J., **7** (1996), 633–647.

[10] J. Faraut and A. Korányi, Analysis on symmetric cones, Clarendon Press, Oxford, 1994.

[11] S. G. Gindikin, *Analysis in homogeneous domains*, Russian Math. Surveys, **19-4** (1964), 1–89.

[12] S. Helgason, Geometric analysis on symmetric spaces, Amer. Math. Soc. Providence, Rhode Island, 1994.

[13] L. K. Hua and K. H. Look, *Theory of harmonic functions in classical domains*, Scientia Sinica, **8** (1959), 1031–1094.

[14] H. Ishi, *Representations of the affine transformation groups acting simply transitively on Siegel domains*, J. Funct. Anal., **167** (1999), 425–462.

[15] A. Korányi, *The Poisson integral for generalized half-planes and bounded symmetric domains*, Ann. of Math., **82** (1965), 332–350.

[16] A. Korányi, *Holomorphic and harmonic functions on bounded symmetric domains*, In "Geometry of Homogeneous Bounded Domains (C.I.M.E., 3 Ciclo, Urbino, 1967)", 1968, 127–197, Edizioni Cremonese, Rome.

[17] A. Korányi and E. M. Stein, *H^2 spaces of generalized half-planes*, Studia Math., **44** (1972), 379–388.

[18] J. L. Koszul, *Sur la forme hermitienne canonique des espaces homogènes complexes*, Canad. J. Math., **7** (1955), 562–576.

[19] T. Nomura, *On Penney's Cayley transform of a homogeneous Siegel domain*, J. Lie Theory, **11** (2001), 185–206.

[20] T. Nomura, *A characterization of symmetric Siegel domains through a Cayley transform*, Transform. Groups, **6** (2001), 227–260.

[21] T. Nomura, *Berezin transforms and Laplace-Beltrami operators on homogeneous Siegel domains*, Diff. Geom. Appl., **15** (2001), 90–106.

[22] T. Nomura, *Family of Cayley transforms of a homogeneous Siegel domain parametrized by admissible linear forms*, to appear in Diff. Geom. Appl.

[23] T. Nomura, *Geometric norm equality related to the harmonicity of the Poisson kernel for homogeneous Siegel domains*, to appear in J. Funct. Anal.

[24] R. Penney, *The Harish-Chandra realization for non-symmetric domains in* $\mathbb{C}^n$, in "Topics in geometry in memory of Joseph D'Atri", Ed. by S. Gindikin, Birkhäuser, Boston, 1996, 295–313.

[25] I. I. Pyatetskii-Shapiro, Automorphic functions and the geometry of classical domains, Gordon and Breach, New York, 1969.

[26] H. Rossi, *Lectures on representations of groups of holomorphic transformations of Siegel domains*, Lecture Notes, Brandeis Univ., 1972.

[27] H. Rossi and M. Vergne, *Representations of certain solvable Lie groups on Hilbert spaces of holomorphic functions and the application to the holomorphic discrete series of a semisimple Lie group*, J. Funct. Anal., **13** (1973), 324–389.

[28] I. Satake, Algebraic structures of symmetric domains, Iwanami Shoten and Princeton Univ. Press, Tokyo-Princeton, 1980.

[29] H. Urakawa, *On the least positive eigenvalue of the Laplacian for compact group manifolds*, J. Math. Soc. Japan, **31** (1979), 209–226.

[30] Xu Yichao, *On the Bergman kernel function of homogeneous bounded domains*, Scientia Sinica, Special Issue (II) (1979), 80–90.

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The Canonical Contact Form

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Abstract.

The structure group and the involutive differential system that characterize the pseudo-group of contact transformations on a jet space are determined.

§1. Introduction

The canonical form on the coframe bundle over a smooth manifold originally arose as the natural generalization of the canonical form on the cotangent bundle, which plays an essential role in Hamiltonian mechanics, [19, §III.7]. The coframe bundle $\mathcal{F}^*M \to M$ forms a principal $\mathrm{GL}(m)$ bundle over the m-dimensional manifold M. The canonical form on the coframe bundle serves to characterize the diffeomorphism pseudo-group of the manifold, or, more correctly, its lift to the coframe bundle. Indeed, the invariance of the canonical form forms an involutive differential system, whose general solution, guaranteed by the Cartan-Kähler Theorem, is the lifted diffeomorphism pseudo-group. Kobayashi, [11], introduces a vector-valued canonical form on the higher order frame bundles over the manifold. He demonstrates that the components of the canonical form constitute an involutive differential system that characterizes the higher order lifts of the diffeomorphism group.

The geometrical study of differential equations relies on the jet space first introduced by Ehresmann, [6]. In the jet bundle framework, the pseudo-group of contact transformations, [13], [16], assumes the role of the diffeomorphism pseudo-group. Contact transformations are characterized by the fact that they preserve the contact ideal generated by the contact forms on the jet bundle. Thus, the characterization of the contact pseudo-group by an involutive differential system should rely on

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a "canonical contact form" constructed on a suitable principal bundle lying over the jet bundle. This canonical contact form should play the same basic role in the study of the geometry of jet bundles and differential equations that the canonical form over the coframe bundle plays in the ordinary differential geometry of manifolds and submanifolds. In [21], Yamaguchi uses the theory of exterior differential systems to conduct a detailed investigation of the contact geometry of higher order jet space, but does not provide a general construction of the required principal bundle or canonical form. This is more complicated than the frame bundle construction, since the definition of a contact transformation via the contact ideal does not directly yield an involutive differential system; see [4], [16]. One must apply the Cartan procedure of absorption and normalization of torsion in order to reduce the original structure group to the appropriate involutive version, and this in turn will yield the "minimal, involutive" version of the canonical contact form.

A crucial theorem, due to Bäcklund, [2], demonstrates that every contact transformation is either a prolonged point transformation, or, in the case of a single dependent variable, a prolonged first order contact transformation; see also [16], [20]. This allows us to restrict the structure group associated with the contact pseudo-group to one of block upper triangular form, but this still is not enough to produce an involutive differential system, and further normalizations must be imposed. In this paper, we find the complete system of normalizations, thereby constructing an involutive differential system on a certain principal bundle over the jet bundle that characterizes the contact pseudo-group.

A significant source of applications of this construction can be found in a variety of equivalence problems defined on the jet bundle, including differential equations, variational problems, and others. In such situations, one needs to incorporate the contact structure into the problem via the contact forms. The canonical contact form will provide the minimal lift that can be imposed on the contact component of the lifted coframe, and thus help avoid normalizations that are universally valid for all contact transformations. Examples include equivalence problems for differential equations, for differential operators, and for variational problems. See [10], [16] for typical problems and applications. Additional applications to the method of moving frames developed by Mark Fels and the author, [8], [9], [17], will appear elsewhere.

§2. Contact Forms on Jet Bundles

We will work with the smooth category of manifolds and maps throughout this paper. Let $E \to X$ be a smooth vector bundle over a p-dimensional base manifold X, with q-dimensional fibers. We use $x = (x^1, \ldots, x^p)$ to denote local coordinates on X, and $u = (u^1, \ldots, u^q)$ to denote the fiber coordinates, so that sections of E are prescribed by smooth functions $u = f(x)$. Let $\mathrm{J}^n = \mathrm{J}^n E$ denote the n^{th} jet bundle of E, with associated local coordinates $z^{(n)} = (x, u^{(n)}) = (\ldots x^i \ldots u^\alpha_J \ldots)$, where the derivative coordinates u^α_J are indexed by unordered multi-indices $J = (j_1, \ldots, j_k)$, with $1 \leq j_\kappa \leq p$, of orders $0 \leq k = \#J \leq n$. Given a (local) section $f\colon X \to E$, we let $\mathrm{j}_n f\colon X \to \mathrm{J}^n$ denote its n-jet, which forms a section of the n^{th} order jet bundle.

Definition 2.1. A differential form θ on the jet space J^n is called a *contact form* if it is annihilated by all jets: $(\mathrm{j}_n f)^* \theta = 0$.

The space of contact forms on J^n forms differential ideal $\mathcal{I}^{(n)}$, called the *contact ideal*, over J^n.

Theorem 2.2. *In local coordinates, every contact one-form on* J^n *can be written as a linear combination of the* basic contact forms

$$\theta^\alpha_J = du^\alpha_J - \sum_{i=1}^{p} u^\alpha_{J,i}\, dx^i, \qquad \alpha = 1, \ldots, q, \quad 0 \leq \#J < n. \tag{2.1}$$

These one-forms constitute a basis for the contact ideal $\mathcal{I}^{(n)}$.

For instance, in the case of one independent and one dependent variable, the basic contact forms are

$$\begin{aligned} \theta_0 &= du - u_x\, dx, \\ \theta_1 &= du_x - u_{xx}\, dx, \\ \theta_2 &= du_{xx} - u_{xxx}\, dx, \quad \ldots \end{aligned} \tag{2.2}$$

In (2.1), we call $\#J$ the *order* of the contact form θ^α_J. The reader should note that the contact forms on J^n have orders at most $n - 1$.

Lemma 2.3. *A section* $F\colon X \to \mathrm{J}^n$ *locally coincides with the n-jet of a section* $f\colon X \to E$, *meaning* $F = \mathrm{j}_n f$ *on an open subset of* X, *if and only if* F *annihilates all the contact forms on* J^n:

$$F^* \theta^\alpha_J = 0, \qquad \alpha = 1, \ldots, q, \quad 0 \leq \#J < n. \tag{2.3}$$

Definition 2.4. A local diffeomorphism $\Psi\colon \mathrm{J}^n \to \mathrm{J}^n$ defines a *contact transformation* of order n if it preserves the contact ideal, meaning that if θ is any *contact form* on J^n, then $\Psi^*\theta$ is also a contact form.

Definition 2.5. The $(n + k)^{\text{th}}$ order *prolongation* of the contact transformation $\Psi^{(n)}$ is the unique contact transformation

$\Psi^{(n+k)}: \mathrm{J}^{n+k} \to \mathrm{J}^{n+k}$ satisfying $\pi_n^{n+k} \circ \Psi^{(n+k)} = \Psi^{(n)} \circ \pi_n^{n+k}$, where $\pi_n^{n+k}: \mathrm{J}^{n+k} \to \mathrm{J}^n$ is the usual projection.

In local coordinates, a local diffeomorphism Ψ defines a contact transformation if and only if

$$\Psi^* \theta_J^\alpha = \sum_{\beta,K} A_{J,\beta}^{\alpha,K} \theta_K^\beta, \tag{2.4}$$

for suitable coefficient functions $A_{J,\beta}^{\alpha,K}: \mathrm{J}^n \to \mathbb{R}$. There are nontrivial constraints on these coefficients resulting from Bäcklund's Theorem, [2].

Theorem 2.6. *If the number of dependent variables is greater than one, $q > 1$, then every contact transformation is the prolongation of a point transformation $\psi: E \to E$. If $q = 1$, then every n^{th} order contact transformation is the prolongation of a first order contact transformation $\psi: \mathrm{J}^1 \to \mathrm{J}^1$.*

Remark. Interestingly, if one restricts to a submanifold of the jet space defined by system of differential equations, additional "internal" higher order contact transformations can exist; see [1] for a Bäcklund-style classification of these transformations.

§3. The Prolonged General Linear Group

There are two fundamental transformation groups that lie at the foundation of the geometric characterization of contact transformations. The first is the standard prolongation of the general linear group, [12, p. 139], [14]. Let $\mathrm{GL}(p)$ denote the general linear group on $\mathbb{R}^p$ consisting of all real, invertible, $p \times p$ matrices. Let $\mathcal{D}_0(p)$ denote the space of all diffeomorphisms $\varphi: \mathbb{R}^p \to \mathbb{R}^p$ preserving the origin, so $\varphi(0) = 0$. We let $\mathrm{j}_n\varphi(0)$ denote the n-jet (or n^{th} order Taylor expansion) of the diffeomorphism at the origin.

Definition 3.1. The n^{th} prolongation of the general linear group $\mathrm{GL}(p)$ is the group

$$\mathrm{GL}^{(n)}(p) = \{\mathrm{j}_n\varphi(0) \mid \varphi \in \mathcal{D}_0(p)\}. \tag{3.1}$$

The group multiplication is given by composition of diffeomorphisms, so that if $S = \mathrm{j}_n\varphi(0)$, $T = \mathrm{j}_n\psi(0)$, then $S \cdot T = \mathrm{j}_n(\varphi \circ \psi)(0)$.

Note that the one-jet of a diffeomorphism φ at 0 is uniquely determined by its Jacobian matrix $D\varphi(0)$, which can be viewed as an invertible matrix in $\mathrm{GL}(p)$, and, in this way, we identify $\mathrm{GL}(p) = \mathrm{GL}^{(1)}(p)$.

The most convenient method of representing the elements of $\mathrm{GL}^{(n)}(p)$ is via formal Taylor polynomials. We introduce coordinates $t = (t^1, \ldots, t^p)$ in a neighborhood of $0 \in M$. We then identify a group element $S \in \mathrm{GL}^{(n)}(p)$ with the vector-valued Taylor polynomial† $\mathbf{S}(t) = (S^1(t), \ldots, S^p(t))^T$ of any smooth diffeomorphism $\varphi(x)$ that represents it, so

$$S^i(t) = \sum_{1 \le \# J \le n} S^i_J \frac{t^J}{J!}, \quad \text{where} \quad S^i_J = \frac{\partial^{\# J} \varphi^i}{\partial x^J}(0), \quad i = 1, \ldots, p. \tag{3.2}$$

Note that there is no constant (order 0) term in the Taylor polynomial (3.2) since we are assuming that $\varphi(0) = 0$; moreover the first order Taylor coefficients (S^i_j) form an invertible $p \times p$ matrix, whereas the higher order coefficients can be arbitrary. Therefore, $\mathrm{GL}^{(n)}(p)$ forms a Lie group of dimension

$$p^{(n)} = p\left[\binom{p+n}{n} - 1\right]. \tag{3.3}$$

The group multiplication is then given by formal composition of polynomials, so that $U = R \cdot S$ if and only if the corresponding polynomials satisfy

$$\mathbf{U}(t) = \mathbf{R}(\mathbf{S}(t)) \bmod n, \tag{3.4}$$

where $\bmod n$ means that we truncate the resulting polynomial to order n. The explicit formulae can be identified with the Faà di Bruno formula, [7, p. 222], [14], for the derivatives of the composition of two functions.

Example 3.2. In the one-dimensional situation, $p = 1$, the Taylor polynomial of a diffeomorphism $\varphi \to \mathbb{R} \to \mathbb{R}$ that fixes $0 = \varphi(0)$ takes the form

$$\mathbf{S}(t) = s_1 t + \frac{1}{2} s_2 t^2 + \frac{1}{3!} s_3 t^3 + \frac{1}{4!} s_4 t^4 + \cdots, \tag{3.5}$$

with the coefficients $s_1, s_2, \ldots$ representing the derivatives $s_k = \varphi^{(k)}(0)$ of our diffeomorphism $\varphi \colon \mathbb{R} \to \mathbb{R}$ fixing $0 = \varphi(0)$. The composition formula (3.4) gives the explicit rules

$$\begin{aligned} &u_1 = r_1 s_1, \qquad u_2 = r_1 s_2 + r_2 s_1^2, \qquad u_3 = r_1 s_3 + 3 r_2 s_1 s_2 + r_3 s_1^3, \\ &u_4 = r_1 s_4 + r_2(4 s_1 s_3 + 3 s_2^2) + 6 r_3 s_1^2 s_2 + r_4 s_1^4, \end{aligned}$$

†We use a formal variable t here instead of x for later clarity.

and so on. As in [5, §3.4], the one-dimensional Faà di Bruno formula is

$$
\begin{aligned}
u_k &= \sum_{m=1}^{k} r_m B_k^m(s_1,\ldots,s_k), \\
\text{where} \qquad B_k^m(s_1,\ldots,s_k) &= \sum_{\Sigma I = k} \frac{s_{i_1} s_{i_2} \cdots s_{i_m}}{I!(\#I)!}
\end{aligned}
\tag{3.6}
$$

is a Bell polynomial, [3], [18, §2.8]. The sum in (3.6) is over all unordered multi-indices $I = (i_1,\ldots,i_m)$ with $1 \le i_\nu \le k$, $\sum I = i_1 + \cdots + i_m = k$, and where $J = \#I$ denotes the "repetition" multi-index of I, so that $j_r = \#\{i_\nu = r\}$ indicates the number of times that the integer r appears in the multi-index I.

We can explicitly realize $\mathrm{GL}^{(n)}(p)$ as a matrix Lie group, namely a subgroup of $\mathrm{GL}(p^{(n)})$, as follows. The space of vector-valued Taylor polynomials $\mathbf{x}(t)$ of degree at most n without constant term, $\mathbf{x}(0) = 0$, can be identified with $\mathbb{R}^{p^{(n)}}$. Given $S \in \mathrm{GL}^{(n)}(p)$, we define $\rho(S) \in \mathrm{GL}(p^{(n)})$ by

$$
\rho(S)\mathbf{x}(t) = \mathbf{x}(\mathbf{S}(t)),
\tag{3.7}
$$

where $\mathbf{S}(t)$ is the Taylor polynomial (3.2) corresponding to S. The explicit formulae for the *Faà di Bruno injection* ρ can be found in [14, p. 503].

Example 3.3. In the one-dimensional situation described in example 3.2, we identify a fourth order Taylor polynomial (3.5) with its coefficient vector (s_1, s_2, s_3, s_4). The corresponding matrix is

$$
\rho(S) = \begin{pmatrix} s_1 & s_2 & s_3 & s_4 \\ 0 & s_1^2 & 3s_1 s_2 & 4s_1 s_3 + 3s_2^2 \\ 0 & 0 & s_1^3 & 6s_1^2 s_2 \\ 0 & 0 & 0 & s_1^4 \end{pmatrix}.
$$

The reader may enjoy verifying that this forms a subgroup of $\mathrm{GL}(4)$. The k^{th} order version has $\rho(S)$ equal to the upper triangular matrix with entries given by the Bell polynomials $B_j^i(s_1,\ldots,s_j)$ for $i \le j$.

We next determine the left and right-invariant Maurer-Cartan forms on the prolonged general linear group. These will be found by adapting the usual formulae

$$
\mu_L = A^{-1} \cdot dA, \qquad \mu_R = dA \cdot A^{-1},
\tag{3.8}
$$

valid for matrix Lie groups $G \subset \mathrm{GL}(n)$, [16]. In our case, the Maurer-Cartan forms will appear as the coefficients of a formal "Taylor" polynomial

$$\boldsymbol{\sigma}(t) = \sum_{1 \le \#J \le n} \frac{t^J}{J!} \sigma_J, \tag{3.9}$$

where each σ_J is a p vector of one-forms defined on the group $\mathrm{GL}^{(n)}(p)$. Using (3.8) and the multiplication rule (3.4) for the group, we deduce that the right-invariant Maurer-Cartan form polynomial is given by

$$\widetilde{\boldsymbol{\sigma}}(t) = d\mathbf{S}\big[\mathbf{S}^{-1}(t)\big] \bmod n, \tag{3.10}$$

obtained by composing the formal inverse series (or inverse Taylor polynomial) $\mathbf{S}^{-1}(t)$ and the formal series of basis one-forms

$$d\mathbf{S}(t) = \sum_{1 \le \#J \le n} \frac{t^J}{J!} dS_J, \tag{3.11}$$

on the group. On the other hand, the left-invariant Maurer-Cartan form polynomial can be found by first computing the differential of the composition

$$d\big[\mathbf{T}(\mathbf{S}(t))\big] = D\mathbf{T}(\mathbf{S}(t)) \cdot d\mathbf{S}(t), \tag{3.12}$$

of the two power series with respect to the coefficients of $\mathbf{S}$. Here $D\mathbf{T}(t) = (\partial \mathbf{T}^i / \partial t^j)$ denotes the Jacobian matrix series associated with $\mathbf{T}(t)$. Replacing $\mathbf{T}$ in (3.12) by the inverse of $\mathbf{S}(t)$ and truncating produces the left-invariant Maurer-Cartan form polynomial:

$$\boldsymbol{\sigma}(t) = D\mathbf{S}^{-1}(\mathbf{S}(t)) \cdot d\mathbf{S}(t) \bmod n = D\mathbf{S}(t)^{-1} \cdot d\mathbf{S}(t) \bmod n, \tag{3.13}$$

where $D\mathbf{S}(t)^{-1}$ is the inverse of the Jacobian matrix of $\mathbf{S}(t)$.

Example 3.4. For the one-dimensional situation considered above we have

$$\begin{aligned}
\mathbf{S}(t) &= s_1 t + \frac{1}{2} s_2 t^2 + \frac{1}{3!} s_3 t^3 + \frac{1}{4!} s_4 t^4 + \cdots, \\
d\mathbf{S}(t) &= t\, ds_1 + \frac{1}{2} t^2\, ds_2 + \frac{1}{3!} t^3\, ds_3 + \frac{1}{4!} t^4\, ds_4 + \cdots, \\
\mathbf{S}^{-1}(t) &= \frac{1}{s_1} t - \frac{s_2}{2 s_1^3} t^2 - \frac{s_1 s_3 - 3 s_2^2}{6 s_1^5} t^3 - \frac{s_1^2 s_4 - 10 s_1 s_2 s_3 + 12 s_2^3}{24 s_1^7} t^4 + \cdots.
\end{aligned}$$

Therefore, the right-invariant Maurer-Cartan forms on $\mathrm{GL}^{(n)}(1)$ are obtained as the coefficients of the "Maurer-Cartan polynomials"

$$\begin{aligned}
\widetilde{\boldsymbol{\sigma}}(t) &= d\mathbf{S}\left[\frac{1}{\mathbf{S}(t)}\right] = \widetilde{\sigma}_1 t + \frac{1}{2}\widetilde{\sigma}_2 t^2 + \frac{1}{3!}\widetilde{\sigma}_3 t^3 + \frac{1}{4!}\widetilde{\sigma}_4 t^4 + \cdots \\
&= \frac{ds_1}{s_1}t + \frac{s_1\,ds_2 - s_2\,ds_1}{2s_1^3}t^2 - \frac{s_1^2\,ds_3 - 3s_1s_2\,ds_2 - (s_1s_3 - 3s_2^2)\,ds_1}{6s_1^5}t^3 \\
&\quad + \frac{1}{24s_1^7}\Big\{s_1^3\,ds_4 - 6s_1^2s_2\,ds_3 - (4s_1^2s_3 - 15s_1s_2^2)\,ds_2 \\
&\qquad\qquad - (s_1^2s_4 - 10s_1s_2s_3 + 15s_2^3)\,ds_1\Big\}t^4 + \cdots .
\end{aligned}$$

The left-invariant Maurer-Cartan form polynomial (3.10) for $\mathrm{GL}^{(n)}(1)$ is

$$\begin{aligned}
\boldsymbol{\sigma}(t) &= \frac{d\mathbf{S}(t)}{\mathbf{S}'(t)} = \sigma_1 t + \frac{1}{2}\sigma_2 t^2 + \frac{1}{3!}\sigma_3 t^3 + \frac{1}{4!}\sigma_4 t^4 + \cdots \\
&= \frac{ds_1}{s_1}t + \frac{s_1\,ds_2 - 2s_2\,ds_1}{2s_1^2}t^2 \\
&\quad + \frac{s_1^2\,ds_3 - 3s_1s_2\,ds_2 - 3(s_1s_3 - 2s_2^2)\,ds_1}{6s_1^3}t^3 \\
&\quad + \frac{1}{24s_1^4}\Big\{s_1^3\,ds_4 - 4s_1^2s_2\,ds_3 - 6s_1(s_1s_3 - 2s_2^2)\,ds_2 \\
&\qquad\qquad - 4(s_1^2s_4 - 6s_1s_2s_3 + 6s_2^3)\,ds_1\Big\}t^4 + \cdots .
\end{aligned}$$

Let $\rho(\boldsymbol{\sigma}) = \rho(\mathbf{S})^{-1}d\rho(\mathbf{S})$ denote the corresponding left Maurer-Cartan matrix, (3.8). In view of (3.7), (3.13), it acts on the column vector $\mathbf{x}$ according to the power series formulation

$$\begin{aligned}
[\rho(\boldsymbol{\sigma})\mathbf{x}](t) &= \rho(\mathbf{S})^{-1}\,d[\rho(\mathbf{S})\mathbf{x}](t) = \rho(\mathbf{S})^{-1}\,d[\mathbf{x}(\mathbf{S}(t))] \\
&= \rho(\mathbf{S})^{-1}\left(\sum_{i=1}^{p}\frac{\partial\mathbf{x}}{\partial t^i}[\mathbf{S}(t)]\,d\mathbf{S}(t)\right) \\
&= \sum_{i=1}^{p}\frac{\partial\mathbf{x}}{\partial t^i}(t)\,d\mathbf{S}^i[\mathbf{S}^{-1}(t)] \\
&= \sum_{i=1}^{p}\frac{\partial\mathbf{x}}{\partial t^i}(t)\,\boldsymbol{\sigma}^i(t).
\end{aligned}$$

Example 3.5. For the one-dimensional version, we have

$$\begin{aligned}\mathbf{x}(t) &= x_1 t + \frac{1}{2} x_2 t^2 + \frac{1}{3!} x_3 t^3 + \frac{1}{4!} x_4 t^4 + \cdots ,\\ \mathbf{x}'(t) &= x_1 + x_2 t + \frac{1}{2!} x_3 t^2 + \frac{1}{3!} x_4 t^3 + \cdots .\end{aligned}$$

Therefore,

$$\begin{aligned}\rho(\boldsymbol{\sigma})\mathbf{x}(t) &= \mathbf{x}'(t)\boldsymbol{\sigma}(t)\\ &= (\sigma_1 x_1)t + \frac{1}{2}(\sigma_2 x_1 + 2\sigma_1 x_2)t^2\\ &\quad + \frac{1}{3!}(\sigma_3 x_1 + 3\sigma_2 x_2 + 3\sigma_1 x_3)t^3\\ &\quad + \frac{1}{4!}(\sigma_4 x_1 + 4\sigma_3 x_2 + 6\sigma_2 x_3 + 4\sigma_1 x_4)t^4 + \cdots ,\end{aligned}$$

and hence the Maurer-Cartan form matrix for $\mathrm{GL}^{(n)}(1)$ is

$$\rho(\boldsymbol{\sigma}) = \begin{pmatrix} \sigma_1 & \sigma_2 & \sigma_3 & \sigma_4 & \cdots \\ 0 & 2\sigma_1 & 3\sigma_2 & 4\sigma_3 & \cdots \\ 0 & 0 & 3\sigma_1 & 6\sigma_2 & \cdots \\ 0 & 0 & 0 & 4\sigma_1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{3.14}$$

The (i,j) entry of the full $n \times n$ matrix is

$$\rho(S)^i_j = \begin{cases} \binom{j}{i-1}\sigma_{i-j+1}, & i \leq j, \\ 0, & i > j. \end{cases} \tag{3.15}$$

§4. The Leibniz Group

Besides the prolonged general linear group that provides the structure group for jets of diffeomorphisms, we also require a structure group related to the multiplication of jets.

Definition 4.1. The *Leibniz group* $\mathrm{L}^{(n)}(p,q)$ is the Lie group consisting of all n-jets of smooth maps $\Psi\colon \mathbb{R}^p \to \mathrm{GL}(q)$ at the point 0, so

$$\mathrm{L}^{(n)}(p,q) = \{\mathrm{j}_n\Psi(0) \mid \Psi\colon \mathbb{R}^p \to \mathrm{GL}(q)\}. \tag{4.1}$$

The group law is induced by matrix multiplication $\Phi(x) \cdot \Psi(x)$ of the smooth maps.

Given a vector bundle $E \to X$ over a p-dimensional base with q-dimensional fiber, there is an induced representation

$$\tau(L^{(n)}) \cdot z^{(n)} = \mathrm{j}_n[\Psi(x) \cdot f(x)] \qquad \text{whenever} \qquad L^{(n)} = \mathrm{j}_n \Psi(x), \quad z^{(n)} = \mathrm{j}_n f(x) \tag{4.2}$$

of $\mathrm{L}^{(n)}(p, q)$ on the jet fiber $\mathrm{J}^n E$. As with the prolonged general linear group, we identify the elements of the Leibniz group with their Taylor series. Thus, the group element $L^{(n)} = \mathrm{j}_n \Psi(0)$ is identified with the n^{th} order truncation of the power series

$$\mathbf{L}(t) = \sum_{1 \le \# J \le n} \frac{t^J}{J!} L_J, \tag{4.3}$$

where each L_J is a $q \times q$ matrix. The entries $(L_J)^\alpha_\beta$ can be identified with the Taylor coefficients $\partial^k \Psi^\alpha_\beta / \partial x^J(0)$ for the corresponding matrix entry of $\Psi(x)$. Identifying a point $z^{(n)} \in \mathrm{J}^n$ with the corresponding n^{th} order Taylor polynomial $\mathbf{z}(t)$, the action of the Leibniz group is given by

$$[\tau(L^{(n)})\mathbf{z}](t) = \mathbf{L}(t) \cdot \mathbf{z}(t) \bmod n. \tag{4.4}$$

Example 4.2. In the one-dimensional version, $\mathrm{GL}(1) \simeq \mathbb{R}^*$ is just the set of nonzero reals, and so the maps $\Psi\colon \mathbb{R} \to \mathrm{GL}(1)$ are scalar-valued. The Leibniz group is induced by multiplication of Taylor series, and so the product of

$$\mathbf{L}(t) = l_0 + l_1 t + \frac{1}{2} l_2 t^2 + \frac{1}{3!} l_3 t^3 + \cdots ,$$
$$\mathbf{M}(t) = m_0 + m_1 t + \frac{1}{2} m_2 t^2 + \frac{1}{3!} m_3 t^3 + \cdots ,$$

is given by truncating the product series

$$\mathbf{L}(t) \cdot \mathbf{M}(t) = l_0 m_0 + (l_0 m_1 + l_1 m_0) t + \frac{1}{2}(l_0 m_2 + 2 l_1 m_1 + l_2 m_0) t^2 + \frac{1}{3!}(l_0 m_3 + 3 l_1 m_2 + 3 l_2 m_1 + l_3 m_0) t^3 + \cdots ,$$

at order n. The action (4.4) on a series

$$\mathbf{z}(t) = z_0 + z_1 t + \frac{1}{2} z_2 t^2 + \frac{1}{3!} z_3 t^3 + \cdots \tag{4.5}$$

is the same — just replace the m's by z's. Therefore, the matrix representation (4.2) of an element of $\mathrm{L}^{(4)}(1,1)$ is

$$\tau(L^{(4)}) = \begin{pmatrix} l_0 & l_1 & l_2 & l_3 & l_4 \\ 0 & l_0 & 2l_1 & 3l_2 & 4l_3 \\ 0 & 0 & l_0 & 3l_1 & 6l_2 \\ 0 & 0 & 0 & l_0 & 4l_1 \\ 0 & 0 & 0 & 0 & l_0 \end{pmatrix}.$$

The matrix of Maurer-Cartan forms on the Leibniz group are found using the usual formula (3.8), which becomes

$$\boldsymbol{\lambda} = \sum_J \frac{t^J}{J!} \lambda_J, \tag{4.6}$$

where each λ_J is a $q \times q$ matrix of one-forms. We have

$$\boldsymbol{\lambda}(t) = \tau(L)^{-1}\, d\mathbf{L}(t) = \mathbf{L}(t)^{-1}\, d\mathbf{L}(t) = d \log \mathbf{L}(t). \tag{4.7}$$

Let $\tau(\boldsymbol{\lambda}) = \tau(\mathbf{L})^{-1} d\tau(\mathbf{L})$ denote the corresponding Maurer-Cartan matrix. In view of (4.4), it acts on the column vector $\mathbf{z}$ according to the power series formulation
(4.8)

$$[\tau(\boldsymbol{\lambda})\mathbf{z}](t) = \tau(\mathbf{L})^{-1} d[\tau(\mathbf{L})\mathbf{z}](t) = \tau(\mathbf{L})^{-1} d[\mathbf{L}(t)] \cdot \mathbf{z}(t) = \boldsymbol{\lambda}(t) \cdot \mathbf{z}(t).$$

Example 4.3. For the one-dimensional version, we have

$$\begin{aligned} \mathbf{L}(t) &= l_0 + l_1 t + \frac{1}{2} l_2 t^2 + \frac{1}{3!} l_3 t^3 + \cdots, \\ d\mathbf{L}(t) &= dl_0 + t\, dl_1 + \frac{1}{2} t^2\, dl_2 + \frac{1}{3!} t^3\, dl_3 + \cdots, \\ \mathbf{L}(t)^{-1} &= \frac{1}{l_0} - \frac{l_1}{l_0^2} t - \frac{l_0 l_2 - 2l_1^2}{2l_0^3} t^2 - \frac{l_0^2 l_3 - 6 l_0 l_1 l_2 + 6 l_1^3}{6 l_0^4} t^3 + \cdots. \end{aligned}$$

Therefore, the Maurer-Cartan form series for $\mathrm{L}^{(n)}(1,1)$ is

$$\begin{aligned} \boldsymbol{\lambda}(t) = d \log \mathbf{L}(t) &= \lambda_0 + \lambda_1 t + \frac{1}{2} \lambda_2 t^2 + \frac{1}{3!} \lambda_3 t^3 + \cdots, \\ &= \frac{dl_0}{l_0} + \frac{l_0\, dl_1 - l_1\, dl_0}{l_0^2} t + \frac{l_0^2\, dl_2 - 2 l_0 l_1\, dl_1 - (l_0 l_2 - 2l_1^2)\, dl_0}{2l_0^3} t^2 \\ &\quad + \frac{1}{6l_0^4} \Big\{ l_0^3\, dl_3 - 3 l_0^2 l_1\, dl_2 - 3(l_0^2 l_2 - 2 l_0 l_1^2)\, dl_1 \\ &\qquad - (l_0^2 l_3 - 6 l_0 l_1 l_2 + 6 l_1^3)\, dl_0 \Big\} t^3 + \cdots. \end{aligned}$$

Given $\mathbf{z}(t)$ as in (4.5), equation (4.8) implies that

$$\begin{aligned}\tau(\boldsymbol{\lambda})\mathbf{z}(t) &= \boldsymbol{\lambda}(t)\mathbf{z}(t)\\ &= \lambda_0 z_0 + (\lambda_1 z_0 + \lambda_0 z_1)t + \frac{1}{2}(\lambda_2 z_0 + 2\lambda_1 z_1 + \lambda_0 z_2)t^2\\ &\qquad + \frac{1}{3!}(\lambda_3 z_0 + 3\lambda_2 z_1 + 3\lambda_1 z_2 + \lambda_0 z_3)t^3 + \cdots .\end{aligned}$$

Thus, the Maurer-Cartan form matrix for $\mathrm{L}^{(n)}(1,1)$ is

$$\tau(\boldsymbol{\lambda}) = \begin{pmatrix} \lambda_0 & \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \ldots \\ 0 & \lambda_0 & 2\lambda_1 & 3\lambda_2 & 4\lambda_3 & \ldots \\ 0 & 0 & \lambda_0 & 3\lambda_1 & 6\lambda_2 & \ldots \\ 0 & 0 & 0 & \lambda_0 & 4\lambda_1 & \ldots \\ 0 & 0 & 0 & 0 & \lambda_0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{4.9}$$

The (i,j) entry of the full $n \times n$ matrix is

$$\tau(L)^i_j = \begin{cases} \dbinom{j}{i}\lambda_{i-j}, & i \leq j, \\ 0, & i > j. \end{cases} \tag{4.10}$$

Note the remarkable similarity between the Maurer-Cartan form matrices for the prolonged general linear group, (3.14), and for the Leibniz group, (4.9)! The Leibniz version forms a "Pascal upper triangular matrix", whereas the prolonged version is obtained by throwing away the main diagonal of the Pascal matrix.

§5. The Contact Group

We are now in a position to describe the structure group for the pseudo-group of contact transformations on the jet bundle J^n.

Definition 5.1. The n^{th} order *contact group* is the semidirect product group

$$\mathrm{C}^{(n)}(p,q) = \mathrm{GL}^{(n-1)}(p) \ltimes \mathrm{L}^{(n-1)}(p,q). \tag{5.1}$$

The group acts on a Taylor series $\mathbf{z}(t)$ according to

$$\psi(\mathbf{S},\mathbf{L}) \cdot \mathbf{z}(t) = \mathbf{L}(t) \cdot \mathbf{z}(\mathbf{S}^{-1}(t)), \tag{5.2}$$

and then truncating to order n. Therefore, the group multiplication in $\mathrm{C}^{(n)}(p,q)$ is given, in series form, by

$$(5.3)\qquad \big(\mathbf{S}(t),\mathbf{L}(t)\big)\cdot\big(\mathbf{T}(t),\mathbf{M}(t)\big)=\big(\mathbf{S}(\mathbf{T}(t)),\mathbf{L}(t)\cdot\mathbf{M}(\mathbf{S}^{-1}(t))\big).$$

The Maurer-Cartan form matrix for the contact structure group is given by the "difference" between the two Maurer-Cartan form matrices, so $\psi(\boldsymbol{\sigma},\boldsymbol{\lambda})=\rho(\boldsymbol{\sigma})-\tau(\boldsymbol{\lambda})$. Thus, we find

$$[\psi(\sigma,\lambda)\mathbf{z}](t)=\boldsymbol{\lambda}(t)\cdot\mathbf{z}(t)-\sum_{i=1}^{p}\frac{\partial\mathbf{z}}{\partial t^i}(t)\boldsymbol{\sigma}^i(t).$$

Note that the prolonged general linear group acts trivially on the zero$^{\text{th}}$ order coefficient in the power series for $\mathbf{z}$. In the one-dimensional version, we have

(5.4)

$$\psi(\boldsymbol{\lambda},\boldsymbol{\sigma})=\begin{pmatrix}\lambda_0 & \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \cdots\\ 0 & \lambda_0-\sigma_1 & 2\lambda_1-\sigma_2 & 3\lambda_2-\sigma_3 & 4\lambda_3-\sigma_4 & \cdots\\ 0 & 0 & \lambda_0-2\sigma_1 & 3\lambda_1-3\sigma_1 & 6\lambda_2-4\sigma_1 & \cdots\\ 0 & 0 & 0 & \lambda_0-3\sigma_1 & 4\lambda_1-6\sigma_2 & \cdots\\ 0 & 0 & 0 & 0 & \lambda_0-4\sigma_1 & \cdots\\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots\end{pmatrix}.$$

We now introduce the infinite power series of basis contact forms

$$(5.5)\qquad \boldsymbol{\theta}^\alpha(t)=\sum_{0\le\#J}\frac{t^J}{J!}\theta^\alpha_J,\qquad \alpha=1,\ldots,q,$$

in the variable u^α, and let $\boldsymbol{\theta}(t)=(\boldsymbol{\theta}^1(t),\ldots,\boldsymbol{\theta}^q(t))^T$ be the associated column vector-valued series of contact forms. Note that the contact forms on J^n are obtained by truncating the series $\boldsymbol{\theta}(t)$ at order $n-1$ and *not* at order n.

We are now able to introduce the goal of our investigations.

Definition 5.2. The *canonical contact form* is the vector-valued series of one-forms

$$(5.6)\qquad \boldsymbol{\vartheta}(t)=\psi(\mathbf{L},\mathbf{S})\boldsymbol{\theta}(t)=\mathbf{L}(t)\cdot\boldsymbol{\theta}(\mathbf{S}^{-1}(t)),$$

where $\mathbf{L}(t)$ and $\mathbf{S}(t)$ are the associated group series.

Example 5.3. In the one-dimensional situation, the canonical contact form is composed of the following linear combinations of contact forms:

$$
\begin{aligned}
\vartheta_0 &= l_0\theta_0,\\
\vartheta_1 &= \frac{l_0}{s_1}\theta_1 + l_1\theta_0,\\
\vartheta_2 &= \frac{l_0}{s_1^2}\theta_2 + \frac{2s_1^2 l_1 - s_2 l_0}{s_1^3}\theta_1 + l_2\theta_0,\\
\vartheta_3 &= \frac{l_0}{s_1^3}\theta_3 + \frac{3s_1^2 l_1 - 3s_2 l_0}{s_1^4}\theta_2\\
&\quad + \frac{3s_1^4 l_2 - 3s_1^2 s_2 l_1 - (s_1 s_3 - 3s_2^2) l_0}{s_1^5}\theta_1 + l_3\theta_0.
\end{aligned}
\tag{5.7}
$$

Remark. We can compute ϑ_k by repeatedly applying the (formal) differential operator $\mathcal{D} = (1/s_1)D_x$ to ϑ_0, using the identifications $\mathcal{D}(l_j) = l_{j+1}$, $\mathcal{D}(s_j) = s_{j+1}/s_1$. A proof of this observation is left to the reader.

Theorem 5.4. *The canonical contact form of order n defines an involutive differential system. The equivalence maps preserving the canonical contact form are the lifts of contact transformations on* J^n.

The structure equations are found as follows. The usual contact form structure equations

$$d\theta_I^\alpha = \sum_{i=1}^{p} \theta_{I,i}^\alpha \wedge dx^i,$$

can be rewritten in series form

$$d\boldsymbol{\theta}(t) = \boldsymbol{\theta}'(t) \wedge d\mathbf{x} = \sum_{i=1}^{p} \frac{\partial \boldsymbol{\theta}}{\partial t^i} \wedge dx^i, \qquad \alpha = 1, \ldots, q. \tag{5.8}$$

Here $\boldsymbol{\theta}'(t) = \big(\partial\theta^\alpha/\partial t^i\big)$ is the formal $q \times p$ Jacobian matrix of $\boldsymbol{\theta}(t)$ with respect to t. Therefore, using (5.8), we can compute

$$
\begin{aligned}
d\boldsymbol{\vartheta}(t) &= \boldsymbol{\lambda}(t) \wedge \boldsymbol{\vartheta}(t) + \boldsymbol{\vartheta}'(t) \wedge \boldsymbol{\sigma}(t) + \mathbf{L}(t) d\boldsymbol{\theta}(\mathbf{S}^{-1}(t))\\
&= \boldsymbol{\lambda}(t) \wedge \boldsymbol{\vartheta}(t) + \boldsymbol{\vartheta}'(t) \wedge \boldsymbol{\sigma}(t) + \mathbf{L}(t)\boldsymbol{\theta}'(\mathbf{S}^{-1}(t)) \wedge d\mathbf{x}.
\end{aligned}
\tag{5.9}
$$

In the language of the Cartan equivalence method, cf. [10], [16], the first two terms in (5.9) form the group components of the structure equations, while the third term is the torsion.

On the other hand, using the definition (5.6), we can compute

$$\begin{aligned}\frac{\partial}{\partial t}\boldsymbol{\vartheta}(t) &= \mathbf{L}'(t)\cdot\boldsymbol{\theta}(\mathbf{S}^{-1}(t)) + \mathbf{L}(t)\cdot\boldsymbol{\theta}'(\mathbf{S}^{-1}(t))\cdot\frac{\partial}{\partial t}[\mathbf{S}^{-1}(t)]\\ &= \mathbf{L}'(t)\cdot\mathbf{L}(t)^{-1}\cdot\boldsymbol{\vartheta}(t) + \mathbf{L}(t)\cdot\boldsymbol{\vartheta}'(\mathbf{S}^{-1}(t))\cdot\left[\mathbf{S}'(\mathbf{S}^{-1}(t))\right]^{-1},\end{aligned}$$

the last equality following from the chain rule. Therefore,

$$\begin{aligned}d\boldsymbol{\vartheta}(t) &= \boldsymbol{\lambda}(t)\wedge\boldsymbol{\vartheta}(t) + \boldsymbol{\vartheta}'(t)\wedge\boldsymbol{\sigma}(t)\\ &\qquad + \left[\boldsymbol{\vartheta}'(t) - \mathbf{L}'(t)\cdot\mathbf{L}(t)^{-1}\cdot\boldsymbol{\vartheta}(t)\right]\wedge\mathbf{S}'(\mathbf{S}^{-1}(t))\,d\mathbf{x}.\end{aligned}\tag{5.10}$$

Most of the torsion terms can therefore be absorbed by suitably modifying the Maurer-Cartan forms $\boldsymbol{\lambda}(t)$ and $\boldsymbol{\sigma}(t)$; the only exceptions are the constant terms multiplying $\boldsymbol{\vartheta}'(t)$; this is because $\boldsymbol{\sigma}(t)$ does not contain any constant terms, i.e., $\boldsymbol{\sigma}(0) = 0$. If we define the modified Maurer-Cartan forms to be

$$\begin{aligned}\widetilde{\boldsymbol{\lambda}}(t) &= \boldsymbol{\lambda}(t) + \mathbf{L}'(t)\cdot\mathbf{L}(t)^{-1}\cdot\mathbf{S}'(\mathbf{S}^{-1}(t))\,d\mathbf{x},\\ \widetilde{\boldsymbol{\sigma}}(t) &= \boldsymbol{\sigma}(t) + \left[\mathbf{S}'(\mathbf{S}^{-1}(t)) - \mathbf{S}'(0)\right]d\mathbf{x},\end{aligned}\tag{5.11}$$

we can rewrite the structure equations (5.10) in the "semi-absorbed form"

$$d\boldsymbol{\vartheta}(t) = \widetilde{\boldsymbol{\lambda}}(t)\wedge\boldsymbol{\vartheta}(t) + \boldsymbol{\vartheta}'(t)\wedge\widetilde{\boldsymbol{\sigma}}(t) + \boldsymbol{\vartheta}'(t)\wedge\mathbf{S}'(0)\,d\mathbf{x}.\tag{5.12}$$

We now complete the canonical contact form to a coframe on J^n by including the additional p one-forms

$$\boldsymbol{\xi} = \mathbf{S}'(0)d\mathbf{x} + \mathbf{a}\boldsymbol{\vartheta}(0) + \mathbf{B}\boldsymbol{\vartheta}'(0).\tag{5.13}$$

Here $\mathbf{a} = (a^i_\alpha)$ is a $p\times q$ matrix and $\mathbf{B} = (b^{ik}_\alpha)$ a $p\times p\times q$ tensor of parameters. In components, (5.13) reads

$$\xi^i = \sum_{j=1}^{p} S^i_j\,dx^j + \sum_{\alpha=1}^{q} a^i_\alpha\vartheta^\alpha + \sum_{\alpha=1}^{q}\sum_{k=1}^{p} b^{ik}_\alpha\vartheta^\alpha_k,$$

$$\text{where}\qquad \vartheta^\alpha = \boldsymbol{\vartheta}^\alpha(0),\qquad \vartheta^\alpha_k = \frac{\partial\boldsymbol{\vartheta}^\alpha}{\partial t^k}(0),$$

are the lifted zero$^{\text{th}}$ and first order contact forms, which can be written as linear combinations of the ordinary zero$^{\text{th}}$ and first order contact forms θ^α, θ^α_k via (5.6). Bäcklund's Theorem implies that the $\mathbf{x}$ coordinates depend only on x, u, and, if $q = 1$, first order derivatives of u. This implies that the first order contact form coefficients in (5.13) must vanish, $\mathbf{B}=0$, when $q>1$. (Alternatively, one can use a particular unabsorbable torsion term to justify this normalization.) We therefore use (5.13) to rewrite the structure equations (5.12) in the fully absorbed form

$$d\boldsymbol{\vartheta}(t)=\widehat{\boldsymbol{\lambda}}(t)\wedge\boldsymbol{\vartheta}(t)+\boldsymbol{\vartheta}'(t)\wedge\widehat{\boldsymbol{\sigma}}(t)+\boldsymbol{\vartheta}'(t)\wedge\boldsymbol{\xi}, \tag{5.14}$$

where the modified Maurer-Cartan forms are now

$$\begin{aligned}\widehat{\boldsymbol{\lambda}}(t)&=\widetilde{\boldsymbol{\lambda}}(t)-\mathbf{a}\cdot\boldsymbol{\vartheta}'(t),\\ \widehat{\boldsymbol{\sigma}}(t)&=\widetilde{\boldsymbol{\sigma}}(t)+\mathbf{a}\cdot[\boldsymbol{\vartheta}(t)-\boldsymbol{\vartheta}(0)]+\mathbf{B}\cdot[\boldsymbol{\vartheta}'(t)-\boldsymbol{\vartheta}'(0)].\end{aligned} \tag{5.15}$$

(Again, note that $\widehat{\boldsymbol{\sigma}}(0)=\boldsymbol{\sigma}(0)=0$, so that this modification is allowed.) The only term in (5.12) which remains unaccounted for is

$$\boldsymbol{\vartheta}'(t)\wedge\mathbf{B}\cdot\boldsymbol{\vartheta}'(t),$$

but this vanishes because either $q=1$, in which case the wedge product of the two scalar one-forms $\boldsymbol{\vartheta}'(t)$ is zero, or $q>1$, in which case, by Bäcklund's Theorem, $\mathbf{B}=0$. In fact, this is the essential torsion component that provides the equivalence method proof of this part of Bäcklund's Theorem, cf. [16]. Equation (5.14) provides the main constituent of the structure equations for the contact pseudo-group.

We also need to compute the remaining structure equations for the one-forms (5.13). We find

$$d\boldsymbol{\xi}=\boldsymbol{\sigma}'(0)\wedge\boldsymbol{\xi}+\boldsymbol{\alpha}\wedge\boldsymbol{\vartheta}(0)+\boldsymbol{\beta}\wedge\boldsymbol{\vartheta}'(0)+\mathbf{a}\cdot\,d\boldsymbol{\vartheta}(0)+\mathbf{B}\,d\boldsymbol{\vartheta}'(0), \tag{5.16}$$

where $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ are the Maurer-Cartan forms corresponding to the additional group parameters $\mathbf{a}$, $\mathbf{B}$. Note that $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ do not depend on t. Differentiating (5.14) with respect to t, and recalling $\boldsymbol{\sigma}(0)=0$, we find

$$\begin{aligned}d\boldsymbol{\vartheta}(0)&=\widehat{\boldsymbol{\lambda}}(0)\wedge\boldsymbol{\vartheta}(0)+\boldsymbol{\vartheta}'(0)\wedge\boldsymbol{\xi},\\ d\boldsymbol{\vartheta}'(0)&=\widehat{\boldsymbol{\lambda}}'(0)\wedge\boldsymbol{\vartheta}(0)+\widehat{\boldsymbol{\lambda}}(0)\wedge\boldsymbol{\vartheta}'(0)+\boldsymbol{\vartheta}'(0)\wedge\widehat{\boldsymbol{\sigma}}'(0)+\boldsymbol{\vartheta}''(0)\wedge\boldsymbol{\xi}.\end{aligned} \tag{5.17}$$

Moreover, according to (5.11), (5.15), for any constant (column) vector $\mathbf{z}\in\mathbb{R}^p$,

$$\begin{aligned}\widehat{\boldsymbol{\sigma}}'(0)\cdot\mathbf{z}&=\widetilde{\boldsymbol{\sigma}}'(0)\cdot\mathbf{z}+\left[\mathbf{a}\cdot\boldsymbol{\vartheta}'(0)+\mathbf{B}\cdot\boldsymbol{\vartheta}''(0)\right]\cdot\mathbf{z}\\ &=\boldsymbol{\sigma}'(0)+\mathbf{S}''(0)\big(\mathbf{S}'(0)^{-1}\cdot\mathbf{z},\mathbf{x}\big)+[\mathbf{a}\cdot\boldsymbol{\vartheta}'(0)+\mathbf{B}\cdot\boldsymbol{\vartheta}''(0)]\cdot\mathbf{z}.\end{aligned}$$

Wedging the result with $\boldsymbol{\xi}$, and using (5.13), (5.17) and, we find

$$\widehat{\boldsymbol{\sigma}}'(0)\wedge\boldsymbol{\xi}=\boldsymbol{\sigma}'(0)\wedge\boldsymbol{\xi}+\boldsymbol{\pi}\wedge\boldsymbol{\vartheta}(0)+\boldsymbol{\vartheta}'(0)\wedge\boldsymbol{\varpi}+\mathbf{B}\cdot\boldsymbol{\vartheta}''(0)\wedge\boldsymbol{\xi}, \tag{5.18}$$

for certain one-forms $\boldsymbol{\pi}$, $\boldsymbol{\varpi}$, whose precise form is not hard to find, but which is unimportant. Note that we used the fact that the extra term

$$\mathbf{S}''(0)\big(\mathbf{S}'(0)^{-1}\cdot\boldsymbol{\xi},\mathbf{S}'(0)^{-1}\cdot\boldsymbol{\xi}\big)=0$$

vanishes by symmetry of second order derivatives. Finally, substituting (5.18), (5.17) into (5.16), we conclude that

$$d\boldsymbol{\xi}=\widehat{\boldsymbol{\sigma}}'(0)\wedge\boldsymbol{\xi}+\widehat{\boldsymbol{\alpha}}\wedge\boldsymbol{\vartheta}(0)+\widehat{\boldsymbol{\beta}}\wedge\boldsymbol{\vartheta}'(0), \tag{5.19}$$

where $\widehat{\boldsymbol{\sigma}}'(0)$ are the order 1 terms of our earlier modified Maurer-Cartan forms (5.15), while $\widehat{\boldsymbol{\alpha}}$, $\widehat{\boldsymbol{\beta}}$ are suitably modified one-forms corresponding to the additional structure group parameters $\mathbf{a}$, $\mathbf{B}$. Note particularly that (5.19) contains no essential torsion. Equations (5.14) and (5.19) form the complete structure equations for the contact pseudo-group on the infinite jet bundle.

There is one final item to deal with when working on a finite jet bundle J^n. Since the contact forms which are well-defined on J^n have orders at most $n-1$, we must include $q\binom{p+n-1}{n}$ additional one-forms to complete the coframe on J^n. These will clearly be the basis forms du_J^α, $\#J=n$, which must be lifted appropriately. (See [15] for more details.) However, we can most simply accomplish this as follows: First, truncate the canonical contact form series $\boldsymbol{\vartheta}(t)$ at order n. The resulting lifted contact form will depend on $(n+1)^{\mathrm{st}}$ order derivatives of u. These can be eliminated, while retaining the proper lift, by adding in a suitable multiple of the base forms dx^i. Thus, the lifted coframe on J^n consists of the one-forms (5.13) along with the modified canonical contact form

$$\widehat{\boldsymbol{\vartheta}}(t)=\boldsymbol{\vartheta}(t)+\mathbf{e}(t^n)\cdot\boldsymbol{\xi} \bmod n, \tag{5.20}$$

where $\mathbf{e}=(e_I^\alpha)$ is a $q\times\binom{p+n-1}{n}$ matrix of additional group parameters. The corresponding truncated structure equations are now

$$d\widehat{\boldsymbol{\vartheta}}(t)=\widehat{\boldsymbol{\lambda}}(t)\wedge\widehat{\boldsymbol{\vartheta}}(t)+\widehat{\boldsymbol{\vartheta}}'(t)\wedge\widehat{\boldsymbol{\sigma}}(t)+\boldsymbol{\varepsilon}(t^n)\wedge\boldsymbol{\xi}+\widehat{\boldsymbol{\vartheta}}'(t)\wedge\boldsymbol{\xi} \bmod n. \tag{5.21}$$

This completes our proof.

Example 5.5. The structure equations for the one-dimensional situation are as follows:

$$
\begin{aligned}
d\vartheta_0 &= \lambda_0 \wedge \vartheta_0 + \xi \wedge \vartheta_1, \\
d\vartheta_1 &= \lambda_1 \wedge \vartheta_0 + (\lambda_0 - \sigma_1) \wedge \vartheta_1 + \xi \wedge \vartheta_2, \\
d\vartheta_2 &= \lambda_2 \wedge \vartheta_0 + (2\lambda_1 - \sigma_2) \wedge \vartheta_1 + (\lambda_0 - 2\sigma_1) \wedge \vartheta_2 + \xi \wedge \vartheta_3, \\
d\vartheta_3 &= \lambda_3 \wedge \vartheta_0 + (3\lambda_2 - \sigma_3) \wedge \vartheta_1 + (3\lambda_1 - 3\sigma_2) \wedge \vartheta_2 \\
&\qquad + (\lambda_0 - 3\sigma_1) \wedge \vartheta_3 + \xi \wedge \vartheta_4, \\
&\ \ \vdots \qquad\qquad\qquad \vdots \\
d\vartheta_{n-1} &= \sum_{i=0}^{n-1} \left[\binom{n-1}{i} \lambda_{n-1-i} - \binom{n-1}{i-1} \sigma_{n-i} \right] \wedge \vartheta_i + \xi \wedge \widetilde{\vartheta}_n, \\
d\widetilde{\vartheta}_n &= \sum_{i=0}^{n} \left[\binom{n}{i} \lambda_{n-i} - \binom{n}{i-1} \sigma_{n+1-i} \right] \wedge \vartheta_i + \varepsilon \wedge \xi, \\
d\xi &= \sigma_1 \wedge \xi + \varphi \wedge \vartheta_0 + \psi \wedge \vartheta_1.
\end{aligned} \tag{5.22}
$$

Here $\lambda_0, \ldots, \lambda_n$ are the Leibniz Maurer-Cartan forms, $\sigma_1, \ldots, \sigma_n$ the prolonged general linear group Maurer-Cartan forms, and ε, φ, ψ the three additional Maurer-Cartan forms, corresponding to the truncated or non-canonical part of the lifted coframe.

References

[1] I. M. Anderson, N. Kamran and P. J. Olver, *Internal, external and generalized symmetries*, Adv. in Math., **100** (1993), 53–100.

[2] A. V. Bäcklund, *Ueber Flachentransformationen*, Math. Ann., **9** (1876), 297–320.

[3] E. T. Bell, *Exponential polynomials*; Ann. Math., **35** (1934), 258–277.

[4] R. L. Bryant, Some Aspects of the Local and Global Theory of Pfaffian Systems, University of North Carolina, Chapel Hill, NC, 1979.

[5] L. Comtet, Advanced Combinatorics: The Art of Finite and Infinite Expansions, D. Reidel, Boston, 1974.

[6] C. Ehresmann, Introduction a la théorie des structures infinitésimales et des pseudo-groupes de Lie, Geometrie Differentielle, Colloq. Inter. du Centre Nat. de la Recherche Scientifique, Strasbourg, 1953, pp. 97–110.

[7] H. Federer, Geometric Measure Theory, Springer-Verlag, New York, 1969.

[8] M. Fels and P. J. Olver, *Moving coframes.* I. *A practical algorithm*, Acta Appl. Math., **51** (1998) 161–213.

[9] M. Fels and P. J. Olver, *Moving coframes.* II. *Regularization and theoretical foundations*, Acta Appl. Math., **55** (1999), 127–208.

[10] R. B. Gardner, The Method of Equivalence and Its Applications, SIAM, Philadelphia, 1989.

[11] S. Kobayashi, *Canonical forms on frame bundles of higher order contact*, Proc. Sympos. Pure Math., **3** (1961) 186–19.

[12] S. Kobayashi, Transformation Groups in Differential Geometry, Springer-Verlag, New York, 1972.

[13] S. Lie and G. Scheffers, Geometrie der Berührungstransformationen, B.G. Teubner, Leipzig, 1896.

[14] P. J. Olver, *Symmetry groups and group invariant solutions of partial differential equations*, J. Diff. Geom., **14** (1979), 497–542.

[15] P. J. Olver, *Equivalence and the Cartan form*, Acta Appl. Math., **31** (1993) 99–136.

[16] P. J. Olver, Equivalence, Invariants, and Symmetry, Cambridge University Press, Cambridge, 1995.

[17] P. J. Olver, Moving frames, RIMS Kokyuroku, **1150** (2000), 114–124.

[18] J. Riordan, An Introduction to Combinatorial Analysis, Princeton University Press, Princeton, N.J., 1980.

[19] S. Sternberg, Lectures on Differential Geometry, Prentice-Hall, Englewood Cliffs, N.J., 1964.

[20] A. M. Vinogradov and I. S. Krasil'shchik (eds.), Symmetries and Conservation Laws for Differential Equations of Mathematical Physics; American Mathematical Society, Providence, RI, 1998.

[21] K. Yamaguchi, *Contact geometry of higher order*, Japan J. Math., **8** (1982), 109–176.

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Associativity Breaks Down in Deformation Quantization

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§1. Introduction

The Weyl algebra $W_\hbar$ is the associative algebra generated over $\mathbb{C}$ by u, v with the fundamental relation $u * v - v * u = -\hbar i$ where $\hbar$ is a positive constant. (u, v) is called a *canonical conjugate pair*. This is one of the simplest algebra which appears in the theory of deformation quantization [BFLS].

In such a noncommutative algebra, the *ordering problem* may be viewed as the problem of expressing elements of the algebra in a unique way. In the Weyl algebra, three kind of orderings; normal ordering, anti-normal ordering, and Weyl ordering, are mainly used. The normal ordering expression is the way of writing elements in the form $\sum a_{m,n} u^m * v^n$ by arranging u to the left hand side in each term. The anti-normal ordering is in the form $\sum a_{m,n} v^m * u^n$. The Weyl ordering is in the form $\sum a_{m,n} u^m \odot v^n$ by using the symmetric product $\odot$ defined by $u \odot v = \frac{1}{2}(u * v + v * u)$ etc. (See [OMY] for the detail of symmetric product.)

Through such an ordering, one can linearly identify the algebra with the space of all polynomials.

In other words, the Weyl algebra can be viewed, through each ordering mentioned above, as a non commutative associative product structure defined on the space $\mathbb{C}[u, v]$ of all polynomials with the ordinary commutative product. Product formulas are given respectively as follows: (We denote the ordinary commutative product by $\circ$, $\bullet$, $\cdot$ in order to distinguish what ordering expression is used.)

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- In the normal ordering expression: the product $*$ of the Weyl algebra is given by the ΨDO-*product formula* as follows:

$$f(u,v) * g(u,v) = f \exp\{\hbar i(\overleftarrow{\partial_v} \circ \overrightarrow{\partial_u})\} g. \tag{1.1}$$

- In the anti-normal ordering expression: the product $*$ of the Weyl algebra is given by the $\overline{\Psi}$DO-*product formula* as follows:

$$f(u,v) * g(u,v) = f \exp\{-\hbar i(\overleftarrow{\partial_u} \bullet \overrightarrow{\partial_v})\} g. \tag{1.2}$$

- In the Weyl ordering expression: the product $*$ of the Weyl algebra is given by the *Moyal product formula* as follows:

$$f(u,v) * g(u,v) = f \exp \frac{\hbar i}{2}\{\overleftarrow{\partial_v} \overset{\cdot}{\wedge} \overrightarrow{\partial_u}\} g \tag{1.3}$$

where $\overleftarrow{\partial_v} \overset{\cdot}{\wedge} \overrightarrow{\partial_u} = \overleftarrow{\partial_v} \cdot \overrightarrow{\partial_u} - \overleftarrow{\partial_u} \cdot \overrightarrow{\partial_v}$. Every product formula yields $u*v - v*u = -\hbar i$, and hence defines the Weyl algebra. Here, commutative products $\circ$, $\bullet$, $\cdot$ play only a supplementary role to express elements in the unique way. We distinguish these to indicate what ordering expression is used.

Remark that we can change generators. For every $A \in SL(2,\mathbb{C})$, let

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = A \begin{pmatrix} u \\ v \end{pmatrix}, \quad A \in SL(2,\mathbb{C}).$$

Then, it is obvious that $[u', v']_* = -\hbar i$, and hence u', v' may be viewed as generators. The replacement (pull-back) A^* of u, v by u', v' gives an algebra isomorphism of $W_\hbar$. Thus, we may consider the ordering problem by using u', v' instead of u, v.

Moreover, using a suitable canonical conjugate pair u, v, we can extend the algebra by using one of the above product formulas.

Let $Hol(\mathbb{C}^2)$ be the space of all entire functions on $\mathbb{C}^2$ with the compact open topology. In the case that the parameter $\hbar$ is treated as a formal parameter, which has been the usual attitude in the theory of deformation quantization (cf. [O,el.2]), the product $*$ extends associatively in any ordering expression to the space $Hol(\mathbb{C}^2)[[\hbar]]$ of all formal power series of $\hbar$ with coefficients in $Hol(\mathbb{C}^2)$. This is because product formulas mentioned above are bidifferential operators of total order $2k$ at the level of the coefficients of $\hbar^k$. (See [Om], §13 for more general treatment.)

However, it is obvious that $\hbar$ should be a positive parameter in a true quantum theory.

In this paper, we treat $\hbar$ is a positive parameter. Since all product formulas are given by concrete forms, these extend to the following:

- $f * g$ is defined if one of f, g is a polynomial.
- For every polynomial $p = p(u, v)$, the left-(resp. right-) multiplication $p*$ (resp. $*p$) is a continuous linear mapping of $Hol(\mathbb{C}^2)$ into itself under the compact open topology.

We call such a system a $(\mathbb{C}[u, v]; *)$-*bimodule.*

Proposition 1. *In every product formula mentioned above,* $(Hol(\mathbb{C}^2), \mathbb{C}[u, v], *)$ *is a* $(\mathbb{C}[u, v]; *)$*-bimodule.*

By the polynomial approximation theorem, the associativity $f * (g * h) = (f * g) * h$ holds if two of f, g, h are polynomials. We refer this as 2-*p-associativity.*

On the other hand, it is easy to see that the set of all quadratic forms in $W_\hbar$ is closed under the commutator bracket $[\ ,\ ]_*$, hence it forms a Lie algebra. $X = \frac{1}{\hbar}u^2$, $Y = \frac{1}{\hbar}v^2$, $H = \frac{i}{\hbar}uv$, where $uv = u * v + \frac{\hbar i}{2}$, form a basis of the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$: We see

$$\left[\frac{i}{2\hbar}uv, \frac{1}{\hbar\sqrt{8}}u^2\right] = -\frac{1}{\hbar\sqrt{8}}u^2, \quad \left[\frac{i}{2\hbar}uv, \frac{1}{\hbar\sqrt{8}}v^2\right] = \frac{1}{\hbar\sqrt{8}}v^2,$$
$$\left[\frac{1}{\hbar\sqrt{8}}u^2, \frac{1}{\hbar\sqrt{8}}v^2\right] = -\frac{i}{2\hbar}uv.$$

X, Y, H generate an associative algebra in the space $\mathbb{C}[u, v]$ of all polynomials. This is an enveloping algebra of $\mathfrak{sl}(2, \mathbb{C})$.

The Casimir element $C = H^2 + (X * Y + Y * X)$, that is

$$C = \left(\frac{i}{2\hbar}uv\right)^2_* + \frac{1}{\hbar\sqrt{8}}u^2 * \frac{1}{\hbar\sqrt{8}}v^2 + \frac{1}{\hbar\sqrt{8}}v^2 * \frac{1}{\hbar\sqrt{8}}u^2$$

is given by

$$\begin{aligned} 8\hbar^2 C &= u^2 * v^2 + v^2 * u^2 - 2\left(u * v + \frac{\hbar i}{2}\right)^2 \\ &= u^2 * v^2 + v^2 * u^2 - 2u * v * u * v - 2\hbar i u * v + \frac{\hbar^2}{2}. \end{aligned}$$

Hence, $C = -\frac{3}{16}$. This means that our enveloping algebra is constrained in the space $C = -\frac{3}{16}$.

In a $(\mathbb{C}[u, v]; *)$-*bimodule* with an ordering expression mentioned above, we can consider the differential equation

$$\frac{d}{dt}f_t(u, v) = p(u, v) * f_t(u, v), \quad f_0(u, v) = f(u, v)$$

for every polynomial $p(u,v)$. If $p(u,v) = u^2 + (\frac{i}{\hbar}v)^2$, this equation is viewed as that of standard harmonic oscillator. If the complex variable t is considered, the existence of the solution for arbitrary initial function does not hold, but a real analytic solution in t is unique, if exists. If the real analytic solution exists, then we denote this by $e_*^{tp(u,v)} * f(u,v)$, where $e_*^{tp(u,v)}$ is the solution with initial condition 1.

The purpose of this paper is to investigate the group generated by $e_*^{aH+bX+cY}$. It is obvious that the obtained group should be $SL(2,\mathbb{C})$ or $SL(2,\mathbb{C})/\mathbb{Z}_2$.

However, we have to use several ordering expressions to define $e_*^{aH+bX+cY}$ for all a, b, $c \in \mathbb{C}$. This is just like a 2-sphere can not be covered by one coordinate sheet. We need at least three ordering expressions to cover $SL(2,\mathbb{C})$. The precise meaning of the "union" will become clear in the proof.

Moreover, we see that the $*$-product $e_*^{aH+bX+cY} * e_*^{a'H+b'X+c'Y}$ is defined in general with an ambiguity of $\pm$-sign of $\sqrt{\cdot}$, and the ambiguity can not be eliminated. Since the group structure is considered by using $*$-multiplication and the addition is not used, we can calculate the group operation with $\pm$ ambiguity. We show the following in this paper:

Theorem 2. *There is no $(\mathbb{C}[u,v];*)$-bimodule with an ordering expression containing $e_*^{aH+bX+cY}$ for all a, b, $c \in \mathbb{C}$.*

However, if we use several $(\mathbb{C}[u,v];*)$-bimodules with ordering expressions and forget about the ambiguity of $\sqrt{\cdot}$, then the group generated by $\{e_*^{aH+bX+cY}; a,\ b,\ c \in \mathbb{C}\}$ is embedded in the union of such bimodules, and the image is $SL(2,\mathbb{C})$.

Several anomalous phenomena relating this theorem will be also discussed in this paper. Especially, we discuss how the associativity breaks down in the calculation of extended $*$-product.

§2. Extensions of product formula

In this section, we mainly use the Weyl ordering expression. The following is the most useful property of Moyal product formula (1.3):

Proposition 3. *For every $A \in SL(2,\mathbb{C})$, let Φ^* be the replacement (pull-back) of u, v into u', v' by the combination of the linear transformation by the matrix A and the parallel displacement:*

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = A\begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \quad A \in SL(2,\mathbb{C}), \quad (\alpha,\beta) \in \mathbb{C}^2.$$

Then, Φ^* is an isomorphism in both $*$-product and $\cdot$-product.

Remark that other expressions do not have such a property. It is easily seen that

$$(au+bv)^m_* = (au+bv)^m_\cdot, \quad \text{but} \quad (au+bv)^m_* \neq (au+bv)^m_\circ \quad \text{for} \quad ab \neq 0.$$

For the proof of Proposition 3, we have only to remark the following identity:

$$\overleftarrow{\partial_v} \wedge \overrightarrow{\partial_u} = \overleftarrow{\partial_{v'}} \wedge \overrightarrow{\partial_{u'}}.$$

It is clear that if $A = \text{diag}\{\lambda, \lambda^{-1}\}$, then the replacement Φ^* of (u, v) by (u', v') which is given by

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = A\begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \quad \lambda \in \mathbb{C}_*, \ (\alpha, \beta) \in \mathbb{C}^2,$$

gives an isomorphism in both $*$-product and $\circ$-product or in both $*$-product and $\bullet$-product.

Starting from a $(\mathbb{C}[u, v]; *)$-bimodule, $*$-product extends to a wider class of functions. For every positive real number p, we set

$$\text{(2.1)} \quad \mathcal{E}_p(\mathbb{C}^2) = \{f \in Hol(\mathbb{C}^2) \mid \|f\|_{p,s} = \sup |f| e^{-s|\xi|^p} < \infty, \ \forall s > 0\}$$

where $|\xi| = (|u|^2 + |v|^2)^{1/2}$. The family $\{\| \ \|_{p,s}\}_{s>0}$ induces a topology on $\mathcal{E}_p(\mathbb{C}^2)$ and $(\mathcal{E}_p(\mathbb{C}^2), \cdot)$ is an associative commutative Fréchet algebra, where the dot $\cdot$ is the ordinary multiplication for functions in $\mathcal{E}_p(\mathbb{C}^2)$. Thus, $\cdot$ may be replaced by $\circ$ or $\bullet$ to indicate ordering of expression. It is easily seen that for $0 < p < p'$, there is a continuous embedding

$$\text{(2.2)} \qquad \mathcal{E}_p(\mathbb{C}^2) \subset \mathcal{E}_{p'}(\mathbb{C}^2)$$

as commutative Fréchet algebras (cf. [GS]), and that $\mathcal{E}_p(\mathbb{C}^2)$ is $SL(2, \mathbb{C})$-invariant.

It is obvious that every polynomial is contained in $\mathcal{E}_p(\mathbb{C}^2)$ and $\mathbb{C}[u, v]$ is dense in $\mathcal{E}_p(\mathbb{C}^2)$ for any $p > 0$ in the Fréchet topology defined by the family $\{\| \ \|_{p,s}\}_{s>0}$.

Every exponential function $e_\cdot^{\alpha u+\beta v}$ is contained in $\mathcal{E}_p(\mathbb{C}^2)$ for any $p > 1$, but not in $\mathcal{E}_1(\mathbb{C}^2)$, and functions such as $e_\cdot^{au^2+bv^2+2cuv}$ are contained in $\mathcal{E}_p(\mathbb{C}^2)$ for any $p > 2$, but not in $\mathcal{E}_2(\mathbb{C}^2)$. Functions such as $\sum \frac{1}{(n!)^{1/p}} u^k$ is contained in $\mathcal{E}_q(\mathbb{C}^2)$ for any $q > p$, but not in $\mathcal{E}_p(\mathbb{C}^2)$.

$Hol(\mathbb{C}^2)$ is a complete topological linear space under the compact open topology.

The following theorem is the main result of [OMMY]: [1]

[1] In [OMMY], the proof is given in the case of Weyl ordering expression, but the same proof works for other orderings.

Theorem 4. *Any product formula* (1.1), (1.2), (1.3) *extend to give the following*:

- **(i):** *For* $0 < p \leq 2$, *the space* $(\mathcal{E}_p(\mathbb{C}^2), *)$ *forms a topological associative algebra.*
- **(ii):** *For* $p > 2$, *every product formula gives a continuous bi-linear mapping of*

$$\mathcal{E}_p(\mathbb{C}^2) \times \mathcal{E}_{p'}(\mathbb{C}^2) \to \mathcal{E}_p(\mathbb{C}^2), \quad \mathcal{E}_{p'}(\mathbb{C}^2) \times \mathcal{E}_p(\mathbb{C}^2) \to \mathcal{E}_p(\mathbb{C}^2), \tag{2.3}$$

for every p' *such that* $\frac{1}{p} + \frac{1}{p'} \geq 1$.

We remark here about the statement (ii). Since $p > 2$, p' must be $p' < 2$, hence the statement (i) gives that $(\mathcal{E}_{p'}(\mathbb{C}^2); *)$ is a Fréchet algebra. So the statement (ii) means that every $\mathcal{E}_p(\mathbb{C}^2)$, $p > 2$, is a topological $\mathcal{E}_{p'}(\mathbb{C}^2)$-bimodule.

We remark also that if $\hbar > 0$, then $e_{\cdot}^{\pm(1/\hbar)(au^2+bv^2+2cuv)} \in \mathcal{E}_p(\mathbb{C}^2)$ for every $p > 2$. Remark also that such an element does not appear in the theory of *formal* deformation quantization.

Let $\mathcal{E}_{2+}(\mathbb{C}^2) = \bigcap_{p>2} \mathcal{E}_p(\mathbb{C}^2)$. $\mathcal{E}_{2+}(\mathbb{C}^2)$ is a Fréchet space under the natural intersection topology, $e^{\pm(1/\hbar)(au^2+bv^2+2cuv)}$ is continuous in $\mathcal{E}_{2+}(\mathbb{C}^2)$ with respect to $(a, b, c) \in \mathbb{C}^3$.

The following are examples of elements of $\mathcal{E}_{2+}(\mathbb{C}^2)$ which play important role in the later sections:

$$\int_{-\infty}^{\infty} \frac{1}{\cosh t} e^{(\tanh t)uv}\, dt, \quad \frac{1}{u}(1 - e^{(2i/\hbar)uv}), \quad \frac{1}{v}(1 - e^{-(2i/\hbar)uv}).$$

2.1. Intertwiner, or coordinate transformations

We have three kind of $(\mathbb{C}[u, v]; *)$-bimodules according to normal ordering expressions, the anti-normal ordering expression and the Weyl ordering expression.

Let e_*^{su}, e_*^{tv} be $*$-exponential functions defined by $e_*^{su} = \sum \frac{1}{k!}(su)^k$ or equivalently by the solution of $\frac{d}{dt} f_t(u) = u * f_t(u)$ with $f_0(u) = 1$. By each product formula, $e_*^{su} * e_*^{tv}$ is computed as follows:

- $e_*^{su} * e_*^{tv} = e_\circ^{su+tv}$ in the ΨDO-product formula,
- $e_*^{su} * e_*^{tv} = e^{-\hbar ist} e_\bullet^{su+tv}$ in the $\bar{\Psi}$DO-product formula,
- $e_*^{su} * e_*^{tv} = e^{-(\hbar ist/2)} e_\cdot^{su+tv}$ in the Moyal product formula,

where $\circ$, $\bullet$, $\cdot$ indicate the commutative product used in each expression. We have also

- $e_*^{\alpha u+\beta v} = e_\cdot^{\alpha u+\beta v}$ in the Weyl ordering expression, but
- $e_*^{\alpha u+\beta v} = e^{(\hbar i/2)\alpha\beta} e_\circ^{\alpha u+\beta v}$ in the normal ordering expression with respect to (u, v).

Thus, we must identify $e_{\circ}^{su+tv}$, $e^{-\hbar ist}e_{\bullet}^{su+tv}$, $e^{-(\hbar ist/2)}e_{\cdot}^{su+tv}$ through linear transformations. These are obtained by the following

$$e^{\hbar i\partial_u\partial_v}e_{\circ}^{su+tv}\longleftrightarrow e^{-\hbar ist}e_{\bullet}^{su+tv},\quad e^{(\hbar i/2)\partial_u\partial_v}e_{\circ}^{su+tv}\longleftrightarrow e^{-(\hbar ist/2)}e_{\cdot}^{su+tv}$$

Thus we define intertwiners as follows:

$$I_{\circ}^{\bullet}(f)=e^{-\hbar i\partial_u\partial_v}f,\quad I_{\circ}^{\cdot}(f)=e^{-(\hbar i/2)\partial_u\partial_v}f.$$

We consider also the intertwiner between normal ordering expression with respect to (u,v) and the normal ordering expression with respect to (u',v') when (u,v) and (u',v') are related by $u'=au+bv$, $v'=cu+dv$ such that $ad-bc=1$.

The principle of making the intertwiner is that the $*$-exponential functions $e_*^{\alpha u+\beta v}$ and $e_*^{\alpha' u'+\beta' v'}$ coincide if (u,v) and (u',v') are canonical conjugate pairs related linearly by each other and $\alpha u+\beta v=\alpha' u'+\beta' v'$.

Lemma 5. *If $u'=\alpha u+\beta v$, and $v'=\gamma u+\delta v$ is a canonical conjugate pair, then $e_*^{tu'}=e_{\circ'}^{tu'}$ in the normal ordering expression with respect to (u',v').*

Applying Lemma 5 to a canonical conjugate pair (u',v'), we take the normal ordering expression with respect to (u',v'):

$$e_*^{\alpha' u'+\beta' v'}=e^{(\hbar i/2)\alpha'\beta'}e_{\diamond}^{\alpha' u'+\beta' v'}$$

Suppose $\alpha u+\beta v=\alpha' u'+\beta' v'$ and $u'=au+bv$, $v'=cu+dv$, $ad-bc=1$. Then, we must identify $e^{(\hbar i/2)\alpha\beta}e_{\circ}^{\alpha u+\beta v}$ with $e^{(\hbar i/2)\alpha'\beta'}e_{\diamond}^{\alpha' u'+\beta' v'}$.

Hence, we have to define the intertwiner $I_{\circ}^{\diamond}$ as a linear mappings:

$$I_{\circ}^{\diamond}f=e^{(\hbar i/2)\partial_{u'}\partial_{v'}-(\hbar i/2)\partial_u\partial_v}f. \tag{2.4}$$

Precisely speaking, if (u,v) and (u',v') relate by

$$u'=au+bv,\quad v'=cu+dv,\quad ad-bc=1,$$

we first consider the exponential of the operator

$$\partial_{u'}\partial_{v'}-\partial_u\partial_v=-bd\partial_u^2+(ad+bc-1)\partial_u\partial_v-ac\partial_v^2$$

and then we replace the variable (u,v) by $(du'-bv',-cv'+av')$ to obtain $I_{\circ}^{\diamond}f(u,v)$. That is, if

$$e^{-bd\partial_u^2+(ad+bc-1)\partial_u\partial_v-ac\partial_v^2}f(u,v)=g(u,v),$$

then we set $I_{\circ}^{\diamond}f(u,v)=g(du'-bv',-cv'+av')$.

These are first defined on the space $\mathbb{C}[u,v]$, and these give different expressions to a *same element* written by using $*$-product via different commutative algebras.

Theorem 6. *The intertwiners defined above extend to continuous linear isomorphisms of $\mathcal{E}_p(\mathbb{C}^2)$ onto itself for every $0 < p \leq 2$, and to give algebra isomorphisms of $(\mathcal{E}_p(\mathbb{C}^2); *)$ onto $(\mathcal{E}_p(\mathbb{C}^2); *)$.*

However, these do not extend to the space $\mathcal{E}_{2+}(\mathbb{C}^2)$.

Just like a coordinate transformation, the intertwiner is defined only on a part of $\mathcal{E}_{2+}(\mathbb{C}^2)$ onto a part of another $\mathcal{E}_{2+}(\mathbb{C}^2)$.

In spite of this, it is remarkable that the patching property, that is, $I_\diamond^{\diamond'} I_\circ^\diamond(f) = I_\circ^{\diamond'}(f)$ holds for $f \in \mathcal{E}_2(\mathbb{C}^2)$, and this hold also for $f \in \mathcal{E}_{2+}(\mathbb{C}^2)$ if both sides are defined. This is proved by the approximation by elements of $\mathcal{E}_2(\mathbb{C}^2)$. *Intertwiners have the property of gluing maps of bimodules.*

By the above observation we see in particular:

Lemma 7. *The anti-normal ordering expression with respect to (u,v), and the normal ordering expression with respect to $(-v,u)$ coincides.*

By the observation as above, we have to consider the differential equations

$$\frac{\partial}{\partial t} f = \hbar i \partial_u \partial_v f, \quad \frac{d}{d\tau} f = \hbar i \partial_u^2 f. \tag{2.5}$$

The solution with initial function e^{au+bv} is given by $e^{\hbar iabt} e^{au+bv}$, $e^{\hbar a^2 t} e^{au+bv}$. To obtain the solution with the initial function $e^{\alpha u^2 + \beta v^2 + 2\gamma uv}$, we set $f = s(t) e^{\phi_1(t)u^2 + \phi_2(t)v^2 + \phi_3(t)2uv}$. Then, the equations in (2.5) are rewritten respectively as systems of ordinary differential equations:

$$\begin{aligned} s'(t) &= 2\hbar i s(t)\phi_3(t), \quad \phi_1'(t) = 4\hbar i \phi_1(t)\phi_3(t), \\ \phi_2'(t) &= 4\hbar i \phi_2(t)\phi_3(t), \quad \phi_3'(t) = 2\hbar i(\phi_1(t)\phi_2(t) + \phi_3(t)^2). \end{aligned} \tag{2.6}$$

$$\begin{aligned} s'(\tau) &= 2\hbar i \phi_1(\tau) s(\tau), \quad \phi_1'(\tau) = 4\hbar i \phi_1(\tau)^2, \\ \phi_2'(\tau) &= 4\hbar i \phi_3(\tau)^2, \quad \phi_3'(\tau) = 4\hbar i \phi_1(\tau)\phi_3(\tau). \end{aligned} \tag{2.7}$$

Through the solutions, we can patch exponential functions of quadratic forms together, and although the domain and the region are not clearly stated, intertwiners give patching identities of $\mathcal{E}_p(\mathbb{C}^2)$-bimodules for $p < 2$, to define a certain $\mathcal{E}_p(\mathbb{C}^2)$-bimodule as a patched object.

§3. Vacuums, half-inverses and the break down of the associativity

A direct calculation using the Moyal product formula (1.3) shows that the coordinate function v has a right inverse $v^{\circ} = \frac{1}{v}(1 - e^{(2i/\hbar)uv})$, and a left inverse $v^{\bullet} = \frac{1}{v}(1 - e^{-(2i/\hbar)uv})$ in $\mathcal{E}_{2+}(\mathbb{C}^2)$, i.e,

$$v * v^{\circ} = 1 = v^{\bullet} * v, \quad v^{\circ} * v = 1 - 2e^{(2i/\hbar)uv}, \quad v * v^{\bullet} = 1 - 2e^{-(2i/\hbar)uv}.$$

If the associativity holds, then these should be the same genuine inverse. Hence we must set $\frac{1}{v}\sin\frac{2}{\hbar}uv = 0$. Since this is impossible (cf. [O,el.1]), we loose the associativity in $\mathcal{E}_{2+}(\mathbb{C}^2)$. This is one of the most basic phenomenon which breaks the associativity. That is, *coordinate functions have both left- and right-inverses.*

3.1. Star-exponentials of quadratic forms in the Weyl ordering expression

These strange phenomena are deeply related to the $*$-exponential function such as $e_*^{(t/\hbar)u\cdot v}$ defined by the equation $\frac{d}{dt}f_t(u,v) = \frac{1}{\hbar}(u\cdot v) * f_t(u,v)$, $f_0(u,v) = 1$. Recall again that such an element can not appear in the formal deformation theory.

For every point $(a,b,c;s)$ in $\mathbb{C}^4$, consider a curve $s(t)\exp\{\frac{1}{\hbar}(a(t)u^2 + b(t)v^2 + 2c(t)uv)\}$ starting at the point $s\exp\{\frac{1}{\hbar}(au^2+bv^2+2cuv)\}$ then the tangent vector of this curve is given as

$$\left(\frac{1}{\hbar}(a'u^2 + b'v^2 + 2c'uv)s + s'\right) e^{(1/\hbar)(au^2+bv^2+2cuv)}.$$

On the other hand, consider the $*$-product

$$\frac{d}{dt}\Big|_{t=0} e^{(t/\hbar)(a'u^2+b'v^2+2c'uv)} * se^{(1/\hbar)(au^2+bv^2+2cuv)}.$$

This is computed as follows:

$$\begin{aligned}
&\frac{1}{\hbar}(a'u^2 + b'v^2 + 2c'uv) * se^{(1/\hbar)(au^2+bv^2+2cuv)} \\
&\quad = \frac{1}{\hbar}(a'u^2 + b'v^2 + 2c'uv)se^{(1/\hbar)(au^2+bv^2+2cuv)} \\
&\qquad + \frac{2i}{\hbar}\{(b'v + c'u)(au+cv) - (a'u + c'v)(bv+cu)\} \\
&\qquad\qquad \times se^{(1/\hbar)(au^2+bv^2+2cuv)} \\
&\qquad - \frac{1}{2\hbar}\{b'(\hbar a + 2(au+cv)^2) - 2c'(\hbar c + 2(au+cv)(bv+cu)) \\
&\qquad\qquad + a'(\hbar b + 2(bv+cu)^2)\}se^{(1/\hbar)(au^2+bv^2+2cuv)}
\end{aligned}$$

This may be written as

(3.1)

$$\frac{1}{\hbar}(a',b',c')\begin{bmatrix} -(c+i)^2, & -b^2, & -b(c+i), & -\frac{b}{2} \\ -a^2, & -(c-i)^2, & -a(c-i), & -\frac{a}{2} \\ 2a(c+i), & 2b(c-i), & 1+ab+c^2, & c \end{bmatrix}\begin{bmatrix} u^2 \\ v^2 \\ 2uv \\ \hbar \end{bmatrix}$$

$$\times\, s e^{(1/\hbar)(au^2+bv^2+2cuv)}.$$

We denote this matrix by $M(a,b,c;s)$, and by $M(a,b,c)$ the submatrix of first three columns.

Remark that

$$\det M(a,b,c) = (c^2-ab+1)^3. \tag{3.2}$$

The feature of this matrix is that the radial direction is the direction of eigen vector:

$$(a,b,c)M(\tau a,\tau b,\tau c) = (1+(c^2-ab)\tau^2)(a,b,c), \tag{3.3}$$

holds for every (a,b,c).

If $c^2-ab+1=0$, then we can write

$$au^2+bv^2+2cuv = 2i(\alpha u+\beta v)(\gamma u+\delta v), \quad \alpha\delta-\beta\gamma=1.$$

Clearly, $[\alpha u+\beta v, \gamma u+\delta v] = -\hbar i$. Hence, setting $u' = \alpha u+\beta v$, $v' = \gamma u+\delta v$, (u',v') is a canonical conjugate pair, and hence by Proposition 3, we easily see by (1.3) that

$$(\gamma u+\delta v) * e^{(2i/\hbar)(\alpha u+\beta v)(\gamma u+\delta v)} = 0, \quad \text{for} \quad \alpha\delta-\beta\gamma=1. \tag{3.4}$$

It follows that

$$(\gamma u+\delta v)^2_* * e^{(1/\hbar)(au^2+bv^2+2cuv)} = 0,$$
$$(\alpha u+\beta v) * (\gamma u+\delta v) * e^{(1/\hbar)(au^2+bv^2+2cuv)} = 0.$$

The second identity yields $(a,b,c)M(a,b,c)=0$, if $c^2-ab+1=0$, which corresponds to (3.3), and the first one yields

$$(\gamma^2,\delta^2,\gamma\delta)M(a,b,c)=0, \quad c^2-ab+1=0.$$

Hence we see that $M(a,b,c)$ is rank 1 at the point $c^2-ab+1=0$, but the rank of $M(a,b,c;s)$ is 2 at such a point. $2e^{(2i/\hbar)(\alpha u+\beta v)(\gamma u+\delta v)}$ and $2e^{-(2i/\hbar)(\alpha u+\beta v)(\gamma u+\delta v)}$ are called *vacuums*. Remark that $(\alpha u+\beta v)\times(\gamma u+\delta v)$ and $(\gamma u+\delta v)(\alpha u+\beta v)$ are distinguished in the expression of vacuums.

3.2. Horizontal distributions

Using (3.1), we consider a holomorphic singular distribution D given by

$$D(a,b,c;s) = \{(a',b',c')M(a,b,c;s) \mid (a',b',c') \in \mathbb{C}^3\}$$

on the space $\mathbb{C}^3 \times \mathbb{C}_*$. Let $\pi\colon \mathbb{C}^3 \times \mathbb{C}_* \to \mathbb{C}^3$ be the natural projection.

Let $\Sigma = \{(a,b,c); c^2 - ab + 1 = 0\}$. $\Sigma \times \mathbb{C}_*$ is a 3-dimensional complex submanifold of $\mathbb{C}^3 \times \mathbb{C}_*$.

Though $\{D\}$ is singular on $\Sigma \times \mathbb{C}_*$, $\{D\}$ is a strongly involutive distribution in the sense of [Om] p. 51, for $\{D\}$ is given as an infinitesimal action of a Lie group. This gives an ordinary involutive distribution on $(\mathbb{C}^3 - \Sigma) \times \mathbb{C}_*$ and hence there is the 3-dimensional maximal integral holomorphic submanifold M^3 through the origin $(0,0,0;1)$.

A curve $\boldsymbol{g}(t) = (a(t), b(t), c(t); s(t))$ is an *integral curve* of $\{D\}$, if $\frac{d}{dt}\boldsymbol{g}(t) \in D(\boldsymbol{g}(t))$ for every t. For every curve $\boldsymbol{c}(t)$ in $\mathbb{C}^3 - \Sigma$, we have an integral curve $\boldsymbol{g}(t)$ such that $\pi(\boldsymbol{g}(t)) = \boldsymbol{c}(t)$. $\boldsymbol{g}(t)$ is a *lift* of $\boldsymbol{c}(t)$. Remark that $\boldsymbol{g}(1)$ depends only on the homotopy class of curves joining $(0,0,0)$ and $\boldsymbol{c}(1)$.

Points of M^3 is given as the homotopy equivalence class of lift of curves in $\mathbb{C}^3 - \Sigma$ starting at the origin $(0,0,0)$.

Every integral curve $\boldsymbol{g}(t)$ staring at a point of $\Sigma \times \mathbb{C}_*$ remains in this space. The maximal integral submanifold through a point of $\Sigma \times \mathbb{C}_*$ is a 2-dimensional complex submanifold M^2 such that $\pi(M^2)$ is a one dimensional submanifold of Σ. Hence, $\Sigma \times \mathbb{C}_*$ is foliated by maximal integral submanifolds.

3.3. $*$-exponentials and vacuums

In this subsection we define the exponential function $e_*^{t(au^2+bv^2+2cuv)}$. Set $e_*^{t(au^2+bv^2+2cuv)} = F(t,u,v)$, and consider the evolution equation

$$\text{(3.5)}\quad \frac{\partial}{\partial t}F(t,u,v) = (au^2+bv^2+2cuv) * F(t,u,v), \quad F(0,u,v) = 1.$$

The right hand side of (3.5) is computed by the Moyal product formula (1.3) as follows:

$$\begin{aligned}(au^2 &+ bv^2 + 2cuv) * F(t,u,v)\\ &= (au^2+bv^2+2cuv)F + \hbar i\{(bv+cu)\partial_u F - (au+cv)\partial_v F\}\\ &\quad - \frac{\hbar^2}{4}\{b\partial_u^2 F - 2c\partial_v\partial_u F + a\partial_v^2 F\}\end{aligned}$$

This is a partial differential equation. If $ab - c^2 > 0$, then this is the heat equation and the existence of solutions is not ensured in general.

This implies that the mapping $f(u,v) \to e_*^{(t/\hbar)(au^2+bv^2+2cuv)} * f(u,v)$ is not always defined for C^∞-functions.

However, we see that real analytic solution in t is unique, if it exists. Hence we assume that $e_*^{t(au^2+bv^2+2cuv)}$ is a function of au^2+bv^2+2cuv; that is $e_*^{t(au^2+bv^2+2cuv)} = f_t(au^2+bv^2+2cuv)$. Then, setting $x = au^2+bv^2+2cuv$, we have

$$\frac{d}{dt} f_t(x) = x f_t(x) - \hbar^2(ab-c^2)\big(f_t'(x) + x f_t''(x)\big). \tag{3.6}$$

The right hand side is the Bessel operator.

However, there is another method to treat this differential equation. We assume that

$$e_*^{t(au^2+bv^2+2cuv)} = s(t)e^{a(t)u^2+b(t)v^2+2c(t)uv},$$

then we have only to solve the system of ordinary differential equations

$$\begin{aligned} &\frac{d}{dt}(a(t), b(t), c(t); s(t)) = (a,b,c)M(a(t), b(t), c(t); s(t)),\\ &\qquad (a(0), b(0), c(0); s(0)) = (0,0,0;1). \end{aligned} \tag{3.7}$$

Lemma 8. *The solution of* (3.6) *with the initial function* 1 *is given by*

$$f_t(x) = \frac{1}{\cosh(\hbar\sqrt{ab-c^2}\,t)} \exp \frac{x}{\hbar\sqrt{ab-c^2}} \tanh\left(\hbar\sqrt{ab-c^2}\,t\right).$$

If $ab - c^2 = 0$, *then we set*

$$\frac{1}{\hbar\sqrt{ab-c^2}} \tanh\left(\hbar\sqrt{ab-c^2}\,t\right) = t.$$

This shows that e_*^{tuv} cannot be defined for all $t \in \mathbb{C}$, if the equation $\frac{d}{dt} f_t(uv) = (uv) * f_t(uv)$ is considered in the Weyl ordering expression.

We shall show that such singularities appear also in the other ordering expressions. Such an observation gives the first half of Theorem 2.

By Lemma 8, we have

(3.8)

$$\begin{aligned}
&\exp_*\left\{\frac{t}{\hbar}(au^2+bv^2+2cuv)\right\}\\
&=\frac{1}{\cosh(\sqrt{ab-c^2}\,t)}\\
&\quad\times\exp\left\{(au^2+bv^2+2cuv)\left(\frac{1}{\hbar\sqrt{ab-c^2}}\tanh\left(\sqrt{ab-c^2}\,t\right)\right)\right\}\\
&=\frac{1}{\cos(\sqrt{c^2-ab}\,t)}\\
&\quad\times\exp\left\{(au^2+bv^2+2cuv)\left(\frac{1}{\hbar\sqrt{c^2-ab}}\tan\left(\sqrt{c^2-ab}\,t\right)\right)\right\}
\end{aligned}$$

(cf. same formula is seen also in [MS].) Remark here that $e_*^{t(au^2+bv^2+2cuv)} \in M^3$. Though the ambiguity of $\pm\sqrt{ab-c^2}$ makes no difference for the result, the difference of the periodicity of cos and tan gives that if $c^2-ab\neq 0$, then

$$\pi^{-1}\pi\{e_*^{t(au^2+bv^2+2cuv)}; t\in\mathbb{C}\} = \{\pm e_*^{t(au^2+bv^2+2cuv)}; t\in\mathbb{C}\}. \tag{3.9}$$

Since $\tan\theta = \sqrt{c^2-ab}$ gives $\frac{1}{\cos^2\theta} = c^2-ab+1$, (3.8) is equivalent with

$$\begin{aligned}
&\sqrt{c^2-ab+1}\,\exp\left\{\frac{1}{\hbar}(au^2+bv^2+2cuv)\right\}\\
&\quad=\exp_*\left\{\frac{1}{\hbar\sqrt{c^2-ab}}\left(\arctan\sqrt{c^2-ab}\right)(au^2+bv^2+2cuv)\right\}.
\end{aligned}\tag{3.10}$$

Using this, we have the following:

Proposition 9. *If $c^2-ab+1\neq 0$, then $\pm\sqrt{c^2-ab+1}\times\exp\{\frac{1}{\hbar}(au^2+bv^2+2cuv)\}$ are elements of M^3. Conversely, if $\pi(Q) = \exp\{\frac{1}{\hbar}(au^2+bv^2+2cuv)\}$ with $c^2-ab+1\neq 0$ for some $Q\in M^3$, then*

$$\begin{aligned}
Q = \sqrt{c^2-ab+1}\,e^{(1/\hbar)(au^2+bv^2+2cuv)} \quad &or\\
-\sqrt{c^2-ab+1}\,e^{(1/\hbar)(au^2+bv^2+2cuv)}&.
\end{aligned}$$

These are written as $*$-exponential functions written in the form

$$e_*^{(t/\hbar)(au^2+bv^2+2cuv)};\quad a,\ b,\ c\in\mathbb{C}$$

except the case $Q = -e^{(t/\hbar)(au^2+bv^2+2cuv)}$, $c^2-ab=0$.

By (3.8), we have in particular, if $c^2 \neq ab$, then $\exp_*\{\frac{\pi}{\hbar\sqrt{c^2-ab}} \times (au^2+bv^2+2cuv)\} = -1$, but $\exp_*\{\frac{\pi}{2\hbar\sqrt{c^2-ab}}(au^2+bv^2+2cuv)\}$ diverges in the Weyl ordering expression.

Let Π_0 be the subset defined as follows:

$$\Pi_0 = \{(a,b,c) \in \mathbb{C}^3; e_*^{(1/\hbar)(au^2+bv^2+2cuv)} \quad \text{does not defined}\}.$$

Remark that Proposition 9 shows that $\pi\colon M^3 \to \mathbb{C}^3 - \Sigma$ is surjective, but the difference of period of cos and tan, and the ambiguity of the sign of $\sqrt{c^2 - ab + 1}$ of (3.10) shows that π gives a double cover. Hence we have the following result:

Proposition 10. $\exp_*\colon \mathbb{C}^3 - \Pi_0 \to \mathbb{C}^3 - \Sigma$ *is a holomorphic mapping such that*

$$\begin{aligned}&\exp_*(\mathbb{C}^3 - \Pi_0)\\ &\quad = M^3 - \{-e^{(1/\hbar)(au^2+bv^2+2cuv)};\quad c^2 - ab = 0, \quad (a,b,c) \neq (0,0,0)\}.\end{aligned}$$

The element -1 *is on a* $*$*-exponential function as* $\exp_*(\frac{\pi}{\hbar}2uv) = -1$.

By the uniqueness of analytic solutions, the exponential law

$$e_*^{isx} * e_*^{itx} = e_*^{i(s+t)x}$$

holds where both sides are defined.

Lemma 11. *For* s, $\sigma \in \mathbb{C}$ *such that* $1 + s\sigma(ab - c^2) \neq 0$, *we have*

$$\begin{aligned}&\exp\left\{\frac{s}{\hbar}(au^2 + bv^2 + 2cuv)\right\} * \exp\left\{\frac{\sigma}{\hbar}(au^2 + bv^2 + 2cuv)\right\}\\ &\quad = \frac{1}{1 + s\sigma(ab - c^2)} \exp\left\{\frac{s + \sigma}{\hbar(1 + s\sigma(ab - c^2))}(au^2 + bv^2 + 2cuv)\right\}.\end{aligned}$$

Thus, we have idempotent elements

$$\begin{aligned}&2\exp\left\{\pm\frac{1}{\hbar\sqrt{ab - c^2}}(au^2 + bv^2 + 2cuv)\right\}\\ &\qquad * 2\exp\left\{\pm\frac{1}{\hbar\sqrt{ab - c^2}}(au^2 + bv^2 + 2cuv)\right\}\\ &\quad = 2\exp\left\{\pm\frac{1}{\hbar\sqrt{ab - c^2}}(au^2 + bv^2 + 2cuv)\right\}.\end{aligned}$$

Recall $2\exp\{\frac{1}{\hbar\sqrt{ab-c^2}}(au^2 + bv^2 + 2cuv)\}$ is a vacuum.

Corollary 12. *Vacuums are obtained as the limit point of $*$-exponential functions*:

$$2\exp\left\{\frac{1}{\hbar\sqrt{ab-c^2}}(au^2+bv^2+2cuv)\right\}$$
$$=\lim_{t\to\infty}\exp\left\{it\sqrt{ab-c^2}\right\}\exp_*\left\{\frac{t}{\hbar\sqrt{ab-c^2}}(au^2+bv^2+2cuv)\right\}$$

is a vacuum.

This shows that vacuums may be regarded as certain equilibrium states (cf. [BL]).

The following lemma is useful in the computation, and is proved by that both quantities satisfy the same partial differential equation with the same initial condition:

Lemma 13 (Bumping lemma).

$$v*e_*^{itu*v}=e_*^{itv*u}*v,\quad e_*^{itu*v}*u=u*e_*^{itv*u}$$

3.4. Anomarous phenomena

We easily see by the Moyal product formula (1.3) that

$$v*e^{(2i/\hbar)uv}=0=e^{(2i/\hbar)uv}*u,\quad u*e^{-(2i/\hbar)uv}=0=e^{-(2i/\hbar)uv}*v.$$

We call $2e^{(2i/\hbar)uv}$ a *vacuum* and $2e^{-(2i/\hbar)uv}$ a *bar-vacuum* and denote these by $\varpi_{0,0}$, and $\overline{\varpi}_{0,0}$ respectively. By the Moyal product formula and the 2-p-associativity, we see easily

$$\left(uv-\frac{\hbar i}{2}\right)*e^{(2i/\hbar)uv}=u*v*e^{(2i/\hbar)uv}=0.$$

However, $uv-\hbar i/2=u*v$ has the inverse $i\int_0^\infty e_*^{-(it/\hbar)u*v}\,dt$ in $\mathcal{E}_{2+}(\mathbb{C}^2)$. Thus, the associativity fails in $\mathcal{E}_{2+}(\mathbb{C}^2)$:

$$\begin{aligned}&\left(\left(uv-\frac{\hbar i}{2}\right)^{-1}*\left(uv-\frac{\hbar i}{2}\right)\right)*e^{(2i/\hbar)uv}\\ &\neq\left(uv-\frac{\hbar i}{2}\right)^{-1}*\left(\left(uv-\frac{\hbar i}{2}\right)*e^{(2i/\hbar)uv}\right).\end{aligned}\tag{3.11}$$

Furthermore, we see that

$$\int_0^\infty\frac{1}{\cosh(t/2)}\exp\left\{\frac{i}{\hbar}\left(\tanh\frac{t}{2}\right)2u\cdot v\right\}dt,$$
$$\int_{-\infty}^0\frac{1}{\cosh(t/2)}\exp\left\{\frac{i}{\hbar}\left(\tanh\frac{t}{2}\right)2u\cdot v\right\}dt$$

exist in the space $\mathcal{E}_{2+}(\mathbb{C}^2)$. It follows that $u \cdot v$ has *two different* inverses as follows:

$$(u\cdot v)^{-1}_{+i0} = -i\int_0^{\infty} e_*^{(it/\hbar)u\cdot v}\,dt, \quad (u\cdot v)^{-1}_{-i0} = i\int_{-\infty}^{0} e_*^{(it/\hbar)u\cdot v}\,dt.$$

The difference is given as

$$(3.12) \qquad (u\cdot v)^{-1}_{+i0} - (u\cdot v)^{-1}_{-i0} = -i\int_{-\infty}^{\infty} e_*^{(it/\hbar)u\cdot v}\,dt.$$

Since the right hand side of (3.12) can be viewed as the $*$-Fourier transform of 1, this may be written as the *$*$-delta function* $-i\delta_*(u\cdot v)$ (cf. [OMMY]). Hence the associativity must break down again, and it holds $(u\cdot v) * \delta_*(u\cdot v) = \delta_*(u\cdot v) * (u\cdot v) = 0$.

Thus, it is impossible to treat $(u\cdot v)^{-1}_{+i0}$ and $(u\cdot v)^{-1}_{-i0}$ in the same associative algebra. In spite of this, the right hand side of (3.12) has the expression as follows by using Hansen-Bessel formula:

$$\begin{aligned}\int_{-\infty}^{\infty} e_*^{(it/\hbar)u\cdot v}\,dt &= \int_{-\infty}^{\infty} \frac{1}{\cosh(t/2)} \exp\left\{\frac{i}{\hbar}\left(\tanh\frac{t}{2}\right)2u\cdot v\right\} dt \\ &= \frac{\pi}{2} J_0\left(\frac{2}{\hbar}u\cdot v\right).\end{aligned}$$

Hence, $-i\delta_*(u\cdot v)$ is expressed as an entire function by the Weyl ordering expression.

Several fancy relations to Sato's hyper functions [M] can be seen, since $(u\cdot v \pm z)^{-1}_{\pm i0}$ is defined as a holomorphic function with respect to z on the upper half plane, and $-i\delta_*(u\cdot v)$ is viewed as the difference $(u\cdot v + z)^{-1}_{+i0} - (u\cdot v - z)^{-1}_{-i0}$. These will be discussed in another paper.

3.5. Several product formulas

Every quadratic form $Q(u,v)$ is written in the form

- $(\alpha u + \beta v)^2$, if $ab - c^2 = 0$,
- $\lambda(\alpha u + \beta v)(\gamma u + \delta v)$ with $\alpha\delta - \beta\gamma = 1$, if $ab - c^2 \neq 0$.

By Proposition 3, the general product formula for quadratic exponential functions can be obtained from only the two cases as follows:

$$e^{tu^2} * e^{au^2+bv^2+2cuv}, \quad e^{\tau uv} * e^{au^2+bv^2+2cuv}.$$

By solving the system of ordinary equations (3.7) with the general initial condition

$$(a(0), b(0), c(0); s(0)) = (a, b, c; 1),$$

we see that the first one is written as

(3.13)

$$
\begin{aligned}
&\exp_*\left\{\frac{t}{\hbar}u^2\right\} * \exp\left\{\frac{1}{\hbar}(au^2+bv^2+2cuv)\right\}\\
&=\frac{1}{\sqrt{1+bt}}\\
&\times\exp\left\{\frac{1}{\hbar(1+bt)}\{(a+(ab-c^2-2ci+1)t)u^2+bv^2+2(c-ibt)uv\}\right\}.
\end{aligned}
$$

The ambiguity of $\pm\sqrt{1+bt}$ can not be eliminated for all t, b.

The formula (3.13) yields several results for the $*$-product. Remark first that $e_*^{(t/\hbar)u^2}=e^{(t/\hbar)u^2}$.

Lemma 14. *For* $\exp\{\frac{t}{\hbar}u^2\}$, $Q \in M^3$ *such that* $\pi(Q) = \exp\{\frac{1}{\hbar}(au^2+bv^2+2cuv)\}$ *and* $bt \neq -1$, *the product* $\exp\{\frac{t}{\hbar}u^2\} * Q$ *is defined as an element of* M^3 *written as*

$$
\begin{aligned}
&\sqrt{\frac{c^2-ab+1}{1+bt}}\\
&\quad\times\exp\left\{\frac{1}{\hbar(1+bt)}\{(a+(ab-c^2-2ci+1)t)u^2+bv^2+2(c-ibt)uv\}\right\}.
\end{aligned}
$$

Similar to (3.13), we have

(3.14)

$$
\begin{aligned}
&\exp_*\left\{\frac{t}{\hbar}v^2\right\} * \exp\left\{\frac{1}{\hbar}(au^2+bv^2+2cuv)\right\}\\
&=\frac{1}{\sqrt{1+at}}\\
&\quad\times\exp\left\{\frac{1}{\hbar(1+at)}\{au^2+(b+(ab-c^2+2ci+1)t)v^2+2(c+iat)uv\}\right\},
\end{aligned}
$$

and hence we have the similar result as Lemma 14.

Remarking $e_*^{(t/\hbar)2uv}=\sqrt{1+s^2}\,e^{(s/\hbar)2uv}$, and solving carefully the system of ordinary equations (3.7) with the general initial condition, we have

(3.15)

$$\exp\left\{\frac{s}{\hbar}2uv\right\} * \exp\left\{\frac{1}{\hbar}(au^2+bv^2+2cuv)\right\}$$
$$= \frac{1}{\sqrt{1-2cs+(c^2-ab)s^2}}$$
$$\times \exp\left\{\frac{1}{\hbar(1-2cs+(c^2-ab)s^2)}\right.$$
$$\left.\times\ (a(1+is)^2u^2+b(1-is)^2v^2+(c-(c^2-ab-1)s-cs^2)2uv)\right\}.$$

The following identity is useful for the computation of discriminant D:

$$\begin{aligned}&\big(1-2cs+(c^2-ab)s^2\big)^2+\big(c-(c^2-ab-1)s-cs^2\big)^2\\ &\qquad -ab(1+is)^2(1-is)^2\\ &= (c^2-ab+1)(1+s^2)\big((c^2-ab)s^2-2cs+1\big),\end{aligned} \tag{3.16}$$

but the ambiguity of $\pm\sqrt{1-2cs+(c^2-ab)s^2}$ can not be eliminated.

Using (3.15) and (3.10), we have several results as follows:

Lemma 15. *If Q_1, $Q_2 \in M^3$ such that $\pi(Q_1) = e^{(s/\hbar)2uv}$, $\pi(Q_2) = e^{(1/\hbar)(au^2+bv^2+2cuv)}$, then*

$$Q_1 = \pm\sqrt{1+s^2}\,e^{(s/\hbar)2uv}, \quad Q_2 = \pm\sqrt{c^2-ab+1}\,e^{(1/\hbar)(au^2+bv^2+2cuv)}$$

with $1+s^2 \neq 0$, $c^2-ab+1 \neq 0$.

If $1-2cs+(c^2-ab)s^2 \neq 0$, then the $*$-product $Q_1 * Q_2$ is defined as an element of M^3 by

$$\left(\sqrt{1+s^2}\exp\left\{\frac{s}{\hbar}2uv\right\}\right) * \left(\sqrt{c^2-ab+1}\right)\exp\left\{\frac{1}{\hbar}(au^2+bv^2+2cuv)\right\}$$
$$= \sqrt{1+D}\exp\left\{\frac{1}{\hbar(1-2cs+(c^2-ab)s^2)}\right.$$
$$\left.\times\ (a(1+is)^2u^2+b(1-is)^2v^2+(c-(c^2-ab-1)s-cs^2)2uv)\right\}$$

where D is the discriminant of the quadratic form

$$\frac{1}{1-2cs+(c^2-ab)s^2}$$
$$\times\ \big(a(1+is)^2u^2+b(1-is)^2v^2+(c-(c^2-ab-1)s-cs^2)2uv\big).$$

Hence the right hand side is also an element of $*$-exponential function.

By Lemmas 15, 14, we have the following:

Theorem 16. *M^3 forms a local group, which is locally isomorphic to $SL(2,\mathbb{C})$ and M^3 is embedded in $\mathcal{E}_{2+}(\mathbb{C}^2)$ as*

$$M^3 = \left\{\pm\sqrt{c^2-ab+1}\, e^{(1/\hbar)(au^2+bv^2+2cuv)};\quad c^2-ab+1\neq 0\right\}.$$

The open dense subset

$$M^3 - \left\{-e^{(1/\hbar)(au^2+bv^2+2cuv)};\quad c^2-ab=0,\ (a,b,c)\neq(0,0,0)\right\}$$

is covered by $*$-exponential functions $\{e_*^{(1/\hbar)(au^2+bv^2+2cuv)}\}$.

§4. Star exponential functions in the normal ordering expression

Although $e_*^{\pm(\pi/\hbar)uv}$ diverge in the Weyl ordering expression, we prove in this section that such elements make sense in the normal ordering expression.

Since $uv = u\circ v + (\hbar i/2)$, we have $au^2+2cuv+bv^2 = au^2+2cu\circ v+bv^2+\hbar ci$. In this section, we compute $e^{-cit}e_*^{(t/\hbar)(au^2+bv^2+2cuv)} = e_*^{(t/\hbar)(au^2+bv^2+2cu\circ v)}$ by ΨDO-product formula. Thus, we set

$$e_*^{(t/\hbar)(au^2+bv^2+2cu*v)} = s(t)e_\circ^{(1/\hbar)(a(t)u^2+b(t)v^2+2c(t)u\circ v)}.$$

We first compute

$$\begin{aligned}
&\frac{1}{\hbar}(a'u^2+b'v^2+2c'u\circ v) * e_\circ^{(1/\hbar)(au^2+bv^2+2cu\circ v)}\\
&\quad=\left\{\frac{1}{\hbar}(a'u^2+b'v^2+2c'u\circ v)+\frac{i}{\hbar}(2b'v+2c'u)\circ(2au+2cv)\right.\\
&\qquad\left.+\frac{-1}{\hbar}\frac{1}{2}(2b')((2au+2bv)^2+2a\hbar)\right\}\circ e_\circ^{(1/\hbar)(au^2+bv^2+2cu\circ v)}.
\end{aligned}$$

This is

(4.1)

$$\frac{1}{\hbar}(a',b',c')\begin{bmatrix}1, & 0, & 0, & 0\\ -4a^2, & 1+4ci-4c^2, & 2ai-2ac, & -2\hbar a\\ 4ai, & 0, & 1+2ci, & 0\end{bmatrix}\begin{bmatrix}u^2\\ v^2\\ 2u\circ v\\ \hbar\end{bmatrix}$$

$$\circ\, se_\circ^{(1/\hbar)(au^2+bv^2+2cu\circ v)}$$

Submatrix of first 3-columns is singular only at $1+2ci=0$, i.e. at $e_\circ^{(1/\hbar)iu\circ v}$. This is in fact a vacuum computed by ΨDO-product formula (cf. Corollary 12 and (4.3) below).

Hence, setting $e_*^{(t/\hbar)(au^2+bv^2+2cu\circ v)}=\psi(t)e_\circ^{\phi_1(t)u^2+\phi_2(t)v^2+2\phi_3(t)u\circ v}$, we have only to solve the system of ordinary differential equations

$$(4.2)\qquad \begin{aligned} \phi_1'(t) &= \frac{1}{\hbar}a+4ic\phi_1(t)-4\hbar b\phi_1(t)^2\\ \phi_2'(t) &= \frac{1}{\hbar}b+4ib\phi_3(t)-4\hbar b\phi_3(t)^2\\ \phi_3'(t) &= \frac{1}{\hbar}c+2ic\phi_3(t)+2ib\phi_1(t)-4\hbar b\phi_1(t)\phi_3(t)\\ \psi'(t) &= -2\hbar b\phi_1(t)\psi(t) \end{aligned}$$

with the initial condition $\phi_i(0)=0$ and $\psi(0)=1$.

4.1. The case $b=0$ as the simplest case

(4.2) is easily solved if $b=0$, and we have

$$(4.3)\qquad e_*^{(t/\hbar)(au^2+2cu\circ v)}=e_\circ^{(a/4ci\hbar)(e^{4cit}-1)u^2+(1/2i\hbar)(e^{2cit}-1)2u\circ v}.$$

In particular, $\lim_{t\to\infty}e_*^{(it/\hbar)u\circ v}=e_\circ^{(1/\hbar)iu\circ v}$. By Corollary 12, the limit is the vacuum ϖ_{00}.

Note also that the case $1+2ci=0$ in (4.1) is written as follows:

$$\begin{aligned} e_\circ^{(1/\hbar)(au^2+bv^2+iu\circ v)} &= (e_\circ^{(1/\hbar)au^2}*\varpi_{00})*e_\circ^{(1/\hbar)bv^2}\\ &= e_\circ^{(1/\hbar)au^2}*(\varpi_{00}*e_\circ^{(1/\hbar)bv^2}). \end{aligned}$$

We have also the following remarkable fact:

$$(4.4)\qquad e_*^{(\pi/2\hbar)(au^2+2u*v)}=e_\circ^{-(1/\hbar i)2u\circ v}=e_*^{(\pi/\hbar)u*v}$$

that is, $e_*^{(\pi/\hbar)(au^2+u\circ v)}$ does not depend on a.

Using the exponential law we have the following:

Proposition 17. *In the normal ordering expression with respect to (u,v), the exponential law holds*

$$\begin{aligned} &\left(\exp_\circ\left\{\frac{a}{4ci\hbar}(e^{4cis}-1)u^2+\frac{1}{2i\hbar}(e^{2cis}-1)2u\circ v\right\}\right)\\ &\quad*\left(\exp_\circ\left\{\frac{a}{4ci\hbar}(e^{4cit}-1)u^2+\frac{1}{2i\hbar}(e^{2cit}-1)2u\circ v\right\}\right)\\ &=\exp_\circ\left\{\frac{a}{4ci\hbar}(e^{4ci(s+t)}-1)u^2+\frac{1}{2i\hbar}(e^{2ci(s+t)}-1)2u\circ v\right\} \end{aligned}$$

In particular, we have the exponential law:

$$e_{\circ}^{(1/\hbar i)(e^{is}-1)u\circ v} * e_{\circ}^{(1/\hbar i)(e^{it}-1)u\circ v} = e_{\circ}^{(1/\hbar i)(e^{i(s+t)}-1)u\circ v}$$

If we set $\sigma = e^{is} - 1$, $\tau = e^{it} - 1$, then the exponential law gives the following product formula:

$$e_{\circ}^{(1/\hbar i)\sigma u\circ v} * e_{\circ}^{(1/\hbar i)\tau u\circ v} = e_{\circ}^{(1/\hbar i)(\sigma\tau+\sigma+\tau)u\circ v} \tag{4.5}$$

Though the product has no singularity, the inverse has a singular point:

$$(e_{\circ}^{(1/\hbar i)\sigma u\circ v})_{*}^{-1} = e_{\circ}^{-(1/\hbar i)(\sigma/1+\sigma)u\circ v}. \tag{4.6}$$

The singular point $e_{\circ}^{-(1/\hbar i)u\circ v}$ is in fact the normal ordering expression of the vacuum $\varpi_{0,0}$.

4.2. Several facts, concluded from the case $a = 0$

If $a = 0$ in (4.2), then we have

$$e_{*}^{(t/\hbar)(bv^2+2cu\circ v)} = e_{\circ}^{(b/4ci\hbar)(e^{4cit}-1)v^2+(1/2\hbar i)(e^{2cit}-1)2u\circ v} \tag{4.7}$$

The same exponential law as in Proposition 17 holds.

In particular, we see that

$$e_{*}^{(\pi/2\hbar)(bv^2+2u*v)} = e_{\circ}^{-(2/\hbar i)u\circ v} = e_{*}^{(\pi/\hbar)u*v} \tag{4.8}$$

and this quantity does not depend on b.

By (4.4), (4.8), we have the following remarkable fact:

Lemma 18. *In the normal ordering expression with respect to the canonical conjugate pair (u, v), the identities*

$$e_{*}^{(\pi/\hbar)u*(v+au)} = e_{*}^{(\pi/\hbar)u*v} = e_{*}^{(\pi/\hbar)(u+bv)*v} = e_{\circ}^{(2i/\hbar)u\circ v}$$

hold for any $a, b \in \mathbb{C}$.

An element $e_{*}^{(\pi/\hbar)(\alpha u+\beta v)(\gamma u+\delta v)} = ie_{*}^{(\pi/\hbar)(\alpha u+\beta v)*(\gamma u+\delta v)}$ with $\alpha\delta - \beta\gamma = 1$ is called a *polar element.* This element is computed in the normal ordering expression with respect to $u' = \alpha u + \beta v$, $v' = \gamma u + \delta v$. We denote the set of all polar elements by ϵ_{00}. Obviously,

$$\begin{aligned} \epsilon_{00} &= \left\{e_{*}^{(\pi/\hbar)(\alpha u+\beta v)(\gamma u+\delta v)};\ \alpha\delta - \beta\gamma = 1\right\} \\ &= \left\{e_{*}^{(\pi/\hbar)(au^2+bv^2+2cuv)};\ c^2 - ab = \frac{1}{4}\right\} \end{aligned}$$

$e_*^{(\pi/\hbar)uv} = ie_*^{(\pi/\hbar)u*v}$ is a polar element. Though this is not computed in the Moyal product formula, this is computed in the ΨDO-product formula (1.1) as $e_\circ^{(2i/\hbar)u\circ v}$.

Note that $u' = u$, $v' = au + v$ gives a canonical conjugate pair. Hence, by Lemma 18 applied for u', v', we have

$$e_*^{(\pi/\hbar)u'*v'} = e_*^{(\pi/\hbar)(u'+bv')*v'} = e_{\circ'}^{(2i/\hbar)u'\circ v'}$$

in the normal ordering expression with respect to (u', v').

Note that for every $c \neq 0$,

$$(u' + bv') * v' = (u + b(v + au)) * (v + au) = \left(\frac{1+ab}{c}u + \frac{b}{c}v\right) * (cau + cv).$$

Thus, in a first glance, it looks very natural to set as

$$e_*^{(\pi/\hbar)u*v} = e_*^{(\pi/\hbar)u*(v+au)} = e_*^{(\pi/\hbar)u'*v'} = e_*^{(\pi/\hbar)(u'+bv')*v'},$$

hence we have

$$\exp_*\left\{\frac{\pi}{\hbar}u * v\right\} = \exp_*\left\{\frac{\pi}{\hbar}\left(\frac{1+ab}{c}u + \frac{b}{c}v\right) * (cau + cv)\right\}.$$

However, such equalities are dangerous, because quantities of left and right members are computed separately by using different canonical conjugate pairs. Such two elements should be compared through intertwiners mentioned in § 2.1.

Although $e_*^{\pm(\pi/\hbar)uv}$ is defined only by normal ordering expression, the equality (3.15) gives also the following:

Lemma 19. *If $c^2 - ab \neq 0$, then*

$$\begin{aligned}
&\exp_*\left\{\pm\frac{\pi}{\hbar}uv\right\} * \exp\left\{\frac{1}{\hbar}(au^2 + bv^2 + 2cuv)\right\} \\
&= \frac{1}{\sqrt{c^2 - ab}} \exp\left\{\frac{1}{\hbar(ab - c^2)}(au^2 + bv^2 + 2cuv)\right\}.
\end{aligned}$$

Proof. Remark $e_*^{\pm(t/\hbar)2uv} = \sqrt{1+s^2}\, e^{(s/\hbar)2uv}$ and if $t \to \pm\frac{\pi}{2}$, then $s \to \infty$. Multiplying $\sqrt{1+s^2}$ to the both sides of (3.15), and take $s \to \infty$. We have the lemma. Q.E.D.

Since linearly related canonical conjugate pairs form an arcwise connected subset, polar elements look like forming connected complex 2-dimensional manifold. In fact, however, we have the following:

Proposition 20. *For every a, $b \in \epsilon_{00}$, we have $a * b = \pm 1$, $a * a = -1$. Hence $a = -a^{-1}$ and hence $a = \pm b$ by applying a^{-1}. Consequently, ϵ_{00} forms a single point.*

Proof. By Lemma 8, we see easily that $a * a = -1$. To prove $a * b = \pm 1$, we have only to compute

$$\lim_{c^2 - ab \to \frac{\pi^2}{4}} \left[e_*^{(\pi/\hbar)u*v} * \frac{1}{\cos\sqrt{c^2 - ab}} \times \exp\left\{ \left(\frac{1}{\hbar\sqrt{c^2 - ab}} \tan\sqrt{c^2 - ab} \right) (au^2 + bv^2 + 2cuv) \right\} \right]$$

in the Weyl ordering expression. By Lemma 19, this is rewritten as

$$\lim_{c^2 - ab \to \frac{\pi^2}{4}} \frac{1}{\sin\sqrt{c^2 - ab}} \exp\left\{ -\left(\frac{1}{\hbar} \cot^2 \sqrt{c^2 - ab} \right)(au^2 + bv^2 + 2cuv) \right\}.$$

Since $\cos\theta = 0$ implies $\sin\theta = \pm 1$, the above quantity tends to ± 1. This shows $a * b = \pm 1$.

Since the set $\alpha\delta - \beta\gamma = 1$ is connected, we see that $\{\exp_*\{(\pi/\hbar) \times (\alpha u + \beta v)(\gamma u + \delta v)\}\}$ forms a single element and $a * b = 1$ in fact. Q.E.D.

We denote the polar element by the same notation ϵ_{00}.

Remark. This is a little tricky, because $(-v, u)$ is also a canonical conjugate pair. Hence at the first glance the above result looks like insisting $e_*^{(\pi/\hbar)u*v} = e_*^{(\pi/\hbar)(-v)*u}$. If this were true, then since $-v * u = -u * v - \hbar i$, we must have

$$e_*^{(\pi/\hbar)(-v)*u} = -e_*^{-(\pi/\hbar)u*v}.$$

However, we have already seen that $\epsilon_{00} * \epsilon_{00} = -1$. This gives $e_*^{-(\pi/\hbar)u*v} = e_*^{(\pi/\hbar)u*v}$, and hence we have $\epsilon_{00} = -\epsilon_{00}$. This looks like a contradiction. Remark however, that $\epsilon_{00} = -\epsilon_{00}$ does not necessarily imply $2\epsilon_{00} = 0$.

In Lemma 23, we will see that ϵ_{00} is expressed as $e_\circ^{(2i/\hbar)u \circ v}$ and $e_\bullet^{(2i/\hbar)(-v)\bullet u}$ by normal ordering expressions with respect to (u, v) and $(-v, u)$ respectively. Thus, we have to use the intertwiner between canonical conjugate pairs (u, v) and $(-v, u)$ to compare $e_*^{(\pi/\hbar)u*v}$ and $e_*^{(\pi/\hbar)(-v)*u}$. Consequently, we have to set $e_*^{(\pi/\hbar)u*v} = -e_*^{(\pi/\hbar)(-v)*u}$.

Although ϵ_{00} forms a single element and $\epsilon_{00} * \epsilon_{00} = -1$, this does not imply that $\epsilon_{00} = i$, because the following holds by the bumping Lemma 13:

Proposition 21. $u * \epsilon_{00} + \epsilon_{00} * u = 0$, $v * \epsilon_{00} + \epsilon_{00} * v = 0$. *In particular,* ϵ_{00} *commutes with every even element.*

This suggests that ϵ_{00} has some super theoretic character [W]. There are several *odd variables* in our system, but a systematic treatment of these will be given some other paper [O,el.4].

On the contrary, the normal ordering expressions of ϵ_{00} with respect to the canonical conjugate pair (u,v) is $e_{\circ}^{(2i/\hbar)u\circ v}$. Hence the normal ordering expression of $e_*^{(\pi/\hbar)u*v} = e_*^{(\pi/\hbar)u*(v+a'u)}$, with respect to $(u, v+a'u)$ is $e_{\diamond}^{(2i/\hbar)u\diamond(v+a'u)}$. Similarly, the normal ordering expressions of $e_*^{(\pi/\hbar)u*v} = e_*^{(\pi/\hbar)(u+b'v)*v}$ with respect to $(u+b'v, v)$ is $e_{\diamond'}^{(2i/\hbar)(u+b'v)\diamond' v}$.

In the Weyl ordering expression, we have had

$$\begin{aligned}\exp_*\left\{\frac{t}{\hbar}(bv^2+2cu\circ v)\right\} &= \exp_*\left\{\frac{t}{\hbar}(bv^2+2cu*v)\right\}\\ &= \exp\{-c\hbar it\}\exp_*\left\{\frac{t}{\hbar}(bv^2+2cu\cdot v)\right\}\\ &= \frac{\exp\{-c\hbar it\}}{\cos ct}\exp\left\{\left(\frac{1}{\hbar c}\tan ct\right)(bv^2+2cuv)\right\}.\end{aligned}$$

Thus, we have

Proposition 22. *In the normal ordering expression with respect to* (u,v), *the product*

$$e_*^{(t/\hbar)u^2} * e_*^{(1/\hbar)(bv^2+2cu\circ v)} = e_\circ^{(t/\hbar)u^2} * e_\circ^{(b/4ci\hbar)(e^{4ci}-1)v^2+(1/2\hbar i)(e^{2ci}-1)2u\circ v}$$

is welldefined for every t *as* $e_\circ^{(t/\hbar)u^2+(b/4ci\hbar)(e^{4ci}-1)v^2+(1/2\hbar i)(e^{2ci}-1)2u\circ v}$.

Recall this is defined only for $1+bt \neq 0$ in the Weyl ordering expression (cf. Lemma 14).

4.3. The case $ab \neq 0$, Proof of the first half of Theorem 2

If $ab \neq 0$ in (4.2), there appear singularities in the $*$-exponential functions, and this gives the first half of Theorem 2. This is because that the exponential function $e_*^{aH+bX+cY}$ is not defined for all $(a,b,c) \in \mathbb{C}^3$ under any ordering expression.

The first equation of (4.2) is

$$\left(\phi_1(t)-\frac{ci}{2\hbar b}\right)' = \frac{ab-c^2}{\hbar b} - 4\hbar b\left(\phi_1(t)-\frac{ci}{2\hbar b}\right)^2.$$

It follows

$$\phi_1(t) = \frac{ic}{2\hbar b} - \frac{\sqrt{c^2 - ab}}{2\hbar b}\left(\tan 2\sqrt{c^2 - ab}\,(t + t_0)\right),$$

where t_0 is fixed by the initial condition $\phi_1(0) = 0$, i.e.

$$\sqrt{c^2 - ab}\left(\tan 2\sqrt{c^2 - ab}\,t_0\right) = ic.$$

The forth equation gives that

$$\frac{d}{dt}\psi(t) = \left(\sqrt{c^2 - ab}\left(\tan 2\sqrt{c^2 - ab}\,(t + t_0)\right) - ic\right)\psi(t).$$

It follows

$$\psi(t) = e^{-ict}\left(\frac{\cos(2\sqrt{c^2 - ab}\,t_0)}{\cos 2\sqrt{c^2 - ab}\,(t + t_0)}\right)^{1/2}.$$

The third equation

$$\left(\phi_3(t) + \frac{1}{2\hbar i}\right)' = 2\sqrt{c^2 - ab}\left(\tan 2\sqrt{c^2 - ab}\,(t + t_0)\right)\left(\phi_3(t) + \frac{1}{2\hbar i}\right)$$

gives

$$\phi_3(t) + \frac{1}{2\hbar i} = \frac{A}{\cos 2\sqrt{c^2 - ab}\,(t + t_0)}$$

where A is fixed by the initial condition $\phi_3(0) = 0$, i.e. $A = \frac{1}{2\hbar i} \times \cos 2\sqrt{c^2 - ab}\,t_0$.

The second equation is

$$\phi_2'(t) = -4\hbar b\left(\phi_3(t) + \frac{1}{2\hbar i}\right)^2 = -4\hbar b A^2\left(\cos 2\sqrt{c^2 - ab}\,(t + t_0)\right)^{-2}.$$

Hence

$$\phi_2(t) = -\frac{2\hbar b A^2}{\sqrt{c^2 - ab}}\left(\tan 2\sqrt{c^2 - ab}\,(t + t_0) - \frac{ic}{\sqrt{c^2 - ab}}\right).$$

If $c^2 - ab = 0$, then the first equation of (4.2) is

$$\left(\phi_1(t) - \frac{ci}{2\hbar b}\right)' = -4\hbar b\left(\phi_1(t) - \frac{ci}{2\hbar b}\right)^2.$$

It follows

$$\phi_1(t) - \frac{ci}{2\hbar b} = \frac{1}{1 + 4\hbar b t}.$$

$$\psi(t) = e^{2\hbar t}(1 + 4\hbar bt)^{-1/(4b)}.$$

Hence, $e_*^{(t/\hbar)(au^2+bv^2+2cuv)}$ with $ab - c^2 = 0$, $ab \neq 0$, is singular at $1 + 4\hbar bt = 0$ in the ΨDO-expression, while this is computed as $e_\cdot^{(t/\hbar)(au^2+bv^2+2cuv)}$ in the Weyl ordering expression.

Some of $*$-products are easy to compute in the normal ordering expression. Note that $e_*^{(t/\hbar)u^2} = e_\cdot^{(t/\hbar)u^2} = e_\circ^{(t/\hbar)u^2}$ and

$$e_\circ^{(t/\hbar)u^2} * e_\circ^{(1/\hbar)(au^2+bv^2+2cuv)} = e_\circ^{(1/\hbar)((a+t)u^2+bv^2+2cuv)},$$
$$e_\circ^{(t/\hbar)u\circ v} * e_\circ^{(1/\hbar)(bv^2+2cuv)} = e_\circ^{(1/\hbar)(t(1+2ic)u^2+bv^2+2cuv)}$$

in the ΨDO-product formula under the normal ordering expression with respect to the canonical conjugate pair (u, v). Remark these are defined for all t.

By these computations, we see also the following:

Lemma 23. *In the normal ordering expression with respect to* (u, v), $e_*^{(\pi/2\hbar)(au^2+bv^2+cu*v)}$ *with* $c^2 - ab = 1$ *is given identically as* $e_\circ^{(2i/\hbar)u\circ v}$.

§5. Proof of Theorem 2

We have already seen the first half of Theorem 2. To prove the second half, we consider the set obtained by gluing M^3 and $\epsilon_{00} * M^3$ by the mapping $\epsilon_{00}*$. We set

$$M_0^3 = \{e_*^{(1/\hbar)(au^2+bv^2+2cuv)};\ c^2 - ab = 0\}.$$

Since ϵ_{00} commutes with every $e^{(1/\hbar)(au^2+bv^2+2cuv)}$, and $\epsilon_{00}{}_*^2 = -1$, Lemma 19 gives that $\epsilon_{00}*$ gives a diffeomorphism of $M^3 - M_0^3$ onto itself, but this can not extend to the whole space:

For a point P of M_0^3, the computation is represented by setting $P = e^{au^2}$. Since

$$e_*^{(t/\hbar)2uv} * e^{au^2} = \sqrt{1+s^2}\, e^{(1/\hbar)(a(1+is)^2u^2+2suv)}, \quad \tan t = s,$$

this is written in the form of $*$-exponential function and hence this is a member of M^3, if $t \neq \pm\frac{\pi}{2}$. However, if $t \to \pm\frac{\pi}{2}$, then $s \to \infty$. Hence, we see that $e_*^{(\pi/\hbar)uv} * e^{au^2}$ can not be a member of M^3, but of $\epsilon_{00} * M^3$.

We show the following in this section:

For Q_1, $Q_2 \in M^3$, *if* $Q_1 * Q_2$ *is not defined in the Weyl ordering expression, then*

$$Q_1 * (\epsilon_{00} * Q_2) = (Q_1 * \epsilon_{00}) * Q_2$$

is defined in the Weyl ordering expression as an element of M^3.

If $Q_1 * Q_2$ is defined, then the product $Q_1 * (\epsilon_{00} * Q_2)$, $(\epsilon_{00} * Q_1) * (\epsilon_{00} * Q_2)$ are defined by $\epsilon_{00} * (Q_1 * Q_2)$, $-Q_1 * Q_2$ respectively. If $Q_1 * Q_2$ is not defined in the Weyl ordering expression, then $-\epsilon_{00} * (Q_1 * (\epsilon_{00} * Q_2))$ is defined.

This shows that $M^3 \cup (\epsilon_{00} * M^3)$ forms a group. We already know that by Theorem 16, M^3 forms a local group, which is locally isomorphic to $SL(2, \mathbb{C})$ and M^3 is embedded in $\mathcal{E}_{2+}(\mathbb{C}^2)$ as

$$M^3 = \left\{ \pm\sqrt{c^2 - ab + 1} \exp\left\{ \frac{1}{\hbar}(au^2 + bv^2 + 2cuv);\ c^2 - ab + 1 \neq 0 \right\} \right\}.$$

It is well known that $SL(2; \mathbb{C})$ is simply connected with the non-trivial discrete center $\{\pm 1\}$.

Since $\pm 1 \in M^3 \cup (\epsilon_{00} * M^3)$, we see that $M^3 \cup (\epsilon_{00} * M^3)$ is isomorphic to $SL(2, \mathbb{C})$.

By the argument in the first paragraph of §3.5, the case that $Q_1 * Q_2$ is not defined in the Weyl ordering expression is represented by the following two cases: Namely,

$$e^{(t/\hbar)u^2} * \sqrt{c^2 - ab + 1}\, e^{(1/\hbar)(au^2 + bv^2 + 2cuv)},$$
$$\sqrt{1 + s^2}\, e^{(s/\hbar)2uv} * \sqrt{c^2 - ab + 1}\, e^{(1/\hbar)(au^2 + bv^2 + 2cuv)}$$

are not defined only for $1 + bt = 0$, and $1 - 2cs + (c^2 - ab)s^2 = 0$ respectively.

However using the polar element combined with Lemma 19, we show these are defined by Weyl ordering expressions.

By the computation in Lemma14, we remark first the following:

Lemma 24. *Under the condition* $1 + bt \neq 0$, $\exp_*\{\frac{t}{\hbar}u^2\} * \exp\{\frac{1}{\hbar}(au^2 + bv^2 + 2cuv)\}$ *is a vacuum, if and only if* $\exp\{\frac{1}{\hbar}(au^2 + bv^2 + 2cuv)\}$ *is a vacuum, i.e.* $c^2 - ab + 1 = 0$.

Lemma 19 is used for the computation of

$$e^{(t/\hbar)u^2} * \sqrt{c^2 - ab + 1}\, e^{(1/\hbar)(au^2 + bv^2 + 2cuv)},$$
$$\sqrt{1 + s^2}\, e^{(s/\hbar)2uv} * \sqrt{c^2 - ab + 1}\, e^{(1/\hbar)(au^2 + bv^2 + 2cuv)}$$

for $1 + bt = 0$, and $1 - 2cs + (c^2 - ab)s^2 = 0$ respectively.

Corollary 25. *If* $1 + bt = 0$, *then*

$$\exp_* \left\{ \pm\frac{\pi}{\hbar}uv \right\} * \exp\left\{ \frac{t}{\hbar}u^2 \right\} * \sqrt{c^2 - ab + 1} \exp\left\{ \frac{1}{\hbar}(au^2 + bv^2 + 2cuv) \right\}$$
$$= \exp\left\{ \frac{1}{\hbar(c^2 - ab + 1)}((ci - 1)^2 tu^2 + bv^2 + 2(c - ibt)uv) \right\}$$

and the right hand side is written in the form $e^{(1/\hbar)(\alpha u+\beta v)^2}$.

If $1-2cs+(c^2-ab)s^2=0$, *then remarking*

$$\begin{aligned}&\big(c-(c^2-ab-1)s-cs^2\big)^2-ab(1+is)^2(1-is)^2\\&\quad=\big((c^2-ab+1)(1+s^2)-((c^2-ab)s^2-2cs+1)\big)\\&\qquad\times\big((c^2-ab)s^2-2cs+1\big)\end{aligned}$$

we have

$$\begin{aligned}&\exp_*\left\{\pm\frac{\pi}{\hbar}uv\right\}*\sqrt{1+s^2}\exp\left\{\frac{s}{\hbar}2uv\right\}\\&\quad*\sqrt{c^2-ab+1}\exp\left\{\frac{1}{\hbar}(au^2+bv^2+2cuv)\right\}\\&=\exp\left\{\frac{1}{\hbar(c^2-ab+1)(1+s^2)}\right\}\\&\quad\times\left(a(1+is)^2u^2+b(1-is)^2v^2+(c-(c^2-ab-1)s-cs^2)2uv\right).\end{aligned}$$

The discriminant of the right hand side vanishes, and hence it is written in the form $e^{(1/\hbar)(\alpha u+\beta v)^2}$.

This completes the proof of Theorem 2.

References

[BFLS] H. Basart, M. Flato, A. Lichnerowicz and D. Sternheimer, *Deformation theory applied to quantization and statistical mechanics*, Lett. Math. Phys., **8** (1984), 483–494.

[BL] H. Basart and A. Lichnerowicz, *Conformal symplectic geometry, deformations, rigidity and geometrical KMS conditions*, Lett. Math. Phys., **10** (1985), 167–177.

[GS] I. M. Gel'fand and G. E. Shilov, Generalized Functions, 2, Academic Press, 1968.

[M] M. Morimoto, *An introduction to Sato's hyperfunctions*, AMS Trans. Mono., **129**, 1993.

[MS] C. Moreno and J. A. P. da Silva, *Star products and spectral analysis*, Preprint.

[Om] H. Omori, *Infinite dimensional Lie groups*, AMS Trans. Mono., **158**, 1997.

[OMY] H. Omori, Y. Maeda and A. Yoshioka, Weyl manifolds and deformation quantization, Adv. Math., **85** (1991), 224–255.

[O,el.1] H. Omori, Y. Maeda, N. Miyazaki and A. Yoshioka, *Deformation quantization of Fréchet-Poisson algebras —Convergence of the Moyal product—*, Math. Phys. Studies, **22** (2000), 233–246.

[O,el.2] H. Omori, Y. Maeda, N. Miyazaki and A. Yoshioka, *Deformation Quantizations of the Poisson Algebra of Laurent Polynomials*, Lett. Math. Phys. (1998), 1–10.

[O,el.3] H. Omori, Y. Maeda, N. Miyazaki and A. Yoshioka, *Convergent star-products on Fréchet linear Poisson algebras of Heisenberg type*, Contem. Math., **288** (2001), 391–393.

[O,el.4] H. Omori, Y. Maeda, N. Miyazaki and A.Yoshioka, *One must break symmetry to keep associativity*, in preparation.

[OMMY] H. Omori, Y. Maeda, N. Miyazaki and A. Yoshioka, *Singular system of exponential functions*, Math. Phys. Studies, **23** (2001), 169–186.

[R] M. Rieffel, Deformation quantization for actions of $\mathbb{R}^n$, Memoir. A.M.S., **106**, 1993.

[W] B. de Witt, Supermanifolds, Cambridge XSCC Univ. Press, 1984

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Lifting of Holomorphic Actions on Complex Supermanifolds

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Contact Transformations and Their Schwarzian Derivatives

Tetsuya Ozawa and Hajime Sato

Abstract.

The second author introduced Schwarzian derivatives for 3-dimensional contact transformations in [Sat]. Our purposes of this paper are firstly to investigate the fundamental properties of the contact Schwarzian derivatives of 3-dimensional contact transformations that are satisfied by the usual Schwarzian derivatives, secondly to consider systems of linear PDE's with contact Schwarzian derivatives as coefficients and their integrability conditions, and finally to reconstruct the contact transformation from the solutions of the systems of linear PDE's. We obtain the necessary and sufficient condition for functions to be contact Schwarzian derivatives of a 3-dimensional contact transformation.

§1. Introduction

The classical theory of Schwarzian derivative plays an important role in the study of holomorphic equivalence of one-dimensional complex domains and especially in the Gauss-Schwarz theory of hypergeometric differential equation.

The Schwarzian derivative $S(f)$ of an analytic function f on $\mathbb{C}$ is defined by

$$S(f) = -\frac{1}{2}\Big(\frac{f''}{f'}\Big)' + \frac{1}{4}\Big(\frac{f''}{f'}\Big)^2.$$

Among many important properties of classical Schwarzian deirvative, we pay attention to the following three basic facts: (i) for an arbitrary function σ, the quotient $f = \varphi_1/\varphi_0$ of linearly independent solutions φ_0 and φ_1 of the differential equation

$$\varphi'' - \sigma\varphi = 0$$

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has the Schwarzian derivative $S(f)$ equal to σ, (ii) The Schwarzian derivative of a composition $f \circ g$ satisfies the formula

$$S(f \circ g) = S(g) + S(f)(g')^2,$$

which is nothing but the cocycle condition of continuous group cohomology (see, for example, [O-S2]), and (iii) the Schwarzian derivative $S(f)$ of a function f vanishes, if and only if f is a Möbius transformation;

$$f(x) = \frac{\alpha x + \beta}{\gamma x + \delta},$$

where α, β, γ, and δ are complex constants.

The Schwarzian derivative of higher dimensional diffeomorphisms is studied by several authors, and linear partial differential equations whose coefficients are Schwarzian derivatives has been observed to give, as a special case, the Appell and Lauricella hypergeometric differential equations (see, for example, [Yos]).

The purpose of this paper is to develop the theory of Schwarzian derivative for contact transformations, especially study the properties that correspond to the above three facts (i), (ii) and (iii). Originally, the notion of Schwarzian derivative of contact transformation is introduced by the second author in [Sat], through the study of equivalence of third order ordinary differential equations.

The notion of equivalence of ordinary differential equations depends on the transformations employed to reduce a given equation to simpler one. In the most general setting, we deal with the problem through *contact transformations* $\phi\colon \mathbb{K}^3 \to \mathbb{K}^3$, where we regard, using the coordinate system (x, y, z), the three space $\mathbb{K}^3$ as a contact manifold with contact form $dy - z\,dx$. If we transform the simplest third order differential equation $d^3y/dx^3 = 0$ by a contact transformation $\phi\colon (x, y, z) \mapsto (X, Y, Z)$, then the resulting equation is

$$\frac{d^3y}{dx^3} = P + 3Q\Big(\frac{d^2y}{dx^2}\Big) + 3R\Big(\frac{d^2y}{dx^2}\Big)^2 + S\Big(\frac{d^2y}{dx^2}\Big)^3,$$

where P, Q, R and S are functions of $(x, y, z = dy/dx)$. Our definition of the contact Schwarzian derivative of ϕ is to be the quadruple

$$S(\varphi) = (P, Q, R, S)$$

that appears in the above transformed equation. The explicit formula of the Schwarzian derivative is given in Section 2. In this paper, we will not discuss on the equivalence problem of ordinary differential equations.

For the arguments from this point of view, the readers will refer the papers [O-S1] and [Sat] and the references therein.

For the calculus on the contact space $\mathbb{K}^3$ ($= \mathbb{C}^3$ or $\mathbb{R}^3$) with the contact form $dy - z\,dx$, it is convenient to use the vector fields

$$v_1 = \frac{\partial}{\partial x} + z\frac{\partial}{\partial y}, \quad v_2 = \frac{\partial}{\partial z}, \quad v_3 = \frac{\partial}{\partial y},$$

that satisfy the *Heisenberg relation* for Lie bracket; $v_3 = [v_2, v_1]$ and $[v_3, v_1] = [v_3, v_2] = 0$ (for useful formulae with v_1, v_2, v_3 on contact transformations, see Section 3).

In the classical theory of Schwarzian derivative, we could reconstruct an analytic function f, whose Schwarzian derivative is equal to a given function σ, from solutions of the linear ordinary differential equation as stated in the above fact (i). In order to reconstruct a contact transformation ϕ with Schwarzian derivative equal to a given quadruple (P, Q, R, S), we consider the following system of linear partial differential equations (PDE system, for short):

$$\begin{cases} {v_1}^2(\vartheta) = Qv_1(\vartheta) - Pv_2(\vartheta) + M_{11}\vartheta \\ v_4(\vartheta) = 2(Rv_1(\vartheta) - Qv_2(\vartheta) + M_4\vartheta) \\ {v_2}^2(\vartheta) = Sv_1(\vartheta) - Rv_2(\vartheta) + M_{22}\vartheta \end{cases}$$

where v_4 stands for

$$v_4 = v_2v_1 + v_1v_2 = 2\frac{\partial^2}{\partial x\partial z} + 2z\frac{\partial^2}{\partial y\partial z} + \frac{\partial}{\partial y},$$

and M_{11}, M_4 and M_{22} are certain functions of P, Q, R, S and their differentiations by the vector fields v_1 and v_2 (see the formula (2) in Section 4).

The dimension of solution space of this PDE system depends on the quadruple (P, Q, R, S), and attains the maximum equal to 4, if and only if (P, Q, R, S) satisfies the integrability condition (IC) (see Theorem 4.1). And also the solution space carries a natural linear symplectic structure as proved in Proposition 4.4, provided its dimension equals 4. We will prove that the contact transformation ϕ can be reconstructed by using a symplectic basis $\{\vartheta, \xi, \zeta, \eta\}$ of the solution space (Theorem 6.1);

$$\phi(x, y, z) = \left(\frac{\xi}{\vartheta}, \frac{1}{2}\left(\frac{\eta}{\vartheta} + \frac{\xi\zeta}{\vartheta^2}\right), \frac{\zeta}{\vartheta}\right).$$

The quadruple (P, Q, R, S) automatically satisfies the integrability condition (IC), if it is a Schwarzian derivative of contact transformation.

Therefore we find that the condition (IC) is necessary and sufficient for (P, Q, R, S) to be a Schwarzian deirvative of some contact transformation.

For the proof of Theorem 4.1, we introduce a set of operators V_1, V_2 and V_3 acting on a certain module over functions, and verify that it satisfies the Heisenberg relation; $V_3 = [V_2, V_1]$ and $[V_3, V_i] = 0$ ($i = 1$, 2) (see Section 7). The verification is a series of quite long calculations, and we are compelled to use a computer software "Maple". We hope to find a simpler method for the proof.

The contact Schwarzian derivative satisfies a similar formula for compositions of maps as the classical Schwarzian derivative satisfies. The verification is easily performed by using the operators v_1 and v_2 (see Proposition 8.1).

In order to study contact transformations with vanishing Schwarzian derivative, we solve the above PDE system all of whose coefficients equal 0, and find that the functions 1, x, z and $2y - xz$ form a symplectic basis of the solution space. We notice that the embedding $\mathbb{K}^3 \to \mathbb{K}^4$ defined by

$$(x, y, z) \to (1, x, z, 2y - xz)$$

is compatible for both structures, contact and symplectic, on the source and the target. Especially, the pull back of Lagrangian planes are Legendrian curves. Through this embedding, we reduce the $Sp(2)$-action on $\mathbb{K}^4$ to that on $\mathbb{K}^3$, at least locally, and describe the exact set of contact transformations with vanishing Schwarzian derivative, in Theorem 9.5.

The paper is organized as follows. In Section 2, we give explicitly the formula of the contact Schwarzian derivative for three dimensional contact transformations. In Section 3, we prepare elementary calculations about contact transformations. In Section 4, we introduce a fundamental PDE system that has contact Schwarzian derivatives as its coefficients, and show that the solution space of this PDE system has a natural symplectic structure. In Section 5, we derive a PDE system whose solutions are coordinate functions of contact transformations, and a PDE system whose solutions are Jacobians of contact transformations. In Section 6, we show how to construct the contact transformation with a prescribed contact Schwarzian derivative from the solutions of the PDE systems introduced in Sections 4 and 5. The symplectic structure of the solution space of the PDE system introduced in Section 4 plays an essential role here. In Section 7, we show that the integrability conditions of the various PDE systems in this paper are equivalent in a certain sense, and prove Theorem 4.1 that concerns the integrability of the fundamental

PDE system. In Section 8, we prove that our contact Schwarzian derivative satisfies a connection formula with respect to compositions of contact transformations. Finally in Section 9, we show that the set of three dimensional contact transformations with vanishing contact Schwarzian derivatives form a Lie group locally isomorphic to $Sp(2, \mathbb{K})$.

§2. Contact Schwarzian derivative

Throughout the paper, we regard the affine 3-space $\mathbb{K}^3$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) with the usual coordinate (x, y, z) as a contact manifold endowed with the contact form $\alpha = dy - z\,dx$. We use the following notations:

$$v_1 = \frac{\partial}{\partial x} + z\frac{\partial}{\partial y}, \quad v_2 = \frac{\partial}{\partial z}, \quad v_3 = \frac{\partial}{\partial y}, \quad v_4 = v_2 v_1 + v_1 v_2.$$

Notice that the vector fields v_1, v_2 and v_3 form a Heisenberg Lie algebra:

$$v_3 = [v_2, v_1], \quad \text{and} \quad [v_3, v_1] = [v_3, v_2] = 0,$$

and that the vector fields v_1 and v_2 span the contact distribution; $\alpha(v_1) = \alpha(v_2) = 0$.

A local diffeomorphism ϕ is said to be a *contact transformation*, if it preserves the contact distribution, or equivalently, it satisfies $\phi^*(\alpha) = \rho\alpha$ for some nonvanishing function ρ. For a contact transformation ϕ: $(x, y, z) \mapsto (X, Y, Z)$, we define the contact Schwarzian derivatives as follows: for i, j, $k = 1$, 2, set

$$s_{[ij,k]}(\phi) = v_i v_j(X) v_k(Z) - v_i v_j(Z) v_k(X),$$

and

$$S_{\{ijk\}}(\phi) = \frac{1}{3\Delta(\phi)}\big(s_{[ij,k]}(\phi) + s_{[jk,i]}(\phi) + s_{[ki,j]}(\phi)\big),$$

where $\Delta(\phi) = v_1(X)v_2(Z) - v_1(Z)v_2(X)$. We call the functions $S_{\{ijk\}}(\phi)$ the *contact Schwarzian derivatives* of the contact transformation ϕ. We denote the quadruple of functions by

$$S(\phi) = \big(S_{\{111\}}(\phi), S_{\{112\}}(\phi), S_{\{122\}}(\phi), S_{\{222\}}(\phi)\big),$$

which also we call the Schwarzian derivative of ϕ.

§3. Some remarks on contact transformations

The facts stated in the following lemmas will be used in later sections. The affine 3-space $\mathbb{K}^3$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) with the usual coordinate

(x, y, z) is always regarded as a contact manifold endowed with the contact form $\alpha = dy - z\,dx$.

Lemma 3.1. *If a map ϕ: $(x, y, z) \mapsto (X, Y, Z)$ is a contact transformation, then X, Y and Z satisfy the equations*

$$v_1(Y) = Zv_1(X), \quad \textit{and} \quad v_2(Y) = Zv_2(X). \tag{1}$$

Proof. The pull back of the contact form is equal to

$$\begin{aligned}\phi^*(\alpha) &= dY - Z\,dX \\ &= \Big(\frac{\partial}{\partial x}Y - Z\frac{\partial}{\partial x}X\Big)dx + \Big(\frac{\partial}{\partial y}Y - Z\frac{\partial}{\partial y}X\Big)dy \\ &\quad + \Big(\frac{\partial}{\partial z}Y - Z\frac{\partial}{\partial z}X\Big)dz.\end{aligned}$$

Thus, if ϕ is a contact transformation, then it holds that

$$\frac{\partial}{\partial x}Y - Z\frac{\partial}{\partial x}X = -\rho z, \quad \frac{\partial}{\partial y}Y - Z\frac{\partial}{\partial y}X = \rho, \quad \frac{\partial}{\partial z}Y - Z\frac{\partial}{\partial z}X = 0,$$

and equivalently that $v_i(Y) = Zv_i(X)$, $(i = 1,\ 2)$. Q.E.D.

The equations (1) determine the function Y up to additive constant. These equations do not necessarily have a solution Y. The following lemma shows the integrability condition of these equations. The proof is given in the paper [Sat].

Lemma 3.2. *The necessary and sufficient condition for the equations* (1) *to have a solution Y is that the functions X and Z satisfy the relations*

$$v_i(\Delta) = v_i(X)v_3(Z) - v_3(X)v_i(Z), \quad (i = 1,\ 2),$$

where $\Delta = v_1(X)v_2(Z) - v_1(Z)v_2(X)$.

We call $\Delta = v_1(X)v_2(Z) - v_1(Z)v_2(X)$ the *contact Jacobian.* The relation between the contact Jacobian and the usual Jacobian of the map ϕ is given in the following lemma.

Lemma 3.3. *Let ϕ: $(x, y, z) \mapsto (X, Y, Z)$ be a contact transformation, and ρ a scalar function that satisfies $\phi^*(\alpha) = \rho\alpha$. Then ρ is equal to the contact Jacobian $\Delta = v_1(X)v_2(Z) - v_1(Z)v_2(X)$, and the usual Jacobian of ϕ is equal to*

$$\frac{\partial(X, Y, Z)}{\partial(x, y, z)} = \rho^2 = \Delta^2.$$

Proof. Differentiating the equations (1) by v_1 and v_2, we obtain

$$v_2 v_1(Y) = v_2(Z)v_1(X) + Z v_2 v_1(X),$$
$$v_1 v_2(Y) = v_1(Z)v_2(X) + Z v_1 v_2(X),$$

and subtracting them, obtain

$$\rho = \frac{\partial}{\partial y}Y - Z\frac{\partial}{\partial y}X = v_3(Y) - Zv_3(X) = v_2(Z)v_1(X) - v_1(Z)v_2(X),$$

which is equal to Δ.

By definition, the Jacobian satisfies

$$\phi^*(\alpha \wedge d\alpha) = \frac{\partial(X,Y,Z)}{\partial(x,y,z)}\alpha \wedge d\alpha.$$

Since ϕ is contact, we have

$$\phi^*(\alpha \wedge d\alpha) = \rho\alpha \wedge d(\rho\alpha) = \rho^2 \alpha \wedge d\alpha.$$

Therefore we get the required equality $\partial(X,Y,Z)/\partial(x,y,z) = \rho^2$. Q.E.D.

From these lemmas, it follows immediately that

Corollary 3.4. *A map* $\phi\colon (x,y,z) \mapsto (X,Y,Z)$ *is a contact transformation, if and only if the contact Jacobian* $\Delta = v_1(X)v_2(Z) - v_1(Z)v_2(X)$ *does not vanish anywhere, and* X, Y *and* Z *satisfy* $v_i(Y) = Zv_i(X)$ $(i = 1,\ 2)$.

In order to simplify the notation, we sometimes denote by (x^1, x^3, x^2) the coordinate functions of $\mathbb{K}^3$ in place of (x,y,z). In this notation, the contact form is written as $dy - z\,dx = dx^3 - x^2\,dx^1$.

Lemma 3.5. *For any contact transformation* $(x^1, x^3, x^2) \mapsto (y^1, y^3, y^2)$ *and any function* $f(y^1, y^3, y^2)$, *we have, for* $i = 1$ *and* 2,

$$v_i(f) = \sum_{j=1,2} v_i(y^j)u_j(f),$$

where we used the notation

$$u_1 = \frac{\partial}{\partial y^1} + y^2\frac{\partial}{\partial y^3}, \quad u_2 = \frac{\partial}{\partial y^2}.$$

Proof. The differential of f with respect to v_1 is

$$\begin{aligned}
v_1(f) &= \frac{\partial f}{\partial x^1} + x^2 \frac{\partial f}{\partial x^3} \\
&= \frac{\partial f}{\partial y^1}\frac{\partial y^1}{\partial x^1} + \frac{\partial f}{\partial y^2}\frac{\partial y^2}{\partial x^1} + \frac{\partial f}{\partial y^3}\frac{\partial y^3}{\partial x^1} \\
&\quad + x^2 \left(\frac{\partial f}{\partial y^1}\frac{\partial y^1}{\partial x^3} + \frac{\partial f}{\partial y^2}\frac{\partial y^2}{\partial x^3} + \frac{\partial f}{\partial y^3}\frac{\partial y^3}{\partial x^3} \right).
\end{aligned}$$

Reminding that the map is contact; $dy^3 - y^2\, dy^1 = \rho(dx^3 - x^2\, dx^1)$, and thus

$$\frac{\partial y^3}{\partial x^1} - y^2 \frac{\partial y^1}{\partial x^1} = -\rho x^2, \quad \frac{\partial y^3}{\partial x^2} - y^2 \frac{\partial y^1}{\partial x^2} = 0, \quad \frac{\partial y^3}{\partial x^3} - y^2 \frac{\partial y^1}{\partial x^3} = \rho,$$

we get

$$\begin{aligned}
v_1(f) &= \frac{\partial f}{\partial y^1}\frac{\partial y^1}{\partial x^1} + \frac{\partial f}{\partial y^2}\frac{\partial y^2}{\partial x^1} + \frac{\partial f}{\partial y^3}\left(y^2 \frac{\partial y^1}{\partial x^1} - \rho x^2 \right) \\
&\quad + x^2 \left(\frac{\partial f}{\partial y^1}\frac{\partial y^1}{\partial x^3} + \frac{\partial f}{\partial y^2}\frac{\partial y^2}{\partial x^3} + \frac{\partial f}{\partial y^3}\left(y^2 \frac{\partial y^1}{\partial x^3} + \rho \right) \right) \\
&= \frac{\partial f}{\partial y^1} v_1(y^1) + \frac{\partial f}{\partial y^2} v_1(y^2) + y^2 \frac{\partial f}{\partial y^3} v_1(y^1) \\
&= u_1(f) v_1(y^1) + u_2(f) v_1(y^2).
\end{aligned}$$

We prove the equality for the differentiation by v_2 in the same way. Q.E.D.

Lemma 3.6. *For a compositions of contact transformations* $\psi \circ \phi\colon x = (x^1, x^3, x^2) \mapsto y = (y^1, y^3, y^2) \mapsto z = (z^1, z^3, z^2)$, *the contact Jacobians satisfy* $\Delta(z, x) = \Delta(z, y)\Delta(y, x)$.

Proof. Define functions ρ and σ by

$$\phi^*(dy^3 - y^2\, dy^1) = \rho(dx^3 - x^2\, dx^1), \text{ and } \psi^*(dz^3 - z^2\, dz^1) = \sigma(dy^3 - y^2\, dy^1)$$

then we have

$$(\psi \circ \phi)^*(dz^3 - z^2\, dz^1) = \sigma\rho(dx^3 - x^2\, dx^1).$$

The lemma follows immediately from Lemma 3.3. Q.E.D.

§4. Fundamental equation

In this section, we introduce a fundamental PDE system, which will play a central role in this paper. The coefficients P, Q, R and S of this PDE system will be replaced by contact Schwarzian derivatives in later sections.

Given functions $P = P(x,y,z)$, $Q = Q(x,y,z)$, $R = R(x,y,z)$ and $S = S(x,y,z)$, we define functions M_{11}, M_4 and M_{22} by

$$\begin{aligned} M_{11} &= -\frac{1}{4}\big(v_1(Q) - v_2(P) - 2Q^2 + 2PR\big) \\ M_4 &= -\frac{1}{4}\big(v_1(R) - v_2(Q) - QR + PS\big) \\ M_{22} &= -\frac{1}{4}\big(v_1(S) - v_2(R) - 2R^2 + 2QS\big). \end{aligned} \tag{2}$$

Theorem 4.1. *The necessary and sufficient condition for the linear PDE system*

$$\begin{cases} {v_1}^2(\vartheta) = Qv_1(\vartheta) - Pv_2(\vartheta) + M_{11}\vartheta \\ v_4(\vartheta) = 2\big(Rv_1(\vartheta) - Qv_2(\vartheta) + M_4\vartheta\big) \\ {v_2}^2(\vartheta) = Sv_1(\vartheta) - Rv_2(\vartheta) + M_{22}\vartheta \end{cases} \tag{Sp}$$

with unknown function ϑ to have 4-dimensional solution space is that the functions P, Q, R and S satisfy the set of relations

$$\begin{aligned} v_3(P) &= 2(v_1 - 2Q)(M_{11}) + 4PM_4 \\ 3v_3(Q) &= 2(v_2 - 4R)(M_{11}) + 4(v_1 + Q)(M_4) + 4PM_{22} \\ 3v_3(R) &= 2(v_1 + 4Q)(M_{22}) + 4(v_2 - R)(M_4) - 4SM_{11} \\ v_3(S) &= 2(v_2 + 2R)(M_{22}) - 4SM_4. \end{aligned} \tag{IC}$$

Moreover, if it is the case, the solution space is parameterized by initial values $\vartheta(p)$, $v_1(\vartheta)(p)$, $v_2(\vartheta)(p)$ and $v_3(\vartheta)(p)$ at an arbitrarily fixed point p.

The proof will be given in Section 7.

Remark 4.2. The above relations (IC) are automatically satisfied if the functions P, Q, R and S are contact Schwarzian derivatives, $S_{\{111\}}(\phi)$, $S_{\{112\}}(\phi)$, $S_{\{122\}}(\phi)$ and $S_{\{222\}}(\phi)$ respectively, of some contact transformation ϕ. This fact can be proved by a direct calculation. Other aspects of this can be found in the papers [Sat] and [S-Y].

Remark 4.3. It might be useful to express the functions M_{11}, M_4, M_{22} as follows: let w_1 and w_2 be operators defined by

$$w_1 = v_1 - \begin{pmatrix} Q & -P \\ R & -Q \end{pmatrix}, \quad w_2 = v_2 - \begin{pmatrix} R & -Q \\ S & -R \end{pmatrix}.$$

If we put $w_3 = [w_2, w_1]$, then we have

$$w_3 = v_3 - 4\begin{pmatrix} M_4 & -M_{11} \\ M_{22} & -M_4 \end{pmatrix}.$$

Furthermore, we compute the commutators $[w_3, w_1]$ and $[w_3, w_2]$ as follows:

$$[w_3, w_1] = \begin{pmatrix} -v_3(Q) + 4v_1(M_4) - 4RM_{11} + 4PM_{22} & v_3(P) - 4v_1(M_{11}) - 8PM_4 + 8QM_{11} \\ -v_3(R) + 4v_1(M_{22}) + 8QM_{22} - 8RM_4 & v_3(Q) - 4v_1(M_4) + 4RM_{11} - 4PM_{22} \end{pmatrix}$$

$$[w_3, w_2] = \begin{pmatrix} -v_3(R) + 4v_2(M_4) - 4SM_{11} + 4QM_{22} & v_3(Q) - 4v_2(M_{11}) - 8QM_4 + 8RM_{11} \\ -v_3(S) + 4v_2(M_{22}) + 8RM_{22} - 8SM_4 & v_3(R) - 4v_2(M_4) + 4SM_{11} - 4QM_{22} \end{pmatrix}$$

Supposing the PDE system (Sp) has a nonzero solution ϑ, we consider the following PDE system with unknown function ξ;

$$\text{(Sp-s)} \quad \begin{cases} {v_1}^2(\xi) = Qv_1(\xi) - Pv_2(\xi) + M_{11}\xi \\ v_2v_1(\xi) = \left(R + \dfrac{v_2(\vartheta)}{\vartheta}\right) v_1(\xi) \\ \qquad\qquad - \left(Q + \dfrac{v_1(\vartheta)}{\vartheta}\right) v_2(\xi) + \left(M_4 + \dfrac{1}{2}\dfrac{v_3(\vartheta)}{\vartheta}\right)\xi \\ v_1v_2(\xi) = \left(R - \dfrac{v_2(\vartheta)}{\vartheta}\right) v_1(\xi) \\ \qquad\qquad - \left(Q - \dfrac{v_1(\vartheta)}{\vartheta}\right) v_2(\xi) + \left(M_4 - \dfrac{1}{2}\dfrac{v_3(\vartheta)}{\vartheta}\right)\xi \\ {v_2}^2(\xi) = Sv_1(\xi) - Rv_2(\xi) + M_{22}\xi. \end{cases}$$

We denote by $\mathcal{S}(P, Q, R, S)$ and $s(P, Q, R, S; \vartheta)$ the solution spaces of the PDE systems (Sp) and (Sp-s), respectively. Adding the second equation in (Sp-s) to the third, we get the second equation of (Sp). This implies that $s(P, Q, R, S; \vartheta)$ is a subspace of $\mathcal{S}(P, Q, R, S)$ for any ϑ.

As we will see in Section 6, the PDE systems (Sp) and (Sp-s) will play important roles to reconstruct contact transformations with prescribed contact Schwarzian derivatives. In the following propositions, we study the relation of the solution spaces $\mathcal{S}(P, Q, R, S)$ and $s(P, Q, R, S; \vartheta)$.

Proposition 4.4. *For any two solutions α and β of the PDE system* (Sp), *the function $I(\alpha,\beta)$ defined by*

$$I(\alpha,\beta)=\frac{1}{2}\alpha v_3(\beta)-\frac{1}{2}v_3(\alpha)\beta+v_1(\alpha)v_2(\beta)-v_2(\alpha)v_1(\beta) \tag{3}$$

is constant on (x,y,z). Moreover this skew product $I(\alpha,\beta)$ is non-degenerate, and thus it defines a symplectic structure on the solution space $\mathcal{S}(P,Q,R,S)$, provided the dimension of $\mathcal{S}(P,Q,R,S)$ is equal to 4.

Proof. In order to prove the constantness of $I(\alpha,\beta)$, it is sufficient to verify the vanishing of the derivatives

$$v_i(I(\alpha,\beta))=0, \qquad (i=1,\ 2).$$

The derivative $v_1(I(\alpha,\beta))$ is calculated as follows: we use the notation $v_i(f)=f_i$ and $v_iv_j(f)=f_{ji}$. Remarking that $3v_1v_3=3v_3v_1=2v_2{v_1}^2-v_1v_4$, we get

$$\begin{aligned}
&v_1(\alpha\beta_3+2\alpha_1\beta_2)\\
&\quad=\alpha_1\beta_3+\alpha\beta_{31}+2\alpha_{11}\beta_2+2\alpha_1\beta_{21}\\
&\quad=\alpha_1\beta_4+2\alpha_{11}\beta_2+\frac{1}{3}(2(\beta_{11})_2-(\beta_4)_1)\alpha\\
&\quad=2(R\beta_1-Q\beta_2+M_4\beta)\alpha_1+2(Q\alpha_1-P\alpha_2+M_{11}\alpha)\beta_2\\
&\qquad\qquad+\frac{2}{3}\big((Q\beta_1-P\beta_2+M_{11})_2-(R\beta_1-Q\beta_2+M_4\beta)_1\big)\alpha\\
&\quad=2(R\alpha_1\beta_1+M_4\alpha_1\beta-P\alpha_2\beta_2+M_{11}\alpha\beta_2)\\
&\qquad+\frac{2}{3}\big(2Q(R\beta_1-Q\beta_2+M_4\beta)-P(S\beta_1-R\beta_2+M_{22}\beta)\\
&\qquad\qquad-R(Q\beta_1-P\beta_2+M_{11}\beta)+(Q_2-R_1-M_4\beta)\beta_1\\
&\qquad\qquad-(P_2-Q_1-M_{11})\beta_2+((M_{11})_2-(M_4)_1)\beta\big)\alpha\\
&\quad=2(M_4\alpha_1\beta+R\alpha_1\beta_1-P\alpha_2\beta_2)\\
&\qquad+\frac{2}{3}\big((2QM_4-PM_{22}-RM_{11}+(M_{11})_2-(M_4)_1)\alpha\beta\\
&\qquad\qquad+(Q_2-R_1+QR-PS-M_4)\alpha\beta_1\\
&\qquad\qquad-(P_2-Q_1+2Q^2-2PR-4M_{11})\alpha\beta_2\big)\\
&\quad=2(M_4(\alpha\beta_1+\alpha_1\beta)+R\alpha_1\beta_1-P\alpha_2\beta_2)\\
&\qquad\qquad+\frac{2}{3}\big(2QM_4-PM_{22}-RM_{11}+(M_{11})_2-(M_4)_1\big)\alpha\beta.
\end{aligned}$$

Since the last expression is symmetric on α and β, we have

$$v_1(I(\alpha,\beta))=\frac{1}{2}\alpha\beta_3+\alpha_1\beta_2-\Big(\frac{1}{2}\alpha_3\beta+\alpha_2\beta_1\Big)=0.$$

In a similar way, we get the equality $v_2(I(\alpha,\beta)) = 0$.

The non-degeneracy of I is understood from the fact that the solution space of (Sp) is parametrized by the initial values $\vartheta(p)$, $v_1(\vartheta)(p)$, $v_2(\vartheta)(p)$ and $v_3(\vartheta)(p)$, where p is a fixed point in the domain. Q.E.D.

Proposition 4.5. *Let $\vartheta \in \mathcal{S}(P,Q,R,S)$ be a non-zero solution. Then the solution space $s(P,Q,R,S;\vartheta)$ of the PDE system* (Sp-s) *is the skew orthogonal subspace of ϑ in the symplectic space $\mathcal{S}(P,Q,R,S)$;*

$$s(P,Q,R,S;\vartheta) = \{\alpha \in \mathcal{S}(P,Q,R,S) \mid I(\alpha,\vartheta) = 0\}.$$

Proof. Let $\xi \in s(P,Q,R,S;\vartheta)$ be fixed. Subtracting the third equation from the second of (Sp-s) that ξ satisfies, and multiplying it by ϑ, we get the equality $I(\xi,\vartheta) = 0$. And conversely, if a solution $\xi \in \mathcal{S}(P,Q,R,S)$ satisfies $I(\xi,\vartheta) = 0$, then it satisfies automatically the equations (Sp-s). This proves the proposition. Q.E.D.

§5. PDE systems related to contact transformation

If a transformation $\phi\colon (x,y,z) \mapsto (X,Y,Z)$ is contact, the coordinate functions X, Y and Z satisfy certain differential equations. In the following, we study relations of those equations, whose integrability conditions will be discussed in Section 7.

We start with the following PDE system with unknown function Ω and given functions P, Q, R, S and Δ

$$\text{(Ct-s)} \qquad \begin{cases} {v_1}^2(\Omega) = \left(Q + \dfrac{v_1(\Delta)}{\Delta}\right) v_1(\Omega) - Pv_2(\Omega) \\[2ex] v_2v_1(\Omega) = Rv_1(\Omega) - \left(Q - \dfrac{v_1(\Delta)}{\Delta}\right) v_2(\Omega) \\[2ex] v_1v_2(\Omega) = \left(R + \dfrac{v_2(\Delta)}{\Delta}\right) v_1(\Omega) - Qv_2(\Omega) \\[2ex] {v_2}^2(\Omega) = Sv_1(\Omega) - \left(R - \dfrac{v_2(\Delta)}{\Delta}\right) v_2(\Omega). \end{cases}$$

Proposition 5.1. *If a transformation $\phi\colon (x,y,z) \mapsto (X,Y,Z)$ is contact, then X and Z are solutions of the PDE system* (Ct-s) *with $(P,Q,R,S) = S(\phi)$ and Δ any nonzero constant multiple of the contact Jacobian $\Delta(\phi)$.*

Conversely, if 1 *(a constant function), X and Z are mutually linearly independent solutions of* (Ct-s), *then there exists a function Y such that the map $\phi\colon (x,y,z) \mapsto (X,Y,Z)$ is a contact transformation whose* *contact Schwarzian derivative is equal to* $S(\phi) = (P, Q, R, S)$, *and whose contact Jacobian is equal to a constant multiple of* Δ.

Proof. If X and Z satisfy the equations (Ct-s), then we get

$$\begin{pmatrix} X_2 & X_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & X_1 & X_2 & 0 \\ 0 & 0 & 0 & 0 & X_2 & 0 & 0 & X_1 \\ 0 & 0 & X_2 & X_1 & 0 & 0 & 0 & 0 \\ Z_2 & Z_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & Z_1 & Z_2 & 0 \\ 0 & 0 & 0 & 0 & Z_2 & 0 & 0 & Z_1 \\ 0 & 0 & Z_2 & Z_1 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -P \\ Q + \Delta_1/\Delta \\ -R + \Delta_2/\Delta \\ S \\ -Q \\ R \\ -Q + \Delta_1/\Delta \\ R + \Delta_2/\Delta \end{pmatrix} = \begin{pmatrix} X_{11} \\ X_{12} \\ X_{21} \\ X_{22} \\ Z_{11} \\ Z_{12} \\ Z_{21} \\ Z_{22} \end{pmatrix},$$

where $(\ )_i = v_i(\ )$ and $(\ )_{ij} = v_j v_i(\ )$. Put $D = v_1(X)v_2(Z) - v_2(X)v_1(Z)$. The determinant of the above 8×8-matrix equals $-D^4$. The linear independence of 1, X and Z implies that the determinant does not vanish. By multiplying the inverse matrix, and taking linear combinations, we get eight relations, four of which are exactly the definitions of contact Schwarzian derivative, and the others read

$$\begin{aligned} \frac{v_1(\Delta)}{\Delta} &= \frac{1}{D}\big(v_1{}^2(X)v_2(Z) + v_1(X)v_1v_2(Z) \\ &\qquad - v_1v_2(X)v_1(Z) - v_2(X)v_1{}^2(Z)\big) \\ \frac{v_2(\Delta)}{\Delta} &= \frac{1}{D}\big(v_2v_1(X)v_2(Z) + v_1(X)v_2{}^2(Z) \\ &\qquad - v_2{}^2(X)v_1(Z) - v_2(X)v_2v_1(Z)\big) \\ \frac{v_1(\Delta)}{\Delta} &= \frac{1}{D}\big(v_1(X)v_3(Z) - v_3(X)v_1(Z)\big) \\ \frac{v_2(\Delta)}{\Delta} &= \frac{1}{D}\big(v_2(X)v_3(Z) - v_3(X)v_2(Z)\big). \end{aligned}$$

The first two equations are equivalent to

$$\frac{v_1(\Delta)}{\Delta} = \frac{v_1(D)}{D}, \quad \frac{v_2(\Delta)}{\Delta} = \frac{v_2(D)}{D}.$$

These equations imply that Δ is a constant multiple of D. Furthermore, this fact allows us to write the last two equalities as

$$v_i(D) = v_i(X)v_3(Z) - v_3(X)v_i(Z), \quad (i = 1,\ 2).$$

These equations amount to the condition that guarantees the existence of the solution of the equation $v_i(Y) = Zv_i(X)$ $(i = 1, 2)$ (cf. Lemma 3.2).

Q.E.D.

Proposition 5.2. *If a transformation ϕ: $(x,y,z) \mapsto (X,Y,Z)$ is contact, then the functions X, Z and $W = 2Y - XZ$ satisfy*

$$\text{(Ct)} \qquad \begin{cases} {v_1}^2(\Omega) = \left(Q + \dfrac{v_1(\Delta)}{\Delta}\right) v_1(\Omega) - P v_2(\Omega) \\ v_4(\Omega) = \left(2R + \dfrac{v_2(\Delta)}{\Delta}\right) v_1(\Omega) - \left(2Q - \dfrac{v_1(\Delta)}{\Delta}\right) v_2(\Omega) \\ {v_2}^2(\Omega) = S v_1(\Omega) - \left(R - \dfrac{v_2(\Delta)}{\Delta}\right) v_2(\Omega), \end{cases}$$

provided $(P, Q, R, S) = S(\phi)$ and $\Delta = \Delta(\phi)$.

Proof. If $(x,y,z) \mapsto (X,Y,Z)$ is contact, then we have $v_i(Y) = Z v_i(X)$ $(i = 1,\ 2)$ (see Lemma 3.1), and therefore the function $W = 2Y - XZ$ satisfies

$$v_i(W) = Z v_i(X) - X v_i(Z), \quad (i = 1,\ 2).$$

Differentiating it by v_j, we get

$$v_j v_i(W) = Z v_j v_i(X) - X v_j v_i(Z) + \varepsilon_{ij} \Delta,$$

where Δ is the contact Jacobian, and $\varepsilon_{11} = \varepsilon_{22} = 0$ and $\varepsilon_{12} = 1 = -\varepsilon_{21}$. Reminding that X and Z are solutions of (Ct-s), we find that the function W satisfies the equations

$$\begin{cases} {v_1}^2(W) = Q v_1(W) - P v_2(W) + \dfrac{v_1(\Delta)}{\Delta} v_1(W) \\ v_2 v_1(W) = R v_1(W) - Q v_2(W) + \dfrac{v_1(\Delta)}{\Delta} v_2(W) + \Delta \\ v_1 v_2(W) = R v_1(W) - Q v_2(W) + \dfrac{v_2(\Delta)}{\Delta} v_1(W) - \Delta \\ {v_2}^2(W) = S v_1(W) - R v_2(W) + \dfrac{v_2(\Delta)}{\Delta} v_2(W). \end{cases}$$

By adding the second to the third expression, this system is reduced to the system (Ct). Therefore W satisfies the PDE system (Ct). Since the system (Ct-s) also reduces to (Ct), it follows, by Proposition 5.1, that X and Z also satisfy (Ct). Q.E.D.

If the functions $\{1, X, Z, W\}$ in the above proposition form a linear basis of the solution space, then the solution space of the system (Ct-s) is a subspace spanned by $\{1, X, Z\}$, and is characterized by the linear equation

$$\Delta v_3(\Omega) - v_1(\Delta) v_2(\Omega) + v_2(\Delta) v_1(\Omega) = 0. \tag{4}$$

Subtracting the third equation of (Ct-s) from the second, we get (4). The function $W = 2Y - XZ$ stays on the affine subspace parallel to this subspace, and characterized by

$$\Delta v_3(\Omega) - v_1(\Delta)v_2(\Omega) + v_2(\Delta)v_1(\Omega) - 2\Delta^2 = 0. \tag{5}$$

If a function Y corresponds to independent solutions X and Z of (Ct-s) so that $\phi = (X, Y, Z)$ is contact, then the function

$$\tilde{Y} = \frac{1}{2}\big((ad - bc)(2Y - XZ) + (aX + bZ)(cX + dZ)\big).$$

corresponds to the solutions $\tilde{X} = aX + bZ$ and $\tilde{Z} = cX + dZ$ for any constant a, b, c and d with $ad - bc \neq 0$. The resulting contact transformation $\tilde{\phi} = (\tilde{X}, \tilde{Y}, \tilde{Z})$ has the same contact Schwarzian derivatives $S(\tilde{\phi}) = S(\phi)$, and the contact Jacobian is equal to $\Delta(\tilde{\phi}) = (ad - bc)\Delta(\phi)$. Thus the PDE systems (Ct-s) and (Ct) remain the same for the contact transformation $\tilde{\phi}$.

We study equations that contact Jacobians $\Delta = \Delta(\phi)$ satisfy. Suppose (Ct-s) has two linearly independent non-constant solutions X and Z. They necessarily satisfy

$$0 = v_i v_3(\Omega) - v_3 v_i(\Omega), \quad (i = 1,\ 2).$$

Since $\Omega = X$ or Z is a solution of (Ct-s), the above is equivalent to

$$0 = F_i(\Delta)v_1(\Omega) + G_i(\Delta)v_2(\Omega),$$

where F_i and G_i are the operators given by

$$\begin{aligned}
F_2(\Delta) &= -2\Big(-{v_1}^2(\Delta) + Qv_1(\Delta) - Pv_2(\Delta) + \frac{1}{2}(v_1(\Delta))^2 - 2M_{11}\Big)\\
F_1(\Delta) &= G_2(\Delta)\\
&= -v_4(\Delta) + 2Rv_1(\Delta) - 2Qv_2(\Delta) + v_1(\Delta)v_2(\Delta) - 4M_4\\
G_1(\Delta) &= -2\Big(-{v_2}^2(\Delta) + Sv_1(\Delta) - Rv_2(\Delta) + \frac{1}{2}(v_2(\Delta))^2 - 2M_{22}\Big).
\end{aligned}$$

The independence of X and Z implies $F_i(\Delta) = G_i(\Delta) = 0$ $(i = 1,\ 2)$, which amounts to the PDE system

$$\begin{cases}
{v_1}^2(\Delta) = Qv_1(\Delta) - Pv_2(\Delta) + \frac{1}{2}(v_1(\Delta))^2 - 2M_{11}\\
v_4(\Delta) = 2Rv_1(\Delta) - 2Qv_2(\Delta) + v_1(\Delta)v_2(\Delta) - 4M_4\\
{v_2}^2(\Delta) = Sv_1(\Delta) - Rv_2(\Delta) + \frac{1}{2}(v_2(\Delta))^2 - 2M_{22}
\end{cases} \tag{6}$$

This PDE system is nonlinear. The change of variable $\vartheta = 1/\sqrt{\Delta}$ (taking one branch of square root) linearizes it, and the resulting PDE system is exactly the same as (Sp).

Proposition 5.3. *If Δ is a contact Jacobian, then $1/\sqrt{\Delta}$ is a solution of the PDE system* (Sp) *whose coefficients P, Q, R and S are equal to the contact Schwarzian derivatives.*

Remark 5.4. The PDE system (6) is a necessary condition for Δ to have a nonconstant solution of the PDE system (Ct-s), but not sufficient. If, in addition, P, Q, R and S satisfy the relations (IC), then the PDE system (Ct-s) has a nonconstant solution. This fact can be shown after a quite long calculation. In order to perform it, we used the computer program "Maple".

§6. Construction of contact transformation via solutions of PDE system

Let I be the skew product on the solution space $\mathcal{S}(P,Q,R,S)$ of the PDE system (Sp) defined in Proposition 4.4. We prove, in this section, the following

Theorem 6.1. *If a map $\phi\colon (x,y,z) \mapsto (X,Y,Z)$ is contact, then there exists a symplectic basis $\{\vartheta,\xi,\zeta,\eta\}$ of the solution space $\mathcal{S}(S(\phi))$ of the PDE system* (Sp) *such that ϕ is given by*

$$(x,y,z) \mapsto \left(\frac{\xi}{\vartheta}, \frac{1}{2}\left(\frac{\eta}{\vartheta} + \frac{\xi\zeta}{\vartheta^2}\right), \frac{\zeta}{\vartheta}\right). \tag{7}$$

Conversely, given a symplectic basis $\{\vartheta,\xi,\zeta,\eta\}$ of the solution space $\mathcal{S}(P,Q,R,S)$ of (Sp), *the map ϕ defined by* (7) *is a contact transformation whose contact Schwarzian derivatives $S(\phi)$ is equal to (P,Q,R,S).*

Here if we say a linear basis $\{\xi_0,\xi_1,\xi_2,\xi_3\}$ is symplectic, we mean

$$\left(I(\xi_i,\xi_j)\right)_{i,j=0,3} = cJ, \quad \text{where } J = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \tag{8}$$

and c is a nonzero constant. We denote by $Sp_2(\mathbb{K})$ and $CSp_2(\mathbb{K})$ the Lie groups defined by

$$Sp_2(\mathbb{K}) = \{g \in GL(4;\mathbb{K}) \mid gJg^t = J\}$$
$$CSp_2(\mathbb{K}) = \{gJg^t = cJ \text{ for some nonzero constant } c \in \mathbb{K}\}.$$

In order to emphasize the PDE systems (Sp-s), (Ct-s) and (Ct) to contain the functions ϑ and Δ in their coefficients, we use the notations $(\text{Sp-s})^{\vartheta}$, $(\text{Ct-s})^{\Delta}$ and $(\text{Ct})^{\Delta}$. Their solution spaces are denoted by $s(P,Q,R,S;\vartheta)$, $t(P,Q,R,S;\Delta)$ and $\mathcal{T}(P,Q,R,S;\Delta)$, respectively;

$$\begin{aligned}
\mathcal{S}(P,Q,R,S) &= \text{the solution space of the PDE system (Sp)}\\
s(P,Q,R,S;\vartheta) &= \text{the solution space of the PDE system } (\text{Sp-s})^{\vartheta}\\
\mathcal{T}(P,Q,R,S;\Delta) &= \text{the solution space of the PDE system } (\text{Ct})^{\Delta}\\
t(P,Q,R,S;\Delta) &= \text{the solution space of the PDE system } (\text{Ct-s})^{\Delta}
\end{aligned}$$

We find a relation among those solution spaces in the following

Proposition 6.2. *For each nonzero solution ϑ of* (Sp), *the projectification $\xi \mapsto \xi/\vartheta$ gives a linear map between the solution spaces $\mathcal{S}(P,Q,R,S)$ and $\mathcal{T}(P,Q,R,S;\Delta)$, provided Δ is a constant multiple of ϑ^{-2}. The image of the subspace $s(P,Q,R,S;\vartheta)$ is contained in the subspace $t(P,Q,R,S;\Delta)$;*

$$\begin{array}{ccc}
\mathcal{S}(P,Q,R,S) & \longrightarrow & \mathcal{T}(P,Q,R,S;\Delta)\\
\cup & & \cup\\
s(P,Q,R,S;\vartheta) & \longrightarrow & t(P,Q,R,S;\Delta).
\end{array}$$

The largest dimension of the solution spaces $\mathcal{S}(P,Q,R,S)$ and $\mathcal{T}(P,Q,R,S;\Delta)$ are 4, and those of $s(P,Q,R,S;\theta)$ and $t(P,Q,R,S;\Delta)$ are 3. By Proposition 4.5, the codimension of $s(P,Q,R,S;\vartheta)$ in $\mathcal{S}(P,Q,R,S)$ is always 1, provided the PDE system (Sp) has at least one nonzero solution ϑ. Clearly the map $\xi \mapsto \xi/\vartheta$ is injective. Therefore, if the dimension of $\mathcal{S}(P,Q,R,S)$ is 4, that is, if the PDE system (Sp) is integrable, then all the other PDE systems are also integrable. We will manage the integrability of (Sp) in the next section.

Proof. Let a nonzero solution ϑ of (Sp) be fixed. Suppose ξ is a solution of (Sp). Then we have

$$\begin{aligned}
{v_1}^2\Big(\frac{\xi}{\vartheta}\Big) &= v_1\Big(\frac{1}{\vartheta^2}\big(v_1(\xi)\vartheta - \xi v_1(\vartheta)\big)\Big)\\
&= \frac{1}{\vartheta^4}\Big\{\big({v_1}^2(\xi)\vartheta - \xi {v_1}^2(\vartheta)\big)\vartheta^2 - \big(v_1(\xi)\vartheta - \xi v_1(\vartheta)\big)\big(2\vartheta v_1(\vartheta)\big)\Big\}\\
&= \frac{1}{\vartheta^3}\Big\{\big(Qv_1(\xi) - Pv_2(\xi) + M_{11}\xi\big)\vartheta^2\\
&\qquad - \xi\vartheta\big(Qv_1(\vartheta) - Pv_2(\vartheta) + M_{11}\vartheta\big)\\
&\qquad - 2v_1(\xi)v_1(\vartheta)\vartheta + 2\xi v_1(\vartheta)^2\Big\}
\end{aligned}$$

$$
\begin{aligned}
&= Q\frac{v_1(\xi)\vartheta - \xi v_1(\vartheta)}{\vartheta^2} - P\frac{v_2(\xi)\vartheta - \xi v_2(\vartheta)}{\vartheta^2} \\
&\quad -2\frac{v_1(\vartheta)}{\vartheta}\frac{v_1(\xi)\vartheta - \xi v_1(\vartheta)}{\vartheta^2} \\
&= \left(Q - 2\frac{v_1(\vartheta)}{\vartheta}\right) v_1\left(\frac{\xi}{\vartheta}\right) - Pv_2\left(\frac{\xi}{\vartheta}\right).
\end{aligned}
$$

If Δ is equal to $c\vartheta^{-2}$ with c a nonzero constant, then the last expression is equal to

$$
\left(Q + \frac{v_1(\Delta)}{\Delta}\right) v_1\left(\frac{\xi}{\vartheta}\right) - Pv_2\left(\frac{\xi}{\vartheta}\right).
$$

Thus $\Omega = \xi/\vartheta$ satisfies the first equation in (Ct). In a similar way, we verify that $\Omega = \xi/\vartheta$ satisfies other two equalities of $(\mathrm{Ct})^{\Delta}$ with $\Delta = c\vartheta^{-2}$. The subspaces $s(P,Q,R,S;\vartheta)$ and $t(P,Q,R,S;\vartheta)$ are characterized by the equations $I(\vartheta,\xi) = 0$ and (4), respectively. Thus we have

$$
\begin{aligned}
&\frac{1}{c}\big(\Delta v_3(\Omega) - v_1(\Delta)v_2(\Omega) + v_2(\Delta)v_1(\Omega)\big) \\
&\quad = \frac{1}{\vartheta^2}\frac{v_3(\xi)\vartheta - \xi v_3(\vartheta)}{\vartheta^2} + \frac{2v_1(\vartheta)}{\vartheta^3}\frac{v_2(\xi)\vartheta - \xi v_2(\vartheta)}{\vartheta^2} \\
&\quad\quad - \frac{2v_2(\vartheta)}{\vartheta^3}\frac{v_1(\xi)\vartheta - \xi v_1(\vartheta)}{\vartheta^2} \\
&\quad = 2\vartheta^{-4} I(\vartheta,\xi).
\end{aligned}
$$

Therefore the image of $s(P,Q,R,S;\vartheta)$ is contained in $t(P,Q,R,S;\vartheta)$. Q.E.D.

The rest of this section is devoted to the proof of Theorem 6.1. Suppose ϕ: $(x,y,z) \mapsto (X,Y,Z)$ is a contact transformation. From Propositions 5.1 and 5.2, we see that $\{1, X, Z, W = 2Y - XZ\}$ is a linear basis of the solution space $\mathcal{T}(P,Q,R,S;\Delta)$ of the system $(\mathrm{Ct})^{\Delta}$, and $\{1, X, Z\}$ a basis of $t(P,Q,R,S;\Delta)$ of the system $(\mathrm{Ct\text{-}s})^{\Delta}$, where $(P,Q,R,S) = S(\phi)$ and $\Delta = \Delta(\phi)$; the contact Schwarzian derivative and the contact Jacobian of ϕ.

From Proposition 5.3, we see that, if $\Delta(\phi) = c\vartheta^{-2}$ with c a nonzero constant, then ϑ satisfies the equations (Sp). Therefore by Proposition 6.2, we conclude that there exists a linear basis $\{\vartheta, \xi, \zeta, \eta\}$ (where ϑ satisfies $\Delta = c\vartheta^{-2}$) of the solution space $\mathcal{S}(P,Q,R,S)$ such that

$$
X = \frac{\xi}{\vartheta}, \quad Z = \frac{\zeta}{\vartheta}, \quad W = \frac{\eta}{\vartheta}.
$$

It remains to prove that the basis $\{\vartheta, \xi, \zeta, \eta\}$ is symplectic in the sence of Proposition 4.4. We remind that $I(\alpha, \beta)$ is constant, if α and β are solutions of the PDE system (Sp).

By using the functions ϑ, ξ, ζ and η, we calculate the contact Jacobian $\Delta(\phi)$ as follows:

$$
\begin{aligned}
\Delta(\phi) &= v_1(X)v_2(Z) - v_1(Z)v_2(X) \\
&= \frac{1}{\vartheta^4}\Big\{ \big(v_1(\xi)\vartheta - \xi v_1(\vartheta)\big)\big(v_2(\zeta)\vartheta - \zeta v_2(\vartheta)\big) \\
&\qquad\qquad - \big(v_1(\zeta)\vartheta - \zeta v_1(\vartheta)\big)\big(v_2(\xi)\vartheta - \xi v_2(\vartheta)\big)\Big\} \\
&= \frac{1}{\vartheta^3}\Big\{ \big(v_1(\xi)v_2(\zeta) - v_1(\zeta)v_2(\xi)\big)\vartheta \\
&\qquad\qquad + \xi\big(- v_1(\vartheta)v_2(\zeta) + v_1(\zeta)v_2(\vartheta)\big) \\
&\qquad\qquad - \zeta\big(- v_1(\xi)v_2(\vartheta) + v_1(\vartheta)v_2(\xi)\big)\Big\} \\
&= \frac{1}{\vartheta^3}\Big\{ \big(v_1(\xi)v_2(\zeta) - v_1(\zeta)v_2(\xi)\big)\vartheta \\
&\qquad\qquad - \frac{1}{2}\zeta\big(v_3(\xi)\vartheta - \xi v_3(\vartheta)\big) + \frac{1}{2}\xi\big(v_3(\zeta)\vartheta - \zeta v_3(\vartheta)\big)\Big\} \\
&= I(\xi, \zeta)\vartheta^{-2}.
\end{aligned} \tag{9}
$$

Since $\Delta(\phi) = c\vartheta^{-2}$, we conclude

$$I(\xi, \zeta) = c.$$

The function W satisfies (5). If $W = \eta/\vartheta$ and $\Delta = c\vartheta^{-2}$, the equality (5) is replaced by

$$
\begin{aligned}
0 &= \Delta v_3(\Omega) - v_1(\Delta)v_2(\Omega) + v_2(\Delta)v_1(\Omega) - 2\Delta^2 \\
&= c\Big(\vartheta^{-2}v_3\Big(\frac{\eta}{\vartheta}\Big) - v_1(\vartheta^{-2})v_2\Big(\frac{\eta}{\vartheta}\Big) + v_2(\vartheta^{-2})v_1\Big(\frac{\eta}{\vartheta}\Big) - 2c\vartheta^{-4}\Big) \\
&= 2c\vartheta^{-4}(I(\vartheta, \eta) - c),
\end{aligned}
$$

and thus

$$I(\vartheta, \eta) = c.$$

Since $X = \xi/\vartheta$ and $Z = \zeta/\vartheta$ are in the solution space $t(P, Q, R, S; \vartheta)$, ξ and ζ are in the solution space $s(P, Q, R, S; \vartheta)$. Therefore ξ and ζ are skew orthogonal to ϑ;

$$I(\vartheta, \xi) = I(\vartheta, \zeta) = 0$$

(cf. Proposition 4.5).

In order to prove the skew orthogonalities $I(\eta,\xi)=I(\eta,\zeta)=0$, we prove the following

Lemma 6.3. *For solutions ϑ, ξ, ζ and η of* (Sp), *put $X=\xi/\vartheta$, $Z=\zeta/\vartheta$ and $W=\eta/\vartheta$. Then the equalities $v_i(W)=v_i(X)Z-Xv_i(Z)$ $(i=1,\ 2)$ hold, if and only if $I(\eta,\xi)=I(\eta,\zeta)=0$.*

Proof. For each point $p=(x,y,z)$ in the domain, the evaluation map $|_p$:

$$\mathcal{S}(P,Q,R,S)\ni\vartheta\mapsto\left(\vartheta(p),v_1(\vartheta)(p),v_2(\vartheta)(p),\frac{1}{2}v_3(\vartheta)(p)\right)\in\mathbb{K}^4$$

is a symplectic linear map with respect to the skew products I on $\mathcal{S}(P,Q,R,S)$ and J on $\mathbb{K}^4$. If ϑ, ξ, ζ and η is a symplectic basis, then the matrix

$$\left.\begin{pmatrix}\vartheta\\ \xi\\ \zeta\\ \eta\end{pmatrix}\right|_p=\begin{pmatrix}\vartheta(p) & v_1(\vartheta)(p) & v_2(\vartheta)(p) & v_3(\vartheta)(p)/2\\ \xi(p) & v_1(\xi)(p) & v_2(\xi)(p) & v_3(\xi)(p)/2\\ \zeta(p) & v_1(\zeta)(p) & v_2(\zeta)(p) & v_3(\zeta)(p)/2\\ \eta(p) & v_1(\eta)(p) & v_2(\eta)(p) & v_3(\eta)(p)/2\end{pmatrix}$$

is in the symplectic group $Sp_2(\mathbb{K})$. Since the complex structure J satisfies $J^{-1}=J^t=-J$, the transpose of this matrix is also in $Sp_2(\mathbb{K})$. Thus we have

$$0=v_i(\eta)(p)\vartheta(p)-\eta(p)v_i(\vartheta)(p)+\xi(p)v_i(\zeta)(p)-v_i(\xi)(p)\zeta(p)$$

for $i=1,\ 2$. These are equivalent to

$$\begin{aligned}0&=v_i\left(\frac{\eta}{\vartheta}\right)(p)-v_i\left(\frac{\xi}{\vartheta}\right)(p)\frac{\zeta(p)}{\vartheta(p)}+\frac{\xi(p)}{\vartheta(p)}v_i\left(\frac{\zeta}{\vartheta}\right)(p)\\ &=v_i(W)(p)-v_i(X)(p)Z(p)+X(p)v_i(Z)(p).\end{aligned}$$

This completes the proof. Q.E.D.

Since $\phi=(X,Y,Z)$ is contact, $W=2Y-XZ$ satisfies the equality of the above lemma. Therefore we conclude the equalities

$$I(\eta,\xi)=I(\eta,\zeta)=0,$$

and complete the proof of the first half of Theorem 6.1.

Now we prove the latter half of the theorem. Suppose $\{\vartheta,\xi,\zeta,\eta\}$ is a symplectic basis of the solution space $\mathcal{S}(P,Q,R,S)$, and put $X=\xi/\vartheta$, $Z=\zeta/\vartheta$ and $W=\eta/\vartheta$. From Lemma 6.3, it follows that the function $Y=(W+XZ)/2$ satisfies the equalities $v_i(Y)=Zv_i(X)$ $(i=1,\ 2)$. Put

$\Delta = v_1(X)v_2(Z) - v_2(X)v_1(Z)$. Then the same caluculation as (9) shows that Δ is equal to $I(\xi,\zeta)\vartheta^{-2}$, where $I(\xi,\zeta)$ is a nonzero constant, and thus Δ does not vanish. Therefore the map $\phi = (X, Y, Z)$ is a contact transformation (cf. Corollary 3.4).

Since X and Z are solutions of (Ct-s)^{Δ} with $\Delta = c\vartheta^{-2}$ (cf. Proposition 6.2), it follows, from Proposition 5.1, that the given function P, Q, R and S are the contact Schwarzian derivatives of ϕ; $(P, Q, R, S) = S(\phi)$, and also that the contact Jacobian $\Delta(\phi)$ is equal to $c\vartheta^{-2}$.

This completes the proof of Theorem 6.1.

§7. Integrability conditions

The higher derivatives of the solutions of the PDE systems are determined by the lower terms. For the systems (Sp) and (Ct), the initial values $\vartheta(p)$, $v_1(\vartheta)(p)$, $v_2(\vartheta)(p)$ and $v_3(\vartheta)(p)$ at a fixed point $p = (x, y, z)$ in the domain determine all higher derivatives

$$v_{i_1}v_{i_2}\cdots v_{i_n}(\vartheta)(p).$$

Here the coefficient functions are required to satisfy certain compatibility conditions, which we call the *integrability condition* of the PDE system. The solution spaces of (Sp) and (Ct) is of dimension at most 4. The maximality holds, if and only if the integrability condition is fulfilled. On the other hand, the solution spaces of the PDE systems (Sp-s) and (Ct-s) are parameterized by the initial values $\vartheta(p)$, $v_1(\vartheta)(p)$ and $v_2(\vartheta)(p)$. Therefore the maximal dimension is equal to 3. The PDE systems that satisfy the maximality for the dimension of the solution spaces are said to be *integrable*.

As we investigated in the previous two sections, if the system (Sp) is integrable, then for each nonzero solution ϑ of (Sp), the other systems $\text{(Sp-s)}^{\vartheta}$, $\text{(Ct-s)}^{\vartheta^{-2}}$ and $\text{(Ct)}^{\vartheta^{-2}}$ are integrable (see Proposition 6.2). If one of the other systems is integrable, then, by Theorem 6.1, there exists a contact transformation whose contact Schwarzian derivatives are equal to P, Q, R and S, and thus they necessarily satisfy the condition (IC) (see Remark 4.2). Theorem 4.1 says that the condition (IC) is the integrability condition of this system. Therefore, in order to clarify the integrability conditions of those PDE systems, it remains to prove Theorem 4.1.

Denote by $\mathcal{F}$ the ring of quadruples (f_0, f_1, f_2, f_3) of functions f_i of (x, y, z), and define differential operators V_1 and V_2 on $\mathcal{F}$ by

$$V_i(f_0, f_1, f_2, f_3) = (g_{0i}, g_{1i}, g_{2i}, g_{3i}),$$

where g_{ji} are defined by

$$
\begin{aligned}
g_{01} &= v_1(f_0) + M_{11}f_1 + M_4 f_2 + \frac{2}{3}(*)f_3 \\
g_{11} &= (v_1 + Q)(f_1) + f_0 + Rf_2 + 2M_4 f_3 \\
g_{21} &= (v_1 - Q)(f_2) - Pf_1 - 2M_{11}f_3 \\
g_{31} &= v_1(f_3) - \frac{1}{2}f_2 \\
g_{02} &= v_2(f_0) + M_4 f_1 + M_{22}f_2 + \frac{2}{3}(**)f_3 \\
g_{12} &= (v_2 + R)(f_1) + Sf_2 + 2M_{22}f_3 \\
g_{22} &= (v_2 - R)(f_2) + f_0 - Qf_1 - 2M_4 f_3 \\
g_{32} &= v_2(f_3) + \frac{1}{2}f_1.
\end{aligned}
$$

$(*)$ and $(**)$ read as

$$
\begin{aligned}
(*) &= -v_1(M_4) + v_2(M_{11}) + 2QM_4 - RM_{11} - PM_{22} \\
(**) &= -v_1(M_{22}) + v_2(M_4) - SM_{11} + 2RM_4 - QM_{22}.
\end{aligned}
$$

The idea how these operators V_1 and V_2 came out is the following: suppose the equation (Sp) has a 4-dimensional solution space, and fix a non-zero solution ϑ. If we consider the module over the ring of functions on (x, y, z) with ϑ, $v_1(\vartheta)$, $v_2(\vartheta)$, $v_3(\vartheta)$ as a formal basis;

$$\{f_0\vartheta + f_1 v_1(\vartheta) + f_2 v_2(\vartheta) + f_3 v_3(\vartheta) \mid f_i\colon \text{ functions on } (x, y, z)\},$$

then on this module the operators v_1 and v_2 operate naturally. Reminding that ϑ is a solution of (Sp), and using g_{ji}, we actually calculate the operations as

$$
\begin{aligned}
& v_i\big(f_0\vartheta + f_1 v_1(\vartheta) + f_2 v_2(\vartheta) + f_3 v_3(\vartheta)\big) \\
&\quad = g_{0i}\vartheta + g_{1i}v_1(\vartheta) + g_{2i}v_2(\vartheta) + g_{3i}v_3(\vartheta).
\end{aligned}
$$

Here we used the equalities

$$
\begin{aligned}
v_2 v_1(\vartheta) &= \frac{1}{2}\big(v_4(\vartheta) + v_3(\vartheta)\big) \\
v_1 v_2(\vartheta) &= \frac{1}{2}\big(v_4(\vartheta) - v_3(\vartheta)\big) \\
v_1 v_2 v_1(\vartheta) &= \frac{1}{3}\big(v_2 {v_1}^2(\vartheta) + v_1 v_4(\vartheta)\big) \\
v_2 v_1 v_2(\vartheta) &= \frac{1}{3}\big(v_2 {v_1}^2(\vartheta) + v_2 v_4(\vartheta)\big)
\end{aligned}
$$

$$v_3v_1(\vartheta) = v_1v_3(\vartheta) = \frac{1}{3}\big(2v_2{v_1}^2(\vartheta) - v_1v_4(\vartheta)\big)$$
$$v_3v_2(\vartheta) = v_2v_3(\vartheta) = \frac{1}{3}\big(-2v_1{v_2}^2(\vartheta) + v_2v_4(\vartheta)\big).$$

Proposition 7.1. *Define the operator* V_3 *by* $V_3 = [V_2, V_1]$. *The following three conditions on the four functions* P, Q, R *and* S *are equivalent*:

(a) *the PDE system* (Sp) *is integrable, that is, the solution space is of dimension equal to* 4,
(b) $[V_3, V_i] = 0, \quad (i = 1,\ 2)$, *and*
(c) P, Q, R *and* S *satisfy the relations* (IC).

If ϑ_i $(i = 0,\ 1,\ 2,\ 3)$ are 4 independent solutions of (Sp), then the vectors

$$\big(\vartheta_i(p), v_1(\vartheta_i)(p), v_2(\vartheta_i)(p), v_3(\vartheta_i)(p)\big), \quad (i = 0,\ 1,\ 2,\ 3)$$

are linearly independent at each point $p = (x, y, z)$. Thus the dimension of the solution space of (Sp) is at most 4.

The higher derivatives $v_{i_1}v_{i_2}\cdots v_{i_n}(\vartheta)(p)$ can be expressed as unique linear combinations of $\{\vartheta(p), v_1(\vartheta)(p), v_2(\vartheta)(p), v_3(\vartheta)(p)\}$. Those coefficients are uniquely determined, provided the operators V_1 and V_2 satisfy the Heisenberg relation $[V_3, V_i] = 0$.

The equivalence of the conditions (a) and (b) is deduced from these facts and the definition of V_i. The equivalence of (b) and (c) is deduced by the following calculation: for elements

$$e_0 = (1,0,0,0) \qquad e_1 = (0,1,0,0)$$
$$e_2 = (0,0,1,0) \qquad e_3 = (0,0,0,1)$$

of $\mathcal{F}$, we define functions a_{ij}^k by

$$[V_3, V_i](e_j) = \sum_{k=0}^{3} a_{ij}^k e_k, \qquad (i = 1,\ 2, \quad j = 0,\ 1,\ 2,\ 3).$$

All these functions are polynomials of P, Q, R, S and their derivatives.

Let c_i $(i = 0,\ 1,\ 2,\ 3)$ be functions of (x, y, z) defined by

$$c_0 = -6\big(-v_3(P) + 2(v_1 - 2Q)(M_{11}) + 4PM_4\big)$$
$$c_1 = -4\big(-3v_3(Q) + 2(v_2 - 4R)(M_{11}) + 4(v_1 + Q)(M_4) + 4PM_{22}\big)$$
$$c_2 = 6 - 2\big(-3v_3(R) + 2(v_1 + 4Q)(M_{22}) + 4(v_2 - R)(M_4) - 4SM_{11}\big)$$
$$c_3 = -6\big(-v_3(S) + 2(v_2 + 2R)(M_{22}) - 4SM_4\big),$$

so that the condition (IC) amounts to the condition $c_0 = c_1 = c_2 = c_3 = 0$. Our purpose is to explain a_{ij}^k as polynomials of the derivatives of c_0, c_1, c_2 and c_3.

The equalities $a_{i0}^k = 0$ ($\forall i$, $\forall k$) are obvious, since we have $[v_3, v_i] = 0$. The other functions are as follows:

$$
\begin{aligned}
a_{11}^0 &= \frac{1}{12}\big(-v_1(c_1) + v_2(c_0) - 2Rc_0 + 4Qc_1 - 2Pc_2\big),\\
a_{11}^1 &= c_1, \qquad a_{11}^2 = c_0,\\
a_{12}^0 &= \frac{1}{12}\big(-v_1(c_2) + v_2(c_1) - Sc_0 + Rc_1 + Qc_2 - Pc_3\big),\\
a_{12}^1 &= c_2, \qquad a_{12}^2 = c_1,\\
a_{13}^0 &= 2\big(-v_1(a_{12}^0) + v_2(a_{11}^0) - Pa_{22}^0 + 2Qa_{12}^0 - Ra_{11}^0\big)\\
&\quad -\frac{1}{3}\big(M_{22}c_0 - 2M_4c_1 + M_{11}c_2\big),\\
a_{13}^1 &= 2a_{12}^0, \qquad a_{13}^2 = -2a_{11}^0,\\
a_{21}^0 &= a_{12}^0, \qquad a_{21}^1 = c_2, \qquad a_{21}^2 = c_1,\\
a_{22}^0 &= \frac{1}{12}\big(-v_1(c_3) + v_2(c_2) - 2Sc_1 + 4Rc_2 - 2Qc_3\big),\\
a_{22}^1 &= c_3, \qquad a_{22}^2 = c_2,\\
a_{23}^0 &= 2\big(-v_1(a_{22}^0) + v_2(a_{12}^0) - Qa_{22}^0 + 2Ra_{12}^0 - Sa_{11}^0\big)\\
&\quad -\frac{1}{3}\big(M_{22}c_1 - 2M_4c_2 + M_{11}c_3\big),\\
a_{23}^1 &= 2a_{22}^0, \qquad a_{23}^2 = 2a_{12}^0,\\
a_{ij}^3 &= 0 \quad (\forall i,\ j = 0, \ldots, 3),
\end{aligned}
$$

all of which are polynomials of the derivatives of c_0, c_1, c_2, c_3. The necessary and sufficient condition for (b) is the vanishings of all $a_{ij}^k = 0$. Therefore we established the equivalence of (b) and (c).

§8. Connection formula of contact Schwarzian derivatives

In this section, we use the notation $(x, y, z) = (x^1, x^3, x^2)$ for the coordinate functions. The superscript is unusual, but this simplifies the expressions that follow. And then the contact form on $\mathbb{K}^3$ is $dx^3 - x^2\,dx^1$, and three vector fields v_1, v_2 and v_3 are

$$
v_1 = \frac{\partial}{\partial x^1} + x^2\frac{\partial}{\partial x^3}, \quad v_2 = \frac{\partial}{\partial x^2} \quad \text{and} \quad v_3 = \frac{\partial}{\partial x^3}.
$$

Proposition 8.1. *The contact Schwarzian derivatives of a composition of contact transformations* $(x^1, x^3, x^2) \mapsto (y^1, y^3, y^2) \mapsto (z^1, z^3, z^2)$

satisfy the formula

$$(10)\quad S_{\{ijk\}}(z,x) = S_{\{ijk\}}(y,x) + \sum_{p,q,r=1}^{2} S_{\{pqr\}}(z,y)\frac{v_i(y^p)v_j(y^q)v_k(y^r)}{\Delta(y,x)}.$$

In the rest of this section, we prove the above proposition. We use the following notations:

$$v_1 = \frac{\partial}{\partial x^1} + x^2\frac{\partial}{\partial x^3}, \quad v_2 = \frac{\partial}{\partial x^2},$$
$$u_1 = \frac{\partial}{\partial y^1} + y^2\frac{\partial}{\partial y^3}, \quad u_2 = \frac{\partial}{\partial y^2}.$$

For a contact transformation $(x^1, x^3, x^2) \mapsto (y^1, y^3, y^2)$ and a function $f(y^1, y^3, y^2)$, it holds that $v_i(f) = \sum_{p=1}^{2} u_p(f)v_i(y^p)$ (see Lemma 3.5), and differentiating this by v_j, it holds that

$$v_jv_i(f) = \sum_{p=1}^{2} u_p(f)v_jv_i(y^p) + \sum_{p,q=1}^{2} u_qu_p(f)v_j(y^q)v_i(y^p).$$

Using this, we calculate $s_{[ij,k]}(z,x)$ as follows:

$$\begin{aligned} s_{[ij,k]}(z,x) &= v_iv_j(z^1)v_k(z^2) - v_iv_j(z^2)v_k(z^1) \\ &= \sum_{p,q} \big(u_p(z^1)u_q(z^2) - u_p(z^2)u_q(z^1)\big)v_iv_j(y^p)v_k(y^q) \\ &\quad + \sum_{p,q,r} s_{[pq,r]}(z,y)v_i(y^p)v_j(y^q)v_k(y^r). \end{aligned}$$

Concerning the first summation of the last expression, the sum for $(p,q) = (1,1)$ and $(2,2)$ is equal to 0, and for $(p,q) = (1,2)$ and $(2,1)$, it is equal to

$$\Delta(z,y)s_{[ij,k]}(y,x).$$

By summing up them for (i,j,k) cyclically permuted, and dividing it by 3, we thus get

$$(11)\qquad \Delta(z,x)S_{\{ij,k\}}(y,x),$$

which is the first term of the expression (10).

For the second term

$$\sum_{p,q,r} s_{[pq,r]}(z,y)v_i(y^p)v_j(y^q)v_k(y^r),$$

we calculate the sum of them for (i,j,k) cyclically permuted as follows:

(i) for $(p,q,r)=(1,1,1)$, the summand is equal to

$$3\Delta(z,y)S_{\{111\}}(z,y)v_i(y^1)v_j(y^1)v_k(y^1),$$

where we used the formula $\Delta(z,x)=\Delta(z,y)\Delta(y,x)$ (Lemma 3.6).

(ii) the sum over $(p,q,r)=(1,1,2)$, $(1,2,1)$ and $(2,1,1)$ is equal to

$$3\Delta(z,y)S_{\{112\}}(z,y)\times \\ \big(v_i(y^2)v_j(y^1)v_k(y^1)+v_i(y^1)v_j(y^2)v_k(y^1)+v_i(y^1)v_j(y^1)v_k(y^2)\big),$$

(iii) the sum over $(p,q,r)=(1,2,2)$, $(2,1,2)$ and $(2,2,1)$ is calculated in the same way as above, and

(iv) for $(p,q,r)=(2,2,2)$, it is equal to

$$3\Delta(z,y)S_{\{222\}}(z,y)v_i(y^2)v_j(y^2)v_k(y^2).$$

Adding up them with (11), we finally get the formula (10).

§9. Contact transformations with vanishing contact Schwarzian derivatives

We regard $\mathbb{K}^4$ a linear symplectic space with the symplectic form ω defined by

$$\begin{aligned}\omega((a_0,a_1,a_2,a_3),(b_0,b_1,b_2,b_3)) &= (a_0,a_1,a_2,a_3)J(b_0,b_1,b_2,b_3)^t \\ &= a_0b_3-a_3b_0+a_1b_2-a_2b_1,\end{aligned}$$

where the complex structure J is defined in (8).

Let ξ_i $(i=0,\dots,3)$ be functions defined by

$$\begin{aligned}&\xi_0(x,y,z)=1, && \xi_1(x,y,z)=x,\\ &\xi_2(x,y,z)=z, && \xi_3(x,y,z)=2y-xz.\end{aligned}$$

Then we easily verify that these functions form a symplectic basis of the solution space $\mathcal{S}(0,0,0,0)$ of the PDE system (Sp) with $P=Q=R=S=0$. Using these functions ξ_i, we define a map $\varphi\colon \mathbb{K}^3\to\mathbb{K}^4$ by

$$\varphi\colon (x,y,z)\mapsto(\xi_0,\xi_1,\xi_2,\xi_3). \tag{12}$$

The contact structure of $\mathbb{K}^3$ and the symplectic structure of $\mathbb{K}^4$ are related to each other via the map φ as follows: for each point $p\in\mathbb{K}^3$, the pull back of the skew orthogonal plane $\varphi(p)^\perp$ coincides with the contact distribution at p. This follows from the equalities

$$\omega\big(\varphi(p),v_i(\varphi)(p)\big)=0, \qquad (i=1,\ 2),$$

and implies the following

Lemma 9.1. *The projectification of the $Sp_2(\mathbb{K})$-action on $\mathbb{K}^4$ induces a contact transformation of $\mathbb{K}^3$ (or on a domain in $\mathbb{K}^3$) through the map φ.*

The image of φ is an affine 3-space that does not pass the origin of $\mathbb{K}^4$. Therefore, if a Lagrangian plane intersects with the image of φ, the inverse image of the Lagrangian plane by φ is a Legendrian curve in $\mathbb{K}^3$. Moreover it follows that

Lemma 9.2. *The inverse image of a Lagrangian plane by φ is a graph*

$$x \mapsto (x, ax^2 + 2bx + c, 2ax + 2b)$$

of a quadratic function $y = ax^2 + 2bx + c$, provided it is nonempty.

Proof. It suffices to consider Lagrangian planes that are the images of linear maps

$$(u, v) \mapsto \big(u, v, 2(bu + av), 2(cu + bv)\big).$$

The inverse image of those planes by φ are described by

$$x \mapsto \big(x,\ y = ax^2 + 2bx + c,\ z = 2(ax + b)\big).$$

and actually they satisfy $dy/dx = z$. Q.E.D.

The symplectic structure on $\mathbb{K}^4$ and the standard contact structure on $P^3(\mathbb{K})$ are related as follows: let a map $\psi\colon \mathbb{K}^3 \to \mathbb{K}^4$ be central (here "central" mean that, at each point p, the line through the origin of $\mathbb{K}^4$ and $\psi(p)$ is transversal to the tangent space of the image of ψ at $\psi(p)$). If ψ mediates between the contact structure on the source and the symplectic structure on the target as the map φ mediates, then the projectification of ψ is a contact map $\mathbb{K}^3 \to P^3(\mathbb{K})$.

In order to prove the following lemma, it suffices to use the same argument as in the proof of Lemma 6.3.

Lemma 9.3. *Let $\{\eta_0, \eta_1, \eta_2, \eta_3\}$ be a symplectic basis of the solution space $\mathcal{S}(0,0,0,0)$. Then the map*

$$(x, y, z) \mapsto [\eta_0; \eta_1; \eta_2; \eta_3] \in P^3(\mathbb{K})$$

is contact.

We denote by $\mathrm{Cont}(P^3(\mathbb{K}))$ the group of contact isomorphisms of the projective space $P^3(\mathbb{K})$.

$$\Phi\colon CSp_2(\mathbb{K}) \to \mathrm{Cont}(P^3(\mathbb{K})).$$

The kernel of the homomorphism consists of the scalar matrices;

$$\ker(\Phi) = \mathbb{K} \cdot I_4.$$

Thus the image of Φ is isomorphic to

$$\Phi(CSp_2(\mathbb{K})) = \begin{cases} Sp_2(\mathbb{K})/\{\pm I_4\} \times \{I_4, I'\} & (\mathbb{K} = \mathbb{R}) \\ Sp_2(\mathbb{K})/\{\pm I_4\} & (\mathbb{K} = \mathbb{C}), \end{cases}$$

where I' denotes the diagonal matrix $\mathrm{diag}(1, 1, -1, -1)$.

We denote by $[\varphi]\colon \mathbb{K}^3 \to P^3(\mathbb{K})$ the contact map obtained by projectifying the map φ in (12).

Proposition 9.4. *If a map $\phi\colon D \to D'$ is a contact transformation with Schwarzian derivative $S(\phi) = 0$, then there exists a contact transformation $[g] \in \mathrm{Im}(\Phi)$ that commutes the following diagram;*

$$\begin{tikzcd} D \arrow[r, "\phi"] \arrow[d, "{[\varphi]}"'] & D' \arrow[d, "{[\varphi]}"] \\ P^3(\mathbb{K}) \arrow[r, "{[g]}"] & P^3(\mathbb{K}) \end{tikzcd}$$

Proof. Let ϕ be a contact transformation with Schwarzian derivative $S(\phi) = 0$. It follows, from Theorem 6.1, that there exists a symplectic basis $\{\eta_0, \eta_1, \eta_2, \eta_3\}$ of the solution space $\mathcal{S}(0, 0, 0, 0)$ such that ϕ is given by the equation (7) for these functions η_i. Then we find an element $g = (g_i^j) \in CSp_2(\mathbb{K})$ such that

$$\eta_i = \sum g_i^j \xi_j,$$

where ξ_j are the functions given as above. The projectification $\Phi(g) = [g]$ satisfies the required property. Q.E.D.

Thus we obtain the

Theorem 9.5. *The set of all contact transformations whose contact Schwarzian derivatives vanish forms a Lie group isomorphic to $Sp_2(\mathbb{K})/\{\pm I_4\}$ if $\mathbb{K} = \mathbb{C}$, and to $Sp_2(\mathbb{K})/\{\pm I_4\} \times \{I_4, I'\}$ if $\mathbb{K} = \mathbb{R}$.*

As is stated in Introduction, if we transform the ordinary differential equation $y''' = 0$ by a contact transformation $\phi = (X, Y, Z)$, we get the equation

$$y''' = P + 3Q(y'') + 3R(y'')^2 + S(y'')^3,$$

where the functions P, Q, R, S on (x, y, z) form the Schwarzian derivative of ϕ; $S(\phi) = (P, Q, R, S)$, and the variable z stands for $y' = dy/dx$.

Therefore, if the Schwarzian derivative (P, Q, R, S) of $\phi\colon D \to D'$ vanishes, the map ϕ transforms the set of graphs on D of all quadratic functions $y = ax^2 + 2bx + c$ into that on D'. The inverse implication is also true, because the solution space determines the differential equation. Thus we obtain

Corollary 9.6. *If $\phi\colon D \to D'$ is a diffeomorphism between two domains D and D' in $\mathbb{K}^3$ that gives a one-to-one correspondence on the sets of graphs of quadratic functions in D and D', then ϕ is a restriction of the projectification of a linear symplectic isomorphism on $\mathbb{K}^4$ through the embedding $\varphi(x, y, z) = (1, x, z, 2y - xz)$, where we regarded the map gI' with $g \in Sp_2(\mathbb{R})$ to be symplectic in case $\mathbb{K} = \mathbb{R}$.*

Schwarzian derivative of contact transformations on contact manifolds that are projectively flat, like $P^3(\mathbb{K})$, will be discussed in a forthcoming paper [O-S2].

References

[Gun] R. Gunning, On uniformization of complex manifolds: the role of connections, Math. Notes No. 22, Princeton. Princeton University Press, 1978.

[O-S1] T. Ozawa and H. Sato, *Linearizations of ordinary differential equations by area preserving maps*, Nagoya Math. J., **156** (1999), 109–122.

[O-S2] T. Ozawa and H. Sato, *Schwarzian derivative of global transformations and cocycles in continuous group cohomology*, in preparation.

[Sat] H. Sato, *Schwarzian derivatives of contact diffeomorphisms*, *Schwarzian derivatives of contact diffeomorphisms*, Lobachevskii J. Math., **4** (1999), 89–98.

[S-Y] H. Sato and A. Y. Yoshikawa, *Third order ordinary differential equations and Legendre connections*, J. Math. Soc. Japan, **50** (1998), 993–1013.

[Yos] M. Yoshida, Fuchsian differential equations, Aspects of Mathematics, Vieweg, Baunschweig, 1987.

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Isometric Immersions into Complex Projective Space

Per Tomter

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Geometry of Higher Order Differential Equations of Finite Type Associated with Symmetric Spaces

Keizo Yamaguchi and Tomoaki Yatsui

Introduction

This is a half survey article on the geometry of higher order differential equations of finite type associated with symmetric spaces.

Historically, geometric study of differential equations, especially ordinary differential equations, was initiated by Sophus Lie [Lie91]. For linear ordinary differential equations, Laguerre and Forsyth studied the differential invariants of these equations by transforming them to the canonical forms (cf. [Wil06]).

For higher order equations, after Lie, the classification of the second order ordinary differential equations by point transformations was achieved by Tresse [Tre96] and E. Cartan [Car24] studied the case when the equation is associated with paths in projective geometry by his method of the equivalence. The third order equations were studied by S. S. Chern [Ch50], following the method of E. Cartan (cf. [SY98]). Then N. Tanaka [Tan82] studied the equivalence problem for the system of second order ordinary differential equations by point transformations and formulated this geometry in terms of the pseudo-product structures. Furthermore he constructed normal Cartan connections on these systems and utilized the connections to the normal form problem and the integration problem of these systems [Tan79], [Tan89].

For the geometrization problem for the equivalence of ordinary differential equations with some historical comments, we refer the reader to the excellent survey article [DKM99].

In this paper we adopt the point of view initiated by N. Tanaka. Let us consider a system of higher order differential equations of finite

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type of the following form:

$$\frac{\partial^k y^\alpha}{\partial x_{i_1} \cdots \partial x_{i_k}} = F^\alpha_{i_1 \dots i_k}(x_1, \dots, x_n,\ y^1, \dots, y^m, \dots, p^\beta_i, \dots, p^\beta_{j_1 \dots j_{k-1}})$$
$$(1 \leqq \alpha \leqq m,\ 1 \leqq i_1 \leqq \cdots \leqq i_k \leqq n),$$

where $p^\beta_{i_1 \dots i_l} = \frac{\partial^l y^\beta}{\partial x_{i_1} \cdots \partial x_{i_l}}$. Namely let us consider a system R of k-th order equations such that every k-th derivative is expressed in terms of the derivatives of the lower order. If we regard R as a submanifold of the k-jet space J^k with coordinates

$$(x_1, \dots, x_n,\ y^1, \dots, y^m, \dots, p^\beta_i, \dots, p^\beta_{j_1 \dots j_k}),$$

then R is diffeomorphic to J^{k-1}. Moreover R specifies an n-dimensional subspace $E(x)$ of $C^{k-1}(x)$ at each point x of J^{k-1}, where C^{k-1} is the **canonical differential system** on J^{k-1} (for the precise definitions, see §1). Thus R defines a differential system E on J^{k-1} such that $C^{k-1} = E \oplus F$, where $F = \operatorname{Ker}(\pi^{k-1}_{k-2})_*$ and $\pi^{k-1}_{k-2} \colon J^{k-1} \to J^{k-2}$ is the projection. E is completely integrable when the system R is integrable. The triplet $(J^{k-1}; E, F)$ is called the **pseudo-product structure** associated with R.

Our basic strategy to study higher order differential equations R of finite type is to utilize the 'rich' geometry of the differential system $C^{k-1} = E \oplus F$ naturally associated with R. We further pursue this approach to an important class of higher order differential equations of finite type, which is called of type $(\mathfrak{l}, S)$.

Now let us proceed to describe the contents of each section. In §1, we will explain the pseudo-product structure $(R; E, F)$ associated with a system R of higher order differential equations of finite type and give an overview on the Tanaka theory for regular differential systems. Especially we will review the **symbol algebra** $\mathfrak{m}(x) = \bigoplus_{p<0} \mathfrak{g}_p(x)$ of a regular differential system (M, D) and the notion of the (algebraic) **prolongation** $\mathfrak{g}(\mathfrak{m})$ (resp. $\mathfrak{g}(\mathfrak{m}, \mathfrak{g}_0)$) of $\mathfrak{m}$ (resp. $(\mathfrak{m}, \mathfrak{g}_0)$), for a given fundamental graded Lie algebra $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$, where $\mathfrak{g}_0$ is a subalgebra of the (gradation preserving) derivation algebra$\mathfrak{g}_0(\mathfrak{m})$ of $\mathfrak{m}$. $\mathfrak{g}(\mathfrak{m})$ represents the Lie algebra of infinitesimal automorphisms of the standard differential system $(M(\mathfrak{m}), D_\mathfrak{m})$ of type $\mathfrak{m}$, which is the local model differential system of type $\mathfrak{m}$. As an example we will calculate the symbol algebra $\mathfrak{C}^k(n, m)$ of the canonical differential system (J^k, C^k) in §1.3 and will show that this algebra has the following description:

$$\mathfrak{C}^k(n, m) = \mathfrak{C}_{-(k+1)} \oplus \mathfrak{C}_{-k} \oplus \cdots \oplus \mathfrak{C}_{-1},$$

where $\mathfrak{C}_{-(k+1)} = W$, $\mathfrak{C}_p = W \otimes S^{k+p+1}(V^*)$, $\mathfrak{C}_{-1} = V \oplus W \otimes S^k(V^*)$. Here V and W are vector spaces of dimension n and m respectively and the bracket product of $\mathfrak{C}^k(n,m) = \mathfrak{C}^k(V,W)$ is defined accordingly through the pairing between V and V^* such that V and $W \otimes S^k(V^*)$ are both abelian subspaces of $\mathfrak{C}_{-1}$. Here $S^k(V^*)$ denotes the k-th symmetric product of V^*.

Corresponding to the splitting $C^{k-1} = E \oplus F$ of the pseudo-product structure, we have the splitting in the symbol algebra $\mathfrak{C}^{k-1}(n,m)$;

$$\mathfrak{C}_{-1} = \mathfrak{e} \oplus \mathfrak{f},$$

where $\mathfrak{e} = V$, $\mathfrak{f} = W \otimes S^{k-1}(V^*)$. In §2 we first consider the prolongation $\mathfrak{g}^k(n,m)$ of $(\mathfrak{C}^{k-1}(n,m), \check{\mathfrak{g}}_0)$, where $\check{\mathfrak{g}}_0$ is the subalgebra of $\mathfrak{g}_0(\mathfrak{C}^{k-1}(n,m))$ consisting of elements which preserves both $\mathfrak{e}$ and $\mathfrak{f}$. $\mathfrak{g}^k(n,m)$ is called the **pseudo-projective GLA** (graded Lie algebra) of order k of bidegree (n,m). We will give the explicit description of these algebras in §2.1. $\mathfrak{g}^k(n,m)$ gives the Lie algebra of infinitesimal automorphisms of the (local) model k-th order differential equation R_o of finite type, where

$$R_o = \left\{ \frac{\partial^k y^\alpha}{\partial x_{i_1} \cdots \partial x_{i_k}} = 0 \quad (1 \leqq \alpha \leqq m,\ 1 \leqq i_1 \leqq \cdots \leqq i_k \leqq n) \right\}.$$

Generalizing the structure of $\mathfrak{g}^k(n,m)$, we will now introduce the important class of pseudo-product GLA of the irreducible type. Namely, starting from a reductive GLA $\mathfrak{l} = \mathfrak{l}_{-1} \oplus \mathfrak{l}_0 \oplus \mathfrak{l}_1$ and a faithful irreducible $\mathfrak{l}$-module S, we define the pseudo-product GLA $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ of type $(\mathfrak{l}, S)$ as follows: Let $\mathfrak{l} = \mathfrak{l}_{-1} \oplus \mathfrak{l}_0 \oplus \mathfrak{l}_1$ be a finite dimensional **reductive GLA** of the first kind such that

(1) The ideal $\hat{\mathfrak{l}} = \mathfrak{l}_{-1} \oplus [\mathfrak{l}_{-1}, \mathfrak{l}_1] \oplus \mathfrak{l}_1$ of $\mathfrak{l}$ is a simple Lie algebra.
(2) The center $\mathfrak{z}(\mathfrak{l})$ of $\mathfrak{l}$ is contained in $\mathfrak{l}_0$.

Let S be a finite dimensional **faithful irreducible $\mathfrak{l}$-module**. We put

$$S_{-1} = \{ s \in S \mid \mathfrak{l}_1 \cdot s = 0 \}$$

and

$$S_p = \mathrm{ad}(\mathfrak{l}_{-1})^{-p-1} S_{-1} \quad \text{for } p < 0.$$

We form the semi-direct product $\mathfrak{g}$ of $\mathfrak{l}$ by S, and put

$$\mathfrak{g} = S \oplus \mathfrak{l}, \quad [S,S] = 0$$
$$\mathfrak{g}_k = \mathfrak{l}_k \ (k \geqq 0), \quad \mathfrak{g}_{-1} = \mathfrak{l}_{-1} \oplus S_{-1},$$
$$\mathfrak{g}_p = S_p \ (p < -1).$$

Then $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ enjoys the following properties (Lemma 2.1);

(1) $S = \bigoplus_{p=-1}^{-\mu} S_p$, *where* $S_{-\mu} = \{s \in S \mid [\mathfrak{l}_{-1}, s] = 0\}$.
(2) $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ is generated by $\mathfrak{g}_{-1}$.
(3) S_p *is naturally embedded as a subspace of* $W \otimes S^{\mu+p}(\mathfrak{l}_{-1}{}^*)$ *through the bracket operation in* $\mathfrak{m}$, *where* $W = S_{-\mu}$.

Thus $\mathfrak{m}$ is a graded subalgebra of $\mathfrak{C}^{\mu-1}(V, W)$, which has the splitting $\mathfrak{g}_{-1} = \mathfrak{l}_{-1} \oplus S_{-1}$, where $V = \mathfrak{l}_{-1}$ and $W = S_{-\mu}$. Hence $\mathfrak{m}$ is a symbol algebra of μ-th order differential equations of finite type, which is called the typical symbol of type $(\mathfrak{l}, S)$.

This class of higher order (linear) differential equations of finite type were first appeared in the work of Y. Se-ashi [Sea88], who discussed the linear equivalence of this class of equations and gave the complete system of differential invariants of these equations, generalizing the classical theory of Laguerre-Forsyth for linear ordinary differential equations.

We will ask the following questions for the pseudo-product GLA $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ of type $(\mathfrak{l}, S)$:

(A) *When is* $\mathfrak{g}$ *the prolongation of* $\mathfrak{m}$ *or* $(\mathfrak{m}, \mathfrak{g}_0)$?
(B) *Find the fundamental invariants for equations of type* $\mathfrak{m}$.

Utilizing the Tanaka-Morimoto theory of normal Cartan connections [Tan79] [Mor93], these questions will be answered by calculating the first and second **generalized Spencer cohomology spaces**. The complete answer for (A) will be given as Theorem 5.2 in §5 and the problem (B) will be discussed in §6 and §7.

In §3, we will recall the construction of the **model equations** for the typical symbol of type $(\mathfrak{l}, S)$, following [Sea88]. For simplicity, let us explain this construction here in holomorphic category. Assuming the condition $H^1(\mathfrak{G})_{0,0} = 0$ (see §4.5), we see that the Lie algebra $\mathfrak{l}$ coincides with the Lie algebra of infinitesimal (linear) automorphisms of the local model equations of type $\mathfrak{m}$ (see the discussion in §3.1). Regarding $\mathfrak{l}$ as a subalgebra of $\mathfrak{gl}(S)$, let L and L' be the Lie subgroup of $GL(S)$ with Lie algebra $\mathfrak{l}$ and $\mathfrak{l}'$ respectively, where $\mathfrak{l}' = \mathfrak{l}_0 \oplus \mathfrak{l}_1$. Moreover let $\hat{L}$ be the Lie subgroup of $GL(S)$ with Lie algebra $\hat{\mathfrak{l}}$ and put $\hat{L}' = L' \cap \hat{L}$. Then $\hat{L}'$ is a parabolic subgroup of $\hat{L}$ and $M = L/L' = \hat{L}/\hat{L}'$ is an irreducible compact hermitian symmetric space (cf. §3.2). Since L' preserves the filtration $\{S^q\}_{q<0}$ of S, where $S^q = \bigoplus_{p=-1}^{q} S_p$, we get the representation ρ_W of L':

$$\rho_W : L' \to GL(W),$$

through the projection $\pi_0 : S = \bigoplus_{p=-1}^{-\mu} S_p \to S_{-\mu} = W$.

Let E_S be the vector bundle over $M = L/L'$ associated with the representation $\rho_W : L' \to GL(W)$. As is well known, each $s \in S$ defines a global section σ_s of the vector bundle E_S (see the discussion in §3.1).

Let $J^{\mu}(E_S)$ be the bundle of μ-jets of E_S. At each point $x \in M = L/L'$, let $(R_S)_x$ be the subspace of $J^{\mu}_x(E_S)$ defined by

$$(R_S)_x = \{j^{\mu}_x(\sigma_s) \mid s \in S\}$$

where $j^{\mu}_x(\sigma_s)$ is the μ-jet at x of the section σ_s. Then the model (linear) equation R_S for the typical symbol of type $(\mathfrak{l}, S)$ is defined as the subbundle of $J^{\mu}(E_S)$ by

$$R_S = \bigcup_{x \in M} (R_S)_x.$$

R_S is the system of differential equations of finite type which characterizes global sections of E_S locally.

In §3.2, we will discuss the Plücker embedding equations as our examples of model equations. We assume here that $\mathfrak{l} = \hat{\mathfrak{l}} \oplus \mathfrak{z}(\mathfrak{l})$. Then, a little generally, an equivariant projective embedding of the model space $M = L/L' = \hat{L}/\hat{L}'$ can be obtained from an irreducible representation of $\hat{L}$ as follows: Let $\tau \colon \hat{L} \to GL(T)$ be an irreducible representation of $\hat{L}$ with the highest weight Λ. Let t_Λ be a maximal vector in T of the highest weight Λ. Then a stabilizer of the line $[t_\Lambda]$ spanned by t_Λ in T is a parabolic subgroup of $\hat{L}$. When this stabilizer coincides with $\hat{L}'$, we obtain an equivariant projective embedding of $M = \hat{L}/\hat{L}'$ by taking the $\hat{L}$-orbit passing through $[t_\Lambda]$ in the projective space $P(T)$. In this case, it can be shown that this embedding can be obtained by global sections of the line bundle F which is constructed from the dual representation $\rho = \tau^*$ of $\hat{L}$ on $S = T^*$ (see the discussion in §3.2). Then the model equation R_S for the typical symbol of type $(\mathfrak{l}, S)$, which characterizes global sections of F locally, can be called the embedding equation for M. The case when $\mathfrak{l} = \mathfrak{sl}(l+1, \mathbb{C})$ and $S = \bigwedge^{l-k+1} \mathbb{C}^{l+1}$ corresponds to the **Plücker embedding equations** for the Grassmann manifold $M = \mathrm{Gr}(k, l+1)$.

From §4, we will study the cohomology group $H^*(\mathfrak{G}) = H^*(\mathfrak{m}, \mathfrak{g})$ associated with the adjoint representation of $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ on $\mathfrak{g}$ for the pseudo-product GLA $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ of type $(\mathfrak{l}, S)$. Namely we will study the cohomology space of the cochain complex $C^*(\mathfrak{G}) = \bigoplus C^p(\mathfrak{G})$ with the coboundary operator $\partial \colon C^p(\mathfrak{G}) \longrightarrow C^{p+1}(\mathfrak{G})$ given by

$$\begin{aligned} &\partial^p \omega(x_1, \ldots, x_{p+1}) \\ &= \sum_{i=1}^{p+1} (-1)^{i+1} [x_i, \omega(x_1, \ldots, \hat{x}_i, \ldots, x_{p+1})] \end{aligned}$$

$$+\sum_{i<j}(-1)^{i+j}\omega([x_i,x_j],x_1,\ldots,\hat{x}_i,\ldots,\hat{x}_j,\ldots,x_{p+1}),$$

where $\omega \in C^p(\mathfrak{G}) = \mathrm{Hom}(\bigwedge^p \mathfrak{m}, \mathfrak{g})$ and $x_i \in \mathfrak{m}$. Moreover we put $\mathfrak{b}_{-1} = S$, $\mathfrak{b}_0 = \mathfrak{l}$ and $\mathfrak{b}_p = 0$ ($p \neq -1,\ 0$). We utilize the bigradation $(\mathfrak{g}_{p,q})_{p,q\in\mathbb{Z}}$ of $\mathfrak{g} = S \oplus \mathfrak{l}$ given by $\mathfrak{g}_{p,q} = \mathfrak{g}_p \cap \mathfrak{b}_q$. Then $C^p(\mathfrak{G})$ has the following decomposition: $C^p(\mathfrak{G}) = \bigoplus_{r,s} C^p(\mathfrak{G})_{r,s}$, where

$$\begin{aligned} C^p(\mathfrak{G})_{r,s} = \{\omega \in C^p(\mathfrak{G}) \mid \\ &\omega(\mathfrak{g}_{i_1,j_1}\wedge\cdots\wedge\mathfrak{g}_{i_p,j_p}) \subset \mathfrak{g}_{i_1+\cdots+i_p+r,\ j_1+\cdots+j_p+s} \\ &\qquad\qquad \text{for all} \quad i_1,\ldots,i_p,\ j_1,\ldots,j_p\}. \end{aligned}$$

In §4.3, we prepare the fundamental theorem (Theorem 4.1) to calculate the cohomology space $H^p(\mathfrak{G})_{r,s}$ by means of Kostant's theorem on Lie algebra cohomology.

Now let us describe $(\mathfrak{l}, S)$ in detail, utilizing the structure theory of semi-simple Lie algebras over $\mathbb{C}$. First we decompose $\mathfrak{l} = \hat{\mathfrak{l}} \oplus \mathfrak{z}_{\mathfrak{l}}(\hat{\mathfrak{l}})$, $\mathfrak{l}_0 = \hat{\mathfrak{l}}_0 \oplus \mathfrak{z}_{\mathfrak{l}}(\hat{\mathfrak{l}})$. Since S is a faithful irreducible $\mathfrak{l}$-module, there is an irreducible $\hat{\mathfrak{l}}$-module T and $\mathfrak{z}_{\mathfrak{l}}(\hat{\mathfrak{l}})$-module U such that S is isomorphic to $T \otimes U$ as an $\mathfrak{l}$-module. Then we impose the following condition for $(\mathfrak{l}, S)$: $\mathfrak{z}_{\mathfrak{l}}(\hat{\mathfrak{l}})$ *is isomorphic to* $\mathfrak{gl}(U)$. We interpret this condition as the condition $H^1(\mathfrak{G})_{0,0} = 0$ for the first cohomology in Lemma 4.5.

Fixing a Cartan subalgebra $\mathfrak{h}$ (the set Φ of roots relative to $\mathfrak{h}$) and a simple root system $\Delta = \{\alpha_1, \ldots, \alpha_l\}$, we have the root space decomposition of the simple Lie algebra $\hat{\mathfrak{l}}$. Then the gradation $\hat{\mathfrak{l}} = \mathfrak{l}_{-1} \oplus \hat{\mathfrak{l}}_0 \oplus \mathfrak{l}_1$ can be described as (X_l, Δ_1) (see §4.4). Here X_l stands for the Dynkin diagram of $\hat{\mathfrak{l}}$, $\Delta_1 = \{\alpha_{i_o}\} \subset \Delta$ and α_{i_o} satisfies $m_{i_o}(\theta) = 1$, where θ is the highest root and m_i is a $\mathbb{Z}$-valued function on Φ defined by $m_i(\alpha) = k_i$ for $\alpha = \sum_{j=1}^{l} k_j\alpha_j \in \Phi$. Namely the gradation $\hat{\mathfrak{l}} = \mathfrak{l}_{-1} \oplus \hat{\mathfrak{l}}_0 \oplus \mathfrak{l}_1$ is given as the decomposition by root spaces according to the height $m_{i_o}(\alpha)$ of each root α, for suitable choices of $\mathfrak{h}$ and Δ. Thus $(\mathfrak{l}, S)$ can be described by the triplet $(\mathcal{D}(\mathfrak{l}), \Delta_1, \Xi)$ as follows, where $\mathcal{D}(\mathfrak{l})$ denotes the derived algebra $[\mathfrak{l}, \mathfrak{l}]$ of $\mathfrak{l}$: $(\mathcal{D}(\mathfrak{l}), \Delta_1)$ is of type $(X_l \times A_n, \{\alpha_i\})$ and S is an irreducible $D(\mathfrak{l})$-module with highest weight $\Xi = \chi + \pi_1$ when $\dim U > 1$ and $(\mathcal{D}(\mathfrak{l}), \Delta_1)$ is of type $(X_l, \{\alpha_i\})$ and S is an irreducible $\hat{\mathfrak{l}}$-module with highest weight $\Xi = \chi$ when $\dim U = 1$. Here π_1 is the fundamental weight corresponding to the identity representation on U of $\mathfrak{z}_{\mathfrak{l}}(\hat{\mathfrak{l}}) = \mathfrak{gl}(U)$ of type A_n and χ is a dominant integral weight of $\hat{\mathfrak{l}}$ of type X_l.

In §5, we will calculate the first cohomology $H^1(\mathfrak{G})$ and give the answer to the question (A) raised in §3 in Theorem 5.2: *For a pseudo-product GLA* $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ *of type* $(\mathfrak{l}, S)$ *satisfying the condition*

$H^1(\mathfrak{G})_{0,0} = 0$, $\mathfrak{g}$ *is the prolongation of* $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ *except for three cases.* Let $\check{\mathfrak{b}} = \bigoplus_{p\in\mathbb{Z}} \check{\mathfrak{b}}_p$ be the prolongation of $\mathfrak{g} = \mathfrak{b}_{-1} \oplus \mathfrak{b}_0$. Then the three exceptional cases correspond to cases: (a) $\dim \check{\mathfrak{b}} < \infty$ and $\check{\mathfrak{b}}_1 \neq 0$, (b) $\dim \check{\mathfrak{b}} = \infty$, (c) $\mathfrak{g}$ is a pseudo-projective GLA. In case (a), $\check{\mathfrak{b}} = \mathfrak{b}_{-1} \oplus \mathfrak{b}_0 \oplus \check{\mathfrak{b}}_1$ becomes a simple GLA containing $\mathfrak{g} = \mathfrak{b}_{-1} \oplus \mathfrak{b}_0$ as a parabolic subalgebra. Theorem 5.2 lists up all these exceptions explicitly in terms of the data $(\mathcal{D}(\mathfrak{l}), \Delta_1, \Xi)$.

We will calculate the second cohomology $H^2(\mathfrak{G})$ in §6. Especially we will enumerate the cases when $H^2(\mathfrak{G})_r \neq 0$ $(r \geqq 1)$ for the above exceptional cases (a) and (b). Finally in §7, we will show the **rigidity theorem** (Theorem 7.5) for the Plücker embedding equations for $M = \mathrm{Gr}(k, l+1)$, when $k = 2$ $(l \geqq 4)$ or $k = 3$ $(l \geqq 6)$.

§1. Differential Systems and Pseudo-Product Structures

1.1. Differential Equations of Finite Type

Let us consider a higher order ordinary differential equation

$$y^{(k)} = F(x, y, y', \dots, y^{(k-1)}),$$

or more generally, a system of higher order differential equations of finite type

$$\frac{\partial^k y^\alpha}{\partial x_{i_1} \cdots \partial x_{i_k}} = F^\alpha_{i_1 \dots i_k}(x_1, \dots, x_n,\ y^1, \dots, y^m, \dots, p^\beta_i, \dots, p^\beta_{j_1 \dots j_{k-1}})$$
$$(1 \leqq \alpha \leqq m,\ 1 \leqq i_1 \leqq \cdots \leqq i_k \leqq n),$$

where $p^\beta_{i_1 \dots i_l} = \frac{\partial^l y^\beta}{\partial x_{i_1} \cdots \partial x_{i_l}}$. These equations define a submanifold R in k-jets space J^k such that the restriction p to R of the bundle projection $\pi^k_{k-1} \colon J^k \to J^{k-1}$ gives a diffeomorphism;

$$p \colon R \to J^{k-1}; \quad \text{diffeomorphism} \tag{1.1}$$

On J^k, we have the Contact (differential) system C^k defined by

$$C^k = \{\varpi^\alpha = \varpi^\alpha_i = \cdots = \varpi^\alpha_{i_1 \dots i_{k-1}} = 0\},$$

where

$$
(1.2)\qquad
\begin{cases}
\varpi^\alpha = dy^\alpha - \sum_{i=1}^n p_i^\alpha \, dx_i, \quad (1 \leqq \alpha \leqq m) \\
\varpi_i^\alpha = dp_i^\alpha - \sum_{j=1}^n p_{ij}^\alpha \, dx_j, \quad (1 \leqq \alpha \leqq m, \ 1 \leqq i \leqq n) \\
\cdots\cdots\cdots\cdots, \\
\varpi_{i_1\dots i_{k-1}}^\alpha = dp_{i_1\dots i_{k-1}}^\alpha - \displaystyle\sum_{j=1}^n p_{i_1\dots i_{k-1}j}^\alpha \, dx_j \\
\qquad\qquad (1 \leqq \alpha \leqq m, \ 1 \leqq i_1 \leqq \cdots \leqq i_{k-1} \leqq n).
\end{cases}
$$

Then C^k gives a foliation on R when R is integrable. Namely the restriction E' of C^k to R is completely integrable.

Thus, through the diffeomorphism (1.1), R defines a completely integrable differential system $E = p_*(E')$ on J^{k-1} such that

$$C^{k-1} = E \oplus F, \quad F = \operatorname{Ker}(\pi_{k-2}^{k-1})_*$$

where $\pi_{k-2}^{k-1}\colon J^{k-1} \to J^{k-2}$ is the bundle projection.

To treat with this situation, N. Tanaka ([Tan85]) introduced the notion of pseudo-product manifolds as follows.

Pseudo-Product Manifolds $(R; E, F)$

(1) E and F are differential systems on a manifold R.

(2) $E \cap F = 0$, and both E and F are completely integrable.

(3) $D = E \oplus F$ is non-degenerate.

(4) The full derived systems of D coincides with $T(R)$

In fact he studied ([Tan82], [Tan89]), in this setting, the geometry of systems of second order ordinary differential equations in depth, utilizing his theory of Cartan connections associated with simple graded Lie algebras $\mathfrak{sl}(m+2, \mathbb{R}) = \bigoplus_{p=-2}^{2} \mathfrak{g}_p$.

More generally let R be an involutive system of k-th order differential equations of finite type. Let $R^{(1)} \subset J^{k+1}$ be the first prolongation of R (cf. [Yam82] for a precise definition of involutive systems). The condition "R is involutive and of finite type" implies that $p^{(1)}\colon R^{(1)} \to R$ is a diffeomorphism (cf. Lemma 7.3 [Yam82]). This implies that $\operatorname{Ker} p_* = \{0\}$ and E is completely integrable, where $p = \pi_{k-1}^k|_R$ and $E = C^k|_R$. Then, as in the above example, by putting $F = \operatorname{Ker}(\pi)_*$, $(R; E, F)$ enjoys the properties (1) and (2) above, where $\pi = \pi_{k-2}^{k-1} \circ p$. Here $D = E \oplus F$ is the pull back of the canonical system C^{k-1} by p_*. We call $(R; E, F)$ the pseudo-product structure associated with R in the broader sense.

Our basic strategy to study (involutive) higher order differential equations R of finite type is to utilize the 'rich' geometry of the differential system $D = E \oplus F$ naturally associated with R.

1.2. Geometry of Differential Systems (Tanaka Theory)

We summarize here the basic notion for (linear) differential systems following [Tan70] and [Yam93].

1.2.1. Derived Systems and Characteristic Systems. For a manifold M of dimension d, a subbundle $D \subset T(M)$ of rank r $(s+r=d)$ is called a **differential system** of rank r (or codimension s).

$$D = \{\omega_1 = \cdots = \omega_s = 0\}.$$

For two differential systems (M, D) and $(\hat{M}, \hat{D})$, a diffeomorphism φ of M onto $\hat{M}$ is called an **isomorphism** of (M, D) onto $(\hat{M}, \hat{D})$ if the differential map φ_* of φ sends D onto $\hat{D}$.

By the Frobenius theorem, we know that D is **completely integrable** if and only if

$$d\omega_i \equiv 0 \pmod{\omega_1, \ldots, \omega_s} \qquad \text{for } i = 1, \ldots, s,$$

or equivalently, if and only if

$$[\mathcal{D}, \mathcal{D}] \subset \mathcal{D}.$$

where $\mathcal{D} = \Gamma(D)$ denotes the space of sections of D.

Thus, for a non-integrable differential system D, the **derived system** ∂D of D is defined, in terms of sections, by

$$\partial\mathcal{D} = \mathcal{D} + [\mathcal{D}, \mathcal{D}].$$

The **Cauchy characteristic system** $\mathrm{Ch}(D)$ of (M, D) is defined at each point $x \in M$ by

$$\begin{aligned}\mathrm{Ch}(D)(x) = \{X \in D(x) \mid \\ X \rfloor d\omega_i \equiv 0 \pmod{\omega_1, \ldots, \omega_s} \quad \text{for } i = 1, \ldots, s\},\end{aligned}$$

Then $\mathrm{Ch}(D)$ is always completely integrable when it is a subbundle (i.e. has constant rank) (cf. [Yam82]).

Moreover higher derived systems $\partial^k D$ are usually defined successively by

$$\partial^k D = \partial(\partial^{k-1} D),$$

where we put $\partial^0 D = D$ for convention.

On the other hand we define the k-th weak derived system $\partial^{(k)} D$ of D inductively by

$$\partial^{(k)}\mathcal{D} = \partial^{(k-1)}\mathcal{D} + [\mathcal{D}, \partial^{(k-1)}\mathcal{D}],$$

where $\partial^{(0)}D = D$ and $\partial^{(k)}\mathcal{D}$ denotes the space of sections of $\partial^{(k)}D$. This notion is one of the key point in the Tanaka theory ([Tan70]).

A differential system (M, D) is called **regular**, if $D^{-(k+1)} = \partial^{(k)}D$ are subbundles of $T(M)$ for every integer $k \geqq 1$. For a regular differential system (M, D), we have ([Tan70], Proposition 1.1)

$(S1)$ *There exists a unique integer* $\mu > 0$ *such that, for all* $k \geqq \mu$,

$$D^{-k} = \cdots = D^{-\mu} \supsetneqq D^{-\mu+1} \supsetneqq \cdots \supsetneqq D^{-2} \supsetneqq D^{-1} = D,$$

$(S2)$ $[\mathcal{D}^p, \mathcal{D}^q] \subset \mathcal{D}^{p+q}$ *for all* $p,\ q < 0$.

where $\mathcal{D}^p$ denotes the space of sections of D^p. $(S2)$ can be checked easily by induction on q.

Thus $D^{-\mu}$ is the smallest completely integrable differential system, which contains $D = D^{-1}$.

1.2.2. Symbol Algebras. From now on, we will consider **regular** differential systems (M, D) such that $T(M) = D^{-\mu}$. As a first invariant for non-integrable differential system, the **symbol algebra** $\mathfrak{m}(x)$ of (M, D) at x is defined as follows ([Tan70]);

$$\mathfrak{m}(x) = \bigoplus_{p=-1}^{-\mu} \mathfrak{g}_p(x),$$

where $\mathfrak{g}_{-1}(x) = D^{-1}(x)$, $\mathfrak{g}_p(x) = D^p(x)/D^{p+1}(x)$ $(p < -1)$. Let ϖ_p be the projection of $D^p(x)$ onto $\mathfrak{g}_p(x)$. Then, for $X \in \mathfrak{g}_p(x)$ and $Y \in \mathfrak{g}_q(x)$, the bracket product $[X, Y] \in \mathfrak{g}_{p+q}(x)$ is defined by

$$[X, Y] = \varpi_{p+q}([\tilde{X}, \tilde{Y}]_x),$$

where $\tilde{X}$ and $\tilde{Y}$ are any element of $\mathcal{D}^p$ and $\mathcal{D}^q$ respectively such that $\varpi_p(\tilde{X}_x) = X$ and $\varpi_q(\tilde{Y}_x) = Y$.

Endowed with this bracket operation, by $(S2)$ above, $\mathfrak{m}(x)$ becomes a nilpotent graded Lie algebra such that $\dim \mathfrak{m}(x) = \dim M$ and satisfies

$$\mathfrak{g}_p(x) = [\mathfrak{g}_{p+1}(x), \mathfrak{g}_{-1}(x)] \qquad \text{for } p < -1.$$

Furthermore, let $\mathfrak{m}$ be a fundamental graded Lie algebra (**FGLA**) of μ-th kind, that is,

$$\mathfrak{m} = \bigoplus_{p=-1}^{-\mu} \mathfrak{g}_p$$

is a nilpotent graded Lie algebra such that

$$\mathfrak{g}_p = [\mathfrak{g}_{p+1}, \mathfrak{g}_{-1}] \qquad \text{for } p < -1.$$

Then (M, D) is called of type $\mathfrak{m}$ if the symbol algebra $\mathfrak{m}(x)$ is isomorphic with $\mathfrak{m}$ at each $x \in M$.

Conversely, given a FGLA $\mathfrak{m} = \bigoplus_{p=-1}^{-\mu} \mathfrak{g}_p$, we can construct a model differential system of type $\mathfrak{m}$ as follows: Let $M(\mathfrak{m})$ be the simply connected Lie group with Lie algebra $\mathfrak{m}$. Identifying $\mathfrak{m}$ with the Lie algebra of left invariant vector fields on $M(\mathfrak{m})$, $\mathfrak{g}_{-1}$ defines a left invariant subbundle $D_{\mathfrak{m}}$ of $T(M(\mathfrak{m}))$. By definition of symbol algebras, it is easy to see that $(M(\mathfrak{m}), D_{\mathfrak{m}})$ is a regular differential system of type $\mathfrak{m}$. $(M(\mathfrak{m}), D_{\mathfrak{m}})$ is called the **standard differential system** of type $\mathfrak{m}$. The Lie algebra $\mathfrak{g}(\mathfrak{m})$ of all infinitesimal automorphisms of $(M(\mathfrak{m}), D_{\mathfrak{m}})$ can be calculated algebraically as the (algebraic) **prolongation** of $\mathfrak{m}$ ([Tan70], cf. [Yam93]).

1.2.3. Prolongation of $(\mathfrak{m}, \mathfrak{g}_0)$. Here we recall some basic facts on the algebraic prolongation $\mathfrak{g}(\mathfrak{m})$ of a FGLA $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ (see §2 of [Yam93]).

$\mathfrak{g}(\mathfrak{m})$ is first characterized as the graded Lie algebra which satisfies the following conditions:

(1) $\mathfrak{g}_p(\mathfrak{m}) = \mathfrak{g}_p$ for $p < 0$, where $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$.
(2) For $k \geqq 0$, if $X \in \mathfrak{g}_k(\mathfrak{m})$ and $[X, \mathfrak{m}] = \{0\}$, then $X = 0$.
(3) $\mathfrak{g}(\mathfrak{m})$ is maximum among graded algebras satisfying conditions (1) and (2) above.

More precisely, we can define $\mathfrak{g}_k(\mathfrak{m})$ as follows: First we decompose $\bigwedge^2 \mathfrak{m}^* = \bigoplus_{j<-1} \wedge_j^2 \mathfrak{m}^*$ according to the gradation $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$, where

$$\wedge_j^2 \mathfrak{m}^* = \bigoplus_{p+q=j} \mathfrak{g}_p^* \wedge \mathfrak{g}_q^*.$$

Putting $C_k^1 = \bigoplus_{p<0} \mathfrak{g}_{p+k} \otimes \mathfrak{g}_p^*$ and $C_k^2 = \bigoplus_{j<-1} \mathfrak{g}_{j+k} \otimes \wedge_j^2 \mathfrak{m}^*$, we can define $\mathfrak{g}_k = \mathfrak{g}_k(\mathfrak{m})$ for $k \geqq 0$ inductively by the following exact sequence;

$$0 \to \mathfrak{g}_k \to C_k^1 \xrightarrow{\partial} C_k^2,$$

where the coboundary operator $\partial \colon C_k^1 \to C_k^2$ is given by

$$(\partial p)(X, Y) = [X, p(Y)] - [Y, p(X)] - p([X, Y]).$$

Thus $\mathfrak{g}_0(\mathfrak{m})$ is the (gradation preserving) derivation algebra of $\mathfrak{m}$. Moreover, for $u \in \mathfrak{g}_k(\mathfrak{m})$ and $v \in \mathfrak{g}_l(\mathfrak{m})$ $(k, l \geqq 0)$, by induction on the integer $k + l \geqq 0$, we can define $[u, v] \in \mathfrak{g}_{k+l} \subset C_{k+l}^1$ by

$$[u, v](X) = [[u, X], v] + [u, [v, X]] \qquad \text{for } X \in \mathfrak{m}.$$

With this bracket product, $\mathfrak{g}(\mathfrak{m})$ becomes a graded Lie algebra.

Now let $\mathfrak{g}_0$ be a subalgebra of $\mathfrak{g}(\mathfrak{m})$. We define a subspace $\mathfrak{g}_k$ of $\mathfrak{g}_k(\mathfrak{m})$ for $k \geqq 1$ inductively by

$$\mathfrak{g}_k = \{u \in \mathfrak{g}_k(\mathfrak{m}) \mid [u, \mathfrak{g}_{-1}] \subset \mathfrak{g}_{k-1}\}.$$

Then, putting

$$\mathfrak{g}(\mathfrak{m}, \mathfrak{g}_0) = \mathfrak{m} \oplus \bigoplus_{k \geqq 0} \mathfrak{g}_k,$$

we see, with the generating condition of $\mathfrak{m}$, that $\mathfrak{g}(\mathfrak{m}, \mathfrak{g}_0)$ is a graded subalgebra of $\mathfrak{g}(\mathfrak{m})$. $\mathfrak{g}(\mathfrak{m}, \mathfrak{g}_0)$ is called the **prolongation** of $(\mathfrak{m}, \mathfrak{g}_0)$.

By utilizing the above definition of the algebraic prolongation, we will consider the following situation: Let $\mathfrak{h} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{h}_p$ be a graded Lie algebra such that $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{h}_p$ is a FGLA. To check whether $\mathfrak{h}$ is the prolongation of $\mathfrak{m}$ or $(\mathfrak{m}, \mathfrak{h}_0)$, we consider the Lie algebra cohomology $H^q(\mathfrak{m}, \mathfrak{h})$ associated with the adjoint representation ad: $\mathfrak{m} \to \mathfrak{gl}(\mathfrak{h})$. According to the gradation of $\mathfrak{h}$, this cohomology space has a bigradation (for the precise definition see §4);

$$H^q(\mathfrak{m}, \mathfrak{h}) = \bigoplus_r H^q(\mathfrak{m}, \mathfrak{h})_r$$

With this cohomology group, we will utilize the following criterion in §4.

Lemma A (Lemma 2.1 [Yam93]). *Let $\mathfrak{h} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{h}_p$ be a graded Lie algebra such that $\mathfrak{h}_p = [\mathfrak{h}_{p+1}, \mathfrak{h}_{-1}]$ for $p < -1$. Then $\mathfrak{h}$ is the prolongation of $\mathfrak{m}$ (resp. of $(\mathfrak{m}, \mathfrak{h}_0)$) if and only if the following two conditions hold*:

(1) *For $k \geqq 0$, if $X \in \mathfrak{h}_k$ and $[X, \mathfrak{m}] = 0$, then $X = 0$.*
(2) *$H^1(\mathfrak{m}, \mathfrak{h})_r = \{0\}$ for $r \geqq 0$ (resp. $r \geqq 1$).*

1.3. Symbol Algebra of (J^k, C^k)

As an example, we will now calculate the symbol algebra of the canonical differential system (J^k, C^k). First recall that C^k is defined by 1-forms (1.2) on a coordinate system U; $(x^i, y^\alpha, p_i^\alpha, \ldots, p_{i_1 \ldots i_k}^\alpha)$ of J^k;

$$C^k = \{\varpi^\alpha = \varpi_i^\alpha = \cdots = \varpi_{i_1 \ldots i_{k-1}}^\alpha = 0\}$$

Then we have a following coframe

$$\{\varpi^\alpha, \ldots, \varpi_{i_1 \ldots i_l}^\alpha, \ldots, dx_i, dp_{i_1 \ldots i_k}^\alpha\},$$

at each point in U. Now let us take the dual frame of this coframe;

$$\left\{ \frac{\partial}{\partial y^\alpha}, \ldots, \frac{\partial}{\partial p_{i_1 \ldots i_l}^\alpha}, \ldots, \frac{d}{dx_i}, \frac{\partial}{\partial p_{i_1 \ldots i_k}^\alpha} \right\}$$

where

$$\frac{d}{dx_i} = \frac{\partial}{\partial x_i} + \sum_{\alpha=1}^{m} p_i^\alpha \frac{\partial}{\partial y^\alpha} + \sum_{\alpha=1}^{m} \sum_{l=1}^{k-1} \sum_{j_1 \leqq \cdots \leqq j_l} p_{ij_1\ldots j_l}^\alpha \frac{\partial}{\partial p_{j_1\ldots j_l}^\alpha}$$

We have

$$\left[\frac{\partial}{\partial p_j^\alpha}, \frac{d}{dx_i}\right] = \delta_j^i \frac{\partial}{\partial y^\alpha}, \quad \left[\frac{\partial}{\partial p_{j_1\ldots j_l}^\alpha}, \frac{d}{dx_i}\right] = \sum_{k=1}^{l} \delta_{j_k}^i \frac{\partial}{\partial p_{j_1\ldots \hat{j}_k \ldots j_l}^\alpha}.$$

Then we see that (J^k, C^k) is a regular differential system of type $\mathfrak{C}^k(n,m)$:

$$\mathfrak{C}^k(n,m) = \mathfrak{C}_{-(k+1)} \oplus \mathfrak{C}_{-k} \oplus \cdots \oplus \mathfrak{C}_{-1},$$

where $\mathfrak{C}_{-(k+1)} = W$, $\mathfrak{C}_p = W \otimes S^{k+p+1}(V^*)$, $\mathfrak{C}_{-1} = V \oplus W \otimes S^k(V^*)$.

Here V and W are vector spaces of dimension n and m respectively and the bracket product of $\mathfrak{C}^k(n,m) = \mathfrak{C}^k(V,W)$ is defined accordingly through the pairing between V and V^* such that V and $W \otimes S^k(V^*)$ are both abelian subspaces of $\mathfrak{C}_{-1}$. Namely

$$\begin{aligned}
&[W,V] = \{0\}, \qquad [V,V] = \{0\}, \\
&[W \otimes S^r(V^*), W \otimes S^s(V^*)] = \{0\} \quad (r,\ s = 0,\ldots,k), \\
&[W \otimes S^r(V^*), V] = W \otimes (i(V)(S^r(V^*))) \quad (r = 1,\ldots,k),
\end{aligned}$$

i.e., $[w \otimes s, v] = w \otimes (i(v)s)$ for $v \in V$, $w \in W$ and $s \in S^r(V^*)$, where $i(v)$ denotes the interior multiplication.

The subspace $W \otimes S^k(V^*)$ of $\mathfrak{C}_{-1}$ corresponds to the subbundle $\mathrm{Ker}(\pi_{k-1}^k)_* = \mathrm{Ch}(\partial C^k)$ of C^k and the identification $\mathrm{Ker}(\pi_{k-1}^k)_* \to W \otimes S^k(V^*)$ corresponds to the fundamental identification of the jet bundle theory. For the geometry of the higher order contact system (J^k, C^k), we refer the reader to [Yam82].

§2. Pseudo-product GLA $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ of type $(\mathfrak{l}, S)$

We now discuss the prolongation of symbol algebras of the pseudo-product structures associated with higher order differential equations of finite type. Moreover we will generalize this algebra to the notion of the pseudo-product GLA (graded Lie algebras) of irreducible type and introduce the pseudo-product GLA of type $(\mathfrak{l}, S)$.

2.1. Pseudo-projective GLA of order k of bidegree (n, m)

For a k-th order differential equation R of finite type given in §1.1, we have the pseudo-product structure $(R; E, F)$. Corresponding to the splitting $D = E \oplus F$, we have the splitting in the symbol algebra of the regular differential system $(R, D) \cong (J^{k-1}, C^{k-1})$ of type $\mathfrak{C}^{k-1}(n, m)$;

$$\mathfrak{C}_{-1} = \mathfrak{e} \oplus \mathfrak{f},$$

where $\mathfrak{e} = V$, $\mathfrak{f} = W \otimes S^{k-1}(V^*)$. At each point $x \in R$, $\mathfrak{e}$ corresponds to $E'(x)$ (the point in $R^{(1)}$ over x) and $\mathfrak{f}$ corresponds to $\mathrm{Ker}(\pi^{k-1}_{k-2})_*(p(x))$.

Now we put

$$\check{\mathfrak{g}}_0 = \{X \in \mathfrak{g}_0(\mathfrak{C}^{k-1}(n, m)) \mid [X, \mathfrak{e}] \subset \mathfrak{e},\ [X, \mathfrak{f}] \subset \mathfrak{f}\}$$

and consider the (algebraic) prolongation $\mathfrak{g}^k(n, m)$ of $(\mathfrak{C}^{k-1}(n, m), \check{\mathfrak{g}}_0)$, which is called the pseudo-projective GLA of order k of bidegree (n, m) ([Tan89]).

Let $\check{G}_0 \subset GL(\mathfrak{C}^{k-1}(n, m))$ be the (gradation preserving) automorphism group of $\mathfrak{C}^{k-1}(n, m)$ which also preserve the splitting $\mathfrak{C}_{-1} = \mathfrak{e} \oplus \mathfrak{f}$. Then $\check{G}_0$ is the Lie subgroup of $GL(\mathfrak{C}^{k-1}(n, m))$ with Lie algebra $\check{\mathfrak{g}}_0$. The pseudo-product structure on a k-th order differential equation R of finite type given in §1, which is called the pseudo-projective system of order k of bidegree (n, m) in [Tan89], can be formulated as the $\check{G}_0^\sharp$-structure over a regular differential system of type $\mathfrak{C}^{k-1}(n, m)$ ([Tan70], [Tan89], [DKM99]). Thus the prolongation $\mathfrak{g}^k(n, m)$ of $(\mathfrak{C}^{k-1}(n, m), \check{\mathfrak{g}}_0)$ represents the Lie algebra of infinitesimal automorphisms of the (local) model k-th order differential equation R_o of finite type, where

$$R_o = \left\{ \frac{\partial^k y^\alpha}{\partial x_{i_1} \cdots \partial x_{i_k}} = 0 \quad (1 \leqq \alpha \leqq m,\ 1 \leqq i_1 \leqq \cdots \leqq i_k \leqq n) \right\}.$$

The isomorphism ϕ of the pseudo-product structure on R preserves the differential system $D = E \oplus F$, which is equivalent to the canonical system C^{k-1} on J^{k-1}. Hence, by Bäcklund's Theorem (cf. [Yam83]), ϕ is the lift of a point transformation on J^0 when $m \geqq 2$ and $k \geqq 2$ and is the lift of a contact transformation on J^1 when $m = 1$ and $k \geqq 3$. When $(m, k) = (1, 2)$, ϕ is the lift of the point transformation on J^0, since ϕ preserves both D and $F = \mathrm{Ker}(\pi^1_0)_*$. Thus the equivalence of the pseudo-product structure on R is the equivalence of the k-th order equation under point or contact transformations. To settle the equivalence problem for the pseudo-projective systems of order k of bidegree (n, m), N. Tanaka constructed normal Cartan connections of type $\mathfrak{g}^k(n, m)$ ([Tan79], [Tan82], [Tan89]).

It is well known that $\mathfrak{g}^k(n,m)$ $(k \geqq 2)$ has the following structure ([Tan89], [Yam93], [DKM99]);

(1) $k=2$ $\quad \mathfrak{g}^2(n,m)$ is isomorphic to $\mathfrak{sl}(m+n+1,\mathbb{R})$ and has the following gradation:

$$\mathfrak{sl}(m+n+1,\mathbb{R}) = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2,$$

where the gradation is given by subdividing matrices as follows;

$$\mathfrak{g}_{-2} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \xi & 0 & 0 \end{pmatrix} \;\middle|\; \xi \in W \cong \mathbb{R}^m \right\},$$

$$\mathfrak{g}_{-1} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ x & 0 & 0 \\ 0 & A & 0 \end{pmatrix} \;\middle|\; x \in V \cong \mathbb{R}^n,\ A \in M(m,n) = W \otimes V^* \right\},$$

$$\mathfrak{g}_0 = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{pmatrix} \;\middle|\; \begin{array}{l} a \in \mathbb{R},\ B \in \mathfrak{gl}(V),\ C \in \mathfrak{gl}(W), \\ a + \operatorname{tr} B + \operatorname{tr} C = 0 \end{array} \right\},$$

$$\mathfrak{g}_1 = \{{}^tX \mid X \in \mathfrak{g}_{-1}\}, \qquad \mathfrak{g}_2 = \{{}^tX \mid X \in \mathfrak{g}_{-2}\},$$

where $V = M(n,1)$, $W = M(m,1)$ and $M(a,b)$ denotes the set of $a \times b$ matrices.

(2) $k=3$ and $m=1$ $\quad \mathfrak{g}^3(n,1)$ is isomorphic to $\mathfrak{sp}(n+1,\mathbb{R})$ and has the following gradation:

$$\mathfrak{sp}(n+1,\mathbb{R}) = \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3.$$

First we describe

$$\mathfrak{sp}(n+1,\mathbb{R}) = \{X \in \mathfrak{gl}(2n+2,\mathbb{R}) \mid {}^tXJ + JX = 0\},$$

where

$$J = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & I_n & 0 \\ 0 & -I_n & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \in \mathfrak{gl}(2n+2,\mathbb{R}), \quad I_n = (\delta_{ij}) \in \mathfrak{gl}(n,\mathbb{R}).$$

Here $I_n \in \mathfrak{gl}(n,\mathbb{R})$ is the unit matrix and the gradation is given again by subdividing matrices as follows;

$$\mathfrak{g}_{-3} = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2a & 0 & 0 & 0 \end{pmatrix} \middle| \, a \in \mathbb{R} \right\},$$

$$\mathfrak{g}_{-2} = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \xi & 0 & 0 & 0 \\ 0 & {}^t\xi & 0 & 0 \end{pmatrix} \middle| \, \xi \in \mathbb{R}^n \cong V^* \right\},$$

$$\mathfrak{g}_{-1} = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ x & 0 & 0 & 0 \\ 0 & A & 0 & 0 \\ 0 & 0 & -{}^tx & 0 \end{pmatrix} \middle| \, x \in \mathbb{R}^n = V, \ A \in \mathrm{Sym}(n) \cong S^2(V^*) \right\},$$

$$\mathfrak{g}_{0} = \left\{ \begin{pmatrix} b & 0 & 0 & 0 \\ 0 & B & 0 & 0 \\ 0 & 0 & -{}^tB & 0 \\ 0 & 0 & 0 & -b \end{pmatrix} \middle| \, b \in \mathbb{R}, \ B \in \mathfrak{gl}(V) \right\},$$

$$\mathfrak{g}_k = \{ {}^tX \mid X \in \mathfrak{g}_{-k} \}, \ (k = 1, \ 2, \ 3),$$

where $\mathrm{Sym}(n) = \{A \in \mathfrak{gl}(n,\mathbb{R}) \mid {}^tA = A\}$ is the space of symmetric matrices.

(3) otherwise For vector spaces V and W of dimension n and m respectively, $\mathfrak{g}^k(n,m) = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ has the following description:

$$\mathfrak{g}_k = \{0\} \quad (k \geqq 2), \qquad \mathfrak{g}_1 = V^*, \qquad \mathfrak{g}_0 = \mathfrak{gl}(V) \oplus \mathfrak{gl}(W),$$
$$\mathfrak{g}_{-1} = V \oplus W \otimes S^{k-1}(V^*), \quad \mathfrak{g}_p = W \otimes S^{k+p}(V^*) \quad (p < -1).$$

Here the bracket product in $\mathfrak{g}^k(n,m)$ is given through the natural tensor operations.

For the proof of these facts ((1) and (2)), see e.g., Theorem 5.3 in [Yam93]. We refer the reader to [DKM99] for the description of these algebras as the Lie algebras of infinitesimal automorphisms (polynomial vector fields) of the (local) model equations. We will also give the proof of (3) in §5 by calculating the first generalized Spencer cohomology. For this, we observe the following points: We put

$$\begin{aligned} \mathfrak{l} = V \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 &= (V \oplus \mathfrak{gl}(V) \oplus V^*) \oplus \mathfrak{gl}(W) \\ &\cong \mathfrak{sl}(\hat{V}) \oplus \mathfrak{gl}(W), \\ S = W \otimes S^{k-1}(\hat{V}^*), &\qquad \hat{V} = \mathbb{R} \oplus V. \end{aligned} \tag{2.1}$$

where the gradation of the first kind; $\mathfrak{sl}(\hat{V}) = V \oplus \mathfrak{gl}(V) \oplus V^*$ is given by subdividing matrices corresponding to the decomposition $\hat{V} = \mathbb{R} \oplus V$. Then

$$S^{k-1}(\hat{V}^*) \cong \bigoplus_{l=0}^{k-1} S^l(V^*),$$

and S is a faithful irreducible $\mathfrak{l}$-module such that $\mathfrak{l} = \mathfrak{l}_{-1} \oplus \mathfrak{l}_0 \oplus \mathfrak{l}_1$ is a reductive graded Lie algebras, where $\mathfrak{l}_{-1} = V$, $\mathfrak{l}_0 = \mathfrak{g}_0$, $\mathfrak{l}_1 = \mathfrak{g}_1$. Moreover $\mathfrak{g}^k(n,m) \cong S \oplus \mathfrak{l}$ is the semi-direct product of $\mathfrak{l}$ by S.

2.2. Pseudo-product GLA of type $(\mathfrak{l}, S)$

We will now give the notion of the pseudo-product GLA of type $(\mathfrak{l}, S)$, generalizing the pseudo-projective GLA of order k of bidegree (n,m).

Let $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ be a (transitive) graded Lie algebra (GLA) over the field $\mathbb{K}$ such that the negative part $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ is a FGLA, where $\mathbb{K}$ is the field $\mathbb{R}$ of real numbers or the field $\mathbb{C}$ of complex numbers. Let $\mathfrak{e}$ and $\mathfrak{f}$ be subspaces of $\mathfrak{g}_{-1}$. Then the system $\mathfrak{G} = (\mathfrak{g}, (\mathfrak{g}_p)_{p\in\mathbb{Z}}, \mathfrak{e}, \mathfrak{f})$ is called a pseudo-product GLA (PPGLA) of irreducible type if the following conditions hold:

(1) $\mathfrak{g}$ is transitive, i.e., for each $k \geqq 0$, if $X \in \mathfrak{g}_k$ and $[X, \mathfrak{g}_{-1}] = 0$, then $X = 0$.
(2) $\mathfrak{g}_{-1} = \mathfrak{e} \oplus \mathfrak{f}$, $\quad [\mathfrak{e}, \mathfrak{e}] = [\mathfrak{f}, \mathfrak{f}] = 0$.
(3) $[\mathfrak{g}_0, \mathfrak{e}] \subset \mathfrak{e}$ and $[\mathfrak{g}_0, \mathfrak{f}] \subset \mathfrak{f}$.
(4) $\mathfrak{g}_{-2} \neq 0$ and the $\mathfrak{g}_0$-modules $\mathfrak{e}$ and $\mathfrak{f}$ are irreducible.

It is known that $\mathfrak{g}$ becomes finite dimensional under these conditions (see [Tan85], [Yat88]).

As a typical example, starting from a reductive GLA $\mathfrak{l} = \mathfrak{l}_{-1} \oplus \mathfrak{l}_0 \oplus \mathfrak{l}_1$ and a faithful irreducible $\mathfrak{l}$-module S, we define the pseudo-product GLA $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ of type $(\mathfrak{l}, S)$ as follows: Let $\mathfrak{l} = \mathfrak{l}_{-1} \oplus \mathfrak{l}_0 \oplus \mathfrak{l}_1$ be a finite dimensional **reductive GLA** of the first kind such that

(1) The ideal $\hat{\mathfrak{l}} = \mathfrak{l}_{-1} \oplus [\mathfrak{l}_{-1}, \mathfrak{l}_1] \oplus \mathfrak{l}_1$ of $\mathfrak{l}$ is a simple Lie algebra.
(2) The center $\mathfrak{z}(\mathfrak{l})$ of $\mathfrak{l}$ is contained in $\mathfrak{l}_0$.

Let S be a finite dimensional **faithful irreducible $\mathfrak{l}$-module**. We put

$$S_{-1} = \{s \in S \mid \mathfrak{l}_1 \cdot s = 0\}$$

and

$$S_p = \mathrm{ad}(\mathfrak{l}_{-1})^{-p-1} S_{-1} \quad \text{for } p < 0.$$

We form the semi-direct product $\mathfrak{g}$ of $\mathfrak{l}$ by S, and put

$$\mathfrak{g} = S \oplus \mathfrak{l}, \qquad [S, S] = 0,$$

$$\mathfrak{g}_k = \mathfrak{l}_k \ (k \geqq 0), \quad \mathfrak{g}_{-1} = \mathfrak{l}_{-1} \oplus S_{-1},$$

$$\mathfrak{g}_p = S_p \ (p < -1).$$

Namely $\mathfrak{g}$ is a subalgebra of the Lie algebra $\mathfrak{A}(S) = S \oplus \mathfrak{gl}(S)$ of infinitesimal affine transformations of S.

Then we have

Lemma 2.1. *Notations being as above,*

(1) $S = \bigoplus_{p=-1}^{-\mu} S_p$, *where* $S_{-\mu} = \{s \in S \mid [\mathfrak{l}_{-1}, s] = 0\}$.
(2) $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ *is generated by* $\mathfrak{g}_{-1}$.
(3) $[S_p, \mathfrak{l}_1] = S_{p+1}$ *for* $p < -1$.
(4) S_p *is naturally embedded as a subspace of* $W \otimes S^{\mu+p}(\mathfrak{l}_{-1}{}^*)$ *through the bracket operation in* $\mathfrak{m}$, *where* $W = S_{-\mu}$.
(5) S_{-1}, $S_{-\mu}$ *are irreducible* $\mathfrak{l}_0$*-modules.*

Proof. We have the characteristic element $Z \in \hat{\mathfrak{l}}_0$ (Lemma 4.1.1 [Sea88]), which defines the gradation of $\mathfrak{l}$:

$$\mathfrak{l}_p = \{X \in \mathfrak{l} \mid [Z, X] = pX\} \qquad \text{for} \quad p = -1,\ 0,\ 1.$$

Since $\mathrm{ad}(Z)$ is a semi-simple endomorphism with eigenvalues -1, 0, 1. $\mathrm{ad}(Z)$ is a semi-simple endomorphism of S (Corollary 6.4 [Hum72]) with real eigenvalues (see the argument in §4.5). Moreover, for the eigenspaces $S_{(\lambda)} = \{s \in S \mid [Z, s] = \lambda s\}$, by the Jacobi identity, we have

$$[\mathfrak{l}_p, S_{(\lambda)}] \subset S_{(\lambda+p)} \qquad \text{for} \quad p = -1,\ 0,\ 1.$$

For each eigenvalue λ of $\mathrm{ad}(Z)$, we consider the following subspaces $S^1(\lambda)$ and $S^{-1}(\lambda)$ of S:

$$S^1(\lambda) = \bigoplus_{k \geqq 0} S^1_{(\lambda+k)}(\lambda), \qquad S^{-1}(\lambda) = \bigoplus_{k \geqq 0} S^{-1}_{(\lambda-k)}(\lambda),$$

where

$$S^1_{(\lambda)}(\lambda) = \{s \in S_{(\lambda)} \mid [\mathfrak{l}_{-1}, s] = 0\}, \quad S^{-1}_{(\lambda)}(\lambda) = \{s \in S_{(\lambda)} \mid [\mathfrak{l}_1, s] = 0\},$$
$$S^1_{(\lambda+k)}(\lambda) = [\mathfrak{l}_1, S^1_{(\lambda+k-1)}(\lambda)] \subset S_{(\lambda+k)} \qquad \text{for } k \geqq 1$$
$$S^{-1}_{(\lambda-k)}(\lambda) = [\mathfrak{l}_{-1}, S^{-1}_{(\lambda-k+1)}(\lambda)] \subset S_{(\lambda-k)} \qquad \text{for } k \geqq 1$$

Then $S^1_{(\lambda)}(\lambda)$ and $S^{-1}_{(\lambda)}(\lambda)$ are $\mathfrak{l}_0$-invariant subspaces of $S_{(\lambda)}$. One can easily check that $S^1_{(\lambda+k)}(\lambda)$ and $S^{-1}_{(\lambda-k)}(\lambda)$ are $\mathfrak{l}_0$-invariant and

$$[\mathfrak{l}_{-1}, S^1_{(\lambda+k)}(\lambda)] \subset S^1_{(\lambda+k-1)}(\lambda), \qquad [\mathfrak{l}_1, S^{-1}_{(\lambda-k)}(\lambda)] \subset S^{-1}_{(\lambda-k+1)}(\lambda)$$

by induction on $k \geqq 0$. Thus $S^1(\lambda)$ and $S^{-1}(\lambda)$ are $\mathfrak{l}$-submodule of S for each λ.

Let λ_0 and λ_1 be the minimum and maximum eigenvalue of $\mathrm{ad}(Z)$. We have

$$S^1_{(\lambda_0)}(\lambda_0) = S_{(\lambda_0)}, \qquad S^{-1}_{(\lambda_1)}(\lambda_1) = S_{(\lambda_1)},$$

and $S^1(\lambda)$ and $S^{-1}(\lambda)$ are both proper subspaces of S for an intermediate eigenvalue λ. Then, since S is an irreducible $\mathfrak{l}$-module, we get

$$S^1(\lambda_0) = S^{-1}(\lambda_1) = S \qquad \text{and} \qquad S^1(\lambda) = S^{-1}(\lambda) = 0 \quad \text{otherwise.}$$

Especially, from $S^1_{(\lambda)}(\lambda) = S^{-1}_{(\lambda)}(\lambda) = 0$, we get

$$S_{(\lambda_1)} = \{s \in S \mid [\mathfrak{l}_1, s] = 0\} = S_{-1}, \qquad S_{(\lambda_0)} = \{s \in S \mid [\mathfrak{l}_{-1}, s] = 0\}.$$

Hence, from $S^{-1}(\lambda_1) = S$, we obtain (1) and (2). Moreover, from $S^1(\lambda_0) = S$, we get (3).

Now we put $V = \mathfrak{l}_{-1}$ and $W = S_{-\mu}$. Then we have a linear map ι_r of $S_{r-\mu}$ into $W \otimes S^r(V^*)$ $(r = 1, \dots, \mu - 1)$ defined by

$$\iota_r(s)(X_1, \dots, X_r) = [[\dots[s, X_1], \dots], X_r] \in W \quad \text{for } s \in S_{r-\mu},\ X_i \in V.$$

Since $\mathfrak{l}_{-1}$ is abelian, ι_r is well-defined and is injective by (1). Thus we get (4).

Starting from an $\mathfrak{l}_0$-submodule $T_{(\lambda_1)}$ in S_{-1}, similarly as above, we can form the $\mathfrak{l}$-submodule $T^{-1}(\lambda_1)$ of S by putting;

$$T^{-1}(\lambda_1) = \bigoplus_{k \geqq 0} T_{(\lambda_1 - k)},$$

where $T_{(\lambda_1-k)} = [\mathfrak{l}_{-1}, T_{(\lambda_1-k+1)}] \subset S_{-(k+1)}$ for $k \geqq 1$. Hence we get $T^{-1}(\lambda_1) = S$ or 0, which implies $T_{(\lambda_1)} = S_{-1}$ or 0. Thus S_{-1} is an irreducible $\mathfrak{l}_0$-module. By the similar argument, we see that $S_{-\mu}$ is an irreducible $\mathfrak{l}_0$-module, which completes the proof of Lemma. Q.E.D.

Thus $\mathfrak{m}$ is a graded subalgebra of $\mathfrak{C}^{\mu-1}(V, W)$, which has the splitting $\mathfrak{g}_{-1} = \mathfrak{l}_{-1} \oplus S_{-1}$, where $V = \mathfrak{l}_{-1}$ and $W = S_{-\mu}$. Hence $\mathfrak{m}$ is a symbol algebra of μ-th order differential equations of finite type, which is called the typical symbol of type $(\mathfrak{l}, S)$. Moreover the system

$\mathfrak{G} = (\mathfrak{g}, (\mathfrak{g}_p)_{p\in\mathbb{Z}}, \mathfrak{l}_{-1}, S_{-1})$ becomes a PPGLA of irreducible type, which is called the pseudo-product GLA of type $(\mathfrak{l}, S)$.

This class of higher order (linear) differential equations of finite type were first appeared in the work of Y. Se-ashi [Sea88]. We will construct the model (linear) equations for each PPGLA $\mathfrak{G}$ of type $(\mathfrak{l}, S)$ in §3, following [Sea88]. Moreover we remark that the PPGLA $\mathfrak{G}$ of type $(\mathfrak{l}, S)$ also naturally appeared in the classification of PPGLA's of irreducible type under mild conditions in [Yat92].

In this paper we will consider only pseudo-product graded Lie algebras $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ of type $(\mathfrak{l}, S)$. We recall that there exist an anti-linear involution τ of $\mathfrak{l}$ and an hermitian inner product $(\cdot \mid \cdot)$ of $\mathfrak{g}$ having the following properties:

(i) $\tau(\mathfrak{l}_p) = \mathfrak{l}_{-p}$;
(ii) $(\mathfrak{g}_p \mid \mathfrak{g}_q) = 0$ for $p \neq q$;
(iii) $([x, s] \mid s') + (s \mid [\tau(x), s']) = 0$ for all $x \in \mathfrak{l}$ and s, $s' \in S$.

Thus the PPGLA $\mathfrak{G}$ of type $(\mathfrak{l}, S)$ satisfies the criterion (Proposition 3.10.1) in [Mor93]. Hence, when $\mathfrak{g}$ is the prolongation of $\mathfrak{m}$ or $(\mathfrak{m}, \mathfrak{g}_0)$, for the equivalence of the pseudo-product structure associated with $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$, we can utilize the Morimoto's theory of normal Cartan connections [Mor93]. Especially we can utilize the harmonic theory for the curvature of the normal Cartan connections (cf. [DKM99]). Namely, regarding the curvature as $C^2(\mathfrak{m}, \mathfrak{g})$-valued functions, its harmonic parts constitute the fundamental system of invariants of the connection. In particular the curvature vanishes if and only if its harmonic part vanishes (Theorem 3 [DKM99]).

We will ask the following questions for the PPGLA $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ of type $(\mathfrak{l}, S)$ in the subsequent sections:

(A) *When is* $\mathfrak{g}$ *the prolongation of* $\mathfrak{m}$ *or* $(\mathfrak{m}, \mathfrak{g}_0)$?
(B) *Find the fundamental invariants for equations of type* $\mathfrak{m}$.

Utilizing the Tanaka-Morimoto theory of normal Cartan connections, these questions will be answered by calculating the first and second (generalized Spencer) cohomology spaces. The complete answer for (A) will be given in §5 and the problem (B) will be discussed in §6 and §7.

In the rest of this section, as an example to obtain the local model equations, we will realize the negative part $\mathfrak{m}$ of a pseudo-product GLA of type $(\mathfrak{l}, S)$ as the subalgebra of $\mathfrak{C}^{\mu-1}(n, m)$, which is called the typical symbol of type $(\mathfrak{l}, S)$ or $(\mathfrak{l}, \rho)$ in [SYY97]. Here $\mathfrak{l} = \mathfrak{sl}(l+1, \mathbb{K})$ is endowed with the gradation given by;

$$\mathfrak{sl}(l+1, \mathbb{K}) = \mathfrak{l}_{-1} \oplus \mathfrak{l}_0 \oplus \mathfrak{l}_1, \tag{2.2}$$

where

$$\mathfrak{l}_{-1} = \left\{ \begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix} \,\middle|\, C \in M(p,k) \right\}, \quad \mathfrak{l}_1 = \left\{ \begin{pmatrix} 0 & D \\ 0 & 0 \end{pmatrix} \,\middle|\, D \in M(k,p) \right\},$$

$$\mathfrak{l}_0 = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \,\middle|\, A \in \mathfrak{gl}(k,\mathbb{K}),\ B \in \mathfrak{gl}(p,\mathbb{K}) \text{ and } \operatorname{tr} A + \operatorname{tr} B = 0 \right\}.$$

Here $p = l - k + 1$ and $M(a,b)$ denotes the set of $a \times b$ matrices.

And $S = \bigwedge^{l-k+1} \mathbb{K}^{l+1}$ is the faithful irreducible $\mathfrak{l}$-module given by the following exterior representation $\rho = \rho_0$:

$$\rho_0 \colon \mathfrak{sl}(l+1,\mathbb{K}) \to \mathfrak{gl}\left(\bigwedge^{l-k+1} \mathbb{K}^{l+1} \right),$$

where

$$\rho_0(X)(v_1 \wedge \cdots \wedge v_{l-k+1}) = \sum_{i=1}^{l-k+1} v_1 \wedge \cdots \wedge X(v_i) \wedge \cdots \wedge v_{l-k+1}$$

for $X \in \mathfrak{sl}(l+1,\mathbb{K})$ and $v_i \in \mathbb{K}^{l+1}$ $(i = 1,\ 2, \ldots, l-k+1)$.

Let $\{e_1, \ldots, e_{l+1}\}$ be the natural basis of $\mathbb{K}^{l+1}$. Then $\mathfrak{l}' = \mathfrak{l}_0 \oplus \mathfrak{l}_1$ is the isotropy (stabilizer) algebra of the line $[e_1 \wedge \cdots \wedge e_k]$ in $\bigwedge^k \mathbb{K}^{l+1}$. We denote by $E_{ab} \in \mathfrak{gl}(l+1,\mathbb{K})$ $(1 \leqq a,\ b \leqq l+1)$ the matrix whose (a,b)-component is 1 and all of whose other components are 0. From (2.2), we have the following basis for $V = \mathfrak{l}_{-1}$ and $\mathfrak{l}_1$:

$$\begin{aligned} V = \mathfrak{l}_{-1} &= \langle E_{pi} \mid 1 \leqq i \leqq k,\ k+1 \leqq p \leqq l+1 \rangle \\ \mathfrak{l}_1 &= \langle E_{ip} \mid 1 \leqq i \leqq k,\ k+1 \leqq p \leqq l+1 \rangle \end{aligned}$$

Since $E_{pi}(e_j) = \delta_{ij} e_p$ for $1 \leqq j \leqq k$ and, $E_{pi}(e_q) = 0$ for $k+1 \leqq q \leqq l+1$, we have from Lemma 2.1 (1)

$$W = S_{-\mu} = \langle e_{k+1} \wedge \cdots \wedge e_{l+1} \rangle.$$

Hence we have $m = 1$ and $n = k(l-k+1)$. For $1 \leqq i_1 < \cdots < i_r \leqq k$ and $k+1 \leqq p_1 < \cdots < p_r \leqq l+1$, we put

$$e(p_1, \ldots, p_r) = e_{k+1} \wedge \cdots \wedge \widehat{e}_{p_1} \wedge \cdots \wedge \widehat{e}_{p_r} \wedge \cdots \wedge e_{l+1} \in \bigwedge^{l-k-r+1} \mathbb{K}^{l+1},$$

and consider the following element of S:

$$s(i_1, \ldots, i_r,\ p_1, \ldots, p_r) = e_{i_1} \wedge \cdots \wedge e_{i_r} \wedge e(p_1, \ldots, p_r) \in S = \bigwedge^{l-k+1} \mathbb{K}^{l+1}.$$

Then, from Lemma 2.1 (3) and $E_{ip}(e_j) = 0$, $E_{ip}(e_q) = \delta_{pq}e_i$ for $1 \leqq j \leqq k$, $k+1 \leqq q \leqq l+1$, we get

$$S_{r-\mu} = \langle s(i_1,\dots,i_r,\ p_1,\dots,p_r) \mid \\ 1 \leqq i_1 < \cdots < i_r \leqq k,\ k+1 \leqq p_1 < \cdots < p_r \leqq l+1\rangle,$$

for $r = 1,\ 2,\dots,p_0 - 1$ and

$$S_{r-\mu} = \{0\},$$

for $r \geqq p_0 = \min\{k+1, l-k+2\}$. Thus we have $\mu = p_0$. Moreover, for $X = \sum_{ip} X_{ip}E_{pi} \in V$, we have

$$\begin{aligned}
&\iota_r(s(i_1,\dots,i_r,p_1,\dots,p_r))(X,\dots,X) \\
&= r!\,(-1)^r X(e_{i_1}) \wedge \cdots \wedge X(e_{i_r}) \wedge e(p_1,\dots,p_r) \\
&= r!\,(-1)^r \left(\sum_\sigma \operatorname{sgn}\sigma X_{i_1p_{\sigma(1)}} \dots X_{i_rp_{\sigma(r)}}\right) e_{p_1} \wedge \cdots \wedge e_{p_r} \wedge e(p_1,\dots,p_r).
\end{aligned}$$

Thus, by fixing a basis of W and identifying SV^* with the ring of polynomials on V, we see that $S_{1-\mu} = V^*$ and $S_{r-\mu} \subset S^rV^*$ is spanned by the minor determinants of degree r of the matrix (X_{ip}), which are the linear coordinates of V.

Moreover it is known that $S_{r-\mu} \subset S^r(V^*)$ is equal to the $(r-2)$-th prolongation $p^{(r-2)}(S_{2-\mu})$ of $S_{2-\mu} \subset S^2(V^*)$ (Lemma 3.1 [SYY97]). Hence the local model equation in this case is given as the prolongation of the following second order equations:

$$\frac{\partial^2 y}{\partial x_{ip}\partial x_{jq}} + \frac{\partial^2 y}{\partial x_{iq}\partial x_{jp}} = 0, \qquad (1 \leqq i < j \leqq k,\ k+1 \leqq p < q \leqq l+1).$$

§3. Se-ashi's Theory for Linear Equations of Finite Type

We here recall some relevant facts from Se-ashi's theory [Sea88] for the equivalence of higher order linear differential equations of finite type. Especially we will recall the construction of the model equations for the typical symbol of type $(\mathfrak{l}, S)$ and his "Rigidity Theorem". We will also discuss the Plücker embedding equations as our examples of model equations, following [SYY97] rather closely.

3.1. Model equations for the typical symbol of type $(\mathfrak{l}, S)$

Starting from the typical symbol $\mathfrak{m} = S \oplus \mathfrak{l}_{-1}$ of type $(\mathfrak{l}, S)$ in §2.2, where $S = \bigoplus_{p=-\mu}^{-1} S_p \subset \bigoplus_{r=0}^{\mu-1} W \otimes S^rV^*$, $V = \mathfrak{l}_{-1}, W = S_{-\mu}$, we now explain a recipe to construct an integrable system of linear differential equations of finite type of order μ modeled after $\mathfrak{m}$.

The construction of the model system R_S is preceded by the consideration of the Lie algebra $\mathfrak{a}$ of infinitesimal bundle automorphisms of the constant coefficient differential equations modeled after $S = \bigoplus_{r=0}^{\mu-1} S_{r-\mu} \subset \bigoplus_{r=0}^{\mu} W \otimes S^r V^*$.

Let $E_0 = V \times W$ be the trivial vector bundle over the vector space V. Let $J^\mu(E_0)$ be the bundle of μ-jets of E_0. Then the fibre $J_0^\mu(E_0)$ of $J^\mu(E_0)$ at the origin $0 \in V$ is identified with $\bigoplus_{r=0}^{\mu} W \otimes S^r V^*$, where $W \otimes S^r V^*$ can be regarded as the set of W-valued homogeneous polynomials of degree r on V. Thus, starting from the typical symbol $S = \bigoplus_{r=0}^{\mu-1} S_{r-\mu} \subset \bigoplus_{r=0}^{\mu} W \otimes S^r V^*$, our first (local) model is the constant coefficient differential equations given as the subbundle $\hat{R}_S = V \times S$ of $J^\mu(E_0)$, whose solutions consist of W-valued polynomials contained in $S \subset W \otimes SV^*$.

Let us consider an infinitesimal bundle automorphism of E_0 preserving $\hat{R}_S$. An infinitesimal bundle automorphism of E_0 has a form

$$\sum_i \xi^i(x) \frac{\partial}{\partial x^i} + \sum_{\alpha,\beta} A_{\alpha,\beta}(x) y^\beta \frac{\partial}{\partial y^\alpha},$$

where (x^i) and (y^α) are linear coordinates of V and W, respectively. Thus the Lie algebra $\tilde{\mathfrak{a}}$ of (formal) infinitesimal bundle automorphisms of E_0 can be expressed as a graded Lie algebra $\tilde{\mathfrak{a}} = \bigoplus_{r \geqq -1} \tilde{\mathfrak{a}}_r$ by putting

$$\tilde{\mathfrak{a}}_r = S^{r+1}V^* \otimes V \oplus S^r V^* \otimes \mathfrak{gl}(W),$$

where $\tilde{\mathfrak{a}}_{-1} = V$ corresponds to constant coefficient vector fields on V. The bracket operation in $\tilde{\mathfrak{a}}$ is given by

$$\begin{aligned}
[f \otimes v, g \otimes w] &= -f(i(v)g) \otimes w + g(i(w)f) \otimes v, \\
[f \otimes A, g \otimes w] &= g(i(w)f) \otimes A, \\
[f \otimes A, g \otimes B] &= fg \otimes [A, B],
\end{aligned}$$

where f, $g \in SV^*$, v, $w \in V$ and A, $B \in \mathfrak{gl}(W)$; $i(v)$ denotes the inner multiplication. The Lie algebra $\tilde{\mathfrak{a}}$ acts naturally on the space $SV^* \otimes W$ which is regarded as the space of cross sections of E_0:

$$(f \otimes v + g \otimes A)(h \otimes w) = -f(i(v)h) \otimes w + gh \otimes A(w),$$

where f, g, $h \in SV^*$, v, $w \in V$ and $A \in \mathfrak{gl}(W)$.

Then the Lie algebra $\mathfrak{a}$ of infinitesimal automorphisms of $\hat{R}_S$ is given by

$$\mathfrak{a} = \{X \in \tilde{\mathfrak{a}} \mid X(S) \subset S\}.$$

$\mathfrak{a}$ is a graded subalgebra of $\tilde{\mathfrak{a}} = \bigoplus_{r \geqq -1} \tilde{\mathfrak{a}}_r$, i.e., $\mathfrak{a} = \bigoplus_{r \geqq -1} \mathfrak{a}_r$, where $\mathfrak{a}_r = \mathfrak{a} \cap \tilde{\mathfrak{a}}_r$. The Lie algebra $\mathfrak{gl}(S)$ has also the gradation given by

$$\mathfrak{gl}(S)_r = \{X \in \mathfrak{gl}(S) \mid X(S_p) \subset S_{p+r} \quad \text{for any } p\}.$$

Referring the action above we have a restriction homomorphism: $\mathfrak{a} \to \mathfrak{gl}(S)$, which sends $\mathfrak{a}_r$ into $\mathfrak{gl}(S)_r$. Assume here the following two conditions for S, which are satisfied by the typical symbol of type $(\mathfrak{l}, S)$:

(A1) The action of $\tilde{\mathfrak{a}}_{-1} = V$ leave S invariant.

(A2) The action of $\tilde{\mathfrak{a}}_{-1} = V$ on S is faithful.

Then this homomorphism turns out to be injective and we can characterize $\mathfrak{a}_r$ as a subspace of $\mathfrak{gl}(S)_r$ as follows (Proposition 2.2.2 [Sea88]):

$$\mathfrak{a}_{-1} = V, \qquad \mathfrak{a}_r = \{X \in \mathfrak{gl}(S)_r \mid [\mathfrak{a}_{-1}, X] \subset \mathfrak{a}_{r-1}\} \quad (r \geqq 0). \tag{3.1}$$

Put $\tilde{\mathfrak{n}}_r = S^r V^* \otimes \mathfrak{gl}(W) \subset \tilde{\mathfrak{a}}_r$. Then $\tilde{\mathfrak{n}} = \bigoplus_{r \geqq 0} \tilde{\mathfrak{n}}_r$ is an ideal of $\tilde{\mathfrak{a}}$ and $\mathfrak{n} = \tilde{\mathfrak{n}} \cap \mathfrak{a}$ is an ideal of $\mathfrak{a}$. We can see

$$\mathfrak{n}_r = \{X \in \mathfrak{gl}(S)_r \mid [\mathfrak{a}_{-1}, X] \subset \mathfrak{n}_{r-1}\} \qquad (r \geqq 0), \tag{3.2}$$

where we put $\mathfrak{n}_{-1} = \{0\}$ for convention.

In the case of the typical symbol of type $(\mathfrak{l}, S)$, we have the following: Since S is a faithful $\mathfrak{l}$-module, $\mathfrak{l}$ is a subalgebra of $\mathfrak{gl}(S)$. We have $\mathfrak{a}_{-1} = \mathfrak{l}_{-1}$ and it follows from (3.1) that $\mathfrak{a}_0 = \check{\mathfrak{g}}_0$, where $\check{\mathfrak{g}}_0$ is the Lie algebra of derivations of $\mathfrak{m} = S \oplus \mathfrak{l}_{-1}$ such that $D(S_p) \subset S_p = \mathfrak{g}_p$ $(p < -1)$, $D(\mathfrak{l}_{-1}) \subset \mathfrak{l}_{-1}$ and $D(S_{-1}) \subset S_{-1}$.

Let $\mathfrak{z}_S(\mathfrak{l})$ denote the centralizer of $\mathfrak{l}$ in $\mathfrak{gl}(S)$ and $\mathfrak{a}^\perp$ the orthogonal complement of $\mathfrak{a}$ in $\mathfrak{gl}(S)$ with respect to the non-degenerate bilinear form tr given by $\mathrm{tr}(X, Y) = \mathrm{trace}\, XY$ for X, $Y \in \mathfrak{gl}(S)$. Then, from (3.1) and (3.2), we have (Proposition 4.4.1 [Sea88])

$$\begin{aligned} \mathfrak{a} &= [\mathfrak{l}, \mathfrak{l}] \oplus \mathfrak{z}_S(\mathfrak{l}), \qquad \mathfrak{z}_S(\mathfrak{l}) \subset \mathfrak{n}, \\ \mathfrak{gl}(S) &= [\mathfrak{l}, \mathfrak{l}] \oplus \mathfrak{z}_S(\mathfrak{l}) \oplus \mathfrak{a}^\perp \quad \text{(tr-orthogonal)}. \end{aligned} \tag{3.3}$$

We here note that our assumption on $\mathfrak{l}$ is a little different from that in [Sea88]. We will discuss the condition for $\mathfrak{a} = \mathfrak{l}$ in §4.5.

Now the model equation R_S is constructed as follows: We filtrate the space S by subspaces $S^q = \bigoplus_{p=-1}^{q} S_p$. Notice that the group $GL(V) \times GL(W)$ acts on $\tilde{\mathfrak{a}}$ by the adjoint action: for $a \in GL(V) \times GL(W)$ and $X \in \tilde{\mathfrak{a}}$, the action is $(aX)(s) = (a \cdot X \cdot a^{-1})(s)$ for $s \in S$. Let us define groups

$$A_0 = \{a \in GL(V) \times GL(W) \mid a(S) \subset S\},$$
$$GL^{(0)}(S) = \{g \in GL(S) \mid g(S^q) \subset S^q \quad \text{for any } q\}.$$

Let $\widetilde{A}$ be the analytic subgroup of $GL(S)$ with Lie algebra $\mathfrak{a} \subset \mathfrak{gl}(S)$ and put

$$A = \widetilde{A} \cdot A_0,$$
$$A' = A \cap GL^{(0)}(S).$$

We see that the groups A_0 and A' are Lie subgroups of $GL(S)$ with Lie algebras $\mathfrak{a}_0$ and $\mathfrak{a}' = \bigoplus_{r \geqq 0} \mathfrak{a}_r$ respectively. Since A' preserves the filtration $\{S^q\}_{q<0}$ of S, we get the representation ρ_W of A':

$$\rho_W \colon A' \to GL(W),$$

through the projection $\pi_0 \colon S = \bigoplus_{p=-\mu}^{-1} S_p \to S_{-\mu} = W$.

Let E_S be the vector bundle over $M = A/A'$ associated with the representation $\rho_W \colon A' \to GL(W)$; A' acts on $A \times W$ on the right by

$$(a, w)a' = (aa', \rho_W(a')^{-1}(w)),$$

for $a \in A$, $w \in W$ and $a' \in A'$. Then E_S is the vector bundle over $M = A/A'$ defined by $E_S = A \times W/A'$.

As is well known, the space $\Gamma(E_S)$ of global sections of E_S is identified with the space $\mathcal{F}(A, W)_{A'}$ of all W-valued functions f on A satisfying

$$f(aa') = \rho_W(a')^{-1} f(a),$$

for $a \in A$ and $a' \in A'$, via the correspondence $f \in \mathcal{F}(A, W)_{A'} \mapsto \sigma_f \in \Gamma(E_S)$ given by

$$\sigma_f(\pi_1(a)) = \pi_2(a, f(a)),$$

where $\pi_1 \colon A \to M = A/A'$ and $\pi_2 \colon A \times W \to E_S$ denote the natural projections. Then each $s \in S$ defines an element $\sigma_s \in \Gamma(E_S)$ via the above correspondence by

$$f_s(a) = \pi_0(\rho(a^{-1})s)$$

for $a \in A$.

At each point $x \in M = A/A'$, let $(R_S)_x$ be the subspace of $J^\mu_x(E_S)$ defined by

$$(R_S)_x = \{j^\mu_x(\sigma_s) \mid s \in S\}.$$

where $j^\mu_x(\sigma_s)$ is the μ-jet at x of the section σ_s. Let R_S be the subbundle of $J^\mu(E_S)$ defined by

$$R_S = \bigcup_{x\in M}(R_S)_x.$$

Then we have

Proposition A (Proposition 2.4.1 [Sea88]). *R_S is an integrable system of linear differential equations of finite type of order μ of type S and every local solution of R_S is a restriction of σ_s for some $s \in S$.*

We call R_S the system of equations *modeled after S*. R_S is the system of differential equations which characterizes global sections of E_S given by elements of S even locally. By the construction, it follows that R_S is locally isomorphic with the constant coefficient differential equations $\hat{R}_S$.

Here the condition for R_S to be a system of differential equations of type S means that the total symbol S_x of R_S is isomorphic with $S = \bigoplus_{r=0}^{\mu-1} S_{r-\mu} \subset \bigoplus_{r=0}^{\mu} W \otimes S^rV^*$ at each $x \in M$. For the precise definition in terms of the jet bundle theory, we refer the reader to §2 of [Sea88] or §2.1 of [SYY97].

What is important for us here is that, as a submanifold of $J^\mu(E_S)$, R_S has the pseudo-product structure induced from canonical systems C^μ and $C^{\mu-1}$ as in §1.1. Then, when S is the typical symbol of type $(\mathfrak{l}, S)$, the above condition is equivalent to say that the symbol algebra of the pseudo-product structure is isomorphic with $\mathfrak{m} = S \oplus \mathfrak{l}_{-1}$ at each $v \in R_S$ (cf. [Yam82]), which follows from the fact that R_S is locally isomorphic with $\hat{R}_S$. R_S is our (global) model equations for the pseudo-product structures associated with the PPGLA $\mathfrak{G}$ of type $(\mathfrak{l}, S)$.

Moreover it follows from (3.3) that $A/A' = \hat{L}/\hat{L}'$, where $\hat{L}$ is the Lie subgroup of A with Lie algebra $\hat{\mathfrak{l}} = \mathfrak{l}_{-1} \oplus [\mathfrak{l}_{-1}, \mathfrak{l}_1] \oplus \mathfrak{l}_1$. Especially, in the case of pseudo-projective GLA $\mathfrak{g}^k(n,m)$ of order k of bidegree (n,m), we see from (2.1) that the model space M coincides with the projective space $\mathbb{P}^n$ and it is known ([Tan89]) that the vector bundle E_S is the tensor product $\bar{W} \otimes H^{k-1}$ of the trivial bundle $\bar{W} = \mathbb{P}^n \times W$ with the $(k-1)$-th power of the hyperplane bundle H over $\mathbb{P}^n$ (see the discussion in §3.2).

Y. Se-ashi developed in [Sea88] the theory for the linear equivalence of integrable higher order differential equations R of finite type with the typical symbol of type $(\mathfrak{l}, S)$. He gave the complete system of differential invariants of R and interpreted these invariants in terms of Cartan connections constructed over R. Utilizing these invariants he showed the following **"Rigidity Theorem"** in the linear equivalence of these equations.

Theorem A (cf. Corollary 3 [SYY97]). *Let $\mathfrak{l} = \mathfrak{l}_{-1} \oplus \mathfrak{l}_0 \oplus \mathfrak{l}_1$ be a simple graded Lie algebra over $\mathbb{C}$ and let $M = L/L'$ be the model space associated with $\mathfrak{l} = \mathfrak{l}_{-1} \oplus \mathfrak{l}_0 \oplus \mathfrak{l}_1$. Let S be a faithful irreducible $\mathfrak{l}$-module. Then, except when $M = \mathbb{P}^n$ or Q^n, every integrable system R of differential equations of type S is locally isomorphic with the model system R_S of type $(\mathfrak{l}, S)$, where $\mathbb{P}^n$ is the projective space and Q^n is the hyperquadric in $\mathbb{P}^{n+1}$.*

We will discuss the invariants of the pseudo-product structure on these equations in subsequent sections.

3.2. Plücker embedding equations

In order to discuss the Plücker embedding equations, a little generally, we will consider here projective embedding of hermitian symmetric spaces, following §1 in [SYY97].

Group-theoretically, a compact irreducible hermitian symmetric space M corresponds to a simple graded Lie algebra over $\mathbb{C}$ of the first kind as follows: Let $\mathfrak{l} = \mathfrak{l}_{-1} \oplus \mathfrak{l}_0 \oplus \mathfrak{l}_1$ be a simple graded Lie algebra of the first kind, i.e.,

(1) $\mathfrak{l}$ is a simple Lie algebra over $\mathbb{C}$.

(2) $\mathfrak{l} = \mathfrak{l}_{-1} \oplus \mathfrak{l}_0 \oplus \mathfrak{l}_1$ is a vector space direct sum such that $\mathfrak{l}_{-1} \neq \{0\}$.

(3) $[\mathfrak{l}_p, \mathfrak{l}_q] \subset \mathfrak{l}_{p+q}$, where $\mathfrak{l}_p = \{0\}$ for $|p| \geqq 2$.

Let L be the simply connected Lie group with Lie algebra $\mathfrak{l}$ and L' be the analytic subgroup of L with Lie algebra $\mathfrak{l}' = \mathfrak{l}_0 \oplus \mathfrak{l}_1$. Then $M = L/L'$ is a compact (irreducible) hermitian symmetric space and every compact irreducible hermitian symmetric space is obtained in this manner from a simple graded Lie algebra of the first kind. M is called the **model space** associated with $\mathfrak{l} = \mathfrak{l}_{-1} \oplus \mathfrak{l}_0 \oplus \mathfrak{l}_1$. For example, when $M = \mathrm{Gr}(k, l+1)$ is the Grassmann manifold of k-dimensional subspaces in $\mathbb{C}^{l+1}$, we have $\mathfrak{l} = \mathfrak{sl}(l+1, \mathbb{C})$ and the gradation $\mathfrak{l} = \mathfrak{l}_{-1} \oplus \mathfrak{l}_0 \oplus \mathfrak{l}_1$ is given by subdividing matrices as (2.2) in §2.2. As the extreme case $k = 1$, we have the projective space $M = \mathbb{P}^l = P(\hat{V})$. In this case $\mathfrak{l} = \mathfrak{sl}(l+1, \mathbb{C}) = \mathfrak{sl}(\hat{V})$ and the gradation is given as in (2.1) in §2.1.

An equivariant projective embedding of the model space $M = L/L'$ can be obtained from an irreducible representation of L as follows: Let $\tau \colon L \to GL(T)$ be an irreducible representation of L with the highest weight Λ. Let t_Λ be a maximal vector in T of the highest weight Λ. Then a stabilizer of the line $[t_\Lambda]$ spanned by t_Λ in T is a parabolic subgroup of L. When this stabilizer coincides with L', we obtain an equivariant projective embedding of $M = L/L'$ by taking the L-orbit passing through $[t_\Lambda]$ in the projective space $P(T)$ consisting of all lines in T passing through the origin. For example, when $M = \mathrm{Gr}(k, l+1)$, we take the exterior representation τ_0 of $L = SL(l+1,\mathbb{C})$ on $T = \bigwedge^k \mathbb{C}^{l+1}$:

$$\tau_0 \colon SL(l+1,\mathbb{C}) \to GL\left(\bigwedge^k \mathbb{C}^{l+1}\right),$$

where $\tau_0(a)(v_1 \wedge \cdots \wedge v_k) = a(v_1) \wedge \cdots \wedge a(v_k)$ for $a \in SL(l+1,\mathbb{C})$ and $v_i \in \mathbb{C}^{l+1}$ $(i = 1,\ 2,\dots,k)$. Let $\{e_1,\dots,e_{l+1}\}$ be the natural basis of $\mathbb{C}^{l+1}$. Then τ_0 is an irreducible representation of $SL(l+1,\mathbb{C})$ with the maximal vector $e_1 \wedge \cdots \wedge e_k$ for a suitable choice of a Cartan subalgebra $\mathfrak{h}$ and a simple root system Δ of $\mathfrak{sl}(l+1,\mathbb{C})$. τ_0 is the irreducible representation of $SL(l+1,\mathbb{C})$ with the highest weight ϖ_k, where $\{\varpi_1,\dots,\varpi_l\}$ is the set of fundamental dominant weight relative to Δ (see §4.5). From (2.2), we see that the stabilizer of the line $[e_1 \wedge \cdots \wedge e_k]$ coincides with L'. Thus we see that the Plücker embedding of $\mathrm{Gr}(k, l+1)$ is obtained from the irreducible representation τ_0 of $SL(l+1,\mathbb{C})$.

As other examples, we take the symmetric representation ν_k of $L = SL(\hat{V})$ on $T = S^k(\hat{V})$, where $S^k(\hat{V})$ denotes the k-th symmetric product of $\hat{V}$:

$$\nu_k \colon SL(\hat{V}) \to GL(S^k(\hat{V})),$$

where $\nu_k(a)(v_1 \circledcirc \cdots \circledcirc v_k) = a(v_1) \circledcirc \cdots \circledcirc a(v_k)$ for $a \in SL(\hat{V})$ and $v_i \in \hat{V}$ $(i = 1,\ 2,\dots,k)$ and $\circledcirc$ is the symmetric product. Let us take a highest weight vector $v_o \in \hat{V}$ of the identity representation ν_1. Here $\hat{V} = \langle\{v_o\}\rangle \oplus V$ in the notation of (2.1). Then $v_o^k = v_0 \circledcirc \cdots \circledcirc v_o$ is the highest weight vector of ν_k. ν_k is the irreducible representation of $SL(\hat{V})$ with the highest weight $k\varpi_1$. From (2.1), we see that the stabilizer of the line $[v_o^k]$ coincides with L'. Thus we obtain projective embeddings of $P(\hat{V})$ from irreducible representations ν_k. Here we note that the line bundle over $P(\hat{V}) = L/L'$ obtained from the representation of L' on $W_0 = \langle\{v_o\}\rangle$ is isomorphic with the universal bundle U over $P(\hat{V})$. Hence the line bundle over $P(\hat{V})$ obtained from the representation of L' on $W_k = \langle\{v_o^k\}\rangle$ is isomorphic with the k-th power U^k of U.

Next, for an irreducible representation $\tau \colon L \to GL(T)$, we will construct a (positive) line bundle F over M such that the above orbit is obtained as an embedding of M by global sections of F. To construct F, let us take the dual representation $\rho\colon L \to GL(S)$ of τ, i.e., $S = T^*$ is the dual space of T and $\rho = \tau^*$ is defined by

$$\langle \rho(g)(\xi), t\rangle = \langle \xi, \tau(g^{-1})(t)\rangle,$$

for $g \in L$, $t \in T$, $\xi \in T^*$ and $\langle\ ,\ \rangle$ is the canonical pairing between T^* and T. Then, when τ is an irreducible representation with the highest weight Λ (for a fixed choice of a Cartan subalgebra and a simple root system of $\mathfrak{l}$), ρ is the irreducible representation with the lowest weight $-\Lambda$. Let us take a basis $\{t_1, \ldots, t_r\}$ of T consisting of weight vectors of τ such that $t_1 = t_\Lambda$. Then the dual basis $\{s_1, \ldots, s_r\}$ of $\{t_1, \ldots, t_r\}$ in $S = T^*$ consists of weight vectors of ρ and s_1 is a weight vector corresponding to $-\Lambda$. Let W and W' be the subspaces of S spanned by a vector s_1 and by vectors $s_2, \ldots, s_r$, respectively. Then, since s_1 is a lowest weight vector, we have $W = S_{-\mu}$ in the notation of Lemma 2.1. Since L' is the stabilizer of the line $[t_1]$, W' is preserved by L'. Hence we get the representation ρ_W of L':

$$\rho_W \colon L' \to GL(W),$$

through the projection $\pi_0 \colon S = W \oplus W' \to W$.

Relative to the representation ρ_W, L' acts on $L \times W$ on the right by

$$(g, w)g' = (gg', \rho_W(g')^{-1}(w)),$$

for $g \in L$, $w \in W$ and $g' \in L'$. Then $F = L \times W/L'$ is the line bundle over $M = L/L'$.

As in §3.1, the space $\Gamma(F)$ of global sections of F is identified with the space $\mathcal{F}(L, W)_{L'}$ of all W-valued functions f on L satisfying

$$f(gg') = \rho_W(g')^{-1} f(g),$$

for $g \in L$ and $g' \in L'$. Then each $s \in S$ defines an element $\sigma_s \in \Gamma(F)$ via this correspondence by

$$f_s(g) = \pi_0(\rho(g^{-1})s)$$

for $g \in L$.

Now let us check that global sections of F give the desired embedding of M into $P(T)$. We utilize the above basis $\{t_1, \ldots, t_r\}$ and $\{s_1, \ldots, s_r\}$ of T and $S = T^*$. Let us consider a map $\hat{\varphi}$ of L into T defined by

$$\hat{\varphi}(g) = \sum_{i=1}^{r} \langle f_{s_i}(g), t_1 \rangle t_i \tag{3.4}$$

for $g \in L$. Then, from $\langle f_{s_i}(g), t_1 \rangle = \langle \rho(g^{-1}) s_i, t_1 \rangle$, $\hat{\varphi}$ induces a map φ of M into $P(T)$ satisfying the commutative diagram

$$\begin{array}{ccc} L & \xrightarrow{\hat{\varphi}} & T \backslash \{0\} \\ \downarrow & & \downarrow \\ M = L/L' & \xrightarrow{\varphi} & P(T). \end{array}$$

For $g \in L$, if we represent $\tau(g)$ as a matrix A with respect to the basis $\{t_1, \dots, t_r\}$, $\rho(g^{-1})$ is represented by the transposed matrix ${}^tA$ of A with respect to the basis $\{s_1, \dots, s_r\}$. From (3.4), $\hat{\varphi}(g)$ corresponds to the first row vector of ${}^tA$. Hence we obtain

$$\hat{\varphi}(g) = \tau(g)(t_1).$$

Thus the image of φ coincides with the L-orbit passing through $[t_1]$ in $P(T)$.

In particular we see that, for the model equation R_S associated with the pseudo-projective GLA $\mathfrak{g}^k(n,1)$, the line bundle E_S is isomorphic with the $(k-1)$-th power H^{k-1} of the hyperplane bundle H over $P(\hat{V})$, which is dual to the line bundle over $P(\hat{V})$ obtained from the representation of L' on $W_{k-1} = \langle\{v_o^{k-1}\}\rangle$.

Furthermore we see that the Plücker embedding of $\mathrm{Gr}(k, l+1)$ into $P(\bigwedge^k \mathbb{C}^{l+1})$ is obtained by global sections of the line bundle F, which is constructed from the irreducible representation ρ_0 of $SL(l+1, \mathbb{C})$ on $S = \bigwedge^{l-k+1} \mathbb{C}^{l+1}$. Here ρ_0 is the dual representation of τ_0 on $T = \bigwedge^k \mathbb{C}^{l+1}$.

Let $R^{\rho_0} = R_S$ be the system of equations modeled after $S = \bigwedge^{l-k+1} \mathbb{C}^{l+1}$ constructed in §3.1. Then, by Proposition A, R^{ρ_0} is the system of equations of finite type, whose local solution is the restriction of a global section of F and whose projective solution coincides with the Plücker embedding of $\mathrm{Gr}(k, l+1)$ (cf. §1 of [SYY97]). Thus R^{ρ_0} can be called the **Plücker embedding equation**. Theorem A in §3.1 states the rigidity for these equations in the linear equivalence. For the application of these facts to a problem of the hypergeometric systems, we refer the reader to [SYY97]. We will discuss the rigidity property of these equations in the contact equivalence in §7.

In fact, the symbol algebra $\mathfrak{m}$ of R^{ρ_0} is already calculated in the last paragraph in §2.2 and we see that R^{ρ_0} is a system of order $\mu = \min\{k+1, l-k+2\}$ such that $S_{-\mu} = \mathbb{C}$ and $S_{1-\mu} = V^*$. Namely the system R^{ρ_0} has no equation of the first order. Then, since the symbol algebra $\mathfrak{m}$ of R^{ρ_0} is generated by $\mathfrak{g}_{-1}$ (Lemma 2.1.(2)), it follows from Corollary 5.4 [Yam82] that the equivalence of the pseudo-product structure on the equation R of type $\mathfrak{m}$ is the equivalence of the μ-th order equation under contact transformations.

§4. Generalized Spencer Cohomology

From this section, we assume that the ground field is the field $\mathbb{C}$ of complex numbers for the sake of simplicity. For the discussion over $\mathbb{R}$, the corresponding results will be obtained easily through the argument of complexification as in §3.2 in [Yam93]. We use the following notation: For a graded vector space $V = \bigoplus_{p\in\mathbb{Z}} V_p$, we put $V_{\leqq k} = \bigoplus_{p\leqq k} V_p$ and $V_{\geqq k} = \bigoplus_{p\geqq k} V_p$. In particular, we set $V_- = V_{\leqq -1}$ and $V_+ = V_{\geqq 1}$. For a Lie algebra $\mathfrak{g}$ and a subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, $\mathrm{Der}(\mathfrak{g})$ denotes the Lie algebra of all derivations of $\mathfrak{g}$, $\mathcal{D}(\mathfrak{g})$ denotes the derived subalgebra $[\mathfrak{g},\mathfrak{g}]$ of $\mathfrak{g}$, $\mathfrak{z}(\mathfrak{g})$ denotes the center of $\mathfrak{g}$, and $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{h})$ denotes the centralizer of $\mathfrak{h}$ in $\mathfrak{g}$. For a $\mathfrak{g}$-module M, we denote by $\mathrm{ch}_{\mathfrak{g}}(M)$ the isomorphism class of M.

4.1. Cohomology of Lie algebras

Let $\mathfrak{a} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{a}_p$ be a finite dimensional GLA and $V = \bigoplus_{p\in\mathbb{Z}} V_p$ be a graded $\mathfrak{a}$-module. (i.e., V is a vector space with a gradation such that $\mathfrak{a}_p \cdot V_q \subset V_{p+q}$.) Then we have a cohomology space $H^p(\mathfrak{a}, V)$ associated with the cochain complex $(C^p(\mathfrak{a}, V), \partial)$, where $C^p(\mathfrak{a}, V) = \mathrm{Hom}(\bigwedge^p \mathfrak{a}, V)$ and the coboundary operator $\partial^p\colon C^p(\mathfrak{a}, V) \to C^{p+1}(\mathfrak{a}, V)$ is defined by

$$\begin{aligned}
&\partial^p\omega(x_1, \dots, x_{p+1}) \\
&\quad = \sum_{i=1}^{p+1}(-1)^{i+1} x_i \cdot \omega(x_1, \dots, \hat{x}_i, \dots, x_{p+1}) \\
&\qquad + \sum_{i<j}(-1)^{i+j}\omega([x_i, x_j], x_1, \dots, \hat{x}_i, \dots, \hat{x}_j, \dots, x_{p+1}),
\end{aligned}$$

where $x_i \in \mathfrak{a}$ and $\omega \in C^p(\mathfrak{a}, V)$. Since both $\mathfrak{a}$ and V are graded, we have the natural gradation: $C^p(\mathfrak{a}, V) = \bigoplus_r C^p(\mathfrak{a}, V)_r$, where

$$C^p(\mathfrak{a}, V)_r = \{\omega \in C^p(\mathfrak{a}, V) \mid \omega(\mathfrak{a}_{i_1} \wedge \cdots \wedge \mathfrak{a}_{i_p}) \subset V_{i_1+\cdots+i_p+r}\}.$$

It is easy to see that $C^*(\mathfrak{a}, V)_r = \bigoplus C^p(\mathfrak{a}, V)_r$ is a subcomplex of $C^*(\mathfrak{a}, V)$, whose cohomology space (resp. p-th cohomology space) will be denoted by $H(\mathfrak{a}, V)_r$ (resp. $H^p(\mathfrak{a}, V)_r$). Then we have

$$H^*(\mathfrak{a}, V) = \bigoplus_r H^*(\mathfrak{a}, V)_r = \bigoplus_p H^p(\mathfrak{a}, V) = \bigoplus_{p,r} H^p(\mathfrak{a}, V)_r.$$

4.2. Generalized Spencer cohomology $H^*(\mathfrak{G})$ and $H^*(\mathfrak{b}_-, \mathfrak{g})$

Let $\mathfrak{G} = (\mathfrak{g}, (\mathfrak{g}_p)_{p\in\mathbb{Z}}, \mathfrak{l}_{-1}, S_{-1})$ be a PPGLA of type $(\mathfrak{l}, S)$. We set $\mathfrak{b}_{-1} = S$, $\mathfrak{b}_0 = \mathfrak{l}$ and $\mathfrak{b}_p = 0$ $(p \neq -1,\ 0)$; then $\mathfrak{g}$ has a bigradation $(\mathfrak{g}_{p,q})_{p,q\in\mathbb{Z}}$, where $\mathfrak{g}_{p,q} = \mathfrak{g}_p \cap \mathfrak{b}_q$. We have the cohomology group $H^*(\mathfrak{G}) = H^*(\mathfrak{m}, \mathfrak{g})$ associated with the adjoint representation of $\mathfrak{m} = \mathfrak{g}_-$ on $\mathfrak{g}$, that is, the cohomology space of the cochain complex $C^*(\mathfrak{G}) = \bigoplus C^p(\mathfrak{G})$ with the coboundary operator $\partial\colon C^p(\mathfrak{G}) \longrightarrow$

$C^{p+1}(\mathfrak{G})$, where $C^p(\mathfrak{G}) = \mathrm{Hom}(\bigwedge^p \mathfrak{g}_-, \mathfrak{g})$. We put

$$C^p(\mathfrak{G})_{r,s} = \{\omega \in C^p(\mathfrak{G}) \mid \omega(\mathfrak{g}_{i_1,j_1} \wedge \cdots \wedge \mathfrak{g}_{i_p,j_p}) \subset \mathfrak{g}_{i_1+\cdots+i_p+r,\, j_1+\cdots+j_p+s} \quad \text{for all} \quad i_1, \dots, i_p,\ j_1, \dots, j_p\}.$$

As is easily seen, $C^*(\mathfrak{G})_{r,s} = \bigoplus_p C^p(\mathfrak{G})_{r,s}$ is a subcomplex of $C^*(\mathfrak{G})$. Denoting its cohomology space by $H(\mathfrak{G})_{r,s} = \bigoplus H^p(\mathfrak{G})_{r,s}$, we obtain the direct sum decomposition

$$H^*(\mathfrak{G}) = \bigoplus_{p,r,s} H^p(\mathfrak{G})_{r,s}.$$

The cohomology space, endowed with this tri-gradation, is called the generalized Spencer cohomology space of the PPGLA $\mathfrak{G}$ of type $(\mathfrak{l}, S)$. Note that $H^1(\mathfrak{G})_{0,0} = 0$ if and only if $\check{\mathfrak{g}}_0 = \mathfrak{g}_0$, where $\check{\mathfrak{g}}_0$ is the Lie algebra of derivations of $\mathfrak{m}$ such that $D(\mathfrak{g}_p) \subset \mathfrak{g}_p$ $(p < 0)$, $D(\mathfrak{l}_{-1}) \subset \mathfrak{l}_{-1}$ and $D(S_{-1}) \subset S_{-1}$.

Furthermore we have the cohomology space $H^*(\mathfrak{b}_-, \mathfrak{g})$ associated with the adjoint representation of $\mathfrak{b}_-$ on $\mathfrak{g}$, that is, the cohomology space of the cochain complex $C^*(\mathfrak{b}_-, \mathfrak{g}) = \bigoplus C^p(\mathfrak{b}_-, \mathfrak{g})$ with the coboundary operator $\delta^p \colon C^p(\mathfrak{b}_-, \mathfrak{g}) \to C^{p+1}(\mathfrak{b}_-, \mathfrak{g})$, where $C^p(\mathfrak{b}_-, \mathfrak{g}) = \mathrm{Hom}(\bigwedge^p \mathfrak{b}_-, \mathfrak{g})$. Let $C^p(\mathfrak{b}_-, \mathfrak{g})_s$ be the subspace of $C^p(\mathfrak{b}_-, \mathfrak{g})$ consisting of all the elements $\omega \in C^p(\mathfrak{b}_-, \mathfrak{g})$ such that $\omega(\mathfrak{b}_{j_1} \wedge \cdots \wedge \mathfrak{b}_{j_p}) \subset \mathfrak{b}_{j_1+\cdots+j_p+s}$ for all $j_1, \dots, j_p < 0$. $C^*(\mathfrak{b}_-, \mathfrak{g})_s = \bigoplus_p C^p(\mathfrak{b}_-, \mathfrak{g})_s$ is a subcomplex of $C^*(\mathfrak{b}_-, \mathfrak{g})$. Denoting its cohomology space by $H^*(\mathfrak{b}_-, \mathfrak{g})_s = \bigoplus H^p(\mathfrak{b}_-, \mathfrak{g})_s$, we obtain the direct sum decomposition

$$H^*(\mathfrak{b}_-, \mathfrak{g}) = \bigoplus_{p,s} H^p(\mathfrak{b}_-, \mathfrak{g})_s.$$

The cohomology space, endowed with this bi-gradation, is called the Spencer cohomology of the GLA $\mathfrak{g} = \mathfrak{b}_{-1} \oplus \mathfrak{b}_0$.

4.3. Calculation of the cohomology $H^p(\mathfrak{G})_{r,s}$

Here we prepare the fundamental theorem to calculate the cohomology space $H^p(\mathfrak{G})_{r,s}$ by means of Kostant's theorem.

Since $H^i(\mathfrak{b}_-, \mathfrak{g})_s$ naturally has a graded $\mathfrak{l}$-module structure, we obtain the following direct sum decomposition

$$H^p(\mathfrak{l}_-, H^i(\mathfrak{b}_-, \mathfrak{g})_s) = \bigoplus_r H^p(\mathfrak{l}_-, H^i(\mathfrak{b}_-, \mathfrak{g})_s)_r.$$

For $i = s,\ s+1$, we set

$$C_s^{p,i} = \mathrm{Hom}\left(\bigwedge^{p-i} \mathfrak{l}_- \otimes \bigwedge^{i} \mathfrak{b}_{-1}, \mathfrak{b}_{s-i}\right),$$

and $C_s^p = C_s^{p,s} \oplus C_s^{p,s+1}$. For $\omega \in C_s^{p,i}$, we denote by $\partial_{j,i}^p \omega$ the $C_s^{p,j}$-component of $\partial^p \omega$. Then $\partial_{j,i}$ is a $\mathfrak{g}_0$-module homomorphism of $C_s^{p,i}$ into $C_s^{p+1,j}$.

Lemma 4.1. *Under the above assumption, we have*
(1) $\partial_{j,i}^p = 0$ *for* $j \neq i,\ i+1$.
(2) $\partial_{i,i}^{p+1}\partial_{i,i}^p = 0,\ \partial_{i+2,i+1}^{p+1}\partial_{i+1,i}^p = 0$.
(3) $\partial_{i+1,i}^{p+1}\partial_{i,i}^p + \partial_{i+1,i+1}^{p+1}\partial_{i+1,i}^p = 0$.

Proof. Let $\omega \in C_s^{p,i}$. Then, for $x_1, \ldots, x_q \in \mathfrak{l}_-\ m_1, \ldots, m_{k+i} \in \mathfrak{b}_-$ $(k \in \mathbb{Z})$,

$$\begin{aligned}
&\partial\omega(x_1, \ldots, x_q,\ m_1, \ldots, m_{k+i}) \\
&= \sum_{a=1}^{q} (-1)^{a+1}[x_a, \omega(x_1, \ldots, \hat{x}_a, \ldots, x_q,\ m_1, \ldots, m_{k+i})] \\
&\quad + \sum_{a=1}^{i+k} (-1)^{q+a-1}[m_a, \omega(x_1, \ldots, x_q,\ m_1, \ldots, \hat{m}_a, \ldots, m_{k+i})] \\
&\quad + \sum_{a,b} (-1)^{a+q+b}\omega([x_a, m_b], x_1, \ldots, \hat{x}_a, \ldots, x_q,\ m_1, \ldots, \hat{m}_b, \ldots, m_{k+i})
\end{aligned}$$

where $q = p - i + k + 1$. If $k \neq 0,\ 1$, then

$$\partial\omega(x_1, \ldots, x_{p-i-k+1},\ m_1, \ldots, m_{k+i}) = 0.$$

Thus $\partial_{j,i}^p = 0$ for $j \neq i,\ i+1$, which proves (1). Moreover we see that

$$\begin{aligned}
0 = \partial^{p+1}\partial^p\omega &= \partial^{p+1}(\partial_{ii}^p\omega + \partial_{i+1,i}^p\omega) \\
&= \partial_{i,i}^{p+1}\partial_{i,i}^p\omega + \partial_{i+1,i}^{p+1}\partial_{ii}^p\omega + \partial_{i+1,i+1}^{p+1}\partial_{i+1,i}^p\omega + \partial_{i+2,i+1}^{p+1}\partial_{i+1,i}^p\omega.
\end{aligned}$$

This proves (2) and (3). Q.E.D.

We define a linear mapping $\phi_{p,i}$ of $C_s^{p,i}$ onto $C^{p-i}(\mathfrak{l}_-, C^i(\mathfrak{b}_-, \mathfrak{g})_s)$ $(i = s,\ s+1)$ as follows:

$$\phi_{p,i}(\omega)(x_1, \ldots, x_{p-i})(m_1, \ldots, m_i) = \omega(x_1, \ldots, x_{p-i},\ m_1, \ldots, m_i).$$

Then $\phi_{p,i}$ is a $\mathfrak{g}_0$-module isomorphism.

From the proof of Lemma 4.1, we have

$$
\begin{aligned}
&\partial_{i,i}\omega(x_1,\dots,x_q,\ m_1,\dots,m_i)\\
&=\sum_{a=1}^{q}(-1)^{a+1}[x_a,\omega(x_1,\dots,\hat{x}_a,\dots,x_q,\ m_1,\dots,m_i)]\\
&\quad+\sum_{a,b}(-1)^{a+q+b}\omega([x_a,m_b],x_1,\dots,\hat{x}_a,\dots,x_q,\ m_1,\dots,\hat{m}_b,\dots,m_i),
\end{aligned}
$$

where $\omega\in C^{p,i}_s$ and $q=p-i+1$.

$$
\begin{aligned}
&\partial_{i+1,i}\omega(x_1,\dots,x_q,\ m_1,\dots,m_{i+1})\\
&\qquad=\sum_{a=1}^{i+1}(-1)^{r+a-1}[m_a,\omega(x_1,\dots,x_r,\ m_1,\dots,\hat{m}_a,\dots,m_{i+1})],
\end{aligned}
$$

where $\omega\in C^{p,i}_s$ and $r=p-i+2$. Hence we obtain the following lemma.

Lemma 4.2. *Let $\omega\in C^{p,i}_s$ $(i=s,\ s+1)$. Then:*

(1) $\phi_{p+1,i}(\partial^p_{i,i}\omega)=\rho(\phi_{p,i}(\omega))$, *where ρ is the coboundary operator of $C^*(\mathfrak{l}_-,C^i(\mathfrak{b}_-,\mathfrak{g})_s)$.*

(2) *For $x_1,\dots,x_{p-i}\in\mathfrak{l}_{-1}$, we have*

$$\phi_{p+1,i}(\partial^p_{i+1,i}\omega)(x_1,\dots,x_{p-i})=(-1)^{p-i}\delta^i(\phi_{p,i}(\omega)(x_1,\dots,x_{p-i})).$$

We recall that there exist an anti-linear involution τ of $\mathfrak{l}$ and an hermitian inner product $(\cdot\mid\cdot)$ of $\mathfrak{g}$ having the following properties:

(i) $\tau(\mathfrak{l}_p)=\mathfrak{l}_{-p}$;

(ii) $(\mathfrak{b}_p\mid\mathfrak{b}_q)=0$ for $p\neq q$ and, $(\mathfrak{g}_p\mid\mathfrak{g}_q)=0$ for $p\neq q$;

(iii) $([x,s]\mid s')+(s\mid[\tau(x),s'])=0$ for all $x\in\mathfrak{l}$ and $s,\ s'\in\mathfrak{b}_{-1}$.

The inner product $(\cdot\mid\cdot)$ induces inner products on $\mathfrak{b}^*_{-1}$ and $C^*(\mathfrak{b}_-,\mathfrak{g})$, which are denoted by the same symbol. Namely, for $\omega=\alpha\otimes v_1^*\wedge\cdots\wedge v_p^*$, $\omega'=\beta\otimes w_1^*\wedge\cdots\wedge w_p^*$, we see that

$$(\omega\mid\omega')=(\alpha\mid\beta)\det(v_i^*\mid w_j^*),$$

where $\alpha,\ \beta\in\mathfrak{g}$, $v_i^*,\ w_i^*\in\mathfrak{b}^*_{-1}$.

Now we denote by δ^* the adjoint operator of δ with respect to the inner product $(\cdot\mid\cdot)$, i.e.,

$$(\delta\omega\mid\omega')=(\omega\mid\delta^*\omega')\qquad\text{for}\qquad\omega,\ \omega'\in C^*(\mathfrak{b}_-,\mathfrak{g}).$$

Then δ^* is a $\mathfrak{b}_0$-module homomorphism and we have $\delta^*(C^p(\mathfrak{b}_-,\mathfrak{g})_s) \subset C^{p-1}(\mathfrak{b}_-,\mathfrak{g})_s$. As usual, the operator $\Delta = \delta\delta^* + \delta^*\delta$ is called the Laplacian. We set $\mathcal{H}^p(\mathfrak{b}_-,\mathfrak{g})_s = \{\omega \in C^p(\mathfrak{b}_-,\mathfrak{g})_s \mid \Delta\omega = 0\}$; then

$$C^p(\mathfrak{b}_-,\mathfrak{g})_s = \mathcal{H}^p(\mathfrak{b}_-,\mathfrak{g})_s \oplus \Delta(C^p(\mathfrak{b}_-,\mathfrak{g})_s).$$

We put $\mathcal{D}^p(\mathfrak{b}_-,\mathfrak{g})_s = \Delta(C^p(\mathfrak{b}_-,\mathfrak{g})_s)$. Then $\mathcal{H}^p(\mathfrak{b}_-,\mathfrak{g})_s$ is isomorphic to $H^p(\mathfrak{b}_-,\mathfrak{g})_s$ as an $\mathfrak{l}$-module and $\mathcal{D}^p(\mathfrak{b}_-,\mathfrak{g})_s = \delta^2(C^p_s) \oplus (\operatorname{Ker}\delta \mid C^p_s)^\perp$. We put

$$\mathcal{H}^{p,i}_s = \phi^{-1}_{p,i}\left(\operatorname{Hom}\left(\bigwedge^{p-i}\mathfrak{l}_-, \mathcal{H}^i(\mathfrak{b}_-,\mathfrak{g})_s\right)\right),$$

$$\mathcal{D}^{p,i}_s = \phi^{-1}_{p,i}\left(\operatorname{Hom}\left(\bigwedge^{p-i}\mathfrak{l}_-, \mathcal{D}^i(\mathfrak{b}_-,\mathfrak{g})_s\right)\right),$$

$$\mathcal{H}^p_s = \bigoplus_i \mathcal{H}^{p,i}_s, \quad \mathcal{D}^p_s = \bigoplus_i \mathcal{D}^{p,i}_s.$$

Then $C^{p,i}_s = \mathcal{H}^{p,i}_s \oplus \mathcal{D}^{p,i}_s$ and we obtain the following exact sequence:

$$\mathcal{D}^{p-1,i-1}_s \xrightarrow{\partial_{i,i-1}} \mathcal{D}^{p,i}_s \xrightarrow{\partial_{i+1,i}} \mathcal{D}^{p+1,i+1}_s. \tag{4.1}$$

From Lemma 4.2, we obtain the following lemma.

Lemma 4.3.
(1) $\partial^p_{i,i}(\mathcal{H}^{p,i}_s) \subset \mathcal{H}^{p+1,i}_s$, $\partial^p_{i+1,i}(\mathcal{H}^{p,i}_s) = 0$.
(2) $\partial^p_{i,i}(\mathcal{D}^{p,i}_s) \subset \mathcal{D}^{p+1,i}_s$, $\partial^p_{i+1,i}(\mathcal{D}^{p,i}_s) \subset \mathcal{D}^{p+1,i+1}_s$.

Lemma 4.4. *Let $\omega \in \operatorname{Ker}\partial^p \cap C^p_s$. We decompose ω as follows: $\omega = \omega' + \omega''$, where $\omega' \in \mathcal{H}^p_s$ and $\omega'' \in \mathcal{D}^p_s$. Then $\omega'' \in \operatorname{Im}\partial^{p-1}$.*

Proof. We decompose ω'' as follows: $\omega'' = \omega''_s + \omega''_{s+1}$, where $\omega''_s \in \mathcal{D}^{p,s}_s$ and $\omega''_{s+1} \in \mathcal{D}^{p,s+1}_s$. Since $\partial\omega = 0$, we have $\partial^p_{s+2,s+1}\omega''_{s+1} = 0$ and $\partial^p_{s+1,s+1}\omega''_{s+1} + \partial^p_{s+1,s}\omega''_s = 0$. By (4.1), there is an element $c_s \in \mathcal{D}^{p-1,s}_s$ such that $\partial^{p-1}_{s+1,s}c_s = \omega''_{s+1}$. Furthermore $\omega''_s - \partial^{p-1}_{s,s}c_s \in \mathcal{D}^{p,s}_s$ and

$$\begin{aligned}
\partial^p_{s+1,s}(\omega''_s - \partial^{p-1}_{s,s}c_s) &= \partial^p_{s+1,s}\omega''_s - \partial^p_{s+1,s}\partial^{p-1}_{s,s}c_s \\
&= \partial^p_{s+1,s}\omega''_s + \partial^p_{s+1,s+1}\partial^{p-1}_{s+1,s}c_s \\
&= \partial^p_{s+1,s}\omega''_s + \partial^p_{s+1,s+1}\omega''_{s+1} = 0.
\end{aligned}$$

Since $\partial^p_{s+1,s} \mid \mathcal{D}^{p,s}_s$ is injective, we have $\omega''_s = \partial^{p-1}_{s,s}c_s$. Thus $\omega'' = \omega''_s + \omega''_{s+1} = (\partial^{p-1}_{s,s} + \partial^{p-1}_{s+1,s})c_s = \partial c_s$. Q.E.D.

We are ready to state the following fundamental theorem.

Theorem 4.1. *Let $\mathfrak{G}$ be a PPGLA of type $(\mathfrak{l}, S)$. Then $H^p(\mathfrak{G})_{r,s}$ is isomorphic to $\bigoplus_{i=0}^p H^{p-i}(\mathfrak{l}_-, H^i(\mathfrak{b}_-,\mathfrak{g})_s)_r$ as a $\mathfrak{g}_0$-module.*

Proof. We define a linear mapping

$$\Phi\colon C^p(\mathfrak{G})_{r,s} \to \bigoplus_{i=0}^{p} C^{p-i}(\mathfrak{l}_-, \mathcal{H}^i(\mathfrak{b}_-, \mathfrak{g})_s)_r$$

as follows: $\Phi(\omega) = \sum_{i=0}^{p} \phi_{p,i}(\omega_i')$, where $\omega = \sum_{i=0}^{p} \omega_i' + \omega'' \in C^p(\mathfrak{G})_{r,s}$, $\omega_i' \in \mathcal{H}_s^{p,i}$, $\omega'' \in \mathcal{D}_s^p$. By Lemma 4.2, $\Phi\partial = \partial\Phi$ and hence Φ induces a $\mathfrak{g}_0$-module homomorphism

$$\Phi^*\colon H^p(\mathfrak{G})_{r,s} \to \bigoplus_{i=0}^{p} H^{p-i}(\mathfrak{l}_-, \mathcal{H}^i(\mathfrak{b}_-, \mathfrak{g})_s)_r.$$

By Lemma 4.4, Φ^* is an isomorphism. Q.E.D.

4.4. Gradations of semisimple Lie algebras and Kostant's theorem

Here we recall some basic facts on gradations of semisimple Lie algebras following [Yam93] and state Kostant's theorem on Lie algebra cohomology, which is our basic tool in the discussion of subsequent sections.

Let $\mathfrak{s}$ be a complex semisimple Lie algebra. We fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{s}$ and the set Φ of roots of $\mathfrak{s}$ relative to $\mathfrak{h}$. Let us fix a simple root system $\Delta = \{\alpha_1, \dots, \alpha_l\}$ of Φ. For a subset Δ_1 of Δ and an integer p, we put

$$\Phi_p = \left\{ \alpha \in \Phi \;\middle|\; \sum_{\alpha_i \in \Delta_1} m_i(\alpha) = p \right\},$$

where m_i is a $\mathbb{Z}$-valued function on Φ defined by $m_i(\sum_{j=1}^{l} k_j\alpha_j) = k_i$. Then we can construct a gradation $(\mathfrak{s}_p)_{p\in\mathbb{Z}}$ of $\mathfrak{s}$ as follows:

$$\mathfrak{s}_0 = \mathfrak{h} \oplus \sum_{\alpha\in\Phi_0} \mathfrak{s}^\alpha, \quad \mathfrak{s}_p = \sum_{\alpha\in\Phi_p} \mathfrak{s}^\alpha \quad (p \neq 0),$$

where $\mathfrak{s}^\alpha$ is the root space corresponding to a root α of $\mathfrak{s}$. Then $\mathfrak{s} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{s}_p$ is a GLA such that $\mathfrak{s}_-$ is generated by $\mathfrak{s}_{-1}$. If $\mathfrak{s}$ is of type X_l, then the GLA $\mathfrak{s} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{s}_p$ is said to be of type (X_l, Δ_1), where X_l stands for the Dynkin diagram of $\mathfrak{s}$.

Conversely, let $\mathfrak{s} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{s}_p$ be a simple GLA such that $\mathfrak{s}_-$ is generated by $\mathfrak{s}_{-1}$. Assume that $\mathfrak{s}$ is of type X_l. Then $\mathfrak{s} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{s}_p$ is isomorphic to a simple GLA of type (X_l, Δ_1) for some $\Delta_1 \subset \Delta$. Let $\mathfrak{t} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{t}_p$ be an another simple GLA of type (X_l, Δ_1'). Then $\mathfrak{s} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{s}_p$ is isomorphic to $\mathfrak{t} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{t}_p$ if and only if there exists a diagram automorphism ϕ of X_l such that $\phi(\Delta_1) = \Delta_1'$ (Theorem 3.12 [Yam93]).

Let W be the Weyl group of $\mathfrak{s}$, Φ_+ the set of positive roots. Moreover we put

$$T_w = w\Phi_- \cap \Phi_+ \quad (\text{where } \Phi_- = \Phi \backslash \Phi_+),$$
$$W_1^j = \{w \in W \mid \#(T_w) = j, \quad T_w \subset \Phi(\mathfrak{s}_+)\},$$

where $\Phi(\mathfrak{s}_+) = \{\alpha \in \Phi \mid \mathfrak{s}^\alpha \subset \mathfrak{s}_+\}$. For an antidominant integral weight ω of $\mathfrak{s}$ (resp. $\mathfrak{s}_0$) we denote the irreducible $\mathfrak{s}$ (resp. $\mathfrak{s}_0$)-module with lowest weight ω by $M(\omega)$ (resp. $m(\omega)$). Then we have the following theorem due to Kostant.

Theorem B (Theorem 5.14 [Kos61]). *Let $\mathfrak{s} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{s}_p$ be a simple GLA of type (X_l, Δ_1) and $M(\omega)$ be an irreducible $\mathfrak{s}$-module with the lowest weight ω. Then*

$$\operatorname{ch}_{\mathfrak{s}_0}(H^j(\mathfrak{s}_-, M(\omega))) = \sum_{w\in W_1^j} \operatorname{ch}_{\mathfrak{s}_0}(m(w(\omega - \rho) + \rho)),$$

where ρ is the half sum of positive roots.

4.5. Parameterization of pseudo-product GLA of type $(\mathfrak{l}, S)$

Let $\mathfrak{G}$ be a PPGLA of type $(\mathfrak{l}, S)$. We set $\hat{\mathfrak{l}} = \mathfrak{l}_{-1} \oplus [\mathfrak{l}_{-1}, \mathfrak{l}_1] \oplus \mathfrak{l}_1$ and $\mathfrak{u} = \mathcal{D}(\mathfrak{z}_{\mathfrak{l}}(\hat{\mathfrak{l}}))$; then $\mathfrak{l} = \hat{\mathfrak{l}} \oplus \mathfrak{u} \oplus \mathfrak{z}(\mathfrak{l})$ and $\hat{\mathfrak{l}} = \bigoplus_{p\in\mathbb{Z}} \hat{\mathfrak{l}}_p$ is a simple GLA. Let $(\mathfrak{b}_q)_{q\in\mathbb{Z}}$ be as in §4.2.

Let us take a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{l}$ such that $\mathfrak{h} \subset \mathfrak{l}_0$. Then $\mathfrak{h} \cap \hat{\mathfrak{l}}$ (resp. $\mathfrak{h} \cap \mathfrak{u}$) is a Cartan subalgebra of $\hat{\mathfrak{l}}$ (resp. $\mathfrak{u}$). Let $\Delta = \{\alpha_1, \ldots, \alpha_l\}$ (resp. $\Delta' = \{\beta_1, \ldots, \beta_m\}$) be a simple root system of $(\hat{\mathfrak{l}}, \mathfrak{h} \cap \hat{\mathfrak{l}})$ (resp. $(\mathfrak{u}, \mathfrak{h}\cap\mathfrak{u})$) such that $\alpha(Z) \geqq 0$ for all $\alpha \in \Delta$, where Z is the characteristic element of the GLA $\mathfrak{l} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{l}_p$. We assume that $\hat{\mathfrak{l}}$ is a simple Lie algebra of type X_l. We set $\Delta_1 = \{\alpha \in \Delta \mid \alpha(Z) = 1\}$. It is well known that the pair (X_l, Δ_1) is one of the following type (up to a diagram automorphism) (cf. §3 in [Yam93]):

$$(A_l, \{\alpha_i\}) \left(1 \leqq i \leqq \left[\frac{(l+1)}{2}\right]\right), \ (B_l, \{\alpha_1\}) \ (l \geqq 3), \ (C_l, \{\alpha_l\}) \ (l \geqq 2),$$
$$(D_l, \{\alpha_1\}) \ (l \geqq 4), \ (D_l, \{\alpha_{l-1}\}) \ (l \geqq 5), \ (E_6, \{\alpha_1\}), \ (E_7, \{\alpha_7\}).$$

We denote by $\{\varpi_1, \ldots, \varpi_l\}$ (resp. $\{\pi_1, \ldots, \pi_n\}$) the set of fundamental weights relative to Δ (resp. Δ'). Since S is a faithful $\mathfrak{l}$-module, we have $\dim \mathfrak{z}(\mathfrak{l}) \leqq 1$. Assume that $\mathfrak{z}(\mathfrak{l}) \neq \{0\}$. Let σ be the element of $\mathfrak{z}(\mathfrak{l})^*$ such that $\sigma(J) = 1$, where J is the characteristic element of the GLA $\mathfrak{g} = \bigoplus_{q\in\mathbb{Z}} \mathfrak{b}_q$. Namely $J = -id_S \in \mathfrak{z}(\mathfrak{l}) \subset \mathfrak{b}_0 = \mathfrak{l}$ as the element of $\mathfrak{gl}(S)$. There is an irreducible $\hat{\mathfrak{l}}$-module T (resp. $\mathfrak{z}_{\mathfrak{l}}(\hat{\mathfrak{l}})$-module U) with highest weight χ (resp. $\eta-\sigma$) such that $S=\mathfrak{b}_{-1}$ is isomorphic to $T\otimes U$ as an $\mathfrak{l}$-module, where η is a weight of $\mathfrak{u}$. Then we have

Lemma 4.5. *$H^1(\mathfrak{G})_{0,0}=0$ if and only if $\mathfrak{z}_{\mathfrak{l}}(\hat{\mathfrak{l}})$ is isomorphic to $\mathfrak{gl}(U)$ and $\eta=\pi_1$. Especially, when $\mathcal{D}(\mathfrak{l})=\hat{\mathfrak{l}}$, $H^1(\mathfrak{G})_{0,0}=0$ if and only if $\mathfrak{l}=\hat{\mathfrak{l}}\oplus\mathfrak{z}(\mathfrak{l})$, where $\mathfrak{z}(\mathfrak{l})=\langle J\rangle$.*

Proof. We first remark that $H^1(\mathfrak{G})_{0,0}=0$ if and only if $\check{\mathfrak{g}}_0=\mathfrak{g}_0$ and that

$$\mathfrak{g}_0=[\mathfrak{l}_{-1},\mathfrak{l}_1]\oplus\mathfrak{z}_{\mathfrak{l}}(\hat{\mathfrak{l}}),\qquad \check{\mathfrak{g}}_0=[\mathfrak{l}_{-1},\mathfrak{l}_1]\oplus\mathfrak{z}_{\check{\mathfrak{g}}_0}(\hat{\mathfrak{l}}).$$

For $\varphi\in\mathfrak{gl}(U)$, we define $D_\varphi\in\operatorname{Hom}(\mathfrak{g}_-,\mathfrak{g}_-)$ as follows:

$$D_\varphi(\mathfrak{l}_{-1})=0,\quad D_\varphi(t\otimes u)=t\otimes\varphi(u)\text{ for }t\in T\text{ and }u\in U.$$

Then $D_\varphi\in\check{\mathfrak{g}}_0$ and the mapping $\mathfrak{gl}(U)\ni\varphi\mapsto D_\varphi\in\check{\mathfrak{g}}_0$ is injective. By Schur's lemma, this mapping is also surjective. This proves our assertion. Q.E.D.

Thus, when $H^1(\mathfrak{G})_{0,0}=0$, the semisimple GLA $\mathcal{D}(\mathfrak{l})$ is of type $(X_l\times A_n,\{\alpha_i\})$ and S is an irreducible $\mathcal{D}(\mathfrak{l})$-module with highest weight $\Xi=\chi+\pi_1$ when $\dim U>1$ and $\mathcal{D}(\mathfrak{l})$ is of type $(X_l,\{\alpha_i\})$ and S is an irreducible $\hat{\mathfrak{l}}$-module with highest weight χ, when $\mathcal{D}(\mathfrak{l})=\hat{\mathfrak{l}}$ (i.e., when $\dim U=1$). We will impose the condition $H^1(\mathfrak{G})_{0,0}=0$ on $\mathfrak{G}$ in the rest of this paper.

Now let us consider the characteristic element E of the gradation $\mathfrak{g}=\bigoplus_{p\in\mathbb{Z}}\mathfrak{g}_p$. Since S is the irreducible $\mathfrak{l}$-module with highest weight $\Xi=\chi+\pi_1-\sigma$, and $Z\in\mathfrak{h}\cap\hat{\mathfrak{l}}$ is defined by $\alpha_{i_o}(Z)=1$ and $\alpha_i(Z)=0$ $(i\neq i_o)$ for $\Delta_1=\{\alpha_{i_o}\}\subset\Delta=\{\alpha_1,\dots,\alpha_l\}$, we see that the semi-simple endomorphism $\operatorname{ad}(Z)$ has consecutive eigenvalues in S of the following form:

$$\lambda_1-k\qquad\text{for }k=0,\dots,\mu-1.$$

where $\lambda_1=\chi(Z)$ and $\lambda_0=\lambda_1-\mu+1$ is the minimum eigenvalue (see Lemma 2.1). Thus the characteristic element E of $\mathfrak{g}=\bigoplus_{p\in\mathbb{Z}}\mathfrak{g}_p$ is given by

$$E=Z+(\lambda_1+1)J\in\hat{\mathfrak{l}}_0\oplus\mathfrak{z}(\mathfrak{l})\subset\mathfrak{l}_0=\mathfrak{g}_0.$$

Then we note that, utilizing the characteristic elements E and J, the decomposition of $\mathfrak{l}_0$- modules $C^p(\mathfrak{G})$ and $C^p(\mathfrak{b}_-,\mathfrak{g})$:

$$C^p(\mathfrak{G})=\bigoplus_r C^p(\mathfrak{G})_r=\bigoplus_s C^p(\mathfrak{G})_s,\quad C^p(\mathfrak{b}_-,\mathfrak{g})=\bigoplus_s C^p(\mathfrak{b}_-,\mathfrak{g})_s,$$

are given as the eigenspace decomposition of the action of E and J respectively, where r and s denote eigenvalues of E and J respectively. Moreover the decomposition $C^p(\mathfrak{G}) = \bigoplus_{r,s} C^p(\mathfrak{G})_{r,s}$ given in §4.2 is the simultaneous eigenspace decomposition of commuting semisimple elements E and J.

Let Λ be a dominant integral weight of $\mathcal{D}(\mathfrak{l})$. We denote by $L^s(\Lambda)$ the irreducible $\mathfrak{l}$-module with highest weight $\Lambda + s\sigma$. Now we apply the notation of §4.4 to the case when $\mathfrak{s} = \mathcal{D}(\mathfrak{l})$. For $w \in W$, we set

$$\xi_w^s(\Lambda) = w(w_0(\Lambda) - \rho) + \rho + s\sigma,$$

where w_0 is the element of W such that $w_0(\Delta) = -\Delta$, which sends the highest weight to the lowest weight. Since $S = \mathfrak{b}_{-1}$ is an irreducible $\mathfrak{l}$-module with highest weight $\Xi = \chi + \pi_1 - \sigma$, we have $\langle \chi - \sigma, E\rangle = -1$ and hence $\langle \xi_w^s(\Lambda), J\rangle = s$ and

$$\begin{aligned} \langle \xi_w^s(\Lambda), E\rangle &= \langle w(w_0(\Lambda) - \rho) + \rho, E\rangle + s(1 + \langle \chi, E\rangle) \\ &= \langle w(w_0(\Lambda) - \rho) + \rho, Z\rangle + s(\lambda_1 + 1). \end{aligned}$$

By Kostant's Theorem (Theorem B), $H^p(\mathfrak{l}_-, L^s(\Lambda))_r \neq 0$ if and only if $\langle \xi_w^s(\Lambda), E\rangle = r$ for some $w \in W_1^p$.

§5. First cohomology of pseudo-product graded Lie algebras

Let $\mathfrak{G}$ be a PPGLA of type $(\mathfrak{l}, S)$ satisfying the condition $H^1(\mathfrak{G})_{0,0} = 0$. We use the same notation as in §4.5. By Theorem 4.1,

$$H^1(\mathfrak{G})_{r,s} \cong H^1(\mathfrak{l}_-, H^0(\mathfrak{b}_-, \mathfrak{g})_s)_r \oplus H^0(\mathfrak{l}_-, H^1(\mathfrak{b}_-, \mathfrak{g})_s)_r. \tag{5.1}$$

In particular, we see that $H^1(\mathfrak{G})_{r,s} = 0$ for $s \leqq -2$ or $s \geqq 2$. First of all, we consider $H^1(\mathfrak{G})_{r,-1}$. By (5.1),

$$H^1(\mathfrak{G})_{r,-1} \cong H^1(\mathfrak{l}_-, \mathfrak{b}_{-1})_r.$$

By Kostant's Theorem (Theorem B), we see that

(i) $H^1(\mathfrak{G})_{r,-1} = 0$ for $r \geqq 1$;

(ii) $H^1(\mathfrak{G})_{0,-1} \neq 0$ if and only if $\mathcal{D}(\mathfrak{l})$ is of type $(A_l \times A_n, \{\alpha_1\})$ and S is an irreducible $\mathfrak{l}$-module with highest weight $k\varpi_l + \pi_1$. (cf. [Yat92, pp. 323–324].)

Secondly we consider $H^1(\mathfrak{G})_{r,0}$. Clearly

$$H^1(\mathfrak{G})_{r,0} \cong H^0(\mathfrak{l}_-, H^1(\mathfrak{b}_-, \mathfrak{g})_0)_r.$$

Let Λ be the highest weight of an irreducible component of the $\mathfrak{l}$-module $H^1(\mathfrak{b}_-, \mathfrak{g})_0$. Then $\langle \xi_1^0(\Lambda), E\rangle = \langle w_0(\Lambda), \varpi_i{}^\vee\rangle$. If $w_0(\Lambda) = \sum_{i=1}^l c_i\alpha_i$

($c_i \in \mathbb{R}$), then $c_i \leqq 0$ for all i, so $\langle \xi_1^0(\Lambda), E\rangle \leqq 0$. By Kostant's Theorem (Theorem B), we obtain that $H^1(\mathfrak{G})_{r,0} = 0$ for $r \geqq 1$.

Thirdly, we consider $H^1(\mathfrak{G})_{r,1}$. Let $\check{\mathfrak{b}} = \bigoplus_{p\in\mathbb{Z}} \check{\mathfrak{b}}_p$ be the prolongation of $\bigoplus_{p\leq 0} \mathfrak{b}_p$. Clearly

$$H^1(\mathfrak{G})_{r,1} \cong H^0(\mathfrak{l}_-, \check{\mathfrak{b}}_1)_r.$$

If $\dim \check{\mathfrak{b}} < \infty$ and $\check{\mathfrak{b}}_1 \neq 0$, then $\check{\mathfrak{b}}$ is simple and the prolongation of $\mathfrak{G}_-$ is a simple PPGLA. Hence $H^1(\mathfrak{G})_{1,1} \cong S^*_{-1}$ and $H^1(\mathfrak{G})_{r,1} = 0$ ($r \neq 1$). If $\dim \check{\mathfrak{b}} = \infty$, then, by the theorem of Kobayashi and Nagano [KN65], $\check{\mathfrak{b}} = \bigoplus_{p\in\mathbb{Z}} \check{\mathfrak{b}}_p$ is isomorphic to one of $W(n;\mathbf{1})$ and $CH(m;\mathbf{1};2)$, where $n = 2m = \dim \mathfrak{b}_{-1}$. (For the definition of $W(n;\mathbf{1})$ and $CH(m;\mathbf{1};2)$, see [Yat92].) If $\check{\mathfrak{b}} \cong W(n;\mathbf{1})$, then $\mathfrak{l}$ is of type $(A_l, \{\alpha_i\})$ and S is an irreducible $\mathcal{D}(\mathfrak{l})$-module with highest weight ϖ_1. Hence $H^1(\mathfrak{G})_{0,1} \cong S_{-2} \otimes S^2(S^*_{-1})$, $H^1(\mathfrak{G})_{1,1} \cong S^*_{-1}$, and $H^1(\mathfrak{G})_{r,1} = 0$ ($r \neq 0,\ 1$). If $\check{\mathfrak{b}} \cong CH(m;\mathbf{1};2)$, then $\mathcal{D}(\mathfrak{l})$ is of type $(C_l, \{\alpha_l\})$ and S is an irreducible $\mathfrak{l}$-module with highest weight ϖ_1. Hence $H^1(\mathfrak{G})_{0,1} \cong L^{-2}(0) \otimes S^3(S^*_{-1})$, and $H^1(\mathfrak{G})_{r,1} = 0$ ($r \neq 0$).

We summarize the above results in the following theorem.

Theorem 5.1. *Let $\mathfrak{G}$ be a PPGLA of type $(\mathfrak{l}, S)$ satisfying the condition $H^1(\mathfrak{G})_{0,0} = 0$. Let $\check{\mathfrak{b}} = \bigoplus_{p\in\mathbb{Z}} \check{\mathfrak{b}}_p$ be the prolongation of $\bigoplus_{p\leq 0} \mathfrak{b}_p$. Then*

(1) *$H^1(\mathfrak{G})_{r,s} = 0$ for $r \geqq 0$, $s \neq -1,\ 1$.*
(2) *$H^1(\mathfrak{G})_{r,-1} = 0$ for $r \geqq 1$.*
(3) *$H^1(\mathfrak{G})_{0,-1} \neq 0$ if and only if $\mathcal{D}(\mathfrak{l})$ is of type $(A_l \times A_n, \{\alpha_1\})$ and S is an irreducible $\mathcal{D}(\mathfrak{l})$-module with highest weight $k\varpi_l + \pi_1$. In this case,*

$$H^1(\mathfrak{G})_{0,-1} \cong S_{-\mu} \otimes S^\mu(\mathfrak{l}^*_{-1}).$$

(4) *If $\check{\mathfrak{b}}_1 = 0$, then $H^1(\mathfrak{G})_{r,1} = 0$ for all r.*
(5) *If $\dim \check{\mathfrak{b}} < \infty$ and $\check{\mathfrak{b}}_1 \neq 0$, then*

$$H^1(\mathfrak{G})_{r,1} \cong \begin{cases} S^*_{-1} & \text{if} \quad r = 1 \\ 0 & \text{otherwise.} \end{cases}$$

(6) *If $\dim \check{\mathfrak{b}} = \infty$, then $\mathfrak{G}$ is isomorphic to one of PPGLAs of the following types:*
  (i) *$\mathcal{D}(\mathfrak{l})$ is of type $(A_l, \{\alpha_i\})$ $(1 \leqq i \leqq l)$ and S is an irreducible $\mathcal{D}(\mathfrak{l})$-module with highest weight ϖ_1.*
  (ii) *$\mathcal{D}(\mathfrak{l})$ is of type $(C_l, \{\alpha_l\})$ and S is an irreducible $\mathcal{D}(\mathfrak{l})$-module with highest weight ϖ_1.*

In these cases, we have

(i) *If $\mathcal{D}(\mathfrak{l})$ is of type $(A_l, \{\alpha_i\})$ $(1 \leqq i \leqq l)$ and S is an irreducible $\mathcal{D}(\mathfrak{l})$-module with highest weight ϖ_1, then*

$$H^1(\mathfrak{G})_{r,1} \cong \begin{cases} S_{-2} \otimes S^2(S^*_{-1}) & \textit{if} \quad r = 0 \\ S^*_{-1} & \textit{if} \quad r = 1 \\ 0 & \textit{otherwise.} \end{cases}$$

(ii) *If $\mathcal{D}(\mathfrak{l})$ is of type $(C_l, \{\alpha_l\})$ $(l \geqq 2)$ and S is an irreducible $\mathcal{D}(\mathfrak{l})$-module with highest weight ϖ_1, then*

$$H^1(\mathfrak{G})_{r,1} \cong \begin{cases} L^{-2}(0) \otimes S^3(S^*_{-1}) & \textit{if} \quad r = 0 \\ 0 & \textit{otherwise.} \end{cases}$$

Corollary 5.1. *Let $\mathcal{G} = \bigoplus_{p \in \mathbb{Z}} \mathcal{G}_p$ be the prolongation of $\mathfrak{g}_-$. Then $\dim \mathcal{G} = \infty$ if and only if $H^1(\mathfrak{G})_{0,s} \neq 0$ for some $s \neq 0$.*

If $\dim \check{\mathfrak{b}} < \infty$ and $\check{\mathfrak{b}}_1 \neq 0$, then $H^1(\mathfrak{G})_{1,1} \neq 0$. Now we classify pseudo-product GLAs $\mathfrak{G}$ of type $(\mathfrak{l}, S)$ such that $H^1(\mathfrak{G})_{1,1} \neq 0$. Let $\mathfrak{s} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{s}_p$ be the prolongation of $(\mathfrak{m}, \mathfrak{g}_0)$. The condition $H^1(\mathfrak{G})_{1,1} \neq 0$ implies $\mathfrak{g}_1 \neq \mathfrak{s}_1$. Then, by Theorem 3.2 [Yat88], $\mathfrak{s} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{s}_p$ is a simple GLA such that $\mathfrak{g}_p = \mathfrak{s}_p$ for all $p \leqq 0$. Since the $\mathfrak{g}_0$-module $\mathfrak{s}_0$ is completely reducible, there exists a $\mathfrak{g}_0$-submodule $\mathfrak{s}_1^{(1)}$ of $\mathfrak{s}_1$ such that $\mathfrak{s}_1 = \mathfrak{g}_1 \oplus \mathfrak{s}_1^{(1)}$. Since $\mathfrak{s}_1$ is contragredient to $\mathfrak{s}_{-1}$ as a $\mathfrak{s}_0$-module, the $\mathfrak{g}_0$-module $\mathfrak{s}_1^{(1)}$ is irreducible. Also $\mathfrak{s}_1^{(1)}$ is contragredient to S_{-1} as a $\mathfrak{g}_0$-module. For $q \in \mathbb{Z}$, we put $\mathfrak{c}_q = \{x \in \mathfrak{s} \mid [J, x] = qx\}$; then $(\mathfrak{c}_q)_{q \in \mathbb{Z}}$ gives a gradation of $\mathfrak{s}$. Note that each $\mathfrak{c}_q$ is ad E-stable. Since $[\mathfrak{g}_1, \mathfrak{g}_1] = 0$ and $\mathfrak{s}_{\geqq 1}$ is generated by $\mathfrak{s}_1$, we have $\mathfrak{c}_{-1} = S$, $\mathfrak{c}_0 = \mathfrak{l}$ and $\mathfrak{s}_1^{(1)} = \mathfrak{c}_1 \cap \mathfrak{s}_1$. Since $\mathfrak{s}$ is simple, we get $\mathfrak{s} = \mathfrak{c}_{-1} \oplus \mathfrak{c}_0 \oplus \mathfrak{c}_1$. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{s}$ such that $E \in \mathfrak{h}$; then $J \in \mathfrak{h} \subset \mathfrak{s}_0$. Let Φ be the root system of $(\mathfrak{s}, \mathfrak{h})$. There exists a simple root system $\Sigma = \{\gamma_1, \ldots, \gamma_{l+n+1}\}$ of $(\mathfrak{s}, \mathfrak{h}, \Phi)$ such that $\gamma_j(E) \geqq 0$ for all j, where $l = \operatorname{rank} \hat{\mathfrak{l}}$ and $n = \operatorname{rank} \mathfrak{z}_{\mathfrak{l}}(\hat{\mathfrak{l}})$. We assume that $\mathfrak{s} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{s}_p$ is of type (Y_{l+n+1}, Σ_1), $\Delta \subset \Sigma$ and $\Delta_1 \subset \Sigma_1$. For $\gamma_i \in \Sigma_1$, we denote by $\mathfrak{s}_{-1}^{(i)}$ the $\mathfrak{g}_0$ submodule of $\mathfrak{s}_{-1}$ generated by $\mathfrak{s}_{-\gamma_i}$. $\mathfrak{s}_{-1}^{(i)}$ is an irreducible $\mathfrak{s}_0$-submodule of $\mathfrak{s}_{-1}$ with highest weight $-\gamma_i$ (cf. [VOG90, Chapter 2, 3.5]). Since $\mathfrak{l}_{-1}$ and S_{-1} is not isomorphic to each other as a $\mathfrak{g}_0$-module, there exist γ_{i_1}, $\gamma_{i_2} \in \Sigma_1$ satisfying (i) $\mathfrak{l}_{-1} = \mathfrak{s}_{-1}^{(i_1)}$ and $S_{-1} = \mathfrak{s}_{-1}^{(i_2)}$ or (ii) $\mathfrak{l}_{-1} = \mathfrak{s}_{-1}^{(i_2)}$ and $S_{-1} = \mathfrak{s}_{-1}^{(i_1)}$. In particular, Σ_1 consists of two elements γ_{i_1} and γ_{i_2}. We may assume the case (i). Thus the GLA $\mathfrak{s} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{s}_p$ is of type $(Y_{l+n+1}, \{\gamma_{i_1}, \gamma_{i_2}\})$ and the GLA $\mathfrak{s} = \mathfrak{c}_{-1} \oplus \mathfrak{c}_0 \oplus \mathfrak{c}_1$ is of type $(Y_{l+n+1}, \{\gamma_{i_2}\})$. For each $\gamma_i \in \Sigma_1$, we set $\Phi_p^{(i)} = \{\alpha \in \Phi \mid m_i(\alpha) = p \text{ and } m_j(\alpha) = 0 \text{ for } j \in I \setminus \{i\}\}$. Moreover we set $\Phi^{(i)} = \bigcup_{p\in\mathbb{Z}} \Phi_p^{(i)}$, $\Phi_+^{(i)} = \Phi^{(i)} \cap \Phi_+$ and $\mathfrak{s}^{(i)} = \sum_{\alpha\in\Phi^{(i)}} \mathfrak{s}^\alpha + \mathfrak{h}$. Since $\Phi^{(i)} = -\Phi^{(i)}$, we know that $\mathfrak{s}^{(i)}$ is a reductive graded subalgebra of $\mathfrak{s}$ (cf. [Bou75, Chapter 8, §3, no. 4, Proposition 2]), which we write $\mathfrak{s}^{(i)} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{s}_p^{(i)}$. Then $\mathfrak{s}^{(i)} = \mathcal{D}(\mathfrak{s}^{(i)}) \oplus \mathfrak{z}(\mathfrak{s}^{(i)})$, $\mathfrak{s}_0^{(i)} = \mathfrak{s}_0$, $\mathfrak{s}_{-1}^{(i)} = \sum_{\alpha\in\Phi_{-1}^{(i)}} \mathfrak{s}^\alpha$ and $\mathfrak{s}_{-1} = \bigoplus_{i\in I} \mathfrak{s}_{-1}^{(i)}$. Since $[\mathfrak{s}_{-1}^{(i)}, \mathfrak{s}_{-1}^{(i)}] = 0$, $\Phi_p^{(i)} = \emptyset$ for $|p| > 2$. Clearly $\Phi^{(i)}$ is a root system of $(\mathfrak{s}^{(i)}, \mathfrak{h})$. We set $\Sigma^{(i)} = \Sigma \cap \Sigma^{(i)}$. Since $\langle \alpha, h\rangle = 0$ for all $\alpha \in \Phi^{(i)}$ and $h \in \mathfrak{z}(\mathfrak{s}^{(i)})$, we see that $\Sigma^{(i)}$ is a system of simple roots of $\Phi^{(i)}$ (cf. [Bou75, Chapter 8, §1, no. 7, Corollary 3 to Proposition 20]). Here we extend the definition of a root system and a system of simple roots, etc. to the cases of reductive Lie algebras (cf. Chapter 8, §2 in [Bou75]). Thus the derived subalgebra $\mathcal{D}(\mathfrak{s}^{(i)})$ of $\mathfrak{s}^{(i)}$ is a semisimple Lie algebra whose Dynkin diagram is the subdiagram of Y_{l+n+1} consisting of the vertices $(\{1, \dots, l+n+1\} \setminus I) \cup \{i\}$. In particular, $\hat{\mathfrak{s}}^{(i)}$ is a simple Lie algebra whose Dynkin diagram $Y_{l+n+1}^{(i)}$ is the connected component containing $\{i\}$ of the diagram of $\mathcal{D}(\mathfrak{s}^{(i)})$. Let θ (resp. $\theta^{(i)}$) be the highest root of $\mathfrak{s}$ (resp. $\mathfrak{s}^{(i)}$). Then, for $\alpha \in \Phi_+$ and $\beta \in \Phi_+^{(i)}$, we have $m_j(\alpha) \leqq m_j(\theta)$ for all j and $m_k(\beta) \leqq m_k(\theta^{(i)})$ for all k, so $\theta \in \Phi_\mu$ and $\theta^{(i)} \in \Phi_{\mu_i}$, where μ (resp. μ_i) is the depth of $\mathfrak{s}$ (resp. $\mathfrak{s}^{(i)}$). Thus $\mu = \sum_{i\in I} m_i(\theta)$ and $\mu_i = m_i(\theta^{(i)})$. Since $[\mathfrak{l}_{-1}, \mathfrak{l}_{-1}] = [S_{-1}, S_{-1}] = 0$, we see that $\mu_i = 1$ for all i. Also since $[\mathfrak{c}_{-1}, \mathfrak{c}_{-1}] = 0$, $m_{i_2}(\theta) = 1$. Hence (Y_{l+n+1}, Σ_1) is one of the following types:

$$
\begin{aligned}
&(A_{l+n+1}, \{\gamma_i, \gamma_{l+1}\})\ (n \geqq 0,\ 1 \leqq i \leqq l),\ (B_{l+1}, \{\gamma_1, \gamma_2\})\ (l \geqq 2),\\
&(C_{l+1}, \{\gamma_i, \gamma_{l+1}\})\ (l \geqq 1,\ 1 \leqq i \leqq l),\ (D_{l+1}, \{\gamma_1, \gamma_2\})\ (l \geqq 3),\\
&(D_{l+1}, \{\gamma_i, \gamma_{l+1}\})\ (l \geqq 3,\ 1 \leqq i \leqq l),\\
&(E_6, \{\gamma_1, \gamma_6\}),\ (E_6, \{\gamma_1, \gamma_2\}),\ (E_6, \{\gamma_1, \gamma_3\}),\\
&(E_7, \{\gamma_1, \gamma_7\}),\ (E_7, \{\gamma_6, \gamma_7\}).
\end{aligned}
$$

Summarizing the above discussion, we obtain the following answer for our problem (A) cited in §2.

Theorem 5.2. *Let $\mathfrak{G}$ be a pseudo-product GLA of type $(\mathfrak{l}, S)$ satisfying the condition $H^1(\mathfrak{G})_{0,0} = 0$. Let $\check{\mathfrak{b}} = \bigoplus_{p\in\mathbb{Z}} \check{\mathfrak{b}}_p$ be the prolongation of $\bigoplus_{p\leq 0} \mathfrak{b}_p$. Then $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ is the prolongation of $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ except for the following three cases.*

(a) $\dim \check{\mathfrak{b}} < \infty$ *and* $\check{\mathfrak{b}}_1 \neq 0$ $\quad$ ($\check{\mathfrak{b}}$: *simple*)

$\mathcal{D}(\mathfrak{l})$	Δ_1	$\mathfrak{b}_{-1}$	Y_{l+n+1}	Σ_1
$A_l \times A_n$	$\{\alpha_i\}$	$\varpi_l + \pi_1$	A_{l+n+1}	$\{\gamma_i, \gamma_{l+1}\}$
A_l	$\{\alpha_i\}$	$2\varpi_l$	C_{l+1}	$\{\gamma_i, \gamma_{l+1}\}$
A_l $(l \geqq 3)$	$\{\alpha_i\}$	ϖ_{l-1}	D_{l+1}	$\{\gamma_i, \gamma_{l+1}\}$
B_l $(l \geqq 2)$	$\{\alpha_1\}$	ϖ_1	B_{l+1}	$\{\gamma_1, \gamma_2\}$
D_l $(l \geqq 4)$	$\{\alpha_l\}$	ϖ_1	D_{l+1}	$\{\gamma_1, \gamma_{l+1}\}$
D_l $(l \geqq 4)$	$\{\alpha_1\}$	ϖ_1	D_{l+1}	$\{\gamma_1, \gamma_2\}$
D_5	$\{\alpha_5\}$	ϖ_5	E_6	$\{\gamma_1, \gamma_3\}$
D_5	$\{\alpha_4\}$	ϖ_5	E_6	$\{\gamma_1, \gamma_2\}$
D_5	$\{\alpha_1\}$	ϖ_5	E_6	$\{\gamma_1, \gamma_6\}$
E_6	$\{\alpha_1\}$	ϖ_6	E_7	$\{\gamma_1, \gamma_7\}$
E_6	$\{\alpha_6\}$	ϖ_6	E_7	$\{\gamma_6, \gamma_7\}$

In this case (Y_{l+n+1}, Σ_1) *is the prolongation of* $\mathfrak{m}$ *except for* $(A_{l+n+1}, \{\gamma_1, \gamma_{l+1}\})$ *and* $(C_{l+1}, \{\gamma_1, \gamma_{l+1}\})$. *Moreover the latter two are the prolongations of* $(\mathfrak{m}, \mathfrak{g}_0)$.

(b) $\dim \check{\mathfrak{b}} = \infty$

$\mathcal{D}(\mathfrak{l})$	Δ_1	$\mathfrak{b}_{-1}$	$\mathfrak{g}(\mathfrak{m}, \mathfrak{g}_0)$
A_l	$\{\alpha_i\}$	ϖ_l	$(A_{l+1}, \{\gamma_i, \gamma_{l+1}\})$
C_l	$\{\alpha_l\}$	ϖ_1	$\mathfrak{g}$

In $(C_l, \{\alpha_l\})$*-case,* $\mu = 2$

$$S_{-2} = V^*, \quad S_{-1} = V, \quad \mathfrak{l}_{-1} = S^2(V^*),$$
$$\mathfrak{l}_0 = V \otimes V^* \oplus \mathbb{C}, \qquad \mathfrak{l}_1 = S^2(V).$$

(c) $\mathfrak{g}$ *is a pseudo-projective GLA, i.e.,* $\mathcal{D}(\mathfrak{l}) = (A_l \times A_n, \{\alpha_1\})$, $\Xi = k\varpi_l + \pi_1$, $(k \geqq 2,\ n \geqq 1)$, *or* $\mathcal{D}(\mathfrak{l}) = (A_l, \{\alpha_1\})$, $\chi = k\varpi_l$, $(k \geqq 3,\ n = 0)$

$$S_{-\mu} = W, \quad S_p = W \otimes S^{\mu+p}(V^*)\ (-\mu < p < 0),$$
$$\mathfrak{l}_{-1} = V, \quad \mathfrak{l}_0 = \mathfrak{gl}(V) \oplus \mathfrak{gl}(W), \quad \mathfrak{l}_1 = V^*,$$

where $\mu = k + 1$, $\dim V = l$ *and* $\dim W = n + 1$.

In this case $\mathfrak{g}$ *is the prolongation of* $(\mathfrak{m}, \mathfrak{g}_0)$.

§6. Second cohomology of pseudo-product graded Lie algebras

Let $\mathfrak{G}$ be a PPGLA of type $(\mathfrak{l}, S)$ satisfying the condition $H^1(\mathfrak{G})_{0,0} = 0$. We use the same notation as in §4. We define elements $\{\alpha_i^\vee\}$ and $\{\varpi_i^\vee\}$ of $\hat{\mathfrak{l}} \cap \mathfrak{h}$ by $\langle \alpha_j, \alpha_i^\vee \rangle = a_{ij}$ and $\langle \varpi_i^\vee, \alpha_j \rangle = \delta_{ij}$, where (a_{ij}) is the Cartan matrix of $\hat{\mathfrak{l}}$. Also let $\{r_1, \dots, r_l\}$ be the set of fundamental reflections of $\hat{\mathfrak{l}}$. Also we assume $\chi = \sum_{i=1}^l m_i \varpi_i$ $(m_i \in \mathbb{Z})$.

6.1. Computation of $H^2(\mathfrak{G})_{r,-1}$

By Theorem 4.1,

$$H^2(\mathfrak{G})_{r,-1} \cong H^2(\mathfrak{l}_-, \mathfrak{b}_{-1})_r.$$

Note that $H^2(\mathfrak{G})_{r,-1} = 0$ for all r provided that $(X_l, \Delta_1) = (A_1, \{\alpha_1\})$. Since Δ_1 is the form $\{\alpha_i\}$, an element of W_1^2 becomes the form $\{r_i r_k\}$, where $k \neq i$ and $\langle \alpha_k, \alpha_i^\vee \rangle \neq 0$. Then

$$\langle \xi_{r_i r_k}^{-1}(\chi), E \rangle = \langle w_0(\chi) - \chi, \varpi_i^\vee \rangle - \langle w_0(\chi), \alpha_i^\vee - \langle \alpha_k, \alpha_i^\vee \rangle \alpha_k^\vee \rangle - \langle \alpha_k, \alpha_i^\vee \rangle$$

We compute $\langle \xi_{r_i r_k}^{-1}(\chi), E \rangle$ by a case by case analysis. For convenience, we put $\xi_{r_i r_k} = \xi_{r_i r_k}^{-1}(\chi)$.

Case 1: Take $(X_l, \Delta_1) = (A_l, \{\alpha_1\})$ $(l \geqq 2)$. Then $W_1^2 = \{r_1 r_2\}$ and

$$\langle \xi_{r_1 r_2}, E \rangle = -\sum_{j=1}^{l-2} m_j + 1.$$

Case 2: Take $(X_l, \Delta_1) = (A_l, \{\alpha_i\})$ $(1 < i \leqq [(l+1)/2])$. Then $W_1^2 = \{r_i r_{i-1}, r_i r_{i+1}\}$ and

$$\begin{aligned}
\langle \xi_{r_i r_{i-1}}, E \rangle = &-\sum_{j=1}^{i-1} j m_j - \sum_{j=i}^{l-i} i m_j - (i-1) m_{l-i+1} \\
&- (i-2) m_{l-i+2} - \sum_{j=l-i+3}^{l} (l-j+1) m_j + 1, \\
\langle \xi_{r_i r_{i+1}}, E \rangle = &-\sum_{j=1}^{i-1} j m_j - \sum_{j=i}^{l-j-1} i m_j - (i-1) m_{l-i} \\
&- (i-1) m_{l-i+1} - \sum_{j=l-i+2}^{l} (l-j+1) m_j + 1.
\end{aligned}$$

Case 3: Take $(X_l, \Delta_1) = (B_l, \{\alpha_1\})$ $(l \geqq 3)$. Then $W_1^2 = \{r_1 r_2\}$ and

$$\langle \xi_{r_1 r_2}, E \rangle = -m_1 - m_2 - 2\sum_{j=3}^{l-1} m_j - m_l + 1.$$

Case 4: Take $(X_l, \Delta_1) = (C_l, \{\alpha_l\})$ $(l \geqq 2)$. Then $W_l^2 = \{r_l r_{l-1}\}$ and

$$\langle \xi_{r_l r_{l-1}}, E \rangle = -\sum_{j=1}^{l-2} j m_j - (l-2)m_{l-1} - (l-1)m_l + 1.$$

Case 5: Take $(X_l, \Delta_1) = (D_l, \{\alpha_1\})$ $(l \geqq 4)$. Then $W_1^2 = \{r_1 r_2\}$ and

$$\langle \xi_{r_1 r_2}, E \rangle = -m_1 - m_2 - 2\sum_{j=3}^{l-2} m_j - m_{l-1} - m_l + 1.$$

Case 6: Take $(X_l, \Delta_1) = (D_l, \{\alpha_{l-1}\})$ $(l \geqq 5)$. Then $W_1^2 = \{r_{l-1} r_{l-2}\}$ and

$$\begin{aligned}\langle \xi_{r_{l-1} r_{l-2}}, E \rangle = & -\sum_{j=1}^{l-3} j m_j - (l-3)m_{l-2} - \frac{1}{2}(l-2-\delta_0)m_{l-1} \\ & - \frac{1}{2}(l-2+\delta_0)m_l + 1,\end{aligned}$$

where $\delta_0 = 0$ if l is an even number and $\delta_0 = 1$ if l is an odd number.

Case 7: Take $(X_l, \Delta_1) = (E_6, \{\alpha_1\})$. Then $W^2(\alpha_1) = \{r_1 r_3\}$ and

$$\langle \xi_{r_1 r_3}, E \rangle = -m_1 - 2m_2 - 2m_3 - 4m_3 - 3m_5 - 2m_6 + 1.$$

Case 8: Take $(X_l, \Delta_1) = (E_7, \{\alpha_7\})$. Then $W^2(\alpha_7) = \{r_7 r_6\}$ and

$$\langle \xi_{r_7 r_6}, E \rangle = -2m_1 - 3m_2 - 4m_3 - 6m_4 - 5m_5 - 3m_6 - 2m_7 + 1.$$

Hence we obtain the following proposition

Proposition 6.1.

(1) $H^2(\mathfrak{G})_{r,-1} = 0$ *for all* $r \geqq 2$.

(2) $H^2(\mathfrak{G})_{1,-1} \neq 0$ *if and only if the sequence* (X_l, Δ_1, λ) *is one of the following*

$$(A_l, \{\alpha_1\}, j\varpi_{l-1} + k\varpi_l) \quad (l \geqq 2,\ j,\ k \geqq 0,\ j+k \geqq 1),$$

$$(A_l, \{\alpha_2\}, k\varpi_l) \quad (l \geqq 3,\ k \geqq 1), \quad (C_2, \{\alpha_2\}, k\varpi_1) \quad (k \geqq 1)$$

6.2. Computation of $H^2(\mathfrak{G})_{r,0}$

By Theorem 4.1,

$$H^2(\mathfrak{G})_{r,0} \cong H^1(\mathfrak{l}_-, H^1(\mathfrak{b}_-, \mathfrak{g})_0)_r.$$

and

$$\begin{aligned}\mathrm{ch}_{\mathfrak{l}}(H^1(\mathfrak{b}_-, \mathfrak{g})_0) &= \mathrm{ch}_{\mathfrak{l}}(\mathfrak{b}_{-1} \otimes \mathfrak{b}^*_{-1}) - \mathrm{ch}_{\mathfrak{l}}(\mathfrak{b}_0) \\ &= \mathrm{ch}_{\mathfrak{l}}(\mathfrak{sl}(T))\,\mathrm{ch}_{\mathfrak{l}}(\mathfrak{gl}(U)) - \mathrm{ch}_{\mathfrak{l}}(\hat{\mathfrak{l}}).\end{aligned}$$

Note that $H^1(\mathfrak{b}_-, \mathfrak{g})_0 = 0$ if and only if $\mathcal{D}(\mathfrak{l}) = \hat{\mathfrak{l}}$, $X_l = A_l$ and $\chi = \varpi_1$ or ϖ_l. Hence we may assume that $H^1(\mathfrak{b}_-, \mathfrak{g})_0 \neq 0$. Let Λ be the highest weight of an irreducible component of $H^1(\mathfrak{b}_-, \mathfrak{g})_0$; then $\langle \Lambda, \alpha_j^\vee \rangle > 0$ for some j. Assume that $\Delta_1 = \{\alpha_i\}$. Then $W_1^1 = \{r_i\}$ and

$$\langle \xi^0_{r_i}(\Lambda), E \rangle = \langle w_0(\Lambda), \varpi_i^\vee - \alpha_i^\vee \rangle + 1.$$

By the table of [Bou68], $\langle \varpi_j, \varpi_i^\vee - \alpha_i^\vee \rangle \geq 0$ for all j except for the case when $(X_l, \Delta_1) = (A_l, \{\alpha_1\})$ or $(A_l, \{\alpha_l\})$. Also $\langle \varpi_j, \varpi_i^\vee - \alpha_i^\vee \rangle > 0$ for all j if (X_l, Δ_1) is one of $(A_l, \{\alpha_i\})$ $(l \geqq 4,\ 1 < i \leqq [(l+1/2)])$, $(C_l, \{\alpha_l\})$ $(l \geqq 3)$, $(D_l, \{\alpha_{l-1}\})$ $(l \geqq 5)$, $(E_6, \{\alpha_1\})$, $(E_7, \{\alpha_7\})$.

Now we consider the case when $(X_l, \Delta_1) = (A_l, \{\alpha_1\})$. Following [Fis81], we first decompose $T \otimes T^*$ into irreducible $\mathfrak{sl}(l+1, \mathbb{C})$-modules. Since T is an irreducible $\mathfrak{sl}(l+1, \mathbb{C})$-module with highest weight $\lambda = \sum_{i=1}^l m_i \varpi_i$, T^* is an irreducible $\mathfrak{sl}(l+1, \mathbb{C})$-module with highest weight $\sum_{i=1}^l m_{l-i+1} \varpi_i$. Let $\hat{T}$ be a Young tableau corresponding to T, that is, a collection of boxes, arranged in left-justified rows, with $\sum_{i=1}^k m_i$ boxes in the k-th row. Let $\hat{T}^*$ be a Young tableau corresponding to T^*. We add $s_1^{(1)}$ boxes of the top row in $\hat{T}^*$ to the end of the top row in $\hat{T}$; then we add $s_2^{(1)}$ boxes of the top row in $\hat{T}^*$ to the end of the second row in $\hat{T}$, etc. Next we add $s_2^{(2)}$ boxes of the second row in $\hat{T}^*$ to the end of the second row in $\hat{T}$; then we add $s_3^{(2)}$ boxes of the second row in $\hat{T}^*$ to the end of the third row in $\hat{T}$, etc. Repeating this procedure, we get a Young tableau corresponding to an irreducible $\mathfrak{sl}(l+1, \mathbb{C})$-submodule of $T \otimes T^*$ with highest weight

$$\Lambda = \sum_{p=1}^{l} \left(m_p + \sum_{i=1}^{p+1} (s_p^{(i)} - s_{p+1}^{(i)}) \right) \varpi_p.$$

where $s_i^{(l+1)} = 0$ for all i and $s_p^{(k)} = 0$ for $k > p$. Also, in this procedure, we must impose Rules 1, 1a and 2 in [Fis81]. Hence the following conditions must be satisfied:

$$\sum_{i=1}^{l-p+1} m_i = \sum_{i=1}^{l+1} s_i^{(p)}, \qquad (1 \leqq p \leqq l),$$

$$\sum_{i=1}^{j} s_p^{(i)} \leqq m_{p-1} + \sum_{i=1}^{j-1} s_{p-1}^{(i)} \qquad (1 \leqq j \leqq p,\ 2 \leqq p \leqq l+1),$$

$$\sum_{i=1}^{l} s_i^{(j)} \leqq \sum_{i=1}^{l-1} s_i^{(j-1)} \qquad (1 \leqq j \leqq l).$$

From these inequalities, we have

$$\sum_{i=1}^{l} s_l^{(i)} \leqq \min\left\{ \sum_{i=1}^{l} m_i, 2m_1 + \sum_{i=2}^{l-1} m_i \right\}.$$

Thus

$$\langle \xi_{r_1}^0(\Lambda), E\rangle = -\sum_{j=1}^{l-1} m_j + \sum_{i=1}^{l} s_l^{(i)} + 1 \leqq \min\{m_1, m_l\} + 1.$$

Hence we obtain the following proposition.

Proposition 6.2.

(1) *$H^2(\mathfrak{G})_{r,0} = 0$ for $r \geqq 2$ except for the case when $(X_l, \Delta_1) = (A_l, \{\alpha_1\})$.*

(2) *$H^2(\mathfrak{G})_{1,0} = 0$ except when (X_l, Δ_1) is one of $(A_l, \{\alpha_1\})$ $(l \geqq 1)$, $(A_3, \{\alpha_2\})$, $(B_l, \{\alpha_1\})$ $(l \geqq 2)$ or $(D_l, \{\alpha_1\})$ $(l \geqq 4)$.*

(3) *If $(X_l, \Delta_1) = (A_l, \{\alpha_1\})$, then we see that $H^2(\mathfrak{G})_{r,0} = 0$ for $r \geqq \min\{m_1, m_l\} + 2$.*

Remark 6.1.

(1) The contents of (1) and (2) in Proposition 6.2 were first observed by Y. Se-ashi (see Theorem 2 [SYY97]), which essentially constitutes the proof of Theorem A in §3.1.

(2) Let $(X_l, \Delta_1) = (A_3, \{\alpha_2\})$. We set

$$\Gamma(\lambda) = \left\{ (a,b,c) \in (\mathbb{Z}_{\geqq 0})^3 \;\middle|\; \begin{array}{l} b \leqq m_2,\ m_1 - m_3 \leqq c \leqq m_1 \\ a \neq 0,\ a + b + c = m_1 + m_2 \end{array} \right\}.$$

Let M be the sum of irreducible components of the $\hat{\mathfrak{l}}$-module $\mathfrak{sl}(T)$ with highest weight Λ such that $\langle \xi_{r_2}^0(\Lambda), E\rangle = 1$. We get $\mathrm{ch}_{\mathfrak{l}}(M) = \sum_a n_a \,\mathrm{ch}_{\mathfrak{l}}(L(2a\varpi_2))$, by using the Young tableau method, where $n_a = \sharp\{(b,c) \mid (a,b,c) \in \Gamma(\lambda)\}$. Hence $H^2(\mathfrak{G})_{1,0} = 0$ if and only if $\Gamma(\lambda) = \emptyset$.

6.3. Computation of $H^2(\mathfrak{G})_{r,s}$ $(s = 1, 2)$

By Theorem 4.1,

$$H^2(\mathfrak{G})_{r,1} \cong H^0(\mathfrak{l}_-, H^2(\mathfrak{b}_-, \mathfrak{g})_1)_r \oplus H^1(\mathfrak{l}_-, H^1(\mathfrak{b}_-, \mathfrak{g})_1)_r,$$
$$H^2(\mathfrak{G})_{r,2} \cong H^0(\mathfrak{l}_-, H^2(\mathfrak{b}_-, \mathfrak{g})_2)_r.$$

6.3.1. The case $H^1(\mathfrak{G})_{1,1} \neq 0$. In this case, $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ is a parabolic graded subalgebra of a finite dimensional SGLA $\mathfrak{s} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{s}_p$ such that $\mathfrak{g}_p = \mathfrak{s}_p$ for all $p \leqq 0$.

Theorem 6.1. *Under the above assumption, we have*

(1) $H^2(\mathfrak{G})_{r,s} \cong H^2(\mathfrak{m}, \mathfrak{s})_{r,s}$ $(s = -1,\ 0)$, *where* $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$.
(2) $H^2(\mathfrak{G})_{r,1} \cong H^2(\mathfrak{m}, \mathfrak{s})_{r,1} \oplus H^1(\mathfrak{l}_-, \mathfrak{b}^*_{-1})_r$.
(3) $H^2(\mathfrak{G})_{r,2} \cong H^2(\mathfrak{m}, \mathfrak{s})_{r,2} \oplus H^0(\mathfrak{l}_-, \mathfrak{b}^*_{-1} \otimes \mathfrak{b}^*_{-1})_r$.

Proof. There exits a gradation $(\mathfrak{c}_q)_{q\in\mathbb{Z}}$ of $\mathfrak{s}$ such that $\mathfrak{b}_q = \mathfrak{c}_q$ for all $q \leqq 0$. We remark that

$$H^2(\mathfrak{m}, \mathfrak{s})_{r,s} \cong \bigoplus_{i=0}^{2} H^{2-i}(\mathfrak{l}_-, H^i(\mathfrak{b}_-, \mathfrak{s})_s)_{r,s}.$$

The proof is similar to that of Theorem 4.1.

The statement of (1) is easy to check. We consider the following sequence

$$0 \longrightarrow \mathfrak{c}_1 \xrightarrow{\tilde{\delta}_0} \mathrm{Hom}(\mathfrak{b}_{-1}, \mathfrak{b}_0) \xrightarrow{\delta_1} \mathrm{Hom}\left(\bigwedge^2 \mathfrak{b}_{-1}, \mathfrak{b}_{-1}\right) \longrightarrow 0,$$

where $\tilde{\delta}_0$ is the coboundary operator $\tilde{\delta}_0 \colon C^0(\mathfrak{c}_-, \mathfrak{s})_1 \to C^1(\mathfrak{c}_-, \mathfrak{s})_1$. Since $\tilde{\delta}_0$ is injective, we see that $H^1(\mathfrak{b}_-, \mathfrak{g})_1 \cong \mathfrak{c}_1 \oplus H^1(\mathfrak{b}_-, \mathfrak{s})_1$. Also, since $\mathfrak{c}_1$ is isomorphic to $\mathfrak{b}^*_{-1}$ as a $\mathfrak{b}_0$-module,

$$H^2(\mathfrak{G})_{r,1} \cong H^2(\mathfrak{m}, \mathfrak{s})_{r,1} \oplus H^1(\mathfrak{l}_-, \mathfrak{b}^*_{-1})_r,$$

which proves (2). Next consider the following sequence

$$0 \longrightarrow \mathrm{Hom}(\mathfrak{b}_{-1}, \mathfrak{c}_1) \xrightarrow{\tilde{\delta}_1} \mathrm{Hom}\left(\bigwedge^2 \mathfrak{b}_{-1}, \mathfrak{b}_0\right) \xrightarrow{\delta_2} \mathrm{Hom}\left(\bigwedge^3 \mathfrak{b}_-, \mathfrak{b}_{-1}\right) \longrightarrow 0,$$

where $\tilde{\delta}_1$ is the coboundary operator $\tilde{\delta}_1 \colon C^1(\mathfrak{c}_-, \mathfrak{s})_2 \to C^2(\mathfrak{c}_-, \mathfrak{s})_2$. Since $\tilde{\delta}_1$ is injective, we see that $\operatorname{Ker}\delta_2$ is isomorphic to $H^2(\mathfrak{b}_-, \mathfrak{g})_s \oplus \mathrm{Hom}(\mathfrak{b}_{-1}, \mathfrak{c}_1)$. Since $\mathfrak{c}_1$ is isomorphic to $\mathfrak{b}^*_{-1}$, we get

$$H^2(\mathfrak{G})_{r,2} \cong H^2(\mathfrak{m}, \mathfrak{s})_{r,2} \oplus H^0(\mathfrak{l}_-, \mathfrak{b}^*_{-1} \otimes \mathfrak{b}^*_{-1})_r,$$

which proves (3). Q.E.D.

By the above theorem, we need to decompose the $\mathfrak{b}_0$-module $\otimes^2\mathfrak{b}_{-1}^*$ into irreducible $\mathfrak{b}_0$-modules. By the table in (1) of Theorem 5.2 and the table of [OV90], we get the Table 1 of the irreducible decomposition of the $\mathfrak{b}_0$-module $\otimes^2\mathfrak{b}_{-1}^*$.

By Table 1 and Kostant theorem, we get the following theorem.

Theorem 6.2. *Let $\mathfrak{G}$ be a pseudo-product GLA of type $(\mathfrak{l}, S)$ such that:* (i) *$\hat{\mathfrak{l}}$ is an SGLA of type (X_l, Δ_1);* (ii) *S is an irreducible $\mathcal{D}(\mathfrak{l})$-module with highest weight Ξ;* (iii) *$H^1(\mathfrak{G})_{1,1} \neq 0$. Then the following are the triplet $(\mathcal{D}(\mathfrak{l}), \Delta_1, \Xi)$ and the set of r such that $H^2(\mathfrak{G})_r \neq 0$ $(r \geqq 1)$.*

(1) $(\mathcal{D}(\mathfrak{l}), \Delta_1, \Lambda_{-1}) = (A_1, \{\alpha_1\}, \varpi_1)$, $r = 2, 3, 4$.
(2) $(\mathcal{D}(\mathfrak{l}), \Delta_1, \Lambda_{-1}) = (A_1 \times A_1, \{\alpha_1\}, \varpi_1 + \pi_1)$, $r = 1, 2, 3$.
(3) $(\mathcal{D}(\mathfrak{l}), \Delta_1, \Lambda_{-1}) = (A_1 \times A_n, \{\alpha_1\}, \varpi_1 + \pi_1)$ $(n \geqq 2)$, $r = 2, 3$.
(4) $(\mathcal{D}(\mathfrak{l}), \Delta_1, \Lambda_{-1}) = (A_l \times A_n, \{\alpha_1\}, \varpi_l + \pi_1)$ $(l \geqq 2, \ \leqq n \leqq 0)$, $r = 1, 2$.
(5) $(\mathcal{D}(\mathfrak{l}), \Delta_1, \Lambda_{-1}) = (A_l \times A_n, \{\alpha_2\}, \varpi_l + \pi_1)$ $(l \geqq 3, \ n \leqq 0)$, $r = 1, 2$.

Table 1

$\mathcal{D}(\mathfrak{l})$	Δ_1	Ξ	$\mathfrak{b}_{-1}^*$	$\otimes^2\mathfrak{b}_{-1}^*$
$A_l \times A_n$	$\{\alpha_i\}$	$\varpi_l + \pi_1$	$\varpi_1 + \pi_n$	$2\varpi_1$, ϖ_2 $\pmod{\pi_n, \pi_{n-1}}$
A_l	$\{\alpha_i\}$	$2\varpi_l$	$2\varpi_1$	$4\varpi_1$, $2\varpi_2$, $2\varpi_1 + \varpi_2$
A_l $(l \geqq 3)$	$\{\alpha_i\}$	ϖ_{l-1}	ϖ_2	$2\varpi_2$, ϖ_4, $\varpi_1 + \varpi_3$
B_2	$\{\alpha_1\}$	ϖ_1	ϖ_1	0, $2\varpi_2$, $2\varpi_1$
B_l $(l \geqq 3)$	$\{\alpha_1\}$	ϖ_1	ϖ_1	0, ϖ_2, $2\varpi_1$
D_l $(l \geqq 4)$	$\{\alpha_l\}$	ϖ_1	ϖ_1	$0, \varpi_2$, $2\varpi_1$
D_l $(l \geqq 4)$	$\{\alpha_1\}$	ϖ_1	ϖ_1	0, ϖ_2, $2\varpi_1$
D_5	$\{\alpha_5\}$	ϖ_5	ϖ_4	$2\varpi_4$, ϖ_3, ϖ_1
D_5	$\{\alpha_4\}$	ϖ_5	ϖ_4	$2\varpi_4$, ϖ_3, ϖ_1
D_5	$\{\alpha_1\}$	ϖ_5	ϖ_4	$2\varpi_4$, ϖ_3, ϖ_1
E_6	$\{\alpha_1\}$	ϖ_6	ϖ_1	ϖ_3, $2\varpi_1$, ϖ_6
E_6	$\{\alpha_6\}$	ϖ_6	ϖ_1	ϖ_3, $2\varpi_1$, ϖ_6

(6) $(\mathcal{D}(\mathfrak{l}), \Delta_1, \Lambda_{-1}) = (A_l \times A_n, \{\alpha_i\}, \varpi_l + \pi_1)$ $(3 \leqq i \leqq l-1,\ 0 \leqq n \leqq 1)$, $r = 1,\ 2$.
(7) $(\mathcal{D}(\mathfrak{l}), \Delta_1, \Lambda_{-1}) = (A_l \times A_n, \{\alpha_i\}, \varpi_l + \pi_1)$ $(3 \leqq i \leqq l-1,\ 2 \leqq n)$, $r = 2$.
(8) $(\mathcal{D}(\mathfrak{l}), \Delta_1, \Lambda_{-1}) = (A_2, \{\alpha_2\}, \varpi_2)$, $r = 1,\ 2,\ 3$.
(9) $(\mathcal{D}(\mathfrak{l}), \Delta_1, \Lambda_{-1}) = (A_l, \{\alpha_l\}, \varpi_l)$ $(l \geqq 3)$, $r = 2,\ 3$.
(10) $(\mathcal{D}(\mathfrak{l}), \Delta_1, \Lambda_{-1}) = (A_l \times A_n, \{\alpha_l\}, \varpi_l + \pi_1)$ $(l \geqq 2,\ n \geqq 1)$, $r = 1,\ 2,\ 3$.
(11) $(\mathcal{D}(\mathfrak{l}), \Delta_1, \Lambda_{-1}) = (A_1, \{\alpha_1\}, 2\varpi_1)$, $r = 2,\ 3,\ 4$.
(12) $(\mathcal{D}(\mathfrak{l}), \Delta_1, \Lambda_{-1}) = (A_l, \{\alpha_1\}, 2\varpi_l)$ $(l \geqq 2)$, $r = 1,\ 2$.
(13) $(\mathcal{D}(\mathfrak{l}), \Delta_1, \Lambda_{-1}) = (A_2, \{\alpha_2\}, 2\varpi_2)$, $r = 1,\ 2,\ 3,\ 4$.
(14) $(\mathcal{D}(\mathfrak{l}), \Delta_1, \Lambda_{-1}) = (A_l, \{\alpha_2\}, 2\varpi_l)$ $(l \geqq 3)$, $r = 1,\ 2$.
(15) $(\mathcal{D}(\mathfrak{l}), \Delta_1, \Lambda_{-1}) = (A_l, \{\alpha_l\}, 2\varpi_l)$ $(l \geqq 3)$, $r = 1,\ 2,\ 3,\ 4$.
(16) $(\mathcal{D}(\mathfrak{l}), \Delta_1, \Lambda_{-1}) = (A_l, \{\alpha_2\}, 2\varpi_l)$ $(l \geqq 3,\ 3 \leqq i \leqq l-1)$, $r = 2$.
(17) $(\mathcal{D}(\mathfrak{l}), \Delta_1, \Lambda_{-1}) = (A_3, \{\alpha_1\}, \varpi_2)$, $r = 1,\ 2,\ 3$.
(18) $(\mathcal{D}(\mathfrak{l}), \Delta_1, \Lambda_{-1}) = (A_3, \{\alpha_2\}, \varpi_2)$, $r = 1,\ 2,\ 3,\ 4$.
(19) $(\mathcal{D}(\mathfrak{l}), \Delta_1, \Lambda_{-1}) = (A_l, \{\alpha_1\}, \varpi_{l-1})$ $(l \geqq 4)$, $r = 1,\ 2,\ 3$.
(20) $(\mathcal{D}(\mathfrak{l}), \Delta_1, \Lambda_{-1}) = (A_l, \{\alpha_i\}, \varpi_{l-1})$ $(2 \leqq i \leqq l-3)$, $r = 2,\ 3$.
(21) $(\mathcal{D}(\mathfrak{l}), \Delta_1, \Lambda_{-1}) = (A_l, \{\alpha_{l-2}\}, \varpi_{l-1})$, $r = 2,\ 3$.
(22) $(\mathcal{D}(\mathfrak{l}), \Delta_1, \Lambda_{-1}) = (A_l, \{\alpha_{l-1}\}, \varpi_{l-1})$ $(l \geqq 4)$, $r = 2,\ 3,\ 4$.
(23) $(\mathcal{D}(\mathfrak{l}), \Delta_1, \Lambda_{-1}) = (A_l, \{\alpha_l\}, \varpi_{l-1})$ $(l \geqq 4)$, $r = 2,\ 3$.
(24) $(\mathcal{D}(\mathfrak{l}), \Delta_1, \Lambda_{-1}) = (B_l, \{\alpha_1\}, \varpi_1)$ $(l \geqq 2)$, $r = 1,\ 2,\ 3,\ 4$
(25) $(\mathcal{D}(\mathfrak{l}), \Delta_1, \Lambda_{-1}) = (D_l, \{\alpha_l\}, \varpi_1)$ $(l \geqq 4)$, $r = 1,\ 2,\ 3$.
(26) $(\mathcal{D}(\mathfrak{l}), \Delta_1, \Lambda_{-1}) = (D_l, \{\alpha_1\}, \varpi_1)$ $(l \geqq 4)$, $r = 1,\ 2,\ 3,\ 4$.
(27) $(\mathcal{D}(\mathfrak{l}), \Delta_1, \Lambda_{-1}) = (D_5, \{\alpha_5\}, \varpi_5)$, $r = 2,\ 3,\ 4$.
(28) $(\mathcal{D}(\mathfrak{l}), \Delta_1, \Lambda_{-1}) = (D_5, \{\alpha_4\}, \varpi_5)$, $r = 2,\ 3$.
(29) $(\mathcal{D}(\mathfrak{l}), \Delta_1, \Lambda_{-1}) = (D_5, \{\alpha_1\}, \varpi_5)$, $r = 2$.
(30) $(\mathcal{D}(\mathfrak{l}), \Delta_1, \Lambda_{-1}) = (E_6, \{\alpha_1\}, \varpi_6)$, $r = 2$.
(31) $(\mathcal{D}(\mathfrak{l}), \Delta_1, \Lambda_{-1}) = (E_6, \{\alpha_6\}, \varpi_6)$, $r = 2,\ 3,\ 4$.

In this case, by Theorem 5.2, $\mathfrak{s} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{s}_p$ is the prolongation of $\mathfrak{m}$ or $(\mathfrak{m}, \mathfrak{g}_0)$. Hence we include the following theorem (Proposition 6.2 [Yam99]) for $H^2(\mathfrak{m}, \mathfrak{s})$.

Theorem 6.3. *Let $\mathfrak{G}$ be a pseudo-product GLA of type $(\mathfrak{l}, S)$ such that:* (i) *$\hat{\mathfrak{l}}$ is an SGLA of type (X_l, Δ_1);* (ii) *S is an irreducible $\mathcal{D}(\mathfrak{l})$-module with highest weight Ξ;* (iii) *$H^1(\mathfrak{G})_{1,1} \neq 0$. Then the following are the triplet $(\mathcal{D}(\mathfrak{l}), \Delta_1, \Xi)$ and the set of r such that $H^2(\mathfrak{m}, \mathfrak{s})_r \neq 0$ $(r \geqq 1)$.*

(1) $(\mathcal{D}(\mathfrak{l}), \Delta_1, \Xi) = (A_1, \{\alpha_1\}, \varpi_1)$, $r = 4$.
(2) $(\mathcal{D}(\mathfrak{l}), \Delta_1, \Xi) = (A_1 \times A_1, \{\alpha_1\}, \varpi_1 + \pi_1)$, $r = 1,\ 2,\ 3$.
(3) $(\mathcal{D}(\mathfrak{l}), \Delta_1, \Xi) = (A_1 \times A_n, \{\alpha_1\}, \varpi_1 + \pi_1)$ $(n \geqq 2)$, $r = 2,\ 3$.
(4) $(\mathcal{D}(\mathfrak{l}), \Delta_1, \Xi) = (A_l \times A_n, \{\alpha_1\}, \varpi_l + \pi_1)$ $(l \geqq 2,\ n \geqq 0)$, $r = 1,\ 2$.

(5) $(\mathcal{D}(\mathfrak{l}), \Delta_1, \Xi) = (A_l \times A_n, \{\alpha_2\}, \varpi_l + \pi_1)$ $(l \geqq 3,\ n \geqq 0)$, $r = 1$.
(6) $(\mathcal{D}(\mathfrak{l}), \Delta_1, \Xi) = (A_l \times A_n, \{\alpha_i\}, \varpi_l + \pi_1)$
$(3 \leqq i \leqq l-1,\ 0 \leqq n \leqq 1)$, $r = 1$.
(7) $(\mathcal{D}(\mathfrak{l}), \Delta_1, \Xi) = (A_2, \{\alpha_2\}, \varpi_1)$, $r = 1,\ 2,\ 3$.
(8) $(\mathcal{D}(\mathfrak{l}), \Delta_1, \Xi) = (A_l, \{\alpha_l\}, \varpi_1)$ $(l \geqq 3)$, $r = 2,\ 3$.
(9) $(\mathcal{D}(\mathfrak{l}), \Delta_1, \Xi) = (A_l \times A_n, \{\alpha_l\}, \varpi_l + \pi_1)$ $(l \geqq 2,\ n \geqq 1)$, $r = 1$.
(10) $(\mathcal{D}(\mathfrak{l}), \Delta_1, \Xi) = (A_1, \{\alpha_1\}, 2\varpi_1)$, $r = 3,\ 4$.
(11) $(\mathcal{D}(\mathfrak{l}), \Delta_1, \Xi) = (A_l, \{\alpha_1\}, 2\varpi_l)$ $(l \geqq 2)$, $r = 1$.
(12) $(\mathcal{D}(\mathfrak{l}), \Delta_1, \Xi) = (A_2, \{\alpha_2\}, 2\varpi_2)$, $r = 1$.
(13) $(\mathcal{D}(\mathfrak{l}), \Delta_1, \Xi) = (A_l, \{\alpha_2\}, 2\varpi_l)$ $(l \geqq 3)$, $r = 1$.
(14) $(\mathcal{D}(\mathfrak{l}), \Delta_1, \Xi) = (A_l, \{\alpha_l\}, 2\varpi_l)$ $(l \geqq 3)$, $r = 1$.
(15) $(\mathcal{D}(\mathfrak{l}), \Delta_1, \Xi) = (A_3, \{\alpha_1\}, \varpi_2)$, $r = 1$.
(16) $(\mathcal{D}(\mathfrak{l}), \Delta_1, \Xi) = (A_3, \{\alpha_2\}, \varpi_2)$, $r = 1,\ 2$.
(17) $(\mathcal{D}(\mathfrak{l}), \Delta_1, \Xi) = (A_l, \{\alpha_1\}, \varpi_{l-1})$ $(l \geqq 4)$, $r = 1$.
(18) $(\mathcal{D}(\mathfrak{l}), \Delta_1, \Xi) = (B_l, \{\alpha_1\}, \varpi_1)$ $(l \geqq 2)$, $r = 1,\ 2$
(19) $(\mathcal{D}(\mathfrak{l}), \Delta_1, \Xi) = (D_l, \{\alpha_l\}, \varpi_1)$ $(l \geqq 4)$, $r = 1$.
(20) $(\mathcal{D}(\mathfrak{l}), \Delta_1, \Xi) = (D_l, \{\alpha_1\}, \varpi_1)$ $(l \geqq 4)$, $r = 1,\ 2$.

6.3.2. The case $\dim \check{\mathfrak{b}} = \infty$. By the theorem of Kobayashi and Nagano [KN65], $\check{\mathfrak{b}} = \bigoplus_{p \in \mathbb{Z}} \check{\mathfrak{b}}_p$ is isomorphic to one of $W(n; \mathbf{1})$ and $CH(l; \mathbf{1}; 2)$ $(l \geqq 2)$. On the other hand, if $\check{\mathfrak{b}} = \bigoplus_{p \in \mathbb{Z}} \check{\mathfrak{b}}_p$ is isomorphic to $W(n; \mathbf{1})$, then $H^1(\mathfrak{G})_{1,1} \neq 0$ and the triple $(\mathcal{D}(\mathfrak{l}), \Delta_1, \Xi)$ is $(A_l, \{\alpha_i\}, \varpi_1)$. In this case, our problem is reduced to Theorem 6.2. Hence we may consider only the case where $\check{\mathfrak{b}} = \bigoplus_{p \in \mathbb{Z}} \check{\mathfrak{b}}_p \cong CH(l; \mathbf{1}; 2)$ $(l \geqq 2)$. Thus we assume that the triple $(\mathcal{D}(\mathfrak{l}), \Delta_1, \Xi)$ is $(C_l, \{\alpha_l\}, \varpi_1)$. Since $\mathfrak{csp}(\mathfrak{b}_{-1})$ is an involutive subalgebra of $\mathfrak{gl}(\mathfrak{b}_{-1})$ (cf. [KN66]), we have

$$H^i(\mathfrak{b}_-, \check{\mathfrak{b}})_s = 0 \quad \text{for } i \leqq s. \tag{6.1}$$

By Theorem 4.1,

$$H^2(\mathfrak{G})_{r,1} \cong H^0(\mathfrak{l}_-, H^2(\mathfrak{b}_-, \mathfrak{g})_1)_r \oplus H^1(\mathfrak{l}_-, H^1(\mathfrak{b}_-, \mathfrak{g})_1)_r.$$

Clearly, $\mathrm{ch}_{\mathfrak{l}}(H^1(\mathfrak{b}_-, \mathfrak{g})_1) = \mathrm{ch}_{\mathfrak{l}}(\check{\mathfrak{b}}_1)$. By (6.1), we obtain

$$\mathrm{ch}_{\mathfrak{l}}(H^2(\mathfrak{b}_-, \mathfrak{g})_1) = \mathrm{ch}_{\mathfrak{l}}\left(\bigwedge^2 \mathfrak{b}_{-1}^* \otimes \mathfrak{b}_{-1}\right) - \mathrm{ch}_{\mathfrak{l}}(\mathfrak{b}_{-1}^* \otimes \mathfrak{b}_0) + \mathrm{ch}_{\mathfrak{l}}(\check{\mathfrak{b}}_1).$$

Using the tables of [OV90] and [Kac68], we have the following decomposition into irreducible $\mathfrak{l}$-modules.

$$\check{\mathfrak{b}}_1 \cong L(3\varpi_1)$$

$$\mathrm{Hom}\left(\bigwedge^2 \mathfrak{b}_{-1}, \mathfrak{b}_{-1}\right) \cong 2L(\varpi_1) \oplus L(\varpi_1 + \varpi_2) \oplus L(\varpi_3),$$

$$\mathrm{Hom}(\mathfrak{b}_{-1}, \mathfrak{b}_0) \cong L(\varpi_1) \oplus L(\varpi_1 + \varpi_2) \oplus L(3\varpi_1).$$

Therefore $\mathrm{ch}_{\mathfrak{l}}(H^2(\mathfrak{b}_-,\mathfrak{g})_1) = \mathrm{ch}_{\mathfrak{l}}(L(\varpi_3))$. Here $\mathrm{ch}_{\mathfrak{l}}(L(\varpi_3)) = 0$ in case $l = 2$. Moreover

$$\langle \xi^1_{r_l}(3\varpi_1), E\rangle = 1, \qquad \langle \xi^1_1(\varpi_3), E\rangle = 0.$$

By Theorem 4.1,

$$H^2(\mathfrak{G})_{r,2} \cong H^0(\mathfrak{l}_-, H^2(\mathfrak{b}_-,\mathfrak{g})_2)_r.$$

By (6.1), we obtain

$$\mathrm{ch}_{\mathfrak{l}}(H^2(\mathfrak{b}_-,\mathfrak{g})_2) = \mathrm{ch}_{\mathfrak{l}}(\check{\mathfrak{b}}_1 \otimes \mathfrak{b}^*_{-1}) - \mathrm{ch}_{\mathfrak{l}}(\check{\mathfrak{b}}_2).$$

By the tables of [OV90] (also see [AG93]) and [Kac68]),

$$\begin{aligned} \check{\mathfrak{b}}_1 \otimes \mathfrak{b}^*_{-1} &\cong L(2\varpi_1) \oplus L(2\varpi_1 + \varpi_2) \oplus L(4\varpi_1) \\ \check{\mathfrak{b}}_2 &\cong L(4\varpi_1). \end{aligned}$$

Hence

$$\mathrm{ch}_{\mathfrak{l}}(H^2(\mathfrak{b}_-,\mathfrak{g})_2) = \mathrm{ch}_{\mathfrak{l}}(L(2\varpi_1 + \varpi_2)) + \mathrm{ch}_{\mathfrak{l}}(L(2\varpi_1)).$$

Moreover

$$\langle \xi^2_1(2\varpi_1 + \varpi_2), E\rangle = 1, \qquad \langle \xi^2_1(2\varpi_1), E\rangle = 2,$$

where 1 denotes the unit element of Weyl group of $\mathcal{D}(\mathfrak{l})$.

Theorem 6.4. *Let $\mathfrak{G}$ be a PPGLA of type $(\mathfrak{l}, S)$ such that the triple $(\mathcal{D}(\mathfrak{l}), \Delta_1, \Xi)$ is $(C_l, \{\alpha_l\}, \varpi_1)$ $(l \geqq 2)$. Then we obtain the following list of pairs (r, s) such that $H^2(\mathfrak{G})_{r,s} \neq 0$ $(r \geqq 1,\ s = 1,\ 2)$.*

(i) $l = 2$, $(r, s) = (1, 1), (1, 2), (2, 2)$.
(ii) $l \geqq 3$, $(r, s) = (0, 1), (1, 1), (1, 2), (2, 2)$.

6.3.3. The general case. Let Λ be the highest weight of the $\mathfrak{l}$-module $H^2(\mathfrak{b}_-,\mathfrak{g})_s$. Then

$$\langle \xi^s_1(\Lambda), E\rangle \leqq s\langle \chi, \varpi_i^\vee\rangle + s. \tag{6.2}$$

Hence we have

Proposition 6.3. *Let $\mathfrak{G}$ be a PPGLA of type $(\mathfrak{l}, S)$. Then:*

(1) *For $s = 1,\ 2$,*

$$H^0(\mathfrak{l}_-, H^2(\mathfrak{b}_-,\mathfrak{g})_s)_r = 0 \quad \textit{for } r \geqq s(\mu - 1) + 1.$$

(2) *If $X_l = B_l$, C_l or E_7, then*

$$H^0(\mathfrak{l}_-, H^2(\mathfrak{b}_-, \mathfrak{g})_s)_r = 0 \quad \text{for } r \geqq \left[\frac{s(\mu+1)}{2}\right] + 1.$$

Proof. (1) Since $\mu = \langle \chi - w_0(\chi), \varpi_i^\vee \rangle + 1$ and $\langle w_0(\chi), \varpi_i^\vee \rangle < 0$, we have $s(\langle \chi, \varpi_i^\vee \rangle + 1) = s(\mu + \langle w_0(\chi), \varpi_i^\vee \rangle \leqq s(\mu - 1)$, which proves our assertion.

(2) If $X_l = B_l$, C_l or E_7, then $w_0(\chi) = -\chi$, so $\mu = 2\langle \chi, \varpi_i^\vee \rangle + 1$. By (6.1), we have $\langle \xi_1^s(\Lambda), E \rangle \leqq s(\mu+1)/2$, which proves our assertion. Q.E.D.

§7. The symbol algebras of the Plücker embedding equations

In this section we will calculate $H^2(\mathfrak{G})$ when the PPGLA $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ of type $(\mathfrak{l}, S)$ is associated with the Plücker embedding equations for $M = \mathrm{Gr}(i, l+1)$. Namely let $\mathfrak{G}$ be a pseudo-product GLA of type $(\mathfrak{l}, S)$ such that the triple $(\mathcal{D}(\mathfrak{l}), \Delta_1, \Xi)$ is $(A_l, \{\alpha_i\}, \varpi_{l-i+1})$ $(1 \leqq i \leqq l-i+1)$, i.e., $\mathfrak{l} = \mathfrak{l}_{-1} \oplus \mathfrak{l}_0 \oplus \mathfrak{l}_1$ is a reductive GLA such that

(1) $\mathfrak{l}$ is isomorphic to $\mathfrak{gl}(l+1, \mathbb{C})$.

(2) $\hat{\mathfrak{l}}$ is an SGLA of type $(X_l, \Delta_1) = (A_l, \{\alpha_i\})$ $(1 \leqq i \leqq l)$,

and S is an irreducible $\mathfrak{l}$-module with highest weight $\chi = \varpi_{l-i+1}$ (i.e., $S = \bigwedge^{l-i+1}(\mathbb{C}^{l+1})$). We set $\mathfrak{b}_0 = \mathfrak{l}$, $\mathfrak{b}_{-1} = S$; then $\mathfrak{g} = \mathfrak{b}_{-1} \oplus \mathfrak{b}_0$. Let $\check{\mathfrak{b}} = \bigoplus_{p\in\mathbb{Z}} \check{\mathfrak{b}}_p$ be the prolongation of $\mathfrak{b}_{-1} \oplus \mathfrak{b}_0$. Also, we use the notation in §4.

7.1. The case $i = 1$ or 2 (the case $\check{\mathfrak{b}}_1 \neq 0$)

In this subsection, we assume $\check{\mathfrak{b}}_1 \neq 0$. Then by Theorem 5.1 and Table 1, we see that $H^1(\mathfrak{G})_{1,1} \neq 0$ and the triple $(\mathcal{D}(\mathfrak{l}), \Delta_1, \Lambda_{-1})$ is either $(A_l, \{\alpha_1\}, \varpi_l)$ or $(A_l, \{\alpha_2\}, \varpi_{l-1})$ $(l \geqq 3)$. By Theorem 6.2 (1), (4), (18), (20) and (21), we obtain the following theorem.

Theorem 7.1. *Let $\mathfrak{G}$ be a PPGLA of type $(\mathfrak{l}, S)$ such that the triple $(\mathcal{D}(\mathfrak{l}), \Delta_1, \Xi)$ is $(A_l, \{\alpha_i\}, \varpi_{l-i+1})$. If $\check{\mathfrak{b}}_1 \neq 0$, then the triple $(\mathcal{D}(\mathfrak{l}), \Delta_1, \Xi)$ is $(A_l, \{\alpha_1\}, \varpi_l)$ $(l \geqq 1)$ or $(A_l, \{\alpha_2\}, \varpi_{l-1})$ $(l \geqq 3)$ and we have*

(1) *We assume that $(\mathcal{D}(\mathfrak{l}), \Delta_1, \Xi)$ is $(A_l, \{\alpha_1\}, \varpi_l)$ $(l \geqq 1)$. Then the set of integers such that $H^2(\mathfrak{G})_r \neq 0$ $(r \geqq 1)$ is the following.*
   - (i) $l = 1$, $r = 2, 3, 4$.
   - (ii) $l \geqq 2$, $r = 1, 2$.

(2) *We assume that the triple $(\mathcal{D}(\mathfrak{l}), \Delta_1, \Xi)$ is $(A_l, \{\alpha_2\}, \varpi_{l-1})$ $(l \geqq 3)$. Then the set of integers r such that $H^2(\mathfrak{G})_r \neq 0$ $(r \geqq 1)$ is the following.*

(i) $l=3$, $r=1, 2, 3, 4$.
(ii) $l=4$, $r=2, 3$.
(iii) $l \geqq 5$, $r=2$.

In these cases, $\mathfrak{g}=\bigoplus_{p\in\mathbb{Z}}\mathfrak{g}_p$ is a parabolic graded subalgebra of a simple GLA $\mathfrak{s}=\bigoplus_{p\in\mathbb{Z}}\mathfrak{s}_p$ such that $\mathfrak{g}_p=\mathfrak{s}_p$ for all $p \leqq 0$. In fact $\mathfrak{s}=\bigoplus_{p\in\mathbb{Z}}\mathfrak{s}_p$ is of type $(A_{l+1},\{\gamma_1,\gamma_{l+1}\})$ and is the prolongation of $(\mathfrak{m},\mathfrak{g}_0)$ when $i=1$ and $\mathfrak{s}=\bigoplus_{p\in\mathbb{Z}}\mathfrak{s}_p$ is of type $(D_{l+1},\{\gamma_2,\gamma_{l+1}\})$ and is the prolongation of $\mathfrak{m}$ when $i=2$. Hence we obtain the following theorem for $H^2(\mathfrak{m},\mathfrak{s})$ (Proposition 6.2 [Yam99]).

Theorem 7.2. *Notations being as above.*

(1) *We assume that $(\mathcal{D}(\mathfrak{l}),\Delta_1,\Xi)$ is $(A_l,\{\alpha_1\},\varpi_l)$ $(l \geqq 1)$. Then the set of integers such that $H^2(\mathfrak{m},\mathfrak{s})_r \neq 0$ $(r \geqq 1)$ is the following.*
   (i) $l=1$, $r=4$.
   (ii) $l \geqq 2$, $r=1, 2$.
(2) *We assume that the triple $(\mathcal{D}(\mathfrak{l}),\Delta_1,\Xi)$ is $(A_l,\{\alpha_2\},\varpi_{l-1})$ $(l \geqq 3)$. Then the set of integers r such that $H^2(\mathfrak{m},\mathfrak{s})_r \neq 0$ $(r \geqq 1)$ is the following.*
   (i) $l=3$, $r=1, 2$.

7.2. The case $i \geqq 3$ (the case $\check{\mathfrak{b}}_1=0$)

Let $\hat{\mathfrak{l}}$ be a simple GLA of type $(A_l,\{\alpha_i\})$ $(l \geqq 5,\ 3 \leqq i \leqq l-i+1)$.
7.2.1. The computation of $H^2(\mathfrak{G})_{r,-1}$. We have

$$\begin{aligned}\langle \xi^{-1}_{r_ir_{i-1}}(\varpi_{l-i+1}),E\rangle &= \langle \xi^{-1}_{r_ir_{i+1}}(\varpi_{l-i+1}),E\rangle \\ &= -(i-1)+1=-i+2 \leqq 0.\end{aligned}$$

7.2.2. The computation of $H^2(\mathfrak{G})_{r,0}$.
We have already known that

$$H^2(\mathfrak{G})_{r,0} \cong H^1(\mathfrak{l}_-,H^1(\mathfrak{b}_-,\mathfrak{g})_0)_r.$$

Also $H^1(\mathfrak{b}_-,\mathfrak{g})_0$ is isomorphic to $\mathrm{Hom}(\mathfrak{b}_{-1},\mathfrak{b}_{-1})/\delta^0(\mathfrak{b}_0)$ as a $\mathfrak{b}_0$-module. By the table of [OV90], we have

$$\mathrm{Hom}(\mathfrak{b}_{-1},\mathfrak{b}_{-1}) \cong L(\varpi_{l-i+1})\otimes L(\varpi_i)=\bigoplus_{j\geqq 0}L(\varpi_{l-i+j+1}+\varpi_{i-j})$$
$$\mathfrak{b}_0 \cong L(\varpi_1+\varpi_l)\oplus L(0).$$

Hence

$$H^1(\mathfrak{b}_-,\mathfrak{g})_0 \cong \bigoplus_{j=0}^{i-2}L(\varpi_{l-i+j+1}+\varpi_{i-j}).$$

We get

$$\begin{aligned}
&\langle \xi_1^0(\varpi_{l-i+j+1}+\varpi_{i-j}), E\rangle = \langle r_i(w_0(\varpi_{l-i+j+1}+\varpi_{i-j})-\rho)+\rho, \varpi_i^\vee\rangle \\
&\quad = -\langle \varpi_{i-j}+\varpi_{l-i+j+1}, \varpi_i^\vee\rangle + \langle \varpi_{i-j}+\varpi_{l-i+j+1}, \alpha_i\rangle \\
&\quad = \frac{-1}{l+1}((i-j)(l-i+1)+(i-j)i)+\delta_{j,0}+\delta_{l-i+j+1,i}+1 \\
&\quad = j-i+\delta_{j,0}+\delta_{l-i+j+1,i}+1 \leqq 0
\end{aligned}$$

7.2.3. The computation of $H^2(\mathfrak{G})_{r,1}$.

We have already known that

$$H^2(\mathfrak{G})_{r,1} \cong H^0(\mathfrak{l}_-, H^2(\mathfrak{b}_-, \mathfrak{g})_1)_r.$$

Also $H^2(\mathfrak{b}_-,\mathfrak{g})_1$ is isomorphic to $\mathrm{Hom}(\bigwedge^2 \mathfrak{b}_{-1}, \mathfrak{b}_{-1})/\delta^0(\mathrm{Hom}(\mathfrak{b}_{-1},\mathfrak{b}_0))$ as a $\mathfrak{b}_0$-module. By the table [OV90] and the Young tableau method, we have

$$\begin{aligned}
\bigwedge\nolimits^2 \mathfrak{b}_{-1}^* &= \bigoplus_{j=0}^{[(i-1)/2]} L(\varpi_{i+2j+1}+\varpi_{i-2j-1}) \\
\mathfrak{b}_0\otimes\mathfrak{b}_{-1}^* &\cong (L(\varpi_1+\varpi_l)\otimes L(\varpi_i))\oplus L(\varpi_i) \\
&= L(\varpi_1+\varpi_i+\varpi_l)\oplus L(\varpi_1+\varpi_{i-1}) \\
&\quad \oplus L(\varpi_{i+1}+\varpi_l)\oplus 2L(\varpi_i) \\
\mathfrak{b}_{-1}\otimes\bigwedge\nolimits^2 \mathfrak{b}_{-1}^* &\cong \bigoplus_{j=0}^{[(i-1)/2]} \bigoplus_{(a,b,c)\in A(j)} L(\varpi_a+\varpi_{i-2j-1+b}+\varpi_{i+2j+1+c}),
\end{aligned}$$

where $A(j)=\{(a,b,c)\in\mathbb{Z}^3 \mid 0\leqq a\leqq i-2j-1,\ 0\leqq b\leqq 4j+2,\ 0\leqq c\leqq l-i-2j,\ a+b+c=l-i+1\}$.

Case 1: $i<l-i-2j-c\leqq l-i+2j+2-b\leqq l-a+1$.

$$\begin{aligned}
&\langle \xi_1^1(\varpi_a+\varpi_{i-2j-1+b}+\varpi_{i+2j+c+1}), E\rangle \\
&\quad = -\frac{1}{l+1}((l+i+1)i-i^2)+1 = -i+1\leqq 0
\end{aligned}$$

Case 2: $l-i-2j-c<i\leqq l-i+2j+2-b\leqq l-a+1$.

$$\begin{aligned}
&\langle \xi_1^1(\varpi_a+\varpi_{i-2j-1+b}+\varpi_{i+2j+c+1}), E\rangle \\
&\quad = -(l-i-2j-c)+1\leqq -l+i+2j+1+l-i-2j=1
\end{aligned}$$

Case 3: $l-i-2j-c\leqq l-i+2j+2-b<i\leqq l-a+1$.

$$\begin{aligned}
&\langle \xi_1^1(\varpi_a+\varpi_{i-2j-1+b}+\varpi_{i+2j+c+1}), E\rangle \\
&\quad = -l-a+2i\leqq -l-a+i+l-i+1=-a+1\leqq 1
\end{aligned}$$

Case 4: $l-i-2j-c \leqq l-i+2j+2-b \leqq l-a+1 < i$.

$$\begin{aligned}&\langle \xi_1^1(\varpi_a+\varpi_{i-2j-1+b}+\varpi_{i+2j+c+1}), E\rangle\\ &\quad = -2l+3i-1 \leqq -2l+i+2(l-i+1)-1 \leqq -i+1 \leqq 0.\end{aligned}$$

Proposition 7.1.
(1) *$H^2(\mathfrak{G})_{r,-1}=0$ for all $r \geqq 1$.*
(2) *$H^2(\mathfrak{G})_{r,0}=0$ for all $r \geqq 1$.*
(3) *$H^2(\mathfrak{G})_{r,1}=0$ for all $r \geqq 2$. Furthermore $H^2(\mathfrak{G})_{1,1}=0$ when $i=3$.*

Proof. We need to show the case $i=3$. By the above results, we have

$$\begin{aligned}H^2(\mathfrak{b}_-,\mathfrak{g})_1 \cong\ & L(\varpi_2+\varpi_3+\varpi_{l-1}) \oplus L(2\varpi_2+\varpi_l)\\ &\oplus L(\varpi_1+\varpi_4+\varpi_{l-1}) \oplus L(\varpi_4+\varpi_l)\\ &\oplus L(\varpi_2+\varpi_4+\varpi_{l-2})\ (l \geqq 6) \oplus L(\varpi_5+\varpi_{l-1})\ (l \geqq 7)\\ &\oplus L(\varpi_6+\varpi_{l-2})\ (l \geqq 8).\end{aligned}$$

Hence we obtain
$$H^2(\mathfrak{G})_{r,1}=0 \text{ for all } r \geqq 1.$$

Q.E.D.

7.3. The computation of $H^2(\mathfrak{G})_{r,2}$ when $i=3$ and $l \geqq 5$

In what follows, we consider the case where $\mathfrak{b}_0=\mathfrak{gl}(V)$ and $\mathfrak{b}_{-1}=\bigwedge^{l-2}V$, where $V=\mathbb{C}^{l+1}$ $(l \geqq 5)$. We see that

$$H^2(\mathfrak{G})_{r,2} \cong H^0(\mathfrak{l}_-, H^2(\mathfrak{b}_-,\mathfrak{g})_2)_r, \quad H^2(\mathfrak{b}_-,\mathfrak{g})_2 \cong \operatorname{Ker}\delta_2,$$

where δ^2 is the coboundary operator $\delta^2\colon C^2(\mathfrak{b}_-,\mathfrak{g})_2 \to C^3(\mathfrak{b}_-,\mathfrak{g})_2$.

We investigate a relation between the coboundary operators $\delta^0\colon \mathfrak{b}_0 \to \operatorname{Hom}(\mathfrak{b}_{-1},\mathfrak{b}_{-1})$ and $\delta^2\colon \operatorname{Hom}(\bigwedge^2\mathfrak{b}_{-1},\mathfrak{b}_0) \to \operatorname{Hom}(\bigwedge^3\mathfrak{b}_{-1},\mathfrak{b}_{-1})$. Recall that for $\varphi \in \mathfrak{b}_0$, $\omega \in \operatorname{Hom}(\bigwedge^2\mathfrak{b}_{-1},\mathfrak{b}_0)$ and $v_1, v_2, v_3 \in \mathfrak{b}_{-1}$,

$$\delta_0(\varphi)(v_1)=[v_1,\varphi]=-\varphi(v_1)$$

and

$$\begin{aligned}\delta_2(\omega)(v_1,v_2,v_3) &= [v_1,\omega(v_2,v_3)]-[v_2,\omega(v_1,v_3)]+[v_3,\omega(v_1,v_2)]\\ &= -\omega(v_2,v_3)(v_1)+\omega(v_1,v_3)(v_2)-\omega(v_1,v_2)(v_3).\end{aligned}$$

In particular, if $\omega=\varphi\otimes w_1^*\wedge w_2^*$, $\delta^2(\varphi\otimes w_1^*\wedge w_2^*)=\delta^0(\varphi)\wedge w_1^*\wedge w_2^*$.

We use the following notation: Let $\{e_1,\dots,e_{l+1}\}$ be the canonical basis of V and let $\{e_1^*,\dots,e_{l+1}^*\}$ be the dual basis. Let E_{ij} be the element of $\mathfrak{b}_0$ such that $E_{ij}e_k = \delta_{jk}e_i$. Also we put $e_{i_1\dots i_{l-2}} = e_{i_1} \wedge \cdots \wedge e_{i_{l-2}}$ and $e^*_{i_1\dots i_{l-2}} = e^*_{i_1} \wedge \cdots \wedge e^*_{i_{l-2}}$.

Let $\mathfrak{h}$ be the canonical Cartan subalgebra of $\mathfrak{b}_0$ (i.e., $\mathfrak{h} = \sum_{i=1}^{l+1} \mathbb{C}E_{ii}$). We define a linear form λ_i of $\mathfrak{h}$ by $\lambda_i(E_{jj}) = \delta_{ij}$. Then

$$\varpi_i = \sum_{j=1}^{i} \lambda_j - \frac{i}{l+1} \sum_{j=1}^{l+1} \lambda_j.$$

and $e^*_{i_1\dots i_{l-2}}$ is a weight vector of $\mathfrak{l}$-module S with weight $-\lambda_{i_1} - \cdots - \lambda_{i_{l-2}}$. Note that if Λ is a weight of the $\mathcal{D}(\mathfrak{l})$-module $\operatorname{Ker}\delta^2$, then $\Lambda + 2\sigma$ is a weight of the $\mathfrak{l}$-module $\operatorname{Ker}\delta^2$. Since $\operatorname{ad}(I_{l+1})|S = (l-2)1_S$ and $\operatorname{ad}(J)|S = -1_S$, we have $I_{l+1} = -(l-2)J$, where $I_{l+1} = \sum_{i=1}^{l+1} E_{ii}$. Hence $\sigma = -\frac{l-2}{l+1} \sum_{i=1}^{l+1} \lambda_i$ and

$$\Lambda + 2\sigma = \sum_{i=1}^{l} \sum_{j=1}^{i} m_i \lambda_j - \frac{1}{l+1} \left\{ \sum_{i=1}^{l} m_i i + 2(l-2) \right\} \sum_{j=1}^{l+1} \lambda_j.$$

The highest weight vectors v_Λ of $\operatorname{Ker}\delta^2$ with highest weight $\Lambda + 2\sigma$ have the following forms:

$$v_\Lambda = \sum a_{i,j,i_1,\dots,i_{l-2},j_1,\dots,j_{l-2}} E_{ij} \otimes e^*_{i_1,\dots,i_{l-2}} \wedge e^*_{j_1\dots j_{l-2}},$$

where the summation is taken over all i, j, $i_1, \dots, i_{l-2}$, $j_1, \dots, j_{l-2}$ such that $\Lambda + 2\sigma = \lambda_i - \lambda_j - \sum_{k=1}^{l-2}(\lambda_{i_k} + \lambda_{j_k})$. Since v_Λ is a highest weight vector, $E_{k,k+1}v_\Lambda = 0$ for $1 \leqq k \leqq l$, we get the relations between the coefficients $\{a_{i,j,i_1,\dots,i_{l-2},j_1,\dots,j_{l-2}}\}$ of v_Λ. By these methods, we can calculate $\delta^2(v_\Lambda)$.

7.3.1. The case $l = 5$.

We first consider the case $l = 5$. By the table of [OV90] and the Young tableau method, we have

$$\begin{aligned} \mathfrak{b}_0 \otimes \bigwedge^2 \mathfrak{b}^*_{-1} \cong{} & L(\varpi_3 + \varpi_4 + \varpi_5) \oplus L(2\varpi_3) \\ & \oplus 3L(\varpi_2 + \varpi_4) \oplus L(\varpi_1 + \varpi_2 + \varpi_3) \\ & \oplus L(\varpi_2 + 2\varpi_5) \oplus 2L(\varpi_1 + \varpi_5) \\ & \oplus L(\varpi_1 + \varpi_2 + \varpi_4 + \varpi_5) \oplus L(2\varpi_1 + \varpi_4) \\ & \oplus L(0). \end{aligned}$$

By a direct inspection, we see that $\operatorname{Ker}\delta^2 \cong L(\varpi_1 + \varpi_5)$ and

$$\langle \xi_1(\varpi_1 + \varpi_5), E \rangle = -\langle \varpi_1 + \varpi_5, \varpi_3^\vee \rangle + 2(\langle \varpi_3, \varpi_3^\vee \rangle + 1) = 3.$$

Also a lowest weight vector of $H^2(\mathfrak{b}_-, \mathfrak{g})_2$ is the following (This vector is also a generator of $H^2(\mathfrak{G})_{3,2}$).

$$\begin{aligned}
\omega_{15} = {} & E_{65} \otimes e^*_{641} \wedge e^*_{321} - E_{65} \otimes e^*_{631} \wedge e^*_{421} + E_{65} \otimes e^*_{621} \wedge e^*_{431} \\
& - E_{64} \otimes e^*_{651} \wedge e^*_{321} + E_{64} \otimes e^*_{631} \wedge e^*_{521} - E_{64} \otimes e^*_{621} \wedge e^*_{531} \\
& + E_{63} \otimes e^*_{651} \wedge e^*_{421} - E_{63} \otimes e^*_{641} \wedge e^*_{521} + E_{63} \otimes e^*_{621} \wedge e^*_{541} \\
& - E_{62} \otimes e^*_{651} \wedge e^*_{431} + E_{62} \otimes e^*_{641} \wedge e^*_{531} - E_{62} \otimes e^*_{631} \wedge e^*_{541} \\
& + 2E_{61} \otimes e^*_{654} \wedge e^*_{321} - 2E_{61} \otimes e^*_{653} \wedge e^*_{421} + 2E_{61} \otimes e^*_{652} \wedge e^*_{431} \\
& - E_{61} \otimes e^*_{651} \wedge e^*_{432} + 2E_{61} \otimes e^*_{643} \wedge e^*_{521} - 2E_{61} \otimes e^*_{642} \wedge e^*_{531} \\
& + E_{61} \otimes e^*_{641} \wedge e^*_{532} + 2E_{61} \otimes e^*_{632} \wedge e^*_{541} - E_{61} \otimes e^*_{631} \wedge e^*_{542} \\
& + E_{61} \otimes e^*_{621} \wedge e^*_{543} + E_{55} \otimes e^*_{541} \wedge e^*_{321} - E_{55} \otimes e^*_{531} \wedge e^*_{421} \\
& + E_{55} \otimes e^*_{521} \wedge e^*_{431} + 2E_{54} \otimes e^*_{531} \wedge e^*_{521} - 2E_{53} \otimes e^*_{541} \wedge e^*_{521} \\
& + 2E_{52} \otimes e^*_{541} \wedge e^*_{531} + E_{51} \otimes e^*_{543} \wedge e^*_{521} - E_{51} \otimes e^*_{542} \wedge e^*_{531} \\
& - E_{51} \otimes e^*_{541} \wedge e^*_{532} - 2E_{45} \otimes e^*_{431} \wedge e^*_{421} - 2E_{43} \otimes e^*_{541} \wedge e^*_{421} \\
& + 2E_{42} \otimes e^*_{541} \wedge e^*_{431} + E_{44} \otimes e^*_{541} \wedge e^*_{321} + E_{44} \otimes e^*_{531} \wedge e^*_{421} \\
& - E_{44} \otimes e^*_{521} \wedge e^*_{431} + E_{41} \otimes e^*_{543} \wedge e^*_{421} - E_{41} \otimes e^*_{542} \wedge e^*_{431} \\
& - E_{41} \otimes e^*_{541} \wedge e^*_{432} - 2E_{35} \otimes e^*_{431} \wedge e^*_{321} + 2E_{34} \otimes e^*_{531} \wedge e^*_{321} \\
& + 2E_{32} \otimes e^*_{531} \wedge e^*_{431} - E_{33} \otimes e^*_{541} \wedge e^*_{321} - E_{33} \otimes e^*_{531} \wedge e^*_{421} \\
& - E_{33} \otimes e^*_{521} \wedge e^*_{431} + E_{31} \otimes e^*_{543} \wedge e^*_{321} - E_{31} \otimes e^*_{532} \wedge e^*_{431} \\
& - E_{31} \otimes e^*_{531} \wedge e^*_{432} - 2E_{25} \otimes e^*_{421} \wedge e^*_{321} + 2E_{24} \otimes e^*_{521} \wedge e^*_{321} \\
& - 2E_{23} \otimes e^*_{521} \wedge e^*_{421} - E_{22} \otimes e^*_{541} \wedge e^*_{321} + E_{22} \otimes e^*_{531} \wedge e^*_{421} \\
& + E_{22} \otimes e^*_{521} \wedge e^*_{431} + E_{21} \otimes e^*_{542} \wedge e^*_{321} - E_{21} \otimes e^*_{532} \wedge e^*_{421} \\
& - E_{21} \otimes e^*_{521} \wedge e^*_{432}
\end{aligned}$$

We summarize the above results in the following theorem.

Theorem 7.3. *Let $\mathfrak{G}$ be a PPGLA of type $(\mathfrak{l}, S)$ such that the triple $(\mathcal{D}(\mathfrak{l}), \Delta_1, \Xi)$ is $(A_5, \{\alpha_3\}, \varpi_3)$. Then we have*

(1) $H^2(\mathfrak{G})_{r,-1} \neq 0$ *if and only if* $r = -1$.
(2) $H^2(\mathfrak{G})_{r,0} \neq 0$ *if and only if* $r = -1, 0$.
(3) $H^2(\mathfrak{G})_{r,1} \neq 0$ *if and only if* $r = -1, 0$.
(4) $H^2(\mathfrak{G})_{r,2} \neq 0$ *if and only if* $r = 3$.

Consequently $H^2(\mathfrak{G})_r \neq 0$ *if and only if* $r = -1, 0, 3$.

7.3.2. The case $l \geqq 6$. In this case, by the table of [VG90] and the Young tableau method, we have

$$\bigwedge^2 \mathfrak{b}_{-1}^* \cong L(\varpi_2 + \varpi_4) \oplus L(\varpi_6)$$

$$\begin{aligned}\mathfrak{b}_0 \otimes \bigwedge^2 \mathfrak{b}_{-1}^* \cong\ & L(\varpi_3 + \varpi_4 + \varpi_l) \oplus L(2\varpi_3) \\ & \oplus L(\varpi_1 + \varpi_2 + \varpi_3) \oplus L(\varpi_7 + \varpi_l) \\ & \oplus 3L(\varpi_2 + \varpi_4) \oplus L(\varpi_1 + \varpi_6 + \varpi_l) \\ & \oplus L(\varpi_2 + \varpi_5 + \varpi_l) \oplus 2L(\varpi_1 + \varpi_5) \\ & \oplus L(\varpi_1 + \varpi_2 + \varpi_4 + \varpi_l) \oplus L(2\varpi_1 + \varpi_4) \\ & \oplus 2L(\varpi_6).\end{aligned}$$

By a direct inspection, we see that $\operatorname{Ker} \delta^2 = 0$. Hence we obtain the following theorem.

Theorem 7.4. *Let $\mathfrak{G}$ be a PPGLA of type $(\mathfrak{l}, S)$ such that the triple $(\mathcal{D}(\mathfrak{l}), \Delta_1, \Xi)$ is $(A_l, \{\alpha_3\}, \varpi_{l-2})$ $(l \geqq 6)$. Then $H^2(\mathfrak{G})_{r,2} = 0$ for all r.*

Thus, by Proposition 7.1 and Theorem 7.4, we obtain the vanishing $H^2(\mathfrak{G})_r = 0$ $(r \geqq 1)$ for the second cohomology when $l \geqq 6$.

Summarizing the discussion above, we have the following rigidity theorem for the Plücker embedding equations for $M = \mathrm{Gr}(k, l+1)$, when $k = 2$ $(l \geqq 4)$ and $k = 3$ $(l \geqq 6)$: Let $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ be the pseudo-product GLA of type $(\mathfrak{l}, S)$ such that the triple $(\mathcal{D}(\mathfrak{l}), \Delta_1, \Xi)$ is $(A_l, \{\alpha_2\}, \varpi_{l-1})$ or $(A_l, \{\alpha_3\}, \varpi_{l-2})$. In case $k = 2$, the prolongation of $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ becomes a simple GLA $\mathfrak{s} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{s}_p$ of type $(D_{l+1}, \{\gamma_2, \gamma_{l+1}\})$ and we have $H^2(\mathfrak{m}, \mathfrak{s})_r = 0$ for $r \geqq 1$ when $l \geqq 4$. In case $k = 3$, $\mathfrak{g}$ is the prolongation of $\mathfrak{m}$ and we have $H^2(\mathfrak{G})_r = 0$ for $r \geqq 1$ when $l \geqq 6$. Thus in both cases, the pseudo-product structure reduces to that of the regular differential system of type $\mathfrak{m}$. Here $\mathfrak{m}$ is a subalgebra of $\mathfrak{C}^k(V, W)$ as in Lemma 2.1. A submanifold R of J^{k+1} is called a system of differential equation of type $\mathfrak{m}$, when (R, D) is a regular differential system of type $\mathfrak{m}$, where D is the pullback of the canonical differential system C^k on J^k by $p \colon R \to J^k$.

Then, utilizing the Tanaka-Morimoto theory of normal Cartan connections [Tan79] [Mor93], we obtain the following rigidity theorem for the Plücker embedding equations.

Theorem 7.5. *Let $\mathfrak{G}$ be a PPGLA of type $(\mathfrak{l}, S)$ such that the triple $(\mathcal{D}(\mathfrak{l}), \Delta_1, \Xi)$ is $(A_l, \{\alpha_2\}, \varpi_{l-1})$ $(l \geqq 4)$ or $(A_l, \{\alpha_3\}, \varpi_{l-2})$ $(l \geqq 6)$. Then every system R of differential equation of type $\mathfrak{m}$ is locally isomorphic to the model system R_S of type $(\mathfrak{l}, S)$.*

References

[AG93] D. V. Alekseevskij and M. M. Graev, *Twistors and G-structures*, Russian Acad. Sci. Izv. Math., **40** (1993), 1–31.

[Bou68] N. Bourbaki, Groupes et algebres de Lie, Chapitres 4, 5 et 6, Hermann Paris, 1968.

[Bou75] N. Bourbaki, Groupes et algebres de Lie, Chapitres 7 et 8, Hermann Paris, 1975.

[Car24] E. Cartan, *Sur les variétés à connexion projective*, Bull. Soc. Math. France, **52** (1924), 205–241.

[Ch50] S. S. Chern, *The geometry of the differential equation* $y''' = F(x, y, y', y'')$, Sci. Rep. Nat. Tsing Hua Univ., **4** (1950), 97–111.

[DKM99] B. Doubrov, B. Komrakov and T. Morimoto, *Equivalence of holonomic differential equations*, Lobachevskii J. of Math., **3** (1999), 39–71.

[Fis81] M. Fishler, *Young-tableau methods for Kronecker products of representations of the classical groups*, J. Math. Phys., **22**(4) (1981), 637–648.

[Hum72] J. E. Humphreys, Introduction to Lie Algebras and Representation Theory, Springer-Verlag, 1972.

[Kac68] V. G. Kac, *Simple irreducible graded Lie algebras of finite growth*, Math. Izvestija, **2** (1968), 1271–1311.

[KN65] S. Kobayashi and T. Nagano, *On filtered Lie algebras and geometric structures* III, J. of Math. and Mech., **14** (1965), 679–706.

[KN66] S. Kobayashi and T. Nagano, *On filtered Lie algebras and geometric structures* IV, J. of Math. and Mech., **14** (1966), 163–175.

[Kos61] B. Kostant, *Lie algebra cohomology and generalized Borel-Weil theorem*, Ann. of Math., **74** (1961), 329–397.

[Lie91] S. Lie, Vorlesungen über Differentialgleichungen mit bekannten infinitesimalen Transformationen, Teubner, Leipzig, 1891.

[Mor93] T. Morimoto, *Geometric structures on filtered manifolds*, Hokkaido Math. J., **22** (1993), 263–347.

[OV90] A. L. Onishchik and E. B. Vinberg, Lie groups and algebraic groups, Springer-Verlag, 1990.

[Sea88] Y. Se-ashi, *On differential invariants of integrable finite type linear differential equations*, Hokkaido Math. J., **17** (1988), 151–195.

[SYY97] T. Sasaki, K. Yamaguchi and M. Yoshida, *On the Rigidity of Differential Systems modeled on Hermitian Symmetric Spaces and Disproofs of a Conjecture concerning Modular Interpretations of Configuration Spaces*, Advanced Studies in Pure Math., **25** (1997), 318–354.

[SY98] H. Sato and A. Y. Yoshikawa, *Third order ordinary differential equations and Legendre connection*, J. Math. Soc. Japan, **50** (1998), 993–1013.

[Tan70] N. Tanaka, *On differential systems, graded Lie algebras and pseudo-groups*, J. Math. Kyoto Univ., **10** (1970), 1–82.

[Tan79] N. Tanaka, *On the equivalence problems associated with simple graded Lie algebras*, Hokkaido Math. J., **8** (1979), 23–84.

[Tan82] N. Tanaka, *On geometry and integration of systems of second order ordinary differential equations*, Proc. Symposium on Differential Geometry, 1982, 194–205, (in Japanese).

[Tan85] N. Tanaka, *On affine symmetric spaces and the automorphism groups of product manifolds*, Hokkaido Math. J., **14** (1985), 277–351.

[Tan89] N. Tanaka, *Geometric theory of ordinary differential equations*, Report of Grant-in-Aid for Scientific Research MESC Japan, 1989.

[Tre96] M. A. Tresse, Détermination des invariants ponctuels de l'équation différentielle ordinaire du second ordre $y'' = \omega(x, y, y')$, Leipzig, Hirzel, 1896.

[VGO90] E. B. Vinberg, V. V. Gorbatsevich and A. L. Onishchik, *Structure of Lie groups and Lie algebras*, Itogi Nauki i Tekuniki: Soveremennye Problemy Mat. Fundamental'nye Hapravleniya, **41** (1990), VINITI Moscow, 5–258; English transl.: in Encyclopaedia of Math. Sci., **41**, Springer-Verlag.

[Wil06] E. J. Wilczynski, Projective differential geometry of curves and ruled surfaces, B. G. Teubner, Leipzig, 1906.

[Yam82] K. Yamaguchi, *Contact Geometry of Higher Order*, Japan. J. Math., **8** (1982), 109–176.

[Yam83] K. Yamaguchi, *Geometrization of Jet bundles*, Hokkaido Math. J., **12** (1983), 27–40.

[Yam93] K. Yamaguchi, *Differential systems associated with simple graded Lie algebras*, Advanced Studies in Pure Math., **22** (1993), 413–494.

[Yam99] K. Yamaguchi, *G_2-geometry of overdetermined systems of second order*, Trends in Math. (Analysis and geometry in several complex variables) (1999), 289–314.

[Yat88] T. Yatsui, *On pseudo-product graded Lie algebras*, Hokkaido Math. J., **17** (1988), 333–343.

[Yat92] T. Yatsui, *On completely reducible transitive graded Lie algebras of finite depth*, Japan. J. Math., **18** (1992), 291–330.

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Contact Weyl Manifold over a Symplectic Manifold

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Abstract.

We give a brief review on Weyl manifolds and thier Poincaré-Cartan classes. A Weyl manifold is a Weyl algebra bundle over a symplectic manifold which is a geometrization of deformation quantization and the Poincaré-Cartan class is a complete invariant of Weyl manifolds.

We introduce a concept of a contact Weyl manifold, which is a contact algebra bundle over a symplectic manifold containing a Weyl manifold as a subbundle. We show the existence of contact Weyl manifolds for a symplectic manifold.

We construct a connection on a contact Weyl manifold which gives a Fedosov connection when it is restricted to a Weyl manifold. With the help of the connection, we show that the cohomology class given by the curvature of Fedosov connection coincides with the Poincaré-Cartan class.

§1. Introduction

A contact manifold is embedded into a symplectic manifold compatible with the symplectic structure (cf. the concept of symplectification and contactification in [AG], §2). However, as to the converse direction, we have some obstruction as follows. A linear symplectic manifold $(\mathbf{R}^{2n}, \sigma_0)$ is embedded into a canonical contact manifold $(\mathbf{R}^{2n+1}, \theta_0)$ compatible with the contact structure $d\theta_0 = \sigma_0$, but in general an arbitrary symplectic manifold is not necessarily embedded into a contact manifold in a compatible way. For such an embedding, we need at least a vanishing cohomology class of the symplectic structure.

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In this paper, we show in quantum world the embedding is always possible, namely, a quantized symplectic manifold is embedded into a quantized contact manifold. Here a quantized symplectic manifold means a Weyl manifold defined by Omori-Maeda-Yoshioka ([OMY1]), which is a Weyl algebra bundle over a symplectic manifold, and a quantized contact manifold means a contact Weyl manifold, a contact algebra bundle over a symplectic manifold containing the Weyl manifold as a subbundle.

The purpose of this paper is two fold. First we give a brief review on Weyl manifold and its Poincaré-Cartan class, which are given in [OMY1] and in Omori-Maeda-Miyazaki-Yoshioka [OMMY1], respectively.

The second purpose is to give a concept of contact Weyl manifold, and to show the existence of contact Weyl manifold. We also construct a connection on a contact Weyl manifold, which is an extension of Fedosov connection. Using this connection, we show the Poincaré-Cartan class is equal to the cohomology class of the curvature of Fedosov connection.

The Weyl algebra W is the space of all formal power series of elements ν, $Z^1, \ldots, Z^{2n}$ with coefficients in $\mathbf{R}$ having the Moyal type product $\widehat{*}$ (see equation (2.3) below). The algebra W is equipped with the formal power series topology under which $(W, \widehat{*})$ is a complete topological algebra.

In a word, a Weyl manifold W_M is a locally trivial fiber bundle over a symplectic manifold (M, σ) with fibers consisting of the Weyl algebra W and the transition functions of local trivializations are given by Weyl diffeomorphisms (see Definition 2.9). Locally trivial bundles can be regarded as quantized Darboux charts and Weyl diffeomorphisms can be regarded as quantized symplectomorhpisms. In this way, a Weyl manifold W_M is considered as a quantization of symplectic manifold.

It is shown in Theorem 6.1 in [OMY1], a star product is made from sections of a Weyl manifold and conversely a Weyl manifold is constructed by a star product. In this sense a Weyl manifold W_M is also viewed as a geometrization of deformation quantization. It is also proved that every symplectic manifold has Weyl manifolds over itself, which yields the existence of deformation quantization for a symplectic manifold (see, Theorem A and Theorem B in [OMY1]).

For constructing W_M, one needs to handle the center of W. In order to extract information of the center, the contact algebra is introduced in [OMY1]. The *contact algebra* C is a Lie algebra given as the direct sum $C = \mathbf{R}\tau \oplus W$, where τ is an element such that $[\tau, \nu] = 2\nu^2$, $[\tau, Z^i] = \nu Z^i$. In [OMMY1], by means of the contact algebra, it is also shown that the equivalence classes of the bundle W_M have a bijection to the set of all formal power series in ν^2 with coefficients in $H^2(M)$ of the form

$$c = [\sigma] + \nu^2 c_2 + \cdots + \nu^{2k} c_{2k} + \cdots \in H^2(M)[[\nu^2]].$$

The element $c(W_M) \in H^2(M)[[\nu^2]]$ corresponding to W_M is called a *Poincaré-Cartan class* of W_M.

In this paper, we will establish the following. Using a Čech 2-cocycle giving the class $c(W_M)$ we extend the transition functions of W_M to gluing maps of locally trivial contact algebra bundles and construct a contact algebra bundle C_M over M with fiber C in §3. The bundle C_M contains W_M as a subbundle and will be called a *contact Weyl manifold*, and then we regard C_M as a contactification of a Weyl manifold W_M. Thus we have

Theorem A. *For every symplectic manifold there exists a contact Weyl manifold C_M.*

This theorem means that in quantum world contactification is always possible for a symplectic manifold.

On the other hand, we can take a closed 2-form $\Omega_M \in \Lambda^2(M)[[\nu^2]]$ such that $[\Omega_M] = c(W_M) \in H^2(M)[[\nu^2]]$ according to the deRham theorem. We will construct a connection on C_M having a curvature form Ω_M in §3.2. We show the connection gives a Fedosov connection if it is restricted to a subbundle W_M, which indicates that Ω_M is equal to the curvature of Fedosov connection.

Theorem B. *On C_M there is a connection ∂ whose curvature form is Ω_M. When restricted to a Weyl manifold W_M, the connection ∂ gives a Fedosov connection.*

Then, using this connection we prove that a Poincaré-Cartan class $c(W_M)$ coincides with the class given by the curvature form of Fedosov connection (cf. the conjecture in §Introduction of [OMMY1]).

Theorem C. *The deRham cohomomology class of the curvature Ω_M of Fedosov connection is equal to the Poincaré-Cartan class $c(W_M)$ of the Weyl manifold W_M.*

We remark that Theorems A, B and C are already given in Yoshioka [Y1]. Also in [Y2] we gave a proof of Theorem C. In this paper, we describe contructions in more detail. Especially, we improve the description of the transformation from Weyl charts to classical charts of [Y2] explicitly, which shows the restriction of the connection ∂ to W_M is equal to the Fedosov connection in §4.

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§2. Weyl manifold

A Weyl manifold emerges naturally from deformation quantization ([OMY1]) and it is considered as a quantized symplectic manifold. In this section, we give a review on Weyl manifolds.

2.1. Deformation quantization

Deformation quantization is proposed by Bayen-Flato-Fronsdal-Lichnerowicz-Sternheimer ([BFFLS]), which is an idea to quantize classical mechanical systems on a Poisson manifold without using operators in the following way.

2.1.1. Definition. Let $(M, \{\ ,\ \})$ be a Poisson manifold. Introduce a parameter ν and consider the space of formal power series $\mathfrak{a}_\nu(M) = C^\infty(M)[[\nu]]$ with coefficients in $\mathfrak{a}(M) = C^\infty(M)$, the space of all real valued smooth functions on M. Here we only consider real valued smooth functions for simplicity, although the argument is directly extened to complex valued functions. Let us consider a $\mathbf{R}[[\nu]]$-bilinear product $\mathfrak{a}_\nu(M) \times \mathfrak{a}_\nu(M) \to \mathfrak{a}_\nu(M)$.

Definition 2.1. A product $*$ is called a **star product** if

(i) For any f, $g \in \mathfrak{a}(M) = C^\infty(M)$, the product $f * g$ is expanded as

$$f * g = fg + \frac{\nu}{2}\{f, g\} + \cdots + \nu^k \pi_k(f, g) + \cdots \tag{2.1}$$

where fg is the pointwise multiplication of functions on M, $\{f, g\}$ is the Poisson bracket and $\pi_k\colon \mathfrak{a}(M) \times \mathfrak{a}(M) \to \mathfrak{a}(M)$ is a bidifferential operator ($k = 2, 3, \ldots$),

(ii) $f * 1 = 1 * f = f$ for $\forall f \in \mathfrak{a}(M)$,

(iii) $*$ is associative.

For a star product $*$, the associative algebra $(\mathfrak{a}_\nu(M), *)$ is called a **deformation quantization** of the Poisson manifold $(M, \{\ ,\ \})$.

The existence of star products is proved by De Wilde-Lecomte for symplectic manifolds ([DL]) and for general Poisson manifolds by Kontsevich ([K]).

In this paper, we consider the case where M is a symplectic manifold with symplectic structure σ. A typical example is the Moyal product on the canonical symplectic manifold $(\mathbf{R}^{2n}, \sigma_0)$. We write the canonical coordinates as $z = (z^1, \ldots, z^{2n})$ and the canonical symplectic structure as $\sigma_0 = \frac{1}{2}\sum_{i,j} \omega_{ij} dz^i \wedge dz^j$ where ω_{ij} are the components of the constant $2n \times 2n$ matrix $\omega = (\omega_{ij}) = \left(\begin{smallmatrix} 0 & 1_n \\ -1_n & 0 \end{smallmatrix}\right)$. The Poisson bracket is then written as

$$\{f, g\}_0 = \sum_{i,j} \Lambda^{ij} \partial_{z^i} f \partial_{z^j} g, \quad f,\ g \in \mathfrak{a}(\mathbf{R}^{2n}) = C^\infty(\mathbf{R}^{2n})$$

where $\Lambda = (\Lambda^{ij}) = -\omega^{-1}$. With the notation

$$\{f, g\}_0 = f\left(\sum_{i,j} \Lambda^{ij} \overleftarrow{\partial_{z^i}}\, \overrightarrow{\partial_{z^j}}\right) g = f\, \overleftarrow{\partial_z} \wedge \overrightarrow{\partial_z} g,$$

the Moyal product $*_0$ is a star product given by

$$\begin{aligned} (2.2) \qquad f *_0 g &= f\left(\exp \frac{\nu}{2} \overleftarrow{\partial_z} \wedge \overrightarrow{\partial_z}\right) g \\ &= fg + \frac{\nu}{2}\{f, g\}_0 + \cdots + \frac{1}{n!}\left(\frac{\nu}{2}\right)^n f\left(\overleftarrow{\partial_z} \wedge \overrightarrow{\partial_z}\right)^n g + \cdots . \end{aligned}$$

We sometimes refer to the deformation quantization $(\mathfrak{a}_\nu(\mathbf{R}^{2n}), *_0)$ as the *Moyal algebra.*

A star product can be restricted to an open subset $V \subset M$ and gives a star product on $\mathfrak{a}_\nu(V) = C^\infty(V)[[\nu]]$, since π_k in (2.1) are bidifferential operators. Due to the following theorem, star products on a symplectic manifold are locally isomorphic to the Moyal product (cf. Gutt [G], Lichnerowicz [L]).

Theorem 2.2. *Assume U be an open subset of $\mathbf{R}^{2n}$ with $H^2(U) = 0$. Then a star product $*$ on U is equivalent to $*_0$, that is, there exists an $\mathbf{R}[[\nu]]$-linear isomorphism $T\colon \mathfrak{a}_\nu(U) \to \mathfrak{a}_\nu(U)$ satisfying $f * g = T^{-1}(Tf *_0 Tg)$ for $\forall f,\ g \in \mathfrak{a}(U)$, where T is given in the form*

$$Tf = f + \nu T_1(f) + \cdots + \nu^k T_k(f) + \cdots, \quad \forall f \in \mathfrak{a}(U)$$

and $T_k\colon \mathfrak{a}(U) \to \mathfrak{a}(U)$, $(k = 1, 2, \ldots)$ are differential operators. Thus, deformation quantizations $(\mathfrak{a}_\nu(U), *)$ and $(\mathfrak{a}_\nu(U), *_0)$ are isomorphic.

2.1.2. System of the Moyal algebras. By Theorem 2.2, a star product induces a system of local Moyal algebras and their isomorphisms as follows. Let $\{(V_\alpha, \varphi_\alpha)\}_{\alpha \in A}$ be a symplectic atlas: $\bigcup_\alpha V_\alpha = M$ and each $\varphi_\alpha\colon V_\alpha \to U_\alpha \subset \mathbf{R}^{2n}$ is a homeomorphism such that $\varphi_\alpha = (z_\alpha^1, z_\alpha^2, \ldots, z_\alpha^n)$ is a canonical coordinate system of U_α and $\varphi_\alpha^* \sigma_{\alpha,0} = \sigma$ where $\sigma_{\alpha,0} = \frac{1}{2}\sum \omega_{ij} dz_\alpha^i \wedge dz_\alpha^j$ is the canonical symplectic structure. Here one may assume $H^2(V_\alpha) = 0$ for every $\alpha \in A$. A star product $*$ on M is reduced to every V_α and produces a deformation quantization $(\mathfrak{a}_\nu(U_\alpha), *_\alpha)$ of a linear symplectic manifold $(U_\alpha, \sigma_{\alpha,0})$ by the local coordinate expression. Theorem 2.2 yields an isomorphism $T_\alpha \colon (\mathfrak{a}_\nu(U_\alpha), *_\alpha) \to (\mathfrak{a}_\nu(U_\alpha), *_0)$. Then we have a Moyal algebra isomorphism $T_{\alpha\beta} = T_\beta \circ T_\alpha^{-1} \colon (\mathfrak{a}_\nu(U_{\alpha\beta}), *_0) \to (\mathfrak{a}_\nu(U_{\beta\alpha}), *_0)$, where $U_{\alpha\beta} = \varphi_\alpha(V_\alpha \cap V_\beta)$. Thus, a star product $*$ on M gives a picture that a system of local Moyal algebras $\{(\mathfrak{a}_\nu(U_\alpha), *_0)\}_{\alpha \in A}$ is glued together by a system of algebra isomorphisms $\{T_{\alpha\beta} \colon (\mathfrak{a}_\nu(U_{\alpha\beta}), *_0) \to (\mathfrak{a}_\nu(U_{\beta\alpha}), *_0)\}$.

We can consider the local Moyal algebra $(\mathfrak{a}_\nu(U_{\alpha\beta}), *_0)$ as a quantized Darboux coordinate and the transformation $T_{\alpha\beta} \colon (\mathfrak{a}_\nu(U_{\alpha\beta}), *_0) \to (\mathfrak{a}_\nu(U_{\beta\alpha}), *_0)$ as the quantized symplectormorphism.

2.1.3. Motivation of Weyl manifold. Using the Weyl algebra W, we can attach to the system $\{((\mathfrak{a}_\nu(U_\alpha), *_0), T_{\alpha\beta})\}$ a geometric picture, that is, a bundle over M and its sections. Although the details will be given in the next sections, we see here an idea of Weyl manifold. We consider a locally trivial bundle $W_{U_\alpha} = U_\alpha \times W$ and consider the space of all smooth sections of W_{U_α}, which is denoted by $\Gamma(W_{U_\alpha})$. By the pointwise multiplication, $\Gamma(W_{U_\alpha})$ is an associative algebra. We will see that there exists a subalgebra $\mathcal{F}(W_{U_\alpha}) \subset \Gamma(W_{U_\alpha})$, whose elements are called local *Weyl functions*, isomorphic to the local Moyal algebra $(\mathfrak{a}_\nu(U_\alpha), *_0)$. With the identification of $(\mathfrak{a}_\nu(U_\alpha), *_0)$ and $\mathcal{F}(W_{U_\alpha})$, the isomorphism $T_{\alpha\beta}$ of local Moyal algebras naturally induces an algebra isomorphism $\widehat{T}_{\alpha\beta} \colon \mathcal{F}(W_{U_{\alpha\beta}}) \to \mathcal{F}(W_{U_{\beta\alpha}})$. This algebra isomorphism is given as the pullback map of certain algebra bundle isomorphism $\Phi_{\beta\alpha} \colon W_{U_{\beta\alpha}} \to W_{U_{\alpha\beta}}$, $\Phi^*_{\beta\alpha} = \widehat{T}_{\alpha\beta}$ (see for a proof, Lemma 3.2 in [OMY1]). Such a bundle isomorphism will be called a *Weyl diffeomorphism* (see §2.2.3). Then, a Weyl manifold W_M will be given as a bundle over M by gluing trivial bundles $\{W_{U_\alpha}\}$ with Weyl diffeomorphisms $\{\Phi_{\alpha\beta}\}$. Since the local Weyl function algebra $\mathcal{F}(W_{U_\alpha}) \cong (\mathfrak{a}_\nu(U_\alpha), *_0)$ can be regarded as a quantized Darboux chart and also the Weyl diffeomorphism $\Phi_{\alpha\beta}$ can regarded as a quantized symplectomorphism, the Weyl manifold W_M is considered as a quantized symplectic manifold, and also considered as a geometric picture of a star product on M.

2.2. Review of Weyl manifold

In this section, we give a brief review of Weyl manifold defined in [OMY1]. Let us consider a $2n$-dimensional symplectic manifold M with symplectic structure σ.

2.2.1. Weyl algebra. Introducing $2n+1$ elements ν, $Z^1, Z^2, \ldots, Z^{2n}$, we consider a formal power series with coefficients in $\mathbf{R}$, $a = \sum a_{l\alpha} \nu^l Z^\alpha$, where $a_{l\alpha} \in \mathbf{R}$, $l = 0, 1, 2, \ldots$ and $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{2n})$ is a multi-index.

We put the set of all formal power series

$$W = \Big\{ a = \sum a_{l\alpha} \nu^l Z^\alpha \;\Big|\; a_{l\alpha} \in \mathbf{R} \Big\}.$$

We introduce the formal power series topology in W and W is complete under this topology. Using a $2n \times 2n$ constant matrix $\Lambda = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}$ where 1_n is the $n \times n$ identity matrix, we put a $\mathbf{R}[[\nu]]$-bilinear Poisson bracket in W

$$\{a, b\} = \sum_{ij} \Lambda^{ij} \partial_{Z^i} a \partial_{Z^j} b = a \overleftarrow{\partial}_Z \wedge \overrightarrow{\partial}_Z b.$$

Similarly as in (2.2), we consider a $\mathbf{R}[[\nu]]$-bilinear product $\widehat{*}$ in W of Moyal type

$$a \,\widehat{*}\, b = a\Big(\exp \frac{\nu}{2} \overleftarrow{\partial}_Z \wedge \overrightarrow{\partial}_Z\Big) b = ab + \frac{\nu}{2}\{a, b\} + \cdots . \tag{2.3}$$

The product $\widehat{*}$ is continuous. We call the space W with $\widehat{*}$ the *Weyl algebra.*

We can introduce an anti-involution $a \mapsto \overline{a}$ in W by putting

$$\overline{a \,\widehat{*}\, b} = \overline{b} \,\widehat{*}\, \overline{a}, \quad \overline{\nu} = -\nu, \quad \overline{Z}^i = Z^i, \quad \overline{a}_{l\alpha} = a_{l\alpha}. \tag{2.4}$$

It is easy to see the generators satisfy the canonical commutation relations (CCR for short)

$$\begin{cases} [\nu, Z^i] = \nu \,\widehat{*}\, Z^i - Z^i \,\widehat{*}\, \nu = 0 \\ [Z^i, Z^j] = Z^i \,\widehat{*}\, Z^j - Z^j \,\widehat{*}\, Z^i = \nu \Lambda^{ij}. \end{cases} \tag{2.5}$$

Remark 2.3. By (2.3), we have $Z^i Z^j = Z^i \,\widehat{*}\, Z^j - \frac{\nu}{2}\Lambda^{ij}$, and inductively we see that every monomial $\nu^l Z^\alpha$ is written as a linear combination of the $\widehat{*}$ products of the genrators ν, $Z^1, \ldots, Z^{2n}$. Thus, the Weyl algebra W is also described as an algebra over $\mathbf{R}$ formally generated by the elements ν, Z^1, $Z^2, \ldots, Z^{2n}$ satisfying the CCR relations (2.5) (see [OMY1], §1.1–1.2).

We introduce the degrees of monomials of W. We put the degrees for generators and monomials as

$$d(\nu) = 2, \quad d(Z^i) = 1, \ (i = 1, \ldots, 2n), \quad d(\nu^l Z^\alpha) = 2l + |\alpha|, \tag{2.6}$$

respectively. We set $W_0 = W$ and

$$W_k = \Big\{ a \in W \;\Big|\; a = \sum_{2l+|\alpha| \geq k} a_{l\alpha} \nu^l Z^\alpha \Big\}, \quad k = 1, \ 2, \ldots . \tag{2.7}$$

Using the conjugation, we can decompose W as a dirct sum of the set of hermitian, skewhermitian elements, respectively a

$$W = W^+ \oplus W^-, \ W^+ = \{a \in W \mid \bar{a} = a\}, \ W^- = \{a \in W \mid \bar{a} = -a\}.$$

It is obvious the center is $\mathbf{R}[[\nu]]$, the set of all formal power seires in ν. We also put the set of noncentral elements

$$W^\circ = \left\{ a \in W \,\middle|\, a = \sum_{|\alpha|>0} a_{l\alpha} \nu^l Z^\alpha \right\}. \tag{2.8}$$

We set the intersections as

$$W_k^{\circ +} = W_k \cap W^\circ \cap W^+, \quad W_k^{\circ -} = W_k \cap W^\circ \cap W^-. \tag{2.9}$$

Definition 2.4. An $\mathbf{R}$-linear isomorphism $\Phi \colon W \to W$ satisfying

$$\text{(i) } \Phi(a \,\hat{*}\, b) = \Phi(a) \,\hat{*}\, \Phi(b), \quad \forall a,\ b \in W, \qquad \text{(ii) } \Phi(\nu) = \nu$$

is called a **ν-automorphism** of the Weyl algebra W.

A ν-automorphism Φ is **hermitian** if and only if

$$\overline{\Phi(a)} = \Phi(\bar{a}), \quad \forall a \in W. \tag{2.10}$$

We give two basic examples. Let $A = (a_{ij})$ be a $2n \times 2n$ symplectic matrix. We set $AZ^i = \sum_j a_{ij} Z^i$ and $A\nu = \nu$. Then it holds $[AZ^i, AZ^j] = A[Z^i, Z^j]$ and $[A\nu, AZ^i] = A[\nu, Z^i]$ and hence the matrix A naturally acts on W as a ν-automorphism. Also let us consider $F \in W_3$. Notice $[F, W] \subset \nu W_2$, and $[F, G]$ can be divided by ν for every $G \in W$. Then $\frac{1}{\nu} \operatorname{ad} F = \frac{1}{\nu}[F, \ ]$ gives a derivation of W. By exponentiating this derivation we have a ν-automrophism $\exp \frac{1}{\nu} \operatorname{ad} F$, which satisfies $\exp \frac{1}{\nu} \operatorname{ad} F(Z^i) = Z^i + O(2)$ where $O(2)$ means the collection of the terms belonging to W_2.

Now as to the structure of ν-automorphisms, we have

Proposition 2.5. *For a ν-automorphism $\Phi \colon W \to W$, there exist uniquely a $2n \times 2n$ symplectic matrix $A = (a_{ij})$ and $F \in W_3^\circ$ such that*

$$\Phi = A \circ \exp \frac{1}{\nu} \operatorname{ad} F.$$

If Φ has the hermitian property (2.10), *then $F \in W_3^{\circ +}$.*

Proof. Notice W_1 is the maximal ideal of W and then $\Phi(W_1) \subset W_1$. We put

$$\Phi(Z^i) = \sum_{j=1}^{2n} a_{ij} Z^j + O(2), \quad a_{ij} \in \mathbf{R}, \quad i = 1,\ 2, \ldots, 2n.$$

Form the identity $[\Phi(Z^i), \Phi(Z^j)] = \Phi([Z^i, Z^j])$, one sees $A = (a_{ij})$ is a $2n \times 2n$ symplectic matrix. Consider the ν-automorphism $A^{-1} \circ \Phi$ and apply to each Z^i. We have

$$A^{-1} \circ \Phi(Z^i) = Z^i + g^i_{(2)} + O(3), \quad i = 1, \dots, 2n$$

where $g^i_{(2)}$ is the term of homogeneous degree 2. Then the identity $[A^{-1} \circ \Phi(Z^i), A^{-1} \circ \Phi(Z^j)] = A^{-1} \circ \Phi([Z^i, Z^j])$ gives the equation $[Z^i, g^j_{(2)}] = [Z^j, g^i_{(2)}]$, $i,\ j = 1,\ 2, \dots, 2n$. The Poincaré lemma yields an element $F_{(3)}$ of homogenous degree 3 such that $\frac{1}{\nu}[Z^i, F_{(3)}] = g^i_{(2)}$, $i = 1, \dots, 2n$. Thus we have

$$\Phi(Z^i) = A \circ \exp \frac{1}{\nu} \operatorname{ad} F_{(3)}(Z^i) + O(3), \quad i = 1, \dots, 2n.$$

Repeating this process, we have a sequence $\{F_{(k)}\}$ of elements of homogenous degree k such that

$$\Phi(Z^i) = A \circ \exp \frac{1}{\nu} \operatorname{ad} F_{(3)} \circ \cdots \circ \exp \frac{1}{\nu} \operatorname{ad} F_{(k)}(Z^i) + O(k),$$

for $i = 1, \dots, 2n$. By Campbell-Hausdorff formula there is an element $F \in W_3^\circ$ such that $\lim_{k\to\infty} \Pi_{l=3}^k \exp \frac{1}{\nu} \operatorname{ad} F_{(l)} = \exp \frac{1}{\nu} \operatorname{ad} F_3$ which completes the existence proof of A and F. The uniqueness is inductively checked by looking at the lawest degree term of $\exp \frac{1}{\nu} \operatorname{ad} F_{(k)}(Z^i) - Z^i$, $i = 1,\ 2, \dots, 2n$. For the case Φ is hermitian, we obtain a proof by the similar manner. Q.E.D.

2.2.2. Weyl functions. Suppose $U \subset \mathbf{R}^{2n}$ be an open subset and let $W_U = U \times W$ be a trivial bundle over U. We set $\Gamma(W_U)$ the space of all smooth sections of W_U. By the pointwise multiplication, $\Gamma(W_U)$ is equipped with the associative product $\widehat{*}$. Under the smooth topology, $\Gamma(W_U)$ becomes a complete topological algebra. Consider a formal power series in ν with coefficients in $C^\infty(U)$

$$\widetilde{f} = f_0 + \nu f_1 + \cdots \in \mathfrak{a}_\nu(U) = C^\infty(U)[[\nu]].$$

Definition 2.6. We define a section $\widetilde{f}^\# \in \Gamma(W_U)$ in a Taylor expansion fashion

$$\widetilde{f}^\#(z) = \sum_{|\alpha|\geq 0} \frac{1}{\alpha!} \partial_z^\alpha f(z) Z^\alpha, \quad (z \in U).$$

We call $\widetilde{f}^\#$ a **Weyl continuation** of $\widetilde{f}$.

We put the space $\mathcal{F}(W_U) = \{\widetilde{f}^\# \mid \widetilde{f} \in \mathfrak{a}_\nu(U)\}$ and we call an element of $\mathcal{F}(W_U)$ a **Weyl function**.

We have the following (see Theorem 2.6, [OMY1]).

Theorem 2.7.

(i) *The Weyl continuation gives an* $\mathbf{R}[[\nu]]$*-linear isomorphism* $\#\colon \mathfrak{a}_\nu(U) \to \mathcal{F}(W_U)$.

(ii) $(\widetilde{f} *_0 \widetilde{g})^\# = \widetilde{f}^\# \,\widehat{*}\, \widetilde{g}^\#$ *for* $\forall \widetilde{f},\ \widetilde{g} \in \mathfrak{a}_\nu(U)$.

The above theorem indicates $\mathcal{F}(W_U)$ is a subalgebra of $\Gamma(W_U)$ and $\#$ is an algebra isomorphism between the local Moyal algebra $(\mathfrak{a}_\nu(U), *_0)$ and the algebra of local Weyl functions $(\mathcal{F}(W_U), \widehat{*})$. It is easy to see

Proposition 2.8. *Let F be a section of $\Gamma(W_U)$ satisfying*

$$[F, g^\#] \in \mathcal{F}(W_U) \quad \textit{for} \quad \forall g^\#.$$

Then there exist a local Weyl function $f^\# \in \mathcal{F}(W_U)$ and a formal power series $a \in \mathfrak{a}_\nu(U)$ such that

$$F = f^\# + a.$$

If F is a hermitian element, $\overline{F} = F$, then we can take as hermitian, i.e., $f = f(\nu^2)$, $a = a(\nu^2) \in \mathfrak{a}_{\nu^2}(U) = C^\infty(U)[[\nu^2]]$.

Proof. Set $g^{i\#} = \frac{1}{\nu}[F, z^{i\#}]$, $i = 1, \ldots, 2n$. Then the Jacobi identity of the commutator togher with the relation $[z^{i\#}, z^{j\#}] = \nu\Lambda^{ij}$ yields $[z^{i\#}, g^{j\#}] = [z^{j\#}, g^{i\#}]$, which is equivalent to $\sum_l \Lambda^{il}\partial g^i/\partial z^l = \sum_l \Lambda^{jl}\partial g^j/\partial z^l$ for $i,\ j = 1, \ldots, 2n$. Then, the Poincaré lemma shows there exists $f \in \mathfrak{a}_\nu(U)$ such that $g^i = \{f, z^i\}$. Hence $F - f^\# \in \Gamma(W_U)$ belongs to the center and we have $F = f^\# + a$ for certain $a \in \mathfrak{a}_\nu(U)$. If F is hermitian, it is obvious that we can take f and a as elements of $\mathfrak{a}_{\nu^2}(U)$. Q.E.D.

2.2.3. Weyl diffeomorphism. Consider a bundle isomorphism $\Phi\colon W_U \to W_{U'}$.

Definition 2.9. Φ is called a **Weyl diffeomorphism** if and only if it satisfies the following three conditions.

(i) $\Phi_z\colon W_z \to W_{\varphi(z)}$ is a ν-automorphism for every $z \in U$ where W_z is a fiber of W_U at z and $\varphi\colon U \to U'$ is the induced diffeomorphism.

(ii) The pullback map Φ^* satisfies $\Phi^*\mathcal{F}(W_{U'}) = \mathcal{F}(W_U)$.

(iii) Φ has the hermitian property $\overline{\Phi(a)} = \Phi(\overline{a})$, $\forall a \in W_U$.

As to the induced map, we have (cf. Lemma 3.3, [OMY1])

Lemma 2.10. *The induced map $\varphi\colon U \to U'$ of a Weyl diffeomorphism is a symplectic diffeomorphism.*

On the other hand, the converse direction also holds (see Theorem 3.7, [OMY1]).

Theorem 2.11. *For a symplectic diffeomorphism $\varphi\colon U \to U'$, there exists a Weyl diffeomorphism $\Phi\colon W_U \to W_{U'}$ whose induced diffeomorphism is φ.*

2.2.4. Definition of Weyl manifold. Now, gluing $\{W_U = U \times W\}$ with Weyl diffeomorphisms we can define a Weyl algebra bundle over M called a *Weyl manifold* in the following way.

Suppose we have a locally trivial bundle $W_M \to M$ with fibers isomorphic to the Weyl algebra. Let $\{(V_\alpha, \varphi_\alpha)\}_{\alpha\in A}$ be an atlas of M such that $\varphi_\alpha\colon V_\alpha \to U_\alpha \subset \mathbf{R}^{2n}$ is a local canonical coordinate for every $\alpha \in A$. We denote by $\Phi_\alpha\colon W_{V_\alpha} \to W_{U_\alpha} = U_\alpha \times W$ a local bundle chart and by $\Phi_{\alpha\beta} = \Phi_\beta \circ \Phi_\alpha^{-1}\colon W_{U_{\alpha\beta}} \to W_{U_{\beta\alpha}}$ the overlap map, where $W_{V_\alpha = \pi^{-1}(V_\alpha)}$ and $W_{U_{\alpha\beta}} = \Phi_\alpha(V_\alpha \cap V_\beta) = U_{\alpha\beta} \times W$, $U_{\alpha\beta} = \varphi_\alpha(V_\alpha \cap V_\beta)$.

Definition 2.12. A locally trivial Weyl algebra bundle $W_M \xrightarrow{\pi} M$ is called a **Weyl manifold** if the overlap maps $\Phi_{\alpha\beta}$ are Weyl diffeomorphisms.

The sets $\{(W_{V_\alpha}, W_{U_\alpha}, \Phi_\alpha\colon W_{V_\alpha} \to W_{U_\alpha})\}_\alpha$ is called local Weyl charts.

We showed the existence of Weyl manifolds (Theorem A, [OMY1]).

Theorem 2.13. *For a symplectic manifold M, there exists a Weyl manifold W_M.*

The set of all smooth sections $\Gamma(W_M)$ becomes an algebra by the pointwise multiplication at each fiber. The overlap map $\Phi_{\alpha\beta}$ of a Weyl manifold preserves the class of local Weyl functions and then we can introduce a concept of global Weyl functions of W_M as follows.

Definition 2.14. A smooth section $F \in \Gamma(W_M)$ is called a **Weyl function** of W_M if $\Phi_\alpha^{*-1}F \in \mathcal{F}(W_{U_\alpha})$ for every $\alpha \in A$.

We denote by $\mathcal{F}(W_M)$ the set of all Weyl functions of W_M. It is easy to see $\mathcal{F}(W_M)$ forms a subalgebra of $\Gamma(W_M)$.

2.2.5. Weyl manifold and deformation quantization. As is seen in Theorem 2.7, the algebra of local Weyl functions $\mathcal{F}(W_{U_\alpha})$ is isomorphic to the local Moyal algebra $(\mathfrak{a}_\nu(U_\alpha), *_0)$. Also for globally defined

Weyl functions, we have an $\mathbf{R}[[\nu]]$-linear isomorphism with the following property (see, Theorems 3.10 and 6.1, [OMY1]).

Theorem 2.15. *For a Weyl manifold W_M, there exists an $\mathbf{R}[[\nu]]$-linear isomorphism $\rho\colon \mathfrak{a}_\nu(M) \to \mathcal{F}(W_M)$ such that*

$$\rho^{-1}(\rho(f) \hat{*} \rho(g)) = fg + \frac{\nu}{2}\{f,g\} + \cdots + \nu^k \pi_k(f,g) + \cdots,$$

for f, $g \in \mathfrak{a}_\nu(M)$ where $\{f,g\}$ is the Poisson bracket of the symplectic manifold M and π_k is a bidifferential operator on M.

If we define a product on $\mathfrak{a}_\nu(M)$ by

$$f * g = \rho^{-1}\bigl(\rho(f) \hat{*} \rho(g)\bigr), \quad f,\ g \in \mathfrak{a}_\nu(M), \tag{2.11}$$

then $*$ is obviously associative and becomes a star product, which induces an existence of star products on M (Theorem B, [OMY1]). Thus, by virtue of Theorem 2.13 and Theorem 2.15, we have obtained another proof of the existence of deformation quantization for symplectic manifold by De Wilde-Lecomte [DL].

2.3. Poincaré-Cartan class

In [OMMY1], we obtained a complete invariant $c(W_M)$ of Weyl manifolds over a symplectic manifold M, called the *Poincaré-Cartan class.* The invariant $c(W_M)$ of a Weyl manifold W_M is an element of $H^2(M)[[\nu^2]]$, the set of all formal power series in ν^2 with coefficients in $H^2(M)$. We derived a Čech 2-cocycle with values in the hermitian center $\mathbf{R}[[\nu^2]]$ through patching $\{W_{U_\alpha}\}$ and this 2-cocycle gave the Poincaré-Cartan class. Thus, we need to extract certain information from the center $\mathbf{R}[[\nu]]$ of W in order to define $c(W_M)$, and the contact algebra C was indeed introduced as a tool for this purpose. In this section, we recall the definition of the contact algebra C and give a review on the Poincaré-Cartan class.

2.3.1. Contact algebra. Let us introduce an element τ and set relations

$$[\tau, \nu] = 2\nu^2, \quad [\tau, Z^i] = \nu Z^i, \quad (i = 1,\ 2, \ldots, 2n). \tag{2.12}$$

It is easy to see

$$[\tau, [\nu, Z^i]] = [[\tau, \nu], Z^i] + [\nu, [\tau, Z^i]]$$

and

$$[\tau, [Z^i, Z^j]] = [[\tau, Z^i], Z^j] + [Z^i, [\tau, Z^j]].$$

Then the bracket $[\tau,\ ]$ is extended on the Weyl algebra as a derivation $a \mapsto [\tau, a]$, $a \in W$. We then consider a direct sum

$$C = \mathbf{R}\tau \oplus W \tag{2.13}$$

and define a bracket by

$$[\lambda_1\tau + a_1, \lambda_2\tau + a_2] = \lambda_1[\tau, a_2] - \lambda_2[\tau, a_1] + [a_1, a_2] \tag{2.14}$$

where $\lambda_i \in \mathbf{R}$, $a_i \in W$ and $[a_1, a_2] = a_1 \widehat{*} a_2 - a_2 \widehat{*} a_1$ is the commutator of the Weyl algebra W. The derivation property yields the Jacobi identity of $[\ ,\ ]$ and $(C, [\ ,\ ])$ becomes a Lie algebra.

Definition 2.16. The Lie algebra $(C, [\ ,\ ])$ is called a **contact algebra**.

We put the product topology of $\mathbf{R}$ and W into $C = \mathbf{R}\tau \oplus W$ and $(C, [\ ,\ ])$ is a complete topological algebra. We consider an anti-involution of C by setting

$$\overline{\tau} = \tau, \quad \overline{\lambda\tau + F} = \lambda\tau + \overline{F}, \quad \lambda \in \mathbf{R}, \quad F \in W. \tag{2.15}$$

We remark here the derivation $a \mapsto [\tau, a]$ of W counts the degree of monomials, i.e., it holds

$$[\tau, \nu^l Z^\alpha] = \nu(2l + |\alpha|)\nu^l Z^\alpha$$

and hence we have

$$[\tau, F] = 2\nu^2 \partial_\nu F + \nu \sum_{k=1}^{2n} Z^k \partial_{Z^k} F, \quad F \in W. \tag{2.16}$$

Now, we consider an automorphism group of the contact algebra. First we consider a derivation; for $F \in W$, we set

$$\operatorname{ad} \frac{1}{\nu} F(a) = \frac{1}{\nu} \operatorname{ad} F(a), \quad \forall a \in W, \tag{2.17}$$

$$\operatorname{ad} \frac{1}{\nu} F(\tau) = 2F + \frac{1}{\nu}[F, \tau]. \tag{2.18}$$

Definition 2.17. An algebra isomorphsim $\Psi \colon C \to C$ is called a ν-**autormorphism** if it gives a ν-automorphism of Weyl algebra when restricted to W. A ν-automorphism Ψ is **hermitian** if it satisfies $\overline{\Psi(P)} = \Psi(\overline{P})$, $P \in C$.

We have

Proposition 2.18. *For a hermitian ν-automorphism $\Psi\colon C \to C$, there exist uniquely a $2n \times 2n$ symplectic matrix A, a hermitian central element $c(\nu^2) \in \mathbf{R}[[\nu^2]]$ and $F \in W_3^{\circ+}$ such that*

$$\Psi = A \circ \exp \operatorname{ad} \frac{1}{\nu}(F + c(\nu^2)).$$

Proof. For the restriction $\Psi|_W$, Proposition 2.5 gives a symplectic matrix A and $F \in W_3^{\circ+}$ such that $\Psi|_W = A \circ \exp \frac{1}{\nu} \operatorname{ad} F$. Then, the ν-automorphisms Ψ and $\psi = A \circ \exp \operatorname{ad}\big(\frac{1}{\nu}F\big)$ coincide when restricted to W, and it holds $\psi^{-1} \circ \Psi(Z^i) = Z^i$, $i = 1, \dots, 2n$. As to $\psi^{-1} \circ \Psi(\tau) \in C = \mathbf{R}\tau \oplus W$, we apply $\psi^{-1} \circ \Psi$ to the identities $[\tau, Z^i] = \nu Z^i$ and we have $\psi^{-1} \circ \Psi(\tau) = \tau + b$ for certain central element b. The the hermitian property induces

$$b = b(\nu^2) = b_0 + \nu^2 b_2 + \cdots + \nu^{2k} b_{2k} + \cdots \in \mathbf{R}[[\nu^2]].$$

A central element $c(\nu^2) = \sum_{k=0}^{\infty} \nu^{2k} c_{2k}$ with $c_{2k} = 1/(2(1-2k))b_{2k}$ satisfies $\exp \operatorname{ad} \frac{1}{\nu} c(\nu^2)(\tau) = \tau + b(\nu^2)$, which shows the existence. The uniqueness is a direct consequence of Proposition 2.5 and the uniqueness of $c(\nu^2)$. Q.E.D.

2.3.2. Contact Weyl diffeomorphism. Let U be an open subset of $\mathbf{R}^{2n}$ and consider a trivial bundle $C_U = U \times C$. We denote by $\Gamma(C_U)$ the set of all smooth sections of C_U. Then $\Gamma(C_U)$ forms a Lie algebra by the pointwise multiplication and becomes a complete topological Lie algebra under smooth topology. We consider a section $\tau_U \in \Gamma(C_U)$ such that

$$\tau_U(z) = \tau + \sum_{i,j=1}^{2n} \omega_{ij} z^i Z^j, \quad (z \in U). \tag{2.19}$$

Recall the derivation $[\tau, \ ]$ satisfies $[\tau, F] = 2\nu^2 \partial_\nu F + \nu \sum_{k=1}^{2n} Z^k \partial_{Z^k} F$ in (2.16) and notice $[\sum_{ij} \omega_{ij} z^i Z^j, F] = \nu \sum_k z^k \partial_{Z^k} F$, $F \in W$ for each $z \in U$. Then we see easily the fiberwise derivation $[\tau_U, \ ]$ acts on $\Gamma(W_U)$ in the form

$$[\tau_U(z), F(z)] = 2\nu^2 \partial_\nu F(z) + \nu \sum_{k=1}^{2n} z^{k\#} \partial_{Z^k} F(z), \quad F \in \Gamma(W_U). \tag{2.20}$$

The identity $\partial_{Z^k} f^\# = (\partial_{z^k} f)^\#$ yields

$$[\tau_U(z), f^\#(z)] = 2\nu^2 \partial_\nu f^\#(z) + \nu \sum_{k=1}^{2n} (z^k \partial_{z^k} f)^\#(z), \quad f^\# \in \mathcal{F}(W_U).$$

Here we use the identity $z^{k\#}(\partial_{z^k} f)^{\#} = (z^k \partial_{z^k} f)^{\#}$. In fact, using the definition of $\widehat{*}$ we calculate $z^{k\#} \widehat{*} g^{\#} = z^{k\#} g^{\#} + \frac{\nu}{2} \sum_m \Lambda^{km}(\partial_{z^m} g)^{\#}$, and using the formula given in Proposition 2.7 we see

$$z^{k\#} \mathbin{\widehat{*}} g^{\#} = (z^k *_0 g)^{\#} = (z^k g)^{\#} + \frac{\nu}{2} \sum_m \Lambda^{km}(\partial_{z^m} g)^{\#}$$

which shows $z^{k\#} g^{\#} = (z^k g)^{\#}$ for $\forall g^{\#} \in \mathcal{F}(W_U)$. Thus, we have

Lemma 2.19.

$$[\tau_U(z), f^{\#}(z)] = 2\nu^2 \partial_\nu f^{\#}(z) + \nu(Ef)^{\#}(z), \quad f^{\#} \in \mathcal{F}(W_U),$$

where $E = \sum_{k=1}^{2n} z^k \partial_{z^k}$ is the Euler vector field.

Now remark W_U is a subbundle of C_U.

Definition 2.20. We call a bundle isomorphism $\widetilde{\Phi}\colon C_U \to C_{U'}$ a **contact extension of Weyl diffeomorphism**, or **CEWD** for short, if the restriction $\widetilde{\Phi}|_{W_U}$ is a Weyl diffeomorphism.

For a contact extension of a Weyl diffeomorphism $\widetilde{\Phi}$, the pullback $\widetilde{\Phi}^* \tau_{U'}$ is obviously a section of C_U and is written in the form

$$\widetilde{\Phi}^* \tau_{U'} = \lambda \tau_U + H, \quad \lambda \in C^\infty(U) \text{ and } H \in \Gamma(W_U).$$

Applying $\widetilde{\Phi}^*$ to the identity $[\tau_{U'}, \nu] = 2\nu^2$ induces $\lambda \equiv 1$. Applying $\widetilde{\Phi}^*$ to the identity $[\tau_{U'}, z^{i\#}] = \nu z^{i\#}$ shows $[H, \widetilde{\Phi}^* z^{i\#}] = \nu \widetilde{\Phi}^* z^{i\#}$ and hence $[\widetilde{\Phi}^{*-1} H, z^{i\#}] = \nu z^{i\#}$, $i = 1, \ldots, 2n$ which induces $[\widetilde{\Phi}^{*-1} H, \mathcal{F}(W_{U'})] \subset \nu \mathcal{F}(W_{U'})$. Proposition 2.8 then shows $\widetilde{\Phi}^{*-1} H$ is a sum of a certain Weyl function and an element of $\mathfrak{a}_{\nu^2}(U')$. Thus, for a contact extension of Weyl diffeomorphism $\widetilde{\Phi}\colon C_U \to C_{U'}$, we have

Lemma 2.21. *The pullback of $\tau_{U'}$ is written as $\widetilde{\Phi}^* \tau_{U'} = \tau_U + f^{\#} + a(\nu^2)$, for certain $f^{\#} \in \mathcal{F}(W_U)$ with $\overline{f^{\#}} = f^{\#}$ and $a(\nu^2) \in \mathfrak{a}_{\nu^2}(U)$.*

Further, if $a(\nu^2)$ is a constant, i.e., $a(\nu^2) \in \mathbf{R}[[\nu^2]]$, we can view $f^{\#} + a(\nu^2) = \big(f^{\#} + a(\nu^2)\big)^{\#}$ as a Weyl function. We define (cf. Definition 4.6, [OMY1])

Definition 2.22. A contact extension of Weyl diffeomorphism $\Psi\colon C_U \to C_{U'}$ is called a **contact Weyl diffeomorphism** if and only if $\Psi^* \tau_{U'} = \tau_U + f^{\#}$ for certain $f^{\#} \in \mathcal{F}(W_U)$.

Notice a contact Weyl diffeomorphism Ψ yields a Weyl diffeomorphism $\Psi|_{W_U}$, and then by Lemma 2.10 the induced map $\varphi\colon U \to U'$ is a symplectic diffeomorphism. As to the existence we have (see Theorem 4.7, [OMY1])

Theorem 2.23. (i) *For a Weyl diffeomorphism* $\Phi\colon W_U \to W_{U'}$, *there exists a contact Weyl diffeomorphism* $\Psi\colon C_U \to C_{U'}$ *such that* $\Psi|_{W_U} = \Phi$. (ii) *For a symplectic diffeomorphism* $\varphi\colon U \to U'$, *there exists a contact Weyl diffeomorphism* $\Psi\colon C_U \to C_{U'}$ *whose induced map is* φ.

Now we proceed to consider a contact Weyl diffeomorphism with the identity base map. Consider a hermitian, local Weyl function with the constant leading term

$$g^{\#}(\nu^2) = g_0 + \nu^2 g_2^{\#} + \cdots + \nu^{2k} g_{2k}^{\#} + \cdots, \tag{2.21}$$

where $g_0 \in \mathbf{R}$, $g_{2k} \in C^\infty(U)$, $(k = 1, 2, \ldots)$. Obviously we have a decomposition $g^{\#} = a(\nu^2) + F$ with $a(\nu^2) \in \mathfrak{a}_{\nu^2}(U)$, $F \in \Gamma(U \times W_3^{\circ +})$. Then due to Proposition 2.18, $\Psi = \exp\operatorname{ad}\big(\frac{1}{\nu} g^{\#}\big)$ gives a bundle isomorphism of C_U which clearly satisfies $\Psi^*\mathcal{F}(W_U) = \mathcal{F}(W_U)$. It is easy to see

$$\Psi^*\tau_U = \tau_U + 2g_0 + \nu^2 A_{(1)}^{\#}(\nu^2), \quad \Psi^* z^{i\#} = z^{i\#} + \nu^2 A_{(2)}^{\#}(\nu^2)$$

where $A_{(i)}^{\#}(\nu^2) = A_{(i),0}^{\#} + \nu^2 A_{(i),2}^{\#} + \cdots + \nu^{2k} A_{(i),2k}^{\#} + \cdots$, $A_{(i),2k} \in C^\infty(U)$, $i = 1, 2$. Thus, we have contact Weyl diffeomorphism Ψ with the identity base map. Moreover, one sees that a contact Weyl diffeomorphism of the above form is general as follows (cf. Corollary 2.5, [OMMY1]).

Proposition 2.24. (i) *If a contact Weyl diffeomorphism* $\Psi\colon C_U \to C_U$ *indueces the identity base map, there exists uniquely a Weyl function* $g^{\#}(\nu^2)$ *of the form* (2.21) *such that* $\Psi = \exp\operatorname{ad}\big(\frac{1}{\nu} g^{\#}(\nu^2)\big)$. (ii) *If a contact Weyl diffeomorphism* Ψ *yields the identity map* $\widetilde{\Phi}|_{W_U} = 1$ *on* W_U, *then there exists uniquley a hermitian central element*

$$c(\nu^2) = c_0 + \nu^2 c_2 + \cdots + \nu^{2k} c_{2k} + \cdots, \quad c_{2k} \in \mathbf{R} \tag{2.22}$$

such that $\Psi = \exp\operatorname{ad}\big(\frac{1}{\nu} c(\nu^2)\big)$.

In what follows we consider the relation between contact extensions of Weyl diffeomorphism and contact Weyl diffeomorphisms. Let $\widetilde{\Phi}\colon C_U \to C_{U'}$ be a contact extension of Weyl diffeomorphism, CEWD. Then Lemma 2.21 gives that $\widetilde{\Phi}^*\tau_{U'} = \tau_U + f^{\#} + b(\nu^2)$ for certain $f^{\#} \in \mathcal{F}(W_U)$ and $b(\nu^2) = b_0 + \nu^2 b_2 + \cdots + \nu^{2k} b_{2k} + \cdots$, $b_{2k} \in C^\infty(U)$. It is easy to see $a(\nu^2) = \frac{1}{2} b_0 - \frac{\nu^2}{2} b_2 + \cdots + \frac{\nu^{2k}}{2(1-2k)} b_{2k} + \cdots \in \mathfrak{a}_{\nu^2}(U)$ satisfies $\exp\operatorname{ad}\big(\frac{1}{\nu} a(\nu^2)\big)\tau_U = \tau_U + b(\nu^2)$. Then the composition

$\widetilde{\Psi} = \widetilde{\Phi} \circ \exp\bigl(-\operatorname{ad} \frac{1}{\nu} a(\nu^2)\bigr) \colon C_U \to C_U$ satisfies $\widetilde{\Psi}^* \tau_{U'} = \tau_U + f^{\#}$ and $\widetilde{\Psi}$ is a contact Weyl diffeomorphism. Thus, we have

Proposition 2.25. *For a contact extension of Weyl diffeomorphism $\widetilde{\Phi} \colon C_U \to C_{U'}$, there exists a contact Weyl diffeomorphism $\widetilde{\Psi} \colon C_U \to C_{U'}$ and $a(\nu^2) \in \mathfrak{a}_{\nu^2}(U)$ such that $\widetilde{\Phi} = \widetilde{\Psi} \circ \exp\operatorname{ad}\bigl(\frac{1}{\nu} a(\nu^2)\bigr)$.*

2.3.3. Čech 2-cocycle of W_M and the Poincaré-Cartan class. Consider a Weyl manifold $W_M \xrightarrow{\pi} M$. We have a system of local Weyl charts $\Phi_\alpha \colon \pi^{-1}(V_\alpha) = W_{V_\alpha} \to W_{U_\alpha} = U_\alpha \times W$ and a system of overlap maps $\Phi_{\alpha\beta} = \Phi_\beta \circ \Phi_\alpha^{-1} \colon W_{U_{\alpha\beta}} \to W_{U_{\beta\alpha}}$, $U_{\alpha\beta} = \varphi_\alpha(V_\alpha \cap V_\beta)$.

According to Theorem 2.23, we take a contact Weyl diffeomorphism $\widetilde{\Phi}_{\alpha\beta} \colon C_{U_{\alpha\beta}} \to C_{U_{\beta\alpha}}$ such that $\widetilde{\Phi}_{\alpha\beta}|_{W_{U_{\alpha\beta}}} = \Phi_{\alpha\beta}$ for each overlap map. Here we may assume $\widetilde{\Phi}_{\beta\alpha} \circ \widetilde{\Phi}_{\alpha\beta} = 1$ on $C_{U_{\alpha\beta}}$. In fact, $\Phi_{\beta\alpha} \circ \Phi_{\alpha\beta} = 1$ on $W_{U_{\alpha\beta}}$ and Proposition 2.24 yields $\widetilde{\Phi}_{\beta\alpha} \circ \widetilde{\Phi}_{\alpha\beta} = \exp\operatorname{ad}\bigl(\frac{1}{\nu} d_{\alpha\beta}(\nu^2)\bigr)$ for certain $d_{\alpha\beta}(\nu^2) \in \mathbf{R}[[\nu^2]]$. Notice

$$\begin{aligned}\widetilde{\Phi}_{\alpha\beta} \circ \widetilde{\Phi}_{\beta\alpha} \circ \widetilde{\Phi}_{\alpha\beta} &= \exp\operatorname{ad}\Bigl(\frac{1}{\nu} d_{\beta\alpha}(\nu^2)\Bigr) \circ \widetilde{\Phi}_{\alpha\beta} \\ &= \widetilde{\Phi}_{\alpha\beta} \circ \exp\operatorname{ad}\Bigl(\frac{1}{\nu} d_{\alpha\beta}(\nu^2)\Bigr)\end{aligned}$$

which shows $d_{\beta\alpha}(\nu^2) = d_{\alpha\beta}(\nu^2)$. Then we can assume $\widetilde{\Phi}_{\beta\alpha} \circ \widetilde{\Phi}_{\alpha\beta} = 1$ by replacing $\widetilde{\Phi}_{\alpha\beta}$ by $\widetilde{\Phi}_{\alpha\beta} \circ \exp\bigl(-\operatorname{ad} \frac{1}{2\nu} d_{\alpha\beta}(\nu^2)\bigr)$ on $C_{U_{\alpha\beta}}$.

We set a bundle isomorphism $\widetilde{\Phi}_{\alpha\beta\gamma} = \widetilde{\Phi}_{\gamma\alpha} \circ \widetilde{\Phi}_{\beta\gamma} \circ \widetilde{\Phi}_{\alpha\beta}$ of $C_{U_{\alpha\beta\gamma}}$ where $U_{\alpha\beta\gamma} = \varphi_\alpha(V_\alpha \cap V_\beta \cap V_\gamma)$, $V_\alpha \cap V_\beta \cap V_\gamma \neq \emptyset$. Since the restriction $\widetilde{\Phi}_{\alpha\beta\gamma}|_{W_{U_{\alpha\beta\gamma}}} = \Phi_{\gamma\alpha} \circ \Phi_{\beta\gamma} \circ \Phi_{\alpha\beta}$ is the identity of $W_{U_{\alpha\beta\gamma}}$, there exists $c_{\alpha\beta\gamma}(\nu^2) \in \mathbf{R}[[\nu^2]]$ such that $\widetilde{\Phi}_{\alpha\beta\gamma} = \exp\operatorname{ad}\bigl(\frac{1}{\nu} c_{\alpha\beta\gamma}(\nu^2)\bigr)$. The identities $\widetilde{\Phi}_{\alpha\beta\gamma} \circ \widetilde{\Phi}_{\alpha\gamma\beta} = 1$ and $\widetilde{\Phi}_{\alpha\beta\gamma} = \widetilde{\Phi}_{\gamma\alpha} \circ \widetilde{\Phi}_{\gamma\alpha\beta} \circ \widetilde{\Phi}_{\alpha\gamma}$ induce

$$c_{\alpha\beta\gamma}(\nu^2) + c_{\alpha\gamma\beta}(\nu^2) = 0, \quad c_{\alpha\beta\gamma}(\nu^2) = c_{\gamma\alpha\beta}(\nu^2),$$

respectively, which mean $\{c_{\alpha\beta\gamma}(\nu^2)\}$ is a Čech 2-cochain for the covering $\mathcal{U} = \{V_\alpha\}_{\alpha \in A}$. An easy calculation yields

$$\widetilde{\Phi}_{\beta\alpha} \circ \widetilde{\Phi}_{\beta\delta\gamma} \circ \widetilde{\Phi}_{\alpha\beta} \circ \widetilde{\Phi}_{\alpha\delta\beta} \circ \widetilde{\Phi}_{\alpha\gamma\delta} \circ \widetilde{\Phi}_{\alpha\beta\gamma} = 1$$

on $C_{U_{\alpha\beta\gamma\delta}}$, $U_{\alpha\beta\gamma\delta} = \varphi_\alpha(V_\alpha \cap V_\beta \cap V_\gamma \cap V_\delta)$ which induces

$$c_{\beta\delta\gamma}(\nu^2) + c_{\alpha\delta\beta}(\nu^2) + c_{\alpha\gamma\delta}(\nu^2) + c_{\alpha\beta\gamma}(\nu^2) = -\delta c_{\alpha\beta\gamma\delta}(\nu^2) = 0.$$

Thus $\{c_{\alpha\beta\gamma}(\nu^2)\}$ is a Čech 2-cocycle with values in $\mathbf{R}[[\nu^2]]$, or each $\{c_{\alpha\beta\gamma,2k}\}$ of the expansion $c_{\alpha\beta\gamma}(\nu^2) = \sum_{k=0}^{\infty} \nu^{2k} c_{\alpha\beta\gamma,2k}$ is a Čech

2-cocycle with values in $\mathbf{R}$ ($k = 0, 1, 2, \ldots$). Then we obtain an element of $H^2(M)[[\nu^2]]$ which is denoted by $c(W_M)$ and is called the **Poincaré-Cartan class** of the Weyl manifold W_M. We write the expansion as

$$c(W_M) = c_0(W_M) + \nu^2 c_2(W_M) + \cdots + \nu^{2k} c_{2k}(W_M) + \cdots ,$$

where $c_{2k}(W_M) \in H^2(M)$. As to $c(W_M)$ we have the following (Theorem 3.5, [OMMY1]).

Proposition 2.26. (i) *The leading term $c_0(W_M)$ of $c(W_M)$ is equal to $[\sigma]$, the cohomology class of the symplectic structure σ of M.* (ii) *$c(W_M)$ depends on the equivalence class of W_M, i.e., if W'_M is equivalent to W_M as Weyl manifold, then $c(W'_M) = c(W_M)$.*

Theorem 2.27. *The Poincaré-Cartan class is a complete invariant of Weyl manifolds, i.e., the map $[W_M] \mapsto c(W_M)$ is bijective from all equivalence classes of Weyl manifolds to the set of all elements of $H^2(M)[[\nu^2]]$ with the leading term $[\sigma]$.*

§3. Contact algebra bundle and connection

In this section, using a Poincaré-Cartan class $c(W_M)$ of Weyl manifold W_M, we will construct a contact algebra bundle $C_M \to M$ and a connection ∂ on C_M.

Let $\{c_{\alpha\beta\gamma}(\nu^2)\}$ be a Čech 2-cocycle giving $c(W_M)$. Then by definition, contact Weyl diffeomorphisms $\widetilde{\Phi}_{\alpha\beta}\colon C_{U_{\alpha\beta}} \to C_{U_{\beta\alpha}}$ for overlap maps of local Weyl charts $\Phi_{\alpha\beta}\colon W_{U_{\alpha\beta}} \to W_{U_{\beta\alpha}}$ satisfy

$$\widetilde{\Phi}_{\gamma\alpha} \circ \widetilde{\Phi}_{\beta\gamma} \circ \widetilde{\Phi}_{\alpha\beta} = \exp \mathrm{ad}\Big(\frac{1}{\nu} c_{\alpha\beta\gamma}(\nu^2)\Big),$$

which is not the identity transformation in general. Hence the system $\{\widetilde{\Phi}_{\alpha\beta}\}$ is not useful for gluing $\{C_{U_{\alpha\beta}}\}$. Our idea for constructing C_M is to use contact extensions of Weyl diffeomorphisms instead of $\{\widetilde{\Phi}_{\alpha\beta}\}$. By means of $\{c_{\alpha\beta\gamma}(\nu^2)\}$, we construct an appropriate 1-cochain $\{h_{\alpha\beta}(\nu^2)\}$, $h_{\alpha\beta}(\nu^2) \in C^\infty(U_{\alpha\beta})[[\nu^2]]$ and using this cochain and the system of contact Weyl diffeomorphisms $\{\widetilde{\Phi}_{\alpha\beta}\}$, we obtain a certain system of contact extensions of Weyl diffeomorphisms $\{\Psi_{\alpha\beta}\colon C_{U_{\alpha\beta}} \to C_{U_{\beta\alpha}}\}$. We will then have a contact algebra bundle by gluing $\{C_{U_\alpha}\}$ by means of $\{\Psi_{\alpha\beta}\}$.

As to the connection ∂, we first consider a closed 2-form $\Omega_M(\nu^2) \in \Lambda^2_M[[\nu^2]]$ whose cohomology class is equal to the Poincaré-Cartan class $[\Omega_M(\nu^2)] = c(W_M)$. Then we take a system of local 1-forms $\{\xi_\alpha(\nu^2)\}$, $\xi_\alpha \in \Lambda^1(V_\alpha)[[\nu^2]]$ such that $d\xi_\alpha(\nu^2) = \Omega_M(\nu^2)$ on V_α. Using this system $\{\xi_\alpha(\nu^2)\}$, we construct a connection 1-form and then we will obtain a connection ∂ whose curvature form is $\Omega_M(\nu^2)$.

3.1. Construction of C_M; proof of Theorem A

Let W_M be a Weyl manifold and $c(W_M)$ be its Poincaré-Cartan class. Suppose $\{c_{\alpha\beta\gamma}(\nu^2)\}$ is a Čech 2-cocycle giving $c(W_M)$. Recall the cocycle has the form $c_{\alpha\beta\gamma}(\nu^2) = c_{\alpha\beta\gamma,0}+\nu^2 c_{\alpha\beta\gamma,2}+\cdots+\nu^{2k}c_{\alpha\beta\gamma,2k}+\cdots$, $c_{\alpha\beta\gamma,2k} \in \mathbf{R}$. We denote by $\Phi_{\alpha\beta}\colon W_{U_{\alpha\beta}} \to W_{U_{\beta\alpha}}$ the overlap map of local Weyl charts and by $\widetilde{\Phi}_{\alpha\beta}\colon C_{U_{\alpha\beta}} \to C_{U_{\beta\alpha}}$ its lift as a contact Weyl diffeomorphism. Then by definition, we have the identity

$$\widetilde{\Phi}_{\alpha\beta\gamma} = \widetilde{\Phi}_{\gamma\alpha}\circ\widetilde{\Phi}_{\beta\gamma}\circ\widetilde{\Phi}_{\alpha\beta} = \exp\operatorname{ad}\Big(\frac{1}{\nu}c_{\alpha\beta\gamma}(\nu^2)\Big)$$

$$\widetilde{\Phi}_{\alpha\beta}\circ\widetilde{\Phi}_{\beta\alpha} = 1_{U_{\alpha\beta}}.$$

In what follows, we construct Čech 1-cocycle $\{H_{\alpha\beta}(\nu^2)\}$ and a system of 1-forms $\{\xi_\alpha(\nu^2)\}$ on M related to $\{c_{\alpha\beta\gamma}(\nu^2)\}$ by the standard argument. The 1-cocycle $\{H_{\alpha\beta}(\nu^2)\}$ is used in this section to construct a gluing map system of contact extension of Weyl diffeomorphisms $\{\Psi_{\alpha\beta}\colon C_{U_{\alpha\beta}} \to C_{U_{\beta\alpha}}\}$ for the contact algebra bundle C_M. The system $\{\xi_\alpha(\nu^2)\}$ will be used for constructing a connection ∂ on C_M in the next section.

Now we define a formal power seires in ν^2 with coefficients in $\mathfrak{a}(M) = C^\infty(M)$ by

$$H_{\alpha\beta}(\nu^2) = \sum_\lambda c_{\alpha\beta\lambda}(\nu^2)\chi_\lambda \in \mathfrak{a}_M[[\nu^2]], \tag{3.1}$$

where $\{\chi_\lambda\}_\lambda$ is a partition of unity subbordinate to the covering $\{V_\lambda\}_\lambda$ of M. Then we have

$$H_{\alpha\beta}(\nu^2) = -H_{\beta\alpha}(\nu^2) \tag{3.2}$$

$$\delta H_{\alpha\beta\gamma}(\nu^2) = H_{\alpha\beta}(\nu^2) + H_{\beta\gamma}(\nu^2) + H_{\gamma\alpha}(\nu^2) = c_{\alpha\beta\gamma}(\nu^2). \tag{3.3}$$

We set a formal power seiries of one forms on M

$$\Xi_\alpha(\nu^2) = \sum_\lambda dH_{\alpha\lambda}(\nu^2)\chi_\lambda \in \Lambda^1_M[[\nu^2]]. \tag{3.4}$$

Then the identity (3.3) shows

Lemma 3.1. $\Xi_\beta(\nu^2) - \Xi_\alpha(\nu^2) = dH_{\beta\alpha}(\nu^2)$.

We set the local coordinate expressions,

$$h_{\alpha\beta}(\nu^2) = \varphi_\alpha^{-1*}H_{\alpha\beta}(\nu^2) \in \mathfrak{a}_{U_\alpha}[[\nu^2]]. \tag{3.5}$$

Then the identities for $H_{\alpha\beta}$ (3.2) and (3.3) gives

Lemma 3.2.

(i) $$h_{\alpha\beta}(\nu^2) = -\varphi^*_{\alpha\beta} h_{\beta\alpha}(\nu^2)$$

(ii) $$h_{\alpha\beta}(\nu^2) + \varphi^*_{\alpha\beta} h_{\beta\gamma}(\nu^2) + \varphi^*_{\alpha\gamma} h_{\gamma\alpha}(\nu^2) = c_{\alpha\beta\gamma}(\nu^2).$$

We also set the local coordinate expression

$$\xi_\alpha(\nu^2) = \varphi_\alpha^{-1*} \Xi_\alpha(\nu^2) \in \Lambda^1_{U_\alpha}[[\nu^2]]. \tag{3.6}$$

The identity $\Xi_\beta(\nu^2) - \Xi_\alpha(\nu^2) = dH_{\beta\alpha}(\nu^2)$ induces

$$\xi_\beta(\nu^2) = \varphi^*_{\beta\alpha} \xi_\alpha(\nu^2) + dh_{\beta\alpha}(\nu^2). \tag{3.7}$$

Now using $h_{\alpha\beta}(\nu^2) = \varphi_\alpha^{-1*} H_{\alpha\beta}(\nu^2)$ we set a contact extension of Weyl diffeomorphism (CEWD) of $\Phi_{\alpha\beta}$ as

$$\Psi_{\alpha\beta} = \widetilde{\Phi}_{\alpha\beta} \circ \exp\Big(-\operatorname{ad}\frac{1}{\nu} h_{\alpha\beta}(\nu^2)\Big) : C_{U_{\alpha\beta}} \to C_{U_{\beta\alpha}} \tag{3.8}$$

where $\widetilde{\Phi}_{\alpha\beta}$ is a contact Weyl diffeomorhpism lift of $\Phi_{\alpha\beta}$. Then we have

$$\Psi_{\alpha\beta} \circ \Psi_{\beta\alpha} = 1_{U_{\alpha\beta}} : C_{U_{\alpha\beta}} \to C_{U_{\beta\alpha}}. \tag{3.9}$$

In fact, the skew symmetry of $h_{\alpha\beta}(\nu^2)$ in Lemma 3.2 (i) yields

$$\begin{aligned} \Psi_{\alpha\beta} \circ \Psi_{\beta\alpha} &= \widetilde{\Phi}_{\alpha\beta} \circ \exp\Big(-\operatorname{ad}\frac{1}{\nu} h_{\alpha\beta}(\nu^2)\Big) \circ \widetilde{\Phi}_{\beta\alpha} \circ \exp\Big(-\operatorname{ad}\frac{1}{\nu} h_{\beta\alpha}(\nu^2)\Big) \\ &= \widetilde{\Phi}_{\alpha\beta} \circ \widetilde{\Phi}_{\beta\alpha} \circ \exp\Big(-\operatorname{ad}\frac{1}{\nu} h_{\beta\alpha}(\nu^2) + \frac{1}{\nu}\varphi^*_{\beta\alpha} h_{\alpha\beta}(\nu^2)\Big) \\ &= 1_{U_{\alpha\beta}}. \end{aligned}$$

The cyclic condition (ii) in the same lemma also gives

$$\begin{aligned} \Psi_{\alpha\beta\gamma} &= \Psi_{\gamma\alpha} \circ \Psi_{\beta\gamma} \circ \Psi_{\alpha\beta} \\ &= \widetilde{\Phi}_{\alpha\beta\gamma} \circ \exp\Big(-\operatorname{ad}\frac{1}{\nu} h_{\alpha\beta}(\nu^2) + \frac{1}{\nu}\varphi^*_{\alpha\beta} h_{\beta\gamma}(\nu^2) + \frac{1}{\nu}\varphi^*_{\alpha\gamma} h_{\gamma\alpha}(\nu^2)\Big) \\ &= \exp\operatorname{ad}\Big(\frac{1}{\nu} c_{\alpha\beta\gamma}(\nu^2)\Big) \circ \exp\Big(-\operatorname{ad}\frac{1}{\nu} c_{\alpha\beta\gamma}(\nu^2)\Big) = 1_{U_{\alpha\beta\gamma}}. \end{aligned}$$

Thus, by gluing local trivial contact algebra bundles $\{C_{U_\alpha}\}$ by means of the system $\{\Psi_{\alpha\beta}\}$ we have a locally tirivial contact algebra bundle $C_M \to M$. By construction it is obvious that C_M contains the Weyl manifold W_M as a subbundle and hence we have a proof of Theorem A.

Finally we prepare for some basic identities for $\{\tau_{U_\alpha}\}$. The identities will be used in the next sections for contructing a connection on C_M, where $\tau_U = \tau + \sum_{ij} \omega_{ij} z^i Z^j$ (see (2.19)). For simplicity, we write $\tau_\alpha = \tau_{U_\alpha}$.

Proposition 3.3. *We have on $U_{\alpha\beta}$*

$$\Psi^*_{\alpha\beta}\tau_\beta = \tau_\alpha + f^{\#}_{\alpha\beta} - \widehat{h}_{\alpha\beta}(\nu^2)$$

where $f^{\#}_{\alpha\beta}$ is a local Weyl function given by the contact Weyl diffeomorphism $\widetilde{\Phi}_{\alpha\beta}$ as

$$\widetilde{\Phi}^*_{\alpha\beta}\tau_\beta = \tau_\alpha + f^{\#}_{\alpha\beta} \tag{3.10}$$

and $\widehat{h}_{\alpha\beta}(\nu^2) \in \mathfrak{a}_{U_\alpha}[[\nu^2]]$ is given by

$$\mathrm{ad}\Big(\frac{1}{\nu}h_{\alpha\beta}(\nu^2)\Big)\tau_\alpha = \widehat{h}_{\alpha\beta}(\nu^2). \tag{3.11}$$

Proof. It is obvious by $\Psi_{\alpha\beta} = \widetilde{\Phi}_{\alpha\beta} \circ \exp\big(-\,\mathrm{ad}\,\frac{1}{\nu}h_{\alpha\beta}(\nu^2)\big)$. Q.E.D.

3.2. Construction of connection ∂: proof of Thoerem B

The connection ∂ is defined as a twisted exterior derivation. For this, we introduce a tensor product bundle $\Lambda_M \otimes C_M$, where Λ_M is the exterior algebra bundle over M, similarly as Fedosov [F].

3.2.1. Tensor product bundles. We consider the tensor product bundles $\Lambda_M \otimes W_M$ and $\Lambda_M \otimes C_M$, where Λ_M is the exterior algebra bundle over M. Obviously $\Lambda_M \otimes W_M$ is a subbundle of $\Lambda_M \otimes C_M$. Local trivializations are given by $\Lambda_{U_\alpha} \otimes W_{U_\alpha} \subset \Lambda_{U_\alpha} \otimes C_{U_\alpha}$ for each $\{U_\alpha\}$, and gluing maps are given by

$${}^t d\varphi_{\alpha\beta} \otimes \Phi_{\alpha\beta} \colon \Lambda_{U_{\alpha\beta}} \otimes W_{U_{\alpha\beta}} \to \Lambda_{U_{\beta\alpha}} \otimes W_{U_{\beta\alpha}}$$

and

$${}^t d\varphi_{\alpha\beta} \otimes \Psi_{\alpha\beta} \colon \Lambda_{U_{\alpha\beta}} \otimes C_{U_{\alpha\beta}} \to \Lambda_{U_{\beta\alpha}} \otimes C_{U_{\beta\alpha}},$$

respectively.

The algebra structure of these bundles is given in the following way: Let $U \subset \mathbf{R}^{2n}$ be an open subset and let us consider elements $P, Q \in \Lambda_U \otimes C_U$ given by $P = \sum_I dz^I P_I$, $Q = \sum_J dz^J Q_J$, $P_I, Q_J \in C_U$, $dz^I = dz^{i_1} \wedge \cdots \wedge dz^{i_k}$ where $I = \{i_1, \dots, i_k\}$, $k = |I|$, etc. We introduce a bracket

$$[P, Q] = \sum_{I,J} dz^I \wedge dz^J [P_I, Q_J]. \tag{3.12}$$

For $F=\sum_I dz^I F_I$, $G=\sum_J dz^J G_J \in \Lambda_U \otimes W_U$ we also set a product

$$F \mathbin{\widehat{*}} G = \sum_{I,J} dz^I \wedge dz^J F_I \mathbin{\widehat{*}} G_J. \tag{3.13}$$

Then we have the following super algebra identities:

Lemma 3.4. *Let P, Q, R be monomials of $\Lambda_U \otimes C_U$ and $|P|$, $|Q|$, $|R|$ be their degrees as forms, respectively. Then we have the skewsymmetry*

$$[P,Q] = (-1)^{|P|\cdot|Q|+1}[Q,P] \tag{i}$$

and the super-Jacobi identity

$$\begin{aligned}[P,[Q,R]] &+ (-1)^{|P||R|+|Q||R|}[R,[P,Q]] \\ &+ (-1)^{|P||Q|+|Q||R|}[Q,[R,P]] = 0.\end{aligned} \tag{ii}$$

Proof. For (i), consider $dz^I \wedge dz^J = (-1)^{|I||J|} dz^J \wedge dz^I$ and the skewsymmetry of the bracket of C. The second identity (ii) is obtained by using

$$\begin{aligned} dz^P \wedge dz^Q \wedge dz^R &= (-1)^{pr+qr} dz^R \wedge dz^P \wedge dz^Q \\ &= (-1)^{pq+qr} dz^Q \wedge dz^R \wedge dz^P \end{aligned}$$

and the Jacobi identity. Q.E.D.

3.2.2. Derivations. Now we consider derivations on $\Lambda_{U_\alpha} \otimes C_{U_\alpha}$. Let δ_α be a fiberwise derivation defined by

$$\delta_\alpha = \mathrm{ad}\left(\frac{1}{\nu}\sum_{ij} dz_\alpha^i \omega_{ij} Z^j\right) : \Lambda_{U_\alpha}^p \otimes C_{U_\alpha} \to \Lambda_{U_\alpha}^{p+1} \otimes C_{U_\alpha} \tag{3.14}$$

for each $p = 0, 1, \ldots, 2n$. It is easy to see

Lemma 3.5. *For every $dz_\alpha^I = dz_\alpha^{i_1} \wedge \cdots \wedge dz_\alpha^{i_k}$, $I = \{i_1, \ldots, i_k\}$, it holds for any $Q \in \Lambda_{U_\alpha} \otimes C_{U_\alpha}$*

$$\delta_\alpha(dz_\alpha^I \wedge Q) = (-1)^{|I|} dz_\alpha^I \wedge \delta_\alpha Q. \tag{3.15}$$

For $P \in \Lambda_{U_\alpha}^p \otimes C_{U_\alpha}$, it holds for any $Q \in \Lambda_{U_\alpha} \otimes C_{U_\alpha}$

$$\delta_\alpha[P,Q] = [\delta_\alpha P, Q] + (-1)^p [P, \delta_\alpha Q]. \tag{3.16}$$

Proof. First equation is a direct consequence of definition of the bracket (3.12). As to the second, set $P = dz_\alpha^I P_I \in \Lambda_{U_\alpha}^{|I|} \otimes C_{U_\alpha}$,

$Q = dz_\alpha^J \wedge Q_J \in \Lambda_{U_\alpha}^{|J|} \otimes C_{U_\alpha}$ and use also the definiton (3.12) then we have $[P, Q] = dz_\alpha^I \wedge dz_\alpha^J [P_I, Q_J]$. A direct calculation gives

$$\begin{aligned}\delta_\alpha[P,Q] &= (-1)^{|I|+|J|} dz_\alpha^I \wedge dz_\alpha^J \wedge \{[\delta_\alpha P_I, Q_J] + [P_I, \delta_\alpha Q_J]\} \\ &= [(-1)^{|I|} dz_\alpha^I \wedge \delta_\alpha P_I, dz_\alpha^J Q_J] \\ &\quad + (-1)^{|I|}[dz_\alpha^I P_I, (-1)^{|J|} dz_\alpha^J \wedge \delta_\alpha Q_J].\end{aligned}$$

Then the first identity shows the desired relation. Q.E.D.

Now take an arbitrary 1-form $\kappa_\alpha(\nu^2) \in \Gamma(\Lambda_{U_\alpha}^1[[\nu^2]])$ and consider a derivation $\partial_\alpha \colon \Gamma(\Lambda_{U_\alpha}^p \otimes C_{U_\alpha}) \to \Gamma(\Lambda_{U_\alpha}^{p+1} \otimes C_{U_\alpha})$, $(p = 0, 1, \ldots, 2n)$, given by

$$\partial_\alpha = d - \delta_\alpha + \mathrm{ad}\Big(\frac{1}{\nu}\kappa_\alpha(\nu^2)\Big). \tag{3.17}$$

Obviously ∂_α induces a derivation from $\Gamma(\Lambda_{U_\alpha}^p \otimes W_{U_\alpha})$ to $\Gamma(\Lambda_{U_\alpha}^{p+1} \otimes W_{U_\alpha})$ when restricted the subbundle $\Lambda_{U_\alpha} \otimes W_{U_\alpha}$. We see easily

Proposition 3.6.

(a) $$\partial_\alpha f = df, \ f \in C^\infty(U_\alpha)$$

(b) $$\partial_\alpha Z^i = -dz_\alpha^i, \quad i = 1,\ 2, \ldots, 2n$$

(c) $$\partial_\alpha \nu = 0$$

(d) $$\partial_\alpha \tau = -\sum_{ij} dz_\alpha^i \omega_{ij} Z^j + \widehat{\kappa}_\alpha(\nu^2)$$

where $\widehat{\kappa}_\alpha(\nu^2) = \mathrm{ad}\big(\frac{1}{\nu}\kappa_\alpha(\nu^2)\big)\tau \in \Gamma(\Lambda_{U_\alpha}^1[[\nu^2]])$.

Proof. The identities (a) and (c) are obvious. For (b) we calculate as

$$\begin{aligned}\partial_\alpha Z^i &= \Big(d - \delta_\alpha + \mathrm{ad}\Big(\frac{1}{\nu}\kappa_\alpha(\nu^2)\Big)\Big) Z^i = -\delta_\alpha Z^i + \frac{1}{\nu}[\kappa_\alpha(\nu^2), Z^i] \\ &= -\frac{1}{\nu}\sum_{kl} dz_\alpha^k \omega_{kl}[Z^l, Z^i] = -\sum_{kl} dz_\alpha^k \omega_{kl}\Lambda^{li} = -dz_\alpha^i.\end{aligned}$$

As to (d), we remark the identity (2.18) and we see

$$\Big[\frac{1}{\nu}\sum_{kl} dz_\alpha^k \omega_{kl} Z^l, \tau\Big] = \sum_{kl} dz_\alpha^k \omega_{kl} Z^l.$$

Thus, we have

$$\partial_\alpha \tau = -\delta_\alpha \tau + \mathrm{ad}\Big(\frac{1}{\nu}\kappa_\alpha(\nu^2)\Big)\tau = -\sum_{kl} dz_\alpha^k \omega_{kl} Z^l + \widehat{\kappa}_\alpha(\nu^2),$$

which gives (d). Q.E.D.

We have also the following formulae:

Lemma 3.7. *Let us consider a p-form $P \in \Gamma(\Lambda^p_{U_\alpha} \otimes C_{U_\alpha})$. Then it holds for any $Q \in \Gamma(\Lambda_{U_\alpha} \otimes C_{U_\alpha})$*

$$\text{(i)} \qquad \partial_\alpha[P, Q] = [\partial_\alpha P, Q] + (-1)^p [P, \partial_\alpha Q].$$

For $F \in \Gamma(\Lambda^f_{U_\alpha} \otimes W_{U_\alpha})$, it holds for any $G \in \Gamma(\Lambda_{U_\alpha} \otimes W_{U_\alpha})$

$$\text{(ii)} \qquad \partial_\alpha(F \mathbin{\widehat{*}} G) = \partial_\alpha F \mathbin{\widehat{*}} G + (-1)^f F \mathbin{\widehat{*}} \partial_\alpha G.$$

Proof. We remark

$$\partial_\alpha(dz^I_\alpha P_I) = (-1)^{|I|} dz^I_\alpha \wedge \partial_\alpha P_I. \tag{3.18}$$

For (i), we show first for monomials $P = dz^I_\alpha P_I$ and $Q = dz^J_\alpha Q_J$. Using the identity above we calculate as $[P, Q] = dz^I_\alpha \wedge dz^J_\alpha [P_I, Q_J]$ and

$$\partial_\alpha[P, Q] = (-1)^{|I|+|J|} dz^I_\alpha \wedge dz^J_\alpha \wedge \partial_\alpha[P_I, Q_J].$$

For sections of $\Gamma(C_{U_\alpha})$, ∂_α acts as derivation and hence

$$\partial_\alpha[P_I, Q_J] = [\partial_\alpha P_I, Q_J] + [P_I, \partial_\alpha Q_J].$$

Thus,

$$\begin{aligned}\partial_\alpha[P, Q] = {} & [(-1)^{|I|} dz^I_\alpha \wedge \partial_\alpha P_I, dz^J Q_J] \\ & + (-1)^{|I|}[dz^I_\alpha P_I, (-1)^{|J|} dz^J_\alpha \wedge \partial_\alpha Q_J]\end{aligned}$$

which yields the desired result. For (ii), replacing $[P_I, Q_J]$ with $F_I \mathbin{\widehat{*}} G_J$ gives the equation similarly. Q.E.D.

3.2.3. Construction of connection. First we set

$$\widehat{\xi}_\alpha(\nu^2) = \mathrm{ad}\Big(\frac{1}{\nu}\xi_\alpha(\nu^2)\Big)\tau_\alpha \in \Lambda^1_{U_\alpha}[[\nu^2]] \tag{3.19}$$

where we take $\xi_\alpha(\nu^2) = \varphi_\alpha^{-1*}\Xi_\alpha(\nu^2)$ and $\Xi_\alpha(\nu^2) = \sum_\lambda dH_{\alpha\lambda}(\nu^2)\chi_\lambda$ in (3.6), (3.4) respctively. Then we have a relation

Lemma 3.8.

$$\widehat{\xi}_\beta(\nu^2) = \varphi^*_{\beta\alpha}\widehat{\xi}_\alpha(\nu^2) + d\widehat{h}_{\beta\alpha}(\nu^2)$$

where $\widehat{h}_{\beta\alpha}(\nu^2) = \mathrm{ad}\big(\frac{1}{\nu}h_{\beta\alpha}(\nu^2)\big)\tau_\beta \in \mathfrak{a}_{U_{\beta\alpha}}[[\nu^2]]$.

Proof. Recall $\xi_\beta(\nu^2) = \varphi^*_{\beta\alpha}\xi_\alpha(\nu^2) + dh_{\beta\alpha}(\nu^2)$ in (3.7). Then we calculate as

$$\widehat{\xi}_\beta(\nu^2) = \left[\frac{1}{\nu}\xi_\beta(\nu^2), \tau_\beta\right] = \left[\frac{1}{\nu}\varphi^*_{\beta\alpha}\xi_\alpha(\nu^2), \tau_\beta\right] + \left[\frac{1}{\nu}dh_{\beta\alpha}(\nu^2), \tau_\beta\right].$$

Notice that $\left[\frac{1}{\nu}\varphi^*_{\beta\alpha}\xi_\alpha(\nu^2), \tau_\beta\right] = \Psi^*_{\beta\alpha}\left[\frac{1}{\nu}\xi_\alpha(\nu^2), \Psi^*_{\alpha\beta}\tau_\beta\right]$. Since $\xi_\alpha(\nu^2)$ is a one form with values in the center of the algebra W, the identity $\Psi^*_{\alpha\beta}\tau_\beta = \tau_\alpha + f^\#_{\alpha\beta} - h_{\alpha\beta}(\nu^2)$ shows that $[\frac{1}{\nu}\xi_\alpha(\nu^2), \Psi^*_{\alpha\beta}\tau_\beta]$ is equal to

$$\left[\frac{1}{\nu}\xi_\alpha(\nu^2), \tau_\alpha + f^\#_{\alpha\beta} - h_{\alpha\beta}(\nu^2)\right] = \left[\frac{1}{\nu}\xi_\alpha(\nu^2), \tau_\alpha\right] = \widehat{\xi}_\alpha(\nu^2).$$

Using $\left[\frac{1}{\nu}dh_{\beta\alpha}(\nu^2), \tau_\beta\right] = d\widehat{h}_{\beta\alpha}(\nu^2)$, we obtain the desired result. Q.E.D.

For an arbitrary 1-form $\kappa_\alpha(\nu^2) \in \Lambda^1_{U_\alpha}[[\nu^2]]$ we have

Lemma 3.9.

$$\partial_\alpha\tau_\alpha = -2\theta_\alpha + \widehat{\kappa}_\alpha(\nu^2)$$

where $\theta_\alpha = \frac{1}{2}\sum_{ij} z^i_\alpha\omega_{ij}dz^j_\alpha$ *is a canonical* 1*-form and* $\widehat{\kappa}_\alpha(\nu^2) \in \Lambda^1_{U_\alpha}[[\nu^2]]$ *is given by* $\widehat{\kappa}_\alpha(\nu^2) = \mathrm{ad}\big(\frac{1}{\nu}\kappa_\alpha(\nu^2)\big)\tau_\alpha$.

Proof. By Proposition 3.6 (a) and (d), we see $\partial_\alpha\tau_\alpha = \partial_\alpha\tau + \partial_\alpha\sum_{ij} z^i_\alpha\omega_{ij}Z^j$ is computed as

$$\begin{aligned}\partial_\alpha\tau_\alpha = &-\sum_{ij} dz^i_\alpha\omega_{ij}Z^j + \widehat{\kappa}_\alpha(\nu^2)\\ &+\sum_{ij}\partial_\alpha z^i_\alpha\omega_{ij}Z^j + \sum_{ij} z^i_\alpha\omega_{ij}\partial_\alpha Z^j\end{aligned}$$

which gives the desired result. Q.E.D.

Now we can set a connection. For $\widehat{\xi}_\alpha(\nu^2)$ defined in (3.19) we consider to find $\kappa_\alpha(\nu^2)$ so that the derivation $\partial_\alpha = d - \delta_\alpha + \mathrm{ad}\big(\frac{1}{\nu}\kappa_\alpha(\nu^2)\big)$ satisfies

$$\partial_\alpha\tau_\alpha = \widehat{\xi}_\alpha(\nu^2). \tag{3.20}$$

We have

Lemma 3.10. *The equation* (3.20) *has a unique solution* $\kappa_\alpha(\nu^2)$.

Proof. By Lemma 3.9, the equation is equivalent to

$$\widehat{\kappa}_\alpha(\nu^2) = 2\theta_\alpha + \widehat{\xi}_\alpha(\nu^2), \tag{3.21}$$

where $\widehat{\kappa}_\alpha(\nu^2) = \mathrm{ad}\big(\frac{1}{\nu}\kappa_\alpha(\nu^2)\big)\tau_\alpha$. Since

$$\mathrm{ad}\left(\frac{1}{\nu}\sum_{k=0}^{\infty}\nu^{2k}\kappa_{\alpha,2k}\right)\tau_\alpha = \sum_{k=0}^{\infty}(2-4k)\nu^{2k}\kappa_{\alpha,2k},$$

a 1-form $\kappa_\alpha(\nu^2) = \sum_{k=0}^{\infty}\nu^{2k}\kappa_{\alpha,2k}$ is uniquely determined from (3.21).

Q.E.D.

We have

Lemma 3.11.

(i) *The restriction of ∂_α satisfies $\partial_\alpha^2|_{W_{U_\alpha}} = 0$.*

(ii) *A section $F \in \Gamma(W_{U_\alpha})$ satisfies $\partial_\alpha F = 0$ if and only if $F \in \mathcal{F}(W_{U_\alpha})$.*

(iii) *It holds $\partial_\alpha^2\tau_\alpha = \widehat{\Omega}_M(\nu^2)$, where $\widehat{\Omega}_M(\nu^2) = \mathrm{ad}\big(\frac{1}{\nu}\Omega_M(\nu^2)\big)\tau_\alpha$.*

Proof. The first statement is a dirct consequence of Proposition 3.6. For (iii), we use the equation (3.20) and we see

$$\partial_\alpha^2\tau_\alpha = \partial_\alpha\widehat{\xi}_\alpha = d\widehat{\xi}_\alpha = \widehat{\Omega}_M(\nu^2) = \mathrm{ad}\Big(\frac{1}{\nu}\Omega_M(\nu^2)\Big)\tau_\alpha.$$

As to (ii), we calculate for $F = \sum_\mu \frac{1}{\mu!}F_\mu(\nu)Z^\mu$, $F_\mu(\nu) \in \mathfrak{a}_{U_\alpha}[[\nu]]$,

$$\partial_\alpha F = \sum_\mu \frac{1}{\mu!}\,dF_\mu(\nu)Z^\mu + \sum_\mu \frac{1}{\mu!}F_\mu(\nu)\partial_\alpha Z^\mu.$$

Notice here $\partial_\alpha Z^\mu = -\sum_i \mu_i\, dz_\alpha^i Z^{\mu-e_i}$, and hence

$$\sum_\mu \frac{1}{\mu!}F_\mu(\nu)\partial_\alpha Z^\mu = \sum_\mu\sum_{i=1}^{2n}\frac{1}{\mu!}F_{\mu+e_i}(\nu)\,dz_\alpha^i Z^\mu.$$

We then have

$$\partial_\alpha F = \sum_\mu\sum_{i=1}^{2n}\frac{1}{\mu!}\Big(\frac{\partial}{\partial z_\alpha^i}F_\mu(\nu) - F_{\mu+e_i}(\nu)\Big)dz_\alpha^i Z^\mu$$

which yields the desired result. Q.E.D.

In what follows, we fix $\kappa_\alpha(\nu^2)$ given by (3.20) for each α. Then, we have a derivation $\partial_\alpha\colon \Gamma(\Lambda_{U_\alpha}^p \otimes C_{U_\alpha}) \to \Gamma(\Lambda_{U_\alpha}^{p+1} \otimes C_{U_\alpha})$ for each local trivialization.

We set a transformation as

$$\Psi_{\alpha\beta*}\partial_\alpha = \Psi_{\alpha\beta}^{-1*}\partial_\alpha\Psi_{\alpha\beta}^*\colon \Gamma(\Lambda_{U_{\beta\alpha}}^p \otimes C_{U_{\beta\alpha}}) \to \Gamma(\Lambda_{U_{\beta\alpha}}^{p+1} \otimes C_{U_{\beta\alpha}}). \tag{3.22}$$

Then we have

Proposition 3.12. $\Psi^*_{\alpha\beta}\partial_\alpha = \partial_\beta$

A proof follows from the following identities:

Lemma 3.13.

(i) $$(\Psi_{\alpha\beta *}\partial_\alpha)f = df, \quad f \in \mathfrak{a}_{U_{\beta\alpha}}[[\nu^2]],$$

(ii) $$(\Psi_{\alpha\beta *}\partial_\alpha)Z^j = -dz^j_\beta, \quad j = 1,\ 2, \dots, 2n,$$

(iii) $$(\Psi_{\alpha\beta *}\partial_\alpha)\nu = 0,$$

(iv) $$(\Psi_{\alpha\beta *}\partial_\alpha)\tau_\beta = \widehat{\xi}_\beta.$$

Proof. For the first identity, we apply the definiton of the transformation (3.22) and we see

$$(\Psi_{\alpha\beta *}\partial_\alpha)f = \Psi^{-1*}_{\alpha\beta}\partial_\alpha(\varphi^*_{\alpha\beta}f) = \varphi^{-1*}_{\alpha\beta}\, d\varphi^*_{\alpha\beta}f = df.$$

For (ii), we notice $Z^j = z^{j\#}_\beta - z^j_\beta$, $j = 1, \dots, 2n$ and we calculate

$$(\Psi_{\alpha\beta *}\partial_\alpha)Z^j = \Psi^{-1*}_{\alpha\beta}\partial_\alpha\Psi^*_{\alpha\beta}(z^{j\#}_\beta - z^j_\beta) = -\Psi^{-1*}_{\alpha\beta}\partial_\alpha\Psi^*_{\alpha\beta}z^j_\beta = -dz^j_\beta$$

since $\Psi^*_{\alpha\beta}z^{j\#}_\beta$ is a Weyl function and then vanished by ∂_α. The third one is obvious. As to (iv), we notice the identity in Proposition 3.3 and we have

$$(\Psi_{\alpha\beta *}\partial_\alpha)\tau_\beta = \Psi^{-1*}_{\alpha\beta}\partial_\alpha(\tau_\alpha + f^\#_{\alpha\beta} - \widehat{h}_{\alpha\beta}) = \Psi^{-1*}_{\alpha\beta}(\widehat{\xi}_\alpha - d\widehat{h}_{\alpha\beta}).$$

Hence Lemma 3.2 (i) gives $\Psi^{-1*}_{\alpha\beta}(\widehat{\xi}_\alpha - d\widehat{h}_{\alpha\beta}) = \varphi^*_{\beta\alpha}\widehat{\xi}_\alpha + d\widehat{h}_{\beta\alpha}$. Thus, Lemma 3.8 yields the desired equation. Q.E.D.

Now, by virtue of Proposition 3.12 we define

Definition 3.14. We set a globally defined derivation

$$\partial\colon \Gamma(\Lambda^p_M \otimes C_M) \to \Gamma(\Lambda^{p+1}_M \otimes C_M)$$

by

$$\partial F = \Psi^{-1}_{\alpha *}\partial_\alpha F = \Psi^*_\alpha\partial_\alpha\Psi^{-1*}_\alpha F, \quad F \in \Gamma(\Lambda_M \otimes C_M).$$

Now, we consider the curvature of the covariant exterior derivative ∂. From Lemma 3.11, we have

Theorem 3.15.

(i) $\partial^2|_{W_{U_\alpha}} = 0.$

(ii) *A section* $F \in \Gamma(W_M)$ *satisfies* $\partial F = 0$ *if and only if* $F \in \mathcal{F}(W_M)$.
(iii) $\partial^2 = \mathrm{ad}\big(\frac{1}{\nu}\Omega_M(\nu^2)\big)$, *i.e., the curvature form of* ∂ *is equal to* $\Omega_M(\nu^2)$.

As a corollary of the theorem, we have

Corollary 3.16.
(i) *The restriction* $\partial|_{W_M}$ *is a Fedosov connection.*
(ii) *The curvature of the connection* ∂ *is given by the adjoint of a 2-form which is a curvature form of Fedosov connection.*
(iii) *The Poincaré-Cartan class is equal to the cohomology class of Fedosov connection;* $[\Omega_M(\nu^2)] = c(W_M)$.

The statements (ii) and (iii) are direct consequences of the above theorem and (i). A proof of (i) will be given in the next section.

§4. Weyl charts and Classical charts

Recall that a Weyl manifold is obtained by gluing locally trivial bundles $\{W_{U_\alpha}\}$ with Weyl diffeomorphisms $\Phi_{\alpha\beta}\colon W_{U_{\alpha\beta}} \to W_{U_{\beta\alpha}}$ and local trivializations $\Phi_\alpha\colon W_{V_\alpha} \to W_{U_\alpha}$ such that $\Phi_\beta\Phi_\alpha^{-1} = \Phi_{\alpha\beta}$ are called local Weyl charts.

Let $\Psi_\alpha\colon C_{V_\alpha} \to C_{U_\alpha}$ be a locally trivialization of C_M given in §3.1 such that the restriction $\Phi_\alpha = \Psi_\alpha|_{W_{U_\alpha}}$ is a local Weyl chart. For simplicity, we also call $\Psi_\alpha\colon C_{V_\alpha} \to C_{U_\alpha}$ a local **Weyl chart** of C_M.

In this section, we introduce another system of local trivializations of C_M, called *classical charts* of C_M. We also obtain an expression of the connection ∂ with respect to classical charts, which shows the restriction to a Weyl manifold $\partial|_{W_M}$ gives Fedosov connection explicitly.

4.1. Basic Lemma

The essential part of the contruction of classical charts depends on the following lemma for a ν-automorphism of the contact algebra C.

Before stating the basic lemma, we remark first the following. For a ν-automorphism $\Psi\colon C \to C$, there exist $t(\nu^2) \in \mathbf{R}[[\nu^2]]$ and $T \in W_3^{\circ+}$ such that

$$\Psi(\tau) = \tau + T + t(\nu^2)$$

due to Proposition 2.18. We also have the converse direction.

Lemma 4.1. *For* $T \in W_3^{\circ+}$, $t(\nu^2) \in \mathbf{R}[[\nu^2]]$, *there exist uniquely* $F \in W_3^{\circ+}$ *and* $c(\nu^2) \in \mathbf{R}[[\nu^2]]$ *such that*

$$\exp \mathrm{ad}\Big(\frac{1}{\nu}F + \frac{1}{\nu}c(\nu^2)\Big)(\tau) = \tau + T + t(\nu^2).$$

Proof. Notice $\left[\frac{1}{\nu}F_{(3)}, \tau\right] = -F_{(3)}$ for an element $F_{(3)}$ of homogeneous degree 3. Write $T = T_{(3)} + O(4)$ where $T_{(3)}$ is the terms of homogenous degree 3. Then putting $F_{(3)} = T_{(3)}$ we have

$$\exp \operatorname{ad}\Big(\frac{1}{\nu}F_{(3)}\Big)(\tau + T + t(\nu^2)) = \tau + T_4 + t'(\nu^2),$$

where $T_4 \in W_4^{\circ +}$, $t'(\nu^2) \in \mathbf{R}[\nu^2]]$. Similarly we put $T_4 = T_{(4)} + O(5)$ where $T_{(4)}$ is the term of homogeneous degree 4. Then taking a certain $F_{(4)} \in W_4^{\circ +}$ of homogeneous degree 4, we can eliminate $T_{(4)}$ by means of $\exp \operatorname{ad}\big(\frac{1}{\nu}F_{(4)}\big)(\tau + T_4 + t'(\nu^2)) = \tau + T_5 + t''(\nu^2)$ where $T_5 \in W_5^{\circ +}$, $t''(\nu^2) \in \mathbf{R}[\nu^2]]$. Repeating this procedure with the Campbell-Hausdorff formula gives

$$\exp \operatorname{ad}\Big(-\frac{1}{\nu}F\Big)(\tau + T + t(\nu^2)) = \tau + \tilde{t}(\nu^2)$$

for certain $F \in W_3^{\circ +}$ and $\tilde{t}(\nu^2) \in \mathbf{R}[[\nu^2]]$. By the similar argument as in the proof of Proposition 2.18, there exists $c(\nu^2) \in \mathbf{R}[[\nu^2]]$ such that $\exp \operatorname{ad}\big(-\frac{1}{\nu}c(\nu^2)\big)(\tau + \tilde{t}(\nu^2)) = \tau$. Hence $\exp \operatorname{ad}\big(\frac{1}{\nu}F + \frac{1}{\nu}c(\nu^2)\big)$ is the desired ν-automorphism. The uniqueness is obtained by the similar argument in the proof of Proposition 2.18. Q.E.D.

4.2. Section $\widehat{\tau} \in \Gamma(C_M)$ and classical charts

Regarding τ as a constant section for each C_{U_λ}, we set a global section of C_M by

$$\widehat{\tau} = \sum_\lambda \chi_\lambda \Psi_\lambda^* \tau \tag{4.1}$$

where $\{\chi_\lambda\}_\lambda$ is a partition of unity. On each local Weyl chart, we set the local expression as

$$\widehat{\tau}_\mu = \Psi_\mu^{-1*}\widehat{\tau} = \sum_\lambda \varphi_\mu^{-1*}\chi_\lambda \Psi_\mu^{-1*}\Psi_\lambda^*\tau \in \Gamma(C_{U_\mu}). \tag{4.2}$$

Since $\Psi_\mu^{-1*}\Psi_\lambda^*\tau = \Psi_{\mu\lambda}^*\tau$, as we see at the begining of the previous subsection there exist $t_\mu(\nu^2) = \sum_{k=0}^\infty t_\mu^{(2k)}\nu^{2k} \in \mathfrak{a}_{\nu^2}(U_\mu)$ and $T_\mu \in \Gamma(U_\mu \times W_3^{\circ +})$ such that $\widehat{\tau}_\mu = \tau + t_\mu(\nu^2) + T_\mu$. Hence, by Lemma 4.1 there exists an algebra bundle isomorphism $\psi_\mu \colon C_{U_\mu} \to C_{U_\mu}$ such that $\psi_\mu^*\tau = \widehat{\tau}_\mu$. Thus, we have a local trivialization

$$\widetilde{\Psi}_\mu = \psi_\mu \circ \Psi_\mu \colon C_{V_\mu} \to C_{U_\mu} \tag{4.3}$$

giving a contact algebra isomorphism at each fiber and satisfying $\widetilde{\Psi}^*\tau = \widehat{\tau}$. We remark here the algebra bundle isomorphism ψ_μ does not give a Weyl diffeomorphism when restricted to Weyl algebra bundle W_{U_μ} in general. Thus a local trivialization $\widetilde{\Psi}_\mu\colon C_{V_\mu} \to C_{U_\mu}$ is not necessarily a Weyl chart, and hence the algebra of local Weyl functions are not preserved by the transformation $\widetilde{\Psi}_\mu \circ \widetilde{\Psi}_\lambda^{-1}$.

Definition 4.2. The local trivialization $\widetilde{\Psi}_\mu\colon C_{V_\mu} \to C_{U_\mu}$ is referred to as a **classical chart**.

4.3. Expression of ∂ in the classical charts

In this section we will give the explicit form of $\widetilde{\Psi}_{\mu*}\partial$.

We denote transition functions between classical charts by

$$\widetilde{\Psi}_{\lambda\mu} = \widetilde{\Psi}_\mu \circ \widetilde{\Psi}_\lambda^{-1}\colon C_{U_{\lambda\mu}} \to C_{U_{\mu\lambda}}, \quad U_{\lambda\mu} = \varphi_\lambda(V_\lambda \cap V_\mu).$$

We consider the transformation rule for $\widetilde{\Psi}^*_{\lambda\mu}\tau$ and $\widetilde{\Psi}^*_{\lambda\mu}Z^i$, $i = 1, \ldots, 2n$. By the identities (4.2) and (4.3), the constant section $\tau \in \Gamma(C_{U_\lambda})$ is transfered to $\widehat{\tau}$, $\widetilde{\Psi}^*_\lambda\tau = \widehat{\tau}$ which yields $\widetilde{\Psi}^*_{\lambda\mu}\tau = \tau$. We put the expansion

$$\widetilde{\Psi}^*_{\lambda\mu}Z^i = \sum_j A^i_{\lambda\mu,j}Z^j + G_{(2)} + G_{(3)} + \cdots + G_{(k)} + \cdots,$$

where $A^i_{\lambda\mu,j} \in \mathfrak{a}_{\nu^2}(U_{\lambda\mu})$, i, $j = 1, \ldots, 2n$ give a symplectic matirx at each point of $U_{\lambda\mu}$ and $G_{(k)} \in \Gamma(W^+_{U_{\lambda\mu}})$ is the terms of homogeneous degree k. Applying $\widetilde{\Psi}^*_{\lambda\mu}$ to the identities $[\tau, Z^i] = \nu Z^i$, $i = 1, \ldots, 2n$ gives

$$\nu\sum_j A^i_{\lambda\mu,j}Z^j + \cdots + k\nu G_{(k)} + \cdots = \nu\sum_j A^i_{\lambda\mu,j}Z^j + \cdots + \nu G_{(k)} + \cdots$$

since $[\tau, G_{(k)}] = k\nu G_{(k)}$, which shows $G_{(k)} = 0$ for $k \geq 2$. Then we have a transformation formula of classical charts

$$\widetilde{\Psi}^*_{\lambda\mu}\nu = \nu, \quad \widetilde{\Psi}^*_{\lambda\mu}Z^i = \sum_j A^i_{\lambda\mu,j}Z^j, \quad \widetilde{\Psi}^*_{\lambda\mu}\tau = \tau. \tag{4.4}$$

In what follows, we see the functions $A^i_{\lambda\mu,j}$ is expressed by means of the symplectic transformation $\varphi_{\lambda\mu}$. Notice the Weyl chart transformation $\Psi_{\lambda\mu}$ and classical chart transformation $\widetilde{\Psi}_{\lambda\mu}$ have the relation

$$\widetilde{\Psi}_{\lambda\mu}\psi_\lambda = \psi_\mu\Psi_{\lambda\mu}. \tag{4.5}$$

We put the expansion

$$\psi_\mu^* Z^i = Z^i + \frac{1}{2}\sum_{j_1,j_2} B^i_{\mu,j_1j_2} Z^{j_1} Z^{j_2} + O(3) \tag{4.6}$$

where $B^i_{\mu,j_1j_2} \in C^\infty(U_\mu)$ and $O(3)$ indicates the terms of $\Gamma(U_{U_\mu} \times W_3^+)$. As for calculating $\Psi^*_{\lambda\mu} Z^i$ we remark $Z^i = z^{i\#}_\mu - z^i_\mu$ on U_μ and $\Psi^*_{\lambda\mu} z^{i\#}_\mu = \psi^{i\#}_{\lambda\mu} + \nu^2 g^{i\#}$ for some $g^{i\#} \in \mathcal{F}(W_{U_{\lambda\mu}})$. Then we have

$$\begin{aligned}\Psi^*_{\lambda\mu} Z^i &= \Psi^*_{\lambda\mu}(z^{i\#}_\mu - z^i_\mu) \\ &= \sum_j \frac{\partial \varphi^i_{\lambda\mu}}{\partial z^j_\lambda} Z^j + \frac{1}{2}\sum_{j_1,j_2} \frac{\partial^2 \varphi^i_{\lambda\mu}}{\partial z^{j_1}_\lambda \partial z^{j_2}_\lambda} Z^{j_1} Z^{j_2} + O(3).\end{aligned} \tag{4.7}$$

Using the identities (4.6) and (4.7) for $\Psi^*_{\lambda\mu}\psi^*_\mu Z^i$ we have

$$\Psi^*_{\lambda\mu}\psi^*_\mu Z^i = \sum_j \frac{\partial \varphi^i_{\lambda\mu}}{\partial z^j_\lambda} Z^j + \frac{1}{2}\sum_{j_1,j_2} C^i_{\lambda\mu,j_1j_2} Z^{j_1} Z^{j_2} + O(3) \tag{4.8}$$

where

$$C^i_{\lambda\mu,j_1j_2} = \frac{\partial^2 \varphi^i_{\lambda\mu}}{\partial z^{j_1}_\lambda \partial z^{j_2}_\lambda} + \sum_{lm} \varphi^*_{\lambda\mu} B^i_{\mu,lm} \frac{\partial \varphi^l_{\lambda\mu}}{\partial z^{j_1}_\lambda} \frac{\partial \varphi^m_{\lambda\mu}}{\partial z^{j_2}_\lambda}. \tag{4.9}$$

Also we calculate

$$\psi^*_\lambda \widetilde{\Psi}^*_{\lambda\mu} Z^i = \sum_l A^i_{\lambda\mu,l}\left(Z^l + \frac{1}{2}\sum_{j_1,j_2} B^l_{\lambda,j_1j_2} Z^{j_1} Z^{j_2}\right) + O(3), \tag{4.10}$$

which shows $A^i_{\lambda\mu,j} = \partial\varphi^i_{\lambda\mu}/\partial z^j_\lambda$.

Thus we have the transformation formula of classical charts:

Lemma 4.3.

$$\widetilde{\Psi}^*_{\lambda\mu}\nu = \nu, \quad \widetilde{\Psi}^*_{\lambda\mu} Z^i = \sum_j \frac{\partial \varphi^i_{\lambda\mu}}{\partial z^j_\lambda} Z^j, \quad \widetilde{\Psi}^*_{\lambda\mu}\tau = \tau.$$

Remark 4.4. Substituting (4.8), (4.10) into (4.5) shows B_{λ,j_1j_2}'s are transformed as

$$\sum_{j=l}^{2n} \frac{\partial \varphi^i_{\lambda\mu}}{\partial z^l_\lambda} B^l_{\lambda,j_1j_2} = \frac{\partial^2 \varphi^i_{\lambda\mu}}{\partial z^{j_1}_\lambda \partial z^{j_2}_\lambda} + \sum_{l,m=1}^{2n} \varphi^*_{\lambda\mu} B^i_{\mu,lm} \frac{\partial \varphi^l_{\lambda\mu}}{\partial z^{j_1}_\lambda} \frac{\partial \varphi^m_{\lambda\mu}}{\partial z^{j_2}_\lambda}.$$

4.4. Expression of ∂ in classical charts

In this subsection, we give a local expression of the connection with respect to the classical chart.

By the Definition 3.14, the local expression of the connection in the Weyl chart is given by (3.17), $\Psi_{\lambda *}\partial = \partial_\lambda = d - \delta_\lambda + \mathrm{ad}\big(\frac{1}{\nu}\kappa_\lambda(\nu^2)\big)$, where $\delta_\lambda = \mathrm{ad}\big(\frac{1}{\nu}\sum_{ij} dz_\lambda^i \omega_{ij} Z^j\big)$ and $\kappa_\lambda(\nu^2)$ is given in Lemma 3.10. According to the definition (4.3), the local expression in the classical chart is given by $\widetilde{\partial}_\lambda = \widetilde{\Psi}_{\lambda *}\partial = \psi_{\lambda *}\partial_\lambda = \psi_\lambda^{-1*}\partial_\lambda \psi_\lambda^*$. In what follows, we will calculate $\widetilde{\partial}_\lambda f$ for $f \in C^\infty(U_\lambda)$, $\widetilde{\partial}_\lambda \tau$, $\widetilde{\partial}_\lambda \nu$ and $\widetilde{\partial}_\lambda Z^i$ for $i = 1, 2, \ldots, 2n$, in order to determine the form of $\widetilde{\partial}_\lambda$.

Since f belongs to the center of $\Gamma(W_{U_\lambda})$, we have $\widetilde{\partial}_\lambda f = df$. It is obvious that $\widetilde{\partial}_\lambda \nu = 0$ by the same reason.

4.4.1. Calculation of $\widetilde{\partial}_\lambda Z^i$. In this subsection, we will determine the form $\widetilde{\partial}_\lambda$ up to the central component, which calculus is mainly given by $\widetilde{\partial}_\lambda Z^i$.

Now we put $\Gamma^i_{\lambda,jk} = B^i_{\lambda,jk}$. Then the identity in Remark 4.4 means $\{\Gamma^i_{\lambda,jk}\}$ defines a connection ∇. Substituting (4.6) into $[\psi_\lambda^* Z^i, \psi_\lambda^* Z^i] = \nu\Lambda^{ij}$ induces $\sum_l \Lambda^{il} B^j_{\lambda,lk} = \sum_l \Lambda^{jl} B^i_{\lambda,lk}$ which means ∇ is a symplectic connection on M. Hence we have

Lemma 4.5. *The expansion* (4.6) *gives*

$$\psi_\lambda^* Z^i = Z^i + \frac{1}{2}\sum_{j_1,j_2} \Gamma^i_{\lambda,j_1j_2} Z^{j_1} Z^{j_2} + O(3), \tag{4.11}$$

where Γ_{μ,j_1j_2} is the Chirstofell symbol of a symplectic connection ∇ with respect to a canonical coordinate $(z_\lambda^1, \ldots, z_\lambda^{2n})$.

Notice $\widetilde{\partial}_\lambda Z^i = \psi_\lambda^{-1*}\partial_\lambda \psi_\lambda^* Z^i$. Then Lemma 4.5 induces

$$\begin{aligned}\widetilde{\partial}_\lambda Z^i = -dz_\lambda^i - \sum_{jk} \Gamma^i_{\lambda,jk}\, dz_\lambda^j Z^k + \frac{1}{2}\sum_{jj_1j_2} S^i_{\lambda,jj_1j_2}\, dz_\lambda^j Z^{j_1} Z^{j_2} \\ + O(3)\end{aligned} \tag{4.12}$$

where $S^i_{\lambda,jj_1j_2} = \partial\Gamma^i_{\lambda,j_1j_2}/\partial z_\lambda^j + \sum_m \Gamma^i_{\lambda,jm}\Gamma^m_{\lambda,j_1j_2}$.

Set $\widetilde{L}^i_{\lambda,2} = \frac{1}{2}\sum_{jj_1j_2} S^i_{\lambda,jj_1j_2}\, dz_\lambda^j Z^{j_1} Z^{j_2} + O(3)$ and apply $\widetilde{\partial}_\lambda$ to the identity $[Z^i, Z^j] = \nu\Lambda^{ij}$. Then, we have $[\widetilde{\partial}_\lambda Z^i, Z^j] + [Z^i, \widetilde{\partial}_\lambda Z^j] = 0$, which yields $[Z^i, \widetilde{L}^j_{\lambda,2}] = [Z^j, \widetilde{L}^i_{\lambda,2}]$. The Poincaré Lemma for formal power seires gives the unique section γ_λ in the space $\Gamma(\Lambda^1(U_\lambda)\otimes(W^{+\circ}_{3,U_\lambda}))$ such that

$$\widetilde{L}^i_{\lambda,2} = \frac{1}{\nu}\,\mathrm{ad}(\gamma_\lambda)(Z^i) \tag{4.13}$$

where $W^{+\circ}_{3,U_\lambda} = U_\lambda \times W^{+\circ}_3$.

The transformation rule in Lemma 4.3 shows the term $\sum_{ij} dz^i_\lambda \omega_{ij} Z^j$ given in classical charts defines a global section of C_M and hence we have a globally defined fiberwise derivation δ of C_M such that

$$\widetilde{\Psi}_{\lambda *}\delta = \widetilde{\delta}_\lambda = \frac{1}{\nu}\,\mathrm{ad}\Big(\sum_{ij} dz^i_\lambda \omega_{ij} Z^j\Big). \tag{4.14}$$

Notice $\widetilde{\delta}_\lambda Z^i = -dz^i_\lambda$ on C_{U_λ}. Also we can extend the classical connection ∇ as a globally defined derivation of $\Gamma(W_M)$ by

$$\widetilde{\Psi}_{\lambda *}\nabla Z^i = -\sum_m \Gamma^i_{\lambda,jk}\, dz^j_\lambda Z^k. \tag{4.15}$$

Thus, in terms of (4.14) and (4.15), the connection is expressed as

$$\widetilde{\Psi}_{\lambda *}\partial = \widetilde{\partial}_\lambda = \nabla - \widetilde{\delta}_\lambda + \frac{1}{\nu}\,\mathrm{ad}(\gamma_\lambda)$$

on the classical chart W_{U_λ}.

4.4.2. Proof of Corollary 3.16. Notice the transformation $\widetilde{\Psi}_{\lambda\mu *}\widetilde{\partial}_\lambda = \widetilde{\partial}_\mu$, $\widetilde{\Psi}_{\lambda\mu *}\widetilde{\delta}_\lambda = \widetilde{\delta}_\mu$. Since the section γ_λ is unique for each W_{U_λ}, we have the identity $\widetilde{\Psi}^*_{\lambda\mu}\gamma_\mu = \gamma_\lambda$. Thus, we have

Proposition 4.6. *There exists a section γ of W_M such that $\gamma = \widetilde{\Psi}^*_\lambda \gamma_\lambda$ and*

$$\partial|_{W_M} = \nabla - \delta + \frac{1}{\nu}\,\mathrm{ad}(\gamma),$$

where ∇, δ and γ_λ are given by (4.15), (4.14) *and* (4.13), *respectively.*

Now we are in a position to prove Corollary 3.16. By Theorem 3.15, $\partial^2 = 0$ shows $\partial|^2_{W_M} = 0$ where $\partial|_{W_M}$ is of the form in Proposition 4.6 and hence the restricted connection $\partial|_{W_M}$ is a Fedosov connection.

4.4.3. Central components of ∂. We proceed to determine the shape of ∂. First we remark on the classical chart C_{U_λ}

$$\widetilde{\Psi}_{\lambda *}\partial = \widetilde{\partial}_\lambda = \nabla - \widetilde{\delta}_\lambda + \mathrm{ad}\Big(\frac{1}{\nu}\gamma_\lambda\Big) + \mathrm{ad}\Big(\frac{1}{\nu}\widetilde{\sigma}_\lambda(\nu^2)\Big)$$

for certain central section $\widetilde{\sigma}_\lambda(\nu^2) \in \mathfrak{a}_{\nu^2}(U_\lambda)$. In fact, on classical chart C_{U_λ}, we consider the difference

$$\widetilde{\partial}^\circ_\lambda = \widetilde{\partial}_\lambda - \nabla + \widetilde{\delta}_\lambda - \mathrm{ad}\Big(\frac{1}{\nu}\gamma_\lambda\Big)$$

Since $\widetilde{\partial}^\circ_\lambda Z^i = 0$ for $i = 1, \ldots, 2n$, applying $\widetilde{\partial}^\circ_\lambda$ to the identity $[\tau, Z^i] = \nu Z^i$ shows $\widetilde{\partial}^\circ_\lambda \tau$ is a central section. Thus, we see $\widetilde{\partial}^\circ_\lambda \tau = \rho_\lambda(\nu^2)$ for certain $\rho_\lambda(\nu^2) \in \mathfrak{a}_{\nu^2}(U_\lambda)$. Similary as in the proof of Lemma 3.10, we can take $\widetilde{\sigma}_\lambda(\nu^2) \in \mathfrak{a}_{\nu^2}(U_\lambda)$ such that

$$\mathrm{ad}\Big(\frac{1}{\nu}\widetilde{\sigma}_\lambda(\nu^2)\Big)\tau = \rho_\lambda(\nu^2).$$

Notice the connection is written as $\widetilde{\Psi}_{\lambda *}\nabla = d + \mathrm{ad}\big(\frac{1}{\nu}\Gamma_{\lambda,(2)}\big)$ where $\Gamma_{\lambda,(2)} = \frac{1}{2}\sum_m \omega_{jm}\Gamma^m_{\lambda,kl} Z^j Z^k dz^l_\lambda$. Thus, one can easily check $\nabla\tau = 0$. Then, we have

$$\widetilde{\partial}_\lambda \tau = -\sum_{ij} dz^i_\lambda \omega_{ij} Z^j + 2\gamma_\lambda - [\tau, \gamma_\lambda] + 2\widetilde{\sigma}_\lambda(\nu^2) - [\tau, \widetilde{\sigma}_\lambda(\nu^2)].$$

On the other hand, we calculate as

$$\begin{aligned}
\psi^{*-1}_\lambda \partial_\lambda \psi^*_\lambda \tau &= \psi^{*-1}_\lambda \partial_\lambda (\tau + t_\lambda(\nu^2) + T_\lambda) \\
&= \psi^{*-1}_\lambda \partial_\lambda \tau + dt_\lambda(\nu^2) + \psi^{*-1}_\lambda \partial_\lambda T_\lambda \\
&= -\sum_{ij} dz^i_\lambda \omega_{ij} \psi^{*-1}_\lambda Z^j + \widehat{\kappa}_\lambda(\nu^2) + dt_\lambda(\nu^2) + \psi^{*-1}_\lambda \partial_\lambda T_\lambda.
\end{aligned}$$

Comparing the central terms of the both equation we see

$$\begin{aligned}
&2\widetilde{\sigma}_\lambda(\nu^2) - [\tau, \widetilde{\sigma}_\lambda(\nu^2)] \\
&\quad = -\Big(\sum_{ij} dz^i_\lambda \omega_{ij} \psi^{*-1}_\lambda Z^j\Big)^\circ + \widehat{\kappa}_\lambda(\nu^2) + dt_\lambda(\nu^2) + (\psi^{*-1}_\lambda \partial_\lambda T_\lambda)^\circ
\end{aligned}$$

where N° means the central terms of N of $N \in \Gamma(C_{U_\lambda})$. Similary as in the proof of Lemma 3.10, the equation above gives the unique $\widetilde{\sigma}_\lambda(\nu^2)$.

References

[AG] V. I. Arnol'd and A. B. Givental, Symplectic Geometry; Dynamical Systems IV, Encyclopaedia of Mathematical Sciences, **4**, Springer-Verlag, 1990.

[BFFLS] F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowizc and D. Sternheimer, *Deformation theory and quantization*, Ann. Physics, **111** (1978), 61–111.

[F] B. Fedosov, *A simple geometrical construction of deformation quantization*, J. Differential Geom., **40** (1994), 213–238.

[K] M. Kontsevich, *Deformation quantization of Poisson manifolds* I, q-alg/9709040.

[DL] M. De Wilde and P. B. A. Lecomte, *Existence of star-products and of formal deformations of the Poisson Lie algebra of arbitrary symplectic manifolds*, Lett. Math. Phys., **7** (1983), 487–496.

[G] S. Gutt, *Cohomology associated with the Poisson Lie algebra*, Lett. Math. Phys., **3** (1979), 297–310.

[L] A. Lichnerowizc, *Déformaions d'algèbres associèes à une variété symplectique (les $*_\nu$-products)*, Ann. Inst. Fourier, Grenoble **32** (1982), 157–209.

[OMY1] H. Omori, Y. Maeda and A. Yoshioka, *Weyl manifolds and Deformation quantization*, Adv. Math., **85** (1991), 224–255.

[OMY2] H. Omori, Y. Maeda and A. Yoshioka, *Global calculus on Weyl manifolds*, Japanese. J. Math., **17** (1991), 57–82.

[OMMY1] H. Omori, Y. Maeda, N. Miyazaki and A. Yoshioka, *Poincaré-Cartan class and deformation quantization of Kähler manifolds*, Comm. Math. Phys., **194** (1998), 207–230.

[T] B. Tsygan, *Local Forumula for the index of a Fourier integral operator*, DQ/0004022.

[Y1] A. Yoshioka, *Contact structures on Weyl manifolds and Deformation quantization*, Proceedings of the conference on "Dynamical systems and differential geometry", RIMS Kokyuroku, **1119** (1999), 1–17, (in Japanese).

[Y2] A. Yoshioka, *Weyl manifold and quantized connection*, Proceedings of the Workshop "Towards 100 years after Sophus Lie" (Kazan, 1998), Lobachevskii J. Math., **4** (1999), 177–206, (electronic).

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