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Semiprojectivity in Simple C*-Algebras

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Abstract.

We show that certain purely infinite simple C*-algebras, including the Cuntz algebra O_∞, are semiprojective. Some related results and conjectures are discussed, and some crossed product examples constructed.

§1. Introduction

The notions of projectivity and semiprojectivity were introduced in the development of shape theory for C*-algebras ([EK86], [Bla85]) as noncommutative analogs of the topological notions of absolute retract (AR) and absolute neighborhood retract (ANR) respectively. Semiprojective C*-algebras have rigidity properties which make them conceptually and technically important in several aspects of C*-algebra theory; this is reflected especially in the work of Loring and his coauthors (see, for example, [Lor97].) It is not too easy for a C*-algebra to be semiprojective, but there does seem to be a reasonable supply of such algebras.

Most known semiprojective C*-algebras are far from simple. (Indeed, a projective C*-algebra must be contractible, so cannot be simple.) In fact, the only known simple semiprojective C*-algebras have been the finite-dimensional matrix algebras and the (simple) Cuntz-Krieger algebras [Bla85]. In this paper, we will give a few more examples of simple semiprojective C*-algebras (and more are given in [Szy]), but also obtain some structure results which show that the class of infinite-dimensional semiprojective simple C*-algebras may not be too much larger than the class of Cuntz-Krieger algebras (in fact, it might consist exactly of the separable purely infinite simple nuclear C*-algebras with finitely generated K-theory.)

The work of this paper was largely inspired by the remarkable recent classification theorem of Kirchberg, also in part proved independently

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2000 *Mathematics Subject Classification.* Primary 46L05; Secondary 19A10 by Phillips ([Kir00], [KP00a], [KP00b], [Phi00].) The theorem asserts that the following class of C*-algebras is classified up to isomorphism by K-theory:

Definition 1.1. A separable, nuclear, simple, unital, purely infinite C*-algebra in the bootstrap class for the Universal Coefficient Theorem ([RS87], [Bla98, §23]) is called a *Kirchberg algebra.*

It was (and is) hoped that the notion of semiprojectivity, and results such as those of this paper, will lead to a simplification and clarification of the proof of this theorem. Although this hope has yet to be fully realized, there are obvious close connections between semiprojectivity and some of the ingredients of the proof; see 2.15.

Our main results are:

(1.) The Cuntz algebra O_∞ is semiprojective (3.2).
(2.) If A is simple, semiprojective, and properly infinite, then $A \otimes \mathbb{K}$ is also semiprojective (4.1).
(3.) If A is a semiprojective Kirchberg algebra, then $K_*(A)$ is finitely generated (2.11).
(4.) The class of semiprojective (simple) C*-algebras is not closed under crossed products by finite groups, even $\mathbb{Z}_2$ (6.3).

§2. Semiprojective C*-Algebras

We recall the definition of a semiprojective C*-algebra, which first appeared in this form in [Bla85] (a somewhat different, less restrictive, definition previously appeared in [EK86].)

Definition 2.1. A separable C*-algebra A is *semiprojective* if, for any C*-algebra B, increasing sequence $\langle J_n \rangle$ of (closed two-sided) ideals of B, with $J = [\cup J_n]^-$, and *-homomorphism $\phi : A \to B/J$, there is an n and a *-homomorphism $\psi : A \to B/J_n$ such that $\phi = \pi \circ \psi$, where $\pi : B/J_n \to B/J$ is the natural quotient map.

A ϕ for which such a ψ exists is said to be *partially liftable.* If there is a $\psi : A \to B$ with $\phi = \pi \circ \psi$, then ϕ is *liftable*; if every homomorphism from A is liftable, A is said to be *projective.*

Note that for convenience, we have only defined semiprojectivity for separable C*-algebras (although the same definition makes sense also for nonseparable C*-algebras, it is probably not the appropriate one.) Thus in this paper all semiprojective C*-algebras will implicitly be separable. In the definition, B is not required to be separable; however:

Proposition 2.2. *The definition of semiprojectivity does not change if in 2.1 we make any or all of the following restrictions:*

(i) *B is separable.*
(ii) *ϕ is surjective.*
(iii) *ϕ is injective.*

Proof. B can clearly be replaced by the C*-subalgebra D generated by any preimage of a dense set in $\phi(A)$, proving (i) and (ii). (One technical point: $\cup_n(D \cap J_n)$ is dense in $D \cap J$, an easy consequence of the uniqueness of norm on a C*-algebra.) To prove (iii), replace B by $A \oplus B$, J_n by $0 \oplus J_n$, J by $0 \oplus J$, and ϕ by $id_A \oplus \phi$. Q.E.D.

For the convenience of the reader, we recall some standard facts about semiprojective C*-algebras which we will need to use.

Proposition 2.3. [Bla85, 2.18] *Let B, J_n, and J be as in 2.1, and let $q_1, \ldots, q_k$ be mutually orthogonal projections in B/J. Then for sufficiently large n, there are mutually orthogonal projections $p_1, \ldots, p_k$ in B/J_n with $\pi(p_j) = q_j$ for all j. If B (and hence B/J) is unital and $q_1 + \cdots + q_k = 1$, then we may choose the p_j so that $p_1 + \cdots + p_k = 1$.*

Corollary 2.4. [Bla85, 2.16] *If A is unital, then the definition of semiprojectivity for A does not change if in 2.1 B and ϕ are required to be unital. In particular, $\mathbb{C}$ is semiprojective.*

Note that $\mathbb{C}$ is not projective (in the category of general C*-algebras and *-homomorphisms): a *-homomorphism from $\mathbb{C}$ to B/J is effectively just a choice of projection in B/J, and projections do not lift from quotients in general.

Proposition 2.5. [Bla85, 2.23] *Let B, J_n, J be as in 2.1. Let v be a partial isometry in B/J, and set $q_1 = v^*v$, $q_2 = vv^*$. Suppose there are projections $p_1, p_2 \in B/J_n$ for some n with $\pi(p_j) = q_j$. Then, after increasing n if necessary, there is a partial isometry $u \in B/J_n$ with $\pi(u) = v$ and $p_1 = u^*u$, $p_2 = uu^*$.*

Proposition 2.6. *(*[Bla85, 2.19], [Lor97]*) A finite direct sum of semiprojective C*-algebras is semiprojective.*

Proposition 2.7. *(*[Bla85, 2.28-2.29], [Lor97]*) If A is semiprojective, then $M_n(A)$ is semiprojective for all n. If A is semiprojective, then any unital C*-algebra strongly Morita equivalent to A is also semiprojective.*

The unital cases of 2.6 and 2.7 are simple consequences of 2.3 and 2.5, but the nonunital cases are more delicate.

Examples 2.8. Simple repeated applications of 2.3-2.7 show that the following C*-algebras are semiprojective:

(i) $\mathbb{M}_n = M_n(\mathbb{C})$, and more generally any finite-dimensional C*-algebra.

(ii) $C(\mathbb{T})$, where $\mathbb{T}$ is a circle (the universal C*-algebra generated by one unitary.)

(iii) Generalizing (ii), $C^*(\mathbb{F}_n)$, the full C*-algebra of the free group on n generators for n *finite* (the universal C*-algebra generated by n unitaries.)

(iv) The Toeplitz algebra $\mathcal{T}$ (the universal C*-algebra generated by an isometry.)

(v) The Cuntz-Krieger algebras O_A for a finite square $0-1$ matrix A [CK80], and in particular the Cuntz algebras O_n $(n \neq \infty)$.

(vi) Any C*-algebra which is the universal C*-algebra generated by a finite number of partial isometries, where the only relations (finitely many) are order and orthogonality relations among the source and range projections of the partial isometries; this includes all the above examples.

Some potential or actual non-examples are:

(vii) $C^*(\mathbb{F}_\infty)$, the universal C*-algebra generated by a sequence of unitaries. The problem is that, in the setting of 2.1 with B and ϕ unital, the n might have to be increased each time an additional generator is partially lifted. In fact, $C^*(\mathbb{F}_\infty)$ violates the conclusion of 2.10 (and obviously satisfies the hypothesis), so is not semiprojective.

(viii) The Cuntz algebra O_∞, the universal C*-algebra generated by a sequence of isometries with mutually orthogonal range projections, has the same potential difficulty as $C^*(\mathbb{F}_\infty)$. However, it turns out that O_∞ *is* semiprojective (3.2). (Note that $K_*(O_\infty)$ is finitely generated.)

(ix) $C(\mathbb{T}^n)$ for $n \geq 2$ is the universal C*-algebra generated by n commuting unitaries. Commutation relations are difficult to lift in general, and it can be shown that $C(\mathbb{T}^n)$ $(n \geq 2)$ fails to satisfy the conclusion of 2.9 and is thus not semiprojective.

We recall the following important approximate factorization property for semiprojective C*-algebras:

Proposition 2.9. [Bla85, 3.1] *Let A be a semiprojective C*-algebra, and $(B_n, \beta_{m,n})$ be an inductive system of C*-algebras with $B = \lim_\rightarrow(B_n, \beta_{m,n})$. If $\phi : A \to B$ is a homomorphism, then for all sufficiently large n there are homomorphisms $\phi_n : A \to B_n$ such that $\beta_n \circ \phi_n$ is homotopic* *to ϕ and converges pointwise to ϕ as $n \to \infty$, where β_n is the standard map from B_n to B.*

2.9 almost implies that a semiprojective C*-algebra has finitely generated K-theory:

Corollary 2.10. *Let A be a semiprojective C^*-algebra. If A can be written as an inductive limit of C^*-algebras with finitely generated K-theory, then A itself has finitely generated K-theory.*

Corollary 2.11. *If A is a semiprojective Kirchberg algebra, then $K_*(A)$ is finitely generated.*

Proof. If A is a Kirchberg algebra, then by the results of [Kir00] A can be written as an inductive limit of (Kirchberg) algebras with finitely generated K-theory, since $K_*(A)$ can be written as an inductive limit of finitely generated groups and every map on K-theory can be implemented by an algebra homomorphism between the corresponding Kirchberg algebras. Q.E.D.

The pointwise approximation part of 2.9 also applies to inductive limits in the generalized sense of [BK97] (it is unclear how an analog of the homotopy result might be phrased.) This generalization follows from the next fact about continuous fields, using [BK97, 2.2.4].

Proposition 2.12. *Let A be a semiprojective C^*-algebra, $\langle B(t)\rangle$ a continuous field of C^*-algebras over a locally compact Hausdorff space X, and t_0 a point of X with a countable neighborhood base. If ϕ is a homomorphism from A to $B(t_0)$, then there is a compact neighborhood Z of t_0 in X and a homomorphism ψ from A to the continuous field C^*-algebra defined by $\{B(t) : t \in Z\}$ such that $\phi = \pi_{t_0} \circ \psi$. In particular, if $x \in A$ with $\phi(x) \neq 0$, then for each t in some neighborhood of t_0 there is a homomorphism $\phi_t : A \to B(t)$ with $\phi_t(x) \neq 0$.*

Proof. Let (U_n) be a sequence of open sets in X with $Z_n = \bar{U}_n$ compact and contained in U_{n-1} for all n, and $\cap U_n = \{t_0\}$. Let B be the continuous field algebra defined by $\{B(t) : t \in X\}$, J_n the ideal of sections vanishing on Z_n, and J the sections vanishing at t_0. Apply 2.1. Q.E.D.

Corollary 2.13. *Let A be a semiprojective C^*-algebra, and $(B_n, \beta_{m,n})$ be a generalized inductive system of C^*-algebras* [BK97] *with $B = \lim_{\to}(B_n, \beta_{m,n})$. If $\phi : A \to B$ is a homomorphism, then for all sufficiently large n there are homomorphisms $\phi_n : A \to B_n$ such that $\beta_n \circ \phi_n$ converges pointwise to ϕ as $n \to \infty$, where β_n is the standard map from B_n to B.*

Corollary 2.14. *Let A be a semiprojective MF algebra* [BK97]. *Then A is residually finite-dimensional (has a separating family of finite-dimensional representations). If A is simple, then A is a finite-dimensional matrix algebra.*

Proof. Apply 2.12 and [BK97, 3.2.2(v)]. Q.E.D.

Another consequence of 2.12 is that every asymptotic morphism from a semiprojective C*-algebra to any other C*-algebra (in the sense of Connes-Higson E-theory ([CH90], [Bla98, §25])) can be realized up to homotopy by an actual homomorphism. This has potentially important consequences in the classification of purely infinite simple C*-algebras.

Corollary 2.15. [Bla98, 25.1.7] *Let A and B be separable C*-algebras, with A semiprojective. Then the canonical map from the set $[A, B]$ of homotopy classes of homomorphisms into the set $[[A, B]]$ of homotopy classes of asymptotic homomorphisms is a bijection.*

§3. Examples of Semiprojective Simple C*-Algebras

In this section, we show that certain purely infinite simple nuclear C*-algebras such as the Cuntz algebra O_∞ are semiprojective.

The main technical fact used in the proofs of this section and those of section 4 is the following sharpening of a well-known lifting property for unitaries (cf. [Bla98, 3.4.5].) If A is a C*-algebra, we write $A^\dagger$ for its unitization.

Proposition 3.1. *Let B be a C*-algebra, J a (closed 2-sided) ideal of B, and $\pi : B \to B/J$ the quotient map. Let q be a projection in B/J and v a unitary in $(B/J)^\dagger$ such that*

(1) $qv = vq = q$
(2) $(1-q)v = (1-q)v(1-q)$ *is in the connected component of the identity in* $\mathcal{U}((1-q)(B/J)^\dagger(1-q)$.

If there is a projection p in B with $\pi(p) = q$, then there is a unitary u in $B^\dagger$ with $\pi(u) = v$ and $pu = up = p$.

Proof. π maps $(1-p)B(1-p)$ onto $(1-q)(B/J)(1-q)$, so by [Bla98, 3.4.5] there is a unitary w in $(1-p)B^\dagger(1-p)$ with $\pi(w) = (1-q)v$. Set $u = p + w$. Q.E.D.

Theorem 3.2. *O_∞ is semiprojective.*

Proof. Let $\{s_1, s_2, \dots\}$ be the standard generators of O_∞, i.e. the s_j are isometries with mutually orthogonal ranges. Let B, J_n, J, and ϕ be as in 2.1. By 2.2 and 2.4 we may assume B is unital and ϕ is an isomorphism, and identify O_∞ with B/J. Using 2.3 and 2.5, we may partially lift any finite number of the s_j to isometries with mutually orthogonal ranges in B/J_n, for some n; the difficulty is that *a priori* we might have to increase n each time we partially lift another generator. But by using the next lemma inductively on k (with $A = O_\infty$ and $p_0 = q_0 = 0$), once we partially lift the first two generators we can lift all the rest without further increasing the n. Note that at each step we correct the provisional lift of the last of the previous generators, but do not change the lifts of the earlier ones. Q.E.D.

Lemma 3.3. *Let A be a unital C^*-algebra, q_0 a projection in A, and $\{s_1, s_2, \dots\}$ a sequence of isometries in A whose range projections are mutually orthogonal and all orthogonal to q_0. Let D be a unital C^*-algebra, and $\pi : D \to A$ a surjective homomorphism, and let $k \geq 2$. Suppose p_0 is a projection in D and $r_1, \dots, r_{k-1}, t_k$ are isometries in D whose range projections are mutually orthogonal and all orthogonal to p_0, with $\pi(p_0) = q_0$, $\pi(r_j) = s_j$ for $1 \leq j \leq k-1$, and $\pi(t_k) = s_k$. Then there are isometries r_k and t_{k+1} in D, such that the ranges of $r_1, \dots, r_k, t_{k+1}$ are mutually orthogonal and orthogonal to p_0, and $\pi(r_k) = s_k$, $\pi(t_{k+1}) = s_{k+1}$.*

Proof. We may assume A is generated by q_0 and $\{s_n\}$. Then A is isomorphic either to O_∞ (if $q_0 = 0$) or to a split essential extension of O_∞ by $\mathbb{K}$ (if $q_0 \neq 0$.) In either case, the unitary group of A, or any corner in A, is connected: this follows from [Cun81] for O_∞, and if u is a unitary in the extension, let v be the image of $\pi(u^*) \in O_\infty$ under a cross section; then v is in the connected component of 1, and so is vu since it is a unitary in $\mathbb{K}^\dagger$.

Set $p = p_0 + \sum_{j=1}^{k-1} r_j r_j^*$ and $q = q_0 + \sum_{j=1}^{k-1} s_j s_j^*$; then p and q are projections, and $\pi(p) = q$. In the copy of O_∞ in A generated by $\{s_1, s_2, \dots\}$, the range projections of the isometries $s_k s_1$ and s_k^2 are orthogonal to each other and to q, and are equivalent to $s_k s_k^*$ and $s_{k+1} s_{k+1}^*$ via partial isometries $v_1 = s_k s_1^* s_k^*$ and $v_2 = s_{k+1} s_k^{*2}$ respectively. Also, the projections $1 - q - s_k s_1 s_1^* s_k^* - s_k^2 s_k^{*2}$ and $= 1 - q - s_k s_k^* - s_{k+1} s_{k+1}^*$ are equivalent via a partial isometry v_3, since these projections are nonzero and have the same K_0-class. Set $v = q + v_1 + v_2 + v_3$. Then v is a unitary in O_∞, $qv = vq = q$, and $v s_k s_1 = s_k$, $v s_k^2 = s_{k+1}$. Also, the unitary group of $(1-q)A(1-q)$ is connected; thus by 3.1 there is a unitary u in D with $\pi(u) = v$ and $pu = up = p$. Set $r_k = u r_1 t_k$ and $t_{k+1} = u t_k^2$. Q.E.D.

We next consider a non-simple example, which will be used to obtain a generalization of 3.2.

Proposition 3.4. *Let $\mathcal{T}$ be the Toeplitz algebra, the universal C*-algebra generated by a single isometry s. Let ω be a primitive n'th root of unity, and let α be the automorphism of $\mathcal{T}$ which sends s to ωs. Then $\mathcal{T} \times_\alpha \mathbb{Z}_n$ is semiprojective.*

Proof. $A = \mathcal{T} \times_\alpha \mathbb{Z}_n$ is the universal unital C*-algebra generated by $\{s, v\}$, with relations $\{s^*s = 1, v^n = v^*v = 1, v^*sv = \omega s\}$. Let B, J_n, J be as in 2.1; as usual, assume B is unital and ϕ is an isomorphism, and identify A with B/J. We can partially lift v to a unitary $u \in B/J_m$ for some m. If $x \in B/J_m$ is a preimage of s, then $y = n^{-1}\sum_{k=1}^n \omega^k u^{-k} s u^k$ is a preimage for s with $u^*yu = \omega y$. y^*y commutes with u, and since $\pi(y^*y) = 1$ we may assume y^*y is close to 1 and therefore invertible, by increasing m if necessary. Then $t = y(y^*y)^{-1/2}$ is an isometry, $\pi(t) = s$, and $u^*tu = \omega t$, so $\{t, u\}$ generate the partial lift of A. Q.E.D.

Theorem 3.5. *Let ω be a primitive n'th root of unity, and let α be the automorphism of O_∞ such that $\alpha(s_1) = \omega s_1$ and $\alpha(s_k) = s_k$ for all $k > 1$. Let $A = O_\infty \times_\alpha \mathbb{Z}_n$. Then*

(i) *A is the (unique) Kirchberg algebra with $K_0(A) = \mathbb{Z}^n$ (with $[1] = (1, 0, \cdots, 0)$) and $K_1(A) = 0$.*

(ii) *A is semiprojective.*

Proof. (i): This can be proved directly using arguments very similar to those in [CE81]. A more elegant approach, though, is to write $O_\infty \otimes \mathbb{K}$ as a graph C*-algebra as in [Kum98, 2.3(h)]; then A is the graph C*-algebra of the skew product graph [KP99], and hence is purely infinite (and simple) by [KPR98, 3.9] and in the UCT bootstrap class by [KP99, 2.6]. The K-theory can be calculated as in [PR96].
(ii): Let w be the unitary in A or order n implementing α. Let B, J_n, J, and ϕ be as in 2.1, with B unital and ϕ an isomorphism (2.2, 2.4.) Identify A with B/J. By 3.4 we can partially lift s_1 and w to an isometry r_1 and a unitary z of order n in B/J_m for some m, with $z^*r_1z = \omega r_1$. Let D be the commutant of z in B/J_m. Then the image of D in A contains s_k for all $k > 1$, since if $x \in B/J_m$ with $\pi(x) = s_k$, then $y = n^{-1}\sum_{j=1}^n z^{-j} s_k z^j \in D$ satisfies $\pi(y) = s_k$. Also, $r_1r_1^* \in D$. By increasing m if necessary, we can find isometries $r_2, r_3 \in D$ with range projections orthogonal to $r_1r_1^*$ and with $\pi(r_j) = s_j$ for $j = 2, 3$, by 2.3 and 2.5. Now we can, by inductively using 3.3 with $q_0 = s_1s_1^*$ and $p_0 = r_1r_1^*$, find a lift $r_k \in D$ for s_k for each k, such that the range projections are all mutually orthogonal and also orthogonal to $r_1r_1^*$. Q.E.D.

This example has been generalized in [Szy] to include all Kirchberg algebras A where $K_0(A)$ is finitely generated, $K_1(A)$ is finitely generated and torsion-free, and $rank(K_1(A)) \leq rank(K_0(A))$. [See note added in proof.]

The results of this section, [Szy], 2.8, and 2.11 suggest the following conjecture:

Conjecture 3.6. A Kirchberg algebra is semiprojective if and only if its K-theory is finitely generated.

Note that if A is a Kirchberg algebra, $K_0(A)$ is finitely generated, and $K_1(A)$ is isomorphic to the torsion-free part of $K_0(A)$, then A is stably isomorphic to a Cuntz-Krieger algebra (and conversely) [Rør95], and therefore semiprojective. Thus the most important test algebras for this conjecture, besides the examples of this section, include:

$O_n \otimes O_n$ (the Kirchberg algebra B with $K_0(B) \cong= K_1(B) \cong \mathbb{Z}_{n-1}$)
P_∞ (the Kirchberg algebra B with $K_0(B) = 0$ and $K_1(B) = \mathbb{Z}$.)

[See note added in proof.]

The difficulty in proving that $O_n \otimes O_n$ is semiprojective is that the two copies of O_n must be partially lifted so that the lifts exactly commute. One can come frustratingly close to proving that this can be done: for example, inside O_n is a copy of O_∞ containing the first $n-1$ generators of O_n, and the subalgebra $O_\infty \otimes O_n$ can be partially lifted since it is isomorphic to O_n and is therefore semiprojective.

It appears that the results and techniques of [DE] can be used to show that a Kirchberg algebra with finitely generated K-theory is weakly semiprojective in the sense of [Lor97] (I am indebted to M. Dadarlat for this observation.) The best approach to the conjecture might be to solve the following problem (if it has a positive solution.)

Problem 3.7. If B is a Kirchberg algebra with finitely generated K-theory, find a finite presentation for B as in [Bla85], preferably with stable (partially liftable) relations.

The only Kirchberg algebras for which such a presentation is known are the (simple) Cuntz-Krieger algebras O_A and their matrix algebras. Finite tensor products of these, such as $O_n \otimes O_n$, and certain crossed products by finite groups, also have obvious finite presentations, but the relations include ones such as commutation relations, which are not (obviously) stable. No finite presentation for O_∞ or P_∞ is known (B. Neubüser has obtained a non-finite presentation of P_∞ as a graph C*-algebra.)

§4. Stable Semiprojective C*-Algebras and Hereditary Subalgebras

In this section, we examine conditions related to when a stable C*-algebra is semiprojective. Semiprojectivity in stable C*-algebras is fairly exceptional.

Recall that a unital C*-algebra A is *properly infinite* if A contains two isometries with orthogonal range projections; A then contains a unital copy of O_∞. A simple unital C*-algebra which is infinite (contains a nonunitary isometry) is automatically properly infinite ([Cun81], [Bla98, 6.11.3]). The main result of this section is:

Theorem 4.1. *Let A be a semiprojective properly infinite unital C*-algebra. Then its stable algebra $A \otimes \mathbb{K}$ is also semiprojective.*

Proof. The proof is quite similar in spirit to the proof of 3.2, based on the fact that inside A is a nicely embedded copy of $A \otimes \mathbb{K}$. Let $\{s_j\}$ be a sequence of isometries in A with mutually orthogonal ranges, and set $f_{ij} = s_i s_j^*$. = Then $\{f_{ij}\}$ is a set of matrix units in A. Let $\{e_{ij}\}$ be the standard matrix units in $\mathbb{K}$.

Let B, J_n, and J be as in 2.1, with $\phi : A \otimes \mathbb{K} \to B/J$ an isomorphism. Fix an m such that there is a projection $h_{11} \in B/J_m$ with $\pi(h_{11}) = 1 \otimes e_{11}$ and such that ϕ lifts to a unital $\psi : A \otimes e_{11} \cong A \to h_{11}(B/J_m)h_{11}$, using 2.3 and semiprojectivity of A. It suffices to show that the matrix units $\{1 \otimes e_{ij}\}$ lift to matrix units $\{h_{ij}\}$ B/J_m including the chosen h_{11}.

For each i, j, let $r_j = \psi(s_j)$, and $g_{ij} = r_i r_j^* = \psi(f_{ij})$. We now inductively choose unitaries v_k and u_k, and projections h_{kk}, as follows. Let v_1 be a unitary in the connected component of the identity of $\mathcal{U}((A \otimes \mathbb{K})^\dagger)$ with $v_1^* f_{11} v_1 = 1 \otimes e_{11}$ [Bla98, 4.3.1, 4.4.1], and let $u_1 \in \mathcal{U}((B/J_m)^\dagger)$ be a lift of v_1. Increasing m if necessary, we may choose u_1 so that $u_1^* g_{11} u_1 = h_{11}$ (2.5).

If projections $h_{11}, \ldots, h_{kk} \in B/J_m$ and unitaries $v_1, \ldots, v_k \in (A \otimes \mathbb{K})^\dagger$ have been defined, with lifts $u_1, \ldots, u_k \in (B/J_m)^\dagger$, set $q = \sum_{j=1}^k 1 \otimes e_{jj}$ and $p = \sum_{j=1}^k h_{jj}$. We have that

$$v_k^* \cdots v_2^* v_1^* f_{k+1,k+1} v_1 v_2 \cdots v_k$$

is orthogonal to q. Let v_{k+1} be a unitary in the connected component of the identity in $\mathcal{U}((A \otimes \mathbb{K})^\dagger)$ with $v_{k+1} q = q v_{k+1} = q$ and

$$v_{k+1}^* v_k^* \cdots v_1^* f_{k+1,k+1} v_1 \cdots v_k v_{k+1} = 1 \otimes e_{k+1,k+1}$$

and let $u_{k+1} \in \mathcal{U}((B/J_m)^\dagger)$ be a lift of v_{k+1} with $u_{k+1}p = pu_{k+1} = p$ (3.1). Then

$$h_{k+1,k+1} = u_{k+1}^* u_k^* \cdots u_1^* g_{k+1,k+1} u_1 \cdots = u_k u_{k+1}$$

is a lift of $1 \otimes e_{k+1,k+1}$ to a projection in B_m orthogonal to $h_{11}, \ldots, h_{kk}$.

Now for each k let $w_k = g_{kk} u_k u_{k-1} \cdots u_1$. Then w_k is a partial isometry in B/J_m with $w_k^* w_k = h_{kk}$ and $w_k w_k^* = g_{kk}$. Set $z_k = w_k^* g_{k1} w_1$; then z_k is a partial isometry with $z_k^* z_k = h_{11}$ and $z_k z_k^* = h_{kk}$. We have that $(1 \otimes e_{1k})\pi(z_k)$ is a unitary y_k in $A \otimes e_{11} \cong A$, and if $h_{k1} = z_k \psi(y_k)^*$, then h_{k1} is a partial isometry in B/J_m from h_{11} to h_{kk} which is a lift of $1 \otimes e_{k1}$. For each i, j, set $h_{ij} = h_{i1} h_{j1}^*$; then the $\{h_{ij}\}$ are the desired lifts of $\{1 \otimes e_{ij}\}$. Q.E.D.

This result is false if A is stably finite: for example, $\mathbb{K} \cong \mathbb{C} \otimes \mathbb{K}$ is not semiprojective, as is easily seen from 2.9 (or 2.14). In fact, a partial converse to 4.1 (a full converse, at least stably, in the simple unital case) is a special case of the next result.

Proposition 4.2. *Let A be a nonunital semiprojective C^*-algebra. If A has an approximate identity of projections, then A contains an infinite projection.*

Proof. This follows easily from 2.9. Let $\{p_n\}$ be a strictly increasing approximate unit of projections in A. Then $A \cong \lim_\to p_n A p_n$, and so the identity map on A is homotopic to a homomorphism through $p_n A p_n$ for some n. In particular, p_{n+1} is homotopic, hence equivalent, to a subprojection of p_n. Q.E.D.

Corollary 4.3. *Let A be a (separable) unital C^*-algebra. If $A \otimes \mathbb{K}$ is semiprojective, then A is not stably finite.*

This result is probably not the best possible; in fact, $\mathcal{T} \otimes \mathbb{K}$ is not semiprojective (if it were, a nontrivial homomorphism from $\mathcal{T} \otimes \mathbb{K}$ to $M_n(\mathcal{T})$, and hence to $\mathbb{M}_n$, could be constructed for some n by 2.9), and the full converse (stably) of 4.1 may well hold.

Note that if A is unital and $A \otimes \mathbb{K}$ is semiprojective, then A is semiprojective (2.7). This should be true even if A is nonunital. In fact, the following conjecture seems likely:

Conjecture 4.4. Let A be a semiprojective C*-algebra. Then any full corner in A is also semiprojective.

To prove this, it would suffice (and be essentially equivalent) to prove:

Conjecture 4.5. Let $0 \to A \to B \to \mathbb{C} \to 0$ be a split exact sequence of separable C*-algebras. If A is semiprojective, then so is B.

It is plausible, but less clear, that a full hereditary C*-subalgebra of a semiprojective C*-algebra should be semiprojective. Fullness is essential: $\mathcal{T}$ has a hereditary C*-subalgebra (closed ideal) isomorphic to $\mathbb{K}$.

If it is true that full hereditary C*-subalgebras of semiprojective C*-algebras are semiprojective, then a stably finite semiprojective simple C*-algebra must be nearly projectionless: by 4.2, such a C*-algebra could not contain a strictly increasing or decreasing sequence of projections.

It is quite conceivable that there could be semiprojective simple C*-algebras which are projectionless. However, 2.14 strongly suggests that there cannot be such examples which are nuclear. Some evidence is described in the next section to suggest that semiprojective simple C*-algebras are at least exact, if not nuclear; thus there is some modest evidence for a positive answer to the following question:

Question 4.6. Is every semiprojective simple C*-algebra either a finite-dimensional matrix algebra, or a Kirchberg algebra or stabilized Kirchberg algebra with finitely generated K-theory?

Recall that every hereditary C*-subalgebra of a purely infinite simple C*-algebra is either unital or stable [Zha90].

§5. Exactness of Semiprojective Simple C*-Algebras

It is quite possible that every simple semiprojective C*-algebra is C*-exact. Recall that a C*-algebra A is (C*-)exact if forming the minimal tensor products by A preserves exact sequences, and that a C*-subalgebra of an exact C*-algebra is exact [Kir94]. If $\mathbb{F}_2$ is the free group on two generators, then $C^*(\mathbb{F}_2)$ is not exact [Was76], hence cannot be embedded in an exact C*-algebra.

Conjecture 5.1. Every semiprojective simple C*-algebra embeds in O_2 and is therefore exact.

Note that by [KP00b] every separable exact C*-algebra embeds into O_2, so the first conclusion follows from the second. However, the likely proof of 5.1 would show exactness by directly embedding A into O_2, using the following conjecture of Kirchberg, for which there seems to be good evidence:

Conjecture 5.2. Every separable C*-algebra embeds in the corona algebra

$$\left(\prod O_2\right) / \left(\bigoplus O_2\right),$$

the quotient of the bounded sequences in O_2 by the sequences converging to 0.

This conjecture can be slightly modified by replacing the corona algebra with an ultrapower of O_2. In fact, it seems likely that $\mathcal{B}(\mathcal{H})$ for separable $\mathcal{H}$ embeds in these corona algebras or ultraproducts.

To prove 5.1 from 5.2, let $B = \prod O_2$, and J_n the sequences in B which are 0 after the n'th term. Then $[\cup J_n]^- = \oplus O_2$, and $B/J_n \cong \prod O_2$. Let ϕ be an embedding of A into the corona algebra, and partially lift ϕ to an embedding of A into $\prod O_2$. Since A is simple, composing this embedding with a suitable coordinate projection gives an embedding of A into O_2.

Note that a nonsimple semiprojective C*-algebra, e.g. $C^*(\mathbb{F}_2)$, need not be C*-exact. This proof of 5.1 would show that any semiprojective C*-algebra embeds into $\prod O_2$. A separable simple C*-algebra which is not exact cannot embed into $\prod O_2$, although it would embed in the corona algebra if 5.2 is true; such a C*-algebra can be constructed by embedding $C^*(\mathbb{F}_2)$ into the hyperfinite II_1 factor and applying [Bla78, 2.2].

§6. Finite Group Actions

One might hope (or expect) from 2.7 that a crossed product of a semiprojective C*-algebra by a finite group, or more generally a subalgebra of finite Jones index in a semiprojective C*-algebra, would be semiprojective, since these operations are the "square root" of a Morita equivalence. However, we will give examples here of $\mathbb{Z}_2$-actions on a semiprojective C*-algebra such that the crossed product is not semiprojective.

In fact, we show that there are $\mathbb{Z}_2$-actions on O_2 such that the crossed product, which is a Kirchberg algebra, does not have finitely generated K-theory. It is a bit surprising that the crossed product can even have nontrivial K-theory, since O_2 is K-contractible and thus K-theoretically trivial. The first example of a symmetry of O_2 such that the crossed product has nontrivial K-theory appeared in [CE81]. This gives yet another indication that $\mathbb{Z}_2$-actions can be badly behaved from a K-theoretic point of view; cf. [Bla98, 10.7], [Bla90], [Ell95]. It may even be true that the bootstrap class for the Universal Coefficient Theorem is not closed under crossed products by $\mathbb{Z}_2$ (in fact, this appears to be equivalent to the question of whether every separable nuclear C*-algebra is in the bootstrap class).

We will use the next example, which is a special case of [Bla90, 6.3.3].

Proposition 6.1. *Let B be the UHF algebra with $K_0(B) \cong \mathbb{Q}$. Then there is a symmetry σ of B such that, if $D = B \times_\sigma \mathbb{Z}_2$, then $K_0(D) \cong \mathbb{Q}$ and $K_1(D)$ is the dyadic rationals $\mathbb{D}$.*

Theorem 6.2. *Let G_0 and G_1 be countable abelian torsion groups in which every element has odd order. Then there is a symmetry α of O_2 such that $K_n(O_2 \times_\alpha \mathbb{Z}_2) \cong G_n$ for $n = 0, 1$.*

Proof. By [ER95] there is a Kirchberg algebra A such that $K_0(A) \cong G_1$ and $K_1(A) \cong G_0$. If B is the UHF algebra of 6.1, then $A \otimes B$ has trivial K-theory by the Künneth theorem for tensor products ([Sch82], [Bla98, 23.1.3]), and hence is isomorphic to O_2 by [ER95]. Let α be the symmetry $id \otimes \sigma$ of $O_2 \cong A \otimes B$. Then $O_2 \times_\alpha \mathbb{Z}_2$ is isomorphic to $A \otimes D$ (6.1), so $K_0(O_2 \times_\alpha \mathbb{Z}_2) \cong G_0$ and $K_1(O_2 \times_\alpha \mathbb{Z}_2) \cong G_1$ by the Künneth Theorem. Q.E.D.

Corollary 6.3. *There is a simple semiprojective C*-algebra A and a symmetry α of A such that $A \times_\alpha \mathbb{Z}_2$ is not semiprojective.*

Proof. In 6.2, choose G_0 or G_1 to be not finitely generated (i.e. not finite.) Apply 2.10. Q.E.D.

It would be interesting to study and classify symmetries of O_2 (and, more generally, finite group actions on Kirchberg algebras, or subalgebras of finite index in Kirchberg algebras). It is worth noting that 6.2 exhausts the possibilities for the K-theory of the crossed product of O_2 by a symmetry if these K-groups are torsion groups:

Proposition 6.4. *Let α be a symmetry of O_2. Then the K-groups of $O_2 \times_\alpha \mathbb{Z}_2$ are 2-divisible countable abelian groups with no 2-torsion.*

Proof. The proof is very similar to the arguments in [Bla90, 2.2.1]. The groups are countable because $O_2 \times_\alpha \mathbb{Z}_2$ is separable. For the rest, first note that $K_*(O_2 \times_\alpha \mathbb{Z})$ is trivial by the Pimsner-Voiculescu exact sequence [Bla98, 10.2.1]. It then follows from the exact sequence of [Bla98, 10.7.1] that $1 - \hat{\alpha}_* : K_*(O_2 \times_\alpha \mathbb{Z}_2) \to K_*(O_2 \times_\alpha \mathbb{Z}_2)$ is an isomorphism. Then from $0 = 1 - \hat{\alpha}_*^2 = (1 + \hat{\alpha}_*)(1 - \hat{\alpha}_*)$ it follows that $1 + \hat{\alpha}_* = 0$, $\hat{\alpha}_* = -1$, so $1 - \hat{\alpha}_*$ is multiplication by 2. Q.E.D.

Question 6.5. Is the K-theory of the crossed product by $\mathbb{Z}_2$ a complete outer conjugacy or stable conjugacy invariant for symmetries of O_2?

Example 6.6. The K-theory, or even the isomorphism class, of the crossed product is not a complete conjugacy invariant for symmetries of O_2. If α is the symmetry of O_2 which sends each generator to its negative, then the fixed-point algebra is isomorphic to O_4; but if β is the stabilized version of α, i.e. $\beta = ad\, diag(1,-1) \otimes \alpha$ on $\mathbb{M}_2 \otimes O_2 \cong O_2$, then the fixed-point algebra of β is isomorphic to $M_3(O_4)$, so α and β are not conjugate. If $\gamma = id \otimes \alpha$ on $\mathbb{M}_3 \otimes O_2 \cong O_2$, then the fixed-point algebra of γ is also $M_3(O_4)$. We do not know if β and γ are conjugate or outer conjugate. The crossed products of O_2 by each of these symmetries is isomorphic to $M_3(O_4)$.

Note Added in Proof.

J. Spielberg [Spi] has recently shown that Conjecture 3.6 is true if $K_1(A)$ is torsion-free. This includes the test case of P_∞, as well as the examples of 3.5 and [Szy]. The question remains open for such Kirchberg algebras as $O_n \otimes O_n$.

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On quasidiagonal C^*-algebras

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Abstract.

We give a detailed survey of the theory of quasidiagonal C^*-algebras. The main structural results are presented and various functorial questions around quasidiagonality are discussed. In particular we look at what is currently known (and not known) about tensor products, quotients, extensions, free products, etc. of quasidiagonal C^*-algebras. We also point out how quasidiagonality is connected to some important open problems.

§1. Introduction

Quasidiagonal C^*-algebras have now been studied for more than 20 years. They are a large class of algebras which arise naturally in many contexts and include many of the basic examples of finite C^*-algebras. Notions around quasidiagonality have also played an important role in BDF/KK-theory and are connected to some important open questions. For example, whether every nuclear C^*-algebra satisfies the Universal Coefficient Theorem, Elliott's Classification Program and whether or not $Ext(C_r^*(\mathbb{F}_2))$ is a group.

In these notes we give a detailed survey of the basic theory of quasidiagonal C^*-algebras. At present there is only one survey article in the literature which deals with this subject (cf. [Vo4]). While there is certainly overlap between this article and [Vo4], the focus of the present paper is quite different. We will spend a fair amount of time giving detailed proofs of a number of basic facts about quasidiagonal C^*-algebras. Some of these results have appeared in print, some are well known to the experts but have not (explicitly) appeared in print and some of them are new. Moreover, there have been a number of important advances since the writing of [Vo4]. We will not give proofs of most of the more difficult recent results. However, we have tried to at least give precise statements of these results and have included an extensive bibliography so that the interested reader may track down the original papers.

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In this paper we are primarily concerned with basic structural questions. In particular this means that many interesting topics have been left out or only briefly touched upon. For example, we do not explore the connections between (relative) quasidiagonality and BDF/KK-theory (found in the work of Salinas, Kirchberg, S. Wassermann, Dădărlat-Eilers, Schochet and others) or the generalized inductive limit approach (introduced by Blackadar and Kirchberg). But, for the interested reader, we have included a section containing references to a number of these topics.

Throughout the main body of these notes we will only be concerned with separable C^*-algebras and representations on separable Hilbert spaces. It turns out that one can usually reduce to this case so we don't view this as a major loss of generality. However, it causes one problem in that certain nonseparable C^*-algebras naturally arise in the (separable) theory. Hence we have included an appendix which deals with the nonseparable case.

A brief overview of this paper is as follows. In section 2 we collect a number of facts that will be needed in the rest of these notes. This is an attempt to keep the paper self contained, but these results include some of the deepest and most important tools in C^*-algebra theory and no proofs of well known results are given.

Section 3 contains the definitions and some basic properties of quasidiagonal operators, quasidiagonal sets of operators and quasidiagonal (QD) C^*-algebras. We also give some examples of QD and non-QD C^*-algebras. We end the section with the well known fact that quasidiagonality implies stable finiteness.

In section 4 we prove the abstract characterization of QD C^*-algebras which is due to Voiculescu.

Section 5 deals with the local approximation of QD C^*-algebras. We show that every such algebra can be locally approximated by a residually finite dimensional algebra. We also state a result of Dădărlat showing that every exact QD C^*-algebra can be locally approximated by finite dimensional C^*-algebras.

Section 6 contains the simple fact that every unital QD C^*-algebra has a tracial state.

Sections 7 - 11 deal with how quasidiagonality behaves under some of the standard operator algebra constructions. Section 7 discusses the easiest of these questions. Namely what happens when taking subalgebras, direct products and minimal tensor products of QD C^*-algebras. Quotients of QD algebras are treated in section 8, inductive limits in section 9, extensions in section 10 and crossed products in section 11.

Section 12 discusses the relationship between quasidiagonality and nuclearity. We state a result of Popa which led many experts to believe that simple QD C^*-algebras with 'sufficiently many projections' are always nuclear. We then state a result of Dădărlat which shows that this is not the case. We also discuss a certain converse to this question and it's relationship to the classification program. Namely the question (due to Blackadar and Kirchberg) of whether every nuclear stably finite C^*-algebra must be QD.

Section 13 contains miscellaneous results which didn't quite fit anywhere else. We state results of Boca and Voiculescu which concern full free products and homotopy invariance, respectively. We observe that all projective algebras and semiprojective MF algebras must be residually finite dimensional. Finally, we discuss how quasidiagonality relates to the question of when the classical BDF $Ext(\cdot)$ semigroups are actually groups and the question of whether all nuclear C^*-algebras satisfy the Universal Coefficient Theorem.

In section 14 we point out where the interested reader can go to learn more about some of the things that are not covered thoroughly here.

Finally at the end we have an appendix which treats the case of nonseparable QD C^*-algebras. The main result being that a C^*-algebra is QD if and only if all of it's separable C^*-subalgebras are QD.

§2. Preliminaries

Central to much of what will follow is the theory of *completely positive* maps. We refer the reader to [Pa] for a comprehensive treatment of these important maps. Perhaps the single most important result about these maps is Stinespring's Dilation Theorem (cf. [Pa, Thm. 4.1]). We will not state the most general version; for our purposes the following result will suffice.

Theorem 2.1. *(Stinespring) Let A be a unital separable C^*-algebra and $\varphi : A \to B(H)$ be a unital completely positive map. Then there exists a separable Hilbert space K, an isometry $V : H \to K$ and a unital representation $\pi : A \to B(K)$ such that $\varphi(a) = V^*\pi(a)V$ for all $a \in A$.*

Throughout most of [Pa], only the unital case is treated. The following result shows that this is not a serious problem.

Proposition 2.2. *(cf. [CE2, Lem. 3.9]) Let $\varphi : A \to B$ be a contractive completely positive map. Then the unique unital extension $\tilde{\varphi} : \tilde{A} \to \tilde{B}$ is also completely positive, where $\tilde{A}$, $\tilde{B}$ are the C^*-algebras obtained by adjoining new units.*

Another fundamental result concerning completely positive maps is Arveson's Extension Theorem. To state the theorem we need the following definition.

Definition 2.3. Let A be a unital C^*-algebra and $X \subset A$ be a closed linear subspace. Then X is called an *operator system* if $1_A \in X$ and $X = X^*$.

Theorem 2.4. *(Arveson's Extension Theorem) If A is a unital C^*-algebra, $X \subset A$ is an operator system and $\varphi : X \to C$ is a contractive completely positive map with $C = B(H)$ or $dim(C) < \infty$ then there exists a completely positive map $\Phi : A \to C$ which extends φ (i.e. $\Phi|_X = \varphi$). If X is a C^*-subalgebra of A then there always exists a unital completely positive extension of φ (whether or not X contains the unit of A).*

A proof of the unital statement above can be found in [Pa] while the nonunital statement is due to Lance [La1, Thm. 4.2].

Representations of quasidiagonal C^*-algebras will be important in what follows and hence we will need Voiculescu's Theorem (cf. [Vo1]). In fact, we will need a number of different versions of this result. It will be convenient to have Hadwin's formulation in terms of rank.

Definition 2.5. If $T \in B(H)$ then let $rank(T) = dim(\overline{TH})$.

Theorem 2.6. *Let A be a unital C^*-algebra and $\pi_i : A \to B(H_i)$ be unital *-representations for $i = 1, 2$. Then there exists a net of unitaries $U_\lambda : H_1 \to H_2$ such that $\|\pi_2(a) - U_\lambda \pi_1(a) U_\lambda^*\| \to 0$ for all $a \in A$ if and only if $rank(\pi_1(a)) = rank(\pi_2(a))$ for all $a \in A$. If A is nonunital then there exists such a net of unitaries if and only if $rank(\pi_1(a)) = rank(\pi_2(a))$ for all $a \in A$ and $dim(H_1) = dim(H_2)$.*

When such unitaries exist we say that π_1 and π_2 are *approximately unitarily equivalent.* When both A and the underlying Hilbert spaces are separable one can even arrange the stronger condition that $\pi_2(a) - U_n \pi_1(a) U_n^*$ is a compact operator for each $a \in A$, $n \in \mathbb{N}$ (of course, we can take a sequence of unitaries when A is separable). When this is the case we say that π_1 and π_2 are *approximately unitarily equivalent modulo the compacts.* A proof of this stronger (in the separable case) result can be found in [Dav, Thm. II.5.8] or a proof of the general result can be found in [Had1].

It turns out that one can usually reduce to the case of separable C^*-algebras and Hilbert spaces. In this case, the following version of Voiculescu's theorem will be convenient (cf. [Dav, Cor. II.5.5]).

Theorem 2.7. *Let H be a separable Hilbert space and $C \subset B(H)$ be a unital separable C^*-algebra such that $1_H \in C$. Let $\iota : C \hookrightarrow B(H)$ denote the canonical inclusion and let $\rho : C \to B(K)$ be any unital representation such that $\rho(C \cap \mathcal{K}(H)) = 0$. Then ι is approximately unitarily equivalent modulo the compacts to $\iota \oplus \rho$.*

We will be particularly interested in the case that $C \cap \mathcal{K}(H) = 0$.

Definition 2.8. Let $\pi : A \to B(H)$ be a faithful representation of a C^*-algebra A. Then π is called *essential* if $\pi(A)$ contains no nonzero finite rank operators.

Corollary 2.9. *Let A be a separable C^*-algebra and $\pi_i : A \to B(H_i)$ be faithful essential representations with H_i separable for $i = 1, 2$. If A is unital and both π_1, π_2 are unital then π_1 and π_2 are approximately unitarily equivalent modulo the compacts. If A is nonunital then π_1 and π_2 are always approximately unitarily equivalent modulo the compacts.*

We will need one more form of Voiculescu's Theorem. We have not been able to find the following version written explicitly in the literature. However, the main idea is essentially due to Salinas (see the proofs of [Sa1, Thm. 2.9] and [DHS, Thm. 4.2]).

If A is a separable, unital C^*-algebra and $\varphi : A \to B(H)$ (with H separable and infinite dimensional) is a unital completely positive map then we say that φ is a *representation modulo the compacts* if $\pi \circ \varphi : A \to Q(H)$ is a *-homomorphism, where π is the quotient map onto the Calkin algebra. If $\pi \circ \varphi$ is injective then we say that φ is a *faithful representation modulo the compacts.* In this situation we define constants $\eta_\varphi(a)$ by

$$\eta_\varphi(a) = 2\max(\|\varphi(a^*a) - \varphi(a^*)\varphi(a)\|^{1/2}, \|\varphi(aa^*) - \varphi(a)\varphi(a^*)\|^{1/2})$$

for every $a \in A$.

Theorem 2.10. *Let A be a separable, unital C^*-algebra and $\varphi : A \to B(H)$ be a faithful representation modulo the compacts. If $\sigma : A \to B(K)$ is any faithful, unital, essential representation then there exist unitaries $U_n : H \to K$ such that*

$$\limsup_{n\to\infty} \|\sigma(a) - U_n\varphi(a)U_n^*\| \le \eta_\varphi(a)$$

for every $a \in A$.

Proof. Note that by Corollary 2.9 it suffices to show that *there exists* a representation σ satisfying the conclusion of the theorem since all such representations are approximately unitarily equivalent.

Let $\rho : A \to B(L)$ be the Stinespring dilation of φ; i.e. ρ is a unital representation of A and there exists an isometry $V : H \to L$ such that $\varphi(a) = V^*\rho(a)V$, for all $a \in A$. Let $P = VV^* \in B(L)$ and $P^\perp = 1_L - P$. We claim that for every $a \in A$,

$$\|P^\perp \rho(a) P\| \leq \|\varphi(a^*a) - \varphi(a^*)\varphi(a)\|^{1/2}.$$

This follows from a simple calculation:

$$\begin{aligned}(P^\perp\rho(a)P)^*(P^\perp\rho(a)P) &= P\rho(a^*)P^\perp\rho(a)P \\ &= VV^*\rho(a^*a)VV^* - VV^*\rho(a^*)VV^*\rho(a)VV^* \\ &= V\big(\varphi(a^*a) - \varphi(a^*)\varphi(a)\big)V^*.\end{aligned}$$

Now write $L = PL \oplus P^\perp L$ and decompose the representation ρ accordingly. That is, consider the matrix decomposition

$$\rho(a) = \begin{pmatrix} \rho(a)_{11} & \rho(a)_{12} \\ \rho(a)_{21} & \rho(a)_{22} \end{pmatrix},$$

where $\rho(a)_{21} = P^\perp\rho(a)P$ and $\rho(a)_{12} = \rho(a^*)^*_{21}$. Hence the norm of the matrix

$$\begin{pmatrix} 0 & \rho(a)_{12} \\ \rho(a)_{21} & 0 \end{pmatrix}$$

is bounded above by $1/2\eta_\varphi(a)$ because of

$$\|P^\perp \rho(a) P\| \leq \|\varphi(a^*a) - \varphi(a^*)\varphi(a)\|^{1/2}.$$

Now comes the trick. We consider the Hilbert space $P^\perp L \oplus PL$ and the representation $\rho' : A \to B(P^\perp L \oplus PL)$ given in matrix form as

$$\rho'(a) = \begin{pmatrix} \rho(a)_{22} & \rho(a)_{21} \\ \rho(a)_{12} & \rho(a)_{11} \end{pmatrix}.$$

Now using the obvious identification of the Hilbert spaces

$$PL \oplus \Big(\bigoplus_{\mathbb{N}} P^\perp L \oplus PL\Big) \quad \text{and} \quad \bigoplus_{\mathbb{N}} L = \bigoplus_{\mathbb{N}} (PL \oplus P^\perp L)$$

a standard calculation shows that

$$\|\rho(a)_{11} \oplus \rho'^\infty(a) - \rho^\infty(a)\| \leq \eta_\varphi(a)$$

for all $a \in A$, where $\rho'^\infty = \oplus_{\mathbb{N}}\rho'$ and $\rho^\infty = \oplus_{\mathbb{N}}\rho$. Note also that $\rho(a)_{11} = V\varphi(a)V^*$.

Now, let C be the linear space $\varphi(A) + \mathcal{K}(H)$. Note that C is actually a separable, unital C^*-subalgebra of $B(H)$ with $\pi(C) = A$ where $\pi : B(H) \to Q(H)$ is the quotient map onto the Calkin algebra. By Theorem 2.7 we have that $\iota \oplus \rho'^{\infty} \circ \pi$ is approximately unitarily equivalent modulo the compacts to ι, where $\iota : C \hookrightarrow B(H)$ is the inclusion. Let $W_n : H \to H \oplus (\oplus_{\mathbb{N}}(P^{\perp}L \oplus PL))$ be unitaries such that

$$\|\varphi(a) \oplus \rho'^{\infty}(a) - W_n \varphi(a) W_n^*\| \to 0$$

for all $a \in A$.

We now let

$$\tilde{V} : H \oplus (\bigoplus_{\mathbb{N}}(P^{\perp}L \oplus PL)) \to \bigoplus_{\mathbb{N}} L$$

be the unitary $V \oplus 1$ (again using the obvious identification of $PL \oplus \big(\oplus_{\mathbb{N}} (P^{\perp}L \oplus PL)\big)$ and $\oplus_{\mathbb{N}} L$). Note that $\tilde{V}(\varphi(a) \oplus \rho'^{\infty}(a))\tilde{V}^* = V\varphi(a)V^* \oplus \rho'^{\infty}(a) = \rho(a)_{11} \oplus \rho'^{\infty}(a)$. We now complete the proof by defining

$$K = \oplus_{\mathbb{N}} L, \quad \sigma = \rho^{\infty} = \oplus_{\mathbb{N}} \rho, \quad U_n = \tilde{V} W_n : H \to \oplus_{\mathbb{N}} L = K. \qquad \text{Q.E.D.}$$

Finally, we will need a basic result concerning quotient maps of locally reflexive C^*-algebras. The notion of local reflexivity in the category of C^*-algebras was first introduced by Effros and Haagerup (cf. [EH]).

Definition 2.11. A unital C^*-algebra A is called *locally reflexive* if each unital completely positive map $\varphi : X \to A^{**}$ is the limit (in the point-weak* topology) of a net of unital completely positive maps $\varphi_\lambda : X \to A$, where X is an arbitrary finite dimensional operator system and A^{**} denotes the enveloping von Neumann algebra of A.

Definition 2.12. Let $\pi : A \to B$ be a surjective *-homomorphism with A unital. Then π is called *locally liftable* if for each finite dimensional operator system $X \subset B$ there exists a unital completely positive map $\varphi : X \to A$ such that $\pi \circ \varphi = id_X$.

Of course, if either A or B is *nuclear* then the Choi-Effros Lifting Theorem (cf. [CE2, Thm. 3.10]) implies that π is more than just locally liftable; there then exists a completely positive splitting defined on all of B. However, local liftability is usually all we will need. The following result will be used several times and is a consequence of [EH; 3.2, 5.1, 5.3 and 5.5].

Theorem 2.13. *Let $0 \to I \xrightarrow{\iota} E \xrightarrow{\pi} B \to 0$ be an exact sequence with E unital. Then E is locally reflexive if and only if both I and B are locally reflexive and the morphism π is locally liftable.*

Local reflexivity plays an important role in the theory of operator spaces. We will not need any more results about local reflexivity. However, we do wish to point out the following implications:

$$\textit{Nuclear} \Longrightarrow \textit{Exact} \Longrightarrow \textit{Locally Reflexive}.$$

These results (together with the definitions of nuclear and exact C^*-algebras) can essentially be found in S. Wassermann's monograph [Wa3]. ([Wa3] Propositions 5.5 and 5.4 give the first implication while [Wa3, Remark 9.5.2] states that exactness is equivalent to property C of Archbold and Batty. However, property C implies property C$''$, as defined in [EH, pg. 120], which in turn is equivalent to local reflexivity by [EH, Thm. 5.1].) Since the pioneering work of E. Kirchberg, exactness has played a central role in C^*-algebras. However, since we will only need the local liftability statement of Theorem 2.13, we will also consider the class of locally reflexive C^*-algebras. (It is not known if this is really a larger class of algebras – i.e. it is not known if every locally reflexive C^*-algebra is exact [Kir2,pg. 75].)

§3. Definitions, Basic Results and Examples

Recall that throughout the body of these notes all Hilbert spaces and C^*-algebras are assumed to be separable.

We begin this section by recalling the notions of *block diagonal* and *quasidiagonal* operators on a Hilbert space. In Proposition 3.4 we show that the notion of a quasidiagonal operator can be expressed in terms of a local finite dimensional approximation property. This local version then extends to a suitable definition of a quasidiagonal (QD) C^*-algebra (Definition 3.8). In Theorem 3.11 we prove a fundamental result about representations of QD C^*-algebras. At the end of this section we give some examples of QD (and non-QD) C^*-algebras and observe that QD C^*-algebras are always *stably finite* (cf. Proposition 3.19).

Definition 3.1. A bounded linear operator D on a Hilbert space H is called *block diagonal* if there exists an increasing sequence of finite rank projections, $P_1 \leq P_2 \leq P_3 \cdots$, such that $\|[D, P_n]\| = \|DP_n - P_nD\| = 0$ for all $n \in \mathbb{N}$ and $P_n \to 1_H$ (in the strong operator topology) as $n \to \infty$.

Note that if $\|[D, P_n]\| = 0$ then $\|[D, (P_n - P_{n-1})]\| = 0$ as well. Thus the matrix for D with respect to the decomposition $H = P_1H \oplus (P_2 - P_1)H \oplus (P_3 - P_2)H \oplus \cdots$ is block diagonal.

The notion of a quasidiagonal operator is due to Halmos and is a natural generalization of a block diagonal operator.

Definition 3.2. A bounded linear operator T on a Hilbert space H is called *quasidiagonal* if there exists an increasing sequence of finite rank projections, $P_1 \leq P_2 \leq P_3 \cdots$, such that $\|[T, P_n]\| = \|TP_n - P_nT\| \to 0$ and $P_n \to 1_H$ (in the strong operator topology) as $n \to \infty$.

Halmos observed the following relationship between quasidiagonal and block diagonal operators.

Proposition 3.3. *If $T \in B(H)$ then T is quasidiagonal if and only if there exist a block diagonal operator $D \in B(H)$ and a compact operator $K \in \mathcal{K}(H)$ such that $T = D + K$.*

We will not give the proof of this proposition here as it is a special case of Theorem 5.2. Note, however, that one direction is easy. Namely, if $T = D+K$ as above then T must be quasidiagonal since any increasing sequence of finite rank projections converging to 1_H (s.o.t.) will form an approximate identity for $\mathcal{K}(H)$ and hence will asymptotically commute with every compact operator.

It is an important fact that the seemingly global notion of quasidiagonality can be expressed in a local way.

Proposition 3.4. *Let $T \in B(H)$. Then T is quasidiagonal if and only if for each finite set $\chi \subset H$ and $\varepsilon > 0$ there exists a finite rank projection $P \in B(H)$ such that $\|[T, P]\| \leq \varepsilon$ and $\|P(x) - x\| \leq \varepsilon$ for all $x \in \chi$.*

Proof. We may assume that $\|T\| \leq 1$. It is clear that the definition of a quasidiagonal operator implies the condition stated above. To prove the converse, it suffices to show that for each finite set $\chi \subset H$ and $\varepsilon > 0$ there exists a finite rank projection P such that $\|[P, T]\| < \varepsilon$ and $P(x) = x$ for all $x \in \chi$. Having established this it is not hard to construct finite rank projections $P_1 \leq P_2 \leq P_3 \cdots$, such that $\|[T, P_n]\| \to 0$ and $P_n \to 1_H$ in the strong operator topology.

So let $\chi \subset H$ be a finite set, $\varepsilon > 0$ and let R be the orthogonal projection onto $K = span\{\chi\}$. By compactness of the unit ball of K there is a finite set $\tilde{\chi} \subset K$ which is ε-dense in the unit ball of K. Now let Q be a finite rank projection such that $\|[Q, T]\| < \varepsilon$ and $\|Q(x) - x\| < \varepsilon$ for all $x \in \tilde{\chi}$. Then for all $y \in K$ we have $\|Q(y) - y\| < 3\varepsilon\|y\|$ and hence $\|(1 - R)QR\| < 3\varepsilon$.

Now consider the positive contraction $X = RQR + (1-R)Q(1-R)$. Observe that X is actually very close to Q:

$$\begin{aligned} \|Q - X\| &= \|RQ(1-R) + (1-R)QR\| \\ &= max\{ \|RQ(1-R)\|, \|(1-R)QR\| \} \\ &= \|(1-R)QR\| \\ &< 3\varepsilon. \end{aligned}$$

Hence X is almost a projection (i.e. it's spectrum is contained in $[0, 3\varepsilon) \cup (1-3\varepsilon, 1]$). Let P be the projection obtained from functional calculus on X. Then $\|P - Q\| \leq 6\varepsilon$ and hence $\|[P, T]\| \leq 13\varepsilon$. Finally we claim that $P(x) = x$ for all $x \in \chi$. To see this, first note that X commutes with R and hence so does P. This implies that $PR = RPR$ is a projection with support contained in K. However, for each $y \in K$ we also have $\|PR(y) - y\| = \|R(P(y) - y)\| \leq \|P(y) - Q(y)\| + \|Q(y) - y\| \leq 9\varepsilon\|y\|$. Hence the support of PR is all of K; i.e. $PR = R$. Q.E.D.

With this local characterization in hand we now define the following generalization of a quasidiagonal operator.

Definition 3.5. A subset $\Omega \subset B(H)$ is a called a *quasidiagonal set of operators* if for each finite set $\omega \subset \Omega$, finite set $\chi \subset H$ and $\varepsilon > 0$ there exists a finite rank projection $P \in B(H)$ such that $\|[T, P]\| \leq \varepsilon$ and $\|P(x) - x\| \leq \varepsilon$ for all $T \in \omega$ and $x \in \chi$.

It is easy to see that a set $\Omega \subset B(H)$ is a quasidiagonal set of operators if and only if the C^*-algebra generated by Ω, $C^*(\Omega) \subset B(H)$, is a quasidiagonal set of operators.

The proof of the next proposition is a straightforward adaptation of the proof of Proposition 3.4.

Proposition 3.6. *If $A \subset B(H)$ is separable then A is a quasidiagonal set of operators if and only if there exists an increasing sequence of finite rank projections, $P_1 \leq P_2 \leq P_3 \cdots$, such that for all $a \in A$, $\|[a, P_n]\| \to 0$ and $P_n \to 1_H$ (s.o.t.) as $n \to \infty$.*

Remark 3.7. *The previous proposition is often used when defining quasidiagonal C^*-algebras. However, L. Brown has pointed out to us that the previous proposition is not true if A is not separable (even if H is separable). Definition 3.5 allows one to use Zorn's lemma to construct* maximal *quasidiagonal subsets of $B(H)$ and we claim that they provide counterexamples. The proof goes by contradiction. So assume that $\Omega \subset B(H)$ is a maximal quasidiagonal set of operators and there exist* *finite rank projections such that $\|[x, P_n]\| \to 0$ for all $x \in \Omega$. Construct a block diagonal operator T such that $[T, P_{2n}] = 0$ and $\|[T, P_{2n+1}]\| = 1$ for all $n \in \mathbb{N}$. Since $[T, P_{2n}] = 0$ for all n we see that $\Omega \cup \{T\}$ is a quasidiagonal set of operators and hence (by maximality) $T \in \Omega$. This gives the contradiction since $\|[T, P_{2n+1}]\| = 1$ for all $n \in \mathbb{N}$. Hence it is important to take Definition 3.5 for general quasidiagonal questions.*

We finally come to the definition of a quasidiagonal C^*-algebra.

Definition 3.8. Let A be a C^*-algebra. Then A is called *quasidiagonal* (QD) if there exists a faithful representation $\pi : A \to B(H)$ such that $\pi(A)$ is a quasidiagonal set of operators.

Some remarks regarding this definition are in order. First we should point out that some authors (e.g. [Had2]) refer to C^*-algebras satisfying Definition 3.8 as 'weakly quasidiagonal' C^*-algebras. There is good reason for this terminology as it emphasizes the distinction between abstract and concrete C^*-algebras. It is important to make this distinction since every C^*-algebra has a representation π such that $\pi(A)$ is a quasidiagonal set of operators (namely the zero representation). On the other hand, it is possible to give examples of C^*-algebras A and faithful representations $\pi : A \to B(H)$ such that A is QD but $\pi(A)$ is *not* a quasidiagonal set of operators. Perhaps the most extreme case of this is an example of L. Brown. In [BrL2] it was shown that there exists an operator T on a separable Hilbert space such that $T \oplus T$ is quasidiagonal while T is not! Hence $C^*(T)$ is a QD C^*-algebra but is *not* a quasidiagonal set of operators in it's natural representation. Thus it is indeed very important to distinguish between abstract QD C^*-algebras and concrete quasidiagonal sets of operators. (The reader is cautioned that this is not always done carefully in the literature.) Other authors prefer to say that a *representation* is quasidiagonal if it's image is a quasidiagonal set of operators. Definition 3.8 then becomes equivalent to the statement that A admits a faithful quasidiagonal representation.

Definitions 3.5 and 3.8 are the correct definitions in the nonseparable case as well (see the Appendix). We will see that certain nonseparable C^*-algebras (namely $\Pi M_n(\mathbb{C})$) naturally arise in the separable theory and hence it will be logically necessary to treat this case also.

Finally note that Definition 3.8 does not require the representation to be nondegenerate. Of course, this can always be arranged. Note, however, that this is actually a deep fact as the proof of Lemma 3.10 below depends on Voiculescu's Theorem (at least in the nonunital case).

Definition 3.9. If $\pi : A \to B(H)$ is a representation and $L \subset H$ is a $\pi(A)$-invariant subspace then $\pi_L : A \to B(L)$ denotes the restriction representation (i.e. $\pi_L(a) = P_L\pi(a)|_L$, where P_L is the orthogonal projection from $H \to L$).

Lemma 3.10. *Let $\pi : A \to B(H)$ be a faithful representation and $L \subset H$ be the nondegeneracy subspace of $\pi(A)$. Then $\pi(A)$ is a quasidiagonal set of operators if and only if $\pi_L(A)$ is a quasidiagonal set of operators.*

Proof. Assume first that $\pi_L(A)$ is a quasidiagonal set of operators. Then write $H = L \oplus \tilde{L}$. Since $\pi(a) = \pi_L(a) \oplus 0$, any finite rank projection $P \in B(L)$ can be extended to a finite rank projection $P \oplus \tilde{P}$ such that $\|[\pi(a), P \oplus \tilde{P}]\| = \|[\pi_L(a), P]\|$. From this one deduces that $\pi(A)$ must also be a quasidiagonal set of operators.

Now assume that $\pi(A)$ is a quasidiagonal set of operators. If A is unital then $\pi(1_A) = P_L$. If $R \in B(H)$ is any finite rank projection that almost commutes with $\pi(1_A) = P_L$ then $P_LRP_L \in B(L)$ is very close to a projection which does commute with P_L. We leave the details to the reader, but some standard functional calculus then implies that $\pi_L(A)$ is also a quasidiagonal set of operators.

In the case that $\pi(A)$ is a quasidiagonal set of operators and A is nonunital, we have to call on Voiculescu's Theorem (version 2.6). Since it is clear that $rank(\pi(a)) = rank(\pi_L(a))$ for all $a \in A$ we have that π and π_L are approximately unitarily equivalent. However, it is an easy exercise to verify that if ρ and $\tilde{\rho}$ are two approximately unitarily equivalent representations then $\rho(A)$ is a quasidiagonal set of operators if and only if $\tilde{\rho}(A)$ is a quasidiagonal set of operators. Q.E.D.

We now give the fundamental theorem on representations of QD C^*-algebras.

Theorem 3.11. *(cf. [Vo4, 1.7]) Let $\pi : A \to B(H)$ be a faithful* essential *(cf. Definition 2.8) representation. Then A is QD if and only if $\pi(A)$ is a quasidiagonal set of operators.*

Proof. If $\pi(A)$ is a quasidiagonal set of operators then, of course, A is QD. Conversely, if A is QD then there exists a faithful representation $\rho : A \to B(K)$ such that $\rho(A)$ is a quasidiagonal set of operators. In light of Lemma 3.10, we may assume that both π and ρ are nondegenerate. Defining $\rho_\infty = \oplus_{\mathbb{N}}\rho : A \to B(\oplus_{\mathbb{N}}K)$ it is easy to see that $\rho_\infty(A)$ is also a quasidiagonal set of operators. But, since ρ_∞ is an essential representation, Voiculescu's Theorem (version 2.9) implies that π and ρ_∞ are approximately unitarily equivalent. Hence $\pi(A)$ is also a quasidiagonal set of operators. Q.E.D.

We now give some examples of QD C^*-algebras and non-QD C^*-algebras.

Example 3.12. *Every commutative C^*-algebra is QD. Indeed, if $A = C_0(X)$ and for each $x \in X$ we let $ev_x : A \to \mathbb{C}$ be evaluation at x then $\pi = \oplus_{x \in F} ev_x$, where $F \subset X$ is a countable dense set, is a faithful representation and it is easy to see that $\pi(A)$ is a quasidiagonal (in fact, diagonal) set of operators.*

Example 3.13. *Approximately finite dimensional (AF) algebras are QD. Let $A = \overline{\cup_n A_n}$ be AF with each $A_n \subset A_{n+1}$ finite dimensional. Let $\pi : A \to B(H)$ be a faithful nondegenerate representation and write $H = \overline{\cup_n H_n}$ where each $H_n \subset H_{n+1}$ is a finite dimensional subspace. Then define $P_n \in B(H)$ to be the (finite rank) projection onto the subspace $\pi(A_n)H_n$. Then we evidently have that $\|[\pi(a), P_n]\| \to 0$ for all $a \in A$ and $P_n \to 1_H$ in the strong topology.*

Example 3.14. *Irrational rotation algebras are QD. That is, if A_θ is the universal C^*-algebra generated by two unitaries U, V subject to the relation $UV = (\exp(2\pi\theta i))VU$ for some irrational number $\theta \in [0, 1]$ then A_θ is QD. This was first proved by Pimsner and Voiculescu when they showed how to embed A_θ into an AF algebra (cf. [PV2]). This was later generalized by Pimsner in [Pi] (see also section 11 of these notes).*

Example 3.15. *Perhaps the most important class of QD C^*-algebras are the so-called residually finite dimensional (RFD) C^*-algebras. A C^*-algebra R is called RFD if for each $x \in R$ there exists a *-homomorphism $\pi : R \to B$ such that $dim(B) < \infty$ and $\pi(x) \neq 0$. That such algebras have a faithful representation whose image is a quasidiagonal (in fact, block diagonal) set of operators is proved similar to the case of abelian algebras. Often times general questions about QD C^*-algebras can be reduced to the case of RFD algebras.*

Example 3.16. *Both the cone ($CA = C_0((0, 1]) \otimes A$) and suspension ($SA = C_0((0, 1)) \otimes A$) over any C^*-algebra A are QD. Since $SA \subset CA$ and CA is homotopic to $\{0\}$, this can be deduced from the homotopy invariance of quasidiagonality (cf. [Vo3] or Theorem 13.1 of these notes). From this we see that every C^*-algebra is a quotient of a QD C^*-algebra (since $A \cong CA/SA$).*

Example 3.17. *A C^*-algebra which contains a proper (i.e. nonunitary) isometry is not QD. Since it is clear that a subalgebra of a QD C^*-algebra is again QD, it suffices to show that the Toeplitz algebra is not QD. (Recall that Coburn's Theorem states that the C^*-algebras generated by any two proper isometries are isomorphic.) We* *let $C^*(S)$ denote the Toeplitz algebra, where S is a proper isometry, and let $\pi : C^*(S) \to B(H)$ be any faithful unital essential representation. Then $\pi(S)$ is a semi-Fredholm operator with index $-\infty$. On the other hand, it follows from Proposition 3.3 that any semi-Fredholm quasidiagonal operator on H must have index zero (since any semi-Fredholm block diagonal operator must have index zero) and hence $\pi(S)$ is not a quasidiagonal operator. Hence, by Theorem 3.11, $C^*(S)$ is not QD. (See [Hal1] for generalizations of this result.)*

The previous example implies a more general result.

Definition 3.18. Let A be a unital C^*-algebra. Then A is said to be *stably finite* if $A \otimes M_n(\mathbb{C})$ contains no proper isometries for all $n \in \mathbb{N}$. If A is nonunital, then A is called *stably finite* if the unitization $\tilde{A}$ is stably finite.

Proposition 3.19. *QD C^*-algebras are stably finite.*

Proof. It is easy to see that if A is nonunital and QD then the unitization $\tilde{A}$ is also QD. Furthermore, it is a good exercise to verify that if A is QD then for all $n \in \mathbb{N}$, $M_n(\mathbb{C}) \otimes A$ is also QD. From these observations and Example 3.17 we see that if A is QD then $M_n(\mathbb{C}) \otimes A$ (or $M_n(\mathbb{C}) \otimes \tilde{A}$ in the non-unital case) has no proper isometries for all $n \in \mathbb{N}$. Hence A is stable finite. Q.E.D.

The converse is not true. S. Wassermann has given examples of non-QD MF algebras (cf. Definition 9.1 and Example 8.6 of these notes). But every MF algebras is stably finite (cf. [BK1, Prop. 3.3.8]). Hence, in general, QD is not equivalent to stably finite. However, Blackadar and Kirchberg have asked whether or not they are equivalent within the category of nuclear C^*-algebras (see Question 12.5).

§4. Voiculescu's Abstract Characterization

In this section we prove an abstract (i.e. representation free) characterization of QD C^*-algebras (cf. [Vo3, Thm. 1]). This fundamental result will be crucial in sections 8 - 10.

Consider the following property of an arbitrary C^*-algebra A.

$(*)$ For each finite set $\mathcal{F} \subset A$ and $\varepsilon > 0$ there exists a contractive completely positive map $\varphi : A \to B$ such that *i)* $dim(B) < \infty$, *ii)* $\|\varphi(x)\| \geq \|x\| - \varepsilon$ for all $x \in \mathcal{F}$ and *iii)* $\|\varphi(xy) - \varphi(x)\varphi(y)\| \leq \varepsilon$ for all $x, y \in \mathcal{F}$.

For a unital algebra A we have a related property.

$(**)$ For each finite set $\mathcal{F} \subset A$ and $\varepsilon > 0$ there exists a unital completely positive map $\varphi : A \to B$ such that *i)* $B \cong M_n(\mathbb{C})$ for some

$n \in \mathbb{N}$, *ii)* $\|\varphi(x)\| \geq \|x\| - \varepsilon$ for all $x \in \mathcal{F}$ and *iii)* $\|\varphi(xy) - \varphi(x)\varphi(y)\| \leq \varepsilon$ for all $x, y \in \mathcal{F}$.

We will refer to such maps as *ε-isometric* and *ε-multiplicative* on $\mathcal{F}$.

Lemma 4.1. *If A is a unital C^*-algebra then A satisfies $(*)$ if and only if A satisfies $(**)$.*

Proof. $(\Leftarrow)$ This is obvious.

$(\Rightarrow)$ We only sketch the main idea. First, we identify B with a unital subalgebra of $M_m(\mathbb{C}) = B(\mathbb{C}^m)$ for some $m \in \mathbb{N}$. Let 1_A denote the unit of A and let $P \in M_m(\mathbb{C})$ be the projection onto the range of $\varphi(1_A)$. Then one shows that $\varphi(a) = P\varphi(a) = \varphi(a)P$ for all $a \in A$. Moreover, if φ is very multiplicative on 1_A then $\varphi(1_A)$ is close to P.

Now let $\psi : A \to PM_m(\mathbb{C})P \cong M_n(\mathbb{C})$ (for some $n \leq m$) be given by $\psi(a) = P\varphi(a)P$ and clearly ψ has the same multiplicativity and isometric properties (up to ε) that φ does. Moreover, since $\varphi(1_A)$ is close to P, $\psi(1_A)$ is invertible in $PM_m(\mathbb{C})P \cong M_n(\mathbb{C})$. Thus, to get a unital complete positive map into a matrix algebra, we replace ψ with the map $a \mapsto (\psi(1_A))^{-1/2}\psi(a)(\psi(1_A))^{-1/2}$. The multiplicativity and isometric properties of this new map are not quite as good as those of φ, but they are good enough. Q.E.D.

We are now ready for Voiculescu's abstract characterization of QD C^*-algebras. Our proof is based on the proof of [DHS, Thm. 4.2] and, hopefully, is easier to follow than the original. However, the main ideas are the same. We have simply isolated the hard part in Theorem 2.10.

Theorem 4.2 (Voiculescu). *Let A be a C^*-algebra. Then A is QD if and only if A satisfies $(*)$.*

Proof. From Proposition 2.2 it is easy to see that A satisfies $(*)$ if and only if $\tilde{A}$ satisfies $(*)$. Similarly is it clear that A is QD if and only if $\tilde{A}$ is QD and hence we may assume that A is unital.

$(\Rightarrow)$ Let $\pi : A \to B(H)$ be a unital faithful essential representation on a separable Hilbert space. We can then find an increasing sequence of finite rank projections, $P_1 \leq P_2 \leq P_3 \cdots$, such that for all $a \in A$, $\|[\pi(a), P_n]\| \to 0$ and $P_n \to 1_H$ in the strong topology. Then for all n, $P_nB(H)P_n$ is isomorphic to a matrix algebra and the unital completely positive maps $\varphi_n : A \to P_nB(H)P_n$, $a \mapsto P_n\pi(a)P_n$ are easily seen to be asymptotically multiplicative and isometric.

$(\Leftarrow)$ By Lemma 4.1 we can find a sequence of unital completely positive maps $\varphi_i : A \to M_{n(i)}(\mathbb{C})$ which are asymptotically multiplicative and asymptotically isometric. Let

$$H_m = \bigoplus_{i=m}^{\infty} \mathbb{C}^{n(i)}, \quad \Phi_m = \bigoplus_{i=m}^{\infty} \varphi_i : A \to B(H_m).$$

Evidently each Φ_m is a faithful representation modulo the compacts (as in Theorem 2.10). Let $\sigma : A \to B(K)$ be any faithful, unital, essential representation and by Theorem 2.10 we can find unitaries $U_m : H_m \to K$ such that $\|\sigma(a) - U_m\Phi_m(a)U_m^*\| \to 0$ as $m \to \infty$ for all $a \in A$. Since it is clear that $\Phi_m(A)$ is a quasidiagonal (in fact, block diagonal) set of operators for every m it is easy to see that $\sigma(A)$ is also a quasidiagonal set of operators and hence A is QD. Q.E.D.

In addition to being a very useful tool in establishing the quasidiagonality of a given C^*-algebra this result also shows that QD C^*-algebras are a very natural *abstract* class of algebras. Indeed, this result shows that QD C^*-algebras are precisely those which have 'good matrix models' in the sense that all of the relevant structure (order, adjoints, multiplication, norms) can approximately be seen in a matrix.

We wish to note a minor generalization which will be useful later on.

Definition 4.3. If A is a unital C^*-algebra and $\mathcal{F} \subset A$ is a finite set then we will let $X_{\mathcal{F}\mathcal{F}}$ denote the smallest operator system (cf. Definition 2.3) containing $\mathcal{F}$ and $\{ab : a, b \in \mathcal{F}\}$.

Definition 4.4. If $F, B \subset B(H)$ are sets of operators then we say F is *ε-contained in B* if for each $x \in F$ there exists $y \in B$ such that $\|x - y\| < \varepsilon$. When this is the case we write $F \subset^{\varepsilon} B$.

Corollary 4.5. *Assume that A is unital and for every finite subset $\mathcal{F} \subset A$ and $\varepsilon > 0$ there exists a contractive completely positive map $\varphi : X_{\mathcal{F}\mathcal{F}} \to B(H)$ such that φ is ε-isometric and ε-multiplicative on $\mathcal{F}$ and $\varphi(\mathcal{F})$ is ε-contained in a QD C^*-algebra $B \subset B(H)$. Then A is QD.*

Proof. That A satisfies $(*)$ follows from Arveson's Extension Theorem applied to φ and to the almost isometric and multiplicative maps from B to finite dimensional C^*-algebras. Q.E.D.

Remark 4.6. *Note that the hypotheses of the previous corollary can be relaxed further. Indeed, one only needs such ε-isometric and ε-multiplicative maps on a sequence of finite sets which are suitably dense in A (e.g. generate a dense *-subalgebra of A).*

§5. Local Approximation

We observe that every QD C^*-algebra can be locally approximated by a residually finite dimensional (RFD) C^*-algebra (cf. Example 3.15). The proof is a simple adaptation of Halmos' original proof that every quasidiagonal operator can be written as a block diagonal operator plus a compact. We also recall a result of M. Dădărlat which gives a much stronger approximation in the case of exact QD C^*-algebras.

Definition 5.1. *(cf. Definition 3.1)* Let $B \subset B(H)$ be a C^*-algebra. Then B is called a *block diagonal* algebra if there exists an increasing sequence of finite rank projections, $P_1 \leq P_2 \leq P_3 \cdots$, such that $\|[b, P_n]\| = 0$ for all $b \in B$, $n \in \mathbb{N}$ and $P_n \to 1_H$ (s.o.t.).

It is relatively easy to see that a C^*-algebra R is RFD if and only if there exists a faithful representation $\pi : R \to B(H)$ such that $\pi(R)$ is a block diagonal algebra. The next result, which is well known to the experts, shows that every QD C^*-algebra can be locally approximated by an RFD algebra.

Theorem 5.2. *Let $A \subset B(H)$ be a C^*-algebra. Then A is a quasidiagonal set of operators if and only if for every finite set $\mathcal{F} \subset A$ and $\varepsilon > 0$ there exists a block diagonal algebra $B \subset B(H)$ such that $\mathcal{F} \subset^{\varepsilon} B$ (cf. Definition 4.4) and $A + \mathcal{K}(H) = B + \mathcal{K}(H)$.*

Proof. Clearly we only have to prove the necessity since $B + \mathcal{K}(H)$ is a quasidiagonal set of operators. Our proof follows closely the proof of [Ar, Thm. 2] where a similar result is obtained for general quasicentral approximate units.

So let $\mathcal{F} \subset A$ and $\varepsilon > 0$ be given. We may assume that $\mathcal{F}$ is contained in the unit ball of A. Let $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \mathcal{F}_3 \ldots$ be a sequence of finite sets such that $\mathcal{F} \subset \mathcal{F}_1$ and whose union is dense in the unit ball of A. Since A is a quasidiagonal set of operators we can use Proposition 3.6 to find finite rank projections $P_1 \leq P_2 \ldots$ converging to 1_H (strongly) and such that $\|[P_n, a]\| \to 0$ for all $a \in A$. By passing to a subsequence we may assume that $\|[P_n, a]\| < \varepsilon/(2^n)$ for all $a \in \mathcal{F}_n$. Now, let $E_n = P_n - P_{n-1}$ for $n = 1, 2, \ldots$ where $P_0 = 0$. Note that $\sum_{n=1}^{\infty} E_n = 1_H$.

Then one defines completely positive maps $\delta_k : A \to B(H)$ via the formula

$$\delta_k(a) = \sum_{n=1}^{k} E_n a E_n.$$

We leave it to the reader to verify that the δ_k's converge in the point strong operator topology (i.e. $\delta_k(a)$ is strongly convergent for each $a \in A$) and hence

$$\delta(a) = \sum_{n=1}^{\infty} E_n a E_n.$$

is a well defined completely positive map. Now let $B = C^*(\delta(A))$ and clearly B is a block diagonal set of operators. Moreover, for each $a \in A$

we have

$$\begin{aligned} a - \delta(a) &= \sum_{n=1}^{\infty} aE_n - \sum_{n=1}^{\infty} E_n a E_n \\ &= \sum_{n=1}^{\infty} (aE_n - E_n a E_n) \\ &= \sum_{n=1}^{\infty} (aE_n - E_n a) E_n \end{aligned}$$

where convergence of these sums is again taken in the strong operator topology. However, for each $a \in \cup \mathcal{F}_n$ the last summation above is actually convergent in the norm topology and is compact since the E_n's are finite rank. Note that by construction we have $\|a - \delta(a)\| \leq \sum \varepsilon/(2^n) = \varepsilon$ for all $a \in \mathcal{F}_1$. Now since δ is norm continuous (being completely positive) we then conclude that $a - \delta(a)$ is a compact operator for all $a \in A$. It follows that $A + \mathcal{K}(H) = B + \mathcal{K}(H)$. Q.E.D.

Theorem 5.2 fails when A is not separable (cf. Remark 3.7).

Corollary 5.3. *(cf. [GM]) Every (separable) C^*-algebra A is a quotient of an RFD algebra. If A is nuclear (resp. exact) then the RFD algebra can be chosen nuclear (resp. exact).*

Proof. Let $\pi : CA \to B(H)$ be a faithful essential representation of the cone over A (cf. Example 3.16). If A is nuclear (resp. exact) then so is CA and hence so is $\pi(CA)+\mathcal{K}(H)$ (cf. [CE1, Cor. 3.3], [Kir2, Prop. 7.1]). Let $R \subset B(H)$ be an RFD algebra such that $\pi(CA)+\mathcal{K}(H) = R+\mathcal{K}(H)$. Passing to the Calkin algebra we see that CA, and hence A, is a quotient of R. Since exactness passes to subalgebras ([Kir2, Prop. 7.1]), it is clear that R is exact whenever A is exact. When A is nuclear we deduce that R is also nuclear from [CE1, Cor. 3.3] and the exact sequence

$$0 \to R \cap \mathcal{K}(H) \to R \to CA \to 0,$$

since $\mathcal{K}(H)$ is type I and hence all of it's subalgebras are nuclear (cf. [Bl1]). Q.E.D.

The next result of Dădărlat is a vast improvement under the additional assumption of exactness. We will not prove this here; see [Dăd3, Thm. 6]. However we remark that the proof depends in an essential way on Theorem 5.2 as it allows one to reduce to the case of RFD algebras.

Theorem 5.4 (Dădărlat). *Let $A \subset B(H)$ be such that $A \cap \mathcal{K}(H) = 0$. Then A is exact and QD if and only if for every finite set $\mathcal{F} \subset A$ and*

$\varepsilon > 0$ *there exists a finite dimensional subalgebra* $B \subset B(H)$ *such that* $\mathcal{F} \subset^{\varepsilon} B$.

Note the similarity with the definition of an AF algebra. The difference, of course, is that we have had to go outside the algebra to get the finite dimensional approximation. We regard this as very strong evidence in favor of an affirmative answer to the following conjecture. (See also [BK1, Question 7.3.3])

Conjecture 5.5. *Every (separable) exact QD C^*-algebra is isomorphic to a subalgebra of an AF algebra.*[1]

§6. Traces

Proposition 6.1. *(cf. [Vo4, 2.4]) If A is a unital QD C^*-algebra then A has a tracial state.*

Proof. By Theorem 4.2 and Lemma 4.1 we can find a sequence of unital completely positive maps $\varphi_i : A \to M_{n(i)}(\mathbb{C})$ such that $\|a\| = \lim_i \|\varphi_i(a)\|$ and $\|\varphi_i(ab) - \varphi_i(a)\varphi_i(b)\| \to 0$ for all $a, b \in A$. Let $\tau_{n(i)}$ denote the tracial state on $M_{n(i)}(\mathbb{C})$ and let $\tau \in S(A)$ be a weak limit point of the sequence $\{\tau_{n(i)} \circ \varphi_i\} \subset S(A)$. An easy calculation shows that τ is a tracial state. Q.E.D.

One should not be tempted to think that the trace constructed above is faithful. Of course some very nice unital QD C^*-algebras, like the unitization of the compact operators, can't have a faithful tracial state. But we do have the following immediate corollary.

Corollary 6.2. *Every simple unital QD C^*-algebra has a faithful trace.*

§7. Easy Functorial Properties

The following two facts are immediate from the definition.

Proposition 7.1. *A subalgebra of a QD C^*-algebra is also QD.*

Proposition 7.2. *The unitization of a QD C^*-algebra is also QD.*

We need some notation before going further.

[1] Some exciting progress on this conjecture has been made by Ozawa who showed that the cone over any exact algebra is AF-embeddable (cf. [Oz], [Rør3]).

Definition 7.3. Let $\{A_n\}$ be a sequence of C^*-algebras. Then $\Pi_{n\in\mathbb{N}}A_n = \{(a_n) : \sup_n \|a_n\| < \infty\}$, where (a_n) is an element of the set theoretic product of the A_n's. We let $\oplus_{n\in\mathbb{N}}A_n$ denote the ideal of $\Pi_{n\in\mathbb{N}}A_n$ which consists of elements (a_n) with the property that $\lim_{n\to\infty} \|a_n\| = 0$.

If A and B are QD and $\pi : A \to B(H)$, $\rho : B \to B(K)$ are faithful representations whose ranges are quasidiagonal sets of operators then one easily checks that $A \oplus B$ is QD by considering the representation $\pi \oplus \rho$. The following fact is an easy extension of this argument.

Proposition 7.4. *The direct product of QD C^*-algebras is QD. That is, if $\{A_n\}$ is a sequence of C^*-algebras then $\Pi_{n\in\mathbb{N}}A_n$ is QD if and only if each A_n is QD.*

Recall that if A and B are C^*-algebras with faithful representations $\pi : A \to B(H)$ and $\rho : B \to B(K)$ then the minimal (or 'spatial') tensor product is defined to be the C^*-algebra generated by the image of the algebraic tensor product representation $\pi \odot \rho : A \odot B \to B(H) \odot B(K) \subset B(H \otimes K)$. The following result, which appeared first in [Had2], is left as an easy exercise. The proof only depends on the fact that the tensor product of two finite rank projections is again a finite rank projection.

Proposition 7.5. *The minimal tensor product of QD C^*-algebras is again QD.*

If both A and B contain projections and $A \otimes_{min} B$ is QD then both A and B must be QD as well. But in general the converse of Proposition 7.5 is not true (since cones and suspensions are always QD).

When one of the algebras happens to be nuclear then there is only one possible tensor product and hence quasidiagonality is always preserved in this case. In particular this fact implies that quasidiagonality is even invariant under the weaker notion of *stable isomorphism* (cf. [BrL1]). (Recall that A and B are stably isomorphic if $A \otimes \mathcal{K} \cong B \otimes \mathcal{K}$, where $\mathcal{K}$ denotes the compact operators on an infinite dimensional separable Hilbert space. Recall also that for separable algebras this is the same as strong Morita equivalence; cf. [BGR].)

It is not known whether or not Proposition 7.5 holds for other tensor products. In particular the following question is still open.

Question 7.6. *If A and B are QD then is $A \otimes_{max} B$ also QD?*

§8. Quotients

We already pointed out in Example 3.16 that every C^*-algebra is a quotient of a QD C^*-algebra. Thus quasidiagonality does not pass to quotients in general. In this section we give a sufficient condition for a quotient of a QD algebra to be QD. However, this condition is far from necessary and it is not clear what the real obstruction is. Also, at the end of this section we give another proof of Corollary 5.3 (i.e. one which does not depend on the fact that cones are always QD).

To state our result we first need a definition. The notion of relative quasidiagonality was introduced by Salinas in connection with KK-theory (cf. [Sa2]).

Definition 8.1. Let A be a C^*-algebra with (closed, 2-sided) ideal I. Then A is said to be *quasidiagonal relative to I* if I has an approximate unit consisting of projections which is quasicentral in A.

Example 8.2. *In general, an algebra can be quasidiagonal relative to an ideal without itself (or the ideal) being QD. For example, let $\{A_i\}_{i\in\mathbb{N}}$ be a sequence of unital (non-QD) C^*-algebras. Then $A = \Pi_i A_i$ is quasidiagonal relative to the ideal $I = \oplus_i A_i$. But the terminology is inspired by a close connection in the case that the ideal is the compact operators. Indeed, if $B \subset B(H)$ is a C^*-algebra then it is easy to see that B is a quasidiagonal set of operators if and only if $B + \mathcal{K}(H)$ is quasidiagonal relative to $\mathcal{K}(H)$ (cf. Proposition 3.6).*

Proposition 8.3. *Assume A is unital, QD, quasidiagonal relative to an ideal I and $\pi : A \to A/I$ is locally liftable (cf. Definition 2.12). Then A/I is also QD.*

Proof. Let $\mathcal{F} \subset A/I$ be a finite set and $\varepsilon > 0$. In the notation of Corollary 4.5 we let $\varphi : X_{\mathcal{F}\mathcal{F}} \to A$ be a unital completely positive splitting.

Now take a quasicentral approximate unit of projections, say $\{p_n\}$ and consider the (isometric – though no longer unital) completely positive splittings $\varphi_n(x) = (1-p_n)\varphi(x)(1-p_n)$. We claim that for sufficiently large n, these maps are ε-multiplicative on $\mathcal{F}$ and hence from Corollary 4.5 we will have that A/I is QD.

To see the ε-multiplicativity we first recall that if $a \in A$ and $\dot{a}$ denotes it's image in A/I then $\|\dot{a}\| = \lim \|(1-p_n)a\|$ since $\{p_n\}$ is an approximate unit for I. However, since the p_n's are projections and quasicentral for A we see that $\|\dot{a}\| = \lim \|(1-p_n)a(1-p_n)\|$ as well. Now for $a, b \in A$ consider the following estimates (see also the proof of Lem. 3.1 in [Ar] where these estimates are given in greater generality):

$$
\begin{aligned}
&\|\varphi_n(\dot{a}\dot{b}) - \varphi_n(\dot{a})\varphi_n(\dot{b})\| \\
&\quad = \|(1-p_n)\varphi(\dot{a}\dot{b})(1-p_n) - (1-p_n)\varphi(\dot{a})(1-p_n)\varphi(\dot{b})(1-p_n)\| \\
&\quad \leq \|(1-p_n)\Big(\varphi(\dot{a}\dot{b}) - \varphi(\dot{a})\varphi(\dot{b})\Big)(1-p_n)\| \\
&\qquad + \|(1-p_n)\Big((1-p_n)\varphi(\dot{a}) - \varphi(\dot{a})(1-p_n)\Big)\varphi(\dot{b})(1-p_n)\|.
\end{aligned}
$$

Finally since φ is a splitting, $\|\dot{x}\| = \lim \|(1-p_n)x(1-p_n)\|$ and $\{p_n\}$ is quasicentral we see that φ_n is ε-multiplicative on $\mathcal{F}$ for sufficiently large n. Q.E.D.

Corollary 8.4. *If A is unital, locally reflexive (e.g. exact or nuclear), QD and quasidiagonal relative to an ideal I then A/I is also QD.*

Proof. Use the previous proposition together with Theorem 2.13. Q.E.D.

Remark 8.5. *The proof of Proposition 8.3 given here is simply a formalization of a well known argument in the case the ideal is the compact operators. (cf. [Dăd1, Prop. 4.5].)*

Example 8.6. *Proposition 8.3 is no longer true without the 'local liftability' hypothesis. Indeed, S. Wassermann gave the first examples of quasidiagonal sets of operators whose image in the Calkin algebra was a non-QD C^*-algebra (cf. [Was1,2]). Hence, by the remarks in Example 8.2, Wassermann's examples show that the 'local liftability' hypothesis can't be dropped in Proposition 8.3.*

In section 10 we will see that the 'right' obstruction to look at for extensions is probably given in K-theoretic terms (for a large class of algebras). However, the following example shows that this is not the case for the quotient question. Indeed, it is not at all clear what type of obstruction one should be looking at in relation to the quotient question.

Example 8.7. *Let $A = \mathcal{O}_2$ be the Cuntz algebra on two generators (cf. [Cu]). Then we have the short exact sequence $0 \to SA \to CA \to A \to 0$, where SA and CA denote the suspension and cone, respectively. The point we wish to make is that any potential K-theoretic obstruction would vanish for this example since the six term exact sequence is trivial. However, CA is QD while A is not.*

Finally we give another proof of the fact that every separable C^*-algebra is a quotient of an RFD algebra (cf. Corollary 5.3). The simplest proof of this fact is the original (cf. [GM]), however it does not yield the useful fact that this can be done within certain categories (e.g. nuclear, exact, etc.) Rather than use Voiculescu's result on the quasidiagonality of cones, we now exploit a basic fact from noncommutative topology. We will need the following generalization of the Tietze Extension Theorem.

Theorem 8.8. *(cf. [We, Thm. 2.3.9]) Let A be a separable C^*-algebra and $\pi : A \to B$ be a* surjective **-homomorphism. Then π extends to a surjective *-homomorphism $\tilde{\pi} : M(A) \to M(B)$ of multipier algebras.*

Proposition 8.9. *Let R be an RFD algebra. Then the multiplier algebra, $M(R)$, is also RFD.*

Proof. Let $\pi_n : R \to A_n$ be a sequence of *surjective* *-homomorphisms such that each A_n is unital and QD (e.g. finite dimensional) and the map $\oplus_{n\in\mathbb{N}}\pi_n$ is faithful. Construct extending morphisms $\tilde{\pi}_n : M(R) \to A_n$. (In this case the extensions are easy to construct. For each n let $e_n \in R$ be a lift of the unit of A_n and simply define $\tilde{\pi}_n(x) = \pi_n(xe_n)$ for all $x \in M(R)$.)

In general, if $I \subset A$ is an essential ideal and $\varphi : A \to B$ is a *-homomorphism such that $\varphi|_I$ is injective then φ must be injective on all of A (since any nonzero ideal of A must have nonzero intersection with I). Hence we see that the *-homomorphism $\oplus_n\tilde{\pi}_n : M(R) \to \Pi A_n$ is also injective. Q.E.D.

Corollary 8.10. *Let H be a (separable) Hilbert space. Then $B(H)$ is a quotient of a RFD algebra.*

Proof. Since the compact operators on a separable Hilbert space are a quotient of an RFD algebra (note that for QD C^*-algebras we do not have to use cones in the proof of Corollary 5.3), it follows from the Tietze Extension Theorem and proposition 8.9 that $B(H) = M(\mathcal{K})$ is a quotient of an RFD algebra. Q.E.D.

§9. Inductive Limits

It follows easily from Theorem 3.11 that an inductive limit of QD C^*-algebras where the connecting maps are all *injective* will again be a QD C^*-algebra. We will see that, in general, inductive limits of QD algebras need not be QD. However, if the algebras in the sequence are also locally reflexive (e.g. exact or nuclear) then the limit algebra must be QD.

In [BK1] the notion of MF C^*-algebra was introduced. There are a number of characterizations of these algebras and hence we can choose the most convenient as our definition (though it is actually a theorem).

Definition 9.1. *(cf. [BK1, Thm. 3.2.2])* A C^*-algebra A is MF if and only if A is isomorphic to a subalgebra of $\Pi M_{n(i)}(\mathbb{C})/\oplus M_{n(i)}(\mathbb{C})$ for some sequence $\{n_{(i)}\}$.

Proposition 9.2. *(cf. [BK1, Prop. 3.1.3]) Let $C \subset B(H)$ be a C^*-algebra which is also a quasidiagonal set of operators and $\pi : B(H) \to Q(H)$ denote the quotient map onto the Calkin algebra. Then $\pi(C)$ is MF.*

Proof. By Theorem 5.2 we can find a block diagonal algebra $B \subset B(H)$ such that $C + \mathcal{K} = B + \mathcal{K}$. If $P_1 \leq P_2 \leq P_3 \leq \ldots$ are finite rank projetions which commute with B and converge to 1_H then there is a canonical identification $\Pi_i M_{n(i)}(\mathbb{C}) \hookrightarrow B(H) = B(\oplus_{i\in\mathbb{N}}\mathbb{C}^{n(i)})$, where $n(i) = rank(P_i) - rank(P_{i-1})$ (and $P_0 = 0$). Note that under this identification we have $\Pi M_{n(i)}(\mathbb{C}) \cap \mathcal{K} = \oplus M_{n(i)}(\mathbb{C})$ and $B \subset \Pi M_{n(i)}(\mathbb{C})$. Hence

$$\pi(C) \cong B/(B \cap \mathcal{K}) \hookrightarrow \Pi M_{n(i)}(\mathbb{C})/\oplus M_{n(i)}(\mathbb{C}). \qquad \text{Q.E.D.}$$

Evidently every MF algebra has such an extension by the compacts.

It follows that every QD C^*-algebra is MF (cf. Theorem 3.11). The converse is not true by Example 8.6.

The following simple result shows that MF algebras can also be described as the class of C^*-algebras arising as inductive limits of RFD C^*-algebras.

Proposition 9.3. *A C^*-algebra A is MF if and only if A is isomorphic to an inductive limit of RFD algebras.*

Proof. That an inductive limit of MF algebras (e.g. RFD algebras) is again MF is a bit out of the scope of this article. Please see [BK1, Cor. 3.4.4] for the proof. We will prove the converse however.

By pulling back the embedding $A \hookrightarrow \Pi M_{n(i)}(\mathbb{C})/\oplus M_{n(i)}(\mathbb{C})$ we can find an RFD algebra R with ideal $I = \oplus M_{n_{(i)}}(\mathbb{C})$ such that $A \cong R/I$. Consider the finite dimensional ideals $I_k = \oplus_{i=1}^k M_{n_{(i)}}(\mathbb{C})$. Evidently R/I_k is again an RFD algebra (being a direct summand of R) and hence the natural inductive system

$$R \to R/I_1 \to R/I_2 \to R/I_3 \to \cdots$$

consists of RFD algebras. Moreover, since $I = \overline{\cup I_k}$ it is routine to verify that $A \cong R/I$ is isomorphic to the inductive limit of the above sequence. Q.E.D.

Remark 9.4. *It follows that inductive limits of QD C^*-algebras need not be QD. To get such examples, we let $A \subset B(H)$ be a C^*-algebra which is a quasidiagonal set of operators and such that the image, $B \subset Q(H)$, in the Calkin algebra is non-QD. (We mentioned in Example 8.6 that Wassermann has constructed such algebras.) By Propositions 9.2 and 9.3, B is an inductive limit of RFD algebras which is not QD.*

In contrast to the previous remark, the next result shows that mild assumptions will ensure the quasidiagonality of the limit.

Theorem 9.5. *Let $\{A_m, \varphi_{n,m}\}_{m\in\mathbb{N}}$ be an inductive system of unital locally reflexive QD C^*-algebras with limit $A = \varinjlim A_i$. Then A is QD.*

Proof. To clarify our notation, we mean that for each $n \geq m$ there is a *-homomorphism $\varphi_{n,m} : A_m \to A_n$ and we have the usual compatibility condition that $\varphi_{n,m} \circ \varphi_{m,l} = \varphi_{n,l}$ whenever $l \leq m \leq n$. We also let $\Phi_n : A_n \to A$ denote the induced *-homomorphism.

Unitizing the inductive system, if necessary, we may assume that all the connecting maps are unit preserving. Now let $\Psi_m : A_m \to \Pi A_i$ be the *-monomorphism defined by

$$\Psi_m(x) = 0 \oplus \cdots \oplus 0 \oplus x \oplus \varphi_{m+1,m}(x) \oplus \cdots$$

and $B = C^*(\cup\Psi_m(A_m)) + \oplus A_i \subset \Pi A_i$. Then it is easy to see that B is QD, quasidiagonal relative to the ideal $\oplus A_i$ and $A \cong B/(\oplus A_i)$. Thus it suffices to see (by Proposition 8.3 and Remark 4.6) that the quotient map $B \to B/(\oplus A_i)$ is locally liftable on a dense set. But this follows from the fact that each A_n is locally reflexive (cf. Theorem 2.13), the maps Ψ_n are injective, the exact sequences $0 \to (\Psi_n(A_n) \cap \oplus A_i) \to \Psi_n(A_n) \to \Phi_n(A_n) \to 0$ and the fact that the union of the $\Phi_n(A_n)$'s is dense in A. Q.E.D.

Remark 9.6. *Blackadar and Kirchberg have shown that generalized inductive limits (where the connecting maps are completely positive contractions) of nuclear QD algebras are again QD (cf. [BK1, Cor. 5.3.5]).*

Inductive limit decompositions have played a crucial role in (the finite case of) Elliott's Classification Program. The next result of Blackadar and Kirchberg may turn out to have important consequences in this program. This theorem follows immediately from [BK1, Prop. 6.1.6] and [BK2, Cor. 5.1].

Theorem 9.7. *Let A be a unital simple nuclear QD C^*-algebra. Then $A = \overline{\cup R_i}$ where $R_i \subset R_{i+1}$ are nuclear RFD algebras.*

The remarkable point of this theorem is that the connecting maps in the inductive system are all *injective.* Indeed, if one relaxes this condition then we can easily get every nuclear QD C^*-algebra.

Proposition 9.8. *Let A be a nuclear C^*-algebra. Then A is QD if and only if A is isomorphic to an inductive limit of nuclear RFD C^*-algebras.*

Proof. ($\Longleftarrow$) This follows from Theorem 9.5.

($\Longrightarrow$) This follows from the proof of Proposition 9.3 since extensions of nuclear C^*-algebras are nuclear. Q.E.D.

§10. Extensions

Since the Toeplitz algebra is an extension of the compacts by $C(\mathbb{T})$, it follows that extensions of QD algebras need not be QD. Indeed, as with the quotient question, the general extension problem for QD algebras appears to be very hard. As we will see, it is not even clear whether or not a split extension of QD algebras should be QD.

We begin, however, with two simple positive results. The first states that if the *ideal* is sufficiently quasidiagonal then the middle algebra is always QD. The second states that if the *extension* is sufficiently quasidiagonal then the middle algebra is always QD.

Proposition 10.1. *Assume $0 \to I \xrightarrow{\iota} E \xrightarrow{\pi} B \to 0$ is exact with I an RFD algebra and B a QD algebra. Then E is QD.*

Proof. Let $\varphi : E \to M(I)$ be the natural extension of the inclusion $I \hookrightarrow M(I)$ (cf. [We, 2.2.14]). Then the map $\varphi \oplus \pi : E \to M(I) \oplus B$ is injective. But Proposition 8.9 states that $M(I)$ is QD (even RFD) and hence E is a QD C^*-algebra. Q.E.D.

Note that the proof of Proposition 8.9 actually shows that $M(I)$ is QD whenever I has a separating family of *unital* QD quotients. Hence the proposition above remains true for ideals of the form $I = R \otimes_{min} B$ where R is RFD and B is unital and QD. Hence a natural question is the following.

Question 10.2. *Which (nonunital) C^*-algebras have QD multiplier algebras?*

Definition 10.3. Let $0 \to I \xrightarrow{\iota} E \xrightarrow{\pi} B \to 0$ be a short exact sequence of C^*-algebras. Such a sequence is called a *quasidiagonal extension* if E is quasidiagonal relative to $\iota(I)$ (cf. Definition 8.1).

Remark 10.4. *It is important to note that in general an extension being quasidiagonal has nothing to do with whether or not the middle algebra E is QD (see Example 8.2).*

Proposition 10.5. *Let $0 \to I \xrightarrow{\iota} E \xrightarrow{\pi} B \to 0$ be a quasidiagonal extension where both I and B are QD. Then E is QD.*

Proof. To ease notation somewhat, we identify I with $\iota(I)$ and let $\{P_n\} \subset I$ be an approximate unit of projections which is quasicentral in E. Now consider the contractive completely positive maps $\varphi_n : E \to I \oplus B$, $\varphi_n(x) = P_n x P_n \oplus \pi(x)$. Evidently these maps are asymptotically multiplicative. So we may appeal to Corollary 4.5 and deduce that E is QD as soon as we verify the following assertion:

Claim. If $x \in E$ then $\|x\| = max\{ \liminf_n \|P_n x P_n\|, \ \|\pi(x)\| \ \}$.

To prove the claim we pass to the double dual E^{**}. Let $P \in I^{**} \subset E^{**}$ be the (weak) limit of the P_n's. Then P is central in E^{**} and we have a decomposition $E^{**} = I^{**} \oplus B^{**}$. Hence (regarding $E \subset E^{**}$) for each $x \in E$ we have $\|x\| = max\{ \ \|PxP\|, \ \|(1-P)x(1-P)\| \ \}$. But $\|\pi(x)\| = \|(1-P)x(1-P)\|$ and $\|PxP\| \leq \liminf_n \|P_n x P_n\|$ since $P_n x P_n \to PxP$ in the strong operator topology. But this proves the claim since the inequality $\|x\| \geq max\{ \liminf_n \|P_n x P_n\|, \ \|\pi(x)\| \ \}$ is obvious. Q.E.D.

Remark 10.6. *As mentioned previously, Propositions 10.1 and 10.5 can be regarded as saying that quasidiagonality is always preserved, provided that either the* ideal *or the* extension *is sufficiently quasidiagonal. This is not true if only the* quotient *is highly QD (e.g. the Toeplitz algebra). Instead a K-theoretic obstruction appears to govern in general.*

We would now like to discuss the general question of when quasidiagonality is preserved in extensions. However, to illustrate the difficulty of this problem we first pose two basic (open) questions.

Question 10.7. *Let $0 \to I \xrightarrow{\iota} E \xrightarrow{\pi} B \to 0$ be a* split *exact sequence (i.e. there exists a *-homomorphism $\rho : B \to E$ such that $\pi \circ \rho = id_B$) with I and B QD. Is E necessarily QD?*

Question 10.8. *Let I and B be QD C^*-algebras and $\pi : B \to M(I \otimes \mathcal{K})$ be a *-monomorphism such that $\pi(B) \cap (I \otimes \mathcal{K}) = \{0\}$. Is $\pi(B) + I \otimes \mathcal{K}$ necessarily QD?*

Clearly an affirmative answer to Question 10.7 would imply an affirmative answer to Question 10.8. In fact the converse is true.

Lemma 10.9. *Questions 10.7 and 10.8 are equivalent.*

Proof. Assume Question 10.8 has an affirmative answer and let $0 \to I \xrightarrow{\iota} E \xrightarrow{\pi} B \to 0$ be a split exact sequence and $\rho : B \to E$ be such that $\pi \circ \rho = id_B$. Identify I with $\iota(I)$. Let $\eta : E \to B(H)$ be a faithful essential representation. Then from Theorem 3.11, $\eta(I) + \mathcal{K}$ is QD.

Moreover, $\eta(I)+\mathcal{K}$ is an essential ideal in $\eta(E)+\mathcal{K}$. So replacing I by $\eta(I)+\mathcal{K}$ and E by $\eta(E)+\mathcal{K}$ we may further assume that I is essential in E. But then $0 \to I\otimes\mathcal{K} \xrightarrow{\iota} E\otimes\mathcal{K} \xrightarrow{\pi} B\otimes\mathcal{K} \to 0$ is still a split exact sequence with $I\otimes\mathcal{K}$ essential in $E\otimes\mathcal{K}$. Hence $E\otimes\mathcal{K}$ may be regarded as a subalgebra of $M(I\otimes\mathcal{K})$ (cf. [We, 2.2.14]) and thus an affirmative answer to Question 10.8 would imply that $E\otimes\mathcal{K}$ is QD. Q.E.D.

In [BND] it is shown that Question 10.8 has an affirmative answer under the additional hypothesis that either I or B is nuclear. Note, however, that even in the case that $I=\mathbb{C}$, Question 10.8 is not trivial (an affirmative answer still depends on the full power of Voiculescu's Theorem; cf. Theorem 3.11). Hence it is not clear whether or not we should expect an affirmative answer to these questions in general.

If we restrict to the class of nuclear C^*-algebras then some progress can be made on the general extension problem. Blackadar and Kirchberg have asked whether or not every nuclear stably finite C^*-algebra is QD (cf. [BK1, Question 7.3.1]). Hence one may ask whether the extension problem can be solved for stably finite C^*-algebras. J. Spielberg has given a complete answer to this question in his work on the AF embeddability of extensions of C^*-algebras.

Proposition 10.10 (Sp, Lem. 1.5). *Let $0\to I\to E\to B\to 0$ be an exact sequence with both I and B stably finite. If $\partial : K_1(B)\to K_0(I)$ denotes the boundary map of this sequence then E is stably finite if and only if $\partial(K_1(B))\cap K_0^+(I)=\{0\}$, where $K_0^+(I)$ is the canonical positive cone of $K_0(I)$.*

Though the proof is fairly straightforward, we will not prove this result here as we do not wish to introduce the K-theory which is needed.

In light of the previous proposition and the question of whether or not the notions of quasidiagonality and stable finiteness coincide in the class of nuclear C^*-algebras, the following question becomes quite natural.

Question 10.11. *Let $0\to I\to E\to B\to 0$ be an exact sequence with both I and B nuclear QD C^*-algebras. Is it true that E is QD if and only if $\partial(K_1(B))\cap K_0^+(I)=\{0\}$?*

If one approaches this problem via KK-theory then it is probably necessary to further assume that B satisfies the Universal Coefficient Theorem (UCT) of Rosenberg and Schochet (cf. [RS]). In [BND] it is shown that this question is equivalent to some very natural questions concerning the K-theory of nuclear QD C^*-algebras. Moreover, it seems likely that an affirmative answer to the question above could have important consequences in the classification program (specifically to the classification of Lin's TAF algebras; [Li1,2]).

In [BND] we also give a partial solution to the question above. The techniques used to prove the following result are similar to those from [Sp]. (See also [ELP] for the case that the quotient is AF.)

Theorem 10.12 (BND). *Let $0 \to I \to E \to B \to 0$ be an exact sequence with I QD and B nuclear, QD and satisfying the UCT. If $\partial : K_1(B) \to K_0(I)$ is the zero map then E is QD*

§11. Crossed Products

In this section we discuss when crossed products of QD C^*-algebras are again QD. This is not always the case since the (purely infinite) Cuntz algebras are stably isomorphic to crossed products of AF algebras by $\mathbb{Z}$. The basic theory of crossed products by locally compact groups can be found in [Pe1, Chpt. 7]. (See also [Dav, Chpt. 8] for a nice treatment of the discrete case.)

We begin with a corollary of an imprimitivity theorem of P. Green. To state the result we will need to introduce some notation. So, let G be a separable locally compact group and $H \subset G$ be a closed subgroup. Then G/H (the space of left cosets) is a separable locally compact space. There is a natural action γ of G on $C_0(G/H)$ defined by $\gamma_g(f)(xH) = f(g^{-1}xH)$ for all $xH \in G/H$ and $f \in C_0(G/H)$. The crossed products below are the full crossed products and all groups actions $\alpha : G \to Aut(A)$ are assumed to be suitably continuous (i.e. for each $a \in A$ the map $g \mapsto \alpha_g(a)$ is continuous).

Theorem 11.1. *([Gr2, Cor. 2.8]) Let $\alpha : G \to Aut(A)$ be a homomorphism from the separable locally compact group G. For each closed subgroup $H \subset G$ there is an isomorphism*

$$A \otimes C_0(G/H) \rtimes_{\alpha\otimes\gamma} G \cong (A \rtimes_{\alpha|_H} H) \otimes \mathcal{K},$$

where $\mathcal{K}$ denotes the compact operators on a separable (finite dimensional if and only if G/H is finite) Hilbert space.

For the rest of this section we will only be dealing with amenable groups (cf. [Pe1, 7.3]) and hence we do not need to distinguish between reduced and full crossed products (cf. [Pe1, Thm. 7.7.7]).

Corollary 11.2. *Let A be QD and $\alpha : G \to Aut(A)$ be a homomorphism with G a separable compact group. Then $A \rtimes_\alpha G$ is QD.*

Proof. Let $H \subset G$ be the zero subgroup. The previous theorem then asserts that $A \otimes C(G) \rtimes_{\alpha\otimes\gamma} G \cong A \otimes \mathcal{K}$. But $A \otimes \mathcal{K}$ is QD and there is a natural embedding $A \rtimes_\alpha G \hookrightarrow A \otimes C(G) \rtimes_{\alpha\otimes\gamma} G$ since G amenable implies that the full and reduced crossed products are isomorphic (cf. [Pe1, 7.7.7 and 7.7.9]). Q.E.D.

For non-compact discrete groups the problem is considerably harder. However, Rosenberg has shown that we must restrict to the class of amenable groups.

Theorem 11.3 (Ros, Thm. A1). *If G is discrete and $C_r^*(G)$ is QD then G is amenable.*

It is not known whether the converse of this theorem holds (cf. [Vo4, 3.1]), but Følner's characterization of amenable groups in terms of almost shift invariant finite subsets leads one to believe that the converse should be true.

For actions of $\mathbb{Z}$ there are only two classes of C^*-algebras where we currently have complete information on the quasidiagonality of $A \rtimes_\alpha \mathbb{Z}$; when A is abelian or AF. Before stating the theorems we first give a definition.

Definition 11.4. Let A be a C^*-algebra. Then A is called *AF embeddable* if there exists a $*$-monomorphism $\rho : A \to B$ where B is AF.

Of course AF embeddable C^*-algebras are QD. However, it is a non-trivial fact that the converse is not true. In fact, even RFD algebras need not be AF embeddable. The best known example is the full group C^*-algebra $C^*(\mathbb{F}_2)$. This is RFD but is not exact and hence cannot be embed into *any* nuclear (in particular, AF) algebra (cf. [Was3]). However, for crossed products of abelian or AF algebras by $\mathbb{Z}$, quasidiagonality does imply AF embeddability.

Theorem 11.5. *([Pi, Thm. 9]) Let $\varphi : X \to X$ be a homeomorphism of the compact metric space X and $\Phi \in Aut(C(X))$ denote the induced automorphism. Then the following are equivalent:*

1. *$C(X) \rtimes_\Phi \mathbb{Z}$ is AF embeddable,*
2. *$C(X) \rtimes_\Phi \mathbb{Z}$ is QD,*
3. *$C(X) \rtimes_\Phi \mathbb{Z}$ is stably finite,*
4. *'φ compresses no open sets.' (That is, if $U \subset X$ is open and $\varphi(\overline{U}) \subset U$ then $\varphi(U) = U$.)*

Theorem 11.6. *([BrN1, Thm. 0.2]) Let A be AF and $\alpha \in Aut(A)$ be given. Then the following are equivalent:*

1. *$A \rtimes_\alpha \mathbb{Z}$ is AF embeddable,*
2. *$A \rtimes_\alpha \mathbb{Z}$ is QD,*

3. $A \rtimes_\alpha \mathbb{Z}$ *is stably finite,*
4. *'$\alpha_* : K_0(A) \to K_0(A)$ compresses no elements.' (That is, if $x \in K_0(A)$ and $\alpha_*(x) \le x$ in the natural order then $\alpha_*(x) = x$.)*

We have chosen to formulate the above results in a way that illustrates their similarities. In both cases the hard implications are $4 \Rightarrow 1$. Also in both cases it is not at all clear that the techniques in the proof will be of much use in general. Before going beyond actions of $\mathbb{Z}$ we wish to point out that there is no harm in assuming unital algebras.

Proposition 11.7. *Let A be nonunital, $\alpha \in Aut(A)$, $\tilde{A}$ be the unitization of A and $\tilde{\alpha} \in Aut(\tilde{A})$ the unique unital extension of α. Then $A \rtimes_\alpha \mathbb{Z}$ is QD if and only if $\tilde{A} \rtimes_{\tilde{\alpha}} \mathbb{Z}$ is QD.*

Proof. Recall that we always have a *split* exact sequence

$$0 \to A \rtimes_\alpha \mathbb{Z} \to \tilde{A} \rtimes_{\tilde{\alpha}} \mathbb{Z} \to C(\mathbb{T}) \to 0.$$

Thus the implication ($\Leftarrow$) is immediate and ($\Rightarrow$) follows from Theorem 10.12 since abelian algebras are nuclear, QD and satisfy the Universal Coefficient Theorem (cf. [RS]). Q.E.D.

Another natural direction to consider would be to try crossed products of well behaved C^*-algebras by more general groups. (We must stay within the class of amenable groups, though, because of Rosenberg's result; cf. Theorem 11.3) However, even for actions of $\mathbb{Z}^2$ this is a problem. Indeed the following question of Voiculescu remains open even now – more than 15 years after Pimsner's result for $C(X) \rtimes_\Phi \mathbb{Z}$.

Question 11.8. *(cf. [Vo4, 4.6]) When is $C(X) \rtimes_\Phi \mathbb{Z}^2$ AF embeddable?*[2]

For crossed products of certain simple AF algebras the question is more manageable.

Theorem 11.9 (BrN2, Thm. 1). *If A is UHF and $\alpha : \mathbb{Z}^n \to Aut(A)$ is a homomorphism then there always exists a $*$-monomorphism $\rho : A \rtimes_\alpha \mathbb{Z}^n \to B$ where B is AF.*

The proof of this result (and Theorem 11.6 above) depends in an essential way on a technical notion known as the *Rohlin property* for automorphisms. This notion has been used by Connes, Kishimoto, Evans, Nakamura and others (with great success!) in classifying automorphisms of operator algebras. Moreover, Kishimoto has used these

[2]Some nice progress, in the case where X is a Cantor set, has recently been made by Matui [Ma].

ideas to prove that many crossed products of certain simple $A\mathbb{T}$ algebras by automorphisms with the Rohlin property will again be $A\mathbb{T}$ (which is much stronger than just saying they are QD). See, for example, [Kis1-4].

Remark 11.10. *One nice consequence of Green's theorem (Theorem 11.1) is that understanding crossed products by $\mathbb{Z}^n$ gives results about much more general groups. For example, if G is a finitely generated discrete abelian group then $G \cong \mathbb{Z}^n \oplus F$ where F is a finite (hence compact) abelian group then by Green's result we have an embedding $A \rtimes_\alpha G \hookrightarrow (A \rtimes_{\alpha|_{\mathbb{Z}^n}} \mathbb{Z}^n) \otimes \mathcal{K}$. Writing a general discrete abelian group as an inductive limit of finitely generated such groups one can then handle crossed products by arbitrary discrete abelian groups. One can then proceed to take extensions by arbitrary separable compact groups and build a very large class of groups for which it suffices to consider crossed products by $\mathbb{Z}^n$. (See Def. 3.4 and the proof of Thm. 2 in [BrN2] for more details).*

§12. Relationship with Nuclearity

It was an open question for quite some time whether or not quasidiagonality implied nuclearity. In [Had2], Hadwin asked whether or not every 'strongly' quasidiagonal (e.g. simple QD) C^*-algebra was nuclear. Then in [Po], Popa asked whether every simple unital QD C^*-algebra with 'sufficiently many projections' (e.g. real rank zero) was nuclear. There was some evidence supporting a positive answer to these questions. The strongest was the following theorem of Popa.

Theorem 12.1 (Po, Thm. 1.2). *Let A be a simple unital C^*-algebra with 'sufficiently many projections' (e.g. real rank zero). Then A is QD if and only if for each finite set $\mathcal{F} \subset A$ and $\varepsilon > 0$ there exists a (nonzero) finite dimensional subalgebra $B \subset A$ with unit $P = 1_B$ such that $\|[a, P]\| \le \varepsilon$ for all $a \in \mathcal{F}$ and $P\mathcal{F}P \subset^\varepsilon B$ (cf. Definition 4.4).*

The necessity of the technical condition above is quite hard, however the sufficiency is easily seen. Indeed, if one assumes the technical condition then we can find a sequence of finite dimensional subalgebras $B_n \subset A$ with units P_n such that $\|[a, P_n]\| \to 0$ and $d(P_n a P_n, B) \to 0$ for all $a \in A$. Now let $\Phi_n : A \to B_n$ be a conditional expectation and consider the maps $\varphi_n : A \to B_n$ defined by $\varphi_n(a) = \Phi_n(P_n a P_n)$. This sequence of maps is evidently asymptotically multiplicative and hence defines a *-homomorphism

$$A \to \Pi B_n / \oplus B_n.$$

Since A is unital this morphism is nonzero and since A is simple, this morphism is injective. Hence the maps φ_n are also asymptotically isometric which implies (by Theorem 4.2) that A is QD. (Note that the hypothesis of a unit can't be dropped here. Indeed, the stabilization $\mathcal{K} \otimes A$ of any unital C^*-algebra A satisfies the technical condition stated above. Simply take B_n of the form $C_n \otimes 1_A$ where C_n is almost orthogonal to a large part of $\mathcal{K}$.)

The above result gave one hope of deducing nuclearity via the characterization in terms of injective enveloping von Neumann algebras (cf. [CE1]). However, it turns out that this is not possible as the following result of Dădărlat shows.

Theorem 12.2 (Dăd2, Prop. 9). *There exists a unital, separable, simple, QD C^*-algebra with real rank zero, stable rank one and unique tracial state which is not exact (and hence not nuclear).*

The converse of the question we have been considering above is also interesting and worth discussion. Namely, what sort of general conditions on a C^*-algebra imply quasidiagonality?

Example 12.3. *A Cuntz algebra $\mathcal{O}_n$ (cf. [Cu]) is simple, separable, unital, nuclear, has real rank zero and is not QD (since it is purely infinite; cf. Proposition 3.19).*

To get a finite non-QD example is a bit more delicate. Recall that $C_r^*(G)$, where G is a discrete group, is always stably finite since it has a faithful tracial state. Also recall that Rosenberg has shown that if the reduced group C^*-algebra of a discrete group is QD then the group must be amenable (cf. Theorem 11.3).

Example 12.4. *Let $\mathbb{F}_2$ denote the free group on two generators. Then $C_r^*(\mathbb{F}_2)$ is simple, unital, separable, exact, has stable rank one (cf. [DHR]) and a unique tracial state but is not QD since $\mathbb{F}_2$ is not amenable.*

To get an example with the added property of real rank zero one can simply consider $C_r^*(\mathbb{F}_2) \otimes U$, where U is some UHF algebra (cf. [Rør, Thm. 7.2]).

It is also interesting to note that there are no known examples of finite *nuclear* non-QD C^*-algebras. (Recall that $C_r^*(\mathbb{F}_2)$ is only exact.) In fact, as noted in Section 10, Blackadar and Kirchberg have formulated the following question.

Question 12.5 (BK1, Question 7.3.1). *If A is nuclear and stably finite then must A necessarily be QD?*

This question is of particular interest in Elliott's classification program (cf. [Ell]). Indeed, if this question turns out to have an affirmative answer then classifying simple unital nuclear *finite* C^*-algebras may be equivalent to classifying simple unital nuclear QD C^*-algebras. (One would still have to resolve the important open question of whether every simple finite algebra is stably finite - which is equivalent to the open question of whether every simple infinite C^*-algebra is purely infinite.[3]) The point is that for simple QD algebras (with enough projections) one has the structure theorem of Popa to work with. In fact, Lin has introduced a class of C^*-algebras (the so-called TAF algebras; [Li1]) whose definition is inspired by - and looks very similar to - Popa's structure theorem. Moreover, there are classification results for some of these TAF algebras (cf. [Li2,4], [DE1]) and it is not unreasonable to think that someday the general QD case can be handled in ways similar to the current strategies being applied to the TAF case.

§13. More Advanced Topics

In our final section we will present some miscellaneous results which don't quite fit into any of the previous sections. The first is a very important result of Voiculescu which shows that quasidiagonality is a homotopy invariant. Recall that two C^*-algebras A and B are called *homotopic* if there exist *-homomorphisms $\varphi : A \to B$ and $\psi : B \to A$ such that $\varphi \circ \psi$ is homotopic to id_B and $\psi \circ \varphi$ is homotopic to id_A (cf. [Bl2], [We]).

Theorem 13.1. *Let A and B be homotopic C^*-algebras. Then A is QD if and only if B is QD.*

Voiculescu actually proved a more general result (cf. [Vo3, Thm. 5]). In [Dăd1, Thm. 1.1] Dădărlat generalized this to show that quasidiagonality is even an invariant of the weaker notion of 'asymptotic completely positive homotopy equivalence'. As mentioned previously, this result implies that the cone over *any* C^*-algebra is QD since cones are homotopic to $\{0\}$.

Free products of C^*-algebras were introduced in [Av] and independently in [Vo5]. (See also [VDN].) Reduced free products are rarely QD. The standard example of a reduced free product is $C^*_r(\mathbb{F}_2) = C^*(\mathbb{Z}) * C^*(\mathbb{Z})$, where the reduced free product is taken with respect

[3] Rørdam has recently answered this question – there exist simple, nuclear algebras which are infinite (resp. finite) but not purely infinite (resp. stably finite – hence not QD) [Rør2].

to Haar measure on the circle. The next result of F. Boca is in stark contrast. (See also [ExLo] where the class of RFD algebras is shown to be closed under full free products.)

Theorem 13.2 (Bo, Prop. 13). *If A and B are unital QD C^*-algebras, then the full free product (amalgamating over the units) $A * B$ is also QD.*

We next point out the connection between quasidiagonality and the notions of projectivity and semiprojectivity. These notions are studied at length in [Lo].

Definition 13.3. Let A be a C^*-algebra. Then A is called *projective* if for every C^*-algebra B, closed 2-sided ideal $I \subset B$ and $*$-homomorphism $\varphi : A \to B/I$ there exists a lifting $*$-homomorphism $\psi : A \to B$. A is called *semiprojective* if for every C^*-algebra B, closed 2-sided ideal $I \subset B$ such that $I = \overline{\cup_n I_n}$ for ideals $I_1 \subset I_2 \subset \dots$ and $*$-homomorphism $\varphi : A \to B/I$ there exists an n and a lifting $*$-homomorphism $\psi : A \to B/I_n$ (that is, a lifting for the canonical quotient map $B/I_n \to B/I$).

The projective case in our next result is well known. The semiprojective case was pointed out by B. Blackadar, though his proof was different.

Proposition 13.4. *If A is projective then A is RFD. If A is MF and semiprojective then A is RFD.*

Proof. First assume that A is projective. By Corollary 5.3 A is a quotient of an RFD algebra. But then the definition of projectivity implies that A embeds into an RFD algebra and hence is itself RFD.

Now assume that A is semiprojective and MF. By the proof of Proposition 9.3 we can find an RFD algebra R with finite dimensional ideals $I_n \subset I_{n+1}$ such that $A \cong R/I$ where $I = \overline{\cup_n I_n}$. The definition of semiprojectivity then provides an embedding $A \hookrightarrow R/I_n \subset R$ for some n. Q.E.D.

We now discuss a beautiful connection between quasidiagonality and the question of whether or not 'Ext is a group'. (See also the discussion in [Vo4].) Here we mean the classical BDF Ext semigroups. Recall that if A is nuclear then the Choi-Effros lifting theorem implies that $Ext(A)$ is a group. (See [Ar] for a very nice treatment of this theory.) But it is known that there exist C^*-algebras A for which $Ext(A)$ is not a group (cf. [An], [Was1,2], [Kir1]). For example, Kirchberg has shown that if A is the unitization of the cone over $C^*_r(\mathbb{F}_2)$ then $Ext(A)$ is not a group. However, it has been a long standing open problem to determine whether or not $Ext(C_r^*(\mathbb{F}_2))$ is a group. It is believed that $Ext(C_r^*(\mathbb{F}_2))$ is not a group and we now outline one approach to proving this.

We described the class of MF algebras in Section 9. Recall that these algebras can be characterized as those which appear as the image in the Calkin algebra of a quasidiagonal set of operators in $B(H)$ (cf. Proposition 9.2).

Corollary 13.5. *Let A be MF and assume $Ext(A)$ is a group. Then A is QD.*

Proof. If $Ext(A)$ is a group then every *-monomorphism $\varphi : A \to B(H)/\mathcal{K}$ has a completely positive lifting (cf. [Ar, pg. 353]). But then from Propositions 8.3 we see that A must be QD. Q.E.D.

It follows then that every nuclear MF algebra is QD. Recall, though, that there exist non-QD MF algebras. But it is not known whether Wassermann's examples are exact. The following question remains open.

Question 13.6. *Do there exist exact non-QD MF algebras? In particular, is $C_r^*(\mathbb{F}_2)$ MF?*[4]

Kirchberg has also proved some remarkable results connecting quasi-diagonality, *Ext* and various lifting properties of C^*-algebras (see [Kir1]).

Finally, we wish to point out a connection with one of the most important questions in C^*-algebras. Namely, whether or not the Universal Coefficient Theorem (UCT) holds for all nuclear separable C^*-algebras (cf. [RS]). We will not formulate this question precisely as it is well out of the scope of these notes. However, the experts will have no problem following our argument. The main ingredient is the following 'two out of three principle' for the UCT.

Theorem 13.7. *(cf. [RS, Prop. 2.3 and Thm. 4.1]) Let $0 \to I \to E \to B \to 0$ be a short exact sequence with E nuclear and separable. If any two of $\{I,\ E,\ B\}$ satisfy the UCT then so does the third. In particular, if I and E satisfy the UCT then so does B.*

Our final result has been noticed by several experts.

Corollary 13.8. *If the UCT holds for all separable nuclear RFD algebras then the UCT holds for all separable nuclear C^*-algebras.*

Proof. By the two out of three principle, it suffices to show that every separable nuclear C^*-algebra is a quotient of a separable nuclear RFD algebra. But this is contained in Corollary 5.3 Q.E.D.

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[4] We have been informed that Haagerup and Thorbjørnsen have now resolved this question affirmatively and hence $Ext(C_r^*(\mathbb{F}_2))$ is not a group.

§14. Further Reading

Below are references to some of the topics around quasidiagonality which are only briefly discussed (or not discussed at all) in these notes.

AF embeddability. [BrN1,2], [Dăd4,5], [Ka], [Li3], [Ma], [Oz], [Pi], [PV2], [Rør3], [Sp], [Vo2].

Ext and KK-theory. [BrL2], [DE2], [DHS], [Kir1], [PV1], [Sa1,2], [Sc1,2], [Wa1,2].

Classification. [DE1], [Ell], [Li2,4] and their bibliographies.

MF, (strong) NF, and inner quasidiagonal algebras. [BK1,2], [KW].

General. [Had2], [Th], [Vo4].

§15. Appendix: Nonseparable QD C^*-algebras

In this appendix we treat the case of nonseparable C^*-algebras. Hence we no longer require the Hilbert spaces in this section to be separable either. The results of this section (in particular Corollary 15.7) are necessary for the general case of Voiculescu's characterization of QD C^*-algebras. Though we have seen some of these results stated in the literature, we have been unable to find any proofs and hence complete proofs will be given.

Definition 15.1. A subset $\Omega \subset B(H)$ is a called a *quasidiagonal set of operators* if for each finite set $\omega \subset \Omega$, finite set $\chi \subset H$ and $\varepsilon > 0$ there exists a finite rank projection $P \in B(H)$ such that $\|[T, P]\| \leq \varepsilon$ and $\|P(x) - x\| \leq \varepsilon$ for all $T \in \omega$ and $x \in \chi$.

It is still easy to see that a set $\Omega \subset B(H)$ is a quasidiagonal set of operators if and only if the C^*-algebra generated by Ω, $C^*(\Omega) \subset B(H)$, is a quasidiagonal set of operators.

We may finally give the general definition of a quasidiagonal C^*-algebra.

Definition 15.2. Let A be a C^*-algebra. Then A is called *quasidiagonal* (QD) if there exists a faithful representation $\pi : A \to B(H)$ such that $\pi(A)$ is a quasidiagonal set of operators.

There is one subtle point that needs resolved here. Namely we must show that the previous definition is equivalent to Definition 3.8 in the case that A is a separable C^*-algebra.

Lemma 15.3. *Let A be a separable C^*-algebra and assume that there exists a faithful representation $\pi : A \to B(H)$ such that $\pi(A)$ is a* *quasidiagonal set of operators. Then there exists a faithful representation $\rho : A \to B(K)$ such that K is a separable Hilbert space and $\rho(A)$ is a quasidiagonal set of operators.*

Proof. Let $\pi : A \to B(H)$ be a faithful representation such that $\pi(A)$ is a quasidiagonal set of operators. We will show that there exists a *separable* subspace $K \subset H$ which is $\pi(A)$-invariant and such that the restriction representation $\rho = \pi_K = P_K\pi(\cdot)P_K : A \to B(K)$ (cf. Definition 3.9) is faithful and has the property that $\rho(A)$ is a quasidiagonal set of operators.

The idea is to construct an increasing sequence of separable $\pi(A)$-invariant subspaces $\tilde{K}_1 \subset \tilde{K}_2 \subset \tilde{K}_3 \dots$ and finite rank projections Q_n such that $Q_n(H) \subset \tilde{K}_{n+1}$, $\|[Q_n, \pi(a)]\| \to 0$ for all $a \in A$ and $\|Q_n(\xi) - \xi\| \to 0$ for all $\xi \in \cup\tilde{K}_i$. If we further arrange that the restriction of $\pi(A)$ to $\tilde{K}_1$ is faithful then it is clear that $K = \overline{\cup\tilde{K}_i}$ is the desired subspace.

We begin by choosing a sequence $\{a_i\} \subset A$ which is dense in the unit ball of A. For each $n \in \mathbb{N}$ we then choose a sequence of unit vectors $\{\xi_i^{(n)}\}_{i\in\mathbb{N}} \subset H$ such that $\|\pi(a_i)\xi_i^{(n)}\| > \|a_i\| - 1/2^n$. Let $K_1 \subset H$ be the closure of the span of $\{\xi_i^{(n)}\}_{i,n\in\mathbb{N}}$ and let $\tilde{K}_1$ be the closure of $\pi(A)K_1$. Then it is clear that $\tilde{K}_1$ is separable, $\pi(A)$-invariant and the restriction of $\pi(A)$ to $\tilde{K}_1$ is faithful (since it is isometric on $\{a_i\}$).

One then constructs the desired $\tilde{K}_i$ and Q_i recursively as follows. Let $\{h_i^{(1)}\}$ be an orthonormal basis for $\tilde{K}_1$. Choose a finite rank projection $Q_1 \in B(H)$ such that $\|[Q_1, \pi(a_1)]\| < 1/2$ and $Q_1(h_1^{(1)}) = h_1^{(1)}$. Recall from the proof of Proposition 3.4 that we can always arrange the stronger condition $Q_1(h_1^{(1)}) = h_1^{(1)}$.

Next let $X_2 = span\{Q_1(H), \tilde{K}_1\}$, $\tilde{K}_2$ be the losure of $\pi(A)X_2$ and let $\{h_i^{(2)}\}$ be an orthonormal basis for $\tilde{K}_2$. Now choose a finite rank projection $Q_2 \in B(H)$ such that $\|[Q_2, \pi(a_i)]\| < 1/(2^2)$ for $i = 1, 2$, $Q_2(h_i^{(j)}) = h_i^{(j)}$ for $i, j = 1, 2$ and $Q_1 \leq Q_2$ (this is arranged by requiring that $Q_2(h) = h$ for a (finite) basis of $Q_1(H)$).

Next let $X_3 = span\{Q_2(H), \tilde{K}_2\}$, $\tilde{K}_3$ be the closure of $\pi(A)X_3$ and let $\{h_i^{(3)}\}$ be an orthonormal basis for $\tilde{K}_3$, etc. Proceeding in this way we get an increasing sequence of separable $\pi(A)$-invariant subspaces $\tilde{K}_1 \subset \tilde{K}_2 \subset \tilde{K}_3 \dots$ with orthonormal bases $\{h_i^{(n)}\}_{i\in\mathbb{N}}$ and finite rank projections $Q_n \leq Q_{n+1}$ such that $Q_n(h_i^{(j)}) = h_i^{(j)}$ for $i, j = 1, \dots, n$ and $Q_n(H) \subset \tilde{K}_{n+1}$, $\|[Q_n, \pi(a)]\| \to 0$ for all $a \in A$. Evidently this proves the lemma. Q.E.D.

Hence we see that Definitions 3.8 and 14.2 are equivalent for separable C^*-algebras. Indeed, it clear that if A is separable and satisfies

Definition 3.8 then A also satisfies Definition 14.2. On the other hand, if A is separable and satisfies Definition 14.2 then by the previous lemma we can find a representation of A on a *separable* Hilbert space which gives a quasidiagonal set of operators and hence A satisfies Definition 3.8 as well.

We will need the following elementary, but technical, lemma.

Lemma 15.4. *Let $\pi : A \to B(H)$ be a faithful representation where A is separable (but H is not). Then there exists a separable $\pi(A)$-invariant subspace $K \subset H$ with the property that $\pi_K : A \to B(K)$ is faithful, $\pi_K(a)$ is a finite rank operator if and only if $\pi(a)$ is a finite rank operator and in this case $dim(\pi(a)H) = dim(\pi_K(a)K)$.*

Proof. The idea is to find a sequence of $\pi(A)$-invariant separable subspaces, $\tilde{H}_i$, with the following properties:

1. The restriction of $\pi(A)$ to $\tilde{H}_1$ is faithful.
2. If $a \in A$ is such that $\pi(a)$ is a finite rank operator then $\pi(a)H \subset \tilde{H}_1$.
3. $\tilde{H}_m \perp (\tilde{H}_1 \oplus \tilde{H}_2 \oplus \ldots \oplus \tilde{H}_{m-1})$
4. If $P_{\tilde{H}_m}\pi(a)P_{\tilde{H}_m} = 0$ then
$$(1 - P_{\tilde{H}_1 \oplus \ldots \oplus \tilde{H}_{m-1}})\pi(a)(1 - P_{\tilde{H}_1 \oplus \ldots \oplus \tilde{H}_{m-1}}) = 0$$
for all $a \in A$, where for any subspace $L \subset H$, P_L denotes the orthogonal projection onto L.

Having the subspaces $\{\tilde{H}_i\}$ we define
$$K = \oplus_{i=1}^{\infty} \tilde{H}_i \subset H$$
and note that $\pi_K : A \to B(K)$ is faithful (since this was already arranged on $\tilde{H}_1$). Moreover, condition 2 ensures that if $\pi(a)$ is a finite rank operator then $dim(\pi(a)H) = dim(\pi_K(a)K)$. Finally, note that if $\pi_K(a)$ is a finite rank operator then there exists some integer $m \in \mathbb{N}$ such that $\tilde{\pi}_m(a) = P_{\tilde{H}_m}\pi(a)P_{\tilde{H}_m} = 0$. Hence
$$\begin{aligned}
\pi(a) &= \tilde{\pi}_1(a) \oplus \cdots \oplus \tilde{\pi}_{m-1}(a) \\
&\qquad \oplus (1 - P_{\tilde{H}_1 \oplus \ldots \oplus \tilde{H}_{m-1}})\pi(a)(1 - P_{\tilde{H}_1 \oplus \ldots \oplus \tilde{H}_{m-1}}) \\
&= \tilde{\pi}_1(a) \oplus \cdots \oplus \tilde{\pi}_{m-1}(a) \oplus 0 \\
&= \pi_K(a),
\end{aligned}$$
by condition 4 above. Hence $\pi(a)$ is also a finite rank operator and clearly $dim(\pi(a)\ H) = dim(\pi_K(a)K)$. So we now show how to construct subspaces $\tilde{H}_i$ as above.

Begin by letting $\mathcal{F}(A) = \{a \in A : dim(\pi(a)H) < \infty\}$ and choosing a countable dense subset $\{a_i\}_{i\in\mathbb{N}} \subset \mathcal{F}(A)$. For each $i \in \mathbb{N}$ let $L_i = \pi(a_i)H$ and define H_1 to be the closure of

$$span\{\bigcup_{i=1}^{\infty} \pi(A)L_i\}.$$

By throwing in a countable number of vectors (as in the proof of Lemma 15.3) we can replace H_1 with a larger $\pi(A)$-invariant subspace $\tilde{H}_1$ such that the restriction of $\pi(A)$ to $\tilde{H}_1$ is also faithful. We claim that this $\tilde{H}_1$ also satisfies condition 2 above. Indeed, if $a \in \mathcal{F}(A)$ then we can find a subsequence $a_{i_j} \to a$. But since $\pi(a_{i_j})H \subset \tilde{H}_1$ and $\pi(a_{i_j}) \to \pi(a)$ it is clear that $\pi(a)H \subset \tilde{H}_1$ as well. Hence we have constructed $\tilde{H}_1$ with the desired properties.

Assume now that we have constructed $\tilde{H}_1, \ldots, \tilde{H}_{m-1}$ with the desired properties. To get $\tilde{H}_m$ we simply consider the separable C^*-algebra

$$C = (1 - P_{\tilde{H}_1\oplus\ldots\oplus\tilde{H}_{m-1}})\pi(A)(1 - P_{\tilde{H}_1\oplus\ldots\oplus\tilde{H}_{m-1}}).$$

By the proof of Lemma 15.3 we can find a separable C-invariant subspace $\tilde{H}_m \subset (1 - P_{\tilde{H}_1\oplus\ldots\oplus\tilde{H}_{m-1}})H$ such that the restriction of C to $\tilde{H}_m$ is faithful. Evidently $\tilde{H}_m$ is also $\pi(A)$ invariant, perpendicular to $\tilde{H}_1\oplus\tilde{H}_2\oplus \ldots \oplus \tilde{H}_{m-1}$ and condition 4 above is nothing more than the statement that the map $C \to P_{\tilde{H}_m} C P_{\tilde{H}_m}$ is faithful. Q.E.D.

As in section 3 we want to resolve the technical issue of nondegeneracy of representations.

Lemma 15.5. *Let A be a C^*-algebra and $\pi : A \to B(H)$ be a faithful representation. Let $L \subset H$ be the nondegeneracy subspace of $\pi(A)$ and $\pi_L : A \to B(L)$ denote the restriction. Then $\pi(A)$ is a quasidiagonal set of operators if and only if $\pi_L(A)$ is a quasidiagonal set of operators.*

Proof. The implication ($\Leftarrow$) is proved exactly as in Lemma 3.10. Also, if A is unital, the implication ($\Rightarrow$) is the same and so we only have to show ($\Rightarrow$) in the case that A is nonunital.

So assume that A is nonunital and $\pi(A)$ is a quasidiagonal set of operators. Note that we cannot apply Voiculescu's Theorem in this setting since the dimensions of H and L may be different. To resolve this problem we first note that since quasidiagonality is defined via finite sets it suffices to show that $\pi_L(B)$ is a quasidiagonal set of operators for every *separable* C^*-subalgebra $B \subset A$.

Given a finite set of vectors $\chi \subset L$, by Lemma 15.4, we can find a separable subspace $K \subset L$ with the property that $\chi \subset K$, K is $\pi_L(A)$-invariant, the restriction to K is faithful, $\pi_L(a)$ is finite rank if and only if $\pi_K(a)$ is finite rank and in this case $rank(\pi_L(a)) = rank(\pi_K(a))$. As in the proof of Lemma 15.3 we can now enlarge K to a separable $\pi(A)$-invariant subspace $\tilde{K} \subset H$ (we do not have $\tilde{K} \subset L$, of course) such that $\pi_{\tilde{K}}(A)$ is a quasidiagonal set of operators. Since we have been careful about separability and preservation of rank it now follows from Voiculescu's theorem (version 2.6) that $\pi_{\tilde{K}}$ and π_K are approximately unitarily equivalent and hence $\pi_K(A)$ is a quasidiagonal set of operators. Q.E.D.

Theorem 15.6. *Let $\pi : A \to B(H)$ be a faithful* essential *(cf. Definition 2.8) representation. Then A is QD if and only if $\pi(A)$ is a quasidiagonal set of operators.*

Proof. Clearly we only have to prove the necessity. As in the proof of the previous lemma, it suffices to show that $\pi(B)$ is a quasidiagonal set of operators for every separable subalgebra $B \subset A$.

Let $\chi \subset H$ be an arbitrary finite set and use Lemma 15.4 to construct a separable $\pi(B)$-invariant subspace $K \subset H$ such that $\chi \subset K$ and the restriction to K is both faithful and essential. The remainder of the proof is now similar to that of Theorem 3.11. Q.E.D.

The next corollary shows that with care, one can usually just treat the separable case when dealing with quasidiagonality.

Corollary 15.7. *A is QD if and only if all of it's finitely generated subalgebras are QD.*

Proof. The necessity is obvious from the definition. So assume all finitely generated subalgebras of A are QD and let $\pi : A \to B(H)$ be a faithful essential representation. Then for each finitely generated subalgebra $B \subset A$ the restriction $\pi|_B$ is a faithful essential representation and hence (by Theorem 15.6) $\pi(B)$ is a quasidiagonal set of operators. It then follows that $\pi(A)$ is a quasidiagonal set of operators. Q.E.D.

Finally we observe the nonseparable version of Theorem 4.2.

Corollary 15.8 (Voiculescu). *Let A be a C^*-algebra. Then A is QD if and only if A satisfies $(*)$.*

Proof. As in the proof of Theorem 4.2, we may assume that A is unital. From Arveson's Extension Theorem it follows that A satisfies $(*)$ if and only if every separable unital subalgebra of A satisfies $(*)$. Similarly, from Corollary 15.7 it follows that A is QD if and only if every separable unital subalgebra of A is QD. Hence this corollary follows from the separable case. Q.E.D.

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Extensions of quasidiagonal C^*-algebras and K-theory

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Abstract.

Let $0 \to I \to E \to B \to 0$ be a short exact sequence of C^*-algebras where E is separable, I is quasidiagonal (QD) and B is nuclear, QD and satisfies the UCT. It is shown that if the boundary map $\partial : K_1(B) \to K_0(I)$ vanishes then E must be QD also.

A Hahn-Banach type property for K_0 of QD C^*-algebras is also formulated. It is shown that every nuclear QD C^*-algebra has this K_0-Hahn-Banach property if and only if the boundary map $\partial : K_1(B) \to K_0(I)$ (from above) always completely determines when E is QD in the nuclear case.

§1. Introduction

Quasidiagonal (QD) C^*-algebras are those which enjoy a certain finite dimensional approximation property. (See [Vo2], [Br3] for surveys of the theory of QD C^*-algebras.) While these finite dimensional approximations have certainly lead to a better understanding of the structure of QD C^*-algebras, there are a number of very basic open questions. For example, assume that $0 \to I \to E \xrightarrow{\pi} B \to 0$ is a split exact sequence (i.e. there exists a *-homomorphism $\varphi : B \to E$ such that $\pi \circ \varphi = id_B$) where both I and B are QD. It is not known whether E must be QD (and, in fact, it is not even clear what to expect).

In this paper we study the extension problem for QD C^*-algebras and it's relation to some natural questions concerning K-theory of QD C^*-algebras. Our techniques rely heavily on Kasparov's theory of extensions and thus we will always need some nuclearity assumptions.

For example, adapting techniques found in [Sp] we will show (Theorem 3.4) that if $0 \to I \to E \to B \to 0$ is short exact where E is separable,

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I is QD, B is nuclear, QD and satisfies the Universal Coefficient Theorem (UCT) and the boundary map $\partial : K_1(B) \to K_0(I)$ vanishes then E must be QD also. It follows that if $K_1(B) = 0$ then E is always QD, which generalizes work of Eilers, Loring and Pedersen ([ELP]). As another application we observe that in the case that I is the compact operators our result implies that E is QD if and only if the (class of the) extension is in the kernel of the natural map $Ext(B) \to Hom(K_1(B), \mathbb{Z})$, where $Ext(B)$ denotes the classical BDF group (recall that we are assuming B is nuclear and hence $Ext(B)$ is a group). Also, we verify a conjecture of [BK], stating that an asymptotically split extension of NF algebras is NF, under the additional assumption that the quotient algebra satisfies the UCT of [RS].

We then study the general extension problem. Now let $0 \to I \to E \to B \to 0$ be exact where E is separable and nuclear, I is QD and B is QD and satisfies the UCT. Based on previous work of Spielberg ([Sp]) it is reasonable to expect that in this case E will be QD if and only if $\partial(K_1(B)) \cap K_0^+(I) = \{0\}$, where $K_0^+(I) = \{0\}$ denotes the positive cone of $K_0(I)$. Though we are unable to resolve this question we do show that it is equivalent to some other natural questions concerning the K-theory of QD C^*-algebras and that in order to solve the general extension problem it suffices to prove the special case that $B = C(\mathbb{T})$ (see Theorem 4.11).

The first equivalent K-theory question is: If A is nuclear, separable and QD and $G \subset K_0(A)$ is a subgroup such that $G \cap K_0^+(A) = 0$ then can one always find an embedding $\rho : A \hookrightarrow C$ where C is QD and $\rho_*(G) = 0$? The condition $G \cap K_0^+(A) = 0$ is easily seen to be necessary and hence the question is whether or not it is sufficient. The second K-theory question asks whether every nuclear QD C^*-algebra satisfies what we call the *K_0-Hahn-Banach property* (see Definition 4.7). Roughly speaking this K_0-Hahn-Banach property states that if $x \in K_0(A)$ and $\pm x \notin K_0^+(A)$ then one can always find finite dimensional approximate morphisms (i.e. "functionals") which separate x from $K_0^+(A)$. (Due to possible perforation in $K_0(A)$ this statement is not quite correct, but it conveys the main idea.) Determining whether every nuclear QD algebra satisfies the K_0-Hahn-Banach property is of independent interest as our inability to understand how well finite dimensional approximate morphisms read K-theory has been a major obstacle in the classification program.

In section 2 we review the necessary theory of extensions and prove a few simple results needed later. In section 3 we handle the case when $\partial : K_1(B) \to K_0(I)$ vanishes. In section 4 we turn to the general extension problem and show equivalence with the K-theory questions described above.

The present work is related to work of Salinas [Sa1], [Sa2] and Schochet [Sch]. Those authors study the quasidiagonality of extensions $0 \to I \to E \to B \to 0$ (i.e. the question of whether or not I contains an approximate unit of projections which is quasicentral in E) whereas we study the QD of the C^*-algebra E. The two questions are different even if I is the compact operators. Indeed, while the quasidiagonality of $0 \to \mathcal{K} \to E \to B \to 0$ does imply the QD of E, the converse implication is false (see Section 3).

§2. Preliminaries and Trivial Extensions.

Most of this section is devoted to reviewing definitions, introducing notation and recalling some standard facts about extensions of C^*-algebras. However, at the end we prove a few simple facts which will be needed later. The main result states that quasidiagonality is preserved in split extensions provided that either the ideal or the quotient is a nuclear C^*-algebra (see Proposition 2.5).

For a comprehensive introduction to the aspects of extension theory which we will need we recommend looking at [Bl, Section 15]. For any C^*-algebra I we will let $M(I)$ be it's multiplier algebra and $Q(I) = M(I)/I$ be it's corona algebra. Let $\pi : M(I) \to Q(I)$ be the quotient map.

If E is any C^*-algebra containing I as an ideal and $B = E/I$ then there exists a unique *-homomorphism $\rho : E \to M(I)$ such that $\rho(I) = I$ and hence an induced *-homomorphism $\gamma : B \to Q(I)$. The map γ is injective if and only if ρ is in injective if and only if I sits as an essential ideal in E. Conversely, given a C^*-algebra B and a *-homomorphism $\gamma : B \to Q(I)$ we can construct the pullback which, by definition, is the C^*-algebra

$$E(\gamma) = \{x \oplus b \in M(I) \oplus B : \pi(x) = \gamma(b)\}.$$

This gives a short exact sequence $0 \to I \to E(\gamma) \to B \to 0$. Moreover, if $B = E/I$ with induced map $\gamma : B \to Q(I)$ then there is an induced *-isomorphism $\Phi : E \to E(\gamma)$ with commutativity in the diagram

$$\begin{tikzcd}
0 \arrow[r] & I \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d, "\Phi"'] & B \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & I \arrow[r] & E(\gamma) \arrow[r] & B \arrow[r] & 0.
\end{tikzcd}$$

Hence there is a one to one correspondence between extensions of I by B and *-homomorphisms $\gamma : B \to Q(I)$. As is standard, we will refer to a *-homomorphism $\gamma : B \to Q(I)$ as a *Busby invariant* and freely use the above correspondence between Busby invariants and extensions.

When I is stable (i.e. $I \cong \mathcal{K} \otimes I$, where $\mathcal{K}$ denotes the compact operators on a separable infinite dimensional Hilbert space) there is a natural way of adding two extensions which we now describe. Any isomorphism $M_2(\mathbb{C}) \otimes \mathcal{K} \cong \mathcal{K}$ induces an isomorphism $M_2(\mathbb{C}) \otimes \mathcal{K} \otimes I \cong \mathcal{K} \otimes I$ which then gives isomorphisms $M_2(\mathbb{C}) \otimes M(\mathcal{K} \otimes I) \cong M(\mathcal{K} \otimes I)$ and $M_2(\mathbb{C}) \otimes Q(\mathcal{K} \otimes I) \cong Q(\mathcal{K} \otimes I)$. Thus if we are given two Busby invariants $\gamma_1,\ \gamma_2 : B \to Q(\mathcal{K} \otimes I)$ we can define a new Busby invariant $\gamma_1 \oplus \gamma_2$ by

$$(\gamma_1 \oplus \gamma_2)(b) = \begin{pmatrix} \gamma_1(b) & 0 \\ 0 & \gamma_2(b) \end{pmatrix} \in M_2(\mathbb{C}) \otimes Q(\mathcal{K} \otimes I) \cong Q(\mathcal{K} \otimes I).$$

Of course the Busby invariant $\gamma_1 \oplus \gamma_2$ constructed in this way will depend on the particular isomorphism $M_2(\mathbb{C}) \otimes \mathcal{K} \cong \mathcal{K}$. To remedy this we say that two Busby invariants $\gamma_1,\ \gamma_2$ are *strongly equivalent* if there exists a unitary $u \in M(I)$ such that $\mathrm{Ad}\pi(u)\big(\gamma_1(b)\big) = \pi(u)\gamma_1(b)\pi(u^*) = \gamma_2(b)$, for all $b \in B$, where $\pi : M(I) \to Q(I)$ is the quotient map. Note that if γ_1 and γ_2 are strongly equivalent then $E(\gamma_1)$ and $E(\gamma_2)$ are isomorphic C^*-algebras. Indeed, the map $E(\gamma_1) \to E(\gamma_2)$, $x \oplus b \mapsto uxu^* \oplus b$ is easily seen to be an isomorphism. Since any isomorphism $M_2(\mathbb{C}) \otimes \mathcal{K} \cong \mathcal{K}$ is implemented by a unitary we see that $\gamma_1 \oplus \gamma_2$ is unique up to strong equivalence. In particular, the isomorphism class of the C^*-algebra $E(\gamma_1 \oplus \gamma_2)$ does not depend on the choice of isomorphism $M_2(\mathbb{C}) \otimes \mathcal{K} \cong \mathcal{K}$.

A Busby invariant τ is called *trivial* if it lifts to a *-homomorphism $\varphi : B \to M(I)$ (i.e. $\pi \circ \varphi = \gamma$). A Busby invariant $\gamma : B \to Q(\mathcal{K} \otimes I)$ is called *absorbing* if $\gamma \oplus \tau$ is strongly equivalent to γ for every trivial τ. Note that if γ is absorbing then so is $\tilde{\gamma} \oplus \gamma$ for any $\tilde{\gamma}$. In particular if γ is absorbing then γ is injective. Note also that if τ_1 and τ_2 are both trivial and absorbing then τ_1, $\tau_1 \oplus \tau_2$ and τ_2 are all strongly equivalent. Thus we get the following fact.

Lemma 2.1. *If $\tau_1,\ \tau_2 : B \to Q(\mathcal{K} \otimes I)$ are both trivial and absorbing then $E(\tau_1) \cong E(\tau_2)$.*

Another simple fact we will need is the following.

Lemma 2.2. *If $\gamma,\ \tau : B \to Q(\mathcal{K} \otimes I)$ are Busby invariants with τ trivial then there is a natural embedding $E(\gamma) \hookrightarrow E(\gamma \oplus \tau)$.*

Proof. Let $\varphi : B \to M(I)$ be a lifting of τ. Define a map $E(\gamma) \to E(\gamma \oplus \tau)$ by

$$x \oplus b \mapsto \begin{pmatrix} x & 0 \\ 0 & \varphi(b) \end{pmatrix} \oplus b.$$

Evidently this map is an injective $*$-homomorphism. □

The following generalization of Voiculescu's Theorem, which is due to Kasparov, will be crucial in what follows.

Theorem 2.3. ([Bl, Thm. 15.12.4]) *Assume that B is separable, I is σ-unital and either B or I is nuclear. Let $\rho : B \to B(H)$ be a faithful representation such that H is separable, $\rho(B) \cap \mathcal{K}(H) = \{0\}$ and the orthogonal complement of the nondegeneracy subspace of $\rho(B)$ (i.e. $H \ominus \overline{\rho(B)H}$) is infinite dimensional. Regarding $B(H) \cong B(H) \otimes 1 \subset M(\mathcal{K} \otimes I)$ as scalar operators we get a short exact sequence*

$$0 \to \mathcal{K} \otimes I \to \rho(B) \otimes 1 + \mathcal{K} \otimes I \to B \to 0.$$

If τ denotes the induced Busby invariant then τ is both trivial and absorbing.

We define an equivalence relation on the set of Bubsy invariants $B \to Q(\mathcal{K} \otimes I)$ by saying γ is related to $\tilde{\gamma}$ if there exist trivial Busby invariants $\tau, \tilde{\tau}$ such that $\gamma \oplus \tau$ is strongly equivalent to $\tilde{\gamma} \oplus \tilde{\tau}$. Taking the quotient by this relation yields the semigroup $Ext(B, \mathcal{K} \otimes I)$. The image of a map $\gamma : B \to Q(\mathcal{K} \otimes I)$ in $Ext(B, \mathcal{K} \otimes I)$ is denoted $[\gamma]$. Note that all trivial Busby invariants give rise to the same class denoted by $0 \in Ext(B, \mathcal{K} \otimes I)$ and this class is a neutral element (i.e. identity) for the semigroup. Note also that if $[\gamma] = 0 \in Ext(B, \mathcal{K} \otimes I)$ then it does not follow that γ is trivial. However, it does follow that if τ is a trivial absorbing Busby invariant then so is $\gamma \oplus \tau$.

We are almost ready to prove the main result of this section. We just need one more definition.

Definition 2.4. *If $0 \to I \to E \to B \to 0$ is an exact sequence with Busby invariant γ then we let $\gamma^s : \mathcal{K} \otimes B \to Q(\mathcal{K} \otimes I)$ denote the stabilization of γ. That is, γ^s is the Busby invariant of the exact sequence $0 \to \mathcal{K} \otimes I \to \mathcal{K} \otimes E \to \mathcal{K} \otimes B \to 0$.*

Note that there is always an embedding $E \cong E(\gamma) \hookrightarrow E(\gamma^s)$.

Proposition 2.5. *Let $0 \to I \to E \to B \to 0$ be exact with Busby invariant γ. If both I and B are QD, B is separable, I is σ-unital, either I or B is nuclear and $[\gamma^s] = 0 \in Ext(\mathcal{K} \otimes B, \mathcal{K} \otimes I)$ then E is also QD.*

Proof. Since quasidiagonality passes to subalgebras, it suffices to show that if $\tau : \mathcal{K}\otimes B \to Q(\mathcal{K}\otimes I)$ is a trivial absorbing Busby invariant (which exists by Theorem 2.3) then $E(\tau)$ is QD. Indeed, by Lemmas 2.1, 2.2 and the definition of $Ext(\mathcal{K}\otimes B, \mathcal{K}\otimes I)$ we have the inclusions

$$E \hookrightarrow E(\gamma^s) \hookrightarrow E(\gamma^s \oplus \tau) \cong E(\tau).$$

To prove that $E(\tau)$ is QD we may assume (again by Lemma 2.1) that τ arises from the particular extension described in Theorem 2.3. However for that extension it is easy to see that $E(\tau) \hookrightarrow (\rho(B)+\mathcal{K})\otimes \tilde{I}$, where $\tilde{I}$ is the unitization of I. But since $\rho(B)\cap\mathcal{K} = \{0\}$ it follows that $\rho(B)+\mathcal{K}$ is QD ([Br3, Thm. 3.11]). Hence $(\rho(B)+\mathcal{K})\otimes\tilde{I}$ is also QD as a minimal tensor product QD C^*-algebras ([Br3, Prop. 7.5]). □

Note that the above proposition covers the case of split extensions (i.e. when γ is trivial).

§3. When $\partial : K_1(B) \to K_0(I)$ is zero.

The main result of this section (Theorem 3.4) states that if the boundary map $\partial : K_1(B) \to K_0(I)$ coming from an exact sequence $0 \to I \to E \to B \to 0$ is zero then E will be QD whenever I is QD and B is nuclear, QD and satisfies the Universal Coefficient Theorem (UCT) of Rosenberg and Schochet ([RS]). The main ideas in the proof are inspired by work of Spielberg ([Sp]). We also discuss a few consequences of our result, including generalization of work of Eilers-Loring-Pedersen ([ELP]) and a partial solution to a conjecture of Blackadar and Kirchberg [BK].

Definition 3.1. An embedding $I \hookrightarrow J$ is called *approximately unital* if it takes an approximate unit of I to an approximate unit of J.

In this case there is a natural inclusion $M(I) \hookrightarrow M(J)$ which induces an inclusion $Q(I) \hookrightarrow Q(J)$ [Pe, 3.12.12]. Hence for any Busby invariant $\gamma : B \to Q(I)$ there is an induced Busby invariant $\eta : B \to Q(J)$ with commutativity in the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & I & \longrightarrow & E(\gamma) & \longrightarrow & B & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & J & \longrightarrow & E(\eta) & \longrightarrow & B & \longrightarrow & 0.
\end{array}$$

Moreover, the two vertical maps on the left are injective.

There are two ways of producing approximately unital embeddings which we will need. The first is $I \hookrightarrow I \otimes A$, for some *unital* C^*-algebra

A. If $\{e_\lambda\}$ is an approximate unit of I then, of course, $e_\lambda \otimes 1_A$ will be an approximate unit of $I \otimes A$. The other is to start with an arbitrary embedding $I \hookrightarrow J'$ and define J to be the hereditary subalgebra in J' generated by I. That is, define J to be the closure of $\cup_\lambda e_\lambda J' e_\lambda$. One easily checks that J is then a hereditary subalgebra of J' and the embedding $I \hookrightarrow J$ is approximately unital.

In the theory of separable QD C^*-algebras there are some *nonseparable* algebras which play a key role. The first is the direct product $\Pi_i M_{n_i}(\mathbb{C})$ for some sequence of integers $\{n_i\}$. This algebra is the multiplier algebra of the direct sum $\oplus_i M_{n_i}(\mathbb{C})$. If H is any separable Hilbert space then we can always find a decomposition $H = \oplus_i \mathbb{C}^{n_i}$ and then we have natural inclusions $\oplus_i M_{n_i}(\mathbb{C}) \hookrightarrow \mathcal{K}(H)$, $\Pi_i M_{n_i}(\mathbb{C}) \hookrightarrow B(H)$ and $Q(\oplus_i M_{n_i}(\mathbb{C})) \hookrightarrow Q(\mathcal{K}(H))$. Another algebra which we will need is $\Pi_i M_{n_i}(\mathbb{C}) + \mathcal{K}(H)$.

Lemma 3.2. *Let $J \subset \Pi_i M_{n_i}(\mathbb{C}) + \mathcal{K}(H)$ be a hereditary subalgebra containing $\mathcal{K}(H)$. Then $K_1(J) = 0$.*

Proof. Letting $\pi : B(H) \to Q(H)$ be the quotient map we have that $\pi(J)$ is a hereditary subalgebra of $Q(\oplus_i M_{n_i}(\mathbb{C}))$ (use the fact that if $0 \le a \in J, b \in Q(\oplus_i M_{n_i}(\mathbb{C}))$ and $0 \le b \le \pi(a)$ then there exists $0 \le c \in \Pi_i M_{n_i}(\mathbb{C}) + \mathcal{K}(H)$ such that $c \le a$ and $\pi(c) = b$; [Da, Cor. IX.4.5]. Also, the exact sequence $0 \to \mathcal{K}(H) \to J \to \pi(J) \to 0$ is a *quasidiagonal extension* (i.e. $\mathcal{K}(H)$ contains an approximate unit *of projections which is quasicentral in J*). Hence [BD, Thm. 8], states that we have a short exact sequence

$$0 \to K_1(\mathcal{K}(H)) \to K_1(J) \to K_1(\pi(J)) \to 0.$$

Thus it suffices to show that $K_1(X) = 0$ for any hereditary subalgebra X of $Q(\oplus_i M_{n_i}(\mathbb{C}))$.

But if $X \subset Q(\oplus_i M_{n_i}(\mathbb{C}))$ is a hereditary subalgebra then we can find a quasidiagonal extension

$$0 \to \oplus_i M_{n_i}(\mathbb{C}) \to Y \to X \to 0,$$

where $Y \subset \Pi_i M_{n_i}(\mathbb{C})$ is a hereditary subalgebra. Applying [BD, Thm. 8] again it suffices to show that every hereditary subalgebra of $\Pi_i M_{n_i}(\mathbb{C})$ has trivial K_1-group.

But, if $Y \subset \Pi_i M_{n_i}(\mathbb{C})$ is a hereditary σ-unital subalgebra then Y has an increasing approximate unit consisting of projections, say $\{e_n\}$ ([BP]). Hence

$$K_1(Y) = \lim K_1(e_n \Pi_i M_{n_i}(\mathbb{C}) e_n),$$

since $Y = \lim e_n \Pi_i M_{n_i}(\mathbb{C}) e_n$ (by heredity). But, for each n, it is clear that $e_n \Pi_i M_{n_i}(\mathbb{C}) e_n$ is isomorphic to $\Pi_i M_{k_i}(\mathbb{C})$ for some integers $\{k_i\}$ and consequently $K_1(e_n \Pi_i M_{n_i}(\mathbb{C}) e_n) = 0$. □

Proposition 3.3. *Let I be a separable QD C^*-algebra. Then there exists an approximately unital embedding $I \hookrightarrow J$, where J is a σ-unital QD C^*-algebra with $K_1(J) = 0$.*

Proof. Let $\rho : I \to B(H)$ be a nondegenerate faithful representation such that $\rho(I) \cap \mathcal{K}(H) = \{0\}$. By [Br3, Prop. 5.2], there exists a decomposition $H = \oplus_i \mathbb{C}^{n_i}$ such that $\rho(I) \subset \Pi_i M_{n_i}(\mathbb{C}) + \mathcal{K}(H)$. Let J be the hereditary subalgebra of $\Pi_i M_{n_i}(\mathbb{C}) + \mathcal{K}(H)$ generated by $\rho(I)$. The conclusion follows from the previous lemma. □

For the remainder of this section we will let $\mathcal{U} = \otimes_n M_n(\mathbb{C})$ be the Universal UHF algebra (i.e. the UHF algebra with $K_0(\mathcal{U}) = \mathbb{Q}$). For any Busby invariant $\gamma : B \to Q(J)$ we let $\gamma^{\mathbb{Q}}$ denote the Busby invariant coming from the short exact sequence

$$0 \to J \otimes \mathcal{U} \to E(\gamma) \otimes \mathcal{U} \to B \otimes \mathcal{U} \to 0.$$

Theorem 3.4. *Let $0 \to I \to E \to B \to 0$ be a short exact sequence where E is separable, I is QD and B is nuclear, QD and satisfies the UCT. If the induced map $\partial : K_1(B) \to K_0(I)$ is zero then E is QD.*

Proof. Let γ be the Busby invariant of the exact sequence in question. By the previous proposition we can find an approximately unital embedding $I \hookrightarrow J$, where J is QD with $K_1(J) = 0$. By the remarks following Definition 3.1 we have an inclusion $E \hookrightarrow E(\eta)$ where $\eta : B \to Q(J)$ is the induced Busby invariant. By naturality we then have that both index maps $\partial : K_1(B) \to K_0(J)$ and $\partial : K_0(B) \to K_1(J)$ are zero. Hence the index maps arising from the stabilization $\eta^s : B \otimes \mathcal{K} \to Q(J \otimes \mathcal{K})$ are also zero.

Now, if it happens that $K_0(J)$ is a divisible group then the Universal Coefficient Theorem would imply that $[\eta^s] = 0 \in Ext(B \otimes \mathcal{K}, J \otimes \mathcal{K})$ and so by Proposition 2.5 we would be done. Of course this will not be true in general and so may have to replace η^s with $(\eta^s)^{\mathbb{Q}}$. But applying naturality one more time, both boundary maps on K-theory arising from $(\eta^s)^{\mathbb{Q}}$ will also vanish. Hence the theorem follows from the inclusions $E \hookrightarrow E(\eta) \hookrightarrow E(\eta^s) \hookrightarrow E((\eta^s)^{\mathbb{Q}})$ together with Proposition 2.5 applied to $(\eta^s)^{\mathbb{Q}}$. □

In the case that the ideal is nuclear and the quotient is an AF algebra, the next result was obtained by Eilers, Loring and Pedersen ([ELP, Cor. 4.6]).

Corollary 3.5. *Assume that B is a separable nuclear QD C^*-algebra satisfying the UCT and with $K_1(B) = 0$. For any separable QD C^*-algebra I and Busby invariant $\gamma : B \to Q(I)$ we have that $E(\gamma)$ is QD.*

This corollary actually extends to the case where $K_1(B)$ is a torsion group since we can tensor any short exact sequence with $\mathcal{U}$ and $K_1(B \otimes \mathcal{U}) = 0$ in this case. For example, this would cover the case that $B = C_0(\mathbb{R}) \otimes \mathcal{O}_n$, $(2 \leq n \leq \infty)$, where $\mathcal{O}_n$ denotes the Cuntz algebra on n generators. Similarly, it is clear that Theorem 3.4 is valid under the weaker hypothesis that $\partial(K_1(B))$ is contained in the torsion subgroup of $K_0(I)$.

Definition 3.6. For any two QD C^*-algebras I, B let $Ext_{QD}(B, \mathcal{K} \otimes I) \subset Ext(B, \mathcal{K} \otimes I)$ denote the set of classes of Busby invariants γ such that $E(\gamma)$ is QD.

It is easy to check that if $[\gamma] = [\tilde{\gamma}] \in Ext(B, \mathcal{K} \otimes I)$ then $E(\gamma)$ is QD if and only if $E(\tilde{\gamma})$ is QD and hence $Ext_{QD}(B, \mathcal{K} \otimes I)$ is well defined. It is also easy to see that $Ext_{QD}(B, \mathcal{K} \otimes I)$ is a sub-semigroup of $Ext(B, \mathcal{K} \otimes I)$. Finally, we remark that in the case $I = \mathbb{C}$ we do *not* get the semigroup $Ext_{qd}(B, \mathcal{K})$ defined by Salinas; it follows from Corollary 3.7 below, however, that we do get what he called $Ext_{bqt}(B, \mathcal{K})$ in this case (see [Sal, Definitions 2.7, 2.12 and Thm. 2.14]). One has $Ext_{qd}(B, \mathcal{K}) \subset Ext_{QD}(B, \mathcal{K})$. The elements of $Ext_{QD}(B, \mathcal{K})$ corresponds to C^*-algebras $E(\gamma)$ that are QD whereas $[\gamma] \in Ext_{qd}(B, \mathcal{K})$ if the only if the extension $0 \to \mathcal{K} \to E(\gamma) \to B \to 0$ is QD i.e. the concrete set $E(\gamma) \subset M(\mathcal{K})$ is QD.

Recall that there is a natural group homomorphism $\Phi : Ext(B, \mathcal{K} \otimes I) \to Hom(K_1(B), K_0(I))$ taking a Busby invariant to the corresponding boundary map on K-theory. From Theorem 3.4 it follows that we always have an inclusion $Ker(\Phi) \subset Ext_{QD}(B, \mathcal{K} \otimes I)$, when B is nuclear, QD and satisfies the UCT. In general this inclusion will be proper, but we now describe a class of algebras for which we have equality.

There is a natural semigroup $K_0^+(I) \subset K_0(I)$, called the *positive cone*, given by

$$K_0^+(I) = \bigcup_{n \in \mathbb{N}} \{x \in K_0(I) : x = [p], \text{ for some projection } p \in M_n(I)\}.$$

When I is unital this semigroup generates $K_0(I)$ but can also be trivial in general (e.g. if I is stably projectionless). The natural isomorphism $K_0(I) \cong K_0(\mathcal{K} \otimes I)$ induced by an embedding $I = e_{11} \otimes I \subset \mathcal{K} \otimes I$, where e_{11} is a minimal projection in $\mathcal{K}$, preserves the positive cones. We say that $K_0(I)$ is *totally ordered* if for every $x \in K_0(I)$ either x or $-x$ is an element of $K_0^+(I)$.

Corollary 3.7. *Assume I is separable, QD and $K_0(I)$ is totally ordered. For any separable, nuclear, QD algebra B which satisfies the UCT we have that $Ext_{QD}(B, \mathcal{K} \otimes I) = Ker(\Phi)$.*

Proof. We only have to show $Ext_{QD}(B, \mathcal{K} \otimes I) \subset Ker(\Phi)$. So let $[\gamma] \in Ext(B, \mathcal{K} \otimes I)$. If $E(\gamma)$ is a stably finite C^*-algebra then a result of Spielberg (see Proposition 4.1 of the next section), together with the assumption that $K_0(I)$ is totally ordered, implies that $[\gamma] \in Ker(\Phi)$. But since QD implies stably finite ([Br3, Prop. 3.19]) we have that if $[\gamma] \in Ext_{QD}(B, \mathcal{K} \otimes I)$ then $[\gamma] \in Ker(\Phi)$. □

The classic example for which $K_0(I)$ is totally ordered is the case when $I = \mathcal{K}$. In this setting the corollary above is very similar to a result of Salinas' which describes the closure of $0 \in Ext(B, \mathcal{K})$ in terms of quasidiagonality ([Sa1, Thm. 2.9]). See also [Sa1, Thm. 2.14] for another characterization of $Ext_{QD}(B, \mathcal{K})$ in terms of bi-quasitriangular operators. For a K-theoretical characterization of $Ext_{qd}(B, \mathcal{K})$ see [Sch, Theorem 8.3].

The class of NF algebras introduced in [BK] coincides with the class of separable QD nuclear C^*-algebras. It was conjectured in [BK, Conj. 7.1.6] that an asymptotically split extension of NF algebras is NF. We can verify the conjecture under an additional asumption.

Corollary 3.8. *Let $0 \to I \to E \to B \to 0$ be an asymptotically split extension with I and B NF algebras. If B satisfies the UCT, then E is NF.*

Proof. Both index maps are vanishing since the extension is asymptotically split. The conclusion follows from Theorem 3.4. □

§4. Extensions and K-theory

In this section we show that the general extension problem for nuclear QD C^*-algebras is equivalent to some natural K-theoretic questions.

We begin by recalling a result of Spielberg which solves the extension problem for stably finite C^*-algebras and shows that it is completely governed by K-theory.

Proposition 4.1. [Sp, Lemma 1.5] *Let $0 \to I \to E \to B \to 0$ be short exact where both I and B are stably finite. Then E is stably finite if and only if $\partial(K_1(B)) \cap K_0^+(I) = \{0\}$, where $\partial : K_1(B) \to K_0(I)$ is the boundary map of the sequence.*

In [BK, Question 7.3.1], it is asked whether every nuclear stably finite C^*-algebra is QD. Support for an affirmative answer to this question is provided by a number of nontrivial examples ([Pi], [Sp], [Br1], [Br2]). In fact, Corollary 3.7 above also provides examples since the proof shows the equivalence of quasidiagonality and stable finiteness (in fact we did not even assume nuclearity of E in that corollary). Hence it is natural to wonder if Spielberg's criterion completely determines quasidiagonality in extensions as well. The following result gives some more evidence for an affirmative answer. If I is a C^*-algebra, let $SI = C_0(\mathbb{R}) \otimes I$ denote the suspension of I. Note that $K_0(SI)^+ = \{0\}$ since $SI \otimes \mathcal{K}$ contains no nonzero projections.

Proposition 4.2. *Let $0 \to SI \to E \to B \to 0$ be exact, where I is σ-unital and B is separable, QD, nuclear. Then E is QD.*

Proof. The suspension SI of I is QD by [Vo1]. We may assume that I is stable. Let $\alpha : SI \hookrightarrow SI$ be a null-homotopic approximately unital embedding and let $\widehat{\alpha} : Q(SI) \hookrightarrow Q(SI)$ be the corresponding $*$-monomorphism. Then for any Busby invariant $\gamma : B \to M(SI)$, $[\widehat{\alpha} \circ \gamma] = 0 \in Ext(B, SI)$ by the homotopy invariance of $Ext(B, SI)$ in the second variable [Kas]. It follows that $E(\gamma) \hookrightarrow E(\widehat{\alpha} \circ \gamma)$ is QD by Proposition 2.5. $\square$

Definition 4.3. Say that a QD C^*-algebra A has the *QD extension property* if for every separable, nuclear, QD algebra B which satisfies the UCT and Busby invariant $\gamma : B \to Q(\mathcal{K} \otimes A)$ we have that $E(\gamma)$ is QD if and only if $E(\gamma)$ is stably finite (which is if and only if $\partial(K_1(B)) \cap K_0^+(\mathcal{K} \otimes A) = \{0\}$, by Proposition 4.1).

The QD extension property is closely related to a certain embedding property for the K-theory of A which we now describe. The interest in controlling the K-theory of embeddings of C^*-algebras goes back to the seminal work of Pimsner and Voiculescu on AF embeddings of irrational rotation algebras ([PV]). Since then other authors have studied the K-theory of (AF) embeddings ([Lo], [EL], [DL], [Br1], [Br1]).

Definition 4.4. Say that a QD C^*-algebra A has the *K_0-embedding property* if for every subgroup $G \subset K_0(A)$ such that $G \cap K_0^+(A) = \{0\}$ there exists an embedding $\rho : A \hookrightarrow C$, where C is also QD, such that $\rho_*(G) = 0$.

It is not hard to see that if C is a stably finite C^*-algebra and $p \in C$ is a nonzero projection then $[p]$ must be a nonzero element of $K_0(C)$. From this remark it follows that the condition $G \cap K_0^+(A) = \{0\}$ is necessary. Hence the K_0-embedding property states that this condition is also sufficient.

A number of QD C^*-algebras have the K_0-embedding property. For example, commutative C^*-algebras, AF algebras ([Sp, Lem. 1.14]), crossed products of AF algebras by $\mathbb{Z}$ ([Br1, Thm. 5.5]) and simple nuclear unital C^*-algebras with unique trace.

Our next goal is to connect the QD extension and K_0-embedding properties. But we first need a simple lemma.

Lemma 4.5. *Let C be a hereditary subalgebra of a unital C^*-algebra D. If C has an approximate unit consisting of projections and $K_0(D)$ has cancellation then the inclusion $C \hookrightarrow D$ induces an injective map $K_0(C) \hookrightarrow K_0(D)$.*

Proof. By *cancellation* we mean that if $p, q \in M_n(D)$ are projections with $[p] = [q]$ in $K_0(D)$ then there exists a partial isometry $v \in M_n(D)$ such that $vv^* = p$ and $v^*v = q$.

Let $x = [p] - [q] \in K_0(C)$ be an element such that $x = 0 \in K_0(D)$. Since C has an approximate unit of projections, say $\{e_\lambda\}$, we may assume that p and q are projections in $(e_\lambda \otimes 1)C \otimes M_n(\mathbb{C})(e_\lambda \otimes 1)$ for sufficiently large n and λ. Since $[p] = [q]$ in $K_0(D)$ and this group has cancellation we can find a partial isometry $v \in M_n(D)$ such that $vv^* = p$ and $v^*v = q$.

We claim that actually $v \in M_n(C)$ (which will evidently prove the lemma). To see this we first note that $v = vv^*(v)v^*v = pvq$ and hence

$$v = pvq = (e_\lambda \otimes 1)pvq(e_\lambda \otimes 1) = (e_\lambda \otimes 1)v(e_\lambda \otimes 1).$$

Hence $v \in (e_\lambda \otimes 1)D \otimes M_n(\mathbb{C})(e_\lambda \otimes 1)$. But since C is hereditary in D, $C \otimes M_n(\mathbb{C})$ is hereditary in $D \otimes M_n(\mathbb{C})$ and thus

$$v \in (e_\lambda \otimes 1)D \otimes M_n(\mathbb{C})(e_\lambda \otimes 1) \subset C \otimes M_n(\mathbb{C}).$$

□

Proposition 4.6. *Let A be a separable QD C^*-algebra. Then A satisfies the QD extension property if and only if A satisfies the K_0-embedding property.*

Proof. We begin with the easy direction. Assume that A has the QD extension property and let $G \subset K_0(A)$ be a subgroup such that $G \cap K_0^+(A) = \{0\}$. Since abelian C^*-algebras satisfy the UCT we can construct an extension

$$0 \to \mathcal{K} \otimes A \to E \to \oplus_{\mathbb{N}} C(\mathbb{T}) \to 0,$$

such that $\partial(K_1(\oplus_{\mathbb{N}} C(\mathbb{T}))) = \partial(\oplus_{\mathbb{N}} \mathbb{Z}) = G$. But since A has the QD extension property E must be a QD C^*-algebra. Thus the six-term K-theory exact sequence implies that A has the K_0-embedding property (i.e. the embedding into E will work).

Conversely, assume that A has the K_0-embedding property and let

$$0 \to \mathcal{K} \otimes A \to E \to B \to 0$$

be a short exact sequence where B is separable, nuclear, QD, satisfies the UCT and E is stably finite.

Let $G = \partial(K_1(B)) \subset K(\mathcal{K} \otimes A) \cong K_0(A)$. Since E is stably finite, $G \cap K_0^+(A) = \{0\}$. By the K_0-embedding property we can find a QD C^*-algebra C and an embedding $\rho : A \hookrightarrow C$ such that $\rho_*(G) = 0$. Since A is separable we may assume that C is also separable. Indeed $K_0(A)$ (and hence G) is countable. Thus it only takes a countable number of projections and partial isometries in matrices over C to kill off $\rho_*(G)$. From this observation it is easy to see that we may assume that C is also separable.

Let $\pi : C \hookrightarrow \Pi_i M_{n_i}(\mathbb{C}) + \mathcal{K}(H)$ be an embedding (the existence of which is ensured by the separability of C) as in the proof of Proposition 3.3. Let $J \subset \Pi_i M_{n_i}(\mathbb{C}) + \mathcal{K}(H)$ be the hereditary subalgebra generated by $\pi \circ \rho(A)$. Since $\Pi_i M_{n_i}(\mathbb{C}) + \mathcal{K}(H)$ has real rank zero and stable rank one it follows from Lemma 4.5 that the inclusion $J \hookrightarrow \Pi_i M_{n_i}(\mathbb{C}) + \mathcal{K}(H)$ induces an injective map $K_0(J) \hookrightarrow K_0(\Pi_i M_{n_i}(\mathbb{C}) + \mathcal{K}(H))$. Since G is in the kernel of the K-theory map induced by the embedding $\pi \circ \rho : A \to \Pi_i M_{n_i}(\mathbb{C}) + \mathcal{K}(H)$ it follows that G is also in the kernel of the K-theory map induced by the embedding $\pi \circ \rho : A \to J$. But the embedding into J is approximately unital by construction and so we get a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{K} \otimes A & \longrightarrow & E & \longrightarrow & B & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathcal{K} \otimes J & \longrightarrow & E(\eta) & \longrightarrow & B & \longrightarrow & 0,
\end{array}$$

where η is the induced Busby invariant and the two vertical maps on the left are injective.

Now we are done since naturality of the boundary map implies that the homomorphism $\partial : K_1(B) \to K_0(\mathcal{K} \otimes J)$ is zero and hence $E(\eta)$ is QD by Theorem 3.4. □

We now wish to point out a connection between extensions of QD C^*-algebras and another very natural K-theoretic question. For brevity, we say a linear map $\varphi : A \to B$ is *ccp* if it is contractive and completely positive ([Pa]). We recall a theorem of Voiculescu.

Theorem 4.7. [Vo1, Thm. 1] *Let A be a separable C^*-algebra. Then A is QD if and only if there exists an asymptotically multiplicative, asymptotically isometric sequence of ccp maps $\varphi_n : A \to M_{k_n}(\mathbb{C})$ for* *some sequence of natural numbers k_n (i.e. $\|\varphi_n(ab) - \varphi_n(a)\varphi_n(b)\| \to 0$ and $\|\varphi_n(a)\| \to \|a\|$ for all $a, b \in A$).*

Given this abstract characterization of QD C^*-algebras it is natural to ask how well these approximating maps capture the relevant K-theoretic data.

Definition 4.8. Say that a QD C^*-algebra A has the *K_0-Hahn-Banach property* if for each $x \in K_0(A)$ such that $\mathbb{Z}x \cap K_0^+(A) = \{0\}$, where $\mathbb{Z}x = \{kx : k \in \mathbb{Z}\}$, there exists a sequence of asymptotically multiplicative, asymptotically isometric ccp maps $\varphi_n : A \to M_{k_n}(\mathbb{C})$ such that $(\varphi_n)_*(x) = 0$ for all n large enough.

It is easy to see that if $y \in K_0(A)$ and there exists a nonzero integer k such that $ky \in K_0^+(A)$ then *for every* asymptotically multiplicative, asymptotically isometric sequence of ccp maps $\varphi_n : A \to M_{k_n}(\mathbb{C})$ we have $(\varphi_n)_*(y) > 0$ (if $k > 0$) or $(\varphi_n)_*(y) < 0$ (if $k < 0$), for all sufficiently large n. Hence this K_0-Hahn-Banach property states that one can separate elements $x \in K_0(A)$ such that $\mathbb{Z}x \cap K_0^+(A) = \{0\}$ from (finite subsets of) the positive cone using finite dimensional approximate morphisms.

Another way of thinking about this property is that A has the K_0-Hahn-Banach property if and only if finite dimensional approximate morphisms determine the order on $K_0(A)$ to a large extent. A more precise formulation is contained in the next proposition (not needed for the rest of the paper).

Proposition 4.9. *The K_0-Hahn-Banach property is equivalent to the following property: If $x \in K_0(A)$ and for every sequence of asymptotically multiplicative, asymptotically isometric ccp maps $\varphi_n : A \to M_{k_n}(\mathbb{C})$ we have that $(\varphi_n)_*(x) > 0$ for all large n then there exists a positive integer k such that $kx \in K_0^+(A)$.*

Proof. We first show that the (contrapositive of the) second property above follows from the K_0-Hahn-Banach property. So assume we are given an element $x \in K_0(A)$ and assume that there is *no* positive integer k such that $kx \in K_0^+(A)$. We must exhibit a sequence of asymptotically multiplicative, asymptotically isometric ccp maps $\varphi_n : A \to M_{k_n}(\mathbb{C})$ such that $(\varphi_n)_*(x) \leq 0$ for all sufficiently large n. There are two cases.

If there exists a negative integer k such that $kx \in K_0^+(A)$ then for every sequence $\varphi_n : A \to M_{k_n}(\mathbb{C})$ we have $(\varphi_n)_*(x) < 0$ for all sufficiently large n (see the discussion following definition 4.7). The second case is if $\mathbb{Z}x \cap K_0^+(A) = \{0\}$. This case is obviously handled by the K_0-Hahn-Banach property.

Now we show how the second property above implies the K_0-Hahn-Banach property. So let $x \in K_0(A)$ be such that $\mathbb{Z}x \cap K_0^+(A) = \{0\}$. Since no positive multiple of x is in $K_0^+(A)$ the second property implies that we can find some sequence $\varphi_n : A \to M_{k_n}(\mathbb{C})$ such that $(\varphi_n)_*(x) \leq 0$ for all sufficiently large n. Similarly, since no positive multiple of $-x$ is in $K_0^+(A)$ we can find a sequence $\psi_n : A \to M_{j_n}(\mathbb{C})$ such that $(\psi_n)_*(x) \geq 0$ for all sufficiently large n. If either of $\{\varphi_n\}$ or $\{\psi_n\}$ contains a subsequence with equality at 0 then we are done so we assume that $(\varphi_n)_*(x) = -s_n < 0$ and $(\psi_n)_*(x) = t_n > 0$ for all (sufficiently large) n. It is now clear what to do: we simply add up appropriate numbers of copies of φ_n and ψ_n so that these positive and negative ranks cancel. More precisely we define maps

$$\Phi_n = (\bigoplus_1^{t_n} \varphi_n) \oplus (\bigoplus_1^{s_n} \psi_n)$$

and regard these maps as taking values in the $(t_n k_n + s_n j_n) \times (t_n k_n + s_n j_n)$ matrices. □

Proposition 4.10. *If a separable QD C^*-algebra A has the QD extension property or, equivalently, the K_0-embedding property then A also has the K_0-Hahn-Banach property.*

Proof. Assume that A has the K_0-embedding property and we are given $x \in K_0(A)$ such that $\mathbb{Z}x \cap K_0^+(A) = \{0\}$, where $\mathbb{Z}x = \{kx : k \in \mathbb{Z}\}$. By the K_0-embedding property we can find an embedding $\rho : A \hookrightarrow C$, where C is QD and $\rho_*(x) = 0$. As in the proof of Proposition 4.6 we may assume that C is also separable. But then take any asymptotically multiplicative, asymptotically isometric sequence of contractive completely positive maps $\varphi_n : C \to M_{k_n}(\mathbb{C})$ and we get that $(\varphi_n \circ \rho)_*(x) = 0$ for all sufficiently large n. □

We do not know if the converse of the previous proposition holds. However our final result will complete the circle for the class of nuclear C^*-algebras. Moreover, the next theorem also states that in order to prove that every separable, nuclear, QD C^*-algebra has any of the properties we have been studying, it actually suffices to consider very special cases of either the QD extension problem or K_0-embedding problem.

Theorem 4.11. *The following statements are equivalent.*

1. *Every separable, nuclear, QD C^*-algebra has the QD extension property.*
2. *Every separable, nuclear, QD C^*-algebra has the K_0-embedding property.*

3. *Every separable, nuclear, QD C^*-algebra satisfies the K_0-Hahn-Banach property.*
4. *If A is any separable, nuclear, QD C^*-algebra and $x \in K_0(A)$ is such that $\mathbb{Z}x \cap K_0^+(A) = \{0\}$ then there exists an embedding $\rho : A \hookrightarrow C$, where C is QD (but not necessarily separable or nuclear), such that $\rho_*(x) = 0$.*
5. *If A is any separable, nuclear, QD C^*-algebra and $x \in K_0(A)$ is such that $\mathbb{Z}x \cap K_0^+(A) = \{0\}$ then there exists a short exact sequence $0 \to \mathcal{K} \otimes A \to E \to C(\mathbb{T}) \to 0$ where E is QD and $x \in \partial(K_1(C(\mathbb{T}))) = \partial(\mathbb{Z})$.*

Proof. The proof of Proposition 4.6 carries over verbatim to show the equivalence of 1 and 2. That proof also shows the equivalence of 4 and 5. The previous proposition shows that 2 implies 3 and hence we are left to show that 3 implies 5 and 4 implies 2.

We begin with the easier implication 4 $\implies$ 2. So, let A be any separable, nuclear, QD C^*-algebra and $G \subset K_0(A)$ be a subgroup such that $G \cap K_0^+(A) = \{0\}$. As in the proof of Proposition 4.6 we can construct a short exact sequence

$$0 \to \mathcal{K} \otimes A \to E \to \bigoplus_1^\infty C(\mathbb{T}) \to 0,$$

such that $\partial(K_1(\oplus_{\mathbb{N}} C(\mathbb{T}))) = \partial(\oplus_{\mathbb{N}}\mathbb{Z}) = G$. We will prove that E is QD and, by exactness of $\oplus_{\mathbb{N}}\mathbb{Z} \xrightarrow{\partial} K_0(A) \to K_1(E)$, this will show 2.

For each n there is a short exact sequence

$$0 \to \mathcal{K} \otimes A \to E_n \to \bigoplus_1^n C(\mathbb{T}) \to 0,$$

where each $E_n \subset E$ is an ideal and $E = \overline{\cup_n E_n}$. Note also that each E_n is nuclear since extensions of nuclear algebras are again nuclear. Since a locally QD algebra is actually QD it suffices to show that each E_n is QD. Since E_1 is stably finite (being a subalgebra of E) we have that the boundary map $\partial : K_1(C(\mathbb{T})) \to K_0(E_1)$ takes no positive values. But then the proof of Proposition 4.6 shows that if we assume 4 then E_1 will be QD. Proceeding by induction we may assume that E_{n-1} is QD. Since E_n is also stably finite, E_{n-1} is an ideal in E_n and $E_n/E_{n-1} = C(\mathbb{T})$, applying the same argument to the exact sequence $0 \to E_{n-1} \to E_n \to C(\mathbb{T}) \to 0$ we see that E_n is also QD.

We now show that 3 $\implies$ 5, which will complete the proof. So let A be any separable, nuclear, QD C^*-algebra and $x \in K_0(A)$ be such that

$\mathbb{Z}x \cap K_0^+(A) = \{0\}$. Construct a short exact sequence $0 \to \mathcal{K} \otimes A \to E \to C(\mathbb{T}) \to 0$ such that $\partial(1) = x$. We will show that E must be QD.

We can use the K_0-Hahn-Banach property to construct an embedding $\rho : \mathcal{K} \otimes A \to Q(\oplus_i M_{n_i}(\mathbb{C}))$ such that $\rho_*(x) = 0$. Let $D \subset Q(\oplus_i M_{n_i}(\mathbb{C}))$ be the hereditary subalgebra generated by $\rho(\mathcal{K} \otimes A)$. Let $\pi : C(\mathbb{T}) \to B(H)$ be any faithful unital representation such that $\pi(C(\mathbb{T})) \cap \mathcal{K}(H) = \{0\}$. We first claim that there is an embedding of E into $(\pi(C(\mathbb{T})) + \mathcal{K}(H)) \otimes \tilde{D}$, where $\tilde{D}$ is the unitization of D. Indeed, since the embedding $\rho : \mathcal{K} \otimes A \to D$ is approximately unital we get a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{K} \otimes A & \longrightarrow & E & \longrightarrow & C(\mathbb{T}) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & D & \longrightarrow & F & \longrightarrow & C(\mathbb{T}) & \longrightarrow & 0, \end{array}$$

for some algebra F and the map $E \to F$ is injective. Since $\rho_*(x) = 0 \in K_0(D)$ (by Lemma 4.5) and $K_1(D) = 0$ (by the proof of Lemma 3.2) it follows that both boundary maps arising from the sequence $0 \to D \to F \to C(\mathbb{T}) \to 0$ are zero. Hence we may appeal to the UCT, add on a trivial absorbing extension and eventually find an embedding of F into $\pi(C(\mathbb{T})) \otimes 1 + \mathcal{K}(H) \otimes D \subset (\pi(C(\mathbb{T})) + \mathcal{K}(H)) \otimes \tilde{D}$.

Since E is nuclear it now suffices to show that every nuclear subalgebra of $(\pi(C(\mathbb{T})) + \mathcal{K}(H)) \otimes \tilde{D}$ is QD. Hence, by [Br3, Prop. 8.3] and the Choi-Effros lifting theorem ([CE]) it suffices to show that there exists a short exact sequence

$$0 \to J \to C \to (\pi(C(\mathbb{T})) + \mathcal{K}(H)) \otimes \tilde{D} \to 0,$$

where C is QD and J contains an approximate unit consisting of projections which is quasicentral in C (i.e. the extension is quasidiagonal). However, this is now trivial since $D \subset Q(\oplus_i M_{n_i}(\mathbb{C}))$ implies that there is a quasidiagonal extension

$$0 \to \oplus_i M_{n_i}(\mathbb{C}) \to R \to \tilde{D} \to 0,$$

where $R \subset \Pi_i M_{n_i}(\mathbb{C})$. But since $X = \pi(C(\mathbb{T})) + \mathcal{K}(H)$ is nuclear the sequence

$$0 \to (\oplus_i M_{n_i}(\mathbb{C})) \otimes X \to R \otimes X \to \tilde{D} \otimes X \to 0$$

is exact and since X is unital the extension is also quasidiagonal. $\square$

Though Theorem 4.11 is stated for the class of nuclear QD C^*-algebras a close inspection of the proof shows that this assumption was only used in the proof of $4 \implies 2$. Hence we also have the following result which applies to individual nuclear C^*-algebras.

Theorem 4.12. *Let A be a separable nuclear QD C^*-algebra and consider the following statements.*

1. *A has the QD extension property.*
2. *A has the K_0-embedding property.*
3. *A has the K_0-Hahn-Banach property.*
4. *If $x \in K_0(A)$ is such that $\mathbb{Z}x \cap K_0^+(A) = \{0\}$ then there exists an embedding $\rho : A \hookrightarrow C$, where C is QD (but not necessarily separable or nuclear), such that $\rho_*(x) = 0$.*
5. *If $x \in K_0(A)$ is such that $\mathbb{Z}x \cap K_0^+(A) = \{0\}$ then there exists a short exact sequence $0 \to \mathcal{K} \otimes A \to E \to C(\mathbb{T}) \to 0$ where E is QD and $x \in \partial(K_1(C(\mathbb{T}))) = \partial(\mathbb{Z})$.*

Then $1 \iff 2 \implies 3 \iff 4 \iff 5$.

Remark. There is another version of Theorem 4.11 where the class of nuclear C^*-algebras is replaced by a class $\mathcal{A}$ of separable C^*-algebras with the following closure property. If $0 \to A \otimes \mathcal{K} \to E \to B \to 0$ is exact with $A \in \mathcal{A}$ and B separable abelian, then $E \in \mathcal{A}$. For instance $\mathcal{A}$ can be the class of all separable C^*-algebras or the class of all separable exact C^*-algebras. Then the statements 1-5 of Theorem 4.11 formulated for the class $\mathcal{A}$ (rather then for the class of nuclear C^*-algebras) are related as follows: $1 \iff 2 \iff 4 \iff 5 \implies 3$.

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The ideal structure of graph algebras

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§1. Introduction

For an $n \times n$ $\{0,1\}$-matrix $A = [A(i,j)]$ without zero rows or columns the corresponding Cuntz-Krieger algebra $\mathcal{O}_A$ is defined in [4] as a C^*-algebra generated by partial isometries $\{s_i \mid i = 1, \ldots, n\}$ on a Hilbert space satisfying $s_i^* s_i = \sum_{j=1}^{n} A(i,j) s_j s_j^*$. Almost from the start it was observed [25] that instead of a matrix we can use a directed graph to encode this data. It took a little bit longer though before it was realized that graphical approach may be equally successfully applied to infinite graphs. This extension (cf. [16, 15, 6, 2, 19] and references there) allows us to study by similar tools and within the same framework objects as diverse as classical Cuntz-Krieger algebras $\mathcal{O}_n$, $\mathcal{O}_\infty$, AF-algebras, and many other C^*-algebras.

A variety of methods have been employed in the investigations of graph algebras. The arguments in [16] and several subsequent papers (eg see [17]) rely heavily on the machinery of groupoids. A different approach is based on the realization of graph algebras as Cuntz-Pimsner algebras (cf. [18, 13, 7]) corresponding to suitable Hilbert bimodules over discrete abelian C^*-algebras. However it may well be that the direct approach yields the sharpest results (cf. [2, 19]).

The structure of graph algebras is fairly well-known by now. Indeed, after several earlier partial results a criterion for their simplicity has been found [21] (see also [17]). Their K-theory is readily computable [19, 23]. Their stable rank can be determined from the graph [5]. A number of other questions, like injectivity of their homomorphisms (cf. [24]) or direct sum decomposability (cf. [8]) can now be easily answered. Modelling with graph algebras has been employed in the studies of semiprojectivity (cf. [22, 20]) and pure-infiniteness (cf. [10]).

We begin this article with a brief overview of basic facts about graph algebras, illustrated with a number of examples. Then we move to our

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main point of interest, the discussion of the structure of their ideals. First fundamental results about ideals of Cuntz-Krieger algebras were obtained in [3]. A complete discussion of the primitive spectrum of a Cuntz-Krieger algebra corresponding to a finite $\{0,1\}$-matrix (and hence a finite graph) was later given in [12]. However the ideal structure of graph algebras corresponding to infinite graphs is much more complicated. Most previously obtained results in this direction dealt with the case of ideals of row-finite graphs (ie such that each vertex emits only finitely many edges) which are invariant under the canonical gauge action of the circle group [16, 15, 2].

Similar results for row-finite graph algebras were obtained in [13] by viewing graph algebras as Cuntz-Pimsner algebras of suitable Hilbert bimodules. Very recently a complete description of ideals of all graph algebras has been obtained [1, 9]. That is, all primitive ideals together with the hull-kernel topology on the primitive spectrum are known. These results cover the most general countable directed graphs, with no restrictive assumptions whatever. In this article we present without proofs the description of gauge invariant ideals and then briefly indicate how other ideals arise. For the complete results, see [1, 9].

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§2. Cuntz-Krieger algebras of directed graphs

2.1. Definitions and examples

A directed graph E is a quadruple (E^0, E^1, r, s) with E^0 the set of vertices, E^1 the set of edges, and $r, s : E^1 \to E^0$ the range and source function, respectively. In what follows we always assume that both E^0 and E^1 are at most countable.

If $n \geq 1$ then a path α of length n in E is a sequence $\alpha = (e_1, \dots, e_n)$ with $e_i \in E^1$ and $r(e_i) = s(e_{i+1})$ for $i = 1, \dots, n-1$. Then $s(\alpha) = s(e_1)$, $r(\alpha) = r(e_n)$, and we say that α is a path from $s(\alpha)$ to $r(\alpha)$. A path α (of length at least 1) is a loop if $r(\alpha) = s(\alpha)$. It is a vertex simple loop if the vertices $s(e_i)$ are distinct. The loop has no exits if $s^{-1}(s(e_i)) = \{e_i\}$ for $i = 1, \dots, n$. A vertex v is called sink if $s^{-1}(v) = \emptyset$.

The following concept of a Cuntz-Krieger E-family for a given directed graph E was introduced in [15].

Definition 1. *Let E be a directed graph and let B be a C^*-algebra. A Cuntz-Krieger E-family $\{S_e, P_v\}$ inside B consists of a collection of partial isometries $\{S_e \in B \mid e \in E^1\}$ and a collection of projections $\{P_v \in B \mid v \in E^0\}$ such that the following conditions are satisfied.*

(G1) $P_v P_w = 0$ *if* $v \neq w$.
(G2) $S_e^* S_f = 0$ *if* $e \neq f$.
(G3) $S_e^* S_e = P_{r(e)}$.
(G4) $S_e S_e^* \leq P_{s(e)}$.
(G5) $\sum_{e \in E^1, s(e)=v} S_e S_e^* = P_v$, *if v emits finitely many (and at least one) edges.*

The following definition of graph algebras was given in [6].

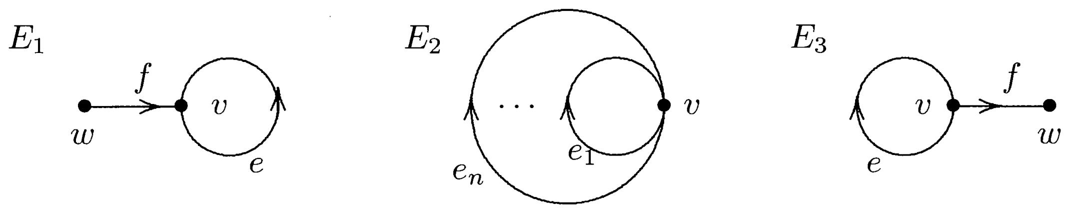
Definition 2. *The C^*-algebra $C^*(E)$ of a directed graph E is a C^*-algebra generated by partial isometries $\{s_e \mid e \in E^1\}$ and projections $\{p_v \mid v \in E^0\}$, which is universal for Cuntz-Krieger E-families. That is, for any Cuntz-Krieger E-family $\{S_e, P_v\}$ inside a C^*-algebra B there exists a unique C^*-algebra homomorphism $\pi_{S,P} : C^*(E) \to B$ such that $\pi_{S,P}(s_e) = S_e$ for all $e \in E^1$ and $\pi_{S,P}(p_v) = P_v$ for all $v \in E^0$.*

Throughout this article we use symbols $\{s_e, p_v\}$ with small s, p for the generators of the C^*-algebra $C^*(E)$. Universality of graph algebras implies that there exists a canonical action γ of the circle group $\mathbf{T}$ on $C^*(E)$, called the gauge action,

$$\gamma : \mathbf{T} \to \mathrm{Aut}(C^*(E))$$

such that $\gamma_t(p_v) = p_v$ and $\gamma_t(s_e) = ts_e$ for all $v \in E^0$, $e \in E^1$, $t \in \mathbf{T}$.

Example 3. Let E_i, i=1,2,3, be the following directed graphs.



We have $C^*(E_1) \cong M_2 \otimes C(\mathbf{T})$ and $C^*(E_3)$ is isomorphic to the Toeplitz algebra generated by a unilateral shift. Also $C^*(E_2) \cong C(\mathbf{T})$ for $n = 1$ and $C^*(E_2) \cong \mathcal{O}_n$ for $n \geq 2$, including $n = \infty$.

2.2. Basic properties of graph algebras

One of the great advantages of working with graph algebras is the ease with which we can read all basic properties of these complicated objects from the underlying graphs. For example, $C^*(E)$ is unital if and only if E^0 is finite. Below we show how to recognize from directed graphs such properties of the corresponding algebras as Cuntz-Krieger uniqueness, simplicity, being AF, pure infiniteness, and K-theory.

Since graph algebras are defined via a universal property, it is not too difficult to construct homomorphisms from these algebras to other C^*-algebras. However, it is usually much more difficult to verify whether such a homomorphism is injective or not. To this end we often use the following gauge invariant uniqueness theorem, which essentially says that the universality in the definition of $C^*(E)$ is equivalent to the existence of the gauge action. This result was proved in [2] for row-finite graphs, and in full generality in [19].

Theorem 4. *Let E be a directed graph, $\{S_e, P_v\}$ be a Cuntz-Krieger E-family, and $\pi_{S,P} : C^*(E) \to C^*(\{S_e, P_v\})$ be a C^*-algebra homomorphism such that $\pi_{S,P}\,(s_e) = S_e$ and $\pi_{S,P}(p_v) = P_v$ for all $e \in E^1$ and all $v \in E^0$. Suppose that each P_v is non-zero, and that there is a strongly continuous action β of $\mathbf{T}$ on $C^*(\{S_e, P_v\})$ such that $\beta_t \circ \pi_{S,P} = \pi_{S,P} \circ \gamma_t$ for all $t \in \mathbf{T}$. Then $\pi_{S,P}$ is injective.*

The classical Cuntz-Krieger uniqueness has also been generalized to the context of graph algebras. The following result was proved in [16] for row-finite graphs, and in full generality in [6].

Theorem 5. *Let E be a directed graph in which every loop has an exit. Then for all Cuntz-Krieger E-family $\{S_e, P_v\}$ such that each P_v is different from 0, the corresponding C^*-algebra homomorphism $\pi_{S,P}$: $C^*(E) \to C^*(\{S_e, P_v\})$ (with $\pi_{S,P}(s_e) = S_e$ and $\pi_{S,P}(p_v) = P_v$ for all $e \in E^1$, $v \in E^0$) is an isomorphism.*

An easy consequence of Theorem 5 is the uniqueness of the C^*-algebra, generated by a proper isometry (ie the classical result due to Coburn), corresponding to the graph E_3. A common generalization of Theorems 4 and 5 is given in [24].

A convenient criterion of simplicity of graph algebras is known [21]. It is formulated in terms of hereditary and saturated sets of vertices, which also play a crucial role in our description of ideals in the next section. A subset $H \subseteq E^0$ is called;

(i) *saturated* if any $v \in E^0$, emitting finitely many (and at least one) edges and such that $r(e) \in H$ for all $e \in E^1$ with $s(e) = v$, belongs to H,

(ii) *hereditary* if $r(e) \in H$ for any $e \in E^1$ such that $s(e) \in H$.

Theorem 6. *Let E be a directed graph. Then $C^*(E)$ is simple if and only if the following two conditions are satisfied.*

1. *All loops in E have exits.*
2. *The only hereditary and saturated subsets of E^0 are $\emptyset$ and E^0.*

The graphs E_2 $(n \geq 2)$ satisfy the conditions of Theorem 6, thus Cuntz algebras $\mathcal{O}_n$ $(n \geq 2)$ and $\mathcal{O}_\infty$ are simple. But the Toeplitz algebra is not, since the graph E_3 has a nontrivial proper hereditary and saturated subset $\{v\}$. Earlier partial results in this direction may be found in [4, 16, 7, 6, 2]. A different (based on the groupoid approach) of an equivalent simplicity criterion has been recently found in [17].

It turns out that all simple graph algebras are either AF or purely infinite (cf. [15, 2, 19]). In fact, $C^*(E)$ is AF if and only if E has no loops, and $C^*(E)$ is purely infinite in the sense of [15] (but not necessarily simple) if and only if all loops in E have exits and every vertex connects to a loop by a directed path (cf. [15, 2, 10]).

The K-theory of graph algebras is readily computable by the Cuntz method. Namely, the crossed product of $C^*(E)$ by the gauge action of the circle group $\mathbf{T}$ is known as an AF-algebra ([14, 19]). Thus $C^*(E)$ is stably isomorphic to a crossed product of an AF-algebra by an action of the integers $\mathbf{Z}$ (dual to the gauge action), which allows us to apply the Pimsner-Voiculescu exact sequence.

The following thoerem was obtained in [19] for row-finite graphs, and then extended to the directed graphs with finitely many edges in [22]. These and several other results about the K-theory of graph or Cuntz-Krieger algebras are all based on the original calculation in [3].

Theorem 7. *Let E be a directed graph and let V denote the collection of all those vertices which emit at least one but at most finitely many edges. Let $\mathbf{Z}V$ and $\mathbf{Z}E^0$ be free abelian groups on free generators V and E^0, respectively. Then*

$$K_0(C^*(E)) \cong \operatorname{coker}(\Delta_E) \quad \text{and} \quad K_1(C^*(E)) \cong \ker(\Delta_E),$$

where $\Delta_E : \mathbf{Z}V \to \mathbf{Z}E^0$ is the map defined as

$$\Delta_E(w) = \sum_{e \in E^1,\, s(e)=w} r(e) - w.$$

It follows from Theorem 7 that K_1 groups of graph algebras must be free abelian. It turns out that this is the only restriction. Namely, for any pair of countable abelian groups A_0, A_1 with A_1 free abelian, there exists a stable, purely infinite and simple graph algebra $C^*(E)$ such that $K_i(C^*(E)) \cong A_i$ for $i = 0, 1$ [23]. An easy way to check criterion for stability of graph algebras is given in [8].

Note that all graph algebras are separable according to our definition, since we only deal with countable graphs. All of them are also nuclear and satisfy the Universal Coefficient Theorem. Therefore purely infinite and simple algebras $C^*(E)$ serve as convenient models of a large subclass of the classifiable algebras. This fact has been recently exploited in [22] and [20] to show that all Kirchberg algebras with K_0 finitely generated and K_1 finitely generated free abelian are semiprojective.

§3. Ideals of graph algebras

In this section we present the ideal structure of graph algebras. We focus primarily on gauge invariant ideals J of $C^*(E)$ such that $\gamma_t(J) = J$ for all $t \in \mathbf{T}$. We begin by recalling the classification of gauge invariant ideals for algebras of row-finite graphs. Then we discuss the general case of arbitrary graphs, and conclude with a brief indication of how other ideals, ie non gauge invariant ideals, arise.

3.1. Gauge invariant ideals

3.1.1. *Row-finite directed graphs* It turns out that that gauge invariant ideals of algebras of row-finite graphs are in a one-to-one correspondence with hereditary and saturated sets of vertices. For a directed graph E we denote by Σ_E the collection of all hereditary and saturated subsets of E^0. If $X \subseteq E^0$ then We denote by $\Sigma(X)$ the smallest hereditary and saturated subset of E^0 containing X. If J is a closed two-sided ideal of $C^*(E)$ then we define $V_J := \{v \in E^0 \mid p_v \in J\}$. It is easy to see that V_J is hereditary and saturated. For a hereditary and saturated set $K \subseteq E^0$ we define J_K to be the closed two-sided ideal of $C^*(E)$ generated by $\{p_v \mid v \in K\}$.

The following theorem is given in [2]. Its earlier versions, with some additional restrictions on the underlying directed graphs, are in [4, 13, 16].

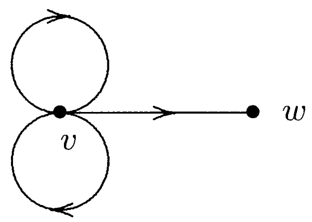
Theorem 8. *If E is a row-finite directed graph then there is a one-to-one correspondence between the collection of closed, two-sided, gauge invariant ideals of $C^*(E)$ and Σ_E, via $J \to V_J$ and $J_K \leftarrow K$.*

The key fact used in the proof of Theorem 8 is that for a gauge invariant ideal J of $C^*(E)$ the quotient $C^*(E)/J$ is again a graph algebra, corresponding to the graph obtained by restriction of E to $E^0 \setminus V_J$.

It is also possible to identify those directed graphs E such that all ideals of $C^*(E)$ are gauge invariant. For Cuntz-Krieger algebras of finite

$\{0,1\}$-matrices, this situation is captured by condition II of Cuntz. Its analogue for row-finite graphs was introduced in [16] by the so-called condition K (an analogue of Cuntz's condition II). Condition K requires that any vertex in E^0 lies on either none or at least two distinct vertex simple loops. If a row-finite graph E satisfies condition K then all ideals of $C^*(E)$ are automatically gauge invariant, and consequently Theorem 8 describes all ideals of $C^*(E)$ in this case.

Example 9. *The graph E below satisfies condition K. Thus Σ_E gives all ideals of $C^*(E)$. Since $\Sigma_E = \{\emptyset, \{w\}, E^0\}$, we have $J_{\{w\}} \cong \mathcal{K}$ (compact operators on a separable Hilbert space) is the only nontrivial ideal of $C^*(E)$. Note that the quotient $C^*(E)/J_{\{w\}}$ is isomorphic to $C^*(E_2)$ (with $n = 2$), which is Cuntz algebra $\mathcal{O}_2$.*



Unfortunately, even for graphs as simple as E_1 or E_3 Theorem 8 does not give the full description of ideals. The reason is that non gauge invariant ideals are present.

3.1.2. *The general case* We now give a brief outline of the results obtained recently by the author in collaboration with the group from the University of Newcastle. Proofs of these results will be published in [1].

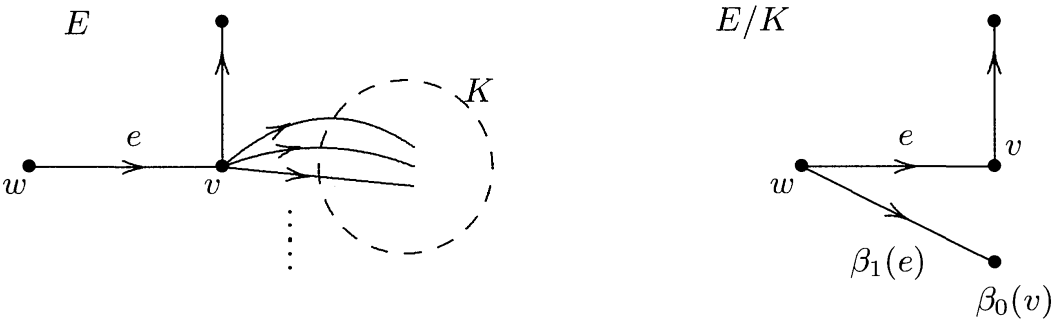
Unlike in the previously discussed much simpler case of row-finite graphs, the collection of hereditary and saturated subsets is not sufficient to describe all gauge inariant ideals in general. In order to do this we must first understand quotients of graph algebras by gauge invariant ideals. We first introduce the notion of the quotient graph.

Let E be an arbitrary directed graph and let $K \subseteq E^0$ be a hereditary and saturated subset. We denote by K_∞^{fin} the collection of all those vertices $v \in E^0 \setminus K$ such that $s^{-1}(v) \cap r^{-1}(K)$ is infinite and $s^{-1}(v) \cap r^{-1}(E^0 \setminus K)$ is finite and non-empty. We then define the quotient graph E/K as follows.

$$\begin{aligned}
(E/K)^0 &= (E^0 \setminus K) \cup \{\beta(v) \mid v \in K_\infty^{\text{fin}}\}, \\
(E/K)^1 &= r^{-1}(E^0 \setminus K) \cup \{\beta(e) \mid e \in E^1,\ r(e) \in K_\infty^{\text{fin}}\},
\end{aligned}$$

with $s(\beta(e)) = s(e)$ and $r(\beta(e)) = \beta(r(e))$. The β is just a symbol helping to distinguish a vertex $v \in E^0$ and an edge $e \in E^1$ from the extra vertex $\beta(v)$ and the extra edge $\beta(e)$ in E/K, respectively. Note that E/K is a subgraph of E if $K_\infty^{\mathrm{fin}} = \emptyset$, and this is always the case when E is row-finite.

Example 10. *Let E be the graph below. It is assumed here that $K \in \Sigma_E$ and v emits infinitely many edges into K. We have $K_\infty^{\mathrm{fin}} = \{v\}$ and the quotient graph E/K looks as follows.*



The following lemma provides a key tool for analyzing gauge invariant ideals of arbitrary graph algebras.

Lemma 11. *Let E be a directed graph and let $K \in \Sigma_E$. Then there is a natural isomorphism*

$$C^*(E)/J_K \cong C^*(E/K).$$

Now let $K \in \Sigma_E$ and $X \subseteq K_\infty^{\mathrm{fin}}$. We define $J_{K,X}$ as the closed two-sided ideal of $C^*(E)$ generated by J_K and $\{p_v - q_v \mid v \in X\}$, where $q_v = \sum_{s(e)=v, r(e)\notin K} s_e s_e^*$ is a subprojection of p_v. As a special case we have $J_{K,\emptyset} = J_K$, which always occurs in row-finite graphs. Clearly the ideal $J_{K,X}$ is gauge invariant. We denote by $\wp(K_\infty^{\mathrm{fin}})$ the collection of all subsets of K_∞^{fin}.

Theorem 12. *If E is an arbitrary directed graph then there is a one-to-one correspondence between $\bigcup_{K\in\Sigma_E}\{K\} \times \wp(K_\infty^{\mathrm{fin}})$ and the collection of all closed, two-sided gauge invariant ideals of $C^*(E)$, given by the map*

$$(K, X) \mapsto J_{K,X}.$$

Theorem 8 is then an immediate consequence of Theorem 12.

Lemma 11 says that quotients of graph algebras by gauge invariant ideals are themselves graph algebras. Thus, in order to describe primitive gauge invarinat ideals it suffices to know what graphs E result in primitive algebras $C^*(E)$, since the quotient of a C^*-algebra by a primitive ideal is a primitive C^*-algebra. We have the following.

Proposition 13. *If E is an arbitrary directed graph then $C^*(E)$ is primitive if and only if E satisfies the following two conditions.*

1. *All loops in E have exits.*
2. *$\Sigma(v) \cap \Sigma(w) \neq \emptyset$ for any $v, w \in E^0$.*

We remark that an easy double application of Lemma 11 gives an isomorphism

$$C^*(E)/J_{K,X} \cong C^*((E/K)/\beta(X)),$$

for all $K \in \Sigma_E$ and $X \subseteq K_\infty^{\mathrm{fin}}$. Thus, combining Theorem 12 with Proposition 13 we get a criterion of primitivity of gauge invariant ideals of $C^*(E)$. Namely, a gauge invariant ideal $J_{K,X}$ of $C^*(E)$ is primitive if and only if the quotient graph $(E/K)/\beta(X)$ satisfies the conditions of Proposition 13.

3.2. Other idelas

In many cases, the graph algebra $C^*(E)$ may contain non gauge invariant ideals. For example, since $\Sigma_{E_3} = \{\emptyset, \{w\}, E^0\}$, $J_{\{w\}} \cong \mathcal{K}$ is the only nontrivial gauge invariant ideal of $C^*(E_3)$. However, since the quotient graph $E_3/\{w\}$ is E_2 with $n = 1$, $C^*(E_3)/J_{\{w\}} \cong C^*(E_3/\{w\}) \cong C(\mathbf{T})$ and consequently we see that $C^*(E_3)$ contains a circle of non gauge invariant primitive ideal.

For an arbitrary directed graph E it may be shown that all non gauge invariant primitive ideals arise essentially in the same way as described in the preceding paragraph. Namely, let J be a non gauge invariant primitive ideal of $C^*(E)$. Then there exists a unique maximal gauge invariant ideal J' of $C^*(E)$ contained in J. Furthermore, the quotient $C^*(E)/J'$ is isomorphic to $C^*(F)$ for a suitable graph F. It is that F must contain a unique vertex simple loop with no exits in F and that the ideal J corresponds to a point on that loop.

A complete discussion of all primitive ideals of $C^*(E)$ for an arbitrary directed graph E, including the hull-kernel topology on the primitive spectrum, are presented in articles [1, 9].

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Stable rank and real rank of graph C^*-algebras

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Abstract.

For any row finite directed graph E there exists a universal C^*-algebra $C^*(E)$ ([KPR, KPRR]) generated by projections and partial isometries satisfying the Cuntz-Krieger E-relations. This class of graph algebras includes the Cuntz-Krieger algebras and all AF algebras up to stable isomorphisms([D]). In this paper we give conditions for E under which the algebra $C^*(E)$ has stable rank one or real rank zero. A simple graph C^*-algebra is either AF or purely infinite, hence it is always extremally rich. We discuss the extremal richness of some graph C^*-algebras and present several examples of prime ones with finitely many closed ideals.

§1. Introduction

As a generalization of Cuntz-Krieger algebras, a class of C^*-algebras generated by projections and partial isometries subject to the relations determined by directed graphs has been studied in [KPRR], [KPR] and later in [BPRS], and these algebras are called graph C^*-algebras. Since they are basically generated by partial isometries and projections one may expect that most of them must have real rank zero like AF algebras or Cuntz algebras. In fact if the associated graph C^*-algebra for a row finite graph E is simple then $C^*(E)$ always has real rank zero since it is either AF or purely infinite ([KPR]). On the other hand, the Toeplitz algebra can occur as a graph C^*-algebra but its real rank is not zero, hence we want to know when the graph algebra has real rank zero. We will answer the question in terms of the loop structure of a graph in Theorem 3.2 and Theorem 3.4.

Recall that a projection p in a C^*-algebra A is said to be *infinite* if it is Murray-von Neumann equivalent to its proper subprojection. We call a unital C^*-algebra A *infinite* if the unit projection is infinite, and *finite* otherwise. An infinite C^*-algebra whose every nonzero hereditary

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subalgebra contains an infinite projection is called *purely infinite.* If a unital C^*-algebra A has stable rank one ($sr(A) = 1$, see [Rf]), that is, the set A^{-1} of all invertible elements is dense in A, then one can see that A should be finite. All AF-algebras ([Rf]), irrational rotation algebras([Pt]) are those known to have stable rank one. We will give a sufficient and necessary condition for a graph E that $C^*(E)$ has stable rank one in Theorem 3.1.

As an attempt to extend notions and results for finite C^*-algebras to infinite cases Brown and Pedersen ([BP2]) considered the quasi-invertible elements A_q^{-1} in a unital C^*-algebra A and call A *extremally rich* if the set A_q^{-1} is dense in A since it turns out in [BP2] that this condition is equivalent to say that the closed unit ball A_1 contains enough extreme points so that the convex hull of its extreme points coincides with the whole A_1;

$$conv(\mathcal{E}(A)) = A_1,$$

where $\mathcal{E}(A)$ denotes the extreme points of A_1. Since $A^{-1} \subset A_q^{-1}$ for any unital C^*-algebra A we see that a unital C^*-algebra A with $sr(A) = 1$ is always extremally rich. On the other hand it is a nontrivial fact that purely infinite simple C^*-algebras (for example, Cuntz algebras) are also extremally rich (see [LO], [Pd]). Therefore a simple graph C^*-algebra is always extremally rich. Recall that a graph C^*-algebra $C^*(E)$ is simple if and only if E is cofinal and satisfies condition (L). We show the cofinality of a graph E is in fact a sufficient condition for the algebra $C^*(E)$ to be extremally rich and also provide an example of non-extremally rich prime graph C^*-algebra that has only three proper ideals and has real rank zero.

§2. Preliminaries

We recall definitions and results from [KPR], [KPRR], and [BPRS] on directed graphs and graph C^*-algebras. A *directed graph* $E = (E^0, E^1, r, s)$ (or simply $E = (E^0, E^1)$) consists of countable vertices E^0, edges E^1 and the range, source maps $r, s : E^1 \to E^0$. E is *row finite* if each vertex $v \in E^0$ emits at most finitely many edges, and a row finite graph is *locally finite* if each vertex receives only finitely many edges. If $e_1, \dots, e_n$ ($n \geq 2$) are edges with $r(e_i) = s(e_{i+1})$, $1 \leq i \leq n-1$, then one can form a (finite) path $\alpha = (e_1, \dots, e_n)$ of *length* $|\alpha| = n$, and extend the maps r, s by $r(\alpha) = r(e_n), s(\alpha) = s(e_1)$. Similarly one can think of infinite paths.

Let E^n be the set of all finite paths of length n (so vertices in E^0 are regarded as finite paths of length zero) and let E^* be the set of all finite paths, and E^∞ the set of infinite paths. A vertex $v \in E^0$ with $s^{-1}(v) = \emptyset$ is called a *sink*.

For a row finite directed graph E, a *Cuntz-Krieger E-family* consists of mutually orthogonal projections $\{P_v \mid v \in E^0\}$ and partial isometries $\{S_e \mid e \in E^1\}$ satisfying the Cuntz-Krieger relations

$$S_e{}^* S_e = P_{r(e)},\ e \in E^1, \text{ and } P_v = \sum_{s(e)=v} S_e S_e{}^*,\ v \in s(E^1).$$

From these relations, it can be shown that every non-zero word in S_e, P_v and S_f^* reduces to a partial isometry of the form $S_\alpha S_\beta^*$ for some $\alpha, \beta \in E^*$ with $r(\alpha) = r(\beta)$ ([KPR], Lemma 1.1).

Theorem 2.1. ([KPR], Theorem 1.2) *For a row finite directed graph $E = (E^0, E^1)$, there exists a C^*-algebra $C^*(E)$ generated by a Cuntz-Krieger E-family $\{s_e, p_v \mid v \in E^0, e \in E^1\}$ of non-zero elements such that for any Cuntz-Krieger E-family $\{S_e, P_v \mid v \in E^0, e \in E^1\}$ of partial isometries acting on a Hilbert space $\mathcal{H}$, there exists a representation $\pi : C^*(E) \to B(\mathcal{H})$ such that*

$$\pi(s_e) = S_e, \text{ and } \pi(p_v) = P_v$$

for all $e \in E^1, v \in E^0$.

Let $\{s_e, p_v \mid e \in E^1, v \in E^0\}$ be a Cuntz-Krieger E-family generating the C^*-algebra $C^*(E)$. Then for each $z \in \mathbb{T}$ we have another Cuntz-Krieger E-family $\{zs_e, p_v \mid e \in E^1,\ v \in E^0\}$ in $C^*(E)$, and by the universal property of $C^*(E)$ there exists an isomorphism $\gamma_z : C^*(E) \to C^*(E)$ such that $\gamma_z(s_e) = zs_e$ and $\gamma_z(p_v) = p_v$. In fact, $\gamma : z \mapsto \gamma_z \in \mathrm{Aut}(C^*(E))$ is a strongly continuous action of $\mathbb{T}$ on $C^*(E)$ and called the *gauge action* ([BPRS]).

A finite path α with $|\alpha| > 0$ is called a *loop* at v if $s(\alpha) = r(\alpha) = v$. It turns out that the distribution of loops in a graph E is very important to understand the structure of a graph C^*-algebra $C^*(E)$, in particular if E has no loops then $C(E)$ is AF.

A graph E is said to satisfy *condition* (L) if every loop in E has an exit, and *condition* (K) if for any vertex v on a loop there exist at least two distinct loops based at v. Note that condition (K) is stronger than (L) and if E has no loops then the two conditions are trivially satisfied.

For two vertices v, w we simply write $v \geq w$ if there is a path $\alpha \in E^*$ from v to w. A subset H of E^0 is said to be *hereditary* if $v \geq w$ and $v \in H$ imply $w \in H$, and a hereditary set H is *saturated* if $s^{-1}(v) \neq \emptyset$ and $\{r(e) \mid s(e) = v\} \subset H$ imply $v \in H$. The *saturation* of a hereditary set H is the smallest saturated subset of E^0 containing H.

Let H be a saturated hereditary subset of E^0. Then the ideal $I(H) = \overline{span}\{s_\alpha s_\beta^* \mid \alpha, \beta \in E^*, r(\alpha) = r(\beta) \in H\}$ is clearly gauge-invariant and $I(H)$ is generated by $\{p_v \mid v \in H\}$.

In case E has no sinks, in [KPRR], an isomorphism of the lattice of saturated hereditary subsets V of E^0 into the lattice of ideals $I(V)$ in $C^*(E)$ was established and it is shown that the quotient algebra $C^*(E)/I(V)$ is isomorphic to a graph algebra $C^*(G)$ for a certain subgraph G of E. More generally, the following was proved in [BPRS].

Theorem 2.2. ([KPRR] [BPRS, Theorem 4.1]) *Let $E = (E^0, E^1, r, s)$ be a row finite directed graph. For each subset H of E^0, let $I(H)$ be the ideal in $C^*(E)$ generated by $\{p_v \mid v \in H\}$.*

(a) *The map $H \mapsto I(H)$ is an isomorphism of the lattice of saturated hereditary subsets of E^0 onto the lattice of closed gauge-invariant ideals of $C^*(E)$.*

(b) *Suppose H is saturated and hereditary. If $G^0 := E^0 \setminus H$, $G^1 := \{e \in E^1 \mid r(e) \notin H\}$, and $G := (G^0, G^1, r, s)$, then $C^*(E)/I(H)$ is canonically isomorphic to $C^*(G)$ and the ideal $I(H)$ is strong Morita equivalent to $C^*(K)$, where $K := (H, \{e \mid s(e) \in H\})$.*

Note that if a graph E satisfies condition (K) then the isomorphism of Theorem 2.2.(a) maps onto the lattice of all closed ideals in $C^*(E)$, that is, every ideal is gauge-invariant. It is known ([BPRS], [JPS]) that for a row-finite graph E, the graph C^*-algebra $C^*(E)$ is simple if and only if E is a cofinal graph satisfying condition (L), here we say that E is *cofinal* if every vertex connects to every infinite path.

Proposition 2.3. ([KPR], Corollary 3.11) *Let E be a locally finite graph which has no sinks, is cofinal, and satisfies condition* (L). *Then $C^*(E)$ is simple, and*

(i) *if E has no loops, then $C^*(E)$ is AF;*

(ii) *if E has a loop, then $C^*(E)$ is purely infinite.*

§3. Stable rank and real rank of graph C^*-algebras

If a graph E has no loops at all then the resulting algebra $C^*(E)$ is AF([KPR, Theorem 2.4]), hence its stable rank is one. To see if a graph with loops can have stable rank one consider the simple graph E consisting of a single vertex v and a single loop at v. Then the graph algebra $C^*(E)$ is the commutative C^*-algebra with the spectrum $\mathbb{T}$, the unit circle, and it also has stable rank one. But if we add an edge ranging at other vertex than v, the resulting graph algebra is the Toeplitz algebra whose stable rank is 2. The following shows precisely when the graph algebra has its stable rank one. Actually if a loop has an exit then there are infinite projections in the graph algebra and so the stable rank is not one anymore.

Theorem 3.1. ([JPS, Theorem 3.3]) *Let $E = (E^0, E^1)$ be a row finite directed graph. Then E has no loop with an exit if and only if $sr(C^*(E)) = 1$.*

Recall from [BP1] that a unital C^*-algebra A (or $\tilde{A}$ if A is non-unital) has real rank zero if the set of invertible self adjoint elements is dense in the whole set of self adjoint elements, or equivalently every non zero hereditary C^*-subalgebra contains a non zero projection. So the C^*-algebras with real rank zero (for example, AF algebras, purely infinite simple C^*-algebras, all von Neumann algebras) have been considered as the ones containing reasonably many projections in some sense.

Theorem 3.2. [JPS, Theorem 4.3] *Let E be a locally finite directed graph with no sinks. If $RR(C^*(E)) = 0$ then E satisfies condition* (K).

Corollary 3.3. *Let E be a locally finite directed graph with no sinks. If $sr(C^*(E)) = 1$ and $RR(C^*(E)) = 0$ then $C^*(E)$ is AF.*

Theorem 3.4. [JPS, Theorem 4.6] *Let E be a locally finite directed graph with no sinks which satisfies condition* (K). *If $C^*(E)$ has only finitely many ideals then $RR(C^*(E)) = 0$. In particular, if E is a finite graph then $RR(C^*(E)) = 0$.*

Let A be a $\{0,1\}$-matrix with no zero row or column. Then A can be viewed as a vertex matrix of a finite graph E with no sinks. If A satisfies Cuntz-Krieger's condition (I) in [CK] then it clearly follows that E satisfies (L) (or, equivalently condition (I) introduced for graphs in [KPR]) from their definitions. By Proposition 4.1 of [KPRR], the graph algebra $C^*(E)$ is also generated by a Cuntz-Krieger A-family of partial isometries, hence the Cuntz-Krieger algebra $\mathcal{O}_A$ is isomorphic to the graph algebra $C^*(E)$. On the other hand, the graph algebra $C^*(E)$ is known to be isomorphic to the Cuntz-Krieger algebra $\mathcal{O}_B$ associated with the edge matrix B of E. Therefore those three algebras are all isomorphic. Furthermore by Theorem 3.2, 3.4, and Lemma 6.1 of [KPRR], we have the following corollary.

Corollary 3.5. *Let A be a $\{0,1\}$-matrix with no zero row or column. Suppose A satisfies Cuntz-Krieger's condition* (I) *and let E be the finite graph having A as its vertex matrix. Then the following are equivalent:*
(i) $RR(\mathcal{O}_A) = 0$,
(ii) *A satisfies Cuntz's condition* (II),

(iii) *E satisfies condition* (K).

§4. Extremal richness of graph C^*-algebras

Let A be a unital C^*-algebra. Then it is well known that an extreme point v in A_1 is characterized as a partial isometry satisfying $(1-vv^*)A(1-v^*v)=0$ ([Pd, Proposition 1.4.7]). Let A_+^{-1} be the set of all positive invertible elements of A. We call elements $x \in \mathcal{E}A_+^{-1}(=A^{-1}\mathcal{E}A^{-1})$ *quasi-invertible* ([BP3]) and denotes the set of all quasi-invertible elements in A by A_q^{-1}. If A_q^{-1} is dense in A A is called *extremally rich.* For a non-unital C^*-algebra A, A is said to be extremally rich when its unitization $\tilde{A}$ is so. Obviously a C^*-algebra A with $sr(A)=1$ is extremally rich since $A^{-1} \subset A_q^{-1}$. In particular, all AF-algebras are extremally rich. Other examples are purely infinite simple C^*-algebras ([Pd, Theorem 10.1], [LO, Lemma 3.3]), the Toeplitz algebra ([Pd, Corollary 9.2]), commutative C^*-algebras $C(X)$ with $\dim(X) \leq 1$ (see [BP3, section 3]), and all von Neumann algebras ([Pd, Theorem 4.2]). Also a simple C^*-algebra A is extremally rich if and only if it is purely infinite or it has stable rank one ([BP2, Corollary 10.5]). Thus from Proposition 2.3 it follows that every simple graph C^*-algebra $C^*(E)$ (hence, E should be cofinal and satisfy condition (L)) is extremally rich. The following shows in fact that every graph C^*-algebra $C^*(E)$ associated to a cofinal graph E is extremally rich.

Proposition 4.1. ([JPS, Proposition 3.7]) *Let G be a locally finite directed graph. If G is cofinal then either $sr(C^*(G))=1$ or it is purely infinite and simple.*

There are extremally rich graph C^*-algebras that are associated to graphs which are not cofinal, for example the Toeplitz algebra (see Example 4.6 below) which is neither purely infinite simple nor of stable rank one ($sr(\mathcal{T})=2$). These graph algebras will arise from directed graphs containing some loops with exits, so that they should have many infinite projections and hence their stable rank are not one any more.

To this end note the following corollary of Theorem 2.2.

Corollary 4.2. ([JPS, Theorem 3.5]) *Let $E=(E^0,E^1,r,s)$ be a row-finite directed graph with the set V of sinks. Then there is a subgraph $G=(E^0\setminus H,\{e\in E^1 \mid r(e)\notin H\})$ of E with no sinks such that $C^*(E)/I(V)$ is isomorphic to $C^*(G)$, where H is the saturation of V and $I(V)=\overline{span}\{s_\alpha s_\beta{}^* \mid \alpha,\beta\in E^*,\ r(\alpha)=r(\beta)\in V\}$.*

We also need to review briefly the following useful results on extremally rich C^*-algebras.

Theorem 4.3. ([BP2],[BP3], [LO]) (a) *Every quotient, every direct sum or direct product and every hereditary C^*-subalgebra of an extremally rich C^*-algebra is again extremally rich.*
(b) *If A is strong Morita equivalent (or stably isomorphic) to an extremally rich C^*-algebra B then A is also extremally rich.*

Let A be a unital C^*-algebra and I be a closed two-sided ideal.
(c) *Suppose $sr(I) = 1$. Then A is extremally rich if and only if A/I is extremally rich and extreme partial isometries lift.*
(d) *$sr(A) = 1$ if and only if $sr(I) = sr(A/I) = 1$ and every invertible elements lifts, that is, $(\tilde{A}/I)^{-1} = \tilde{A}^{-1}/I$.*
(e) *If I and A/I are purely infinite simple C^*-algebras then A is extremally rich.*

For a C^*-algebra A and projections P, Q in A, the extreme points $\mathcal{E}(PAQ)$ of the closed convex set PA_1Q consists of elements $u \in PA_1Q$ which is a partial isometry such that $(P - uu^*)A(Q - u^*u) = \{0\}$. We say that the space PAQ is *extremally rich* if either $\mathcal{E}(PAQ) = \emptyset$ or $\mathcal{E}(PAQ) \neq \emptyset$ and $(PAP)^{-1}\mathcal{E}(PAQ)(QAQ)^{-1}$ is dense in PAQ. If $\mathcal{E}(PAQ) \neq \emptyset$ then PAQ is extremally rich if and only if $PA_1Q = conv(\mathcal{E}(PAQ))$ (see [BP2]).

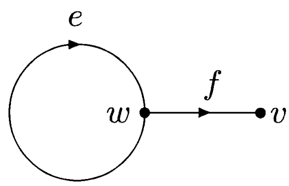
For any non-zero projections P, Q acting on a Hilbert space $\mathcal{H}$, one can show that $\mathcal{E}(PB(\mathcal{H})Q) \neq \emptyset$ and the space $PB(\mathcal{H})Q$ is extremally rich by Proposition 11.4 of [BP2]; if A is a C^*-algebra with real rank zero and $\mathcal{E}(PAQ) \neq \emptyset$ for every pair of projections P, Q in A, then every such a space PAQ is, in fact, extremally rich.

Proposition 4.4. ([BP2, Proposition 11.7]) *Let I be a closed ideal with real rank zero in a unital C^*-algebra A, such that PIQ is extremally rich for any pair of projections such that $P \in A$ and $Q \in I$. If A/I is extremally rich and $\mathcal{E}(A/I)$ consists only of isometries and co-isometries then A is extremally rich.*

Note that the C^*-algebra $B(\mathcal{H})$ and its closed ideal $\mathcal{K}(\mathcal{H})$ of compact operators are known to have real rank zero. The following generalization of Proposition 4.1 can be proved by applying proposition 4.4.

Theorem 4.5. ([J]) *Let $E = (E^0, E^1)$ be a locally finite directed graph and V the set of sinks. If the subgraph G in Corollary 4.2 is cofinal then $C^*(E)$ is extremally rich.*

Example 4.6. Consider the following graph E.

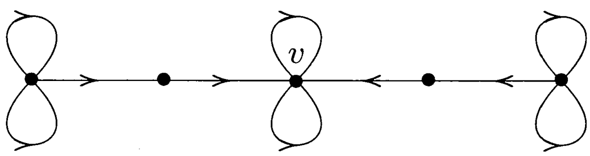


The sink v generates an ideal I which is isomorphic to $\mathcal{K}$, the compact operators acting on an infinite dimensional separable Hilbert space. Set $S = s_e + s_f$. Then $S^*S = 1$ and $SS^* = p_w < 1 = p_w + p_v$. Thus S is a proper isometry. Let $\mathcal{T}$ be the C^*-subalgebra of $C^*(E)$ generated by S. Since $S(1 - SS^*) = s_f$, it follows that $s_f \in \mathcal{T}$, hence $C^*(E) = \mathcal{T}$ is the Toeplitz algebra. Note that the subgraph G in Corollary 4.2 consists of the simple loop e, and so cofinal and by Theorem 4.5 we see that $\mathcal{T}$ is extremally rich, which is known in [Pd]. More generally, if a graph E consists of a simple loop with n vertices and each of the vertices emits an edge then we can conclude that the resulting graph algebra $C^*(E)$ is extremally rich.

Recall from Theorem 4.3(e) that if I is a purely infinite and simple closed ideal of a unital C^*-algebra A such that the quotient algebra A/I is also purely infinite and simple then A is extremally rich. Now suppose a C^*-algebra B has two proper ideals $I_1 \subset I_2$ such that every possible simple quotient is purely infinite. Then one cannot conclude the extremal richness of B. In fact, it is known in [LO, Remark 4.10] that there exists a non-extremally rich unital C^*-algebra B (non-separable) with two proper ideals $I_1 \subset I_2$ such that $I_1, I_2/I_1$ and B/I_2 are all purely infinite and simple.

In the following we give a separable unital graph C^*-algebra B with $RR(B) = 0$ which has exactly three proper ideals and every possible quotient is purely infinite and extremally rich, but B is not.

Example 4.7. Consider the following finite directed graph $E = (E^0, E^1)$.



Since E satisfies condition (K) we see that the graph algebra $C^*(E)$ has real rank zero by Theorem 3.4 and $C^*(E)$ has exactly three proper ideals by Theorem 2.2. Let H be the smallest hereditary saturated vertex subset containing v. Then the ideal $I(H)$ corresponding to H is stably isomorphic to the graph algebra $C^*(G)$, where G is a subgraph of

E with three vertices in the middle of E and four edges connecting them (Theorem 2.2). Since G is cofinal and satisfies (K) (hence (L)) $C^*(G)$ is purely infinite and simple by Proposition 2.3. Thus $I(H)$ is purely infinite and simple since it is well known that being purely infinite and simple is a stable property under a stable isomorphism. Moreover note that $I(H)$ is essential in $C^*(E)$, that is, it has nonzero intersection with every other nonzero closed ideal. Thus the graph algebra $C^*(E)$ is prime and hence its extreme point set of the unit ball consists of isometries or co-isometries. Now consider the quotient algebra $C^*(E)/I(H)$, then it is isomorphic to the graph C^*-algebra $C^*(F)$ by Theorem 2.2, where $F = (E^0 \setminus H, \{e \mid r(e) \notin H\})$. Since $C^*(F)$ is isomorphic to the direct sum $\mathcal{O}_2 \oplus \mathcal{O}_2$ of the Cuntz algebra $\mathcal{O}_2$ the quotient algebra is extremally rich. Let s_1, s_2 be two isometries generating the Cuntz algebra $\mathcal{O}_2$. If $C^*(E)$ were extremally rich then by [BP2, Corollary 9.3] every extreme partial isometry of $C^*(E)/I(H)$ should lift. But the partial isometry $u = s_i \oplus s_i^*$ $(i = 1, 2)$ is extremal in the quotient algebra $C^*(E)/I(H)$ and cannot lift to an isometry or a co-isometry. This proves the assertion. Note that $C^*(E)$ (and so every ideal) is purely infinite since E satisfies (L) and every vertex connects to a loop ([BPRS, Proposition 5.3]).

Example 4.8. *Let $E = (E^0, E^1)$ be a finite graph with $E^0 = \{1, 2, 3\}$ and $E^1 = \{e_{ij} \mid s(e_{ij}) = r(e_{ij}) = i,\ i = 1, 2, 3,\ j = 1, 2\} \cup \{f_i \mid s(f_i) = i, r(f_i) = i + 1, i = 1, 2\}$. Then the ideal generated by the vertex set $\{3\}$ is purely infinite (in fact, isomorphic to the Cuntz algebra $\mathcal{O}_2$) and essential in $C^*(E)$. The quotient (prime) algebra by the ideal is extremally rich by Theorem 4.3 (e) and has isometries and co-isometries as extreme points in its closed unit ball. Thus by* [LO, Theorem 3.6] *we conclude that $C^*(E)$ is extremally rich. More generally one can deduce by induction that for each n there is an extremally rich prime graph C^*-algebra B with precisely n proper ideals $I_1 \subset I_2 \subset \cdots \subset I_n$ and every possible quotient is purely infinite.*

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Direct limit decomposition for C*-algebras of minimal diffeomorphisms

Qing Lin and N. Christopher Phillips

This article outlines the proof that the crossed product $C^*(\mathbf{Z}, M, h)$ of a compact smooth manifold M by a minimal diffeomorphism $h\colon M \to M$ is isomorphic to a direct limit of subhomogeneous C*-algebras belonging to a tractable class. This result is motivated by the Elliott classification program for simple nuclear C*-algebras [9], and the observation that the known classification theorems in the stably finite case mostly apply to certain kinds of direct limits of subhomogeneous C*-algebras, or at least to C*-algebras with related structural conditions. (See Section 1.) This theorem is a generalization, in a sense, of direct limit decompositions for crossed products by minimal homeomorphisms of the Cantor set (Section 2 of [32]), for the irrational rotation algebras ([10]), and for some higher dimensional noncommutative toruses ([13], [14], [24], and [5]). (In [32], only a local approximation result is stated, but the C*-algebras involved are semiprojective.) Our theorem is not a generalization in the strict sense for several reasons; see the discussion in Section 1.

There are four sections. In the first, we state the theorem and discuss some consequences and expected consequences. In the second section, we describe the basic construction in our proof, a modified Rokhlin tower, and show how recursive subhomogeneous algebras appear naturally in our context. The third section describes how to prove local approximation by recursive subhomogeneous algebras, a weak form of the main theorem. In Section 4, we give an outline of how to use the methods of Section 3 to obtain the direct limit decomposition.

This paper is based on a talk given by the second author at the US–Japan Seminar on Operator Algebras and Applications (Fukuoka, June 1999), which roughly covered Sections 2 and 3, and on a talk given by

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the second author at the 28th Canadian Annual Symposium on Operator Algebras (Toronto, June 2000), which roughly covered Sections 1 and 2. At the time of the first talk, only the local approximation result described in Section 3 had been proved. We refer to the earlier survey paper [25] for earlier parts of the story; this paper reports the success of the project described in Section 6 there.

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§1. The main theorem, consequences, and conjectured consequences

The main theorem is as follows. Undefined terminology is discussed after the statement.

Theorem 1.1. Let M be a connected compact smooth manifold with $\dim(M) = d > 0$, and let $h\colon M \to M$ be a minimal diffeomorphism. Then there exists an increasing sequence

$$A_0 \subset A_1 \subset A_2 \subset \cdots \subset C^*(\mathbf{Z}, M, h)$$

of C*-subalgebras of $C^*(\mathbf{Z}, M, h)$ such that

$$\overline{\bigcup_{n=0}^{\infty} A_n} = C^*(\mathbf{Z}, M, h)$$

and such that each A_n has a separable recursive subhomogeneous decomposition with topological dimension at most d and strong covering number at most $d(d+2)$.

A recursive subhomogeneous algebra (a C*-algebra with a recursive subhomogeneous decomposition) is a particularly tractable kind of subhomogeneous C*-algebra. See [29], [30], and [25], and also see the consequences below. We will explain in Section 2 how recursive subhomogeneous algebras arise, and we will recall (informally) the definition there (after Theorem 2.7). A finite direct sum

$$\bigoplus_{k=0}^{l} C\big(X_k, M_{n(k)}\big)$$

of (trivial) homogeneous C*-algebras is a special case of a recursive subhomogeneous algebra, and the topological dimension is simply $\max_{0 \leq k \leq l} \dim(X_k)$. (Dimension is taken to be covering dimension; see Definition 1.6.7 of [15].) The condition in the theorem that A_n have topological dimension at most d for all n thus ensures that the resulting direct limit decomposition $C^*(\mathbf{Z}, M, h) \cong \varinjlim A_n$ has no dimension growth.

In general, it is not possible to find a representation as a direct limit (with no dimension growth) of direct sums of corners of trivial homogeneous C*-algebras. A simple direct limit of this sort must even be approximately divisible in the sense of [4], by Theorem 2.1 of [11]. However, a crossed product by a minimal diffeomorphism may have no nontrivial projections (Corollary 3 and Example 4 of Section 5 of [7]).

We will not define the strong covering number here, although some discussion will be given after Theorem 3.1. We have included it in the conclusion because the proof of Theorem 3.1 suggests that a bound on the strong covering number might be necessary for some classification results.

The requirement that we have a diffeomorphism of a manifold is connected with the appearance of a condition on the strong covering number in the hypotheses of Theorem 3.1. This also will be discussed after that theorem. We certainly expect that the theorem will be true for minimal homeomorphisms of finite dimensional compact metric spaces (even, presumably, compact metric spaces with infinite covering dimension).

We point out here that our theorem does not directly imply the Elliott-Evans direct limit representation for the irrational rotation algebras [10]. Our theorem gives a representation of an irrational rotation algebra as a direct limit of recursive subhomogeneous algebras with topological dimension at most 1, while the Elliott-Evans theorem gives a representation as a direct limit of direct sums of homogeneous C*-algebras with topological dimension at most 1 (in fact, circle algebras). We do not recover the results of [13], [14], and [24] (for certain higher dimensional noncommutative toruses), not only because the algebras in our direct system are more complicated but also because not all the algebras considered there are even crossed products by diffeomorphisms. We also do not recover the direct limit decomposition for crossed products by minimal homeomorphisms of the Cantor set (see Section 2 of [32] for the local approximation result), because the Cantor set is not a manifold. (Our methods do specialize to this case, but that would be silly, since our argument is much more complicated.)

Theorem 1.1 has the following consequences for crossed products by minimal diffeomorphisms. These consequences all hold for an arbitrary simple unital direct limit of recursive subhomogeneous algebras, assuming no dimension growth and that the maps of the system are unital and injective. (Most don't require the full strength of these hypotheses, but all require some restriction on dimension growth. None require any hypotheses on the strong covering number.) The proofs are in [30], and the statements can be found in Section 4 of [25] (except for the last one, which is actually a consequence of stable rank one). In all of these, M is a connected compact smooth manifold with $\dim(M) > 0$, and $h\colon M \to M$ is a minimal diffeomorphism.

Corollary 1.2. (Theorem 3.6 of [30].) The algebra $C^*(\mathbf{Z}, M, h)$ has stable rank one in the sense of [33]. That is, the invertible group $\mathrm{inv}(C^*(\mathbf{Z}, M, h))$ is dense in $C^*(\mathbf{Z}, M, h)$.

Corollary 1.3. (Theorem 2.2 of [30].) The projections in

$$M_\infty(C^*(\mathbf{Z}, M, h)) = \bigcup_{n=1}^{\infty} M_n(C^*(\mathbf{Z}, M, h))$$

satisfy cancellation. That is, if e, p, $q \in M_\infty(C^*(\mathbf{Z}, M, h))$ are projections, and if $p \oplus e \sim q \oplus e$, then $p \sim q$.

Corollary 1.4. (Theorem 2.3 of [30].) The algebra $C^*(\mathbf{Z}, M, h)$ satisfies Blackadar's Second Fundamental Comparability Question ([2], 1.3.1). That is, if p, $q \in M_\infty(C^*(\mathbf{Z}, M, h))$ are projections, and if $\tau(p) < \tau(q)$ for every normalized trace τ on $C^*(\mathbf{Z}, M, h)$, then $p \precsim q$.

Corollary 1.5. (Theorem 2.4 of [30].) The group $K_0(C^*(\mathbf{Z}, M, h))$ is unperforated for the strict order. That is, if $\eta \in K_0(C^*(\mathbf{Z}, M, h))$ and if there is $n > 0$ such that $n\eta > 0$, then $\eta > 0$.

(In the simple case, this is the same as saying that $K_0(C^*(\mathbf{Z}, M, h))$ is weakly unperforated in the sense of 2.1 of [8].)

Corollary 1.6. (Theorem 2.1 of [30].) The canonical map

$$U(C^*(\mathbf{Z}, M, h))/U_0(C^*(\mathbf{Z}, M, h)) \to K_1(C^*(\mathbf{Z}, M, h))$$

is an isomorphism.

A small part of these results could already be obtained using the weaker (and much simpler) methods described in Sections 1 and 5 of [25]. It had already been shown that the order on $K_0(C^*(\mathbf{Z}, M, h))$ is determined by traces (a weak form of Corollary 1.4), and hence that $K_0(C^*(\mathbf{Z}, M, h))$ is unperforated for the strict order (Corollary 1.5). Also, surjectivity in Corollary 1.6 (but not injectivity) was known.

The criterion in [3], for when a simple direct limit of direct sums of trivial homogeneous C*-algebras with slow dimension growth has real rank zero, is known to fail for simple direct limits of recursive subhomogeneous algebras with no dimension growth. (Indeed, it even fails for crossed products by minimal diffeomorphisms; see Example 5.7 of [25].) Nevertheless, it appears likely that a suitable strengthening of the condition will be equivalent to real rank zero for such direct limits, and that the proof will not be difficult. Specializing (for simplicity) to the case of a unique trace, we obtain the following, which we state as a conjecture.

Conjecture 1.7. Let M be a connected compact smooth manifold with $\dim(M) > 0$, and let $h\colon M \to M$ be a uniquely ergodic minimal diffeomorphism. Let

$$\tau\colon C^*(\mathbf{Z}, M, h) \to \mathbf{C}$$

be the trace induced by the unique invariant probability measure. Then $C^*(\mathbf{Z}, M, h)$ has real rank zero ([6]) if and only if $\tau_*(K_0(C^*(\mathbf{Z}, M, h)))$ is dense in $\mathbf{R}$.

For methods for computing the ranges of traces on the K-theory of crossed products by $\mathbf{Z}$, we refer to [16].

It might not be terribly difficult to prove that if a simple C*-algebra A is a direct limit of a system of recursive subhomogeneous algebras with no dimension growth, and possibly also assuming that the maps of the system are unital and injective, then real rank zero implies tracial rank zero in the sense of H. Lin [20]. If so, then the following result of H. Lin (Theorem 3.9 of [23]) implies classifiability:

Theorem 1.8. Suppose A and B are separable simple unital C*-algebras with tracial rank zero in the sense of [20]. Suppose that each has local approximation by subalgebras with bounded dimensions of irreducible representations. That is, for every finite subset $F \subset A$ and every $\varepsilon > 0$, there is a C*-subalgebra $D \subset A$ and an integer N such that every element of F is within ε of an element of D and every irreducible representation of D has dimension at most N; and similarly for B. Then

$$(K_0(A),\, K_0(A)_+,\, [1_A],\, K_1(A)) \cong (K_0(B),\, K_0(B)_+,\, [1_B],\, K_1(B))$$

implies $A \cong B$.

In particular, one would have a proof of the following conjecture:

Conjecture 1.9. Let M be a connected compact smooth manifold with $\dim(M) > 0$, and let $h\colon M \to M$ be a uniquely ergodic minimal diffeomorphism. Let

$$\tau \colon C^*(\mathbf{Z}, M, h) \to \mathbf{C}$$

be the trace induced by the unique invariant probability measure, and assume that $\tau_*(K_0(C^*(\mathbf{Z}, M, h)))$ is dense in $\mathbf{R}$. Then the crossed product C*-algebra $C^*(\mathbf{Z}, M, h)$ is classifiable.

We will not give a precise definition of "classifiable" here.

We note that H. Lin's classification theorem has no hypotheses involving slow dimension growth, and does not even require a direct limit representation; only local approximation is needed, and the condition on the approximating algebras is weak. (Indeed, H. Lin has other classification theorems which don't even require local approximation, but do require further restrictions on the K-theory.) However, at least with our current state of knowledge, the direct limit representation in Theorem 1.1, including the no dimension growth condition, seems to be needed to verify the other hypotheses of Theorem 1.8. For example, simple direct limits that don't have slow dimension growth need not even have stable rank one [35].

Since $C^*(\mathbf{Z}, M, h)$ always has stable rank one, if it doesn't have real rank zero then it has real rank one. However, most of the currently known general classification theorems apply only to algebras with many projections, and those that don't are much too restrictive in other ways (such as assuming trivial K-theory). In particular, the C*-algebras covered by [17] and [12] are approximately divisible (as discussed above), and the theorems of H. Lin (see [21] and [22]) require a finite value of the tracial rank, the definition of which again requires the existence of many nontrivial projections. However, as mentioned above, the example of Connes shows that $C^*(\mathbf{Z}, M, h)$ may have no nontrivial projections. There is a classification theorem [19] for a special class of direct limits which includes simple C*-algebras with no nontrivial projections, but the building blocks there are much more special than those appearing in our theorem.

We are hopeful that the approach of [17] and [12], which now covers simple direct limits, with no dimension growth, of direct sums of homogeneous C*-algebras (actually, a slightly larger class), can be generalized to cover simple direct limits, with no dimension growth, of recursive subhomogeneous algebras, possibly with the added restriction of no growth of the strong covering number. One reason for optimism (as well as for the belief that conditions on the strong covering number might be necessary) is the successful generalization of exponential length results from the case of trivial homogeneous C*-algebras to recursive subhomogeneous algebras; see Theorem 3.1 below. The related results for the trivial homogeneous case (see Theorems 3.3 and 4.5 of [28]) depended heavily on the existence of many projections, but in the proof of Theorem 3.1 we had to learn to handle situations with no nontrivial projections at all. However, we do not know whether Theorem 3.1 is even true without the condition on the strong covering number. (See the discussion after the statement of that theorem.) We included the bound on the strong covering number in Theorem 1.1 because of the possibility that it might be necessary for our suggested approach to proving a classification result, or perhaps even for a classification result to hold.

In any case, a generalization of the methods of [17] and [12] is likely to be very difficult. Possibly the situation will be improved by a generalization of H. Lin's methods that is strong enough to apply to simple C*-algebras which contain no nontrivial projections.

§2. Modified Rokhlin towers

Throughout this section, M is a compact metric space and $h\colon M \to M$ is a minimal homeomorphism. (The requirement that M be a manifold will not be needed until the next section.) We let u denote the implementing unitary in $C^*(\mathbf{Z}, M, h)$, so that $ufu^* = f \circ h^{-1}$ for $f \in C(M)$.

We start with a definition.

Definition 2.1. Let $Y \subset M$, and let $x \in Y$. The *first return time* $\lambda_Y(x)$ (or $\lambda(x)$ if Y is understood) of x to Y is the smallest integer $n \geq 1$ such that $h^n(x) \in Y$. We set $\lambda(x) = \infty$ if no such n exists.

The following result is well known in the area, and is easily proved:

Lemma 2.2. If $\mathrm{int}(Y) \neq \varnothing$, then $\sup_{x \in Y} \lambda(x) < \infty$.

Let $Y \subset M$ with $\mathrm{int}(Y) \neq \varnothing$. Let $n(0) < n(1) < \cdots < n(l)$ (or, if the dependence on Y must be made explicit, $n_Y(0) < n_Y(1) < \cdots < n_Y(l_Y)$) be the distinct values of $\lambda(x)$ for $x \in Y$. The Rokhlin tower based on a subset $Y \subset M$ with $\mathrm{int}(Y) \neq \varnothing$ consists of the partition

$$Y = \coprod_{k=0}^{l} \{x \in Y \colon \lambda(x) = n(k)\}$$

of Y (the sets here are the base sets), and the corresponding partition

$$M = \coprod_{k=0}^{l} \coprod_{j=0}^{n(k)-1} h^j\big(\{x \in Y \colon \lambda(x) = n(k)\}\big)$$

of M. Note that h acts like a cyclic shift except on the top space

$$h^{n(k)-1}\big(\{x \in Y \colon \lambda(x) = n(k)\}\big)$$

of each "tower"

$$\coprod_{j=0}^{n(k)-1} h^j\big(\{x \in Y \colon \lambda(x) = n(k)\}\big).$$

Actually, for our purposes it is more convenient to use the partition

$$M = \coprod_{k=0}^{l} \coprod_{j=1}^{n(k)} h^j\big(\{x \in Y \colon \lambda(x) = n(k)\}\big).$$

Note that

$$Y = \coprod_{k=0}^{l} h^{n(k)}\big(\{x \in Y \colon \lambda(x) = n(k)\}\big),$$

so that h now acts like a cyclic shift on the towers, except on Y itself.

We will be interested in arbitrarily small choices for Y, in particular with arbitrarily small diameter and for which the smallest first return time $n_Y(0)$ is arbitrarily large. If M is totally disconnected, then we may choose Y to be both closed and open. In this case, the sets

$$Y_k = \{x \in Y \colon \lambda(x) = n(k)\}$$

are all closed, and there is a composite homomorphism γ_0 given by

$$C(M) \longrightarrow \bigoplus_{k=0}^{l} \bigoplus_{j=1}^{n(k)} C(h^j(Y_k)) \xrightarrow{\cong} \bigoplus_{k=0}^{l} C(Y_k)^{n(k)},$$

which is in fact an isomorphism. The formula is

$$\gamma_0(f) = \big(\big(f \circ h|_{Y_0}, \ldots, f \circ h^{n(0)}|_{Y_0}\big), \ldots, \big(f \circ h|_{Y_l}, \ldots, f \circ h^{n(l)}|_{Y_l}\big)\big).$$

See [31] for the exploitation of this idea.

In order to have a C*-algebraically sensible codomain for γ_0, we must insist that the sets Y_k be closed. However, the spaces M we are interested in are connected, so we are forced to choose

$$Y_k = \overline{\{x \in Y \colon \lambda(x) = n(k)\}}$$

instead. The sets $h^j(Y_k)$ are no longer disjoint (although they certainly cover M), so our map

$$\gamma_0 \colon C(M) \to \bigoplus_{k=0}^{l} C(Y_k)^{n(k)},$$

while still injective, is no longer an isomorphism.

Next, define

$$s_k \in M_{n(k)} \subset C\left(Y_k, M_{n(k)}\right)$$

by

$$s_k = \begin{pmatrix} 0 & 0 & \cdots & \cdots & 0 & 0 & 1 \\ 1 & 0 & \cdots & \cdots & 0 & 0 & 0 \\ 0 & 1 & \cdots & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & & \vdots & \vdots & \vdots \\ \vdots & \vdots & & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & 1 & 0 & 0 \\ 0 & 0 & \cdots & \cdots & 0 & 1 & 0 \end{pmatrix},$$

and define

$$s = (s_0, s_1, \dots, s_l) \in \bigoplus_{k=0}^{l} C\left(Y_k, M_{n(k)}\right).$$

Then s is unitary. Identifying $C(Y_k)^{n(k)}$ with the diagonal matrices in $C\left(Y_k, M_{n(k)}\right)$ in the obvious way, one can check that if $f \in C(M)$ vanishes on Y, then

$$\gamma_0(ufu^*) = \gamma_0(f \circ h^{-1}) = s\gamma_0(f)s^*.$$

The calculation uses the fact that

$$Y = \bigcup_{k=0}^{l} h^{n(k)}(Y_k),$$

and in fact our choice to start our towers at $h(Y_k)$ rather than at Y_k was made to have this formula work correctly when f vanishes on Y (rather than when f vanishes on $h^{-1}(Y)$).

This relation allows us to extend γ_0 to the following subalgebra of $C^*(\mathbf{Z}, M, h)$:

Definition 2.3. For any closed subset $Y \subset M$, we define

$$A(Y) = C^*\big(C(M),\, uC_0(M \setminus Y)\big) \subset C^*(\mathbf{Z}, M, h),$$

the C*-subalgebra of $C^*(\mathbf{Z}, M, h)$ generated by $C(M)$ and $uC_0(M \setminus Y)$. Here, we identify $C_0(M \setminus Y)$ in the obvious way with the subalgebra of $C(M)$ consisting of those functions vanishing on Y. We use the analogous convention throughout the paper.

Proposition 2.4. Suppose M is a compact metric space, and let $h\colon M \to M$ be a minimal homeomorphism. Let $Y \subset M$ be closed with $\mathrm{int}(Y) \neq \varnothing$. Then there exists a unique homomorphism

$$\gamma_Y : A(Y) \to \bigoplus_{k=0}^{l} C\left(Y_k, M_{n(k)}\right)$$

such that if $f \in C(M)$, then

$$\gamma_Y(f)_k = \mathrm{diag}\left(f \circ h|_{Y_k},\, f \circ h^2|_{Y_k},\, \ldots,\, f \circ h^{n(k)}|_{Y_k}\right)$$

and if $f \in C_0(M \setminus Y)$, then

$$(\gamma_Y(uf))_k = s_k \gamma_Y(f)_k.$$

Moreover, γ_Y is unital and injective.

We now introduce a slight twist on these ideas.

Definition 2.5. Let $Y \subset M$ be closed with $\mathrm{int}(Y) \neq \varnothing$. Let $S \subset \mathrm{int}(Y_0)$ be closed. Define

$$\mathrm{ex}(S) = \left\{h(S), h^2(S), \ldots, h^{n(0)}(S)\right\},$$

which is a collection of disjoint closed subsets of M. Define $C(M)_{\mathrm{ex}(S)}$ to be the set of all $f \in C(M)$ such that f is constant on T for every $T \in \mathrm{ex}(S)$. (The constant value is allowed to depend on T.) Define $A(Y, S)$ to be the C*-subalgebra of $C^*(\mathbf{Z}, M, h)$ given by

$$A(Y, S) = C^*\left(C(M)_{\mathrm{ex}(S)},\, u\left[C_0(M \setminus Y) \cap C(M)_{\mathrm{ex}(S)}\right]\right) \subset A(Y).$$

As we will see below, the point of this definition is that (when $\mathrm{int}(S) \neq \varnothing$) we can construct useful unitaries in $C^*(\mathbf{Z}, M, h)$ which commute with (most of) $A(Y, S)$. (See Step 9 in the proof outline in Section 3.)

It is not obvious what the image

$$\gamma_Y(A(Y,S)) \subset \bigoplus_{k=0}^{l} C\left(Y_k, M_{n(k)}\right)$$

looks like, and working with it directly threatens to be very complicated. Fortunately, the essential properties can be abstracted in a tractable way; the result is what we call a recursive subhomogeneous algebra. (The definition of a recursive subhomogeneous algebra was in fact invented for exactly this purpose.) First, we recall the notion of a pullback.

Definition 2.6. Let A and B be C*-algebras, and let a third C*-algebra C and homomorphisms $\varphi\colon A \to C$ and $\psi\colon B \to C$ be given. The *pullback* (also called fibered product or restricted direct sum) is

$$A \oplus_C B = A \oplus_{C,\varphi,\psi} B = \{(a,b) \in A \oplus B \colon \varphi(a) = \psi(b)\}.$$

If the maps φ and ψ are understood, we will write $A \oplus_C B$.

Theorem 2.7. Let M be a compact metric space, and let $h\colon M \to M$ be a minimal homeomorphism. Let $Y \subset M$ be closed with $\operatorname{int}(Y) \neq \varnothing$. Let $S \subset \operatorname{int}(Y_0)$ be closed. Then there exist closed subsets

$$Y_k^{(0)} \subset \partial Y_k \subset Y_k$$

for $1 \leq k \leq l$, and homomorphisms φ_k and ψ_k (with ψ_k being just the restriction map) such that the image $\gamma_Y(A(Y,S))$ is equal to the subalgebra

$$\begin{aligned}\Big[\cdots \Big[\left[C\left(Y_0, M_{n(0)}\right)_S \oplus_{C\left(Y_1^{(0)}, M_{n(1)}\right),\varphi_1,\psi_1} C\left(Y_1, M_{n(1)}\right)\right] \\ \oplus_{C\left(Y_2^{(0)}, M_{n(2)}\right),\varphi_2,\psi_2} C\left(Y_2, M_{n(2)}\right)\Big] \cdots \Big] \\ \oplus_{C\left(Y_l^{(0)}, M_{n(l)}\right),\varphi_l,\psi_l} C\left(Y_l, M_{n(l)}\right)\end{aligned}$$

of $\bigoplus_{k=0}^{l} C\left(Y_k, M_{n(k)}\right)$. Here, by analogy with Definition 2.5, we set

$$C\left(Y_0, M_{n(0)}\right)_S = \left\{f \in C\left(Y_0, M_{n(0)}\right) : f \text{ is constant on } S\right\}.$$

A C*-algebra given as an iterated pullback as in the conclusion of this theorem, in which the algebras have the form $C\left(X_k, M_{n(k)}\right)$, the maps φ_k are unital, and the maps ψ_k are unital and surjective, is called a *recursive subhomogeneous algebra*. We refer to Section 2 of [25] for a more careful definition, for some useful associated terminology, and examples; to Section 3 of [25] for a discussion of the proof of Theorem 2.7 (in the case $S = \varnothing$); and to Section 4 of [25] for a discussion of why the concept of a recursive subhomogeneous decomposition is useful and what can be done with it. We recall here that the topological dimension is the largest dimension $\dim(X_k)$. Unfortunately, it depends on the particular decomposition; see Example 2.9 of [25]. We will always have a decomposition in mind, usually coming from Theorem 2.7.

The next difficulty we face is that the unitary

$$s = (s_0, s_1, \dots, s_l) \in \bigoplus_{k=0}^{l} C\left(Y_k, M_{n(k)}\right)$$

is not in the image of $A(Y)$. (When M is totally disconnected and Y is both closed and open, there is no problem: the image of γ_Y is all of $\bigoplus_{k=0}^{l} C\left(Y_k, M_{n(k)}\right)$.) The cure for this problem is the following lemma, which however requires that we look at two nested subsets Y and Z, along with the associated subalgebras $A(Y)$ and $A(Z)$.

Lemma 2.8. Let M be a compact metric space with finite covering dimension d, and let $h\colon M \to M$ be a minimal homeomorphism. Let $Y \subset M$ be closed with $\mathrm{int}(Y) \neq \varnothing$. Then every point of $\mathrm{int}(Y)$ has a neighborhood $U \subset \mathrm{int}(Y)$ such that for every closed set $Z \subset U$ with $\mathrm{int}(Z) \neq \varnothing$, and every closed subset $S \subset \mathrm{int}(Z_0)$, there is a unitary $v \in A(Z, S)$ such that $vf = uf$ in $C^*(\mathbf{Z}, M, h)$ whenever $f \in C(M)$ vanishes on Y.

The condition on U used in the proof is that there are at least $\max\left(1, \frac{1}{2}d\right)$ images of $\overline{U}$ under positive powers h^r of h, with r less than the smallest first return time of $\overline{U}$ to itself, which are contained in $\mathrm{int}(Y)$. Under this condition, the first step in the construction of v is an approximate polar decomposition, in the recursive subhomogeneous algebra $\gamma_Z(A(Z, S))$, of ug for a suitable function $g \in C(M)_{\mathrm{ex}_Z(S)}$ which, in particular, is required to be equal to 1 on $M \setminus \mathrm{int}(Y)$ and to vanish on Z.

It isn't in general true that $\mathrm{int}(Z) \neq \varnothing$ implies $\mathrm{int}(Z_0) \neq \varnothing$, although it happens that the sets we use in the diffeomorphism case automatically have $\mathrm{int}(Z_k) \neq \varnothing$ for all k.

To sum up: We have what might be called the "basic construction" for weak approximation in $C^*(\mathbf{Z}, M, h)$ (not to be confused with the basic construction of subfactor theory), namely a triple (Y, Z, v) (or a quadruple (Y, Z, S, v)) consisting of closed subsets with

$$S \subset \mathrm{int}(Z_0) \subset Z \subset \mathrm{int}(Y) \subset Y \subset M$$

(or, if S is not present, at least $\mathrm{int}(Z) \neq \varnothing$), and a unitary $v \in A(Z, S)$ ($A(Z)$ if S is not present) such that $vf = uf$ in $C^*(\mathbf{Z}, M, h)$ whenever $f \in C(M)$ vanishes on Y. We say weak approximation here because we have not approximated u in norm; rather, we have a unitary $v \in A(Z, S)$ which "acts like u" (that is, like h) on most of the space M. In particular, this construction is not the same as what we call a "basic approximation" in [26]. The basic approximation, of which we describe an easier form in the next section, does permit the norm approximation of u, but requires two nested basic constructions and an additional unitary.

§3. An outline of the proof of local approximation

In this section, we outline the proof of a weak form of Theorem 1.1, namely that if $h\colon M \to M$ is a minimal diffeomorphism of a connected compact smooth manifold M with $\dim(M) > 0$, and if $F \subset C^*(\mathbf{Z}, M, h)$ is a finite set and $\varepsilon > 0$, then there is a recursive subhomogeneous algebra $A \subset C^*(\mathbf{Z}, M, h)$ which approximately contains F to within ε. This result requires most of the machinery needed for the proof of the full direct limit decomposition result.

The crucial ingredient not yet mentioned is related to Loring's version [27] of Berg's technique [1]. This method (described in Step 7 below) requires a priori bounds on the lengths of paths connecting certain elements in the unitary groups of hereditary subalgebras of recursive subhomogeneous algebras. This is an exponential length problem in the sense of [34]. We therefore begin by stating our exponential length result; we require some terminology.

First, if A is a unital C*-algebra and $B \subset A$ is a hereditary subalgebra, we define the unitary group $U(B)$ to be

$$U(B) = \{u \in U(A)\colon u - 1 \in B\}.$$

(This is the same as a common definition in terms of the unitization B^+ of B, namely

$$U(B) = \{u \in U(B^+)\colon u - 1 \in B\}.$$

Moreover, if B is actually a corner, then this group can be canonically identified with the usual unitary group of B.) Further, let

$$A = \left[\cdots \left[\left[C\left(X_0, M_{n(0)}\right) \oplus_{C\left(X_1^{(0)}, M_{n(1)}\right)} C\left(X_1, M_{n(1)}\right) \right] \right.\right.$$
$$\left.\left. \oplus_{C\left(X_2^{(0)}, M_{n(2)}\right)} C\left(X_2, M_{n(2)}\right) \right] \cdots \right] \oplus_{C\left(X_l^{(0)}, M_{n(l)}\right)} C\left(X_l, M_{n(l)}\right)$$

be a recursive subhomogeneous algebra. If $B \subset A$ is a hereditary subalgebra and $x \in X_k$ for some k, then we define $\mathrm{rank}_x(B)$ to be the rank of the identity in the image of B in the finite dimensional C*-algebra $M_{n(k)}$ under the map ev_x given by point evaluation at $x \in X_k$. If $v \in U(A)$, then we say that $\det(v) = 1$ if $\det(\mathrm{ev}_x(v)) = 1$ for all k and all $x \in X_k$. (Although determinants are not well defined in recursive subhomogeneous algebras, one can show that the condition $\det(v) = 1$ is well defined.)

Theorem 3.1. Let d, $d' \geq 0$ be integers. Then there is an integer R such that the following holds.

Let A be a recursive subhomogeneous algebra which has a separable recursive subhomogeneous decomposition with topological dimension at most d and strong covering number at most d'. Let $B \subset A$ be a hereditary subalgebra such that $\mathrm{rank}_x(B) \geq R$ for every x in the total space of A. Let $v \in U(B)$ satisfy $\det(v) = 1$ and be connected to 1 by a path $t \mapsto v_t$ in $U(B)$ such that $\det(v_t) = 1$ for all t. Then there is a continuous path from v to 1 in $U(B)$ with length less than $4\pi(d' + 2)$.

At this point, we should give a brief indication of the significance of the strong covering number. We explained in Section 4 of [25] how to use relative versions of the subprojection and cancellation theorems for $C(X, M_n)$ to obtain analogous theorems for recursive subhomogeneous algebras. Theorem 3.1, however, is an exponential length theorem, and, at a crucial step in its proof, we have only been able to prove an approximate relative theorem for $C(X, M_n)$. (See Theorem 6.2 of [25].) Roughly speaking, errors accumulate everywhere that the recursive subhomogeneous decomposition of A specifies that two algebras be glued together. The strong covering number gives a limit on how often a neighborhood of a particular point in one of the base spaces is involved in such a gluing. It is a strengthened version of the most obvious notion (the "covering number"); the more obvious version proved to be technically too weak.

The definition of the strong covering number is somewhat complicated, and is omitted; instead, we illustrate with an example. Let X be a compact metric space, let E be a locally trivial continuous field over X with fiber M_n, and let $\Gamma(E)$ be the corresponding section algebra. Then any finite cover $X_0, X_1, \ldots, X_l$ of X by closed subsets, such that $E|_{X_k}$ is trivial for each k, induces a recursive subhomogeneous decomposition of $\Gamma(E)$. (See the proof of Proposition 1.7 of [29] and Example 2.8 of [25].) It can be shown that the strong covering number of this recursive subhomogeneous decomposition is the order (as in Definition 1.6.6 of [15]) of the cover of X by the sets $X_0, X_1, \ldots, X_l$, that is, the largest number d such that there are distinct $r_0, r_1, \ldots, r_d$ for which

$$\bigcap_{j=0}^{d} X_{r_j} \neq \varnothing.$$

Note the parallel with the definition of the covering dimension (Definition 1.6.7 of [15]).

At this point, we can explain how we use the condition that we have a diffeomorphism of a manifold. Let $Y \subset M$ satisfy $\mathrm{int}(Y) \neq \varnothing$. Our method for bounding the strong covering number requires that there be an integer m such that, for any $m+1$ distinct integers $r_0, r_1, \ldots, r_m \in \mathbf{Z}$, we have

$$\bigcap_{j=0}^{m} h^{r_j}(\partial Y) = \varnothing.$$

When h is a minimal diffeomorphism of a compact manifold, this is arranged as follows. First, require that ∂Y be a smooth submanifold (of codimension 1). Then perturb ∂Y by an arbitrarily small amount, so that all finite sets

$$h^{r_0}(\partial Y),\ h^{r_1}(\partial Y),\ \ldots, h^{r_m}(\partial Y)$$

of distinct images of ∂Y under powers of h are jointly mutually transverse. This means, first, that $h^{r_0}(\partial Y)$ and $h^{r_1}(\partial Y)$ are transverse (see pages 28–30 of [18]) whenever $r_0 \neq r_1$, so that $h^{r_0}(\partial Y) \cap h^{r_1}(\partial Y)$ is a smooth submanifold (of codimension 2; see the theorem on page 30 of [18]); that $h^{r_2}(\partial Y)$ and $h^{r_0}(\partial Y) \cap h^{r_1}(\partial Y)$ are transverse whenever r_0, r_1, and r_2 are all distinct, so that $h^{r_0}(\partial Y) \cap h^{r_1}(\partial Y) \cap h^{r_2}(\partial Y)$ is a smooth submanifold (of codimension 3); etc. These conditions guarantee that the intersection of any $\dim(M)+1$ distinct images of ∂Y under powers of h will be empty. (Note, however, that the resulting upper bound on the strong covering number turns out to be $\dim(M)(\dim(M)+2)$, not $\dim(M)$. The situation is much more complicated than for section algebras of locally trivial continuous fields.) We thus have:

Proposition 3.2. Let M be a connected compact smooth manifold with $\dim(M) = d > 0$, and let $h \colon M \to M$ be a minimal diffeomorphism. For every $x \in M$ and open $U \subset M$ with $x \in U$, there is a closed set $Y \subset M$ with $x \in \mathrm{int}(Y) \subset Y \subset U$ such that for every closed set $S \subset \mathrm{int}(Y_0)$ (notation as in Section 2) which is homeomorphic to a closed ball in $\mathbf{R}^d$, the subalgebra $A(Y, S)$ satisfies the following properties:

- The recursive subhomogeneous decomposition of Theorem 2.7 has topological dimension equal to d.
- The decomposition of Theorem 2.7 has strong covering number at most $d(d+2)$.
- In the notation of Theorem 2.7, we have $Y_k^{(0)} \subset \partial Y_k$ for all k.

We hope that if h is a minimal homeomorphism of a finite dimensional compact metric space, then one might be able to substitute a dimension theory argument for transversality in the above. We have not yet had time to look into this. What to do about infinite dimensional compact metric spaces (such as $(S^1)^{\mathbf{Z}}$) is less clear.

Now we start the outline of the proof of local approximation. We fix a connected compact smooth manifold M with $\dim(M) > 0$ and a minimal diffeomorphism $h: M \to M$.

Step 1. It suffices to prove the following: Let

$$f_1, f_2, \ldots, f_m \in C(M) \subset C^*(\mathbf{Z}, M, h)$$

be a finite collection of functions, and let $\varepsilon > 0$. Then there is a recursive subhomogeneous algebra $A \subset C^*(\mathbf{Z}, M, h)$ which approximately contains $\{f_1, f_2, \ldots, f_m, u\}$ to within ε. (The reason is that $C(M)$ and u generate $C^*(\mathbf{Z}, M, h)$ as a C*-algebra.)

Step 2. Choose $\delta > 0$ so small that the functions $f_1, f_2, \ldots, f_m$ are all approximately constant to within $\frac{1}{2}\varepsilon$ on every subset of M with diameter less than δ. Choose an integer R following Theorem 3.1 for the number $d = \dim(M)$ and for $d' = d(d+2)$, and also with $R \geq \max\left(1, \frac{1}{2}d\right)$. Choose an integer N so large that

$$\frac{4\pi(d'+2)}{N} < \varepsilon.$$

Step 3. Choose a quadruple $\left(Y^{(1)}, Z^{(1)}, S, v_1\right)$, as described at the end of the previous section, consisting of closed subsets with

$$\varnothing \neq \operatorname{int}(S) \subset S \subset \operatorname{int}\left(Z_0^{(1)}\right) \subset Z^{(1)} \subset \operatorname{int}\left(Y^{(1)}\right) \subset Y^{(1)} \subset M$$

and a unitary $v_1 \in A\left(Z^{(1)}, S\right)$ such that $v_1 f = uf$ in $C^*(\mathbf{Z}, M, h)$ whenever $f \in C(M)$ vanishes on $Y^{(1)}$. We also require that the conclusions of Proposition 3.2 be satisfied. Let $n_1(0) < n_1(1) < \cdots < n_1(l_1)$ be the first return times $n_{Y^{(1)}}(0) < n_{Y^{(1)}}(1) < \cdots < n_{Y^{(1)}}\left(l_{Y^{(1)}}\right)$. We then further require that the sets involved be so small that:

- The sets $Y^{(1)}, h^{-1}\left(Y^{(1)}\right), \ldots, h^{-N}\left(Y^{(1)}\right)$ are pairwise disjoint (whence $n_1(0) > N$).

- The sets $Y^{(1)}$, $h^{-1}\left(Y^{(1)}\right)$, ..., $h^{-N}\left(Y^{(1)}\right)$ all have diameter less than δ.
- The sets $h(S)$, $h^2(S)$, ..., $h^{n_1(0)}(S)$ all have diameter less than δ.
- Each of the sets $h(S)$, $h^2(S)$, ..., $h^{n_1(0)}(S)$ is either contained in one of $Y^{(1)}$, $h^{-1}\left(Y^{(1)}\right)$, ..., $h^{-N}\left(Y^{(1)}\right)$ or is disjoint from all of them.

(Note that we choose S after having chosen $Y^{(1)}$.)

Step 4. Choose a triple $\left(Y^{(2)}, Z^{(2)}, v_2\right)$, as described at the end of the previous section, consisting of closed subsets with

$$\varnothing \neq \operatorname{int}\left(Z^{(2)}\right) \subset Z^{(2)} \subset \operatorname{int}\left(Y^{(2)}\right) \subset Y^{(2)} \subset \operatorname{int}(S)$$

and a unitary $v_2 \in A\left(Z^{(2)}, S\right)$ such that $v_2 f = uf$ in $C^*(\mathbf{Z}, M, h)$ whenever $f \in C(M)$ vanishes on $Y^{(2)}$. Again, we also require that the conclusions of Proposition 3.2 be satisfied. Let $n_2(0) < n_2(1) < \cdots < n_2(l_2)$ be the first return times $n_{Y^{(2)}}(0) < n_{Y^{(2)}}(1) < \cdots < n_{Y^{(2)}}\left(l_{Y^{(2)}}\right)$. Let $B \subset A\left(Z^{(2)}\right)$ be the hereditary subalgebra generated by $C_0\left(\operatorname{int}\left(Y^{(1)}\right)\right) \subset C(M)$. We then further require that $Z^{(2)}$ be so small that $\gamma_{Z^{(2)}}(B)$, as a hereditary subalgebra of the recursive subhomogeneous algebra $\gamma_{Z^{(2)}}\left(A\left(Z^{(2)}\right)\right)$, satisfies $\operatorname{rank}_x\left(\gamma_{Z^{(2)}}(B)\right) \geq R$ for all x (in the sense discussed before Theorem 3.1). (This is accomplished by requiring that there be at least R images of $Z^{(2)}$ under positive powers h^r of h, with $r < n_2(0)$, which are contained in $\operatorname{int}\left(Y^{(1)}\right)$.)

Step 5. Observe that the relations $v_j f = uf$ in $C^*(\mathbf{Z}, M, h)$ whenever $f \in C(M)$ vanishes on $Y^{(j)}$ imply that $v_1^* v_2 f = f$ whenever $f \in C(M)$ vanishes on $Y^{(1)}$. From this one can deduce that $v_1^* v_2 \in U(B)$. With the help of the condition $\operatorname{rank}_x\left(\gamma_{Z^{(2)}}(B)\right) \geq \max\left(1, \frac{1}{2}d\right)$, it is possible to alter the choice of v_2 so that, in addition to the conditions we already have, also $z = \gamma_{Z^{(2)}}\left(v_1^* v_2\right) \in U\left(\gamma_{Z^{(2)}}(B)\right)$ satisfies $\det(z) = 1$ and is connected to 1 by a path $t \mapsto z_t$ in $U\left(\gamma_{Z^{(2)}}(B)\right)$ such that $\det(z_t) = 1$ for all t. (For the meaning of these conditions, see the discussion before Theorem 3.1.) Then also $v_2^* v_1 = \left(v_1^* v_2\right)^*$ satisfies these properties.

Step 6. Apply Theorem 3.1 to find a path in $U(B)$ from $v_1^* v_2$ to 1 with total length less than $4\pi(d' + 2)$. Using a suitable subdivision of the domain of this path, find unitaries

$$v_2^* v_1 = w_0,\ w_1,\ \ldots,\ w_{N-1},\ w_N = 1 \in U(B)$$

such that

$$\|w_j - w_{j-1}\| < \frac{4\pi(d'+2)}{N} < \varepsilon$$

for $1 \leq j \leq N$.

Step 7. Define

$$w = w_0 \big(u^{-1} w_1 u\big) \big(u^{-2} w_2 u^2\big) \cdots \big(u^{-N} w_N u^N\big).$$

Then w is a unitary in $C^*(\mathbf{Z}, M, h)$ with the following properties:

(1) w commutes with every $f \in C(M)$ which is constant on each of the sets

$$Y^{(1)}, \, h^{-1}\left(Y^{(1)}\right), \, \ldots, \, h^{-N}\left(Y^{(1)}\right).$$

(2) w commutes with uv_2^*.

(3) $\|wv_1w^* - v_2\| < \varepsilon$.

We will say something below about how these results follow. Some of the ideas are related to calculations in Section 6 of [31] and Section 2 of [32].

Step 8. Set

$$D = C^*\left(uv_2^*, \, A\left(Z^{(1)}, \, S\right)\right) \subset C^*(\mathbf{Z}, M, h) \quad \text{and} \quad A = wDw^*.$$

We show that A approximately contains $f_1, f_2, \ldots, f_m$, and u to within ε.

Let $T_1, T_2, \ldots, T_r$ be the sets $Y^{(1)}, \, h^{-1}\left(Y^{(1)}\right), \, \ldots, \, h^{-N}\left(Y^{(1)}\right)$, together with all of the sets $h(S), \, h^2(S), \, \ldots, \, h^{n_1(0)}(S)$ which are not contained in any of the images of $Y^{(1)}$ listed above. By the construction in Step (3), the sets $T_1, T_2, \ldots, T_r$ are pairwise disjoint and have diameter less than δ. The functions $f_1, f_2, \ldots, f_m$ are all approximately constant to within $\frac{1}{2}\varepsilon$ on every subset of M with diameter less than δ (by Step 2), so there exist functions $g_1, g_2, \ldots, g_m \in C(M)$ which are actually constant on the sets $T_1, T_2, \ldots, T_r$ and satisfy $\|g_i - f_1\| < \varepsilon$ for $1 \leq i \leq m$. These functions are then constant on all of

$$Y^{(1)}, \, h^{-1}\left(Y^{(1)}\right), \, \ldots, \, h^{-N}\left(Y^{(1)}\right) \quad \text{and} \quad h(S), \, h^2(S), \, \ldots, \, h^{n_1(0)}(S).$$

Now $g_i \in A\left(Z^{(1)}, \, S\right) \subset D$ and (by Step 7 (1)) w commutes with $g_1, g_2, \ldots, g_m$, so $g_1, g_2, \ldots, g_m \in wDw^* = A$.

We also have $w\left(uv_2^* \cdot v_1\right) w^* \in A$. Using the relations $w\left(uv_2^*\right) w^* = uv_2^*$ and $\|wv_1w^* - v_2\| < \varepsilon$ from Step 7, we get

$$\left\|w\left(uv_2^* \cdot v_1\right) w^* - u\right\| = \left\|w\left(uv_2^*\right) w^* \cdot wv_1w^* - uv_2^* \cdot v_2\right\| < \varepsilon.$$

So u is approximately in A.

Step 9. The algebra D, and hence $A = wDw^*$, is a recursive subhomogeneous algebra with topological dimension d and strong covering number at most $d' = d(d+2)$ (that is, no more complicated than $A\left(Z^{(1)}, S\right)$). This step is where S is used in an essential way.

Let's assume for simplicity that $\mathrm{sp}\,(uv_2^*)$ is the whole unit circle S^1. Then it turns out that D is a pullback

$$D \cong A\big(Z^{(1)}, S\big) \oplus_{M_{n_1(0)}, \varphi, \psi} C\big(S^1, M_{n_1(0)}\big).$$

The map $\psi\colon C\big(S^1, M_{n_1(0)}\big) \to M_{n_1(0)}$ is evaluation at $1 \in S^1$. The map $\varphi\colon A\big(Z^{(1)}, S\big) \to M_{n_1(0)}$ is the evaluation on the set S in the recursive subhomogeneous decomposition described in Theorem 2.7. (This is really a point evaluation, because the elements of $A\big(Z^{(1)}, S\big)$ are constant on S.) The unitary uv_2^* corresponds to the pair

$$(1,\ \mathrm{diag}\,(z, 1, \ldots, 1))$$

in which z is the identity function $\zeta \mapsto \zeta$ in $C(S^1)$.

The key relation here is that uv_2^* acts as 1 off $h(S)$. Thus, if $f \in C(M)$ vanishes on $h(S)$, then $(uv_2^*)\,f = f\,(uv_2^*) = f$. If in addition f vanishes on $Z^{(1)}$, then $(uv_2^*)\,(uf) = (uf)\,(uv_2^*) = uf$. These relations imply, for example, that uv_2^* commutes with all elements of $\mathrm{Ker}(\varphi) \subset A\big(Z^{(1)}, S\big)$.

The verification of the isomorphism with the pullback requires lots of functional calculus. For example, one needs to define suitable homomorphisms with domain $D = C^*\left(uv_2^*,\, A\left(Z^{(1)}, S\right)\right)$, or at least determine somehow all the elements of this C*-algebra. We omit further discussion, except to note that it is much easier to demonstrate that there is an exact sequence

$$0 \longrightarrow \mathrm{Ker}(\varphi) \longrightarrow D \longrightarrow C\big(S^1, M_{n_1(0)}\big) \longrightarrow 0,$$

as should certainly happen for a pullback with surjective maps. This exact sequence implies (using Theorem 2.16 of [29]) that D is a recursive subhomogeneous algebra with topological dimension d, but doesn't give anything about the strong covering number.

This finishes the outline of the proof of local approximation.

Let us now return to the explanation of Step 7. We first explain the significance of w, in a greatly simplified context—so much simplified that it does not satisfy the hypotheses of this section. Then we give an outline of how to prove the claimed properties in our case.

For the simple context, let us assume that

$$Z^{(1)} = Y^{(1)} \quad \text{and} \quad M = \coprod_{j=1}^{n} h^j(Z^{(1)}).$$

(We ignore S, since it is not relevant for this step.) In this case, note that $Z_0^{(1)} = Z^{(1)}$, that $n = n_1(0)$, and that $\gamma_{Z^{(1)}}$ induces an isomorphism $A\left(Z^{(1)}\right) \cong M_n\left(C\left(Z^{(1)}\right)\right)$, under which functions constant on each of the sets

$$Y^{(1)},\ h^{-1}\left(Y^{(1)}\right),\ \ldots,\ h^{-N}\left(Y^{(1)}\right)$$

are sent to the diagonal matrices in $M_n\left(C\left(Z^{(1)}\right)\right)$, the last $N+1$ diagonal entries of which are constants. (Our simplifying assumptions imply that $h^{-j}\left(Z^{(1)}\right) = h^{n-j}\left(Z^{(1)}\right)$.)

Let us further assume we have an h-invariant Borel probability measure μ on M, and that $C^*(\mathbf{Z}, M, h)$ is represented faithfully on $L^2(M, \mu)$ with $C(M)$ acting as multiplication operators and u acting as $u\xi = \xi \circ h^{-1}$. There is a direct sum decomposition

$$L^2(M, \mu) = \bigoplus_{j=1}^{n} L^2\left(h^j\left(Z^{(1)}\right)\right),$$

which determines an identification of $L(L^2(M, \mu))$ with $M_n\left(L^2\left(Z^{(1)}\right)\right)$ which is compatible in a suitable sense with the isomorphism $\gamma_{Z^{(1)}}$. Further let e_j be the projection onto $L^2\left(h^j\left(Z^{(1)}\right)\right)$. With respect to this identification, we can write

$$u = \begin{pmatrix} 0 & 0 & \cdots & \cdots & 0 & 0 & u^{(0)} \\ 1 & 0 & \cdots & \cdots & 0 & 0 & 0 \\ 0 & 1 & \cdots & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & & \vdots & \vdots & \vdots \\ \vdots & \vdots & & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & 1 & 0 & 0 \\ 0 & 0 & \cdots & \cdots & 0 & 1 & 0 \end{pmatrix},$$

with $u^{(0)} \in e_1 L(L^2(M, \mu)) e_n$. (Note that it is equal to the shift matrix s_0 considered in Section 2, except for the upper right corner.) Similarly, we can write

$$v_j = \begin{pmatrix} 0 & 0 & \cdots & \cdots & 0 & 0 & v_j^{(0)} \\ 1 & 0 & \cdots & \cdots & 0 & 0 & 0 \\ 0 & 1 & \cdots & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & & \vdots & \vdots & \vdots \\ \vdots & \vdots & & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & 1 & 0 & 0 \\ 0 & 0 & \cdots & \cdots & 0 & 1 & 0 \end{pmatrix}.$$

Again, the difference is in the the upper right corner, but note that v_1 and v_2 are now in $A\left(Z^{(2)}\right)$.

In this situation, we let $w_i' = e_0 w_i e_0$, and identify w as

$$w = \operatorname{diag}\left(1, 1, \ldots, 1, w_N', w_{N-1}', \ldots, w_1', w_0'\right).$$

(We have used the fact that $n \geq N+1$.) Now Condition (1) of Step 7 follows from the fact that w is block diagonal and that functions in $C(M)$ constant on each of the sets

$$Y^{(1)}, h^{-1}\left(Y^{(1)}\right), \ldots, h^{-N}\left(Y^{(1)}\right)$$

are diagonal matrices, the last $N+1$ diagonal entries of which are constants. For Condition (2) of Step 7, we calculate:

$$uv_2^* = \operatorname{diag}\left(u^{(0)}\left(v_2^{(0)}\right)^*, 1, 1, \ldots, 1\right).$$

This element clearly commutes with w. (The worst case is $n = N+1$; then, recall that $w_N = 1$.) For Condition (3) of Step 7, we estimate instead $\|w - v_2 w v_1^*\|$. (This is easily seen to be equivalent.) A computation shows that

$$\begin{aligned} v_2 w v_1^* &= \operatorname{diag}\left(v_2^{(0)} w_0' \left(v_1^{(0)}\right)^*, 1, \ldots, 1, 1, w_N', \ldots, w_2', w_1'\right) \\ &= \operatorname{diag}\left(1, 1, \ldots, 1, 1, w_N', \ldots, w_2', w_1'\right). \end{aligned}$$

(The entries of w have all been moved one space down the diagonal. In addition, the new first entry has been modified. Since $w_0 = v_2^* v_1$, we have $w_0' = \left(v_2^{(0)}\right)^* v_1^{(0)}$.) Therefore, using $w_N = 1$, we get

$$\left\|w - v_2 w v_1^*\right\| = \max_{1 \leq j \leq N} \|w_j - w_{j-1}\| < \varepsilon.$$

In the actual situation, we work inside $C^*(\mathbf{Z}, M, h)$. Let $B \subset C^*(\mathbf{Z}, M, h)$ be the hereditary subalgebra of Step 4. For the matrix decomposition, we substitute the fact that the hereditary subalgebras

$$B,\ u^{-1}Bu,\ u^{-2}Bu^2,\ \ldots,\ u^{-N}Bu^N$$

are orthogonal in $C^*(\mathbf{Z}, M, h)$. This follows from the fact that the sets

$$Y^{(1)},\ h^{-1}\left(Y^{(1)}\right),\ \ldots,\ h^{-N}\left(Y^{(1)}\right)$$

are pairwise disjoint. As a consequence, the factors

$$w_0,\ u^{-1}w_1u,\ u^{-2}w_2u^2,\ \ldots,\ u^{-N}w_Nu^N$$

of w, which are in the unitary groups of these hereditary subalgebras, all commute with each other, and also with any function $f \in C(M)$ which is constant on each of the sets

$$Y^{(1)},\ h^{-1}\left(Y^{(1)}\right),\ \ldots,\ h^{-N}\left(Y^{(1)}\right).$$

When proving that w commutes with uv_2^*, it helps to show first that

$$u^{-j}w_ju^j = v_2^{-j}w_jv_2^j$$

for $0 \leq j \leq N$. In fact, this is true if w_j is replaced by any $b \in C^*(\mathbf{Z}, M, h)$ which differs by a scalar from an element of B. For the verification of the norm estimate in Condition (3) of Step 7, one needs in addition the following fact, which is the analog of the estimate on the difference of diagonal matrices above: if $C_0, C_1, \ldots, C_N$ are orthogonal hereditary subalgebras in a C*-algebra A, and if $y_j,\ z_j \in U(C_j)$ for $0 \leq j \leq N$, then

$$\|y_0y_1\cdots y_N - z_0z_1\cdots z_N\| = \max_{0\leq j\leq N} \|y_j - z_j\|.$$

§4. Direct limit decomposition

We give here a very brief approximate outline of the modifications necessary to achieve the direct limit decomposition of Theorem 1.1, as opposed to merely local approximation. The previous section describes the construction of a (simple version of) a single "basic approximation", and the problem is to arrange successively better ones so as to obtain an increasing sequence of subalgebras of $C^*(\mathbf{Z}, M, h)$. As will be clear, putting everything together requires complicated notation, and there are interactions between the modifications described below which we do not have room to discuss here.

First, the unitary corresponding to w in each new basic approximation must commute with all elements of the subalgebra $A\left(Z^{(2)}\right)$ from the previous one. This requires two changes. The old subalgebra $A\left(Z^{(2)}\right)$ must be replaced by $A\left(Z^{(2)}, T\right)$ for some suitable T, and the new set $Y^{(1)}$ must be contained in T. Also, the sequence

$$v_2^* v_1 = w_0,\ w_1,\ \ldots,\ w_{N-1},\ w_N = 1 \in U(B)$$

used to construct the new w must now consist of constant subsequences, the lengths of which are certain return times associated with the old $Z^{(2)}$.

Second, having constructed one approximating subalgebra, say A_0, the next one, say A_1, will be slightly "twisted" with respect to A_0, even with the adjustment above. To straighten this out, it is necessary to modify A_0 by replacing v_2 in the construction by a nearby unitary. Then, after constructing A_2, one must further modify the unitaries v_2 associated with both A_1 and A_0, etc. Enough control must be maintained that the sequences of modifications converge to unitaries not too far from the original choices.

Third, even apart from the "twisting" referred to in the previous paragraph, the use of the subsets S leads to problems with the expected inclusion relations between subalgebras. Suppose, for example, we have closed subsets Y and Z, satisfying the conclusions of Proposition 3.2, with associated first return times

$$n_Y(0) < n_Y(1) < \cdots < n_Y(l_Y) \quad \text{and} \quad n_Z(0) < n_Z(1) < \cdots < n_Z(l_Z),$$

and with corresponding subsets

$$Y_0,\ Y_1,\ \ldots,\ Y_{l_Y} \subset Y \quad \text{and} \quad Z_0,\ Z_1,\ \ldots,\ Z_{l_Z} \subset Z.$$

Suppose that

$$\varnothing \neq S \subset \operatorname{int}(Z_0) \subset Z \subset \operatorname{int}(Y_0)$$

(in particular, $Z \subset Y$), and that $n_Z(0) > n_Y(0)$ (this is the relevant situation, because arbitrarily good approximations require arbitrarily large values of the smallest first return time). We have $A(Y) \subset A(Z)$, because every function in $C(M)$ which vanishes on Y also vanishes on Z. However, it is not true that $A(Y, S) \subset A(Z, S)$. In fact, $C(M) \cap A(Z, S)$ consists of those functions in $C(M)$ that are constant on the sets

$$h(S),\ h^2(S),\ \ldots,\ h^{n_Z(0)}(S),$$

$C(M) \cap A(Y, S)$ consists of those functions in $C(M)$ that are constant on the sets

$$h(S),\ h^2(S),\ \ldots,\ h^{n_Y(0)}(S),$$

and $n_Y(0) < n_Z(0)$, so $C(M) \cap A(Y, S) \subsetneqq C(M) \cap A(Z, S)$.

To fix this problem, it is necessary to replace the single set S in the construction of $A(Y, S)$ by a whole family of subsets. One must require that whenever $h^j(S) \subset Y$, with $0 < j < n_Z(0)$, then there is k with $h^j(S) \subset \operatorname{int}(Y_k)$. Then one uses the collection of all such $h^j(S)$, rather than just S itself, with the obvious modification to account for the fact that they are no longer all subsets of $\operatorname{int}(Y_0)$. The resulting subalgebra is a proper subalgebra of $A(Y, S)$.

In the inductive construction of an increasing sequence of approximating subalgebras of $C^*(\mathbf{Z}, M, h)$, this works out as follows. First, one constructs an approximating algebra $A_0^{(0)}$. Then one constructs an approximating algebra $A_1^{(1)}$, incorporating the first two modifications discussed above, and using a sufficiently small set S. Next, one replaces $A_0^{(0)}$ by a smaller algebra $A_0^{(1)}$, using the approach outlined in the previous paragraph on the algebra $A\left(Z^{(1)},\, S\right)$ appearing in the definition of $A_0^{(0)}$, but with the set S from the construction of $A_1^{(1)}$. That done, one constructs $A_2^{(2)}$. Then it is necessary to go back and replace both $A_1^{(1)}$ and $A_0^{(1)}$ (in that order) by smaller subalgebras $A_1^{(2)}$ and $A_0^{(2)}$, in a similar way. This procedure continues for all n.

There are two problems. First, $\bigcap_{k=n}^{\infty} A_n^{(k)}$ must still be large enough to approximate not too badly the finite set that the first algebra $A_n^{(n)}$ was constructed to approximate. Second, $\bigcap_{k=n}^{\infty} A_n^{(k)}$ must still be a recursive subhomogeneous algebra with topological dimension at most d and strong covering number at most $d(d+2)$. Since subalgebras of recursive subhomogeneous algebras need not even be recursive subhomogeneous algebras (see Example 3.6 of [29]), this requires work. The construction of the subalgebra $A(Y, S)$ can be viewed as identifying the subset S of Y to a point. By the time the inductive process of the previous paragraph is complete, one must identify infinitely many subsets of Y to (distinct) points, in such a way that the resulting space is not only Hausdorff (there is trouble even here) but in fact has dimension no greater than $\dim(Y)$. The details are quite messy.

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Single generation and rank of C*-algebras

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§1. Introduction

We mainly treat a separable C*-algebra A in this article. Let S be a subset of A_{sa}. We call S a generator of A when any C*-subalgebra B of A containing S is equal to A, and we denote $A = C^*(S)$. If S is finite, then we call A finitely generated and we define the number of generators $\mathfrak{gen}(A)$ by the minimum cardinality of S which generates A. We denote $\mathfrak{gen}(A) = \infty$ unless A is finitely generated. We call a C*-algebra A singly generated if $\mathfrak{gen}(A) \leq 2$. Indeed, if $A = C^*(x, y)$ for $x, y \in A_{sa}$, then any C*-subalgebra B of A containing the element $x + \sqrt{-1}y$ is equal to A.

There are many works on single generation of operator algebras. Many of them concern to von Neumann algebras ([2],[6],[17], [19], [20], [24]). Concerning to C*-algebras, there are interesting works of D. Topping([22]), C. L. Olsen and W. R. Zame([15]). With related to them, we introduce the recent work ([11],[12]) of singly generated C*-algebras in the next section and mention the relation between singly generated C*-algebras and their ranks in the last section.

§2. Single generation of C*-algebras

Let S be a subset of a C*-algebra A satisfying $A = C^*(S)$. If A is unital, then $\{s + 2\|s\| \mid s \in S\}$ also generates A. So we may assume that an element of S is invertible. We mention about the fundamental property of $\mathfrak{gen}(\cdot)$ without the proof.

Lemma 1. [12] *Let A and B be C*-algebras.*

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(1) $\mathfrak{gen}(A) = \mathfrak{gen}(\tilde{A})$, *where* $\tilde{A}$ *is the* C^**-algebraic unitization of* A.
(2) *If* A *and* B *are subalgebras of a* C^**-algebra* C, *then we have*

$$\mathfrak{gen}(C^*(A,B)) \leq \mathfrak{gen}(A) + \mathfrak{gen}(B).$$

(3) *If one of* A *and* B *has a unit, then we have*

$$\mathfrak{gen}(A \oplus B) = \max\{\mathfrak{gen}(A), \mathfrak{gen}(B)\}.$$

For a commutative C*-algebra A, we can make clear the meaning of $\mathfrak{gen}(A)$ as follows:

Proposition 2. [12] *Let* A *be a unital commutative* C^**-algebra and* Ω *the spectrum of* A. *Then we have*

$$\mathfrak{gen}(A) = \min\{m \in \mathbb{N} \mid \text{ there is an embedding of } \Omega \text{ into } \mathbb{R}^m\}.$$

Thanks to this statement, we can consider $\mathfrak{gen}(A)$ as a sort of noncommutative topological dimension of a C*-algebra A. So we investigate the relation of $\mathfrak{gen}(A)$ and $\mathfrak{gen}(M_n(A))$, where $M_n(A) \cong M_n(\mathbb{C}) \otimes A$.

Theorem 3. [12] *Let* A *be a unital* C^**-algebra with* $\mathfrak{gen}(A) \leq n^2 + 1$ $(n \in \mathbb{N})$. *Then we have* $\mathfrak{gen}(M_n(A)) \leq 2$.

Outline of Proof. Let $a_1, a_2, \dots, a_{(n-1)^2}, b, c_1, c_2, \dots, c_n, d_1, d_2, \dots, d_{n-1}$ be self-adjoint elements of A. We assume that they generate A and satisfy the following condition:

$$b \geq 1 \text{ and } d_1, d_2, \dots, d_{n-1} \geq \delta \text{ for some } \delta > 0.$$

We define two self-adjoint elements x, y in $M_n(A)$ as follows:

$$x = \begin{pmatrix} a_1 & a_2 + \sqrt{-1}a_3 & \cdots & a_{2n-4} + \sqrt{-1}a_{2n-3} & 0 \\ a_2 - \sqrt{-1}a_3 & a_{2n-2} & \cdots & a_{4n-9} + \sqrt{-1}a_{4n-8} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{2n-4} - \sqrt{-1}a_{2n-3} & a_{4n-9} - \sqrt{-1}a_{4n-8} & \cdots & a_{(n-1)^2} & 0 \\ 0 & 0 & \cdots & 0 & b \end{pmatrix}$$

and

$$y = \begin{pmatrix} c_1 & d_1 & & & \\ d_1 & c_2 & d_2 & & \\ & d_2 & \ddots & \ddots & \\ & & \ddots & \ddots & d_{n-1} \\ & & & d_{n-1} & c_n \end{pmatrix}.$$

If we assume that

$$\varepsilon 1 \leq (x_{ij})_{i,j=1}^{n-1} \leq (1-\varepsilon)1 \quad \text{for some } \varepsilon > 0,$$

then x and y generate A. Q.E.D.

It is proved that $M_n(A)$ is singly generated if $\mathfrak{gen}(A) \leq (n^2+3n)/2$ ([15]), and if $\mathfrak{gen}(A) \leq (n-1)^2$ ([14]). The above result implies the following estimation for unital C*-algebra A:

$$\mathfrak{gen}(M_n(A)) \leq \lceil \frac{\mathfrak{gen}(A)-1}{n^2} + 1 \rceil,$$

where $\lceil \cdot \rceil$ means "the least integer greater than or equal to". We can see that the above estimation is best possible. C. L. Olsen and W. R. Zame [15] prove that $M_2(C([0,1]^n))$ is singly generated if and only if $n \leq 5$.

Theorem 4. [12] *Let n and m be positive integers. Then we have*

$$\mathfrak{gen}(M_m(C[0,1]^n)) = \lceil \frac{n-1}{m^2} + 1 \rceil.$$

Let Ω be an n-dimensional compact manifold. By Whitney's theorem, Ω is embeddable to $\mathbb{R}^{2n}$, so we have

$$\mathfrak{gen}(M_m(C(\Omega))) \leq \lceil \frac{2n-1}{m^2} + 1 \rceil.$$

Now we shall investigate generators for a simple C*-algebra or a C*-algebra which is tensored with a simple C*-algebra.

Theorem 5. [12] *Let A be a simple, infinitely dimensional C^*-algebra. Then we have*

$$\mathfrak{gen}(A \otimes_{max} B) \leq \mathfrak{gen}(A) + 1$$

for any unital C^*-algebra B.

Outline of Proof. We assume that A is unital and $x_1, x_2, \ldots, x_n \in A_{sa}$ generate A. We choose $\{y_k | k = 1, 2, \ldots\} \subset B_{sa}$ such that $\{y_k | k = 1, 2, \ldots\}$ generates B and $\|y_k\| = 1$. By the infinite dimensionality of A, we can choose a family of positive elements in A satisfying $p_i p_j = 0$ if $i \neq j$. We set

$$s_i = x_i \otimes 1, \; t = \sum_{k=1}^{\infty} \frac{1}{k} p_k \otimes y_k.$$

Then we have $p_k^2 \otimes y_k = k(p_k \otimes 1)t \in C^*(s_1, \dots, s_n, t)$. By the simplicity of A, we have

$$\sum_{i=1}^{m} a_i p_k^2 b_i = 1$$

for suitable elements a_i, b_i in A. This means that $\{s_1, \dots, s_n, t\}$ generates $A \otimes_{max} B$. Q.E.D.

Corollary 6. *Let A be a simple, singly generated, infinitely dimensional C^*-algebra. Then we have*

$$\mathfrak{gen}(A \otimes_{max} B) \leq 3$$

for any unital C^*-algebra B.

In particular, $M_k(\mathbb{C}) \otimes A \otimes_{max} B$ is singly generated for $k \geq 2$.

Examples. (1) Let $\mathbb{K}$ be a C*-algebra of all compact operators on a separable Hilbert space. Then $\mathbb{K}$ is singly generated, and

$$\begin{pmatrix} 1 & & & \\ & 1/2 & & \\ & & 1/3 & \\ & & & \ddots \end{pmatrix} + \sqrt{-1} \begin{pmatrix} 0 & 1 & & \\ 1 & 0 & 1/2 & \\ & 1/2 & 0 & \ddots \\ & & \ddots & \ddots \end{pmatrix}$$

is a generator of $\mathbb{K}$.

Every UHF C*-algebra is also singly generated([22]). So we have $A \otimes_{min} \mathbb{K}$ and $A \otimes_{min} (\mathrm{UHF})$ are singly generated for any unital C*-algebra A ([15]) by Corollary 6.

(2) Let A be a unital C*-algebra with a unitary $u \in A$ and $h \in A_{sa}$ satisfying $A = C^*(u, h)$. Then A is singly generated and $u(h + 2\|h\|)$ is a generator of A.

For any compact subspace Ω of $\mathbb{R}$, the C*-crossed product $C(\Omega) \rtimes_\alpha \mathbb{Z}$ is also singly generated.

Let $A_\theta = C^*(u, v)$ be an irrational rotation C*-algebra. Then A_θ is singly generated([10]) and $u(v + v^* + 3)$ is a generator of A_θ.

(3) Every simple AF C*-algebra is singly generated([9]).

(4) The Cuntz algebra $\mathcal{O}_n$ has the property $M_n(\mathcal{O}_n) \cong \mathcal{O}_n$. So we have $A \otimes_{min} \mathcal{O}_n$ is singly generated for any unital C*-algebras. In general, E. Kirchberg ([13],[12]) shows that a C*-algebras A is singly generated if A has two isometries with orthogonal ranges.

(5) By Proposition 2, $C(\mathbb{T} \times \mathbb{T})$ is not singly generated, so the enveloping group C*-algebra $C^*(F_2)$ of the free group F_2 with two generators is not singly generated. By Theorem 3, $M_2(C^*(F_2))$ is singly generated.

§3. Rank of C*-algebras

In this section, we assume that a C*-algebra A has a unit. The notion of real rank is defined by L. G. Brown and G. K. Pedersen [4], and that of stable rank is defined by M. A. Rieffel [18] as follows:

$$\begin{aligned}\mathrm{RR}(A) = \min\{n \in \mathbb{N} \cup \{0\} \mid\ & \\ & \{(a_1, a_2, \ldots, a_{n+1}) \in (A_{sa})^{n+1} \mid Aa_1 + Aa_2 + \cdots + Aa_n = A\} \\ & \text{is dense in } (A_{sa})^{n+1}\}, \\ & \text{or } \infty,\end{aligned}$$

$$\begin{aligned}\mathrm{sr}(A) = \min\{n \in \mathbb{N} \mid\ & \\ & \{(a_1, a_2, \ldots, a_n) \in A^n \mid Aa_1 + Aa_2 + \cdots + Aa_n = A\} \\ & \text{is dense in } A^n\}, \\ & \text{or } \infty.\end{aligned}$$

If A is commutative, then $\mathrm{RR}(A)$ is equal to the covering dimension $\dim(\Omega)$ of its spectrum Ω, and

$$\mathrm{sr}(A) = \lceil \frac{\dim(\Omega) + 1}{2} \rceil.$$

In the case that A is not commutative, we have

$$\mathrm{RR}(A) \leq 2\mathrm{sr}(A) - 1.$$

E. J. Beggs and D. E. Evans [1] show that the following formula:

$$\mathrm{RR}(M_m(C(\Omega))) = \lceil \frac{\dim(\Omega)}{2m - 1} \rceil.$$

We shall construct an example of C*-algebra whose rank is infinite using free products of C*-algebras([23]).

Theorem 7. [14] *If C^*-algebras A and B have surjective $*$-homomorphisms to $C[0,1]$, then we have*

$$\mathrm{RR}(A * B) = \infty,$$

where $A * B$ is the enveloping C^*-algebra of the free product of A and B.

Proof. For any $n \in \mathbb{N}$, by Theorem 4, we have

$$\mathfrak{gen}(M_n(C[0,1]^{n^2})) = 2.$$

Let a, b be invertible self-adjoint generators of $M_n(C[0,1]^{n^2})$. There are surjective C*-homomorphisms from A (resp. B) to $C^*(a)$ (resp.$C^*(b)$). This means that there exists a surjective C*-homomorphism from $A * B$ to $M_n(C[0,1]^{n^2})$. By Beggs-Evans' formula, we have

$$\mathrm{RR}(A * B) \geq \lceil \frac{n^2}{2n-1} \rceil$$

for any n, that is, $\mathrm{RR}(A * B) = \infty$. Q.E.D.

Both $C[0,1] * C[0,1]$ and $C^*(F_2)$ have their real rank ∞ (in particular, their stable rank ∞). The former is singly generated and the latter is not as we have shown. M. A. Rieffel [18] show that $\mathrm{sr}(A) = \infty$ when A contains two isometries with orthogonal ranges. But, for unital C*-algebras $A \subset B$, it is not necessarily true that $\mathrm{sr}(A) = \infty$ implies $\mathrm{sr}(B) = \infty$. We give here such an example.

Lemma 8. *Let A be a unital, separable, residually finite C^*-algebra and M a factor of type II_1. Then there exists a unital embedding of A to M.*

Proof. Since A is residually finite, there exists a countable family $\{\pi_n\}_{n=1}^{\infty}$ of finite-dimensional $*$-representation of A such that $\oplus_{n=1}^{\infty}\pi_n$ is a faithful representation of A. We can choose a family $\{p_n\}_{n=1}^{\infty}$ of orthogonal projections of M such that

$$\sum_{n=1}^{\infty} p_n = 1.$$

For each n, $p_n M p_n$ contains a unital $*$-subalgebra which isomorphic to $M_{\dim \pi_n}(\mathbb{C})$. Using these isomorphisms, we can construct an embedding of A to $\sum_{n=1}^{\infty} p_n M p_n \subset M$. Q.E.D.

M.-D. Choi [5] prove that $C^*(F_2)$ is residually finite. So $C^*(F_2)$ can be embedded in a factor M of type II_1. Every finite factor M is simple and has $\mathrm{RR}(M) = 0$ and $\mathrm{sr}(M) = 1$. More precisely, using N. C. Phillips' argument [16], we can choose a unital, separable, simple C*-algebra A which contains $C^*(F_2)$ and $\mathrm{RR}(A) = 0$ and $\mathrm{sr}(A) = 1$.

Indeed, there exists a simple, separable C*-algebra A_1 such that $C^*(F_2) \subset A_1 \subset M$ [3]. Let $\{\epsilon_n\}_{n=1}^{\infty}$ be a positive decreasing sequence tending to 0. We can choose a countable sequence $\{a_n\}_{n=1}^{\infty} \subset (A_1)_{sa}$ and $\{b_n\}_{n=1}^{\infty} \subset A_1$ such that $\{a_n\}_{n=1}^{\infty}$ (resp.$\{b_n\}_{n=1}^{\infty}$) is dense in the unit ball of $(A_1)_{sa}$ (resp. A_1). By the fact $\mathrm{RR}(M) = 0$ and $\mathrm{sr}(M) = 1$, we can choose invertible elements $a'_n \in M_{sa}$, $b'_n \in M$ such that

$$\|a'_n\|, \|b'_n\| \leq 1, \; \|a_n - a'_n\| < \epsilon_1, \; \|b_n - b'_n\| < \epsilon_1.$$

We put A_2 the C*-algebra generated by A_1, a'_n and b'_n. Then there exists a simple, separable C*-algebra A_3 such that $A_2 \subset A_3 \subset M$. We also choose a countable sequence $\{a''_n\}_{n=1}^{\infty} \subset (A_3)_{sa}$ and $\{b''_n\}_{n=1}^{\infty} \subset A_3$ such that $\{a''_n\}_{n=1}^{\infty}$ (resp.$\{b''_n\}_{n=1}^{\infty}$) is dense in the unit ball of $(A_3)_{sa}$ (resp. A_3), and invertible elements $a'''_n \in M_{sa}$, $b'''_n \in M$ such that

$$\|a'''_n\|, \|b'''_n\| \leq 1, \; \|a''_n - a'''_n\| < \epsilon_2, \; \|b''_n - b'''_n\| < \epsilon_2.$$

We put A_4 the C*-algebra generated by A_3, a'''_n and b'''_n. Repeating this argument, we can construct

$$C^*(F_2) \subset A_1 \subset A_2 \subset \cdots \subset A_n \subset \cdots \subset M.$$

Then the inductive limit C*-algebra $\lim_{n\to\infty} A_n$ is the desired one.

We have no example that a separable C*-algebra A is simple and is not singly generated. Many researcher consider the reduced group C*-algebra $C^*_{red}(F_2)$ as a candidate of such a C*-algebra. K. Dykema, U. Haagerup and M. Rørdam [7] prove that $\mathrm{sr}(C^*_{red}(F_2)) = 1$. Since $C^*_{red}(F_2)$ does not have non-trivial projections, its real rank is one. We do not know whether a separable, simple, C*-algebra of real rank zero is singly generated. This fact is related to the problem of a singly generated factor of type II_1 .

We remark that, if any separable, simple C*-algebra of real rank zero is singly generated, then every factor of type II_1 with the separable predual is singly generated as a von Neumann algebra. Indeed, we choose elements $a_1, a_2, \ldots \in M$ such that $\{a_1, a_2, \ldots\}$ generates M as a von Neumann algebra. For the C*-algebra A generated by $\{a_1, a_2, \ldots\}$, by the above argument, we can choose a separable, simple C*-algebra B of real rank zero such that

$$A \subset B \subset M.$$

By the assumption, there exists an element $x \in B$ such that x generates B. Then x generates M as a von Neumann algebras.

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Aperiodic automorphisms of certain simple C^*-algebras

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C^*-algebras over spheres with fibres noncommutative tori

Chun-Gil Park

Abstract.

All C^*-algebras of sections of locally trivial C^*-algebra bundles over $\prod_{i=1}^{e} S^{2n_i} \times \prod_{j=1}^{s} S^{2k_j-1}$ with fibres $M_c(A_\omega)$ are constructed under the assumption that each completely irrational noncommutative torus is realized as an inductive limit of circle algebras. It is shown that each C^*-algebra of sections of a locally trivial C^*-algebra bundle over $\prod_{i=1}^{e} S^{2n_i} \times \prod_{j=1}^{s} S^{2k_j-1}$ with fibres $M_c(A_\omega)$ is stably isomorphic to $C(\prod_{i=1}^{e} S^{2n_i} \times \prod_{j=1}^{s} S^{2k_j-1}) \otimes M_c(A_\omega)$.

Let A_{cd} be a cd-homogeneous C^*-algebra over $\prod_{i=1}^{e} S^{2n_i} \times \prod_{j=1}^{s} S^{2k_j-1} \times \mathbb{T}^{r+2}$ of which no non-trivial matrix algebra can be factored out. The spherical noncommutative torus $\mathbb{S}_\rho^{cd}$ is defined by twisting $C^*(\widehat{\mathbb{T}^{r+2}} \times \mathbb{Z}^{m-2})$ in $A_{cd} \otimes C^*(\mathbb{Z}^{m-2})$ by a totally skew multiplier ρ on $\widehat{\mathbb{T}^{r+2}} \times \mathbb{Z}^{m-2}$. We prove that $\mathbb{S}_\rho^{cd} \otimes M_{p^\infty}$ is isomorphic to $C(\prod_{i=1}^{e} S^{2n_i} \times \prod_{j=1}^{s} S^{2k_j-1}) \otimes C^*(\widehat{\mathbb{T}^{r+2}} \times \mathbb{Z}^{m-2}, \rho) \otimes M_{cd}(\mathbb{C}) \otimes M_{p^\infty}$ if and only if the set of prime factors of cd is a subset of the set of those of p.

§0. Introduction

Given a locally compact abelian group G and a multiplier ω on G, one can associate to them the twisted group C^*-algebra $C^*(G, \omega)$, which is the universal object for unitary ω-representations of G. $C^*(\mathbb{Z}^m, \omega)$ is said to be a *noncommutative torus of rank* m and denoted by A_ω. The multiplier ω determines a subgroup S_ω of G, called its *symmetry group*, and the multiplier ω is called *totally skew* if the symmetry group S_ω is trivial. And A_ω is called *completely irrational* if ω is totally skew (see [1, 12]). It was shown in [1] that if G is a locally compact abelian group and ω is a totally skew multiplier on G, then $C^*(G, \omega)$ is a simple C^*-algebra. The noncommutative torus A_ω of rank m is the universal object

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for unitary ω-representations of $\mathbb{Z}^m$, so A_ω is realized as $C^*(u_1, \cdots, u_m \mid u_i u_j = e^{2\pi i\theta_{ji}} u_j u_i)$, where u_i are unitaries and θ_{ji} are real numbers for $1 \leq i, j \leq m$.

Boca [4] showed that almost all completely irrational noncommutative tori are isomorphic to inductive limits of circle algebras, where the term "*circle algebra*" denotes a C^*-algebra which is a finite direct sum of C^*-algebras of the form $C(\mathbb{T}^1) \otimes M_q(\mathbb{C})$. We will assume that each completely irrational noncommutative torus appearing in this paper is an inductive limit of circle algebras.

Each cd-homogeneous C^*-algebra A over M is isomorphic to the C^*-algebra $\Gamma(\eta)$ of sections of a locally trivial C^*-algebra bundle η with base space M, fibres $M_{cd}(\mathbb{C})$, and structure group $\mathrm{Aut}(M_{cd}(\mathbb{C})) \cong PU(cd)$ (see [15, 18]). So each cd-homogeneous C^*-algebra over $\prod_{i=1}^e S^{2n_i} \times \prod_{j=1}^s S^{2k_j-1} \times \mathbb{T}^{r+2}$ is realized as the C^*-algebra $\Gamma(\zeta)$ of sections of a locally trivial C^*-algebra bundle ζ over $\prod_{i=1}^e S^{2n_i} \times \prod_{j=1}^s S^{2k_j-1} \times \mathbb{T}^{r+2}$ with fibres $M_{cd}(\mathbb{C})$. Thus the spherical noncommutative torus $\mathbb{S}_\rho^{cd}$, defined in Section 2, is realized as the C^*-algebra of sections of a locally trivial C^*-algebra bundle over $\prod_{i=1}^e S^{2n_i} \times \prod_{j=1}^s S^{2k_j-1}$ with fibres $P_\rho^d \otimes M_c(\mathbb{C})$, where P_ρ^d is defined in Section 2.

We are going to show that the set of all C^*-algebras of sections of locally trivial C^*-algebra bundles over $\prod_{i=1}^e S^{2n_i} \times \prod_{j=1}^s S^{2k_j-1}$ with fibres $P_\rho^d \otimes M_c(\mathbb{C})$ is in bijective correspondence with the set of all spherical noncommutative tori with primitive ideal space $\prod_{i=1}^e S^{2n_i} \times \prod_{j=1}^s S^{2k_j-1}$ and fibres $P_\rho^d \otimes M_c(\mathbb{C})$, that $\mathbb{S}_\rho^{cd} \otimes M_{p^\infty}$ is isomorphic to $C(\prod_{i=1}^e S^{2n_i} \times \prod_{j=1}^s S^{2k_j-1}) \otimes C^*(\widehat{\mathbb{T}^{r+2}} \times \mathbb{Z}^{m-2}, \rho) \otimes M_{cd}(\mathbb{C}) \otimes M_{p^\infty}$ if and only if the set of prime factors of cd is a subset of the set of prime factors of p, and that $\mathbb{S}_\rho^{cd}$ is stably isomorphic to $C(\prod_{i=1}^e S^{2n_i} \times \prod_{j=1}^s S^{2k_j-1}) \otimes C^*(\widehat{\mathbb{T}^{r+2}} \times \mathbb{Z}^{m-2}, \rho) \otimes M_{cd}(\mathbb{C})$.

§1. Homogeneous C^*-algebras over a product space of spheres

An important problem, in the bundle theory of geometry, is to compute the set $[M, BPU(cd)]$ of homotopy classes of continuous maps of a compact CW-complex M into the classifying space $BPU(cd)$ of the Lie group $PU(cd)$. The set $[M, BPU(cd)]$ is in bijective correspondence with the set of equivalence classes of principal $PU(cd)$-bundles over M, which is in bijective correspondence with the set of cd-homogeneous C^*-algebras over M (see [15, 18]). $[S^{2n}, BPU(cd)] = [S^{2n-1}, PU(cd)] \cong \mathbb{Z}$ if $n > 1$, $\cong \mathbb{Z}_{cd}$ if $n = 1$, which are the cyclic groups. So each group has a generator, and there is a unitary $U(z) \in PU(cd)$ such that the generating cd-homogeneous C^*-algebra over S^{2n} can be realized as the C^*-algebra of sections of a locally trivial C^*-algebra bundle over S^{2n} with fibres $M_{cd}(\mathbb{C})$ characterized by the unitary $U(z) \in PU(cd)$ over S^{2n-1}. If $(cd, k) = p$ $(p > 1)$, then consider the cd-homogeneous C^*-algebra over S^{2n} corresponding to each $k \in \mathbb{Z}$ or $\mathbb{Z}_{cd}$ as the tensor product of $M_p(\mathbb{C})$ with a $\frac{cd}{p}$-homogeneous C^*-algebra over S^{2n}, which is given by $U(z)^{\frac{k}{p}} \in PU(\frac{cd}{p})$. Consider $U(z)^k$ as $U(z)^{\frac{k}{p}} \otimes I_p \in PU(cd)$, where I_p denotes the $p \times p$ identity matrix. Then each cd-homogeneous C^*-algebra $B_{cd,k}$ over S^{2n} can be realized as the C^*-algebra of sections of a locally trivial C^*-algebra bundle over S^{2n} with fibres $M_{cd}(\mathbb{C})$ characterized by the unitary $U(z)^k \in PU(cd)$ over S^{2n-1} for some $k \in \mathbb{Z}$ or $\mathbb{Z}_{cd}$ (see [15]).

Lemma 1.1. *Every cd-homogeneous C^*-algebra over $S^{2n-1} \times S^1$, whose cd-homogeneous C^*-subalgebra restricted to the subspace $S^{2n-1} \hookrightarrow S^{2n-1} \times S^1$ has the trivial bundle structure, is isomorphic to one of the C^*-subalgebras $A_{cd,k}$, $k \in \mathbb{Z}$ or $\mathbb{Z}_{cd}$, of $C(S^{2n-1} \times [0,1], M_{cd}(\mathbb{C}))$ given as follows: if $f \in A_{cd,k}$, then the following condition is satisfied*

$$f(z,1) = U(z)^k f(z,0) U(z)^{-k}$$

for all $z \in S^{2n-1}$, where $U(z) \in PU(cd)$ is the unitary given above.

Proof. Let A be a cd-homogeneous C^*-algebra over $S^{2n-1} \times S^1$ whose cd-homogeneous C^*-subalgebra restricted to the subspace $S^{2n-1} \hookrightarrow S^{2n-1} \times S^1$ has the trivial bundle structure. Since there is a map of degree 1 from $S^{2n-1} \times S^1$ to S^{2n}, the composite of the map of degree 1 and the map representing each element of $[S^{2n}, BPU(cd)]$ gives an element of $[S^{2n-1} \times S^1, BPU(cd)]$. Hence each element of $[S^{2n}, BPU(cd)] \cong [S^{2n-1}, PU(cd)]$ representing a cd-homogeneous C^*-algebra over S^{2n} induces an element of $[S^{2n-1}, PU(cd)] \subset [S^{2n-1} \times S^1, BPU(cd)]$, and the cd-homogeneous C^*-algebras $A_{cd,k}$ over $S^{2n-1} \times S^1$ corresponding to the cd-homogeneous C^*-algebras $B_{cd,k}$ over S^{2n} are constructed in the statement. By the assumption, the cd-homogeneous C^*-subalgebra of A restricted to the subspace $S^{2n-1} \times (0,1)$ of $S^{2n-1} \times S^1$ has the trivial bundle structure. Hence A corresponds to an element of $[S^{2n-1}, PU(cd)]$, and A is characterized by the unitary $U(z)^k \in PU(cd)$ over S^{2n-1} for some $k \in \mathbb{Z}$ or $\mathbb{Z}_{cd}$. Q.E.D.

Lemma 1.2. *Let n and k be integers greater than 1. Each cd-homogeneous C^*-algebra over $S^n \times S^k$ is isomorphic to a cd-homogeneous C^*-algebra characterized by the unitary $U(z)^a$ over S^{n-1} in a cd-homogeneous C^*-algebra P_c over $e^n_+ \times S^k$ and $e^n_- \times S^k$, where*

$U(z) \in PU(cd)$ or $PU(c)$ if $M_c(\mathbb{C})$ is factored out of P_c, and e^n_+ (resp. e^n_-) is the n-dimensional northern (resp. southern) hemisphere.

Proof. Since e^n_+, e^n_- are contractible, each cd-homogeneous C^*-algebra over $e^n_+ \times S^k$ and $e^n_- \times S^k$ is essentially induced by a cd-homogeneous C^*-algebra over S^k. Each cd-homogeneous C^*-algebra over $S^n \times S^k$ is characterized by a projective unitary over the boundaries $S^{n-1} \times S^k$ of $e^n_+ \times S^k$ and $e^n_- \times S^k$. But $\pi_1(S^n) = \{0\}$ and so the identification of the boundaries $S^k \hookrightarrow e^n_+ \times S^k$ and $S^k \hookrightarrow e^n_- \times S^k$ does give the trivial bundle structure. Hence the cd-homogeneous C^*-algebra over $S^n \times S^k$ is characterized by the unitary $U(z)^a$, $a \in \mathbb{Z}$ or $a \in \mathbb{Z}_{cd}$, over S^{n-1} in the cd-homogeneous C^*-algebra over $e^n_+ \times S^k$ and $e^n_- \times S^k$, where $U(z) \in PU(cd)$ or $PU(c)$. Q.E.D.

For a cd-homogeneous C^*-algebra A over S^{2n-1} there is a matrix algebra $M_q(\mathbb{C})$ such that $A \otimes M_q(\mathbb{C})$ is isomorphic to $C(S^{2n-1}) \otimes M_{cdq}(\mathbb{C})$. Since there is a map of degree 1 from S^{2n+1} to $S^{2n} \times S^1$, there are cd-homogeneous C^*-algebras over $S^{2n} \times S^1$ induced from cd-homogeneous C^*-algebras over S^{2n+1}. Also there are cd-homogeneous C^*-algebras over $S^{2n} \times S^1$ induced from cd-homogeneous C^*-algebras over S^{2n}. But the tensor product of each cd-homogeneous C^*-algebra over $S^{2n} \times S^1$ induced from a cd-homogeneous C^*-algebra over S^{2n+1} with $M_q(\mathbb{C})$ has the trivial bundle structure for some integer q big enough since $[S^{2n+1}, BPU(cdq)] \cong \{0\}$. And there is a map of degree 1 from S^{2n} to $S^{2n-1} \times S^1$, and so there are cd-homogeneous C^*-algebras over $S^{2n-1} \times S^1$ induced from cd-homogeneous C^*-algebras over S^{2n}. Also there are cd-homogeneous C^*-algebras over $S^{2n-1} \times S^1$ induced from cd-homogeneous C^*-algebras over S^{2n-1}. But $[S^{2n-1} \times S^1, BPU(cdq)]$ and $[S^{2n}, BPU(dq)]$ are the same for some integer q since $[S^{2n-1}, BPU(cdq)] \cong \{0\}$. So the cd-homogeneous C^*-subalgebra of the tensor product of a cd-homogeneous C^*-algebra over $S^{2n-1} \times S^1$ with $M_q(\mathbb{C})$ restricted to the subspace $S^{2n-1} \hookrightarrow S^{2n-1} \times S^1$ has the trivial bundle structure (see [17, 18]). From now on, we assume that each cd-homogeneous C^*-algebra over $S^{2n} \times S^1$ is isomorphic to the tensor product of a cd-homogeneous C^*-algebra over S^{2n} with $C(S^1)$, and that the cd-homogeneous C^*-subalgebra of a cd-homogeneous C^*-algebra over $S^{2n-1} \times S^1$ restricted to the subspace $S^{2n-1} \hookrightarrow S^{2n-1} \times S^1$ has the trivial bundle structure.

Thomsen [19, Theorem 1.15] computed $\pi_{2n-1}(\mathrm{Aut}(M_{cdp}(\mathbb{C}) \otimes M_{q^\infty})) \cong \mathbb{Z}/cdp\mathbb{Z}$ for M_{q^∞} a UHF-algebra of type q^∞, and cdp and q relatively prime integers. Let $A_{cd,k}$ be a cd-homogeneous C^*-algebra over $S^{2n-1} \times S^1$ of which no non-trivial matrix algebra can be factored out. This result implies that for any positive integer p no matrix algebra bigger than $M_p(\mathbb{C})$ can be factored out of $A_{cd,k} \otimes M_p(\mathbb{C})$. So the natural inclusion $C(S^1) \hookrightarrow A_{cd,k}$ induces the canonical homomorphism $K_0(C(S^1)) \to K_0(A_{cd,k})$ such that $[1_{C(S^1)}]$ maps to $[1_{A_{cd,k}}]$.

Lemma 1.3. *Let $A_{cd,k}$ be a cd-homogeneous C^*-algebra over $S^{2n-1} \times S^1$ of which no non-trivial matrix algebra can be factored out. Then $K_0(A_{cd,k}) \cong K_1(A_{cd,k}) \cong \mathbb{Z}^2$, and $[1_{A_{cd,k}}] \in K_0(A_{cd,k})$ is primitive.*

Proof. We will show later that $A_{cd,k}$ is stably isomorphic to $C(S^{2n-1} \times S^1)$. Since $K_0(C(S^{2n-1} \times S^1)) \cong K_1(C(S^{2n-1} \times S^1)) \cong \mathbb{Z}^2$, $K_0(A_{cd,k}) \cong K_1(A_{cd,k}) \cong \mathbb{Z}^2$. Hence it is enough to show that $[1_{A_{cd,k}}] \in K_0(A_{cd,k})$ is primitive.

No matrix algebra bigger than $M_q(\mathbb{C})$ can be factored out of $A_{cd,k} \otimes M_q(\mathbb{C})$, and so $C(S^{2n-1})$ cannot be factored out of $A_{cd,k} \otimes M_q(\mathbb{C})$. Hence the canonical embedding ϕ of $C(S^{2n-1})$ into $A_{cd,k}$ induces an isomorphism μ of $K_0(C(S^{2n-1} \times S^1))$ into $K_0(A_{cd,k})$. But the unit $1_{C(S^{2n-1})}$ maps to the unit $1_{C(S^{2n-1} \times S^1)}$ under the canonical embedding ψ of $C(S^{2n-1})$ into $C(S^{2n-1} \times S^1)$. Thus $[1_{C(S^{2n-1})}] \in K_0(C(S^{2n-1})) \cong \mathbb{Z}$ maps to $[1_{C(S^{2n-1} \times S^1)}] \in K_0(C(S^{2n-1} \times S^1)) \cong \mathbb{Z}^2$, primitive in $K_0(C(S^{2n-1} \times S^1))$ (see [20, 13.3.1]). In the commutative diagram

$$\begin{array}{ccc} K_0(C(S^{2n-1})) & \xrightarrow{\psi_*} & K_0(C(S^{2n-1} \times S^1)) \\ {\scriptstyle (\text{identity})_*}\downarrow & & \downarrow{\scriptstyle \mu(\cong)} \\ K_0(C(S^{2n-1})) & \xrightarrow{\phi_*} & K_0(A_{cd,k}), \end{array}$$

$\mu([1_{C(S^{2n-1} \times S^1)}]) = \phi_* \circ (\text{identity})_* \circ \psi_*^{-1}([1_{C(S^{2n-1} \times S^1)}]) = [1_{A_{cd,k}}]$. Consequently $[1_{A_{cd,k}}]$ is the image of the primitive element $[1_{C(S^{2n-1} \times S^1)}] \in K_0(C(S^{2n-1} \times S^1))$ under the isomorphism μ. Therefore, $[1_{A_{cd,k}}] \in K_0(A_{cd,k}) \cong \mathbb{Z}^2$ is primitive.

Thus, $K_0(A_{cd,k}) \cong \mathbb{Z}^2$, $K_1(A_{cd,k}) \cong \mathbb{Z}^2$, and $[1_{A_{cd,k}}] \in K_0(A_{cd,k})$ is primitive. Q.E.D.

Lemma 1.4. *Let $B_{cd,k}$ be a cd-homogeneous C^*-algebra over S^{2n} of which no non-trivial matrix algebra can be factored out. Then $[1_{B_{cd,k}}] \in K_0(B_{cd,k}) \cong \mathbb{Z}^2$ is primitive.*

Proof. We will show later that $B_{cd,k}$ is stably isomorphic to $C(S^{2n}) \otimes M_{cd}(\mathbb{C})$. So $K_0(B_{cd,k}) \cong K_0(C(S^{2n})) \cong \mathbb{Z} \oplus \mathbb{Z}$. But $B_{cd,k}$ corresponds to $A_{cd,k}$ with respect to the conditions on sections over the boundaries S^{2n-1} of $e_+^{2n} \amalg e_-^{2n}$ and $S^{2n-1} \times [0,1]$, and the canonical embedding of $C(S^{2n-1})$ into $A_{cd,k}$ which induces the isomorphism of $K_0(C(S^{2n-1} \times S^1))$ into $K_0(A_{cd,k})$ corresponds to the imbedding ϕ of $C(S^{2n-1})$ into

$B_{cd,k}$. The canonical imbedding ϕ of $C(S^{2n-1})$ into $B_{cd,k}$ induces an isomorphism μ of $K_0(C(S^{2n}))$ into $K_0(B_{cd,k})$, where $S^{2n-1} = \partial e_{\pm}^{2n}$. The unit $1_{C(S^{2n-1})}$ maps to the unit $1_{C(S^{2n})}$ under the canonical embedding ψ of $C(S^{2n-1})$ into $C(S^{2n})$. $[1_{C(S^{2n-1})}] \in K_0(C(S^{2n-1})) \cong \mathbb{Z}$ maps to $[1_{C(S^{2n})}] \in K_0(C(S^{2n})) \cong \mathbb{Z}^2$, primitive in $K_0(C(S^{2n}))$ (see [20, 13.3.1]). In the commutative diagram

$$\begin{array}{ccc} K_0(C(S^{2n-1})) & \xrightarrow{\psi_*} & K_0(C(S^{2n})) \\ {\scriptstyle (\text{identity})_*}\downarrow & & \downarrow{\scriptstyle \mu(\cong)} \\ K_0(C(S^{2n-1})) & \xrightarrow{\phi_*} & K_0(B_{cd,k}), \end{array}$$

$\mu([1_{C(S^{2n})}]) = \phi_* \circ (\text{identity})_* \circ \psi_*^{-1}([1_{C(S^{2n})}]) = [1_{B_{cd,k}}]$. So $[1_{B_{cd,k}}]$ is the image of the primitive element $[1_{C(S^{2n})}] \in K_0(C(S^{2n}))$ under the isomorphism μ. Hence $[1_{B_{cd,k}}] \in K_0(B_{cd,k})$ is primitive.

Therefore, $[1_{B_{cd,k}}] \in K_0(B_{cd,k}) \cong \mathbb{Z}^2$ is primitive. Q.E.D.

For each 4-dimensional factor S of $\prod^e S^2 \times \prod^{s+r+2} S^1$ every d-homogeneous C^*-algebra over S can be constructed by combining Lemma 1.1 and Lemma 1.2. If $s+r$ is odd, one can make the integer even by tensoring with $C(S^1)$. So one can assume that $s+r$ is even, and that s is greater than or equals to r and big enough. And one can rearrange $\prod_{j=1}^s S^{2k_j-1}$ and $\mathbb{T}^r$ if needed.

Theorem 1.5. *Let A_{cd} be a cd-homogeneous C^*-algebra over $\prod_{i=1}^e S^{2n_i} \times \prod_{j=1}^s S^{2k_j-1} \times \mathbb{T}^r \times \mathbb{T}^2$ whose cd-homogeneous C^*-subalgebra restricted to the subspace $\mathbb{T}^r \times \mathbb{T}^2 \hookrightarrow \prod_{i=1}^e S^{2n_i} \times \prod_{j=1}^s S^{2k_j-1} \times \mathbb{T}^r \times \mathbb{T}^2$ is realized as $C(\mathbb{T}^r) \otimes A_{\frac{l}{d}} \otimes M_c(\mathbb{C})$ for $A_{\frac{l}{d}}$ a rational rotation algebra. Then A_{cd} is isomorphic to one of the C^*-subalgebras $A_{b_1,b_2,\cdots,b_{\frac{s+r}{2}}}^{a_1,a_2,\cdots,a_e}$, $a_1,\cdots,a_e,b_1,\cdots,b_{\frac{s+r}{2}} \in \mathbb{Z}$, of*

$$C(\prod_{i=1}^e (e_+^{2n_i} \amalg e_-^{2n_i}) \times \prod_{j=1}^{\frac{s+r}{2}} (S^{2k_j-1} \times [0,1]) \times \mathbb{T}^1 \times [0,1], M_{cd}(\mathbb{C}))$$

consisting of those functions f that satisfy

$$(f|_{e_+^{2n_i} \amalg e_-^{2n_i}})_+(z_i) = U(z_i)^{a_i} (f|_{e_+^{2n_i} \amalg e_-^{2n_i}})_-(z_i) U(z_i)^{-a_i}$$
$$(f|_{S^{2k_j-1} \times [0,1]})(w_j, 1) = U(w_j)^{b_j} (f|_{S^{2k_j-1} \times [0,1]})(w_j, 0) U(w_j)^{-b_j}$$
$$(f|_{\mathbb{T}^1 \times [0,1]})(x, 1) = U(x)^{cl} (f|_{\mathbb{T}^1 \times [0,1]})(x, 0) U(x)^{-cl}$$

for all $(z_1, \cdots, z_e, w_1, \cdots, w_{\frac{s+r}{2}}, x) \in \prod_{i=1}^e S^{2n_i-1} \times \prod_{j=1}^{\frac{s+r}{2}} S^{2k_j-1} \times \mathbb{T}^1$, *one of the tensor products of homogeneous* C^**-algebras of the type above, or one of the* C^**-algebras given by replacing* $(\prod_{i=1}^e S^{2n_i} \times \prod_{j=1}^s S^{2k_j-1} \times \mathbb{T}^r \times \mathbb{T}^2)$ *in* $A_{b_1,\cdots,b_{\frac{s+r}{2}}}^{a_1,\cdots,a_e}$ *or the tensor products with suitable* $c'd'$*-homogeneous* C^**-algebras in the same sense as above, when* $M_{c'd'}(\mathbb{C})$ *are factored out of* $A_{b_1,\cdots,b_{\frac{s+r}{2}}}^{a_1,\cdots,a_e}$ *or the tensor products, where* $U(z_i), U(w_j)$*, and* $U(x) \in PU(cd)$ *are defined in the statement of Lemma 1.1.*

Proof. By Lemma 1.1, each cd-homogeneous C^*-algebra over $S^{2k_j-1} \times S^1$ corresponds to a cd-homogeneous C^*-algebra over S^{2k_j}. By Lemma 1.2, each cd-homogeneous C^*-algebra over the product space of two even dimensional spheres can be constructed. Combining Lemma 1.1 and Lemma 1.2 yields that replacing S^{2n_i} and S^{2k_j-1} with S^2 and S^1 does not give any change in the relation, associated with bundle structure, among the factors of $\prod_{i=1}^e S^{2n_i} \times \prod_{j=1}^s S^{2k_j-1} \times \mathbb{T}^r \times \mathbb{T}^2$. Hence each cd-homogeneous C^*-algebra over $\prod_{i=1}^e S^{2n_i} \times \prod_{j=1}^s S^{2k_j-1} \times \mathbb{T}^r \times \mathbb{T}^2$ can be given by [5, Theorem 2.5], which is exactly stated in the statement for the case $n_i = 1$ and $k_j = 1$. Q.E.D.

Theorem 1.6. *Let* A_{cd} *be a* C^**-algebra over* $\prod_{i=1}^e S^{2n_i} \times \prod_{j=1}^s S^{2k_j-1} \times \mathbb{T}^r \times \mathbb{T}^2$ *constructed in Theorem 1.5. Assume that no non-trivial matrix algebra can be factored of* A_{cd}*. Then* $K_0(A_{cd}) \cong K_1(A_{cd}) \cong \mathbb{Z}^{2^{e+s+r+1}}$*, and* $[1_{A_{cd}}] \in K_0(A_{cd})$ *is primitive.*

Proof. We are going to show in Lemma 3.1 that A_{cd} is stably isomorphic to $C(\prod_{i=1}^e S^{2n_i} \times \prod_{j=1}^s S^{2k_j-1} \times \mathbb{T}^r \times \mathbb{T}^2) \otimes M_{cd}(\mathbb{C})$. By the Künneth theorem [2, Theorem 23.1.3]

$$\begin{aligned}
K_0(A_{cd}) &\cong K_0(C(\prod_{i=1}^e S^{2n_i} \times \prod_{j=1}^s S^{2k_j-1} \times \mathbb{T}^r \times \mathbb{T}^2)) \\
&\cong K_0(C(\prod_{i=1}^e S^{2n_i})) \otimes K_0(C(\prod_{j=1}^s S^{2k_j-1} \times \mathbb{T}^r \times \mathbb{T}^2)) \\
&\qquad \oplus K_1(C(\prod_{i=1}^e S^{2n_i})) \otimes K_1(C(\prod_{j=1}^s S^{2k_j-1} \times \mathbb{T}^r \times \mathbb{T}^2)) \\
&\cong \mathbb{Z}^{2^e} \otimes \mathbb{Z}^{2^{s+r+1}} \oplus \{0\} \cong \mathbb{Z}^{2^{e+s+r+1}}.
\end{aligned}$$

Similarly, one obtains that $K_1(A_{cd}) \cong \mathbb{Z}^{2^{e+s+r+1}}$.

It is enough to show that $[1_{A_{cd}}] \in K_0(A_{cd})$ is primitive. But the proof is similar to the proof given in [17, Theorem 1.2]. Since the

cd-homogeneous C^*-algebra A_{cd} is just given by replacing each C^*-subalgebra $C(S^2)$ (resp. $C(S^1)$) of the cd-homogeneous C^*-algebra over $\prod_{i=1}^{e} S^2 \times \prod_{i=1}^{s} S^1 \times \mathbb{T}^r \times \mathbb{T}^2$ given in [17] with $C(S^{2n_i})$ (resp. $C(S^{2k_j-1})$), the proof is just given by replacing $C(S^2)$ and $C(S^1)$ given in the proof of [17, Theorem 1.2] with $C(S^{2n_i})$ and $C(S^{2k_j-1})$.

Therefore, $K_0(A_{cd}) \cong K_1(A_{cd}) \cong \mathbb{Z}^{2^{e+s+r+1}}$, and $[1_{A_{cd}}] \in K_0(A_{cd})$ is primitive. Q.E.D.

§2. Spherical noncommutative tori

The noncommutative torus A_ω of rank m is obtained by an iteration of $m-1$ crossed products by actions of $\mathbb{Z}$, the first action on $C(\mathbb{T}^1)$. When A_ω is not simple, by a change of basis, A_ω is obtained by an iteration of $m-2$ crossed products by actions of $\mathbb{Z}$, the first action on a rational rotation algebra $A_{\frac{l}{d}}$. Since the fibre $M_d(\mathbb{C})$ of $A_{\frac{l}{d}}$ is factored out of the fibre of A_ω, A_ω can be obtained by an iteration of $m-2$ crossed products by actions of $\mathbb{Z}$, the first action on $A_{\frac{l}{d}}$, where the actions of $\mathbb{Z}$ on the fibre $M_d(\mathbb{C})$ of $A_{\frac{l}{d}}$ are trivial. This assures us of the existence of such actions α_i in the definition of P_ρ^d below. So one can assume that A_ω is given by twisting $C^*(d\mathbb{Z} \times d\mathbb{Z} \times \mathbb{Z}^{m-2})$ in $A_{\frac{l}{d}} \otimes C^*(\mathbb{Z}^{m-2})$ by the restriction of the multiplier ω to $d\mathbb{Z} \times d\mathbb{Z} \times \mathbb{Z}^{m-2}$, where $\widehat{d\mathbb{Z}} \times \widehat{d\mathbb{Z}}$ is the primitive ideal space of $A_{\frac{l}{d}}$ and $C^*(d\mathbb{Z} \times d\mathbb{Z}, \text{res of } \omega) = C^*(d\mathbb{Z} \times d\mathbb{Z})$ (see [5] for details).

Definition 2.1. *Let A_{cd} be a cd-homogeneous C^*-algebra over $\prod_{i=1}^{e} S^{2n_i} \times \prod_{j=1}^{s} S^{2k_j-1} \times \mathbb{T}^r \times \mathbb{T}^2$ whose cd-homogeneous C^*-subalgebra restricted to the subspace $\mathbb{T}^r \times \mathbb{T}^2 \hookrightarrow \prod_{i=1}^{e} S^{2n_i} \times \prod_{j=1}^{s} S^{2k_j-1} \times \mathbb{T}^r \times \mathbb{T}^2$ is realized as $C(\mathbb{T}^r) \otimes A_{\frac{l}{d}} \otimes M_c(\mathbb{C})$ for $A_{\frac{l}{d}}$ a rational rotation algebra. The C^*-algebra which is given by twisting $C^*(\widehat{\mathbb{T}^r} \times \widehat{\mathbb{T}^2} \times \mathbb{Z}^{m-2})$ in $A_{cd} \otimes C^*(\mathbb{Z}^{m-2})$ by a totally skew multiplier ρ on $\widehat{\mathbb{T}^r} \times \widehat{\mathbb{T}^2} \times \mathbb{Z}^{m-2}$ is said to be a spherical noncommutative torus of rank $(e, s+r, m)$ and denoted by $\mathbb{S}_\rho^{cd}$, where $C^*(\widehat{\mathbb{T}^2}, \text{res of } \rho) = C^*(\widehat{\mathbb{T}^2})$, $\mathbb{T}^2$ is the primitive ideal space of $A_{\frac{l}{d}}$, and $C^*(\widehat{\mathbb{T}^r} \times \widehat{\mathbb{T}^2} \times \mathbb{Z}^{m-2}, \rho)$ is a completely irrational noncommutative torus A_ρ.*

Then the fibre of $\mathbb{S}_\rho^d$, which is called a *generalized noncommutative torus of rank* $r+m$ and denoted by P_ρ^d, can be obtained by an iteration of $r+m-2$ crossed products by actions α_i of $\mathbb{Z}$, the first action on the rational rotation algebra $A_{\frac{l}{d}}$, where the actions α_i on the fibre $M_d(\mathbb{C})$ of $A_{\frac{l}{d}}$ are trivial. Thus the spherical noncommutative torus $\mathbb{S}_\rho^{cd}$ is realized as the C^*-algebra of sections of a locally trivial C^*-algebra bundle over $\prod_{i=1}^{e} S^{2n_i} \times \prod_{j=1}^{s} S^{2k_j-1}$ with fibres $P_\rho^d \otimes M_c(\mathbb{C})$.

We are going to show that $[1_{\mathbb{S}_\rho^{cd}}] \in K_0(\mathbb{S}_\rho^{cd})$ is primitive.

Theorem 2.2. *Let $\mathbb{S}_\rho^{cd}$ be a spherical noncommutative torus of rank $(e, s+r, m)$. Assume no non-trivial matrix algebra can be factored out of A_{cd}. Then $K_0(\mathbb{S}_\rho^{cd}) \cong K_1(\mathbb{S}_\rho^{cd}) \cong \mathbb{Z}^{2^{e+s+r+m-1}}$, and $[1_{\mathbb{S}_\rho^{cd}}] \in K_0(\mathbb{S}_\rho^{cd})$ is primitive.*

Proof. The proof is by induction on m. Assume that $m = 2$. We will show later that $\mathbb{S}_\rho^{cd}$ is stably isomorphic to $C(\prod_{i=1}^{e} S^{2n_i} \times \prod_{j=1}^{s} S^{2k_j-1}) \otimes A_\rho \otimes M_{cd}(\mathbb{C})$, where A_ρ is a noncommutative torus of rank $r+2$. By the Künneth theorem

$$\begin{aligned}
&K_0(C(\prod_{i=1}^{e} S^{2n_i} \times \prod_{j=1}^{s} S^{2k_j-1}) \otimes A_\rho) \\
&\cong K_0(C(\prod_{i=1}^{e} S^{2n_i} \times \prod_{j=1}^{s} S^{2k_j-1})) \otimes K_0(A_\rho) \\
&\qquad \oplus K_1(C(\prod_{i=1}^{e} S^{2n_i} \times \prod_{j=1}^{s} S^{2k_j-1})) \otimes K_1(A_\rho) \\
&\cong \mathbb{Z}^{2^{e+s}} \otimes \mathbb{Z}^{2^{r+1}} \cong \mathbb{Z}^{2^{e+s+r+1}}.
\end{aligned}$$

Similarly, one obtains that $K_1(C(\prod_{i=1}^{e} S^{2n_i} \times \prod_{j=1}^{s} S^{2k_j-1}) \otimes A_\rho) \cong \mathbb{Z}^{2^{e+s+r+1}}$. So $K_0(\mathbb{S}_\rho^{cd}) \cong K_1(\mathbb{S}_\rho^{cd}) \cong K_0(C(\prod_{i=1}^{e} S^{2n_i} \times \prod_{j=1}^{s} S^{2k_j-1}) \otimes A_\rho) \cong \mathbb{Z}^{2^{e+s+r+1}}$. It is enough to show that $[1_{\mathbb{S}_\rho^{cd}}] \in K_0(\mathbb{S}_\rho^{cd})$ is primitive. Combining the tricks given in Theorem 1.6 and [17, Theorem 2.2] yields that $[1_{\mathbb{S}_\rho^{cd}}] \in K_0(\mathbb{S}_\rho^{cd})$ is primitive. So $K_0(\mathbb{S}_\rho^{cd}) \cong K_1(\mathbb{S}_\rho^{cd}) \cong \mathbb{Z}^{2^{e+s+r+1}}$, and $[1_{\mathbb{S}_\rho^{cd}}] \in K_0(\mathbb{S}_\rho^{cd})$ is primitive.

Next, assume that the result is true for all spherical noncommutative tori with $m = i-1$. Write $\mathbb{S}_i = C^*(\mathbb{S}_{i-1}, u_i)$, where $\mathbb{S}_i = C^*(\mathbb{S}_\rho^{cd}, u_3, \ldots, u_i)$, where $\mathbb{S}_\rho^{cd}$ is the case above, $m = 2$. Then the inductive hypothesis applies to $\mathbb{S}_{i-1}$. Also, we can think of $\mathbb{S}_i$ as the crossed product by an action α of $\mathbb{Z}$ on $\mathbb{S}_{i-1}$, where the generator of $\mathbb{Z}$ corresponds to u_i, which acts on $C^*(v_1, \cdots, v_r, u_1^d, u_2^d, u_3, \cdots, u_{i-1})$ by conjugation (sending u_j to $u_i u_j u_i^{-1} = e^{2\pi i \theta_{ji}} u_j, j \neq 1, 2$, sending u_j^d to $u_i u_j^d u_i^{-1} = e^{2\pi i d\theta_{ji}} u_j^d, j = 1, 2$, and sending v_j to $u_i v_j u_i^{-1} = e^{2\pi i \beta_{ji}} v_j$), and which acts trivially on $C(\prod_{i=1}^{e} S^{2n_i} \times \prod_{j=1}^{s} S^{2k_j-1}) \otimes M_{cd}(\mathbb{C})$. Here $C^*(\widehat{\mathbb{T}^r} \times \widehat{\mathbb{T}^2}, \text{res of } \rho) \cong C^*(v_1, v_2, \cdots, v_r, u_1^d, u_2^d)$. Note that this action is homotopic to the trivial action, since we can homotope θ_{ji} and β_{ji} to 0. Hence $\mathbb{Z}$ acts trivially on the K-theory of $\mathbb{S}_{i-1}$. The Pimsner-Voiculescu exact sequence for a crossed product gives an exact sequence

$$K_0(\mathbb{S}_{i-1}) \xrightarrow{1-\alpha_*} K_0(\mathbb{S}_{i-1}) \xrightarrow{\Phi} K_0(\mathbb{S}_i) \longrightarrow K_1(\mathbb{S}_{i-1}) \xrightarrow{1-\alpha_*} K_1(\mathbb{S}_{i-1})$$

and similarly for K_1, where the map Φ is induced by inclusion. Since $\alpha_* = 1$ and since the K-groups of $\mathbb{S}_{i-1}$ are free abelian, this reduces a split short exact sequence

$$\{0\} \longrightarrow K_0(\mathbb{S}_{i-1}) \xrightarrow{\Phi} K_0(\mathbb{S}_i) \longrightarrow K_1(\mathbb{S}_{i-1}) \longrightarrow \{0\}$$

and similarly for K_1. So $K_0(\mathbb{S}_i)$ and $K_1(\mathbb{S}_i)$ are free abelian of rank $2 \cdot 2^{e+s+r+i-2} = 2^{e+s+r+i-1}$. Furthermore, since the inclusion $\mathbb{S}_{i-1} \to \mathbb{S}_i$ sends $1_{\mathbb{S}_{i-1}}$ to $1_{\mathbb{S}_i}$, $[1_{\mathbb{S}_i}]$ is the image of $[1_{\mathbb{S}_{i-1}}]$, which is primitive in $K_0(\mathbb{S}_{i-1})$ by inductive hypothesis. Hence the image is primitive, since the Pimsner-Voiculescu exact sequence is a split short exact sequence of torsion-free groups.

Therefore, $K_0(\mathbb{S}_\rho^{cd}) \cong K_1(\mathbb{S}_\rho^{cd}) \cong \mathbb{Z}^{2^{e+s+r+m-1}}$, and $[1_{\mathbb{S}_\rho^{cd}}] \in K_0(\mathbb{S}_\rho^{cd})$ is primitive. Q.E.D.

Corollary 2.3. *Let q be a positive integer. Assume that no non-trivial matrix algebra can be factored out of A_{cd}. Then $\mathbb{S}_\rho^{cd} \otimes M_q(\mathbb{C})$ is not isomorphic to $A \otimes M_{pq}(\mathbb{C})$ for any C^*-algebra A and any integer p greater than 1. In particular, no non-trivial matrix algebra can be factored out of $\mathbb{S}_\rho^{cd}$, P_ρ^{cd} and A_ρ.*

Proof. Assume $\mathbb{S}_\rho^{cd} \otimes M_q(\mathbb{C})$ is isomorphic to $A \otimes M_{pq}(\mathbb{C})$. Then the unit $1_{\mathbb{S}_\rho^{cd}} \otimes I_q$ maps to the unit $1_A \otimes I_{pq}$. So $[1_{\mathbb{S}_\rho^{cd}} \otimes I_q] = [1_A \otimes I_{pq}]$. Thus there is a projection $e \in \mathbb{S}_\rho^{cd}$ such that $q[1_{\mathbb{S}_\rho^{cd}}] = (pq)[e]$. But $K_0(\mathbb{S}_\rho^{cd})$ is torsion-free, so $[1_{\mathbb{S}_\rho^{cd}}] = p[e]$. This contradicts Theorem 2.2 if $p > 1$.

Therefore, $\mathbb{S}_\rho^{cd} \otimes M_q(\mathbb{C})$ is not isomorphic to $A \otimes M_{pq}(\mathbb{C})$. Q.E.D.

§3. The bundle structure of spherical noncommutative tori

For M a compact CW-complex the Čech cohomology group $H^3(M, \mathbb{Z})$ classifies the tensor products of *cd*-homogeneous C^*-algebras over M with the C^*-algebra $\mathcal{K}(\mathcal{H})$ of compact operators on a separable Hilbert space $\mathcal{H}$ (see [9]). The Čech cohomology group $H^3(M, \mathbb{Z})$ is isomorphic to the singular cohomology group $H^3(M, \mathbb{Z})$ when M is *triangularizable* (see [7, Theorem15.8]).

Lemma 3.1. *Each cd-homogeneous C^*-algebra over $\prod_{i=1}^{e} S^{2n_i} \times \prod_{j=1}^{s+r+2} S^{2k_j-1}$ is stably isomorphic to $C(\prod_{i=1}^{e} S^{2n_i} \times \prod_{j=1}^{s+r+2} S^{2k_j-1}) \otimes M_{cd}(\mathbb{C})$.*

Proof. Each non-trivial element in the Čech cohomology group $H^3(\prod_{i=1}^{e} S^{2n_i} \times \prod_{j=1}^{s+r+2} S^{2k_j-1}, \mathbb{Z})$ can be given by a non-trivial element in $H^3((S^1)^3, \mathbb{Z})$, $H^3(S^2 \times S^1, \mathbb{Z})$, or $H^3(S^3, \mathbb{Z})$ if there exist such factors.

First, $H^3(S^2 \times S^1, \mathbb{Z}) = \mathbb{Z}$. By the Woodward theorem [21], $[S^2 \times S^1, BPU(cd)]$ is embedded into $H^2(S^2 \times S^1, \mathbb{Z}_{cd}) \oplus H^4(S^2 \times S^1, \mathbb{Z}) \cong H^2(S^2, \mathbb{Z}_{cd}) \cong \mathbb{Z}_{cd}$. So each cd-homogeneous C^*-algebra over $S^2 \times S^1$ is isomorphic to the tensor product of a cd-homogeneous C^*-algebra over S^2 with $C(S^1)$, which is stably isomorphic to $C(S^2) \otimes C(S^1) \otimes M_{cd}(\mathbb{C})$, since $H^3(S^2, \mathbb{Z}) = \{0\}$. Thus each cd-homogeneous C^*-algebra over $S^2 \times S^1$ is stably isomorphic to $C(S^2 \times S^1) \otimes M_{cd}(\mathbb{C})$.

Similarly, one obtains the same result for the other cases.

Therefore, each cd-homogeneous C^*-algebra over $\prod_{i=1}^{e} S^{2n_i} \times \prod_{j=1}^{s+r+2} S^{2k_j-1}$ is stably isomorphic to $C(\prod_{i=1}^{e} S^{2n_i} \times \prod_{j=1}^{s+r+2} S^{2k_j-1}) \otimes M_{cd}(\mathbb{C})$.

Q.E.D.

We are going to show that $\mathbb{S}_\rho^{cd} \otimes \mathcal{K}(\mathcal{H})$ has the trivial bundle structure.

Theorem 3.2. *The spherical noncommutative torus $\mathbb{S}_\rho^{cd}$ is stably isomorphic to $C(\prod_{i=1}^{e} S^{2n_i} \times \prod_{j=1}^{s} S^{2k_j-1}) \otimes A_\rho \otimes M_{cd}(\mathbb{C})$. In particular, P_ρ^d is stably isomorphic to $A_\rho \otimes M_d(\mathbb{C})$.*

Proof. Let $\mathbb{S}_\rho^{cd}$ be defined by twisting $C^*(\widehat{\mathbb{T}^r} \times \widehat{\mathbb{T}^2} \times \mathbb{Z}^{m-2})$ in $A_{cd} \otimes C^*(\mathbb{Z}^{m-2})$ by a totally skew multiplier ρ on $\widehat{\mathbb{T}^r} \times \widehat{\mathbb{T}^2} \times \mathbb{Z}^{m-2}$, where $C^*(\widehat{\mathbb{T}^2}, \text{res of } \rho) = C^*(\widehat{\mathbb{T}^2})$. By Lemma 3.1, the cd-homogeneous C^*-algebra A_{cd} is stably isomorphic to $C(\prod_{i=1}^{e} S^{2n_i} \times \prod_{j=1}^{s} S^{2k_j-1} \times \mathbb{T}^r \times \mathbb{T}^2) \otimes M_{cd}(\mathbb{C})$. In particular, $C(\prod_{i=1}^{e} S^{2n_i} \times \prod_{j=1}^{s} S^{2k_j-1})$ is factored out of $A_{cd} \otimes \mathcal{K}(\mathcal{H})$. By the definition of $\mathbb{S}_\rho^{cd}$, $C(\prod_{i=1}^{e} S^{2n_i} \times \prod_{j=1}^{s} S^{2k_j-1})$ is factored out of $\mathbb{S}_\rho^{cd} \otimes \mathcal{K}(\mathcal{H})$. So $\mathbb{S}_\rho^{cd}$ is stably isomorphic to $C(\prod_{i=1}^{e} S^{2n_i} \times \prod_{j=1}^{s} S^{2k_j-1}) \otimes P_\rho^d \otimes M_c(\mathbb{C})$. But it was shown in [5, Theorem 3.4] that P_ρ^d is stably isomorphic to $A_\rho \otimes M_d(\mathbb{C})$.

Therefore, $\mathbb{S}_\rho^{cd}$ is stably isomorphic to $C(\prod_{i=1}^{e} S^{2n_i} \times \prod_{j=1}^{s} S^{2k_j-1}) \otimes A_\rho \otimes M_{cd}(\mathbb{C})$.

Q.E.D.

Using the fact that $[1_{\mathbb{S}_\rho^{cd}}] \in K_0(\mathbb{S}_\rho^{cd})$ is primitive, we are going to investigate the bundle structure of the tensor products of spherical noncommutative tori $\mathbb{S}_\rho^{cd}$ with UHF-algebras M_{p^∞} of type p^∞.

Theorem 3.3. *Let $\mathbb{S}_\rho^{cd}$ be a spherical noncommutative torus. Assume that no non-trivial matrix algebra can be factored out of A_{cd}. Then $\mathbb{S}_\rho^{cd} \otimes M_{p^\infty}$ is isomorphic to $C(\prod_{i=1}^e S^{2n_i} \times \prod_{j=1}^s S^{2k_j-1}) \otimes A_\rho \otimes M_{cd}(\mathbb{C}) \otimes M_{p^\infty}$ if and only if the set of prime factors of cd is a subset of the set of prime factors of p.*

Proof. Assume that the set of prime factors of cd is a subset of the set of prime factors of p. To show that $\mathbb{S}_\rho^{cd} \otimes M_{p^\infty}$ is isomorphic to $C(\prod_{i=1}^e S^{2n_i} \times \prod_{j=1}^s S^{2k_j-1}) \otimes A_\rho \otimes M_{cd}(\mathbb{C}) \otimes M_{p^\infty}$, it is enough to show that $\mathbb{S}_\rho^{cd} \otimes M_{(cd)^\infty}$ is isomorphic to $C(\prod_{i=1}^e S^{2n_i} \times \prod_{j=1}^s S^{2k_j-1}) \otimes A_\rho \otimes M_{cd}(\mathbb{C}) \otimes M_{(cd)^\infty}$. However, there exist the C^*-algebra homomorphisms which are the canonical inclusions

$$\mathbb{S}_\rho^{cd} \otimes M_{(cd)^g}(\mathbb{C}) \hookrightarrow C(\prod_{i=1}^e S^{2n_i} \times \prod_{j=1}^s S^{2k_j-1}) \otimes A_\rho \otimes M_{cd}(\mathbb{C}) \otimes M_{(cd)^g}(\mathbb{C})$$

and the $C(\prod_{i=1}^e S^{2n_i} \times \prod_{j=1}^s S^{2k_j-1}) \otimes A_\rho$-module maps $C(\prod_{i=1}^e S^{2n_i} \times \prod_{j=1}^s S^{2k_j-1}) \otimes A_\rho \otimes M_{(cd)^g}(\mathbb{C}) \hookrightarrow \mathbb{S}_\rho^{cd} \otimes M_{(cd)^g}(\mathbb{C})$:

$$\begin{aligned} \mathbb{S}_\rho^{cd} &\hookrightarrow C(\prod_{i=1}^e S^{2n_i} \times \prod_{j=1}^s S^{2k_j-1}) \otimes A_\rho \otimes M_{cd}(\mathbb{C}) \hookrightarrow \mathbb{S}_\rho^{cd} \otimes M_{cd}(\mathbb{C}) \\ &\hookrightarrow C(\prod_{i=1}^e S^{2n_i} \times \prod_{j=1}^s S^{2k_j-1}) \otimes A_\rho \otimes M_{(cd)^2}(\mathbb{C}) \hookrightarrow \cdots . \end{aligned}$$

The inductive limit of the odd terms

$$\cdots \to \mathbb{S}_\rho^{cd} \otimes M_{(cd)^g}(\mathbb{C}) \to \mathbb{S}_\rho^{cd} \otimes M_{(cd)^{g+1}}(\mathbb{C}) \to \cdots$$

is $\mathbb{S}_\rho^{cd} \otimes M_{(cd)^\infty}$, and the inductive limit of the even terms

$$\begin{aligned} \cdots &\to C(\prod_{i=1}^e S^{2n_i} \times \prod_{j=1}^s S^{2k_j-1}) \otimes A_\rho \otimes M_{(cd)^g}(\mathbb{C}) \\ &\to C(\prod_{i=1}^e S^{2n_i} \times \prod_{j=1}^s S^{2k_j-1}) \otimes A_\rho \otimes M_{(cd)^{g+1}}(\mathbb{C}) \to \cdots \end{aligned}$$

is $C(\prod_{i=1}^e S^{2n_i} \times \prod_{j=1}^s S^{2k_j-1}) \otimes A_\rho \otimes M_{(cd)^\infty}$. Thus by the Elliott theorem [11, Theorem 2.1], $\mathbb{S}_\rho^{cd} \otimes M_{(cd)^\infty}$ is isomorphic to $C(\prod_{i=1}^e S^{2n_i} \times \prod_{j=1}^s S^{2k_j-1}) \otimes A_\rho \otimes M_{(cd)^\infty}$.

Conversely, assume that

$$\mathbb{S}_\rho^{cd} \otimes M_{p^\infty} \cong C(\prod_{i=1}^e S^{2n_i} \times \prod_{j=1}^s S^{2k_j-1}) \otimes A_\rho \otimes M_{cd}(\mathbb{C}) \otimes M_{p^\infty}.$$

Then the unit $1_{\mathbb{S}_\rho^{cd}} \otimes 1_{M_{p^\infty}}$ maps to the unit $1_{C(\prod_{i=1}^e S^{2n_i} \times \prod_{j=1}^s S^{2k_j-1}) \otimes A_\rho}$ $\otimes 1_{M_{p^\infty}} \otimes I_{cd}$. So

$$[1_{\mathbb{S}_\rho^{cd}} \otimes 1_{M_{p^\infty}}] = [1_{C(\prod_{i=1}^e S^{2n_i} \times \prod_{j=1}^s S^{2k_j-1}) \otimes A_\rho} \otimes 1_{M_{p^\infty}} \otimes I_{cd}]$$

$$[1_{\mathbb{S}_\rho^{cd}} \otimes 1_{M_{p^\infty}}] = [1_{\mathbb{S}_\rho^{cd}}] \otimes [1_{M_{p^\infty}}]$$

$$[1_{C(\prod_{i=1}^e S^{2n_i} \times \prod_{j=1}^s S^{2k_j-1}) \otimes A_\rho} \otimes 1_{M_{p^\infty}} \otimes I_{cd}] = cd([1_{C(\prod_{i=1}^e S^{2n_i} \times \prod_{j=1}^s S^{2k_j-1}) \otimes A_\rho}] \otimes [1_{M_{p^\infty}}]).$$

Under the assumption that $1_{\mathbb{S}_\rho^{cd}} \otimes 1_{M_{p^\infty}}$ maps to

$$1_{C(\prod_{i=1}^e S^{2n_i} \times \prod_{j=1}^s S^{2k_j-1}) \otimes A_\rho} \otimes 1_{M_{p^\infty}} \otimes I_{cd},$$

if there is a prime factor q of cd such that $q \nmid p$, then $[1_{M_{p^\infty}}] \neq q[e_\infty]$ for e_∞ a projection in M_{p^∞}. So there is a projection $e \in \mathbb{S}_\rho^{cd}$ such that $[1_{\mathbb{S}_\rho^{cd}}] = q[e]$. This contradicts Theorem 2.2. Thus the set of prime factors of cd is a subset of the set of prime factors of p.

Therefore, $\mathbb{S}_\rho^{cd} \otimes M_{p^\infty}$ is isomorphic to $C(\prod_{i=1}^e S^{2n_i} \times \prod_{j=1}^s S^{2k_j-1}) \otimes A_\rho \otimes M_{cd}(\mathbb{C}) \otimes M_{p^\infty}$ if and only if the set of prime factors of cd is a subset of the set of prime factors of p. Q.E.D.

§4. Completely irrational noncommutative tori

It was proved in [3, Theorem 1.5] that every completely irrational noncommutative torus has real rank 0, where the "*real rank* 0" means that the set of invertible self-adjoint elements is dense in the set of self-adjoint elements. Combining Theorem 3.2 and [8, Corollary 3.3] yields that the generalized noncommutative torus P_ρ^d has real rank 0 since the noncommutative torus A_ρ has real rank 0. The Lin and Rørdam theorem [16, Proposition 3] says that the generalized noncommutative torus P_ρ^d is an inductive limit of circle algebras, since $P_\rho^d \otimes \mathcal{K}(\mathcal{H}) \cong A_\rho \otimes \mathcal{K}(\mathcal{H})$ is an inductive limit of circle algebras [16, Proposition]. Combining [11, Theorem 7.1] and [13, Theorem 1.3] yields that the completely irrational noncommutative tori A_ω of rank $r+m$ and the generalized noncommutative tori P_ρ^d of rank $r+m$ are isomorphic if the ranges of the traces equal.

Lemma 4.1. ([6, Lemma 4.1]) $\mathrm{tr}(K_0(P_\rho^d)) = \frac{1}{d} \cdot \mathrm{tr}(K_0(A_\rho))$.

Theorem 4.2. ([6, Theorem 4.2]) *Let A_ω be a completely irrational noncommutative torus of rank $r+m$ with $\mathrm{tr}(K_0(A_\omega)) = \frac{1}{d} \cdot \mathrm{tr}(K_0(A_\rho))$ for A_ρ a completely irrational noncommutative torus of rank $r+m$. Then A_ω is isomorphic to P_ρ^d.*

§5. C^*-algebras over spheres with fibres noncommutative tori

We are going to show that the set of all spherical noncommutative tori with primitive ideal space $\prod_{i=1}^{e} S^{2n_i} \times \prod_{j=1}^{s} S^{2k_j-1}$ and fibres $A_\omega \otimes M_c(\mathbb{C})$ is in bijective correspondence with the set of all C^*-algebras of sections of locally trivial C^*-algebra bundles over $\prod_{i=1}^{e} S^{2n_i} \times \prod_{j=1}^{s} S^{2k_j-1}$ with fibres $A_\omega \otimes M_c(\mathbb{C})$ for A_ω a completely irrational noncommutative torus.

Let A_ω be a noncommutative torus of rank m with $\widehat{S_\omega} \cong \mathbb{T}^1$. Then A_ω is realized as the C^*-algebra of sections of a locally trivial C^*-algebra bundle over $\widehat{d\mathbb{Z}}$ and fibres $C^*(\mathbb{Z}^m/S_\omega, \omega_1)$ for some totally skew multiplier ω_1, where $C^*(\mathbb{Z}^m/S_\omega, \omega_1) \cong A_\rho \otimes M_d(\mathbb{C})$ for A_ρ a completely irrational noncommutative torus of rank $m-1$ (see [1, 12]). By the definition of A_ω, $C(\mathbb{T}^1)$ and A_ρ split. Since $[\mathbb{T}^1, BPU(d)] \cong \{0\}$, $C(\mathbb{T}^1)$ and $M_d(\mathbb{C})$ split. And $M_d(\mathbb{C})$ and A_ρ also split. But by Corollary 2.3, A_ω has a non-trivial bundle structure if $d > 1$. This implies that a C^*-subalgebra of A_ρ plays a role as a base space in the bundle structure. In fact, A_ω can be obtained by an iteration of $m-2$ crossed products by actions of $\mathbb{Z}$, the first action on a rational rotation algebra $A_{\frac{l}{d}}$, and the non-triviality of the bundle structure is given by a non-trivial element of $[\mathbb{T}^2, BPU(d)] \cong [\mathbb{T}^1, PU(d)] \cong \mathbb{Z}_d$, which represents $A_{\frac{l}{d}}$ canonically embedded into A_ω.

Let d be the biggest integer among the possible integers satisfying the condition $\operatorname{tr}(K_0(A_\omega)) = \frac{1}{d} \cdot \operatorname{tr}(K_0(A_\rho))$, i.e., $A_\omega \cong P_\rho^d$. For a d-homogeneous C^*-algebra A over S^{2n+1}, there is a matrix algebra $M_q(\mathbb{C})$ such that $A \otimes M_q(\mathbb{C})$ is isomorphic to $C(S^{2n+1}) \otimes M_{dq}(\mathbb{C})$. But there is a matrix subalgebra $M_q(\mathbb{C})$ big enough satisfying the above condition such that $M_q(\mathbb{C})$ is embedded into P_ρ^d, since P_ρ^d is an inductive limit of circle algebras, which is simple.

Lemma 5.1. *Each C^*-algebra $\Gamma(\eta)$ of sections of a locally trivial C^*-algebra bundle η over S^{2n+1} with fibres $P_\rho^1 = A_\rho$ has the trivial bundle structure.*

Proof. Let $P_\rho^1 = \varinjlim(\bigoplus_{j=1} C(\mathbb{T}^1) \otimes M_{p_{i(j)}}(\mathbb{C}))$. The C^*-algebra $\Gamma(\eta)$ is isomorphic to an inductive limit of direct sums of $p_{i(j)}$-homogeneous C^*-algebras over $S^{2n+1} \times \mathbb{T}^1$, and each $C(S^{2n+1} \times \mathbb{T}^1)$ is canonically embedded into $\Gamma(\eta)$. So there could be a canonical homomorphism of $C(S^{2n+1}) \otimes M_d(\mathbb{C})$ into the C^*-algebra $\Gamma(\eta)$ of sections of a locally trivial C^*-algebra bundle η over S^{2n+1} with fibres P_ρ^1 such that the non-triviality can be given by a d-homogeneous C^*-algebra over $S^{2n+1} \times \mathbb{T}^1$. Then $M_d(\mathbb{C})$ must be factored out of the circle algebra in each inductive step, and so the range of the trace of P_ρ^1 would be the form $\frac{1}{d} \cdot \operatorname{tr}(A)$ for

A a simple unital C^*-algebra, which is impossible by the assumption. We have two cases; one of them is the case that a C^*-subalgebra of P_ρ^1 plays a role as a base space in the bundle structure, and the other is not.

For the first case, when a C^*-subalgebra of P_ρ^1 plays a role as a base space in the bundle structure and P_ρ^1 is realized as a tensor product of non-trivial completely irrational noncommutative tori, the torsion-free groups in $P_\rho^1 = A_\rho$ giving simple noncommutative tori which are given by twisting the torsion-free groups by totally skew multipliers must split, so all factors of P_ρ^1 must split. The relation among factors of P_ρ^1 is different from the relation between fibres $M_d(\mathbb{C})$ and base A_ρ in the fibres of the non-simple noncommutative torus A_ω given above, and so one can assume that all factors of P_ρ^1 play roles as a base space in the bundle structure. Hence P_ρ^1 plays a role as a base space in the bundle structure, and so $\Gamma(\eta)$ is isomorphic to $C(S^{2n+1}) \otimes P_\rho^1$.

For the other case, since $P_\rho^1 = \varinjlim(\bigoplus_{j=1} C(\mathbb{T}^1) \otimes M_{p_{i(j)}}(\mathbb{C}))$, there is a matrix algebra $M_p(\mathbb{C})$ big enough which is embedded into P_ρ^1. Since $[S^{2n+1}, BPU(p)] \cong \{0\}$, $C(S^{2n+1})$ and $M_p(\mathbb{C})$ split, i.e., any p-homogeneous C^*-algebra over S^{2n+1} has the trivial bundle structure. By the same reasoning as above, $M_p(\mathbb{C})$ cannot be factored out of the circle algebras in all inductive steps. But $\Gamma(\eta)$ has a locally trivial bundle structure. Hence $C(S^{2n+1})$ and $(M_p(\mathbb{C}) \hookrightarrow)$ P_ρ^1 must split, and so $\Gamma(\eta)$ has the trivial bundle structure.

Therefore, each C^*-algebra $\Gamma(\eta)$ of sections of a locally trivial C^*-algebra bundle η over S^{2n+1} with fibres P_ρ^1 has the trivial bundle structure. Q.E.D.

Now we want to show that each C^*-algebra of sections of a locally trivial C^*-algebra bundle over $\prod_{i=1}^e S^{2n_i} \times \prod_{j=1}^s S^{2k_j-1}$ with fibres $P_\rho^1 = A_\rho$ has the trivial bundle structure.

Proposition 5.2. *Each C^*-algebra $\Gamma(\eta)$ of sections of a locally trivial C^*-algebra bundle η over $\prod_{i=1}^e S^{2n_i} \times \prod_{j=1}^s S^{2k_j-1}$ with fibres $P_\rho^1 = A_\rho$ has the trivial bundle structure.*

Proof. Let P_ρ^1 be an inductive limit of $\bigoplus_{j=1} C(\mathbb{T}^1) \otimes M_{p_{i(j)}}(\mathbb{C})$. For some pair $(2k_j - 1, 2k_{j'} - 1) = (2k_j - 1, 1)$, if the C^*-subalgebra of sections of a locally trivial C^*-algebra bundle over $S^{2k_j-1} \times S^1$ with fibres P_ρ^1, which is canonically embedded into $\Gamma(\eta)$, has a non-trivial bundle structure, then the factor $S^{2k_j-1} \times S^1$ can be replaced by S^{2k_j}, since there is a map of degree 1 from $S^{2k_j-1} \times S^1$ to S^{2k_j}. For each j, there is a canonical homomorphism of the C^*-subalgebra $\Gamma(\eta_j)$ of sections of a locally trivial C^*-algebra bundle η_j over S^{2k_j-1} with fibres P_ρ^1 into $\Gamma(\eta)$.

By Lemma 5.1, the C^*-algebra of sections of a locally trivial C^*-algebra bundle over S^{2k_j-1} with fibres P_ρ^1 has the trivial bundle structure. Thus $C(S^{2k_j-1})$ are factored out of $\Gamma(\eta)$, and so $C(\prod_{j=1}^s S^{2k_j-1})$ is factored out of $\Gamma(\eta)$.

Next, $[S^{2n_i}, B(\mathrm{Aut}(P_\rho^1))] = [S^{2n_i-1}, \mathrm{Aut}(P_\rho^1)]$. But there is a map of degree 1 from S^{2n_i} to $S^{2n_i-1} \times S^1$. So for each i each C^*-algebra of sections of a locally trivial C^*-algebra bundle over S^{2n_i} with fibres P_ρ^1 is induced from the C^*-algebra $\Gamma(\zeta_i)$ of sections of a locally trivial C^*-algebra bundle ζ_i over $S^{2n_i-1} \times \mathbb{T}^1$ with fibres P_ρ^1. Consider the crossed product by the action α_θ of $\mathbb{Z}$ on $\Gamma(\zeta_i)$ for a suitable irrational number θ such that the range of the trace of $P_\rho^1 \otimes A_\theta$ is not $\frac{1}{w}\times$ the range of the trace of any simple irrational noncommutative torus of rank $m+1$ for any positive integer w greater than 1, where the action α_θ on $C(S^{2n_i-1}) \otimes P_\rho^1$ is trivial and $C(\mathbb{T}^1) \times_{\alpha_\theta} \mathbb{Z}$ is the irrational rotation algebra A_θ. Then $\Gamma(\zeta_i) \times_{\alpha_\theta} \mathbb{Z}$ is obviously realized as the C^*-algebra of sections of a locally trivial C^*-algebra bundle over S^{2n_i-1} with fibres $P_\rho^1 \otimes A_\theta$. But $\Gamma(\zeta_i) \times_{\alpha_\theta} \mathbb{Z}$ has the trivial bundle structure. So each C^*-algebra of sections of a locally trivial C^*-algebra bundle over S^{2n_i} with fibres P_ρ^1 has the trivial bundle structure. Thus $C(S^{2n_i})$ are factored out of $\Gamma(\eta)$. Hence $C(\prod_{i=1}^e S^{2n_i})$ is factored out of $\Gamma(\eta)$, and so $C(\prod_{i=1}^e S^{2n_i} \times \prod_{j=1}^s S^{2k_j-1})$ is factored out of $\Gamma(\eta)$, as desired. Q.E.D.

Each *cd*-homogeneous C^*-algebra over $\prod_{i=1}^e S^{2n_i} \times \prod_{j=1}^s S^{2k_j-1} \times \mathbb{T}^r \times \mathbb{T}^2$ is realized as the C^*-algebra $\Gamma(\eta)$ of sections of a locally trivial C^*-algebra bundle η over $\prod_{i=1}^e S^{2n_i} \times \prod_{j=1}^s S^{2k_j-1} \times \mathbb{T}^r \times \mathbb{T}^2$ with fibres $M_{cd}(\mathbb{C})$, and hence $\mathbb{S}_\rho^{cd}$ is realized as the C^*-algebra of sections of a locally trivial C^*-algebra bundle over $\prod_{i=1}^e S^{2n_i} \times \prod_{j=1}^s S^{2k_j-1}$ with fibres $P_\rho^d \otimes M_c(\mathbb{C})$.

Theorem 5.3. *The set of spherical noncommutative tori with primitive ideal space* $\prod_{i=1}^e S^{2n_i} \times \prod_{j=1}^s S^{2k_j-1}$ *and fibres* $P_\rho^d \otimes M_c(\mathbb{C})$ *is in bijective correspondence with the set of* C^**-algebras of sections of locally trivial* C^**-algebra bundles over* $\prod_{i=1}^e S^{2n_i} \times \prod_{j=1}^s S^{2k_j-1}$ *with fibres* $P_\rho^d \otimes M_c(\mathbb{C})$.

Proof. If $cd = 1$, we have obtained the result in Proposition 5.2. So assume that $cd > 1$. Then one can assume that there is a matrix subalgebra $M_{cd}(\mathbb{C})$ which is factored out of each inductive step, even though $M_d(\mathbb{C})$ is not factored out of P_ρ^d. And P_ρ^d is isomorphic to $A_{\frac{1}{d}} \times_{\alpha_3} \mathbb{Z} \times_{\alpha_4} \cdots \times_{\alpha_{r+m}} \mathbb{Z}$. By Proposition 5.2, each C^*-algebra of sections of a locally trivial C^*-algebra bundle over $\prod_{i=1}^e S^{2n_i} \times \prod_{j=1}^s S^{2k_j-1}$ with fibres $C^*(d\mathbb{Z} \times d\mathbb{Z}) \times_{\alpha_3} \mathbb{Z} \times_{\alpha_4} \cdots \times_{\alpha_{r+m}} \mathbb{Z}$ has the trivial bundle structure.

Hence each C^*-algebra of sections of a locally trivial C^*-algebra bundle over $\prod_{i=1}^{e} S^{2n_i} \times \prod_{j=1}^{s} S^{2k_j-1}$ with fibres $P_\rho^d \otimes M_c(\mathbb{C})$ is given by twisting $C^*(\widehat{\mathbb{T}^r} \times \widehat{\mathbb{T}^2} \times \mathbb{Z}^{m-2})$ in $A_{cd} \otimes C^*(\mathbb{Z}^{m-2})$ by the totally skew multiplier ρ on $\widehat{\mathbb{T}^r} \times \widehat{\mathbb{T}^2} \times \mathbb{Z}^{m-2}$, which is a spherical noncommutative torus.

Therefore, the set of spherical noncommutative tori with primitive ideal space $\prod_{i=1}^{e} S^{2n_i} \times \prod_{j=1}^{s} S^{2k_j-1}$ and fibres $P_\rho^d \otimes M_c(\mathbb{C})$ is in bijective correspondence with the set of C^*-algebras of sections of locally trivial C^*-algebra bundles over $\prod_{i=1}^{e} S^{2n_i} \times \prod_{j=1}^{s} S^{2k_j-1}$ with fibres $P_\rho^d \otimes M_c(\mathbb{C})$. Q.E.D.

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Stable C^*-algebras

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Abstract.

We give a survey of known and a few new results on stable C^*-algebras. Characterizations of stable C^*-algebras are described, it is decided for a number of operations on C^*-algebras whether or not they leave the class of stable C^*-algebras invariant, and the relation between this topic and the structure of simple C^*-algebras is discussed.

§1. Introduction

This article contains some new results and a survey of older results, mostly from the articles [12], [16], [17], and [19], on stable C^*-algebras. Recall that a C^*-algebra A is *stable* if it is isomorphic to $A \otimes \mathcal{K}$, where $\mathcal{K}$ denotes the C^*-algebra of compact operators on a separable Hilbert space. Since $\mathcal{K} \otimes \mathcal{K} \cong \mathcal{K}$ it follows that $A \otimes \mathcal{K}$ is stable for every C^*-algebra A. If B_1 and B_2 are full hereditary sub-C^*-algebras of a C^*-algebra A, then $B_1 \otimes \mathcal{K} \cong B_2 \otimes \mathcal{K}$ by Brown's theorem, [4]. In other words, among full hereditary sub-C^*-algebras the stable ones have the distinguished property that they all are isomorphic to each other.

In BDF-theory, [5], extensions $0 \to \mathcal{K} \to A \to B \to 0$ (for fixed (abelian) C^*-algebras B) are classified, and it is contained in this theory that A is stable if and only if B is stable in any such extension. The extension question for stable C^*-algebras asked if for any extension $0 \to I \to A \to B \to 0$ of (separable) C^*-algebras one has that A is stable if and only if I and B are stable. This question has recently been answered in the negative in [19] (see Theorem 6.1). Some partial positive results do however hold (see Section 6).

Blackadar has shown that an AF-algebra is stable if and only if it admits no bounded non-zero traces. This results can be generalized (see Section 3), but the existence (established in [16], see Theorem 4.3) of a

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simple, stably finite, non-stable C^*-algebra A such that $M_2(A)$ is stable shows that Blackadar's result is not valid for all (stably finite, simple) C^*-algebras.

The negative answer to the extension problem for stable C^*-algebras was obtained using methods similar to those used in the recent article [18] were an example of a simple C^*-algebra with a finite and an infinite projection was constructed. It is no surprise that these two problems are linked. In both cases one seeks C^*-algebras exhibiting exotic comparison properties (as first found by Villadsen in [20]). Another link is given in the observation by Kirchberg that a simple C^*-algebra is purely infinite if and only if all its hereditary sub-C^*-algebras contain a stable sub-C^*-algebra.

The first version of this paper was written in December, 2000. The paper was revised in July, 2001, to include the results from the papers [18] and [19].

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§2. Characterizing stable C^*-algebras

We begin this section by stating a result from [12] by Hjelmborg and the author that characterizes stable C^*-algebras. We need some notation to state the result.

In a C^*-algebra A, let $F(A)$ denote the set of positive elements a in A for which there exists e in A such that $ea = ae = a$. (Every element in $F(A)$ belongs to the Pedersen ideal of A; but the Pedersen ideal can in some cases contain positive elements not in $F(A)$. This is for example the case whenever A is an algebraically simple, non-unital C^*-algebra.)

A C^*-algebra is said to be *σ-unital* if it contains a countable approximate unit; and it is called *σ_{p}-unital* if it contains a countable approximate unit consisting of projections. One can show that an approximate unit of projections always can always be taken to be increasing and to dominate any fixed projection in the C^*-algebra.

Remark 2.1. **(Equivalence of positive elements)** Two positive elements a, b in a C^*-algebra A are said to be equivalent, written $a \sim b$, if there is an element x in A such that $x^*x = a$ and $xx^* = b$. Let $x = u(x^*x)^{1/2}$ be the polar decomposition for x in A^{**}. Then uc belongs to A for every c in $\overline{aAa}$, and the map $c \mapsto ucu^*$ defines an isomorphism from $\overline{aAa}$ onto $\overline{bAb}$ which maps a to b. Moreover, for each positive element c in $\overline{aAa}$ we have $c \sim ucu^*$ because $y = uc^{1/2}$ belongs to A, $y^*y = c$, and $yy^* = ucu^*$.

Write $a \precsim b$ if a and b are positive elements in A such that $x_n^* b x_n \to a$ for some sequence $\{x_n\}$ in A. For a in A^+ and $\varepsilon > 0$ let $(a-\varepsilon)_+$ denote the positive part of the self-adjoint element $a - \varepsilon \cdot 1$ in the unitization of $\widetilde{A}$. Then $(a-\varepsilon)_+$ belongs to A, and $a \precsim b$ if and only if $(a-\varepsilon)_+ \sim b_\varepsilon$ for some b_ε in $\overline{bAb}$ for each $\varepsilon > 0$ (cf. [15, Proposition 2.4]).

Theorem 2.2. **(Theorems 2.1 and 3.3 of** [12]) *The following conditions are equivalent for every σ-unital C^*-algebra A:*

(i) *A is stable,*
(ii) *for every positive element a in A and for every positive $\varepsilon > 0$ there are positive elements b, c in A such that $\|a-b\| \le \varepsilon$, $b \sim c$, and $\|ac\| \le \varepsilon$,*
(iii) *for every a in $F(A)$ there is a positive element b in A such that $a \precsim b$ and $a \perp b$,*
(iv) *for every a in $F(A)$ there is a unitary element u in the unitization of A such that $a \perp uau^*$,*
(v) *there is a sequence $\{E_n\}_{n=1}^\infty$ of mutually orthogonal, mutually equivalent projections in the multiplier algebra $\mathcal{M}(A)$ of A such that $\sum_{n=1}^\infty E_n = 1$ (the sum converges in the strict topology).*

If A is further assumed to be σ_{p}-unital, then (i) – (v) *above are equivalent to:*

(vi) *for every projection p in A there is a projection q in A such that $p \sim q$ and $p \perp q$.*

Corollary 2.3. **(Permanence)**

(i) *If A is a σ-unital C^*-algebra and if A is the inductive limit of an inductive system of σ-unital stable C^*-algebras, then A is stable.*
(ii) *If A is stable, then so is every ideal in A and every quotient of A.*
(iii) *If A is a σ-unital, stable C^*-algebra and if a is a positive contraction in A, then $\overline{(1-a)A(1-a)}$ is stable.*
(iv) *If B is a sub-C^*-algebra of a σ-unital, stable C^*-algebra A and if B contains an approximate unit for A, then B is stable.*
(v) *If A is a σ-unital, stable C^*-algebra and if G is a countable discrete group acting on A, then $A \rtimes G$ is stable.*

Parts (i), (iii), (iv), and (v) are proved in [12] (and the proof of (i) and (iii) uses Theorem 2.2). To see that (ii) holds we may assume that $A = A_0 \otimes \mathcal{K}$ for some C^*-algebra A_0. If I is a closed two-sided ideal in $A_0 \otimes \mathcal{K}$, then $I = I_0 \otimes \mathcal{K}$ for some closed two-sided ideal I_0 of A_0, and it follows that I and A/I are stable.

Extension of two (σ-unital) stable C^*-algebras need not be stable, cf. Section 6. If A is stable, then so is $A \otimes B$ for every C^*-algebra B.

In the converse direction one can clearly not conclude that A is stable knowing that $A \otimes B$ is stable for some C^*-algebra B, perhaps surprisingly not even in the case when $B = M_2(\mathbb{C})$, cf. Theorem 4.3.

No stable C^*-algebra can admit a bounded trace nor can it have a unital quotient. The converse does not hold in general (see Corollary 4.4), but it does hold for certain well-behaved C^*-algebras, cf. Proposition 2.7 below. Hjelmborg proved in [11] that Cuntz–Krieger algebras arising from infinite graphs are stable if and only if they admit no bounded trace and have no unital quotient.

For a particularly well-behaved class of finite C^*-algebras, absence of bounded traces is equivalent to stability (see Section 3); and absence of unital quotients is equivalent to stability for purely infinite C^*-algebras in the sense of [14] (see Section 5). A precursor to these results is given in Proposition 2.7 below.

Definition 2.4. (**Large subalgebras**) *A hereditary sub-C^*-algebra B of a C^*-algebra A is said to be* large in A *if for every positive element a in A and for every $\varepsilon > 0$ there is x in A such that $\|x^*x - a\| \le \varepsilon$ and xx^* belongs to B.*

Any large hereditary sub-C^*-algebra is necessarily full, i.e., not contained in any proper ideal.

Every C^*-algebra A is large in itself.

If B is a large hereditary sub-C^*-algebra of A, then for each a in $F(A)$ there is x in A such that $x^*x = a$ and xx^* belongs to B. Indeed, if e is a positive contraction in A such that $ea = ae = a$ then $z^*(e - 1/2)_+ z = a$ for some z in A (in the notation of Remark 2.1). Find x in A such that $\|x^*x - e\| < 1/2$ and xx^* belongs to B. By [13, Lemma 2.2] there is y in A such that $y^*x^*xy = (e - 1/2)_+$. Put $w = xyz$. Then ww^* belongs to B and $w^*w = a$.

The argument above also shows that for every projection p in A there is a projection q in B such that $p \sim q$ (whenever B is large in A).

Recall that a (possibly non-simple) C^*-algebra A is called *purely infinite* if for every pair of positive elements a, b in A such that b belongs to the closed two-sided ideal generated by a there is a sequence $\{x_n\}$ of elements in A with $x_n^* a x_n \to b$ (see [14]).

Lemma 2.5. *Every full, hereditary sub-C^*-algebra of a purely infinite C^*-algebra is large.*

Proof. Suppose that B be a full, hereditary sub-C^*-algebra of a purely infinite C^*-algebra A. Let a be a positive element in A and let $\varepsilon > 0$ be given. Then a belongs to the closed two-sided ideal generated by B, hence $(a - \varepsilon/3)_+$ belongs to the algebraic ideal generated by B, and hence $(a - 2\varepsilon/3)_+$ belongs to the algebraic ideal generated by some positive element b in B. Since A is purely infinite, $(a - \varepsilon)_+ = y^* b y$ for some y in A. This shows that $(a - \varepsilon)_+ = x^* x$ and $xx^* \in B$ when $x = b^{1/2} y$. Q.E.D.

Lemma 2.6. *Any full, stable, hereditary sub-C^*-algebra of a separable C^*-algebra is large.*

Proof. Let B be a full, stable, hereditary sub-C^*-algebra of a C^*-algebra A, let a be a positive element in A and let $\varepsilon > 0$ be given. Since $F(B)$ is dense in B^+ and since B is full in A, the algebraic ideal in A generated by $F(B)$ is dense in A. It follows that we can find b in $F(B)$ and $x_1, \dots, x_n$ in A such that

$$\|\sum_{j=1}^{n} x_j^* b x_j - a\| \le \varepsilon.$$

It follows from Theorem 2.2 that there are mutually orthogonal and mutually equivalent positive elements $b_1 = b, b_2, \dots, b_n$ in B. Find $u_1, \dots, u_n$ in B such that $u_j^* u_j = b$ and $u_j u_j^* = b_j$, so that $u_i^* u_j = 0$ when $i \neq j$. Put $x = \sum_{j=1}^{n} u_j x_j$. Then xx^* belongs to B and $x^* x = \sum_{j=1}^{n} x_j^* u_j^* u_j x_j = \sum_{j=1}^{n} x_j^* b x_j$. Q.E.D.

Proposition 2.7. **(Proposition 5.1 of** [12]) *Let A be a σ-unital C^*-algebra that has the property that any full, hereditary sub-C^*-algebra of A is large if it admits no non-zero bounded trace. Then A is stable if and only if A has no non-zero bounded trace and no non-trivial unital quotient.*

Section 4 contains an example of a non-stable σ_{p}-unital C^*-algebra A without bounded traces and unital quotients. Consequently, this C^*-algebra has a full, hereditary sub-C^*-algebra which is not large in A and which does not have a bounded trace.

The example below is due to Ken Dykema.

Example 2.8. The full free product $\mathcal{K} * \mathcal{K}$ is not stable; hence the class of stable C^*-algebras is not closed under forming free products.

Indeed, $\mathcal{K} * \mathcal{K}$ has a unital quotient. To see this, let $\{e_{ij}\}_{i,j=1}^{\infty}$ be the standard matrix units for $\mathcal{K}$. Observe that if D is a C^*-algebra and if $f_1, f_2, \dots$ is a sequence of mutually orthogonal and equivalent projections in D, then there is an embedding $\varphi\colon \mathcal{K} \to D$ such that $\varphi(e_{jj}) = f_j$. Take the Cuntz algebra $\mathcal{O}_2$ with its two canonical generators s_1 and s_2. Since every pair of non-zero projections in $\mathcal{O}_2$ are equivalent and any non-zero projection in $\mathcal{O}_2$ has countably many mutually orthogonal non-zero sub-projections, there are embeddings $\varphi_1, \varphi_2\colon \mathcal{K} \to \mathcal{O}_2$ such that

$\varphi_1(e_{11}) = s_1 s_1^*$ and $\varphi_2(e_{11}) = s_2 s_2^*$. By the universal property of free products there is a *-homomorphism $\varphi\colon \mathcal{K} * \mathcal{K} \to \mathcal{O}_2$ whose restriction to the first and the second copy of $\mathcal{K}$ is φ_1, respectively, φ_2. Accordingly, $1 = s_1 s_1^* + s_2 s_2^*$ belongs to the image of φ. Hence $\mathcal{K} * \mathcal{K}$ has a unital quotient.

§3. Stability of finite C^*-algebras

Blackadar proved in [1] that a (simple) AF-algebra is stable if and only if it admits no bounded trace. We shall in this section pursue generalizations of this result. Let us first remark that any unital, properly infinite C^*-algebra is traceless but not stable. One will therefore expect the two properties, being stable and being traceless, to be equivalent only for finite C^*-algebras; and even here the equivalence does not hold without qualifications.

As in [1] it is convenient to consider also a third property of a C^*-algebra that the scale of its K_0-group equals the entire positive cone. The positive cone and the scale of the K_0-group of a C^*-algebra A are given by

$$K_0(A)^+ = \{[p]_0 : p \in \mathcal{P}(A \otimes \mathcal{K})\}, \qquad \mathcal{D}_0(A) = \{[p]_0 : p \in \mathcal{P}(A)\}.$$

It follows from Lemma 2.6 that $\mathcal{D}_0(A) = K_0(A)^+$ for all stable C^*-algebras.

An axiomatic description of a scaled ordered Abelian group is given in the following:

Definition 3.1. *A triple (G, G^+, Σ) will be called a* scaled, ordered Abelian group *if (G, G^+) is an ordered Abelian group and Σ is an upper directed, hereditary, full subset of G^+, i.e.,*

(i) $\forall x_1, x_2 \in \Sigma\ \exists x \in \Sigma : x_1 \le x,\ x_2 \le x,$
(ii) $\forall x \in G^+\ \forall y \in \Sigma : x \le y \Rightarrow x \in \Sigma,$
(iii) $\forall x \in G^+\ \exists y \in \Sigma\ \exists k \in \mathbb{N} : x \le ky.$

A (σ_{p}-unital) C^*-algebra A is said to be *finite* if it contains no infinite projections, and A is *stably finite* if $M_n(A)$ is finite for every n. (A projection is infinite if it is Murray–von Neumann equivalent to a proper subprojection of itself.) If

$$\forall p, q \in \mathcal{P}(A \otimes \mathcal{K}) : [p]_0 = [q]_0 \text{ in } K_0(A) \implies p \sim q,$$

then A is said to have *cancellation.* We have

$$\mathrm{sr}(A) = 1 \implies A \text{ has cancellation } \implies A \text{ is stably finite},$$

for all C^*-algebras A.

Lemma 3.2. *Let A be a C^*-algebra with the cancellation property, let p be a projection in A, and let g be an element in $K_0(A)$.*

(i) *If $0 \le g \le [p]_0$, then there is a projection q in A such that $q \le p$ and $[q]_0 = g$.*

(ii) *If A is σ_{p}-unital, if $[p]_0 \le g$, and if g belongs to $\mathcal{D}_0(A)$, then there is a projection q in A such that $p \le q$ and $[q]_0 = g$.*

Proof. (i). Find projections e, f in matrix algebras over A such that $[e]_0 = g$ and $[f]_0 = [p]_0 - g$. Then $[e \oplus f]_0 = [p]_0$ and because A is assumed to have the cancellation property we conclude that $e \oplus f \sim p$. Find a rectangular matrix v over A such that $v^*v = e \oplus f$ and $vv^* = p$, and set $q = v(e \oplus 0)v^*$. Then q belongs to A, $q \le p$, and $[q]_0 = g$.

(ii). There is an approximate unit $\{p_n\}_{n=1}^{\infty}$ for A where each p_n is a projection dominating p. Now, $[p_n]_0 \ge g$ for some n. Indeed, take a projection q' in A such that $g = [q']_0$ and choose n such that $\|(1 - p_n)q'\| < 1$. Then $q' \precsim p_n$, and so $[p_n]_0 \ge [q']_0 = g$. Use (i) to find a projection e in A such that $e \le p_n - p$ and $[e]_0 = g - [p]_0$. The projection $q = p + e$ will then be as desired. Q.E.D.

Lemma 3.3. *The triple $\big(K_0(A), K_0(A)^+, \mathcal{D}_0(A)\big)$ is a scaled, ordered, Abelian group if A is a σ_{p}-unital C^*-algebra with the cancellation property.*

Conversely, if A is a stable, σ_{p}-unital C^*-algebra with the cancellation property, and if Σ is a subset of $K_0(A)^+$ for which the triple $\big(K_0(A), K_0(A)^+, \Sigma\big)$ is a scaled, ordered, Abelian group, then there is a full, σ_{p}-unital, hereditary sub-C^*-algebra B of A such that

$$\big(K_0(B), K_0(B)^+, \mathcal{D}_0(B)\big) \cong \big(K_0(A), K_0(A)^+, \Sigma\big).$$

Proof. Assume that A is a σ_{p}-unital C^*-algebra with the cancellation property. Let $\{p_n\}_{n=1}^{\infty}$ be an approximate unit for A consisting of projections. (i) in Definition 3.1 holds as we can take x to be $[p_n]_0$ for some large enough n. (ii) follows from Lemma 3.2 (i). If q is a projection in $M_k(A)$, then q is equivalent to a projection in $M_k(p_nAp_n)$ for some large enough n whence $[q]_0 \le k[p_n]_0$. Hence (iii) in Definition 3.1 holds.

To prove the second part of the lemma, use (i), (ii), and (iii) in Definition 3.1 to find $0 \le x_1 \le x_2 \le x_3 \le \ldots$ in Σ such that for every g in $K_0(A)^+$ the following two conditions are satisfied:

- $g \le kx_n$ for some positive integers k and n, and
- g belongs to Σ if and only if $g \le x_n$ for some n.

Use Lemma 3.2 (ii) to find an increasing sequence $\{q_n\}_{n=1}^{\infty}$ of projections in A such that $[q_n]_0 = x_n$. Let B be the closure of $\bigcup_{n=1}^{\infty} q_nAq_n$. Then B is a full σ_{p}-unital sub-C^*-algebra of A. By construction of B, if g

is an element in $K_0(A)$, then g belongs to Σ if and only if there is a projection e in B such that $g = [e]_0$. It follows that the isomorphism $K_0(B) \to K_0(A)$ induced by the inclusion mapping $B \hookrightarrow A$ maps $\mathcal{D}_0(B)$ onto Σ. Q.E.D.

An ordered Abelian group (G, G^+) is said to be *weakly unperforated* if $ng > 0$ implies $g > 0$ for every g in G and for every positive integer n. (Other texts have assigned other meanings to the term weak unperforation.)

Proposition 3.4. *Let A be a σ_{p}-unital C^*-algebra with the cancellation property, and consider the following three conditions:*

(i) *A is stable,*
(ii) *$\mathcal{D}_0(A) = K_0(A)^+$,*
(iii) *A admits no bounded trace.*

Then

$$\text{(i)} \iff \text{(ii)} \implies \text{(iii)},$$

and (iii) $\Rightarrow$ (ii) *if A is exact, $K_0(A)$ is weakly unperforated, and every ideal in A is σ_{p}-unital.*

Proof. The implication (i) $\Rightarrow$ (ii) holds for all C^*-algebras (as noted above). The assumption that A is σ_{p}-unital implies that every non-zero, densely defined trace τ on A induces a non-zero state $\widehat{\tau}$ on $K_0(A)$, and

$$\|\tau\| \geq \sup\{\widehat{\tau}(g) : g \in \mathcal{D}_0(A)\} = \sup\{\widehat{\tau}(g) : g \in K_0(A)^+\} = \infty,$$

when (ii) holds. Therefore (ii) $\Rightarrow$ (iii).

(ii) $\Rightarrow$ (i): Assume that (ii) holds. Let p be a projection in A. Then $2[p]_0$ belongs to $\mathcal{D}_0(A)$, and so it follows from Lemma 3.2 (ii) that there is a projection q in A with $p \leq q$ and $[q]_0 = 2[p]_0$. Using again that A has the cancellation property we find that $q - p \sim p$. It now follows from Theorem 2.2 that A is stable.

(iii) $\Rightarrow$ (ii): Assume next that $K_0(A)$ is weakly unperforated, each ideal in A is σ_{p}-unital, A is exact, and that (iii) holds. Take g in $K_0(A)^+$ and find a projection p in $A \otimes \mathcal{K}$ such that $g = [p]_0$. Let $I \otimes \mathcal{K}$ be the closed two-sided ideal in $A \otimes \mathcal{K}$ generated by p, and take an increasing approximate unit $\{p_n\}_{n=1}^{\infty}$ of projections for I. Let T be the compact set of traces τ on I such that $\tau(p) = 1$. Then

$$\sup_{n \in \mathbb{N}} \tau(p_n) = \infty$$

for every τ in T (otherwise τ would extend to a bounded trace on I and in turns to a bounded trace on A).

Each projection q in I (or in $I \otimes \mathcal{K}$) defines a continuous affine function $\widehat{q}\colon T \to \mathbb{R}$, and $\{\widehat{p}_n\}$ is an increasing sequence of functions tending pointwise to infinity. Since T is compact we have $\widehat{p}_n > 1$ for some n. In other words, $\tau(p) < \tau(p_n)$ for all τ in T. We infer that $f([p]_0) < f([p_n]_0)$ for all states f on $(K_0(I), K_0(I)^+)$ with $f([p]_0) = 1$. Indeed, each such state f lifts to a quasitrace τ on I (by [3]) and each quasitrace on an exact C^*-algebra is a trace (by Haagerup's theorem in [10]). By Goodearl–Handelman's extension theorem (see [9]), $k[p]_0 < k[p_n]_0$ in $K_0(I)$ (and hence in $K_0(A)$) for some natural number k. Since $K_0(A)$ is weakly unperforated we can conclude that $[p]_0 < [p_n]_0$. This entails that $g = [p]_0$ belongs to $\mathcal{D}_0(A)$ using Lemma 3.3 and Definition 3.1 (ii). Q.E.D.

The three conditions of Proposition 3.4 are equivalent for all separable, exact, real rank zero C^*-algebras with the cancellation property and with weakly unperforated K_0-group. This is a lot to ask for, but many commonly encountered C^*-algebras satisfy these properties. For example, all AF-algebras, and more generally, all AH-algebras of real rank zero and of slow dimension growth have these properties (see [8] and [2]).

Stability of a finite C^*-algebra can also be expressed in terms of properties of its multiplier algebra as in the proposition below from [17]. Recall that a unital C^*-algebra is *properly infinite* if it contains two mutually orthogonal projections p, q such that $1 \sim p \sim q$.

Proposition 3.5. *Let A be a C^*-algebra and let $\mathcal{M}(A)$ denote its multiplier algebra.*

(i) *If A is stable, then $\mathcal{M}(A)$ is properly infinite.*
(ii) *If A is σ-unital,* $\mathrm{sr}(A) = 1$, *and A is not stable, then $\mathcal{M}(A)$ is not properly infinite.*
(iii) *If A is σ-unital, simple,* $\mathrm{sr}(A) = 1$, *and A is not stable, then $\mathcal{M}(A)$ is finite.*

Part (i) is standard and follows from the fact that $\mathcal{M}(A) \otimes \mathcal{M}(\mathcal{K})$ (maximal tensor product) maps into $\mathcal{M}(A \otimes \mathcal{K})$. Parts (i) and (ii) say that for σ-unital C^*-algebras A of stable rank one, A is stable if and only if $\mathcal{M}(A)$ is properly infinite.

If A is a unital, properly infinite C^*-algebra, then $\mathcal{M}(A) = A$, and hence $\mathcal{M}(A)$ is properly infinite. On the other hand, A is not stable. We can therefore not in general deduce that A is stable knowing that $\mathcal{M}(A)$ is properly infinite.

§4. Stability is not a stable property

One often refers to a property of C^*-algebras as being stable if it is preserved by passing from A to $M_n(A)$ and vice versa for each n. Being stable is not a stable property in this sense, as shown by the author in [16] using techniques of Villadsen from [20].

We first state a result that limits how exotic this behavior can be:

Proposition 4.1. (**Proposition 2.1 of** [16]) *Let A be a σ-unital C^*-algebra. If $M_n(A)$ is stable for some integer n, then $M_k(A)$ is stable for all $k \geq n$.*

The proof uses Theorem 2.2.

Let us indicate at the level of scaled, ordered Abelian groups why there should exists a non-stable C^*-algebra A such that $M_2(A)$ is stable:

Example 4.2. (**Example 3.4 of** [16]) Let $\mathbb{Z}_2$ denote the group $\mathbb{Z}/2\mathbb{Z}$, and let $\mathbb{Z}_2^{(\infty)}$ denote the group of all sequences $t = (t_j)_{j=1}^{\infty}$, with $t_j \in \mathbb{Z}_2$ and where $t_j \neq 0$ for at most finitely many j. For each $t \in \mathbb{Z}_2^{(\infty)}$ let $d(t)$ be the number of elements in the set $\{j \in \mathbb{N} \mid t_j \neq 0\}$. Set

$$G = \mathbb{Z} \oplus \mathbb{Z}_2^{(\infty)}, \qquad G^+ = \{(k,t) \mid d(t) \leq k\}, \qquad \Sigma = \{(k,t) \mid d(t) = k\}.$$

Then (G, G^+, Σ) is a scaled, ordered Abelian group, cf. Definition 3.1. To see this, let $e_j \in \mathbb{Z}_2^{(\infty)}$ be the generator of the jth copy of $\mathbb{Z}_2$ and set $g_j = (1, e_j) \in G^+$. Then

$$\Sigma = \bigcup_{j=1}^{\infty} \{x \in G^+ \mid x \leq g_1 + g_2 + \cdots + g_j\},$$

and in this picture it is easy to see that Σ satisfies the axioms of Definition 3.1.

The element $(2, e_1)$ belongs to G^+ but not to Σ, and so $\Sigma \neq G^+$.

If A is a C^*-algebra whose scaled ordered K_0-group is isomorphic to (G, G^+, Σ), then the scaled ordered K_0-group of $M_2(A)$ is isomorphic to $(G, G^+, \Sigma \hat{+} \Sigma)$, where $\Sigma \hat{+} \Sigma$ is the set of elements x in G^+ for which there exist y_1, y_2 in Σ such that $x \leq y_1 + y_2$. In the given example, $\Sigma \hat{+} \Sigma = G^+$, because if $g = (k, t)$ belongs to G^+, then

$$g \leq g + g = (2k, 0) = 2(g_1 + g_2 + \cdots + g_k).$$

If we can find a σ_{p}-unital C^*-algebra A with the cancellation property such that the scaled ordered K_0-group of A is isomorphic to (G, G^+, Σ), then A will be non-stable and $M_2(A)$ will be stable by Proposition 3.4. The C^*-algebra found in Theorem 4.3 below, corresponding to

$n = 2$, has the property that a *subgroup* of its K_0-group is isomorphic to (G, G^+, Σ).

For the formulation of the next result, recall that an AH-algebra is a C^*-algebra that is the inductive limit of a sequence of C^*-algebras of the form $p(C(X) \otimes \mathcal{K})p$, where X is a (not necessarily connected) compact Hausdorff space and p is a projection in $C(X) \otimes \mathcal{K}$.

Theorem 4.3. (**Theorem 5.3 and Corollary 4.2 of** [16])

(i) *For each natural number n there is a simple, separable, σ_{p}-unital AH-algebra A of stable rank one such that $M_n(A)$ is stable but $M_{n-1}(A)$ is not stable.*

(ii) *For each natural number n there is continuous field C^*-algebra $A = (A_x)_{x \in X}$, where X is a compact Hausdorff space and where each fiber A_x is isomorphic to $\mathcal{K}$, such that $M_{n-1}(A)$ is not stable and $M_n(A)$ is stable.*

We indicate here the proof of part (ii) in the case where $n = 2$. As mentioned above, the proof follows ideas of Villadsen.

Let $Y = \mathbb{RP}^2$ be the real projective plane and recall that its cohomology (over $\mathbb{Z}$) is given as:

$$H^0(Y;\mathbb{Z}) \cong \mathbb{Z}, \qquad H^1(Y;\mathbb{Z}) = 0, \qquad H^2(Y;\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}.$$

We have

$$C(Y) = \{f \in C(\mathbb{D}) : f(z) = f(-z) \text{ for all } z \in \mathbb{T}\}.$$

Let ξ_0 be a complex line bundle over Y with non-trivial Euler class $\mathrm{e}(\xi_0)$ in $H^2(Y;\mathbb{Z})$. This line bundle corresponds to the projection p in $M_2(C(Y))$ given by

$$p(re^{it}) = \begin{pmatrix} r & e^{it}\sqrt{r(1-r)} \\ e^{-it}\sqrt{r(1-r)} & 1-r \end{pmatrix}, \qquad r \in [0,1],\ t \in [0, 2\pi].$$

Also, $\xi_0 \oplus \xi_0 \cong \theta_2$, the trivial 2-dimensional complex bundle over Y.

Put $X = \prod_{n=1}^{\infty} Y$ and let $\pi_n \colon X \to Y$ be the coordinate map onto the nth copy of Y. Put $\xi_n = \pi_n^*(\xi_0)$, so that each ξ_n is a complex line bundle over X. We have $\xi_n \oplus \xi_n = \pi_n^*(\xi_0 \oplus \xi_0) \cong \theta_2$ for every n. An application of Künneth's theorem shows that $\xi_1 \oplus \xi_2 \oplus \cdots \oplus \xi_n$ has non-trivial Euler class for every n. It follows that there for no n is a complex bundle η such that $\xi_1 \oplus \eta \cong \xi_2 \oplus \cdots \oplus \xi_n$ since that would entail

$$\theta_2 \oplus \eta \cong \xi_1 \oplus \xi_1 \oplus \eta \cong \xi_1 \oplus \xi_2 \oplus \cdots \oplus \xi_n.$$

This cannot be because $\theta_2 \oplus \eta$ has trivial Euler class, whereas $\xi_1 \oplus \xi_2 \oplus \cdots \oplus \xi_n$ was constructed to have non-trivial Euler class.

Choose mutually orthogonal projections $p_1, p_2, \dots$ in $C(X) \otimes \mathcal{K}$ such that p_n corresponds to the line bundle ξ_n, and let e be a projection in $C(X) \otimes \mathcal{K}$ corresponding to the trivial bundle θ_1. Then

(a) $p_n \oplus p_n \sim e \oplus e$ for all n, and
(b) p_1 is not equivalent to a sub-projection of $p_2 + p_3 + \cdots + p_n$ for any n.

Put $q_n = p_1 + \cdots + p_n$ and set

$$A = \overline{\bigcup_{n=1}^{\infty} q_n (C(X) \otimes \mathcal{K}) q_n}.$$

With $\rho_x \colon A \to \mathcal{K}$ the restriction to A of the evaluation mapping $C(X) \otimes \mathcal{K} \to \mathcal{K}$ at x, A gets the structure of a continuous field C^*-algebra with base space X and with each fiber isomorphic to $\mathcal{K}$.

By (b) above, there is no projection q in A such that $q \sim p_1$ and $q \perp p_1$, and it follows from (a) that $M_2(A)$ is stable. □

We can now conclude that there are non-stable C^*-algebras that do not have bounded traces or unital quotients:

Corollary 4.4. *There is a non-stable, non-unital, separable, nuclear, simple, σ_{p}-unital C^*-algebra A that admits no bounded traces.*

Proof. Take A as in Theorem 4.3 (i) corresponding to $n = 2$. Then A is non-stable, separable, nuclear, simple and σ_{p}-unital. Since $M_2(A)$ is stable, A is not unital, nor can it have a bounded trace. Q.E.D.

The corollary below (or a modification of it) was in [18] used to construct a simple, unital, finite C^*-algebra B such that $M_2(B)$ is infinite. Cuntz has shown that every infinite simple C^*-algebra is properly infinite, so $M_2(B)$ is necessarily properly infinite. A non-simple unital, finite C^*-algebra A such that $M_2(A)$ is infinite has been known to exist for a long time (see [6]), but in this (and related) examples, $M_2(A)$ is not properly infinite.

Corollary 4.5. *For each natural number n there is a unital C^*-algebra B such that $M_n(B)$ is properly infinite, but $M_k(B)$ is finite for $k < n$.*

Proof. Take A to be the C^*-algebra constructed in Theorem 4.3 (i). Let $B = \mathcal{M}(A)$ be the multiplier algebra of A. Then $M_k(B) \cong \mathcal{M}(M_k(A))$. We can now apply Proposition 3.5 to conclude that B is as desired. Q.E.D.

The C^*-algebra B constructed in Corollary 4.5 is not separable, not simple, and not nuclear. It is easy to make B separable: Take two isometries s_1, s_2 in $M_n(B)$ such that $s_1 s_1^* \perp s_2 s_2^*$. Let $s_k(i,j) \in B$ be the matrix entries for s_k, $k = 1, 2$, and let B_0 be the separable sub-C^*-algebra of B generated by the $2n^2$ elements $s_k(i,j)$. Then s_1, s_2 belong to $M_n(B_0)$, and this makes $M_n(B_0)$ properly infinite. Being a sub-C^*-algebra of the finite C^*-algebra $M_k(B)$, $M_k(B_0)$ is finite when $k < n$.

We can rephrase Corollary 4.5 as follows: There is a unital, properly infinite C^*-algebra A such that $(1-e)A(1-e)$ is finite for some projection $e \neq 1$ in A, and e can be chosen to have size $1/n$. The next corollary says that the example can be sharpened in that e can be chosen to have infinitesimal size.

Corollary 4.6. *There is a properly infinite, unital C^*-algebra A and an embedding $\varphi\colon \mathcal{K} \to A$ such that for every non-zero projection e in $\mathcal{K}$, the corner C^*-algebra $(1-\varphi(e))A(1-\varphi(e))$ is finite.*

Proof. By Corollary 4.5 there is for each natural number n a unital C^*-algebra B_n such that $M_n(B_n)$ is properly infinite and $M_{n-1}(B_n)$ is finite. Put

$$A = \prod_{n=1}^{\infty} M_n(B_n) / \sum_{n=1}^{\infty} M_n(B_n),$$

where $\prod_{n=1}^{\infty} M_n(B_n)$ is the C^*-algebra of all bounded sequences $\{x_n\}_{n=1}^{\infty}$ such that $x_n \in M_n(B_n)$, and $\sum_{n=1}^{\infty} M_n(B_n)$ is the ideal of those $\{x_n\}_{n=1}^{\infty}$ for which $\|x_n\| \to 0$. Let $\pi\colon \prod_{n=1}^{\infty} M_n(B_n) \to A$ denote the quotient mapping.

Since each $M_n(B_n)$ is properly infinite, $\prod_{n=1}^{\infty} M_n(B_n)$ and hence A are properly infinite.

Let $\{e_{ij}\}_{i,j=1}^{\infty}$ be a set of matrix units for the compact operators $\mathcal{K}$. For n in $\mathbb{N}$ and for $1 \le i, j \le n$, let $g_{ij}^{(n)} \in M_n(\mathbb{C}) \subseteq M_n(B_n)$ be the (i,j)th standard matrix unit (wrt. the natural embedding of $M_n(\mathbb{C})$ into $M_n(B_n)$ defined by the unit of B_n). Set $g_{ij}^{(n)} = 0$ if i or j is greater than n. Put

$$g_{ij} = (g_{ij}^{(1)}, g_{ij}^{(2)}, g_{ij}^{(3)}, \dots), \qquad f_{ij} = \pi(g_{ij}).$$

Then $\{f_{ij}\}_{i,j=1}^{\infty}$ are matrix units for $\mathcal{K}$, and so there is a *-homomorphism $\varphi\colon \mathcal{K} \to A$ given by $\varphi(e_{ij}) = f_{ij}$. We proceed to check that $(1-\varphi(e))A(1-\varphi(e))$ is finite for all non-zero projections e in $\mathcal{K}$. It suffices to consider the case $e = e_{11}$.

Suppose, to reach a contradiction, that $(1-\varphi(e_{11}))A(1-\varphi(e_{11}))$ is infinite and take a non-unitary isometry s in that algebra. Lift s to an element $x = (x_1, x_2, \dots)$ in $\prod_{n=1}^{\infty} M_n(B_n)$. Upon replacing each x_n by $(1-g_{11}^{(n)})x_n(1-g_{11}^{(n)})$ we may assume that each $x_n = (1-g_{11}^{(n)})x_n(1-g_{11}^{(n)})$. Since

$$\pi(1 - g_{11}^{(1)}, 1 - g_{11}^{(2)}, \dots) = 1 - \varphi(e_{11}) = \pi(x_1^* x_1, x_2^* x_2, \dots),$$

we conclude that $\|x_n^* x_n - (1 - g_{11}^{(n)})\| \to 0$, and so $x_n^* x_n$ is invertible (in the corner algebra $(1 - g_{11}^{(n)})M_n(B_n)(1 - g_{11}^{(n)})$) for all sufficiently large n. As $(1 - g_{11}^{(n)})M_n(B_n)(1 - g_{11}^{(n)}) \cong M_{n-1}(B_n)$ and this C^*-algebra is finite, we can further conclude that x_n is invertible for all large enough n. But then s is invertible, a contradiction. Q.E.D.

By an argument similar to the one outlined below Corollary 4.5, the C^*-algebra A in Corollary 4.6 can be taken to be separable. One cannot take A to be simple: any simple, unital C^*-algebra that admits an embedding of $\mathcal{K}$ is properly infinite (cf. [7]); and there are embeddings

$$\mathcal{K} \hookrightarrow (1 - e)\mathcal{K}(1 - e) \hookrightarrow (1 - \varphi(e))A(1 - \varphi(e)).$$

§5. Stability of infinite C^*-algebras

A (simple or non-simple) C^*-algebra A is said to be *purely infinite* if it has no Abelian quotient and if for every pair of positive elements a, b in A, such that b belongs to the closed two-sided ideal generated by a, there is a sequence $\{x_n\}$ of elements in A with $x_n^* a x_n \to b$ (see [14]). This notion was introduced by Cuntz for simple C^*-algebras, and he defined, in agreement with the definition above, a *simple* C^*-algebra to be purely infinite if each of its non-zero hereditary sub-C^*-algebras contain an infinite projection.

There are nice characterizations of stability for purely infinite C^*-algebras, and conversely, one can characterize pure infiniteness in terms of stability.

We look first at the case of simple C^*-algebras. Here we have the following classical result of of S. Zhang from [21] (that also can be derived from Theorem 2.2 using that every purely infinite, simple, σ-unital C^*-algebra has an (increasing) approximate unit consisting of projections, and that for any pair of non-zero projections p, q in such a C^*-algebra one has $p \precsim q$):

Proposition 5.1. (**Zhang's Dichotomy**) *A σ-unital, purely infinite, simple C^*-algebra is either unital or stable.*

The result below is an observation of Kirchberg and it is a special case of Proposition 5.4 below for which we include a proof.

Proposition 5.2. *A simple C^*-algebra A is purely infinite if and only if every non-zero hereditary sub-C^*-algebra of A contains a (non-zero) stable sub-C^*-algebra.*

Purely infinite C^*-algebras (simple and non-simple alike) have no traces. The proposition below, proved in [14, Theorem 4.24] and which is an easy consequence of Proposition 2.7, extends Zhang's Dichotomy. There are (non-simple) purely infinite C^*-algebras that are neither stable nor unital. Take for example $C_0(\mathbb{R}) \otimes \mathcal{O}_2$.

Proposition 5.3. *A (possibly non-simple) purely infinite, σ-unital C^*-algebra is stable if and only if it has no unital quotients.*

George Elliott suggested that the following result holds:

Proposition 5.4. *Let A be a (possibly non-simple) separable C^*-algebra A. Then the following three conditions are equivalent:*

(i) *A is purely infinite,*
(ii) *every non-zero hereditary sub-C^*-algebra of A contains a full, stable, hereditary sub-C^*-algebra,*
(iii) *every non-zero hereditary sub-C^*-algebra of A contains a full, stable (not necessarily hereditary) sub-C^*-algebra.*

Proof. (i) $\Rightarrow$ (ii): Let B be a non-zero hereditary sub-C^*-algebra of A. Take a countable dense subset X of the unit ball of B^+ and put

$$Y = \{(b - 1/n)_+ : b \in X, \ \ n \in \mathbb{N}\},$$

cf. Remark 2.1. Let $Y = \{b_1, b_2, \dots\}$ be an enumeration of Y. We proceed to find mutually orthogonal positive elements $c_1, c_2, \dots$ in B such that $c_j \sim b_j$ (cf. Remark 2.1) and $B \cap \{c_1, \dots, c_n\}^\perp$ is full in B for every n. The set $\{c_n, c_{n+1}, \dots\}$ will then be full in B for every natural number n. We construct the sequence $\{c_n\}_{n=1}^\infty$ by induction and to do so it suffices to justify the first step, i.e., to find c_1.

By construction, $b_1 = (b - \varepsilon)_+$ for some $\varepsilon > 0$ and some positive contraction b in B. The element b is properly infinite because A is purely infinite and we can therefore find x, y in $\overline{bAb}$ with

$$x^*x = y^*y = (b - \varepsilon/2)_+, \qquad xx^* \perp yy^*,$$

(see [14, Lemma 3.2]). Let $x = u|x|$ be the polar decomposition for x as in Remark 2.1. There is a positive contraction f in the hereditary sub-C^*-algebra generated by $x^*x = (b-\varepsilon/2)_+$ such that $fb_1 = b_1 f = b_1$. Put $c_1 = ub_1u^*$, put $e = ufu^*$, and let I be the closed two-sided ideal in B generated by $B \cap \{c_1\}^\perp$. Then $c_1 \sim b_1$, cf. Remark 2.1, and it remains to show that $I = B$. Because yy^* belongs to $B \cap \{c_1\}^\perp$ we conclude that $y^*y = (b - \varepsilon/2)_+$ belongs to I. It follows that f and hence e belong to I. By construction, $ec_1 = c_1 e = c_1$ and so $(1-e)a(1-e)$ belongs to $B \cap \{c_1\}^\perp$ for all a in B. Now, each element a in B belongs to the ideal generated by $\{eaa^*e, (1-e)aa^*(1-e)\}$ and hence to I. This proves that $I = B$.

Let D and D_n be hereditary sub-C^*-algebras of B generated by $c_1, c_2, \dots$, respectively, by $c_1, \dots, c_n$. Then $D_1 \subseteq D_2 \subseteq \cdots$ and $D = \overline{\bigcup_{n=1}^\infty D_n}$. Since D contains $c_1, c_2, \dots$, the closed two-sided ideal of B generated by D contains $b_1, b_2, \dots$, and this set generates B. Therefore D is full in B. We must also show that D is stable. This follows by an application of Theorem 2.2, but it can be seen more easily by first noting that D is purely infinite, being a hereditary sub-C^*-algebra of A, and D has no unital quotient. Indeed, assume that J is a proper ideal in D and that D/J is unital. The unit of D/J will then belong to $D_n/(J \cap D_n)$ for some sufficiently large n. In that case c_k belongs to J for all $k > n$; but $c_{n+1}, c_{n+2}, \dots$ is full in D (by construction of b_n and c_n), and hence $J = D$, a contradiction. Proposition 5.3 now yields that D is stable.

(ii) $\Rightarrow$ (iii) is trivial.

(iii) $\Rightarrow$ (i): Suppose that (iii) holds. Take a positive element a in A and find a full, stable sub-C^*-algebra D of $\overline{aAa}$. Let $\varepsilon > 0$ be given. Being separable and stable, D contains a sequence of mutually orthogonal and equivalent elements $c_1, c_2, \dots$ so that a belongs to the ideal generated c_1. (To see this, write $D = D_0 \otimes \mathcal{K}$, take a strictly positive element c in D_0 and put $c_j = c \otimes e_{jj}$.) Let u_j, $j \geq 2$, be partial isometries in A^{**} implementing the equivalence between c_1 and c_j so that $u_j^* c_j u_j = c_1$, cf. Remark 2.1, and such that $u_i u_i^* \perp c_j$ when $i \neq j$. Find n and elements $x_1, \dots, x_n$ in D such that $(a - \varepsilon)_+ = \sum_{j=1}^n x_j^* c_1 x_j$. Put

$$e = \sum_{j=1}^{n} c_j, \qquad f = \sum_{j=n+1}^{2n} c_j, \qquad x = \sum_{j=1}^{n} u_j x_j, \qquad y = \sum_{j=1}^{n} u_{n+j} x_j.$$

Then e and f are mutually orthogonal positive elements in $\overline{aAa}$ and $x^*ex = y^*fy = (a-\varepsilon)_+$. This shows that a is properly infinite, cf. [14, Proposition 3.3], and since a was arbitrary we conclude that A is purely infinite. Q.E.D.

§6. Extensions of stable C^*-algebras

Extensions of two stable C^*-algebras need not be stable as the following theorem, proved recently in [19], shows:

Theorem 6.1. *There is an extension*

$$0 \longrightarrow C(Z) \otimes \mathcal{K} \longrightarrow A \longrightarrow \mathcal{K} \longrightarrow 0$$

of C^*-algebras, where $Z = \prod_{n=1}^{\infty} S^2$, such that A is non-stable. Moreover, A can be chosen to be σ_{p}-unital.

The proof of Theorem 6.1 is somewhat similar to the proof of Theorem 4.3. Some special cases of the extension problem for stable C^*-algebras remain open:

Question 6.2. *Given a split-exact sequence of (separable) C^*-algebras*

$$0 \longrightarrow J \longrightarrow A \underset{\lambda}{\overset{\pi}{\rightleftarrows}} B \longrightarrow 0$$

Does it follow that A is stable if I and B are known to be stable?

Question 6.3. *Given two stable closed two-sided ideals I and J in a (separable) C^*-algebra A. Does it follow that their sum $I + J$ is stable?*

If I and J are stable ideals in a C^*-algebra A, then $I + J$ is an extension of two stable ideals:

$$0 \longrightarrow I \longrightarrow I + J \longrightarrow (I + J)/I \longrightarrow 0.$$

(Note that $(I + J)/I \cong J/(I \cap J)$ is stable being (isomorphic) to a quotient of the stable C^*-algebra J.)

Given a partially ordered set $(P, \leq)$. An element x in P is called *maximal* if $x \leq y$ implies $x = y$ for all y in X. An element x is called a *greatest element* if $y \leq x$ for every y in X. A greatest element is also a maximal element (but not conversely); a partially ordered set can have at most one greatest element, but it can have several maximal elements.

Proposition 6.4. *Every separable C^*-algebra has a maximal stable ideal (i.e., a stable ideal not properly contained in any other stable ideal).*

Proof. Use Zorn's Lemma to choose a maximal totally ordered family $\{I_i\}_{i \in \mathbb{I}}$ of stable ideals in A (counting 0 as a stable ideal) and set $I = \overline{\bigcup_{i \in \mathbb{I}} I_i}$. Then I is an ideal in A and I is not properly contained in any stable ideal in A by maximality of the set $\{I_i\}_{i \in \mathbb{I}}$. It follows from Corollary 2.3 (i) that I is stable. Q.E.D.

Question 6.5. *Does every (separable) C^*-algebra A have a greatest stable ideal (i.e., a stable ideal that contains all other stable ideals)?*

It can be shown that the canonical ideal $C(Z) \otimes \mathcal{K}$ is a greatest stable ideal in the C^*-algebra A from Theorem 6.1. Notice that the quotient by this ideal is stable. Hence the quotient of a separable C^*-algebra by its greatest stable ideal (whenever it exists) can have stable ideals.

It follows from Proposition 6.4 (and its proof) that any stable ideal of a separable C^*-algebra is contained in a maximal stable ideal. We can therefore rephrase Question 6.5 as follows: Does every (separable) C^*-algebra have a *unique* maximal stable ideal?

For separable C^*-algebras, Question 6.5 is equivalent to Question 6.3. It is trivial that Question 6.3 will have affirmative answer if Question 6.5 has affirmative answer. To see the converse direction, let A be a separable C^*-algebra, and let $\{I_i\}_{i \in \mathbb{I}}$ be the collection of all stable ideals in A (including 0). If Question 6.3 has affirmative answer, then $I_{i_1} + I_{i_2}$ belongs to this collection for all $i_1, i_2 \in \mathbb{I}$. It follows that $I = \overline{\bigcup_{i \in \mathbb{I}} I_i}$ is an ideal in A, and every stable ideal in A is contained in I. Corollary 2.3 (i) shows that I is stable.

Consider the continuous field C^*-algebra $A = (A_x)_{x \in X}$ constructed in Theorem 4.3 (ii). Each open subset U of X defines an ideal $A_U = (A_x)_{x \in U}$ of A consisting of those section $a = (a_x)$ in A such that $a_x = 0$ whenever $x \notin U$; and every ideal in A is of this form. In the given case, each fiber A_x is isomorphic to $\mathcal{K}$ and hence is stable, but no ideal A_U is stable — roughly because each non-empty open subset U of X contains an open cylinder set:

$$V_1 \times V_2 \times \cdots \times V_n \times Y \times Y \times \cdots \subseteq U, \qquad V_j \subseteq Y.$$

We give in Propositions 6.8 and 6.12 below a partial positive answer to Question 6.2.

Lemma 6.6. *Let A be a C^*-algebra and let I be a closed two-sided ideal in A. If I and A/I have no (non-trivial) unital quotients, then neither has A.*

Proof. Suppose, to reach a contradiction, that J is a proper closed two-sided ideal in A such that A/J is unital. Then $A/(I+J)$ is a unital quotient of A/I and therefore $I + J = A$. Hence

$$\frac{I}{I \cap J} \cong \frac{I+J}{J} = \frac{A}{J},$$

so that $I/(I \cap J)$ is unital. This entails that $I \cap J = I$. It follows that $I \subseteq J$ and consequently $J = A$, a contradiction. Q.E.D.

Lemma 6.7. *Let A be a C^*-algebra, let I be a closed two-sided ideal in A, and assume that neither I nor A/I have (non-trivial) unital quotients. Then for each a in A, the C^*-algebra $\overline{(1-a)I(1-a^*)}$ is full in I and has no (non-trivial) unital quotients.*

Proof. Let $\widetilde{A}$ denote the unitization of A. Let J be the closed two-sided ideal in $\widetilde{A}$ generated by $1-a$, let J_0 be the closed two-sided ideal in A generated by $\overline{(1-a)A(1-a^*)}$, and let I_0 be the closed two-sided ideal in I generated by $\overline{(1-a)I(1-a^*)}$. Then $J_0 = J \cap A$ and $I_0 = J \cap I = J_0 \cap I$. Let $\pi\colon \widetilde{A} \to \widetilde{A}/J$ be the quotient mapping. Then $\pi(a) = \pi(1)$, and so $\pi(A)$ is unital. The kernel of the restriction of π to A is equal to J_0. Hence A/J_0 is unital. By Lemma 6.6 and the assumption that I and A/I have no unital quotients we conclude that $J_0 = A$. It follows that $I_0 = I$ so that $\overline{(1-a)I(1-a^*)}$ is full in I.

Assume next, to reach a contradiction, that L_0 is a proper ideal in $\overline{(1-a)I(1-a^*)}$ such that $\overline{(1-a)I(1-a^*)}/L_0$ is unital. Let L be the closed two-sided ideal in I generated by L_0 so that

$$L_0 = \overline{(1-a)I(1-a^*)} \cap L.$$

Let $\pi\colon A \to A/L$ be the quotient mapping. Find e in $\overline{(1-a)I(1-a^*)}$ such that $\pi(e)$ is the unit for $\overline{(1-a)I(1-a^*)}/L_0$, and put $y = e+a-ea$. Then y belongs to A and

$$(1-y)I(1-y^*) = (1-e)(1-a)I(1-a^*)(1-e^*) \subseteq L,$$

contradicting the first part of the lemma saying that $\overline{(1-y)I(1-y^*)}$ is full in I. Q.E.D.

Proposition 6.8. *Let I be a stable, closed, two-sided ideal in a separable C^*-algebra A, and suppose that A/I is stable. Then the following three conditions are equivalent:*

(i) *A is stable,*
(ii) *for each positive contraction a in A, the hereditary sub-C^*-algebra $\overline{(1-a)I(1-a)}$ is large in I (cf. Definition 2.4),*
(iii) *$\overline{(1-a)I(1-a)}$ is stable for each positive contraction a in A.*

Proof. (i) $\Rightarrow$ (iii). If A is stable, then so is $\overline{(1-a)A(1-a)}$ by Corollary 2.3 (iii). Hence $\overline{(1-a)I(1-a)}$ is stable by Corollary 2.3 (ii) being an ideal in a stable C^*-algebra.

(iii) $\Rightarrow$ (ii) follows from Lemmas 2.6 and 6.7.

(ii) $\Rightarrow$ (i). Suppose that (ii) holds. To show that A is stable we use Theorem 2.2 and find to each a in $F(A)$ a positive element a_1 in A such that $a \perp a_1$ and $a \precsim a_1$ (cf. Remark 2.1). Let $\pi\colon A \to A/I$ denote the quotient mapping.

There is a positive contractions e in $F(A)$ such that $ea = a = ae$. Set $f = \pi(e)$. Since A/I is stable and f belongs to $F(A/I)$ there is f' in $F(A/I)$ with $f \sim f'$ and $f \perp f'$ (by Theorem 2.2). Because $f' = (1-f)f'(1-f)$ we get

$$f' \in (1-f)A/I(1-f) = \pi(\overline{(1-e)A(1-e)}),$$

and we can therefore find a positive contraction e' in $\overline{(1-e)A(1-e)}$ such that $\pi(e') = f'$. Since $\pi(e') \sim \pi(e)$ there is a positive element c in I such that $(e-1/3)_+ \precsim e' \oplus c$, cf. [14, Lemma 4.2]. It follows that $(e-2/3)_+ \precsim (e'-\delta)_+ \oplus (c-\delta)_+$ for some $\delta > 0$, cf. [15, Proposition 2.4]. Put $c_0 = (c-\delta)_+ \in F(I)$ and $e'_0 = (e'-\delta)_+ \in F(A)$. Then $a \precsim (e-2/3)_+ \precsim e'_0 \oplus c_0$. Let g be a positive contraction in A such that $ge'_0 = e'_0 g = e'_0$. By assumption (and by the remarks below Definition 2.4) there is a positive element c_1 in $\overline{(1-e-g)I(1-e-g)}$ such that $c_0 \sim c_1$. Now, a, e'_0, and c_1 are mutually orthogonal, positive elements in A, and

$$a \precsim e'_0 \oplus c_0 \precsim e'_0 \oplus c_1 \precsim e'_0 + c_1.$$

We can therefore take a_1 to be $e'_0 + c_1$. Q.E.D.

Lemma 6.9. *Let A be a C^*-algebra, and let I be a stable, closed two-sided ideal in A such that the quotient A/I does not have (non-trivial) unital quotients. Let a be a positive contraction in A. Then $\overline{(1-a)I(1-a)}$ admits no non-zero bounded trace.*

Proof. Assume to reach a contradiction that τ is a bounded (positive) trace on the hereditary sub-C^*-algebra $\overline{(1-a)I(1-a)}$. This hereditary sub-C^*-algebra is full in I by Lemma 6.7. We can therefore extend τ to an unbounded (because I is stable) densely defined trace τ on I. Now, I is an ideal in the unitization $\widetilde{A}$ of A, and we can extend τ to a lower semi-continuous trace function $\bar{\tau}\colon \widetilde{A}^+ \to [0,\infty]$. Let J be the closed two-sided ideal in $\widetilde{A}$ generated by all positive elements b in $\widetilde{A}$ with $\bar{\tau}(b) < \infty$. A positive element b in $\widetilde{A}$ will then belong to J if and only if $\bar{\tau}((b-\varepsilon)_+) < \infty$ for all $\varepsilon > 0$.

Now, I is contained in J because τ is densely defined on I. Since τ is not bounded on I we cannot have $\tau(1) < \infty$; thus $J \neq \widetilde{A}$. The assumption that τ is bounded on $\overline{(1-a)I(1-a)}$ leads to $\tau(1-a) < \infty$, and hence $1-a$ belongs to J.

Let $\psi\colon \widetilde{A} \to \widetilde{A}/J$ and $\pi\colon \widetilde{A}/I \to \widetilde{A}/J$ be the quotient mappings. Then $\psi(1) = \psi(a)$ because $1-a$ belongs to J, and it follows that $\psi(A)$

is unital. Since $\pi(A/I) = \psi(A)$, A/I has a unital quotient contrary to our assumptions. Q.E.D.

To state Proposition 6.12 in general terms the following definition is convenient.

Definition 6.10. *A C^*-algebra I is called* regular *if every full, hereditary sub-C^*-algebra of I, that has no unital quotients and no bounded traces, is stable*

It follows from Corollary 4.4 that not all C^*-algebras are regular. On the other hand, many C^*-algebras are regular:

Lemma 6.11. *A C^*-algebra I is regular*

(i) *if I is an exact C^*-algebra with the cancellation property,* $\mathrm{RR}(I) = 0$, *and $K_0(I)$ is weakly unperforated,* or

(ii) *if I is purely infinite.*

Proof. (i). Let I_0 be a full, hereditary sub-C^*-algebra of I. Then I_0 is σ_{p}-unital because I has real rank zero. The cancellation property, exactness, and having weakly unperforated K_0-group are all properties that pass to full hereditary sub-C^*-algebras, so I_0 has these properties. Proposition 3.4 therefore yields that I_0 is stable if I_0 has no bounded trace.

(ii). Every hereditary sub-C^*-algebra of a purely infinite C^*-algebra is again purely infinite ([14, Proposition 4.17]) and hence is stable if it has no unital quotient, cf. Proposition 5.3. Q.E.D.

Proposition 6.12. *Let*

$$0 \longrightarrow I \longrightarrow A \longrightarrow B \longrightarrow 0$$

be a short exact sequence of separable C^*-algebras and suppose that I is regular. Then A is stable if and only if I and B are stable.

All AF-algebras, and more generally all AH-algebras of real rank zero and of slow dimension growth, are regular (see the comments below Proposition 3.4). In particular, for every extension $0 \to \mathcal{K} \to A \to B \to 0$ of separable C^*-algebras one has that A is stable if and only if B is stable, a fact that implicitly is contained in the BDF-paper [5].

Proof. If A is stable, then so are I and A/I (by Corollary 2.3 (ii)). Assume now that I and A/I are stable and that I is regular. As $\overline{(1-a)I(1-a)}$ is a full hereditary sub-C^*-algebra of I that has no unital quotient (by Lemma 6.7) and no bounded traces (by Lemma 6.9) for every positive contraction a in A, the assumption that I is regular implies that $\overline{(1-a)I(1-a)}$ is stable. Proposition 6.8 then yields that A is stable. Q.E.D.

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Non-commutative Markov operators arising from subfactors

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§1. Introduction

It is well-known that there exists a close relationship between subfactor theory and (ordinary or non-commutative) probability theory. Indeed, one may observe it already in V. F. R. Jones' original paper [12], where L^1-estimate of conditional expectations plays an important role in his proof of reducibility of Jones subfactors of index larger than 4. Since then, several authors discussed the relationship between these two fields [1] [2] [8] [9] [10] [15] [16] [17] [18]. Among other notions in probability theory, the most suitable one for subfactors so far is the theory of Poisson boundaries of random walks. It is well-known that the center of the core of a subfactor can be identified with the L^∞-space of the Poisson boundary of some random walk on the principal graph.

In [11], the author obtained a precise description of the relative commutant of the fixed point subalgebra under the infinite tensor product action of the quantum group $SU_q(2)$ on the Powers factor. Indeed, it may be regarded as "the function algebra" over "the Poisson boundary" of a non-commutative Markov operator (synonymously, a unital completely positive operator) on "the group algebra" of $SU_q(2)$.

Following the same philosophy, in this note we provide a general machinery to determine the structure of the (higher) relative commutants of the core inclusions of (not necessarily strongly amenable) subfactors. These relative commutants also may be regarded as "the function algebras" of "the Poisson boundaries" of some non-commutative Markov operators of finite type I von Neumann algebras. As an easy application, we give a new proof, based on a random walk on some ladder-like graph, to the above mentioned fact about Jones inclusions.

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§2. Preliminaries

In this section, we give a quick introduction to two main ingredients of this note: (1) Poisson boundaries for Markov operators (2) a bimodule description of higher relative commutants of subfactors. Our basic reference for the boundary theory of (ordinary) random walks is V.A. Kaimanovich's review article [13]. Here, we give an algebraic description of the Poisson boundaries, and also give their extension to the non-commutative setting. For subfactors, we freely use definitions and notations in D. E. Evans and Y. Kawahigashi's book [5].

2.1. Poisson Boundaries

We start with a simple and classical case. Let $\mathcal{X}$ be a countable set. A Markov operator P on the state space $\mathcal{X}$ is a unital normal positive map from $\ell^\infty(\mathcal{X})$ to itself. For a given Markov operator, the corresponding transition probability $p(s,t)$ from $s \in \mathcal{X}$ to $t \in \mathcal{X}$ is given by

$$P(\delta_t) = \sum_{s\in\mathcal{X}} p(s,t)\delta_s,$$

where δ_s is the characteristic function of the one point set $\{s\}$. A function f is called *harmonic* if the right-hand side of the following makes sense and it is satisfied:

$$f(s) = \sum_{t\in\mathcal{X}} p(s,t)f(t),$$

which is equivalent to $P(f) = f$ for bounded f. We denote by $H^\infty(\mathcal{X}, P)$ the set all bounded harmonic functions.

The Poisson boundary of $(\mathcal{X}, P)$ is, roughly speaking, a measure space (Ω, μ) describing $H^\infty(\mathcal{X}, P)$, as in an analogous manner that the boundary values on the unit circle determines harmonic functions on the unit disc through the classical Poisson integral formula. Though one can find in [13] a decent measure theoretic construction of the Poisson boundary of $(\mathcal{X}, P)$, in this note we adopt the following characterization as a local definition [14, pp. 462], which is more suitable for the non-commutative situation: For every pair $f, g \in H^\infty(\mathcal{X}, P)$, strong limit

$$s-\lim_{n\to\infty} P^n(fg)$$

always exists and harmonic. This introduces a new associative product into $H^\infty(\mathcal{X}, P)$, and equips it with abelian von Neumann algebra structure. The Poisson boundary is characterized as a measure space (Ω, μ)

such that $L^\infty(\Omega, \mu)$ is isomorphic to the abelian von Neumann algebra $H^\infty(\mathcal{X}, P)$.

Now we consider the notion of "Poisson boundaries" in a more general situation. Let A be a von Neumann algebra and P be a normal unital completely positive map from A to itself. Sometimes, we call P a *non-commutative Markov operator* for an obvious reason. We say that $x \in A$ is *P-harmonic* or *harmonic with respect to P* if x is fixed by P. $H^\infty(A, P)$ denotes the set of P-harmonic elements. Note that $H^\infty(A, P)$ is a weakly closed operator system [4]: namely it is a unital self-adjoint subspace of A.

We show that $H^\infty(A, P)$ has a von Neumann algebra structure as in the classical case, though it is in general non-commutative and no underlying measure theoretic object exists. We fix a free ultrafilter $\omega \in \beta\mathbf{N} \setminus \mathbf{N}$ and define a norm one projection E_ω from A to $H^\infty(A, P)$ by the weak limit

$$E_\omega(x) = w - \lim_{n \to \omega} \frac{1}{n} \sum_{k=0}^{n-1} P^k(x).$$

Then, we can introduce von Neumann algebra structure into $H^\infty(A, P)$ by using the Choi-Effros product $E_\omega(xy)$ for $x, y \in H^\infty(A, P)$ [4]. The resulting von Neumann algebra may be considered as a non-commutative analogue of the function algebra over "the Poisson boundary" associated with (A, P). Note that the Choi-Effros product $E_\omega(xy)$ for $x, y \in H^\infty(A, P)$ does not depend on ω because every completely positive surjective isometry between two von Neumann algebras is actually an isomorphism.

As in the classical case, a natural and tempting question would be to identify this von Neumann algebra with known one for a given concrete example of P. The goal of this note is to show that some von Neumann algebra naturally appearing in a subfactor problem happens to be "the function algebra" of "the Poisson boundary" of some non-commutative Markov operator, and $H^\infty(A, P)$ with the Choi-Effros product gives a better description of the algebra.

2.2. Core Inclusions

Throughout this note, $N \subset M$ denotes an extremal inclusion of type II_1 factors with a finite Jones index $[M : N]$. Let

$$N = M_{-1} \subset M = M_0 \subset M_1 \subset M_2 \subset M_3 \subset \cdots,$$

be the Jones tower for $N \subset M$. We set $A_n := M' \cap M_n$, $n = 0, 1, 2, \cdots$, and $B_n := N' \cap M_n$, $n = -1, 0, 1, \cdots$. Then, the standard invariant of the inclusion introduced by S. Popa [17] is the following nested commuting squares:

$$\begin{array}{ccccccccc} & & A_0 & \subset & A_1 & \subset & A_2 & \subset & \cdots \\ & & \cap & & \cap & & \cap & & \\ B_{-1} & \subset & B_0 & \subset & B_1 & \subset & B_2 & \subset & \cdots \end{array}$$

We denote by A_∞ and B_∞ the weak closures of $\bigcup_n A_n$ and $\bigcup_n B_n$ respectively in the GNS representations with respect to the natural traces. The inclusion $A_\infty \subset B_\infty$ is called the core of $M \subset M_1$, which is known to be anti-isomorphic to the original one if M is hyperfinite and $N \subset M$ is strongly amenable (See [17] for these terms). However, we focus on the non-strongly amenable case in this note.

As in [5], we identify A_n and B_n with appropriate endomorphism spaces of bimodules M_j; more precisely, we have the following identification:

$$A_{2n} = \mathrm{End}_M(M_n)_M,$$

$$A_{2n+1} = \mathrm{End}_M(M_n)_N,$$

$$B_{2n} = \mathrm{End}_N(M_n)_M,$$

$$B_{2n+1} = \mathrm{End}_N(M_n)_N.$$

These spaces have natural inclusion relations coming from taking tensor products with the basic bimodules ${}_N M_M$ and ${}_M M_N$ from either left or right, which are of course compatible with the inclusion relations of $\{A_n\}_n$ and $\{B_n\}_n$.

Let $\mathcal{G}$ and $\mathcal{H}$ be the principal graphs of $N \subset M$. We denote by $\mathcal{G}^0$ and $\mathcal{H}^0$ the set of vertices of $\mathcal{G}$ and $\mathcal{H}$ respectively. $\mathcal{G}$ and $\mathcal{H}$ are bipartite graphs and we denote by $\mathcal{G}^{\text{even}}$, $\mathcal{G}^{\text{odd}}$, $\mathcal{H}^{\text{even}}$, $\mathcal{H}^{\text{odd}}$ their even and odd vertices respectively. We identify $\mathcal{G}^{\text{even}}$ (respectively $\mathcal{G}^{\text{odd}}$, $\mathcal{H}^{\text{even}}$, $\mathcal{H}^{\text{odd}}$) with the set of irreducible $M-M$ (respectively $M-N$, $N-N$, $N-M$) bimodules contained in ${}_M {M_n}_M$ (respectively ${}_M {M_n}_N$, ${}_N {M_n}_N$, ${}_N {M_n}_M$) for some n. For even (respectively odd) n, we denote by $\mathcal{G}^0_n \subset \mathcal{G}^0$ the set of even (respectively odd) vertices with distance from the distinguished vertex $*_M = {}_M M_M$ less than or equal to n. Note that each element of $\mathcal{G}^0_n$ is identified with a simple component of A_n.

It is well-known that the centers $Z(A_\infty)$ (respectively $Z(B_\infty)$) of A_∞ (respectively B_∞) can be identified with the L^∞-space of the Poisson boundaries of some random walk on $\mathcal{G}$ (respectively $\mathcal{H}$) [10] [17]. In this note, we give a similar description of the relative commutant $A'_\infty \cap B_\infty$ using a non-commutative Markov operator. While the random walk on $\mathcal{G}$ is determined only by the trace vector [10] [17], the Markov operator describing $A'_\infty \cap B_\infty$ is much more involved. Indeed, it is described in terms of intertwiners of bimodules.

In the rest of this section, we collect notations for bimodules and string algebras that will be used in this note.

Let A, B, and C be II_1 factors. For an $A-B$ bimodules ${}_AX_B$, we defined the statistical dimension of X by

$$d({}_AX_B) = \sqrt{\dim_A X \dim X_B}.$$

For irreducible bimodules ${}_AX_B$, ${}_BY_C$, and ${}_AZ_C$ with finite statistical dimensions, we denote by $\mathcal{H}^Z_{X,Y}$ the space of bimodule maps

$$\mathcal{H}^Z_{X,Y} = \mathrm{Hom}({}_AX \otimes_B Y_C, {}_AZ_C),$$

and by $N^Z_{X,Y}$ the multiplicity of ${}_AZ_C$ in ${}_AX \otimes_B Y_C$. In particular, we set $\beta := d({}_NM_M) = d({}_MM_N)$, and

$$\Gamma_{Z,X} := N^Z_{X,{}_MM_N}$$

for $A = B = M$, $C = N$ and bimodules X and Z associated with the inclusion $N \subset M$. Let $r_X \in \mathcal{H}^A_{X,\overline{X}}$ be the element defined by

$$r_X(\xi \otimes \overline{\eta}) = \langle \xi, \eta \rangle_A,$$

where $\xi, \eta \in_A X$ are A-bounded elements and $\langle \cdot, \cdot \rangle_A$ is the A-valued inner product. Note that this is, up to constant, the Frobenius dual of the identity map $1_X \in \mathcal{H}^X_{A,X}$. Then, the right hand side Frobenius dual of $\sigma \in \mathcal{H}^Z_{X,Y}$ is expressed as

$$\sqrt{\frac{\dim X_B}{\dim Z_C}}(1_X \otimes r_Y) \cdot (\sigma^* \otimes 1_{\overline{Y}}).$$

For a graph $\mathcal{G}$ and a path ξ on $\mathcal{G}$, $s(\xi)$, $r(\xi)$, and $|\xi|$ denote the source, the range, and the length of ξ respectively. For a vertex $v \in \mathcal{G}^0$ and $n \in \mathbf{N}$, we denote by $\mathrm{Path}^n_v(\mathcal{G})$ the set of paths on $\mathcal{G}$ with source v and length n. We denote by $A^n_v(\mathcal{G})$ the string algebra spanned by the strings (ξ, η) with $\xi, \eta \in \mathrm{Path}^n_v(\mathcal{G})$, $r(\xi) = r(\eta)$.

§3. Main Result

Let $C_\infty := A'_\infty \cap B_\infty$, $C_n := A'_n \cap B_n$, $n = 0, 1, 2, \cdots$. We denote by E_n the trace preserving conditional expectation from B_∞ onto B_n.

Thanks to the commuting square condition, for a given $x \in C_\infty$, $x_n := E_n(x)$ belongs to C_n. The sequence $\{x_n\}_{n=0}^\infty$ converges to x in strong $*$-topology. On the other hand, if $\{x_n\}_{n=0}^{\infty}$ is a bounded sequence satisfying $x_n \in C_n$ and $E_n(x_{n+1}) = x_n$, the sequence converges to some element $x \in C_\infty$ such that $x_n = E_n(x)$. Therefore, all information of $x \in C_\infty$ is encoded in the sequence $\{x_n\}_{n=1}^{\infty}$. Here, a possible difficulty in analyzing this sequence would be that all members of $\{x_n\}_{n=1}^{\infty}$ belong to different algebras C_n, $n = 0, 1, 2, \cdots$. We start with a description of C_n in terms of bimodules. The following lemma is just a translation from an algebra language to a bimodule language:

Lemma 3.1. *With the above notation, we have*

$$C_{2n} \cong \bigoplus_{X \in \mathcal{G}^0_{2n}} \mathrm{End}_N({}_N M \otimes_M X_M)_M,$$

$$C_{2n+1} \cong \bigoplus_{X \in \mathcal{G}^0_{2n+1}} \mathrm{End}_N({}_N M \otimes_M X_N)_N.$$

Proof. Using the string algebra expression of C_n with respect to the inclusions

$$A_0 \subset A_1 \subset \cdots \subset A_n \subset B_n,$$

we can see that every element in C_n has the following form:

$$\sum_{|\sigma_+|=|\sigma_-|=1} \sum_{|\xi|=n} c_{\sigma_+,\sigma_-}(\xi \cdot \sigma_+, \xi \cdot \sigma_-), \quad c_{\sigma_+,\sigma_-} \in \mathbf{C}.$$

This means that we have isomorphisms

$$C_{2n} \cong \bigoplus_{X \in \mathcal{G}^0_{2n}} A^1_X(\mathcal{G}),$$

$$C_{2n+1} \cong \bigoplus_{X \in \mathcal{G}^0_{2n+1}} A^1_X(\mathcal{H}),$$

where $\mathcal{G}^{\mathrm{odd}}$ is identified with $\mathcal{H}^{\mathrm{odd}}$ though the contragredient map in the second equation. Thus, we get the result. Q.E.D.

In view of the above lemma, we set

$$D_X := \mathrm{End}_N({}_N M \otimes_M X_M)_M, \quad X \in \mathcal{G}^{\mathrm{even}},$$

$$D_X := \mathrm{End}_N({}_N M \otimes_M X_N)_N, \quad X \in \mathcal{G}^{\mathrm{odd}},$$

$$D_n := \bigoplus_{X \in \mathcal{G}^0_n} D_X,$$

$$D^{\text{even}} := \bigoplus_{X \in \mathcal{G}^{\text{even}}} D_X,$$

$$D^{\text{odd}} := \bigoplus_{X \in \mathcal{G}^{\text{odd}}} D_X,$$

$$D := D^{\text{even}} \oplus D^{\text{odd}},$$

where the direct sums are understood as von Neumann algebra direct sums. We regard D_n as a subalgebra of D in a natural way, and denote by $\pi_n : D \longrightarrow D_n$ the natural projection. Let $\theta_n : D_n \longrightarrow C_n$ be the isomorphism established in the above lemma. Note that θ_n is *not* compatible with the inclusion relations of $\{D_n\}$ and $\{C_n\}$ (in fact, there exists no inclusion relation between D_n and D_{n+1}).

We introduce a Markov operator P of D. For simplicity, the bimodule ${}_NM_M$ and ${}_MM_N$ will be denote by ρ and $\overline{\rho}$. For $x \in D_X$, $X \in \mathcal{G}^{\text{even}}$, we set

$$P(x) = \frac{d(X)}{\beta} \bigoplus_{Y \in \mathcal{G}^{\text{odd}}} \frac{1}{d(Y)} \sum_{i=1}^{\Gamma_{Y,X}} (1_\rho \otimes \nu_{Y,i}) \cdot (x \otimes 1_{\overline{\rho}}) \cdot (1_\rho \otimes \nu_{Y,i}^*),$$

where $\{\nu_{Y,i}\}_{i=1}^{\Gamma_{Y,X}}$ is an orthonormal basis of $\mathcal{H}_{X,\overline{\rho}}^{Y}$. In a similar way, for $x \in D_X$ and $X \in \mathcal{G}^{\text{odd}}$, we set

$$P(x) = \frac{d(X)}{\beta} \bigoplus_{Y \in \mathcal{G}^{\text{even}}} \frac{1}{d(Y)} \sum_{i=1}^{\Gamma_{X,Y}} (1_\rho \otimes \nu_{Y,i}) \cdot (x \otimes 1_\rho) \cdot (1_\rho \otimes \nu_{Y,i}^*),$$

where $\{\nu_{Y,i}\}_{i=1}^{\Gamma_{X,Y}}$ is an orthonormal basis of $\mathcal{H}_{X,\rho}^{Y}$. It is easy to show that P restricted to D^{even} and D^{odd} are unital normal completely positive maps from one to the other. In fact, this is the right Markov operator that gives C_∞ as "the function algebra" of "the Poisson boundary".

Lemma 3.2. *Let E_n, θ_n, π_n, and P be as above. Then, they satisfy*

$$\theta_{n-1} \cdot \pi_{n-1} \cdot P = E_{n-1} \cdot \theta_n \cdot \pi_n, \quad n = 1, 2, \cdots .$$

Proof. It suffices to show the equality for $x \in D_X$, $X \in \mathcal{G}_n^0$. We may and do further assume that n is even, (the odd case can be treated in a similar way), and x has the form $x = \sigma_+^* \cdot \sigma_-$, where $\sigma_+, \sigma_- \in \mathcal{H}_{\rho,X}^{W}$, for some irreducible ${}_NW_M$. Let $\{\xi\}_i$ be an orthonormal basis of $\mathcal{H}_{Y,\rho}^{X}$. Since $N \subset M$ is extremal, we have

$$\dim {}_MX = \dim X_M = d(X),$$

$$\dim_M Y = \frac{d(Y)}{\beta}, \quad \dim Y_N = \beta d(Y).$$

Thus, the right hand side Frobenius dual of ξ_i is given by

$$\widetilde{\xi_i} = \sqrt{\frac{\beta d(Y)}{d(X)}}(1_Y \otimes r_\rho) \cdot (\xi_i^* \otimes 1_{\overline{\rho}}).$$

Therefore, the D_Y-component of $P(x)$ is given by

$$\begin{aligned}
&\frac{d(X)}{\beta d(Y)} \sum_{i=1}^{\Gamma_{Y,X}} (1_\rho \otimes \widetilde{\xi_i}) \cdot (x \otimes 1_{\overline{\rho}}) \cdot (1_\rho \otimes \widetilde{\xi_i}^*) \\
&= \sum_{i=1}^{\Gamma_{Y,X}} (1_{\rho\otimes Y} \otimes r_\rho) \cdot (((1_\rho \otimes \xi_i^*) \cdot \sigma_+^* \cdot \sigma_- \cdot (1_\rho \otimes \xi_i)) \otimes 1_{\overline{\rho}}) \\
&\qquad \cdot (1_{\rho\otimes Y} \otimes r_\rho^*),
\end{aligned}$$

Let ${}_N V_N$ be an irreducible $N-N$ bimodule contained in ${}_N M \otimes_M Y_N$. We choose orthonormal bases $\{\eta_j\}_j$ and $\{\zeta_k\}_k$ of $\mathcal{H}_{\rho,Y}^V$ and $\mathcal{H}_{V,\rho}^W$ respectively. Using the connection and the basis of $\mathrm{Hom}({}_N M \otimes_M Y \otimes_N M_M, {}_N W_N)$ coming from these, we get

$$\begin{aligned}
&(1_\rho \otimes \xi_i^*) \cdot \sigma_+^* \cdot \sigma_- \cdot (1_\rho \otimes \xi_i) \\
&= \sum_{V,j,j',k,k'} \begin{array}{ccc} Y & \stackrel{\xi_i}{\rightarrow} & X \\ \eta_j \downarrow & & \downarrow \sigma_+ \\ V & \underset{\zeta_k}{\rightarrow} & W \end{array} \overline{\begin{array}{ccc} Y & \stackrel{\xi_i}{\rightarrow} & X \\ \eta_{j'} \downarrow & & \downarrow \sigma_- \\ V & \underset{\zeta_{k'}}{\rightarrow} & W \end{array}} (\eta_j^* \otimes 1_\rho) \cdot \zeta_k^* \cdot \zeta_{k'} \\
&\qquad \cdot (\eta_{j'} \otimes 1_\rho).
\end{aligned}$$

Using the Frobenius reciprocity again, we get

$$\begin{aligned}
&(1_{\rho\otimes Y} \otimes r_\rho) \cdot (((\eta_j^* \otimes 1_\rho) \cdot \zeta_k^* \cdot \zeta_{k'} \cdot (\eta_{j'} \otimes 1_\rho)) \otimes 1_{\overline{\rho}}) \cdot (1_{\rho\otimes Y} \otimes r_\rho^*) \\
&= \eta_j^* \cdot (1_V \otimes r_\rho) \cdot (\zeta_k^* \cdot \zeta_{k'} \otimes 1_{\overline{\rho}}) \cdot (1_V \otimes r_\rho^*) \cdot \eta_{j'} \\
&= \frac{d(W)}{\beta d(V)} \eta_j^* \cdot \widetilde{\zeta_k} \cdot \widetilde{\zeta_{k'}}^* \cdot \eta_{j'} \\
&= \frac{\delta_{k,k'} d(W)}{\beta d(V)} \eta_j^* \cdot \eta_{j'},
\end{aligned}$$

where $\widetilde{\zeta_k}$ is the right hand side Frobenius dual of ζ_k. Thus, the D_Y-component of $P(x)$ is

$$\sum_{V,i,j,j',k} \frac{d(W)}{\beta d(V)} \begin{array}{ccc} Y & \xrightarrow{\xi_i} & X \\ \eta_j \downarrow & & \downarrow \sigma_+ \\ V & \xrightarrow[\zeta_k]{} & W \end{array} \quad \overline{\begin{array}{ccc} Y & \xrightarrow{\xi_i} & X \\ \eta_{j'} \downarrow & & \downarrow \sigma_- \\ V & \xrightarrow[\zeta_k]{} & W \end{array}} \quad \eta_j^* \cdot \eta_{j'}.$$

Now, we compute $E_{n-1} \cdot \theta_n \cdot \pi_n(x)$. The string algebra expression of $\theta_n \cdot \pi_n(x)$ in terms of the inclusions $A_{n-1} \subset A_n \subset B_n$ is

$$\sum_{|\xi|=n} (\xi \cdot \sigma_+, \xi \cdot \sigma_-).$$

The same element can be expressed in terms of $A_{n-1} \subset B_{n-1} \subset B_n$ as

$$\sum_{|\nu|=n-1} \sum_{V,i,j,j',k,k'} \begin{array}{ccc} Y & \xrightarrow{\xi_i} & X \\ \eta_j \downarrow & & \downarrow \sigma_+ \\ V & \xrightarrow[\zeta_k]{} & W \end{array} \quad \overline{\begin{array}{ccc} Y & \xrightarrow{\xi_i} & X \\ \eta_{j'} \downarrow & & \downarrow \sigma_- \\ V & \xrightarrow[\zeta_{k'}]{} & W \end{array}} \quad (\nu\cdot\eta_j\cdot\zeta_k, \nu\cdot\eta_{j'}\cdot\zeta_{k'}).$$

Therefore, we can get the statement from the explicit formula of the conditional expectation from B_n to B_{n-1} in terms of the string algebra [5, Lemma 11.7]. Q.E.D.

Theorem 3.3. *There exists a unital completely positive surjective isometry $\theta_\infty : H^\infty(D,P) \longrightarrow C_\infty$ satisfying*
(1) *For every $x \in H^\infty(D,P)$, $\theta_\infty(x)$ is given by*

$$\theta_\infty(x) = s - \lim_{n\to\infty} \theta_n \cdot \pi_n(x).$$

(2) *For every pair $x, y \in H^\infty(D,P)$, $\{P^n(xy)\}_{n=1}^\infty$ converges to an element in $H^\infty(D,P)$ in strong operator topology, and*

$$\theta_\infty(x)\theta_\infty(y) = \theta_\infty(s - \lim_n P^n(xy)).$$

Except for surjectivity of θ_∞, Theorem 3.3 is a direct consequence of Lemma 3.2 and the non-commutative martingale convergence theorem mentioned at the beginning of this section. To show that θ_∞ is surjective, we need the following Fougel'type estimate as usual:

Lemma 3.4. *For the Markov operator P as above, we have*

$$\lim_{n\to\infty} ||P^{n+2} - P^n|| = 0.$$

In consequence, for every bounded sequence $\{x_n\}_{n=1}^\infty$ in D satisfying $x_n \in D_n$, $\pi_n \cdot P(x_{n+1}) = x_n$, $n = 0, 1, 2, \cdots$, there exists $x \in H^\infty(D,P)$ such that $\pi_n(x) = x_n$ for all n.

Proof. For $V \in \mathcal{G}^{\mathrm{even}}$, we define a normal completely positive map Φ_V from D^{even} to itself in a similar way as P; for $x \in D_X$, we set

$$\Phi_V(x) = \bigoplus_{Y \in \mathcal{G}^{\mathrm{even}}} \frac{d(X)}{d(Y)} \sum_{i=1}^{N_{X,\overline{V}}^{Y}} (1_\rho \otimes \xi_{Y,i}) \cdot (x \otimes 1_{\overline{\rho}}) \cdot (1_\rho \otimes \xi_{Y,i}^*),$$

where $\{\xi_{Y,i}\}_{i=1}^{N_{X,\overline{V}}^{Y}}$ is an orthonormal basis of $\mathcal{H}_{X,\overline{V}}^{Y}$. Then, it is a routine work to show $\Phi_V(1) = d(V)$ and

$$\Phi_V \cdot \Phi_W = \sum_Z N_{V,W}^Z \Phi_Z.$$

In the same way, for $V \in \mathcal{H}^{\mathrm{even}}$ we define a normal completely positive map Φ_V from D^{odd} to itself.

For a probability measure μ on $\mathcal{G}^{\mathrm{even}}$ or on $\mathcal{H}^{\mathrm{even}}$, we set

$$\phi_\mu = \sum_V \frac{\mu(V)}{d(V)} \Phi_V,$$

which is a non-commutative Markov operator. Then, we get $\phi_\mu \cdot \phi_\nu = \phi_{\mu * \nu}$, where $\mu * \nu$ is the convolution product of two probability measure μ and ν introduced in [10]. Moreover, the following holds:

$$P^2 = \frac{1}{\beta^2}(\Phi_{\overline{\rho}\rho} \oplus \Phi_{\rho\overline{\rho}}).$$

If we define two probability measures μ on $\mathcal{G}^{\mathrm{even}}$ and ν on $\mathcal{H}^{\mathrm{even}}$ by

$$\mu = \sum_V \frac{d(V) N_{\overline{\rho},\rho}^V}{\beta^2} \delta_V, \qquad \nu = \sum_V \frac{d(V) N_{\rho,\overline{\rho}}^V}{\beta^2} \delta_V,$$

we get $P^{2n} = \phi_{\mu^n} \oplus \phi_{\nu^n}$, where μ^n and ν^n are the n-fold convolution product of μ and ν. Thanks to Fougel's theorem [6] [10, Lemma 3.1], we have the following ℓ^1-norm estimate:

$$\lim_{n\to\infty} ||\mu^{n+1} - \mu^n||_1 = 0, \qquad \lim_{n\to\infty} ||\nu^{n+1} - \nu^n||_1 = 0.$$

Therefore, we get $\lim_{n\to\infty} ||P^{n+2} - P^n|| = 0$. The rest of the statements is standard (see [13] for example). Q.E.D.

(2) of Theorem 3.3 implies the following:

Corollary 3.5. *If $\mathcal{G}$ or $\mathcal{H}$ has no multi-edges, C_∞ is abelian.*

Remark.

(1) Assume neither $\mathcal{G}$ or $\mathcal{H}$ has multi-edges. Then, since D is abelian, P induces a random walk on a graph. Let ${}_MX_M$, ${}_MY_N$, ${}_NV_N$, and ${}_NW_M$ be irreducible bimodules associated with the inclusion $N \subset M$ and $\xi \in \mathcal{H}^X_{Y,\rho}$, $\eta \in \mathcal{H}^V_{\bar{\rho},Y}$, $\zeta \in \mathcal{H}^W_{V,\rho}$, $\sigma \in \mathcal{H}^W_{\rho,X}$ be normalized unique (up to phase) intertwiners. Then, we have

$$P(\sigma^*\sigma) = \sum_\eta \frac{d(W)}{\beta d(V)} \left| \begin{array}{ccc} Y & \xrightarrow{\xi} & X \\ \eta\downarrow & & \downarrow\sigma \\ V & \xrightarrow[\zeta]{} & W \end{array} \right|^2 \eta^* \cdot \eta.$$

Therefore, if we regard D as the ℓ^∞-space over the set of vertical paths $\mathcal{X}$, the transition probability from η to σ is given by

$$p(\eta,\sigma) = \frac{d(W)}{\beta d(V)} \left| \begin{array}{ccc} Y & \xrightarrow{\xi} & X \\ \eta\downarrow & & \downarrow\sigma \\ V & \xrightarrow[\zeta]{} & W \end{array} \right|^2 .$$

This is a reversible random walk in the sense of [19] thanks to the renormalization rule of the connection. Let τ be the natural trace on B_∞. Then, for $f \in H^\infty(D,P)$ we get

$$\tau(\theta_\infty(f)) = \tau(E_0(\theta_\infty(f))) = \tau(\theta_0 \cdot \pi_0(f)).$$

Now we assume, for simplicity, that $N \subset M$ is irreducible and $\sigma_0 \in \mathcal{X}$ is the path corresponding to the intertwiner in $\mathrm{Hom}(\rho \otimes_M M_M, \rho)$. Since the dimension of D_0 is one, the above equation means that $\tau(\theta_\infty(f))$ is given by the evaluation of f at σ_0. Thus, the measure corresponding to the restriction of τ to C_∞ is nothing but the harmonic measure on the Poisson boundary of the Markov chain induce by P with the initial distribution δ_{σ_0}.

(2) Let

$$\cdots M_{-2} \subset M_{-1} = N \subset M_0 = M$$

be a tunnel. We set $A_{i,j} := M'_{-i} \cap M_j$ and define $A_{i,\infty}$ to be the weak closure of $\cup_j A_{i,j}$. Then, the same machinery works in order to obtain $C_{n,\infty} := A'_{0,\infty} \cap A_{n,\infty}$. Indeed, there are obvious elements in $C_{n,\infty}$ coming from $A_{n,0}$, and what we really need to obtain is $pC_{n,\infty}q$ where p and q are minimal projections in $A_{n,0}$. Let ${}_AY_M$ and ${}_AZ_M$ be bimodules corresponding p and q respectively, where A is either M or N depending on the parity of n. Instead of D, we need to work on

$$\begin{aligned} D^{Y,Z} : &= \bigoplus_{X \in \mathcal{G}^{\mathrm{even}}} \mathrm{Hom}_A(Y \otimes_M X, Z \otimes_M X)_M \\ &\oplus \bigoplus_{X \in \mathcal{G}^{\mathrm{odd}}} \mathrm{Hom}_A(Y \otimes_M X, Z \otimes_M X)_N, \end{aligned}$$

as an object that the Markov operator P acts on. Or to make it an algebra, we can put it into

$$\bigoplus_{Y,Z} D^{Y,Z},$$

where the product is given by the composition (the product of not composable two elements is understood as 0).

(3) Let A be a von Neumann algebra and P be a unital normal completely positive map form A to itself. As we stated in the last section, we can always discuss the "Poisson boundary" using the Choi-Effros product. However, those P coming from natural examples, such as the classical examples or the ones discussed here, seem to have an additional property: namely, for every pair $x, y \in H^\infty(A, P)$, the sequence $\{P^n(xy)\}_n$ converges to an element in $H^\infty(A, P)$ in the strong operator topology. Does this hold for every unital normal completely positive map? If it is not the case, only those with the above property maybe deserve to be called "non-commutative Markov operators".

§4. Examples

In this section, we take the most fundamental example among non-trivial ones: a subfactor with the principal graph A_∞ and index larger than 4 (A_∞ should not be confused with the algebra A_∞ in the previous two sections). Another example, for which $H^\infty(D, P)$ may be explicitly obtained, would be the free composition of the A_3 and A_4 subfactors [3] [7] (see also [10]), though computation would be more complicated.

Let $N \subset M$ be a subfactor with the principal graph A_∞ and index larger than 4. Then, the core of this subfactor is the Jones inclusion, whose reducibility was first proven in Jones paper [12]. Another proof is available in Pimsner and Popa's paper [15]. We choose $0 < q < 1$ satisfying $\beta = [2]_q$, where

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}, \quad n = 1, 2, \cdots .$$

For the edges of $\mathcal{G}$ and $\mathcal{H}$, we use the following labeling:

$$\mathcal{G}:\ 1 — 2 — 3 — 4 — \cdots,$$

$$\mathcal{H}:\ 1' — 2' — 3' — 4' — \cdots,$$

The statistical dimension of the bimodules corresponding to n and n' is $[n]_q$.

There exists only one connection, up to gauge freedom, for A_∞ graph, which is given by

$$\begin{array}{ccc} n & \rightarrow & n+1 \\ \downarrow & & \downarrow \\ n+1' & \rightarrow & n' \end{array} = \frac{(-1)^{n+1}}{[n]_q},$$

$$\begin{array}{ccc} n & \rightarrow & n-1 \\ \downarrow & & \downarrow \\ n-1' & \rightarrow & n' \end{array} = \frac{(-1)^{n}}{[n]_q},$$

$$\begin{array}{ccc} n & \rightarrow & n+1 \\ \downarrow & & \downarrow \\ n+1' & \rightarrow & n+2' \end{array} = \begin{array}{ccc} n+2 & \rightarrow & n+1 \\ \downarrow & & \downarrow \\ n+1' & \rightarrow & n' \end{array} = 1,$$

$$\begin{array}{ccc} n & \rightarrow & n+1 \\ \downarrow & & \downarrow \\ n-1' & \rightarrow & n' \end{array} = \begin{array}{ccc} n & \rightarrow & n-1 \\ \downarrow & & \downarrow \\ n+1' & \rightarrow & n' \end{array} = \frac{\sqrt{[n-1]_q[n+1]_q}}{[n]_q}.$$

We use the following labeling of the vertical paths of length 1:

$$a_n = \begin{array}{c} n \\ \downarrow \\ n+1' \end{array}, \qquad b_n = \begin{array}{c} n+1 \\ \downarrow \\ n' \end{array}.$$

Then, D is identified with the ℓ^∞-space over

$$\mathcal{X} = \{a_n\}_{n=1}^\infty \cup \{b_n\}_{n=1}^\infty.$$

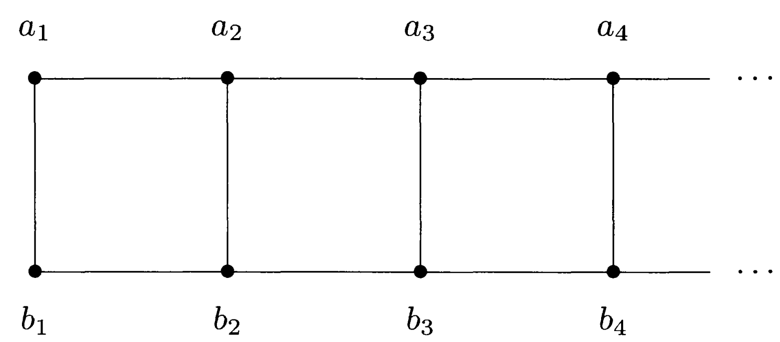

Fig. 1. Graph $\widetilde{\mathcal{X}}$

Thanks to the formula obtained in the remark of the last section, the transition probabilities corresponding to P are given as follows:

$$\begin{aligned}
p(a_n, b_n) &= \frac{1}{[2]_q[n]_q[n+1]_q}, \quad n \geq 1, \\
p(a_n, a_{n+1}) &= \frac{[n+2]_q}{[2]_q[n+1]_q}, \quad n \geq 1, \\
p(a_n, a_{n-1}) &= \frac{[n-1]_q}{[2]_q[n]_q}, \quad n \geq 2, \\
p(b_n, a_n) &= \frac{1}{[2]_q[n]_q[n+1]_q}, \quad n \geq 1, \\
p(b_n, b_{n+1}) &= \frac{[n+2]_q}{[2]_q[n+1]_q}, \quad n \geq 1, \\
p(b_n, b_{n-1}) &= \frac{[n-1]_q}{[2]_q[n]_q}, \quad n \geq 2.
\end{aligned}$$

All the other transition probabilities are 0. Therefore, we can regard $\mathcal{X}$ as the vertex set of the graph $\widetilde{\mathcal{X}}$ as in Figure 1, such that transitions occur only to the nearest neighbors.

An important feature of this random walk is that the vertical bonds decay exponentially fast as n tends to infinity, while we have asymptotics

$$p(a_n, a_{n+1}) = p(b_n, b_{n+1}) \sim \frac{1}{1+q^2} > \frac{1}{2}, \quad (n \to \infty).$$

More intuitively, when n is sufficiently large, the graph looks like splitting into two straight lines, while our random walk quickly goes to infinity. In consequence, we get exactly two points in the Poisson boundary, or in other words, $C_\infty \cong \mathbf{C} \oplus \mathbf{C}$.

To make the above intuitive argument rigorous, we explicitly calculate the harmonic functions. There are exactly two independent (not necessary bounded) harmonic functions (even when $q = 1$). We choose a basis of them consisting of the constant function 1 and h satisfying $h(a_n) = -h(b_n)$, $n = 1, 2, \cdots$, $h(a_1) = 1$. Let $x_n = h(a_n)$ and $x_0 = 0$. Then, the sequence $\{x_n\}_{n=0}^{\infty}$ is determined by the following three-term recurrence relation:

$$x_0 = 0, \qquad x_1 = 1.$$

$$\begin{aligned}(1)\qquad &(1 + [2]_q[n]_q[n+1]_q)x_n \\ &= [n+2]_q[n]_q x_{n+1} + [n+1]_q[n-1]_q x_{n-1}, \quad n \geq 1,\end{aligned}$$

We show that this sequence is monotone increasing, and obtain the limit $\lim_n a_n$ for $0 < q < 1$. Equation (1) can be expressed as

$$\begin{aligned}(2)\qquad 2x_n &= [n+2]_q[n]_q(x_{n+1} - x_n) \\ &\quad -[n+1]_q[n-1]_q(x_n - x_{n-1}), \quad n \geq 1.\end{aligned}$$

Thus, by induction we can show that $\{x_n\}_{n=0}^{\infty}$ is positive and monotone increasing. We set $y_n := x_{n+1} - x_n$, $n = 0, 1, 2, \cdots$. Then, Equation (2) implies

$$y_0 = 1, \qquad y_1 = \frac{2}{[3]_q},$$

$$(3)\qquad 2[n+1]_q y_n = [n+3]_q y_{n+1} + [n-1]_q y_{n-1}, \quad n \geq 1.$$

When $q = 1$, it is easy to solve Equation (3), and we can see that $\{x_n\}_{n=1}^{\infty}$ is not bounded and C_∞ is trivial. When, $0 < q < 1$, we introduce an analytic function $g(z)$ defined on a neighborhood of 0 as follows (it is easy to show that the radius of convergence is positive):

$$g(z) = \sum_{n=0}^{\infty} y_n z^n.$$

Equation (3) implies that the following function equation holds:

$$(4)\qquad (z-q)^2 g(qz) - (z-q^{-1})^2 g(q^{-1}z) = (q^2 - q^{-2}).$$

This means that the radius of convergence of $g(z)$ is larger than or equal to q^{-2} (in fact it is q^{-2}). Setting $z = q$ in Equation (4), we get

$$g(1) = \frac{q + q^{-1}}{q^{-1} - q},$$

and so,

$$\lim_{n\to\infty} x_n = \sum_{n=0}^{\infty} y_n = g(1) = \frac{q + q^{-1}}{q^{-1} - q}.$$

Therefore, h is bounded and $\dim C_\infty = 2$.

We set

$$f_1 = \frac{g(1) + h}{2g(1)}, \quad f_2 = \frac{g(1) - h}{2g(1)}.$$

Then, f_1 and f_2 are two extremal positive harmonic functions of norm 1, and so $\theta_\infty(f_1)$ and $\theta_\infty(f_2)$ are two minimal projections in C_∞. The trace evaluations of these projections are given by

$$\tau(\theta_\infty(f_1)) = f_1(a_1) = \frac{q^{-1}}{q + q^{-1}},$$

$$\tau(\theta_\infty(f_2)) = f_2(a_1) = \frac{q}{q + q^{-1}}.$$

Of course, this agrees with the result in [15].

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Braiding and nets of factors on the circle

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Abstract.

We review various properties of braiding in subfactor theory and their connection to nets of factors on S^1 particularly.

§1. Introduction

The notion of braiding has recently caught much attention in theory of quantum groups, 3-dimensional topological quantum field theory, and conformal field theory. Here we review the current status of results related to braiding in subfactor theory. We particularly focus on nets of factors on S^1, or chiral conformal field theories on S^1 here.

§2. Braiding in subfactor theory

Braiding plays an important role in subfactor theory. Rehren's early work [26] sets a fundamental base in the theory of braiding in the setting of subfactors and algebraic quantum field theory. He defined the notion of braiding and its non-degeneracy for a system of endomorphisms of a factor and showed that we have a unitary representation of $SL(2, \mathbf{Z})$ if and only if a braiding on a finite system of irreducible endomorphisms is non-degenerate.

In subfactor theory, we work on a certain algebraic system which is closed under algebraic operations such as "tensor product" and "conjugation". In an axiomatic approach, our "object" is just something satisfying certain set of axioms and one can study algebraic systems of such objects independently from operator algebras, but we are interested in operator algebraic viewpoints here. Then an object we study in such a theory is an M-N bimodule or a $*$-homomorphism from N into M where M and N are appropriate von Neumann algebras, usually factors of type II_1 or type III. Considering bimodules over factors of type II_1

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and $*$-homomorphisms form a factor of type III into another are essentially the same from a viewpoint of algebraic/combinatorial structures, but in this paper we deal with type III factors in connection to algebraic quantum field theory.

Let N be a factor of type III and $\Delta \subset \mathrm{End}(N)$ a finite system of endomorphisms of N in the following sense.

1. Each $\lambda \in \Delta$ is an irreducible endomorphism of N and has a finite statistical dimension.
2. Endomorphisms in Δ are mutually inequivalent.
3. The identity morphism is in Δ.
4. For any $\lambda \in \Delta$, we have a conjugate morphism $\bar{\lambda}$ in Δ.
5. For any $\lambda, \mu \in \Delta$, we have non-negative integers $N_{\lambda,\mu}^{\nu}$ satisfying $[\lambda][\mu] = \sum_{\nu\in\Delta} N_{\lambda,\mu}^{\nu}[\nu]$, where $[\lambda]$ denotes the unitary equivalence class of λ which is also called a sector.

A system of endomorphism naturally gives a fusion rule algebra with composition of endomorphisms as its multiplication, but there is no reason this multiplication is commutative (up to inner automorphisms) and it is very easy to construct a non-commutative example from an action of a finite non-commutative group, for example. But here we are interested in the commutative case.

When the composition of the endomorphisms in the system is commutative up to inner automorphism of N, a braiding, roughly speaking, means a "compatible choice" of such unitary intertwiners in each space $\mathrm{Hom}(\lambda\mu, \mu\lambda)$, $\lambda, \mu \in \Delta$. The following gives the precise definition of a braiding on a system of endomorphisms. (Even when such a commutative system is given, we do not have existence nor uniqueness of a braiding in general.)

Definition 2.1. We say that a system Δ of endomorphisms of N has a *braiding* if for any pair $\lambda, \mu \in \Delta$ there is a unitary operator $\varepsilon(\lambda, \mu) \in \mathrm{Hom}(\lambda\mu, \mu\lambda)$ satisfying the following properties.

1. We have $\varepsilon(\mathrm{id}_N, \mu) = \varepsilon(\lambda, \mathrm{id}_N) = 1$, for any $\lambda, \mu \in \Delta$.
2. Whenever $t \in \mathrm{Hom}(\lambda, \mu\nu)$ we have

$$\begin{aligned} \rho(t)\varepsilon(\lambda,\rho) &= \varepsilon(\mu,\rho)\mu(\varepsilon(\nu,\rho))t, \\ t\varepsilon(\rho,\lambda) &= \mu(\varepsilon(\rho,\nu))\varepsilon(\rho,\mu)\rho(t), \\ \rho(t)^*\varepsilon(\mu,\rho)\mu(\varepsilon(\nu,\rho)) &= \varepsilon(\lambda,\rho)t^*, \\ t^*\mu(\varepsilon(\rho,\nu))\varepsilon(\rho,\mu) &= \varepsilon(\rho,\lambda)\rho(t)^*, \end{aligned}$$

   for any $\lambda, \mu, \nu \in \Delta$.

The unitaries $\varepsilon(\lambda, \mu)$ are called *braiding operators.* We sometimes write ε^+ for ε with convention $\varepsilon^-(\lambda, \mu) = (\varepsilon(\mu, \lambda))^*$ for the opposite braiding. The following definition of non-degeneracy of a braiding means that ε^+ and ε^- are "really different". This notion is quite important for our study as well as study of topological invariants, since a braiding corresponds to a crossing of a planar picture of a link. (If overcrossing and undercrossing are not really distinguished, one can easily imagine that such a topological study is rather limited.)

Definition 2.2. We say that a braiding ε on a system Δ of endomorphisms of N is *non-degenerate*, if the equalities $\varepsilon^+(\lambda,\mu)=\varepsilon^-(\lambda,\mu)$ for all endomorphisms $\mu\in\Delta$ imply $\lambda=\mathrm{id}_N$.

If we have a braiding on a finite system Δ, we can define S- and T-matrices whose sizes are the number of endomorphisms in Δ, as in [26]. The above non-degeneracy is equivalent to unitarity of the S-matrix as proved in [26], and if it is non-degenerate, the S- and T-matrices give a unitary representation of $SL(2,\mathbf{Z})$.

The above setting is for endomorphisms of a single operator algebra N. We now discuss subfactors $N\subset M$. Suppose we start with an arbitrary subfactor $N\subset M$ of type III with finite index. Let $\iota: N\to M$ be the embedding map and $\bar\iota: M\to N$ be its conjugate morphism. We choose sets of morphisms ${}_N\mathcal{X}_N\subset\mathrm{Mor}(N,N)$, ${}_N\mathcal{X}_M\subset\mathrm{Mor}(M,N)$, ${}_M\mathcal{X}_N\subset\mathrm{Mor}(N,M)$ and ${}_M\mathcal{X}_M\subset\mathrm{Mor}(M,M)$ consisting of representative morphisms of irreducible subsectors of sectors of the form $[\bar\iota\iota\cdots\bar\iota\iota]$, $[\bar\iota\iota\cdots\bar\iota]$, $[\iota\cdots\bar\iota\iota]$ and $[\iota\bar\iota\cdots\iota\bar\iota]$ respectively. (We may and do choose $\mathrm{id}_M,\mathrm{id}_N$ in ${}_N\mathcal{X}_N, {}_M\mathcal{X}_M$ as the endomorphisms representing the trivial sectors.) Then ${}_N\mathcal{X}_N$ and ${}_M\mathcal{X}_M$ are systems of endomorphisms of N and M, respectively, in the above sense. We also assume that ${}_N\mathcal{X}_N$ is finite. This automatically implies that the subfactor $N\subset M$ is of finite depth. If ${}_N\mathcal{X}_N$ is braided in the above sense, we say that the subfactor $N\subset M$ is braided. (Note that this is not equivalent to the condition that ${}_M\mathcal{X}_M$ is braided.) More generally, we also consider a finite system of endomorphism containing ${}_N\mathcal{X}_N$ strictly as a subsystem, and such an extension is important in many aspects, but we do not care this matter very much in this article. Even when a subfactor $N\subset M$ is of type II, we can consider a subfactor $N\otimes R\subset M\otimes R$ for any type III factor R and this tensoring does not change any abstract structure of bimodules arising from the subfactor in which we are interested, so if the resulting subfactor $N\otimes R\subset M\otimes R$ is braided, we also say that $N\subset M$ is braided.

If we arbitrarily construct a subfactor, it is highly unlikely that it is braided. However, natural constructions of a braiding are well-known in theory of quantum groups and conformal field theory. We also have natural appearance of braided subfactors in theory of subfactors as follows.

1. Ocneanu's asymptotic inclusions in [22, 23].
2. Longo-Rehren subfactors in [18].
3. Goodman-de la Harpe-Jones subfactors in [11, Sect. 4.5].
4. Wassermann's loop group construction in [29].

The first and second construction give a new subfactor from a given one, and from a categorical viewpoint, they are identified as in [19]. They are very general constructions to produce a braiding from an arbitrary finite system of endomorphisms. In this sense, these constructions can be regarded as an analogue of the quantum double construction [7] in subfactor theory. (See [20, 21] for a more precise interpretation as a quantum double construction.) Both of these are special cases of Popa's construction of symmetric enveloping inclusion [25]. For the third, we need results from conformal field theory or quantum group theory in order to show that the system of N-N bimodules is indeed braided. For the fourth construction, we get more interesting examples in connection to conformal inclusions as in [30, 31, 2]. The non-degeneracy of the resulting braiding was claimed for (1) in [23] and proved for (2) in [12]. (Strictly speaking, we need connectedness of the fusion graph as in [9, Theorem 12.29]. Otherwise, we need to extend the system of endomorphisms in order to get the non-degeneracy. See [12] for more on this matter.) For (3), if we just consider the usual system of N-N bimodules, then the braiding on it is possibly degenerate, since the N-N bimodules correspond to the even vertices of the Dynkin diagram A_n. We need to extend the system of bimodules so that we have N-N bimodules corresponding to the odd vertices of A_n. Then the braiding there is non-degenerate.) For (4), non-degeneracy of the braiding is proved in [29].

If we have a non-degenerate braiding on a finite system of endomorphism, we can produce an invariant of colored links up to regular isotopy and a 3-dimensional topological quantum field theory of Reshetikhin-Turaev type [27]. See [28] for more details on topological quantum field theory. It has been extensively studied these years.

In subfactor theory, one of the most important applications of braiding is theory of α-induction. This construction was defined by [18] and used systematically in [30, 31]. For further development and unification with Ocneanu's graphical method in [24], see [1, 2, 3, 4, 5, 6]. With this method, one can pass from a braided system to a new system which is not braided in general. Other studies of non-degeneracy of braiding in subfactor theory can be found in [8, 12, 13]. Izumi [12] found that study of the Longo-Rehren subfactors can also be made from a viewpoint of extension (or restriction) of endomorphisms.

§3. Completely rational nets of factors on S^1

Longo [16, 17] has found a deep relation of algebraic quantum field theory to the Jones theory [14] of subfactors. Such a relation was also studied in [10]. Here we explain algebraic quantum field theory on S^1, which is regarded as a compactification of $\mathbf{R}$, and results in [15] in connection to theory of braiding as described above.

We denote by $\mathcal{I}$ the set of non-empty open connected proper subsets of S^1. Such a set is simply called an *interval* here. We study a local irreducible conformal net $\mathcal{A}$ of factors on S^1, which is axiomatized as follows.

For each interval I, we have a factor $\mathcal{A}(I)$ on a fixed Hilbert space H. We also have a strongly continuous unitary representation U on H of the Möbius group $PSU(1,1) = SU(1,1)/\{\pm 1\}$ which acts on S^1 as fractional linear transformations. For an arbitrary set $E \subset S^1$, we define $\mathcal{A}(E)$ to be the von Neumann algebra generated by all the $\mathcal{A}(I)$'s with I contained in E. For $E \subset S^1$, we denote the interior of the complement of E by E'. We then require that they satisfy the following properties. (Though there are slightly different versions of requirements, here we just list a simple set of axioms. Our results in [15] actually hold under a weaker set of assumptions.)

- Isotony: For intervals $I \subset J$, we have $\mathcal{A}(I) \subset \mathcal{A}(J)$.
- Locality: For disjoint intervals I and J, we have $\mathcal{A}(I) \subset \mathcal{A}(J)'$.
- Irreducibility: The von Neumann algebra generated by all $\mathcal{A}(I)$'s is $B(H)$.
- Covariance: For $g \in PSU(1,1)$ and an interval I, we have

$$U(g)\mathcal{A}(I)U(g)^* = \mathcal{A}(gI).$$

- Positive energy: The generator of the rotation subgroup of PSU $(1,1)$ is positive.
- Split property: If $\bar{I}$ and $\bar{J}$ do not intersect for intervals I and J, then $\mathcal{A}(I) \otimes \mathcal{A}(J)$ are naturally isomorphic to $\mathcal{A}(I) \vee \mathcal{A}(J)$.
- Strong additivity: For an interval I and its interior point p, we have $\mathcal{A}(I) = \mathcal{A}(I \setminus \{p\})$.
- Unique existence of vacuum: All the vectors in H fixed by the action of $PSU(1,1)$ are multiples of a fixed non-zero vector Ω.

We then acutally have a stronger form of locality, Haag duality, which says that for an interval I, we have $\mathcal{A}(I') = \mathcal{A}(I)'$. Factors $\mathcal{A}(I)$ are then automatically injective and of type III_1. Important examples of such nets of factors on S^1 have been constructed by A. Wassermann [29] using loop groups of $SU(n)$.

For an arbitrary set $E \subset S^1$, locality implies that $\mathcal{A}(E)$ and $\mathcal{A}(E')$ commute, thus we naturally have an inclusion $\mathcal{A}(E) \subset \mathcal{A}(E')'$. This inclusion can be non-trivial if E is not an interval. We are interested in this inclusion for the case E is a union of two intervals whose closures have no intersection.

A representation π of a net $\mathcal{A}$ on a Hilbert space K is a family $\pi = \{\pi_I\}_{I \subset S^1}$, where π_I is a representation of $\mathcal{A}(I)$ on K and we require that π_J is an extension of π_I for intervals $I \subset J$. A representation π is called *locally normal* if each π_I is normal. Since we deal with only representations on separable Hilbert spaces, the local normality automatically holds. There is also a notion of covariance for such a representation, which is defined as obvious compatibility with a unitary representation of the Möbius group on K, but we do not assume such a property on representations of a net. It turns out that this covariance property automatically holds for representations of a net which we are interested in.

Such a representation of a net is described as a localized transportable endomorphism λ of the quasi-local C^*-algebra as usual in the DHR-framework. See [10] for example. A unitary equivalence class of such representations (or localized endomorphisms) is called a *(superselection) sector* of the net $\mathcal{A}$. For an interval I, such λ gives a sector of $\mathcal{A}(I)$, which is a unitary equivalence class of endomorphisms of $\mathcal{A}(I)$. We are interested in structure of superselection sectors of a net $\mathcal{A}$.

Let E be any union of two intervals on S^1 whose closures have no intersection. Let $\hat{\mathcal{A}}(E) = \mathcal{A}(E')'$ for such E and consider the subfactor $\mathcal{A}(E) \subset \hat{\mathcal{A}}(E)$. It turns out that if this subfactor has a finite index for some E, then we always have the same finite index for any E. When this finiteness holds, we say that the net $\mathcal{A}$ is *completely rational* and write $\mu_{\mathcal{A}}$ for the index value.

Let $\mathcal{A}$ be a completely rational net of factors on S^1 as above. Let E be a disjoint union of two intervals I, J whose closures have no intersection. Let λ and μ be irreducible endomorphisms of $\mathcal{A}$ localized in I and in J, respectively. Then $\lambda\mu$ restricts to an endomorphism of $\mathcal{A}(E)$. Let γ_E be the canonical endomorphism of $\hat{\mathcal{A}}(E)$ into $\mathcal{A}(E)$ and θ_E its restriction on $\mathcal{A}(E)$. We can prove as in [15] that $\lambda\mu$ restricted on $\mathcal{A}(E)$ is contained in θ_E if and only if λ and μ are mutually conjugate. Moreover, in this case, the multiplicity of $\lambda\mu|_{\mathcal{A}(E)}$ in θ_E is one. Using this, we can prove the following result as in [15], which gives a reason for the terminology "completely rational".

Theorem 3.1. *Let $\mathcal{A}$ be a completely rational net on S^1 as above. Then the net $\mathcal{A}$ is rational in the sense that we have only finitely many irreducible superselection sectors $[\lambda_0], [\lambda_1], \ldots, [\lambda_n]$ with finite dimension, and furthermore, we have $\sum_{i=0}^n d(\lambda_i)^2 = \mu_{\mathcal{A}}$.*

Fix an interval I and regard $[\lambda_0], [\lambda_1], \ldots, [\lambda_n]$ as sectors of $\mathcal{A}(I)$. Then $\{\lambda_0, \lambda_1, \ldots, \lambda_n\}$ gives a system of endomorphisms of $\mathcal{A}(I)$ in the sense defined above. The Longo-Rehren construction [18] applies to such a system and we have a factor $\mathcal{A}(I) \otimes \mathcal{A}(I)^{\mathrm{opp}} \subset B$. The index of this Longo-Rehren subfactor is equal to $\sum_{i=0}^n d(\lambda_i)^2 = \mu_{\mathcal{A}}$ and this equality suggests some relation between $\mathcal{A}(E) \subset \hat{\mathcal{A}}(E)$ and the Longo-Rehren subfactor. Actually, we have the following result [15], where the symbol "opp" means the opposite algebra.

Theorem 3.2. *The subfactor $\mathcal{A}(E) \subset \hat{\mathcal{A}}(E)$ is isomorphic to the Longo-Rehren subfactor $\mathcal{A}(I) \otimes \mathcal{A}(I)^{\mathrm{opp}} \subset B$.*

Now we discuss a relation of this result to theory of braiding. It is well-known that we naturally have a braiding on the system of endomorphisms $\{\lambda_0, \lambda_1, \ldots, \lambda_n\}$ of $\mathcal{A}(I)$, and the construction of the braiding goes roughly as follows. (See [1, Section 2.2], for example.)

Take endomorphisms λ_j, λ_k localized in an interval I. Choose two intervals I_1, I_2 with empty intersection, and Then there are unitaries U_1 and U_2 such that $\lambda'_j = \mathrm{Ad}(U_1) \circ \lambda_j$ and $\lambda'_k = \mathrm{Ad}(U_2) \circ \lambda_k$ are localized in I_1 and I_2, respectively. Set $\varepsilon(\lambda_j, \lambda_k) = \lambda_k(U_1^*) U_2^* U_1 \lambda_j(U_2)$. This unitary does not depend on choices of U_1, U_2, and it depends only on the "order" of I_1 and I_2 on S^1. In this way, we get two unitaries $\varepsilon^{\pm}(\lambda_j, \lambda_k)$ and these give a braiding on the system of endomorphisms $\{\lambda_0, \lambda_1, \ldots, \lambda_n\}$ of a type III factor $\mathcal{A}(I)$.

On one hand, the above theorem says that the subfactor $\mathcal{A}(E) \subset \hat{\mathcal{A}}(E)$ is isomorphic to the Longo-Rehren subfactor arising from a braided system of endomorphisms. As mentioned above, the Longo-Rehren construction produces a non-degenerate braiding, but if we have a non-degenerate braiding from the beginning, the Longo-Rehren construction just produces a direct product system of the original braided system and its opposite system as in [23, 8, 12]. So if the original system $\{\lambda_0, \lambda_1, \ldots, \lambda_n\}$ has a non-degenerate braiding, then the systems of endomorphisms of $\mathcal{A}(E)$ and $\hat{\mathcal{A}}(E)$ are isomorphic for the subfactor $\mathcal{A}(E) \subset \hat{\mathcal{A}}(E)$. On the other hand, it is trivial from the construction that the subfactor $\mathcal{A}(E) \subset \hat{\mathcal{A}}(E)$ is self-dual. In comparison to the study of the Longo-Rehren subfactors (or asymptotic inclusions) arising from a non-degenerate system as mentioned above, this self-duality suggests that the braiding on the system $\{\lambda_0, \lambda_1, \ldots, \lambda_n\}$ is non-degenerate. We have proved in [15] that this is indeed the case.

Theorem 3.3. *The braiding on the system $\{\lambda_0, \lambda_1, \ldots, \lambda_n\}$ is non-degenerate and thus we have a unitary representation of $SL(2, \mathbf{Z})$.*

As a final remark, we note that it is not very easy to verify the complete rationality since it involves the index computation, but Xu has verified this condition in several cases. In the case of Wassermann's net [29] arising from loop groups of $SU(n)$, Xu [32] computed the index of the subfactor $\mathcal{A}(E) \subset \hat{\mathcal{A}}(E)$ using a brilliant idea and thus verified the complete rationality. He then also applied the above our results in various other contexts in [33, 34] by verifying the complete rationality.

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A notion of Morita equivalence between subfactors

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Abstract.

We will review a notion of Morita equivalence between subfactors, which is a variation of Morita equivalence in ring and module theory. The main result is stated as follows: for arbitrary two Morita equivalent subfactors of hyperfinite II_1 factors with finite Jones index and finite depth we can always choose a finite dimensional non-degenerate commuting square which gives rise to the subfactors isomorphic to the original ones. As an application of Morita equivalence between subfactors in connection with recent developments of theory of finite dimensional weak C^*-Hopf algebras, we will make a brief comment about the 3-dimensional topological quantum field theories obtained from subfactors with finite index and finite depth.

§1. Introduction

A basic tool to construct inclusions of hyperfinite II_1 factors with finite Jones index would be the method of the commuting squares. From a finite dimensional non-degenerate commuting square

$$\begin{array}{ccc} R_{00} & \subset & R_{01} \\ \cap & & \cap \\ R_{10} & \subset & R_{11} \end{array}$$

we can make the double sequence of finite dimensional C^*-algebras $\{R_{ij}\}$ by iterating the Jones' basic construction. Then we get horizontal and vertical inclusions of hyperfinite II_1 factors with finite Jones indices $N = \vee_{n=0}^{\infty} R_{0n} \subset \vee_{n=0}^{\infty} R_{1n} = M$ and $P = \vee_{n=0}^{\infty} R_{n0} \subset \vee_{n=0}^{\infty} R_{n1} = Q$ under the assumption that the Bratteli diagrams of the initial inclusions in the commuting square are connected.

A natural question is whether there is any relationship between these two inclusions. In particular, when one of the inclusions is of finite depth, then it is of great interest to know whether so is the other. Such questions were first raised by V. Jones in 1995. This question was answered

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affirmatively using techniques of paragroup theory and the answer says these inclusions have the same "size as C^*-tensor categories" [S1].

When we have an inclusion of hyperfinite II_1 factors with finite Jones index and finite depth, we can construct the topological quantum field theory in three dimensions (TQFT) based on the triangulation of the given 3-dimensional manifold V. This method was first done by Ocneanu. In the case of a closed manifold, we get a complex number. The simplest example will be the three dimensional sphere S^3. In this case, the value of the theory is given by 1/(the size of C^*-tensor category). With this observation and the above relationship between the vertical and horizontal inclusions, we can prove TQFT's constructed from the above vertical and horizontal inclusions are complex conjugate to each other [S2]. This gives a finer answer to Jones' question.

There should be a reasonable explanation for the above phenomenon. We will see this can be achieved by the notion of "Morita equivalence" between subfactors, which tells us that equivalent inclusions have "equivalent representation theory" to each other. Actually, the notion of Morita equivalence between subfactors is formulated in a quite similar way as the one in ring and module theory. With this equivalence, we will see there always exists some symmetry described by Morita equivalence on the commuting squares in question.

§2. Definition and examples of Morita equivalence between subfactors

The following observation is fundamental for our formalism.

Let $N \subset M$ be an inclusion of II_1 factors with finite Jones index and finite depth and let $N \subset M \subset M_1 \subset M_2 \subset \cdots \subset M_k \subset \cdots$ be the Jones' tower of II_1 factors obtained by the basic construction. By the assumption of finite depth, we have finitely many irreducible bimodules when decomposing a P-Q bimodule ${}_PL^2(M_k)_Q$, where P and Q are either of M or N. Hence, one gets a finite system of graded bimodules (called a graded fusion rule algebra) consisting of irreducible N-N, N-M, M-N and M-M bimodules in the sense that it is closed under the operations of the relative tensor product, conjugation and direct sum decomposition of bimodules. We call a pair of these finite systems of four kinds of bimodules and additional information about homomorphisms between bimodules (called quantum $6j$-symbols) a finite paragroup of (M, N)-type. In other words, for finite systems of M-M and N-N bimodules, a finite paragroup of (M, N)-type is defined when there exists an N-M bimodule which generates a finite system of four kinds of graded bimodules.

This gives rise to the following definition. We denote a finite system of P-Q bimodules with finite index by ${}_P\mathcal{M}_Q$.

Definition 2.1. *Let A and B be II_1 factors. We say that a finite system of A-A bimodules ${}_A\mathcal{M}_A$ is* Morita equivalent *to that of B-B bimodules ${}_B\mathcal{M}_B$, if there exist A-B bimodules and the four kinds of bimodules A-A, B-B, A-B, B-A make a finite system. We denote this relation by ${}_A\mathcal{M}_A \sim {}_B\mathcal{M}_B$. (This is an equivalence relation.)*

Note that finite system of four kinds of bimodules in Definition 2.1 gives a finite paragroup of $(A,\ B)$-type.

We are ready to introduce the notion of Morita equivalence between subfactors. See [K] in the case of strongly amenable paragroups.

Definition 2.2. *Let $A \subset B$ and $C \subset D$ be inclusions of II_1 factors with finite Jones index and finite depth. We say that these two subfactors are* Morita equivalent *if ${}_B\mathcal{M}_B \sim {}_C\mathcal{M}_C$.*

Since we have finite paragroups of (A, B)- and (C, D)-type from the inclusions $A \subset B$ and $C \subset D$ respectively, we get ${}_A\mathcal{M}_A \sim {}_B\mathcal{M}_B \sim {}_C\mathcal{M}_C \sim {}_D\mathcal{M}_D$. This means that one may use, for instance, ${}_A\mathcal{M}_A \sim {}_D\mathcal{M}_D$ instead in Definition 2.2. And the standard arguments in subfactor theory give the equalities $\dim_A\mathcal{M}_A = \dim_B\mathcal{M}_B = \dim_C\mathcal{M}_C = \dim_D\mathcal{M}_D$, where $\dim_A\mathcal{M}_A = \sum_{X\in{}_A\mathcal{M}_A}(\dim_A X_A)^2$ (Here, summation is taken over a representative set of irreducible A-A bimodules) is a global index (or dimension) for ${}_A\mathcal{M}_A$.

Typical examples are in order.

Example 1 Let G be a finite group and H be its subgroup which is relatively simple. Two subfactors $N \subset N \rtimes G$ and $N \rtimes H \subset N \rtimes G$ obtained by the crossed product of an outer action of G on a II_1 factor N are Morita equivalent because the systems of the $N \rtimes G$-$N \rtimes G$ bimodules arising from both subfactors are identified by the Mackey machine of Ocneanu [KY].

Example 2 More generally, let $P \subset Q \subset R$ be inclusions of II_1 factors. Assume that the inclusion $P \subset R$ has finite Jones index and finite depth. It is known that the intermediate inclusion of $P \subset R$ is also of finite depth. Then the system of the R-R bimodules arising from $Q \subset R$ is a subsystem of that arising from $P \subset R$. Moreover, assume that both of the system of the R -R bimodules have the same global indices. Then these two subfactors are Morita equivalent since both systems of the R-R bimodules are identified. Hence, $P \subset R$ and $Q \subset R$ are Morita equivalent in this case.

Example 3 Let $N \subset M$ be an inclusion of II_1 factors with finite index and finite depth. Then, inclusions $N \subset M$ and $N \subset M_k$ are clearly Morita equivalent.

When we take k to be the least integer such that $N \subset M$ becomes of depth two (this is possible since we are dealing with a finite depth subfactor), one finds that there is an action of a finite dimensional biconnected weak C^*-Hopf algebra A on N and a crossed product algebra $N \rtimes A$ is isomorphic to M_k, by a characterization of a depth two subfactor due to Nikshych and Vainerman [NV1]. Roughly speaking, a finite dimensional weak C^*-Hopf algebra is a finite dimensional C^*-algebra which satisfies axioms of a C^*-Hopf algebra with non-unital coproduct and counit. See [BNS] for details. In [NV2], they proved a finite system of N-N bimodules arising from $N \subset N \rtimes A$ is equivalent to the category of unitary representations $\mathrm{Rep}(A^*)$ as monoidal categories. Hence, a finite system of M-M bimodules arising from $N \subset M$ and the finite system of $\mathrm{Rep}(A^*)$ are Morita equivalent in a broad sense.

Morita equivalence of subfactors is naturally associated with a finite dimensional non-degenerate commuting square

$$\begin{array}{ccc} R_{00} & \subset & R_{01} \\ \cap & & \cap \\ R_{10} & \subset & R_{11}. \end{array}$$

As in Introduction, iterating the basic construction in the horizontal and vertical directions, we get two subfactors $M_0 \subset M_1$ and $Q_0 \subset Q_1$, respectively. Instead of the language of commuting squares, we use the paragroup theoretical one. Then, we have the biunitary connection on the four finite bipartite connected graphs corresponding to the inclusion matrices in the initial commuting square. And by the string algebra construction, we have the following double sequence of finite dimensional C^*-algebras.

$$\begin{array}{ccccccc} \mathbb{C} = A_{0,0} & \subset & A_{0,1} & \subset & A_{0,2} & \subset \cdots & \subset M_0 = A_{0,\infty} \\ \cap & & \cap & & \cap & & \cap \\ A_{1,0} & \subset & A_{1,1} & \subset & A_{1,2} & \subset \cdots & \subset M_1 = A_{1,\infty} \\ \cap & & \cap & & \cap & & \cap \\ A_{2,0} & \subset & A_{2,1} & \subset & A_{2,2} & \subset \cdots & \subset M_2 = A_{2,\infty} \\ \cap & & \cap & & \cap & & \cap \\ \vdots & & \vdots & & \vdots & & \vdots \\ \cap & & \cap & & \cap & & \\ Q_0 = A_{\infty,0} & \subset & Q_1 = A_{\infty,1} & \subset & Q_2 = A_{\infty,2} & \subset \cdots & \end{array}$$

Assume that either $M_0 \subset M_1$ or $Q_0 \subset Q_1$ is of finite depth. (Hence, by [S1, Corollary 2.2], both of them are of finite depth.) By Ocneanu's compactness argument, the higher relative commutant $M_0' \cap M_k$ of the inclusion $M_0 \subset M_1$ is contained in $A_{k,0}$ for each k. Denote $\vee_{k=0}^{\infty} M_0' \cap M_k$ by B_k. See the following diagram.

$$
\begin{array}{ccccccccccc}
B_0 & \subset & A_{0,0} & \subset & A_{0,1} & \subset & A_{0,2} & \subset & \cdots & \subset & M_0 \\
\cap & & \cap & & \cap & & \cap & & & & \cap \\
B_1 & \subset & A_{1,0} & \subset & A_{1,1} & \subset & A_{1,2} & \subset & \cdots & \subset & M_1 \\
\cap & & \cap & & \cap & & \cap & & & & \cap \\
B_2 & \subset & A_{2,0} & \subset & A_{2,1} & \subset & A_{2,2} & \subset & \cdots & \subset & M_2 \\
\cap & & \cap & & \cap & & \cap & & & & \cap \\
\vdots & & \vdots & & \vdots & & \vdots & & & & \vdots \\
\cap & & \cap & & \cap & & \cap & & & & \\
B_\infty & \subset & Q_0 & \subset & Q_1 & \subset & Q_2 & \subset & \cdots & &
\end{array}
$$

Then one can show that the system of the B_∞^{op}-B_∞^{op} bimodules arising from $B_\infty^{op} \subset Q_1^{op}$ is identified with that of the M_0-M_0 bimodules arising from $M_0 \subset M_1$. This implies that the subfactor $B_\infty^{op} \subset Q_1^{op}$ is Morita equivalent to $M_0 \subset M_1$. Since one can verify that $B_\infty^{op} \subset Q_0^{op} \subset Q_1^{op}$ satisfy the assumption of example 2, $Q_0^{op} \subset Q_1^{op}$ is Morita equivalent to $M_0 \subset M_1$.

§3. Reconstruction of a commuting square from the equivalent subfactors

In Section 2, we saw that horizontal and vertical inclusions of hyperfinite II_1 factors obtained from a finite dimensional non-degenerate commuting square are "opposite equivalent". Now, our main theorem claims that opposite equivalence of subfactors has enough information to insure that they certainly come from a finite dimensional non-degenerate commuting square.

Let ${}_D g_C$ be the D-C bimodule ${}_D D_C$ of the finite paragroup of (C, D)-type which canonically arises from $C \subset D$, ${}_A h_B$ be the A-B bimodule ${}_A B_B$ of the A-B paragroup which canonically arises from $A \subset B$. Denote the direct sum of the unitarily inequivalent irreducible C-A bimodules by ${}_C X_A$. Construct the following inclusions of finite dimensional C^*-algebras.

$$
\begin{array}{ccc}
\mathrm{End}(g(\bar{g}g)^{n-1} \otimes_C X_A \otimes (h\bar{h})^{n-1}h) & \subset & \mathrm{End}(g(\bar{g}g)^{n-1} \otimes_C X_A \otimes (h\bar{h})^{n}) \\
\cap & & \cap \\
\mathrm{End}((\bar{g}g)^{n} \otimes_C X_A \otimes (h\bar{h})^{n-1}h) & \subset & \mathrm{End}((\bar{g}g)^{n} \otimes_C X_A \otimes (h\bar{h})^{n})
\end{array}
$$

Then, one can show that this is a non-degenerate commuting square of period two when n is large enough. Thus we have the biunitary connection arising from the above commuting square. By the string algebra construction, we get the following double sequences of finite dimensional C^*-algebras $A_{k,l} = \mathrm{End}(\underbrace{\cdots \otimes \bar{g} \otimes g}_{k-\text{folds}} \otimes {}_C *_A \otimes \underbrace{h \otimes \bar{h} \cdots}_{l-\text{folds}})$, where ${}_C *_A$ is an irreducible C-A bimodule. See the following diagram.

$$\begin{array}{ccccccccc}
A_{0,0}(= \mathrm{End}({}_C *_A)) & \subset & A_{0,1} & \subset & A_{0,2} & \subset & \cdots & \subset M_0 = A_{0,\infty} \\
\cap & & \cap & & \cap & & & \cap \\
A_{1,0} & \subset & A_{1,1} & \subset & A_{1,2} & \subset & \cdots & \subset M_1 = A_{1,\infty} \\
\cap & & \cap & & \cap & & & \cap \\
A_{2,0} & \subset & A_{2,1} & \subset & A_{2,2} & \subset & \cdots & \subset M_2 = A_{2,\infty} \\
\vdots & & \vdots & & \vdots & & & \vdots \\
\cap & & \cap & & \cap & & & \\
Q_0 = A_{\infty,0} & \subset & Q_1 = A_{\infty,1} & \subset & Q_2 = A_{\infty,2} & \subset & \cdots &
\end{array}$$

Then we have our main theorem.

Theorem 3.1. [S3] *In the above notations, the subfactors $M_0 \subset M_1$ and $Q_0 \subset Q_1$ are isomorphic to $C^{op} \subset D^{op}$ and $A \subset B$, respectively.*

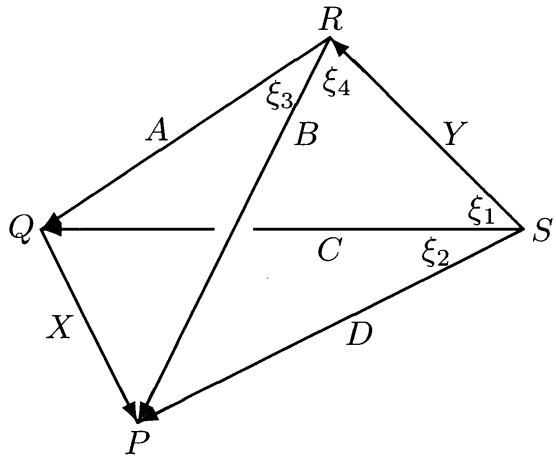
Example Assume that the inclusion $M_0 \subset M_1$ obtained in the theorem is isomorphic to the inclusion $R^G \subset R$, where R is the hyperfinite II_1 factor and G is a finite group acting freely on R by $\alpha \in \mathrm{Aut}(R)$. Then, we know that the finite system of B_∞^{op}- B_∞^{op} bimodules arising from $B_\infty^{op} \subset Q_1^{op}$ is isomorphic to that of R^G-R^G bimodules arising from $R^G \subset R$. In such a situation, Schaflitzel proved that $B_\infty^{op} \subset Q_1^{op}$ is isomorphic to $R^G \subset (R \otimes M_n(\mathbb{C}))^H$, where H $(\subset G)$ is a subgroup acting on $R \otimes M_n(\mathbb{C})$ by $\alpha|_H \otimes \mathrm{Ad}\Psi$ (Ψ is a projective representation of H on $M_n(\mathbb{C})$) [Sc]. This characterization gives a restriction of the inclusion $Q_0^{op} \subset Q_1^{op}$. Namely, it is isomorphic to $R^G \subset (R \otimes M_n(\mathbb{C}))^H$ in a good situation (i.e., $*$-flat case) and in general it is an intermediate inclusion of $R^G \subset (R \otimes M_n(\mathbb{C}))^H$, i.e., $R^G \subset Q_0^{op} \subset Q_1^{op} = (R \otimes M_n(\mathbb{C}))^H$.

§4. A comment on topological quantum field theories in 3 dimensions constructed from subfactors

From an inclusion $N \subset M$ of type II_1 factors with finite index and finite depth, we have two kinds of 3-dimensional topological quantum field theories (TQFT for short). Let us recall briefly how they are obtained.

Since we have a finite paragroup of (M, N)-type, we have a graded fusion rule algebra. A space of homomorphisms $\mathrm{Hom}({}_P X_Q \otimes_Q Y_R, {}_P Z_R)$ for irreducible X, Y and Z is a finite dimensional Hilbert space and has an orthonormal basis with respect to the inner product defined by $(\xi, \eta) = \xi \cdot \eta^* \in \mathrm{End}(Z) = \mathbb{C}$. Then, for each orthonormal basis of homomorphisms $\xi_1 \in \mathrm{Hom}({}_Q A_R \otimes_R Y_S, {}_Q C_S)$, $\xi_2 \in \mathrm{Hom}({}_P X_Q \otimes_Q C_S, {}_P D_S)$,

$\xi_3 \in \mathrm{Hom}({}_PX_Q \otimes {}_QA_R, {}_PB_R)$, $\xi_4 \in \mathrm{Hom}({}_PB_R \otimes {}_RY_S, {}_PD_S)$, where P, Q, R, S are either of M or N and ${}_PX_Q$, ${}_RY_S$, ${}_QA_R$, ${}_PB_R$, ${}_QC_S$, ${}_PD_S$ are irreducible bimodules, we have a complex number $W(A,B,C,D,X,Y|\xi_1, \xi_2,\xi_3,\xi_4)$ defined by $\xi_4 \cdot (\xi_3 \otimes \mathrm{id}) \cdot (\mathrm{id} \otimes \xi_1)^* \cdot \xi_2^* \in \mathrm{End}({}_PD_S) = \mathbb{C}$. Then, the quantum $6j$-symbol Z is defined by $Z(A,B,C,D,X,Y|\xi_1,\xi_2,\xi_3,\xi_4) = [B]^{-1/4}[C]^{-1/4}W(A,B,C,D,X,Y|\xi_1,\xi_2,\xi_3,\xi_4)$. Now, we associate each tetrahedron with a quantum $6j$-symbol. See the following figure.



Let V be a compact 3-dimensional manifold without boundaries and $\mathcal{T}$ be a triangulation of V. For each vertex of $(V,\mathcal{T})$, we assign the label either of M or N and fix it. Let C_e be the possible assignment of irreducible bimodules in a finite paragroup to each edge of triangulation. Namely, we consider a face of a tetrahedron as a space of homomorphisms. For each assignment of C_e, let C_f be the assignment of an orthonormal basis of homomorphisms to each face of a tetrahedron. Now, $\zeta(V,\mathcal{T})$ is defined to be a complex number $(\dim\mathcal{P})^{-a}\sum_{C_e}\sum_{C_f}\Pi_{X:\mathrm{edges}}[X]^{\frac{1}{2}}\Pi Z$ (tetrahedron), where $\dim\mathcal{P}$ is the global index of the finite paragroup $\mathcal{P}$, a is the number of vertices and $[X] = (\dim_N X)(\dim X_N)$. The important point here is that $\zeta(V,\mathcal{T})$ does not depend on neither the triangulation or labelings of N and M on each vertex. Hence, we get a topological invariant of V and we may write $\zeta(V)$ instead of $\zeta(V,\mathcal{T})$. (When V has a boundary, some modification is needed.) Moreover, $\zeta(V)$ can be extended to satisfy the axioms of a (unitary) TQFT in the sense of M. Atiyah [A]. Namely, ζ is a functor from the category of cobordisms of surfaces to the category of finite dimensional Hilbert spaces with some properties for cut and gluing. This type of construction was first achieved by Turaev and Viro in the case of data from a quantum group of $U_q(sl_2)$ [TV] and the present formalism using a paragroup is due to A. Ocneanu [O1]. We call our ζ the Turaev-Viro type TQFT obtained from a finite paragroup.

The notion of Morita equivalence between subfactors is remarkably efficient when one constructs topological quantum field theory of Turaev-Viro type. Denote by ζ_{M-M} (resp. ζ_{N-N}) Turaev-Viro type TQFT constructed from the data of M-M(resp. N-N) fusion rules and associated quantum $6j$-symbols obtained from $N \subset M$. Then, ζ_{M-M} and ζ_{N-N} give rise to the same TQFT's. More generally, we have the same TQFT's for finite paragroups arising from Morita equivalent subfactors.

Ocneanu introduced a new construction of an inclusion of II_1 factors with finite depth and finite index, the asymptotic inclusion $M \vee (M' \cap M_\infty) \subset M_\infty$, from an inclusion of II_1 factors $N \subset M$ with finite index and finite depth, where $M_\infty = \vee_{n=1}^{\infty} M_n$. He had noticed that the finite system $\mathcal{M}_\infty$ of M_∞-M_∞ bimodules arising from the asymptotic inclusion was an analogue of the quantum double in quantum group theory. Actually, he claimed that the finite system $\mathcal{M}_\infty$ has a non-degenerate braiding through TQFT of Turaev-Viro type [O1], [EK]. In fact, one can prove that $\mathcal{M}_\infty$ satisfies the axioms of modular tensor category. Hence, with a general machinery of Turaev, one can construct a Reshetikhin-Turaev type TQFT based on Dehn surgery of 3-dimensional manifolds [O2], [T].

By the definition of the Morita equivalence of two finite depth subfactors, they give rise to the isomorphic Turaev-Viro type topological quantum field theories. Since, by Example 3 in Section 2, a finite depth subfactor $N \subset M$ is always Morita equivalent to an inclusion of the form $N \subset N \rtimes A$, where A is a finite dimensional weak C^*-Hopf algebra, one knows that TQFT's obtained from the former and the latter are isomorphic on one hand. On the other hand, a system of N-N bimodules arising from the latter inclusion is equivalent to the category of finite dimensional unitary representations $\mathrm{Rep}(A^*)$ of the dual weak C^*-Hopf algebra A^* of A. Hence, we may conclude that Turaev-Viro type TQFT obtained from a finite depth subfactor is always obtained from the data of the category of finite dimensional unitary representations of a weak C^*-Hopf algebra with quantum $6j$-symbols.

Keeping the situation in the previous paragraph, when we take the asymptotic inclusions for both $N \subset M$ and $N \subset N \rtimes A$, one can prove that the tensor category of M_∞-M_∞ bimodules arising from the former inclusion is isomorphic to that of the latter because the original subfactors are Morita equivalent and modular tensor categories obtained from the asymptotic inclusions are described by TQFT's of Turaev-Viro type [O2], [EK]. Moreover, one can prove the latter tensor category is isomorphic to the category $\mathrm{Rep}D(A^*)$ of finite dimensional unitary representations of $D(A^*)$ as unitary modular tensor categories, where $D(A^*)$ is the quantum double weak C^*-Hopf algebra of A^* [BS], [NTV]. As a consequence, the Reshetikhin-Turaev type TQFT obtained from $N \subset M$ through the asymptotic inclusion is isomorphic to the one obtained from the category of finite dimensional unitary representations of the quantum double of a weak C^*-Hopf algebra. Hence, TQFT's of both Turaev-Viro and Reshetikhin-Turaev type constructed from subfactors with finite Jones index and finite depth are obtained within the category of finite dimensional unitary representations of finite dimensional weak C^*-Hopf algebras. See [S4] for more details.

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Amalgamated free product over Cartan subalgebra, II Supplementary Results & Examples

Yoshimichi Ueda

§1 Introduction

Let $A \supseteq D \subseteq B$ be two von Neumann algebras together with a common Cartan subalgebra. Then the amalgamated free product $M = A *_D B$ with respect to the unique conditional expectations from A, B onto D can be considered. In our previous paper [U1], the questions of its factoriality and type classification were discussed in detail, which will be reviewed in §4. The main purpose of the paper is to give further supplementary results obtained after the completion of the previous paper together with discussions on some examples.

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§2 Amalgamated Free Products of von Neumann algebras

Let $A \supseteq D \subseteq B$ be σ-finite von Neumann algebras, and let $E_D^A : A \to D$, $E_D^B : B \to D$ be faithful normal conditional expectations. Then one can consider the amalgamated free product of A and B over D with respect to the conditional expectations E_D^A, E_D^B:

$$(M, E_D^M) = (A, E_D^A) *_D (B, E_D^B).$$

It is defined as a pair of a von Neumann algebra M into which the triple $A \supseteq D \subseteq B$ is embedded and a faithful normal conditional expectation $E_D^M : M \to D$, and characterized by the following three conditions:

- M is generated by the subalgebras A, B;

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- $E_D^M|_A = E_D^A$, $E_D^M|_B = E_D^B$;
- A, B are free with amalgamation over D in the D-probability space $(M \supseteq D, E_D^M)$, see [VDN], i.e.,

$$E_D^M(\{\text{alternating words in } A^\circ, B^\circ\}) = 0,$$

where we denote $A^\circ = \mathrm{Ker} E_D^A$, $B^\circ = \mathrm{Ker} E_D^B$ as usual.

For the details, we refer to [P2],[VDN],[U1] (see also [BD]).

In analysis on type III factors, modular automorphisms are of central importance so that we need to compute the modular automorphisms $\sigma_t^{\varphi\circ E_D^M}$ $(t \in \mathbf{R})$ for a faithful normal state φ on D.

Theorem 2.1. **([U1, Theorem 2.6])** *We have*

$$\sigma_t^{\varphi\circ E_D^M}|_A = \sigma_t^{\varphi\circ E_D^A}, \quad \sigma_t^{\varphi\circ E_D^M}|_B = \sigma_t^{\varphi\circ E_D^B}. \tag{2.1}$$

The modular operator $\Delta_{\varphi\circ E_D^M}$ and the modular conjugation J^M can be also computed explicitly. (See [U1, Appendix I].)

§3 Amalgamated Free Products over Cartan Subalgebras

Let A and B be von Neumann algebras with separable preduals, and we suppose that they have a common Cartan subalgebra D or equivalently that there is a common subalgebra D satisfying:

- D is a MASA in both A and B;
- there are (automatically unique faithful) normal conditional expectations $E_D^A : A \to D$, $E_D^B : B \to D$;
- the normalizers $\mathcal{N}_A(D)$, $\mathcal{N}_B(D)$ generate the whole A, B, respectively.

(See [FM].) Let

$$(M, E_D^M) = (A, E_D^A) *_D (B, E_D^B),$$

and we will write $M = A *_D B$ since there is no other choice of normal conditional expectations from A, B onto D.

The triple $A \supseteq D \subseteq B$ produces two countable non-singular Borel equivalence relations $\mathcal{R}_A$, $\mathcal{R}_B$ over a common standard Borel probability space (X, μ) in such a way that

$$A = W^*_{\sigma_A}(\mathcal{R}_A), \quad B = W^*_{\sigma_B}(\mathcal{R}_B), \quad D = L^\infty(X, \mu), \tag{3.1}$$

where $W^*_{\sigma_A}(\mathcal{R}_A)$, $W^*_{\sigma_B}(\mathcal{R}_B)$ denote the von Neumann algebras constructed from $\mathcal{R}_A$, $\mathcal{R}_B$ together with relevant 2-cocycles σ_A, σ_B, respectively, by the Feldman-Moore construction ([FM]). After fixing a point realization $D = L^\infty(X, \mu)$, such a pair $(\mathcal{R}_A, \mathcal{R}_B)$ is uniquely determined up to null set. (This fact will be discussed in [U2] in detail.) Therefore, the countable non-singular Borel equivalence relation

$$\mathcal{R}_M := \mathcal{R}_A \vee \mathcal{R}_B \ (\subseteq X \times X)$$

is a canonical object attached to the triple $A \supseteq D \subseteq B$, and we call this equivalence relation the *canonical equivalence relation* associated with the amalgamated free product $M = A *_D B$ or the triple $A \supseteq D \subseteq B$.

§4 Factoriality & Type Classification ([U1])

In this section, we review our previous paper [U1].

Let $A \supseteq D \subseteq B$ be as in §3, i.e., two von Neumann algebras (with separable preduals) and a common Cartan subalgebra. We here discuss the amalgamated free product $M = A *_D B$. The first problem is its factoriality, namely, to find a suitable sufficient condition for the amalgamated free product M to be a factor. A satisfactory answer to the problem was given in our previous paper.

Theorem 4.1. **([U1, Theorem 4.3])** *If either A or B is a factor of non-type* I*, then the amalgamated free product $M = A *_D B$ over a common Cartan subalgebra D becomes a factor. More precisely, if A (or B) is a factor of non-type* I*, then there is a faithful normal state φ on D such that*

$$(4.1)\qquad (A_{\varphi\circ E_D^A})' \cap M \subseteq A \quad \left(\textit{resp. } (B_{\varphi\circ E_D^B})' \cap M \subseteq B\right).$$

Furthermore, if A (or B) is further assumed to be of type III_λ $(0 < \lambda \leq 1)$*, then the state φ can be chosen in such a way that*

$$(4.2)\qquad (A_{\varphi\circ E_D^A})' \cap A = \mathbf{C}1 \quad \left(\textit{resp. } (B_{\varphi\circ E_D^B})' \cap B = \mathbf{C}1\right).$$

This result can be generalized further. Such a generalization will be discussed later, see Remark 4.8 (2),(3).

The second problem seems Murray-von Neumann-Connes' type classification of the amalgamated free product M. In this direction, we obtained the following corollaries of Theorem 4.1:

Corollary 4.2. **[U1, Corollary 4.5])** *Suppose that both A and B are factors of non-type* I. *If $M = A *_D B$ is of type* III_0, *then both A and B must also be of type* III_0.

Corollary 4.3. **([U1, p.377])** *Suppose that both A and B are factors of non-type* I.
(1) *If either A or B of type* II_1 *and if $M = A *_D B$ is semi-finite* (*i.e., has a faithful semi-finite normal trace*), *then M must be of type* II_1.
(2) *If either A or B is of type* III_λ $(0 < \lambda < 1)$, *then $M = A *_D B$ must be of type* $\mathrm{III}_{\lambda^{1/n}}$ *or of type* III_1.
(3) *If either A or B is of type* III_1, *then $M = A *_D B$ must be of type* III_1.
(4) *If A is of type* III_λ *and B of type* III_μ *with* $\dfrac{\log\lambda}{\log\mu} \notin \mathbb{Q}$, *then $M = A *_D B$ must be of type* III_1.

For a while, we assume that $A \supseteq D \subseteq B$ is a general triple of σ-finite von Neumann algebras together with faithful normal conditional expectations $E_D^A : A \to D$, $E_D^B : B \to D$, and let

$$(M, E_D^M) = (A, E_D^A) *_D (B, E_D^B)$$

be the amalgamated free product. Choose and fix a faithful normal state φ on D, and we set:

$$\widetilde{A} := A \rtimes_{\sigma^{\varphi\circ E_D^A}} \mathbf{R} \supseteq \widetilde{D} := D \rtimes_{\sigma^\varphi} \mathbf{R} \subseteq \widetilde{B} := B \rtimes_{\sigma^{\varphi\circ E_D^B}} \mathbf{R},$$

and

$$\widetilde{M} := M \rtimes_{\sigma^{\varphi\circ E_D^M}} \mathbf{R}.$$

Then there are faithful normal conditional expectations

$$\widehat{E_D^A} : \widetilde{A} \to \widetilde{D}\ ; \qquad \widehat{E_D^A}\left(\int_{-\infty}^{\infty} a(t)\lambda(t)dt\right) := \int_{-\infty}^{\infty} E_D^A(a(t))\lambda(t)dt,$$

$$\widehat{E_D^B} : \widetilde{B} \to \widetilde{D}\ ; \qquad \widehat{E_D^B}\left(\int_{-\infty}^{\infty} b(t)\lambda(t)dt\right) := \int_{-\infty}^{\infty} E_D^B(b(t))\lambda(t)dt,$$

$$\widehat{E_D^M} : \widetilde{M} \to \widetilde{D}\ ; \qquad \widehat{E_D^M}\left(\int_{-\infty}^{\infty} m(t)\lambda(t)dt\right) := \int_{-\infty}^{\infty} E_D^M(m(t))\lambda(t)dt.$$

Theorem 4.4. **([U1, Theorem 5.1])** *In the current general setting, we have*

$$\left(\widetilde{M}, \widehat{E_D^M}\right) \cong \left(\widetilde{A}, \widehat{E_D^A}\right) *_{\widetilde{D}} \left(\widetilde{B}, \widehat{E_D^B}\right). \tag{4.3}$$

Moreover, the dual action θ_t^M ($t \in \mathbf{R}$) associated with M is determined by those θ_t^A, θ_t^B associated with A, B, respectively, in such a way that

$$\theta_t^M|_{\widetilde{A}} = \theta_t^A, \quad \theta_t^M|_{\widetilde{B}} = \theta_t^B. \tag{4.4}$$

Let us return to the original setting, namely, the triple $A \supseteq D \subseteq B$ consists of two von Neumann algebras with separable preduals and a common Cartan subalgebra. By Theorem 4.4, we have

$$\widetilde{M}\left(= \widetilde{A *_D B}\right) \cong \widetilde{A} *_{\widetilde{D}} \widetilde{B}. \tag{4.5}$$

Since $\widetilde{D}$ is also a common Cartan subalgebra in both $\widetilde{A}$ and $\widetilde{B}$, we write $\widetilde{A} *_{\widetilde{D}} \widetilde{B}$ as the amalgamated free product von Neumann algebra of $\widetilde{A}$ and $\widetilde{B}$ over $\widetilde{D}$ with respect to the conditional expectations $\widehat{E_D^A}$, $\widehat{E_D^B}$ since no confusion is possible.

We further suppose that both A and B are factors of non-type I in what follows. Theorem 4.4 (or (4.5)) together with the proof of Theorem 4.2 implies

Theorem 4.5. **([U1, Theorem 5.4])** *In the current setting, we have*

$$\mathcal{Z}(\widetilde{M}) = \mathcal{Z}(\widetilde{A}) \cap \mathcal{Z}(\widetilde{B}) \subseteq \widetilde{D}. \tag{4.6}$$

Let (X_A, F_t^A), (X_B, F_t^B) be the flows of weights ([CT]) of A, B, respectively. Fix a point realization $D = L^\infty(X, \mu)$, and set $X_D := X \times \mathbf{R}$ equipped with the usual product measure $d\mu \otimes e^{-t}dt$ and $F_t^D(x, s) := (x, s+t)$. Then there are two factor maps

$$\pi_A^D : (X_D, F_t^D) \to (X_A, F_t^A), \quad \pi_B^D : (X_D, F_t^D) \to (X_B, F_t^B)$$

since $\widetilde{D}$ is a common Cartan subalgebra in both $\widetilde{A}$, $\widetilde{B}$. Let (X_M, F_t^M) be the flow of weights of M. Theorem 4.5 says that there are three factor maps

$$\pi_M^A : (X_A, F_t^A) \to (X_M, F_t^M), \quad \pi_M^B : (X_B, F_t^D) \to (X_M, F_t^M),$$

$$\pi_M^D : (X_D, F_t^D) \to (X_M, F_t^M).$$

Corollary 4.6. **([U1, Corollary 5.6])** *The flow* (X_M, F_t^M) *is determined as the unique maximal common factor flow of those* (X_D, F_t^D), (X_A, F_t^A), *and* (X_B, F_t^B).

This corollary explains all the type classification results mentioned before. Indeed, if A (or B) is a factor of type III_1, then the flow of weights (X_A, F_t^A) (resp. (X_B, F_t^B)) is trivial (see [CT],[T2]), i.e., the one-point flow, and hence the corollary says that so is the flow of weights (X_M, F_t^M), which means that M is of type III_1. The others can be also explained similarly.

Corollary 4.7. **([U1, Corollary 5.8])** *If A and B coincide with each other, i.e., $A = B$, then the amalgamated free product $M = A *_D B$ ($= A *_D A$) and A ($= B$) have the same flow of weights. In particular, any ergodic flow can be realized as the flow of weights of a certain amalgamated free product.*

Remarks 4.8. A few remarks are in order.
(1) Corollary 4.6 has the trivial reformulation: The flow of weights (X_M, F_t^M) coincides with the associated flow ([HOO],[Kr],[FM]) of the canonical equivalence relation $\mathcal{R}_M$ introduced in §3.
(2) Based on Theorem 4.4, (4.3) (or (4.5)) together with the proof of Theorem 4.1, we can show the following: Let A and B be von Neumann algebras (with separable preduals) having no type I direct summand, and let D be a common Cartan subalgebra. Then we have

$$\mathcal{Z}(\widetilde{A *_D B}) = \mathcal{Z}(\widetilde{A}) \cap \mathcal{Z}(\widetilde{B}) \ (\subseteq \widetilde{D}). \tag{4.7}$$

This implies
(4.8)
$$\mathcal{Z}(A *_D B) = \mathcal{Z}(\widetilde{A *_D B})^{\theta^M} = \mathcal{Z}(\widetilde{A})^{\theta^A} \cap \mathcal{Z}(\widetilde{B})^{\theta^B} = \mathcal{Z}(A) \cap \mathcal{Z}(B) \ (\subseteq D)$$

thanks to (4.4) and the continuous decomposition theorem [T2].
(3) The (4.8) in the above (2) can be reformulated as follows: Under the assumption that both A and B have no type I direct summand, the amalgamated free product $M = A *_D B$ is a factor if and only if the canonical equivalence relation $\mathcal{R}_M$ is ergodic.

§5 Miscellaneous Results

5.1. Let $A \supseteq D \subseteq B$ and $E_D^A : A \to D$, $E_D^B : B \to D$ be as in §2, and let

$$(M, E_D^M) = (A, E_D^A) *_D (B, E_D^B)$$

be the amalgamated free product. Suppose that M has separable predual, or equivalently that so do both A and B, and further that we have known that

$$\mathcal{Z}(M) = \mathcal{Z}(A) \cap \mathcal{Z}(B) \subseteq \mathcal{Z}(D). \tag{5.1.1}$$

Then we have the following simultaneous direct integral decompositions:

$$M = \int_\Omega^\oplus M(\omega)\, d\nu(\omega) \supseteq A = \int_\Omega^\oplus A(\omega)\, d\nu(\omega) \supseteq D = \int_\Omega^\oplus D(\omega)\, d\nu(\omega)$$

$$M = \int_\Omega^\oplus M(\omega)\, d\nu(\omega) \supseteq B = \int_\Omega^\oplus B(\omega)\, d\nu(\omega) \supseteq D = \int_\Omega^\oplus D(\omega)\, d\nu(\omega)$$

with $\mathcal{Z}(M) = L^\infty(\Omega, \nu)$. Let

$$\mathcal{H} = \int_\Omega^\oplus \mathcal{H}(\omega)\, d\nu(\omega)$$

be the corresponding direct integral decomposition of $\mathcal{H} = L^2(M)$. We may and do assume that $\omega \mapsto \mathcal{H}(\omega)$ is a constant field of the separable infinite dimensional Hilbert space. The conditional expectations E_D^M, E_D^A, E_D^B are also decomposed as follows:

$$E_D^M = \int_\Omega^\oplus (E_D^M)_\omega\, d\nu(\omega),$$

$$E_D^A = \int_\Omega^\oplus (E_D^A)_\omega\, d\nu(\omega), \quad E_D^B = \int_\Omega^\oplus (E_D^B)_\omega\, d\nu(\omega),$$

Theorem 5.1. *Under the hypothesis (5.1.1), we have*

$$(M(\omega), (E_D^M)_\omega) \cong (A(\omega), (E_D^A)_\omega) *_{D(\omega)} (B(\omega), (E_D^B)_\omega) \tag{5.1.2}$$

for almost every $\omega \in \Omega$.

Before going to the proof, we provide a suitable (for our purpose) reformulation of freeness. Let $(N \supseteq L,\ E : N \to L)$ be a L-probability space, i.e., $N \supseteq L$ is an inclusion of unital algebras with the same unit and $E : N \to L$ is a conditional expectation in the purely algebraic sense. Assume that N_1, N_2 are unital algebras containing L in common. We introduce the operation $x \in N \mapsto [x]^\circ := x - E(x)$, and the freeness (with amalgamation over L) of the pair N_1, N_2 can be interpreted as follows:

Lemma 5.2. *The pair N_1, N_2 are free with amalgamation over L if and only if the mapping*

$$\Phi_{(N_1,N_2)} : (x_1, x_2, \cdots, x_n) \in \Lambda(N_1, N_2) \mapsto E([x_1]^\circ [x_2]^\circ \cdots [x_n]^\circ)$$

is identically zero. Here, $\Lambda(N_1, N_2)$ is the set of those finite alternating sequences $(x_1, x_2, \cdots, x_n)$ of elements in $N_1 \cup N_2$ with $x_i \in N_{j(i)}$, $j(1) \neq j(2) \neq \cdots \neq j(n)$.

In our case, the operation $m \in M \mapsto [m]^\circ := m - E_D^M(m) \in M^\circ := \mathrm{Ker} E_D^M$ is normal and linear, and can be (direct integral) decomposed as follows:

$$[\,\cdot\,]^\circ = \int_\Omega^\oplus [\,\cdot\,]_\omega^\circ \; d\nu(\omega),$$

where $[m(\omega)]_\omega^\circ = m(\omega) - (E_D^M)_\omega(m(\omega))$, a normal linear map, for almost every $\omega \in \Omega$.

Proof of Theorem 5.1. Since both A and B have separable preduals, one can choose countable families $\{a_k\}_{k\in\mathbb{N}}$, $\{b_k\}_{k\in\mathbb{N}}$ in such a way that they generate A and B, respectively. Let

$$a_k = \int_\Omega^\oplus a_k(\omega)\; d\nu(\omega), \quad b_k = \int_\Omega^\oplus b_k(\omega)\; d\nu(\omega).$$

Then we can choose a co-null Borel subset Ω_1 of Ω in such a way that for every $\omega \in \Omega_1$

(a) $M(\omega)$ is generated by $A(\omega)$ and $B(\omega)$;

(b) $(E_D^M)_\omega : M(\omega) \to D(\omega)$, $(E_D^A)_\omega : A(\omega) \to D(\omega)$, $(E_D^B)_\omega : B(\omega) \to D(\omega)$ are faithful normal conditional expectations and

$$(E_D^M)_\omega|_{A(\omega)} = (E_D^A)_\omega, \quad (E_D^M)_\omega|_{B(\omega)} = (E_D^B)_\omega;$$

(c) $A(\omega)$ and $B(\omega)$ are generated by the $a_k(\omega)$'s and the $b_k(\omega)$'s, respectively.

Replacing the a_k's (resp. the b_k's) by all the finite products of them and their adjoints from the beginning, we may and do assume, instead of (c), that

(c)' the linear span of the $a_k(\omega)$'s (resp. the $b_k(\omega)$'s) forms a σ-weakly dense $*$-subalgebra of $A(\omega)$ (resp. $B(\omega)$).

The restriction of the map $\Phi_{(A,B)}$ to the subset $\Lambda(\{a_k\}_{k\in\mathbb{N}}, \{b_k\}_{k\in\mathbb{N}})$ is identically zero by the freeness of the pair A, B (see Lemma 5.2).

Therefore, there is a co-null Borel subset Ω_0 of Ω_1 such that

$$\begin{aligned}\Phi_{(A(\omega),B(\omega))}&((m_1(\omega), m_2(\omega), \cdots, m_n(\omega))) \\ &= (E_D^M)_\omega([m_1(\omega)]_\omega^\circ [m_2(\omega)]_\omega^\circ \cdots [m_n(\omega)]_\omega^\circ) = 0\end{aligned}$$

for every $(m_1(x), m_2(x), \cdots, m_n(x)) \in \Lambda(\{a_k(\omega)\}_{k\in\mathbb{N}}, \{b_k(\omega)\}_{k\in\mathbb{N}})$ and for every $\omega \in \Omega_0$. Since the operation $y \mapsto [y]_\omega^\circ$ is normal and linear, the mapping $\Phi_{(A(\omega),B(\omega))}$ itself is identically zero thanks to (c)' with the aid of the Kaplansky density theorem. Hence, $A(\omega)$ and $B(\omega)$ are free with amalgamation over $D(\omega)$ for every $\omega \in \Omega_0$ by Lemma 5.2. Hence we have proved the assertion. □

5.2. We would like to apply Theorem 5.1 to amalgamated free products over Cartan subalgebras. In what follows, we suppose that the triple $A \supseteq D \subseteq B$ consists of factors (with separable preduals) of non-type I and a common Cartan subalgebra. The starting point of the discussion is Theorem 4.5, (4.6):

$$\mathcal{Z}(\widetilde{M}) = \mathcal{Z}(\widetilde{A}) \cap \mathcal{Z}(\widetilde{B}) \subseteq \widetilde{D}.$$

Theorem 5.1 implies

Corollary 5.3. *Almost every factor $\widetilde{M}(x)$ in the central decomposition*

$$\widetilde{M} = \int_{X_M}^{\oplus} \widetilde{M}(x) d\mu(x)$$

of the continuous core $\widetilde{M}$ can be written as an amalgamated free product over a common Cartan subalgebra.

Here, we further suppose that M is of type III_λ $(0 < \lambda < 1)$, or equivalently that the flow of weights (X_M, F_t^M) is an (essentially) transitive flow with period $-\log\lambda$ (see [CT],[T2]) so that we may and do assume that

$$X_M = [0, -\log\lambda), \quad F_t^M = \text{the translation by } t \ (\text{mod} : -\log\lambda).$$

Since F_t^M is transitive, we have $\widetilde{M}(0) \cong \widetilde{M}(x)$ for every $x \in X_M = [0, -\log\lambda)$, and hence we may and do assume that $x \in X_M \mapsto \widetilde{M}(x)$ is a constant field of the type II_∞ factor $\widetilde{M}(0)$. Hence, we conclude

Corollary 5.4. *In the current setting, the (unique) type* II_∞ *factor appearing in the discrete decomposition ([C1]) of M is written as an amalgamated free product over a common Cartan subalgebra.*

5.3. We keep the setting and the notations as in §§5.2. It is known that an ergodic free action of a non-amenable discrete group may or may not be amenable ([Z], see also §6), or equivalently the associated von Neumann factor may or may not be injective (see [C2]) (or hyperfinite), and hence it is somewhat non-trivial whether or not the amalgamated free product $M = A *_D B$ is non-injective.

Theorem 5.5. *In the current setting, there is a copy of the free group factor $L(\mathbb{F}_2)$ in the continuous core $\widetilde{M}$ which is the range of a faithful normal conditional expectation. In particular, M is not injective.*

The non-injectivity result follows also from a result in our resent work [U2], where we have shown that the amalgamated free product $M = A *_D B$ is not a McDuff factor (under the assumption that both A and B are factors of non-type I). However, the proof below is still valid even in the case that both A and B have no type I direct summand. (See Remarks 4.8, (2),(3).)

Proof. Let $\mathrm{Tr}_{\widetilde{M}}$, $\mathrm{Tr}_{\widetilde{A}}$, $\mathrm{Tr}_{\widetilde{B}}$, $\mathrm{Tr}_{\widetilde{D}}$ be the canonical traces on $\widetilde{M}$, $\widetilde{A}$, $\widetilde{B}$, $\widetilde{D}$, respectively, (scaled in the usual way under the dual actions). It can be checked that

$$\mathrm{Tr}_{\widetilde{M}} = \mathrm{Tr}_{\widetilde{D}} \circ \widehat{E_D^M}, \quad \mathrm{Tr}_{\widetilde{A}} = \mathrm{Tr}_{\widetilde{D}} \circ \widehat{E_D^A}, \quad \mathrm{Tr}_{\widetilde{B}} = \mathrm{Tr}_{\widetilde{D}} \circ \widehat{E_D^B},$$

where $\widehat{E_D^M} : \widetilde{M} \to \widetilde{D}$, $\widehat{E_D^A} : \widetilde{A} \to \widetilde{D}$, $\widehat{E_D^B} : \widetilde{B} \to \widetilde{D}$ are as in §4. As in §§6.1, we consider the simultaneous direct integral decompositions of the inclusions $\widetilde{M} \supseteq \widetilde{A}$, $\widetilde{B} \supseteq \widetilde{D}$ and the conditional expectations $\widehat{E_D^M} : \widetilde{M} \to \widetilde{D}$, $\widehat{E_D^A} : \widetilde{A} \to \widetilde{D}$, $\widehat{E_D^B} : \widetilde{B} \to \widetilde{D}$ subject to the central decomposition

$$\widetilde{M} = \int_{X_M}^{\oplus} \widetilde{M}(x)\, d\mu(x)$$

thanks to Theorem 4.5, (4.6), and let

$$\mathrm{Tr}_{\widetilde{D}} = \int_{X_M}^{\oplus} \left(\mathrm{Tr}_{\widetilde{D}}\right)_x \, d\mu(x)$$

be the corresponding direct integral decomposition.

Note that the continuous core $\widetilde{M}$ is of type II_∞ or of type II_1. (Recall that, the continuous core of a von Neumann algebra of type III must be of type II_∞, while there is no type change for the other types.) We assume that $\widetilde{M}$ is of type II_∞ in what follows, since the quite similar (actually simpler) argument as below apparently works in the type II_1 case.

Let X be a co-null Borel subset of X_M satisfying the same conditions (a), (b), (c) as in the proof of Theorem 5.1 with the additional ones:

(e) Both $\widetilde{A}(x)$ and $\widetilde{B}(x)$ are von Neumann algebras of type II for every $x \in X$. (This follows from the assumption that A and B have no type I direct summand.)

(f) $\left(\mathrm{Tr}_{\widetilde{D}}\right)_x \circ \left(\widehat{E_D^M}\right)_x$ is a faithful normal semi-finite trace, and thus so are both $\left(\mathrm{Tr}_{\widetilde{D}}\right)_x \circ \left(\widehat{E_D^A}\right)_x$, $\left(\mathrm{Tr}_{\widetilde{D}}\right)_x \circ \left(\widehat{E_D^B}\right)_x$.

By choosing a much smaller co-null subset instead of X if necessary, the set of those $(x, p) \in X \times B(\mathcal{H}_0)$ with the separable Hilbert space $\mathcal{H}_0$ (on which almost every $\widetilde{D}(\omega)$ act), satisfying

- $p = p^2 = p^* \in \widetilde{D}(x)$;
- $\left(\mathrm{Tr}_{\widetilde{D}}\right)_x (p) = 1$

can be assumed to be Borel. Therefore, the measurable selection principle enables us to choose a measurable field $x \in X \mapsto p_x \in \widetilde{D}(x)$ of projections such that $\left(\mathrm{Tr}_{\widetilde{D}}\right)_x (p_x) = 1$ for every $x \in X$, and we set

$$p := \int_{X_M}^{\oplus} p_x d\mu(x).$$

Since $\widetilde{M}$ is of type II_∞, we have

$$\begin{aligned}\int_{X_M}^{\oplus} \widetilde{M}(x)\, d\mu(x) = \widetilde{M} &= \left(p\widetilde{M}p\right) \otimes B(\mathcal{H}_0) \\ &= \int_{X_M}^{\oplus} \left(p_x\widetilde{M}(x)p_x\right) \otimes B(\mathcal{H}_0)\, d\mu(x)\end{aligned}$$

with the separable Hilbert space $\mathcal{H}_0$. Then we see that, for every $x \in X$,

- both $p_x\widetilde{A}(x)p_x$ and $p_x\widetilde{B}(x)p_x$ are of type II_1 (thanks to (e),(f));

- $p_x\widetilde{A}(x)p_x$, $p_x\widetilde{B}(x)p_x$ are free with amalgamation over $\widetilde{D}(x)p_x$ in the $\widetilde{D}(x)p_x$-probability space $\left(p_x\widetilde{M}(x)p_x, \left(\widehat{E_D^M}\right)_x|_{p_x\widetilde{M}(x)p_x}\right)$ (use (b),(c) and the proof of Theorem 5.1).

Repeating the argument of the type III_0 case in the proof of [U1, Lemma 4.2], we choose two measurable fields $x \in X \mapsto u_x \in p_x\widetilde{A}(x)p_x$, $x \in X \mapsto v_x \in p_x\widetilde{B}(x)p_x$ of unitaries satisfying

$$\left(\widehat{E_D^A}\right)_x((u_x)^n) = 0, \quad \left(\widehat{E_D^B}\right)_x((v_x)^n) = 0$$

for every $n(\neq 0) \in \mathbb{Z}$, and set

$$u := \int_{X_M}^{\oplus} u_x d\mu(x), \quad v := \int_{X_M}^{\oplus} v_x d\mu(x).$$

Then u, v are free Haar unitaries, and hence the von Neumann subalgebra $N := \{u, v\}''$ of $p\widetilde{M}p$ is isomorphic to the free group factor $L(\mathbb{F}_2)$. Notice that N is decomposable relative to $\mathcal{Z}(\widetilde{M}) = L^\infty(X_M, \mu)$ and that u_x, v_x also are free Haar unitaries for almost every $x \in X_M$. Therefore, we have

$$L(\mathbb{F}_2) \cong N(x) = \{u_x, v_x\}'' \subseteq p_x\widetilde{M}(x)p_x \quad \text{for almost every } x \in X_M,$$

and thus

$$\begin{aligned} L(\mathbb{F}_2) \otimes B(\mathcal{H}_0) \otimes \mathbf{C}1 &\subseteq L(\mathbb{F}_2) \otimes B(\mathcal{H}_0) \otimes \mathcal{Z}(\widetilde{M}) \\ &= \int_{X_M}^{\oplus} L(\mathbb{F}_2) \otimes B(\mathcal{H}_0)\ d\mu(x) \\ &\subseteq \int_{X_M}^{\oplus} \left(p_x\widetilde{M}(x)p_x\right) \otimes B(\mathcal{H})\ d\mu(x) \\ &= \int_{X_M}^{\oplus} \widetilde{M}(x)\ d\mu(x). \end{aligned}$$

Since $p\widetilde{M}p$ is of type II_1, the copy of $L(\mathbb{F}_2) \otimes B(\mathcal{H}_0)$ in $\widetilde{M}$ (or in $\widetilde{M}(x)$ for almost every $x \in X_M$) is clearly the range of a faithful normal conditional expectation. Hence we are done. □

Since the copy of $L(\mathbb{F}_2)$ constructed in the proof is well-behaved with the central decomposition of $\widetilde{M}$, we have

Corollary 5.6. *We enjoy in the same setting as in Theorem 5.5. If the amalgamated free product $M = A *_D B$ is of type* III_λ*, then the type* II_∞ *factor appearing in the discrete decomposition of M contains a copy of the free group factor $L(\mathbb{F}_2)$ which is the range of a faithful normal conditional expectation. Therefore, so does the M itself.*

Proof. The first part of the assertion is clear from the proof of Theorem 5.5 together with Corollary 5.4. The latter follows from the first half and the discrete decomposition theorem ([C1]) for type III_λ factors. □

For a while, we assume that both A and B are general factors (not necessary of non-type I) and that D is a common Cartan subalgebra. If either A or B is of type I_n with possibly $n = \infty$, then both must coincide, i.e., $A = B = M_n(\mathbf{C})$ or $B(\mathcal{H})$. In this case, we can see that $M = A *_D B$ $(= A *_D A)$ is isomorphic to $L(\mathbb{F}_{n-1}) \otimes A$. Therefore, we obtain

Corollary 5.7. *The amalgamated free product $M = A *_D B$ of factors (with separable preduals) over a common Cartan subalgebra is injective if and only if either A or B is of type* I_2 *(and hence $A = B = M_2(\mathbf{C})$ and D is the diagonals).*

This corollary can be thought of as an analogue of the following classical group theoretical fact: A free product group $G * H$ is amenable if and only if $G = H = \mathbb{Z}_2$.

5.4. We keep the same setting and the notations as in §§5.2 even in this subsection. We would like here to show that the amalgamated free product $M = A *_D B$ is not related to any free group factor as a simple application of the striking result [V2] of D. Voiculescu with the aid of Theorem 5.1 (or Corollary 5.3, Corollary 5.4). A similar application of Voiculescu's result was also given by D. Shlyakhtenko [S1] in a different context.

When the amalgamated free product $M = A *_D B$ is of type II_1, we can apply directly Voiculescu's theorem to the case since the normalizer $\mathcal{N}_M(D)$ generates the whole M, and hence M is not isomorphic to any (interpolated) free group factor $L(\mathbb{F}_r)$ with $0 < r \leq \infty$. Thus it suffices to consider only the infinite cases, and we start with the following lemma:

Lemma 5.8. *Let $N \supseteq C$ be a factor of type* II_∞ *and an abelian von Neumann subalgebra. Let* Tr *be a faithful normal semi-finite trace on N such that* $\mathrm{Tr}|_C$ *is semi-finite. Suppose that the normalizer $\mathcal{N}_N(C)$*

generates the whole N. Then, for each finite (in N) non-zero projection $p \in C$ (such a projection indeed exists since $\mathrm{Tr}|_C$ is semi-finite), the normalizer $\mathcal{N}_{pNp}(Cp)$ generates the whole pNp.

Proof. By assumption, we see that the linear span of $\mathcal{N}_N(C)$ forms a σ-weakly dense $*$-subalgebra in N. Hence, the linear span of elements of the form pup with $u \in \mathcal{N}_N(C)$ also is σ-weakly dense in pNp. Since p is finite in N, pNp is a factor of type II_1. Let us denote q and r the support and the range projections of pup, respectively. Then the projections q, r are in pNp since $q, r \leq p$. We here need the following fact:

Fact. *If $p \neq q$ (or equivalently $p \neq r$ thanks to the fact that pNp is finite), then there is an element $w \in \mathcal{GN}_{pNp}(Cp)$ with $w^*w = p - q$, $ww^* = p - r$. Here, $\mathcal{GN}_{pNp}(Cp)$ denotes the normalizing groupoid, i.e., the set of those partial isometries $v \in N$ such that $v^*v, vv^* \in C$ and $vCv^* = Cvv^*$, $v^*Cv = Cv^*v$.*

Proof of Fact. Since N is a factor, we have $(p-r)N(p-q) \neq \{0\}$, and hence there is a unitary $v \in \mathcal{N}_N(C)$ such that $(p-r)v(p-q)$ is not equal zero. Thus, there is a non-zero element $v_0 \in \mathcal{GN}_{pNp}(Cp)$ such that $v_0^*v_0 \leq p - q$ and $v_0v_0^* \leq p - r$. We can do the standard exhaustion argument thanks to the fact that pNp is finite. Hence we get a desired partial isometry. □

Let $w := pup$ with $u \in \mathcal{N}_N(C)$, and the above fact says that we can choose a unitary $\widetilde{w} \in \mathcal{N}_{pNp}(Cp)$ in such a way that $w = r\widetilde{w}q$. Moreover, q, r can be written as finite linear combinations of unitries in Cp, and hence $r\widetilde{w}q$ is a finite linear combinations of elements in $\mathcal{N}_{pNp}(Cp)$. Therefore, any element in pNp can be approximated σ-weakly by finite linear combinations of elements in $\mathcal{N}_{pNp}(Cp)$. Hence we complete the proof of the lemma. □

Proposition 5.9. *We enjoy in the same setting as in Theorem 5.5.*
(1) *If the amalgamated free product $M = A *_D B$ is of type II_∞, then M is not isomorphic to any $L(\mathbb{F}_r) \otimes B(\mathcal{H})$ with $0 < r \leq \infty$.*
(2) *If the amalgamated free product $M = A *_D B$ is of type III, then almost every type II_∞ factor appearing in the central decomposition of the continuous core $\widetilde{M}$ is not isomorphic to any $L(\mathbb{F}_r) \otimes B(\mathcal{H})$ with $0 < r \leq \infty$.*
(3) *If the amalgamated free product $M = A *_D B$ is of type III_λ ($0 < \lambda < 1$), then the type II_∞ factor appearing in the discrete decomposition is not isomorphic to any $L(\mathbb{F}_r) \otimes B(\mathcal{H})$ with $0 < r \leq \infty$.*

Proof. All the assertions follow from [V2, 5,3 Theorem, 7.4 Corollary] with the aid of Lemma 5.8. When showing the assertions (2), (3), we further need Corollary 5.3, Corollary 5.4, respectively. □

§6 Example I. Boundary Actions of Free Groups

6.1. Let $\mathfrak{X}$ be a finite set with $|\mathfrak{X}| \geq 2$, and we set $\mathfrak{X}^{-1} := \{x^{-1}; x \in \mathfrak{X}\}$. We consider the free group $\Gamma := \mathbb{F}(\mathfrak{X})$ over the generators $\mathfrak{X}$ and its boundary $\partial\Gamma$. In this case, the boundary $\partial\Gamma$ is defined as the one-sided shift space of the alphabets $\mathfrak{X} \cup \mathfrak{X}^{-1}$ determined by the forbidden blocks $(x^{-1}x)$, (xx^{-1}), equipped with the usual product topology. It is plain to see that $\partial\Gamma$ is identified with the set of semi-infinite reduced words in $\mathfrak{X} \cup \mathfrak{X}^{-1}$. We will freely use these two different descriptions in what follows. The group Γ acts topologically on the boundary $\partial\Gamma$ by the left multiplication, i.e., for $\gamma \in \Gamma$ and for $\omega = \omega_1\omega_2\cdots \in \partial\Gamma$,

$$\gamma \cdot \omega := \text{the reduced form of the word } \gamma\omega_1\omega_2\cdots.$$

6.2. We decompose the set $\mathfrak{X}$ into two disjoint non-empty subsets $\mathfrak{X}_1$, $\mathfrak{X}_2$ with $\mathfrak{X} = \mathfrak{X}_1 \sqcup \mathfrak{X}_2$. Then we have $\Gamma = \Gamma_1 * \Gamma_2$ with $\Gamma_1 = \mathbb{F}(\mathfrak{X}_1)$, $\Gamma_2 = \mathbb{F}(\mathfrak{X}_2)$. The boundaries $\partial\Gamma_1$, $\partial\Gamma_2$ can be (topologically) embedded into $\partial\Gamma$ as follows:

$$\begin{aligned}\partial\Gamma_1 =& \text{the shift subspace of } \partial\Gamma \\ & \quad \text{with extra forbidden blocks } (x),\ x \in \mathfrak{X}_2, \\ \partial\Gamma_2 =& \text{the subset of } \partial\Gamma \\ & \quad \text{with extra forbidden blocks } (x),\ x \in \mathfrak{X}_1.\end{aligned}$$

Therefore, the subspaces $\partial\Gamma_1$, $\partial\Gamma_2$ are closed and invariant under the actions of Γ_1, Γ_2, respectively. We consider the following disjoint decompositions:

$$\partial\Gamma = (\partial\Gamma_1)^c \sqcup \partial\Gamma_1, \quad \partial\Gamma = (\partial\Gamma_2)^c \sqcup \partial\Gamma_2,$$

and note that $(\partial\Gamma_1)^c$, $(\partial\Gamma_2)^c$ are open and invariant under the actions of Γ_1, Γ_2, respectively. We define

$$\begin{aligned}(\partial\Gamma_1)^{\perp} &:= \{\omega = (\omega_n)_{n=1}^{\infty} \in (\partial\Gamma)^c \ ; \ \omega_1 \in \mathfrak{X}_2 \cup (\mathfrak{X}_2)^{-1}\}, \\ (\partial\Gamma_2)^{\perp} &:= \{\omega = (\omega_n)_{n=1}^{\infty} \in (\partial\Gamma)^c \ ; \ \omega_1 \in \mathfrak{X}_1 \cup (\mathfrak{X}_1)^{-1}\}.\end{aligned}$$

Let us define the map $\Phi_1 : \Gamma_1 \times (\partial\Gamma_1)^\perp \to (\partial\Gamma_1)^c$ as the restriction of the action map $\Gamma \times \partial\Gamma \to \partial\Gamma$, i.e.,

$$\Phi_1 \ : \ \begin{cases} (e,\omega) \mapsto \omega, \\ (\gamma_1 \cdots \gamma_2, (\omega_n)_{n=1}^\infty) \mapsto (\gamma_1, \dots, \gamma_n, \omega_1, \omega_2, \dots) \end{cases} \tag{6.2.1}$$

for each reduced word $\gamma_1 \cdots \gamma_n$. Similarly, the map $\Phi_2 : \Gamma_2 \times (\partial\Gamma_2)^\perp \to (\partial\Gamma_2)^c$ is defined as the restriction of the action map $\Gamma \times \partial\Gamma \to \partial\Gamma$. We here note that the topology on $\partial\Gamma$ is generated by the family of clopen sets of the form:

$$\Omega(\gamma) = \{\omega = (\omega_n)_{n=1}^\infty \in \partial\Gamma \ ; \ \omega_1 = \gamma_1, \dots, \omega_n = \gamma_n\}$$

with a reduced word $\gamma = \gamma_1 \cdots \gamma_n$.

Lemma 6.1. *We have*

$$(\partial\Gamma_1)^c = \bigsqcup_{\gamma \in \Gamma_1} \bigsqcup_{\omega \in \mathfrak{X}_2 \cup (\mathfrak{X}_2)^{-1}} \Omega(\gamma\omega), \tag{6.2.2}$$

$$(\partial\Gamma_1)^c = \bigsqcup_{\gamma \in \Gamma_2} \bigsqcup_{\omega \in \mathfrak{X}_1 \cup (\mathfrak{X}_1)^{-1}} \Omega(\gamma\omega). \tag{6.2.3}$$

It is plain to check that

(1) the maps Φ_1, Φ_2 are bijections;
(2) $\Phi_k(\{\gamma\} \times \Omega(\omega_1 \cdots \omega_n)) = \Omega(\gamma\omega_1 \cdots \omega_n)$ $(k = 1, 2)$ with a reduced word $\omega_1 \cdots \omega_n$.

Therefore, thanks to Lemma 6.1, we see that Φ_1, Φ_2 send all the basic clopen sets to all those, when the product topologies of the discrete one and the induced one from $\partial\Gamma$ are considered on both $\Gamma_1 \times (\partial\Gamma_1)^\perp$, $\Gamma_2 \times (\partial\Gamma_2)^\perp$. It is also plain to check that

$$\Phi_k(\gamma_1\gamma_2, \omega) = \gamma_1 \cdot \Phi_k(\gamma_2, \omega) \quad (k = 1, 2),$$

and hence we conclude

Proposition 6.2. *The maps $\Phi_1 : \Gamma_1 \times (\partial\Gamma_1)^\perp \to (\partial\Gamma_1)^c$, $\Phi_2 : \Gamma_2 \times (\partial\Gamma_2)^\perp \to (\partial\Gamma_2)^c$ are homeomorphisms, and via these homeomorphisms, the actions of Γ_1, Γ_2 on $(\partial\Gamma_1)^c$, $(\partial\Gamma_2)^c$ are conjugate to those of Γ_1, Γ_2 on $\Gamma_1 \times (\partial\Gamma_1)^\perp$, $\Gamma_1 \times (\partial\Gamma_2)^\perp$ which are defined as the product action of the translation and the trivial one.*

6.3. Let $n = |\mathfrak{X}|$ and $n_1 = |\mathfrak{X}_1|$, $n_2 = |\mathfrak{X}_2|$. We discuss here the probability measure μ on the boundary $\partial\Gamma$ defined in such a way that

$$\mu(\Omega(\gamma)) := \frac{1}{2n}\left(\frac{1}{2n-1}\right)^{\ell(\gamma)-1}$$

with the word length function $\ell(\,\cdot\,)$. It is known that the measure μ is quasi-invariant under the action of Γ. (See [KS].) The non-singular action of Γ on the probability space $(\partial\Gamma, \mu)$ can be checked to be free and ergodic (see [RR],[KS],[PS]). Moreover, J. Ramagge & G. Robertson [RR] showed that the action of Γ is of type $\mathrm{III}_{\frac{1}{2n-1}}$ so that the crossed-product $M = L^\infty(\partial\Gamma, \mu) \rtimes \Gamma$ is a factor of type $\mathrm{III}_{\frac{1}{2n-1}}$. Moreover, S. Adams' result [A] (see also [Ver, Example 2 in p. 89]) implies that the factor is injective. (It should be remarked that the Cuntz-Krieger algebra interpretation for boundary actions provided by J. Spielberg [Sp] together with M. Enomoto, M. Fujii & Y. Watatani [EFW] also shows that the crossed-product is the injective factor of type $\mathrm{III}_{\frac{1}{2n-1}}$.) Set $A := L^\infty(\partial\Gamma, \mu) \rtimes \Gamma_1$, $B := L^\infty(\partial\Gamma, \mu) \rtimes \Gamma_2$, and it is plain to see that, the crossed-product is written as an amalgamated free product over a common Cartan subalgebra, that is, we have $M \cong A *_D B$ with $D := L^\infty(\partial\Gamma, \mu)$. Therefore, the boundary action of the free group Γ provides an example of an injective factor arising as an amalgamated free product over a common Cartan subalgebra.

6.4. Since

$$|\{\gamma \in \Gamma_k \; ; \; \ell(\gamma) = m\}| = 2n_k(2n_k - 1)^{m-1} \quad (k = 1, 2),$$

we have

$$\begin{aligned}
\mu((\partial\Gamma_1)^c) &= \mu\left(\bigsqcup_{\gamma\in\Gamma_1}\ \bigsqcup_{\omega\in\mathfrak{X}_2\cup(\mathfrak{X}_2)^{-1}} \Omega(\gamma\omega)\right) \qquad \text{(Lemma 6.1)} \\
&= \sum_{\omega\in\mathfrak{X}_2\cup(\mathfrak{X}_2)^{-1}} \mu(\Omega(\omega)) + \sum_{m=1}^{\infty} \sum_{\substack{\gamma\in\Gamma_1 \\ \ell(\gamma)=m}} \sum_{\omega\in\mathfrak{X}_2\cup(\mathfrak{X}_2)^{-1}} \mu(\Omega(\gamma\omega)) \\
&= 2n_2\frac{1}{2n} + \sum_{m=1}^{\infty} \left(2n_1(2n_1-1)^{m-1}\right) 2n_2 \left(\frac{1}{2n}\left(\frac{1}{2n-1}\right)^m\right) \\
&= \frac{n_2}{n} + \frac{2n_1n_2}{n(2n-1)} \cdot \sum_{m=1}^{\infty}\left(\frac{2n_1-1}{2n-1}\right)^{m-1} \\
&= 1 \quad (\text{by } n = n_1 + n_2).
\end{aligned}$$

Similarly, we have $\mu((\partial\Gamma_2)^c) = 1$. Hence, we obtain

$$\mu(\partial\Gamma_1) = \mu(\partial\Gamma_2) = 0. \tag{6.4.1}$$

Furthermore, we have

$$\begin{aligned}\mu(\Phi_k(\{\gamma\}\times\Omega(\omega_1\cdots\omega_m))) &= \frac{1}{2n}\left(\frac{1}{2n-1}\right)^{\ell(\gamma)+m-1}\\ &= \left(\frac{1}{2n-1}\right)^{\ell(\gamma)} \times \frac{1}{2n}\left(\frac{1}{2n-1}\right)^{m-1},\end{aligned}$$

and hence

$$(\mu|_{(\partial\Gamma_k)^c}) \circ \Phi_k = \delta_k \otimes (\mu|_{(\partial\Gamma_k)^\perp}) \quad (k = 1, 2) \tag{6.4.2}$$

with the measure $\delta_k(\{\gamma\}) = \left(\dfrac{1}{2n-1}\right)^{\ell(\gamma)}$ equivalent to the counting measure. From the discussions above, we conclude

Proposition 6.3. *We have*

$$\begin{aligned}L^\infty(\partial\Gamma,\mu) \rtimes \Gamma_1 &= L^\infty((\partial\Gamma_1)^c,\mu) \rtimes \Gamma_1\\ &\cong (\ell^\infty(\Gamma_1) \rtimes \Gamma_1) \otimes L^\infty((\partial\Gamma_1)^\perp, \mu|_{(\partial\Gamma_1)^\perp}),\end{aligned} \tag{6.4.3}$$

$$\begin{aligned}L^\infty(\partial\Gamma,\mu) \rtimes \Gamma_2 &= L^\infty((\partial\Gamma_2)^c,\mu) \rtimes \Gamma_2\\ &\cong (\ell^\infty(\Gamma_2) \rtimes \Gamma_2) \otimes L^\infty((\partial\Gamma_2)^\perp, \mu|_{(\partial\Gamma_2)^\perp}).\end{aligned} \tag{6.4.4}$$

The isomorphisms are induced from the maps Φ_1, Φ_2, respectively. In particular, the crossed-products both are of homogeneous type I_∞.

Therefore, we have seen that the free components of our injective amalgamated free product $M = A *_D B$ both are of homogeneous type I_∞.

6.5. At the end of this section, we give a criterion on injectivity of amalgamated free products over Cartan subalgebras. Let $M = A *_D B$ be an amalgamated free product over a common Cartan subalgebra. Here, we do not assume that A and B are factors nor that they have no type I direct summand. We choose central projections p_A, p_B of A, B in such a way that both Ap_A and Bp_B have no type I direct summand. Since D is a Cartan subalgebra in both A and B, the projections p_A, p_B are in D so that $p := p_Ap_B = p_Bp_A$ is also a projection in D. Suppose here that p is non-zero. Then the reduced von Neumann algebra pMp

contains both pAp and pBp, and their freeness can be easily checked with respect to the conditional expectation

$$(E_D^M)_p := E_D^M|_{pMp} : pMp \to Dp.$$

It can be easily checked that the continuous cores satisfy

$$\widetilde{pMp} = p\widetilde{M}p, \quad \widetilde{pAp} = p\widetilde{A}p, \quad \widetilde{pBp} = p\widetilde{B}p, \quad \widetilde{Dp} = \widetilde{D}p. \tag{6.5.1}$$

Hence we get the inclusion relations

$$\widetilde{pMp} \supseteq \widetilde{pAp} \supseteq \widetilde{Dp}, \quad \widetilde{pMp} \supseteq \widetilde{pBp} \supseteq \widetilde{Dp}. \tag{6.5.2}$$

We can easily see that the conditional expectation

$$\widetilde{(E_D^M)_p} : \widetilde{pMp} \to \widetilde{Dp}$$

coincides with

$$\left(\widetilde{E_D^M}\right)_p := \widetilde{E_D^M}|_{p\widetilde{M}p} : p\widetilde{M}p \to \widetilde{D}p.$$

Hence we can show that, the von Neumann subalgebra

$$N := \widetilde{pAp} \vee \widetilde{pBp} \; (\subseteq \widetilde{pMp})$$

is identified with the amalgamated free product over a common Cartan subalgebra

$$\left(\widetilde{pAp}\right) *_{\widetilde{Dp}} \left(\widetilde{pBp}\right).$$

Notice here that

- both pAp and pBp have no type I direct summand;
- there is a faithful normal conditional expectation from $\widetilde{pMp}$ onto N since N is invariant under the modular action $\sigma_t^{\psi\circ\widetilde{(E_D^M)_p}}$ ($t \in \mathbf{R}$) with a faithful normal state ψ on $\widetilde{Dp}$ thanks to [T1] and Theorem 1.1.

Thus we apply the same argument as in the proof of Theorem 5.5 (see after the statement of that theorem) to the amalgamated free product N, and as a consequence we get a copy of the free group factor in $\widetilde{pMp}$ as the range of a faithful normal conditional expectation from N (and hence from $\widetilde{pMp}$). Therefore, pMp is not injective, and neither is M. Therefore, we conclude

Proposition 6.4. *In the current setting, if the amalgamated free product $M = A *_D B$ is injective, then the non-type I direct summands in A and B need not meet (in D), i.e., their support central projections are disjoint (in D).*

Remark 6.5. One can construct two von Neumann algebras A, B with a common Cartan subalgebra D in such a way that (i) A has the non-type I direct summand and $B \cong L^\infty(\Omega) \otimes B(\mathcal{H})$ (possibly with any dimension $\dim\mathcal{H} \geq 2$); (ii) the amalgamated free product $A *_D B$ is injective. (Compare with Proposition 6.3, 6.4.)

§7 Example II. Number of Free Components

Let $A \supseteq D \subseteq B$ be σ-finite von Neumann algebras with faithful normal conditional expectations $E_D^A : A \to D$, $E_D^B : B \to D$, which are assumed to be of the form:

$$A = A_0 \otimes B(\ell^2(\mathbb{N})), \quad B = B_0 \otimes B(\ell^2(\mathbb{N})), \quad D = D_0 \otimes \ell^\infty(\mathbb{N}),$$

$$E_D^A(a \otimes e_{ij}) = \delta_{ij} E_{D_0}^{A_0}(a), \quad E_D^A(b \otimes e_{ij}) = \delta_{ij} E_{D_0}^{B_0}(b)$$

with

$$B(\ell^2(\mathbb{N})) = \{e_{ij}\}'' \supseteq \ell^\infty(\mathbb{N}) = \{e_{ii}\}'',$$

where the e_{ij}'s are the natural matrix units and will be denoted by e_{ij}^A or e_{ij}^B instead of e_{ij} when regarded as elements in A or B, to avoid any confusion. We further suppose that B_0 (and hence B itself) is injective or hyperfinite, has no type I direct summand, and that D_0 is a Cartan subalgebra. Thanks to A. Connes, J. Feldman & B. Weiss [CFW], we may and do assume that there is a unitary $u \in B_0$ such that

$$B_0 =< D_0, u >'', \quad E_{D_0}^{B_0}(u^n) = 0 \text{ as long as } n \neq 0,$$

and the automorphism $\mathrm{Ad}u \in \mathrm{Aut}(D_0)$ is denoted by α.

In this setting, we will investigate the reduced von Neumann algebra pMp of the amalgamated free product:

$$(M, E_D^M) = (A, E_D^A) *_D (B, E_D^B)$$

by a minimal projection $p := 1 \otimes e_{11}$ in the common subalgebra $\mathbf{C}1 \otimes \ell^\infty(\mathbb{N})$.

We introduce the following notation rule:

$$[a]_{ij}^A := a \otimes e_{ij}^A \quad \text{in } A, \quad \text{and} \quad [b]_{ij}^B := b \otimes e_{ij}^B \quad \text{in } B,$$

and, in what follows, will freely use the identification:

$$e_{ij}^A = 1 \otimes e_{ij}^A = [1_{A_0}]_{ij}^A, \quad e_{ij}^B = 1 \otimes e_{ij}^B = [1_{B_0}]_{ij}^B.$$

Lemma 7.1. *We have*

(7.1) $[a_1]_{ij}^A \cdot [a_2]_{k\ell}^A = \delta_{jk} \cdot [a_1 a_2]_{i\ell}^A, \quad [b_1]_{ij}^B \cdot [b_2]_{k\ell}^B = \delta_{jk} \cdot [b_1 b_2]_{i\ell}^B.$
(7.2) $e_{ij}^A \cdot [a]_{k\ell}^A = \delta_{jk} \cdot [a]_{i\ell}^A = [a]_{ij}^A \cdot e_{k\ell}^A, \quad e_{ij}^B \cdot [b]_{k\ell}^B = \delta_{jk} \cdot [b]_{i\ell}^B = [b]_{ij}^B \cdot e_{k\ell}^B.$
(7.3) $[a]_{ij}^{A*} = [a^*]_{ji}^A, \quad [b]_{ij}^{B*} = [b^*]_{ji}^B.$

Lemma 7.2. *The von Neumann algebra B is generated by $D_0 \otimes \mathbf{C}1$ and partial isometries $[u^n]_{i1}^B$, $n \in \mathbb{Z}$, $i = 2, 3 \ldots$*

Set $u(n,i) := e_{1i}^A \cdot [u^n]_{i1}^B$, $n \in \mathbb{Z}$, $i = 2, 3, \ldots$, a unitary in pMp.

Lemma 7.3. *We have, for each $n \in \mathbb{Z}$, $i = 2, 3, \ldots$,*

$$E_D^M(u(n,i)^k) = 0$$

whenever $k \neq 0$ $(\in \mathbb{Z})$.

Proof. We may and do assume $k > 0$ since $E_D^M(u(n,i)^{-k}) = E_D^M(u(n,i)^k)^*$. Notice that

$$\begin{cases} E_D^M(e_{1i}^A) = E_D^A(e_{1i}^A) = 0 & \text{as long as } i \neq 1, \\ E_D^M([u^n]_{i1}^B) = E_D^B(u^n \otimes e_{i1}^B) = \delta_{i1} \cdot E_{D_0}^{B_0}(u^n) = 0 & \text{as long as } i \neq 1. \end{cases}$$

Therefore, by the freeness, we have, for $n \in \mathbb{Z}$, $i \neq 1$,

$$E_D^M(u(n,i)^k) = E_D^M(e_{1i}^A \cdot [u^n]_{i1}^B \cdots e_{1i}^A \cdot [u^n]_{i1}^B) = 0.$$

Hence we are done. □

We define the faithful normal conditional expectation

$$(E_D^M)_p := E_D^M|_{pMp} : pMp \to Dp = D_0 \otimes \mathbf{C}p$$

(which is well-defined since p is in the smaller algebra D).

Lemma 7.4. *The family*

$$\{A_0 \otimes \mathbf{C}p\} \cup \{u(n,i) : n \in \mathbb{Z}, i = 2, 3, \ldots\}$$

is free with amalgamation over $D_0 \otimes \mathbf{C}p$ with respect to $(E_D^M)_p$.

Proof. Since all the $u(n,i)$'s normalize the subalgebra $D_0 \otimes \mathbf{C}p$ and since $(D_0 \otimes \mathbf{C}p)(A_0^\circ \otimes \mathbf{C}p)$ and $(A_0^\circ \otimes \mathbf{C}1)(D_0 \otimes \mathbf{C}1)$ are contained in $(A_0^\circ \otimes \mathbf{C}p)$, it suffices to show that

$$\begin{aligned} E_D^M([a_1]_{11}^A u(n_1,i_1)^{k_1}[a_2]_{11}^A u(n_2,i_2)^{k_2} \\ \cdots [a_m]_{11}^A u(n_m,i_m)^{k_m}[a_{m+1}]_{11}^A) = 0 \end{aligned} \tag{7.4}$$

whenever all k_j's are not equal to 0, the beginning and the ending letters a_1,a_{m+1} are the identity 1 or in $A_0^\circ := \mathrm{Ker} E_{D_0}^{A_0}$, and the other a_j's are

$$a_j \text{ is } \begin{cases} \text{in } A^\circ \text{ or the identity } 1 & \text{if } (n_{j-1},i_{j-1}) \neq (n_j,i_j), \\ \text{in } A_0^\circ & \text{if } (n_{j-1},i_{j-1}) = (n_j,i_j). \end{cases} \tag{7.5}$$

We have, for $a \in A$,

$$\begin{aligned} &u(n_1,i_1)^{-k_1} \cdot [a]_{11}^A u(n_2,i_2)^{k_2} \\ &\quad = [u^{-n_1}]_{1i_1}^B e_{i_1 1}^A \cdots [u^{-n_1}]_{1i_1}^B [a]_{i_1 i_2}^A [u^{n_2}]_{i_2 1}^B \cdots e_{1i_2}^A \cdot [u^{n_2}]_{i_2 1}^B, \\ &u(n_1,i_1)^{k_1}[a]_{11} u(n_2,i_2)^{-k_2} \\ &\quad = e_{1i_1}^A [u^{n_1}]_{i_1 1}^B \cdots e_{1i_1}^A [u^{n_1}]_{i_1 1}^B [a]_{11}^A [u^{-n_2}]_{1i_2}^B e_{i_2 1}^A \cdots [u^{-n_2}]_{1i_2}^B e_{i_2 1}^A, \end{aligned}$$

Thus, if $k_1, k_2 \neq 0$ and if a is such as in (7.5), we see that

$$\begin{aligned} u(n_1,i_1)^{-k_1}[a]_{11}^A u(n_2,i_2)^{k_2} &\in [u^{-n_1}]_{1i_1}^B \underbrace{A^\circ \cdots A^\circ}_{\text{alternating}} [u^{n_2}]_{i_2 1}^B \\ &\subseteq B^\circ A^\circ \cdots A^\circ B^\circ, \\ u(n_1,i_1)^{k_1}[a]_{11} u(n_2,i_2)^{-k_2} &\in u(n_1,i_1) \underbrace{A^\circ \cdots A^\circ}_{\text{alternating}} u(n_2,i_2)^* \\ &\subseteq A^\circ \cdot B^\circ \cdot A^\circ \cdots A^\circ \cdot B^\circ \cdot A^\circ. \end{aligned}$$

Notice that

$$\begin{aligned} &[u^{n_1}]_{1i_1}^B [u^{-n_2}]_{1i_2}^B = [u^{n_1-n_2}]_{i_1 i_2}^B \in B^\circ, \\ &u(n_1,i_1)^* u(n_2,i_2) = \begin{cases} [u^{-n_1}]_{1i_1}^B e_{i_1 i_2}^A [u^{n_2}]_{i_2 1}^B \in B^\circ A^\circ B^\circ & (i_1 \neq i_2), \\ [u^{n_2-n_1}]_{11}^B \in B^\circ & (i_1 = i_2) \end{cases} \end{aligned}$$

as long as $(n_1,i_1) \neq (n_2,i_2)$, and one can easily check the desired equality (7.4) based on the above facts. $\square$

Thanks to Lemma 7.2 together with [V1, 3.1.Lemma], we see that the reduced von Neumann algebra pMp is generated by

$$e^A_{1i} \cdot (a \otimes 1) \cdot e^A_{j1} = \delta_{ij} \cdot (a \otimes p),\ a \in A;$$
$$e^A_{1i} \cdot e^A_{k\ell} \cdot e^A_{j1} = \delta_{ik} \cdot \delta_{\ell j} \cdot p;$$
$$e^A_{1i} \cdot [u^n]^B_{k1} \cdot e^A_{j1} = \delta_{ik} \cdot \delta_{j1} \cdot u(n,i),\ n \in \mathbb{Z},\ i = 2, 3, \ldots.$$

We set

$$N(n,i) := \{D_0 \otimes \mathbf{C}p, u(n,i)\}'' \cong D_0 \rtimes_{\alpha^n} \mathbb{Z} \quad \text{(thanks to Lemma 7.3)}$$

with the conditional expectation

$$E_{(n,i)} = E^M_D|_{N(n,i)} : N(n,i) \to D_0 \otimes \mathbf{C}p \cong D_0,$$

which coincides with the canonical one from $D_0 \rtimes_{\alpha^n} \mathbb{Z}$ onto D_0.

Summing up the discussions above, we conclude

Theorem 7.5. *We have*

$$(7.6) \qquad (pMp, (E^M_D)_p) \cong (A_0, E^{A_0}_{D_0}) *_{D_0} \left(\mathop{*}_{\substack{n \in \mathbb{Z} \\ i=2,3,\ldots}} (N(n,i), E_{(n,i)}) \right).$$

Here, the amalgamated free product

$$\mathop{*}_{\substack{n \in \mathbb{Z} \\ i=2,3,\ldots}} (N(n,i), E_{(n,i)})$$

is noting less than the crossed product of D_0 by the free group $\mathbb{F}_\infty$ with countably many generators, whose action is defined as follows:

$$(7.7) \qquad (\mathrm{Id})^{*\mathbb{N}} * (\alpha)^{*\mathbb{N}} * (\alpha^{-1})^{*\mathbb{N}} * \cdots * (\alpha^n)^{*\mathbb{N}} * (\alpha^{-n})^{*\mathbb{N}} * \cdots.$$

Here, $(\beta)^{*\mathbb{N}}$ means the free product of countably infinite copies of an automorphism β.

We further suppose that $A = B$, that is, $A_0 = B_0 = D_0 \rtimes_\alpha \mathbb{Z}$. Theorem 7.5 says that the reduced von Neumann algebra pMp is isomorphic to the crossed product of D_0 by the free group $\mathbb{F}_\infty$ whose action is

$$\begin{aligned} &\alpha * \left((\mathrm{Id})^{*\mathbb{N}} * (\alpha)^{*\mathbb{N}} * (\alpha^{-1})^{*\mathbb{N}} * \cdots * (\alpha^n)^{*\mathbb{N}} * (\alpha^{-n})^{*\mathbb{N}} * \cdots\right) \\ &\qquad = (\mathrm{Id})^{*\mathbb{N}} * (\alpha)^{*\mathbb{N}} * (\alpha^{-1})^{*\mathbb{N}} * \cdots * (\alpha^n)^{*\mathbb{N}} * (\alpha^{-n})^{*\mathbb{N}} * \cdots. \end{aligned}$$

Here, this equality follows from the simple fact: $\alpha * (\alpha)^{*\mathbb{N}} = (\alpha)^{*\mathbb{N}}$.

Remark 7.6. The result obtained in this section is thought of as a negative evidence towards generalizing the work [G] on the invariant "cost" of D. Gaboriau to general non-singular discrete measured groupoids. Roughly speaking, the "cost" counts the number of free components in a given finite-measure preserving countable equivalence relation, and recently D. Shlyakhtenko [S2] generalized further to finite-measure preserving discrete groupoids from the free entropic viewpoint. Our result here says that the number of free components cannot be determined in the general non-singular case. Indeed, we suppose that our $A = B$ is a factor of type III (or of type II_∞) and that D is a Cartan subalgebra as before. Then the amalgamated free product M is also a factor of type III and captured as a groupoid von Neumann algebra (see [Ks]). We can then choose an isometry $v \in A$ in such a way that $vDv^* = Dp$ with $vv^* = p \in D$. The $\mathrm{Ad}v$ gives rise to an isomorphism between $M \supseteq D$ and $pMp \supseteq Dp$. Moreover, we can show

$$(E_D^M)_p \circ \mathrm{Ad}v = \mathrm{Ad}v \circ E_D^M, \tag{7.8}$$

and hence $(M \supseteq D, E_D^M)$ can be identified with $(pMp \supseteq Dp, (E_D^M)_p)$, and the former has **two** free components, but the latter has **infinite** ones.

The discussions here (with trivial changes) also implies

Corollary 7.7. *Let N be an infinite injective factor of non-type* I *with a Cartan subalgebra D. Then we have*

$$N *_D N \cong N *_D N *_D N \cong \cdots \cong N *_D N *_D N *_D \cdots. \tag{7.9}$$

Remark 7.8. One may replace $B(\ell^2(\mathbb{N}))$ by $k \times k$ matrix algebra $M_k(\mathbf{C})$ in the setting, and the discussion here still works without any essential change and the assertion (7.6) should be changed to

$$(pMp, (E_D^M)_p) \cong (A_0, E_{D_0}^{A_0}) *_{D_0} \left(\mathop{*}_{\substack{n\in\mathbb{Z} \\ i=2,3,\dots,k}} (N(n,i), E_{(n,i)}) \right) \tag{7.10}$$

so that if $A_0 = B_0$ then pMp is the crossed-product of D_0 by the free group $\mathbb{F}_\infty$ whose action is:

$$(\mathrm{Id})^{*(k-1)} * (\alpha)^{*k} * (\alpha^2)^{*(k-1)} * \cdots * (\alpha^n)^{*(k-1)} * \cdots. \tag{7.11}$$

This is in particular thought of as a reduction formula of the amalgamated free product $R *_D R$ of two copies of the injective II_1 factor R

over a common Cartan subalgebra D thanks to [CFW]. The result says that, if the amalgamated free product $R *_D R$ had the whole fundamental group (defined as in [P1]) $\mathcal{F}(R *_D R \supseteq D) = \mathbf{R}_+^{\times}$, then the number of its free components would not be able to be determined uniquely. This is completely analogous to the situation of free group factors $L(\mathbb{F}_n)$ with finite n (see [V2, 6.13 *Remark*]). This analogy is very natural in a certain sense, because the amalgamated free product $R *_D R$ can be regarded as one candidate of the true generalizations of the free group factor $L(\mathbb{F}_2)$ from the view-point of the idea generalizing the group von Neumann algebra construction to the group-measure space construction. (D. Shlyakhtenko [S1] provided another candidate "A-valued semicircular systems" from the view-point of Voiculecu's free Gaussian functor (see [VDN]).)

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Finite approximations and physics over unconventional fields

Trond Digernes

In this talk we will discuss some ideas and results from 'unconventional physics', partly from the point of view of finite approximations.

Finite approximations play an important role in many areas of mathematics. In operator algebras there are several notions of approximate finiteness, for example *hyperfinite* algebras, AF-algebras, *residually finite* algebras, to mention some.

In the context of locally compact abelian groups there is a useful notion of closeness which takes into account the Weyl structures of the groups. It was shown in [4] that – with respect to this concept of closeness – any (separable) locally compact abelian group is a limit of finite abelian groups. This notion of convergence – called convergence of Weyl systems – involves *approximation from the outside*, i.e., the approximating groups need not be subgroups of the given group.

Convergence of Weyl systems takes place at the kinematical level. The deeper problem of approximating dynamical operators requires a more detailed analysis, and was treated in [6] for the case $\mathbf{R}^n$. Here it was shown that for quantum systems with potentials of 'oscillator type' (essentially those with discrete Hamilton spectrum), the finite approximands converge to the continuous system in the strongest possible sense: eigenvalues and eigenfunctions for the finite systems converge to the corresponding objects for the continuous system. These results have later been generalized to the setting of a general locally compact group [1]. (In this general setting, though, the position and momentum operators do not have obvious interpretations.)

The above approximation results may serve as motivation for studying quantum systems over fields other than $\mathbf{R}$ and $\mathbf{C}$ – like $\mathbf{Q}_p$, for instance – since, after all, such systems, too, can be obtained as limits of finite systems (where most computations will have to take place). This is,

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however, a rather modest approach to the subject of 'physics over unconventional fields and rings' (or 'unconventional physics' for short). There are other, and more profound, reasons, of which we mention just a couple here: 1) It can be argued that the structure of space-time below the Planck scale is best described by a number field with a non-archimedean metric (like $\mathbf{Q}_p$; see [13] and further references there). 2) Regularization: the well-known divergences in quantum field theory disappear when the underlying field is non-archimedean; and unbounded operators become bounded [2]. -It should be mentioned that simple quantum mechanical models, like the harmonic oscillator, can be successfully formulated over $\mathbf{Q}_p$, although it is not obvious what the nature of 'time' should be: should it be p-adic, real, or discrete? [13, 10, 7]

Since there does not seem to be a preferred prime p in nature, one eventually has to work with all the primes at the same time, which leads to the *adelic* theories (the ring of *adeles* is the restricted product of all the $\mathbf{Q}_p$'s, including $\mathbf{Q}_\infty = \mathbf{R}$). An adelic string theory has been formulated (see the excellent review article by Brekke and Freund [3] and the references therein), and Manin gives a beautiful argument for the adelic nature of our physical world in [8].

In a forthcoming paper [5] we study some other phenomena which may occur in general models. More specifically, we study irreducible models for Heisenberg groups based on compact maximal isotropic subgroups. It is shown that if both the Heisenberg group and the subgroup are 2-regular, the "vacuum sector" of the associated representation is 1-dimensional and thus gives rise to a unique vacuum state (this generalizes a result in [13] for $\mathbf{Q}_p$ with $p \neq 2$). On the other hand, if the Heisenberg group is 2-regular, but the subgroup is not, the vacuum sector exhibits a fermionic structure. This will be the case, for instance, in a quantum mechanical model built on the 2-adic numbers, with a maximal isotropic subgroup constructed from the 2-adic integers.

Finally it should be mentioned that the idea of doing physics in a setting other than ordinary space-time appears already in the works of Weyl [14, 15] and Schwinger (see, e.g., [11] and further references there; this article served as a starting point and motivation for the paper [6]). In fact, some of these ideas go back to Riemann [9] (for an English translation, see [12, page 135]).

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Operator means and their norms

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§1. Introduction

In his very interesting (unpublished) 1979 notes [17] A. McIntosh obtained the following arithmetic-geometric mean inequality for Hilbert space operators H, K, X:

$$||HXK|| \leq \frac{1}{2}||H^*HX + XKK^*||. \tag{1}$$

Among other things he also pointed out that simple alternative proofs for so-called Heinz-type inequalities ([9], and see also the discussions in §2) are possible based on this inequality. Then, about 15 years later Bhatia and Davis ([4]) noticed that the inequality remains valid for all unitarily invariant norms (including the Schatten norms $||\cdot||_p$ and so on). Recall that a norm $|||\cdot|||$ for Hilbert space operators is called unitarily invariant when $|||UXV||| = |||X|||$ for unitary operators U, V, and basic facts on these norms can be found for example in [8, 10, 19]. In recent years the arithmetic-geometric mean and related inequalities have been under active investigation by several authors, and very readable accounts on this subject can be found in [1, 3].

Motivated by all of the above, the authors have investigated simple unified proofs for known (as well as some new) norm inequalities, some refinement of the norm inequality (1) (such as the arithmetic-logarithmic-geometric mean inequality), and a general theory on operator (and/or matrix) means in a series of recent articles [15, 11, 12]. The purpose of the present notes is to give a brief survey on the topics dealt in these articles.

We will derive a variety of integral expressions for relevant operators to establish desired norm inequalities. This means that our arguments are not just for proving norm inequalities, but we are actually solving

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certain operator equations in a very explicit form. We will briefly touch this viewpoint at the end of the article, and more details were worked out in [12, §4,(A)]. Some related analysis can be found in [18], where the notion of a differential is investigated in detail. We also point out that the recent article [6] is technically closely related to our works although the main emphasis there may be different from ours.

§2. Arithmetic-geometric mean and related inequalities

As was observed in [15] one can obtain simple and unified proofs for the norm inequalities mentioned in §1 based on the Poisson integral formula for the strip

$$S = \{z \in \mathbf{C};\ 0 \le \mathbf{Im}\ z \le 1\}.$$

Namely, for $0 < \theta < 1$ we set $d\mu_\theta(t) = a_\theta(t)dt$ and $d\nu_\theta(t) = b_\theta(t)dt$ with

$$a_\theta(t) = \frac{\sin(\pi\theta)}{2(\cosh(\pi t) - \cos(\pi\theta))} \quad \text{and} \quad b_\theta(t) = \frac{\sin(\pi\theta)}{2(\cosh(\pi t) + \cos(\pi\theta))}.$$

Then, for a bounded continuous function $f(z)$ on the strip S which is analytic in the interior, the well-known Poisson integral formula

$$f(i\theta) = \int_{-\infty}^{\infty} f(t)d\mu_\theta(t) + \int_{-\infty}^{\infty} f(i+t)d\nu_\theta(t)$$

is valid (see [20] for example). We point out that the total masses of the measures $d\mu_\theta(t), d\nu_\theta(t)$ are $1-\theta, \theta$ respectively.

We begin with a simple proof for the arithmetic-geometric mean inequality (1). To this end, we may and do assume the positivity of H, K (by the standard argument on the polar decomposition). The function $f(t) = H^{1+it}XK^{-it}$ ($t \in \mathbf{R}$) extends to a bounded continuous (in the strong operator topology) function on the strip S which is analytic in the interior. Here, H^{it}, K^{-it} are understood as unitaries on the support spaces of H, K respectively. Since $d\mu_{\frac{1}{2}}(t) = d\nu_{\frac{1}{2}}(t) = \dfrac{dt}{2\cosh(\pi t)}$ (with total mass $\frac{1}{2}$), we have

$$\begin{aligned} H^{\frac{1}{2}}XK^{\frac{1}{2}} &= f(\tfrac{i}{2}) = \int_{-\infty}^{\infty} f(t)\ d\mu_{\frac{1}{2}}(t) + \int_{-\infty}^{\infty} f(i+t)\ d\nu_{\frac{1}{2}}(t) \\ &= \int_{-\infty}^{\infty} K^{it}(KX + XK)K^{-it}\frac{dt}{2\cosh(\pi t)}. \end{aligned}$$

The unitary invariance of $|||\cdot|||$ thus implies

$$|||K^{\frac{1}{2}}XK^{\frac{1}{2}}||| \leq |||HX+XK||| \times \int_{-\infty}^{\infty} \frac{dt}{2\cosh(\pi t)} = \frac{1}{2}|||HX+XK|||.$$

Heinz-type inequalities ([9]) deal with operators of forms $H^{\frac{1}{p}}XK^{\frac{1}{q}} \pm H^{\frac{1}{q}}XK^{\frac{1}{p}}$, where $p, q \geq 1$ with $1/p + 1/q = 1$. Note that the preceding argument also shows

$$\text{(2)} \quad H^{\frac{1}{p}}XK^{\frac{1}{q}} = \int_{-\infty}^{\infty} H^{it}HXK^{-it}d\mu_{\frac{1}{q}}(t) + \int_{-\infty}^{\infty} H^{it}XKK^{-it}d\nu_{\frac{1}{q}}(t),$$

$$\text{(3)} \quad H^{\frac{1}{q}}XK^{\frac{1}{p}} = \int_{-\infty}^{\infty} H^{it}HXK^{-it}d\mu_{\frac{1}{p}}(t) + \int_{-\infty}^{\infty} H^{it}XKK^{-it}d\nu_{\frac{1}{p}}(t).$$

We note $d\mu_{\frac{1}{q}} = d\nu_{\frac{1}{p}}$ and $d\mu_{\frac{1}{p}} = d\nu_{\frac{1}{q}}$. Hence, by summing up (2) and (3), we get

$$\begin{aligned} H^{\frac{1}{p}}XK^{\frac{1}{q}} + H^{\frac{1}{q}}XK^{\frac{1}{p}} &= \int_{-\infty}^{\infty} H^{it}(HX+XK)K^{-it}d\mu_{\frac{1}{q}}(t) \\ &\quad + \int_{-\infty}^{\infty} H^{it}(HX+XK)K^{-it}d\nu_{\frac{1}{q}}(t) \\ &= \int_{-\infty}^{\infty} K^{it}(HX+XK)K^{-it}d(\mu_{\frac{1}{q}}+\nu_{\frac{1}{q}})(t). \end{aligned}$$

This expression obviously shows

$$|||H^{\frac{1}{p}}XK^{\frac{1}{q}} + H^{\frac{1}{q}}XK^{\frac{1}{p}}||| \leq |||HX+XK|||$$

since the total mass of the measure $d(\mu_{\frac{1}{q}}+\nu_{\frac{1}{q}})(t)$ is $\frac{1}{p}+\frac{1}{q}=1$. The "difference version"

$$\text{(4)} \qquad |||H^{\frac{1}{p}}XK^{\frac{1}{q}} - H^{\frac{1}{q}}XK^{\frac{1}{p}}||| \leq |\tfrac{2}{p}-1| \times |||HX-XK|||.$$

is also valid. Indeed, by subtracting (3) from (2), we have

$$H^{\frac{1}{p}}XK^{\frac{1}{q}} - H^{\frac{1}{q}}XK^{\frac{1}{p}} = \int_{-\infty}^{\infty} H^{it}(HX-XK)K^{-it}d(\mu_{\frac{1}{q}}-\nu_{\frac{1}{q}})(t).$$

It is plain to see

$$a_{\frac{1}{q}}(t) - b_{\frac{1}{q}}(t) = \frac{\sin(\frac{\pi}{q}) \times 2\cos(\frac{\pi}{q})}{2\left(\cosh^2(\pi t) - \cos^2(\frac{\pi}{q})\right)} = \frac{\sin(\frac{2\pi}{q})}{\cosh(2\pi t) - \cos(\frac{2\pi}{q})},$$

and consequently we have

$$\begin{aligned}
&H^{\frac{1}{p}}XK^{\frac{1}{q}} - H^{\frac{1}{q}}XK^{\frac{1}{p}} \\
&\quad = \int_{-\infty}^{\infty} H^{it}(HX - XK)K^{-it} \frac{\sin(\frac{2\pi}{q})}{\cosh(2\pi t) - \cos(\frac{2\pi}{q})}\, dt \\
&\quad = \int_{-\infty}^{\infty} H^{\frac{is}{2}}(HX - XK)K^{-\frac{is}{2}} \frac{\sin(\frac{2\pi}{q})}{\cosh(\pi s) - \cos(\frac{2\pi}{q})} \times \frac{ds}{2}.
\end{aligned}$$

Let us assume $1 < p < 2$. Then, the above measure is exactly $d\mu_{\frac{q}{2}}(s)$ with the total mass $1 - \frac{2}{q} = 1 - 2(1 - \frac{1}{p}) = \frac{2}{p} - 1$ (> 0), showing (4) in this case. The opposite case $2 < p < \infty$ can be handled simply by switching H and K.

It follows from (4) that

$$|||H^{\frac{1}{2}+\varepsilon}XK^{\frac{1}{2}-\varepsilon} - H^{\frac{1}{2}-\varepsilon}XK^{\frac{1}{2}+\varepsilon}||| \le 2\varepsilon \times |||HX - XK|||$$

is valid for $0 < \varepsilon < \frac{1}{2}$. Let us assume the invertibility of $H, K \ge 0$ here. By dividing the above by ε and then by letting $\varepsilon \searrow 0$, we easily see

$$|||(\log H)(H^{\frac{1}{2}}XK^{\frac{1}{2}}) - (H^{\frac{1}{2}}XK^{\frac{1}{2}})(\log K)||| \le |||HX - XK|||.$$

From this we obtain the following commutator estimate:

Theorem 1 (Theorem 4, [15]). *For operators A, B, X with A, B self-adjoint, we have*

$$|||AX - XB||| \le |||\exp(\tfrac{A}{2})X\exp(-\tfrac{B}{2}) - \exp(-\tfrac{A}{2})X\exp(\tfrac{B}{2})|||$$

for each unitarily invariant norm $|||\cdot|||$.

A somewhat related topic is the "matrix Young inequality" due to T. Ando. In [2] he showed that for each positive matrices H, K and $1 < p < \infty$ (with the conjugate exponent q) one can find a unitary matrix U satisfying

$$|HK| \le U(\tfrac{1}{p}H^p + \tfrac{1}{q}K^q)U^*$$

(the special case $p = q = 2$ was dealt in [5]). In particular,

$$|||HK||| \le |||\tfrac{1}{p}H^p + \tfrac{1}{q}K^q|||$$

is valid, however in [2, §7] he pointed out that

$$(5) \qquad |||HXK||| \le |||\tfrac{1}{p}H^pX + \tfrac{1}{q}XK^q||| \quad \text{is } \textbf{false}$$

(unless $p=2$) for example for the operator norm $|||\cdot||| = ||\cdot||$.

Let us try to understand this phenomenon. Assume that H $(=\exp A)$, X are matrices, and let A be a diagonal matrix with diagonal entries $\lambda_1, \lambda_2, \cdots, \lambda_n \in \mathbf{R}$. We set

$$Y = \int_{-\infty}^{\infty} \exp(itA)(\tfrac{1}{p}\exp(A)X + \tfrac{1}{q}X\exp(A))\exp(-itA)f(t)dt$$

with $f(t)$ to be determined. The (j,k)-component of Y is

$$Y_{j,k} = (\tfrac{1}{p}\exp(\lambda_j) + \tfrac{1}{q}\exp(\lambda_k))(\mathcal{F}f)(\lambda_j - \lambda_k)X_{j,k}.$$

Therefore, if one wants $Y = \exp(\frac{A}{p})X\exp(\frac{A}{q})$, then one must have

$$\left(\frac{\exp(\lambda_j)}{p} + \frac{\exp(\lambda_k)}{q}\right)(\mathcal{F}f)(\lambda_j - \lambda_k) = \exp(\tfrac{\lambda_j}{p})\exp(\tfrac{\lambda_k}{q}).$$

This requirement is the same as

$$(6) \qquad (\mathcal{F}f)(\lambda_j - \lambda_k) = \frac{\exp(\frac{\lambda_j}{p})\exp(\frac{\lambda_k}{q})}{\frac{\exp(\lambda_j)}{p} + \frac{\exp(\lambda_k)}{q}} = \frac{1}{\frac{1}{p}\exp(\frac{\lambda_j-\lambda_k}{q}) + \frac{1}{q}\exp(-\frac{\lambda_j-\lambda_k}{p})},$$

that is,

$$(\mathcal{F}f)(s) = \left(\tfrac{1}{p}\exp(\tfrac{s}{q}) + \tfrac{1}{q}\exp(-\tfrac{s}{p})\right)^{-1}.$$

It is possible to compute explicitly the inverse Fourier transform of this function, and indeed we can prove

$$(7) \qquad f(t) = \frac{p^{\frac{1}{p}}q^{\frac{1}{q}}(\frac{p}{q})^{-it}}{2\cosh\left(\pi t + \frac{\pi i}{2}\left(\frac{1}{p} - \frac{1}{q}\right)\right)}.$$

By the standard approximation argument we get the next result in the special case $H = K$. Then, the general case can be handled by the well-known 2×2-matrix trick: by applying the special case to

$$\tilde{H} = \begin{bmatrix} H & 0 \\ 0 & K \end{bmatrix} \quad \text{and} \quad \tilde{X} = \begin{bmatrix} 0 & X \\ 0 & 0 \end{bmatrix},$$

one can look at the $(1,2)$-component to get the desired conclusion.

Theorem 2 (Theorem 6, [15]). *For operators H, K, X with H, K positive and $p \in (1, \infty)$ with the conjugate exponent q we have*

$$H^{\frac{1}{p}} X K^{\frac{1}{q}} = \int_{-\infty}^{\infty} H^{it}(\tfrac{1}{p}HX + \tfrac{1}{q}XK)K^{-it} f(t)dt$$

with the function $f(t)$ defined by (7). In particular, for each unitarily invariant norm $|||\cdot|||$ we have

$$|||H^{\frac{1}{p}} X K^{\frac{1}{q}}||| \le k_p |||\tfrac{1}{p}HX + \tfrac{1}{q}XK|||$$

with $k_p = \int_{-\infty}^{\infty} |f(t)|dt$ $(< \infty)$.

Notice $\int_{-\infty}^{\infty} f(t)dt = (\mathcal{F}f)(0) = 1$. However, $f(t)$ is complex-valued and consequently $k_p = \int_{-\infty}^{\infty} |f(t)|dt > 1$ (unless $p = 2$). This fact corresponds to the failure of (5). The constant k_p can be rewritten as

$$k_p = \frac{p^{\frac{1}{p}} q^{\frac{1}{q}}}{\pi} \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1 - \kappa^2 \sin^2\theta}} \quad \text{with } \kappa = \sin\left(\tfrac{\pi}{2}\left(\tfrac{1}{p} - \tfrac{1}{q}\right)\right).$$

Note that k_p depends only on $p \in (1, \infty)$ (independent of the choice of $|||\cdot|||$), but unfortunately k_p blows up when either $p \searrow 1$ or $p \nearrow \infty$. On the other hand, unitarily invariant norms under which the map $A \to |A|$ is Lipschitz continuous were thoroughly analyzed in [7, 14]. For such a unitarily invariant norm $|||\cdot|||$ a constant $k = k_{|||\cdot|||}$ can be chosen in such a way that

$$|||H^{\frac{1}{p}} X K^{\frac{1}{q}}||| \le k |||\tfrac{1}{p}HX + \tfrac{1}{q}XK|||$$

is valid for all $p \in (1, \infty)$ (see [12, Proposition 3.1]).

§3. Refinement of the arithmetic-geometric mean inequality

The logarithmic mean of positive scalars λ, μ is

$$\frac{\lambda - \mu}{\log\lambda - \log\mu} = \int_0^1 \lambda^t \mu^{1-t} dt.$$

The second integral form indicates that for operators H, K, X with $H, K \ge 0$ one can introduce their logarithmic mean by

$$L = \int_0^1 H^t X K^{1-t} dt.$$

The above right side should be understood in the weak sense, i.e.,

$$(L\xi,\eta)=\int_0^1 (H^tXK^{1-t}\xi,\eta)dt \quad \text{(for each vectors } \xi,\eta).$$

For simplicity, we set

$$\begin{aligned} G &= H^{\frac{1}{2}}XK^{\frac{1}{2}} \quad \text{(geometric mean)},\\ A &= \frac{1}{2}\,(HX+XK) \quad \text{(arithmetic mean)}, \end{aligned}$$

and we would like to compare the three means.

The ratios (between the relevant scalar means) are

$$\begin{aligned} \frac{\log\lambda-\log\mu}{\lambda-\mu}\times\sqrt{\lambda\mu} &= g_1(\log\lambda-\log\mu),\\ \frac{\lambda-\mu}{\log\lambda-\log\mu}\times\frac{2}{\lambda+\mu} &= g_2(\log\lambda-\log\mu) \end{aligned}$$

with

$$g_1(s)=\frac{s/2}{\sinh(s/2)} \quad \text{and} \quad g_2(s)=\frac{\tanh(s/2)}{s/2}.$$

By repeating the argument (recall (6)) before Theorem 2 with H^{it} instead of e^{itA}, we arrive at the integral expressions

$$\begin{aligned} G &= \int_{-\infty}^{\infty} H^{it}LK^{-it}\,\frac{\pi}{2\cosh^2(\pi t)}\,dt,\\ L &= \int_{-\infty}^{\infty} H^{it}AK^{-it}\log\left|\coth\left(\frac{\pi t}{2}\right)\right|\,\frac{2dt}{\pi}. \end{aligned}$$

The densities $\dfrac{\pi}{2\cosh^2(\pi t)}$ and $\dfrac{2}{\pi}\log\left|\coth\left(\dfrac{\pi t}{2}\right)\right|$ here arise as the inverse Fourier transforms of $g_1(s)$ and $g_2(s)$. They are positive functions (i.e., $g_i(s)$'s are positive definite thanks to the Bochner theorem) with total mass $g_i(0)=1$. Consequently, we get the following strengthening of (1) (i.e., arithmetic-logarithmic-geometric mean inequality):

Proposition 3 (Proposition 1, [11]). *Let H,K,X be Hilbert space operators with $H,K\geq 0$. For any unitarily invariant norm $|||\cdot|||$ we have*

$$|||H^{\frac{1}{2}}XK^{\frac{1}{2}}|||\leq |||\int_0^1 H^sXK^{1-s}ds|||\leq \frac{1}{2}|||HX+XK|||.$$

Actually, further refinement is possible by introducing the two series of operator means corresponding to the following natural scalar means:

$$\frac{1}{m}\sum_{k=0}^{m-1}\lambda^{\frac{k}{m-1}}\mu^{\frac{m-1-k}{m-1}}, \quad \frac{1}{n}\sum_{k=1}^{n}\lambda^{\frac{k}{n+1}}\mu^{\frac{n+1-k}{n+1}}.$$

The cases $m = 2$ and $n = 1$ correspond to the arithmetic and geometric means respectively. Note that what was important in the proof of Proposition 3 is the positive definiteness of ratios between relevant scalar means, and this reasoning (together with some others) enables us to prove

Theorem 4 (Theorem 5, [11]). *Let H, K, X be Hilbert space operators with H, K positive, and $|||\cdot|||$ be a unitarily invariant norm.*
(i) For each m (≥ 1) and n (≥ 2), the following inequalities are valid:

$$\begin{aligned} |||H^{\frac{1}{2}}XK^{\frac{1}{2}}||| &\leq \frac{1}{m}|||\sum_{k=1}^{m}H^{\frac{k}{m+1}}XK^{\frac{m+1-k}{m+1}}||| \leq |||\int_0^1 H^tXK^{1-t}dt||| \\ &\leq \frac{1}{n}|||\sum_{k=0}^{n-1}H^{\frac{k}{n-1}}XK^{\frac{n-1-k}{n-1}}||| \leq \frac{1}{2}|||HX+XK|||. \end{aligned}$$

(ii) The quantity $\frac{1}{m}|||\sum_{k=1}^{m}H^{\frac{k}{m+1}}XK^{\frac{m+1-k}{m+1}}|||$ is monotone increasing in m, and furthermore we have the monotone convergence

$$\lim_{m\to\infty}\frac{1}{m}|||\sum_{k=1}^{m}H^{\frac{k}{m+1}}XK^{\frac{m+1-k}{m+1}}||| = |||\int_0^1 H^tXK^{1-t}dt|||.$$

(iii) The quantity $\frac{1}{n}|||\sum_{k=0}^{n-1}H^{\frac{k}{n-1}}XK^{\frac{n-1-k}{n-1}}|||$ is monotone decreasing in n.

Notice that the assertion (ii) in the theorem is a certain monotone convergence theorem for a norm, and more precise convergence results (for operators) for various means are investigated in our recent article [13].

§4. General means for matrices

It is clear from the discussions so far that the positive definiteness of ratios between involved scalar means is a key to establish norm inequalities. This viewpoint in fact enables us to investigate norm comparison of means in a more axiomatic fashion, which makes it possible to handle various other means. In this section we explain this approach, but for simplicity we will mainly restrict ourselves to finite-dimensional operators (see Remark 6 for the infinite-dimensional case). Namely, we introduce a certain class of binary means (for positive scalars), to each of which one can associate a matrix mean in a natural way.

By a symmetric homogeneous mean we shall mean a continuous positive function on $[0,\infty) \times [0,\infty)$ satisfying

(a) $M(\lambda,\mu) = M(\mu,\lambda)$,
(b) $M(\alpha\lambda,\alpha\mu) = \alpha M(\lambda,\mu)$ for any $\alpha > 0$,
(c) $M(\lambda,\mu)$ is non-decreasing in λ and μ,
(d) $\min\{\lambda,\mu\} \le M(\lambda,\mu) \le \max\{\lambda,\mu\}$.

We denote by $\mathfrak{M}$ the set of all such means.

For $H \in M_n(\mathbf{C})$, the $n \times n$ matrices, we write $H \ge 0$ if H is positive semi-definite, and $H > 0$ if $H \ge 0$ is invertible. We regard $M_n(\mathbf{C})$ as a (finite-dimensional) Hilbert space equipped with the inner product $\langle X, Y\rangle = \mathrm{Tr}(XY^*)$ $(X, Y \in M_n(\mathbf{C}))$. For $H, K \ge 0$ let $\mathbf{L}_H, \mathbf{R}_K$ be the left multiplication by H and the right multiplication by K respectively, i.e., $\mathbf{L}_H X = HX$ and $\mathbf{R}_K X = XK$ for $X \in M_n(\mathbf{C})$. Note that they are commuting positive operators acting on $M_n(\mathbf{C})$, and for each $M \in \mathfrak{M}$ one can perform the functional calculus $M(\mathbf{L}_H, \mathbf{R}_K)$ (which is a positive operator acting on $M_n(\mathbf{C})$). Thus, for each $X \in M_n(\mathbf{C})$ we can consider $M(\mathbf{L}_H, \mathbf{R}_K)X$ $(\in M_n(\mathbf{C}))$, which will be simply denoted by $M(H,K)X$.

Assume that the spectral decompositions of $H, K \in M_n(\mathbf{C})$ are

$$H = \sum_{i=1}^{n} \lambda_i P_i, \quad K = \sum_{j=1}^{n} \mu_j Q_j$$

with eigenvalue lists $\{\lambda_i\}, \{\mu_j\}$ and rank-one projections $\{P_i\}, \{Q_j\}$ respectively. Then, $M(H,K)$ is obviously given by

$$M(H,K)X = \sum_{i,j=1}^{n} M(\lambda_i,\mu_j) P_i X Q_j. \tag{8}$$

This means that with the diagonalization

$$H = U\mathrm{diag}(\lambda_1,\lambda_2,\dots,\lambda_n)U^*, \quad K = V\mathrm{diag}(\mu_1,\mu_2\dots,\mu_n)V^*$$

via unitary matrices U, V we have

$$M(H,K)X = U\Big(\big[M(\lambda_i,\mu_j)\big] \circ (U^*XV)\Big)V^*, \tag{9}$$

where $\circ$ means the Hadamard product (i.e., the entry-wise product).

With the interpretation of $M(H,K)X$ explained so far, we can prove

Theorem 5 (Theorem 1.1, [12]). *For means $M, N \in \mathfrak{M}$ the following conditions are equivalent:*

(i) *one can find a symmetric probability measure ν on $\mathbf{R}$ satisfying*

$$M(H,K)X = \int_{-\infty}^{\infty} H^{is}(N(H,K)X)K^{-is}\,d\nu(s)$$

for all matrices H, K, X (of any size) with $H, K > 0$;

(ii) *one has $|||M(H,K)X||| \le |||N(H,K)X|||$ for each matrices H, K, X (of any size) with $H, K \ge 0$ and for each unitarily invariant norm $|||\cdot|||$;*

(iii) *one has $\|M(H,H)X\| \le \|N(H,H)X\|$ for each matrices H, X (of any size) with $H \ge 0$;*

(iv) *for each $\lambda_1, \lambda_2, \dots, \lambda_n > 0$ (with any n), $\left[\dfrac{M(\lambda_i,\lambda_j)}{N(\lambda_i,\lambda_j)}\right]_{1\le i,j\le n}$ is a positive semi-definite matrix;*

(v) *the function $M(e^t,1)/N(e^t,1)$ is positive definite on $\mathbf{R}$.*

In the above, the measure ν in (i) is the representing one for the ratio $M(e^t,1)/N(e^t,1)$ in the Bochner theorem, i.e.,

$$M(e^t,1)/N(e^t,1) = \int_{-\infty}^{\infty} e^{its}\,d\nu(s).$$

We consider the following typical one-parameter families of means:

$$\begin{aligned} M_\alpha(\lambda,\mu) &= \begin{cases} \frac{\alpha-1}{\alpha} \times \frac{\lambda^\alpha - \mu^\alpha}{\lambda^{\alpha-1}-\mu^{\alpha-1}} & (\lambda \ne \mu) \\ \lambda & (\lambda = \mu), \end{cases} \\ B_\alpha(\lambda,\mu) &= \left(\frac{\lambda^\alpha + \mu^\alpha}{2}\right)^{1/\alpha} \end{aligned}$$

with $-\infty \le \alpha \le \infty$. The arithmetic, logarithmic and geometric means appear as

$$\begin{aligned} M_2(\lambda,\mu) &= \frac{\lambda+\mu}{2}, \\ M_1(\lambda,\mu) &= \frac{\lambda-\mu}{\log\lambda - \log\mu} \left(= \lim_{\alpha\to 1} M_\alpha(\lambda,\mu)\right), \\ M_{1/2}(\lambda,\mu) &= \sqrt{\lambda\mu}, \end{aligned}$$

while it is easy to see

$$
\begin{aligned}
M_{\frac{n}{n-1}}(\lambda,\mu) &= \frac{1}{n}\sum_{k=0}^{n-1}\lambda^{\frac{k}{n-1}}\mu^{\frac{k-1}{n-1}} \quad (n=2,3,\cdots),\\
M_{\frac{m}{m+1}}(\lambda,\mu) &= \frac{1}{m}\sum_{k=1}^{m}\lambda^{\frac{k}{m+1}}\mu^{\frac{m+1-k}{m+1}} \quad (m=1,2,\cdots)
\end{aligned}
$$

(which correspond to the operator means appeared in Theorem 4). On the other hand, with the special choice $\alpha = 1/n$

$$B_{1/n}(\lambda,\mu) = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\lambda^{\frac{k}{n}}\mu^{\frac{n-k}{n}}$$

is the usual binomial mean.

For $-\infty \le \alpha \le \beta \le \infty$ one can prove the positive definiteness of the ratio

$$
\begin{aligned}
\frac{M_\alpha(e^t,1)}{M_\beta(e^t,1)} &= \frac{(\alpha-1)\beta}{\alpha(\beta-1)} \times \frac{(e^{\alpha t}-1)(e^{(\beta-1)t}-1)}{(e^{(\alpha-1)t}-1)(e^{\beta t}-1)}\\
&= \frac{(\alpha-1)\beta}{\alpha(\beta-1)} \times \frac{\sinh(\alpha t/2)\sinh((\beta-1)t/2)}{\sinh((\alpha-1)t/2)\sinh(\beta t/2)}
\end{aligned}
$$

(see [12, Theorem 2.1]). Therefore, thanks to Theorem 5 we can obtain further generalization of Theorem 4 in §3. One can also prove the positive definiteness of ratios such as

$$
\begin{aligned}
\frac{M_{1/2}(e^t,1)}{B_\alpha(e^t,1)} &= \left(\frac{1}{\cosh(\alpha t/2)}\right)^{1/\alpha} \quad (\alpha > 0),\\
\frac{B_{1/n}(e^t,1)}{M_2(e^t,1)} &= \frac{\cosh^n(t/2n)}{\cosh(t/2)}.
\end{aligned}
$$

Note

$$\frac{1}{\cosh t} = \int_{-\infty}^{\infty} e^{its}\,\frac{ds}{2\cosh(\pi s/2)},$$

and the positive definiteness of the former is indeed a consequence of the infinite divisibility of the probability measure $(2\cosh(\pi s/2))^{-1}ds$. From these we conclude

$$|||H^{1/2}XK^{1/2}||| \le \frac{1}{2^n}|||\sum_{k=0}^{n}\binom{n}{k}H^{\frac{k}{n}}K^{\frac{n-k}{n}}||| \le \frac{1}{2}|||HX+XK|||$$

for instance (see [12, Proposition 3.3]). Some other means as well as a variety of comparison results for their norms (based on Theorem 5) are obtained in [12].

The idea behind Theorem 5 (especially the integral representation for matrix means) can be also adopted to obtain solutions to certain matrix equations in a very explicit way. To see a flavor of this application, as a typical example we consider the matrix equation

$$\int_0^1 H^t Y K^{1-t} dt = X$$

for a unknown matrix Y with positive invertible matrices H, K. The equation means $M_1(H,K)Y = X$ with the logarithmic mean $M_1(\lambda,\mu) = \dfrac{\lambda-\mu}{\log\lambda - \log\mu}$. It is plain to see that the reciprocal is $M_0(\lambda^{-1},\mu^{-1})$, and it follows from the expression (9) that the unique solution Y is given by

$$Y = M_0(H^{-1}, K^{-1})X.$$

The comparison of M_0 with $M_{1/2}$, for instance, supplies the integral expression

$$Y = \int_{-\infty}^{\infty} H^{-\frac{1}{2}+is} X K^{-\frac{1}{2}-is} \,\frac{\pi ds}{2\cosh^2(\pi s)}$$

for this solution. Furthermore, the different integral expressions

$$Y = \int_0^\infty (H+tI)^{-1} X (K+tI)^{-1} dt$$

and

$$Y = \int_0^\infty \int_0^\infty e^{-sH} X e^{-tK} \,\frac{dsdt}{s+t}$$

for the same Y are also possible based on some other tools (see [12]).

Remark 6. *It is possible to generalize Theorem 5 to infinite-dimensional operators. Namely, we simply replace $M_n(\mathbf{C})$ by the Hilbert space $\mathcal{C}_2(\mathcal{H})$ of Hilbert-Schmidt class operators. In this setting the multiplication operators $\mathbf{L}_H, \mathbf{R}_K$ (positive operators in $B(\mathcal{C}_2(\mathcal{H}))$) can be also considered for arbitrary positive operators $H, K \geq 0$. Consequently, as long as X is taken from $\mathcal{C}_2(\mathcal{H})$, the mean $M(H,K)X = M(\mathbf{L}_H, \mathbf{R}_K)X$ ($\in \mathcal{C}_2(\mathcal{H})$) makes a perfect sense. With this interpretation the theorem remains valid for Hilbert space operators. In [12, §4,(C)] the requirement $X \in \mathcal{C}_2(\mathcal{H})$ was not explicitly mentioned, and we apologize for this inaccuracy.*

A theory of means $M(H,K)X$ *with* $X \in B(\mathcal{H})$ *is more preferable. Such a theory is developed in our recent article* [13] *based on the theory of double integral transformations. Roughly speaking it is a continuous version of* (8), *and for a very wide class of scalar means* $M(\lambda,\mu)$ (*including all the examples in* [12]) *corresponding operator means* $M(H,K)X$ ($\in B(\mathcal{H})$) *are completely justified for each* $X \in B(\mathcal{H})$. *Moreover, in the forthcoming article* [16] *we will obtain a variety of new norm inequalities not covered here* (*nor in* [11, 12, 13, 15]).

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Quantum spin chain and Popescu systems

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§1. Introduction

In this article, we explain how Popescu systems and their dilation to representations of the Cuntz algebra are related to some problems of quantum statistical mechanics. The physics we discuss here is the quasi one-dimensional material, closely related to an unsolved problem of anti-ferromagnetic Heisenberg models. First we begin by stating our notation and the mathematical problem precisely. Our quantum spin models with an infinite degree of freedom are described as a C^*-dynamical system on a UHF C^*-algebra. The standard references of this mathematical approach are [9] and [10]. The algebra of local observables is the infinite tensor product $\mathfrak{A}_{loc}$ of the full matrix algebras. For the usual quantum system with spin s ($s = 1/2, 1, 3/2, \cdots$), the one site algebra is $M_{2s+1}(\mathbf{C})$, the set of $2s+1$ by $2s+1$ matrices, and in this case

$$\mathfrak{A}_{loc} = \bigotimes_{\mathbf{Z}} M_{2s+1}(\mathbf{C}).$$

Each component of the tensor product above is specified with a lattice site $j \in \mathbf{Z}$. The C^*-completion of $\mathfrak{A}_{loc}$ is denoted by $\mathfrak{A}$.

For any integer j and any matrix Q in $M_{2s+1}(\mathbf{C})$, $Q^{(j)}$ will be an observable Q located at the lattice site j. Thus, by $Q^{(j)}$ we denote the following element of $\mathfrak{A}$:

$$\cdots \otimes 1 \otimes 1 \otimes \underbrace{Q}_{\text{the j-th component}} \otimes 1 \otimes 1 \otimes \cdots \in \mathfrak{A}.$$

Given a subset Λ of $\mathbf{Z}$, $\mathfrak{A}_\Lambda$ is defined as the C^*-subalgebra of $\mathfrak{A}$ generated by all $Q^{(j)}$ with $Q \in M_n(\mathbf{C})$, $j \in \Lambda$. If φ is a state of $\mathfrak{A}$ the restriction of φ to $\mathfrak{A}_\Lambda$ will be denoted by φ_Λ:

$$\varphi_\Lambda = \varphi|_{\mathfrak{A}_\Lambda}.$$

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The translation τ_k (shift on the integer lattice $\mathbf{Z}$) is an automorphism of $\mathfrak{A}$ determined by

$$\tau_k(Q^{(j)}) = Q^{(j+k)}.$$

As the Lie group $SU(2)$ acts on the 2s+1 dimensional vector space irreducibly, each one site algebra $M_{2s+1}(\mathbf{C})$ has the adjoint action of $SU(2)$. From this action we obtain the product type action β_g of $SU(2)$ on $\mathfrak{A}$ which commutes with the lattice translation τ_k.

The time evolution of the system is governed by the one parameter group of automorphisms α_t on $\mathfrak{A}$. The generator δ of α_t is an approximate inner derivation obtained by the infinite volume limit of local Hamiltonians H_Λ on the finite subset Λ of $\mathbf{Z}$:

$$\frac{d}{dt}\alpha_t(Q)|_{t=0} = \delta(Q) = \lim_{\Lambda\to\mathbf{Z}} \ [H_\Lambda, Q]$$

for Q in $\mathfrak{A}_{loc}$. A standard Hamiltonian of the spin s antiferromagnetic chain is the Heisenberg Hamiltonian H_Λ

$$H_\Lambda = \sum_{j\in\Lambda} (S^{(j)}, S^{(j+1)}), \tag{1.1}$$

where $S^{(j)}_\alpha$ is the spin operator at the site j and

$$(S^{(j)}, S^{(j+1)}) = \sum_{\alpha=x,y,z} S^{(j)}_\alpha S^{(j+1)}_\alpha. \tag{1.2}$$

Another Hamiltonian frequently used in Sold State Physics is the following spin 1 Hamiltonian ($s = 1$):

$$H_\Lambda = \sum_{j\in\Lambda} \left\{ J_1(S^{(j)}, S^{(j+1)}) + J_2(S^{(j)}, S^{(j+1)})^2 \right\}. \tag{1.3}$$

The above Hamiltonians of (1.1) and (1.3) and the time evolution α_t associated with them are obviously $SU(2)$ invariant:

$$\beta_g \circ \alpha_t = \alpha_t \circ \beta_g.$$

Mathematically we may consider more general Hamiltonians as well. Although these two Hamiltonians are approximation of more complicated interactions, it is believed that the qualitative feature of these models represent "universal property" of $SU(2)$ invariant antiferromagnetic systems.

Decay of correlation and the spectrum property of these C^*-dynamical system $\{\mathfrak{A}, \alpha_t\}$ are of prime interest in mathematical physics. Next we formulate the problem more precisely.

States considered for the above models are ground states. As far as the spectrum and decay of correlation are concerned, KMS states are of no interest as they always have exponentially decay of correlation and each model does not exhibit any individual character.

By definition, a state φ is a ground state of the C^*-dynamical system $\{\mathfrak{A}, \alpha_t\}$ if

$$\frac{1}{i}\frac{d}{dt}\varphi(Q^*\alpha_t(Q)) \geq 0$$

for any Q of $\mathfrak{A}_{loc}$. Let $\{\pi_\varphi, \mathfrak{H}_\varphi, \Omega_\varphi\}$ be the GNS representation of a ground state φ where π_φ is the representation, and $\mathfrak{H}_\varphi$ is the Hilbert space and Ω_φ is the GNS cyclic vector. When the state φ is a ground state, there exists a positive (unbounded) selfadjoint operator H_φ on $\mathfrak{H}_\varphi$ such that

$$e^{itH_\varphi}\pi_\varphi(Q)e^{-itH_\varphi} = \pi_\varphi\alpha_t((Q)), \qquad H_\varphi\Omega_\varphi = 0. \tag{1.4}$$

By spectrum of the infinite volume Heisenberg model, we mean the spectrum of H_φ.

When the spin s is 1/2, the Heisenberg model of (1.1) is exactly solved and eigenvectors and their eigenvalues of (1.1) have been found by Bethe ansatz. Even though the completeness of eigenvectors is not yet proved, a lot of heuristic argument has been published. On the other hand, for higher spin, nothing is rigorously proved for the Hamiltonian (1.1) so far. Nevertheless due to the heuristic arguments and numerical simulations, the following are now believed.

Conjecture 1.1. *(i) The ground state of the antiferromagnetic Heisenberg model obtained by the infinite volume of (1.1) is unique.*
(iia) If the spin s is a half odd integer ($s = (2n-1)/2$) the spectrum of H_φ for the unique ground state φ has no gap. Namely for any positive number δ $\mathrm{Spec}\, H_\varphi \cap (0,\delta) \neq \emptyset$. The decay of correlation has the power law, so there exists Q and Q' in $\mathfrak{A}_{loc}$ such that the limit $\lim_{n\to\infty} |\varphi(Q\tau_n(Q')) - \varphi(Q)\varphi(Q')| = 0$ decays in a negative power of n.
(iib) If the spin s is an integer, the spectrum of H_φ has the gap in the sense that $\mathrm{Spec}\, H_\varphi \cap (0,\delta) = \emptyset$ for a positive number δ.

It is possible to show that the decay of correlation is exponential if the spectral gap of H_φ is open. In the case of integer spin Heisenberg models, the conjecture suggests the exponential decay of correlation of $|\varphi(Q\tau_n(Q')) - \varphi(Q)\varphi(Q')|$ for any Q and Q' in $\mathfrak{A}_{loc}$ (c.f. Theorem 1.4 below).

Note that when the ground state is unique, it is pure and SU(2) invariant (invariant under the product type action β_g). The conjecture (iia) and (iib) was proposed by Haldane in the beginning of 1980's. The spin 1/2 case of Haldane's conjecture is supported by the exact (non-rigorous) solution. For the half odd integer spin I.Affleck and E.Lieb have shown that the gap of the spectrum of the finite volume Hamiltonian H_Λ vanishes in the infinite volume limit (see [2]). When the spin is integer nothing is known but in [3] I.Affleck, T.Kennedy, E.Lieb and H.Tasaki found an example of Hamiltonian with the similar property to the property Haldane's conjecture claims. In fact, they have shown that when the spin is one and $J_1 = 1$, $J_2 = 1/3$ the Hamiltonian (1.3) has the unique infinite volume ground state which has the exponential decay correlation and the spectral gap. Their Hamiltonian H_{AKLT} is referred to as the AKLT model or the AKLT Hamiltonian:

$$H_{AKLT} = \sum_j \left\{ (S^{(j)}, S^{(j+1)}) + \frac{1}{3}(S^{(j)}, S^{(j+1)})^2 \right\}. \tag{1.5}$$

To prove their results, I.Affleck, T.Kennedy, E.Lieb and H.Tasaki constructed the ground state of the AKLT Hamiltonian with a finite algebraic manipulation (the iteration of complete positive maps on finite matrix algebras). The state they obtained in [3] is named Valence Bond Solid state. Since the paper [3] was published, Valence Bond Solid states was studied extensively. We will return "VBS" states later.

Unlike the integer spin case, we do not have any definite result on the decay of correlation for the half odd integer spin case. However, M.Aizenman and B.Nachtergaele obtained a measure theoretic representation of ground states for a class of Hamiltonians. Within their class of Hamiltonians, M.Aizenman and B.Nachtergaele have found that translation symmetry breaking occurs if the decay of correlation is fast. It is interesting to ask if the result which M.Aizenman and B.Nachtergaele proved is a universal phenomenon or due to special choice of Hamiltonians. The theorem below is related to this question.

Theorem 1.2. *Suppose that the spin s is a half odd integer and consider a translationally invariant pure state φ of $\mathfrak{A}$. If φ is $SU(2)$ invariant, φ cannot have the following uniform cluster property:*

$$\lim_{k\to\infty} \sup_{\|Q\|\leq 1} \left| \sum_i (\varphi(Q_i \tau_k(R_i)) - \varphi(Q_i)\varphi(R_i)) \right| = 0, \tag{1.6}$$

where Q is any local observable in $\mathfrak{A}_{loc}$ written in the finite sum

$$Q = \sum_i \gamma_i Q_i R_i, \qquad Q_j \in \mathfrak{A}_{(-\infty,-1]} \quad R_i \in \mathfrak{A}_{[0,\infty)}.$$

We can formulate the same kind of results as above for an arbitrary compact semisimple Lie group.

Purity and translation invariance of φ implies the cluster property

$$\lim_{k\to\infty}\left|\sum_{i,j}(\varphi(Q_j\tau_k(R_i)) - \varphi(Q_j)\varphi(R_i))\right| = 0.$$

Let $\{\pi_\varphi, \mathfrak{H}_\varphi \Omega_\varphi\}$ be the GNS representation of a translation invariant factor state φ. Consider the von Neumann algebras $M_1 = \pi_\varphi(\mathfrak{A}_{(-\infty,-1]})''$ and $M_2 = \pi_\varphi(\mathfrak{A}_{[0,\infty)})'$. The uniformity of the cluster property in (1.6) is equivalent to the condition that the inclusion $M_1 \subset M_2$ contains an intermediate type I factor $\mathfrak{N}$. Following R.Longo in [18] we call such an inclusion $M_1 \subset \mathfrak{N} \subset M_2$ split.

Definition 1.3. *Let φ be a translation invariant factor state of $\mathfrak{A}$. We say that the state φ is split if the uniform cluster condition (1.6) is satisfied.*

Any Gibbs state for a finite range interaction is split and so is any VBS state. The construction of non-split pure states is a non trivial mathematical problem. We believe that the exponential decay correlation implies the split property of state, as we are not aware of any counter-example to this claim. Moreover Theorem 1.4 tells us that the spectral gap implies the exponential decay of correlation. If the exponential decay of correlation implies the split property, Theorem 1.1 is a solution to a part of Haldane's conjecture.

Theorem 1.4. *Consider a finite range translationally invariant Hamiltonian H_Λ. Suppose that the state φ is a pure ground state, and that the ground state energy of H_φ is non-degenerate and the spectral gap opens in the sense specified in Conjecture 1.1. For any Q and Q' in $\mathfrak{A}_{loc}$,*

$$|\varphi(Q\tau_n(Q')) - \varphi(Q)\varphi(Q')| \leq C(Q,Q')e^{-Mn},$$

where $C(Q,Q')$, M are positive constants dependent on Q and Q'

This theorem is a lattice version of Cluster Theorem of K.Fredenhagen (c.f. [15]). A crucial assumption for Cluster Theorem of K.Fredenhagen is strict locality for the time evolution, which is not valid in our case. However we have entire analyticity and quasi-locality of our time evolution, with which we obtain Theorem 1.4. Unfortunately we do not have good estimates for constants $C(Q,Q')$ and M.

About the question whether there exist any pure state without split property, we have the following answer.

Theorem 1.5. *Consider the spin $\frac{1}{2}$ chain. Let φ be a translationally invariant pure state. Suppose that it is invariant under the torus $U(1)$ in $SU(2)$: $\varphi \circ \beta_z = \varphi$ for any $z \in U(1)$. Then φ is either a product state or a non-split state.*

Due to this result, the ground state of the one-dimensional XY model is non-split. We will also discuss this model later.

Next we sketch the key point of our proof for Theorem 1.2, 1.4. The proof is based on notions of split property, the shift of $\mathfrak{B}(\mathfrak{H})$ and Popescu systems.

A standard argument of quasi-local algebra implies that a state φ is split if and only if φ is quasi-equivalent to the state $\varphi_{(-\infty,-1]} \otimes \varphi_{[0,\infty)}$ where $\varphi_{(-\infty,-1]}$ and $\varphi_{[0,\infty)}$ are the restriction of φ to $\mathfrak{A}_{(-\infty,-1]}$ and $\mathfrak{A}_{[0,\infty)}$. The split property of states is one of basic concepts in the local quantum field theory (see [16], [11] and [12] and the references therein).

When φ is a translation invariant pure split state, the restriction $\varphi_{[0,\infty)}$ is a type I factor state. Passing to the GNS representation and the restriction of τ_j $(j \geq 0)$ to $\mathfrak{A}_R$ we obtain the shift of $\mathfrak{B}(\mathfrak{H})$ in the sense of R.Powers ([21]). Any shift of $\mathfrak{B}(\mathfrak{H})$ with Powers index d is implemented by the generator of the Cuntz algebra O_d. Thus the translation invariant state φ is extendible to a state of the Cuntz algebra O_d. The Popescu system describes the connection of symmetry of φ and the representation of the Cuntz algebra O_d. We will explain this connection more explicitly in the next section.

§2. Popescu Systems

Here we begin with the definition of the Popescu system. This notion naturally appears when a translationally invariant state $\varphi_{[0,\infty)}$ of $\mathfrak{A}_{[0,\infty)}$ is extended to a state on the Cuntz algebra O_d $(d = 2s + 1)$.

Definition 2.1. *Let $\mathfrak{K}$ be a separable Hilbert space. By Popescu system on $\mathfrak{K}$ we mean the triple $\{\mathfrak{M}, V, \psi\}$ satisfying the following conditions.*

$\mathfrak{M}$ is a von Neumann algebra acting on $\mathfrak{K}$ non-degenerately. V is an isometry from $\mathfrak{K}$ to $\mathbf{C}^d \otimes \mathfrak{K}$. ψ is a normal faithful state of $\mathfrak{M}$ satisfying the invariance

$$\psi(R) = \psi(E(1_{\mathbf{C}^d} \otimes R)), \tag{2.1}$$

where R is any element of $\mathfrak{M}$ and $E(Q)$ is the unital completely positive map from $M_d(\mathbf{C}) \otimes \mathfrak{B}(\mathfrak{K})$ to $\mathfrak{B}(\mathfrak{K})$ determined by

$$E(Q) = V^*QV \qquad \text{for any } Q \text{ in } M_d(\mathbf{C}) \otimes \mathfrak{B}(\mathfrak{K}). \tag{2.2}$$

Given a Popescu system $\{\mathfrak{M}, V, \psi\}$ on $\mathfrak{K}$, we can construct a translationally invariant state of the UHF algebra $\mathfrak{A}$ (d=2s+1) by the formula

$$(2.3) \qquad \begin{aligned} &\varphi(Q_0^{(j)} Q_1^{(j+1)} Q_2^{(j+2)} \cdots Q_l^{(j+l)}) \\ &\qquad = \psi(E(Q_0 \otimes E(Q_1 \otimes \cdots E(Q_l \otimes 1_{\mathfrak{K}})) \cdots)). \end{aligned}$$

When the dimension of $\mathfrak{K}$ is finite, the state φ determined by (2.3) is called the finitely correlated state or quantum Markov state or Matrix product state in mathematical physics. L.Accardi introduced quantum Markov states as a non-commutative extension of Markov measures in [1]. After the discovery of the AKLT Hamiltonian, M.Fannes, B.Nachtergaele and R,Werner found the relationship between VBS states and quantum Markov states in [13]. (See also [14].) The ground state of the AKLT Hamiltonian is described by the Popescu system where $\mathfrak{K}$ is two dimensional and V is the intertwiner of the representations of SU(2).

Any translationally invariant state is described by the Popescu system if we allow the dimension of $\mathfrak{K}$ to be infinite.

Lemma 2.2. *Let φ be a translationally invariant state of $\mathfrak{A}$. There exists a Popescu system on $\mathfrak{K}$, $\{\mathfrak{M}, V, \psi\}$ such that the state φ is described by (2.3). Furthermore when φ is a factor state, the complete positive map E of (2.2) has the following ergodicity:*
If $E(1_{\mathbf{C}^d} \otimes Q) = Q$ for Q in $\mathfrak{M}$, Q is a scalar $Q = c1$.

The lemma is based on the following observation. Let $\{\pi_\varphi, \mathfrak{H}_\varphi, \Omega_\varphi\}$ be the GNS triple of φ. Consider the von Neumann algebra $\pi(\mathfrak{A}_{[0,\infty)})''$ and the shift τ_1 restricted to $\mathfrak{A}_{[0,\infty)}$. τ_1 is extendible to the endomorphism Θ on the von Neumann algebra $\pi(\mathfrak{A}_{[0,\infty)})''$. Θ is implemented by a representation of the Cuntz algebra. Namely, on $\mathfrak{H}_\varphi$ there exists isometries S_k $(k = 1, 2, ..d)$ such that

$$S_k^* S_j = \delta_{kj} 1, \quad \sum_{k=1}^{d} S_k S_k^* = 1$$

and

$$\Theta(Q) = \sum_{k=1}^{d} S_k Q S_k^* \qquad Q \in \pi(\mathfrak{A}_{[0,\infty)})''.$$

Let P be the support projection of φ as the state of $\pi(\mathfrak{A}_{[0,\infty)})''$. Let $\mathfrak{K}$ be the range of P. Then we set $\mathfrak{M} = P\pi(\mathfrak{A}_{[0,\infty)})''P$, $V = (V_1, V_2, ...V_d) = (PS_1^*P, PS_2^*P, ...PS_d^*P)$ and we obtain the claim of Lemma.

In [8] O.Bratteli, P.Jorgensen, A.Kishimoto and R.Werner studied the connection of the Popescu system and the shift of the von Neumann algebra $\mathfrak{M} = P\pi(\mathfrak{A}_{[0,\infty)})''P$ when it is of type I. We also consider the same situation.

Lemma 2.3. *Let φ be a translationally invariant pure state of $\mathfrak{A}$. The following conditions are equivalent:*
(i) The state φ is split;
(ii) The GNS representation of $\mathfrak{A}_{[0,\infty)}$ associated with $\varphi_{[0,\infty)}$ is type I factor.

An endomorphism Θ on a von Neumann algebra $\mathfrak{M}$ is shift if it satisfies the following condition:

$$\cap_{n=01,2,..}\Theta^n(\mathfrak{M}) = \mathbf{C1}.$$

It is known that the shift of a type I factor is inner in the sense that Θ is implemented by the canonical endomorphism of the Cuntz algebra O_d in $\mathfrak{M}$.

Let E be the completely positive map of (2.2) and set

$$E_Q(R) = E(Q \otimes R)$$

for $Q \in M_d(\mathbf{C})$. E_Q is a complete bounded map from $\mathfrak{M}$ to $\mathfrak{M}$.

Lemma 2.4. *Let φ be a translationally invariant factor state of $\mathfrak{A}$ and $\{\mathfrak{M}, V, \psi\}$ be the associated Popescu system. If $\mathfrak{M}_1$ is a von Neumann subalgebra of $\mathfrak{M}$ containing* 1 *and invariant under the operator E_Q with any $Q \in M_d(\mathbf{C})$, then $\mathfrak{M}_1 = 1$ or $\mathfrak{M}_1 = \mathfrak{M}$.*

We call that the above condition the minimality condition. This minimality is crucial in our proof of our Theorem.

Now we proceed to study of the symmetry property of pure states with split property. Here we consider $\mathfrak{A}_{[0,\infty)}$ and the Cuntz algebra implementing the shift. The gauge action γ_U of the group $U(d)$ of d by d unitary matrices is defined via the formula

$$\gamma_U(S_k) \;=\; \sum_{l=1}^{d} U_{lk}S_l,$$

where U_{kl} is the k l matrix element for U in $U(d)$. Consider the diagonal circle group $U(1) = \{z \;\; |z| = 1 \; z \in \mathbf{C}\,\}$. We identify the fixed point algebra ${O_d}^{U(1)}$ with $\mathfrak{A}_{[0,\infty)}$.

Let G be a compact group and $v(g)$ be a d-dimensional unitary representation of G. By β_g we denote the product action of G on the infinite tensor product $\mathfrak{A}$ induced by $v(g)$:

$$\beta_g(Q) = (....\otimes v(g)\otimes v(g)\otimes v(g)\otimes ...)Q(....\otimes v(g)^{-1}\otimes v(g)^{-1}\otimes v(g)^{-1}\otimes ...)$$

for any Q in $\mathfrak{A}$. On $\mathfrak{A}_{[0,\infty)}$ the gauge action $\gamma_{v(g)}$ of the Cuntz algebra O_d and β_g coincide, $\gamma_{v(g)}(Q) = \beta_g(Q)$ for any Q in $\mathfrak{A}_{[0,\infty)}$.

Proposition 2.5. *Let φ be a translationally invariant factor state of $\mathfrak{A}$. Suppose that φ and G invariant:*

$$\varphi(\beta_g(Q)) = \varphi(Q) \quad \textit{for any } g \textit{ in } G \textit{ and any } Q \textit{ in } \mathfrak{A}.$$

Let $\{\mathfrak{M}, V, \psi\}$ be the canonical Popescu system for φ.
(i)There exist a projective unitary representation $u(g)$ of G on $\mathfrak{K}$ and a one-dimensional unitary representation $\xi(g)$ such that V intertwines the representation as follows:

$$(\xi(g)v(g)) \otimes u(g)V = Vu(g). \tag{2.4}$$

(ii) $Ad(u(g))$ leaves $\mathfrak{M}$ invariant.
(iii) The normal state of ψ of $\mathfrak{M}$ is invariant under $Ad(u(g))$.

Proposition 2.6. *Suppose that φ is translationally invariant, pure and split. The automorphism $Ad(u(g))$ on $\mathfrak{M}$ is the inner ($u(g) \in \mathfrak{M}$).*

Suppose, further, that G is the one-dimensional torus $U(1)$ or a connected, simply connected compact semisimple Lie group. The projective representation $u(g)$ is a unitary representation.

Corollary 2.7. *Let G be a connected, simply connected compact semisimple Lie group. Let $\hat{G}$ be the set of irreducible unitary duals of the compact group G. Let $\{F_\alpha\}$ be a partition of $\hat{G}$; $F_\alpha \subset \hat{G}$, $F_\alpha \cap F_\beta =$ $(\alpha \neq \beta)$, $\cup_\alpha F_\alpha = \hat{G}$.*

Suppose that any irreducible component in the tensor of the representation $v(g)$ and F_α is contained in another different F_β.

Then there exists no translationally invariant pure split state.

Using the above Proposition we obtain our main results Theorem 1.2 and 1.4.

§3. Examples

In this section we present a few examples of ground states, which illustrate some aspects of previous results. We consider the translationally invariant Hamiltonian. For simplicity we assume that the interaction is of nearest neighbor. So suppose a selfadjoint element $h_0 = h_0^*$ in $\mathfrak{A}_{\{0,1\}}$ is given and consider the finite volume Hamiltonian $H_{[n,m]}$ in $\mathfrak{A}_{[n,m]}$ on the interval $[n, m]$ is determined by

$$H_{[n,m]} = \sum_{j=n}^{m-1} h_j,$$

where we set $\tau_j(h_0) = h_j$.

We begin with the exactly solvable XY model as an example of Theorem 1.4. The Hamiltonian H_{XY} of the XY model is determined by the equation

$$H_{XY} = -\sum_{j\in\mathbf{Z}} \left\{\sigma_x^{(j)}\sigma_x^{(j+1)} + \sigma_y^{(j)}\sigma_y^{(j+1)}\right\} - 2\lambda \sum_{j\in\mathbf{Z}} \sigma_z^{(j)}, \tag{3.1}$$

where λ is a real parameter (an external magnetic field), $\sigma_x^{(j)}$, $\sigma_y^{(j)}$ and $\sigma_z^{(j)}$ are the Pauli spin matrices at the site j. For the finite chain the XY model is equivalent to the free Fermion via the Jordan Wigner transformation. For the infinite chain the equivalence is not literally correct due to the infinite product of $\sigma_z^{(j)}$ in the Jordan Wigner transformation. Nevertheless we have obtained the following results in [5].

Theorem 3.1. *The ground state of the XY model (3.1) is unique for any real λ.*
(i) $|\lambda| \geq 1$ it is a product state. The spectral gap is open if $|\lambda| > 1$.
(ii) $|\lambda| < 1$ the ground state is not a product state. The spectrum is purely absolutely continuous without gap.

The Hamiltonian H_{XY} of the XY model is invariant under the rotation around the z axis where the infinitesimal generator of the rotation is

$$N = \sum_{j\in\mathbf{Z}} \sigma_z^{(j)}.$$

Due to uniqueness the ground state is invariant under the rotation around the z axis. As a consequence, Theorem 1.4 implies the following.

Corollary 3.2. *For the case $|\lambda| < 1$, the unique ground state of the XY model (3.1) is a pure state without split property.*

Next we consider a generalization of the AKLT model of [3]. Here we only give an abstract condition for Hamiltonians.

Assumption 3.3. *(i) We assume that h_j is positive: $H_{[n,m]} \geq 0$ for any $[n,m]$.*
(ii) The dimension of the kernel of $H_{[n,m]}$ (the multiplicity of zero eigenvalue of $H_{[n,m]}$ as a finite matrix) is greater than one and uniformly bounded in n and m:

$$1 \leq \sup_{n,m} \, dim \, ker \, H_{[n,m]} \, < \, \infty. \tag{3.2}$$

If a suitable constant is added to the Hamiltonian, AKLT model (1.5) satisfies the Assumption 3.3.

Theorem 3.4. *Suppose that Assumption 3.3 is valid. Let φ be a translationally invariant pure ground state.*

Then the state φ is split and pure. In fact, the auxiliary Hilbert space $\mathfrak{K}$ of the canonical Popescu system is finite dimensional. The two point function decays exponentially fast:

$$\sup_{j \in \mathbf{Z}} |\varphi(Q_1 \tau_j(Q_2)) - \varphi(Q_1)\varphi(Q_2)| \, e^{m|j|} \, < \, \infty$$

for any local Q_1 and Q_2.

This result is a converse to a result of M.Fannes, B.Nachetergaele and R.Werner in [13]. They have shown that if φ is a translationally invariant pure state with the finite dimensional auxiliary Hilbert space $\mathfrak{K}$ of the associated Popescu system, there exists a projection P in $\mathfrak{A}_{loc}$ such that

$$\varphi(P_j) = 0 \;\; \text{with} \;\; P_j = \tau_j(P).$$

Thus φ is a ground state for the Hamiltonian $H = \sum_j P_j$.

Thus our Theorem 1.2 asserts impossibility of the construction of SU(2) spin half odd integer models satisfying the Assumption 3.3.

Next we consider another variant of the Heisenberg model, the antiferromagnetic XXZ model. The Hamiltonian of the XXZ model is defined by

$$H_{XXZ} = \sum_j \{\sigma_x^{(j)}\sigma_x^{(j+1)} + \sigma_y^{(j)}\sigma_y^{(j+1)} + \Delta\sigma_z^{(j)}\sigma_z^{(j+1)}\}, \tag{3.3}$$

where Δ is a real parameter. We consider the antiferromagnetic region of the model, i.e., $-1 < \Delta$. The following is the standard picture on the

ground state commonly accepted by physicists.
(a) If $-1 < \Delta \leq 1$, the ground state is unique. The correlation function decays in power. There is no spectral gap of Hamiltonian.
(b) When $1 < \Delta$, there exists precisely two pure ground states φ_{even} and φ_{odd}. They are not translationally invariant, however, periodic with period two, $\varphi_{even} = \varphi_{odd} \circ \tau_1$. The Hamiltonian has the spectral gap and their two point correlation functions decay exponentially fast.

The XXZ model is exactly solved by the Yang-Baxter machinery, though none of the above assertions has not be yet proved rigorously except the case where Δ is extremely large $1 << \Delta$. The reason why the large Δ case is well understood is that the standard cluster expansion (convergent perturbation theory) works. We can verify the above claim as well as the split property of ground states. It is easy to see that any product state is not the ground state of the XXZ model. Theorem 1.4 suggests the following implication.

Theorem 3.5. *Consider the XXZ model H_{XXZ} of (3.3) in the antiferromagnetic region. Then, one of the following is valid:*
(i) The ground state is unique and it is not split;
(ii) There exists two pure ground states which are not translationally invariant.

Theorem 3.5 is a complementary result to the one due to I.Affleck and E.Lieb ([2]) saying: there is no spectral gap if the ground state is unique. Their argument does not yield any information on correlation function while our Theorem 3.5 (i) asserts lack of certain uniform clustering.

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Topological conjugacy invariants of symbolic dynamics arising from C^*-algebra K-theory

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§1 Introduction

In [Wi], R. F. Williams introduced the notions of strong shift equivalence and shift equivalence between nonnegative square matrices and showed that two topological Markov shifts are topologically conjugate if and only if the associated matrices are strong shift equivalent. He also showed that strong shift equivalence implies shift equivalence (cf. [KimR]). There is a class of subshifts called sofic subshifts that are generalized class of Markov shifts and determined by square matrices with entries in formal sums of symbols (see [Kit],[Kr4],[LM],[We],etc.). Such a square matrix is called a symbolic matrix. It is an equivalent object to a labeled graph called a λ-graph. M. Nasu in [N],[N2] generalized the notion of strong shift equivalence to symbolic matrices. He showed that two sofic subshifts are topologically conjugate if and only if their canonical symbolic matrices are strong shift equivalent ([N],[N2],see also [HN]). M. Boyle and W. Krieger in [BK] introduced the notion of shift equivalence for symbolic matrices and studied topologically conjugacy for sofic subshifts.

In [Ma6], the notions of symbolic matrix system and λ-graph system have been introduced as presentations of subshifts. They are generalized notions of symbolic matrix and λ-graph for sofic subshifts. Let Σ be a finite set. A symbolic matrix system over Σ consists of two sequences of rectangular matrices $(\mathcal{M}_{l,l+1}, I_{l,l+1}), l \in \mathbb{N}$. The matrices $\mathcal{M}_{l,l+1}$ have entries in formal sums of Σ and the matrices $I_{l,l+1}$ have entries in $\{0,1\}$. They satisfy the following commutation relations

$$I_{l,l+1}\mathcal{M}_{l+1,l+2} = \mathcal{M}_{l,l+1}I_{l+1,l+2}, \qquad l \in \mathbb{N}.$$

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We assume that for i there exists j such that the (i,j)-component $I_{l,l+1}(i,j) = 1$ and for j there uniquely exists i such that $I_{l,l+1}(i,j) = 1$. We denote it by $(\mathcal{M}, I)$.

A λ-graph system $\mathfrak{L} = (V, E, \lambda, \iota)$ is a labeled Bratteli diagrams with vertex set $V = V_1 \cup V_2 \cup \cdots$ and edge set $E = E_{1,2} \cup E_{2,3} \cup \cdots$ that naturally arises from a symbolic matrix system $(\mathcal{M}, I)$. The matrix $\mathcal{M}_{l,l+1}$ defines an edge e in $E_{l,l+1}$ from a vertex in V_l to a vertex in V_{l+1} whose label is denoted by $\lambda(e) \in \Sigma$. The matrix $I_{l,l+1}$ defines a surjection from V_{l+1} to V_l. The symbolic matrix systems and the λ-graph systems are the same objects and give rise to subshifts. There is a canonical method to construct a symbolic matrix system from an arbitrary subshift. The obtained symbolic matrix system is said to be canonical for the subshift. If a subshift is sofic, the canonical symbolic matrix system corresponds to the symbolic matrix of its left Krieger cover graph. The notion of strong shift equivalence for nonnegative matrices and symbolic matrices has been generalized to symbolic matrix systems ([Ma6], cf. [Ma11]). We have proved (cf. [N],[Wi])

Theorem A ([Ma6]). *Two subshifts are topologically conjugate if and only if their canonical symbolic matrix systems are strong shift equivalent.*

Shift equivalence between two symbolic matrix systems has been defined in [Ma6] as a generalization of the corresponding notion for symbolic matrices defined by Boyle-Krieger in [BK].

For a symbolic matrix system $(\mathcal{M}, I)$, let $M_{l,l+1}$ be the nonnegative rectangular matrix obtained from $\mathcal{M}_{l,l+1}$ by setting all the symbols equal to 1 for each $l \in \mathbb{N}$. Then the resulting pair (M, I) still satisfies the following relations.

$$I_{l,l+1}M_{l+1,l+2} = M_{l,l+1}I_{l+1,l+2}, \qquad l \in \mathbb{N}.$$

We call (M, I) the nonnegative matrix system for $(\mathcal{M}, I)$. We say (M, I) to be canonical when $(\mathcal{M}, I)$ is canonical. More generally, for a sequence $M_{l,l+1}, l \in \mathbb{N}$ of rectangular matrices with entries in nonnegative integers and a sequence $I_{l,l+1}, l \in \mathbb{N}$ of rectangular matrices with entries in $\{0, 1\}$, the pair (M, I) is called a nonnegative matrix system if they satisfy the relations above. For a single $n \times n$ nonnegative square matrix A, if we set $M_{l,l+1} = A$ and $I_{l,l+1} = I_n$: the $n \times n$ identity matrix for all $l \in \mathbb{N}$, the pair (M, I) is a nonnegative matrix system. We similarly formulate strong shift equivalence and shift equivalence between nonnegative matrix systems as generalizations of the corresponding equivalences for single nonnegative square matrices.

For nonnegative matrices, the dimension groups defined by Krieger in [Kr2], [Kr3] and the Bowen-Franks groups considered in [BF] are crucial shift equivalence invariants. They induce topological conjugacy invariants for the associated topological Markov shifts. We generalize them to nonnegative matrix systems. The following three kinds of objects for a nonnegative matrix system (M, I) are defined:

$$(\Delta_{(M,I)}, \Delta^+_{(M,I)}, \delta_{(M,I)}), \qquad K_i(M,I), \qquad BF^i(M,I), \quad i=0,1.$$

The dimension triple $(\Delta_{(M,I)}, \Delta^+_{(M,I)}, \delta_{(M,I)})$ consists of an abelian group $\Delta_{(M,I)}$ with positive cone $\Delta^+_{(M,I)}$ and an ordered automorphism $\delta_{(M,I)}$ on it. The K-groups $K_i(M,I), i=0,1$ and the Bowen-Franks groups $BF^i(M,I), i=0,1$ consist of a pair of abelian groups for each. The three kinds of objects above are invariant under shift equivalence in nonnegative matrix systems. Hence they are naturally induce topological conjugacy invariants for subshifts by taking their canonical nonnegative matrix systems. Relationships among the equivalences and the invariants for the matrix systems are as in the following way:

Theorem B ([Ma6]). *For two symbolic matrix systems $(\mathcal{M}, I)$, $(\mathcal{M}', I')$ and their respect nonnegative matrix systems $(M,I), (M',I')$, consider the following situations:*

(S1) $(\mathcal{M}, I) \approx (\mathcal{M}', I)$: *strong shift equivalence,*
(N1) $(M, I) \approx (M', I)$: *strong shift equivalence,*
(S2) $(\mathcal{M}, I) \sim (\mathcal{M}', I)$: *shift equivalence,*
(N2) $(M, I) \sim (M', I)$: *shift equivalence,*
(3) $(\Delta_{(M,I)}, \Delta^+_{(M,I)}, \delta_{(M,I)}) \cong (\Delta_{(M',I')}, \Delta^+_{(M',I')}, \delta_{(M',I')})$: *isomorphic dimension triples,*
(4) $(\Delta_{(M,I)}, \delta_{(M,I)}) \cong (\Delta_{(M',I')}, \delta_{(M',I')})$: *isomorphic dimension pairs,*
(5) $K_*(M,I) \cong K_*(M',I)$: *isomorphic K-groups,*
(6) $BF^*(M,I) \cong BF^*(M',I)$: *isomorphic Bowen-Franks groups.*

Then we have the following implications:

$$\begin{array}{ccccccccccc} (S1) & \Longrightarrow & (S2) & & & & & & & & \\ \Downarrow & & \Downarrow & & & & & & & & \\ (N1) & \Longrightarrow & (N2) & \Longrightarrow & (3) & \Longrightarrow & (4) & \Longrightarrow & (5) & \Longrightarrow & (6). \end{array}$$

Two subshifts are said to be flow equivalent if their suspension flows act on homeomorphic spaces under some homeomorphism that preserves orbits in an orientation preserving way (cf.[PS]). The Bowen-Franks group $\mathbb{Z}^n/(1-A)\mathbb{Z}^n$ for nonnegative matrix A is known to be not only a topological conjugacy invariant but also a flow equivalence invariant for the associated topological Markov shift ([BF], cf. [Fr], [PS]). We generalize it to subshifts.

Theorem C ([Ma8], cf.[Ma3]). *The K-groups and the Bowen-Franks groups for canonical nonnegative matrix systems for subshifts are invariant under flow equivalence.*

In [Ma6], the eigenvalues and eigenvectors of a nonnegative matrix system (M, I) have been defined as a generalization of the nonzero spectrum of a single nonnegative matrix. We denote by $Sp^{\times}(M, I)$ the set of all nonzero eigenvalues of (M, I). Let $Sp_b^{\times}(M, I)$ be the set of all nonzero eigenvalues of (M, I) having a certain boundedness condition on the corresponding eigenvectors. We know that the both $Sp^{\times}(M, I)$ and $Sp_b^{\times}(M, I)$ are not empty and invariant under shift equivalence of (M, I). In particular, if (M, I) is the canonical nonnegative matrix system for a subshift, the set $Sp_b^{\times}(M, I)$ is bounded by the topological entropy of the subshift.

The author has constructed the C^*-algebra $\mathcal{O}_\Lambda$ associated with subshift Λ ([Ma], cf. [CaM], [Ma10]) as a generalization of the Cuntz-Krieger algebra $\mathcal{O}_A$ associated with topological Markov shift Λ_A for matrix A with entries in $\{0, 1\}$. The C^*-algebra $\mathcal{O}_\Lambda$ has a canonical action of the one dimensional torus group, called gauge action and written as α_Λ. Let (M, I) be the canonical nonnegative matrix system for the subshift Λ. The invariants mentioned above are described in terms of the K-theoretic objects for the C^*-algebra as in the following way:

Theorem D ([Ma2], [Ma3], [Ma4], cf.[C3],[CK]).

$$\begin{aligned}(\Delta_{(M,I)}, \Delta^+_{(M,I)}, \delta_{(M,I)}) &\cong (K_0(\mathcal{F}_\Lambda), K_0(\mathcal{F}_\Lambda)_+, \hat{\alpha_\Lambda}_*),\\ K_i(M,I) &\cong K_i(\mathcal{O}_\Lambda), \qquad i = 0,1,\\ BF^i(M,I) &\cong \mathrm{Ext}^{i+1}(\mathcal{O}_\Lambda), \qquad i = 0,1\end{aligned}$$

where $\hat{\alpha_\Lambda}$ *denotes the dual action of* α_Λ *and* $\mathrm{Ext}^1(\mathcal{O}_\Lambda) = \mathrm{Ext}(\mathcal{O}_\Lambda)$, $\mathrm{Ext}^0(\mathcal{O}_\Lambda) = \mathrm{Ext}(\mathcal{O}_\Lambda \otimes C_0(\mathbb{R}))$.

The normalized nonnegative eigenvectors of (M, I) exactly correspond to the KMS-states for α_Λ on the C^*-algebra $\mathcal{O}_\Lambda$. Hence the set of bounded spectrums with nonnegative eigenvectors are the set of inverse temperatures for the admitted KMS states ([MWY],cf.[EFW]).

§2 Symbolic matrix systems and λ-graph systems as presentations of subshifts

We fix a finite set Σ and call it the alphabet. Each element of Σ is called a symbol. We write the empty symbol $\emptyset$ in Σ as 0. We denote by $\mathfrak{S}_\Sigma$ the set of all finite formal sums of elements of Σ.

For two symbolic matrices $\mathcal{A}$ over alphabet Σ and $\mathcal{A}'$ over alphabet Σ' and bijection ϕ from a subset of Σ onto a subset of Σ', we say that $\mathcal{A}$ and $\mathcal{A}'$ are specified equivalence under specification ϕ if $\mathcal{A}'$ can be obtained from $\mathcal{A}$ by replacing every symbol a appearing in $\mathcal{A}$ by $\phi(a)$. We write it as $\mathcal{A} \overset{\phi}{\simeq} \mathcal{A}'$. We call ϕ a specification from Σ to Σ'. These notions are due to M. Nasu in [N],[N2].

Two symbolic matrix systems $(\mathcal{M}, I)$ over Σ and $(\mathcal{M}', I')$ over Σ' are said to be isomorphic if there exists a specification ϕ from Σ to Σ' and an $m(l) \times m(l)$-square permutation matrix P_l for each $l \in \mathbb{N}$ such that

$$P_l \mathcal{M}_{l,l+1} \overset{\phi}{\simeq} \mathcal{M}'_{l,l+1} P_{l+1}, \qquad P_l I_{l,l+1} = I'_{l,l+1} P_{l+1} \qquad \text{for} \quad l \in \mathbb{N}.$$

Two λ-graph systems (V, E, λ, ι) over alphabet Σ and $(V', E', \lambda', \iota')$ over alphabet Σ' are said to be isomorphic if there exist bijections $\Phi_V : V \to V'$, $\Phi_E : E \to E'$ and a specification $\phi : \Sigma \to \Sigma'$ such that

(1) $\Phi_V(V_l) = V'_l \quad$ and $\quad \Phi_E(E_{l,l+1}) = E'_{l,l+1} \quad$ for $l \in \mathbb{N}$,
(2) $\Phi_V(s(e)) = s(\Phi_E(e)) \quad$ and $\quad \Phi_V(r(e)) = r(\Phi_E(e)) \quad$ for $e \in E$,
(3) $\iota'(\Phi_V(v)) = \Phi_V(\iota(v)) \quad$ for $v \in V$,
(4) $\lambda'(\Phi_E(e)) = \phi(\lambda(e)) \quad$ for $e \in E$

where for an edge $e \in E_{l,l+1}$, $s(e) \in V_l$ and $r(e) \in V_{l+1}$ denote the source vertex of e and the range vertex of e respectively. Then we know that there exists a bijective correspondence between the set of all isomorphism classes of symbolic matrix systems and the set of all isomorphism classes of λ-graph systems.

We will see that any subshift comes from a symbolic matrix system and equivalently from a λ-graph system. We review on subshifts. Let Σ be an alphabet. Let $\Sigma^{\mathbb{Z}}$ be the infinite product spaces $\prod_{i=-\infty}^{\infty} \Sigma_i$, where $\Sigma_i = \Sigma$, endowed with the product topology. The transformation σ on $\Sigma^{\mathbb{Z}}$ given by $(\sigma(x_i)) = (x_{i+1}), i \in \mathbb{Z}$ is called the (full) shift. Let Λ be a shift invariant closed subset of $\Sigma^{\mathbb{Z}}$ i.e. $\sigma(\Lambda) = \Lambda$. The topological dynamical system $(\Lambda, \sigma|_\Lambda)$ is called a subshift. We denote $\sigma|_\Lambda$ by σ and write the subshift as Λ for short. We denote by $X_\Lambda (\subset \prod_{i=1}^{\infty} \Sigma_i)$ the set of all right-infinite sequences that appear in Λ. A finite sequence $\mu = (\mu_1, ..., \mu_k)$ of elements $\mu_j \in \Sigma$ is called a block or a word of length

k. A block $\mu = (\mu_1, ..., \mu_k)$ is said to occur in $x = (x_i) \in \Sigma^{\mathbb{Z}}$ if $x_m = \mu_1, ..., x_{m+k-1} = \mu_k$ for some $m \in \mathbb{Z}$.

We will construct subshifts from symbolic matrix systems.

Let $(\mathcal{M}, I)$ be a symbolic matrix system over Σ and $\mathfrak{L} = (V, E, \lambda, \iota)$ its corresponding λ-graph system. Let L_l for $l \in \mathbb{N}$ be the set of all label sequences of paths from V_1 to V_l, that is,

$$L_l = \{(\lambda(e_1), \lambda(e_2), \dots, \lambda(e_l)) \in \Sigma^l \mid e_i \in E_{i,i+1}, r(e_i) = s(e_{i+1}) \text{ for } i = 1, 2, \dots, l-1\}.$$

We set

$$X_{(\mathcal{M},I)} = \{(\lambda(e_1), \lambda(e_2), \dots) \in \Sigma^{\mathbb{N}} \mid e_i \in E_{i,i+1}, r(e_i) = s(e_{i+1}) \text{ for } i \in \mathbb{N}\}$$

the set of all right infinite sequences consisting of labels along infinite paths. The topology on $X_{(\mathcal{M},I)}$ is defined from open sets of the form

$$U_{(\mu_1,\dots,\mu_k)} = \{(\alpha_1, \alpha_2, \dots) \in X_{(\mathcal{M},I)} \mid \alpha_i = \mu_i \text{ for } i = 1, \dots, k\}$$

for $(\mu_1, \dots, \mu_k) \in L_k$ so that $X_{(\mathcal{M},I)}$ is a compact Hausdorff space. For $(\alpha_1, \alpha_2, \dots) \in X_{(\mathcal{M},I)}$, we have $(\alpha_2, \alpha_3, \dots) \in X_{(\mathcal{M},I)}$. For $(\alpha_1, \alpha_2, \dots) \in X_{(\mathcal{M},I)}$, we may find a symbol $\alpha_0 \in \Sigma$ such that $(\alpha_0, \alpha_1, \alpha_2, \dots) \in X_{(\mathcal{M},I)}$. Hence the following map

$$S : (\alpha_1, \alpha_2, \alpha_3, \dots) \in X_{(\mathcal{M},I)} \to (\alpha_2, \alpha_3, \dots) \in X_{(\mathcal{M},I)}$$

is well-defined, continuous and surjective. We set

$$\Lambda_{(\mathcal{M},I)} = \varprojlim \{S : X_{(\mathcal{M},I)} \to X_{(\mathcal{M},I)}\}$$

the projective limit in the category of compact Hausdorff spaces. Thus $\Lambda_{(\mathcal{M},I)}$ is identified with the set of all biinfinite sequences arising from the sequences in $X_{(\mathcal{M},I)}$. That is

$$\Lambda_{(\mathcal{M},I)} = \{(\dots, \alpha_2, \alpha_1, \alpha_0, \alpha_1, \alpha_2, \dots) \mid (\alpha_n, \alpha_{n+1}, \dots) \in X_{(\mathcal{M},I)} \text{ for } n \in \mathbb{Z}\}.$$

The map S induces a homeomorphism on it. We denote it by σ that satisfies $\sigma((\alpha_i)_{i\in\mathbb{Z}}) = (\alpha_{i+1})_{i\in\mathbb{Z}}$. Therefore we obtain a subshift $(\Lambda_{(\mathcal{M},I)}, \sigma)$.

We next construct symbolic matrix systems from subshifts.

For a subshift (Λ, σ) over Σ and a number $k \in \mathbb{N}$, let Λ^k be the set of all words of length k in $\Sigma^{\mathbb{Z}}$ occurring in some $x \in \Lambda$. For $l \in \mathbb{N}$, two points

$x, y \in X_\Lambda$ are said to be l-past equivalent if $\{\mu \in \Lambda^l | \mu x \in X_\Lambda\} = \{\nu \in \Lambda^l | \nu y \in X_\Lambda\}$. Let $F_i^l, i = 1, 2, \dots, m(l)$ be the set of all l-past equivalence classes of X_Λ. We define two rectangular $m(l) \times m(l+1)$ matrices $I_{l,l+1}^\Lambda, \mathcal{M}_{l,l+1}^\Lambda$ with entries in $\{0,1\}$ and entries in $\mathfrak{S}_\Sigma$ respectively as in the following way. For $i = 1, 2, \dots, m(l), j = 1, 2, \dots, m(l+1)$, the (i,j)-component $I_{l,l+1}^\Lambda(i,j)$ of $I_{l,l+1}^\Lambda$ is one if F_i^l contains F_j^{l+1} otherwise zero. Let $a_1, \dots, a_n$ be the set of all symbols in Σ for which $a_k x \in F_i^l$ for some $x \in F_j^{l+1}$. We then define the (i,j)-component of the matrix $\mathcal{M}_{l,l+1}^\Lambda(i,j)$ as $\mathcal{M}_{l,l+1}^\Lambda(i,j) = a_1 + \cdots + a_n$: the formal sum of $a_1, \dots, a_n$. We can show that the pair $(\mathcal{M}^\Lambda, I^\Lambda)$ becomes a symbolic matrix system. We call it the *canonical* symbolic matrix system for Λ. We denote its λ-graph system by $\mathfrak{L}^\Lambda = (V^\Lambda, E^\Lambda, \lambda^\Lambda, \iota^\Lambda)$ and call it the *canonical* λ-graph system for Λ. The subshift $\Lambda_{(\mathcal{M}^\Lambda, I^\Lambda)}$ associated with $(\mathcal{M}^\Lambda, I^\Lambda)$ coincides with the original subshift Λ.

§3 Strong shift equivalence and shift equivalence

In this section, we define strong shift equivalence and shift equivalence between two symbolic matrix systems. For alphabets C, D, put $C \cdot D = \{cd | c \in C, d \in D\}$. For $x = \sum_j c_j \in \mathfrak{S}_C$ and $y = \sum_k d_k \in \mathfrak{S}_D$, define $xy = \sum_{j,k} c_j d_k \in \mathfrak{S}_{C \cdot D}$.

Let $(\mathcal{M}, I)$ and $(\mathcal{M}', I')$ be symbolic matrix systems over alphabets Σ, Σ' respectively, where $\mathcal{M}_{l,l+1}, I_{l,l+1}$ are $m(l) \times m(l+1)$ matrices and $\mathcal{M}'_{l,l+1}, I'_{l,l+1}$ are $m'(l) \times m'(l+1)$ matrices.

Definition. Two symbolic matrix systems $(\mathcal{M}, I), (\mathcal{M}', I)$ are said to be *strong shift equivalent in 1-step* and written as $(\mathcal{M}, I) \underset{1-st}{\approx} (\mathcal{M}', I')$ if there exist alphabets C, D and specifications $\varphi : \Sigma \to C \cdot D$ and $\phi : \Sigma' \to D \cdot C$ such that for each $l \in \mathbb{N}$, there exist an $m(l-1) \times m'(l)$ matrix $\mathcal{H}_l$ over C and an $m'(l-1) \times m(l)$ matrix $\mathcal{K}_l$ over D satisfying the following equations:

$$I_{l-1,l}\mathcal{M}_{l,l+1} \overset{\varphi}{\simeq} \mathcal{H}_l \mathcal{K}_{l+1}, \qquad I'_{l-1,l}\mathcal{M}'_{l,l+1} \overset{\phi}{\simeq} \mathcal{K}_l \mathcal{H}_{l+1}$$

and

$$\mathcal{H}_l I'_{l,l+1} = I_{l-1,l}\mathcal{H}_{l+1}, \qquad \mathcal{K}_l I_{l,l+1} = I'_{l-1,l}\mathcal{K}_{l+1}.$$

Two symbolic matrix systems $(\mathcal{M}, I)$ and $(\mathcal{M}', I')$ are said to be *strong shift equivalent in N-step* and written as $(\mathcal{M}, I) \underset{N-st}{\approx} (\mathcal{M}', I')$ if there exist symbolic matrix systems $(\mathcal{M}^{(i)}, I^{(i)}), i = 1, 2, \dots, N-1$ such that

$$(\mathcal{M}, I) \underset{1-st}{\approx} (\mathcal{M}^{(1)}, I^{(1)}) \underset{1-st}{\approx} (\mathcal{M}^{(2)}, I^{(2)}) \underset{1-st}{\approx} \cdots$$
$$\cdots \underset{1-st}{\approx} (\mathcal{M}^{(N-1)}, I^{(N-1)}) \underset{1-st}{\approx} (\mathcal{M}', I').$$

We simply call it a *strong shift equivalence.*

We see the following theorem.

Theorem 3.1 ([Ma6]). *Two subshifts Λ and Λ' are topologically conjugate if and only if their respect canonical symbolic matrix systems $(\mathcal{M}^{\Lambda}, I^{\Lambda})$ and $(\mathcal{M}^{\Lambda'}, I^{\Lambda'})$ are strong shift equivalent.*

In the proof given in [Ma6] of the only if part of Theorem 3.1, the bipartite λ-graph system has been introduced and M. Nasu's factorization theorem for topological conjugacy between subshifts into bipartite codes and symbolic conjugacies has been used. We can also prove Theorem 3.1 without using the Nasu's result, by considering the state splitting of λ-graph systems. Let $\mathfrak{L} = (V, E, \lambda, \iota)$ be a λ-graph system over Σ. Let $\mathcal{P}$ be a partition of Σ. We put $\Sigma^{[\mathcal{P}]} = \Sigma \times \Sigma/\mathcal{P}$ and $\Sigma_{[\mathcal{P}]} = \Sigma/\mathcal{P} \times \Sigma$ where $\Sigma/\mathcal{P}$ denotes the equivalence classes of Σ by the partition $\mathcal{P}$. Then we can construct the out-split λ-graph system $\mathfrak{L}^{[\mathcal{P}]} = (V^{[\mathcal{P}]}, E^{[\mathcal{P}]}, \lambda^{[\mathcal{P}]}, \iota^{[\mathcal{P}]})$ over $\Sigma^{[\mathcal{P}]}$ and the in-split λ-graph system $\mathfrak{L}_{[\mathcal{P}]} = (V_{[\mathcal{P}]}, E_{[\mathcal{P}]}, \lambda_{[\mathcal{P}]}, \iota_{[\mathcal{P}]})$ over $\Sigma_{[\mathcal{P}]}$ such that

$$(\mathcal{M}, I) \underset{1-st}{\approx} (\mathcal{M}^{[\mathcal{P}]}, I^{[\mathcal{P}]}) \quad \text{and} \quad (\mathcal{M}, I) \underset{1-st}{\approx} (\mathcal{M}_{[\mathcal{P}]}, I_{[\mathcal{P}]})$$

where $(\mathcal{M}, I), (\mathcal{M}^{[\mathcal{P}]}, I^{[\mathcal{P}]})$ and $(\mathcal{M}_{[\mathcal{P}]}, I_{[\mathcal{P}]})$ are the symbolic matrix systems for the λ-graph systems $\mathfrak{L}, \mathfrak{L}^{[\mathcal{P}]}$ and $\mathfrak{L}_{[\mathcal{P}]}$ respectively. Full detail of the construction will appear in [Ma11].

We will state the notion of shift equivalence between two symbolic matrix systems as a generalization of Williams's notion for nonnegative matrices and Boyle-Krieger's notion for symbolic matrices. Let $(\mathcal{M}, I), (\mathcal{M}', I')$ be two symbolic matrix systems over alphabets Σ, Σ' respectively. For $N \in \mathbb{N}$, we put $(\Sigma)^N = \Sigma \cdots \Sigma, (\Sigma')^N = \Sigma' \cdots \Sigma'$: the N-times products.

Definition. For $N \in \mathbb{N}$, two symbolic matrix systems $(\mathcal{M}, I), (\mathcal{M}', I')$ are said to be *shift equivalent of lag N* if there exist alphabets C_N, D_N and specifications

$$\varphi_1 : \Sigma \cdot C_N \to C_N \cdot \Sigma', \qquad \varphi_2 : \Sigma' \cdot D_N \to D_N \cdot \Sigma$$

and

$$\psi_1 : (\Sigma)^N \to C_N \cdot D_N, \qquad \psi_2 : (\Sigma')^N \to D_N \cdot C_N$$

such that for each $l \in \mathbb{N}$, there exist an $m(l) \times m'(l+N)$ matrix $\mathcal{H}_l$ over C_N and an $m'(l) \times m(l+N)$ matrix $\mathcal{K}_l$ over D_N satisfying the following equations:

$$\mathcal{M}_{l,l+1}\mathcal{H}_{l+1} \overset{\varphi_1}{\simeq} \mathcal{H}_l\mathcal{M}'_{l+N,l+N+1}, \qquad \mathcal{M}'_{l,l+1}\mathcal{K}_{l+1} \overset{\varphi_2}{\simeq} \mathcal{K}_l\mathcal{M}_{l+N,l+N+1},$$

$$I_{l,l+N}\mathcal{M}_{l+N,l+2N} \overset{\psi_1}{\simeq} \mathcal{H}_l\mathcal{K}_{l+N}, \qquad I'_{l,l+N}\mathcal{M}'_{l+N,l+2N} \overset{\psi_2}{\simeq} \mathcal{K}_l\mathcal{H}_{l+N}$$

and

$$I_{l,l+1}\mathcal{H}_{l+1} = \mathcal{H}_l I'_{l+N,l+N+1}, \qquad I'_{l,l+1}\mathcal{K}_{l+1} = \mathcal{K}_l I_{l+N,l+N+1}.$$

We denote this situation by

$$(\mathcal{M}, I) \underset{lagN}{\sim} (\mathcal{M}', I') \quad \text{or} \quad (\mathcal{H}, \mathcal{K}) : (\mathcal{M}, I) \underset{lagN}{\sim} (\mathcal{M}', I')$$

and simply call it a *shift equivalence.*

Proposition 3.2. *Strong shift equivalence in N-step implies shift equivalence of lag N.*

§4 Nonnegative matrix systems and dimension groups

A nonnegative matrix system consists of two sequences of rectangular matrices $(A_{l,l+1}, I_{l,l+1}), l \in \mathbb{N}$. The matrices $A_{l,l+1}$ have entries in nonnegative integers and the matrices $I_{l,l+1}$ have entries in $\{0,1\}$. They satisfy the following commutation relations

$$I_{l,l+1}A_{l+1,l+2} = A_{l,l+1}I_{l+1,l+2}, \qquad l \in \mathbb{N}.$$

We assume that for i there exists j such that the (i,j)-component $I_{l,l+1}(i,j) = 1$ and for j there uniquely exists i such that $I_{l,l+1}(i,j) = 1$. We denote it by (A, I).

Lemma 4.1. *For a symbolic matrix system $(\mathcal{M}, I)$, let $M_{l,l+1}$ be the $m(l) \times m(l+1)$ rectangular matrix obtained from $\mathcal{M}_{l,l+1}$ by setting all the symbols equal to 1. Then the resulting pair (M, I) becomes a nonnegative matrix system.*

For nonnegative matrix systems we formulate strong shift equivalence and shift equivalence as follows.

Definition. Two nonnegative matrix systems $(A, I), (A', I')$ are said to be *strong shift equivalent in 1-step* and written as $(A, I) \underset{1-st}{\approx} (A', I')$ if for each $l \in \mathbb{N}$, there exist an $m(l-1) \times m'(l)$ matrix H_l with entries in nonnegative integers and an $m'(l-1) \times m(l)$ matrix K_l with entries in nonnegative integers satisfying the following equations:

$$I_{l-1,l} A_{l,l+1} = H_l K_{l+1}, \qquad I'_{l-1,l} A'_{l,l+1} = K_l H_{l+1}$$

and

$$H_l I'_{l,l+1} = I_{l-1,l} H_{l+1}, \qquad K_l I_{l,l+1} = I'_{l-1,l} K_{l+1}.$$

Two nonnegative matrix systems (A, I) and (A', I') are said to be *strong shift equivalent in N-step* $(A, I) \underset{N-st}{\approx} (A', I')$ if there exist nonnegative matrix systems $(A^{(i)}, I^{(i)}), i = 1, 2, \dots, N-1$ such that

$$(A, I) \underset{1-st}{\approx} (A^{(1)}, I^{(1)}) \underset{1-st}{\approx} (A^{(2)}, I^{(2)}) \underset{1-st}{\approx} \cdots$$
$$\cdots \underset{1-st}{\approx} (A^{(N-1)}, I^{(N-1)}) \underset{1-st}{\approx} (A', I').$$

We simply call it a *strong shift equivalence.*

For a nonnegative matrix system (A, I), we set the $m(l) \times m(l+k)$ matrices:

$$I_{l,l+k} = I_{l,l+1} \cdot I_{l+1,l+2} \cdots I_{l+k-1,l+k},$$
$$A_{l,l+k} = A_{l,l+1} \cdot A_{l+1,l+2} \cdots A_{l+k-1,l+k}.$$

Definition. Two nonnegative matrix systems $(A, I), (A', I')$ are said to be *shift equivalent of lag N* if for each $l \in \mathbb{N}$, there exist an $m(l) \times m'(l+N)$ matrix H_l with entries in nonnegative integers and an $m'(l) \times m(l+N)$ matrix K_l with entries in nonnegative integers satisfying the following equations:

$$A_{l,l+1} H_{l+1} = H_l A'_{l+N,l+N+1}, \qquad A'_{l,l+1} K_{l+1} = K_l A_{l+N,l+N+1},$$
$$H_l K_{l+N} = I_{l,l+N} A_{l+N,l+2N}, \qquad K_l H_{l+N} = I'_{l,l+N} A'_{l+N,l+2N}$$

and

$$I_{l,l+1} H_{l+1} = H_l I'_{l+N,l+N+1}, \qquad I'_{l,l+1} K_{l+1} = K_l I_{l+N,l+N+1}.$$

We write this situation as

$$(A, I) \underset{lagN}{\sim} (A', I') \quad \text{or} \quad (H, K) : (A, I) \underset{lagN}{\sim} (A', I')$$

and simply call it a *shift equivalence.*

Proposition 4.2. *If two symbolic matrix systems are strong shift equivalent in N-step (resp. shift equivalent of lag N), then the associated nonnegative matrix systems are strong shift equivalent in N-step (resp. shift equivalent of lag N).*

We describe the matrix relations appearing in the formulations of strong shift equivalence and shift equivalence between nonnegative matrix systems in terms of certain single homomorphisms between inductive limits of associated abelian groups. For a nonnegative matrix system (A, I), the transpose $I^t_{l,l+1}$ of the matrix $I_{l,l+1}$ naturally induces an ordered homomorphism from $\mathbb{Z}^{m(l)}$ to $\mathbb{Z}^{m(l+1)}$, where the positive cone $\mathbb{Z}^{m(l)}_+$ of the group $\mathbb{Z}^{m(l)}$ is defined by

$$\mathbb{Z}^{m(l)}_+ = \{(n_1, n_2, \ldots, n_{m(l)}) \in \mathbb{Z}^{m(l)} | n_i \geq 0, i = 1, 2 \ldots m(l)\}.$$

We put the inductive limits:

$$\mathbb{Z}_{I^t} = \varinjlim_{l} \{I^t_{l,l+1} : \mathbb{Z}^{m(l)} \to \mathbb{Z}^{m(l+1)}\},$$
$$\mathbb{Z}^+_{I^t} = \varinjlim_{l} \{I^t_{l,l+1} : \mathbb{Z}^{m(l)}_+ \to \mathbb{Z}^{m(l+1)}_+\}.$$

The canonical homomorphism $\iota_l : \mathbb{Z}^{m(l)} \to \mathbb{Z}_{I^t}$ is injective. By the relation: $I_{l,l+1}A_{l+1,l+2} = A_{l,l+1}I_{l+1,l+2}$, the sequence of the transposed matrices $A^t_{l,l+1}, l \in \mathbb{N}$ of the matrices $A_{l,l+1}, l \in \mathbb{N}$ yields an endomorphism of the ordered group $\mathbb{Z}_{I^t}$. We write it as $\lambda_{(A,I)}$. For nonnegative matrix systems $(A, I), (A', I')$ and $L \in \mathbb{N}$, a homomorphism ξ from the group $\mathbb{Z}_{I^t}$ to the group $\mathbb{Z}_{I'^t}$ is said to be *finite homomorphism of lag L* if it satisfies the condition

$$\xi(\mathbb{Z}^{m(l)}) \subset \mathbb{Z}^{m'(l+L)} \qquad \text{for all } l \in \mathbb{N}$$

where $\mathbb{Z}^{m(l)}$ and $\mathbb{Z}^{m'(l)}$ are naturally imbedded into $\mathbb{Z}_{I^t}$ and $\mathbb{Z}_{I'^t}$ respectively.

Lemma 4.3. *Two nonnegative matrix systems (A, I) and (A', I') are shift equivalent of lag N if and only if there exist order preserving finite homomorphisms of lag N: $\xi : \mathbb{Z}_{I^t} \to \mathbb{Z}_{I'^t}$ and $\eta : \mathbb{Z}_{I'^t} \to \mathbb{Z}_{I^t}$ such that*

$$\lambda_{(A',I')} \circ \xi = \xi \circ \lambda_{(A,I)}, \qquad \lambda_{(A,I)} \circ \eta = \eta \circ \lambda_{(A',I')}$$

and

$$\eta \circ \xi = \lambda^N_{(A,I)}, \qquad \xi \circ \eta = \lambda^N_{(A',I')}.$$

In particular, (A,I) *and* (A',I') *are strong shift equivalent in 1-step if and only if there exist order preserving finite homomorphisms of lag* 1*:* $\xi : \mathbb{Z}_{I^t} \to \mathbb{Z}_{I'^t}$ *and* $\eta : \mathbb{Z}_{I'^t} \to \mathbb{Z}_{I^t}$ *such that*

$$\eta \circ \xi = \lambda_{(A,I)}, \qquad \xi \circ \eta = \lambda_{(A',I')}.$$

For nonnegative matrices, W. Krieger in [Kr2],[Kr3] showed that shift equivalence relation is the complete relation that defines the same dimension triples. We next formulate dimension groups for nonnegative matrix systems. Let (A,I) be a nonnegative matrix system. We set $\mathbb{Z}_{I^t}(k) = \mathbb{Z}_{I^t}$ and $\mathbb{Z}^+_{I^t}(k) = \mathbb{Z}^+_{I^t}$ for $k \in \mathbb{N}$. We define an abelian group and its positive cone by the following inductive limits:

$$\Delta_{(A,I)} = \varinjlim_k \{\lambda_{(A,I)} : \mathbb{Z}_{I^t}(k) \to \mathbb{Z}_{I^t}(k+1)\},$$
$$\Delta^+_{(A,I)} = \varinjlim_k \{\lambda_{(A,I)} : \mathbb{Z}^+_{I^t}(k) \to \mathbb{Z}^+_{I^t}(k+1)\}.$$

The ordered group $(\Delta_{(A,I)}, \Delta^+_{(A,I)})$ is called the *dimension group for* (A,I). The map $\delta_{(A,I)} : \mathbb{Z}_{I^t}(k) \to \mathbb{Z}_{I^t}(k+1)$ defined by $\delta_{(A,I)}([X,k]) = ([X,k+1])$ for $X \in \mathbb{Z}_{I^t}$ yields an automorphism on $\Delta_{(A,I)}$ that preserves the positive cone $\Delta^+_{(A,I)}$. We also denote it by $\delta_{(A,I)}$ and call it the *dimension automorphism*. We call the triple $(\Delta_{(A,I)}, \Delta^+_{(A,I)}, \delta_{(A,I)})$ the *dimension triple for* (A,I) and the pair $(\Delta_{(A,I)}, \delta_{(A,I)})$ the *dimension pair for* (A,I).

Proposition 4.4. *If two nonnegative matrix systems are shift equivalent, their dimension triples are isomorphic.*

§5 K-groups, Bowen-Franks groups and flow equivalence

Let (A,I) be a nonnegative matrix system. For $l \in \mathbb{N}$, we set the abelian groups

$$K^l_0(A,I) = \mathbb{Z}^{m(l+1)}/(I^t_{l,l+1} - A^t_{l,l+1})\mathbb{Z}^{m(l)},$$
$$K^l_1(A,I) = \mathrm{Ker}(I^t_{l,l+1} - A^t_{l,l+1}) \text{ in } \mathbb{Z}^{m(l)}.$$

Then the map $I^t_{l,l+1} : \mathbb{Z}^{m(l)} \to \mathbb{Z}^{m(l+1)}$ naturally induces homomorphisms between the groups:

$$i^l_* : K^l_*(A,I) \to K^{l+1}_*(A,I) \qquad \text{for} \quad * = 0, 1.$$

Definition. The K-groups for (A, I) are defined as the following inductive limits of the abelian groups:

$$K_0(A, I) = \varinjlim_{l}\{i_0^l : K_0^l(A, I) \to K_0^{l+1}(A, I)\},$$
$$K_1(A, I) = \varinjlim_{l}\{i_1^l : K_1^l(A, I) \to K_1^{l+1}(A, I)\}.$$

The groups $K_*(A, I)$ are also represented as in the following way

Proposition 5.1.
(i) $K_0(A, I) = \mathbb{Z}_{I^t}/(\mathrm{id} - \lambda_{(A,I)})\mathbb{Z}_{I^t}$,
(ii) $K_1(A, I) = \mathrm{Ker}(\mathrm{id} - \lambda_{(A,I)})$ *in* $\mathbb{Z}_{I^t}$.

As the automorphism $\delta_{(A,I)}$ is given by $\lambda_{(A,I)} = \{A_{l,l+1}^t\}$ on $\Delta_{(A,I)}$, we have

Proposition 5.2.
(i) $K_0(A, I) = \Delta_{(A,I)}/(\mathrm{id} - \delta_{(A,I)})\Delta_{(A,I)}$,
(ii) $K_1(A, I) = \mathrm{Ker}(\mathrm{id} - \delta_{(A,I)})$ *in* $\Delta_{(A,I)}$.

Set the abelian group

$$\mathbb{Z}_I = \varprojlim_{l}\{I_{l,l+1} : \mathbb{Z}^{m(l+1)} \to \mathbb{Z}^{m(l)}\}$$

the projective limit of the system: $I_{l,l+1} : \mathbb{Z}^{m(l+1)} \to \mathbb{Z}^{m(l)}, l \in \mathbb{N}$. The sequence $A_{l,l+1}, l \in \mathbb{N}$ naturally acts on $\mathbb{Z}_I$ as an endomorphism that we denote by A. The identity on $\mathbb{Z}_I$ is denoted by I.
Definition. For a nonnegative matrix system (A, I),

$$BF^0(A, I) = \mathbb{Z}_I/(I - A)\mathbb{Z}_I, \qquad BF^1(A, I) = \mathrm{Ker}(I - A) \text{ in } \mathbb{Z}_I.$$

We call $BF^i(A, I), i = 0, 1$ the Bowen-Franks groups for (A, I).

Theorem 5.3. *The K-groups and the Bowen-Franks groups are invariant under shift equivalence of nonnegative matrix systems. Hence the K-groups and the Bowen-Franks groups of the canonical symbolic matrix systems for subshifts are invariant under topological conjugacy of the subshifts.*

The following formulation of the universal coefficient type theorem comes from the Universal Coefficient Theorem for K-theory of C^*-algebras (cf.[Bro],[Rs]). It says that the Bowen-Franks groups are determined by the K-groups.

Theorem 5.4 ([Ma6]).

(i) *There exists a short exact sequence*

$$0 \longrightarrow \mathrm{Ext}^1_{\mathbb{Z}}(K_0(A,I),\mathbb{Z}) \xrightarrow{\delta} BF^0(A,I) \xrightarrow{\gamma} \mathrm{Hom}_{\mathbb{Z}}(K_1(A,I),\mathbb{Z}) \longrightarrow 0$$

that splits unnaturally.

(ii)

$$BF^1(A,I) \cong \mathrm{Hom}_{\mathbb{Z}}(K_0(A,I),\mathbb{Z}).$$

Parry-Sullivan showed that the flow equivalence relation on homeomorphisms of Cantor sets is generated by topological conjugacies and operations called expansions ([PS]). For the case of topological Markov shifts, they also gave a description of the expansions in terms of a matrix operation. By using their result, Bowen-Franks in [BF] proved that for an $n \times n$ nonnegative matrix A the groups $\mathbb{Z}^n/(1-A)\mathbb{Z}^n$ and $\mathrm{Ker}(1-A)$ are invariant under flow equivalence of the topological Markov shifts Λ_A for the matrix A. We can generalize the Parry-Sullivan's argument and the Bowen-Franks's proof to the canonical nonnegative matrix systems for subshifts.

Theorem 5.5 ([Ma8]). *The K-groups and Bowen-Franks groups of the canonical nonnegative matrix systems for subshifts are invariant under flow equivalence of the subshifts.*

This result first has been shown by using a C^*-algebra technique under some conditions on subshifts in [Ma3] (cf.[Ma7]). For the case of topological Markov shifts, Cuntz-Krieger [CK] and Cuntz [C3] had discussed their flow equivalence by C^*-algebra approach and obtained the corresponding result to the above theorem (cf. [H], [H2]).

§6 Spectrum

We fix a nonnegative matrix system (A,I). A sequence $\{v^l\}_{l\in\mathbb{N}}$ of vectors $v^l = (v^l_1,\dots,v^l_{m(l)}) \in \mathbb{C}^{m(l)}, l \in \mathbb{N}$ is called an I-compatible vector if it satisfies the conditions:

$$v^l = I_{l,l+1}v^{l+1} \qquad \text{for all} \quad l \in \mathbb{N}.$$

An I-compatible vector $\{v^l\}_{l\in\mathbb{N}}$ is said to be nonzero if v^l is a nonzero vector for some l. If $v^l_i \geq 0$ for all $i = 1,\dots,m(l)$ and $l \in \mathbb{N}$, the sequence $\{v^l\}_{l\in\mathbb{N}}$ of vectors is said to be nonnegative. If there exists a number M such that $\sum_{i=1}^{m(l)} |v^l_i| \leq M$ for all $l \in \mathbb{N}$, $\{v^l\}_{l\in\mathbb{N}}$ is said to be bounded.

Definition. For a complex number β, a nonzero I-compatible vector $\{v^l\}$ is called an *eigenvector* of (A, I) for *eigenvalue* β if it satisfies the conditions:

$$A_{l,l+1}v^{l+1} = \beta v^l \qquad \text{for all} \quad l \in \mathbb{N}.$$

An eigenvalue β is said to be bounded if it is an eigenvalue for a bounded eigenvector. Let $Sp^{\times}(A, I)$ be the set of all nonzero eigenvalues of (A, I) and $Sp_b^{\times}(A, I)$ the set of all nonzero bounded eigenvalues of (A, I). We call them the *nonzero spectrum* of (A, I) and the *nonzero bounded spectrum* of (A, I) respectively.

Proposition 6.1. *The nonzero spectrum and the nonzero bounded spectrum are invariant under shift equivalence of nonnegative matrix systems.*

We denote by $\mathfrak{B}_I$ the set of all bounded I-compatible vectors. It is a complex Banach space with norm $\|\cdot\|_1$ where $\|v\|_1 = \sup_l \Sigma_{i=1}^{m(l)}|v_i^l|$ for $v = \{v^l\}_{l\in\mathbb{N}}, v^l = (v_i^l)_{i=1,\cdots,m(l)}$. The sequence $A_{l,l+1}, l \in \mathbb{N}$ of matrices gives rise to a bounded linear operator on the Banach space $\mathfrak{B}_I$. We denote it by L_A.

Proposition 6.2. *A complex number β belongs to $Sp_b(A, I)$ if and only if it satisfies $L_A v = \beta v$ for some nonzero $v \in \mathfrak{B}_I$. In particular, the spectral radius of the operator L_A on $\mathfrak{B}_I$ belongs to $Sp_b^{\times}(A, I)$.*

We say a symbolic matrix system $(\mathcal{M}, I)$ to be left resolving if a symbol appearing in $\mathcal{M}_{l,l+1}(i,j)$ can not appear in $\mathcal{M}_{l,l+1}(i',j)$ for other $i' \neq i$. A canonical symbolic matrix system is left resolving. The following proposition states a relation between spectrum and the topological entropy of subshift ([MWY], cf.[EFW]).

Proposition 6.3. *Let $(\mathcal{M}, I)$ be a left resolving symbolic matrix system and (M, I) its associated nonnegative matrix system. For any $\beta \in Sp_b(M, I)$, we have the inequalities:*

$$\log|\beta| \le \log r_M \le h_{\text{top}}(\Lambda_{(\mathcal{M},I)})$$

where r_M is the spectral radius of the operator L_M on $\mathfrak{B}_I$ and $\Lambda_{(\mathcal{M},I)}$ is the associated subshift with $(\mathcal{M}, I)$.

§7 Examples

Let M be an $n \times n$ nonnegative matrix . Put for each $l \in \mathbb{N}$

$$A_{l,l+1} = M, \qquad I_{l,l+1} = \text{ the } n \times n \text{ identity matrix.}$$

Then (A, I) is a nonnegative matrix system. We know

$$\begin{aligned} K_0(A,I) &= \mathbb{Z}^n/(1-M^t)\mathbb{Z}^n, & K_1(A,I) &= \operatorname{Ker}(1-M^t) \text{ in } \mathbb{Z}^n, \\ BF^0(A,I) &= \mathbb{Z}^n/(1-M)\mathbb{Z}^n, & BF^1(A,I) &= \operatorname{Ker}(1-M) \text{ in } \mathbb{Z}^n. \end{aligned}$$

Hence we have

$$\begin{aligned} K_0(A,I) \cong & BF^0(A,I) \\ & = BF(M) : \text{the original Bowen-Franks group for } M, \\ K_1(A,I) \cong & BF^1(A,I) = \text{ the torsion-free part of } BF(M). \end{aligned}$$

Note that for a general nonnegative matrix systems (A, I), $BF^1(A, I)$ is not necessarily the torsion-free part of $BF^0(A, I)$ as in the following examples.

We will next present examples of the groups K_*, BF^* for canonical nonnegative matrix system of nonsofic subshifts (cf. [KMW]). Let Z be the subshift over $\{1, 2, 3\}$ whose forbidden words are $\{32^m1^k3 | m \neq k\}$ where the word 32^m1^k3 means $3\underbrace{2\cdots2}_{m \text{ times}}\underbrace{1\cdots1}_{k \text{ times}}3$. Let D be the Dyck shift over brackets $(,),[,]$ whose forbidden words consist of words that do not obey the standard bracket rules (cf. [AU], [Kr]). We denote by $(A^Z, I^Z), (A^D, I^D)$ and $(A^{D\times[n]}, I^{D\times[n]})$ the canonical nonnegative matrix systems of the subshifts Z, D and the product subshift between D and the full n-shift $[n]$ respectively.

Proposition 7.1 ([Ma5], [Ma9]).

(i)

$$\begin{aligned} K_0(A^Z, I^Z) &= BF^1(A^Z, I^Z) = \mathbb{Z}, \\ K_1(A^Z, I^Z) &= BF^0(A^Z, I^Z) = 0. \end{aligned}$$

(ii)

$$K_0(A^D, I^D) = \sum^{\infty} \mathbb{Z}, \qquad BF^1(A^D, I^D) = \prod^{\infty} \mathbb{Z},$$
$$K_1(A^D, I^D) = BF^0(A^D, I^D) = 0.$$

(iii)

$$
\begin{aligned}
K_0(A^{D\times[n]}, I^{D\times[n]}) &\cong \mathbb{Z}[\frac{1}{n}]^\infty, \quad K_1(A^{D\times[n]}, I^{D\times[n]}) \cong 0,\\
BF^0(A^{D\times[n]}, I^{D\times[n]}) &\cong \prod_{\mathbb{N}} (\varprojlim_i \mathbb{Z}/n^i\mathbb{Z})/\mathbb{Z}\\
&\cong \prod_{\mathbb{N}} \frac{n-\textit{adic infinite polynomials}}{n-\textit{adic finite polynomials}},\\
BF^1(A^{D\times[n]}, I^{D\times[n]}) &\cong 0
\end{aligned}
$$

where $\prod_{\mathbb{N}}(\varprojlim_i \mathbb{Z}/n^i\mathbb{Z})/\mathbb{Z}$ is the countable infinite product of the quotient group by $\mathbb{Z}$ of the natural projective limit: $\mathbb{Z}/n\mathbb{Z} \leftarrow \mathbb{Z}/n^2\mathbb{Z} \leftarrow \cdots$.

Corollary 7.2.

(i) *The subshift Z is not flow equivalent to any of the product subshifts $D \times [n], n = 1, 2, \cdots$.*

(ii) *$D \times [n]$ is not flow equivalent to $D \times [m]$ for $n \neq m$.*

§8 Connection to C^*-algebra K-theory

The author in [Ma] (cf.[CaM],[Ma10]) has constructed the C^*-algebra $\mathcal{O}_\Lambda$ associated with subshift Λ as a generalization of the Cuntz-Krieger algebra $\mathcal{O}_A$ associated with topological Markov shift Λ_A for matrix A with entries in $\{0,1\}$ ([CK]). The C^*-algebra $\mathcal{O}_\Lambda$ has a canonical action of the one dimensional torus group, called gauge action and written as α_Λ. The fixed point algebra $\mathcal{F}_\Lambda$ of $\mathcal{O}_\Lambda$ under α_Λ is an AF-algebra which is stably isomorphic to the crossed product $\mathcal{O}_\Lambda \times_{\alpha_\Lambda} \mathbb{T}$ ([Ma2]).

Proposition 8.1 ([Ma7], [Ma12], [CK]). *If two subshifts Λ and Λ' are topologically conjugate, we have*

$$(\mathcal{O}_\Lambda \otimes \mathcal{K}, \alpha_\Lambda \otimes \mathrm{id}) \cong (\mathcal{O}_{\Lambda'} \otimes \mathcal{K}, \alpha_{\Lambda'} \otimes \mathrm{id})$$

where $\mathcal{K}$ is the C^*-algebra of all compact operators on separable infinite dimensional Hilbert space.

Let (M, I) be the canonical nonnegative matrix system for the subshift Λ. The invariants mentioned above are described in terms of the K-theoretic objects for the C^*-algebras as in the following way, where if Λ is a topological Markov shift Λ_A the corresponding results have been seen in [C3], [CK].

Theorem 8.2.

$$(\Delta_{(M,I)}, \Delta^+_{(M,I)}, \delta_{(M,I)}) = (K_0(\mathcal{F}_\Lambda), K_0(\mathcal{F}_\Lambda)_+, \hat{\alpha_\Lambda}_*),$$
$$K_i(M,I) = K_i(\mathcal{O}_\Lambda), \qquad i = 0,1,$$
$$BF^i(M,I) = \mathrm{Ext}^{i+1}(\mathcal{O}_\Lambda), \qquad i = 0,1$$

where $\hat{\alpha_\Lambda}$ *denotes the dual action of* α_Λ *and* $\mathrm{Ext}^1(\mathcal{O}_\Lambda) = \mathrm{Ext}(\mathcal{O}_\Lambda)$, $\mathrm{Ext}^0(\mathcal{O}_\Lambda) = \mathrm{Ext}(\mathcal{O}_\Lambda \otimes C_0(\mathbb{R}))$.

The normalized nonnegative eigenvectors of (M,I) exactly correspond to the KMS-states for α_Λ on the C^*-algebra $\mathcal{O}_\Lambda$. Hence the set of all bounded spectrums with nonnegative eigenvectors are the set of all inverse temperatures for the admitted KMS states ([MWY],cf.[EFW]).

As the K-groups and the Ext-groups for C^*-algebras are stably isomorphic invariant, it is possible to know that the dimension triples, the K-groups and the Bowen-Franks groups for the canonical nonnegative matrix systems for subshifts are topological conjugacy invariants of the subshifts by using Proposition 8.1 and Theorem 8.2 under some conditions on subshifts.

In [Ma10], as a generalization of the C^*-algebras associated with subshifts, construction of C^*-algebras from symbolic matrix systems are introduced.

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Relative positions of four subspaces in a Hilbert space and subfactors

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Abstract.

We study relative positions of four subspaces in a Hilbert space. Gelfand-Ponomarev gave a complete classification of indecomposable systems of four subspaces in a finite-dimensional space. In this note we show that there exist uncountably many indecomposable systems of four subspaces in an infinite-dimesional Hilbert space. We extend a numerical invariant, called defect, for a certain class of systems of four subspaces using Fredholm index. We show that the set of possible values of the defect is $\{\frac{n}{3};\ n \in \mathbf{Z}\}$.

§1. Introduction

This is an announcement of the joint work [EW] with M. Enomoto.

The relative position of one subfactor in a factor has been proved quite rich after the work [J] of Jones. But the relative position of one subspace of a Hilbert space is extremely poor and simply determined by its dimension and co-dimension. The aim of the paper is to cover up the poorness by considering the relative position of several subspaces.

It is a well-known fact that the relative position of two subspaces E and F in a Hilbert space H can be described completely up to unitary equivalence as in Dixmier [D] and Halmos [H]. The Hilbert space is the direct sum of five subspaces:

$$H = (E \cap F) \oplus (\text{the rest}) \oplus (E \cap F^{\perp}) \oplus (E^{\perp} \cap F) \oplus (E^{\perp} \cap F^{\perp}).$$

In the "rest part", E and F are in generic position and the relative position is described only by the "angles" between them. In fact, let e and f be the projections onto E and F respectively. Then e and f look like

$$e = I_{(e\wedge f)H} \oplus \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \oplus I_{(e\wedge f^{\perp})H} \oplus 0 \oplus 0,$$

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$$f = I_{(e\wedge f)H} \oplus \begin{pmatrix} c^2 & cs \\ cs & s^2 \end{pmatrix} \oplus 0 \oplus I_{(e^{\perp}\wedge f)H} \oplus 0,$$

where c and s are two positive operators with null kernels and $c^2+s^2=1$. By the functional calculus, there exists a unique positive operator θ, called the angle operator, such that $c = \cos\theta$ and $s = \sin\theta$ with $0 \leq \theta \leq \frac{\pi}{2}$.

Consider two self-adjoint unitaries $u = 2e-1$ and $v = 2f-1$. It is obvious that there is a bijective correspondence between the set of two subspaces in a Hilbert space H and the set of unitary representations π of the free product $G_2 = \mathbf{Z}/2\mathbf{Z} * \mathbf{Z}/2\mathbf{Z} = \langle a, b\rangle$ of the cyclic groups of order two on H through $\pi(a) = u$ and $\pi(b) = v$. Similarly there is a bijective correspondence between the set of n subspaces in a Hilbert space H and the set of unitary representations of the free product $G_n = \mathbf{Z}/2\mathbf{Z} * \cdots * \mathbf{Z}/2\mathbf{Z}$ (n-times) of the cyclic groups of order two. It is well-known that for $n \geq 3$ the group G_n is non-type I and non-amenable. Therefore it seems brave and stupid to study the relative positions of n subspaces for $n \geq 3$ up to unitary equivalence. To avoid the difficulty, we forget the angles and consider a weaker equivalent relation for the systems of n-subspaces as topological vector spaces.

We say that two systems $\mathcal{S} = (H; E_1, \cdots, E_n)$ and $\mathcal{S}' = (H'; E_1', \cdots, E_n')$ of n subspaces in Hilbert spaces H and H' are similar if their exists a bounded invertible operator $T : H \to H'$ satisfying $TE_i = E_i'$ for $i = 1, \cdots, n$.

We should study an *indecomposable* system $\mathcal{S} = (H; E_1, \cdots, E_n)$ of n-subspaces in the sense that the system $\mathcal{S}$ can not be similar to a direct sum of two non-zero systems. Consider the case that the Hilbert space H is finite-dimensional. Then we have four indecomposable systems of two subspaces. We have nine indecomposable systems of three subspaces. They are trivial ones, that is, H is one dimensional, except one system. But, in the old paper [GP], Gelfand and Ponomarev showed that there exist infinitely many indecomposable systems of four subspaces with higher finite dimensions and surprisingly they completely classified them.

We shall show that there exist infinitely many indecomposable systems of four subspaces in an infinite-dimensional Hilbert space H.

The most important numerical invariant of a subfactor $N \subset M$ is the Jones index $[M : N]$ introduced in [J]. Similarly the most important numerical invariant of a system $\mathcal{S} = (H; E_1, E_2, E_3, E_4)$ of four subspaces in a finite-dimensional Hilbert space H is the defect

$$\rho(\mathcal{S}) = \sum_{i=1}^{4} dim\ E_i - 2dim\ H$$

introduced by Gelfand-Ponomarev in [GP]. We shall extend their notion of defect $\rho(\mathcal{S})$ for a certain class of systems $\mathcal{S}$ of four subspaces in an infinite-dimensional Hilbert space H using Fredholm index. If a pair $N \subset M$ of factor-subfactor is finite-dimensional, then Jones index $[M : N]$ is an integer. But if $N \subset M$ is infinite-dimensional, then Jones index $[M : N]$ is a non-integer in general. One of the amazing facts was that the possible value of Jones index is in $\{4cos^2\frac{\pi}{n} \mid n = 3, 4, \cdots\} \cup [4, \infty]$. Similarly if a system $\mathcal{S} = (H; E_1, E_2, E_3, E_4)$ of four subspaces is finite-dimensional, then the defect $\rho(\mathcal{S})$ is an integer. Gelfand-Ponpmarev showed that if a system $\mathcal{S}$ is indecomposable and finite-dimensional, then the possible value of defect $\rho(\mathcal{S})$ is exactly in $\{-2, -1, 0, 1, 2\}$. We show that the set of values of defect for indecomposable systems of four subspaces in an infinite-dimesional Hilbert space is $\{\frac{n}{3}; n \in \mathbf{Z}\}$.

Sunder also considered n subspaces in [S]. But his interest is opposite to ours. In fact he studied the case that the Hilbert space H is the algebraic sum of the n subspaces and solved the statistical problem of computing the canonical partial correlation coefficients between three sets of random variables.

§2. Systems of n subspaces

Our purpose is to study relative positions of n subspaces in a Hilbert space. Let H be a (separable) Hilbert space and $E_1, \cdots, E_n$ be a finite family of subspaces of H. We shall write $\mathcal{S} = (H; E_1, \cdots, E_n)$ for such a system of n subspaces. Let $\mathcal{S} = (H; E_1, \cdots, E_n)$ and $\mathcal{S}' = (H'; E_1', \cdots, E_n')$ be systems of n subspaces. We say that $\mathcal{S}$ and $\mathcal{S}'$ are *similar*, denoted by $\mathcal{S} \sim \mathcal{S}'$, if there exists a bounded linear operator $T : H \to H'$ such that $E_i' = TE_i$ for $i = 1, \cdots, n$. We say that $\mathcal{S}$ and $\mathcal{S}'$ are *unitary equivalent* if there exists a unitary operator $u : H \to H'$ such that $E_i' = uE_i$ for $i = 1, \cdots, n$. We study relative positions of n subspaces up to similarity to ignore angles between subspaces in a certain sense. We denote by $\mathcal{S} \oplus \mathcal{S}'$ the direct sum $(H \oplus H'; E_1 \oplus E_1', \cdots, E_n \oplus E_n')$ of two systems $\mathcal{S}$ and $\mathcal{S}'$. We write $\mathcal{S} = 0$ if $H = 0$.

Lemma 1. *Let H be a Hilbert space and $\mathcal{S} = (H; E_1, E_2)$ a system of two subspaces. Then the following are equivalent:*

1. *There exists a closed subspace $M \subset H$ such that*

$$(H; E_1, E_2) \sim (H; M, M^\perp)$$

2. $H = E_1 + E_2$ *and* $E_1 \cap E_2 = 0$.

Definition. Let $\mathcal{S} = (H; E_1, \cdots, E_n)$ be a system of n subspaces in a Hilbert space H. We say that $\mathcal{S}$ is *decomposable* if there exists non-zero systems $\mathcal{S}_1$ and $\mathcal{S}_2$ of n subspaces such that $\mathcal{S} \sim \mathcal{S}_1 \oplus \mathcal{S}_2$. It is useful to note that $\mathcal{S}$ is decomposable if and only if there exist non-zero closed subspaces H_1 and H_2 of H such that $H_1 + H_2 = H$, $H_1 \cap H_2 = 0$ and $E_i = E_i \cap H_1 + E_i \cap H_2$ for $i = 1, \cdots, n$. We say $\mathcal{S}$ is *indecomposable* if $\mathcal{S}$ is not decomposable.

Example 1. Let $H = \mathbf{C}^2$. Fix an angle θ with $0 < \theta < \pi/2$. Put $E_1 = \mathbf{C}(1,0)$ and $E_2 = \mathbf{C}(cos\theta, sin\theta)$. Then

$$(H; E_1, E_2) \sim (\mathbf{C}; \mathbf{C}, 0) \oplus (\mathbf{C}; 0, \mathbf{C})$$

Hence $(H; E_1, E_2)$ is decomposable. Let e_1 and e_2 be the projections onto E_1 and E_2. Then the C^*-algebra $C^*(\{e_1, e_2\})$ generated by e_1 and e_2 is exactly $B(H) \cong M_2(\mathbf{C})$. Thus the irreducibility of $C^*(\{e_1, e_2\})$ does not imply the indecomposability of $\mathcal{S} = (H; E_1, E_2)$.

Remark. Let $\mathcal{S} = (H; E_1, \cdots, E_n)$ be a system of n subspaces in a Hilbert space H. Let e_i be the projection of H onto E_i for $i = 1, \cdots, n$. If $\mathcal{S} = (H; E_1, \cdots, E_n)$ is indecomposable, then the $C^*(\{e_1, \cdots, e_n\})$ generated by $e_1, \cdots, e_n$ is irreducible. But the converse is not true as in Example 1.

Example 2. Let $H = \mathbf{C}^2$. Put $E_1 = \mathbf{C}(1,0)$, $E_2 = \mathbf{C}(0,1)$ and $E_3 = \mathbf{C}(1,1)$. Then $\mathcal{S} = (H; E_1, E_2, E_3)$ is indecomposable.

Example 3. Let $H = \mathbf{C}^3$ and $\{a_1, a_2, a_3\}$ be a linearly independent subset of H. Put $E_1 = \mathbf{C}a_1$, $E_2 = \mathbf{C}a_2$ and $E_3 = \mathbf{C}a_3$. Then $\mathcal{S} = (H; E_1, E_2, E_3)$ is decomposable. In fact, let $H_1 = E_1 \vee E_2 \neq 0$ and $H_2 = E_3 \neq 0$. Then $H_1 + H_2 = H$, $H_1 \cap H_2 = 0$ and $E_i = E_i \cap H_1 + E_i \cap H_2$, for $i = 1, 2, 3$.

Example 4. Let $H = \mathbf{C}^3$ and $\{b_1, b_2, b_3, b_4\}$ be a subset of H. Put $E_i = \mathbf{C}b_i$ for $i = 1, \cdots, 4$. Consider a system $\mathcal{S} = (H; E_1, E_2, E_3, E_4)$ of four subspaces. Then the following are equivalent:

1. $\mathcal{S}$ is indecomposable.
2. Any three vectors of $\{b_1, b_2, b_3, b_4\}$ is linearly independent.
3. The set $\{b_1, b_2, b_3\}$ is linearly independent and $b_4 = \lambda_1 b_1 + \lambda_2 b_2 + \lambda_3 b_3$ for some scalars $\lambda_i \neq 0$ $(i = 1, 2, 3)$.

Assume that $\{u_1, u_2, u_3, u_4\} \subset H$ and $\{v_1, v_2, v_3, v_4\} \subset H$ satisfy the above condition (2). Then $\mathcal{S} = (H; \mathbf{C}u_1, \mathbf{C}u_2, \mathbf{C}u_3, \mathbf{C}u_4)$ and $\mathcal{T} = (H; \mathbf{C}v_1, \mathbf{C}v_2, \mathbf{C}v_3, \mathbf{C}v_4)$ are similar.

Example 5. Let $H = \mathbf{C}^3$. Put $E_1 = \mathbf{C} \oplus \mathbf{C} \oplus 0$, $E_2 = \mathbf{C}(1,1,1)$ and $E_3 = \mathbf{C}(1,2,3)$. Then a system $\mathcal{S} = (H; E_1, E_2, E_3)$ is decomposable. In fact, let $E_1' = (E_2 \vee E_3) \cap E_1$ and $H_1 = E_1 \cap (E_1')^{\perp} \neq 0$. Let

$H_2 = E_2 \vee E_3 \neq 0$. Then $H_1 + H_2 = H$, $H_1 \cap H_2 = H$ and $E_i = E_i \cap H_1 + E_i \cap H_2$, for $i = 1, 2, 3$.

Example 6. Let $H = \mathbf{C}^3$. Put $E_1 = \mathbf{C} \oplus \mathbf{C} \oplus 0$, $E_2 = \mathbf{C}(0,0,1)$, $E_3 = \mathbf{C}(0,1,1)$ and $E_4 = \mathbf{C}(1,0,1)$. Then a system $\mathcal{S}_1 = (H; E_1, E_2, E_3, E_4)$ of four subspaces is indecomposable.

Example 7. Let $H = \mathbf{C}^3$. Put $E_1 = \mathbf{C} \oplus \mathbf{C} \oplus 0$, $E_2 = \mathbf{C}(0,0,1)$, $E_3 = \mathbf{C}(1,0,0) \oplus \mathbf{C}(0,1,1)$ and $E_4 = \mathbf{C}(1,0,1)$. Then a system $\mathcal{S}_2 = (H; E_1, E_2, E_3, E_4)$ of four subspaces is indecomposable.

Example 8. Let $H = \mathbf{C}^3$. Put $E_1 = \mathbf{C} \oplus \mathbf{C} \oplus 0$, $E_2 = \mathbf{C}(0,0,1)$, $E_3 = \mathbf{C}(1,0,0) \oplus \mathbf{C}(0,1,1)$ and $E_4 = \mathbf{C}(1,0,1) \oplus \mathbf{C}(0,1,0)$. Then a system $\mathcal{S}_3 = (H; E_1, E_2, E_3, E_4)$ of four subspaces is indecomposable.

Example 9. Let $H = \mathbf{C}^3$. Put $E_1 = \mathbf{C}(1,0,0) \oplus \mathbf{C}(0,1,0)$, $E_2 = \mathbf{C}(0,1,0) \oplus \mathbf{C}(0,0,1)$, $E_3 = \mathbf{C}(1,0,0) \oplus \mathbf{C}(0,1,1)$ and $E_4 = \mathbf{C}(0,0,1) \oplus \mathbf{C}(1,1,0)$. Then a system $\mathcal{S}_4 = (H; E_1, E_2, E_3, E_4)$ of four subspaces is indecomposable.

Remark Any two of the above indecomposable systems $\mathcal{S}_1, \cdots, \mathcal{S}_4$ of four subspaces are not similar.

Example 10. Let $K = \ell^2(\mathbf{N})$ and $H = K \oplus K$. Consider a unilateral shift $S : K \to K$. Let $E_1 = K \oplus 0$, $E_2 = 0 \oplus K$, $E_3 = \{(x, Sx) \in H | x \in K\}$ and $E_4 = \{(x, x) \in H | x \in K\}$. Then a system $\mathcal{S}_4 = (H; E_1, E_2, E_3, E_4)$ of four subspaces in H is indecomposable.

Example 11.(Harrison-Radjavi-Rosental [HRR]) Let $K = \ell^2(\mathbf{Z})$ and $H = K \oplus K$. Consider a sequence $(\alpha_n)_n$ given by $\alpha_n = 1$ for $n \leq 0$ and $\alpha_n = exp((-1)^n n!)$ for $n > 1$. Consider a bilateral weighted shift $S : \mathcal{D}_T \to K$ such that $T(x_n)_n = (\alpha_{n-1} x_{n-1})_n$ with the domain $\mathcal{D}_T = \{(x_n)_n \in \ell^2(\mathbf{Z}) | \sum_n |\alpha_n x_n|^2 < \infty\}$. Let $E_1 = K \oplus 0$, $E_2 = 0 \oplus K$, $E_3 = \{(x, Tx) \in H | x \in \mathcal{D}_T\}$ and $E_4 = \{(x, x) \in H | x \in K\}$. Since $\{0, H, E_1, E_2, E_3, E_4\}$ is a transitive lattice, a system $\mathcal{S}_4 = (H; E_1, E_2, E_3, E_4)$ of four subspaces in H is indecomposable.

Definition. Let $\mathcal{S} = (H; E_1, \cdots, E_n)$ be a system of n subspaces in a Hilbert space H. Then the orthogonal complement of $\mathcal{S}$, denoted by $\mathcal{S}^\perp$, is defined by $\mathcal{S}^\perp = (H; E_1^\perp, \cdots, E_n^\perp)$.

Proposition 2. *Let H be a Hilbert space and $\mathcal{S} = (H; E_1, \cdots, E_n)$ a system of four subspaces in H. Then $\mathcal{S}$ is indecomposable if and only if $\mathcal{S}^\perp$ is indecomposable.*

§3. Classification of two subspaces

Let $\mathcal{S} = (H; E_1, \cdots, E_n)$ be a system of n subspaces in H. We say that $\mathcal{S}$ is *trivial* if *dim* $H = 1$.

Gelfand-Ponomarev [GP] claim that if H is finite-dimensional, then every indecomposable system of $\mathcal{S} = (H; E_1, E_2)$ of two subspaces is trivial and similar to one of the following four systems:

$$\mathcal{S}_1 = (\mathbf{C}; \mathbf{C}, 0), \quad \mathcal{S}_2 = (\mathbf{C}; 0, \mathbf{C}), \quad \mathcal{S}_3 = (\mathbf{C}; \mathbf{C}, \mathbf{C}), \quad \mathcal{S}_4 = (\mathbf{C}; 0, 0).$$

Any system of two subspaces is similar to a direct sum of a finite number of indecomposable systems above.

We consider the case that H is infinite-dimensional.

Proposition 3. *Let H be a separable infinite-dimesional Hilbert space and $\mathcal{S} = (H; E_1, E_2)$ a system of two subspaces in H. If $\mathcal{S}$ is indecomposable, then $\mathcal{S}$ is similar to one of the following four systems:*

$$\mathcal{S}_1 = (\mathbf{C}; \mathbf{C}, 0), \quad \mathcal{S}_2 = (\mathbf{C}; 0, \mathbf{C}), \quad \mathcal{S}_3 = (\mathbf{C}; \mathbf{C}, \mathbf{C}), \quad \mathcal{S}_4 = (\mathbf{C}; 0, 0).$$

§4. Classification of three subspaces

Gelfand-Ponomarev ([GP]) also claim that if H is finite-dimensional, then there exist nine different indecomposable system $\mathcal{S} = (H; E_1, E_2, E_3)$ of three subspaces in H. The eight of them are trivial and similar to one of the following systems:

$$\mathcal{S}_1 = (\mathbf{C}; 0, 0, 0), \quad \mathcal{S}_2 = (\mathbf{C}; \mathbf{C}, 0, 0), \quad \mathcal{S}_3 = (\mathbf{C}; 0, \mathbf{C}, 0),$$

$$\mathcal{S}_4 = (\mathbf{C}; 0, 0, \mathbf{C}), \quad \mathcal{S}_5 = (\mathbf{C}; \mathbf{C}, \mathbf{C}, 0), \quad \mathcal{S}_6 = (\mathbf{C}; \mathbf{C}, 0, \mathbf{C}),$$

$$\mathcal{S}_7 = (\mathbf{C}; 0, \mathbf{C}, \mathbf{C}), \quad \mathcal{S}_8 = (\mathbf{C}; \mathbf{C}, \mathbf{C}, \mathbf{C})$$

The only non-trivial indecomposable system of three subspaces is

$$\mathcal{S} = (\mathbf{C}^2; \mathbf{C}(1,0), \mathbf{C}(0,1), \mathbf{C}(1,1))$$

up to similarity.

§5. Classification of four subspaces

The classification of indecomposable systems $\mathcal{S} = (H; E_1, E_2, E_3, E_4)$ of four subspaces in a Hilbert space H is a central problem. If H is finite-dimensional, Gelfand-Ponomarev [GP] completely classified them and gave a complete list of their canonical forms. Their important numerical invariants are $dim\ H$ and the defect

$$\rho(\mathcal{S}) = \sum_{i=1}^{4} dim\ E_i - 2dim\ H.$$

Proposition 4 (Gelfand-Ponomarev [GP]). *If a system $\mathcal{S}$ of four subspaces in a finite-dimensional H is indecomposable, then a possible value of the defect $\rho(\mathcal{S})$ is exactly in the set* $\{-2,-1,0,1,2\}$.

The defect characterizes an essential feature of the system. If $\rho(\mathcal{S}) = 0$, then there exists a pair of linear operators $A : E \to F$ and $B : F \to E$ and the system $\mathcal{S} = (H; E_1, E_2, E_3, E_4)$ is described up to permutation by $H = E \oplus F$, $E_1 = E \oplus 0$, $E_2 = 0 \oplus F$, $E_3 = \{(x, Ax) \in H | x \in E\}$ and $E_4 = \{(By, y) \in H | y \in F\}$. If $\rho(\mathcal{S}) = \pm 1$, then $\mathcal{S}$ is given up to permutation by $H = E \oplus F$, $E_1 = E \oplus 0$, $E_2 = 0 \oplus F$, E_3 and E_4 are subspaces of H that do not reduced to the graphs of the operators as in the case that $\rho(\mathcal{S}) = 0$. A system with $\rho(\mathcal{S}) = \pm 2$ cannot be described in the above forms.

Following [GP], we write down the canonical forms of indecomposable systems $\mathcal{S} = (H; E_1, E_2, E_3, E_4)$ of four subspaces in an finite-dimensional space H up to permutation. We first consider the case when $dim\ H$ is even and $2k$ for some positive integer k. There exist no indecomposable systems $\mathcal{S}$ with $\rho(\mathcal{S}) = \pm 2$. Let H be a space with a basis $\{e_1, \cdots, e_k, f_1, \cdots, f_k\}$.

The system $\mathcal{S}_3(2k, -1) = (H; E_1, E_2, E_3, E_4)$ has the defect $\rho(\mathcal{S}) = -1$ and given by

$$E_1 = [e_1, \cdots, e_k],\ E_2 = [f_1, \cdots, f_k],$$

$$E_3 = [(e_2 + f_1), \cdots, (e_k + f_{k-1})],\ E_4 = [(e_1 + f_1), \cdots, (e_k + f_k)].$$

The system $\mathcal{S}_3(2k, 1) = (H; E_1, E_2, E_3, E_4)$ has the defect $\rho(\mathcal{S}) = 1$ and given by

$$E_1 = [e_1, \cdots, e_k],\ E_2 = [f_1, \cdots, f_k],$$

$$E_3 = [e_1, (e_2 + f_1), \cdots, (e_k + f_{k-1}), f_k],\ E_4 = [(e_1 + f_1), \cdots, (e_k + f_k)].$$

The system $\mathcal{S}_{1,3}(2k, 0) = (H; E_1, E_2, E_3, E_4)$ has the defect $\rho(\mathcal{S}) = 0$ and given by

$$E_1 = [e_1, \cdots, e_k],\ E_2 = [f_1, \cdots, f_k],$$

$$E_3 = [e_1, (e_2 + f_1), \cdots, (e_k + f_{k-1})],\ E_4 = [(e_1 + f_1), \cdots, (e_k + f_k)].$$

The system $\mathcal{S}(2k,0;\lambda) = (H;E_1,E_2,E_3,E_4)$ has the defect $\rho(\mathcal{S}) = 0$ and given by

$$\begin{aligned}
&E_1 = [e_1,\cdots,e_k],\ E_2 = [f_1,\cdots,f_k],\\
&E_3 = [(e_1+\lambda f_1),(e_2+f_1+\lambda f_2),\cdots,(e_k+f_{k-1}+\lambda f_k)],\\
&E_4 = [(e_1+f_1),\cdots,(e_k+f_k)].
\end{aligned}$$

Every other system $\mathcal{S}_i(2k,\rho)$, $\mathcal{S}_{i,j}(2k,0)$ can be obtained from the systems $\mathcal{S}_3(2k,\rho)$, $\mathcal{S}_{i,3}(2k,0)$ by a suitable permutation of the subspaces. Let σ be a permutation on the set $\{1,2,3,4\}$ and $\mathcal{S} = (H;E_1,E_2,E_3,E_4)$ a system of four subspaces. We define

$$\sigma\mathcal{S} = (H;E_{\sigma^{-1}(1)},E_{\sigma^{-1}(2)},E_{\sigma^{-1}(3)},E_{\sigma^{-1}(4)}).$$

Let $\sigma_{i,j}$ be the transposition (i,j). We put $\mathcal{S}_i(2k,\rho) = \sigma_{3,i}\mathcal{S}_3(2k,\rho)$ for $\rho = -1,1$. We also define $\mathcal{S}_{i,j}(2k,0) = \sigma_{1,i}\sigma_{3,j}\mathcal{S}_{1,3}(2k,0)$ for $i,j \in \{1,2,3,4\}$.

We next consider the case $dim\ H = 2k+1$, odd (for some positive integer k). Let H be a space with a basis $\{e_1,\cdots,e_k,e_{k+1},f_1,\cdots,f_k\}$.

The system $\mathcal{S}_1(2k+1,-1) = (H;E_1,E_2,E_3,E_4)$ has the defect $\rho(\mathcal{S}) = -1$ and given by

$$E_1 = [e_1,\cdots,e_k,e_{k+1}],\ E_2 = [f_1,\cdots,f_k],$$

$$E_3 = [(e_2+f_1),\cdots,(e_{k+1}+f_k)],\ E_4 = [(e_1+f_1),\cdots,(e_k+f_k)].$$

The system $\mathcal{S}_2(2k+1,1) = (H;E_1,E_2,E_3,E_4)$ has the defect $\rho(\mathcal{S}) = 1$ and given by

$$E_1 = [e_1,\cdots,e_k,e_{k+1}],\ E_2 = [f_1,\cdots,f_k],$$

$$E_3 = [e_1,(e_2+f_1),\cdots,(e_{k+1}+f_k)],\ E_4 = [(e_1+f_1),\cdots,(e_k+f_k),e_{k+1}].$$

The system $\mathcal{S}_{1,3}(2k+1,0) = (H;E_1,E_2,E_3,E_4)$ has the defect $\rho(\mathcal{S}) = 0$ and given by

$$E_1 = [e_1,\cdots,e_k,e_{k+1}],\ E_2 = [f_1,\cdots,f_k],$$

$$E_3 = [e_1,(e_2+f_1),\cdots,(e_{k+1}+f_k)],\ E_4 = [(e_1+f_1),\cdots,(e_k+f_k)].$$

The system $\mathcal{S}(2k+1,-2) = (H;E_1,E_2,E_3,E_4)$ has the defect $\rho(\mathcal{S}) = -2$ and given by

$$\begin{aligned}
&E_1 = [e_1,\cdots,e_k,e_{k+1}],\quad E_2 = [f_1,\cdots,f_k],\\
&E_3 = [(e_2+f_1),\cdots,(e_{k+1}+f_k)],\\
&E_4 = [(e_1+f_2),\cdots,(e_{k-1}+f_k),(e_k+e_{k+1})].
\end{aligned}$$

The system $\mathcal{S}(2k+1,2) = (H; E_1, E_2, E_3, E_4)$ has the defect $\rho(\mathcal{S}) = 2$ and given by

$$E_1 = [e_1, \cdots, e_k, e_{k+1}], \quad E_2 = [f_1, \cdots, f_k],$$
$$E_3 = [e_1, (e_2 + f_1), \cdots, (e_{k+1} + f_k)],$$
$$E_4 = [f_1, (e_1 + f_2), \cdots, (e_{k-1} + f_k), (e_k + e_{k+1})].$$

We put $\mathcal{S}_i(2k+1,-1) = \sigma_{1,i}\mathcal{S}_1(2k+1,-1)$, $\mathcal{S}_i(2k+1,+1) = \sigma_{2,i}\mathcal{S}_2(2k+1,1)$, $\mathcal{S}_{i,j}(2k+1,0) = \sigma_{1,i}\sigma_{3,j}\mathcal{S}_{1,3}(2k+1,0)$ for $i,j \in \{1,2,3,4\}$.

Theorem 5 (Gelfand-Ponomarev [GP]). *If a system $\mathcal{S}$ of four subspaces in a finite-dimensional H is indecomposable, then $\mathcal{S}$ is similar to one of the systems $\mathcal{S}_{i,j}(m,0)$, $(i<j, i,j \in \{1,2,3,4\}, m = 1,2,\cdots)$; $\mathcal{S}(2k,0,\lambda)$, $(\lambda \in \mathbf{C}, \lambda \neq 0, \lambda \neq 1, k = 1,2,\cdots)$; $\mathcal{S}_i(m,-1)$, $\mathcal{S}_i(m,1)$, $(i \in \{1,2,3,4\}, m = 1,2,\cdots)$; $\mathcal{S}(2k+1,-2)$, $\mathcal{S}(2k+1,+2), k = 0,1,\cdots)$*

We would like to investigate the case when H is infinite-dimensional. The complete classification is at present far from being solved. But we can show the existence of plenty of examples.

Theorem 6 ([EW]). *There exist uncountably many indecomposable systems $\mathcal{S} = (H; E_1, E_2, E_3, E_4)$ of four subspaces in an infinite-dimensional Hilbert space H.*

We shall extend the notion of the defect for a certain class of systems using Fredholm index.

Definition. Let $\mathcal{S} = (H; E_1, E_2, E_3, E_4)$ be a system of four subspaces in a Hilbert space H . For any $i \neq j \in \{1,2,3,4\}$, define a bounded linear operator $T_{ij} = E_i \oplus E_j \to H$ by $T_{ij}(x,y) = x+y$. If T_{ij} is a Fredholm operator, then $ind\ T_{ij} = dim\ (E_i \cap E_j) - dim\ (E_i + E_j)^{\perp}$. We say that $\mathcal{S}$ is a *Fredholm system* if T_{ij} is a Fredholm operator for any $i \neq j \in \{1,2,3,4\}$. We also say that $\mathcal{S}$ is a *weak Fredholm system* if $ker\ T_{ij}$ and $ker\ T_{ij}^*$ is finite-dimensional for any $i \neq j \in \{1,2,3,4\}$. It is clear that if $\mathcal{S}$ is a Fredholm system, then $\mathcal{S}$ is a weak Fredholm system. For any weak Fredholm system $\mathcal{S}$ we define the defect of $\mathcal{S}$, denoted by $\rho(\mathcal{S})$, by

$$\rho(\mathcal{S}) = \frac{1}{3} \sum_{1 \leq i < j \leq 4} Ind\ T_{i,j}.$$

The new definition of the defect agrees with the original one when H is finite-dimensional. In that case the value of the defect is an integer.

Proposition 7 ([EW]). *If $\mathcal{S}$ is a weak Fredholm system, then the orthogonal complement $\mathcal{S}^{\perp}$ is also a weak Fredholm system and $\rho(\mathcal{S}^{\perp}) = -\rho(\mathcal{S})$.*

Recall that one of the amazing fact in subfactor theory was that the possible value of the Jones index for a subfactor is in $\{4cos^2\frac{\pi}{n} \mid n = 3, 4, \cdots\} \cup [4, \infty]$. We shall determine the possible value of the defect for an indecomposable system $\mathcal{S}$ of four subspaces in an infinite-dimesional Hilbert space.

Theorem 8 ([EW]). *The set of possible values of the defect for indecomposable systems of four subspaces in an infinite-dimesional Hilbert space is* $\{\frac{n}{3}; n \in \mathbf{Z}\}$.

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